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The Theory of Discrete Fractional Calculus: Development and Application

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THE THEORY OF DISCRETE FRACTIONAL CALCULUS: DEVELOPMENT AND APPLICATION

by

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THE THEORY OF DISCRETE FRACTIONAL CALCULUS:

DEVELOPMENT AND APPLICATION

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The author's purpose in this dissertation is to introduce, develop and apply the tools of discrete fractional calculus to the arena of fractional difference equations. To this end, we develop the Fractional Composition Rules and the Fractional Laplace Transform Method to solve a linear, fractional initial value problem in Chapters 2 and 3. We then apply fixed point strategies of Krasnosel'skii and Banach to study a nonlinear, fractional boundary value problem in Chapter 4.

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Introduction

1.1 Discrete Fractional Calculus

Gottfried Leibniz and Guilliaume L'Hôpital sparked initial curiosity into the theory of fractional calculus during a 1695 correspondence on the possible value and meaning of noninteger-order derivatives. In one exchange, L'Hôpital inquired, "then what would be the one-half derivative of x?" to which Leibniz responded that the answer "leads to an apparent paradox, from which one day useful consequences will be drawn" (see [15] and [16]). Leibniz may well have toyed with several seemingly correct ways to define a one-half order derivative but was forced to cede they lead to unequivalent results. In any case, by the late nineteenth century, the combined efforts of a number of mathematicians—most notably Liouville, Grünwald, Letnikov and Riemann—produced a fairly solid theory of fractional calculus for functions of a real variable. Though several viable fractional derivatives were proposed, the so-called Riemann-Liouville and Caputo derivatives are the two most commonly used today. Mathematicians have employed this fractional calculus in recent years to model and solve a variety of applied problems. Indeed, as Podlubney outlines in [17], fractional calculus aids significantly in the fields of viscoelasticity, capacitor theory, electrical circuits, electro-analytical chemistry, neurology, diffusion, control theory and statistics.

The theory of fractional calculus for functions of the natural numbers, however, is far less developed. To the author's knowledge, significant work did not appear in this area until the mid-1950's, with the majority of interest shown within the past thirty years. Diaz and Osler published their 1974 paper [9] introducing a discrete fractional difference operator defined as an infinite series, a generalization of the binomial formula for the N^{th}-order difference operator Δ^N. However, their definition differs fundamentally from the one presented in this dissertation (they agree only for integer order differences). In 1988, Gray and Zhang [12] introduced the type of fractional difference operator used here; they developed Leibniz' formula, a limited composition rule and a version of a power rule for differentiation. However, they dealt exclusively with the nabla (backward) difference operator and therefore offer results distinct from those presented in this dissertation, where the delta (forward) difference operator is used exclusively.

A recent interest in discrete fractional calculus has been shown by Atici and Eloe, who in [2] discuss properties of the generalized falling function, a corresponding power rule for fractional delta-operators and the commutivity of fractional sums. They present in [3] more rules for composing fractional sums and differences but leave many important cases unresolved. Moreover, Atici and Eloe pay little attention in [2] and [3] to function domains or to lower limits of summation and differentiation, two details vital for a rigorous and correct treatment of the power rule and the fractional composition rules. Their neglect leads to domain confusion and, worse, to false or ambiguous claims.

The goal of this dissertation is to further develop the theory of fractional calculus on the natural numbers. In Chapter 2, we present a full and rigorous theory for composing fractional sum and difference operators, including correcting and broadening the power rule stated incorrectly in [2], [3] and [4]. We then apply these rules to solve a certain fractional initial value problem. In Chapter 3, we develop a Fractional Laplace Transform Method (paying close attention to correct domains of convergence) and apply this to resolve the fractional initial value problem introduced in the second chapter. In Chapter 4, we shift our attention to a certain fractional boundary value problem of arbitrary real order. Specifically, we take $N-1$ boundary conditions at the left endpoint and 1 boundary condition at the right endpoint and show the existence of a solution using two well-known fixed point theorems, those of Krasnosel'skii and Banach.

1.1.1 Whole-Order Sums

We consider throughout this dissertation real-valued functions defined on a shift of the natural numbers:

$$f : \mathbb{N}_a \to \mathbb{R}, \text{ where } \mathbb{N}_a := \{a\} + \mathbb{N}_0 = \{a, a+1, a+2, ...\} \qquad (a \in \mathbb{R} \text{ fixed}).$$

In the continuous setting, we know that n repeated definite integrals of a function f yields

$$\begin{aligned} y(t) &= \int_a^t \int_a^s \int_a^{\tau_1} \cdots \int_a^{\tau_{n-2}} f(\tau_{n-1})\,(d\tau_{n-1} \cdots d\tau_2 d\tau_1 ds) \\ &= \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds,\ t \in [a, \infty), \end{aligned} \tag{1.1}$$

the unique solution to the continuous n^{th}-order initial-value problem

$$\begin{cases} y^{(n)}(t) = f(t), & t \in [a, \infty) \\ y^{(i)}(a) = 0, & i = 0, 1, ..., n-1. \end{cases}$$

We call the kernel of integral (1.1) the Continuous Cauchy Function.

Likewise, n repeated definite sums of a discrete function f yields

$$\begin{aligned} y(t) &= \sum_{s=a}^{t-1} \sum_{\tau_1=a}^{s-1} \cdots \sum_{\tau_{n-1}=a}^{\tau_{n-2}-1} f(\tau_{n-1}) \\ &= \sum_{s=a}^{t-n} \frac{\prod_{j=0}^{n-1} (t-s-1-j)}{(n-1)!} f(s), \ \ t \in \mathbb{N}_a, \end{aligned} \tag{1.2}$$

the unique solution to the discrete n^{th}-order initial-value problem

$$\begin{cases} \Delta^n y(t) = f(t), & t \in \mathbb{N}_a \\ \Delta^i y(a) = 0, & i = 0, 1, ..., n-1. \end{cases}$$

In this case, the kernel of summation (1.2) is the Discrete Cauchy Function, and we define

$$(t-s-1)^{\underline{n-1}} := \prod_{j=0}^{n-1} (t-s-1-j)\,.$$

Notice that writing summation (1.2) as a single definite integral carries the additional information

$$y\,(a) = y\,(a+1) = \cdots = y\,(a+n-1) = 0,$$

from which we immediately obtain the desired initial conditions

$$y(a) = \Delta y(a) = \cdots = \Delta^{n-1} y(a) = 0.$$

We call summation (1.2) the n^{th}-*order sum of* f (denoted $\Delta_a^{-n} f$) and write

$$y(t) = \left(\Delta_a^{-n} f\right)(t) = \sum_{s=a}^{t-n} \frac{(t-s-1)^{\underline{n-1}}}{(n-1)!} f(s), \ t \in \mathbb{N}_a.$$

1.1.2 Fractional-Order Sums and Differences

The above discussion motivates the following definition for an arbitrary real-order sum.

Definition 1 *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $\nu > 0$ *be given. Then the* ν^{th}-order fractional sum of f *is given by*

$$(\Delta_a^{-\nu} f)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{\underline{\nu-1}} f(s), \quad \text{for } t \in \mathbb{N}_{a+\nu}.$$

Also, we define the trivial sum by $\Delta_a^{-0} f(t) := f(t)$, *for* $t \in \mathbb{N}_a$.

Remark 1

- *The name 'fractional sum' is a misnomer, strictly speaking. Early mathematicians penned the name with rational-order sums in mind, but both in the general theory and here, we allow sums of arbitrary real-order. Hence,* $\Delta_a^{-5} f$, $\Delta_a^{-\frac{7}{3}} f$, $\Delta_a^{-\sqrt{2}} f$ *and* $\Delta_a^{-\pi} f$ *are all legitimate fractional sums.*

- *For ease of notation, we use throughout this dissertation the symbol* $\Delta_a^{-\nu} f(t)$ *in place of the technically proper but less convenient symbol* $(\Delta_a^{-\nu} f)(t)$. *Therefore, the* t *in* $\Delta_a^{-\nu} f(t)$ *represents an input for the fractional sum* $\Delta_a^{-\nu} f$ *and not for the function* f.

- *The fractional sum* $\Delta_a^{-\nu} f$ *is a definite integral and therefore depends (in addition to its variable argument* t*) on its fixed lower summation limit* a. *In fact, it only* *makes sense to write* $\Delta_a^{-\nu} f$ *if we know a priori that the function* f *is defined on* $\mathbb{N}_a$. *The lower summation limit, therefore, provides us with an important tool for keeping track of function domains throughout our calculations; omitting this lower limit, as some authors do, leads to domain confusion and general ambiguity.*

- Euler's Gamma Function *in Definition 1 is given by*

$$\Gamma(\nu) := \int_0^\infty e^{-t} t^{\nu-1} dt, \quad \nu \in \mathbb{C} \backslash (-\mathbb{N}_0)$$

and is used, along with many of its properties, extensively throughout this dissertation. Most notable are its properties

(i) $\Gamma(\nu) > 0$, *for* $\nu > 0$.

(ii) $\Gamma(\nu+1) = \nu\Gamma(\nu)$, *for* $\nu \in \mathbb{C} \backslash (-\mathbb{N}_0)$.

(iii) $\Gamma(n+1) = n!$, *for* $n \in \mathbb{N}_0$.

(iv) $\frac{\Gamma(\nu+k)}{\Gamma(\nu)} = (\nu+k-1) \cdots (\nu-1)\nu$, *for* $\nu \in \mathbb{C} \backslash (-\mathbb{N}_0)$ *and* $k \in \mathbb{N}$.

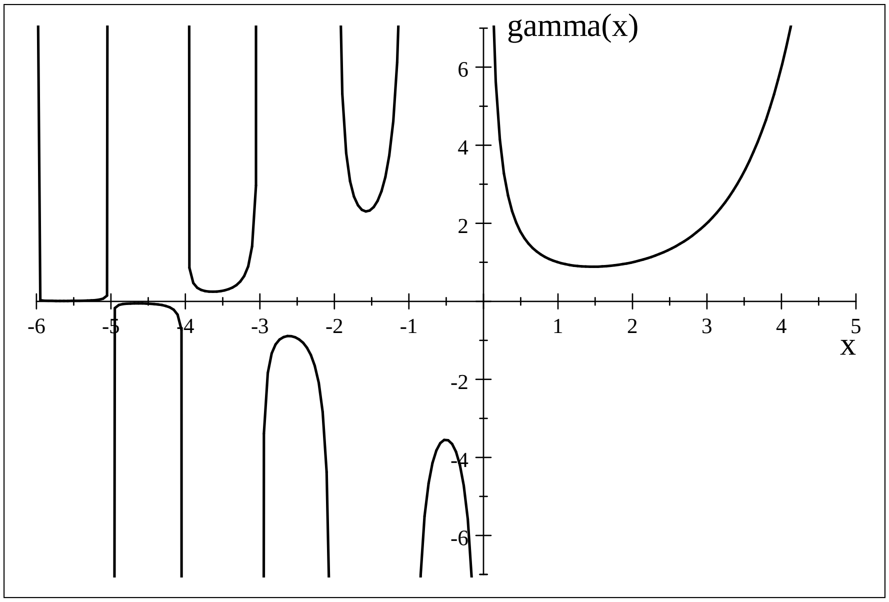

Figure 1.1: The Real Gamma Function $\Gamma : \mathbb{R}\backslash(-\mathbb{N}_0) \to \mathbb{R}$

- *The* σ-function *in Definition 1 comes from the general theory of time scales and is used here to assist in connecting with the general theory. For a discrete time scale such as* $\mathbb{N}_a$*,* $\sigma(s)$ *denotes the next point in the time scale after* s*. In this case,* $\sigma(s) = s+1$*, for all* $s \in \mathbb{N}_a$*.*

- *The term* $(t-\sigma(s))^{\underline{\nu-1}}$ *in Definition 1 is the so-called* generalized falling function*, defined by*

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any $t, \nu \in \mathbb{R}$ *for which the right-hand side is well-defined. Hence,*

$$(t-\sigma(s))^{\underline{\nu-1}} = \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)}.$$

- *The following identities, holding whenever the generalized falling functions are well defined, will be used extensively throughout this dissertation.*

(i) $\nu^{\underline{\nu}} = \Gamma(\nu + 1)$

(ii) $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu - 1}}$

(iii) $t^{\underline{\nu+1}} = (t - \nu) t^{\underline{\nu}}$

With fractional sums in hand, we are prepared to introduce fractional differences, as traditionally defined, such as in [15].

Definition 2 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu \geq 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \nu \leq N$. Then the ν^{th}*-order fractional difference of f *is given by*

$$(\Delta_a^\nu f)(t) = \Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \text{ for } t \in \mathbb{N}_{a+N-\nu}.$$

Remark 2

- *This traditional definition defines fractional-order differences as the next higher whole-order difference acting on a small-order fractional sum. In Section 1.1.4 (Theorem 1), we derive an equivalent definition for fractional differences which mirrors that for fractional sums and which proves essential in several applications.*

- *Observe that Definition 2 agrees with the standard whole-order difference definition: For any $\nu = N \in \mathbb{N}_0$,*

$$\Delta_a^v f(t) = \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t), \text{ for } t \in \mathbb{N}_a.$$

- *When applying Definition 2, we often wish to use the following well-known binomial formula for the whole-order difference Δ^N:*

$$\Delta^N f(t) = \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(t + N - i).$$

The important extension to a fractional-order binomial formula is presented at the conclusion of this chapter (Theorem 3).

- *It is important to note that, whereas whole-order differences do not depend on any 'starting point' or lower limit a, fractional differences do. To demonstrate this point, the second-order difference of a function $f : \mathbb{N}_a \to \mathbb{R}$ at a point $t \in \mathbb{N}_a$ is given by*

$$\Delta^2 f(t) = f(t+2) - 2f(t+1) + f(t),$$

a value which in no way depends on the starting point a. However, the fractional difference

$$\begin{aligned} \Delta_a^{1.5} f(t) &= \Delta^2 \Delta_a^{-0.5} f(t) \\ &= \Delta_a^{-0.5} f(t+2) - 2\Delta_a^{-0.5} f(t+1) + \Delta_a^{-0.5} f(t) \\ &= 2f\,(t+1.5) - 3f\,(t+0.5) + \frac{3}{4\sqrt{\pi}} \sum_{s=a}^{t-0.5} (t-\sigma(s))^{\underline{-2.5}} f(s) \end{aligned}$$

certainly does depend on the lower summation limit a. This dependence frustrates our traditional notion of a difference; fortunately, the dependence on a vanishes as $\nu \to N^-$ (see Theorem 2 in Section 1.1.4). For this reason, we write $\Delta^N f$ for whole-order differences but must write $\Delta_a^\nu f$ for fractional-order differences.

- *In the above setting, it is important to think of $\nu > 0$ as being situated between two natural numbers. For such a ν with $N - 1 < \nu \leq N$ ($N \in \mathbb{N}_0$), we call the fractional difference equation $\Delta_{a+\nu-N}^\nu y(t) = f(t)$ a ν^{th}*-order equation, *and we identify it with the whole-order difference equation $\Delta^N y(t) = f(t)$. For* *example, an initial value problem for the equation* $\Delta^{7.4}_{a-0.6}y(t) = f(t)$ *must have* 8 *initial conditions to be well posed.*

1.1.3 Domains

When working with fractional sums and differences, it is crucial to characterize and track their domains correctly. We begin by considering the domain of a fractional sum. Consider the first-order sum of a function f at a point $t \in \mathbb{N}_a$, for which we sum up the values of f from $\tau = a$ to $\tau = t-1$. As always, the definite sum $\Delta_a^{-1} f(t)$ represents the area under the curve f from a to t, where the height on the interval $[t, t+1]$ is given by the value $f(t)$.

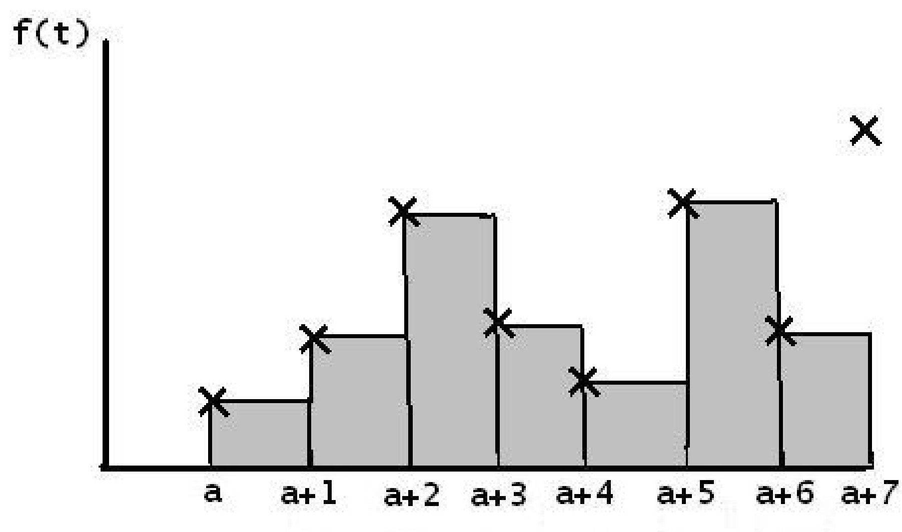

Figure 1.2: A First Order Sum

One may consider the value of $\Delta_a^{-1} f$ at $t = a$, thinking of this as the non-existent area under f from $t = a$ to $t = a$. Using Definition 1 for $\Delta_a^{-1} f(a)$, we find that

$$\Delta_a^{-1} f(a) = \frac{1}{\Gamma(1)} \sum_{s=a}^{t-1} (t - \sigma(s))^{\underline{1-1}} f(s) \bigg|_{t=a} = \sum_{s=a}^{a-1} f(s).$$

Therefore, if we insist on considering $\Delta_a^{-1} f(a)$ as a legitimate value, then we must hold to the convention that $\sum_{s=a}^{a-1} f(s) = 0$.

Likewise, if we recognize the values of $\Delta_a^{-N} f$ at the N points $t = a, a+1, ..., a + N - 1$, we must insist that

$$\Delta_a^{-N} f(a) = \Delta_a^{-N} f(a+1) = \cdots = \Delta_a^{-N} f(a + N - 1) = 0.$$

This reasoning leads us to following sensible convention on fractional-order sums: For any $\nu > 0$ with $N - 1 < \nu \leq N$, $\Delta_a^{-\nu} f$ satisfies

$$\Delta_a^{-\nu} f(a + \nu - N) = \Delta_a^{-\nu} f(a + \nu - N + 1) = \cdots = \Delta_a^{-\nu} f(a + \nu - 1) = 0.$$

Moreover, the first nontrivial value of $\Delta_a^{-\nu} f$ occurs at the point $t = a + \nu$:

$$\Delta_a^{-\nu} f(a + \nu) = f(a).$$

In light of this, it is convenient to ignore the initial N zeroes by defining the domain of $\Delta_a^{-\nu} f$ to be

$$\mathcal{D}\left\{\Delta_a^{-\nu} f\right\} := \mathbb{N}_{a+\nu},$$

as given in Definition 2.

We now utilize fractional sum domains to easily determine fractional difference domains. Whereas whole-order differences are domain *preserving* operators (i.e. $\mathcal{D}\left(\Delta^N f\right) = \mathcal{D}(f)$, for all $N \in \mathbb{N}_0$), we find that fractional differences are domain *shifting* operators. Using Definition 2 together with a function $f : \mathbb{N}_a \to \mathbb{R}$ and an order $\nu > 0$ with $N-1 < \nu \leq N$, we may calculate the domain of the ν^{th}-order fractional difference as

$$\mathcal{D}\left\{\Delta_a^\nu f\right\} = \mathcal{D}\left\{\Delta^N \Delta_a^{-(N-\nu)} f\right\} = \mathcal{D}\left\{\Delta_a^{-(N-\nu)} f\right\} = \mathbb{N}_{a+N-\nu}.$$

Note that whereas the domain shift for a fractional sum is a large shift by ν to the right, the domain shift for a fractional difference is a relatively small shift by $N-\nu$ to the right.

Let us next consider the domains for the four *two-operator sum and difference compositions.* For example, the composition of two arbitrary fractional sums is

$$\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu} f(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}\left(\frac{1}{\Gamma(\mu)}\sum_{r=a}^{s-\mu}(s-\sigma(r))^{\underline{\mu-1}} f(r)\right),$$

which has domain $\mathbb{N}_{a+\mu+\nu}$. Notice that the lower limit of the outer operator $\Delta_{a+\mu}^{-\nu}$ matches the starting point for the domain of the inner function $\Delta_a^{-\mu} f(t)$, which is $t = a+\mu$. The domains of all four sum and difference compositions are given below.

Summary 1 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu, \mu > 0$ be given. Let $N, M \in \mathbb{N}_0$ be chosen so that $N-1 < \nu \leq N$ and $M-1 < \mu \leq M$. Then*

- $\mathcal{D}\left\{\Delta_a^{-\nu} f\right\} = \mathbb{N}_{a+\nu}$
- $\mathcal{D}\left\{\Delta_a^{\nu} f\right\} = \mathbb{N}_{a+N-\nu}$
- $\mathcal{D}\left\{\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu} f\right\} = \mathbb{N}_{a+\mu+\nu}$
- $\mathcal{D}\left\{\Delta_{a+\mu}^{\nu}\Delta_a^{-\mu} f\right\} = \mathbb{N}_{a+\mu+N-\nu}$
- $\mathcal{D}\left\{\Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu} f\right\} = \mathbb{N}_{a+M-\mu+\nu}$
- $\mathcal{D}\left\{\Delta_{a+M-\mu}^{\nu}\Delta_a^{\mu} f\right\} = \mathbb{N}_{a+M-\mu+N-\nu}$

1.1.4 Unifying Fractional Sums and Differences

We demonstrate in this section that fractional sums and differences may be unified via a common definition. To accomplish this, we need the following version of Leibniz' Rule, easily proved below.

Let $g : \mathbb{N}_{a+\nu} \times \mathbb{N}_a \to \mathbb{R}$ be given. Then for $t \in \mathbb{N}_{a+\nu}$,

$$\begin{aligned}
\Delta\left(\sum_{s=a}^{t-\nu} g(t,s)\right) &= \sum_{s=a}^{t+1-\nu} g(t+1,s) - \sum_{s=a}^{t-\nu} g(t,s) \\
&= \sum_{s=a}^{t-\nu}\Big(g(t+1,s) - g(t,s)\Big) + g(t+1,t+1-\nu) \\
&= \sum_{s=a}^{t-\nu} \Delta_t g(t,s) + g(t+1,t+1-\nu). \qquad (1.3)
\end{aligned}$$

Theorem 1 below effectively unifies fractional sums and differences, allowing us to substantially extend several results from previous publications—most notably the power rule and composition rules found in [2] and [3].

Theorem 1 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, with $N-1 < \nu \leq N$. The following two definitions for the fractional difference $\Delta_a^\nu f : \mathbb{N}_{a+N-\nu} \to \mathbb{R}$ are equivalent:*

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \qquad (1.4)$$

$$\Delta_a^\nu f(t) := \begin{cases} \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & N-1 < \nu < N \\ \Delta^N f(t), & \nu = N \end{cases} \qquad (1.5)$$

Proof. Let f and ν be given as in the statement of the theorem. We suppose that (1.4) is the correct fractional difference definition and show that (1.5) is equivalent to (1.4) on $\mathbb{N}_{a+N-\nu}$.

If $\nu = N$, then (1.4) and (1.5) are clearly equivalent, since in this case,

$$\Delta_a^\nu f(t) = \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t).$$

If $N-1 < \nu < N$, then a direct application of (1.4) yields

$$\begin{aligned}
&\Delta_a^\nu f(t) \\
= \;& \Delta^N \Delta_a^{-(N-\nu)} f(t) \\
= \;& \Delta^N \left[\frac{1}{\Gamma(N-\nu)} \sum_{s=a}^{t-(N-\nu)} (t-\sigma(s))^{\underline{N-\nu-1}} f(s) \right] \\
= \;& \frac{\Delta^{N-1}}{\Gamma(N-\nu)} \cdot \Delta \left[\sum_{s=a}^{t-(N-\nu)} (t-\sigma(s))^{\underline{N-\nu-1}} f(s) \right] \text{ (now apply (1.3)),} \\
= \;& \frac{\Delta^{N-1}}{\Gamma(N-\nu)} \left[\sum_{s=a}^{t-(N-\nu)} \Big((N-\nu-1)(t-\sigma(s))^{\underline{N-\nu-2}} f(s) \Big) \right. \\
& \left. \qquad + (t+1-\sigma(t+1-(N-\nu)))^{\underline{N-\nu-1}} f(t+1-(N-\nu)) \right] \\
= \;& \Delta^{N-1} \left[\sum_{s=a}^{t-(N-\nu)} \frac{(t-\sigma(s))^{\underline{N-\nu-2}}}{\Gamma(N-\nu-1)} f(s) + f(t+1-(N-\nu)) \right] \\
= \;& \Delta^{N-1} \left[\frac{1}{\Gamma(N-\nu-1)} \sum_{s=a}^{t+1-(N-\nu)} (t-\sigma\,(s))^{\underline{N-\nu-2}} f(s) \right] \\
= \;& \Delta^{N-1} \left[\frac{1}{\Gamma(N-\nu-1)} \sum_{s=a}^{t-(N-\nu-1)} (t-\sigma\,(s))^{\underline{N-\nu-2}} f(s) \right].
\end{aligned}$$

Repeating these steps $N-2$ times yields

$$\Delta_a^\nu f(t) \;=\; \Delta^{N-1} \left[\frac{1}{\Gamma(N-\nu-1)} \sum_{s=a}^{t-(N-\nu-1)} (t-\sigma\,(s))^{\underline{N-\nu-2}} f(s) \right]$$

$$
\begin{aligned}
&= \Delta^{N-2}\left[\frac{1}{\Gamma(N-\nu-2)}\sum_{s=a}^{t-(N-\nu-2)}(t-\sigma(s))^{\underline{N-\nu-3}}f(s)\right] \\
&= \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\
&= \Delta^{N-N}\left[\frac{1}{\Gamma(N-\nu-N)}\sum_{s=a}^{t-(N-\nu-N)}(t-\sigma(s))^{\underline{N-\nu-(N+1)}}f(s)\right] \\
&= \frac{1}{\Gamma(-\nu)}\sum_{s=a}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}f(s).
\end{aligned}
$$

Note that since $N-1<\nu<N$ in the above work, the term $\frac{1}{\Gamma(N-\nu-k)}$ exists for each $k=1,2,...,N$. Furthermore, for each point $t\in\mathbb{N}_{a+N-\nu}$ (say $t=a+N-\nu+m$, for some $m\in\mathbb{N}_0$), we see that

$$
(t-\sigma(s))^{\underline{N-\nu-k-1}}=\frac{\Gamma(t-s)}{\Gamma(t-s-N+\nu+k+1)}=\frac{\Gamma(a+N-\nu+m-s)}{\Gamma(a+m+k+1-s)},
$$

which exists for each $k\in\{1,2,...,N\}$ and for each

$$
s\in\{a,a+1,...,t-(N-\nu-k)\}=\{a,a+1,...,a+m+k\}.
$$

Finally, note that though (1.5) at first glance appears to be valid for all $t\in\mathbb{N}_{a-\nu}$, it only defines the ν^{th}-fractional difference on $\mathbb{N}_{a+N-\nu}$. $\blacksquare$

In (1.5), one ought to wonder about the continuity of $\Delta_a^\nu f$ with respect to ν. We would certainly wish, for example, that $\Delta_a^{1.99}f$ be very close to $\Delta^2 f$, for every function $f:\mathbb{N}_a\to\mathbb{R}$. Seemingly to the contrary, however, the term $\frac{1}{\Gamma(-\nu)}$ in (1.5) blows up as $\nu\to N^-$! The following theorem fully clarifies this matter.

