5-2011

Global Well-Posedness for a Nonlinear Wave Equation with p-Laplacian Damping

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GLOBAL WELL-POSEDNESS FOR A NONLINEAR WAVE EQUATION WITH P-LAPLACIAN DAMPING

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Mohammad A. Rammaha

Lincoln, Nebraska

May, 2011

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University of Nebraska, 2011

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This dissertation deals with the global well-posedness of the nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u - \Delta_p u_t = f(u) \quad \text{in} \quad \Omega \times (0,T), \\ \{u(0), u_t(0)\} = \{u_0, u_1\} \in \mathrm{H}^1_0(\Omega) \times L^2(\Omega), \\ u = 0 \quad \text{on} \quad \Gamma \times (0,T), \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlét boundary conditions. The nonlinearities $f(u)$ acts as a strong source, which is allowed to have, in some cases, a super-supercritical exponent. Under suitable restrictions on the parameters and with careful analysis involving the theory of monotone operators, we prove the existence and uniqueness of local solutions. We also provide two types of restrictions on either the power of the source or the initial energy that give global existence of solutions. Finally, we give decay rates for the energy of the system for suitable initial data, with the proof of the decay and decay rates the focus of the talk.

ACKNOWLEDGMENTS

I would first like to thank my advisor, Mohammad Rammaha, for dedicating immeasurable time and energy to my education and development as a mathematician and as a person. His guidance, advice and friendship have been invaluable and I have greatly enjoyed the time we have spent together. The Partial Differential Equations group at the University of Nebraska asked and answered countless questions of and for me. In particular, Daniel Toundykov provided considerable help and guidance and was instrumental in achieving some of the results of this dissertation. I would also like to thank my committee: Dr. Steve Cohn, Dr. Allan Peterson, Dr. David Pitts and Dr. Berthe Choueiry.

The faculty, staff and graduate students of the Mathematics Department at UNL were incredibly supportive of me during my schooling and made the last four years a wonderful time that I will always treasure. In particular, Michael Uhrig and Joe and Kayde Geisbauer became my family in Lincoln and helped me in more ways than I can ever list. I also thank the Mathematics Department at North Carolina State University for its support as I earned my masters degree and sparking my interest in differential equations.

Finally, I thank my loving family in Utah, New Zealand and California. They have always provided endless support and encouragement in all my endeavors–regardless of how nerdy they may be!

GRANT INFORMATION

National Science Foundation Grant # DMS-0838463

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Chapter 1

Introduction

This dissertation is concerned with the local and global well-posedness, as well as energy decay rates, of the following problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta_p u_t = f(u) \quad \text{in} \quad \Omega \times (0,T), \\ \{u(0), u_t(0)\} = \{u_0, u_1\} \in \mathrm{H}^1_0(\Omega) \times L^2(\Omega), \\ u = 0 \quad \text{on} \quad \Gamma \times (0,T), \end{cases} \tag{1.0.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set with sufficiently smooth boundary Γ. Here, the p-Laplacian is given by:

$$\begin{cases} -\Delta_p : W_0^{1,p}(\Omega) \longrightarrow W^{-1,p'}(\Omega), \\ \langle -\Delta_p v, \phi \rangle = \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \phi dx, \quad 2 \le p < \infty, \end{cases} \tag{1.0.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality paring between $\mathrm{W}^{-1,p'}(\Omega)$ and $\mathrm{W}_0^{1,p}(\Omega)$, $\frac{1}{p} + \frac{1}{p'} = 1$. In addition, we assume that $f \in C^1(\mathbb{R})$ enjoys a general polynomial growth at infinity, namely, $|f(u)| \le c|u|^r$ for all $|u| \ge 1$, where $1 \le r < 6$.

In order to simplify the exposition, we restrict our analysis to the physically more relevant case when $\Omega \subset \mathbb{R}^3$. Our results easily extend to bounded domains in $\mathbb{R}^n$ by accounting for the corresponding Sobolev imbeddings and accordingly adjusting the conditions imposed on the parameters.

In view of the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ (in 3D), the Nemytski operator $f(u)$ is locally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$ for the values $1 \leq r \leq 3$. Hence, we call the exponents of the source $1 \leq r < 3$ **sub-critical** and $r = 3$ is **critical**. The values $3 < r \leq 5$ are called **supercritical**, and in this case the operator $f(u)$ is **not** locally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$. However, for $3 < r \leq 5$, the potential energy $\int_\Omega F(u(t))dx$ induced by the source, where F is the primitive of f, is well defined in the finite energy space. The values $5 < r < 6$ are called **super-supercritical**. In this case, the potential energy may not be defined in the finite energy space and the problem itself is no longer within the framework of potential well theory (see [3, 20, 22, 28, 30]).

In recent years, wave equations under the influence of nonlinear damping and nonlinear sources have generated considerable interest. However, the majority of the work that has been done deals with sources that are at most critical, where standard fixed point theorems and Galerkin approximations can be employed [1, 2, 3, 4, 14, 23, 29, 31]. Indeed, few papers [8, 9, 10] have dealt with supercritical sources. The authors of [8, 9, 10] provided a comprehensive study for a semilinear wave equation under the influence of boundary/interior damping and nonlinear boundary/interior sources (where the interior damping term is of the order $|u_t|^{m-1}u_t$, $m \geq 1$). The main tool used in [8, 10] is the powerful theory of monotone operators [5, 27] in combination with important ideas from [12].

It is worth noting here that when the damping term $-\Delta_p u_t$ is absent, the source term of the form $|u|^{r-1}u$ should drive the solution of (1.0.1) to blow-up in finite time. In such a case, by appealing to a variety of methods (going back to the work of Glassey

[15], Levine [19], and others) one can show that *most* solutions to the problem blow up in finite time. In addition, if the source term $f(u)$ is removed from the equation, then damping terms of various forms should yield existence of global solutions (cf. [2, 4, 5, 16]). However, when both damping and source terms are present, then the analysis of their interaction and their influence on the global behavior of solutions becomes more difficult (see for instance [7, 14, 20, 23, 25] and the references therein). We finally note here that, when $p = 2$, the term $-\Delta u_t$ provides very strong dissipation. However, for $p > 2$, this effect is diminished by the fact that such a damping is quasilinear and is, in some sense, degenerate. The case when $-\Delta_p u_t$ is replaced by $-\Delta u_t$ (when p=2), has been studied by Webb [31], but only for the case of a good source $f(u)$ that is globally Lipschitz continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$. Here, the source in (1.0.1) is allowed to be **super-supercritical**. As an additional complication, the degenerate nature of the p-Laplacian as an elliptic operator is known to cause serious difficulties, as one can see from the work of DiBenedetto [13]. Nonetheless, the problem is still monotonic and is treatable with the theory of monotone operators, at least for the case when $f : H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is globally Lipschitz.

In our work we are also able to provide a decay rate for the energy of the system, provided that the initial data lies in the good part of the potential well (see Section 1.1.1). When $p = 2$, it is well-known that the energy decays exponentially (see Webb [31]), however the situation becomes much more difficult for $p > 2$. We follow the method presented in [21] and further refined in [3, 11] and compare the energy of the system to a suitable ordinary differential equation. Difficulties arise in constructing an observability-stability inequality due to the lack of a uniform bound for $\|\nabla u(t)\|_p$. As a result, our estimate for $\|\nabla u(t)\|_p$ is growing in time (see (5.1.23)). This leads to a non-autonomous ODE (in contrast to the time-independent ODEs of [3, 11, 21]) and careful analysis is required to show both decay to zero and a subsequent decay rate of the solution.

1.1 Preliminaries

We begin by introducing some basic notation that will be used in the subsequent discussion:

$$\|u\|_r = \|u\|_{L^r(\Omega)} \quad \text{and} \quad (u,v)_\Omega = (u,v)_{L^2(\Omega)}.$$

For duality pairing between $\mathrm{H}^{-1}(\Omega)$ and $\mathrm{H}_0^1(\Omega)$ and between $\mathrm{W}^{-1,p'}(\Omega)$ and $\mathrm{W}_0^{1,p}(\Omega)$, we shall use the simple notation $\langle\cdot,\cdot\rangle$. Also, due to Poincaré's inequality, the standard norms $\|u\|_{H_0^1(\Omega)}$ and $\|u\|_{W_0^{1,p}(\Omega)}$ are equivalent to the norms $\|\nabla u\|_2$ and $\|\nabla u\|_p$, respectvely. Hence, we put

$$\|u\|_{\mathrm{H}_0^1(\Omega)} = \|\nabla u\|_2, \quad \text{and} \quad \|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p .$$

Also, the following Sobolev imbeddings (in $\mathbb{R}^3$) will used frequently, and sometimes without mention:

$$\begin{cases} W_0^{1,p}(\Omega) \hookrightarrow L^{\frac{3p}{3-p}}(\Omega), & \text{for } 2 \le p < 3, \\ W_0^{1,3}(\Omega) \hookrightarrow L^s(\Omega), & \text{for } 1 \le s < \infty, \\ W_0^{1,p}(\Omega) \hookrightarrow C_B^0(\Omega), & \text{for } p > 3. \end{cases} \tag{1.1.1}$$

In addition, the following parameter q will be fixed throughout the disseration:

$$q = \begin{cases} \frac{3p}{4p-3}, \ \text{if } 2 \le p < 3, \\ 1+\delta, \ \text{if } p = 3, \\ 1, \ \text{if } p > 3, \end{cases} \tag{1.1.2}$$

where $\delta > 0$ can be taken arbitrary small. In view of the imbeddings in (1.1.1) we always have

$$W_0^{1,p}(\Omega) \hookrightarrow L^{q'}(\Omega), \quad 2 \leq p < \infty, \tag{1.1.3}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

1.1.1 Defining the Potential Well

A powerful tool in the study of the global existence of solutions to partial differential equations is potential well theory, first developed by Payne and Sattinger [22]. The energy of a PDE system is, in some sense, split into kinetic and potential energy. By examining the functional J, defined below in (1.1.4), along scalings of functions in $\mathrm{H}_0^1(\Omega)$ we discover that there is a valley or a "well" of height d created in the potential energy. Because this height d is strictly positive, we find that, for solutions with initial data in the "good part" of the well (see (1.1.21), the potential energy of the solution can never escape the well. In general, it is possible for the energy from the source term to cause the magnitude of the total energy to go to $-\infty$ in finite time (i.e., blow-up in finite time). However in the good part of the well it remains bounded by the quadratic potential energy, $\|\nabla u\|_2^2$, which is bounded in time. As a result, the total energy of the solution remains finite on any time interval $[0, T)$, providing the global existence of the solution.

We proceed by defining the functional $J : \mathrm{H}_0^1(\Omega) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{r+1}\|u\|_{r+1}^{r+1}. \tag{1.1.4}$$

Then, we write the total energy of the system as

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)) \tag{1.1.5}$$

and write the positive quadratic energy as

$$\mathcal{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2. \tag{1.1.6}$$

The Gâteaux[1] derivative, $J'(u,v)$, of J at u in the direction v is given by

$$\begin{aligned} J'(u,v) &:= \lim_{\epsilon\to 0}\frac{J(u+\epsilon v)-J(u)}{\epsilon} = \frac{d}{d\epsilon}J(u+\epsilon v)\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\left(\frac{1}{2}\|\nabla u+\epsilon\nabla v\|_2^2 - \frac{1}{r+1}\|u+\epsilon v\|_{r+1}^{r+1}\right)\Big|_{\epsilon=0} \\ &= \big(\nabla u,\nabla v\big) - \big(|u|^{r-1}u, v\big). \end{aligned} \tag{1.1.7}$$

Now, the critical points of J (the u for which $J'(u,v)=0$ for all smooth v of compact support) are the weak solutions of the elliptic problem

$$\begin{cases} -\Delta u = |u|^{r-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.8}$$

We define the Nehari Manifold,

$$\mathcal{N} := \{u \in \mathrm{H}_0^1(\Omega)\setminus\{0\} : \langle J'(u), u\rangle = 0\}, \tag{1.1.9}$$

[1]For the functional J defined in (1.1.4), the Gâteaux derivative (1.1.7) is linear in v and also bounded (hence continuous) whenever $u \in \mathrm{H}_0^1(\Omega)\cap L^{r+1}(\Omega)$. In this case, $J'(u) = \big(\nabla u, \nabla v\big) - \big(|u|^{r-1}u, v\big)$ is also the Frechét dervivative.

where $J'(u)$ is the Gâteaux derivative at u. Equivalently,

$$\mathcal{N} = \{u \in \mathrm{H}^1_0(\Omega) \setminus \{0\} : \|\nabla u\|_2^2 = \|u\|_{r+1}^{r+1}\}. \tag{1.1.10}$$

We define, as in the Mountain Pass Theorem,

$$d := \inf_{u \in \mathrm{H}^1_0(\Omega)\setminus\{0\}} \sup_{\lambda \geq 0} J(\lambda u). \tag{1.1.11}$$

We first show the following lemma.

Lemma 1.1.1. *For $1 < r \leq 5$ and J as defined above, we have that*

$$d = \inf_{u \in \mathcal{N}} J(u) \tag{1.1.12}$$

and $d > 0$.

Proof. The statement in (1.1.12) follows from a scaling argument. First, let $u \in \mathrm{H}^1_0(\Omega) \setminus \{0\}$ be fixed. Then, for any $\lambda \geq 0$, we have

$$\begin{aligned} J(\lambda u) &= \frac{1}{2}\|\lambda \nabla u\|_2^2 - \frac{1}{r+1}\|\lambda u\|_{r+1}^{r+1} \\ &= \frac{1}{2}\lambda^2\|\nabla u\|_2^2 - \frac{1}{r+1}\lambda^{r+1}\|u\|_{r+1}^{r+1}. \end{aligned} \tag{1.1.13}$$

Taking the derivative of the right hand side of (1.1.13) with respect to λ yields

$$\frac{d}{d\lambda}J(\lambda u) = \lambda\|\nabla u\|_2^2 - \lambda^r\|u\|_{r+1}^{r+1} = \lambda\left(\|\nabla u\|_2^2 - \lambda^{r-1}\|u\|_{r+1}^{r+1}\right) \tag{1.1.14}$$

and so the critical values for λ are $\lambda_0 = 0$ and $\lambda_1 = \lambda_1(u) = \left(\frac{\|\nabla u\|_2^2}{\|u\|_{r+1}^{r+1}}\right)^{\frac{1}{r-1}}$. Also,

$$\frac{d^2}{d\lambda^2}J(\lambda u) = \|\nabla u\|_2^2 - r\lambda^{r-1}\|u\|_{r+1}^{r+1},$$
$$\frac{d^2}{d\lambda^2}J(\lambda_0 u) \geq 0 \quad \text{and} \quad \frac{d^2}{d\lambda^2}J(\lambda_1 u) \leq 0. \tag{1.1.15}$$

Hence, $\sup_{\lambda \geq 0} J(\lambda u) = J(\lambda_1 u)$.

We now observe that $\lambda_1 u \in \mathcal{N}$ because

$$\begin{aligned}\|\nabla(\lambda_1 u)\|_2^2 &= \left(\frac{\|\nabla u\|_2^2}{\|u\|_{r+1}^{r+1}}\right)^{\frac{2}{r-1}} \|\nabla u\|_2^2 = \left(\frac{\|\nabla u\|_2}{\|u\|_{r+1}}\right)^{\frac{2(r+1)}{r-1}} \\ &= \left(\frac{\|\nabla u\|_2^2}{\|u\|_{r+1}^{r+1}}\right)^{\frac{r+1}{r-1}} \|u\|_{r+1}^{r+1} = \|\lambda_1 u\|_{r+1}^{r+1}.\end{aligned} \tag{1.1.16}$$

Also, for any $u \in \mathcal{N}$, note that $\lambda_1(u) = \left(\frac{\|\nabla u\|_2^2}{\|u\|_{r+1}^{r+1}}\right)^{\frac{1}{r-1}} = 1$ and so $\lambda_1(u)u = u$ for all $u \in \mathcal{N}$. Thus, we have

$$\inf_{u \in \mathrm{H}_0^1(\Omega)\setminus\{0\}} \sup_{\lambda \geq 0} J(\lambda u) = \inf_{u \in \mathrm{H}_0^1(\Omega)\setminus\{0\}} J(\lambda_1(u)u) = \inf_{u \in \mathcal{N}} J(u) \tag{1.1.17}$$

and (1.1.12) is shown.

We now show $d > 0$. From the Sobolev Imbedding (1.1.1), because $r + 1 \leq 6$, for all $u \in \mathcal{N}$ we have,

$$\|\nabla u\|_2^2 = \|u\|_{r+1}^{r+1} \leq C\|\nabla u\|_2^{r+1}, \tag{1.1.18}$$

which, in turn, implies that $\|\nabla u\|_2 \geq C^{\frac{-1}{r-1}}$. Thus, for all $u \in \mathcal{N}$,

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{r+1}\|u||_{r+1}^{r+1} = \frac{r-1}{2(r+1)}\|\nabla u\|_2^2 \geq \frac{r-1}{2(r+1)}C^{\frac{-2}{r-1}} \tag{1.1.19}$$

and thus

$$d = \inf_{u \in \mathcal{N}} J(u) \geq \frac{r-1}{2(r+1)} C^{\frac{-2}{r-1}} > 0, \tag{1.1.20}$$

proving Lemma 1.1.1. □

We now define the potential well $\mathcal{W}$:

$$\mathcal{W} := \{u \in \mathrm{H}_0^1(\Omega) : J(u) < d\}$$

and partition it into the two sets:

$$\begin{aligned} \mathcal{W}_1 &:= \{u \in \mathcal{W} : \|\nabla u\|_2^2 > \|u\|_{r+1}^{r+1}\} \cup \{0\} \\ \mathcal{W}_2 &:= \{u \in \mathcal{W} : \|\nabla u\|_2^2 < \|u\|_{r+1}^{r+1}\}. \end{aligned} \tag{1.1.21}$$

We will refer to $\mathcal{W}_1$ as the "good" part of the well and $\mathcal{W}_2$ as the "bad" part of the well.

1.1.2 A Specific Product Rule in Banach Space

Symbolically, deriving an energy identity for a partial differential equation is as simple as testing the equation against the solution u and integrating by parts. However, it is often the case that the solution does not have enough regularity to allow for this, or even for u to be a test function in the variational formulation of the problem, and we must use a different line of analysis. This is the case with our wave equation (1.0.1) and so we next give a specific case of a product rule for derivatives of functions in Banach Spaces that allows us to provide the energy identity (1.2.1). Following the presentation of [26], this product rule is shown in this subsection through two propositions, Proposition 1.1.2 and Proposition 1.1.3.

Let X be a Banach space. For any $y \in C_w([0,T];X)$[2] and any $h > 0$, define the symmetric difference quotient by:

$$D_h y(t) := \frac{1}{2h}(y_e(t+h) - y_e(t-h)), \tag{1.1.22}$$

where

$$y_e(t) = \begin{cases} y(0), & \text{for } t \leq 0, \\ y(t), & \text{for } 0 < t < T, \\ y(T), & \text{for } t \geq T. \end{cases} \tag{1.1.23}$$

With this notation, we have the following technical results.

Proposition 1.1.2. *Let H be a Hilbert space and X a Banach space with its dual X^* such that $X \subset H \subset X^*$ where the injections are continuous and each space is dense in the following one. Assume that $f, g \in C([0,T], X)$, $f' \in L^{\beta'}(0,T,X^*)$, $g' \in L^{\alpha'}(0,T,X^*)$ where $1 < \alpha', \beta' < \infty$. Then, $\psi(t) := \big(f(t), g(t)\big)$ coincides with an absolutely continuous function a.e. $[0,T]$ and*

$$\frac{d}{dt}\big(f(t), g(t)\big) = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle \qquad a.e.\ [0,T], \tag{1.1.24}$$

where $\big(\cdot,\cdot\big)$ is the inner product in H and $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X.

[2]The space of weakly continuous operators from $[0,T]$ into a Banach space X, $C_w([0,T];X)$, is the set of operators $y : [0,T] \to X$ such that the map $t \mapsto \langle z, y(t)\rangle$ is continuous on $[0,T]$ for every $z \in X'$.

Proof. Extend f and g as in (1.1.23). Then, f_e and g_e are bounded and uniformly continuous from $\mathbb{R}$ into X. Let $\phi(t) := \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle$ and $\psi(t) := \big(f(t), g(t)\big)$. It is well-known that

$$D_h f \to f' \in L^{\beta'}(0,T,X^*) \quad \text{and} \quad D_h g \to g' \in L^{\alpha'}(0,T,X^*). \tag{1.1.25}$$

Let us note here that

$$D_h\psi(t) = \big(D_h f(t), g_e(t+h)\big) + \big(f_e(t-h), D_h g(t)\big). \tag{1.1.26}$$

By elementary calculations, we see that

$$\begin{aligned} D_h\psi(t) - \phi(t) &= \langle D_h f(t) - f'(t), g_e(t+h)\rangle + \langle f'(t), g_e(t+h) - g(t)\rangle \\ &\quad + \langle f_e(t-h), D_h g(t) - g'(t)\rangle + \langle f_e(t-h) - f(t), g'(t)\rangle . \end{aligned} \tag{1.1.27}$$

Therefore,

$$\begin{aligned} \int_0^T |D_h\psi(t) - \phi(t)|dt \le &\int_0^T \|D_h f(t) - f'(t)\|_{X^*}\|g_e(t+h)\|_X dt \\ &+ \int_0^T \|f'(t)\|_{X^*}\|g_e(t+h) - g(t)\|_X dt \\ &+ \int_0^T \|f_e(t-h)\|_X\|D_h g(t) - g'(t)\|_{X^*} dt \\ &+ \int_0^T \|f_e(t-h) - f(t)\|_X\|g'(t)\|_{X^*} dt. \end{aligned} \tag{1.1.28}$$

By using Hölder's inequality with β' and $\beta = \frac{\beta'}{\beta'-1}$ on the first two terms and α' and

$\alpha = \frac{\alpha'}{\alpha'-1}$ on the last two terms of (1.1.28), we have

$$\begin{aligned}\int_0^T |D_h\psi(t) - \phi(t)|dt \leq\ & \|D_h f(t) - f'(t)\|_{L^{\beta'}(0,T,X^*)}\|g_e(t+h)\|_{L^\beta(0,T,X)} \\ & + \|f'(t)\|_{L^{\beta'}(0,T,X^*)}\|g_e(t+h) - g(t)\|_{L^\beta(0,T,X)} \\ & + \|f_e(t-h)\|_{L^\alpha(0,T,X)}\|D_h g(t) - g'(t)\|_{L^{\alpha'}(0,T,X^*)} \\ & + \|f_e(t-h) - f(t)\|_{L^\alpha(0,T,X)}\|g'(t)\|_{L^{\alpha'}(0,T,X^*)}. \end{aligned} \tag{1.1.29}$$

Since f_e and g_e are bounded from $\mathbb{R}$ into X, then $\|g_e(t+h)\|_{L^\beta(0,T,X)}$ and $\|f_e(t-h)\|_{L^\alpha(0,T,X)}$ are uniformly bounded for all $h > 0$. By (1.1.25) the first and third terms in (1.1.29) converge to 0, as $h \to 0$. Also, f_e and g_e are in $C(\mathbb{R}, X)$, so $\|f_e(t-h) - f(t)\|_X^\alpha$ and $\|g_e(t+h) - g(t)\|_X^\beta \to 0$, as $h \to 0$, for all $0 \leq t \leq T$. Given that f_e and g_e are bounded from $\mathbb{R}$ into X, then by the Dominated Convergence Theorem $\|f_e(t-h) - f(t)\|_{L^\alpha(0,T,X)}, \|g_e(t+h) - g(t)\|_{L^\beta(0,T,X)} \to 0$, as $h \to 0$. As $\|f'(t)\|_{L^{\beta'}(0,T,X^*)}$, $\|g'(t)\|_{L^{\alpha'}(0,T,X^*)} < \infty$, the second and fourth terms in (1.1.29) converge to 0 as $h \to 0$. Therefore,

$$\lim_{h\to 0} \int_0^T |D_h\psi(t) - \phi(t)| = 0. \tag{1.1.30}$$

Let us note that $\phi(t) \in L^1(0,T)$, because

$$\begin{aligned}\int_0^T |\phi(t)|dt &= \int_0^T |\langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle|dt \\ &\leq \int_0^T |\langle f'(t), g(t)\rangle|dt + \int_0^T |\langle f(t), g'(t)\rangle|dt \\ &\leq \|f'\|_{L^{\beta'}(0,T,X^*)}\|g\|_{L^\beta(0,T,X)} + \|g'\|_{L^{\alpha'}(0,T,X^*)}\|f\|_{L^\alpha(0,T,X)} \\ &< \infty. \end{aligned} \tag{1.1.31}$$

Combining, (1.1.30) and (1.1.31) we have that, $\frac{d}{dt}\psi \in L^1(0,T)$ and thus $\psi \in W^{1,1}(0,T,\mathbb{R})$. By a standard result, ψ is a.e. equal to an absolutely continuous function and $\frac{d}{dt}\psi(t) = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle$ a.e. $[0,T]$, which completes the proof. □

We now relax the conditions on f and g in the following proposition.

Proposition 1.1.3. *Let X, H and X^* be as in Proposition 1.1.2. Assume that $f \in L^\alpha(0,T,X) \cap L^2(0,T,H)$, $g \in L^\beta(0,T,X) \cap L^2(0,T,H)$, $f' \in L^{\beta'}(0,T,X^*)$, $g' \in L^{\alpha'}(0,T,X^*)$ where $1 < \alpha, \beta < \infty$, $\alpha' = \frac{\alpha}{\alpha-1}$ and $\beta' = \frac{\beta}{\beta-1}$. Then $\psi(t) := \big(f(t), g(t)\big)$ coincides with an absolutely continuous function a.e. $[0,T]$ and*

$$\frac{d}{dt}\big(f(t), g(t)\big) = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle \qquad a.e.\ [0,T]. \tag{1.1.32}$$

Proof. Extend f and g to be zero outside of $[0,T]$ and regularize the extensions by mollifying with the sequence of functions $\{\sigma_j\} \in \mathcal{D}(\mathbb{R})$ such that $\sigma_j \geq 0$, $\int_{\mathbb{R}} \sigma_j dt = 1$, $\sigma_j(t) = \sigma_j(-t)$ and supp $\sigma_j \subset (-\frac{1}{j}, \frac{1}{j})$. This gives the sequences of C^∞-functions, $\{f_j\}$ and $\{g_j\}$ defined by

$$f_j(\tau) = \int_{\mathbb{R}} \sigma_j(\tau - s) f(s) ds \quad \text{and} \quad g_j(\tau) = \int_{\mathbb{R}} \sigma_j(\tau - s) g(s) ds. \tag{1.1.33}$$

We denote $f_j|_{[0,T]}$ by f_j and $g_j|_{[0,T]}$ by g_j. It is well-known that

$$\begin{cases} f_j \to f \text{ in } L^2(0,T,H), \quad g_j \to g \text{ in } L^2(0,T,H), \\ f_j \to f \text{ in } L^\alpha(0,T,X), \quad f_j' \to f' \text{ in } L^{\beta'}(0,T,X^*), \\ g_j \to g \text{ in } L^\beta(0,T,X), \quad g_j' \to g' \text{ in } L^{\alpha'}(0,T,X^*). \end{cases} \tag{1.1.34}$$

We now show the following limits as $j \to \infty$:

$$\begin{cases} \big(f_j, g_j\big) \longrightarrow \big(f, g\big) & \text{in } L^1(0,T), \\ \big\langle f_j', g_j\big\rangle \longrightarrow \langle f', g\rangle & \text{in } L^1(0,T), \\ \big\langle f_j, g_j'\big\rangle \longrightarrow \langle f, g'\rangle & \text{in } L^1(0,T). \end{cases} \tag{1.1.35}$$

For the first convergence in (1.1.35) we have,

$$\begin{aligned} &\int_0^T \big|\big(f_j(t), g_j(t)\big) - \big(f(t), g(t)\big)\big|\, dt \\ &\quad \le \int_0^T |\big(f_j(t), g_j(t) - g(t)\big)| dt + \int_0^T |\big(f_j(t) - f(t), g(t)\big)| dt \\ &\quad \le \int_0^T \|f_j(t)\|_H \|g_j(t) - g(t)\|_H dt + \int_0^T \|f_j(t) - f(t)\|_H \|g(t)\|_H dt \\ &\quad \le \|f_j\|_{L^2(0,T,H)} \|g_j - g\|_{L^2(0,T,H)} + \|g\|_{L^2(0,T,H)} \|f_j - f\|_{L^2(0,T,H)} \\ &\quad \longrightarrow 0, \ \text{as } j \longrightarrow \infty, \end{aligned} \tag{1.1.36}$$

where we have used the Cauchy-Schwarz inequality and convergences in (1.1.34). As for the second convergence in (1.1.35), we have

$$\begin{aligned} &\int_0^T |\big\langle f_j'(t), g_j(t)\big\rangle - \langle f'(t), g(t)\rangle|dt \\ &\quad \le \int_0^T \|f_j'(t)\|_{X^*} \|g_j(t) - g(t)\|_X dt + \int_0^T \|f_j'(t) - f'(t)\|_{X^*} \|g(t)\|_X dt \\ &\quad \le \|f_j'\|_{L^{\beta'}(0,T,X^*)} \|g_j - g\|_{L^\beta(0,T,X)} + \|f_j' - f'\|_{L^{\beta'}(0,T,X^*)} \|g\|_{L^\beta(0,T,X)} \\ &\quad \longrightarrow 0, \ \text{as } j \longrightarrow \infty. \end{aligned} \tag{1.1.37}$$

A similar argument gives the third convergence in (1.1.35).

