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Homological characterizations of quasi-complete intersections

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HOMOLOGICAL CHARACTERIZATIONS OF QUASI-COMPLETE INTERSECTIONS

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Let R be a commutative ring, $(\boldsymbol{f})$ an ideal of R, and $E = K(\boldsymbol{f}; R)$ the Koszul complex. We investigate the structure of the Tate construction T associated with E. In particular, we study the relationship between the homology of T, the quasi-complete intersection property of ideals, and the complete intersection property of (local) rings.

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DEDICATION

To Kim

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Chapter 1

Introduction

Let R be a commutative ring and I an ideal of R. For a generating set $\boldsymbol{f}$ of I let E denote the Koszul complex $K(\boldsymbol{f}; R)$; its homology $H_*(E)$ is naturally an algebra over the quotient ring R/I. The ideal I is said to be a *quasi-complete intersection* if $H_1(E)$ is free over R/I and $H_*(E)$ has the structure of an exterior algebra on $H_1(E)$; see Definition 3.2.

In the case where $(R, \mathfrak{m})$ is a local (Noetherian) ring and E is the Koszul complex on a minimal generating set of $\mathfrak{m}$, such an algebra structure on $H_*(E)$ appears in [1]. It is shown that this algebra structure is equivalent to the *complete intersection* property of R, and is further related to the homological properties of the *Tate construction*; see Chapter 4. The Tate construction is the second step in a *Tate resolution* of S over R, i.e, it is the result of adjoining (to the Koszul complex E) variables of degree two to annihilate the degree one homology of E; see [25, §2].

As Avramov, Henriques, and Şega [5] note, these ideals were first introduced in Rodicio's paper [22] and in his joint work with Blanco and Majadas [9] as ideals having *free exterior Koszul homology.* The quasi-complete intersection nomenclature is due to Avramov et al. [5, 1.1].

Let $\boldsymbol{z}$ be a set of degree one cycles whose homology classes generate $H_1(E)$ and let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$ (see Construction 2.10). Blanco, Majadas, and Rodicio characterize quasi-complete intersection ideals as follows:

Theorem 1.1 ([10, Theorem 1])**.** *The following conditions are equivalent:*

(1) I is a quasi-complete intersection and $\boldsymbol{z}$ represents a basis for the free S-module $H_1(E)$.

(2) $H_i(T) = 0$ for all $i > 0$.

This dissertation builds on the work in [1], [10], and [17] and makes a contribution to the study of quasi-complete intersection ideals, with applications to the study of (local) complete intersection rings. In particular, we establish results in the following two themes:

(I) *The quasi-complete intersection property of I can be detected from a finite band of vanishing of $H_*(T)$.*

The size and location of the band of vanishing depend on computable numerical invariants of the ideal I. Moreover, more flexibility in both components is possible given mild assumptions on I; see Proposition 3.20 and Theorem 4.9.

(II) *The quasi-complete intersection property of the maximal ideal $\mathfrak{m}$ of a local ring can be detected from the vanishing in a single degree of $H_i(T)$.*

The case $i = 2$ is a result of Assmus [1, Theorem 2.7]; the case $i = 2$ or 3 is addressed in Theorem 4.11. We study the case $i \geq 5$ in the context of rings which are *Golod away from a complete intersection*; see Section 5.2.

This project connects to two earlier bodies of work in which the (eventual) vanishing of the homology of a complex is determined by the vanishing of the complex in a single degree, namely the theory of Koszul rigidity and the vanishing of the deviations of a local ring. These topics and their connections to this project will be outlined in Section 2.5 and Section 2.6, respectively.

Avramov, Henriques, and Şega [5] present one direction of an ideal-theoretic characterization of quasi-complete intersection ideals, namely that an exact ideal (i.e., an ideal generated by a sequence of *exact elements*) is a quasi-complete intersection; see Section 3.1. The converse does not hold: In [20, Example 4.1] Kustin, Şega, and Vraciu give an example of a quasi-complete intersection which cannot be generated by a sequence of exact elements. In addition, [20, Lemma 1.7] shows that for two-generated ideals, a finite band of vanishing of the homology of the Tate construction is related to the quasi-complete intersection property.; see Section 3.2.1.

Notation. Throughout this work, the following assumptions and notations are in force. All rings are assumed to be commutative and unitary. Our ubiqutous commutative ring is denoted R, I is an ideal of R, and $\boldsymbol{f}$ denotes a generating set of I.

Chapter 2

Homological tools

In this chapter, we present background material and the relevant tools from homological algebra. The results and constructions of Section 2.1 through Section 2.6 are classical; new results are presented in Section 2.7.

2.1 Differential graded algebras

In this section, recall the definition and properties of differential graded (DG) algebras. We use [4] as our primary reference.

Definition 2.1 ([4, §1.3])**.** A *differential graded (DG) algebra* A is a complex (A, ∂) with an element $1 \in A_0$ (the *unit*) and a morphism of complexes (the *product*)

$$A \otimes_R A \to A, \quad a \otimes b \to ab,$$

that is unitary and associative. In addition, we require that A be *(graded) commutative*:

$$ab = (-1)^{|a||b|}ba \text{ for } a, b \in A \qquad \text{and} \qquad a^2 = 0 \text{ when } |a| \text{ is odd.}$$

We further require that $A_i = 0$ for $i < 0$. The underlying R-module $\{A_n\}_{n\geq 0}$ is denoted $A^\natural$.