Theorem 2 *Let $f:\mathbb{N}_a\to\mathbb{R}$ be given. Then the fractional difference $\Delta_a^\nu f$ is continuous with respect to $\nu\geq 0$. More explicitly, for each $\nu>0$ and $m\in\mathbb{N}_0$, let $t_{\nu,m}:=a+\lceil\nu\rceil-\nu+m$ be a fixed but arbitrary point in $\mathcal{D}\{\Delta_a^\nu f\}$. Then for each* *fixed* $m \in \mathbb{N}_0$,

$$\nu \mapsto \Delta_a^\nu f(t_{\nu,m}) \textit{ is continuous on } [0,\infty).$$

Proof. Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, and fix $N \in \mathbb{N}$ and $m \in \mathbb{N}_0$. It is enough to show that

$$\Delta_a^\nu f(a+N-\nu+m) \text{ is continuous with respect to } \nu \text{ on } (N-1,N) \tag{1.6}$$

$$\Delta_a^\nu f(a+N-\nu+m) \to \Delta^N f(a+m) \text{ as } \nu \to N^- \tag{1.7}$$

$$\Delta_a^\nu f(a+N-\nu+m) \to \Delta^{N-1} f(a+m+1) \text{ as } \nu \to (N-1)^+ \tag{1.8}$$

To show (1.6), note that for any fixed $\nu \in (N-1,N)$,

$$\begin{aligned}
&\Delta_a^\nu f(a+N-\nu+m)\\
= \;& \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s) \Bigg|_{t=a+N-\nu+m}\\
= \;& \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{a+N+m} (a+N-\nu+m-\sigma(s))^{\underline{-\nu-1}} f(s)\\
= \;& \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{a+N+m} \frac{\Gamma(a+N-\nu+m-s)}{\Gamma(a+N+m-s+1)} f(s)\\
= \;& \sum_{s=a}^{a+N+m} \frac{1}{\Gamma(a+N+m-s+1)} \frac{\Gamma(a+N-\nu+m-s)}{\Gamma(-\nu)} f(s)\\
= \;& \sum_{s=a}^{a+N+m-1} \left(\frac{(a+N-\nu+m-s-1)\cdots(-\nu)}{(a+N+m-s)!} f(s) \right) + f(a+N+m)\\
= \;& \sum_{i=1}^{N+m} \left(\frac{(i-1-\nu)\cdots(-\nu)}{i!} f(a+N+m-i) \right) + f(a+N+m).
\end{aligned} \tag{1.9}$$

Since (1.9), an expression for $\Delta_a^\nu f(a+N-\nu+m)$, is clearly continuous with respect to ν on $(N-1,N)$, we have demonstrated (1.6). Next, we take the limit of (1.9) as

$\nu \to N^-$ to show (1.7):

$$
\begin{aligned}
&\lim_{\nu \to N^-} \Delta_a^\nu f(a+N-\nu+m) \\
&= \lim_{\nu \to N^-} \left[\sum_{i=1}^{N+m} \left(\frac{(i-1-\nu)\cdots(-\nu)}{i!} f(a+N+m-i) \right) + f(a+N+m) \right] \\
&= \sum_{i=1}^{N+m} \left(\frac{(i-1-N)\cdots(-N)}{i!} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=1}^{N} \left(\frac{(i-1-N)\cdots(-N)}{i!} f(a+N+m-i) \right) + f(a+N+m) \qquad (1.10) \\
&= \sum_{i=1}^{N} \left((-1)^i \frac{(N)\cdots(N-i+1)}{i!} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=1}^{N} \left((-1)^i \binom{N}{i} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(a+N+m-i) \\
&= \sum_{i=0}^{N} (-1)^i \binom{N}{i} f((a+m)+N-i) \\
&= \Delta^N f(a+m).
\end{aligned}
$$

Finally, we take the limit of (1.9) as $\nu \to (N-1)^+$ to show (1.8):

$$
\begin{aligned}
&\lim_{\nu \to (N-1)^+} \Delta_a^\nu f(a+N-\nu+m) \\
&= \lim_{\nu \to (N-1)^+} \left[\sum_{i=1}^{N+m} \left(\frac{(i-1-\nu)\cdot\cdot(-\nu)}{i!} f(a+N+m-i) \right) + f(a+N+m) \right] \\
&= \sum_{i=1}^{N+m} \left(\frac{(i-N)\cdots(-N+1)}{i!} f(a+N+m-i) \right) + f(a+N+m) \\
&= \sum_{i=1}^{N-1} \left((-1)^i \frac{(N-1)\cdots(N-i)}{i!} f(a+N+m-i) \right) + f(a+N+m)
\end{aligned}
$$

$$= \sum_{i=1}^{N-1} \left((-1)^i \binom{N-1}{i} f(a+N+m-i) \right) + f(a+N+m)$$
$$= \sum_{i=0}^{N-1} \left((-1)^i \binom{N-1}{i} f(a+m+1+(N-1)-i) \right)$$
$$= \Delta^{N-1} f(a+m+1).$$

■

Remark 3 • *Step (1.10) in the above proof shows explicitly why the dependence of a fractional difference on its starting point a vanishes as the order of differentiation approaches a whole number. We may, of course, still write t in terms of a, but a whole-order fractional difference has a fixed number of terms, regardless of how far t lies away from a.*

• *Theorem 2 implies that for any $f : \mathbb{N}_a \to \mathbb{R}$ and $m \in \mathbb{N}_0$, the sequence*

$$\Delta_a^{1.9} f(a+m+0.1), \Delta_a^{1.99} f(a+m+0.01), \Delta_a^{1.999} f(a+m+0.001), ...$$

approaches the value $\Delta^2 f(a+m)$. This notion of "order continuity" adds a beauty to fractional calculus that is absent from the standard whole-order calculus.

Theorem 2 shows that (1.5) in Theorem 1 is a legitimate and reasonable definition for the fractional difference. Hence, we may unify fractional sums and differences by a single definition:

Definition 3 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then*

(i) *the* ν^{th}-order fractional sum of f *is given by*

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s), \ t \in \mathbb{N}_{a+\nu}.$$

(ii) *the* ν^{th}-order fractional difference of f *is given by*

$$\Delta_a^{\nu} f(t) := \begin{cases} \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & \nu \notin \mathbb{N} \\ \Delta^N f(t), & \nu = N \in \mathbb{N} \end{cases}, \quad t \in \mathbb{N}_{a+N-\nu}$$

The following theorem generalizing the binomial representation to both fractional sums and differences demonstrates the relevance of Definition 3.

Theorem 3 *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $\nu > 0$ *be given, with* $N - 1 < \nu \leq N$.

For each $t \in \mathbb{N}_{a+N-\nu}$,

$$\Delta_a^{\nu} f(t) = \sum_{k=0}^{\nu+t-a} (-1)^k \binom{\nu}{k} f(t+\nu-k). \tag{1.11}$$

For each $t \in \mathbb{N}_{a+\nu}$,

$$\Delta_a^{-\nu} f(t) = \sum_{k=0}^{-\nu+t-a} (-1)^k \binom{-\nu}{k} f(t-\nu-k) \tag{1.12}$$

$$= \sum_{k=0}^{-\nu+t-a} \binom{\nu+k-1}{k} f(t-\nu-k). \tag{1.13}$$

Proof. Let f, ν and N be given as in the statement of the theorem and let

$t \in \mathbb{N}_{a+N-\nu}$ be given by $t = a + N - \nu + m$, for some $m \in \mathbb{N}_0$. Then

$$\begin{aligned}
\Delta_a^\nu f(t) &= \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s) \\
&= \sum_{s=a}^{t+\nu} \frac{\Gamma(t-s)}{\Gamma(t-s+\nu+1)\Gamma(-\nu)} f(s) \\
&= \sum_{s=a}^{a+N+m} \frac{\Gamma(a+N-\nu+m-s)}{\Gamma(a+N+m-s+1)\Gamma(-\nu)} f(s) \\
&= \sum_{s=0}^{N+m} \frac{\Gamma(N+m-s-\nu)}{\Gamma(N+m-s+1)\Gamma(-\nu)} f(s+a) \\
&= f(a+N+m) + \sum_{s=0}^{N+m-1} \frac{(N+m-1-s-\nu)\cdots(-\nu)}{\Gamma(N+m-s+1)} f(s+a) \\
&= f(a+N+m) + \sum_{s=0}^{N+m-1} (-1)^{N+m-s} \frac{\nu\cdots(\nu-(N+m-s)+1)}{\Gamma(N+m-s+1)} f(s+a) \\
&= \sum_{s=0}^{N+m} (-1)^{N+m-s} \binom{\nu}{N+m-s} f(s+a) \\
&= \sum_{k=0}^{N+m} (-1)^k \binom{\nu}{k} f(a+N+m-k) \\
&= \sum_{k=0}^{N+m} (-1)^k \binom{\nu}{k} f((a+N-\nu+m)+\nu-k) \\
&= \sum_{k=0}^{\nu+t-a} (-1)^k \binom{\nu}{k} f(t+\nu-k), \text{ proving (1.11)}
\end{aligned}$$

Note that when $\nu = N$, (1.11) reduces to the traditional binomial formula

$$\Delta^N f(t) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(t+N-k), \text{ for } t \in \mathbb{N}_a.$$

We prove (1.12) via a quite similar argument. We must note, however, that when

$\nu = N$, we interpret the problematic expression $\binom{-N}{k}$ as

$$\binom{-N}{k} = \frac{\Gamma(-N+1)}{k!\Gamma(-N-k+1)} = \frac{(-N)\cdots(-N-k+1)}{k!} = (-1)^k \binom{N+k-1}{k}.$$

We may, therefore, write generally for $\nu > 0$

$$\binom{-\nu}{k} = (-1)^k \binom{\nu+k-1}{k},$$

whose substitution into (1.12) yields (1.13). Although (1.13) is probably more useful, (1.12) offers a closer resemblance the traditional binomial formula. ■

Chapter 2

The Fractional Composition Rules

Having set the stage by outlining and developing many properties of fractional sums and differences, we have the necessary tools to study the following four compositions

$$\begin{cases} \Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu}f(t) = \frac{1}{\Gamma(\nu)}\displaystyle\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}\sum_{r=a}^{s-\mu}\tfrac{(s-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)}f(r), \\ \Delta_{a+\mu}^{\nu}\Delta_a^{-\mu}f(t) = \frac{1}{\Gamma(-\nu)}\displaystyle\sum_{s=a+\mu}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}\sum_{r=a}^{s-\mu}\tfrac{(s-\sigma(r))^{\underline{\mu-1}}}{\Gamma(\mu)}f(r), \\ \Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu}f(t) = \frac{1}{\Gamma(\nu)}\displaystyle\sum_{s=a+M-\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}\sum_{r=a}^{s+\mu}\tfrac{(s-\sigma(r))^{\underline{-\mu-1}}}{\Gamma(-\mu)}f(r), \\ \Delta_{a+M-\mu}^{\nu}\Delta_a^{\mu}f(t) = \frac{1}{\Gamma(-\nu)}\displaystyle\sum_{s=a+M-\mu}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}\sum_{r=a}^{s+\mu}\tfrac{(s-\sigma(r))^{\underline{-\mu-1}}}{\Gamma(-\mu)}f(r), \end{cases}$$

whose domains are as given in Summary 1. Definition 3 is the tool that allows us to write these four compositions in a uniform manner. It will be helpful to keep the above representations and their domains in mind as we develop a rule to govern each composition.

2.1 The Fractional Power Rule

We first, however, need a precise fractional power rule for sums and differences. Much of the proof for this power rule may be found in [2], though the precise power rule presented in Lemma 4 corrects ambiguity and significantly extends this previous version.

Lemma 4 *Let $a \in \mathbb{R}$ and $\mu > 0$ be given. Then,*

$$\Delta(t-a)^{\underline{\mu}} = \mu(t-a)^{\underline{\mu-1}}, \tag{2.1}$$

for any t for which both sides are well-defined. Furthermore, for $\nu > 0$,

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{-\nu}}(t-a)^{\underline{\mu+\nu}}, \text{ for } t \in \mathbb{N}_{a+\mu+\nu} \tag{2.2}$$

and

$$\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{\nu}}(t-a)^{\underline{\mu-\nu}}, \text{ for } t \in \mathbb{N}_{a+\mu+N-\nu}. \tag{2.3}$$

Proof. It is easy to show (2.1) using the definition of the delta difference and properties of the gamma function. For (2.2) and (2.3), we first note that $(t-a)^{\underline{\mu}}$, $(t-a)^{\underline{\mu+\nu}}$ and $(t-a)^{\underline{\mu-\nu}}$ are all well-defined and positive on their respective domains $\mathbb{N}_{a+\mu}$, $\mathbb{N}_{a+\mu+\nu}$ and $\mathbb{N}_{a+\mu+N-\nu}$.

To prove (2.2), we consider the two cases $\nu = 1$ and $\nu \in (0,1) \cup (1,\infty)$ separately.

For $\nu = 1$, we see from direct calculation that

$$\begin{aligned}
\Delta_{a-\mu}^{-1}(t-a)^{\underline{\mu}} &= \Delta_{a-\mu}^{-1}\Delta\left(\frac{(t-a)^{\underline{\mu+1}}}{\mu+1}\right), \text{ by (2.1)} \\
&= \sum_{s=a+\mu}^{t-1}\left(\frac{(s+1-a)^{\underline{\mu+1}}}{\mu+1} - \frac{(s-a)^{\underline{\mu+1}}}{\mu+1}\right) \\
&= \frac{(t-a)^{\underline{\mu+1}}}{\mu+1} - \frac{\mu^{\underline{\mu+1}}}{\mu+1} \\
&= \mu^{\underline{-1}}\,(t-a)^{\underline{\mu+1}}.
\end{aligned}$$

For $\nu \in (0,1) \cup (1,\infty)$, define for $t \in \mathbb{N}_{a+\mu+\nu}$ the functions

$$\begin{aligned}
g_1(t) &:= \Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} \text{ and} \\
g_2(t) &:= \mu^{\underline{-\nu}}\,(t-a)^{\underline{\mu+\nu}}.
\end{aligned}$$

We will show that both g_1 and g_2 solve the well-posed, first-order initial value problem

$$\begin{cases} (t-a-(\mu+\nu)+1)\Delta g(t) = (\mu+\nu)g(t), \text{ for } t \in \mathbb{N}_{a+\mu+\nu} \\ g(a+\mu+\nu) = \Gamma(\mu+1). \end{cases} \tag{2.4}$$

Since

$$\begin{aligned}
g_1(a+\mu+\nu) &= \frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{a+\mu}(a+\mu+\nu-\sigma(s))^{\underline{\nu-1}}(s-a)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)}(\nu-1)^{\underline{\nu-1}}\mu^{\underline{\mu}} \\
&= \Gamma(\mu+1),
\end{aligned}$$

and

$$
\begin{aligned}
g_2(a+\mu+\nu) &= \mu^{\underline{-\nu}}(\mu+\nu)^{\underline{\mu+\nu}} \\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)}\Gamma(\mu+\nu+1) \\
&= \Gamma(\mu+1),
\end{aligned}
$$

both g_1 and g_2 satisfy the initial condition in (2.4).

Our greatest effort is required to show that g_1 satisfies the difference equation in (2.4). For $t \in \mathbb{N}_{a+\mu+\nu}$,

$$
\begin{aligned}
\Delta g_1(t) &= \Delta\left[\frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}(s-a)^{\underline{\mu}}\right] \quad \text{(now apply (1.3))} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(\nu-1)(t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} \\
&\qquad + \frac{(t+1-(t+2-\nu))^{\underline{\nu-1}}}{\Gamma(\nu)}(t+1-\nu-a)^{\underline{\mu}} \\
&= \frac{\nu-1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} + (t+1-\nu-a)^{\underline{\mu}}.
\end{aligned}
$$

Also, we may manipulate g_1 directly to obtain

$$
\begin{aligned}
& g_1(t) \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}(s-a)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s)-(\nu-2))(t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}\left[(t-a-(\mu+\nu)+1)-(s-a-\mu)\right](t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}} \\
&= \frac{t-a-(\mu+\nu)+1}{\Gamma(\nu)}\sum_{s=a+\mu}^{t-\nu}(t-\sigma(s))^{\underline{\nu-2}}(s-a)^{\underline{\mu}}
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}} (s-a)^{\underline{\mu+1}} \\
= \quad & h(t) - k(t),
\end{aligned}
$$

where

$$
\begin{cases}
h(t) := \frac{t-a-(\mu+\nu)+1}{\Gamma(\nu)} \displaystyle\sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}} (s-a)^{\underline{\mu}} \\
k(t) := \frac{1}{\Gamma(\nu)} \displaystyle\sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-2}} (s-a)^{\underline{\mu+1}}.
\end{cases}
$$

Integrating k by parts, we obtain

$$
\begin{aligned}
k(t) &= \frac{1}{\Gamma(\nu)} \left[\sum_{s=a+\mu}^{(t-\nu+1)-1} (s-a)^{\underline{\mu+1}} \Delta \left(-\frac{(t-s)^{\underline{\nu-1}}}{\nu-1} \right) \right] \\
&= \frac{1}{\Gamma(\nu)} \left[\left((s-a)^{\underline{\mu+1}} \left(-\frac{(t-s)^{\underline{\nu-1}}}{\nu-1} \right) \right) \Bigg|_{s=a+\mu}^{s=t-\nu+1} \right. \\
&\qquad \left. - \sum_{s=a+\mu}^{(t-\nu+1)-1} \left(-\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\nu-1} (\mu+1)(s-a)^{\underline{\mu}} \right) \right] \\
&= \frac{1}{\Gamma(\nu)} \left[-\frac{\Gamma(\nu)}{\nu-1} (t-\nu+1-a)^{\underline{\mu+1}} \right. \\
&\qquad \left. + \frac{\mu+1}{\nu-1} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} (s-a)^{\underline{\mu}} \right] \\
&= \frac{1}{\nu-1} \left[\frac{\mu+1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} (s-a)^{\underline{\mu}} - (t-\nu+1-a)^{\underline{\mu+1}} \right].
\end{aligned}
$$

It follows from the above work that

- $(t-a-(\mu+\nu)+1)\Delta g_1(t) = (\nu-1)h(t) + (t+1-\nu-a)^{\underline{\mu+1}}$
- $(\mu+1)g_1(t) - (\nu-1)k(t) = (t+1-\nu-a)^{\underline{\mu+1}}$.

Hence,

$$
\begin{aligned}
(t-a-(\mu+\nu)+1)\Delta g_1(t) &= (\nu-1)h(t)+(\mu+1)g_1(t)-(\nu-1)k(t) \\
&= (\nu-1)g_1(t)+(\mu+1)g_1(t) \\
&= (\mu+\nu)g_1(t).
\end{aligned}
$$

Finally, g_2 also satisfies the difference equation in (2.4):

$$
\begin{aligned}
&(t-a-(\mu+\nu)+1)\Delta g_2(t) \\
= \ & (t-a-(\mu+\nu)+1)\left[\mu^{\underline{-\nu}}(\mu+\nu)\,(t-a)^{\underline{\mu+\nu-1}}\right], \text{ by (2.1)} \\
= \ & (\mu+\nu)\mu^{\underline{-\nu}}\left[(t-a-(\mu+\nu-1))\,(t-a)^{\underline{\mu+\nu-1}}\right] \\
= \ & (\mu+\nu)\mu^{\underline{-\nu}}(t-a)^{\underline{\mu+\nu}} \\
= \ & (\mu+\nu)g_2(t).
\end{aligned}
$$

By uniqueness of solutions to the well-posed initial value problem (2.4), we conclude that $g_1 \equiv g_2$ on $\mathbb{N}_{a+\mu+\nu}$.

We next employ (2.1) and (2.2) to show (2.3) as follows: For $t \in \mathbb{N}_{a+\mu+N-\nu}$,

$$
\begin{aligned}
&\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} \\
= \ & \Delta^N\left[\Delta_{a+\mu}^{-(N-\nu)}(t-a)^{\underline{\mu}}\right] \\
= \ & \Delta^N\left[\frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}(t-a)^{\underline{\mu+N-\nu}}\right], \text{ by (2.2)} \\
= \ & \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}\Big((\mu+N-\nu)\cdots(\mu+1-\nu)\Big)(t-a)^{\underline{\mu-\nu}}, \text{ by (2.1)} \\
= \ & \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+N-\nu)}\frac{\Gamma(\mu+N-\nu+1)}{\Gamma(\mu+1-\nu)}(t-a)^{\underline{\mu-\nu}} \\
= \ & \mu^{\underline{\nu}}(t-a)^{\underline{\mu-\nu}}.
\end{aligned}
$$

In the special case where $\nu \in \{\mu+1, \mu+2, ...\}$, we have $\mu+1-\nu \in (-\mathbb{N}_0)$, and so the term

$$\mu^{\underline{\nu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}$$

from (2.3) is ill-defined. In this case, we naturally interpret the right hand side of (2.3) as zero, which is exactly as we desire. ■

2.2 Composing Fractional Sums and Differences

2.2.1 Composing a Sum with a Sum

The rule for composing two fractional sums depends on an appropriate application of power rule (2.2) from Lemma 4 (see [2]).

Theorem 5 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$. Then*

$$\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu} f(t) = \Delta_a^{-\nu-\mu} f(t) = \Delta_{a+\nu}^{-\mu}\Delta_a^{-\nu} f(t), \text{ for } t \in \mathbb{N}_{a+\mu+\nu}.$$

Proof. Suppose $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu, \mu > 0$. Then for $t \in \mathbb{N}_{a+\mu+\nu}$,

$$\begin{aligned}
\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu} f(t) &= \frac{1}{\Gamma(\nu)} \sum_{s=a+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} \left(\frac{1}{\Gamma(\mu)} \sum_{r=a}^{s-\mu} (s-\sigma(r))^{\underline{\mu-1}} f(r) \right) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{s=a+\mu}^{t-\nu} \sum_{r=a}^{s-\mu} (t-\sigma(s))^{\underline{\nu-1}} (s-\sigma(r))^{\underline{\mu-1}} f(r) \\
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a}^{t-(\nu+\mu)} \sum_{s=r+\mu}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} (s-\sigma(r))^{\underline{\mu-1}} f(r) .
\end{aligned}$$

Let $x = s - \sigma(r)$ and continue with

$$
\begin{aligned}
&= \frac{1}{\Gamma(\nu)\Gamma(\mu)} \sum_{r=a}^{t-(\nu+\mu)} \left[\sum_{x=\mu-1}^{t-\nu-r-1} (t-x-r-2)^{\underline{\nu-1}} x^{\underline{\mu-1}} \right] f(r) \\
&= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(\nu+\mu)} \left[\frac{1}{\Gamma(\nu)} \sum_{x=\mu-1}^{(t-r-1)-\nu} ((t-r-1)-\sigma(x))^{\underline{\nu-1}} x^{\underline{\mu-1}} \right] f(r) \\
&= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(\nu+\mu)} \left(\left[\Delta_{\mu-1}^{-\nu} (t^{\underline{\mu-1}}) \right] \Big|_{t-r-1} f(r) \right) \\
&= \frac{1}{\Gamma(\mu)} \sum_{r=a}^{t-(\nu+\mu)} \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} (t-r-1)^{\underline{\mu-1+\nu}} f(r), \text{ using (2.2)} \\
&= \frac{1}{\Gamma(\nu+\mu)} \sum_{r=a}^{t-(\nu+\mu)} (t-\sigma(r))^{\underline{(\nu+\mu)-1}} f(r) \\
&= \Delta_a^{-(\nu+\mu)} f(t) \\
&= \Delta_a^{-\nu-\mu} f(t).
\end{aligned}
$$

Since ν and μ are arbitrary, we conclude more generally that

$$
\Delta_{a+\mu}^{-\nu} \Delta_a^{-\mu} f(t) = \Delta_a^{-\nu-\mu} f(t) = \Delta_{a+\nu}^{-\mu} \Delta_a^{-\nu} f(t), \text{ for } t \in \mathbb{N}_{a+\nu+\mu}.
$$

■

Remark 4 *We apply (2.2) above to write*

$$
\Delta_{\mu-1}^{-\nu} \left[\tau^{\underline{\mu-1}} \right] = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)} \tau^{\underline{\mu-1+\nu}}, \ \textit{for } \tau \in \mathbb{N}_{\mu-1+\nu}.
$$

Since we are working with $t \in \mathbb{N}_{a+\mu+\nu}$ *and since* $r \in \{a, ..., t-\mu-\nu\}$, *it is indeed appropriate to evaluate these terms at* $t-r-1 \in \mathbb{N}_{\mu+\nu-1}$.

2.2.2 Composing a Difference with a Sum

Before considering the general composition $\Delta_{a+\mu}^{\nu} \circ \Delta_a^{-\mu}$, we first restrict ν to be a natural number. For this special case, Atici and Eloe in [3] show the identity (2.5) for the even stricter case of $\mu > k$.

Lemma 6 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. For any $k \in \mathbb{N}_0$ and $\mu > 0$ with $M-1 < \mu \leq M$, we have*

$$\Delta^k \Delta_a^{-\mu} f(t) = \Delta_a^{k-\mu} f(t), \;\; \textit{for } t \in \mathbb{N}_{a+\mu} \tag{2.5}$$

$$\Delta^k \Delta_a^{\mu} f(t) = \Delta_a^{k+\mu} f(t), \;\; \textit{for } t \in \mathbb{N}_{a+M-\mu} \tag{2.6}$$

Proof. Let f, μ, M and k be as given in the statement of the lemma. We consider the following two cases.

Case 1 $(\mu = M)$

Observe that for $t \in \mathbb{N}_{a+1}$,

$$\Delta \Delta_a^{-1} f(t) = \Delta \left[\sum_{s=a}^{t-1} f(s) \right] = \sum_{s=a}^{t} f(s) - \sum_{s=a}^{t-1} f(s) = f(t).$$

This is, of course, the discrete analogue of the second part of the Fundamental Theorem of Calculus. Furthermore, for any $k \in \mathbb{N}$ and $t \in \mathbb{N}_{a+k}$,

$$\begin{aligned}
\Delta^k \Delta_a^{-k} f(t) &= \Delta^{k-1} \left[\Delta \Delta_{a+k-1}^{-1} \left(\Delta_a^{-(k-1)} f(t) \right) \right] \\
&= \Delta^{k-1} \Delta_a^{-(k-1)} f(t) \\
&= \cdots\cdots\cdots\cdots\cdots \\
&= f(t).
\end{aligned}$$

Therefore, for any $t \in \mathbb{N}_{a+M}$, we have

$$\Delta^k \Delta_a^{-M} f(t) = \begin{cases} \Delta^{k-M} \left[\Delta^M \Delta_a^{-M} f(t)\right] = \Delta^{k-M} f(t), & \text{provided } k \geq M \\ \Delta^k \Delta_{a+M-k}^{-k} \left[\Delta_a^{-(M-k)} f(t)\right] = \Delta_a^{k-M} f(t), & \text{provided } k < M. \end{cases}$$

Considering (2.6) for the case $\mu = M$, it is already well accepted that whole order differences commute.

Case 2 $(M - 1 < \mu < M)$

We first show that $\Delta\Delta_a^{\mu} f(t) = \Delta_a^{1+\mu} f(t)$, *for* $t \in \mathbb{N}_{a+M-\mu}$.