At this point, we note that $\big(f_j, g_j\big)$, $\big\langle f_j', g_j\big\rangle$ and $\big\langle f_j, g_j'\big\rangle$ are all elements of $L^1(0,T)$. Hence they define regular distributions in $\mathcal{D}'(0,T)$. Thus, the convergences in (1.1.35) hold in $\mathcal{D}'(0,T)$ as well. In addition, we have

$$\frac{d}{dt}\big(f_j(t), g_j(t)\big) \longrightarrow \frac{d}{dt}\big(f(t), g(t)\big) \quad \text{in } \mathcal{D}'(0,T), \text{ as } j \longrightarrow \infty. \tag{1.1.38}$$

To see this fact, let $\phi \in \mathcal{D}(0,T)$. Recall that since $\big(f_j, g_j\big), \big(f, g\big) \in L^1(0,T)$, they define regular distributions. So, it follows from (1.1.35) that,

$$\begin{aligned}\left\langle \frac{d}{dt}\big(f_j(t), g_j(t)\big), \phi \right\rangle &= -\int_0^T \big(f_j(t), g_j(t)\big)\phi'(t)dt \\ &\longrightarrow -\int_0^T \big(f(t), g(t)\big)\phi'(t)dt \\ &= \left\langle \frac{d}{dt}\big(f(t), g(t)\big), \phi \right\rangle,\end{aligned} \tag{1.1.39}$$

and (1.1.38) is shown. Now, because f_j, g_j, f_j' and g_j' satisfy Proposition 1.1.2, then for each $j \in \mathbb{N}$, we have

$$\frac{d}{dt}\big(f_j(t), g_j(t)\big) = \big\langle f_j'(t), g_j(t)\big\rangle + \big\langle f_j(t), g_j'(t)\big\rangle \qquad \text{a.e. } [0,T]. \tag{1.1.40}$$

By letting $j \longrightarrow \infty$ in (1.1.40) and using (1.1.39), (1.1.35), we have

$$\frac{d}{dt}\big(f(t), g(t)\big) = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle \quad \text{in } \mathcal{D}'(0,T). \tag{1.1.41}$$

Because the right-hand side of (1.1.41) is in $L^1(0,T)$, it follows that $\big(f(t), g(t)\big)$ coincides with an absolutely continuous function on $[0,T]$. Moreover,

$$\big(f(t), g(t)\big) - \big(f(0), g(0)\big) = \int_0^t \Big(\langle f'(s), g(s)\rangle + \langle f(s), g'(s)\rangle \Big) ds, \tag{1.1.42}$$

for all $t \in [0,T]$. $\square$

1.1.3 Assumptions and Definition of Weak Solution

Throughout the dissertation, we assume the validity of the following assumption.

Assumption 1.1.1. We assume that $2 \leq p < \infty$ and $f \in C^1(\mathbb{R})$ with the following growth conditions for $|u| \geq 1$:

- $|f(u)| \leq c_0|u|^r$, $|f'(u)| \leq c_1|u|^{r-1}$, $1 \leq r < 6$, for some positive constants $c_0,\ c_1$.
- In addition, for the values $3 < r < 6$, we further require $f \in C^2(\mathbb{R})$ with the growth condition $|f''(u)| \leq c_2|u|^{r-2}$, for $|u| \geq 1$.
- Throughout, the exponent of the source satisfies:

$$\begin{cases} 1 \leq r < 8 - \frac{6}{p}, & \text{if } 2 \leq p < 3, \\ 1 \leq r < 6, & \text{if } p \geq 3. \end{cases} \tag{1.1.43}$$

- $u_0 \in \mathrm{H}_0^1(\Omega), \quad u_1 \in L^2(\Omega)$.

At times we will also employ the following assumption.

Assumption 1.1.2. Assume that $E(0) < d$, where d is the depth of the potential well (see (1.1.11)), and that $u_0 \in \mathcal{W}_1$.

We now give a precise definition of weak solutions of (1.0.1).

Definition 1.1.4. A function u is said to be a weak solution of (1.0.1) on $[0,T]$ if $u \in C([0,T],\mathrm{H}_0^1(\Omega))$, $u_t \in C([0,T],L^2(\Omega)) \cap L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$, $(u(0),u_t(0)) = (u_0,u_1)$ and, for all $0 \leq t \leq T$, u satisfies

$$\int_0^t \int_\Omega (-u_t\phi_t + \nabla u \cdot \nabla\phi)dxds + \int_\Omega u_t\phi\big|_{s=0}^{s=t}dx + \int_0^t \int_\Omega |\nabla u_t|^{p-2}\nabla u_t \cdot \nabla\phi dxds$$
$$= \int_0^t \int_\Omega f(u)\phi dxds, \tag{1.1.44}$$

for all test functions $\phi \in H^1(0,T;L^2(\Omega)) \cap L^p(0,T;\mathrm{W}_0^{1,p}(\Omega))$.

1.2 Main Results

Our first result establishes the existence and uniqueness of a weak solution of (1.0.1).

Theorem 1.2.1. ***(Local Solutions)*** *Under the validity of Assumption 1.1.1, problem (1.0.1) has a local weak solution u defined on $[0,T]$ (in the sense of Definition 1.1.4) for some $T>0$. Moreover, if we further assume that $u_0 \in L^{\frac{3}{2}(r-1)}(\Omega)$, whenever $r>5$, then the said weak solution is unique. In addition, u satisfies the following energy identity:*

$$E(t) + \int_0^t \|\nabla u_t(s)\|_p^p \, ds = E(0), \tag{1.2.1}$$

for all $t \in [0,T]$, where $E(t)$ denotes the total energy of the system

$$E(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \int_\Omega F(u(t))dx, \tag{1.2.2}$$

and $F(u) = \int_0^u f(s)ds$.

Remark 1.2.1. Clearly, the energy identity (1.2.1) shows dissipation of energy in the system.

Our next result shows that the weak solution furnished by Theorem 1.2.1 is a global solution provided the exponent of damping is more dominant than the exponent of the source. More precisely, we have the following theorem.

Theorem 1.2.2. ***(Global Solutions)*** *In addition to Assumption 1.1.1, assume that $r \le p-1$ and $u_0 \in L^{r+1}(\Omega)$, if $r>5$. Then, the said weak solution u in Theorem 1.2.1 is a global solution and T can be taken arbitrarily large.*

Remark 1.2.2. Notice that for $r \leq 5$ the condition that $u_0 \in L^{r+1}(\Omega)$ in Theorem 1.2.2 is not an additional restriction, due to the imbedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$.

Corollary 1.2.3. ***(Continuous Dependence on Initial Data)*** *Under the validity of Assumption 1.1.1, then the weak solution of (1.0.1) depends continuously on the initial data. More precisely, let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, if $1 \leq r \leq 5$, or $(u_0, u_1) \in H_0^1(\Omega) \cap L^k(\Omega) \times L^2(\Omega)$, if $r > 5$, where $k = \frac{3}{2}(r-1)$. Further, let $\{(u_0^n, u_1^n)\}$ be a sequence of initial data such that, as $n \to \infty$,*

$$\begin{cases} (u_0^n, u_1^n) \longrightarrow (u_0, u_1) \ \textit{in} \ H_0^1(\Omega) \times L^2(\Omega), \ \textit{if} \ 1 \leq r \leq 5, \\ \\ (u_0^n, u_1^n) \longrightarrow (u_0, u_1) \ \textit{in} \ H_0^1(\Omega) \cap L^k(\Omega) \times L^2(\Omega), \ \textit{if} \ r > 5. \end{cases} \tag{1.2.3}$$

Then, the corresponding solutions u^n and u of (1.0.1) satisfy:

$$(u^n, u_t^n) \longrightarrow (u, u_t) \ \textit{in} \ C([0,T], H_0^1(\Omega) \times L^2(\Omega)), \ \textit{as} \ n \to \infty. \tag{1.2.4}$$

Let us note here that the allowable range of the parameters is as shown in Figure 1.1 below.

We next provide different criteria for the global existence of solutions via potential well theory. Here, we may allow for a much higher exponent on the source $(1 < r \leq 5)$ by restricting the size of the initial data. Note that, in contrast to Theorem 1.2.2, the allowable values of r do not depend on the value of p.

Theorem 1.2.4. ***(Global Solutions in the Potential Well)*** *In addition to Assumption 1.1.1, assume that $u_0 \in \mathcal{W}_1$, $E(0) < d$, $1 < r \leq 5$ and $f(u) = |u|^{r-1}u$. Then, the unique weak solution u provided by Theorem 1.2.1 can be extended to $[0, \infty)$.*

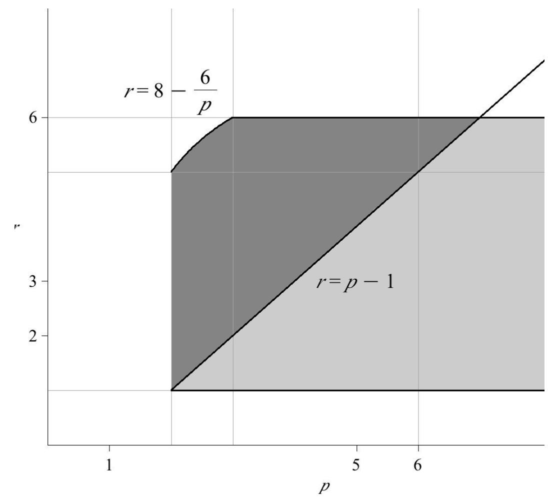

Figure 1.1: Local solutions in the grey and black regions. Global solutions in the grey region.

Furthermore, for all $t \in [0, T)$, $u(t)$ *remains in* $\mathcal{W}_1$,

$$\mathscr{E}(t) \leq d\left(\frac{r+1}{r-1}\right) \tag{1.2.5}$$

and

$$E_1(t) \leq d\left(\frac{r+3}{r-1}\right). \tag{1.2.6}$$

Remark 1.2.3. The source $f(u)$ is chosen as $|u|^{r-1}u$ in Theorem 1.2.4 to clarify the exposition, however the result can be shown for a more general source $f(u)$ satisfying the assumptions.

A final theorem provides a decay rate for the global solution provided by Theorem 1.2.4.

Theorem 1.2.5. ***(Decay of Global Solutions in the Potential Well)*** *Let (1.0.1) satisfy the hypothesis of Theorem 1.2.4,* $u_0 \in W_0^{1,p}(\Omega)$ *and let* u *be the unique* *global solution with total energy $E(t)$ as in (1.2.2). Then,*

$$0 \leq E(t) \leq \frac{C_1}{\left(\ln(C_2 t)\right)^{p-1}}, \qquad t \geq T_0, \tag{1.2.7}$$

where T_0 is chosen in the proof.

Chapter 2

Existence of Local Solutions

2.1 Local Solutions

This chapter is devoted to the proof of the local existence statement in Theorem 1.2.1, which will be carried out in five subsections. At this point, some comments are in order regarding the local solvability of (1.0.1). It is important to point out that nonlinear semi-groups and Kato's Theorem [6, 27] can only accommodate a globally Lipschitz perturbation of a monotone problem. Thus, moving from globally Lipschitz sources to the full generality of super-supercritical sources requires a great effort. Our strategy in handling this major problem is summarized as follows:

Step 1: For a globally Lipschitz source from $\mathrm{H}^1_0(\Omega)$ into $L^2(\Omega)$, construct global solutions using nonlinear semigroup theory (Lemma 2.1.1).

Step 2: Extend the existence result in Step 1 to obtain local existence in the case of sources that are locally Lipschitz from $\mathrm{H}^1_0(\Omega)$ into $L^2(\Omega)$. In the process, derive a priori bounds that do not depend on the locally Lipschitz constant of the source as a mapping from $\mathrm{H}_0^1(\Omega)$ into $L^2(\Omega)$, but rather on the locally Lipschitz constant of the source as a mapping from $\mathrm{H}^{1-\epsilon}(\Omega)$ into $L^q(\Omega)$, where q is given in (1.1.2). In particular, show that the local existence time, T, is independent of the properties required in Step 1 (Lemma 2.1.2).

Step 3: Construct approximations of the original source that obey the requirements in Step 2. This step is accomplished by using a certain truncation of the source, which was employed in [24]. Finally, pass to the limit on the weak variational form to obtain a local weak solution. Handling the corresponding approximations of the damping and sources is a major technical step due to the lack of compactness. Here, special estimates involving the so-called dissipativity kernels, which were introduced in [12], play a critical role. Another important ingredient in this process is the availability of an energy identity. This strategy allows us to pass to the limit without the use of compactness (Section 2.1.3-Section 2.1.5).

We proceed by beginning with some notation. Throughout the dissertation, we let

$$H := \mathrm{H}_0^1(\Omega) \times L^2(\Omega)$$

with the usual inner product, i.e., if $X_1 = (y_1, z_1)$, $X_2 = (y_2, z_2) \in H$, then

$$(X_1, X_2)_H = (\nabla y_1, \nabla y_2)_\Omega + (z_1, z_2)_\Omega.$$

We also define the nonlinear operator $\mathscr{A}$ by:

$$\mathscr{A}\begin{pmatrix} y \\ z \end{pmatrix}^{tr} = \begin{pmatrix} -z \\ -\Delta y - \Delta_p z - f(y) \end{pmatrix}^{tr}$$

with $\mathcal{D}(\mathscr{A}) = \{(y, z) \in \mathrm{H}_0^1(\Omega) \times \mathrm{W}_0^{1,p}(\Omega) : -\Delta y - \Delta_p z - f(y) \in L^2(\Omega)\}$. It is clear that $\mathscr{A}:\mathcal{D}(\mathscr{A})\subseteq H\to H$.

Our first goal is to show that if $(u,v)\in W^{1,\infty}(0,T,H)$ is a solution to the Cauchy problem

$$\begin{cases} \frac{d}{dt}\binom{u}{v}^{tr}+\mathscr{A}\binom{u}{v}^{tr}=0, \quad \text{a.e. } t\in[0,T], \\ (u,v)(0)=(u_0,v_0)\in\mathcal{D}(\mathscr{A}), \end{cases} \tag{2.1.1}$$

then u is a weak solution to (1.0.1) on $[0,T]$ in the case when $(u_0,v_0)\in\mathcal{D}(\mathscr{A})$.

2.1.1 Globally Lipschitz Sources

Our first step toward proving Theorem 1.2.1 is to prove Lemma 2.1.1 below, which deals with the case of a globally Lipschitz source.

Lemma 2.1.1. *Suppose that $f:H_0^1(\Omega)\to L^2(\Omega)$ is globally Lipschitz continuous and $(u_0,v_0)\in\mathcal{D}(\mathscr{A})$. Then Problem (2.1.1) has a unique global solution $(u,v)\in W^{1,\infty}(0,T,H)$, $(u(t),v(t))\in\mathcal{D}(\mathscr{A})$ a.e. $[0,T]$, where $T>0$ is arbitrary.*

Remark 2.1.1. Indeed, Lemma 2.1.1 asserts that Problem (1.0.1) has a unique global solution provided $f:\mathrm{H}_0^1(\Omega)\to L^2(\Omega)$ is globally Lipschitz continuous and $(u_0,v_0)\in\mathcal{D}(\mathscr{A})$. More precisely, Lemma 2.1.1 provides the existence of a unique function u such that, $u\in C^0([0,T],\mathrm{H}_0^1(\Omega))$, $u_t\in C^0([0,T],L^2(\Omega))$, $u_{tt}\in L^\infty(0,T,L^2(\Omega))$, $v(t)=u_t(t)\in \mathrm{W}_0^{1,p}(\Omega)$, a.e. $t\in[0,T]$, and u satisfies:

$$\begin{cases} u_{tt}=\Delta u+\Delta_p u_t+f(u)\in L^2(\Omega), \text{ a.e. } [0,T], \\ (u(0),v(0))=(u_0,v_0)\in\mathcal{D}(\mathscr{A}), \end{cases} \tag{2.1.2}$$

where $T>0$ can be taken arbitrarily large.

Proof. The conclusions of Lemma 2.1.1 follow from Kato's Theorem (e.g., [27]). Thus, it suffices to show that the operator $\mathscr{A} + \omega I$ is m-accretive for some $\omega > 0$.

Step 1: $\mathscr{A} + \omega I$ is accretive for some $\omega > 0$. By assumption, there exists a constant $L_f > 0$ such that $\|f(y_1) - f(y_2)\|_2 \le L_f \|\nabla(y_1 - y_2)\|_2$, for all $y_1, y_2 \in \mathrm{H}_0^1(\Omega)$. Let $X_1, X_2 \in \mathcal{D}(\mathscr{A})$ with $X_i = (y_i, z_i)$, $i = 1, 2$. We aim to show that $\big((\mathscr{A} + \omega I)X_1 - (\mathscr{A} + \omega I)X_2, X_1 - X_2\big)_H \ge 0$ for some $\omega > 0$. By straightforward calculations, we obtain

$$\begin{aligned}
&\big((\mathscr{A} + \omega I)X_1 - (\mathscr{A} + \omega I)X_2, X_1 - X_2\big)_H \\
&\qquad = \big(\mathscr{A}(X_1) - A(X_2), X_1 - X_2\big)_H + \omega\|X_1 - X_2\|_H^2 \\
&\qquad = \big(-z_1 + z_2, y_1 - y_2\big)_{\mathrm{H}_0^1(\Omega)} - \big(f(y_1) - f(y_2), z_1 - z_2\big)_\Omega \\
&\qquad \big(-\Delta y_1 - \Delta_p z_1 + \Delta y_2 + \Delta_p z_2, z_1 - z_2\big)_\Omega + \omega\|X_1 - X_2\|_H^2. \qquad (2.1.3)
\end{aligned}$$

Given that f is globally Lipschitz, the second term in (2.1.3) is estimated as follows:

$$\begin{aligned}
\big(f(y_1) - f(y_2), z_1 - z_2\big)_\Omega &\le \|f(y_1) - f(y_2)\|_2 \|z_1 - z_2\|_2 \\
&\le L_f \|\nabla(y_1 - y_2)\|_2 \|z_1 - z_2\|_2. \qquad (2.1.4)
\end{aligned}$$

By noting that $\Delta y_i + \Delta_p z_i \in L^2(\Omega)$ for $i = 1, 2$, as implied by the definition of

$\mathcal{D}(\mathscr{A})$, the third term in (2.1.3) becomes

$$\begin{aligned}
&\big(- \Delta y_1 - \Delta_p z_1 + \Delta y_2 + \Delta_p z_2, z_1 - z_2 \big)_\Omega \\
&\qquad = \langle -\Delta(y_1 - y_2), z_1 - z_2 \rangle + \langle -(\Delta_p z_1 - \Delta_p z_2), z_1 - z_2 \rangle \\
&\qquad = \big(\nabla(y_1 - y_2), \nabla(z_1 - z_2) \big)_\Omega \\
&\qquad + \int_\Omega (|\nabla z_1|^p + |\nabla z_2|^p - |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla z_2 - |\nabla z_2|^{p-2} \nabla z_2 \cdot \nabla z_1) dx. \qquad (2.1.5)
\end{aligned}$$

Recalling that $z_1, z_2 \in \mathrm{W}_0^{1,p}(\Omega)$, it follows from Hölder's inequality that

$$\begin{aligned}
&\int_\Omega (|\nabla z_1|^p + |\nabla z_2|^p - |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla z_2 - |\nabla z_2|^{p-2} \nabla z_2 \cdot \nabla z_1) dx \\
&\qquad \geq \|\nabla z_1\|_p^p + \|\nabla z_2\|_p^p - \|\nabla z_1\|_p^{p-1} \|\nabla z_2\|_p - \|\nabla z_2\|_p^{p-1} \|\nabla z_1\|_p \\
&\qquad = (\|\nabla z_1\|_p^{p-1} - \|\nabla z_2\|_p^{p-1})(\|\nabla z_1\|_p - \|\nabla z_2\|_p) \geq 0. \qquad (2.1.6)
\end{aligned}$$

Therefore, from (2.1.3)-(2.1.4) and Young's inequality we have

$$\begin{aligned}
&\big((\mathscr{A} + \omega I) X_1 - (\mathscr{A} + \omega I) X_2, X_1 - X_2 \big)_H \\
&\qquad \geq -\big(\nabla(z_1 - z_2), \nabla(y_1 - y_2) \big)_\Omega + \big(\nabla(z_1 - z_2), \nabla(y_1 - y_2) \big)_\Omega \\
&\qquad - L_f \|\nabla(y_1 - y_2)\|_2 \|z_1 - z_2\|_2 + \omega \|X_1 - X_2\|_H^2 \\
&\qquad = (\omega - \frac{L_f}{2}) \Big[\|\nabla(y_1 - y_2)\|_2^2 + \|z_1 - z_2\|_{L^2(\Omega)}^2 \Big] \geq 0, \qquad (2.1.7)
\end{aligned}$$

whenever $\omega \geq \frac{L_f}{2}$. Thus, for such ω, $\mathscr{A} + \omega I$ is accretive.

Step 2: $\mathscr{A} + \omega I$ is m-accretive. It suffices to show that $\mathcal{R}(\mathscr{A} + \omega I + \lambda I) = \mathrm{H}_0^1(\Omega) \times L^2(\Omega)$ for some $\lambda > 0$. Rename $\omega + \lambda$ as λ and let $(a, b) \in \mathrm{H}_0^1(\Omega) \times L^2(\Omega)$ be given. We now find $(y,z) \in \mathcal{D}(\mathscr{A})$ such that

$$(\mathscr{A} + \lambda I)\begin{pmatrix} y \\ z \end{pmatrix}^{tr} = \begin{pmatrix} a \\ b \end{pmatrix}^{tr},$$

or

$$\begin{pmatrix} -z + \lambda y \\ -\Delta y - \Delta_p z - f(y) + \lambda z \end{pmatrix}^{tr} = \begin{pmatrix} a \\ b \end{pmatrix}^{tr}. \tag{2.1.8}$$

Then, it must be that $y = \frac{a+z}{\lambda}$ and

$$-\Delta(\frac{a+z}{\lambda}) - \Delta_p z - f(\frac{a+z}{\lambda}) + \lambda z = b,$$

or

$$-\frac{1}{\lambda}\Delta z - \Delta_p z - f(\frac{a+z}{\lambda}) + \lambda z = b + \frac{1}{\lambda}\Delta a =: \tilde{b}. \tag{2.1.9}$$

Let us note that because $a \in \mathrm{H}_0^1(\Omega)$ and $b \in L^2(\Omega)$, then $\tilde{b} := b + \frac{1}{\lambda}\Delta a \in H^{-1}(\Omega) \subset W_0^{-1,p'}(\Omega)$. Therefore, if we define an operator $T : \mathrm{W}_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ by

$$T(z) := -\frac{1}{\lambda}\Delta z - \Delta_p z - f(\frac{a+z}{\lambda}) + \lambda z, \tag{2.1.10}$$

then (2.1.9) holds for some $z \in \mathrm{W}_0^{1,p}(\Omega)$ if we can show that T is surjective. In order to show that $T : \mathrm{W}_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$ is surjective, it suffices to show that T is maximal monotone and coercive.

Show T is monotone for some $\lambda > 0$: Let $z_1, z_2 \in \mathrm{W}_0^{1,p}(\Omega)$. Then, straightforward computation shows that

$$
\begin{aligned}
&\langle T(z_1) - T(z_2), z_1 - z_2 \rangle \\
&\quad = \lambda \|z_1 - z_2\|_2^2 - \int_\Omega (f(\frac{a+z_1}{\lambda}) - f(\frac{a+z_2}{\lambda}))(z_1 - z_2)dx \\
&\quad + \frac{1}{\lambda}\|\nabla(z_1 - z_2)\|_2^2 - \langle \Delta_p z_1 - \Delta_p z_2, z_1 - z_2 \rangle . \qquad (2.1.11)
\end{aligned}
$$

Employing the fact that f is globally Lipschitz from $\mathrm{H}_0^1(\Omega)$ into $L^2(\Omega)$ and $\frac{a+z_i}{\lambda} \in \mathrm{H}_0^1(\Omega)$, $i = 1, 2$, we have

$$
\begin{aligned}
\langle T(z_1) - T(z_2), z_1 - z_2 \rangle &\geq \lambda \|z_1 - z_2\|_2^2 - \frac{L_f}{\lambda}\|\nabla(z_1 - z_2)\|_2 \|z_1 - z_2\|_2 \\
&\quad + \frac{1}{\lambda}\|\nabla(z_1 - z_2)\|_2^2 - \langle \Delta_p z_1 - \Delta_p z_2, z_1 - z_2 \rangle . \qquad (2.1.12)
\end{aligned}
$$

By the same calculation as in (2.1.5)-(2.1.6), the last term in (2.1.12) is nonnegative. Using Young's inequality, we find

$$
\langle T(z_1) - T(z_2), z_1 - z_2 \rangle \geq (\lambda - \frac{L_f^2}{2\lambda})\|z_1 - z_2\|_2^2 + \frac{1}{2\lambda}\|\nabla(z_1 - z_2)\|_2^2 \geq 0,
$$

provided $\lambda \geq \frac{L_f}{\sqrt{2}}$. Thus, T is monotone for such values of λ.

In order to show the maximality of T, it suffices to show that T is hemicontinuous.

Show T is hemicontinuous: We need to prove that $\text{w-}\lim_{\mu \to 0} T(z_1 + \mu z_2) = T(z_1)$ for every $z_1, z_2 \in \mathrm{W}_0^{1,p}(\Omega)$. To see this, let $\xi \in \mathrm{W}_0^{1,p}(\Omega)$, then

$$
\begin{aligned}
\langle T(z_1 + \mu z_2), \xi \rangle &= -\frac{1}{\lambda} \langle \Delta(z_1 + \mu z_2), \xi \rangle - \langle \Delta_p(z_1 + \mu z_2), \xi \rangle \\
&\quad - \left\langle f(\frac{a + z_1 + \mu z_2}{\lambda}), \xi \right\rangle + \lambda \langle z_1 + \mu z_2, \xi \rangle . \qquad (2.1.13)
\end{aligned}
$$

Since $z_2, \Delta z_2 \in \mathrm{W}^{-1,p'}(\Omega)$, then for the first and fourth terms in (2.1.13) we easily have

$$-\frac{1}{\lambda}\left\langle \Delta(z_1+\mu z_2),\xi\right\rangle = -\frac{1}{\lambda}(\left\langle \Delta z_1,\xi\right\rangle + \mu\left\langle \Delta z_2,\xi\right\rangle) \longrightarrow -\frac{1}{\lambda}\left\langle \Delta z_1,\xi\right\rangle \tag{2.1.14}$$

and

$$\lambda\left\langle z_1+\mu z_2,\xi\right\rangle \longrightarrow \lambda\left\langle z_1,\xi\right\rangle, \text{ as } \mu\to 0. \tag{2.1.15}$$

For the second term in (2.1.13) we utilize the Dominated Convergence Theorem. We first note that,

$$-\left\langle \Delta_p(z_1+\mu z_2),\xi\right\rangle = \int_\Omega |\nabla(z_1+\mu z_2)|^{p-2}\nabla(z_1+\mu z_2)\cdot\nabla\xi dx.$$

Clearly,

$$\lim_{\mu\to 0}|\nabla(z_1+\mu z_2)|^{p-2}\nabla(z_1+\mu z_2)\cdot\nabla\xi = |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\xi \quad \text{a.e. } \Omega, \tag{2.1.16}$$

and for $|\mu|<1$, we have

$$\left||\nabla(z_1+\mu z_2)|^{p-2}\nabla(z_1+\mu z_2)\cdot\nabla\xi\right| \le 2^{p-2}\left(|\nabla z_1|^{p-1}+|\nabla z_2|^{p-1}\right)|\nabla\xi|. \tag{2.1.17}$$

In addition, $|\nabla z_i|^{p-1}|\nabla\xi| \in L^1(\Omega)$, $i=1,2$, because

$$\int_\Omega |\nabla z_i|^{p-1}|\nabla\xi|dx \le \|\nabla z_i\|_p^{p-1}\left\|\nabla\xi\right\|_p < \infty.$$

Therefore, by the Dominated Convergence Theorem, we have

$$
\begin{aligned}
-\lim_{\mu\to 0}\langle \Delta_p(z_1+\mu z_2),\xi\rangle &= \lim_{\mu\to 0}\int_\Omega |\nabla(z_1+\mu z_2)|^{p-2}\nabla(z_1+\mu z_2)\cdot\nabla\xi dx \\
&= \int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\xi dx = -\langle \Delta_p z_1,\xi\rangle . \qquad (2.1.18)
\end{aligned}
$$

Finally, for the third term in (2.1.13) we shall show that

$$
\lim_{\mu\to 0}\left\langle f(\frac{a+z_1+\mu z_2}{\lambda}),\xi\right\rangle = \left\langle f(\frac{a+z_1}{\lambda}),\xi\right\rangle . \qquad (2.1.19)
$$

To see this, we note that

$$
\begin{aligned}
&\left|\left\langle f(\frac{a+z_1+\mu z_2}{\lambda}) - f(\frac{a+z_1}{\lambda}),\xi\right\rangle\right| \\
&\quad\le \left\| f(\frac{a+z_1+\mu z_2}{\lambda}) - f(\frac{a+z_1}{\lambda})\right\|_2 \|\xi\|_2 \\
&\quad\le |\mu|\frac{L_f}{\lambda}\|\nabla z_2\|_2\|\xi\|_2 \longrightarrow 0, \quad \text{as } \mu\to 0. \qquad (2.1.20)
\end{aligned}
$$

Hence, (2.1.19) follows. Combining (2.1.14)-(2.1.15) and (2.1.18)-(2.1.19), we have

$$
\begin{aligned}
\lim_{\mu\to 0}\langle T(z_1+\mu z_2),\xi\rangle &= -\frac{1}{\lambda}\langle \Delta z_1,\xi\rangle - \langle \Delta_p z_1,\xi\rangle - \left\langle f(\frac{a+z_1}{\lambda}),\xi\right\rangle \\
&\quad + \lambda\langle z_1,\xi\rangle = \langle T(z_1),\xi\rangle , \qquad (2.1.21)
\end{aligned}
$$

and thus, T is hemicontinuous. Because T is monotone and hemicontinuous, then by Theorem 1.3 in [5], we conclude that T is maximal monotone.