Given $t \in \mathbb{N}_{a+M-\mu}$ *and using our new Definition 1 for* $\Delta_a^{\mu} f$, *we calculate*

$$\begin{aligned}
& \Delta\Delta_a^{\mu} f(t) \\
= \;& \Delta\left[\frac{1}{\Gamma(-\mu)} \sum_{s=a}^{t+\mu} (t-\sigma(s))^{\underline{-\mu-1}} f(s)\right] \; \textit{(now apply (1.3))} \\
= \;& \frac{1}{\Gamma(-\mu)} \sum_{s=a}^{t+\mu} (-\mu-1)(t-\sigma(s))^{\underline{-\mu-2}} f(s) + f(t+\mu+1) \\
= \;& \frac{1}{\Gamma(-\mu-1)} \sum_{s=a}^{t+\mu} (t-\sigma(s))^{\underline{-\mu-2}} f(s) + f(t+\mu+1) \\
= \;& \frac{1}{\Gamma(-\mu-1)} \sum_{s=a}^{t+\mu+1} (t-\sigma(s))^{\underline{-\mu-2}} f(s) \\
= \;& \frac{1}{\Gamma(-\mu-1)} \sum_{s=a}^{t-(-\mu-1)} (t-\sigma(s))^{\underline{(-\mu-1)-1}} f(s) \\
= \;& \Delta_a^{-(-\mu-1)} f(t) \\
= \;& \Delta_a^{1+\mu} f(t).
\end{aligned}$$

Therefore, we obtain for any $k \in \mathbb{N}$,

$$\Delta^k \Delta_a^{\mu} f(t) = \Delta^{k-1}\left[\Delta\Delta_a^{\mu} f(t)\right]$$

$$= \Delta^{k-1}\Delta_a^{1+\mu} f(t)$$

$$= \cdots\cdots\cdots\cdots$$

$$= \Delta_a^{k+\mu} f(t), \textit{ for } t \in \mathbb{N}_{a+M-\mu}, \textit{ thus proving (2.6)}$$

Identity (2.5) is proved in a nearly identical manner.

■

We have now collected all the necessary tools to obtain a rule for composing fractional differences with fractional sums.

Theorem 7 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, and suppose $\nu, \mu > 0$ with $N - 1 < \nu \leq N$. Then*

$$\Delta_{a+\mu}^{\nu}\Delta_a^{-\mu} f(t) = \Delta_a^{\nu-\mu} f(t), \quad \textit{for } t \in \mathbb{N}_{a+\mu+N-\nu}.$$

Proof. Let f, ν, N and μ be given as in the statement of the theorem and let $t \in \mathbb{N}_{a+\mu+\mathbb{N}-\nu}$. Then

$$\begin{aligned}
\Delta_{a+\mu}^{\nu}\Delta_a^{-\mu} f(t) &= \Delta^N \Delta_{a+\mu}^{-(N-\nu)}\Delta_a^{-\mu} f(t) \\
&= \Delta^N \Delta_a^{-(N-\nu+\mu)} f(t), \text{ by Theorem 5,} \\
&= \Delta_a^{N-(N-\nu+\mu)} f(t), \text{ by Lemma 6,} \\
&= \Delta_a^{\nu-\mu} f(t).
\end{aligned}$$

■

Remark 5 *One ought to wonder if the correct domain has been chosen for the result* *in Theorem 7. To justify the given domain, we observe that*

$$\begin{cases} \mathcal{D}\left\{\Delta^{\nu}_{a+\mu}\Delta_a^{-\mu} f\right\} = \mathbb{N}_{a+\mu+N-\nu} \\ \mathcal{D}\left\{\Delta_a^{\nu-\mu} f(t)\right\} = \mathbb{N}_{a+\mu-\nu}, \text{ if } \nu < \mu \\ \mathcal{D}\left\{\Delta_a^{\nu-\mu} f(t)\right\} = \mathbb{N}_{a+\lceil \nu-\mu\rceil-(\nu-\mu)}, \text{ if } \nu \geq \mu, \end{cases}$$

and note that in both latter cases, we have $\mathcal{D}\left\{\Delta^{\nu}_{a+\mu}\Delta_a^{-\mu} f\right\} \subseteq \mathcal{D}\left\{\Delta_a^{\nu-\mu} f(t)\right\}$. *The composition rule holds on* $\mathbb{N}_{a+\mu+N-\nu}$, *therefore, the intersection of the domains for the left and right hand sides of the equation.*

2.2.3 Composing a Sum with a Difference

For the remaining two composition rules—whose inner operators are differences—we may not simply combine two operators by adding their orders. This should come as no surprise, however, in light of the first part of the Discrete Fundamental Theorem of Calculus:

$$\sum_{s=a}^{t-1} (\Delta f(s)) = f(t) - f(a).$$

Although (2.7) below is merely a special case of (2.8), it is significant in its own right and its proof (found in [3]) provides a natural stepping stone to (2.8).

Theorem 8 *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *be given and suppose* $k \in \mathbb{N}_0$ *and* $\nu > 0$. *Then for* $t \in \mathbb{N}_{a+\nu}$,

$$\Delta_a^{-\nu}\Delta^k f(t) = \Delta_a^{k-\nu} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(\nu-k+j+1)}(t-a)^{\underline{\nu-k+j}}. \tag{2.7}$$

Moreover, if $\mu > 0$ *with* $M-1 < \mu \leq M$, *then for* $t \in \mathbb{N}_{a+M-\mu+\nu}$,

$$\Delta^{-\nu}_{a+M-\mu}\Delta_a^{\mu} f(t) =$$

$$\Delta_a^{\mu-\nu} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-(M-\mu)} f(a+M-\mu)}{\Gamma(\nu-M+j+1)} (t-a-M+\mu)^{\underline{\nu-M+j}}. \tag{2.8}$$

Proof. We first consider (2.7).

Let $k \in \mathbb{N}_0$ be given and suppose ν is nonnegative with $\nu \notin \{1, 2, ..., k-1\}$. Then we may sum by parts, with $t \in \mathbb{N}_{a+\nu}$, to obtain

$$\begin{aligned}
&\Delta_a^{-\nu} \Delta^k f(t) \\
= \;& \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} \left[\Delta^k f(s)\right] \\
= \;& \frac{1}{\Gamma(\nu)} \sum_{s=a}^{(t-\nu+1)-1} (t-\sigma(s))^{\underline{\nu-1}} \Delta\left(\Delta^{k-1} f(s)\right) \\
= \;& \frac{1}{\Gamma(\nu)} \Bigg[(t-s)^{\underline{\nu-1}} \Delta^{k-1} f(s) \Bigg|_{s=a}^{s=t-\nu+1} \\
& \qquad\qquad - \sum_{s=a}^{t-\nu} \Big(-(\nu-1)(t-\sigma(s))^{\underline{\nu-2}} \Delta^{k-1} f(s) \Big) \Bigg] \\
= \;& \Delta^{k-1} f(t-\nu+1) - \frac{(t-a)^{\underline{\nu-1}}}{\Gamma(\nu)} \Delta^{k-1} f(a) \\
& \qquad\qquad + \frac{1}{\Gamma(\nu-1)} \sum_{s=a}^{t-\nu} \Big((t-\sigma(s))^{\underline{\nu-2}} \Delta^{k-1} f(s) \Big) \\
= \;& \frac{1}{\Gamma(\nu-1)} \sum_{s=a}^{t-\nu+1} \Big((t-\sigma(s))^{\underline{\nu-2}} \Delta^{k-1} f(s) \Big) - \frac{\Delta^{k-1} f(a)}{\Gamma(\nu)} (t-a)^{\underline{\nu-1}} \\
= \;& \Delta_a^{-(\nu-1)} \left[\Delta^{k-1} f(t)\right] - \frac{\Delta^{k-1} f(a)}{\Gamma(\nu)} (t-a)^{\underline{\nu-1}} \\
= \;& \Delta_a^{1-\nu} \Delta^{k-1} f(t) - \frac{\Delta^{k-1} f(a)}{\Gamma(\nu)} (t-a)^{\underline{\nu-1}}.
\end{aligned}$$

Summing by parts $(k-1)$-more times yields

$$\Delta_a^{-\nu} \Delta^k f(t)$$

$$
\begin{aligned}
&= \Delta_a^{1-\nu}\Delta^{k-1}f(t) - \frac{\Delta^{k-1}f(a)}{\Gamma(\nu)}(t-a)^{\underline{\nu-1}} \\
&= \Delta_a^{2-\nu}\Delta^{k-2}f(t) - \frac{\Delta^{k-1}f(a)}{\Gamma(\nu)}(t-a)^{\underline{\nu-1}} - \frac{\Delta^{k-2}f(a)}{\Gamma(\nu-1)}(t-a)^{\underline{\nu-2}} \\
&= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \Delta_a^{k-\nu}f(t) - \sum_{i=1}^{k}\frac{\Delta^{k-i}f(a)}{\Gamma(\nu-i+1)}(t-a)^{\underline{\nu-i}} \\
&= \Delta_a^{k-\nu}f(t) - \sum_{j=0}^{k-1}\frac{\Delta^{j}f(a)}{\Gamma(\nu-k+j+1)}(t-a)^{\underline{\nu-k+j}}, \text{ for } t\in\mathbb{N}_{a+\nu}.
\end{aligned}
$$

Note that our assumption $\nu \notin \{1,2,...,k-1\}$ implies that $\nu-k+j+1 \notin (-\mathbb{N}_0)$, leaving the above expression well-defined.

Next, suppose that $\nu \in \{1,2,...,k-1\}$. Then $k-\nu\in\mathbb{N}$, and so we have for $t\in\mathbb{N}_{a+\nu}$,

$$
\begin{aligned}
&\Delta_a^{-\nu}\Delta^k f(t) \\
&= \Delta^{k-\nu}\Delta_{a+\nu}^{-(k-\nu)}\Delta_a^{-\nu}\Delta^k f(t), \text{ by Theorem 7} \\
&= \Delta^{k-\nu}\left[\Delta_a^{-k}\Delta^k f(t)\right], \text{ by Theorem 5} \\
&= \Delta^{k-\nu}\left[f(t) - \sum_{j=0}^{k-1}\frac{\Delta^j f(a)}{\Gamma(j+1)}(t-a)^{\underline{j}}\right], \text{ by the previous case} \\
&= \Delta^{k-\nu}f(t) - \sum_{j=0}^{k-1}\frac{\Delta^j f(a)}{\Gamma(j+1)}\left[\Delta^{k-\nu}(t-a)^{\underline{j}}\right] \\
&= \Delta^{k-\nu}f(t) - \sum_{j=k-\nu}^{k-1}\frac{\Delta^j f(a)}{\Gamma(j+1)}\frac{\Gamma(j+1)}{\Gamma(j+1-k+\nu)}(t-a)^{\underline{j-k+\nu}}, \text{ by (2.1)} \\
&= \Delta^{k-\nu}f(t) - \sum_{j=0}^{k-1}\frac{\Delta^j f(a)}{\Gamma(j+1-k+\nu)}(t-a)^{\underline{j-k+\nu}},
\end{aligned}
$$

with allowance for the convention $\frac{1}{\Gamma(-k)}=0$, for $k\in\mathbb{N}_0$. Combining the above two cases, (2.7) is proved.

We next consider (2.8). Suppose that $\nu,\mu>0$ with $M-1<\mu\le M$. Defining

$$g(t) := \Delta_a^{-(M-\mu)} f(t) \text{ and } b := a + M - \mu,$$

where b is the first point g's domain, we have for $t \in \mathbb{N}_{a+M-\mu+\nu}$,

$$\begin{aligned}
&\Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu} f(t) \\
= \;& \Delta_{a+M-\mu}^{-\nu}\Delta^{M}\left(g(t)\right), \text{ by Theorem 7} \\
= \;& \Delta_{a+M-\mu}^{M-\nu} g(t) - \sum_{j=0}^{M-1} \frac{\Delta^j g(b)}{\Gamma(\nu - M + j + 1)} (t-b)^{\underline{\nu - M + j}}, \text{ by (2.7)} \\
= \;& \Delta_{a+M-\mu}^{M-\nu}\Delta_a^{-(M-\mu)} f(t) - \sum_{j=0}^{M-1} \frac{\Delta^j \Delta_a^{-(M-\mu)} f(b)}{\Gamma(\nu - M + j + 1)} (t-b)^{\underline{\nu - M + j}} \\
= \;& \Delta_a^{\mu-\nu} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-M+\mu} f(a + M - \mu)}{\Gamma(\nu - M + j + 1)} (t - a - M + \mu)^{\underline{\nu - M + j}},
\end{aligned}$$

where in this last step, we applied Theorem 7 and, in the case $\nu > M$, Theorem 5. ■

Remark 6
- *Theorem 7 allows us to write (2.8) in the equivalent form*

$$\begin{aligned}
&\Delta_{a+M-\mu}^{-\nu}\Delta_a^{\mu} f(t) = \\
&\Delta_{a+\nu}^{\mu}\Delta_a^{-\nu} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-(M-\mu)} f(a + M - \mu)}{\Gamma(\nu - M + j + 1)} (t - a - M + \mu)^{\underline{\nu - M + j}},
\end{aligned}$$

  for $t \in \mathbb{N}_{a+M-\mu+\nu}$.

- *When $0 < \mu \leq 1$, the M terms $\left\{\Delta_a^{j-(M-\mu)} f(a + M - \mu)\right\}_{j=0}^{M-1}$ in (2.8) simplify nicely to the single term $f(a)$. More generally for $M - 1 < \mu \leq M$, however, we may alternately write $\Delta_a^{j-(M-\mu)} f(a + M - \mu)$ for each $j \in \{0, ..., M-1\}$ as*

$$\Delta_a^{j-(M-\mu)} f(a + M - \mu)$$

$$= \frac{1}{\Gamma(M-\mu-j)} \sum_{s=a}^{t+j-(M-\mu)} (t-\sigma(s))^{\underline{M-\mu-j-1}} f(s) \Big|_{t=a+M-\mu}$$

$$= \frac{1}{\Gamma(M-\mu-j)} \sum_{s=a}^{a+j} (a+M-\mu-\sigma(s))^{\underline{M-\mu-j-1}} f(s)$$

$$= \frac{1}{\Gamma(M-\mu-j)} \sum_{k=0}^{j} (M-\mu-\sigma(k))^{\underline{M-\mu-j-1}} f(k+a)$$

$$= \sum_{k=0}^{j} \binom{M-\mu-\sigma(k)}{j-k} f(k+a).$$

2.2.4 Composing a Difference with a Difference

We conclude our presentation of the four fractional composition rules with a Theorem governing the composition of two fractional differences. Atici and Eloe in [3] proved the special case of (9) below where μ is a natural number—we extend their result here to the fully fractional case with μ any positive, real number. One quickly observes the similarity between composition rule (2.9) and composition rule (2.8). In fact, though their proofs are different, the two could easily be stated as a single rule governing all cases with an inner fractional differentiation.

Theorem 9 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu, \mu > 0$ with $N-1 < \nu \leq N$ and $M-1 < \mu \leq M$. Then for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$,*

$$\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a} f(t) =$$

$$\Delta^{\nu+\mu}_{a} f(t) - \sum_{j=0}^{M-1} \frac{\Delta^{j-M+\mu}_{a} f(a+M-\mu)}{\Gamma(-\nu-M+j+1)} (t-a-M+\mu)^{\underline{-\nu-M+j}}, \qquad (2.9)$$

where, in agreement with both rule (2.6) and the standard convention on Γ, the terms in the summation vanish in the case $\nu \in \mathbb{N}_0$.

Proof. Let f, ν and μ be given as in the statement of the theorem. Recall that Lemma 6 proves (2.9) in the case when $\nu = N$.

On the other hand, if $N-1<\nu<N$, then we have for $t \in \mathbb{N}_{a+M-\mu+N-\nu}$

$$\begin{aligned}
&\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a}f(t)\\
=\;& \Delta^{N}\left[\Delta^{-(N-\nu)}_{a+M-\mu}\Delta^{\mu}_{a}f(t)\right]\\
=\;& \Delta^{N}\Bigg[\Delta_a^{-N+\nu+\mu}f(t)-\\
&\sum_{j=0}^{M-1}\frac{\Delta_a^{j-M+\mu}f(a+M-\mu)}{\Gamma(N-\nu-M+j+1)}(t-a-M+\mu)^{\underline{N-\nu-M+j}}\Bigg], \text{ by (2.8)}\\
=\;& \Delta_a^{\nu+\mu}f(t)-\\
&\sum_{j=0}^{M-1}\frac{\Delta_a^{j-M+\mu}f(a+M-\mu)}{\Gamma(N-\nu-M+j+1)}\Delta^{N}\left[(t-a-M+\mu)^{\underline{N-\nu-M+j}}\right], \text{ by (2.5)}\\
=\;& \Delta_a^{\nu+\mu}f(t)-\sum_{j=0}^{M-1}\frac{\Delta_a^{j-M+\mu}f(a+M-\mu)}{\Gamma(-\nu-M+j+1)}(t-a-M+\mu)^{\underline{-\nu-M+j}}.
\end{aligned}$$

■

By the same token as (2.9), we may write in reverse order

$$\begin{aligned}
&\Delta^{\mu}_{a+N-\nu}\Delta^{\nu}_{a}f(t)=\\
&\qquad\Delta_a^{\mu+\nu}f(t)-\sum_{j=0}^{N-1}\frac{\Delta_a^{j-N+\nu}f(a+N-\nu)}{\Gamma(-\mu-N+j+1)}(t-a-N+\nu)^{\underline{-\mu-N+j}},
\end{aligned}$$

where the terms in the summation vanish if $\mu \in \mathbb{N}_0$. Combining the above with (2.9), we obtain the following two additional rules for composing fractional differences.

Corollary 10 *Let $f:\mathbb{N}_a \to \mathbb{R}$ and $\nu, \mu > 0$ be given with $N-1<\nu\le N$ and*

$M-1<\mu\leq M$. *Then, for* $t\in\mathbb{N}_{a+M-\mu+N-\nu}$,

$$\begin{aligned}\Delta^{\nu}_{a+M-\mu}\Delta^{\mu}_{a}f(t) = \Delta^{\mu}_{a+N-\nu}\Delta^{\nu}_{a}f(t)\\ &+\sum_{j=0}^{N-1}\frac{\Delta_a^{j-N+\nu}f(a+N-\nu)}{\Gamma(-\mu-N+j+1)}(t-a-N+\nu)^{\underline{-\mu-N+j}}\\ &-\sum_{j=0}^{M-1}\frac{\Delta_a^{j-M+\mu}f(a+M-\mu)}{\Gamma(-\nu-M+j+1)}(t-a-M+\mu)^{\underline{-\nu-M+j}}.\end{aligned}$$

Moreover, for $t\in\mathbb{N}_{a+2(N-\nu)}$,

$$\Delta^{\nu}_{a+N-\nu}\Delta^{\nu}_{a}f(t) = \Delta^{2\nu}_{a}f(t)-\sum_{j=0}^{N-1}\frac{\Delta_a^{j-N+\nu}f(a+N-\nu)}{\Gamma(-\nu-N+j+1)}(t-a-N+\nu)^{\underline{-\nu-N+j}}.$$

2.3 Application

In order to explicitly solve a ν^{th}-order fractional initial value problem, we need many of the tools developed thus far in Chapters 1 and 2. Most importantly, we apply the general power rule (2.3) from Lemma 4 and the two composition rules from Theorems 7 and 8.

Theorem 11 *Let* $f:\mathbb{N}_a\to\mathbb{R}$ *and* $\nu>0$ *be given with* $N-1<\nu\leq N$. *Consider the* ν^{th}*-order fractional difference equation*

$$\Delta^{\nu}_{a+\nu-N}y(t)=f(t),\quad t\in\mathbb{N}_a \tag{2.10}$$

and the corresponding ν^{th}*-order fractional initial value problem*

$$\begin{cases}\Delta^{\nu}_{a+\nu-N}y(t)=f(t), & t\in\mathbb{N}_a\\ \Delta^{i}y(a+\nu-N)=A_i, & i\in\{0,1,...,N-1\},\ A_i\in\mathbb{R}.\end{cases} \tag{2.11}$$

The general solution to (2.10) is

$$y(t) = \sum_{i=0}^{N-1} \alpha_i \, (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \; t \in \mathbb{N}_{a+\nu-N}, \tag{2.12}$$

where $\{\alpha_i\}_{i=0}^{N-1}$ *are* N *real constants. Moreover, the unique solution to (2.11) is (2.12) with particular constants*

$$\alpha_i = \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!} (i-k)^{\underline{N-\nu}} \binom{i}{p} \binom{i-p}{k} A_p,$$

for $i \in \{0, 1, ..., N-1\}$.

Proof. Let f and ν be as given in the statement of the theorem. For arbitrary but fixed constants $\{\alpha_i\}_{i=0}^{N-1} \subseteq \mathbb{R}$, define $y : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$ by

$$y(t) := \sum_{i=0}^{N-1} \alpha_i \, (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t).$$

Here, we extend the usual domain of the fractional sum $\Delta_a^{-\nu} f$ from $\mathbb{N}_{a+\nu}$ to the larger set $\mathbb{N}_{a+\nu-N}$ by including the N zeros of $\Delta_a^{-\nu} f$ discussed in Section 1.1.3.

We will show that any function of y's form is a solution to (2.10) and that every solution to (2.10) must be of y's form. Beginning with the former, observe that for $t \in \mathbb{N}_a$,

$$\begin{aligned} & \Delta_{a+\nu-N}^{\nu} y(t) \\ = \; & \Delta_{a+\nu-N}^{\nu} \left[\sum_{i=0}^{N-1} \alpha_i \, (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t) \right] \\ = \; & \sum_{i=0}^{N-1} \alpha_i \Delta_{a+\nu-N}^{\nu} \, (t-a)^{\underline{i+\nu-N}} + \Delta_{a+\nu-N}^{\nu} \Delta_a^{-\nu} f(t). \end{aligned}$$

At this point, we would like to apply power rule (2.3) within the summation and Theorem 7 on the second term, but neither may be applied directly as is due to the incorrect lower limit on the operator $\Delta^{\nu}_{a+\nu-N}$. However, in this case, we obtain the correct lower limit by throwing away the zero terms involved. Writing each term out by definition, we have

$$\begin{aligned}
\Delta^{\nu}_{a+\nu-N}\,(t-a)^{\underline{i+\nu-N}} &= \frac{1}{\Gamma(-\nu)}\sum_{s=a+\nu-N}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}(s-a)^{\underline{i+\nu-N}} \\
&= \frac{1}{\Gamma(-\nu)}\sum_{s=a+\nu-N+i}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}(s-a)^{\underline{i+\nu-N}} \\
&= \Delta^{\nu}_{a+i+\nu-N}\,(t-a)^{\underline{i+\nu-N}},\ \text{for } i\in\{0,...,N-1\}
\end{aligned}$$

and

$$\begin{aligned}
\Delta^{\nu}_{a+\nu-N}\Delta^{-\nu}_{a}f(t) &= \frac{1}{\Gamma(-\nu)}\sum_{s=a+\nu-N}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}\Delta^{-\nu}_{a}f(s) \\
&= \frac{1}{\Gamma(-\nu)}\sum_{s=a+\nu}^{t+\nu}(t-\sigma(s))^{\underline{-\nu-1}}\Delta^{-\nu}_{a}f(s) \\
&= \Delta^{\nu}_{a+\nu}\Delta^{-\nu}_{a}f(t).
\end{aligned}$$

Therefore, applying (2.3) and Theorem 7, we have

$$\begin{aligned}
\Delta^{\nu}_{a+\nu-N}y(t) &= \sum_{i=0}^{N-1}\alpha_i\Delta^{\nu}_{a+i+\nu-N}\,(t-a)^{\underline{i+\nu-N}}+\Delta^{\nu}_{a+\nu}\Delta^{-\nu}_{a}f(t) \\
&= \sum_{i=0}^{N-1}\alpha_i\frac{\Gamma(i+\nu-N+1)}{\Gamma(i-N+1)}\,(t-a)^{\underline{i-N}}+\Delta^{\nu-\nu}_{a}f(t) \\
&= f(t).
\end{aligned}$$

Next, we show that every solution of (2.10) has y's form. Suppose that $z:\mathbb{N}_{a+\nu-N}\to\mathbb{R}$ is a solution to (2.10). Then we may apply Theorem 8 to directly solve

(2.10) for z:

$$\begin{aligned} & \Delta^{\nu}_{a+\nu-N} z(t) = f(t), \text{ for } t \in \mathbb{N}_a \\ \Rightarrow \quad & \Delta_a^{-\nu} \Delta^{\nu}_{a+\nu-N} z(t) = \Delta_a^{-\nu} f(t), \text{ for } t \in \mathbb{N}_{a+\nu-N} \\ \Rightarrow \quad & \Delta^{0}_{a+\nu-N} z(t) - \sum_{i=0}^{N-1} \frac{\Delta^{i-(N-\nu)}_{a+\nu-N} z(a)}{\Gamma(\nu - N + i + 1)} (t-a)^{\underline{\nu-N+i}} = \Delta_a^{-\nu} f(t) \\ \Rightarrow \quad & z(t) = \sum_{i=0}^{N-1} \left(\frac{\Delta^{i+\nu-N}_{a+\nu-N} y(a)}{\Gamma(i+\nu-N+1)} \right) (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t). \qquad (2.13) \end{aligned}$$

Since z has the same form as (2.12), we have shown that

$$y(t) = \sum_{i=0}^{N-1} \alpha_i \, (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \; t \in \mathbb{N}_{a+\nu-N}$$

is the general solution of (2.10).

The next task is to find the particular $\{\alpha_i\}_{i=0}^{N-1} \subseteq \mathbb{R}$ which make y a solution to (2.11). From (2.13), we already know that these constants have the form

$$\alpha_i = \frac{\Delta^{i+\nu-N}_{a+\nu-N} y(a)}{\Gamma(i+\nu-N+1)}, \text{ for } i \in \{0, ..., N-1\},$$

but to be relevant, we must write each α_i in terms of $\{A_p\}_{p=0}^{N-1}$. In other words, we must find a way to write each fractional initial condition $\Delta^{i+\nu-N}_{a+\nu-N} y(a)$ in terms of the given whole-order initial conditions
$\{\Delta^p y(a+\nu-N)\}_{p=0}^{N-1}$. To accomplish this, we need two tools:

- the fractional binomial representation (1.11) from Theorem 3:

$$\Delta_a^{\nu} g(t) = \sum_{k=0}^{t-a+\nu} (-1)^k \binom{\nu}{k} g(t+\nu-k), \text{ for } t \in \mathbb{N}_{a+N-\nu}.$$

- the following well-known formula (see [13]):

$$g(t+m)=\sum_{k=0}^{m}\binom{m}{k}\Delta^k g(t),\text{ for } m\in\mathbb{N}_0. \tag{2.14}$$

Applying the above two formulas directly yields

$$\begin{aligned}
&\Delta_{a+\nu-N}^{i+\nu-N}y(a)\\
=\ &\sum_{k=0}^{i}(-1)^k\binom{i+\nu-N}{k}y(a+i+\nu-N-k)\\
=\ &\sum_{k=0}^{i}(-1)^k\binom{i+\nu-N}{k}y((a+\nu-N)+(i-k))\\
=\ &\sum_{k=0}^{i}(-1)^k\binom{i+\nu-N}{k}\sum_{p=0}^{i-k}\binom{i-k}{p}\Delta^p y(a+\nu-N)\\
=\ &\sum_{k=0}^{i}\sum_{p=0}^{i-k}(-1)^k\binom{i+\nu-N}{k}\binom{i-k}{p}A_p\\
=\ &\sum_{p=0}^{i}\sum_{k=0}^{i-p}(-1)^k\binom{i+\nu-N}{k}\binom{i-k}{p}A_p\\
=\ &\sum_{p=0}^{i}\sum_{k=0}^{i-p}(-1)^k\frac{\Gamma(i+\nu-N+1)}{k!\Gamma(i+\nu-N-k+1)}\frac{(i-k)!}{p!(i-k-p)!}A_p\\
=\ &\Gamma(i+\nu-N+1)\\
&\quad\cdot\sum_{p=0}^{i}\sum_{k=0}^{i-p}\frac{(-1)^k}{i!}\frac{\Gamma(i-k+1)}{\Gamma(i-k+1+\nu-N)}\frac{i!}{p!(i-p)!}\frac{(i-p)!}{k!(i-p-k)!}A_p\\
=\ &\Gamma(i+\nu-N+1)\sum_{p=0}^{i}\sum_{k=0}^{i-p}\frac{(-1)^k}{i!}(i-k)^{\underline{N-\nu}}\binom{i}{p}\binom{i-p}{k}A_p.
\end{aligned}$$

It follows that

$$\alpha_i=\sum_{p=0}^{i}\left(\sum_{k=0}^{i-p}\frac{(-1)^k}{i!}(i-k)^{\underline{N-\nu}}\binom{i}{p}\binom{i-p}{k}\right)A_p,\quad \text{for } i\in\{0,...,N-1\}.$$

Moreover, this solution is unique since initial value problem (2.11) was solved directly, with no restrictions or lost information. ■

Calculating the constants $\{\alpha_i\}_{i=0}^{N-1}$ is cumbersome at best, especially when done by hand. However, when solving lower order problems, one may use the following user-friendly expressions for the first several α_i. Define

$$g : \mathbb{N}_0 \to \mathbb{R} \text{ by } g(t) := t^{\underline{N-\nu}}.$$

Then the first five constants α_i (provided they are necessary) are always given by:

$$\begin{aligned}
\alpha_0 &= g(0)\, A_0, \\
\alpha_1 &= g(1)\, A_1 - (g(0) - g(1))\, A_0, \\
\alpha_2 &= \frac{g(2)}{2} A_2 - (g(1) - g(2))\, A_1 + \frac{g(0) - 2g(1) + g(2)}{2} A_0, \\
\alpha_3 &= \frac{g(3)}{6} A_3 - \frac{g(2) - g(3)}{2} A_2 + \frac{g(1) - 2g(2) + g(3)}{2} A_1 \\
&\quad - \frac{g(0) - 3g(1) + 3g(2) - g(3)}{6} A_0, \\
\alpha_4 &= \frac{g(4)}{24} A_4 - \frac{g(3) - g(4)}{6} A_3 + \frac{g(2) - 2g(3) + g(4)}{4} A_2 \\
&\quad - \frac{g(1) - 3g(2) + 3g(3) - g(4)}{6} A_1 \\
&\quad + \frac{g(0) - 4g(1) + 6g(2) - 4g(3) + g(4)}{24} A_0.
\end{aligned}$$

Note that if $\nu = N$ in (2.11), we have the well-studied whole-order initial value problem

$$\begin{cases} \Delta^N y(t) = f(t), & t \in \mathbb{N}_a \\ \Delta^i y(a) = A_i, & i \in \{0, 1, ..., N-1\},\ A_i \in \mathbb{R}. \end{cases}$$

In this case, the solution given in Theorem 11 simplifies considerably as

$$\begin{aligned}
y(t) &= \sum_{i=0}^{N-1}\left[\sum_{p=0}^{i}\left(\sum_{k=0}^{i-p}\frac{(-1)^k}{i!}\binom{i}{p}\binom{i-p}{k}\right)A_p\right](t-a)^{\underline{i}}+\Delta_a^{-N}f(t)\\
&= \sum_{i=0}^{N-1}\left[\sum_{p=0}^{i}\frac{A_p}{i!}\binom{i}{p}\sum_{k=0}^{i-p}(-1)^k\binom{i-p}{k}\right](t-a)^{\underline{i}}+\Delta_a^{-N}f(t)\\
&= \sum_{i=0}^{N-1}\frac{A_i}{i!}(t-a)^{\underline{i}}+\Delta_a^{-N}f(t),\ \text{ for } t\in\mathbb{N}_a, \qquad (2.15)
\end{aligned}$$

the well-known solution to the whole-order initial value problem. The substantial simplification prior to (2.15) follows from the result

$$\sum_{k=0}^{i-p}(-1)^k\binom{i-p}{k}=\begin{cases}0, & i-p>0\\ 1, & i-p=0.\end{cases}$$

Next, one may prefer to write everything in solution (2.15) in terms of y:

$$y(t)=\sum_{i=0}^{N-1}\frac{\Delta^i y(a)}{i!}(t-a)^{\underline{i}}+\Delta_a^{-N}\Delta^N y(t),$$

which yields a version of Taylor's Theorem for functions $y:\mathbb{N}_a\to\mathbb{R}$. More specifically, since

$$\Delta_a^{-N}\Delta^N y(t)=\frac{1}{\Gamma(N)}\sum_{s=a}^{t-N}(t-\sigma(s))^{\underline{N-1}}\Delta^N y(s)\to 0 \text{ pointwise as } N\to\infty,$$

we may write

$$y(t)=\sum_{i=0}^{\infty}\frac{\Delta^i y(a)}{i!}(t-a)^{\underline{i}},\ \text{ for } t\in\mathbb{N}_a. \qquad (2.16)$$

However, (2.16) turns out to be just a disguised version of formula (2.14). To see this, let $t \in \mathbb{N}_a$ be given by $t = a + m$, for some $m \in \mathbb{N}_0$. Then (2.16) becomes

$$y(t) = y(a+m) = \sum_{i=0}^{\infty} \frac{\Delta^i y(a)}{i!} m^{\underline{i}} = \sum_{i=0}^{m} \binom{m}{i} \Delta^i y(a).$$

2.4 Examples

Example 1 *Consider the following 2.7^{th}-order initial value problem*

$$\begin{cases} \Delta_{-0.3}^{2.7} y(t) = t^{\underline{2}},\ t \in \mathbb{N}_0 \\ y(-0.3) = 2,\ \ \Delta y(-0.3) = 3,\ \ \Delta^2 y(-0.3) = 5. \end{cases} \tag{2.17}$$

Note that (2.17) is a specific instance of (2.11) from Theorem 11, with

$$\begin{aligned} &a = 0, \quad \nu = 2.7, \quad N = 3, \quad f(t) = t^{\underline{2}}, \\ &A_0 = 2, \quad A_1 = 3, \quad A_2 = 5. \end{aligned}$$

Therefore, the solution to (2.17) is given by

$$\begin{aligned} y(t) &= \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \quad \text{for } t \in \mathbb{N}_{a+\nu-N} \\ &= \sum_{i=0}^{2} \alpha_i t^{\underline{i-0.3}} + \Delta_0^{-2.7} t^{\underline{2}}, \quad \text{for } t \in \mathbb{N}_{-0.3}, \end{aligned}$$

where,

$$\begin{aligned} \alpha_0 &= 0^{\underline{0.3}} A_0 \\ \alpha_1 &= 1^{\underline{0.3}} A_1 - \left(0^{\underline{0.3}} - 1^{\underline{0.3}}\right) A_0, \\ \alpha_2 &= \frac{2^{\underline{0.3}}}{2} A_2 - \left(1^{\underline{0.3}} - 2^{\underline{0.3}}\right) A_1 + \frac{0^{\underline{0.3}} - 2 \cdot 1^{\underline{0.3}} + 2^{\underline{0.3}}}{2} A_0, \end{aligned}$$

$$\Longrightarrow \alpha_0 \approx 1.541, \quad \alpha_1 \approx 3.962, \quad \alpha_2 \approx 3.684.$$

Our only remaining task is to calculate, via power rule (2.2), the sum

$$\begin{aligned} & \Delta_0^{-2.7} t^{\underline{2}} \\ = \; & \Delta_2^{-2.7} t^{\underline{2}}, \textit{ since } 0^{\underline{2}} = 1^{\underline{2}} = 0 \\ = \; & \frac{\Gamma(3)}{\Gamma(5.7)} t^{\underline{4.7}} \\ \approx \; & 0.0276 t^{\underline{4.7}}. \end{aligned}$$

Therefore, the unique solution to (2.17) may be approximated as

$$y(t) \approx 1.541 t^{\underline{-0.3}} + 3.962 t^{\underline{0.7}} + 3.684 t^{\underline{1.7}} + 0.0276 t^{\underline{4.7}}, \textit{ for } t \in \mathbb{N}_{-0.3}.$$

Example 2 *Consider the composed difference operator $\Delta_{a+M-\mu}^{\nu} \circ \Delta_a^{\mu}$, where ν and μ are two positive non-integers who sum to an integer. One ought to wonder by how much the fractional composition $\Delta_{a+M-\mu}^{\nu} \circ \Delta_a^{\mu}$ differs from the corresponding whole-order operator $\Delta^{\nu+\mu}$.*

Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and find $M, N, P \in \mathbb{N}$ so that $N-1 < \nu < N$, $M-1 < \mu < M$ and $\nu + \mu = P$. Then $N + M = P + 1$ and so

$$\mathcal{D}\left\{\Delta_{a+M-\mu}^{\nu} \Delta_a^{\mu} f\right\} = \mathbb{N}_{a+M-\mu+N-\nu} = \mathbb{N}_{a+(N+M)-(\nu+\mu)} = \mathbb{N}_{a+1} \subseteq \mathcal{D}\left\{\Delta^P f\right\}.$$

Applying composition rule (2.9) from Theorem 9, we have for $t \in \mathbb{N}_{a+1}$,

$$\begin{aligned} & \Delta_{a+M-\mu}^{\nu} \Delta_a^{\mu} f(t) \\ = \; & \Delta_a^P f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-M+\mu} f(a+M-\mu)}{\Gamma(-\nu-M+j+1)} (t-a-M+\mu)^{\underline{-\nu-M+j}}. \end{aligned}$$

Observe that for $j \in \{0, ..., M-1\}$,

$$\begin{aligned}
\lim_{\substack{\nu \to N^- \\ \mu \to (M-1)^+}} (t-a-M+\mu)^{\underline{-\nu-M+j}} &= (t-a-1)^{\underline{j-P-1}} \\
&= \frac{\Gamma(t-a)}{\Gamma(t-a+P+1-j)} \in (0,\infty), \\
\lim_{\mu \to (M-1)^+} \Delta_a^{j-M+\mu} f(a+M-\mu) &= \Delta_a^{j-1} f(a+1) \in [0,\infty),
\end{aligned}$$

and

$$\lim_{\nu \to N^-} \frac{1}{\Gamma(-\nu-M+j+1)} = 0.$$

It follows that

$$\lim_{\substack{\nu \to N^- \\ \mu \to (M-1)^+}} \left(\Delta_a^P f(t) - \Delta_{a+M-\mu}^{\nu} \Delta_a^{\mu} f(t)\right) = 0,$$

as expected. Compare this result to (2.6) from Lemma 6.

We also observe how these two operators compare as t *grows large:*

$$\begin{aligned}
&\lim_{t \to \infty} \left(\Delta_a^P f(t) - \Delta_{a+M-\mu}^{\nu} \Delta_a^{\mu} f(t)\right) \\
= \; &\lim_{t \to \infty} \left(\sum_{j=0}^{M-1} \frac{\Delta_a^{j-M+\mu} f(a+M-\mu)}{\Gamma(-\nu-M+j+1)} (t-a-M+\mu)^{\underline{-\nu-M+j}} \right) \\
= \; &\sum_{j=0}^{M-1} \frac{\Delta_a^{j-M+\mu} f(a+M-\mu)}{\Gamma(-\nu-M+j+1)} \lim_{t \to \infty} \frac{\Gamma(t-a-M+\mu+1)}{\Gamma(t-a+P+1-j)} \\
= \; &0,
\end{aligned}$$

after applying the Squeeze Theorem, since we have for $t \in \mathbb{N}_{a+2}$ *that*

$$\begin{aligned}
&t-a-M+\mu+1 \geq 2 \text{ and } t-a+P+1-j \geq 2 \\
\implies \; &0 < \frac{\Gamma(t-a-M+\mu+1)}{\Gamma(t-a+P+1-j)} \leq \frac{\Gamma(t-a+1)}{\Gamma(t-a+2)} = \frac{1}{t-a+1} \\
\implies \; &\lim_{t \to \infty} \frac{\Gamma(t-a-M+\mu+1)}{\Gamma(t-a+P+1-j)} = 0.
\end{aligned}$$

We glean here that the discrepancy between a single P^{th}-order difference and two composed fractional differences whose orders sum to P depends explicitly on how far the point t is away from the first point in their common domain, $\mathbb{N}_{a+1}$. Furthermore, we see this discrepancy vanish as t grows large.

The following table shows the first seven of these differences for the specific case $f(t) = e^t$, $a = 0$ and $\nu = \mu = \frac{1}{2}$.

t	1	2	3	4	5	6	7
$\Delta^{\frac{1}{2}}_{\frac{1}{2}}\Delta^{\frac{1}{2}}_{0}e^t - \Delta e^t$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{5}{128}$	$\frac{7}{256}$	$\frac{21}{1{,}024}$	$\frac{33}{2{,}048}$	$\frac{429}{32{,}768}$

Chapter 3

The Fractional Laplace Transform Method

3.1 Introduction

Pierre Laplace's Transform has been a key tool in solving differential equations since the late eighteenth century. However, since many problems are solved on domains other than the reals, such as on

$$\mathbb{N}_0 := \{0, 1, 2, ...\} \text{ or } \left(\frac{1}{2}\right)^{\mathbb{N}_0} := \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...\right\} \cup \{0\},$$

mathematicians have developed a general theory of analysis to govern all these domains simultaneously. In fact, the theory of *time scales* considers functions on any non-empty, closed subset of the reals. Bohner and Peterson provide a great introduction to time scales theory in [8].

When the general time scales' Laplace Transform from Bohner and Peterson's book [8] is considered for the domain $\mathbb{N}_a$, we have the so-called Discrete Laplace

Transform (Note that the popular $\mathbb{Z}$-transform, which has dominated discrete work for years, is similar but distinct). Several mathematicians have worked to apply the Discrete Laplace Transform to the fractional calculus setting in recent years. In particular, Atici and Eloe offer results similar to some presented in this chapter in [2] and [3]. However, several important definitions from [2] and [3] differ from those made here (including the transform itself and the convolution), so a direct comparison of results cannot be made.

The author's purpose in this chapter is to provide a rigorous development of the Discrete Laplace Transform in the fractional calculus setting. The end goal is to apply the resulting Fractional Laplace Transform Method to resolve the fractional initial value problem first introduced in Chapter 2 (Theorem 11).

3.1.1 The Laplace Transform

In the general time scale setting, the Laplace Transform of a regulated function $f : \mathbb{T}_a \to \mathbb{R}$ is given by

$$\mathcal{L}_a \{f\} (s) := \int_a^\infty e^\sigma_{\ominus s}(t,a) f(t) \Delta t, \text{ for all } s \in \mathcal{D}\{f\}, \tag{3.1}$$

where $a \in \mathbb{R}$ is fixed, $\mathbb{T}_a$ is an unbounded time scale with infimum a and $\mathcal{D}\{f\}$ is the set of all regressive, complex constants for which the integral converges. Laplace Transform (3.1) is a slight generalization of the one introduced by Bohner and Peterson in [8].

We again consider the discrete domain

$$\mathbb{N}_a := \mathbb{N}_0 + \{a\} = \{a, a+1, a+2, ...\}, \text{ where } a \in \mathbb{R} \text{ is fixed.}$$

The uniformly positive graininess of $\mathbb{N}_a$ (i.e. $\mu(t) \equiv 1$ on $\mathbb{N}_a$) implies that every function $f : \mathbb{N}_a \to \mathbb{R}$ is regulated, meaning that left and right-hand limits exist for every point $t \in \mathbb{N}_a$. Moreover, the only nonregressive, complex constant for the domain $\mathbb{N}_a$ is $s = -1$, since

$$1 + \mu(t)s = 0 \Longleftrightarrow s = -1.$$

Therefore, given any $s \in \mathbb{C}\backslash\{-1\}$, we may simplify the exponential term in (3.1) to

$$e^{\sigma}_{\ominus s}(t,a) = e_{\ominus s}(t+1,a) = (1 + \ominus s)^{t+1-a} = \left(1 - \frac{s}{1+s}\right)^{t-a+1} = \frac{1}{(s+1)^{t-a+1}}.$$

It follows that for any function $f : \mathbb{N}_a \to \mathbb{R}$, the Laplace Transform (3.1) can be written as

$$\begin{aligned} \mathcal{L}_a\{f\}(s) &= \int_a^\infty \frac{f(t)}{(s+1)^{t-a+1}} \Delta t = \int_0^\infty \frac{f(t+a)}{(s+1)^{t+1}} \Delta t \\ &= \sum_{k=0}^{\infty} \frac{f(k+a)}{(s+1)^{k+1}}, \end{aligned} \tag{3.2}$$

for each $s \in \mathbb{C}\backslash\{-1\}$ for which the above series converges. Series (3.2) is the Discrete Laplace Transform of a function $f : \mathbb{N}_a \to \mathbb{R}$. However, an important question remains: For which $s \in \mathbb{C}\backslash\{-1\}$ does (3.2) converge? This depends greatly, of course, upon the function f being transformed.

Definition 4 *We say that a function $f : \mathbb{N}_a \to \mathbb{R}$ is of* exponential order r *($r > 0$) if there exists a constant $A > 0$ such that*

$$|f(t)| \leq Ar^t, \textit{ for } t \in \mathbb{N}_a \textit{ sufficiently large.}$$

Suppose that a given function $f : \mathbb{N}_a \to \mathbb{R}$ is of some exponential order $r > 0$. Then there must be some constant $A > 0$ and natural number $m \in \mathbb{N}_0$ such that for each $t \in \mathbb{N}_{a+m}$, we have $|f(t)| \leq Ar^t$. We may write, therefore, for any $s \in \mathbb{C}$ outside of the ball $\overline{B_{-1}(r)}$,

$$\begin{aligned}
\mathcal{L}_a \{f\}(s) &= \sum_{k=0}^{\infty} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| \\
&= \sum_{k=0}^{m-1} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| + \sum_{k=m}^{\infty} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| \\
&\leq \sum_{k=0}^{m-1} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| + \sum_{k=m}^{\infty} \frac{Ar^{k+a}}{|s+1|^{k+1}} \\
&= \sum_{k=0}^{m-1} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| + \frac{Ar^a}{|s+1|} \sum_{k=m}^{\infty} \left(\frac{r}{|s+1|} \right)^k \\
&= \sum_{k=0}^{m-1} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| + \frac{Ar^a}{|s+1|} \frac{\left(\frac{r}{|s+1|}\right)^m}{1 - \left(\frac{r}{|s+1|}\right)} \\
&= \sum_{k=0}^{m-1} \left| \frac{f(k+a)}{(s+1)^{k+1}} \right| + \frac{A}{|s+1|^m} \frac{r^{a+m}}{|s+1| - r} \\
&< \infty.
\end{aligned}$$

This fact allows us to guarantee the convergence of $\mathcal{L}_a \{f\}(s)$ by restricting its domain, whenever f is of some exponential order.

Lemma 12 *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$. Then*

$$\mathcal{L}_a \{f\}(s) \text{ exists for } s \in \mathbb{C} \backslash \overline{B_{-1}(r)}.$$

For the remainder of this chapter, we restrict ourselves to functions of exponential order, insuring that their Laplace Transforms $\mathcal{L}_a \{f\}(s)$ exist on respective domains $\mathbb{C} \backslash \overline{B_{-1}(r)}$, as pictured below.

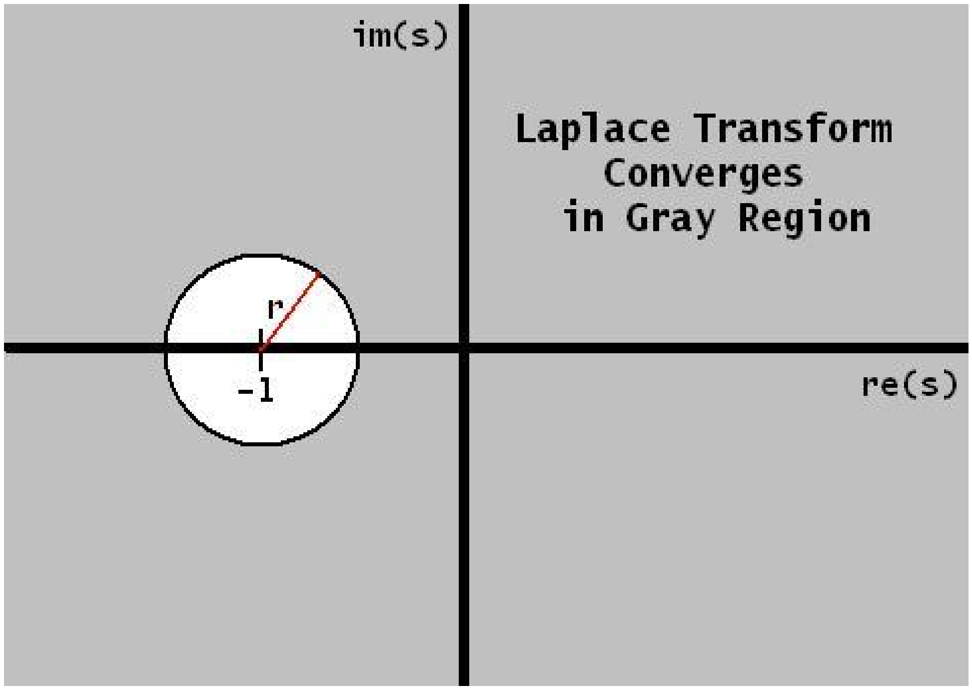

Figure 3.1: Convergence for the Laplace Transform

Of course, the exponential function $e_p(t,a)$ is itself of exponential order, providing the following important example.

Example 3 *Let $p \in \mathbb{C}$ be given. Recall that the exponential function for the time scale $\mathbb{N}_a$ is given by*

$$e_p(t,a) = (1+p)^{t-a}, \; t \in \mathbb{N}_a.$$

Clearly, $(1+p)^{t-a}$ is of exponential order $1+p$. Therefore, we have

$$\begin{aligned} \mathcal{L}_a\left\{(1+p)^{t-a}\right\}(s) = \sum_{k=0}^{\infty} \frac{(1+p)^k}{(s+1)^{k+1}} &= \frac{1}{s+1}\sum_{k=0}^{\infty}\left(\frac{p+1}{s+1}\right)^k \\ &= \frac{1}{s+1}\left(\frac{1}{1-\frac{p+1}{s+1}}\right) \\ &= \frac{1}{s-p}, \end{aligned}$$

for $s \in \mathbb{C}\backslash\overline{B_{-1}(1+p)}$. *An important special case of the above formula is*

$$\mathcal{L}_a\{1\}(s) = \frac{1}{s}, \text{ for } s \in \mathbb{C}\backslash\overline{B_{-1}(1)}.$$

Whenever the Laplace Transform does exist, it is linear and one-to-one, vital properties for obtaining results developed in this chapter. The proof of the following lemma is left to the reader.

Lemma 13 *Suppose* $f, g : \mathbb{N}_a \to \mathbb{R}$ *are of exponential order* $r > 0$ *and let* $c_1, c_2 \in \mathbb{C}$. *Then*

$$\mathcal{L}_a\{c_1 f + c_2 g\}(s) = c_1\mathcal{L}_a\{f\}(s) + c_2\mathcal{L}_a\{g\}(s), \text{ for } s \in \mathbb{C}\backslash\overline{B_{-1}(r)}, \tag{3.3}$$

and

$$\mathcal{L}_a\{f\}(s) \equiv \mathcal{L}_a\{g\}(s) \text{ on } s \in \mathbb{C}\backslash\overline{B_{-1}(r)} \Longleftrightarrow f(t) \equiv g(t) \text{ on } \mathbb{N}_a. \tag{3.4}$$

Also, it is often necessary to track how a shifted Laplace Transform relates to the original, as described in the following lemma.

Lemma 14 *Let* $m \in \mathbb{N}_0$ *be given and suppose* $f : \mathbb{N}_{a-m} \to \mathbb{R}$ *and* $g : \mathbb{N}_a \to \mathbb{R}$ *are of exponential order* $r > 0$. *Then for* $s \in \mathbb{C}\backslash\overline{B_{-1}(r)}$,

$$\mathcal{L}_{a-m}\{f\}(s) = \frac{1}{(s+1)^m}\mathcal{L}_a\{f\}(s) + \sum_{k=0}^{m-1}\frac{f(k+a-m)}{(s+1)^{k+1}} \tag{3.5}$$

and

$$\mathcal{L}_{a+m}\{g\}(s) = (s+1)^m\mathcal{L}_a\{g\}(s) - \sum_{k=0}^{m-1}(s+1)^{m-1-k}g(k+a). \tag{3.6}$$

Proof. Let f, g, r and m be as given in the statement of the lemma. Then for $s \in \mathbb{C} \backslash \overline{B_{-1}(r)}$,

$$\begin{aligned}
\mathcal{L}_{a-m}\{f\}(s) &= \sum_{k=0}^{\infty} \frac{f(k+a-m)}{(s+1)^{k+1}} \\
&= \sum_{k=m}^{\infty} \frac{f(k+a-m)}{(s+1)^{k+1}} + \sum_{k=0}^{m-1} \frac{f(k+a-m)}{(s+1)^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{f(k+a)}{(s+1)^{k+m+1}} + \sum_{k=0}^{m-1} \frac{f(k+a-m)}{(s+1)^{k+1}} \\
&= \frac{1}{(s+1)^m} \mathcal{L}_a\{f\}(s) + \sum_{k=0}^{m-1} \frac{f(k+a-m)}{(s+1)^{k+1}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{a+m}\{g\}(s) &= \sum_{k=0}^{\infty} \frac{g(k+a+m)}{(s+1)^{k+1}} \\
&= \sum_{k=m}^{\infty} \frac{g(k+a)}{(s+1)^{k-m+1}} \\
&= \sum_{k=0}^{\infty} \frac{g(k+a)}{(s+1)^{k-m+1}} - \sum_{k=0}^{m-1} \frac{g(k+a)}{(s+1)^{k-m+1}} \\
&= (s+1)^m \mathcal{L}_a\{g\}(s) - \sum_{k=0}^{m-1} (s+1)^{m-1-k} g(k+a).
\end{aligned}$$

■

We leave it as an exercise to verify that applying formulas (3.5) and (3.6) consecutively yields the identity

$$\mathcal{L}_{(a+m)-m}\{f\}(s) = \mathcal{L}_{(a-m)+m}\{f\}(s) = \mathcal{L}_a\{f\}(s),$$

for $s \in \mathbb{C} \backslash \overline{B_{-1}(r)}$.

3.1.2 The Taylor Monomial

Quite useful for applying the Laplace Transform in discrete fractional calculus are the Taylor Monomials, which are defined and developed in the general time scale setting in [8]. Briefly, the *Taylor Monomials* are defined recursively as

$$\begin{cases} h_0(t,a) := 1 \\ h_{n+1}(t,a) := \int_a^t h_n(s,a)\,\Delta s, \text{ for } n \in \mathbb{N}_0. \end{cases} \tag{3.7}$$

For the specific domain $\mathbb{N}_a$, the Taylor Monomials can be written explicitly as

$$h_n(t,a) = \frac{(t-a)^{\underline{n}}}{n!}, \text{ for } n \in \mathbb{N}_0,\ t \in \mathbb{N}_a,$$

where the above *generalized falling function* is given by

$$t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}, \text{ for } t, \mu \in \mathbb{R}.$$

Here, we take the convention that $t^{\underline{\mu}} = 0$ whenever $t+1-\mu \in -\mathbb{N}_0$. This generalized falling function allows us to extend (3.7) to define a general Taylor Monomial that will serve us well in the discrete fractional calculus setting.