Show T is coercive for some $\lambda > 0$: We now need to show that

$$
\frac{1}{\|\nabla z\|_p}\langle T(z),z\rangle \to \infty, \quad \text{as } \|\nabla z\|_p \to \infty.
$$

For $z \in \mathrm{W}_0^{1,p}(\Omega)$ we have,

$$\begin{aligned}
\langle T(z), z\rangle &= \left\langle -\frac{1}{\lambda}\Delta z - \Delta_p z - f(\frac{a+z}{\lambda}) + \lambda z, z\right\rangle \\
&= \frac{1}{\lambda}\int_\Omega |\nabla z|^2 dx + \int_\Omega |\nabla z|^{p-2}\nabla z\cdot\nabla z dx - \int_\Omega f(\frac{a+z}{\lambda}) z dx + \lambda\|z\|_2^2 \\
&\geq \frac{1}{\lambda}\|\nabla z\|_2^2 + \|\nabla z\|_p^p - \|f(\frac{a+z}{\lambda})\|_2\|z\|_2 + \lambda\|z\|_2^2. \qquad (2.1.22)
\end{aligned}$$

We estimate the third term in (2.1.22) as follows,

$$\begin{aligned}
\|f\left(\frac{a+z}{\lambda}\right)\|_2\|z\|_2 &\leq \left[\|f\left(\frac{a+z}{\lambda}\right) - f\left(\frac{a}{\lambda}\right)\|_2 + \|f\left(\frac{a}{\lambda}\right)\|_2\right]\|z\|_2 \\
&\leq \left[\frac{L_f}{\lambda}\|\nabla z\|_2 + \|f\left(\frac{a}{\lambda}\right)\|_2\right]\|z\|_2 \\
&\leq \frac{1}{2\lambda}\|\nabla z\|_2^2 + \frac{L_f^2}{2\lambda}\|z\|_2^2 + \frac{1}{2}\|f\left(\frac{a}{\lambda}\right)\|_2^2 + \frac{1}{2}\|z\|_2^2, \qquad (2.1.23)
\end{aligned}$$

where we have used Young's inequality in (2.1.23). It follows from (2.1.22)-(2.1.23) that,

$$\begin{aligned}
\langle T(z), z\rangle &\geq \frac{1}{2\lambda}\|\nabla z\|_2^2 + \|\nabla z\|_p^p + (\lambda - \frac{L_f^2}{2\lambda} - \frac{1}{2})\|z\|_2^2 - \frac{1}{2}\|f(\frac{a}{\lambda})\|_2^2 \\
&\geq \|\nabla z\|_p^p - \frac{1}{2}\|f(\frac{a}{\lambda})\|_2^2, \qquad (2.1.24)
\end{aligned}$$

provided $\lambda > 0$ is sufficiently large. Because $p \geq 2$ and $\|f(\frac{a}{\lambda})\|_2^2$ is a constant, we have

$$\frac{1}{\|\nabla z\|_p}\langle T(z), z\rangle \geq \|\nabla z\|_p^{p-1} - \frac{\|f(\frac{a}{\lambda})\|_2^2}{2\|\nabla z\|_p} \longrightarrow \infty, \quad \text{as } \|\nabla z\|_p \to \infty. \qquad (2.1.25)$$

Hence, T is coercive.

Now, given that we have shown that T is maximal monotone and coercive, then by Corollary 1.3 in [5], the operator $T : \mathrm{W}_0^{1,p}(\Omega) \to \mathrm{W}^{-1,p'}(\Omega)$ is surjective. Consequently, given $(a,b) \in H$, and subsequently $\tilde{b} = b + \frac{1}{\lambda}\Delta a \in \mathrm{W}^{-1,p'}(\Omega)$, we find $z \in \mathrm{W}_0^{1,p}(\Omega)$ such that $T(z) = \tilde{b}$. Choosing $y = \frac{a+z}{\lambda} \in \mathrm{H}_0^1(\Omega)$, we obtain

$$-\frac{1}{\lambda}\Delta z - \Delta_p z - f(\frac{a+z}{\lambda}) + \lambda z = b + \frac{1}{\lambda}\Delta a, \tag{2.1.26}$$

which is equivalent to

$$-\Delta(\frac{a+z}{\lambda}) - \Delta_p z - f(y) + \lambda z = b. \tag{2.1.27}$$

Rearranging the terms in (2.1.27) gives

$$-\Delta y - \Delta_p z - f(y) = b - \lambda z \in L^2(\Omega). \tag{2.1.28}$$

Hence, (y,z) is indeed in $\mathcal{D}(\mathscr{A})$ and, therefore, $\mathscr{A} + \omega I$ is m-accretive. By Kato's Theorem (see [27]), there is a unique function $U = (u,v) \in W^{1,\infty}(0,T,H)$, where $T > 0$ is arbitrary, that solves

$$\begin{cases} \frac{d}{dt}\binom{u}{v}^{tr} + (\mathscr{A} + \omega I)\binom{u}{v}^{tr} = \omega\binom{u}{v}^{tr}, \quad \text{a.e. } t \in [0,T], \\ (u(0), v(0)) = (u_0, v_0) \in \mathcal{D}(\mathscr{A}), \end{cases} \tag{2.1.29}$$

or equivalently, $U = (u,v) \in W^{1,\infty}(0,T,H)$ satisfies:

$$\begin{cases} \frac{d}{dt}\binom{u}{v}^{tr} + \binom{-v}{-\Delta u - \Delta_p v - f(u)}^{tr} = 0, \quad \text{a.e. } t \in [0,T], \\ (u(0), v(0)) = (u_0, v_0) \in \mathcal{D}(\mathscr{A}). \end{cases} \tag{2.1.30}$$

This completes the proof and also furnishes the conclusions of Remark 2.1.1. $\square$

2.1.2 Locally Lipschitz Sources

In this subsection we relax the conditions on the source term and allow f to be locally Lipschitz from $\mathrm{H}_0^1(\Omega)$ into $L^2(\Omega)$.

Lemma 2.1.2. *Assume that $f : H_0^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz and $(u_0, v_0) \in \mathcal{D}(\mathscr{A})$. Then, Problem (1.0.1) has a unique solution u such that $u \in C([0,T], H_0^1(\Omega))$, $u_t \in C([0,T], L^2(\Omega)) \cap L^p(0,T, W_0^{1,p}(\Omega))$, $u_{tt} \in L^\infty(0,T,L^2(\Omega))$, for some $T > 0$, where T depends on $\|U(0)\|_H = \|(u_0,v_0)\|_H$, $f(0)$, and the local Lipschitz constant of the mapping $f : H_0^1(\Omega) \to L^q(\Omega)$, where q is as defined in (1.1.2). Moreover, u satisfies the energy identity (1.2.1).*

Remark 2.1.2. The values of the parameter q in (1.1.2) are inherited from the Sobolev imbeddings in (1.1.3). Moreover, $1 \leq q \leq \frac{6}{5}$, and so, by assumption, the mapping $f : \mathrm{H}_0^1(\Omega) \to L^q(\Omega)$ is automatically locally Lipschitz. However, it is essential to note here that the local existence time, T, in Lemma 2.1.2 **does not** depend on the local Lipschitz constant of f as a map from $\mathrm{H}_0^1(\Omega)$ into $L^2(\Omega)$.

Proof. We use a standard truncation of the source (for instance, see [5, 10]). Put

$$f_K(u) := \begin{cases} f(u), & \text{if } \|\nabla u\|_2 \leq K, \\ \\ f\left(\frac{Ku}{\|\nabla u\|_2}\right), & \text{if } \|\nabla u\|_2 > K, \end{cases} \tag{2.1.31}$$

where $K^2 > 2[\|u_t(0)\|_2^2 + \|\nabla u(0)\|_2^2]$. In particular, the mapping $f_K : \mathrm{H}_0^1(\Omega) \to L^2(\Omega)$

is globally Lipschitz continuous. We consider the truncated problem:

$$\begin{cases} u_{tt} - \Delta u - \Delta_p u_t = f_K(u) \quad \text{in} \quad \Omega \times (0,T), \\ \{u(0), u_t(0)\} = \{u_0, u_1\} \in \mathcal{D}(\mathscr{A}), \\ u = 0 \quad \text{on} \quad \Gamma \times (0,T). \end{cases} \tag{2.1.32}$$

By the results of Lemma 2.1.1 and, more precisely, the conclusions of Remark 2.1.1, (2.1.32) has a unique global solution u^K such that, $u^K \in C^0([0,T], \mathrm{H}_0^1(\Omega))$, $u_t^K \in C^0([0,T], L^2(\Omega))$, $u_{tt}^K \in L^\infty(0,T,L^2(\Omega))$, and $u_t^K(t) \in \mathrm{W}_0^{1,p}(\Omega)$, a.e. $t \in [0,T]$, where $T > 0$ is arbitrarily large. Note that f_K is also globally Lipschitz from $\mathrm{H}_0^1(\Omega) \to L^q(\Omega)$, where q is as defined in (1.1.2). That is, there exists a constant $L_f(K) > 0$ such that

$$\|f_K(u) - f_K(v)\|_q \leq L_f(K) \|\nabla(u-v)\|_2, \text{ for all } u, v \in \mathrm{H}_0^1(\Omega).$$

In what follows, we shall write u for the solution, u^K, to (2.1.32). The strong regularity of $u = u^K$ allows us to test the PDE in (2.1.32) with u_t. By multiplying the PDE in (2.1.32) by u_t and integrating in space and time, one easily obtains the following energy identity:

$$\begin{aligned} &\frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2) + \int_0^t \int_\Omega |\nabla u_t(s)|^p dx ds \\ &\quad = \frac{1}{2}(\|u_t(0)\|_2^2 + \|\nabla u(0)\|_2^2) + \int_0^t \int_\Omega f_K(u(s)) u_t(s) dx ds, \end{aligned} \tag{2.1.33}$$

for all $t > 0$. Let $\mathscr{E}(t)$ denote the quadratic the energy, that is,

$$\mathscr{E}(t) := \frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2).$$

Then, (2.1.33) becomes

$$\mathscr{E}(t) + \int_0^t \int_\Omega |\nabla u_t(s)|^p dxds = \mathscr{E}(0) + \int_0^t \int_\Omega f_K(u(s))u_t(s)dxds. \tag{2.1.34}$$

By Hölder's inequality with q and q', where q is defined in (1.1.2) and $\frac{1}{q} + \frac{1}{q'} = 1$, along with Young's inequality with p and $p' = \frac{p}{p-1}$, the last term in (2.1.34) is estimated as follows:

$$\begin{aligned}\int_\Omega f_K(u(s))u_t(s)dx &\leq \|f_K(u(s))\|_q \|u_t(s)\|_{q'} \\ &\leq C_\epsilon \|f_K(u(s))\|_q^{p'} + \epsilon \|u_t(s)\|_{q'}^p \\ &\leq C_\epsilon \big(\|f_K(u(s)) - f_K(0)\|_q + \|f_K(0)\|_q \big)^{p'} + \epsilon \|u_t(s)\|_{q'}^p \\ &\leq C_{\epsilon,p} \|f_K(u(s)) - f_K(0)\|_q^{p'} + C_{\epsilon,p} \|f(0)\|_q^{p'} + \epsilon \|u_t(s)\|_{q'}^p \\ &\leq C_{\epsilon,p} (L_f(K))^{p'} \|\nabla u(s)\|_2^{p'} + C_f + \epsilon \|u_t(s)\|_{q'}^p, \end{aligned} \tag{2.1.35}$$

where $C_f = C_{\epsilon,p}\|f(0)\|_q^{p'}$. Note that we have used the fact that $f_K : \mathrm{H}_0^1(\Omega) \to L^q(\Omega)$ is globally Lipschitz continuous with Lipschitz constant $L_f(K)$. By recalling the imbeddings in (1.1.3), we obtain

$$\begin{aligned}\int_\Omega f_K(u(s))u_t(s)dxdt &\leq C_{\epsilon,p}(L_f(K))^{p'} \|\nabla u(s)\|_2^{p'} + C_f + \epsilon C \|\nabla u_t(s)\|_p^p \\ &\leq C_K \mathscr{E}(s)^{\frac{p}{2(p-1)}} + C_f + \epsilon C \|\nabla u_t(s)\|_p^p, \end{aligned} \tag{2.1.36}$$

where $C_K = C_{\epsilon,p}(L_f(K))^{p'}$. It follows from (2.1.34), (2.1.36) and the fact $\frac{p}{2(p-1)} \leq 1$

that

$$
\begin{aligned}
&\mathscr{E}(t) + \int_0^t \|\nabla u_t(s)\|_p^p dt \\
&\quad \le \mathscr{E}(0) + C_K \int_0^t \mathscr{E}(s)^{\frac{p}{2(p-1)}} ds + C_f t + \epsilon C \int_0^t \|\nabla u_t(s)\|_p^p ds \\
&\quad \le \mathscr{E}(0) + C_K \int_0^t \mathscr{E}(s) ds + C_{K,f} T + \epsilon C \int_0^t \|\nabla u_t(s)\|_p^p ds, \qquad (2.1.37)
\end{aligned}
$$

for all $0 \le t \le T$, where $T > 0$ will chosen below and $C_{K,f} = C_K + C_f$. By choosing $\epsilon > 0$ sufficiently small, we obtain

$$
\mathscr{E}(t) + c_\epsilon \int_0^t \|\nabla u_t(s)\|_p^p ds \le \mathscr{E}(0) + C_{K,f} T + C_K \int_0^t \mathscr{E}(s) ds, \qquad (2.1.38)
$$

for all $0 \le t \le T$. By Gronwall's inequality, we have

$$
\mathscr{E}(t) \le (\mathscr{E}(0) + C_{K,f} T) e^{C_K t}, \qquad (2.1.39)
$$

for all $0 \le t \le T$.

Now, we recall $K^2 > 4\mathscr{E}(0)$. Then we can choose $T > 0$ small enough so that $\mathscr{E}(0) + C_{K,f} T \le \frac{1}{4} K^2$, say $T = \frac{K^2 - 4\mathscr{E}(0)}{4C_{K,f}}$. Therefore, by requiring $t \le \frac{1}{C_K} \ln 2$, one has

$$
\mathscr{E}(t) \le (\mathscr{E}(0) + C_{K,f} T) e^{C_K t} \le \frac{1}{4} K^2 e^{C_K t} \le \frac{1}{2} K^2. \qquad (2.1.40)
$$

Consequently, by taking $T = \min\{\frac{K^2 - 4\mathscr{E}(0)}{4C_{K,f}}, \frac{1}{C_K} \ln 2\}$, then $\mathscr{E}(t) \le \frac{1}{2} K^2$ for all $t \in [0,T]$. Thus, $f_K(u(t)) = f(u(t))$ on the interval $[0,T]$. Because u solves (2.1.32), by the uniqueness of the solution to (2.1.32), we have that u solves the original problem (1.0.1) on $[0,T]$ with the regularity enjoyed by u^K. The fact that $u_t \in L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$ follows immediately from (2.1.38), and the fact that u satisfies the energy identity (1.2.1) follows trivially from (2.1.33), completing the proof. $\square$

2.1.3 More General Source Term

In this subsection, we relax the conditions on the source. Specifically, we allow $f \in C^1(\mathbb{R})$ with the following growth conditions for $|u| \geq 1$: $|f(u)| \leq c_0|u|^r$, $|f'(u)| \leq c_1|u|^{r-1}$, $1 \leq r < 6$, for some positive constants $c_0,\ c_1$. Furthermore, for the values $3 < r < 6$, we require $f \in C^2(\mathbb{R})$ with the growth condition $|f''(u)| \leq c_2|u|^{r-2}$, for $|u| \geq 1$. Throughout this dissertation, the exponent of the source satisfies

$$\begin{cases} 1 \leq r < 8 - \frac{6}{p}, & \text{if } 2 \leq p < 3, \\ 1 \leq r < 6, & \text{if } p \geq 3. \end{cases} \tag{2.1.41}$$

Before reaching a complete proof of Theorem 1.2.1, we need some preparation.

Lemma 2.1.3. *$f : H^{1-\epsilon}(\Omega) \to L^q(\Omega)$ is locally Lipschitz for some $\epsilon > 0$, where q is defined in (1.1.2).*

Proof. From the restriction on r in (2.1.41) we can choose $0 < \epsilon < 1$ such that

$$\epsilon \leq \begin{cases} \frac{8-(6/p)-r}{2r}, & \text{if } 2 \leq p < 3, \\ \frac{6-r}{2r}, & \text{if } p \geq 3. \end{cases} \tag{2.1.42}$$

It is easy to see that $r \leq \frac{8p-6}{p(1+2\epsilon)}$, if $2 \leq p < 3$, and $r \leq \frac{6}{1+2\epsilon}$, if $p \geq 3$. Now, let $u, v \in \mathrm{H}^{1-\epsilon}(\Omega)$ with $\|u\|_{\mathrm{H}^{1-\epsilon}(\Omega)}, \|v\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \leq R$. Then, by the Mean Value Theorem, we have for some $\xi_{u,v}$ between u and v,

$$\begin{aligned}\|f(u)-f(v)\|_q^q &= \int_\Omega |f(u)-f(v)|^q dx \le \int_\Omega |f'(\xi_{u,v})(u-v)|^q dx \\ &\le C\int_\Omega [(|\xi_{u,v}|^{r-1}+1)|u-v|]^q dx \\ &\le C\int_\Omega |u-v|^q(|u|^{q(r-1)}+|v|^{q(r-1)}+1)dx. \end{aligned} \tag{2.1.43}$$

Using Hölder's inequality on (2.1.43) with $\frac{6}{q(1+2\epsilon)}$ and $\frac{6}{6-q(1+2\epsilon)}$ gives

$$\begin{aligned}&\|f(u)-f(v)\|_q^q \\ &\qquad \le C\|u-v\|^q_{\frac{6}{1+2\epsilon}}\big(\|u\|^{q(r-1)}_{q(r-1)\frac{6}{6-q(1+2\epsilon)}}+\|v\|^{q(r-1)}_{q(r-1)\frac{6}{6-q(1+2\epsilon)}}+C_1\big),\end{aligned} \tag{2.1.44}$$

where $C_1>0$ depends on Ω. Notice that $q(r-1)\frac{6}{6-q(1+2\epsilon)}\le 6$. Indeed, for $2\le p<3$, a quick calculation yields

$$\begin{aligned}q(r-1)\frac{6}{6-q(1+2\epsilon)} &\le \frac{3p}{4p-3}\Big(\frac{8p-6}{p(1+2\epsilon)}-1\Big)\frac{6}{6-\frac{3p}{4p-3}(1+2\epsilon)} \\ &= \frac{6}{1+2\epsilon}.\end{aligned}$$

For $p>3$ (recalling $q=1$), we have

$$q(r-1)\frac{6}{6-q(1+2\epsilon)}\le\frac{6}{1+2\epsilon}.$$

For $p=3$, we may choose δ in the definition of q in (1.1.2) small enough so that $q(r-1)\frac{6}{6-q(1+2\epsilon)}\le 6$. By using the imbedding $\mathrm{H}^{1-\epsilon}(\Omega)\hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$, it follows from (2.1.44) that

$$\|f(u)-f(v)\|_q^q\le C\|u-v\|^q_{\mathrm{H}^{1-\epsilon}(\Omega)}\big(\|u\|^{q(r-1)}_{\mathrm{H}^{1-\epsilon}(\Omega)}+\|v\|^{q(r-1)}_{\mathrm{H}^{1-\epsilon}(\Omega)}+1\big). \tag{2.1.45}$$

Because $\|u\|_{\mathrm{H}^{1-\epsilon}(\Omega)}, \|v\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \leq R$, we obtain

$$\|f(u) - f(v)\|_q^q \leq C_R^q \|u - v\|_{\mathrm{H}^{1-\epsilon}(\Omega)}^q, \tag{2.1.46}$$

where $C_R^q = C(2R^{q(r-1)} + 1)$ and so $f : \mathrm{H}^{1-\epsilon}(\Omega) \to L^q(\Omega)$ is locally Lipschitz. $\square$

Since f is not in general locally Lipschitz from $\mathrm{H}_0^1(\Omega)$ into $L^2(\Omega)$, we will construct a Lipschitz approximation of f. We consider a sequence of smooth cut-off functions η_n introduced in [24], where $0 \leq \eta_n \leq 1$, $\eta_n(u) = 1$ if $|u| \leq n$, $\eta_n(u) = 0$ if $|u| > 2n$, and $|\eta_n'(u)| \leq C/n$ for some constant C (independent of n). Define

$$f_n(u) := f(u)\eta_n(u). \tag{2.1.47}$$

Lemma 2.1.4. *For each $n \in \mathbb{N}$, the function f_n has the following properties:*

- *$f_n : H_0^1(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous with Lipschitz constant depending on n.*
- *$f_n : H^{1-\epsilon}(\Omega) \to L^q(\Omega)$ is locally Lipschitz continuous with Lipschitz constant that* ***does not*** *depend on n, where q and ϵ are as defined in Lemma 2.1.3.*

Proof. Let $u, v \in \mathrm{H}_0^1(\Omega)$. Consider the four regions

$$\begin{aligned}
\Omega_1 &:= \{x \in \Omega : |u(x)|, |v(x)| \leq 2n\}, \\
\Omega_2 &:= \{x \in \Omega : |u(x)| \leq 2n, |v(x)| > 2n\}, \\
\Omega_3 &:= \{x \in \Omega : |v(x)| \leq 2n, |u(x)| > 2n\}, \\
\Omega_4 &:= \{x \in \Omega : |u(x)|, |v(x)| > 2n\},
\end{aligned} \tag{2.1.48}$$

and observe that $\Omega = \bigcup_{i=1}^{4} \Omega_i$. Then,

$$\|f_n(u) - f_n(v)\|_2^2 = \sum_{i=1}^{4} \int_{\Omega_i} |f_n(u) - f_n(v)|^2 \, dx. \tag{2.1.49}$$

Now, by the Mean Value Theorem applied to f and η_n we find

$$\int_{\Omega_1} |f_n(u) - f_n(v)|^2 \, dx = \int_{\Omega_1} |f(u)\eta_n(u) - f(v)\eta_n(v)|^2 \, dx$$

$$= \int_{\Omega_1} |f(u)\eta_n(u) - f(v)\eta_n(u) + f(v)\eta_n(u) - f(v)\eta_n(v)|^2 \, dx \tag{2.1.50}$$

$$\leq \int_{\Omega_1} \{|f'(\xi_{u,v})||u - v| + |f(v)||\eta_n'(\tilde{\xi}_{u,v})||u - v|\}^2 dx, \tag{2.1.51}$$

where $\xi_{u,v}$ and $\tilde{\xi}_{u,v}$ lie between u and v. Because $|u|, |v| \leq 2n$, implied by the definition of Ω_1, then $|\xi_{u,v}|, |\tilde{\xi}_{u,v}| \leq 2n$. Recalling properties of f (from Assumption 1.1.1) and η, we have

$$\int_{\Omega_1} \left|f_n(u) - f_n(v)\right|^2 dx$$

$$\leq C \int_{\Omega_1} \{(|\xi_{u,v}|^{r-1} + 1) + (|v|^r + 1)\frac{1}{n}\}^2 |u - v|^2 dx$$

$$\leq C n^{r-1} \int_{\Omega_1} |u - v|^2 dx. \tag{2.1.52}$$

For the second region Ω_2, we notice here that $f_n(v) = 0$ (as $\eta_n(v) = 0$). So,

$$\int_{\Omega_2} |f_n(u) - f_n(v)|^2 \, dx = \int_{\Omega_2} |f(u)\eta_n(u)|^2 \, dx$$

$$= \int_{\Omega_2} |f(u)(\eta_n(u) - \eta_n(v))|^2 \, dx. \tag{2.1.53}$$

Notice that, by switching the roles of u and v in the third and fourth terms of (2.1.50), we can estimate (2.1.53) as we have done in (2.1.51) and (2.1.52) so that

$$\int_{\Omega_2} |f_n(u) - f_n(v)|^2 \, dx \leq Cn^{r-1} \int_{\Omega_2} |u - v|^2 dx. \tag{2.1.54}$$

The integral over Ω_3 is estimated the same way as Ω_2 by reversing the roles of u and v. Finally, in region Ω_4, $f_n(u) = f_n(v) = 0$ and the estimate is trivial. Combining these facts with (2.1.52) and (2.1.54) yields

$$\|f_n(u) - f_n(v)\|_2^2 \leq Cn^{r-1} \|u - v\|_2^2. \tag{2.1.55}$$

Thus, for each $n \in \mathbb{N}$, $f_n : L^2(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous with Lipschitz constant $C_n = Cn^{r-1}$, verifying the first statement of the lemma.

To prove the second statement, let $u, v \in \mathrm{H}^{1-\epsilon}(\Omega)$ with $\|u\|_{\mathrm{H}^{1-\epsilon}(\Omega)}, \|v\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \leq R$ and recall the four regions Ω_i introduced in (2.1.48). Then,

$$\|f_n(u) - f_n(v)\|_q^q = \sum_{i=1}^{4} \|f_n(u) - f_n(v)\|_{L^q(\Omega_i)}^q.$$

We begin by looking at the L^q norm of $f_n(u) - f_n(v)$ on the region Ω_1;

$$\begin{aligned} &\|f_n(u) - f_n(v)\|_{L^q(\Omega_1)} \\ &\quad \leq \|(f(u) - f(v))\eta_n(u)\|_{L^q(\Omega_1)} + \|f(v)\eta_n(u) - f(v)\eta_n(v)\|_{L^q(\Omega_1)} \\ &\quad \leq \|f(u) - f(v)\|_{L^q(\Omega_1)} + \Big(\int_{\Omega_1} \{|f(v)||\eta_n(u) - \eta_n(v)|\}^q dx \Big)^{\frac{1}{q}}. \end{aligned} \tag{2.1.56}$$

By the Mean Value Theorem applied to η_n, the fact that f is locally Lipschitz from

$H^{1-\epsilon}(\Omega) \to L^q(\Omega)$ and $|f(s)| \leq c|s|^r$ for $|s| \geq 1$, we have,

$$\|f_n(u) - f_n(v)\|_{L^q(\Omega_1)} \leq C_R\|u - v\|_{H^{1-\epsilon}(\Omega)} + C\Big(\int_{\Omega_1} \{(|v|^r + 1)|\eta_n'(\xi_{u,v})||u - v|\}^q dx\Big)^{\frac{1}{q}}. \tag{2.1.57}$$

Because $|\eta_n'| \leq C/n$ and $|v| \leq 2n$ in Ω_1, we obtain $|v||\eta_n'(\xi_{u,v})| \leq 2C$ and thus

$$\|f_n(u) - f_n(v)\|_{L^q(\Omega_1)} \leq C_R\|u - v\|_{H^{1-\epsilon}(\Omega)} + C\Big(\int_{\Omega_1} (|v|^{q(r-1)} + 1)|u - v|^q dx\Big)^{\frac{1}{q}}. \tag{2.1.58}$$

The same analysis as in (2.1.43) through (2.1.46) applied to the second term of (2.1.58) yields

$$\Big(\int_{\Omega_1} (|v|^{q(r-1)} + 1)|u - v|^q dx\Big)^{\frac{1}{q}} \leq C_R\|u - v\|_{H^{1-\epsilon}(\Omega)}, \tag{2.1.59}$$

where C_R is as in (2.1.46). Combining (2.1.58) and (2.1.59) gives

$$\|f_n(u)-f_n(v)\|_{L^q(\Omega_1)} \leq C_R'\|u - v\|_{H^{1-\epsilon}(\Omega)}, \tag{2.1.60}$$

where $C_R' > 0$ depends on R. It is important to note here that the estimate for Ω_1 does not rely on the bound for $|u|$ in the region, but only on the fact that $|v| \leq 2n$ and so the estimate for Ω_1 also holds on Ω_3. Furthermore, by switching the roles of u and v, we easily obtain the same bound for Ω_2. That is, for $i = 1, 2, 3$,

$$\|f_n(u) - f_n(v)\|_{L^q(\Omega_i)} \leq C_R'\|u - v\|_{H^{1-\epsilon}(\Omega)}. \tag{2.1.61}$$

Finally, since $f_n(u) = f_n(v) = 0$ in Ω_4, it follows that

$$\|f_n(u) - f_n(v)\|_q \leq C'_R \|u - v\|_{\mathrm{H}^{1-\epsilon}(\Omega)}, \tag{2.1.62}$$

concluding the proof of Lemma 2.1.4. □

2.1.4 The Approximated Problem

In order to prove the existence statement in Theorem 1.2.1, we approximate the original problem (1.0.1) by using the cut-off functions η_m, introduced previously. More precisely, we consider the mth problem given by:

$$\begin{cases} u^m_{tt} - \Delta u^m - \Delta_p u^m_t = f_m(u^m) \quad \text{in} \quad \Omega \times (0,T), \\ (u^m(0), u^m_t(0)) = (u_{m,0}, u_{m,1}) \in \mathcal{D}(\mathscr{A}), \\ u^m = 0 \quad \text{on} \quad \Gamma \times (0,T), \end{cases} \tag{2.1.63}$$

where $(u_{m,0}, u_{m,1}) \to (u_0, u_1)$ in H, as $m \to \infty$, with $\|(u_{m,0}, u_{m,1})\|_H < \|(u_0, u_1)\|_H + 1$, for all $m \in \mathbb{N}$, and $f_m = f\eta_m$ as defined in (2.1.47). We wish to apply Lemma 2.1.2 to the mth problem (2.1.63). In order to do so, we recall the second statement in Lemma 2.1.4, which assures us that the local Lipschitz constants of $f_m : \mathrm{H}^1_0(\Omega) \to L^q(\Omega)$ are independent of m. In addition, by the proof of Lemma 2.1.2, the local existence time depends on the choice $K^2 > 2\|(u_{m,0}, u_{m,1})\|^2_H$. However, by choosing $K^2 > 2(\|(u_0, u_1)\|_H + 1)^2$ in the proof of Lemma 2.1.2, we have one K that properly bounds the norms of the initial data for each $m \in \mathbb{N}$. Therefore, it follows from Lemma 2.1.2 that, for each $m \in \mathbb{N}$, the mth Problem (2.1.63) has a unique solution u^m such that $u^m \in C^0([0,T], \mathrm{H}^1_0(\Omega))$, $u^m_t \in C^0([0,T], L^2(\Omega))$, $u^m_{tt} \in L^\infty(0,T,L^2(\Omega))$, $u^m_t \in L^p(0,T,\mathrm{W}^{1,p}_0(\Omega))$, for some $T > 0$ (independent of m), and u^m satisfies the energy identity,

$$\mathscr{E}_m(t) + \int_0^t \int_\Omega |\nabla u_t^m(s)|^p dxds = \mathscr{E}_m(0) + \int_0^t \int_\Omega f_m(u^m(s))u_t^m(s)dxds, \tag{2.1.64}$$

for all $t \in [0,T]$, where $\mathscr{E}_m(t) := \frac{1}{2}(\|u_t^m(t)\|_2^2 + \|\nabla u^m(t)\|_2^2)$. By the same analysis used to obtain (2.1.38) and (2.1.39), we conclude that there exists $C_T > 0$ such that

$$\mathscr{E}_m(t) + \int_0^t \|\nabla u_t^m(s)\|_p^p ds \leq C_T \quad \text{for all } t \in [0,T]. \tag{2.1.65}$$

Also, with $\frac{1}{p} + \frac{1}{p'} = 1$, we note that

$$\int_0^t \int_\Omega \left||\nabla u_t^m|^{p-2}\nabla u_t^m\right|^{p'} dxds = \int_0^t \|\nabla u_t^m\|_p^p ds \leq C_T, \tag{2.1.66}$$

for all $t \in [0,T]$. That is, $\{|\nabla u_t^m|^{p-2}\nabla u_t^m\}$ is a bounded sequence in $\left(L^{p'}(0,T,L^{p'}(\Omega))\right)^3$. Therefore, it follows from (2.1.65)–(2.1.66) that there exists a subsequence of $\{u^m\}$, which we still denote by $\{u^m\}$, such that

$$\begin{cases} u^m \longrightarrow u \text{ weak}^* \text{ in } L^\infty(0,T,\mathrm{H}_0^1(\Omega)), \\ u_t^m \longrightarrow u_t \text{ weak}^* \text{ in } L^\infty(0,T,L^2(\Omega)), \\ u_t^m \longrightarrow u_t \text{ weakly in } L^p(0,T,\mathrm{W}_0^{1,p}(\Omega)), \\ |\nabla u_t^m|^{p-2}\nabla u_t^m \longrightarrow \psi \text{ weakly in } \left(L^{p'}(0,T,L^{p'}(\Omega))\right)^3, \end{cases} \tag{2.1.67}$$

for some $\psi \in \left(L^{p'}(0,T,L^{p'}(\Omega))\right)^3$.

At this point, we note that $\Delta_p u_t^m \longrightarrow \eta$ weakly in X^*, for some $\eta \in X^*$, where $X = L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$ and $X^* = L^{p'}(0,T,\mathrm{W}^{-1,p'}(\Omega))$ is the dual of X. To see this fact, let $\phi \in X$. Then, from the last convergence in (2.1.67), we have

$$\langle \Delta_p u_t^m, \phi \rangle_{X^*,X} = \int_0^T \int_\Omega |\nabla u_t^m|^{p-2} \nabla u_t^m \cdot \nabla \phi dx ds \longrightarrow \int_0^T \int_\Omega \psi \cdot \nabla \phi dx ds. \quad (2.1.68)$$

Thus, $\Delta_p u_t^m$ is weakly convergent in X^*. Given that X^* is a reflexive Banach space, and by a standard theorem (e.g., [32]), X^* is sequentially weakly complete. Hence, there exists an $\eta \in X^*$ such that

$$\Delta_p u_t^m \longrightarrow \eta \quad \text{weakly in } L^{p'}(0, T, \mathrm{W}^{-1,p'}(\Omega)). \quad (2.1.69)$$

Let us note here that $H_0^1(\Omega) \subset \mathrm{H}^{1-\epsilon}(\Omega) \subset L^{p'}(\Omega)$ where each injection is continuous and the first injection is compact. Also, given that $\{u^m\}$ is bounded in $L^\infty(0, T, \mathrm{H}_0^1(\Omega))$, then, in particular, $\{u^m\}$ is also bounded in $L^{p'}(0, T, \mathrm{H}_0^1(\Omega))$. We also know that $\{u_t^m\}$ is bounded in $L^p(0, T, \mathrm{W}_0^{1,p}(\Omega))$, and thus, in particular, $\{u_t^m\}$ is bounded in $L^{p'}(0, T, L^{p'}(\Omega))$ by Hölder's inequality. Hence, by Aubin's Compactness Theorem there exists a subsequence, labeled again by $\{u^m\}$, such that

$$u^m \to u \text{ strongly in } L^{p'}(0, T, \mathrm{H}^{1-\epsilon}(\Omega)), \quad (2.1.70)$$

where $\epsilon > 0$ is as in (2.1.42).

Our goal now is to identify the function η and pass to the limit. For this purpose, let $\tilde{u}(t) = u^m(t) - u^n(t)$ and $\tilde{u}_t(t) = u_t^m(t) - u_t^n(t)$. Straightforward calculations show that $\tilde{u}$ satisfies the following energy identity:

$$\begin{aligned} \tilde{\mathscr{E}}(t) &+ \int_0^t \int_\Omega \left(|\nabla u_t^m|^{p-2} \nabla u_t^m - |\nabla u_t^n|^{p-2} \nabla u_t^n \right) \cdot \nabla \tilde{u}_t dx ds \\ &= \tilde{\mathscr{E}}(0) + \int_0^t \int_\Omega \left(f_m(u^m) - f_n(u^n) \right) \tilde{u}_t dx ds, \end{aligned} \quad (2.1.71)$$

for all $t \in [0,T]$, where $\tilde{\mathscr{E}}(t) = \frac{1}{2}(\|\tilde{u}_t(t)\|_2^2 + \|\nabla \tilde{u}(t)\|_2^2)$. Since $(u_{m,0}, u_{m,1}) \to (u_0, u_1)$ in H by assumption, $\{(u_{m,0}, u_{m,1})\}$ is Cauchy in H and so,

$$\tilde{\mathscr{E}}(0) = \frac{1}{2}(\|u_{m,1} - u_{n,1}\|_2^2 + \|\nabla(u_{m,0} - u_{m,1})\|_2^2) \longrightarrow 0, \tag{2.1.72}$$

as $m, n \longrightarrow \infty$.

Next, we shall show that the last term in (2.1.71) converges to 0 as $m, n \longrightarrow \infty$. To do so, we recall the bounds in (2.1.65) and the convergences in (2.1.67). Thus, we can choose $R > 0$ such that $\|u^m\|_{\mathrm{H}^{1-\epsilon}(\Omega)}, \|u\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \leq R$, for all $m \in \mathbb{N}$. The last term in (2.1.71) is estimated as follows

$$\begin{aligned} I &:= \left| \int_0^t \int_\Omega \big(f_m(u^m) - f_n(u^n)\big)\tilde{u}_t dxds \right| \leq \int_0^t \int_\Omega \big|f_m(u^m) - f_m(u)||\tilde{u}_t|dxds \\ &+ \int_0^t \int_\Omega \big|f_m(u) - f(u)||\tilde{u}_t|dxds + \int_0^t \int_\Omega |f(u) - f_n(u)||\tilde{u}_t|dxds \\ &+ \int_0^t \int_\Omega |f_n(u) - f_n(u^n)||\tilde{u}_t|dxds. \end{aligned} \tag{2.1.73}$$

By using Hölder's inequality with the exponents q and q', where q is as defined in (1.1.2), followed by the Sobolev imbeddings in (1.1.3) applied to $\tilde{u}_t$, we have

$$\begin{aligned} I &\leq C\Big[\int_0^t \|f_m(u^m) - f_m(u)\|_q \|\nabla \tilde{u}_t\|_p ds + \int_0^t \|f_m(u) - f(u)\|_q \|\nabla \tilde{u}_t\|_p ds \\ &+ \int_0^t \|f(u) - f_n(u)\|_q \|\nabla \tilde{u}_t\|_p ds + \int_0^t \|f_n(u) - f_n(u^n)\|_q \|\nabla \tilde{u}_t\|_p ds\Big] \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{2.1.74}$$

Because f_m and f_n are locally Lipschitz from $\mathrm{H}^{1-\epsilon}(\Omega) \to L^q(\Omega)$, as furnished by

Lemma 2.1.4, we have

$$I_1 \leq C_R \int_0^T \|u^m - u\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \|\nabla \tilde{u}_t\|_p ds \tag{2.1.75}$$

and

$$I_4 \leq C_R \int_0^T \|u^n - u\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \|\nabla \tilde{u}_t\|_p ds, \tag{2.1.76}$$

where C_R is the local Lipschitz constant of f_m, which can be taken to be the same for all $m \in \mathbb{N}$. Combining (2.1.75)-(2.1.76) and employing Hölder's inequality with p and p', we have

$$\begin{aligned} I_1 + I_4 \leq C_R \Big[\|u^m - u\|_{L^{p'}(0,T,\mathrm{H}^{1-\epsilon}(\Omega))} + \|u^n - u\|_{L^{p'}(0,T,\mathrm{H}^{1-\epsilon}(\Omega))} \Big] \\ \cdot \quad \|\tilde{u}_t\|_{L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))}. \end{aligned} \tag{2.1.77}$$

Because u_t^m is bounded in $L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$, so is $\tilde{u}_t$. Hence, it follows from (2.1.77) and the strong convergence in (2.1.70) that

$$I_1 + I_4 \longrightarrow 0, \text{ as } m, n \to \infty. \tag{2.1.78}$$

As for I_2 and I_3, we also employ Hölder's inequality with p and p' to obtain

$$\begin{aligned} I_2 + I_3 \leq C \Big[\Big(\int_0^T \|f_m(u) - f(u)\|_q^{p'} ds \Big)^{\frac{1}{p'}} + \Big(\int_0^T \|f(u) - f_n(u)\|_q^{p'} ds \Big)^{\frac{1}{p'}} \Big] \\ \cdot \quad \|\tilde{u}_t\|_{L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))}. \end{aligned} \tag{2.1.79}$$

Now, because $\eta_m(u) \longrightarrow 1$ as $m \longrightarrow \infty$, we have

$$|f_m(u) - f(u)|^q = |f(u)(\eta_m(u) - 1)|^q \longrightarrow 0 \quad \text{a.e.} \;\in \Omega \times [0,T],$$

as $m \to \infty$. Also, given that $f(u) \in L^q(\Omega)$ for all $u \in \mathrm{H}_0^1(\Omega)$, as furnished by Lemma 2.1.3, then $|f_m(u) - f(u)|^q \leq 2^q|f(u)|^q \in L^1(\Omega)$. Therefore, by the Lebesgue Dominated Convergence Theorem, $f_m(u) \to f(u)$ in $L^q(\Omega)$, or $\|f_m(u) - f(u)\|_q^{p'} \to 0$ a.e. $t \in [0,T]$. Additionally, because f is locally Lipschitz from $\mathrm{H}^{1-\epsilon}(\Omega) \to L^q(\Omega)$ (see Lemma 2.1.3), we have

$$\begin{aligned} \|f_m(u) - f(u)\|_q^{p'} &\leq 2^{p'}\|f(u)\|_q^{p'} \leq C(\|f(u) - f(0)\|_q^{p'} + \|f(0)\|_q^{p'}) \\ &\leq C_R^{p'}\|u\|_{\mathrm{H}^{1-\epsilon}(\Omega)}^{p'} + C_f \in L^1(0,T), \end{aligned} \tag{2.1.80}$$

Given that $u \in L^{p'}(0,T,\mathrm{H}^{1-\epsilon}(\Omega))$, from (2.1.70). Again, by the Lebesgue Dominated Convergence Theorem, we have

$$\Big(\int_0^T \|f_m(u) - f(u)\|_q^{p'} ds\Big)^{\frac{1}{p'}} \longrightarrow 0, \tag{2.1.81}$$

as $m \longrightarrow \infty$ (and similarly for $n \to \infty$). It follows from the boundedness of $\tilde{u}_t$ in $L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$, (2.1.81) and (2.1.79) that

$$I_2 + I_3 \longrightarrow 0, \text{ as } m,n \to \infty. \tag{2.1.82}$$

Moreover, from (2.1.6) it is clear that,

$$\int_0^t \int_\Omega \big(|\nabla u_t^m|^{p-2}\nabla u_t^m - |\nabla u_t^n|^{p-2}\nabla u_t^n\big) \cdot \nabla(u_t^m - u_t^n)dxds \geq 0. \tag{2.1.83}$$

Therefore, it follows from (2.1.72), (2.1.78) and (2.1.82) that, for all $t \in [0,T]$

$$\tilde{\mathscr{E}}(t) + \int_0^t \int_\Omega \left(|\nabla u_t^m|^{p-2}\nabla u_t^m - |\nabla u_t^n|^{p-2}\nabla u_t^n\right) \cdot \nabla(u_t^m - u_t^n) dxds \longrightarrow 0, \quad (2.1.84)$$

as $m, n \longrightarrow \infty$. This implies that $\tilde{\mathscr{E}}(t) \longrightarrow 0$, i.e. (u^m, u_t^m) is uniformly Cauchy in H on $[0,T]$, and so $(u^m, u_t^m) \longrightarrow (u, u_t)$ in H uniformly on $[0,T]$.

2.1.5 Proper Proof of the Existence Statement in Theorem 1.2.1

We recall the regularity of u^m, the solution of the m-th problem, namely, $u^m \in C([0,T], \mathrm{H}_0^1(\Omega))$, $u_t^m \in C([0,T], L^2(\Omega))$, $u_{tt}^m \in L^\infty(0,T,L^2(\Omega))$, $u_t^m \in L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$. In particular, u^m verifies,

$$\int_0^t \int_\Omega (-u_t^m \phi_t + \nabla u^m \cdot \nabla\phi) dxds + \int_\Omega u_t^m \phi|_{s=0}^{s=t} dx + \int_0^t \int_\Omega |\nabla u_t^m|^{p-2}\nabla u_t^m \cdot \nabla\phi dxds$$
$$= \int_0^t \int_\Omega f_m(u^m)\phi dxds, \quad (2.1.85)$$

for all $t \in [0,T]$ and for all test functions $H^1(0,T,L^2(\Omega)) \cap L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$.

Let us first note that because $(u^m, u_t^m) \longrightarrow (u, u_t)$ in H uniformly on $[0,T]$, then $u \in C([0,T], \mathrm{H}_0^1(\Omega))$ and $u_t \in C([0,T], L^2(\Omega))$. In addition, we have

$$u(0) = \lim_{t\to 0^+} u(t) = \lim_{t\to 0^+}\lim_{m\to\infty} u^m(t) = \lim_{m\to\infty}\lim_{t\to 0^+} u^m(t) = \lim_{m\to\infty} u_{m,0} = u_0,$$
$$u_t(0) = \lim_{t\to 0^+} u_t(t) = \lim_{t\to 0^+}\lim_{m\to\infty} u_t^m(t) = \lim_{m\to\infty}\lim_{t\to 0^+} u_t^m(t) = \lim_{m\to\infty} u_{m,1} = u_1, \quad (2.1.86)$$

where the limits in m are strong limits in $\mathrm{H}_0^1(\Omega)$ and $L^2(\Omega)$, respectively. In view of the convergences in (2.1.67), passing to the limit in the first three terms of (2.1.85) is trivial. However, passing to the limit in the last two terms of (2.1.85) requires some care. As for the last term in (2.1.85), we recall the fact that $f_m : \mathrm{H}^{1-\epsilon}(\Omega) \to L^q(\Omega)$ is locally Lipschitz with the same Lipschitz constant for all $m \in \mathbb{N}$. By the imbeddings in (1.1.3), we have

$$\left|\int_0^t \int_\Omega (f_m(u^m) - f(u))\phi dxds\right| \leq \int_0^T \|(f_m(u^m) - f_m(u))\|_q \|\nabla\phi\|_p ds + \int_0^T \|(f_m(u) - f(u))\|_q \|\nabla\phi\|_p ds, \tag{2.1.87}$$

for all $t \in [0,T]$. By going back to (2.1.74) we see that the right-hand side of (2.1.87) is exactly the term $I_1 + I_2$ in (2.1.74), but with $\tilde{u}_t$ is being replaced by ϕ. However, (2.1.77) and (2.1.82) show that the right-hand side of (2.1.87) converges to zero, as $m \to \infty$. That is, for all $t \in [0,T]$,

$$\lim_{m\to\infty} \int_0^t \int_\Omega f_m(u^m)\phi dxds = \int_0^t \int_\Omega f(u)\phi dxds. \tag{2.1.88}$$

As for the term due to the p-Laplacian in (2.1.85), we first recall (2.1.83) and (2.1.84), which yield

$$\begin{aligned} &-\langle \Delta_p u_t^m - \Delta_p u_t^n, u_t^m - u_t^n\rangle_{X^*,X} \\ &\quad= \int_0^t \int_\Omega \left(|\nabla u_t^m|^{p-2}\nabla u_t^m - |\nabla u_t^n|^{p-2}\nabla u_t^n\right) \cdot \nabla(u_t^m - u_t^n) dxds \longrightarrow 0, \end{aligned} \tag{2.1.89}$$

where $X = L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$ and $X^* = L^{p'}(0,T,\mathrm{W}^{-1,p'}(\Omega))$ is the dual of X. In addition, we recall the weak convergences from (2.1.67) and (2.1.69);

$$\begin{cases} u_t^m \to u_t \quad \text{weakly in } L^p(0,T,\mathrm{W}_0^{1,p}(\Omega)), \\ \Delta_p u_t^m \to \eta \quad \text{weakly in } L^{p'}(0,T,\mathrm{W}^{-1,p'}(\Omega)). \end{cases} \tag{2.1.90}$$

Therefore, it follows from Lemma 1.3 of [5] that $\eta = \Delta_p u_t$, provided we show that

$\Delta_p : X \to X^*$ is maximal monotone. Clearly, (2.1.5) and (2.1.6) show that Δ_p is monotone from X into X^*. To show it is maximal, it is enough to show that Δ_p is hemi-continuous from X into X^*. To this end, let $u, v, \xi \in L^p(0, T, \mathrm{W}_0^{1,p}(\Omega))$ and $\mu \in \mathbb{R}$. Then,

$$-\left\langle \Delta_p(u + \mu v), \xi \right\rangle_{X^*,X} = \int_0^T \int_\Omega |\nabla(u + \mu v)|^{p-2} \nabla(u + \mu v) \cdot \nabla\xi dxds.$$

Clearly,

$$\lim_{\mu\to 0} |\nabla(u + \mu v)|^{p-2}\nabla(u + \mu v) \cdot \nabla\xi = |\nabla u|^{p-2}\nabla u \cdot \nabla\xi \quad \text{a.e. } \Omega \times (0, T), \tag{2.1.91}$$

and for $|\mu| < 1$, we have

$$\left||\nabla(u + \mu v)|^{p-2}\nabla(u + \mu v) \cdot \nabla\xi\right| \leq 2^{p-2} \left(|\nabla u|^{p-1} + |\nabla u|^{p-1}\right) |\nabla\xi|. \tag{2.1.92}$$

In addition, $|\nabla u|^{p-1}|\nabla\xi|,\ \ |\nabla v|^{p-1}|\nabla\xi| \in L^1(\Omega \times (0, T))$, because (similarly for v),

$$\int_0^T \int_\Omega |\nabla u|^{p-1}|\nabla\xi|dx \leq \int_0^T \|\nabla u\|_p^{p-1} \|\nabla\xi\|_p \, ds \leq \|u\|_X^{p-1} \|\xi\|_X < \infty.$$

Therefore, by the Dominated Convergence Theorem we have

$$\begin{aligned} -\lim_{\mu\to 0} \left\langle \Delta_p(u + \mu v), \xi \right\rangle_{X^*,X} &= \lim_{\mu\to 0} \int_0^T \int_\Omega |\nabla(u + \mu v)|^{p-2}\nabla(u + \mu v) \cdot \nabla\xi dx \\ &= \int_0^T \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\xi dx = -\left\langle \Delta_p u, \xi \right\rangle_{X^*,X}. \end{aligned} \tag{2.1.93}$$

Hence, Δ_p is hemi-continuous from X into X^* and we conclude that $\eta = \Delta_p u_t$. Now, we have all the ingredients to pass to the limit in (2.1.85) to obtain that u satisfies the variational identity (1.1.44) and u has the required regularity in Definition 1.1.4. This concludes the proof of the local existence statement in Theorem 1.2.1.

2.2 Energy Identity

Let u be a local weak solution to (1.0.1) on $[0,T]$, as furnished by Section 2.1. Our goal here is to prove that u satisfies the following energy identity:

$$E(t) + \int_0^t \|\nabla u_t(s)\|_p^p \, ds = E(0), \tag{2.2.1}$$

where

$$E(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 - \int_\Omega F(u(t))dx, \tag{2.2.2}$$

and $F(u) = \int_0^u f(s)ds$. We begin by proving that $u_{tt} \in L^{p'}(0,T,\mathrm{W}^{-1,p'}(\Omega))$. To see this, let $\phi \in \mathrm{W}_0^{1,p}(\Omega)$ and recall that u satisfies (1.1.44) in Definition 1.1.4. Then, for this particular ϕ, we have

$$\begin{aligned}\big(u_t(t),\phi\big)_\Omega &= \big(u_t(0),\phi\big)_\Omega - \int_0^t\int_\Omega \nabla u\cdot\nabla\phi dxds \\ &\quad - \int_0^t\int_\Omega |\nabla u_t|^{p-2}\nabla u_t\cdot\nabla\phi dxds + \int_0^t\int_\Omega f(u)\phi dxds.\end{aligned} \tag{2.2.3}$$

Thus, the mapping $t \mapsto \big(u_t(t),\phi\big)_\Omega$ is absolutely continuous on $[0,T]$ and for all $\phi \in \mathrm{W}_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\langle u_{tt}(t),\phi\rangle &:= \frac{d}{dt}\langle u_t(t),\phi\rangle = \frac{d}{dt}\big(u_t(t),\phi\big)_\Omega = -\int_\Omega \nabla u(t)\cdot\nabla\phi dx \\ &\quad - \int_\Omega |\nabla u_t(t)|^{p-2}\nabla u_t(t)\cdot\nabla\phi dx + \int_\Omega f(u(t))\phi dx, \ \text{a.e. } [0,T].\end{aligned} \tag{2.2.4}$$

Applying Hölders inequality to the three terms in the right hand side of (2.2.4), one obtains

$$
\begin{aligned}
|\langle u_{tt}(t),\phi\rangle| &\le \|\nabla u(t)\|_2\|\nabla\phi\|_2 + \|\nabla u_t(t)\|_p^{p-1}\|\nabla\phi\|_p + \|f(u(t))\|_q\|\phi\|_{q'} \\
&\le \|\nabla u(t)\|_2\|\nabla\phi\|_p + \|\nabla u_t(t)\|_p^{p-1}\|\nabla\phi\|_p + C\|f(u(t))\|_q\|\nabla\phi\|_p \\
&\le C\|\nabla\phi\|_p\Big(\|\nabla u(t)\|_2 + \|\nabla u_t(t)\|_p^{p-1} + \|f(u(t))\|_q\Big), \qquad (2.2.5)
\end{aligned}
$$

for all $\phi \in \mathrm{W}_0^{1,p}(\Omega)$. Therefore,

$$
\|u_{tt}(t)\|_{\mathrm{W}^{-1,p'}(\Omega)} \le C\big(\|\nabla u(t)\|_2 + \|\nabla u_t(t)\|_p^{p-1} + \|f(u(t))\|_q\big) \text{ a.e. } [0,T]. \qquad (2.2.6)
$$

Hence,

$$
\begin{aligned}
\|u_{tt}\|^{p'}_{L^{p'}(0,T,\mathrm{W}^{-1,p'}(\Omega))} &\le C\int_0^T \Big(\|\nabla u(t)\|_2 + \|\nabla u_t(t)\|_p^{p-1} + \|f(u(t))\|_q\Big)^{p'} dt \\
&\le C\Big(\int_0^T \|\nabla u(t)\|_2^{p'} dt + \int_0^T \|\nabla u_t(t)\|_p^p dt \\
&\qquad + \int_0^T \|f(u(t))\|_q^{p'} dt\Big). \qquad (2.2.7)
\end{aligned}
$$

Because $1 < p' \le 2$, $u \in C([0,T], \mathrm{H}_0^1(\Omega))$ and $u_t \in L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$, the first two terms on the right-hand side of (2.2.7) are finite. As for the third term of (2.2.7), we have

$$
\begin{aligned}
\int_0^T \|f(u(t))\|_q^{p'} dt &\le C\Big(\int_0^T \|f(u(t)) - f(0)\|_q^{p'} dt + \int_0^T \|f(0)\|_q^{p'} dt\Big) \\
&\le C_f \int_0^T \|\nabla u(t)\|_2^{p'} dt + C_{f,T} < \infty, \qquad (2.2.8)
\end{aligned}
$$

where we have used the fact that $f : \mathrm{H}_0^1(\Omega) \to L^q(\Omega)$ is locally lipschitz (see Lemma 2.1.3). Hence, $u_{tt} \in L^{p'}(0,T,\mathrm{W}^{-1,p'}(\Omega))$.

Now, because $u_t(t) \in \mathrm{W}_0^{1,p}(\Omega)$ a.e. $[0,T]$, we can replace ϕ by $u_t(t)$ in (2.2.4) to obtain

$$\langle u_{tt}(t), u_t(t) \rangle = -\frac{1}{2}\int_\Omega \frac{d}{dt}|\nabla u(t)|^2 dx - \|\nabla u_t(t)\|_p^p + \int_\Omega \frac{d}{dt}F(u(t))dx, \tag{2.2.9}$$

for almost all $t \in [0,T]$. Employing Proposition 1.1.3 with $f = g = u_t$, $X = \mathrm{W}_0^{1,p}(\Omega)$, $X^* = \mathrm{W}^{-1,p'}(\Omega)$, $H = L^2(\Omega)$, $\alpha = \beta = p$ and $\alpha' = \beta' = p'$, we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|_2^2 = \langle u_{tt}(t), u_t(t) \rangle\,, \quad \text{a.e. } [0,T] \tag{2.2.10}$$

and

$$\frac{1}{2}\big(\|u_t(t)\|_2^2 - \|u_t(0)\|_2^2\big) = \int_0^t \langle u_{tt}(s), u_t(s) \rangle\, ds \quad \text{for all } t \in [0,T]. \tag{2.2.11}$$

Integrating (2.2.9) from 0 to t and using (2.2.11) we obtain,

$$\begin{aligned}\frac{1}{2}\big(\|u_t(t)\|_2^2 - \|u_t(0)\|_2^2\big) = -\frac{1}{2}\big(\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2\big) - \int_0^t \|\nabla u_t(s)\|_p^p dxds \\ + \int_\Omega F(u(t))dx - \int_\Omega F(u(0))dx.\end{aligned} \tag{2.2.12}$$

Rearranging the terms in (2.2.12) and recalling the definition of $E(s)$ gives the energy identity (2.2.1).

Chapter 3

Uniqueness of Weak Solutions

This chapter is devoted to the proof of the uniqueness statement in Theorem 1.2.1. Precisely, we aim to prove the following proposition.

3.1 Proof of Uniqueness

Proposition 3.1.1. *Given the validity of Assumption 1.1.1, and assuming that* $u_0 \in L^k(\Omega)$*, if* $r \geq 5$*, where* $k = \frac{3(r-1)}{2}$*, local weak solutions to (1.0.1) are unique.*

Proof. Suppose that u and v are weak solutions to (1.0.1) on $[0,T]$ in the sense of Definition 1.1.4. Let $\tilde{u} := u - v$. We aim to show that $\tilde{u} = 0$. Observe that because u and v are weak solutions, then $\tilde{u} \in C([0,T], \mathrm{H}^1_0(\Omega))$ and $\tilde{u}_t \in C([0,T], L^2(\Omega)) \cap L^p(0,T,\mathrm{W}^{1,p}_0(\Omega))$. Thus, there exists $R > 0$ such that for all $t \in [0,T]$,

$$\begin{cases} \|\nabla u(t)\|_2,\ \|\nabla v(t)\|_2,\ \|u_t(t)\|_2,\ \|v_t(t)\|_2 \leq R, \\ \int_0^T \|\nabla u_t\|_p^p dxdt,\ \int_0^T \|\nabla v_t\|_p^p dxdt \leq R. \end{cases} \tag{3.1.1}$$

Let us note here that $\tilde{u}$ satisfies the problem

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} - \Delta_p u_t + \Delta_p v_t = f(u) - f(v) & \text{in } \Omega \times (0,T) \\ \tilde{u} = 0 & \text{on } \Gamma \times (0,T) \\ \tilde{u}(0) = \tilde{u}_t(0) = 0. & \text{in } \Omega \end{cases} \tag{3.1.2}$$

Furthermore, as u and v satisfy their corresponding energy identities (2.2.1), $\tilde{u}$ does as well. That is, $\tilde{u}$ satisfies:

$$\begin{aligned} \tilde{\mathscr{E}}(t) + &\int_0^t \int_\Omega (|\nabla u_t|^{p-2}\nabla u_t - |\nabla v_t|^{p-2}\nabla v_t) \cdot \nabla \tilde{u}_t dxds \\ &= \int_0^t \int_\Omega (f(u) - f(v))\tilde{u}_t dxds, \end{aligned} \tag{3.1.3}$$

for all $t \in [0,T]$, where $\tilde{\mathscr{E}}(t) := \frac{1}{2}\Big(\|\tilde{u}_t(t)\|_2^2 + \|\nabla \tilde{u}(t)\|_2^2\Big)$. Recalling the monotonicity property (2.1.6) of the p-Laplacian, we have

$$\tilde{\mathscr{E}}(t) \leq \int_0^t \int_\Omega (f(u) - f(v))\tilde{u}_t dxds, \text{ for all } t \in [0,T]. \tag{3.1.4}$$

As in [9], our main goal is to estimate the term due to the source. We start with the simple case when $f : \mathrm{H}_0^1(\Omega) \to L^2(\Omega)$ is locally Lipschitz (i.e., when $1 \leq r \leq 3$). For these values of r, we have by the Cauchy-Schwarz inequality and Young's inequality,

$$\begin{aligned} \left| \int_0^t \int_\Omega (f(u) - f(v))\tilde{u}_t dxds \right| &\leq \int_0^t \|f(u) - f(v)\|_2 \|\tilde{u}_t\|_2 ds \\ &\leq \frac{1}{2} L_f \int_0^t \Big(\|\nabla(\tilde{u})\|_2^2 + \|\tilde{u}_t\|_2^2\Big) ds \\ &= L_f \int_0^t \tilde{\mathscr{E}}(s) ds. \end{aligned} \tag{3.1.5}$$

Now, for $r > 3$ we recall Assumption 1.1.1, namely, we further require that $f \in C^2(\mathbb{R})$

and $|f''(u)| \le c_2|u|^{r-2}$ for all $|u| \ge 1$. This yields the following growth conditions on f:

$$\begin{cases} |f'(u)| \le c_1|u|^{r-1}, \quad |f(u)| \le c_0|u|^r, \text{ for } |u| \ge 1, \\ |f'(u) - f'(v)| \le C|u-v|(|u|^{r-2} + |v|^{r-2} + 1), \qquad u, v \in \mathbb{R}, \\ |f(u) - f(v)| \le C|u-v|(|u|^{r-1} + |v|^{r-1} + 1), \qquad u, v \in \mathbb{R}. \end{cases} \tag{3.1.6}$$

Integration by parts, the bounds in (3.1.6), and the fact that $\tilde{u}(0) = 0$ yield

$$\begin{aligned} &\left| \int_0^t \int_\Omega (f(u) - f(v))\tilde{u}_t dx ds \right| \\ &\quad \le \left| \int_\Omega (f(u(t)) - f(v(t)))\tilde{u}(t) dx \right| + \left| \int_0^t \int_\Omega (f'(u)u_t - f'(v)v_t)\tilde{u} dx ds \right| \\ &\quad \le C \int_\Omega |\tilde{u}(t)|^2 (|u(t)|^{r-1} + |v(t)|^{r-1} + 1) dx + \left| \int_0^t \int_\Omega f'(u)\tilde{u}\tilde{u}_t dx ds \right| \\ &\quad + \left| \int_0^t \int_\Omega (f'(u) - f'(v))v_t \tilde{u} dx ds \right| \\ &\quad \le C \int_\Omega |\tilde{u}(t)|^2 \Big(|u(t)|^{r-1} + |v(t)|^{r-1} + 1 \Big) dx + \frac{1}{2} \left| \int_0^t \int_\Omega f'(u) \frac{d}{dt} (\tilde{u})^2 dx ds \right| \\ &\quad + C \int_0^t \int_\Omega (|u|^{r-2} + |v|^{r-2} + 1)|v_t||\tilde{u}|^2 dx ds. \end{aligned} \tag{3.1.7}$$

One more integration by parts in (3.1.7) yields

$$\begin{aligned}
&|\int_0^t \int_\Omega (f(u) - f(v))\tilde{u}_t dxds| \\
&\quad \leq C \int_\Omega |\tilde{u}(t)|^2 (|u(t)|^{r-1} + |v(t)|^{r-1} + 1) dx + \frac{1}{2} | \int_\Omega f'(u(t))\tilde{u}^2(t) dx| \\
&\quad + C \int_0^t \int_\Omega (|u|^{r-2} + |v|^{r-2} + 1)|v_t||\tilde{u}|^2 dxds + \frac{1}{2} | \int_0^t \int_\Omega f''(u) u_t \tilde{u}^2 dxds| \\
&\quad \leq C \Big[\int_\Omega |\tilde{u}(t)|^2 (|u(t)|^{r-1} + |v(t)|^{r-1} + 1) dx + \int_0^t \int_\Omega \tilde{u}^2 (|u_t| + |v_t|) dxds \\
&\quad + \int_0^t \int_\Omega (|u|^{r-2} + |v|^{r-2})(|u_t| + |v_t|)|\tilde{u}|^2 dxds \Big]. \qquad (3.1.8)
\end{aligned}$$

Our task here is to estimate each term on the right-hand side of (3.1.8).