Definition 5 *For each $\mu \in \mathbb{R}\setminus(-\mathbb{N})$, define the μ^{th}-*Taylor Monomial *to be*

$$h_\mu(t,a) := \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)}, \textit{ for } t \in \mathbb{N}_a.$$

Lemma 15 *Let $\mu \in \mathbb{R}\setminus(-\mathbb{N})$ be given and suppose $a, b \in \mathbb{R}$ such that $b - a = \mu$. Then for $s \in \mathbb{C}\setminus\overline{B_{-1}(1)}$,*

$$\mathcal{L}_b\left\{h_\mu(\cdot,a)\right\}(s) = \frac{(s+1)^\mu}{s^{\mu+1}}. \tag{3.8}$$

Proof. Recall the general binomial formula

$$(x+y)^\nu = \sum_{k=0}^{\infty} \binom{\nu}{k} x^k y^{\nu-k}, \text{ for } \nu, x, y \in \mathbb{R} \text{ such that } |x| < |y|,$$

where

$$\binom{\nu}{k} := \frac{\nu^{\underline{k}}}{k!}.$$

Observe that for $k \in \mathbb{N}_0$ and $\nu > 0$, we have

$$\begin{aligned}
\binom{-\nu}{k} = \frac{(-\nu)^{\underline{k}}}{k!} &= \frac{(-\nu)\cdots(-\nu-k+1)}{k!} \\
&= (-1)^k \frac{(k+\nu-1)^{\underline{k}}}{k!} \\
&= (-1)^k \binom{k+\nu-1}{\nu-1}.
\end{aligned}$$

Combining the above two facts, we may write the following for $\nu \in \mathbb{R}$ and $|y| < 1$:

$$\begin{aligned}
\frac{1}{(1-y)^\nu} = ((-y)+1)^{-\nu} &= \sum_{k=0}^{\infty} \binom{-\nu}{k} (-y)^k \, 1^{-\nu-k} \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{-\nu}{k} y^k \\
&= \sum_{k=0}^{\infty} \binom{k+\nu-1}{\nu-1} y^k.
\end{aligned}$$

Therefore, since $b - a = \mu$, we have for $s \in \mathbb{C} \backslash \overline{B_{-1}(1)}$,

$$\begin{aligned}
\frac{(s+1)^\mu}{s^{\mu+1}} &= \frac{1}{s+1} \frac{1}{\left(1-\frac{1}{s+1}\right)^{\mu+1}} \\
&= \frac{1}{s+1} \sum_{k=0}^{\infty} \binom{k+\mu}{\mu} \frac{1}{(s+1)^k} \\
&= \sum_{k=0}^{\infty} \frac{(k+\mu)^{\underline{\mu}}}{\Gamma(\mu+1)} \frac{1}{(s+1)^{k+1}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} h_\mu(k+b,a) \frac{1}{(s+1)^{k+1}} \\
&= \mathcal{L}_b \{h_\mu(\cdot,a)\}(s).
\end{aligned}
$$

■

Remark 7 *We know from Lemma 12 that if $h_\mu(t,a)$ is of some exponential order $r>0$, then $\mathcal{L}_b\{h_\mu(\cdot,a)\}(s)$ exists for certain s in the complex plane. We will show that for each $\mu \in \mathbb{R}\setminus(-\mathbb{N})$, h_μ is of exponential order one.*

First, suppose that $\mu > 0$ and choose $M \in \mathbb{N}$ so that $M-1<\mu\leq M$. Then for any fixed $r>1$,

$$
\begin{aligned}
h_\mu(t,a) = \frac{(t-a)^{\underline{\mu}}}{\Gamma(\mu+1)} &= \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-\mu)} \\
&< \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-M)} \\
&= \frac{(t-a)\cdots(t-a-M+1)}{\Gamma(\mu+1)} \\
&< \frac{t^M}{\Gamma(\mu+1)} \\
&< \frac{r^t}{\Gamma(\mu+1)},
\end{aligned}
$$

for sufficiently large $t\in\mathbb{N}_a$. On the other hand, if $\mu\leq 0$, then we have the inequality

$$
h_\mu(t,a) = \frac{\Gamma(t-a+1)}{\Gamma(\mu+1)\Gamma(t-a+1-\mu)} \leq \frac{1}{\Gamma(\mu+1)},
$$

for sufficiently large $t\in\mathbb{N}_a$.

Combining these two cases, we have that $h_\mu(t,a)$ is of exponential order $1+\epsilon$, for *each* $\epsilon > 0$. *Hence, Lemma 12 implies that* $\mathcal{L}_a \{h_\mu(\cdot, a)\}(s)$ *exists on*

$$\bigcup_{\epsilon>0} \left(\mathbb{C}\backslash\overline{B_{-1}(1+\epsilon)}\right) = \mathbb{C}\backslash \left(\bigcap_{\epsilon>0} \overline{B_{-1}(1+\epsilon)}\right) = \mathbb{C}\backslash\overline{B_{-1}(1)},$$

as indicated in Lemma 15.

3.1.3 The Convolution

Although the following convolution differs from the one defined for general time scales in [8] and from that defined by Atici and Eloe in [2], the author believes (3.9) below best fits our current setting.

Definition 6 *Define* the convolution *of two functions* $f, g : \mathbb{N}_a \to \mathbb{R}$ *by*

$$(f * g)(t) := \sum_{r=a}^{t} f(r)g(t-r+a), \text{ for } t \in \mathbb{N}_a. \tag{3.9}$$

The following lemma composes the Laplace Transform with convolution (3.9) and proves a useful tool for solving fractional initial value problems.

Lemma 16 *Let* $f, g : \mathbb{N}_a \to \mathbb{R}$ *be of exponential order* $r > 0$. *Then*

$$\mathcal{L}_a \{f * g\}(s) = (s+1)\mathcal{L}_a\{f\}(s)\mathcal{L}_a\{g\}(s), \tag{3.10}$$

for $s \in \mathbb{C}\backslash\overline{B_{-1}(r)}$.

Proof. Let f, g and r be as in the statement of the lemma. Then

$$\mathcal{L}_a\{f * g\}(s) = \sum_{k=0}^{\infty} \frac{(f*g)(k+a)}{(s+1)^{k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{k+a} f(r)g(k+a-r+a)$$
$$= \sum_{k=0}^{\infty} \sum_{r=0}^{k} \frac{f(r+a)g(k-r+a)}{(s+1)^{k+1}},$$

whereby applying the change of variables $\tau = k - r$ yields the two independent summations

$$\sum_{\tau=0}^{\infty} \sum_{r=0}^{\infty} \frac{f(r+a)g(\tau+a)}{(s+1)^{\tau+r+1}}$$
$$= (s+1) \sum_{r=0}^{\infty} \frac{f(r+a)}{(s+1)^{r+1}} \sum_{\tau=0}^{\infty} \frac{g(\tau+a)}{(s+1)^{\tau+1}}$$
$$= (s+1)\, \mathcal{L}_a \{f\}(s)\, \mathcal{L}_a \{g\}(s), \text{ for } s \in \mathbb{C} \backslash \overline{B_{-1}(r)}.$$

■

3.2 The Laplace Transform in Discrete Fractional Calculus

Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $\nu > 0$ with $N \in \mathbb{N}$ chosen so that $N - 1 < \nu \le N$. Recall the ν^{th}-*order fractional sum of* f

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{r=a}^{t-\nu} (t - \sigma(r))^{\underline{\nu-1}} f(r), \text{ for } t \in \mathbb{N}_{a+\nu-N},$$

where $\Delta_a^{-\nu} f$ has its N initial zeros on the set $\{a + \nu - N, ..., a + \nu - 1\}$. Likewise, recall the ν^{th}-*order fractional difference of* f

$$\Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \text{ for } t \in \mathbb{N}_{a+N-\nu},$$

which we established in Theorem 1 is equivalent to

$$\Delta_a^\nu f(t) := \begin{cases} \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & \nu \in (N-1, N) \\ \Delta^N f(t), & \nu = N, \end{cases}$$

for $t \in \mathbb{N}_{a+N-\nu}$.

Results analogous to (3.11) and (3.12) below are widely known and used today.

$$\mathcal{L}_a \left\{ \Delta_a^{-N} f \right\}(s) = \frac{\mathcal{L}_a \{f\}(s)}{s^N} \tag{3.11}$$

and

$$\mathcal{L}_a \left\{ \Delta^N f \right\}(s) = s^N \mathcal{L}_a \{f\}(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a), \tag{3.12}$$

for $N \in \mathbb{N}_0$. We wish to generalize (3.11) and (3.12) to Laplace Transforms of fractional-order operators.

3.2.1 The Exponential Order of Fractional Operators

Before we may apply the Laplace Transform on fractional operators, we must discuss the exponential order of fractional sums $\Delta_a^{-\nu} f$ and differences $\Delta_a^\nu f$.

Lemma 17 *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$ and let $\nu > 0$ be given. Then for each fixed $\epsilon > 0$,*

$$\Delta_a^{-\nu} f \text{ and } \Delta_a^\nu f \text{ are of exponential order } r + \epsilon.$$

Proof. Since f is of exponential order r, there exists an $A > 0$ and a $T \in \mathbb{N}_a$ such that

$$|f(t)| \leq Ar^t, \text{ for all } t \in \mathbb{N}_a \text{ with } t \geq T.$$

The following estimates require a good understanding of the gamma function, especially that $\Gamma(x)$ is positive on $(0, \infty)$ and monotone increasing on $[2, \infty)$. See Remark 1 for additional properties and a graph of the gamma function.

We first examine the exponential order of the fractional sum $\Delta_a^{-\nu} f$. Let $\epsilon > 0$ be fixed and let $t \in \mathbb{N}_{a+\nu}$ be given with $t - \nu \geq T + 2$. Then

$$\begin{aligned} \left|\Delta_a^{-\nu} f(t)\right| &= \left|\sum_{s=a}^{T-1} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} f(s) + \sum_{s=T}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} f(s)\right| \\ &\leq \sum_{s=a}^{T-1} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} |f(s)| + \sum_{s=T}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} Ar^s. \end{aligned} \tag{3.13}$$

We consider the two terms in (3.13) separately, beginning with the second and potentially much larger sum:

$$\begin{aligned} & \sum_{s=T}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} Ar^s \\ = \; & \frac{A}{\Gamma(\nu)} \sum_{s=T}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)} r^s \\ = \; & \frac{A}{\Gamma(\nu)} \left(\Gamma(\nu) r^{t-\nu} + \Gamma(\nu+1) r^{t-\nu-1} + \sum_{s=T}^{t-\nu-2} \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+1)} r^s\right) \\ < \; & \frac{A}{r^\nu}\left(1 + \frac{\nu}{r}\right) r^t + \frac{A}{\Gamma(\nu)} \sum_{s=T}^{t-\nu-2} \frac{\Gamma(t-s)}{\Gamma(t-s-N+1)} r^s \\ = \; & \frac{A}{r^\nu}\left(1 + \frac{\nu}{r}\right) r^t + \frac{A}{\Gamma(\nu)} \sum_{s=0}^{t-\nu-T-2} (t-T-s-1) \cdots (t-T-s-N+1)\, r^s \\ < \; & \frac{A}{r^\nu}\left(1 + \frac{\nu}{r}\right) r^t + \frac{At^{N-1}}{\Gamma(\nu)} \sum_{s=0}^{t-\nu-T-2} r^s. \end{aligned}$$

If $r \in (0,1]$, we have

$$\begin{aligned}
\sum_{s=T}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} A r^s &< \frac{A}{r^\nu}\left(1+\frac{\nu}{r}\right) r^t + \frac{A t^{N-1}}{\Gamma(\nu)} \\
&\leq \left(\frac{A}{r^\nu}\left(1+\frac{\nu}{r}\right) + \frac{A}{\Gamma(\nu)}\right) r^t \\
&\leq \frac{1}{2}(r+\epsilon)^t,
\end{aligned}$$

for sufficiently large $t \in \mathbb{N}_{a+\nu}$. Likewise, if $r \in (1,\infty)$, then

$$\begin{aligned}
\sum_{s=T}^{t-\nu} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} A r^s &< \frac{A}{r^\nu}\left(1+\frac{\nu}{r}\right) r^t + \frac{A t^{N-1}}{\Gamma(\nu)}(t-\nu-T-1)\, r^{t-\nu-T-2} \\
&< \left(\frac{A}{r^\nu}\left(1+\frac{\nu}{r}\right) + \frac{A t^N}{\Gamma(\nu)}\right) r^t \\
&\leq \frac{1}{2}(r+\epsilon)^t,
\end{aligned}$$

for sufficiently large $t \in \mathbb{N}_{a+\nu}$, since the exponential $(r+\epsilon)^t$ will eventually dominate the power function t^N.

We employ an analogous argument as above to the second term in (3.13) to show that for any $t \in \mathbb{N}_{a+\nu}$ with $t > T+\nu$,

$$\sum_{s=a}^{T-1} \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)} |f(s)| < \left(\sum_{s=a}^{T-1} \frac{|f(s)|}{\Gamma(\nu)}\right) t^{N-1} \leq \frac{1}{2}(r+\epsilon)^t,$$

for sufficiently large $t \in \mathbb{N}_{a+\nu}$. Combining the above steps, we see that there exists a $T_\epsilon \in \mathbb{N}_{a+\nu}$ (with $T_\epsilon \geq T+\nu+2$) such that for all $t \in \mathbb{N}_{a+\nu}$ with $t \geq T_\epsilon$,

$$\left|\Delta_a^{-\nu} f(t)\right| \leq (r+\epsilon)^t.$$

In other words, $\Delta_a^{-\nu} f$ is of exponential order $r+\epsilon$, for each $\epsilon > 0$.

We turn our attention now to the fractional difference $\Delta_a^\nu f = \Delta^N \Delta_a^{-(N-\nu)} f$. By the first part of the proof, we know that given any $\epsilon > 0$, the fractional sum $\Delta_a^{-(N-\nu)} f$ is of exponential order $r+\epsilon$, with coefficient $A = 1$. Hence, there exists a $T_\epsilon \in \mathbb{N}_{a+N-\nu}$ (with $T_\epsilon > T + N - \nu + 2$) such that for all $t \in \mathbb{N}_{a+N-\nu}$ with $t \geq T_\epsilon$,

$$\left|\Delta_a^{-(N-\nu)} f(t)\right| \leq (r+\epsilon)^t .$$

It follows that for each such $t \in \mathbb{N}_{a+N-\nu}$ with $t \geq T_\epsilon$,

$$\begin{aligned}
|\Delta_a^\nu f(t)| &= \left|\Delta^N \Delta_a^{-(N-\nu)} f(t)\right| \\
&= \left|\sum_{k=0}^{N} (-1)^k \binom{N}{k} \Delta_a^{-(N-\nu)} f(t+N-k)\right| \\
&\leq \sum_{k=0}^{N} \binom{N}{k} \left|\Delta_a^{-(N-\nu)} f(t+N-k)\right| \\
&\leq \left(\sum_{k=0}^{N} \binom{N}{k} (r+\epsilon)^{N-k}\right) \cdot (r+\epsilon)^t .
\end{aligned}$$

We conclude, therefore, that the fractional difference $\Delta_a^\nu f$ is also of exponential order $r+\epsilon$, for every $\epsilon > 0$. ■

It can be shown, though by a longer and more technical proof, that if f is of exponential order $r > 0$, then $\Delta_a^\nu f$ is also of exponential order r, for any $\nu > 0$. No analogous result holds, however, for the fractional sum $\Delta_a^{-\nu} f$. Nevertheless, the distinction between this stronger property for $\Delta_a^\nu f$ and the weaker version presented in Lemma 17 is of little significance practically. In fact, the two results lead to identical domains of convergence for the corresponding Laplace Transform $\mathcal{L}_{a+N-\nu}\{\Delta_a^\nu f\}$, as justified below.

Corollary 18 *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$ and let $\nu > 0$*

be given, with $N-1<\nu\leq N$. Then

$$\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s) \text{ and } \mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu}f\right\}(s)$$

both exist for $s\in\mathbb{C}\backslash\overline{B_{-1}(r)}$.

Proof. Let f,r and ν be as given in the statement of the corollary. By Lemma 17, we know that for each $\epsilon>0$, both $\Delta_a^{-\nu}f$ and $\Delta_a^{\nu}f$ are of exponential order $r+\epsilon$.

Now, fix an arbitrary point $s_0\in\mathbb{C}\backslash\overline{B_{-1}(r)}$. Since $dist(s_0,\overline{B_{-1}(r)})>0$, there exists an $\epsilon_0>0$ small enough so that $s_0\in\mathbb{C}\backslash\overline{B_{-1}(r+\epsilon_0)}$. Since $\Delta_a^{-\nu}f$ and $\Delta_a^{\nu}f$ are both of exponential order $r+\epsilon_0$, it follows that both series $\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s)$ and $\mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu}f\right\}(s)$ converge at $s=s_0$. ■

3.2.2 The Laplace Transform of Fractional Operators

With Corollary 18 in hand to insure correct domains of convergence, we now safely develop formulas for applying the Laplace Transform to fractional sums and differences.

Theorem 19 *Suppose $f:\mathbb{N}_a\to\mathbb{R}$ is of exponential order $r\geq 1$ and let $\nu>0$ be given, with $N-1<\nu\leq N$. Then for $s\in\mathbb{C}\backslash\overline{B_{-1}(r)}$,*

$$\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu}f\right\}(s)=\frac{(s+1)^{\nu}}{s^{\nu}}\mathcal{L}_a\left\{f\right\}(s) \tag{3.14}$$

and

$$\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu}f\right\}(s)=\frac{(s+1)^{\nu-N}}{s^{\nu}}\mathcal{L}_a\left\{f\right\}(s). \tag{3.15}$$

Proof. Let f, r, ν and N be as given in the statement of the theorem. Note that though we assume f is of exponential order $r \geq 1$, it does hold that

$$f \text{ is of exponential order } r \in (0,1) \Longrightarrow f \text{ is of exponential order } 1.$$

Therefore, the assumption $r \geq 1$ is not for excluding functions of exponential order $r \in (0,1)$, but rather for insuring that Lemma 15, applied below, will hold whenever $s \in \mathbb{C} \backslash \overline{B_{-1}(r)}$.

We first apply Lemma 14 to our current setting to reveal the relationship between (3.14) and (3.15). Indeed, shift formula (3.5) implies that for each $s \in \mathbb{C} \backslash \overline{B_{-1}(r)}$,

$$\begin{aligned} & \mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) \\ = \ & \frac{1}{(s+1)^N} \mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) + \sum_{k=0}^{N-1} \frac{\Delta_a^{-\nu} f(k+a+\nu-N)}{(s+1)^{k+1}} \\ = \ & \frac{1}{(s+1)^N} \mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s), \end{aligned}$$

taking the N zeros of $\Delta_a^{-\nu} f$ into account. Moreover,

$$\begin{aligned} & \mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) \\ = \ & \sum_{k=0}^{\infty} \frac{\Delta_a^{-\nu} f(k+a+\nu)}{(s+1)^{k+1}} \\ = \ & \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{k+a} \frac{(k+a+\nu-\sigma(r))^{\underline{\nu-1}}}{\Gamma(\nu)} f(r) \\ = \ & \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} \sum_{r=a}^{k+a} f(r) h_{\nu-1}((k+a)-r+a, a-(\nu-1)) \\ = \ & \sum_{k=0}^{\infty} \frac{(f * h_{\nu-1}(\cdot, a-(\nu-1)))(k+a)}{(s+1)^{k+1}}, \ \text{applying (3.9)} \\ = \ & \mathcal{L}_a\left\{f * h_{\nu-1}(\cdot, a-(\nu-1))\right\}(s) \end{aligned}$$

$$
\begin{aligned}
&= (s+1)\,\mathcal{L}_a\{f\}(s)\,\mathcal{L}_a\{h_{\nu-1}(\cdot, a-(\nu-1))\}(s)\,, \text{ using rule (3.10)} \\
&= (s+1)\,\frac{(s+1)^{\nu-1}}{s^{\nu}}\mathcal{L}_a\{f\}(s)\,, \text{ applying (3.8), since } r \geq 1 \\
&= \frac{(s+1)^{\nu}}{s^{\nu}}\mathcal{L}_a\{f\}(s)\,,
\end{aligned}
$$

thus proving (3.14). We then obtain (3.15) as a consequence via

$$
\begin{aligned}
\mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) &= \frac{1}{(s+1)^N}\mathcal{L}_{a+\nu}\left\{\Delta_a^{-\nu} f\right\}(s) \\
&= \frac{(s+1)^{\nu-N}}{s^{\nu}}\mathcal{L}_a\{f\}(s)\,, \text{ for } s \in \mathbb{C}\backslash\overline{B_{-1}(r)}.
\end{aligned}
$$

■

Remark 8 *Note that when $\nu = N$ in (3.15), the well-known formula (3.11) is obtained. An analogous extension holds true for the Laplace Transform of a fractional difference.*

Theorem 20 *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ be given, with $N-1 < \nu \leq N$. Then for $s \in \mathbb{C}\backslash\overline{B_{-1}(r)}$,*

$$
\mathcal{L}_{a+N-\nu}\{\Delta_a^{\nu} f\}(s) = \frac{s^{\nu}}{(s+1)^{\nu-N}}\mathcal{L}_a\{f\}(s) - \sum_{j=0}^{N-1} s^j \Delta_a^{\nu-1-j} f(a+N-\nu). \tag{3.16}
$$

Proof. Let f, r, ν and N be as given in the statement of the theorem. We already know that (3.16) holds when $\nu = N$, since this agrees with the well-known formula (3.12). If $N-1 < \nu < N$, on the other hand, then $0 < N-\nu < 1$ and we may apply

(3.12), (3.14) and composition rule (2.5) in succession as follows:

$$\begin{aligned}
& \mathcal{L}_{a+N-\nu}\left\{\Delta_a^{\nu} f\right\}(s) \\
= {} & \mathcal{L}_{a+N-\nu}\left\{\Delta^N \Delta_a^{-(N-\nu)} f\right\}(s) \\
= {} & s^N \mathcal{L}_{a+N-\nu}\left\{\Delta_a^{-(N-\nu)} f\right\}(s)-\sum_{j=0}^{N-1} s^j \Delta^{N-1-j} \Delta_a^{-(N-\nu)} f(a+N-\nu) \\
= {} & s^N \frac{(s+1)^{N-\nu}}{s^{N-\nu}} \mathcal{L}_a\{f\}(s)-\sum_{j=0}^{N-1} s^j \Delta^{N-1-j} \Delta_a^{-(N-\nu)} f(a+N-\nu) \\
= {} & \frac{s^\nu}{(s+1)^{\nu-N}} \mathcal{L}_a\{f\}(s)-\sum_{j=0}^{N-1} s^j \Delta_a^{\nu-1-j} f(a+N-\nu) .
\end{aligned}$$

■

Remark 9 *We may certainly compose the results from Theorems 19 and 20. In particular, observe that under the same assumptions as in these two theorems, we have for* $s \in \mathbb{C} \backslash \overline{B_{-1}(r)}$,

$$\begin{aligned}
& \mathcal{L}_{(a+\nu-N)+N-\nu}\left\{\Delta_{a+\nu-N}^{\nu} \Delta_a^{-\nu} f\right\}(s) \\
= {} & \mathcal{L}_a\left\{\Delta_{a+\nu-N}^{\nu}\left(\Delta_a^{-\nu} f\right)\right\}(s) \\
= {} & \frac{s^\nu \mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s)}{(s+1)^{\nu-N}}-\sum_{j=0}^{N-1} s^j \Delta_{a+\nu-N}^{\nu-1-j} \Delta_a^{-\nu} f((a+\nu-N)+N-\nu) \\
= {} & \frac{s^\nu}{(s+1)^{\nu-N}}\left[\frac{(s+1)^{\nu-N}}{s^\nu} \mathcal{L}_a\{f\}(s)\right]-\sum_{j=0}^{N-1} s^j \Delta_a^{-(j+1)} f(a) \\
= {} & \mathcal{L}_a\{f\}(s),
\end{aligned}$$

a result confirmed by the composition rule in Theorem 7.

Example 4 *Define*

$$f(t):=(t-5)^{\underline{\pi}}=\Gamma(\pi+1) h_\pi(t, 5), \text{ for } t \in \mathbb{N}_{5+\pi}.$$

Recalling Remark 7 and formula 3.8, we have for $s \in \mathbb{C}\backslash\overline{B_{-1}(1)}$,

$$\mathcal{L}_{5+\pi}\left\{f\right\}(s) = \Gamma(\pi+1)\mathcal{L}_{5+\pi}\left\{h_\pi(\cdot,5)\right\}(s) = \Gamma(\pi+1)\frac{(s+1)^\pi}{s^{\pi+1}}.$$

Moreover, (3.15) allows us to calculate

$$\begin{aligned}\mathcal{L}_{2+\pi+e}\left\{\Delta_{5+\pi}^{-e}f\right\}(s) &= \frac{(s+1)^{e-3}}{s^e}\left(\Gamma(\pi+1)\frac{(s+1)^\pi}{s^{\pi+1}}\right)\\ &= \Gamma(\pi+1)\frac{(s+1)^{\pi+e-3}}{s^{\pi+e+1}}\end{aligned}$$

and (3.16) together with power rules (2.2) and (2.3) allow us to calculate

$$\begin{aligned}&\mathcal{L}_{8+\pi-e}\left\{\Delta_{5+\pi}^{e}f\right\}(s)\\ =\ & s^e(s+1)^{3-e}\left(\Gamma(\pi+1)\frac{(s+1)^\pi}{s^{\pi+1}}\right) - \sum_{j=0}^{2} s^j\Delta_{5+\pi}^{e-1-j}f(8+\pi-e)\\ =\ & \Gamma(\pi+1)\left(\frac{(s+1)^{\pi-e+3}}{s^{\pi-e+1}} - \sum_{j=0}^{2} s^j\frac{(3+\pi-e)^{\underline{\pi-e+j+1}}}{\Gamma(\pi-e+j+2)}\right)\\ =\ & \Gamma(\pi+1)\left(\frac{(s+1)^{\pi-e+3}}{s^{\pi-e+1}} - \frac{(3+\pi-e)(2+\pi-e)}{2} - (3+\pi-e)s - s^2\right),\end{aligned}$$

both valid for $s \in \mathbb{C}\backslash\overline{B_{-1}(1)}$.

3.3 The Fractional Laplace Transform Method

3.3.1 A Power Rule and Composition Rule

In Chapter 2, a number of formulas concerning fractional sum and difference operators were introduced and developed. These include the four fractional composition rules and the fractional power rules, whose proofs did not employ the Discrete

Laplace Transform presented in this chapter. Several of these results can also be proved, with varying efficacy, using the Laplace Transform. Though this is not always the case, the following are two results for which the Laplace Transform proof is a significant improvement over the original.

Remark 10 *To prove the power rule in Theorem 21 below, we show that for* $s \in \mathbb{C}\backslash\overline{B_{-1}(1)}$, *the Laplace Transform of the left and right-hand sides are equal. To see why* $\mathbb{C}\backslash\overline{B_{-1}(1)}$ *is the correct domain of convergence for these Laplace Transforms, we combine Remark 7 with Lemma 17 to conclude that for each* $\epsilon > 0$, $(t-a)^{\underline{\mu}}$ *is of exponential order* $1+\epsilon$ *and, therefore, that* $\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}}$ *is of exponential order* $1+2\epsilon$. *We then apply an argument identical to that given in Corollary 18 to show that* $\mathcal{L}_{a+\mu}\{(t-a)^{\underline{\mu}}\}$ *and* $\mathcal{L}_{a+\mu+\nu}\left\{\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}}\right\}$ *exist for* $s \in \mathbb{C}\backslash\overline{B_{-1}(1)}$.