1. Estimate for $\int_\Omega |\tilde{u}(t)|^2 dx$: First, notice that the regularity of $\tilde{u}$ and the fact that $\tilde{u}(0) = 0$ allow us to write

$$\begin{aligned}
\int_\Omega |\tilde{u}(t)|^2 dx &= \int_\Omega \left| \int_0^t \tilde{u}_t(s) ds \right|^2 dx \leq \int_\Omega \int_0^t |\tilde{u}_t(s)|^2 ds \cdot t dx \\
&\leq T \int_0^t \|\tilde{u}_t(s)\|_2^2 ds \leq 2T \int_0^t \tilde{\mathscr{E}}(s) ds. \qquad (3.1.9)
\end{aligned}$$

2. Estimate for $\int_0^t \int_\Omega \tilde{u}^2 |u_t| dxds$ and $\int_0^t \int_\Omega \tilde{u}^2 |v_t| dxds$: Using Hölder's inequality with 3 and $\frac{3}{2}$, the imbedding in (1.1.1), and (3.1.1), we have

$$\begin{aligned}
\int_0^t \int_\Omega \tilde{u}^2 |u_t| dxds &\leq \int_0^t \|\tilde{u}(s)\|_6^2 \|u_t(s)\|_{3/2} ds \\
&\leq C \int_0^t \|\nabla \tilde{u}(s)\|_2^2 \|u_t(s)\|_2 ds \leq CR \int_0^t \|\nabla \tilde{u}(s)\|_2^2 ds \\
&\leq C_R \int_0^t \tilde{\mathscr{E}}(s) ds. \qquad (3.1.10)
\end{aligned}$$

Similarly, $\int_0^t \int_\Omega \tilde{u}^2 |v_t| dxds \leq C_R \int_0^t \tilde{\mathscr{E}}(s) ds$.

3. Estimate for $\int_\Omega |\tilde{u}(t)|^2(|u(t)|^{r-1} + |v(t)|^{r-1})dx$: Here, both terms are estimated in the same manner. We have,

$$\int_\Omega |\tilde{u}(t)|^2 |u(t)|^{r-1} dx \leq \|\tilde{u}\|_2^2 + \int_{\Omega_t'} |u(t)|^{r-1}\tilde{u}^2(t)dx, \tag{3.1.11}$$

where $\Omega_t' = \{x \in \Omega : \ |u(t)| > 1\}$. The first term on the right hand side of (3.1.11) was estimated in (3.1.9). As for the second term, we consider two cases.

<u>**Case 1:** $3 < r < 5$</u>**.** Given that $|u(t)| > 1$ on Ω_t', there exists $\epsilon_0 > 0$ (say, $\epsilon_0 = \frac{1}{2}(5-r)$), such that $|u(t)|^{r-1} \leq |u(t)|^{4-\epsilon_0}$ on Ω_t'. Now, choose $0 < \epsilon < \epsilon_0/4$. By Hölder's inequality with $\frac{3}{2(1-\epsilon)}$ and $\frac{3}{1+2\epsilon}$, we obtain

$$\begin{aligned}\int_{\Omega_t'} |u(t)|^{r-1}\tilde{u}^2(t)dx &\leq \int_{\Omega_t'} |u(t)|^{4-\epsilon_0}\tilde{u}^2(t)dx \\ &\leq \|u(t)\|^{(4-\epsilon_0)}_{(4-\epsilon_0)\frac{3}{2(1-\epsilon)}} \|\tilde{u}(t)\|^2_{\frac{6}{1+2\epsilon}}.\end{aligned} \tag{3.1.12}$$

Now, notice that because $\epsilon < \epsilon_0/4$, we find that $\frac{3(4-\epsilon_0)}{2(1-\epsilon)} < 6$. Also, because both $\tilde{u}(t)$ and $u(t)$ are in $\mathrm{H}_0^1(\Omega)$, we have,

$$\int_{\Omega_t'} |u(t)|^{r-1}\tilde{u}^2(t)dx \leq C\|\nabla u(t)\|_2^{(4-\epsilon_0)} \|\tilde{u}(t)\|^2_{\mathrm{H}^{1-\epsilon}(\Omega)}, \tag{3.1.13}$$

where we have used the imbeddings $\mathrm{H}_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and $\mathrm{H}^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$ for $0 < \epsilon < 1$. It follows from a standard interpolation inequality combined with Young's inequality that

$$\|w\|^2_{\mathrm{H}^{1-\epsilon}(\Omega)} \leq \epsilon\|\nabla w\|_2^2 + C_\epsilon\|w\|_2^2, \tag{3.1.14}$$

for all $w \in \mathrm{H}^{1-\epsilon}(\Omega)$ and any $0 < \epsilon < 1$. Recalling that $\|\nabla u(t)\|_2 \leq R$ for all $t \in [0, T]$, we have

$$\int_{\Omega_t'} |u(t)|^{r-1}\tilde{u}^2(t)dx \le C_R(\epsilon\|\nabla\tilde{u}(t)\|_2^2 + C_\epsilon\|\tilde{u}(t)\|_2^2)$$
$$\le C_R(\epsilon\tilde{\mathscr{E}}(t) + C_\epsilon\|\tilde{u}(t)\|_2^2). \quad (3.1.15)$$

Finally, applying the estimate (3.1.9) to the second term of (3.1.15) yields

$$\int_{\Omega_t'} |u(t)|^{r-1}\tilde{u}^2(t)dx \le C_R\Big(\epsilon\tilde{\mathscr{E}}(t) + 2C_\epsilon T\int_0^t \tilde{\mathscr{E}}(s)ds\Big). \quad (3.1.16)$$

Case 2: $r \ge 5$**.** Here, recall the additional assumption that $u_0 \in H_0^1(\Omega)\cap L^k(\Omega)$, where $k = \frac{3}{2}(r-1)$. Because $C_0^\infty(\Omega)$ dense in $L^k(\Omega)$, there exists a function $\phi \in C_0^\infty(\Omega)$ such that

$$\|u_0 - \phi\|_k \le \eta^{\frac{1}{r-1}}, \quad (3.1.17)$$

where $\eta > 0$ will be chosen below. Now,

$$\int_\Omega |u(t)|^{r-1}\tilde{u}^2(t)dx \le C\Big(\int_\Omega |u(t)-u_0|^{r-1}\tilde{u}^2(t)dx + \int_\Omega |u_0-\phi|^{r-1}\tilde{u}^2(t)dx$$
$$+ \int_\Omega |\phi|^{r-1}\tilde{u}^2(t)dx\Big). \quad (3.1.18)$$

Hölder's inequality and the Sobolev Imbedding Theorem yield

$$\int_\Omega |u(t)|^{r-1}\tilde{u}^2(t)dx \le C\Big(\|u(t)-u_0\|_k^{r-1}\|\tilde{u}(t)\|_6^2$$
$$+ \|u_0-\phi\|_k^{r-1}\|\tilde{u}(t)\|_6^2 + \int_\Omega |\phi|^{r-1}\tilde{u}^2(t)dx\Big)$$
$$\le C\Big(\|u(t)-u_0\|_k^{r-1} + \|u_0-\phi\|_k^{r-1}\Big)\tilde{\mathscr{E}}(t) + \int_\Omega |\phi|^{r-1}\tilde{u}^2(t)dx. \quad (3.1.19)$$

Consider the first term on the right-hand side of (3.1.19). The regularity of u, namely, $u \in C([0,T], \mathrm{H}_0^1(\Omega))$, $u_t \in C([0,T], L^2(\Omega)) \cap L^p(0,T,\mathrm{W}_0^{1,p}(\Omega))$, implies that $\nabla u, \nabla u_t \in L^2(0,T,L^2(\Omega))^3$. This allows us to write

$$\nabla(u(t) - u_0) = \int_0^t \nabla u_t(s)ds, \text{ a.e. } \Omega \times [0,T]. \tag{3.1.20}$$

Also, recall the definition of q and q' in (1.1.2). We slightly modify the choice of q' as follows. Choose q' sufficiently large so that $k \le q'$ when $p \ge 3$. For the values of $p \in [2,3)$, note the restriction $r < 8 - \frac{6}{p}$ automatically implies that $k \le q'$. Therefore, with this choice of q', and by using (1.1.3), (3.1.20), we obtain

$$\begin{aligned} \|u(t) - u_0\|_k^p \le C\|u(t) - u_0\|_{q'}^p \le C\|\nabla(u(t) - u_0)\|_p^p = C\int_\Omega |\nabla(u(t) - u_0)|^p \, dx \\ = C\int_\Omega \left|\int_0^t \nabla u_t(s)ds\right|^p dx \le C\int_\Omega \int_0^t |\nabla u_t(s)|^p \, t^{p-1} ds dx. \end{aligned} \tag{3.1.21}$$

Given $\int_0^T \|\nabla u_t(s)\|_p^p ds \le R$, we have,

$$\|u(t) - u_0\|_k^{r-1} \le C t^{\frac{p-1}{p}(r-1)} R^{\frac{r-1}{p}} \le T^{\frac{r-1}{p'}} C_R, \tag{3.1.22}$$

for all $t \in [0,T]$. Because $\phi \in C_0^\infty(\Omega)$, the third term on the right-hand side of (3.1.19) is estimated by

$$\int_\Omega |\phi|^{r-1} \tilde{u}^2(t)dx \le C_\eta \int_\Omega \tilde{u}^2(t)dx \le TC_\eta \int_0^t \tilde{\mathscr{E}}(s)ds, \tag{3.1.23}$$

where we have used (3.1.9). Recalling that $\|u_0 - \phi\|_k^{r-1} < \eta$, it follows from (3.1.19), (3.1.22), and (3.1.23) that

$$\int_\Omega |u(t)|^{r-1}\tilde{u}^2(t)dx \le \big(T^{\frac{r-1}{p'}}C_R + C\eta)\tilde{\mathscr{E}}(t) + TC_\eta \int_0^t \tilde{\mathscr{E}}(s)ds, \tag{3.1.24}$$

where $\eta > 0$ is arbitrary. A similar estimate holds for $\int_\Omega |v(t)|^{r-1}\tilde{u}^2(t)dx$. Combining Cases 1 and 2 with (3.1.9), we obtain,

$$\begin{aligned}\int_\Omega |\tilde{u}(t)|^2(|u(t)|^{r-1} + |v(t)|^{r-1} + 1)dx \le (\epsilon C_R + T^{\frac{r-1}{p'}}C_R + C\eta)\tilde{\mathscr{E}}(t)\\ + TC_{\epsilon,\eta,R}\int_0^t \tilde{\mathscr{E}}(s)ds.\end{aligned} \tag{3.1.25}$$

4. Estimate for $\int_0^t\int_\Omega(|u|^{r-2} + |v|^{r-2})(|u_t| + |v_t|)|\tilde{u}|^2dxds$**:** Here it is enough to consider only one term because the other terms are estimated in the same manner. Using Hölder's inequality with 3 and 3/2, and then again with $\frac{4}{r-2}$ and $\frac{4}{6-r}$, we obtain

$$\begin{aligned}\int_0^t\int_\Omega |u|^{r-2}|u_t||\tilde{u}|^2dxds &\le \int_0^t \|\tilde{u}(s)\|_6^2\Big(\int_\Omega |u(s)|^{\frac{3(r-2)}{2}}|u_t(s)|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}}ds\\ &\le \int_0^t \|\tilde{u}(s)\|_6^2\|u(s)\|_6^{r-2}\|u_t(s)\|_{\frac{6}{6-r}}ds.\end{aligned} \tag{3.1.26}$$

Now notice that for the values of $p \in [2,3)$, our restriction $r < 8 - \frac{6}{p}$ is equivalent to $\frac{6}{6-r} < \frac{3p}{3-p}$. Then, by the Sobolev imbeddings and using (1.1.3), it follows from (3.1.26) that

$$\begin{aligned}\int_0^t\int_\Omega |u|^{r-2}|u_t||\tilde{u}|^2dxds &\le C\int_0^t \|\nabla\tilde{u}(s)\|_2^2\|\nabla u(s)\|_2^{r-2}\|\nabla u_t(s)\|_p ds\\ &\le C_R\int_0^t \tilde{\mathscr{E}}(s)\|\nabla u_t(s)\|_p ds,\end{aligned} \tag{3.1.27}$$

where we have used the fact that $\|\nabla u(s)\|_2 \le R$ for each $0 \le s \le T$. Accounting for the remaining terms in this step, we have

$$\int_0^t \int_\Omega \Big(|u|^{r-2} + |v|^{r-2}\Big)\Big(|u_t| + |v_t|\Big)|\tilde{u}|^2 dxds$$
$$\leq C_R \int_0^t \tilde{\mathscr{E}}(s)\Big(\|\nabla u_t(s)\|_p + \|\nabla v_t(s)\|_p\Big)ds. \tag{3.1.28}$$

Combining (3.1.5), (3.1.9)-(3.1.10), (3.1.25), and (3.1.28), we have

$$|\int_0^t \int_\Omega (f(u) - f(v))\tilde{u}_t dxds| \leq \Big(\epsilon C_R + T^{\frac{1}{p'}(r-1)} C_R + C\eta\Big)\tilde{\mathscr{E}}(t)$$
$$+ C_{T,\epsilon,\eta,R} \int_0^t \tilde{\mathscr{E}}(s)\Big(\|\nabla u_t(s)\|_p + \|\nabla v_t(s)\|_p + 1\Big)ds. \tag{3.1.29}$$

Hence, it follows from (3.1.4) and (3.1.29) that

$$\tilde{\mathscr{E}}(t) \leq \Big(\epsilon C_R + T^{\frac{1}{p'}(r-1)} C_R + C\eta\Big)\tilde{\mathscr{E}}(t)$$
$$+ C_{T,\epsilon,\eta,R} \int_0^t \tilde{\mathscr{E}}(s)\Big(\|\nabla u_t(s)\|_p + \|\nabla v_t(s)\|_p + 1\Big)ds. \tag{3.1.30}$$

We can choose $\epsilon > 0, \eta > 0$ and T sufficiently small [1] so that $\epsilon_0 := \epsilon C_R + T^{\frac{1}{p'}(r-1)} C_R + C\eta < 1$. Then (3.1.30) implies

$$\tilde{\mathscr{E}}(t) \leq C_{T,\epsilon,\eta,R} \int_0^t \tilde{\mathscr{E}}(s)\Big(\|\nabla u_t(s)\|_p + \|\nabla v_t(s)\|_p + 1\Big)ds. \tag{3.1.31}$$

Because $(\|\nabla u_t(s)\|_p + \|\nabla v_t(s)\|_p + 1) \in L^1[0,T]$, Gronwall's inequality yields

$$\tilde{\mathscr{E}}(t) = 0, \text{ for } t \in [0,T], \tag{3.1.32}$$

which completes the proof of the uniqueness statement in Theorem 1.2.1. □

[1] We can choose a small T, since the argument can reiterated to cover the whole local existence interval.

Chapter 4

Global Existence

This chapter is devoted to the proofs of Theorem 1.2.2, Corollary 1.2.3 and Theorem 1.2.4.

4.1 Damping Exponent Dominates Source

In this section we prove the global existence of solutions to (1.0.1), as well as the continuous dependence of solutions on initial data.

4.1.1 Proof of First Global Existence Result

Here, we use a standard continuation argument to conclude that u, the weak solution of (1.0.1), is a global solution or else, for some $0 < T < \infty$, one has

$$\limsup_{t \to T^-} \; E_1(t) = +\infty, \tag{4.1.1}$$

where $\mathscr{E}(t) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2) + \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1}$. Our goal is to show the latter cannot happen under the assumptions of Theorem 1.2.2. Indeed, this assertion is contained in the following proposition.

Proposition 4.1.1. *Let u be a weak solution to (1.0.1) on $[0,T]$ as furnished by Theorem 1.2.1 and assume that $u_0 \in L^{r+1}(\Omega)$ whenever $r > 5$. We have*

- *if $r \leq p-1$, then for all $t \in [0,T]$, u satisfies:*

$$\mathscr{E}(t) + \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1} + \int_0^t \|\nabla u_t\|_p^p ds \leq C_T(\|u_0\|_{H_0^1(\Omega)\cap L^{r+1}(\Omega)}, \|u_1\|_{L^2(\Omega)}), \tag{4.1.2}$$

  where $T > 0$ is arbitrary.

- *If $r > p-1$, then the bound in (4.1.2) holds for $0 \leq t \leq T < T_0$, for some $T_0 > 0$ where T_0 may be finite and depends on $\|u_0\|_{H_0^1(\Omega)\cap L^{r+1}(\Omega)}$ and $\|u_1\|_{L^2(\Omega)}$.*

Proof. As in [7, 10], we introduce the modified energy

$$E_1(t) := \mathscr{E}(t) + \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1}, \tag{4.1.3}$$

where $\mathscr{E}(t) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2)$. Recalling the energy identity (1.2.1) we note that $E_1(t)$ satisfies

$$E_1(t) + \int_0^t \|\nabla u_t\|_p^p ds = E_1(0) + \int_0^t \int_\Omega f(u) u_t dx ds + \int_0^t \int_\Omega |u|^{r-1} u u_t dx ds. \tag{4.1.4}$$

First, we note that

$$\left|\int_0^t \int_\Omega f(u) u_t dx ds\right| \leq C\Big(|Q_t| + \int_0^t E_1(s) ds + \int_0^t \int_\Omega |u|^r |u_t| dx ds\Big) \tag{4.1.5}$$

where $Q_t = \Omega \times (0,t)$ and $|Q_t|$ denotes its Lebesgue measure. To see this, put $Q_t' := \{(x,s) \in \Omega \times (0,t) : |u(x,s)| \leq 1\}$ and $Q_t'' := \{(x,s) \in \Omega \times (0,t) : |u(x,s)| > 1\}$.

Because f is continuous, $f(u)$ is bounded on Q'_t for each $0 \leq t \leq T$. Consequently, after using Young's inequality, we obtain

$$\begin{aligned}
|\int_0^t \int_\Omega f(u)u_t dxds| &\leq \int_{Q'_t} |f(u)u_t| dxds + \int_{Q''_t} |f(u)u_t| dxds \\
&\leq C\Big(\int_{Q'_t} |u_t| dxds + \int_{Q''_t} |u|^r |u_t| dxds \Big) \\
&\leq C\Big(\int_{Q'_t} |u_t|^2 dxds + |Q'_t| + \int_{Q''_t} |u|^r |u_t| dxds \Big) \\
&\leq C\Big(\int_0^t E_1(s)ds + |Q_t| + \int_0^t \int_\Omega |u|^r |u_t| dxds \Big), \qquad (4.1.6)
\end{aligned}$$

for all $t \in [0,T]$. We now estimate $\int_0^t \int_\Omega |u|^r |u_t| dxds$. Recalling the definition of q in (1.1.2), it is easy to check that under Assumption 1.1.1 we have $qr \leq r+1$. By Hölder's inequality with q and q' followed by the Sobolev imbeddings in (1.1.3), we obtain

$$\begin{aligned}
\int_0^t \int_\Omega |u|^r |u_t| dxds \leq \int_0^t \|u_t(s)\|_{q'} \|u(s)\|_{qr}^r ds &\leq C \int_0^t \|\nabla u_t(s)\|_p \|u(s)\|_{qr}^r ds \\
&\leq C \int_0^t \|\nabla u_t(s)\|_p \|u(s)\|_{r+1}^r ds. \quad (4.1.7)
\end{aligned}$$

By Young's inequality we have

$$\int_0^t \int_\Omega |u|^r |u_t| dxds \leq \epsilon \int_0^t \|\nabla u_t(s)\|_p^p ds + C_\epsilon \int_0^t \|u(s)\|_{r+1}^{rp'} ds. \qquad (4.1.8)$$

If $r = p-1$, then $rp' = r+1$ and we have the estimate in (4.1.9) below. Otherwise, $r < p-1$ and, in this case, it is clear that $\frac{r+1}{rp'} > 1$. Thus, we may apply Young's inequality to the integrand of the second term in (4.1.8) with $\frac{r+1}{rp'}$ and $\frac{(p-1)(r+1)}{p-r-1}$ to get

$$\int_0^t \int_\Omega |u|^r |u_t| dx ds \le \epsilon \int_0^t \|\nabla u_t(s)\|_p^p ds + C_1 \int_0^t \|u(s)\|_{r+1}^{r+1} ds + C_1'$$
$$\le \epsilon \int_0^t \|\nabla u_t(s)\|_p^p ds + C_1 \int_0^t E_1(s) ds + C_1', \tag{4.1.9}$$

where C_1' depends on t. Now, returning to (4.1.4) and using (4.1.5) and (4.1.9) we conclude that

$$E_1(t) + \int_0^t \|\nabla u_t\|_p^p ds \le C_2 + C_0 \int_0^t E_1(s) ds + \epsilon C \int_0^t \|\nabla u_t(s)\|_p^p dx ds, \tag{4.1.10}$$

where $C_2 = C(t, |\Omega|, \epsilon, r, E_1(0))$. Choosing $\epsilon > 0$ sufficiently small, (4.1.10) implies

$$E_1(t) + \frac{1}{2} \int_0^t \|\nabla u_t\|_p^p ds \le C_2 + C_0 \int_0^t E_1(s) ds, \tag{4.1.11}$$

and by Gronwall's inequality, we obtain

$$E_1(t) \le C_2 e^{C_0 T}, \tag{4.1.12}$$

for all $0 \le t \le T$, and (4.1.2) follows from (4.1.11).

When $r > p - 1$, we use (4.1.7). Because $qr < 6$, by assumption (when $p = 3$ we may choose $\delta > 0$ sufficiently small so that $qr = (1 + \delta)r < 6$), we have

$$\int_0^t \int_\Omega |u|^r |u_t| dx ds \le \int_0^t \|u_t(s)\|_{q'} \|u(s)\|_{qr}^r ds$$
$$\le C \int_0^t \|\nabla u_t(s)\|_p \|\nabla u(s)\|_2^r ds. \tag{4.1.13}$$

Therefore, instead of (4.1.9), we obtain

$$\begin{aligned}\int_0^t\int_\Omega |u|^r|u_t|dxds &\le \epsilon\int_0^t \|\nabla u_t(s)\|_p^p dxds + C_\epsilon\int_0^t \|\nabla u(s)\|_2^{rp'}dxds\\ &\le \epsilon\int_0^t \|\nabla u_t(s)\|_p^p dxds + C_\epsilon\int_0^t E_1(s)^{\frac{rp'}{2}}dxds. \qquad (4.1.14)\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small and combining (4.1.4), (4.1.5), (4.1.14), we have

$$E_1(t) + \frac{1}{2}\int_0^t \|\nabla u_t\|_p^p ds \le C_2 + C'\int_0^t \left(E_1(s) + E_1(s)^{\frac{rp'}{2}}\right)ds, \qquad (4.1.15)$$

where $C_2 = C(T, |\Omega|, \epsilon, r, E_1(0))$. Now, Put

$$Y(t) = 1 + E_1(t),$$

and notice that $\frac{rp'}{2} > 1$, since $r > p-1$ and $p \ge 2$. Therefore, it follows from (4.1.15) that,

$$Y(t) + \frac{1}{2}\int_0^t \|\nabla u_t\|_p^p ds \le C_3 + 2C'\int_0^t Y(s)^{\frac{rp'}{2}}ds. \qquad (4.1.16)$$

In particular,

$$Y(t) \le C_3 + 2C'\int_0^t Y(s)^\sigma ds, \qquad (4.1.17)$$

where $\sigma := \frac{rp'}{2} > 1$. By using a standard comparison theorem (e.g., [18]), then (4.1.17) yields that $Y(t) \le z(t)$, where $z(t) = [C_3^{1-\sigma} - 2C'(\sigma-1)t]^{-\frac{1}{\sigma-1}}$ is the solution of the Volterra integral equation

$$z(t) = C_3 + 2C'\int_0^t z(s)^\sigma ds. \qquad (4.1.18)$$

Although $z(t)$ blows up in a finite time $T_0 > 0$ (since $\sigma > 1$), nonetheless, whenever $0 < T < T_0$, then $Y(t) \leq z(t) \leq C_T(\|u_0\|_{H_0^1(\Omega)\cap L^{r+1}(\Omega)}, \|u_1\|_{L^2(\Omega)})$ for all $t \in [0,T]$. Hence, the proof of the proposition is complete. □

4.1.2 Continuous Dependence on Initial Data

We now provide the proof of Corollary 1.2.3, which follows from the energy identity, uniqueness of solutions and the bounds given in Proposition 4.1.1.

Proof. As in the proof of uniqueness, let $(u_0, u_1) \in \mathrm{H}_0^1(\Omega) \times L^2(\Omega)$, when $1 \leq r \leq 5$, or $(u_0, u_1) \in \mathrm{H}_0^1(\Omega) \cap L^k(\Omega) \times L^2(\Omega)$, when $r > 5$, where $k = \frac{3}{2}(r-1)$. Let $\{(u_0^n, u_1^n)\} \subset \mathrm{H}_0^1(\Omega) \times L^2(\Omega)$ be a sequence of initial data such that, as $n \to \infty$,

$$\begin{cases} (u_0^n, u_1^n) \longrightarrow (u_0, u_1) \text{ in } \mathrm{H}_0^1(\Omega) \times L^2(\Omega), \text{ if } 1 \leq r \leq 5, \\ (u_0^n, u_1^n) \longrightarrow (u_0, u_1) \text{ in } \mathrm{H}_0^1(\Omega) \cap L^k(\Omega) \times L^2(\Omega), \text{ if } r > 5. \end{cases} \tag{4.1.19}$$

Let $\{u^n\}$ and u be the unique solutions to (1.0.1), in the sense of Definition 1.1.4, corresponding to the initial data $\{(u_0^n, u_1^n)\}$ and $\{(u_0, u_1)\}$, respectively. We first point out that, when $r > 5$, our assumption (4.1.19) implies that $u_0^n, u_0 \in L^{r+1}(\Omega)$, which is required in Proposition 4.1.1. Therefore, in view of Proposition 4.1.1 and (4.1.19), there exists a constant $R > 0$ such that, for all $n \in \mathbb{N}$

$$\begin{cases} \|\nabla u(t)\|_2, \ \|\nabla u^n(t)\|_2, \ \|u_t(t)\|_2, \ \|u_t^n(t)\|_2 \leq R, \\ \int_0^T \|\nabla u_t(s)\|_p^p dxdt, \ \int_0^T \|\nabla u_t^n(s)\|_p^p dt \leq R, \end{cases} \tag{4.1.20}$$

for all $t \in [0,T]$, where $T > 0$ can be chosen arbitrarily large if $r \leq p-1$ and is chosen sufficiently small, if $r > p-1$.

As in the proof of uniqueness of solutions, put $\tilde{u}(t) := u(t) - u^n(t)$ (suppressing the dependence on n and with v is being replaced by u^n). Then, as in (3.1.3), accounting for the now non-zero initial data, $\tilde{u}$ satisfies:

$$\begin{aligned} \tilde{\mathscr{E}}_n(t) &+ \int_0^t \int_\Omega (|\nabla u_t|^{p-2} \nabla u_t - |\nabla u_t^n|^{p-2} \nabla u_t^n) \cdot \nabla \tilde{u} dx ds \\ &= \tilde{\mathscr{E}}_n(0) + \int_0^t \int_\Omega (f(u) - f(u^n)) \tilde{u}_t dx ds, \end{aligned} \tag{4.1.21}$$

for all $t \in [0,T]$, where $\tilde{\mathscr{E}}_n(t) := \frac{1}{2}\Big(\|\tilde{u}_t(t)\|_2^2 + \|\nabla \tilde{u}(t)\|_2^2\Big)$. Our goal here is to prove that $\tilde{\mathscr{E}}_n(t) \longrightarrow 0$ uniformly on $[0,T]$, where $T > 0$ is arbitrarily large (but fixed), if $r \leq p-1$, or $T > 0$ is chosen sufficiently small if $r > p-1$.

Now, similar to (3.1.4), we have

$$\tilde{\mathscr{E}}_n(t) \leq \tilde{\mathscr{E}}_n(0) + \int_0^t \int_\Omega (f(u) - f(u^n)) \tilde{u}_t dx ds, \text{ for all } t \in [0,T]. \tag{4.1.22}$$

We employ the same calculations as in the proof of uniqueness, but account for the extra terms contributed by the non-zero initial data. First, integration by parts in (3.1.7)-(3.1.8) yields two extra terms

$$\left| \int_\Omega (f(u_0) - f(u_0^n)) \tilde{u}(0) dx \right| + \frac{1}{2} \left| \int_\Omega f'(u_0) \tilde{u}^2(0) dx \right|, \tag{4.1.23}$$

which must be added to the right hand side of (3.1.8). Using the properties of (3.1.6), we have

$$\begin{aligned} \left| \int_\Omega (f(u_0) - f(u_0^n)) \tilde{u}(0) dx \right| &+ \frac{1}{2} \left| \int_\Omega f'(u_0) \tilde{u}^2(0) dx \right| \\ &\leq C \int_\Omega |\tilde{u}(0)|^2 \Big(|u_0|^{r-1} + |u_0^n|^{r-1} + 1\Big) dx. \end{aligned} \tag{4.1.24}$$

Each term on the right-hand side of (4.1.24) is estimated in the manner that follows.

$$\int_\Omega |\tilde{u}(0)|^2 |u_0^n|^{r-1} dx \le \|\tilde{u}(0)\|_6^2 \|u_0^n\|_k^{r-1} \le C_R \tilde{\mathscr{E}}_n(0), \tag{4.1.25}$$

where we have used (4.1.19)-(4.1.20).

The non-zero initial data, $\tilde{u}(0) \neq 0$, also changes the estimates in (3.1.9). We find that

$$\begin{aligned}
\int_\Omega |\tilde{u}(t)|^2 dx &= \int_\Omega |\tilde{u}(0) + \int_0^t \tilde{u}_t(s) ds|^2 dx \\
&\le C\Big(\|\tilde{u}(0)\|_2^2 + \int_\Omega \int_0^t |\tilde{u}_t(s)|^2 ds \cdot t dx\Big) \\
&\le C\Big(\|\tilde{u}(0)\|_2^2 + T\int_0^t \tilde{\mathscr{E}}_n(s) ds\Big) \\
&\le C\Big(\tilde{\mathscr{E}}_n(0) + T\int_0^t \tilde{\mathscr{E}}_n(s) ds\Big).
\end{aligned} \tag{4.1.26}$$

Now, for the case $1 \le r < 5$, we can simply perform the estimates in (3.1.11) through (3.1.16). Accounting for (3.1.28) and the additional terms due to the non-zero initial data, we find

$$\begin{aligned}
\tilde{\mathscr{E}}_n(t) &\le C_R \tilde{\mathscr{E}}_n(0) + \epsilon C_R \tilde{\mathscr{E}}_n(t) \\
&\quad + C_{R,T,\epsilon} \int_0^t \Big(\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1\Big) \tilde{\mathscr{E}}_n(s) ds,
\end{aligned} \tag{4.1.27}$$

for all $t \in [0,T]$. By choosing $\epsilon = \frac{1}{2C_R}$ and applying Gronwall's inequality we conclude that

$$\tilde{\mathscr{E}}_n(t) \le C_{R,T,\epsilon} \tilde{\mathscr{E}}_n(0) \exp\Big(\int_0^t (\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1) ds\Big), \tag{4.1.28}$$

for all $t \in [0,T]$. In light of (4.1.20), $\int_0^t (\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1) ds \le C_{T,R}$, for all

$n \in \mathbb{N}$ and all $t \in [0,T]$. Because $\tilde{\mathscr{E}}_n(0) \to 0$, $\tilde{\mathscr{E}}_n(t) \to 0$ as $n \to \infty$ for all $t \in [0,T]$. This establishes the continuous dependence of solutions on initial data in the case $1 \le r < 5$ without a restriction on T.

For the case $r \ge 5$, we must exercise caution in estimating the term $\int_\Omega (|u(t)|^{r-1} + |u^n(t)|^{r-1})\tilde{u}(t)^2 dx$. The bound in (3.1.24) is still valid, but care should be taken in estimating the term $\int_\Omega |u^n(t)|^{r-1}\tilde{u}(t)^2 dx$. Recall the choice of $\phi \in C_0^\infty(\Omega)$ such that,

$$\|u_0 - \phi\|_k \le \eta^{\frac{1}{r-1}}, \tag{4.1.29}$$

where $\eta > 0$ is arbitrary. Indeed, as in (3.1.18)-(3.1.19), we have

$$\begin{aligned}
\int_\Omega |u^n(t)|^{r-1}\tilde{u}^2(t)dx &\le C\Big(\int_\Omega |u^n(t) - u_0^n|^{r-1}\tilde{u}^2(t)dx \\
&+ \int_\Omega |u_0^n - u_0|^{r-1}\tilde{u}^2(t)dx + \int_\Omega |u_0 - \phi|^{r-1}\tilde{u}^2(t)dx + \int_\Omega |\phi|^{r-1}\tilde{u}^2(t)dx\Big) \\
&\le C\Big(\|u^n(t) - u_0^n\|_k^{r-1} + \|u_0^n - u_0\|_k^{r-1}\| + \|u_0 - \phi\|_k^{r-1}\Big)\tilde{\mathscr{E}}_n(t) \\
&+ \int_\Omega |\phi|^{r-1}\tilde{u}^2(t)dx.
\end{aligned} \tag{4.1.30}$$

As in (3.1.20)-(3.1.22), we obtain

$$\|u^n(t) - u_0^n\|_k^{r-1} \le T^{\frac{r-1}{p'}} C_R, \qquad t \in [0,T]. \tag{4.1.31}$$

Also, because $u_0^n \to u_0$ in $L^k(\Omega)$ by assumption,

$$\|u_0^n - u_0\|_k^{r-1} \le \eta, \tag{4.1.32}$$

for all sufficiently large n. Therefore, it follows from (4.1.30)-(4.1.32) and (4.1.26) that

$$\int_\Omega |u(t)|^{r-1}\tilde{u}^2(t)dx \le \Big(T^{\frac{r-1}{p'}}C_R + C\eta\Big)\tilde{\mathscr{E}}_n(t) + C_\eta\Big(\tilde{\mathscr{E}}_n(0) + T\int_0^t \tilde{\mathscr{E}}_n(s)ds\Big). \quad (4.1.33)$$

The remaining estimates in the proof of uniqueness remain valid, and one finally obtains

$$\begin{aligned}\tilde{\mathscr{E}}_n(t) &\le C_{R,\eta}\tilde{\mathscr{E}}_n(0) + \Big(\epsilon C_R + C_R T^{\frac{r-1}{p'}} + C\eta\Big)\tilde{\mathscr{E}}_n(t)\\ &\quad + C_{T,\epsilon,\eta,R}\int_0^t \tilde{\mathscr{E}}_n(s)\Big(\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1\Big)ds. \end{aligned} \quad (4.1.34)$$

Again, we can choose $\epsilon > 0, \eta > 0$ and T sufficiently small so that $\epsilon_0 := \epsilon C_R + T^{\frac{p-1}{p}(r-1)}C_R + C\eta < 1$. Then (4.1.34) implies

$$\tilde{\mathscr{E}}_n(t) \le C_{R,\eta}\tilde{\mathscr{E}}_n(0) + C_{T,\epsilon,\eta,R}\int_0^t \tilde{\mathscr{E}}_n(s)\Big(\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1\Big)ds. \quad (4.1.35)$$

By Gronwall's inequality we conclude that

$$\tilde{\mathscr{E}}_n(t) \le C_{T,\epsilon,\eta,R}\tilde{\mathscr{E}}_n(0)\exp\Big(\int_0^t (\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1)ds\Big). \quad (4.1.36)$$

Since $\int_0^t (\|\nabla u_t^n(s)\|_p + \|\nabla u_t(s)\|_p + 1)ds \le C_{T,R}$, for all $n \in \mathbb{N}$, then $\tilde{\mathscr{E}}_n(t) \to 0$ as $n \to \infty$, for all $t \in [0,T]$, where $T > 0$ is sufficiently small. □

4.2 Global Existence via the Potential Well

In this section, we provide the global existence of solutions to (1.0.1) under different hypotheses than those considered in Theorem 1.2.2. We again use a standard continuation argument on the weak solution u of (1.0.1) and show that it is not possible for

$$\limsup_{t\to T^-} E_1(t) = +\infty, \tag{4.2.1}$$

to occur. However, in this section we will find a bound for the energy that is uniform in t by examining solutions in the "good" part of the potential well.

Following the method of [3] and [11] we proceed in three steps. We begin by recalling the functional J, where

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{r+1}\|u\|_{r+1}^{r+1} \tag{4.2.2}$$

and the potential well

$$\mathcal{W} := \{u \in \mathrm{H}_0^1(\Omega) : J(u) < d\},$$

which is divided into the two sets

$$\mathcal{W}_1 := \{u \in \mathcal{W} : \|\nabla u\|_2^2 > \|u\|_{r+1}^{r+1}\} \cup \{0\}$$
$$\mathcal{W}_2 := \{u \in \mathcal{W} : \|\nabla u\|_2^2 < \|u\|_{r+1}^{r+1}\}.$$

Step 1: $\mathcal{W}_1$ is invariant with respect to (1.0.1). Recall the energy identity given by Theorem 1.2.1,

$$E(t) + \int_0^t \|\nabla u_t(s)\|_p^p \, ds = E(0). \tag{4.2.3}$$

Differentiation with respect to t implies

$$E'(t) \leq 0. \tag{4.2.4}$$

Consequently

$$E(t) \leq E(0) < d, \quad \forall t \in [0, T). \tag{4.2.5}$$

As a result, $J(u(t)) < d$ for all $t \in [0, T)$. Hence $u(t) \in \mathcal{W}$ for all $t \in [0, T)$ since $J(u(t)) \leq E(t)$.

To show that $u(t) \in \mathcal{W}_1$ on $[0, T)$ we proceed by contradiction. Assume there exists $t_0 \in [0, T)$ such that $u(t_0) \notin \mathcal{W}_1$. Since $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, $u(t_0) \in \mathcal{W}_2$ and thus $\|\nabla u(t_0)\|_2^2 < \|u(t_0)\|_{r+1}^{r+1}$. Because $u \in C([0, T], \mathrm{H}_0^1(\Omega))$ and $\mathrm{H}_0^1(\Omega) \hookrightarrow L^{r+1}(\Omega)$ (recall $r + 1 \leq 6$), $\|\nabla u(t)\|_2^2 - \|u(t)\|_{r+1}^{r+1}$ is continuous. Since $\|\nabla u(0)\|_2^2 - \|u(0)\|_{r+1}^{r+1} > 0$, as $u_0 \in \mathcal{W}_1$, and $\|\nabla u(t_0)\|_2^2 - \|u(t_0)\|_{r+1}^{r+1} < 0$, it follows that there exists $s \in (0, t_0)$ such that $\|\nabla u(s)\|_2^2 = \|u(s)\|_{r+1}^{r+1}$. Now, we define

$$t^* = \sup\{s \in (0, t_0) : \|\nabla u(s)\|_2^2 = \|u(s)\|_{r+1}^{r+1}\}. \tag{4.2.6}$$

In particular, $\|\nabla u(t^*)\|_2^2 = \|u(t^*)\|_{r+1}^{r+1}$, and $u(t) \in \mathcal{W}_2$ for all $t^* < t \leq t_0$.

We consider two cases:

Case 1: Suppose that $\|\nabla u(t^*)\|_2^2 \neq 0$. Then $u(t^*) \in \mathcal{N}$, the Nehari Manifold (see (1.1.10)). From (1.1.20) we know that d is the infimum of J over all functions u in the Nehari Manifold and thus we have that $J(u(t^*)) \geq d$. Clearly, $E(t^*) = \frac{1}{2}\|u_t(t^*)\|_2^2 + J(u(t^*)) \geq d$, contradicting (4.2.5).

Case 2: Suppose that $\|\nabla u(t^*)\|_2^2 = 0$. Since $u(t) \in \mathcal{W}_2$ for all $t^* < t \le t_0$,

$$\|\nabla u(t)\|_2^2 < \|u(t)\|_{r+1}^{r+1}, \qquad \text{for all } t^* < t \le t_0. \tag{4.2.7}$$

By the regularity of u, we have

$$\lim_{t \to t^{*+}} \|\nabla u(t)\|_2^2 = 0. \tag{4.2.8}$$

Applying the Sobolev Imbedding to (4.2.7) gives

$$\|\nabla u(t)\|_2^2 < \|u(t)\|_{r+1}^{r+1} \le C\|\nabla u(t)\|_2^{r+1}, \qquad \forall t^* < t \le t_0.$$

Therefore,

$$\|\nabla u(t)\|_2^2\big(1 - C\|\nabla u(t)\|_2^{r-1}\big) < 0, \qquad \forall t^* < t \le t_0, \tag{4.2.9}$$

and thus

$$\|\nabla u(t)\|_2 > C^{\frac{1}{r-1}}, \qquad \forall t^* < t \le t_0,$$

contradicting (4.2.8). Thus, $u(t) \in \mathcal{W}_1$ for all $t \in [0, T)$ and $\mathcal{W}_1$ is invariant under (1.0.1).

Step 2: $\|\nabla u(t)\|_2$ is controlled by the depth of the well. In particular, we will show that $\|\nabla u(t)\|_2^2 < 2d(\frac{r+1}{r-1})$ for all $t \in [0, T)$. Since $E(t) < d$ and $u(t) \in \mathcal{W}_1$ on $[0, T)$,

$$d > J(u(t)) = \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1} > \frac{r-1}{2(r+1)}\|u(t)\|_{r+1}^{r+1}, \tag{4.2.10}$$

for all $t \in [0,T)$, and thus

$$\|u(t)\|_{r+1}^{r+1} < 2d\Big(\frac{r+1}{r-1}\Big), \qquad \forall t \in [0,T). \tag{4.2.11}$$

Now, because $J(u(t)) < d$ for all $t \in [0,T)$,

$$\frac{1}{2}\|\nabla u(t)\|_2^2 < d + \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1} < d + \frac{2d}{r-1} = d\Big(\frac{r+1}{r-1}\Big), \tag{4.2.12}$$

on $[0,T)$, thus concluding Step 2.

Step 3: The solution is a global solution. Rearranging the terms in the energy identity gives

$$\mathscr{E}(t) + \int_0^t \|\nabla u_t(s)\|_p^p ds = E(0) + \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1}. \tag{4.2.13}$$

Thus, for all $t \in [0,T)$, by (4.2.11) implies

$$\mathscr{E}(t) + \int_0^t \|\nabla u_t(s)\|_p^p ds < d + \frac{2d}{r-1} = d\Big(\frac{r+1}{r-1}\Big)$$

and so

$$\mathscr{E}(t) \le d\Big(\frac{r+1}{r-1}\Big), \qquad \forall t \in [0,T). \tag{4.2.14}$$

Also,

$$E_1(t) := \mathscr{E}(t) + \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1} \le d\Big(\frac{r+1}{r-1}\Big) + \frac{2d}{r-1} = d\Big(\frac{r+3}{r-1}\Big), \tag{4.2.15}$$

for all $t \in [0,T)$. By a standard continuation argument, we conclude that the solution is global.

Chapter 5

Energy Decay

We begin this chapter by establishing the observability-stability estimate, whereby the total energy of the system is controlled by a function of damping as in [3, 11, 21]. This estimate, combined with the energy identity, allows us to derive an inequality that compares the energy at time t with the energy at time 0. From here, we construct an ordinary differential equation, related to this inequality, whose solution will bound the total energy for all sufficiently large t. A significant difficulty arises as this ordinary differential equation is non-autonomous and we cannot find its solution explicitly. This problem is overcome via a careful comparison to a new ordinary differential equation, which is also non-autonomous, however the simplicity of its form does allow us to find the explicit solution, along with its decay rate. From here, we may provide a decay rate for the total energy of the system (1.0.1).

5.1 Observability-Stability Estimate

We start by showing Lemma 5.1.1) that provides the equivalence of the quadratic energy, $\mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2$ and the total energy $E(t) = \mathscr{E}(t) - \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1}$.

Lemma 5.1.1. *Under Assumptions 1.1.1 and 1.1.2, we have*

$$\frac{r-1}{2(r+1)}\|\nabla u(t)\|_2^2 \leq J(u(t)), \tag{5.1.1}$$

$$\frac{r-1}{r+1}\mathscr{E}(t) \leq E(t) \leq \mathscr{E}(t), \tag{5.1.2}$$

and

$$\frac{r-1}{r+1}\mathscr{E}(t) + \int_0^t \|\nabla u_t(s)\|_p^p ds \leq \mathscr{E}(0) \leq \frac{r+1}{r-1}\big(\mathscr{E}(t) + \int_0^t \|\nabla u_t(s)\|_p^p ds\big). \tag{5.1.3}$$

Proof. By Theorem 1.2.4, $u(t) \in \mathcal{W}_1$ for all t and so for each fixed $t \in [0, \infty)$ either

(i) $\|\nabla u(t)\|_2^2 = 0$ or

(ii) $\|\nabla u(t)\|_2^2 > \|u(t)\|_{r+1}^{r+1}$.

If (i) holds, then $0 \leq \|u(t)\|_{r+1}^{r+1} \leq C\|\nabla u(t)\|_2^{r+1} = 0$ and $J(u(t)) = 0$. So trivially, $\frac{r-1}{2(r+1)}\|\nabla u(t)\|_2^2 \leq J(u(t))$. Additionally, $E(t) = \mathscr{E}(t) = \frac{1}{2}\|u_t(t)\|_2^2$ and so (5.1.2) is verified.

If (ii) holds, then

$$\begin{aligned} J(u(t)) &= \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1} \qquad \geq \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{r+1}\|\nabla u(t)\|_2^2 \\ &= \frac{r-1}{2(r+1)}\|\nabla u(t)\|_2^2. \end{aligned}$$

and so (5.1.1) holds. Now note that since $\|\nabla u(t)\|_2^2 > \|u(t)\|_{r+1}^{r+1}$ for each t

$$\begin{aligned} \frac{r-1}{(r+1)}\mathscr{E}(t) &= \mathscr{E}(t) - \frac{2}{r+1}\mathscr{E}(t) \leq \mathscr{E}(t) - \frac{1}{r+1}\|\nabla u(t)\|_2^2 \\ &< \mathscr{E}(t) - \frac{1}{r+1}\|u(t)\|_{r+1}^{r+1} = E(t) \leq \mathscr{E}(t), \end{aligned} \tag{5.1.4}$$

and (5.1.2) holds. Utilizing (5.1.2) and recalling the energy identity (1.2.1), we have

$$\frac{r-1}{r+1}\mathscr{E}(t)+\int_0^t\|\nabla u_t(s)\|_p^p ds\leq E(t)+\int_0^t\|\nabla u_t(s)\|_p^p ds=E(0)\leq\mathscr{E}(0), \qquad (5.1.5)$$

which gives the left-hand inequality of (5.1.3). Taking $t=0$ in (5.1.4) and again employing the energy identity, we get

$$\frac{r-1}{(r+1)}\mathscr{E}(0)\leq E(0)=E(t)+\int_0^t\|\nabla u_t(s)\|_p^p ds. \qquad (5.1.6)$$

Given that $E(t)\leq\mathscr{E}(t)$,

$$\frac{r-1}{(r+1)}\mathscr{E}(0)\leq\mathscr{E}(t)+\int_0^t\|\nabla u_t(s)\|_p^p ds. \qquad (5.1.7)$$

Multiplying (5.1.7) by $\frac{r+1}{r-1}$ gives the right-hand side of (5.1.3).

□

Proposition 5.1.2. (Observability-Stabilization Inequality) Under the validity of Assumptions 1.1.1 and 1.1.2 and assuming that $u_0\in \mathrm{W}_0^{1,p}(\Omega)$ and $1<r<5$, there exists $T_0>0$ such that the global solution of (1.0.1) given by Theorem 1.2.4 satisfies

$$\begin{aligned}\frac{r-1}{r+1}\mathscr{E}(t)\leq E(t)\leq C(p,\Omega)\Big\{&\frac{D_{t_0}^t}{t-t_0}+\left(\frac{D_{t_0}^t}{t-t_0}\right)^{\frac{2}{p}}\\&+\left(\frac{D_{t_0}^t}{t-t_0}\right)^{\frac{p-1}{p}}\left(\|\nabla u(0)\|_p+t^{\frac{p-1}{p}}(D_0^t)^{1/p}\right)\Big\},\end{aligned} \qquad (5.1.8)$$

for all $t,t_0\geq 1$ with $t-t_0>T_0$, where $D_\tau^t:=\int_\tau^t\|\nabla u(s)\|_p^p ds$.

Note that the left hand inequality in (5.1.8) provides the non-negativity of the total energy $E(t)$ in Theorem 1.2.5.

Proof. We begin with establishing the equipartition of energy. Testing in (1.1.44) with u (and replacing 0 by $t_0 \geq 0$) gives

$$
\begin{aligned}
-\int_{t_0}^{t} \|u_t(s)\|_2^2 ds + \int_{t_0}^{t} \|\nabla u(s)\|_2^2 ds + \big(u_t(t), u(t)\big) - \big(u_t(t_0), u(t_0)\big) \\
+ \int_{t_0}^{t} \int_\Omega |\nabla u_t|^{p-2} \nabla u_t \cdot \nabla u dx ds = \int_{t_0}^{t} \int_\Omega f(u) u dx ds. \qquad (5.1.9)
\end{aligned}
$$

Adding $2\int_{t_0}^{t} \|u_t(s)\|_2^2 ds$ to both sides of (5.1.9) and dividing by 2 to gives

$$
\begin{aligned}
\int_{t_0}^{t} \mathscr{E}(s) ds = \int_{t_0}^{t} \|u_t(s)\|_2^2 ds - \frac{1}{2}\big(u_t(t), u(t)\big) + \frac{1}{2}\big(u_t(t_0), u(t_0)\big) \\
- \frac{1}{2}\int_{t_0}^{t} \int_\Omega |\nabla u_t|^{p-2} \nabla u_t \cdot \nabla u dx ds + \frac{1}{2}\int_{t_0}^{t} \int_\Omega f(u) u dx ds. \qquad (5.1.10)
\end{aligned}
$$

We proceed by estimating the terms on the right hand side of (5.1.10). We begin by bounding the initial and terminal energies. Let $s \geq 0$. Then by Young's inequality,

$$
|\big(u_t(s), u(s)\big)| \leq \int_\Omega |u_t(s)||u(s)| ds \leq \frac{1}{2}\big(\|u_t(s)\|_2^2 + \|u(s)\|_2^2\big) \leq C\mathscr{E}(s). \qquad (5.1.11)
$$

Since $q' \geq 6$ (from (1.1.2)), $q'/2 > 1$. By Hölder's inequality with $q'/2$ and $\frac{q'}{q'-2}$, followed by (1.1.3) and Hölder's inequality with $p/2$ and $\frac{p}{p-2}$ gives

$$
\begin{aligned}
\int_{t_0}^{t} \|u_t(s)\|_2^2 ds &= \int_{t_0}^{t} \int_\Omega |u_t(s)|^2 dx ds \\
&\leq C_\Omega \int_{t_0}^{t} \|u_t(s)\|_{q'}^2 ds \\
&\leq C_\Omega \int_{t_0}^{t} \|\nabla u_t(s)\|_p^2 ds \\
&\leq C_\Omega (t - t_0)^{\frac{p-2}{p}} (D_{t_0}^t)^{2/p}. \qquad (5.1.12)
\end{aligned}
$$

Now, to bound the term due to damping we apply Hölder's inequality with $\frac{p}{p-1}$

and p twice.

$$
\begin{aligned}
|\int_{t_0}^{t}\int_{\Omega}|\nabla u_t|^{p-2}\nabla u_t\cdot\nabla u dxds| &\leq \int_{t_0}^{t}\|\nabla u_t\|_p^{p-1}\|\nabla u\|_p ds \\
&\leq (D_{t_0}^t)^{\frac{p-1}{p}}(\int_{t_0}^{t}\|\nabla u\|_p^p ds)^{1/p}. \qquad (5.1.13)
\end{aligned}
$$

Finally, we estimate the source term. Recalling that $1 < r < 5$, by Hölder's inequality with $\frac{4}{5-r}$ and $\frac{4}{r-1}$,

$$
\begin{aligned}
\int_{t_0}^{t}\int_{\Omega}f(u)u dxds = \int_{t_0}^{t}\int_{\Omega}|u|^{r+1}dxds &= \int_{t_0}^{t}\int_{\Omega}|u|^{\frac{5-r}{2}}|u|^{\frac{3}{2}(r-1)}dxds \\
&\leq \int_{t_0}^{t}\|u(s)\|_2^{\frac{5-r}{2}}\|u(s)\|_6^{\frac{3}{2}(r-1)}ds. \qquad (5.1.14)
\end{aligned}
$$

Now applying Young's inequality with $\frac{4}{5-r}$ and $\frac{4}{r-1}$, for $\epsilon > 0$, followed by the Sobolev Imbeddings,

$$
\begin{aligned}
\int_{t_0}^{t}\int_{\Omega}f(u)u dxds &\leq C_\epsilon\int_{t_0}^{t}\|u(s)\|_2^2 ds + \epsilon\int_{t_0}^{t}\|u(s)\|_6^6 ds \\
&\leq C_\epsilon\int_{t_0}^{t}\|u(s)\|_2^2 ds + \epsilon C\int_{t_0}^{t}\|\nabla u(s)\|_2^6 ds \\
&\leq C_\epsilon\int_{t_0}^{t}\|u(s)\|_2^2 ds + \epsilon C_{d,r}\int_{t_0}^{t}\|\nabla u(s)\|_2^2 ds, \qquad (5.1.15)
\end{aligned}
$$

by the bound in (4.2.12). Using the estimates of (5.1.11) - (5.1.13) and (5.1.15) with (5.1.10) yields

$$
\begin{aligned}
\int_{t_0}^{t}\mathscr{E}(s)ds \leq C_\Omega(t-t_0)^{\frac{p-2}{p}}(D_{t_0}^t)^{2/p} + C(\mathscr{E}(t_0)+\mathscr{E}(t)) + (D_{t_0}^t)^{\frac{p-1}{p}}(\int_{t_0}^{t}\|\nabla u\|_p^p ds)^{1/p} \\
+ C_\epsilon\int_{t_0}^{t}\|u(s)\|_2^2 ds + \epsilon C_{d,r}\int_{t_0}^{t}\|\nabla u(s)\|_2^2 ds. \qquad (5.1.16)
\end{aligned}
$$

Now, recall that for solutions in the good part of the well, we have the estimate in (5.1.2), which we can multiply by $\frac{r+1}{r-1}$ so that

$$\mathscr{E}(t_0)+\mathscr{E}(t)\le \frac{r+1}{r-1}\big(E(t_0)+E(t)\big)=\frac{r+1}{r-1}\big(2E(t)-\int_{t_0}^{t}\|\nabla u_t(s)\|_p^p ds\big) \quad (5.1.17)$$

by the energy identity (1.2.1).

By choosing $\epsilon>0$ sufficiently small so that $\epsilon C_{d,r}<1/4$ in (5.1.15) and using (5.1.17) we get

$$\begin{aligned}\int_{t_0}^{t}\mathscr{E}(s)ds \le C\big\{&E(t)+D_{t_0}^t + C_\epsilon\int_{t_0}^{t}\|u(s)\|_2^2 ds\\ &+C_\Omega(t-t_0)^{\frac{p-2}{p}}\big(D_{t_0}^t\big)^{2/p}+\big(D_{t_0}^t\big)^{\frac{p-1}{p}}\left(\int_{t_0}^{t}\|\nabla u\|_p^p ds\right)^{1/p}\big\}.\end{aligned} \quad (5.1.18)$$

Now, since $E(t)$ is monotonically decreasing, for all $t_0\le s\le t$,

$$(t-t_0)E(t_0)\ge\int_{t_0}^{t}E(s)ds\ge(t-t_0)E(t), \quad (5.1.19)$$

and for $t-t_0>1$ we have

$$E(t)\le\frac{1}{t-t_0}\int_{t_0}^{t}E(s)ds\le\frac{1}{t-t_0}\int_{t_0}^{t}\mathscr{E}(s)ds\le\int_{t_0}^{t}\mathscr{E}(s)ds. \quad (5.1.20)$$

Thus, we can estimate $\int_{t_0}^{t}\mathscr{E}(s)ds$ as

$$\begin{aligned}\int_{t_0}^{t}\mathscr{E}(s)ds \le C\Big[&D_{t_0}^t + C_\epsilon\int_{t_0}^{t}\|u(s)\|_2^2 ds\\ &+C_\Omega(t-t_0)^{\frac{p-2}{p}}(D_{t_0}^t)^{2/p}+(D_{t_0}^t)^{\frac{p-1}{p}}\left(\int_{t_0}^{t}\|\nabla u\|_p^p ds\right)^{1/p}\Big],\end{aligned} \quad (5.1.21)$$

for $t-t_0$ sufficiently large, say, $t-t_0>2C$.