Theorem 21 *Let* $\nu, \mu > 0$ *be given. Then for* $t \in \mathbb{N}_{a+\mu+\nu}$,

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)}(t-a)^{\underline{\mu+\nu}}.$$

Proof. With Remark 10 in hand, we have for $s \in \mathbb{C}\backslash\overline{B_{-1}(1)}$,

$$\begin{aligned}
\mathcal{L}_{a+\mu+\nu}\left\{\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}}\right\}(s) &= \frac{(s+1)^{\nu}}{s^{\nu}}\mathcal{L}_{a+\mu}\left\{(t-a)^{\underline{\mu}}\right\}(s), \text{ using (3.14)} \\
&= \frac{(s+1)^{\nu}}{s^{\nu}}\Gamma(\mu+1)\mathcal{L}_{a+\mu}\left\{h_{\mu}(\cdot,a)\right\}(s) \\
&= \frac{(s+1)^{\nu}}{s^{\nu}}\Gamma(\mu+1)\frac{(s+1)^{\mu}}{s^{\mu+1}}, \text{ applying (3.8)} \\
&= \Gamma(\mu+1)\frac{(s+1)^{\mu+\nu}}{s^{\mu+\nu+1}} \\
&= \Gamma(\mu+1)\mathcal{L}_{a+\mu+\nu}\left\{h_{\mu+\nu}(\cdot,a)\right\}(s) \\
&= \mathcal{L}_{a+\mu+\nu}\left\{\frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\underline{\mu+\nu}}\right\}(s).
\end{aligned}$$

By the one-to-one property of the Laplace Transform (3.4), it follows that

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)}(t-a)^{\underline{\mu+\nu}}, \text{ for } t \in \mathbb{N}_{a+\mu+\nu}.$$

■

Theorem 22 *Suppose that $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r > 0$ and let $\nu, \mu > 0$ be given. Then for $t \in \mathbb{N}_{a+\mu+\nu}$,*

$$\Delta_a^{-\nu}\Delta_a^{-\mu}f(t) = \Delta_a^{-\nu-\mu}f(t) = \Delta_a^{-\mu}\Delta_a^{-\nu}f(t).$$

Proof. Let f, r, μ and ν be as given in the statement of the theorem. It follows from Corollary 18 that

$$\mathcal{L}_{a+\mu+\nu}\left\{\Delta_a^{-\nu}\Delta_a^{-\mu}f\right\},\ \mathcal{L}_{a+\mu}\left\{\Delta_a^{-\mu}f\right\} \text{ and } \mathcal{L}_{a+(\nu+\mu)}\left\{\Delta_a^{-(\nu+\mu)}f\right\}$$

all exist on $\mathbb{C}\backslash\overline{B_{-1}(r)}$. Therefore, we may apply (3.14) multiple times to write for $s \in \mathbb{C}\backslash\overline{B_{-1}(r)}$,

$$\begin{aligned}
\mathcal{L}_{a+\mu+\nu}\left\{\Delta_a^{-\nu}\Delta_a^{-\mu}f\right\}(s) &= \frac{(s+1)^{\nu}}{s^{\nu}}\mathcal{L}_{a+\mu}\left\{\Delta_a^{-\mu}f\right\}(s) \\
&= \frac{(s+1)^{\nu}}{s^{\nu}}\frac{(s+1)^{\mu}}{s^{\mu}}\mathcal{L}_a\{f\}(s) \\
&= \frac{(s+1)^{\nu+\mu}}{s^{\nu+\mu}}\mathcal{L}_a\{f\}(s) \\
&= \mathcal{L}_{a+(\nu+\mu)}\left\{\Delta_a^{-(\nu+\mu)}f\right\}(s) \\
&= \mathcal{L}_{a+\mu+\nu}\left\{\Delta_a^{-\nu-\mu}f\right\}(s).
\end{aligned}$$

The result then follows from symmetry and the one-to-one property of the Laplace Transform. ■

3.3.2 A Fractional Initial Value Problem

By far the most substantial application of the Laplace Transform is presented in Theorem 23 below. Note that the fractional initial value problem (3.17) solved below by the *Fractional Laplace Transform Method* is identical to problem (2.11) solved in Chapter 2 using the Fractional Composition Rules.

Theorem 23 *Suppose $f : \mathbb{N}_a \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ be given with $N-1 < \nu \leq N$. The unique solution to the fractional initial value problem*

$$\begin{cases} \Delta^{\nu}_{a+\nu-N} y(t) = f(t), & t \in \mathbb{N}_a \\ \Delta^{i} y(a+\nu-N) = A_i, & i \in \{0,1,...,N-1\}\,;\ A_i \in \mathbb{R} \end{cases} \tag{3.17}$$

is given by

$$y(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \ \textit{for } t \in \mathbb{N}_{a+\nu-N},$$

where

$$\alpha_i := \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!} (i-k)^{\underline{N-\nu}} \binom{i}{p} \binom{i-p}{k} A_p,$$

for $i \in \{0,1,...,N-1\}$.

Proof. Since we already solved problem (3.17) once using composition rules in Theorem 11, we present here only that portion of the proof involving the Laplace Transform Method. Since f is of exponential order r, we know by Lemma 12 that $\mathcal{L}_a\{f\}$ exists on $\mathbb{C}\backslash\overline{B_{-1}(r)}$. So, applying the Laplace Transform to both sides of the fractional difference equation in (3.17), we have for $s \in \mathbb{C}\backslash\overline{B_{-1}(r)}$,

$$\mathcal{L}_a\left\{\Delta^{\nu}_{a+\nu-N} y\right\}(s) = \mathcal{L}_a\{f\}(s)$$

$$\implies \frac{s^{\nu}\mathcal{L}_{a+\nu-N}\{y\}(s)}{(s+1)^{\nu-N}} - \sum_{j=0}^{N-1} s^j \Delta_{a+\nu-N}^{\nu-j-1} y(a) = \mathcal{L}_a\{f\}(s), \text{ by (3.16)}$$

$$\implies \mathcal{L}_{a+\nu-N}\{y\}(s) = \frac{\mathcal{L}_a\{f\}(s)}{s^{\nu}(s+1)^{N-\nu}} + \sum_{j=0}^{N-1} \frac{\Delta_{a+\nu-N}^{\nu-j-1} y(a)}{s^{\nu-j}(s+1)^{N-\nu}}.$$

It follows immediately from (3.15) that

$$\frac{\mathcal{L}_a\{f\}(s)}{s^{\nu}(s+1)^{N-\nu}} = \mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s).$$

Considering next the summation terms, we have for each $j \in \{0, ..., N-1\}$,

$$\begin{aligned}
& \frac{1}{s^{\nu-j}(s+1)^{N-\nu}} \\
= \; & \frac{1}{(s+1)^{N-j-1}} \frac{(s+1)^{\nu-j-1}}{s^{\nu-j}} \\
= \; & \frac{1}{(s+1)^{N-j-1}} \mathcal{L}_{a+\nu-j-1}\{h_{\nu-j-1}(\cdot, a)\}(s), \text{ by (3.8)} \\
= \; & \mathcal{L}_{a+\nu-N}\{h_{\nu-j-1}(\cdot, a)\}(s) - \sum_{k=0}^{N-j-2} \frac{h_{\nu-j-1}(k+a+\nu-N, a)}{(s+1)^{k+1}}, \text{ by (3.5)} \\
= \; & \mathcal{L}_{a+\nu-N}\{h_{\nu-j-1}(\cdot, a)\}(s),
\end{aligned}$$

since

$$\begin{aligned}
h_{\nu-j-1}(k+a+\nu-N, a) &= \frac{(k+\nu-N)^{\underline{\nu-j-1}}}{\Gamma(\nu-j)} \\
&= \frac{\Gamma(k+\nu-N+1)}{\Gamma(k-(N-j-1))\,\Gamma(\nu-j)} \\
&= 0,
\end{aligned}$$

for $k \in \{0, ..., N-j-2\}$.

Putting these steps together, we have for $s \in \mathbb{C} \backslash \overline{B_{-1}(r)}$,

$$\begin{aligned}
& \mathcal{L}_{a+\nu-N}\{y\}(s) \\
= \; & \mathcal{L}_{a+\nu-N}\left\{\Delta_a^{-\nu} f\right\}(s) + \sum_{j=0}^{N-1} \Delta_{a+\nu-N}^{\nu-j-1} y(a) \mathcal{L}_{a+\nu-N}\left\{h_{\nu-j-1}(\cdot, a)\right\}(s) \\
= \; & \mathcal{L}_{a+\nu-N}\left\{\sum_{j=0}^{N-1} \Delta_{a+\nu-N}^{\nu-j-1} y(a) h_{\nu-j-1}(\cdot, a) + \Delta_a^{-\nu} f\right\}(s).
\end{aligned}$$

Since the Laplace Transform is a one-to-one operator, we conclude that for $t \in \mathbb{N}_{a+\nu-N}$,

$$\begin{aligned}
y(t) & = \sum_{j=0}^{N-1} \Delta_{a+\nu-N}^{\nu-j-1} y(a) h_{\nu-j-1}(t, a) + \Delta_a^{-\nu} f(t) \\
& = \sum_{j=0}^{N-1} \frac{\Delta_{a+\nu-N}^{\nu-j-1} y(a)}{\Gamma(\nu-j)}(t-a)^{\underline{\nu-j-1}} + \Delta_a^{-\nu} f(t) \\
& = \sum_{i=0}^{N-1}\left(\frac{\Delta_{a+\nu-N}^{i+\nu-N} y(a)}{\Gamma(i+\nu-N+1)}\right)(t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t).
\end{aligned}$$

We already demonstrated in the proof of Theorem 11 that for $i \in \{0, 1, ..., N-1\}$,

$$\frac{\Delta_{a+\nu-N}^{i+\nu-N} y(a)}{\Gamma(i+\nu-N+1)} = \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!}(i-k)^{\underline{N-\nu}} \binom{i}{p} \binom{i-p}{k} \Delta^i y(a+\nu-N),$$

concluding the proof. ■

Example 5 *Consider the* π^{th}*-order, fractional initial value problem*

$$\begin{cases} \Delta_{\pi-4}^{\pi} y(t) = \pi^4 t^{\underline{2}}, \quad t \in \mathbb{N}_0 \\ y(\pi-4) = 2, \; \Delta y(\pi-4) = 3, \; \Delta^2 y(\pi-4) = 5, \; \Delta^3 y(\pi-4) = 7. \end{cases} \tag{3.18}$$

Note that (3.18) is a specific instance of problem (3.17) from Theorem 23, with

$$a = 0, \quad \nu = \pi, \quad N = 4, \quad f(t) = \pi^4 t^{\underline{2}},$$
$$A_0 = 2, \quad A_1 = 3, \quad A_2 = 5 \quad A_3 = 7.$$

After applying the Discrete Laplace Transform Method as outlined in the proof of Theorem 23, we have

$$\begin{aligned} y(t) &= \sum_{i=0}^{3} \alpha_i t^{\underline{i+\pi-4}} + \Delta_0^{-\pi}\left(\pi^4 t^{\underline{2}}\right) \\ &= \sum_{i=0}^{3} \alpha_i t^{\underline{i+\pi-4}} + \Delta_2^{-\pi}\left(\pi^4 t^{\underline{2}}\right), \ \textit{since } t^{\underline{2}} = t(t-1), \\ &\approx 0.303 t^{\underline{\pi-4}} + 5.040 t^{\underline{\pi-3}} + 6.977 t^{\underline{\pi-2}} + 4.876 t^{\underline{\pi-1}} + 3.272 t^{\underline{\pi+2}}, \end{aligned}$$

where in this last step,

$$\alpha_i = \sum_{p=0}^{i} \sum_{k=0}^{i-p} \frac{(-1)^k}{i!} (i-k)^{\underline{4-\pi}} \binom{i}{p} \binom{i-p}{k} A_p,$$

for $i \in \{0, 1, 2, 3\}$, *and we applied power rule (2.2) on the last term.*

Chapter 4

The $(N-1,1)$ Fractional Boundary Value Problem

Given a boundary value problem, its corresponding Green's function is mathematically vital. Indeed, the approach of deriving a Green's function and using its properties to show solution existence and other solution attributes dates back a long time. British mathematician George Green developed the general strategy in the 1830's, and others have broadened its reach since then. In this chapter, we apply Green's strategy to a discrete, nonlinear, $(N-1,1)$ fractional boundary value problem with a fractional right boundary condition. The Green's function allows us to employ Krasnoselllski's Theorem and Banach's Contraction Mapping Theorem to demonstrate the existence of a positive solution.

4.1 The Boundary Value Problem

Specifically, we consider the nonlinear, $(N-1,1)$ fractional boundary value problem

$$\begin{cases} -\Delta_{\nu-N}^{\nu} y(t) = f(t, y(t+\nu-1)), & \text{for } t \in \{0, ..., b+M\}, \\ \Delta^{i} y(\nu - N) = 0, & \text{for } i \in \{0, ..., N-2\}, \\ \Delta_{\nu-N}^{\mu} y(b+M+\nu-\mu) = 0. \end{cases} \tag{4.1}$$

where

- $\nu \geq 2$, with $N \in \mathbb{N}$ chosen so that $N - 1 < \nu \leq N$.
- $1 \leq \mu < \nu$, with $M \in \mathbb{N}$ chosen so that $M - 1 < \mu \leq M$.
- $b \in \mathbb{N}$.
- $f : \{0, ..., b + M\} \times \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $y \geq 0$.

Take note of the domain for each function appearing in problem (4.1):

- $\mathcal{D}\left\{\Delta_{\nu-N}^{\nu} y\right\} = \{0, ..., b + M\}$,
- $\mathcal{D}\{y\} = \{\nu - N, ..., b + M + \nu\}$,
- $\mathcal{D}\{\Delta^{i} y\} = \{\nu - N, ..., b + M + \nu - i\}$, for each $i \in \{0, ..., N - 2\}$,
- $\mathcal{D}\left\{\Delta_{\nu-N}^{\mu} y\right\} = \{\nu - N + M - \mu, ..., b + M + \nu - \mu\}$.

In particular, the 'unknown' function in problem (4.1) has the form

$$y : \{\nu - N, ..., b + M + \nu\} \to \mathbb{R}.$$

A similar boundary value problem was studied by Goodrich in [10]—however, problem (4.1) differs from Goodrich's work in two significant ways:

- In [10], the order of the fractional difference equation is restricted to the case $\nu \in (1, 2]$, requiring only two boundary conditions. Here, we consider a difference equation of order $\nu \in [2, \infty)$, requiring $N := \lceil \nu \rceil$ boundary conditions. The corresponding Green's function is more complicated and behaves quite differently for $\nu \in [2, \infty)$, and the required analysis justifies a separate study of problem (4.1).

- In [10], the right boundary condition is focal (i.e. the first derivative is specified). Here, however, we consider a fractional (μ^{th}-order) right boundary condition, where $\mu \in [1, \nu)$. The fractional boundary condition requires more attention but offers greater flexibility and the potential for broader application.

In addition to the definitions and rules presented in previous chapters, we employ the following two properties of the generalized falling function in our work to come:

$$\bullet \text{ Given } \alpha \in \mathbb{R},\ t \mapsto t^{\underline{\alpha}} \text{ is positive for } t > \max\{-1, \alpha - 1\}. \tag{4.2}$$

$$\bullet \text{ Given } \alpha > 0,\ t \mapsto t^{\underline{\alpha}} \text{ is increasing for } t \geq \alpha - 1. \tag{4.3}$$

4.2 The Green's Function

Following in the footsteps of George Green, our strategy for analyzing problem (4.1) centers on understanding its Green's function. To accomplish this, let us temporarily suppose that the nonlinear term $f(t, y)$ does not depend on y. We then derive the Green's function for (4.1) by solving the corresponding linear boundary value problem

$$\begin{cases} -\Delta_{\nu-N}^{\nu} y(t) = h(t), & \text{for } t \in \{0, ..., b+M\}, \\ \Delta^i y(\nu - N) = 0, & \text{for } i \in \{0, ..., N-2\}, \\ \Delta_{\nu-N}^{\mu} y(b + M + \nu - \mu) = 0. \end{cases} \tag{4.4}$$

As derived in Theorem 11 from Chapter 2, the general solution to the fractional difference equation in (4.4) is

$$y(t) = \sum_{i=0}^{N-1} \alpha_i t^{\underline{i+\nu-N}} - \Delta_0^{-\nu} h(t),$$

where $\alpha_i \in \mathbb{R}$ and $t \in \{\nu - N, ..., b + M + \nu\}$.

Notice that when taken together, the left $N - 1$ boundary conditions

$$\Delta^i y(\nu - N) = 0, \text{ for } i \in \{0, ..., N-2\}$$

imply that

$$y(\nu - N) = y(\nu - N + 1) = \cdots = y(\nu - 2) = 0.$$

Furthermore, since

$$\begin{aligned} \Delta_0^{-\nu} h(\nu - N + k) &= \left(\frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t - \sigma(s))^{\underline{\nu-1}} h(s) \right) \Bigg|_{t=\nu-N+k} \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{-N+k} (\nu - N + k - \sigma(s))^{\underline{\nu-1}} h(s) \\ &= 0, \end{aligned}$$

for each $k \in \{0, ..., N-2\}$, we conclude that

$$0 = y(\nu - N + k) = \sum_{i=0}^{N-1} \alpha_i (\nu - N + k)^{\underline{i+\nu-N}} - \Delta_0^{-\nu} h(\nu - N + k)$$

$$\begin{aligned} &= \sum_{i=0}^{N-1} \alpha_i \frac{\Gamma(\nu - N + k + 1)}{\Gamma(k - i + 1)} \\ &= \sum_{i=0}^{k} \alpha_i \frac{\Gamma(\nu - N + k + 1)}{(k - i)!}. \end{aligned}$$

Solving the above system of $N-1$ equations for $\{\alpha_i\}_{i=0}^{N-2}$, we obtain

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{N-2} = 0.$$

Hence, the general solution to (4.4) simplifies nicely to

$$y(t) = \alpha_{N-1} t^{\underline{\nu-1}} - \Delta_0^{-\nu} h(t).$$

Next, we apply the right fractional boundary condition

$$\Delta_{\nu-N}^{\mu} y(b + M + \nu - \mu) = 0$$

and solve for α_{N-1}.

$$\begin{aligned} \Delta_{\nu-N}^{\mu} y(t) &= \alpha_{N-1} \Delta_{\nu-N}^{\mu} t^{\underline{\nu-1}} - \Delta_{\nu-N}^{\mu} \Delta_0^{-\nu} h(t) \\ &= \alpha_{N-1} \Delta_{\nu-1}^{\mu} t^{\underline{\nu-1}} - \Delta_{\nu}^{\mu} \Delta_0^{-\nu} h(t) \\ &= \alpha_{N-1} \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} t^{\underline{\nu-1-\mu}} - \Delta_0^{\mu-\nu} h(t), \end{aligned}$$

applying power rule (2.3) and the composition rule from Theorem 7,

$$\begin{aligned} \implies \qquad 0 &= \Delta_{\nu-N}^{\mu} y(b + M + \nu - \mu) \\ &= \alpha_{N-1} \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} (b + M + \nu - \mu)^{\underline{\nu-1-\mu}} - \Delta_0^{\mu-\nu} h(b + M + \nu - \mu), \end{aligned}$$

$$\Longrightarrow \alpha_{N-1} \;=\; \frac{\Gamma(\nu-\mu)}{\Gamma(\nu)} \cdot \frac{\Delta_0^{\mu-\nu} h(b+M+\nu-\mu)}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}},$$

noting that $(b+M+\nu-\mu)^{\underline{\nu-\mu-1}} > 0$ by (4.2). Therefore, every solution of (4.4) has the form

$$\begin{aligned}
& y(t) \\
= \;& \frac{\Gamma(\nu-\mu)}{\Gamma(\nu)} \frac{\Delta_0^{\mu-\nu} h(b+M+\nu-\mu)}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} \cdot t^{\underline{\nu-1}} - \Delta_0^{-\nu} h(t) \\
= \;& \frac{1}{\Gamma(\nu)} \left(\sum_{s=0}^{t+\mu-\nu} (t-\sigma(s))^{\underline{\nu-\mu-1}} h(s) \right) \Bigg|_{t=b+M+\nu-\mu} \cdot \frac{t^{\underline{\nu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} \\
& \quad - \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} h(s) \\
= \;& \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b+M} \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} t^{\underline{\nu-1}} h(s) - \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} h(s) \\
= \;& \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t-\nu} \left\{ \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} t^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}} \right\} h(s) \\
& \quad + \frac{1}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{b+M} \left\{ \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} t^{\underline{\nu-1}} \right\} h(s) \\
= \;& \sum_{s=0}^{b+M} G(t,s) h(s),
\end{aligned}$$

where $G : \{\nu - N, ..., b + M + \nu\} \times \{0, ..., b + M\} \to \mathbb{R}$ is given by

$$G(t,s) := \frac{1}{\Gamma(\nu)} \begin{cases} \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} t^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}}, \\ \qquad\qquad 0 \le s \le t-\nu \le b+M, \\ \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} t^{\underline{\nu-1}}, \\ \qquad\qquad 0 \le t-\nu+1 \le s \le b+M, \\ 0, \\ \qquad\qquad t \in \{\nu-N, ..., \nu-2\}, \ s \in \{0, ..., b+M\}, \end{cases}$$

is the Green's function for the homogeneous problem corresponding to (4.1) and (4.4).

In summary, we have found the appropriate Green's function $G(t, s)$ for studying the nonlinear problem (4.1), and

$$y(t) = \sum_{s=0}^{b+M} G(t,s)h(s), \text{ for } t \in \{\nu - N, ..., b + M + \nu\}$$

is the unique solution to the corresponding linear problem (4.4).

We now establish three important properties of this Green's function.

Lemma 24 *The Green's function $G(t, s)$ corresponding to problem (4.1) has the following three properties: For each fixed $s_0 \in \{0, ..., b + M\}$,*

- $$\min_{t\in\{\nu-1,...,b+M+\nu\}} G(t, s_0) > 0. \tag{4.5}$$
- $$\max_{t\in\{\nu-N,...,b+M+\nu\}} G(t, s_0) = G(b + M + \nu, s_0). \tag{4.6}$$
- $$\min_{t\in\{a+M+\nu-1,...,b+M+\nu-1\}} G(t, s_0) \ge \gamma \max_{t\in\{\nu-N,...,b+M+\nu\}} G(t, s_0), \tag{4.7}$$

where $0 < \gamma < 1$ is a fixed constant independent of s_0 and $a := \left\lceil \frac{3}{4}(b + M) \right\rceil$.

Proof. For convenience, let us define the two piecewise components of the Green's function as

$$\begin{aligned} G_1(t,s) &:= \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}t^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}}, \\ G_2(t,s) &:= \frac{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}t^{\underline{\nu-1}}. \end{aligned}$$

Observe first that we may apply rule (4.2) to show that for each fixed $s_0 \in \{0, ..., b+M\}$,

$$G(\nu-1, s_0) = \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} > 0.$$

With this in mind, our strategy to prove (4.5) and (4.6) lies in showing that

$$\Delta_t G(t, s_0) \geq 0, \text{ for all } t \in \{\nu - N, ..., b+M+\nu-1\}. \tag{4.8}$$

To this end, let $s_0 \in \{0, ..., b+M\}$ be chosen and fixed for the remainder of the proof. We demonstrate (4.8) through the following five cases.

- For $t \in \{\nu - N, ..., \nu - 3\}$ (this step not necessary if $\nu = N = 2$),

$$\Delta_t G(t, s_0) = G(t+1, s_0) - G(t, s_0) = 0 - 0 = 0.$$

- For $t = \nu - 2$,

$$\begin{aligned} \Delta_t G(t, s_0) &= G_2(\nu-1, s_0) - 0 \\ &= \frac{1}{\Gamma(\nu)} \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(\nu-1)^{\underline{\nu-1}} \\ &= \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} \end{aligned}$$

$$> \quad 0, \text{ using rule (4.2).}$$

- For $t \in \{\nu - 1, ..., \nu + s_0 - 2\}$,

$$\begin{aligned}
\Delta_t G(t, s_0) &= \frac{1}{\Gamma(\nu)} \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} \Delta t^{\underline{\nu-1}} \\
&= \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{\Gamma(\nu)\,(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} (\nu - 1)\, t^{\underline{\nu-2}} \\
&> 0,
\end{aligned}$$

applying rule (4.2) again, since we also have that $t > \max\{-1, \nu - 3\}$.

- For $t = \nu + s_0 - 1$,

$$\begin{aligned}
&\Gamma(\nu)\Delta_t G(t, s_0) \\
= \;& G_1(\nu + s_0, s_0) - G_2(\nu + s_0 - 1, s_0) \\
= \;& \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} (\nu + s_0)^{\underline{\nu-1}} - (\nu - 1)^{\underline{\nu-1}} \\
& - \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} (\nu + s_0 - 1)^{\underline{\nu-1}} \\
= \;& \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} \left((\nu + s_0)^{\underline{\nu-1}} - (\nu + s_0 - 1)^{\underline{\nu-1}}\right) \\
& - \Gamma(\nu) \\
= \;& \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} \left((\nu - 1)\,(\nu + s_0 - 1)^{\underline{\nu-2}}\right) - \Gamma(\nu).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \Delta_t G(\nu + s_0 - 1, s_0) \geq 0 \\
\Longleftrightarrow \;& \frac{(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu-\mu-1}}}{\Gamma(\nu)\,(b + M + \nu - \mu)^{\underline{\nu-\mu-1}}} \left((\nu - 1)\,(\nu + s_0 - 1)^{\underline{\nu-2}}\right) - 1 \geq 0
\end{aligned}$$

$$
\begin{aligned}
&\iff \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}\frac{(\nu+s_0-1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} \geq 1 \\
&\iff \frac{\Gamma(b+M+\nu-\mu-s_0)\cdot\Gamma(b+M+2)\cdot\Gamma(\nu+s_0)}{\Gamma(b+M-s_0+1)\cdot\Gamma(b+M+\nu-\mu+1)\cdot\Gamma(s_0+2)\cdot\Gamma(\nu-1)} \geq 1 \\
&\iff \frac{\Gamma(b+M+2)}{\Gamma(b+M-s_0+1)}\frac{\Gamma(b+M+\nu-\mu-s_0)}{\Gamma(b+M+\nu-\mu+1)}\frac{\Gamma(\nu+s_0)}{\Gamma(\nu-1)}\frac{1}{(s_0+1)!} \geq 1 \\
&\iff \frac{[(b+M+1)\cdots(b+M-s_0+1)]\,[(\nu+s_0-1)\cdots(\nu-1)]}{[(b+M+\nu-\mu)\cdots(b+M+\nu-\mu-s_0)]\,[(s_0+1)\cdots 1]} \geq 1 \\
&\iff \prod_{p=0}^{s_0}\left(\frac{b+M+1-p}{b+M+\nu-\mu-p}\cdot\frac{\nu-1+s_0-p}{1+s_0-p}\right) \geq 1 \\
&\Longleftarrow \frac{b+M+1-p}{b+M+\nu-\mu-p}\cdot\frac{\nu-1+s_0-p}{1+s_0-p} \geq 1, \text{ for each } p\in\{0,...,s_0\} \\
&\iff \frac{b+M+1-p}{b+M+\nu-\mu-p} \geq \frac{1+s_0-p}{\nu-1+s_0-p}, \text{ for each } p\in\{0,...,s_0\}.
\end{aligned}
$$

Furthermore, since $\nu \geq 2$ and $\max\{s_0\} = b+M$, the above inequality holds

$$
\begin{aligned}
&\iff \quad \frac{b+M+1-p}{b+M+\nu-\mu-p} \geq \frac{1+b+M-p}{\nu-1+b+M-p}, \text{ for each } p\in\{0,...,s_0\} \\
&\iff \quad \nu-1+b+M-p \geq b+M+\nu-\mu-p \\
&\iff \quad \mu \geq 1,
\end{aligned}
$$

which is certainly true. It follows that $\Delta_t G(s_0+\nu-1, s_0) \geq 0$.