We next estimate $\int_{t_0}^{t}\|\nabla u\|_p^p ds$ in terms of $D_{t_0}^t$ and D_0^t. By the regularity of u

provided in Theorem 1.2.1, along with the assumption that $u_0 \in W_0^{1,p}(\Omega)$, we have that $\nabla u, \nabla u_t \in L^2(0,T;L^2(\Omega))^3$ and so for $s \geq 0$,

$$\begin{aligned} |\nabla u(s)|^p &= |\nabla u(0) + \int_0^s \nabla u_t(\tau) d\tau|^p \\ &\leq 2^{p-1}(|\nabla u(0)|^p + |\int_0^s \nabla u_t(\tau) d\tau|^p) \\ &\leq 2^{p-1}(|\nabla u(0)|^p + s^{p-1} \int_0^s |\nabla u_t(\tau)|^p d\tau), \end{aligned} \tag{5.1.22}$$

by Jensen's inequality. Integrating (5.1.22) over Ω yields,

$$\|\nabla u(s)\|_p^p \leq 2^{p-1}\big(\|\nabla u(0)\|_p^p + s^{p-1} \int_0^s \|\nabla u_t(\tau)\|_p^p d\tau\big). \tag{5.1.23}$$

Therefore,

$$\begin{aligned} \int_{t_0}^t \|\nabla u(s)\|_p^p ds &= 2^{p-1} \int_{t_0}^t \Big\{\|\nabla u(0)\|_p^p + s^{p-1} \int_0^s \|\nabla u_t(\tau)\|_p^p d\tau\Big\} ds, \\ &= 2^{p-1}\Big((t-t_0)\|\nabla u(0)\|_p^p + \int_{t_0}^t s^{p-1} \int_0^t \|\nabla u_t(\tau)\|_p^p d\tau ds\Big) \\ &\leq 2^{p-1}\Big((t-t_0)\|\nabla u(0)\|_p^p + t^{p-1} \int_{t_0}^t \int_0^t \|\nabla u_t(\tau)\|_p^p d\tau ds\Big) \\ &\leq 2^{p-1}\Big((t-t_0)\|\nabla u(0)\|_p^p + t^{p-1}(t-t_0) \int_0^t \|\nabla u_t(\tau)\|_p^p d\tau\Big) \\ &\leq 2^p (t-t_0)\Big(\|\nabla u(0)\|_p^p + t^{p-1} D_0^t\Big). \end{aligned} \tag{5.1.24}$$

As a result, we have

$$\begin{aligned} &C(D_{t_0}^t)^{\frac{p-1}{p}} \left(\int_{t_0}^t \|\nabla u\|_p^p ds\right)^{1/p} \\ &\qquad \leq C_p \left(D_{t_0}^t\right)^{\frac{p-1}{p}} (t-t_0)^{1/p} \big[\|\nabla u(0)\|_p + t^{\frac{p-1}{p}} (D_0^t)^{1/p}\big]. \end{aligned} \tag{5.1.25}$$

We now choose $T_0 = 2C + 1$ and employ the bounds in (5.1.25) and (5.1.21) to conclude that

$$\int_{t_0}^{t} \mathscr{E}(s)ds \le C_\epsilon \int_{t_0}^{t} \|u(s)\|_2^2 ds + C_\Omega (t-t_0)^{\frac{p-2}{p}} (D_{t_0}^t)^{2/p} + C D_{t_0}^t$$
$$+ C_p (D_{t_0}^t)^{\frac{p-1}{p}} (t-t_0)^{1/p} \big[\|\nabla u(0)\|_p + t^{\frac{p-1}{p}} (D_0^t)^{1/p}\big]. \tag{5.1.26}$$

Proposition 5.1.3. (Estimate of lower order term) Assume the hypotheses of Proposition 5.1.2 hold, with $T_0 = 2C+1$. Then, for $t-t_0 > T_0$ there exists a constant $C(p,\Omega)$ such that

$$\int_{t_0}^{t} \|u(s)\|_2^2 ds \le C(p,\Omega)\big\{D_{t_0}^t + C_\Omega (t-t_0)^{\frac{p-2}{p}} (D_{t_0}^t)^{2/p}$$
$$+ (D_{t_0}^t)^{\frac{p-1}{p}} (t-t_0)^{1/p} \big[\|\nabla u(0)\|_p + t^{\frac{p-1}{p}} (D_0^t)^{1/p}\big]\big\}. \tag{5.1.27}$$

If we assume for a moment the validity of Proposition 5.1.3, we have from (5.1.26)

$$\int_{t_0}^{t} \mathscr{E}(s)ds \le C_\epsilon \int_{t_0}^{t} \|u(s)\|_2^2 ds + C_\Omega (t-t_0)^{\frac{p-2}{p}} (D_{t_0}^t)^{2/p} + C D_{t_0}^t$$
$$+ C_p (D_{t_0}^t)^{\frac{p-1}{p}} (t-t_0)^{1/p} \big[\|\nabla u(0)\|_p + t^{\frac{p-1}{p}} (D_0^t)^{1/p}\big]$$
$$\le C(p,\Omega,\epsilon)\{D_{t_0}^t + C_\Omega (t-t_0)^{\frac{p-2}{p}} (D_{t_0}^t)^{2/p}$$
$$+ (D_{t_0}^t)^{\frac{p-1}{p}} (t-t_0)^{1/p} \big[\|\nabla u(0)\|_p + t^{\frac{p-1}{p}} (D_0^t)^{1/p}\big]\}.$$

By recalling (5.1.20) we find the following estimate for the total energy

$$E(t) \le C(p,\Omega)\Big\{\frac{D_{t_0}^t}{t-t_0} + \Big(\frac{D_{t_0}^t}{t-t_0}\Big)^{\frac{2}{p}}$$
$$+ \Big(\frac{D_{t_0}^t}{t-t_0}\Big)^{\frac{p-1}{p}} \big[\|\nabla u(0)\|_p + t^{\frac{p-1}{p}} (D_0^t)^{1/p}\big]\Big\}. \tag{5.1.28}$$

Proof. To prove (5.1.27) we proceed by the standard compactness-uniqueness argument (see, for instance, [3, 11, 21]) . Suppose, for the sake of a contradiction, that (5.1.27) is false. Then there exists a sequence of initial data $\{(u^k(0), u_t^k(0))\}$ satisfying Assumptions 1.1.1 and 1.1.2 such that the corresponding solutions of (1.0.1) satisfy

$$\lim_{k\to\infty} \frac{B(u^k)}{\int_{t_0}^t \|u^k(s)\|_2^2 ds} = 0, \tag{5.1.29}$$

where $B(u^k)$ is defined as,

$$\begin{aligned} B(u^k) :=& \int_{t_0}^t \|\nabla u_t^k(s)\|_p^p ds + \left(\int_{t_0}^t \|\nabla u_t^k(s)\|_p^p ds\right)^{2/p} (t-t_0)^{\frac{p-2}{p}} \\ &+ \left(\int_{t_0}^t \|\nabla u_t^k(s)\|_p^p ds\right)^{\frac{p-1}{p}} (t-t_0)^{1/p} \\ &\times \left\{\|\nabla u^k(0)\|_p + t^{\frac{p-1}{p}} \left(\int_0^t \|\nabla u_t^k(s)\|_p^p ds\right)^{1/p}\right\}. \end{aligned} \tag{5.1.30}$$

Now, since the sequence solutions lie in $\mathcal{W}_1$ for each $t > 0$, by Theorem 1.2.4, $\mathscr{E}^k(s)$ and $E_1^k(s)$ are uniformly bounded on $[t_0, t]$ for all k, where

$$\mathscr{E}^k(s) := \frac{1}{2}\|u_t^k(s)\|_2^2 + \frac{1}{2}\|\nabla u^k(s)\|_2^2 \tag{5.1.31}$$

and

$$E_1^k(s) := \mathscr{E}^k(s) + \frac{1}{r+1}\|u^k(s)\|_{r+1}^{r+1}. \tag{5.1.32}$$

Thus, u^k is bounded in $L^\infty(t_0, t; \mathrm{H}_0^1(\Omega))$, u_t^k is bounded in $L^\infty(t_0, t; L^2(\Omega))$, and u^k is bounded in $L^{r+1}(t_0, t; L^{r+1}(\Omega))$. Hence, there exists u such that, on a relabeled subsequence, we have

$$\begin{cases} u^k \to u \text{ weak}^* \text{ in } L^\infty(t_0, t; \mathrm{H}_0^1(\Omega)), \\ u_t^k \to u_t \text{ weak}^* \text{ in } L^\infty(t_0, t; L^2(\Omega)), \\ u_t^k \to u_t \text{ weakly in } L^{r+1}(t_0, t; L^{r+1}(\Omega)). \end{cases} \tag{5.1.33}$$

Also, in light of (5.1.29) and (5.1.33), it must be that

$$\lim_{k\to\infty} \int_{t_0}^{t} \|\nabla u_t^k(s)\|_p^p ds = 0. \tag{5.1.34}$$

Note here that $H_0^1(\Omega) \subset \mathrm{H}^{1-\epsilon}(\Omega) \subset L^2(\Omega)$, where $0 < \epsilon < 1$, each injection is continuous, and the first injection is compact. Also, because $\{u^k\}$ is bounded in $L^\infty(t_0, t; \mathrm{H}_0^1(\Omega))$, then in particular, $\{u^k\}$ is also bounded in $L^2(t_0, t; \mathrm{H}_0^1(\Omega))$. We also know that $\{u_t^k\}$ is bounded in $L^\infty(t_0, t; L^2(\Omega))$. Hence, by Aubin's Compactness Theorem, there exists a subsequence, labeled again by $\{u^k\}$, such that

$$u^k \to u \text{ strongly in } L^2(t_0, t; \mathrm{H}^{1-\epsilon}(\Omega)), \tag{5.1.35}$$

where $\epsilon > 0$ is defined by

$$\epsilon \le \begin{cases} \frac{8-(6/p)-r}{2r}, & \text{if } 2 \le p < 3, \\ \frac{6-r}{2r}, & \text{if } p \ge 3. \end{cases} \tag{5.1.36}$$

Now, we test the kth solution, u^k, against $\phi \in C_0^\infty(\Omega \times (t_0, t))$ to get

$$\int_{t_0}^{t} \int_\Omega (-u_t^k \phi_t + \nabla u^k \cdot \nabla\phi) dxds + \int_{t_0}^{t} \int_\Omega |\nabla u_t^k|^{p-2} \nabla u_t^k \cdot \nabla\phi dxds$$
$$= \int_{t_0}^{t} \int_\Omega f(u^k)\phi dxds. \tag{5.1.37}$$

Note that by Hölder's inequality and the Sobolev Imbedding we have

$$\int_{t_0}^{t} \|u_t^k(s)\|_2^2 ds \le C_\Omega (t-t_0)^{\frac{p-2}{p}} (\int_{t_0}^{t} \|\nabla u_t^k(s)\|_p^p ds)^{2/p} ds \to 0, \quad \text{as } k \to \infty, \quad (5.1.38)$$

by (5.1.34). That is, $u_t^k \to 0$ strongly in $L^2(t_0, t; L^2(\Omega))$ as $k \to 0$ and hence, $\int_{t_0}^{t} \int_\Omega u_t^k \phi_t dxds \to 0$ as $k \to \infty$. Next, by (5.1.33), $\int_{t_0}^{t} \int_\Omega \nabla u^k \cdot \nabla \phi dxds \to \int_{t_0}^{t} \int_\Omega \nabla u \cdot \nabla \phi dxds$. Further, as in (5.1.13),

$$\begin{aligned} &|\int_{t_0}^{t} \int_\Omega |\nabla u_t^k|^{p-2} \nabla u_t^k \cdot \nabla \phi dxds| \\ &\qquad \le (\int_{t_0}^{t} \|\nabla u_t^k(s)\|_p^p ds)^{\frac{p-1}{p}} (\int_{t_0}^{t} \|\nabla \phi\|_p^p ds)^{1/p} \to 0, \quad \text{as } k \to \infty, \end{aligned}$$

by (5.1.34). Finally, we show that $\int_{t_0}^{t} \int_\Omega f(u^k) \phi dxds \to \int_{t_0}^{t} \int_\Omega f(u) \phi dxds$. Let L_f be the local Lipschitz constant for $f : \mathrm{H}^{1-\epsilon}(\Omega) \to L^q(\Omega)$ (see Lemma 2.1.3). By Hölder's inequality with q and q' as in (1.1.2),

$$\begin{aligned} |\int_{t_0}^{t} \int_\Omega (f(u^k) - f(u)) \phi dxds| &\le \int_{t_0}^{t} \|f(u^k) - f(u)\|_q \|\phi\|_{q'} dxds \\ &\le L_f \int_{t_0}^{t} \|u^k - u\|_{\mathrm{H}^{1-\epsilon}(\Omega)} \|\phi\|_{q'} ds \\ &\le L_f (\int_{t_0}^{t} \|u^k - u\|_{\mathrm{H}^{1-\epsilon}(\Omega)}^2 ds)^{1/2} (\int_{t_0}^{t} \|\phi\|_{q'}^2 ds)^{1/2} \quad (5.1.39) \end{aligned}$$

Since $u^k \to u$ strongly in $L^2(t_0, t; \mathrm{H}^{1-\epsilon}(\Omega))$, we have $\int_{t_0}^{t} \int_\Omega f(u^k) \phi dxds \to \int_{t_0}^{t} \int_\Omega f(u) \phi dxds$. Thus, passing to the limit in (5.1.37) shows that u satisfies (in the sense of distributions),

$$\begin{cases} -\Delta u = |u|^{r-1} u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (5.1.40)$$

which implies that $\|\nabla u\|_2^2 = \|u\|_{r+1}^{r+1}$. Since $u(t) \in \mathcal{W}_1$, it must be that $u = 0$ a.e.

$(t_0, t) \times \Omega$.

We now renormalize the sequence $\{u^k\}$. Define

$$d_k := \left(\int_{t_0}^{t} \|u^k(s)\|_2^2 ds\right)^{1/2}. \tag{5.1.41}$$

Without loss of generality, assume that $d_k \neq 0$ for each k and put

$$\overline{u}^k := \frac{u^k}{d_k}. \tag{5.1.42}$$

Observe that (5.1.35) combined with the conclusion that $u = 0$ a.e $(t_0, t) \times \Omega$ implies that $d_k \to 0$. Also, note that

$$\left(\int_{t_0}^{t} \|\overline{u}^k(s)\|_2^2 ds\right)^{1/2} = 1, \qquad \text{for all } k. \tag{5.1.43}$$

Put $E^k(t) := \mathscr{E}^k(s) - \frac{1}{r+1}\|u^k(s)\|_{r+1}^{r+1}$ and $\overline{\mathscr{E}}^k(t) := \frac{1}{2}\|\overline{u}_t^k(t)\|_2^2 + \frac{1}{2}\|\nabla \overline{u}^k\|_2^2$. Then by the equivalence of $\mathscr{E}^k(t)$ and $E^k(t)$ supplied by Lemma 5.1.1, for all $k \in \mathbb{N}$ and $s \in [0, t]$ we have

$$\begin{aligned} \overline{\mathscr{E}}^k(t) \leq \frac{C}{d_k^2} E^k(t) \leq \frac{C}{d_k^2}\{C_\epsilon \int_{t_0}^{t} \|u^k(s)\|_2^2 ds + C(p, \Omega) B(u^k)\} \\ \leq M \end{aligned} \tag{5.1.44}$$

for some number $M > 0$, by (5.1.20), (5.1.26) and (5.1.29). Thus, again we have that $\overline{u}^k$ is bounded in $L^\infty(t_0, t; \mathrm{H}_0^1(\Omega))$, $\overline{u}_t^k$ is bounded in $L^\infty(t_0, t; L^2(\Omega))$ and $\overline{u}^k$ is bounded in $L^{r+1}(t_0, t; L^{r+1}(\Omega))$ for $1 < r < 5$ by the Sobolev Imbedding Theorem.

Then on a relabeled subsequence we have

$$\begin{cases} \overline{u}^k \to \overline{u} \text{ weak}^* \text{ in } L^\infty(t_0, t; \mathrm{H}_0^1(\Omega)), \\ \\ \overline{u}_t^k \to \overline{u}_t \text{ weak}^* \text{ in } L^\infty(t_0, t; L^2(\Omega)). \end{cases} \tag{5.1.45}$$

Also, as in (5.1.35), by Aubin's Compactness Theorem

$$\overline{u}^k \to \overline{u} \text{ strongly in } L^2(t_0, t; \mathrm{H}^{1-\epsilon}(\Omega)), \tag{5.1.46}$$

on a relabeled subsequence.

Let us now examine d_k more closely. By applying Hölder's inequality with $\frac{p}{p-2}$ and $p/2$ twice we have

$$d_k^2 = \int_{t_0}^t \|u^k(s)\|_2^2 ds \le C_\Omega (t - t_0)^{\frac{p-2}{p}} \left(\int_{t_0}^t \|\nabla u^k(s)\|_p^p ds \right)^{2/p}. \tag{5.1.47}$$

Now, recalling (5.1.24) we have

$$d_k^2 \le C_\Omega (t - t_0) \Big\{ \|\nabla u_k(0)\|_p + t^{\frac{p-1}{p}} \left(\int_{t_0}^t \|\nabla u_t^k(s)\|_p^p ds \right)^{1/p} \Big\}^2. \tag{5.1.48}$$

We next divide (5.1.37) by d_k and look at the limit as $k \to \infty$. As in (5.1.12),

$$\begin{aligned} \int_{t_0}^t \int_\Omega |\overline{u}_t^k|^2 dx ds &= \frac{1}{d_k^2} \int_{t_0}^t \|u_t^k(s)\|_2^2 ds \\ &\le \frac{C_\Omega}{d_k^2} (t - t_0)^{\frac{p-2}{p}} (\int_{t_0}^t \|\nabla u_t^k(s)\|_p^p ds)^{2/p} ds, \end{aligned} \tag{5.1.49}$$

which converges to 0 as $k \to \infty$ by (5.1.29). From (5.1.45) it is clear that $\int_{t_0}^t \int_\Omega \nabla \overline{u}^k \cdot$

$\nabla \phi dxds \to \int_{t_0}^{t} \int_\Omega \nabla \overline{u} \cdot \nabla \phi dxds$. Now for the term due to damping, as before

$$|\frac{1}{d_k} \int_{t_0}^{t} \int_\Omega |\nabla u_t^k|^{p-2} \nabla u_t^k \cdot \nabla \phi dxds| \\ \leq \frac{1}{d_k} (\int_{t_0}^{t} \|\nabla u_t^k(s)\|_p^p ds)^{\frac{p-1}{p}} (\int_{t_0}^{t} \|\nabla \phi\|_p^p ds)^{1/p}. \tag{5.1.50}$$

Then by (5.1.48),

$$|\frac{1}{d_k} \int_{t_0}^{t} \int_\Omega |\nabla u_t^k|^{p-2} \nabla u_t^k \cdot \nabla \phi dxds| \\ \leq \frac{C_\Omega C_\phi}{d_k^2} (t - t_0) \left(\int_{t_0}^{t} \|\nabla u_t^k(s)\|_p^p ds \right)^{\frac{p-1}{p}} \Big\{ \|\nabla u^k(0)\|_p \\ + t^{\frac{p-1}{p}} \left(\int_0^t \|\nabla u_t^k(s)\|_p^p ds \right)^{1/p} \Big\}. \tag{5.1.51}$$

where C_ϕ is a constant depending on ϕ. Now by (5.1.29) we have that $\frac{1}{d_k} \int_{t_0}^{t} \int_\Omega |\nabla u_t^k|^{p-2} \nabla u_t^k \cdot \nabla \phi dxds \to 0$ as $k \to \infty$.

Finally, examining the source term gives

$$\frac{1}{d_k} \int_{t_0}^{t} \int_\Omega f(u^k) \phi dxds = \frac{C_\phi}{d_k} \int_{t_0}^{t} \int_\Omega |u^k|^r dxds \\ = C_\phi d_k^{r-1} \int_{t_0}^{t} \int_\Omega |\overline{u}^k|^r dxds, \tag{5.1.52}$$

where $r > 1$. Since $\overline{u}^k$ is bounded in $L^{r+1}(t_0, t; L^{r+1}(\Omega))$ and $d_k \to 0$, we have that $\frac{1}{d_k} \int_{t_0}^{t} \int_\Omega f(u^k) \phi dxds \to 0$. Thus, we now have that $\overline{u}$ solves

$$\int_{t_0}^{t} \int_\Omega \nabla \overline{u} \cdot \nabla \phi dxds = 0, \tag{5.1.53}$$

for all sufficiently smooth $\phi \in C_0^\infty(\Omega \times (t_0, t))$, and so $\overline{u} = 0$ a.e. $\Omega \times (t_0, t)$. However, this contradicts (5.1.43). Thus (5.1.27) must hold and Proposition 5.1.3 is verified. □

□

Recall the energy identity (1.2.1) and rearrange to get

$$D_{t_0}^t = \int_{t_0}^t \|\nabla u_t(s)\|_p^p ds = E(t_0) - E(t). \tag{5.1.54}$$

Now, define

$$T = t - t_0, \quad t > t_0 \tag{5.1.55}$$

and require $t_0 \geq T > T_0$ where T_0 is as in Proposition 5.1.3. Also note that by (5.1.2),

$$D_0^t = E(0) - E(t) \leq E(0) - \frac{r-1}{r+1}\mathscr{E}(t) \leq E(0). \tag{5.1.56}$$

Then substituting (5.1.54) and (5.1.55) into (5.1.28) yields

$$E(t) \leq C(p,\Omega)\{\frac{D_{t_0}^t}{T} + \left(\frac{D_{t_0}^t}{T}\right)^{\frac{2}{p}} + \left(\frac{D_{t_0}^t}{T}\right)^{\frac{p-1}{p}} (\|\nabla u(0)\|_p + t^{\frac{p-1}{p}}(E(0))^{1/p})\}. \tag{5.1.57}$$

Since $T \geq 1$ we can obtain the estimate

$$\begin{aligned} E(t) &\leq C(p,\Omega,\|\nabla u(0)\|_p, E(0))\{\frac{D_{t_0}^t}{T} + \left(\frac{D_{t_0}^t}{T}\right)^{\frac{2}{p}} + t^{\frac{p-1}{p}}\left(\frac{D_{t_0}^t}{T}\right)^{\frac{p-1}{p}}\} \\ &\leq C(p,\Omega,\|\nabla u(0)\|_p, E(0))\{D_{t_0}^t + \left(D_{t_0}^t\right)^{\frac{2}{p}} + t^{\frac{p-1}{p}}\left(D_{t_0}^t\right)^{\frac{p-1}{p}}\} \\ &= H_t(D_{t_0}^t), \end{aligned} \tag{5.1.58}$$

where the function H_t is defined by

$$H_t(\rho) = C\left(\rho + (t\rho)^{\frac{p-1}{p}} + \rho^{2/p}\right), \tag{5.1.59}$$

and $C = C(p, \Omega, \|\nabla u(0)\|_p, E(0))$. Now, assuming that $t_0 \geq T$, we have that $t^{\frac{p-1}{p}} = (t_0 + T)^{\frac{p-1}{p}} \leq 2t_0^{\frac{p-1}{p}}$, implying that $H_t(\rho) \leq 2H_{t_0}(\rho)$. Thus, we estimate

$$E(t) \leq H_t(D_{t_0}^t) \leq 2H_{t_0}(D_{t_0}^t). \tag{5.1.60}$$

At this point we incorporate the multiple of 2 above into the constant C from the definition of H_t. Then (5.1.60) becomes

$$E(t) \leq H_{t_0}(E(t_0) - E(t)). \tag{5.1.61}$$

Now, for fixed t_0, $H_{t_0}(\rho)$ is concave and monotone increasing in ρ and passes through the origin. We denote the inverse of the mapping $\rho \mapsto H_{t_0}(\rho)$ by $H_{t_0}^{-1}$ and note that $H_{t_0}^{-1}$ is convex and monotone increasing in ρ. We then have

$$H_{t_0}^{-1}(E(t)) \leq E(t_0) - E(t),$$

and so

$$E(t) + H_{t_0}^{-1}(E(t)) \leq E(t_0). \tag{5.1.62}$$

Choosing $t_0 = mT$ and $t = (m+1)T$ provides the family of inequalities:

$$E((m+1)T) + H_{t_0}^{-1}(E((m+1)T)) \leq E(mT), \tag{5.1.63}$$

for $m = 1, 2, 3, \ldots$

5.2 Comparable ODE

We now define the sequence s_m,

$$s_m := E(mT) \tag{5.2.1}$$

and put $\phi_m(\cdot) := H_{t_0}^{-1}(\cdot) = H_{mT}^{-1}(\cdot)$. Then (5.1.63) becomes

$$s_{m+1} + \phi_m(s_{m+1}) \le s_m.$$

Since H_{mT}^{-1} is strictly increasing it is clear that ϕ_m and $(I + \phi_m(\cdot))^{-1}$ are strictly increasing as well and so,

$$s_{m+1} \le (I + \phi_m(\cdot))^{-1} s_m. \tag{5.2.2}$$

By multiplying both sides of (5.2.2) by -1 and adding s_m to each side, we conclude that

$$s_m - s_{m+1} \ge s_m - (I + \phi_m(\cdot))^{-1} s_m. \tag{5.2.3}$$

Thusly, we define the function q by

$$q(t,s) := s - (I + \phi_t(\cdot))^{-1}\, s, \tag{5.2.4}$$

and note its following properties:

q is increasing in s and nonnegative: Since $H_{t_0}^{-1}(\cdot)$ is increasing and passes through the origin, $I + H_{t_0}^{-1}(\cdot)$ is strictly increasing (and therefore invertible), passes through the origin, and lies above the graph of the identity. Thus, $\left(I + H_{t_0}^{-1}(\cdot)\right)^{-1}$

is also increasing, passes through the origin and lies below the graph of the identity, implying that q is nonnegative.

q **is decreasing in** t**:** Begin by observing that H_t (see (5.1.59)) maps $[0,\infty)$ onto $[0,\infty)$, for every $t \geq 0$. Let $0 \leq t_1 \leq t_2$. Clearly $H_{t_1T}(\rho) \leq H_{t_2T}(\rho)$ for all $\rho \geq 0$. Thus

$$H^{-1}_{t_1T}(s) \geq H^{-1}_{t_2T}(s), \qquad \text{for all } s \geq 0 \tag{5.2.5}$$

and further,

$$s + H^{-1}_{t_1T}(s) \geq s + H^{-1}_{t_2T}(s), \qquad \text{for all } s \geq 0. \tag{5.2.6}$$

Now as above, for any $t \geq 0$, $I + H_t^{-1}$ maps $[0,\infty)$ onto $[0,\infty)$, is strictly increasing, and invertible. Therefore, from (5.2.6) we conclude

$$\Big(I + H^{-1}_{t_1T}(\cdot)\Big)^{-1}(s) \leq \Big(I + H^{-1}_{t_2T}(\cdot)\Big)^{-1}(s), \qquad \text{for all } s \geq 0 \tag{5.2.7}$$

and so

$$s - \Big(I + H^{-1}_{t_1T}(\cdot)\Big)^{-1}(s) \geq s - \Big(I + H^{-1}_{t_2T}(\cdot)\Big)^{-1}(s), \qquad \text{for all } s \geq 0. \tag{5.2.8}$$

Finally,

$$\begin{aligned} q(t_1, s) &= s - (I + \phi_{t_1}(\cdot))^{-1}s = s - (I + H^{-1}_{t_1T}(\cdot))^{-1}s \\ &\geq s - \Big(I + H^{-1}_{t_2T}(\cdot)\Big)^{-1}s = s - (I + \phi_{t_2}(\cdot))^{-1}s = q(t_2, s) \end{aligned} \tag{5.2.9}$$

and so q is decreasing in t.

In what proceeds, we will consider the non-autonomous ordinary differential equation

$$\frac{d}{dt}S(t) + q(t, S(t)) = 0, \qquad S(1) = s_1 = E(T). \tag{5.2.10}$$

We first rewrite the function q. Let t be fixed, put $\Phi(s) = \phi_t(s) = H_{tT}^{-1}$ and note that $(I + \Phi), \Phi, H_{tT}, q : [0, \infty) \to [0, \infty)$. Then,

$$\begin{aligned} q &= I - (I + \Phi)^{-1} = \left[(I + \Phi) \circ (I + \Phi)^{-1}\right] - (I + \Phi)^{-1} \\ &= \Phi \circ (I + \Phi)^{-1} = \Phi \circ (\Phi^{-1} \circ \Phi + \Phi)^{-1} = \Phi \circ (\left[\Phi^{-1} + I\right] \circ \Phi)^{-1} \\ &= \Phi \circ \Phi^{-1} \circ (\Phi^{-1} + I)^{-1} = (\Phi^{-1} + I)^{-1} = (H_{tT} + I)^{-1}. \end{aligned} \tag{5.2.11}$$

Thus, if we define

$$G_t(\rho) := (C + 1)\rho + C(\rho t T)^{\frac{p-1}{p}} + C\rho^{2/p}, \tag{5.2.12}$$

then $q(t, s) = G_t^{-1}(s)$ and the ODE in (5.2.10) is rewritten as

$$\frac{dS}{dt} + G_t^{-1}(S) = 0, \qquad S(1) = s_1 = E(T). \tag{5.2.13}$$

Proposition 5.2.1. The initial value problem

$$\frac{dS}{dt} + G_t^{-1}(S) = 0, \qquad S(1) = s_1 = E(T). \tag{5.2.14}$$

has a unique solution that exists for all time $t \geq 1$.

Proof. We begin this proof by showing that $\tilde{G} : (1/2, \infty) \times \mathbb{R}$, defined by

$$\tilde{G}(t,s) = \begin{cases} G_t^{-1}(s), & s \geq 0, \\ 0, & s < 0. \end{cases} \tag{5.2.15}$$

is continuous.

Let $(t_0, s_0) \in (1/2, \infty) \times \mathbb{R}$ and $\epsilon > 0$. If $s_0 > 0$, choose $\delta < \min\left\{\frac{2t_0-1}{4}, \frac{s_0}{2}, \frac{\epsilon(C+1)}{1+CT^{\frac{p-1}{p}}\frac{p-1}{p}2^{1/p}\rho_0^{\frac{p-1}{p}}}\right\}$ and if $s_0 \leq 0$, choose $\delta < \min\left\{\frac{2t_0-1}{4}, \frac{s_0}{2}, \epsilon(C+1)\right\}$. Let $(t,s) \in B_\delta(t_0, s_0)$.

Suppose first that $s_0 > 0$. Note that for all $t_0 \in (1/2, \infty)$,

$$\frac{\epsilon(C+1)}{1 + CT^{\frac{p-1}{p}}\frac{p-1}{p}2^{1/p}\rho_0^{\frac{p-1}{p}}} < \frac{\epsilon(C+1)}{1 + CT^{\frac{p-1}{p}}\frac{p-1}{p}t_0^{-1/p}\rho_0^{\frac{p-1}{p}}}.$$

Put $\rho_0 := \tilde{G}(t_0, s_0)$ and $\rho := \tilde{G}(t,s)$. Then, $s_0 = G_{t_0}(\rho_0)$ and $s = G_t(\rho)$ and

$$s_0 - s = (C+1)(\rho_0 - \rho) + CT^{\frac{p-1}{p}}((t_0\rho_0)^{\frac{p-1}{p}} - (t\rho)^{\frac{p-1}{p}}) + C(\rho_0^{2/p} - \rho^{2/p}). \tag{5.2.16}$$

If $\rho_0 \geq \rho$, then the first and last terms of the right-hand side of (5.2.16) are non-negative. Now,

$$(t_0\rho_0)^{\frac{p-1}{p}} - (t\rho)^{\frac{p-1}{p}} = (t_0^{\frac{p-1}{p}} - t^{\frac{p-1}{p}})\rho_0^{\frac{p-1}{p}} + t^{\frac{p-1}{p}}(\rho_0^{\frac{p-1}{p}} - \rho^{\frac{p-1}{p}}). \tag{5.2.17}$$

Since $\frac{p-1}{p} < 1$, $(\cdot)^{\frac{p-1}{p}}$ is a concave function, $t_0^{\frac{p-1}{p}} - t\frac{p-1}{p} \geq \frac{p-1}{p}(t_0 - t)t_0^{-1/p}$. Thus, from

(5.2.16),

$$\begin{aligned} s_0 - s &= (C+1)(\rho_0 - \rho) + C(\rho_0^{2/p} - \rho^{2/p}) \\ &\quad + CT^{\frac{p-1}{p}}((t_0^{\frac{p-1}{p}} - t^{\frac{p-1}{p}})\rho_0^{\frac{p-1}{p}} + t^{\frac{p-1}{p}}(\rho_0^{\frac{p-1}{p}} - \rho^{\frac{p-1}{p}})) \\ &\geq (C+1)(\rho_0 - \rho) + C(\rho_0^{2/p} - \rho^{2/p}) \\ &\quad + CT^{\frac{p-1}{p}}\Big[\frac{p-1}{p}(t_0 - t)t_0^{-1/p}\rho_0^{\frac{p-1}{p}} + t^{\frac{p-1}{p}}(\rho_0^{\frac{p-1}{p}} - \rho^{\frac{p-1}{p}})\Big]. \end{aligned} \tag{5.2.18}$$

The fact that $\rho_0 - \rho \geq 0$ implies

$$s_0 - s \geq (C+1)(\rho_0 - \rho) + CT^{\frac{p-1}{p}}\frac{p-1}{p}(t_0 - t)t_0^{-1/p}\rho_0^{\frac{p-1}{p}}. \tag{5.2.19}$$

Thus,

$$\begin{aligned} \delta + &\delta CT^{\frac{p-1}{p}}\frac{p-1}{p}t_0^{-1/p}\rho_0^{\frac{p-1}{p}} \\ &\geq |s_0 - s| - CT^{\frac{p-1}{p}}\frac{p-1}{p}|t_0 - t|t_0^{-1/p}\rho_0^{\frac{p-1}{p}} \\ &\geq (C+1)(\rho_0 - \rho). \end{aligned} \tag{5.2.20}$$

Therefore,

$$\epsilon > \delta\left(\frac{1 + CT^{\frac{p-1}{p}}\frac{p-1}{p}t_0^{-1/p}\rho_0^{\frac{p-1}{p}}}{C+1}\right) \geq |\rho_0 - \rho|. \tag{5.2.21}$$

If $\rho_0 < \rho$, then consider the difference

$$s - s_0 = (C+1)(\rho - \rho_0) + CT^{\frac{p-1}{p}}((t\rho)^{\frac{p-1}{p}} - (t_0\rho_0)^{\frac{p-1}{p}}) + C(\rho^{2/p} - \rho_0^{2/p}). \tag{5.2.22}$$

Again, since $(\cdot)^{\frac{p-1}{p}}$ is a concave function, $t^{\frac{p-1}{p}} - t_0\frac{p-1}{p} \geq \frac{p-1}{p}(t - t_0)t^{-1/p}$ and as in

(5.2.18),

$$
\begin{aligned}
s - s_0 &\geq (C+1)(\rho - \rho_0) + C(\rho^{2/p} - \rho_0^{2/p}) \\
&\quad + CT^{\frac{p-1}{p}}\Big[\frac{p-1}{p}(t-t_0)t^{-1/p}\rho^{\frac{p-1}{p}} + t_0^{\frac{p-1}{p}}(\rho_0^{\frac{p-1}{p}} - \rho_0^{\frac{p-1}{p}})\Big] \\
&\geq (C+1)(\rho - \rho_0) + CT^{\frac{p-1}{p}}\Big[\frac{p-1}{p}(t-t_0)t^{-1/p}\rho^{\frac{p-1}{p}})\Big]. \qquad (5.2.23)
\end{aligned}
$$

If $t \geq t_0$, we have

$$
\delta > |s - s_0| = s - s_0 \geq (C+1)(\rho - \rho_0) = (C+1)|\rho - \rho_0|, \qquad (5.2.24)
$$

implying

$$
\epsilon > \delta(C+1) \geq |\rho - \rho_0|. \qquad (5.2.25)
$$

If $t < t_0$, then

$$
\begin{aligned}
&\delta + \delta CT^{\frac{p-1}{p}}\Big[\frac{p-1}{p}t^{-1/p}\rho^{\frac{p-1}{p}})\Big] \\
&\qquad \geq |s - s_0| - CT^{\frac{p-1}{p}}\Big[\frac{p-1}{p}(t-t_0)t^{-1/p}\rho^{\frac{p-1}{p}})\Big] \\
&\qquad \geq (C+1)(\rho - \rho_0). \qquad (5.2.26)
\end{aligned}
$$

Recalling that $t \in (1, \infty)$, we know that $t^{-1/p} \leq 1$. This implies

$$\begin{aligned}
\epsilon &> \delta \frac{1 + CT^{\frac{p-1}{p}} \frac{p-1}{p} 2^{1/p} \rho_0^{\frac{p-1}{p}}}{(C+1)} \\
&\geq \delta \frac{1 + CT^{\frac{p-1}{p}} \frac{p-1}{p} t^{-1/p} \rho_0^{\frac{p-1}{p}}}{(C+1)} \\
&\geq \rho - \rho_0 \\
&= |\rho - \rho_0|. \qquad (5.2.27)
\end{aligned}$$

If $s_0 < 0$, then $s < 0$ and thus $|\tilde{G}(t_0, s_0) - \tilde{G}(t, s)| = |0 - 0| = 0 < \epsilon$. Now suppose $s_0 = 0$. If $s \leq 0$ then again, $|\tilde{G}(t_0, s_0) - \tilde{G}(t, s)| < \epsilon$ as above. If $s > 0$, then clearly $\rho > \rho_0$ and

$$s - s_0 = s = (C+1)\rho + CT^{\frac{p-1}{p}} (t\rho)^{\frac{p-1}{p}} + C\rho^{2/p} \geq (C+1)\rho. \qquad (5.2.28)$$

By the definition of δ,

$$\epsilon > \frac{\delta}{C+1} \geq \frac{|s - s_0|}{C+1} \geq \rho = |\rho_0 - \rho|. \qquad (5.2.29)$$

Therefore, given any $\epsilon > 0$ we have found $\delta > 0$ such that $|\tilde{G}(t_0, s_0) - \tilde{G}(t, s)| < \epsilon$ for all $(t, s) \in B_\delta(t_0, s_0)$ and so $\tilde{G}$ is continuous.

Now, observe that for each fixed $t \in [1, \infty)$, $-\tilde{G}(t, s)$ is non-increasing in s on the rectangle $Q := \{(t, s) : 1 \leq t \leq M, |s - E(t)| \leq 2E(T)\}$. By Corollary 8.37 of [17], the Initial Value Problem 5.2.14 has a unique solution in Q.

We proceed to extend our unique solution to $[1, \infty)$. Note that the unique solution to the initial value problem

$$\frac{dS}{dt} + G_t^{-1}(S) = 0, \qquad S(1) = 0 \tag{5.2.30}$$

is identically 0. Note further that the solution S to Problem 5.2.14 is decreasing as G_t^{-1} is non-negative. These two facts imply that $0 \leq S(t) \leq E(T)$ for $t \in [1, M]$. Thus, the unique solution $S(t)$ can be extended, giving the maximal interval of existence $[1, \infty)$. □

Now that we have the global existence of S, we will show that the sequence $\{s_m\}_1^\infty$ is bounded by the sequence $\{S(m)\}_1^\infty$, i.e., $s_m \leq S(m)$. Since q is positive we know that $S(t)$ is non-increasing. We proceed by induction. Note that $S(1) \geq s_1$ and assume that $S(m) \geq s_m$ for some $m \geq 1$. From (5.2.10), $S(m) - S(m+1) = \int_m^{m+1} q(\tau, S(\tau))d\tau$ and therefore,

$$S(m+1) = S(m) - \int_m^{m+1} q(\tau, S(\tau))d\tau$$

Now, note that $m \leq \tau$ and $S(m) \geq S(\tau)$. Then, since q is decreasing in its first argument and increasing in its second argument, $q(\tau, S(\tau)) \leq q(m, S(m))$ for each $\tau \geq m$. Thus,

$$\begin{aligned} S(m+1) &\geq S(m) - \int_m^{m+1} q(m, S(m))d\tau \\ &= S(m) - q(m, S(m)) \geq s_m - q(m, s_m) \geq s_{m+1}, \end{aligned} \tag{5.2.31}$$

since $s - q(t, s) = (I + \phi_t(\cdot))^{-1}s$ is increasing in s.

Since

$$S(m) \geq s_m = E(mT) \tag{5.2.32}$$

and S and E are both decreasing functions, for any $\tau \in [m, m+1]$,

$$S(\tau - 1) \geq S(m) \geq E(mT) \geq E(\tau T) \tag{5.2.33}$$

Now, by relabeling $t = \tau T$ in (5.2.33) we conclude

$$S(t/T - 1) \geq E(t) \quad \text{for } t \geq T. \tag{5.2.34}$$

Thus, $S(t)$ bounds the energy of (1.0.1) for all $t \geq T$ and subsequently will provide a decay rate of the energy.

5.3 Full Decay of ODE

In what follows we will compare G_t to the related function F_t, given by

$$F_t(\rho) := (C_T + \gamma)(\rho t T)^{\frac{p-1}{p}}, \tag{5.3.1}$$

where $\gamma > 0$ and C_T is a constant to be defined later. By showing that $F_t^{-1}(s) \leq G_t^{-1}(s)$ we can use the solution to the ordinary differential equation,

$$\frac{dS}{dt} + F_t^{-1}(S) = 0, \qquad S(1) = s_1 = E(T), \tag{5.3.2}$$

which we will find explicitly, to provide decay rates for the solution to (5.2.14) and, subsequently, the energy $E(t)$ of our wave equation.

We now show that S decreases to zero over time in the following lemma.

Lemma 5.3.1. *Let $S(t)$ be the solution to the ordinary differential equation*

$$\frac{dS}{dt} + G_t^{-1}(S) = 0, \qquad S(1) = s_1 = E(T), \tag{5.3.3}$$

where G_t is defined in (5.2.12). Then $S(t)$ is decreasing and

$$\lim_{t \to \infty} S(t) = 0. \tag{5.3.4}$$

Proof. Since $G_t : [0, \infty) \to [0, \infty)$, G_t^{-1} is non-negative. It is clear from this that S is non-increasing. Put $E_{in} = E(T)$. For any fixed $0 < c < E_{in}$, we will find a time t_c for which $S(t_c) < c$. Let $\gamma > 0$ be any fixed number, $C_T := CT^{\frac{p-1}{p}} + 1$ (recall that C is the constant in the definition of G_t) and define

$$k := \frac{1}{T}\left(\frac{1}{C_T + \gamma}\right)^{\frac{p}{p-1}}, \qquad A_c = A(c, p, \gamma) := \frac{\exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)}{\exp\left(\frac{p-1}{kE_{in}^{1/(p-1)}}\right)}. \tag{5.3.5}$$

Note that $A_c > 1$ since $c < E_{in}$. Now choose a time τ_c as

$$\begin{aligned}\tau_c \geq \max\Bigg\{ T, \frac{1}{T}\left(\frac{2C_T}{\gamma}\left[E_{in}^{1/p} + E_{in}^{\frac{3-p}{p}}\right]\right)^{\frac{p}{p-1}}, \left(\frac{4C_T}{\gamma}\right)^{\frac{p}{p-1}} \frac{E_{in}^{\frac{1}{p-1}}}{T}, \\ \left(\frac{4C_T}{\gamma}\right)^{\frac{p}{2}} \frac{A_c^{\frac{p-3}{2}}}{T}\left(\frac{C_T + \gamma}{c}\right)^{\frac{p(p-3)}{2(p-1)}} \Bigg\}.\end{aligned} \tag{5.3.6}$$

We will show that $t_c := A_c\tau_c$ provides a time for $S(t_c) < c$. Define

$$\rho_0 = \rho_0(\tau_c) := \left(\frac{c}{C_T + \gamma}\right)^{\frac{p}{p-1}} \frac{1}{A_c\tau_c T}. \tag{5.3.7}$$

Observe that $\rho_0 < E_{in}$. As $A_c > 1$,

$$\rho_0 < c^{\frac{1}{p-1}+1}\left(\frac{1}{C_T+\gamma}\right)^{\frac{p}{p-1}}\frac{1}{\tau_c T}$$
$$< c\left(\frac{c}{E_{in}}\right)^{\frac{1}{p-1}}\frac{E_{in}^{\frac{1}{p-1}}}{T}\left(\frac{1}{C_T+\gamma}\right)^{\frac{p}{p-1}}\frac{1}{\tau_c}. \tag{5.3.8}$$

Because $C_T > 1$ (and thus $1 - 4C_T < 0$), for $\gamma > 0$,

$$\frac{1}{C_T+\gamma} - \frac{4C_T}{\gamma} = \frac{\gamma - 4C_T(C_T+\gamma)}{\gamma(C_T+\gamma)} = \frac{\gamma(1-4C_T) - 4C_T^2}{\gamma(C_T+\gamma)} \leq 0. \tag{5.3.9}$$

As a result, we have $\frac{1}{C_T+\gamma} \leq \frac{4C_T}{\gamma}$. From (5.3.8) and the definition of τ_c, for $0 < c < E_{in}$,

$$\rho_o < c\frac{E_{in}^{\frac{1}{p-1}}}{T}\left(\frac{4C_T}{\gamma}\right)^{\frac{p}{p-1}}\frac{1}{\tau_c} < c\tau_c\frac{1}{\tau_c} = c < E_{in}. \tag{5.3.10}$$

We will return to the proof of Lemma 5.3.1 after proving the following propositions.

Proposition 5.3.2. For any $t > \tau_c$ and any $\rho \in [\rho_0, E_{in}]$ we have,

$$G_t(\rho) < F_t(\rho), \tag{5.3.11}$$

where F_t is as given in (5.3.1).

Proof. Since $C < C_T = CT^{\frac{p-1}{p}} + 1$ (as $T \geq 1$), it sufficies to show the inequality

$$(C_T)(\rho + (\rho t T)^{\frac{p-1}{p}} + \rho^{2/p}) < (C_T+\gamma)(\rho t T)^{\frac{p-1}{p}},$$

which is equivalent to the inequality,

$$(C_T)(\rho + \rho^{2/p}) < \gamma(\rho t T)^{\frac{p-1}{p}}. \tag{5.3.12}$$

Since $\rho > 0$ we can divide both sides of the inequality (5.3.12) by $\rho^{\frac{p-1}{p}}$. Thus, we may prove (5.3.11) by showing,

$$(C_T)(\rho^{1/p} + \rho^{\frac{3-p}{p}}) < \gamma(tT)^{\frac{p-1}{p}}. \tag{5.3.13}$$

Indeed, it is enough to show the stronger result (for $p \geq 3$),

$$(C_T)(E_{in}^{1/p} + \rho_0^{\frac{3-p}{p}}) \leq \frac{\gamma}{2}(\tau_c T)^{\frac{p-1}{p}}. \tag{5.3.14}$$

Note, if $2 \leq p \leq 3$, then $\frac{3-p}{p} > 0$ and (5.3.13) is satisfied simply by noting that

$$\tau_c \geq \frac{1}{T}\left(\frac{2C_T}{\gamma}\left[E_{in}^{1/p} + E_{in}^{\frac{3-p}{p}}\right]\right)^{\frac{p}{p-1}} \tag{5.3.15}$$

implies, for $t \geq \tau_c$,

$$\gamma(tT)^{\frac{p-1}{p}} > \frac{\gamma}{2}(\tau_c T)^{\frac{p-1}{p}} \geq (C_T)(E_{in}^{1/p} + E_{in}^{\frac{3-p}{p}}) \geq (C_T)(\rho^{1/p} + \rho^{\frac{3-p}{p}}), \tag{5.3.16}$$

whenever $\rho_0 \leq \rho \leq E_{in}$.

To prove (5.3.14) we note that from the definition of τ_c,

$$\left(\frac{4C_T}{\gamma}\right)^{\frac{p}{p-1}} \frac{E_{in}^{\frac{1}{p-1}}}{T} \leq \tau_c \tag{5.3.17}$$

implies

$$C_T E_{in}^{1/p} \leq \frac{\gamma}{4}(\tau_c T)^{\frac{p-1}{p}}, \tag{5.3.18}$$

and also,

$$\left(\frac{4C_T}{\gamma}\right)^{\frac{p}{2}} \frac{A_c^{\frac{p-3}{2}}}{T} \left(\frac{C_T+\gamma}{c}\right)^{\frac{p(p-3)}{2(p-1)}} \leq \tau_c \tag{5.3.19}$$

implies

$$(\tau_c T)^{-2/p} \leq \frac{\gamma}{4C_T} A_c^{\frac{-(p-3)}{p}} \left(\frac{c}{C_T+\gamma}\right)^{\frac{-(3-p)}{(p-1)}}. \tag{5.3.20}$$

Recalling the definition of ρ_0, we have,

$$C_T \rho_0^{\frac{3-p}{p}} = C_T \left(\frac{c}{C_T+\gamma}\right)^{\frac{3-p}{p-1}} A_c^{\frac{p-3}{p}} (\tau_c T)^{-2/p} (\tau_c T)^{\frac{p-1}{p}} \leq \frac{\gamma}{4}(\tau_c T)^{\frac{p-1}{p}}. \tag{5.3.21}$$

Combining the estimates of (5.3.18) and (5.3.21) provides the inequality of (5.3.14) and concludes the proof of Proposition 5.3.2. □

Proposition 5.3.3. For $\tau_c \leq t \leq A_c \tau_c$ and $c \leq s \leq E_{in}$,

$$F_t^{-1}(s) < G_t^{-1}(s). \tag{5.3.22}$$

Proof. Observe that since F_t and G_t are both continuous and increasing, then F_t maps the interval $[\rho_0, E_{in}]$ onto $[F_t(\rho_0), F_t(E_{in})]$ and G_t maps $[\rho_0, E_{in}]$ onto $[G_t(\rho_0), G_t(E_{in})]$. By Proposition 5.3.2, $\rho < G_t(\rho) < F_t(\rho)$ for $\rho_0 \leq \rho \leq E_{in}$ and in particular, $G_t(\rho_0) <$

$F_t(\rho_0)$ and $G_t(E_{in}) < F_t(E_{in})$. Furthermore, since $F_t(\rho_0)$ is increasing in t,

$$\begin{aligned} F_t(\rho_0) &\leq F_{A_c\tau_c}(\rho_0) = (C_T+\gamma)(\rho_0 A_c\tau_c T)^{\frac{p-1}{p}} \\ &= (C_T+\gamma)\left(\left(\frac{c}{C_T+\gamma}\right)^{\frac{p}{p-1}} \frac{1}{A_c\tau_c T} A_c\tau_c T\right)^{\frac{p-1}{p}} = c, \end{aligned} \tag{5.3.23}$$

for all $t \in [\tau_c, A_c\tau_c]$. Recalling the definition of G_t in (5.2.12), it is clear that $G_t(E_{in}) > E_{in}$. Hence, we have that

$$G_t(\rho_0) < F_t(\rho_0) \leq c < E_{in} < G_t(E_{in}) < F_t(E_{in}). \tag{5.3.24}$$

Therefore,

$$[c, E_{in}] \subset [G_t(\rho_0), G_t(E_{in})] \cap [F_t(\rho_0), F_t(E_{in})], \tag{5.3.25}$$

which is to say that each s, with $c \leq s \leq E_{in}$, is in the common range of F_t and G_t. Now, for fixed $s \in [c, E_{in}]$, $s = G_t(\rho_1) = F_t(\rho_2)$ for some $\rho_1, \rho_2 \in [\rho_0, E_{in}]$. Since $G_t(\rho) < F_t(\rho)$, for all $\rho \in [\rho_0, E_{in}]$ and F_t and G_t are increasing, this implies that $\rho_1 > \rho_2$. Thus $G_t^{-1}(s) = \rho_1 > \rho_2 = F_t^{-1}(s)$ and (5.3.22) is shown. □

We conclude by showing that $S(A_c\tau_c) < c$. We can compare the solutions of the initial value problems:

$$\begin{cases} \frac{d}{dt}S_1 + G_t^{-1}(S_1) = 0 \\ S_1(\tau_c) = S(\tau_c) \end{cases} \tag{5.3.26}$$

and

$$\begin{cases} \frac{d\tilde{S}}{dt} + F_t^{-1}(\tilde{S}) = 0 \\ \tilde{S}(\tau_c) = E_{in}. \end{cases} \tag{5.3.27}$$

Notice, that $S(t)$, the unique solution to (5.3.3) is also the unique solution to the IVP (5.3.26). Furthermore, the solutions $S_1 = S$ and $\tilde{S}$ are continuous and $\tilde{S}(\tau_c) = E_{in} \geq S(\tau_c)$. By Proposition 5.3.3, we have

$$S(t) < \tilde{S}(t) \qquad \text{on} \quad (\tau_c, A_c\tau_c], \tag{5.3.28}$$

so long as $S(t) \geq c$. This is true because whenever the solutions S and $\tilde{S}$ coincide, the magnitude of the rate of change of $\tilde{S}(t)$, which is given by $F_t^{-1}(S(t))$ is strictly below the magnitude of the rate of change of $S(t)$, which is given by $G_t^{-1}(S(t))$.

Since $F_t(\rho) = (C_T + \gamma)(\rho t T)^{\frac{p-1}{p}}$, then

$$F_t^{-1}(s) = \frac{1}{tT}\left(\frac{s}{C_T + \gamma}\right)^{\frac{p}{p-1}} = \frac{k}{t} s^{\frac{p}{p-1}}. \tag{5.3.29}$$

It is then straightforward to verify that the solution to (5.3.27) is given by

$$\tilde{S}(t) = \left(\frac{1}{p-1}\left[k \ln\left(\frac{t}{\tau_c}\right) + (E_{in})^{\frac{1}{1-p}}(p-1)\right]\right)^{1-p}. \tag{5.3.30}$$

If $S(t)$ drops below c at some time before $A_c\tau_c$, then we are done. Otherwise, we are still guaranteed that $S(t) < \tilde{S}(t)$ and evaluation at $A_c\tau_c$ shows

$$\begin{aligned} S(A_c\tau_c) < \tilde{S}(A_c\tau_c) &= \left(\frac{1}{p-1}\left[k\ln A_c + (E_{in})^{\frac{1}{1-p}}(p-1)\right]\right)^{1-p} \\ &= \left(\frac{1}{p-1}\left[\frac{p-1}{c^{1/(p-1)}} - \frac{p-1}{E_{in}^{1/(p-1)}} + \frac{p-1}{E_{in}^{1/(p-1)}}\right]\right)^{1-p} \\ &= \left(\frac{1}{c^{1/(p-1)}}\right)^{1-p} \\ &\leq c. \end{aligned} \tag{5.3.31}$$

This concludes the proof of Lemma 5.3.1.

5.4 Decay Rate

We now conclude the proof of Theorem 1.2.5 by showing the rate of the decay of the energy $E(t)$. In addition to showing that $S(t)$ decreases to zero as t increases to infinity, the proof of Lemma 5.3.1 also provides a time, $t_c = A_c\tau_c$, by which we can guarantee that the energy level is below c, for any $0 < c < E_{in}$. From this information we can find a decay rate for the solution S. So, for $0 < c < E_{in}$ we are assured that $S(A_c\tau_c) < c$, where τ_c satisfies (5.3.6). Indeed, we can specifically choose

$$\begin{aligned} \tau_c = T + \frac{1}{T}\left(\frac{2C_T}{\gamma}\left[E_{in}^{1/p} + E_{in}^{\frac{3-p}{p}}\right]\right)^{\frac{p}{p-1}} + \left(\frac{4C_T}{\gamma}\right)^{\frac{p}{p-1}} \frac{E_{in}^{\frac{1}{p-1}}}{T} \\ + \left(\frac{4C_T}{\gamma}\right)^{\frac{p}{2}} \frac{A_c^{\frac{p-3}{2}}}{T}\left(\frac{1+\gamma}{c}\right)^{\frac{p(p-3)}{2(p-1)}} \\ = B_1 + B_2 \exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)^{\frac{p-3}{2}} c^{-\frac{p(p-3)}{2(p-2)}} \end{aligned} \tag{5.4.1}$$

where $B_1 = T + \frac{1}{T}\left(\frac{2C_T}{\gamma}\left[E_{in}^{1/p} + E_{in}^{\frac{3-p}{p}}\right]\right)^{\frac{p}{p-1}} + \left(\frac{4C_T}{\gamma}\right)^{\frac{p}{p-1}} \frac{E_{in}^{\frac{1}{p-1}}}{T}$
and $B_2 = \frac{1}{T}\left(\frac{4C_T}{\gamma}\right)^{\frac{p}{2}} \exp\left(\frac{(p-3)(1-p)}{2kE_{in}^{1/(p-1)}}\right)$. Then,

$$\begin{aligned} A_c\tau_c &= \frac{\exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)}{\exp\left(\frac{p-1}{kE_{in}^{1/(p-1)}}\right)}\left(B_1 + B_2 \exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)^{\frac{p-3}{2}} c^{-\frac{p(p-3)}{2(p-2)}}\right) \\ &= B_3 \exp\left(\frac{p-1}{kc^{1/(p-1)}}\right) + B_4 \exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)^{\frac{p-1}{2}} c^{-\frac{p(p-3)}{2(p-2)}}. \end{aligned} \tag{5.4.2}$$

We estimate $A_c\tau_c$ in the following two cases.

Case of $2 \le p \le 3$**:** Here, $\frac{p(3-p)}{2(p-1)} \ge 0$ and hence, for all c with $E_{in} > c > 0$ we have that

$$c^{\frac{p(3-p)}{2(p-1)}} \le E_{in}^{\frac{p(3-p)}{2(p-1)}}. \tag{5.4.3}$$

Furthermore, $\frac{p-1}{2} \le 1$, implying that $\frac{p-1}{kc^{\frac{1}{p-1}}} \ge \frac{(p-1)^2}{2kc^{\frac{1}{p-1}}}$ and thus

$$\exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right) \ge \exp\left(\frac{(p-1)^2}{2kc^{\frac{1}{p-1}}}\right). \tag{5.4.4}$$

Therefore,

$$\begin{aligned} t_c &\le B_3 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right) + B_4 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right) E_{in}^{\frac{p(3-p)}{2(p-1)}} \\ &\le B_5 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right) \le B_5 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right)^{p-1}, \end{aligned} \tag{5.4.5}$$

where $B_5 = B_3 + B_4 E_{in}^{\frac{p(3-p)}{2(p-1)}}$. That is, time $t_c = B_5 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right)$ guarantees that $S(t_c) \le c$.

Case of $p > 3$**:** Here, $p-3 > 0$ and $p-1 > 2$, so $\frac{p-1}{2} > 1$ and $\frac{p(p-3)}{2(p-1)} > 0$. Now, for sufficiently small $c > 0$ we know that, for some constant C',

$$c^{-\frac{p(p-3)}{2(p-2)}} \leq C' B_4 \exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)^{\frac{p-1}{2}}, \tag{5.4.6}$$

and also

$$\exp\left(\frac{p-1}{kc^{1/(p-1)}}\right) \leq \exp\left(\frac{p-1}{kc^{1/(p-1)}}\right)^{\frac{p-1}{2}}. \tag{5.4.7}$$

Thus, for all sufficiently small $c > 0$,

$$t_c \leq B_3 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right) + B_4 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right)^{\frac{p-1}{2}} c^{\frac{p(3-p)}{2(p-1)}} \leq B_6 \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right)^{p-1}, \tag{5.4.8}$$

where $B_6 = B_3 + C' B_4$. By choosing $B = \max\{B_5, B_6\}$, we have by (5.4.5) and (5.4.8) that for all $p \geq 2$,

$$t_c \leq B \exp\left(\frac{p-1}{kc^{\frac{1}{p-1}}}\right)^{p-1} = B \exp\left(\frac{(p-1)^2}{kc^{\frac{1}{p-1}}}\right) \tag{5.4.9}$$

Now, let

$$t > B \exp\left(\frac{(p-1)^2}{kE_{in}^{\frac{1}{p-1}}}\right) \tag{5.4.10}$$

and choose $c = c(t)$ to be

$$c = \left(\frac{(p-1)^2}{k \ln\left(t/B\right)}\right)^{p-1}. \tag{5.4.11}$$

Note that $c \leq E_{in}$ is ensured by (5.4.10). Then,

$$t_c = t_{c(t)} = B \exp\left(\frac{(p-1)^2}{k\left[\left(\frac{(p-1)^2}{k\ln\left(t/B\right)}\right)^{p-1}\right]^{\frac{1}{p-1}}}\right) = t \tag{5.4.12}$$

and

$$S(t) = S(t_c) \leq c = c(t) = \left(\frac{(p-1)^2}{k\ln\left(t/B\right)}\right)^{p-1}. \tag{5.4.13}$$

Now, recalling (5.2.34), we have that

$$E(t) \leq S(\frac{t}{T} - 1) \leq \left(\frac{(p-1)^2}{k\ln\left(\frac{t-T}{TB}\right)}\right)^{p-1} \tag{5.4.14}$$

□

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