- For $t \in \{\nu+s_0, ..., b+M+\nu-1\}$, let us find $k \in \{0, ..., b+M-s_0-1\}$ so that $t = \nu+s_0+k$. Then,

$$
\begin{aligned}
&\Delta_t G(t,s_0) \geq 0 \\
&\iff \Gamma(\nu)\Delta_t G_1(t,s_0) \geq 0 \\
&\iff \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}\Delta t^{\underline{\nu-1}} - \Delta_t(t-\sigma(s_0))^{\underline{\nu-1}} \geq 0 \\
&\iff \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(\nu-1)\,t^{\underline{\nu-2}} \geq (\nu-1)\,(t-\sigma(s_0))^{\underline{\nu-2}}
\end{aligned}
$$

$$\iff \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} t^{\underline{\nu-2}} \geq (t-\sigma(s_0))^{\underline{\nu-2}}$$
$$\iff \frac{\Gamma(b+M+\nu-\mu-s_0)\Gamma(b+M+2)\Gamma(t+1)}{\Gamma(b+M-s_0+1)\Gamma(b+M+\nu-\mu+1)\Gamma(t-\nu+3)} \geq \frac{\Gamma(t-s_0)}{\Gamma(t-s_0-\nu+2)}$$
$$\iff \frac{\Gamma(b+M+\nu-\mu-s_0)\Gamma(b+M+2)}{\Gamma(b+M+\nu-\mu+1)\Gamma(b+M-s_0+1)} \geq \frac{\Gamma(t-\nu+3)\Gamma(t-s_0)}{\Gamma(t-\nu-s_0+2)\Gamma(t+1)}$$
$$\iff \frac{(b+M+1)\cdot\cdot(b+M-s_0+1)}{(b+M+\nu-\mu)\cdot\cdot(b+M+\nu-\mu-s_0)} \geq \frac{(t-\nu+2)\cdot\cdot(t-\nu-s_0+2)}{(t)\cdot\cdot(t-s_0)}$$
$$\iff \frac{(b+M+1)\cdot\cdot(b+M+1-s_0)}{(b+M+\nu-\mu)\cdot\cdot(b+M+\nu-\mu-s_0)} \geq \frac{(s_0+k+2)\cdot\cdot(k+2)}{(s_0+k+\nu)\cdot\cdot(k+\nu)},$$

and since $\nu \geq 2$ and $\max\{k\} = b+M-s_0-1$, the above inequality holds

$$\iff \frac{(b+M+1)\cdot\cdot(b+M+1-s_0)}{(b+M+\nu-\mu)\cdot\cdot(b+M+\nu-\mu-s_0)} \geq \frac{(b+M+1)\cdot\cdot(b+M+1-s_0)}{(b+M-1+\nu)\cdot\cdot(b+M-s_0-1+\nu)}$$
$$\iff (b+M+\nu-1)\cdot\cdot(b+M+\nu-s_0-1) \geq (b+M+\nu-\mu)\cdot\cdot(b+M+\nu-s_0-\mu)$$
$$\iff \mu \geq 1,$$

which is true.

In summary, we have shown (4.8): For each fixed $s_0 \in \{0, ..., b+M\}$,

$$\Delta_t G(t, s_0) \geq 0, \text{ for all } t \in \{\nu - N, ..., b+M+\nu-1\}.$$

Since $G(\nu-1, s_0) > 0$, we conclude additionally that for each $s_0 \in \{0, ..., b+M\}$,

i) $G(t, s_0) > 0$ on $\{\nu-1, ..., b+M+\nu\}$,

ii) $G(t, s_0)$ is nondecreasing on $\{\nu-N, ..., b+M+\nu\}$,

from which (4.5) and (4.6) easily follow.

We turn our attention now to proving (4.7). Let a be defined as in the statement of the lemma. Notice that (4.6) allows us to write

$$
\begin{aligned}
&\min_{t\in\{a+M+\nu-1,\dots,b+M+\nu-1\}} G(t,s_0)\\
&= G(a+M+\nu-1,s_0)\\
&= \frac{1}{\Gamma(\nu)}\begin{cases} \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(a+M+\nu-1)^{\underline{\nu-1}}, & \\ -(a+M+\nu-1-\sigma(s_0))^{\underline{\nu-1}}, & 0\le s_0\le a+M-1,\\ \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(a+M+\nu-1)^{\underline{\nu-1}}, & a+M\le s_0\le b+M. \end{cases}
\end{aligned}
$$

Furthermore, (4.6) implies

$$
\begin{aligned}
&\max_{t\in\{\nu-N,\dots,b+M+\nu\}} G(t,s_0)\\
&= G(b+M+\nu,s_0)\\
&= \frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(b+M+\nu)^{\underline{\nu-1}}-(b+M+\nu-\sigma(s_0))^{\underline{\nu-1}}\\
&= \frac{\Gamma(b+M+\nu-\mu-s_0)}{\Gamma(b+M-s_0+1)}\frac{\Gamma(b+M+2)}{\Gamma(b+M+\nu-\mu+1)}\frac{\Gamma(b+M+\nu+1)}{\Gamma(b+M+2)}\\
&\qquad-\frac{\Gamma(b+M+\nu-s_0)}{\Gamma(b+M-s_0+1)}\\
&= \frac{\Gamma(b+M+\nu-\mu-s_0)}{\Gamma(b+M-s_0+1)}\left[\frac{\Gamma(b+M+\nu+1)}{\Gamma(b+M+\nu-\mu+1)}-\frac{\Gamma(b+M+\nu-s_0)}{\Gamma(b+M+\nu-\mu-s_0)}\right]\\
&= (b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}\left[(b+M+\nu)^{\underline{\mu}}-(b+M+\nu-\sigma(s_0))^{\underline{\mu}}\right]\\
&= (b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}\mathcal{P}(s_0),
\end{aligned}
$$

where

$$\mathcal{P}(s):=(b+M+\nu)^{\underline{\mu}}-(b+M+\nu-\sigma(s))^{\underline{\mu}}.$$

Note that since $\mu>0$ and $\nu-1>\mu-1$, we may apply (4.3) to conclude that $t^{\underline{\mu}}$

is increasing for $t \in \{\nu - 1, ..., b + M + \nu\}$. It follows that $(b + M + \nu - \sigma(s))^{\underline{\mu}}$ is decreasing and hence that $\mathcal{P}(s)$ is increasing for $s \in \{0, ..., b + M\}$. Moreover, since

$$\begin{aligned}
\mathcal{P}(0) &= (b + M + \nu)^{\underline{\mu}} - (b + M + \nu - 1)^{\underline{\mu}} \\
&= \mu(b + M + \nu - 1)^{\underline{\mu - 1}}, \text{ using power rule (2.1),} \\
&= \mu \frac{\Gamma(b + M + \nu)}{\Gamma(b + M + \nu - \mu + 1)} \\
&\geq \mu \\
&\geq 1,
\end{aligned}$$

we have that

$$1 \leq \mathcal{P}(0) \leq \mathcal{P}(s_0) \leq \mathcal{P}(b + M), \text{ for all } s_0 \in \{0, ..., b + M\}.$$

So, finding conditions on $0 < \gamma < 1$ such that, for all $s_0 \in \{0, ..., b + M\}$,

$$\min_{t \in \{a+M+\nu-1,...,b+M+\nu-1\}} G(t, s_0) \geq \gamma \max_{t \in \{\nu-N,...,b+M+\nu\}} G(t, s_0)$$

is equivalent to finding conditions on γ such that

$$G(a + M + \nu - 1, s_0) \geq \gamma(b + M + \nu - \mu - \sigma(s_0))^{\underline{\nu - \mu - 1}} \mathcal{P}(s_0), \tag{4.9}$$

for all $s_0 \in \{0, ..., b + M\}$. To find such an appropriate γ, we consider the following two cases.

Case 1 $[s_0 \in \{a + M, ..., b + M\}]$

In this case, (4.9) becomes

$$\begin{aligned}
&\frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(a+M+\nu-1)^{\underline{\nu-1}} \\
&\geq \gamma(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}\mathcal{P}(s_0) \\
\iff\ & \frac{(a+M+\nu-1)^{\underline{\nu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} \geq \gamma\mathcal{P}(s_0) \\
\iff\ & 0<\gamma \leq \frac{1}{\mathcal{P}(s_0)}\frac{(a+M+\nu-1)^{\underline{\nu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} \\
\Longleftarrow\ & 0<\gamma \leq \frac{1}{\mathcal{P}(b+M)}\frac{(a+M+\nu-1)^{\underline{\nu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}.
\end{aligned}$$

Since (4.2) implies that both $(a+M+\nu-1)^{\underline{\nu-1}}$ and $(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}$ are positive, this is indeed the condition on $0<\gamma<1$ that we seek.

Case 2 $[s_0 \in \{0,...,a+M-1\}]$

In this case, (4.9) becomes

$$\begin{aligned}
&\frac{(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}(a+M+\nu-1)^{\underline{\nu-1}} \\
&\quad -(a+M+\nu-1-\sigma(s_0))^{\underline{\nu-1}} \\
&\geq \gamma(b+M+\nu-\mu-\sigma(s_0))^{\underline{\nu-\mu-1}}\mathcal{P}(s_0) \\
\iff\ & 0<\gamma\leq\frac{\mathcal{Q}(s_0)}{\mathcal{P}(s_0)},
\end{aligned}$$

where the function $\mathcal{Q}$ is defined by

$$\mathcal{Q}(s) := \frac{(a+M+\nu-1)^{\underline{\nu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}} - \frac{(a+M+\nu-1-\sigma(s))^{\underline{\nu-1}}}{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}},$$

for $s \in \{-1,0,...,a+M-1\}$. Note that although we do not actually consider the value $s=-1$ as an input for the Green's function, we define $\mathcal{Q}$ at $s=-1$ in order to show that $\mathcal{Q}$ is positive and increasing on $\{0,...,a+M-1\}$.

To see this, observe that $\mathcal{Q}(-1)=0$ and that for $s\in\{-1,0,...,a+M-2\}$,

$$\begin{aligned}
&\Delta\mathcal{Q}(s)\\
=\;& \mathcal{Q}(s+1)-\mathcal{Q}(s)\\
=\;& \frac{(a+M+\nu-1-\sigma(s))^{\underline{\nu-1}}}{(b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}}}-\frac{(a+M+\nu-1-\sigma(s+1))^{\underline{\nu-1}}}{(b+M+\nu-\mu-\sigma(s+1))^{\underline{\nu-\mu-1}}}\\
=\;& \frac{\Gamma(a+M+\nu-1-s)}{\Gamma(a+M-s)}\frac{\Gamma(b+M+1-s)}{\Gamma(b+M+\nu-\mu-s)}\\
&\quad-\frac{\Gamma(a+M+\nu-2-s)}{\Gamma(a+M-s-1)}\frac{\Gamma(b+M-s)}{\Gamma(b+M+\nu-\mu-1-s)}.
\end{aligned}$$

Therefore, noting that all the terms in the above equation are positive for $s\in\{-1,0,...,a+M-2\}$, we have

$$\begin{aligned}
\Delta\mathcal{Q}(s)>0 \iff\;& \frac{\Gamma(a+M+\nu-1-s)}{\Gamma(a+M+\nu-2-s)}\frac{\Gamma(b+M+1-s)}{\Gamma(b+M-s)}\\
&\quad>\frac{\Gamma(a+M-s)}{\Gamma(a+M-s-1)}\frac{\Gamma(b+M+\nu-\mu-s)}{\Gamma(b+M+\nu-\mu-1-s)}\\
\iff\;& (b+M-s)\,(a+M+\nu-2-s)\\
&\quad>(a+M-s-1)\,(b+M+\nu-\mu-1-s)\\
\iff\;& (b+M-s)\,(a+M+\nu-2-s)\\
&\quad>(a+M-s-1)\,(b+M+\nu-\mu-1-s)\\
\iff\;& (\nu-1)\,(b-a+1)+\mu\,(a+M-s)>0,
\end{aligned}$$

which is certainly true, since both terms on the left hand side are positive. Since $\mathcal{Q}(-1)=0$ and $\mathcal{Q}(s)$ is strictly increasing for $s\in\{-1,0,...,a+M-1\}$, the condition on $0 < \gamma < 1$ we seek is

$$0 < \gamma \leq \frac{\mathcal{Q}(0)}{\mathcal{P}(a+M-1)}.$$

Combining Case 1 with Case 2, we have shown that any choice of $0 < \gamma < 1$ satisfying

$$\gamma \leq \min\left\{\frac{1}{\mathcal{P}(b+M)}\frac{(a+M+\nu-1)^{\underline{\nu-1}}}{(b+M+\nu-\mu)^{\underline{\nu-\mu-1}}}, \frac{\mathcal{Q}(0)}{\mathcal{P}(a+M-1)}\right\}$$

will also satisfy—for each $s_0 \in \{0, ..., b+M\}$—the inequality

$$\min_{t\in\{a+M+\nu-1,...,b+M+\nu-1\}} G(t,s_0) \geq \gamma \max_{t\in\{\nu-N,...,b+M+\nu\}} G(t,s_0).$$

Notice that since $(a+M+\nu-2)^{\underline{\nu-1}} > 0$, $(b+M+\nu-\mu-1)^{\underline{\nu-\mu-1}} > 0$ and $\mathcal{P}$ is increasing, a simpler but less sharp requirement for choosing $0 < \gamma < 1$ is

$$0 < \gamma \leq \frac{\mathcal{Q}(0)}{\mathcal{P}(b+M)}.$$

■

4.3 Solutions to the Nonlinear Problem

Now that we have derived and developed properties of the Green's function corresponding to the nonlinear boundary value problem (4.1), we have the necessary tools to discuss solutions to (4.1) when the non-homogeneous term $f(\cdot, y(\cdot+\nu-1))$ does in fact depend on the solution y. Let us recall the problem at hand:

$$\begin{cases} -\Delta^{\nu}_{\nu-N} y(t) = f(t, y(t+\nu-1)), & \text{for } t \in \{0, ..., b+M\}, \\ \Delta^i y(\nu - N) = 0, & \text{for } i \in \{0, ..., N-2\}, \\ \Delta^{\mu}_{\nu-N} y(b+M+\nu-\mu) = 0, \end{cases}$$

where

- $\nu \geq 2$, with $N \in \mathbb{N}$ chosen so that $N - 1 < \nu \leq N$.
- $1 \leq \mu < \nu$, with $M \in \mathbb{N}$ chosen so that $M - 1 < \mu \leq M$.
- $b \in \mathbb{N}$.
- $f : \{0, ..., b + M\} \times \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $y \geq 0$.

Define the Banach space $(\mathbb{B}, \|\cdot\|)$ by

$$\mathbb{B} := \left\{ \begin{array}{l} y : \{\nu - N, ..., b + M + \nu\} \to \mathbb{R} \text{ such that} \\ \quad \Delta^i y(\nu - N) = 0 \text{ for } i \in \{0, ..., N-2\} \text{ and} \\ \quad \Delta^{\mu}_{\nu-N} y(b + M + \nu - \mu) = 0, \end{array} \right\},$$

together with norm

$$\|y\|_{\mathbb{B}} = \|y\| := \max_{t \in \{\nu - N, ..., b+M+\nu\}} |y(t)|.$$

Define also the completely continuous operator

$$T : \mathbb{B} \to \mathbb{B} \text{ by } Ty(t) := \sum_{s=0}^{b+M} G(t, s) f(s, y(s + \nu - 1)).$$

Our work from Section 4.2 allows us to conclude that every fixed point of T is a solution to (4.1). In the following work, we consider two different settings and apply theorems of Krasnosel'skii and Banach to demonstrate the existence of fixed points for T, and hence of solutions for (4.1). Moreover, applying Banach's Theorem in an appropriate setting yields a unique solution to (4.1).

4.3.1 Krasnosel'skii

The following theorem is attributed to Soviet mathematician Mark Krasnosel'skii (1920-1997), who worked primarily in nonlinear functional analysis in the Ukraine and Russia.

Theorem 25 *Let $\mathbb{B}$ be a Banach space, $K \subseteq \mathbb{B}$ a cone and let Ω_1 and Ω_2 be open sets contained in $\mathbb{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Suppose that $T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K$ is a completely continuous operator. Then T has at least one fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$ if either*

i) $\|Ty\| \leq \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|Ty\| \geq \|y\|$ *for* $y \in K \cap \partial\Omega_2$.

ii) $\|Ty\| \geq \|y\|$ *for* $y \in K \cap \partial\Omega_1$ *and* $\|Ty\| \leq \|y\|$ *for* $y \in K \cap \partial\Omega_2$.

To help us apply Krasnosel'skii's Theorem 25 to (4.1), let us define

$$\begin{aligned} a &:= \left\lceil \frac{3(b+M)}{4} \right\rceil \text{ (as in Lemma 24)} \\ \eta &:= \left(\sum_{s=0}^{b+M} G(b+M+\nu, s) \right)^{-1} = \left\| \sum_{s=0}^{b+M} G(\cdot, s) \right\|^{-1} \\ \lambda &:= \left(\sum_{s=a+M}^{b+M} G(b+M+\nu, s) \right)^{-1}. \end{aligned}$$

Also, we make the following bound assumptions on the nonlinearity f:

(A1) There exists an $r_1 > 0$ such that for each $0 \leq y \leq r_1$,

$$f(t, y) \leq \eta r_1, \text{ for all } t \in \{0, ..., b + M\}.$$

(A2) There exists an $r_2 > 0$ such that for each $\gamma r_2 \leq y \leq r_2$,

$$f(t, y) \geq \lambda r_2, \text{ for all } t \in \{a + M, ..., b + M\}.$$

(A3) Either $r_1 < \gamma r_2$ or $\lambda r_2 \leq \eta r_1$.

Note that $\eta < \lambda$ and $0 < \gamma < 1$ (where γ is the constant found in Lemma 24). With these two facts in mind, we make assumption (A3) to avoid the case where assumptions (A1) and (A2) contradict each other.

Theorem 26 *Suppose that assumptions (A1), (A2) and (A3) hold for the nonlinearity $f(t, y)$. Then problem (4.1) has at least one positive solution y_0 such that* $\min\{r_1, r_2\} \leq \|y_0\| \leq \max\{r_1, r_2\}$.

Proof. Suppose assumptions (A1), (A2) and (A3) hold for $f(t, y)$ and define the cone $K \subseteq \mathbb{B}$ by

$$K := \left\{ \begin{array}{c} y \in \mathbb{B} : y(t) \geq 0 \text{ on } \{\nu - N, ..., b + M + \nu\} \text{ and} \\ \min\limits_{t \in \{a+M+\nu-1, ..., b+M+\nu-1\}} y(t) \geq \gamma \|y\|. \end{array} \right\}$$

We first show that the operator T maps K into K. To see this, observe that for any $y \in K$ and $t \in \{\nu - N, ..., b + M + \nu\}$,

$$Ty(t) = \sum_{s=0}^{b+M} G(t, s) f(s, y(s + \nu - 1)) \geq 0,$$

since G is nonnegative everywhere and f is nonnegative for $y \geq 0$. Moreover,

$$
\begin{aligned}
& \min_{t \in \{a+M+\nu-1,\ldots,b+M+\nu-1\}} Ty(t) \\
= \; & \sum_{s=0}^{b+M} \left(\min_{t \in \{a+M+\nu-1,\ldots,b+M+\nu-1\}} G(t,s) \right) f(s, y(s+\nu-1)) \\
\geq \; & \sum_{s=0}^{b+M} \left(\gamma \max_{t \in \{\nu-N,\ldots,b+M+\nu\}} G(t,s) \right) f(s, y(s+\nu-1)) \,, \text{ by property (4.7)} \\
= \; & \gamma \max_{t \in \{\nu-N,\ldots,b+M+\nu\}} \sum_{s=0}^{b+M} G(t,s) f(s, y(s+\nu-1)) \\
= \; & \gamma \left\| Ty \right\| .
\end{aligned}
$$

Hence, $T : K \to K$.

Next, define for all $r > 0$ the open subsets

$$\Omega_r := \{ y \in \mathbb{B} : \|y\| < r \} .$$

Note, in particular, regarding constants r_1 and r_2 from assumptions (A1) and (A2), that

$$y \in \partial\Omega_{r_1} \Longrightarrow \|y\| = r_1 \text{ and } y \in \partial\Omega_{r_2} \Longrightarrow \|y\| = r_2.$$

Assumption (A3) then leads us to the following two cases.

Case 1 $[r_1 < \gamma r_2]$

If $r_1 < \gamma r_2$, we may apply Theorem 25(i) to show the existence of a positive solution to problem (4.1), as follows:

- Suppose $y \in K \cap \partial\Omega_{r_1}$. Then

$$\|Ty\| \;=\; \max_{t \in \{\nu-N,\ldots,b+M+\nu\}} \sum_{s=0}^{b+M} G(t,s) f(s, y(s+\nu-1))$$

$$
\begin{aligned}
&= \sum_{s=0}^{b+M} G(b+M+\nu,s) f(s,y(s+\nu-1))\,, \text{ by property (4.6),} \\
&\leq \eta r_1 \left(\sum_{s=0}^{b+M} G(b+M+\nu,s) \right), \text{ by assumption (A1),} \\
&= r_1 \\
&= \|y\|\,.
\end{aligned}
$$

- Suppose $y \in K \cap \partial\Omega_{r_2}$. Then

$$
\begin{aligned}
\|Ty\| &= Ty(b+M+\nu) \\
&= \sum_{s=0}^{b+M} G(b+M+\nu,s) f(s,y(s+\nu-1)) \\
&\geq \sum_{s=a+M}^{b+M} G(b+M+\nu,s) f(s,y(s+\nu-1)).
\end{aligned}
$$

Now, since $y \in K \cap \partial\Omega_{r_2}$, we have that

$$
\begin{aligned}
&\min_{t\in\{a+M+\nu-1,\ldots,b+M+\nu-1\}} y(t) \geq \gamma \|y\| \\
\implies \quad & \gamma r_2 \leq y(t) \leq r_2, \text{ for } t \in \{a+M+\nu-1,\ldots,b+M+\nu-1\}\,.
\end{aligned}
$$

Assumption (A2) thus implies that

$$
\begin{aligned}
& f(t-\nu+1,y(t)) \geq \lambda r_2, \text{ for } t \in \{a+M+\nu-1,\ldots,b+M+\nu-1\} \\
\implies \quad & f(s,y(s+\nu-1)) \geq \lambda r_2, \text{ for } s \in \{a+M,\ldots,b+M\}\,.
\end{aligned}
$$

Therefore,

$$
\|Ty\| \geq \sum_{s=a+M}^{b+M} G(b+M+\nu,s) f(s,y(s+\nu-1))
$$

$$\begin{aligned} &\geq \lambda r_2 \sum_{s=a+M}^{b+M} G(b+M+\nu, s) \\ &= r_2 \\ &= \|y\| . \end{aligned}$$

In summary, we have shown that

- $\|Ty\| \leq \|y\|$, for all $y \in K \cap \partial\Omega_{r_1}$.
- $\|Ty\| \geq \|y\|$, for all $y \in K \cap \partial\Omega_{r_2}$.

Hence, Theorem 25(i)—with $\Omega_1 := \Omega_{r_1}$ and $\Omega_2 := \Omega_{r_2}$—implies that T has a fixed point y_0 in $K \cap \left(\overline{\Omega}_{r_2} \backslash \Omega_{r_1}\right)$. It follows that y_0 is a solution to (4.1) with $r_1 \leq \|y_0\| \leq r_2$.

Case 2 $[\lambda r_2 \leq \eta r_1]$

If $\lambda r_2 \leq \eta r_1$, we may apply Theorem 25(ii) with $\Omega_1 := \Omega_{r_2}$ and $\Omega_2 := \Omega_{r_1}$ to show the existence of a positive solution y_0 in $K \cap \left(\overline{\Omega}_{r_1} \backslash \Omega_{r_2}\right)$ such that $r_2 \leq \|y_0\| \leq r_1$, following the same procedure as in Case 1.

Since r_1 and r_2 from assumptions (A1) and (A2) are positive, we see that in either case, y_0 is a nontrivial solution to (4.1) satisfying

$$\min\{r_1, r_2\} \leq \|y_0\| \leq \max\{r_1, r_2\} .$$

■

Example 6 *Consider the fractional* $(3,1)$ *boundary value problem*

$$\begin{cases} -\Delta^{\pi}_{\pi-4}y(t) = \left(\frac{y(t+\pi-1)}{e^t}\right)^4, \quad t \in \{0, ..., 12\}, \\ y(\pi-4) = \Delta y(\pi-4) = \Delta^2 y(\pi-4) = 0, \\ \Delta^{e/2}_{\pi-4} y\left(12+\pi-\frac{e}{2}\right) = 0. \end{cases} \tag{4.10}$$

Here, we have the nonlinear problem (4.1) with

$\nu = \pi,$	$N = 4,$	$f(t,y) = \left(\frac{y}{e^t}\right)^4,$
$\mu = \frac{e}{2},$	$M = 2,$	$b = 10,\ a = 9$

and

$$\mathcal{D}\{y\} = \{\pi - 4, ..., \pi + 12\}.$$

A graph of the Green's function corresponding to (4.10) is shown below in three dimensions, together with several two dimensional slices for sample values of s*.*

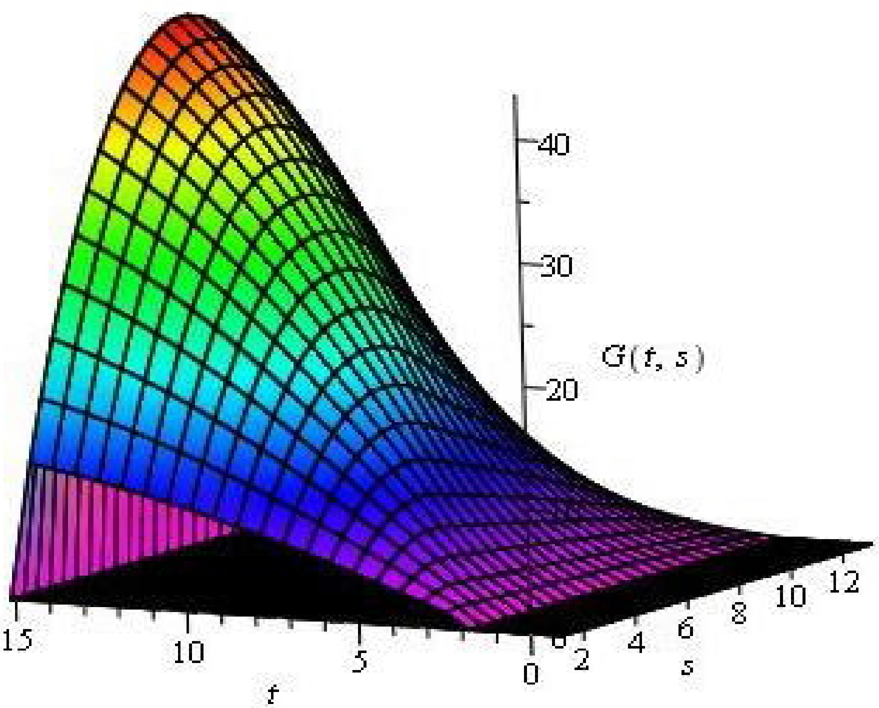

Figure 4.1: The Green's Function for Problem (4.10)

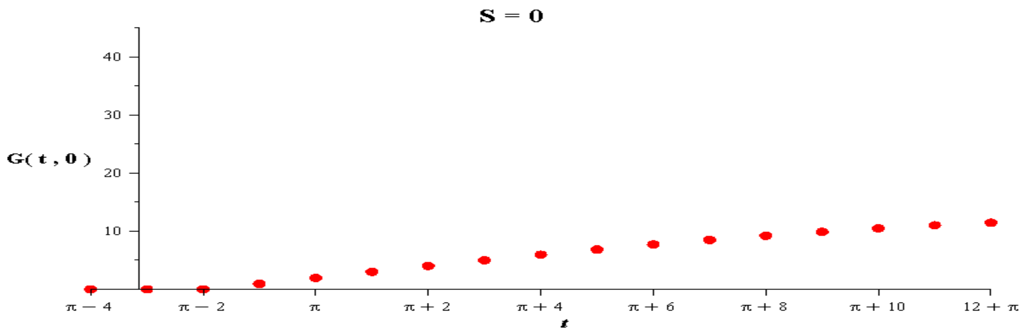

Figure 4.2: Green's Function Slice at $s = 0$

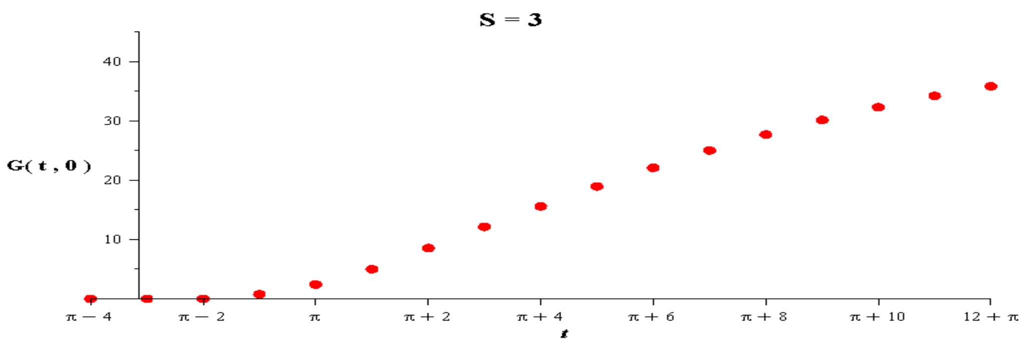

Figure 4.3: Green's Function Slice at $s = 3$

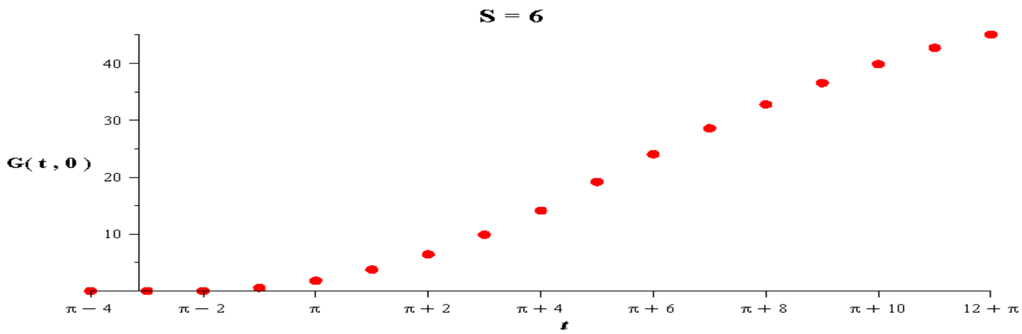

Figure 4.4: Green's Function Slice at $s = 6$

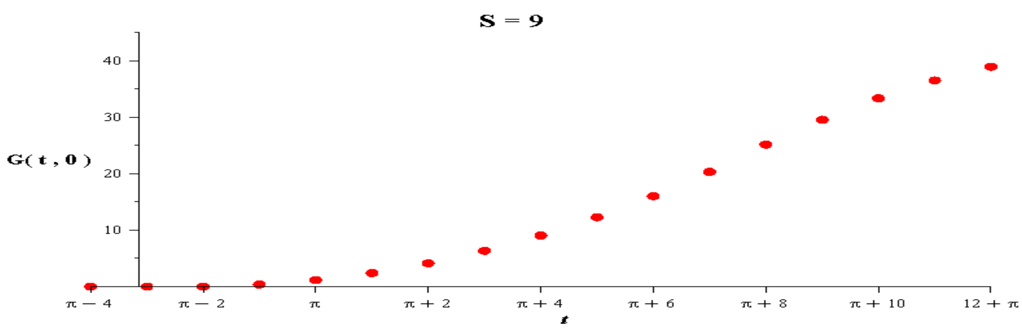

Figure 4.5: Green's Function Slice at $s = 9$

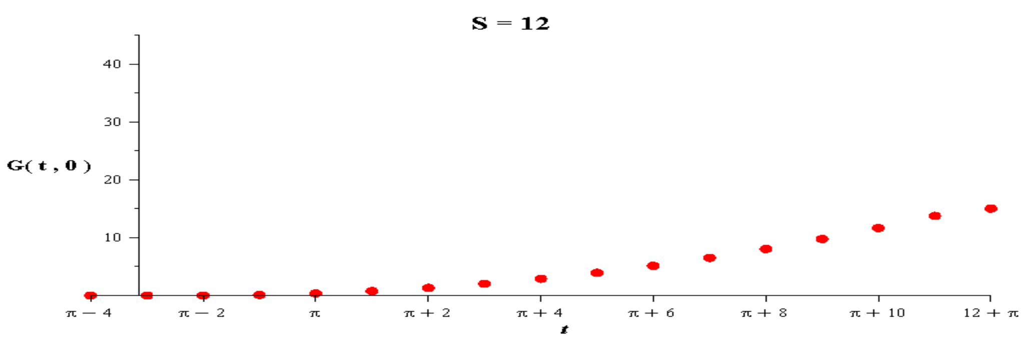

Figure 4.6: Green's Function Slice at $s = 12$

Let us apply Theorem 26(i) to show that (4.10) has at least one positive solution. To do this, we must find constants $r_1, r_2 > 0$ *so that assumptions (A1)–(A3) are* *satisfied. First, we calculate the following three constants from Theorem 26:*

$$
\begin{aligned}
\gamma &:= \frac{\mathcal{Q}(0)}{\mathcal{P}(12)} \approx 0.0875 \\
\eta &:= \left\| \sum_{s=0}^{12} G(\cdot, s) \right\|^{-1} \approx 0.00103 \\
\lambda &:= \left\| \sum_{s=11}^{12} G(\cdot, s) \right\|^{-1} \approx 0.0108
\end{aligned}
$$

- *Satisfying (A1):*

  We wish to find $r_1 > 0$ *such that*

$$
\begin{aligned}
& 0 \le f(t, y) \le \eta r_1, \text{ for } 0 \le y \le r_1 \text{ and } t \in \{0, ..., b+M\}, \\
\iff & 0 \le \left(\frac{y}{e^t}\right)^4 \le \eta r_1, \text{ for } 0 \le y \le r_1 \text{ and } t \in \{0, ..., 12\}, \\
\Longleftarrow & 0 \le r_1^4 \le \eta r_1 \\
\iff & 0 \le r_1 \le \eta^{\frac{1}{3}} \approx 0.1010.
\end{aligned}
$$

  Hence, (A1) is satisfied with $r_1 := \frac{1}{10}$.

- *Satisfying (A2):*

  We wish to find $r_2 > 0$ *such that*

$$
\begin{aligned}
& f(t, y) \ge \lambda r_2, \text{ for } \gamma r_2 \le y \le r_2 \text{ and } t \in \{a+M, ..., b+M\}, \\
\iff & \left(\frac{y}{e^t}\right)^4 \ge 0.0108 r_2, \text{ for } 0.0875 r_2 \le y \le r_2 \text{ and } t \in \{11, 12\}, \\
\Longleftarrow & \left(\frac{0.0875 r_2}{e^{11}}\right)^4 \ge 0.0108 r_2 \\
\iff & r_2 \ge 13,328,472.
\end{aligned}
$$

  Hence, (A2) is satisfied with $r_2 := 13,330,000$.

- *Assumption (A3) is also satisfied, since* $\frac{1}{10} < 0.0875 \cdot 13,330,000$.

Therefore, we may apply Theorem 26(i) to conclude that fractional boundary value problem (4.10) has a positive solution y_0 *on* $\{\pi - 4, ..., \pi + 12\}$. *Moreover, for what it is worth, we know that*

$$\frac{1}{10} \leq \|y_0\| \leq 13,330,000.$$

4.3.2 Banach

Polish born Stefan Banach (1892-1945) produced his well-known Contraction Mapping Theorem in 1922, the same year he received his doctoral degree in mathematics from the King John II Casimir University of Lwów, Ukraine.

Theorem 27 *Let* (X, d) *be a complete metric space and suppose* $f : X \to X$ *is a contraction mapping, where* $0 \leq L < 1$ *is the constant such that*

$$|f(x) - f(y)| \leq L\,|x - y|\,, \text{ for all } x, y \in X.$$

Then

(i) f *has a unique fixed point* $u \in X$.

(ii) $\lim\limits_{n\to\infty} f^n(x) = u$, *for all* $x \in X$.

(iii) $d(f^n(x), u) \leq \frac{L^n}{1-L} d(x, f(x))$, *for all* $x \in X$ *and* $n \in \mathbb{N}$.

Since ($\mathbb{B}$,$\|\cdot\|$) is a complete metric space (with $d(y_1, y_2) := \|y_1 - y_2\|$), we may apply Banach's Contraction Mapping Theorem 27 to show the existence of a unique solution to problem (4.1), provided the nonlinearity f satisfies a certain Lipschitz condition. In Theorem 28 below, $\eta := \left\|\sum_{s=0}^{b+M} G(\cdot, s)\right\|^{-1}$ is the fixed constant involving the Green's function, as defined earlier.

Theorem 28 *The nonlinear boundary value problem (4.1) has a unique solution provided there exists a* $0 \leq \theta < \eta$ *such that* f *satisfies the uniform Lipschitz condition*

$$|f(t,y_1) - f(t,y_2)| \leq \theta \, |y_1 - y_2| \, ,$$

for all $t \in \{0, ..., b+M\}$ *and* $y_1, y_2 \in \mathbb{R}$.

Proof. Suppose there exists a $0 \leq \theta < \eta$ such that $f(t,y)$ satisfies the uniform Lipschitz condition stated in the theorem. Then given any $y_1, y_2 \in \mathbb{B}$, we may estimate

$$\begin{aligned}
& \|Ty_1 - Ty_2\| \\
= & \left\| \sum_{s=0}^{b+M} G(\cdot,s) f(s, y_1(s+\nu-1)) - \sum_{s=0}^{b+M} G(\cdot,s) f(s, y_2(s+\nu-1)) \right\| \\
\leq & \left\| \sum_{s=0}^{b+M} G(\cdot,s) \big| f(s, y_1(s+\nu-1)) - f(s, y_2(s+\nu-1)) \big| \right\| \\
\leq & \left\| \sum_{s=0}^{b+M} G(\cdot,s) \theta \big| y_1(s+\nu-1) - y_2(s+\nu-1) \big| \right\| \\
\leq & \; \theta \, \|y_1 - y_2\| \left\| \sum_{s=0}^{b+M} G(\cdot,s) \right\| \\
= & \; \frac{\theta}{\eta} \, \|y_1 - y_2\| \, .
\end{aligned}$$

Then T is a contraction mapping with Lipschitz constant $\frac{\theta}{\eta} \in [0,1)$, whereby Banach's Contraction Mapping Theorem 27 implies that T has a unique fixed point $y_0 \in \mathbb{B}$. It follows that y_0 is the unique solution to (4.1). Moreover, we may obtain this unique solution y_0 by calculating for any $y \in \mathbb{B}$ the function

$$\lim_{n \to \infty} T^n y.$$

Furthermore, for any chosen initial function $y \in \mathbb{B}$, Theorem 27(iii) allows us to calculate the error of estimation at any step $n \in \mathbb{N}$ via

$$\|T^n y - y_0\| \leq \frac{\theta^n}{\eta^{n-1}(\eta - \theta)} \|Ty - y\|.$$

■

Remark 11 *Since*

$$\begin{aligned}
\frac{1}{\eta} &= \sum_{s=0}^{b+M} G(b+M+\nu, s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b+M} (b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}} \\
&\qquad\qquad \cdot \left[(b+M+\nu)^{\underline{\mu}} - (b+M+\nu-\sigma(s))^{\underline{\mu}}\right] \\
&< \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b+M} (b+M+\nu-\mu-\sigma(s))^{\underline{\nu-\mu-1}} (b+M+\nu)^{\underline{\mu}} \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b+M} \frac{\Gamma(b+M+\nu-\mu-s)}{\Gamma(b+M-s+1)} \frac{\Gamma(b+M+\nu+1)}{\Gamma(b+M+\nu-\mu+1)} \\
&\leq \sum_{s=0}^{b+M} \frac{\Gamma(b+M+\nu+1)}{\Gamma(\nu)} \frac{\Gamma(\nu-\mu)}{\Gamma(b+M+\nu-\mu+1)} \\
&= (b+M+1) \frac{(b+M+\nu)\cdots(\nu)}{(b+M+\nu-\mu)\cdots(\nu-\mu)} \\
&= (b+M+1) \frac{(b+M+\nu)^{\underline{b+M+1}}}{(b+M+\nu-\mu)^{\underline{b+M+1}}},
\end{aligned}$$

we obtain the inequality

$$0 \leq \frac{(b+M+\nu-\mu)^{\underline{b+M+1}}}{(b+M+1)\,(b+M+\nu)^{\underline{b+M+1}}} < \eta.$$

Hence, it is natural to check whether f satisfies a Uniform Lipschitz Condition with *constant*

$$\theta := \frac{(b+M+\nu-\mu)^{\underline{b+M+1}}}{(b+M+1)\,(b+M+\nu)^{\underline{b+M+1}}},$$

since, in this case, Theorem 28 would guarantee a unique solution to the boundary value problem (4.1). Moreover, this conclusion would be reached without ever needing to make the messier calculation for η.

Example 7 *Consider the fractional boundary value problem*

$$\begin{cases} -\Delta^{\pi}_{\pi-4}y(t) = \frac{\ln\left(|y(t+\pi-1)|+1\right)}{(t+4)^5}, \text{ for } t \in \{0,...,12\}, \\ y(\pi-4) = \Delta y(\pi-4) = \Delta^2 y(\pi-4) = 0, \\ \Delta^{e/2}_{\pi-4}y\left(12+\pi-\frac{e}{2}\right) = 0. \end{cases} \tag{4.11}$$

Here, quite similar to problem (4.10), we have

$\nu = \pi$	$N = 4$	$f(t,y) = \frac{\ln(\|y\|+1)}{(t+4)^5}$
$\mu = \frac{e}{2}$	$M = 2$	$b = 10,\ a = 9$

In this case, let us apply Theorem 28 to show that problem (4.11) has a unique solution on $\{\pi-4,...,\pi+12\}$. Observe that for $t \in \{0,...,12\}$ and $y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= \left|\frac{\ln(|y_1|+1)}{(t+4)^5} - \frac{\ln(|y_2|+1)}{(t+4)^5}\right| \\ &= \frac{1}{(t+4)^5}\big|\ln(|y_1|+1) - \ln(|y_2|+1)\big| \\ &\leq \frac{1}{4^5}\big|(|y_1|+1) - (|y_2|+1)\big| \\ &= \frac{1}{1024}\big|\,|y_1| - |y_2|\,\big| \\ &\leq \frac{|y_1 - y_2|}{1024}. \end{aligned}$$

Since

$$0 \leq \theta := \frac{1}{1024} \approx 0.00098 < 0.00103 \approx \eta,$$

Theorem 28 implies the existence of a unique solution y_0 *to (4.11) on* $\{\pi - 4, ..., \pi + 12\}$.

Appendix A

Extending to the Domain $\mathbb{N}_a^h$

All the results presented in this dissertation may be extended from the domain $\mathbb{N}_a$ to the more general domain

$$\mathbb{N}_a^h := a + h\mathbb{N}_0 = \{a, a+h, a+2h, ...\}, \qquad a \in \mathbb{R},\ h > 0 \text{ fixed.}$$

Though this extension may prove useful in some cases, the process itself is of little mathematical interest. Since $\mathbb{N}_a$ and $\mathbb{N}_a^h$ are both discrete domains with constant forward jump operators, proving an extension amounts to nothing more than 'scaling' a current result by a factor of h and, therefore, involves no new mathematical ideas. In light of this, the author has chosen to restrict this dissertation to the domain $\mathbb{N}_a$ in order to avoid the messy calculations required in work with $\mathbb{N}_a^h$.

Nonetheless, we present one such example here, that of extending part of the Fractional Laplace Transform Method to problems defined on $\mathbb{N}_a^h$. Note that Bohner and Guseinov have results similar to some that follow in [7]. Also, for convenience and easier comparison with the main body of this dissertation, some of the summations below come with instructions to sum by increments of h instead of by the standard unit increment. All of the definitions and lemmas leading up to the main theorem (Theorem 32) will be given without motivation or proof.

Definition 7 *Let $f : \mathbb{N}_a^h \to \mathbb{R}$ and $\nu > 0$ with $N - 1 < \nu \leq N$. Define the ν^{th}-order sum and the ν^{th}-order difference of f by*

$$ {}_h\Delta_a^{-\nu} f(t) := \frac{h^\nu}{\Gamma(\nu)} \sum_{\substack{s=a \\ by\ h}}^{t-\nu h} \left(\frac{t - \sigma(s)}{h} \right)^{\underline{\nu - 1}} f(s), $$

for $t \in \mathbb{N}_{a+\nu h}^h$, and

$$ {}_h\Delta_a^{\nu} f(t) := \Delta^N \left({}_h\Delta_a^{-(N-\nu)} \right) f(t), $$

for $t \in \mathbb{N}_{a+(N-\nu)h}^h$.

Similar to the derivation presented in Chapter 3, the Laplace Transform as taken from the general theory of time scales simplifies to

$$ \mathcal{L}_a \{f\}(s) = h \sum_{k=0}^{\infty} \frac{f(kh + a)}{(1 + hs)^{k+1}}, $$

for those $s \in \mathbb{C} \backslash \{-1/h\}$ for which the series converges.

Lemma 29 *Suppose $f : \mathbb{N}_a^h \to \mathbb{R}$ is of exponential order $r > 0$. Then*

$$ \mathcal{L}_a \{f\}(s) \text{ converges for } s \in \mathbb{C} \backslash \overline{B_{-1/h}(r^h/h)}. $$

Next, the Whole-Order Taylor Monomials are

$$ \begin{aligned} h_N(t, s) = & \frac{(t - s)(t - s - h) \cdots (t - s - (N - 1) h)}{\Gamma(N + 1)} \\ = & \frac{h^N}{\Gamma(N + 1)} \left(\frac{t - s}{h} \right) \left(\frac{t - s}{h} - 1 \right) \cdots \left(\frac{t - s}{h} - (N - 1) \right) \end{aligned} $$

$$= \frac{h^N}{\Gamma(N+1)} \left(\frac{t-s}{h}\right)^{\underline{N}}.$$

Definition 8 *For $\nu > 0$, define the ν^{th}-order Taylor Monomial by*

$$h_\nu(t,s) := \frac{h^\nu}{\Gamma(\nu+1)} \left(\frac{t-s}{h}\right)^{\underline{\nu}}.$$

Definition 9 *Let $f, g : \mathbb{N}_a^h \to \mathbb{R}$ be given. Define the convolution of f and g to be*

$$(f * g)(t) := \sum_{\substack{r=a \\ by\ h}}^{t} f(r) g(t-r+a), \text{ for } t \in \mathbb{N}_a^h.$$

Lemma 30 *Let $\mu > 0$ and $a, b \in \mathbb{R}$ such that $b - a = \mu h$. Then*

$$\mathcal{L}_b \{h_\mu(\cdot, a)\}(s) = \frac{(1+hs)^\mu}{s^{\mu+1}},$$

for $s \in \mathbb{C} \backslash \overline{B_{-1/h}(1/h)}$.

Lemma 31 *Let $f, g : \mathbb{N}_a^h \to \mathbb{R}$ be of exponential order $r > 0$. Then*

$$\mathcal{L}_a \{f * g\}(s) = \frac{(1+hs)}{h} \mathcal{L}_a\{f\}(s) \mathcal{L}_a\{g\}(s),$$

for $s \in \mathbb{C} \backslash \overline{B_{-1/h}(r^h/h)}$.

As in Chapter 3, the above definitions and lemmas may be combined to prove Theorem 32 for the Laplace Transform of a fractional sum.

Theorem 32 *Suppose $f : \mathbb{N}_a^h \to \mathbb{R}$ is of exponential order $r \geq 1$ and let $\nu > 0$ be given with $N - 1 < \nu \leq N$. Then for $s \in \mathbb{C} \backslash \overline{B_{-1/h}(r^h/h)}$,*

$$\mathcal{L}_{a+\nu h} \left\{ {}_h\Delta_a^{-\nu} f \right\}(s) = \left(\frac{1+hs}{s}\right)^\nu \mathcal{L}_a\{f\}(s).$$

Proof. Let f, r and ν be as given in the statement of the theorem. Then

$$
\begin{aligned}
& \mathcal{L}_{a+\nu h}\left\{{}_h\Delta_a^{-\nu} f\right\}(s) \\
= \;& h \sum_{k=0}^{\infty} \frac{{}_h\Delta_a^{-\nu} f(kh+a+\nu h)}{(1+hs)^{k+1}} \\
= \;& h \sum_{k=0}^{\infty} \frac{1}{(1+hs)^{k+1}} \cdot \frac{h^{\nu}}{\Gamma(\nu)} \sum_{\substack{r=a \\ \text{by } h}}^{a+kh} \left(\frac{(k+\nu)\,h+a-r-h}{h}\right)^{\underline{\nu-1}} f(r) \\
= \;& h^2 \sum_{k=0}^{\infty} \frac{1}{(1+hs)^{k+1}} \sum_{\substack{r=a \\ \text{by } h}}^{a+kh} f(r) h_{\nu-1}\left((a+kh)-r+a, a-(\nu-1)\,h\right) \\
= \;& h^2 \sum_{k=0}^{\infty} \frac{\left(f * h_{\nu-1}(\cdot, a-(\nu-1)h)\right)(a+kh)}{(1+hs)^{k+1}} \\
= \;& h \cdot \mathcal{L}_a\left\{f * h_{\nu-1}(\cdot, a-(\nu-1)h)\right\}(s) \\
= \;& h\frac{(1+hs)}{h} \mathcal{L}_a\{f\}(s)\, \mathcal{L}_a\left\{h_{\nu-1}(\cdot, a-(\nu-1))\right\}(s) \\
= \;& (1+hs)\frac{(1+hs)^{\nu-1}}{s^{\nu}} \mathcal{L}_a\{f\}(s) \\
= \;& \left(\frac{1+hs}{s}\right)^{\nu} \mathcal{L}_a\{f\}(s),
\end{aligned}
$$

for $s \in \mathbb{C} \backslash \overline{B_{-1/h}(r^h/h)}$. ■

Appendix B

Further Work

The ideas and results discussed in this dissertation are fairly close to the beginnings of calculus, so there are many directions to further develop and explore. One direction that appeals to the author is to extend the Fractional Laplace Transform Method to solve the more general initial value problem

$$\begin{cases} \Delta^{\nu}_{a+\nu-N} y(t) + py\left(t+\nu-N\right) = f(t), & p \in \mathbb{R},\, t \in \mathbb{N}_a \\ \Delta^{i} y(a+\nu-N) = A_i, & i \in \{0,1,...,N-1\}\,,\, A_i \in \mathbb{R}. \end{cases} \tag{B.1}$$

Attempting to solve problem (B.1) by the Laplace Transform Method, we arrive at the equation

$$\mathcal{L}_{a+\nu-N}\left\{y\right\}(s) = \frac{\mathcal{L}_a\left\{f\right\}(s)}{s^{\nu}\left(s+1\right)^{N-\nu}+p} + \sum_{j=0}^{N-1} \frac{s^j}{s^{\nu}\left(s+1\right)^{N-\nu}+p} \Delta^{\nu-j-1}_{a+\nu-N} y(a).$$

Hence, solving problem (B.1) requires one to find, for each $j \in \{0,...,N-1\}$, the inverse Laplace Transforms

$$\mathcal{L}_a^{-1}\left\{\frac{s^j}{s^{\nu}\left(s+1\right)^{N-\nu}+p}\right\}(s).$$

Needless to say, this is much easier accomplished for $p = 0$, as is true for problem (2.11) presented in this dissertation.

A second direction for further work is to develop results analogous to those presented in this dissertation for functions $f : q^{\mathbb{N}_0} \to \mathbb{R}$, where the geometric domain $q^{\mathbb{N}_0}$ is given by

$$q^{\mathbb{N}_0} := \left\{1, q, q^2, q^3, ...\right\}, \text{ where } q > 1 \text{ is fixed.}$$

Though analogous calculations and results are much more involved than for the domain $\mathbb{N}_a$, progress can still been made. In fact, mathematicians Atici, Bohner, Eloe and Guseinov, all referenced in this dissertation, have recent publications in this area.

Bibliography

[1] R Agarwal, M Meehan and D O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, Cambridge, UK, 2001.

[2] F Atici and P Eloe, *A Transform Method in Discrete Fractional Calculus*, International Journal of Difference Equations, 2 2007 165–176.

[3] F Atici and P Eloe, *Initial Value Problems in Discrete Fractional Calculus*, Proc. Amer. Math. Soc., 137 2009 981–989.

[4] FM Atici and PW Eloe, *Discrete Fractional Calculus with the Nabla Operator*, Electronic Journal of Qualitative Theory of Differential Equations, Spec. Ed. I 2009 1–12.

[5] FM Atici and PW Eloe, *Two-point boundary value problems for finite fractional difference equations*, Journal of Difference Equations and Appl. (to appear), doi: 10.1080/10236190903029241.

[6] Z Bai and H Lü, *Positive Solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., 311 2005, 495–505.

[7] M Bohner and G Guseinov, *The h-Laplace and q-Laplace Transforms*, Journal of Mathematical Analysis and Applications, 365 2010, 75-92.

[8] M Bohner and A Peterson, Dynamic Equations on Time Scales, Birkhaeuser, Boston, 2001.

[9] JB Diaz and TJ Olser, *Differences of Fractional Order*, Mathematics of Computation, 28 1974.

[10] CS Goodrich, *Solutions to a Discrete Right-Focal Fractional Boundary Value Problem,* Int. J. Difference Equations, 5 2010, in press.

[11] CS Goodrich, *Existence of a Positive Solution to a Class of Fractional Differential Equations,* Applied Mathematics Letters, 23 2010, 1050-1055.

[12] H Gray and N Zhang, *On a New Definition of the Fractional Difference*, Mathematics of Computation, 50 1988 513–529.

[13] W Kelley and A Peterson, *Difference Equations: An Introduction with Application*, Second Edition, Academic Press, New York, New York, 2000.

[14] B Kuttner, *On Differences of Fractional Order*, Proc. LMS, 7 1957 453–466.

[15] KS Miller and B Ross, *Fractional Difference Calculus*, Proceedings of the International Symposium on Univalent Functions, Fractional Calculus and their Applications, Nihon University, Koriyama, Japan, 1988 139–152.

[16] K Oldham and J Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Publications, Inc., Mineola, New York, 2002.

[17] I Podlubny, *Fractional Differential Equations*, AP, New York, NY, 1999.

[18] E Zeidler, *Nonlinear Functional Analysis and its Applications: Part 1: Fixed-Point Theorems*, Springer, New York, New York, 1986.