Remark 2.2. For a DG algebra A, the fact that the product is a morphism of complexes means precisely that the product satisfies the *Leibniz rule*:

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b) \quad \text{for} \quad a, b \in A.$$

Example 2.3. The ring R is a DG R-algebra (concentrated in degree 0) with differential $\partial^R = 0$.

Definition 2.4. A *morphism* of DG-algebras A and A' is a morphism of complexes $\psi : A \to A'$ such that $\psi(1) = 1$ and $\psi(ab) = \psi(a)\psi(b)$. That is, ψ is compatible with the algebra structures on A and A'. A morphism $\psi : A \to A'$ which induces an isomorphism on homology is said to be a *quasi-isomorphism*.

Definition 2.5. We say that two DG algebras X and X' are *equivalent* if there exists a chain of quasi-isomorphisms

$$X \longleftarrow Y^{(1)} \longrightarrow X^{(1)} \longleftarrow Y^{(2)} \longrightarrow \cdots \longrightarrow X^{(n-1)} \longleftarrow Y^{(n)} \longrightarrow X'$$

2.2 The Koszul complex

In this section, we recall the construction and relevant properties of the Koszul complex. We use [11, §1.6] as a reference.

Consider a sequence of elements $\boldsymbol{x} = x_1, \ldots, x_n$ in R, and let L denote a free R-module with basis $e_1, \ldots, e_n$. The *Koszul complex on* $\boldsymbol{x}$, denoted $K(\boldsymbol{x}; R)$, is defined as follows:

$$K_p(\boldsymbol{x}; R) = \wedge_p^R L,$$

$$\partial_p(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{q=1}^{p} (-1)^{q+1} x_{i_q} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_q}} \wedge \cdots \wedge e_{i_p}.$$

(By $\widehat{e_{i_q}}$ we indicate that the basis element e_{i_q} is omitted from the product.)

Example 2.6. The Koszul complex $K(\boldsymbol{x}; R)$ is a DG R-algebra with multiplication given by the wedge product.

We recall properties of the homology of the Koszul complex, specifically those that relate to the algebraic structure of its homology. Let E denote the Koszul complex $K(\boldsymbol{x}; R)$.

(1) $(\boldsymbol{x}) \cdot H_*(E) = 0$, so that $H_*(E)$ is naturally an $R/(\boldsymbol{x})$-module.

(2) $H_*(E)$ inherits the structure of an algebra over $R/(\boldsymbol{x})$ with multiplication given by

$$\operatorname{cls}(z) \cdot \operatorname{cls}(z') = \operatorname{cls}(z \wedge z').$$

(3) Set $b = \max\{i : H_i(E) \neq 0\}$. If R is Noetherian and $(\boldsymbol{x}) \neq (\boldsymbol{x})^2$, then $b = n - \operatorname{depth}((\boldsymbol{x}), R)$; see [21, Theorem 16.8].

2.3 Tate's "adjunction of variables"

We now present a procedure (developed by Tate [25]) for constructing resolutions of DG algebras. This process utilizes the adjunction of familiar exterior variables to annihilate cycles of even degree; cycles of odd degree are annihilated by *divided powers* variables. We now adopt the notation of [4, §6]. Let A be a DG R-algebra.

Construction 2.7 ([4, Construction 6.1.1]). Suppose that z is a cycle in A of positive odd degree. The DG algebra $A\langle w \mid \partial w = z \rangle$ is defined as follows:

Let $R\langle w\rangle$ denote the free R-algebra on a *divided powers variable* w of degree $|z|+1$; it is the free R-module with a basis

$$\left\{w^{(i)} \,:\, \left|w^{(i)}\right| = i|w|\right\}_{i\geq 0}$$

and multiplication

$$w^{(i)}w^{(j)} = \binom{i+j}{j} w^{(i+j)} \qquad \text{for } \; i,j \geq 0.$$

The underlying algebra $A\langle w\rangle^\natural$ is $A^\natural \otimes_R R\langle w\rangle$, and the differential on $A\langle w\rangle$ is given by

$$\partial\left(\sum_i a_i w^{(i)}\right) = \sum_i \partial(a_i)w^{(i)} + \sum_i (-1)^{|a_i|} a_i z w^{(i-1)};$$

this differential extends the differential of A and satisfies the Leibniz rule.

The divided powers variables represent one of the two types of variables used to annihilate cycles; the second is exterior variables. An extension $A\langle X\rangle$ obtained by the iterated adjunction of exterior and/or divided powers variables is called a *semi-free*[1] *Γ-extension*.

We now present the exact homology triangle due to Tate, and highlight its precise form, which depends on the parity of the degree of the adjoined variable.

Let $A \hookrightarrow A\langle w \,|\, \partial w = z\rangle$ be an extension with $|z| = d$ and let $\iota : A \to A\langle w\rangle$ denote the natural injection.

Remark 2.8. [[4, Remark 6.1.5]] When d is even, we have a short exact sequence of chain maps

$$0 \longrightarrow A \xrightarrow{\iota} A\langle w\rangle \xrightarrow{\vartheta} A \longrightarrow 0$$

[1]A *semi-free* DG-module is not a free object in the category of DG R-modules, yet it does retain a lifting property which is characteristic of free objects; see [4, Proposition 1.3.1].

where $\vartheta(a + xb) = b$ is of degree $-d - 1$.

The resulting long exact sequence in homology has the form

$$\cdots \longrightarrow H_{n-d}(A) \xrightarrow{\cdot\,\mathrm{cls}(z)} H_n(A) \xrightarrow{H_n(\iota)} H_n(A\langle w\rangle) \xrightarrow{H_n(\vartheta)} H_{n-d-1}(A) \longrightarrow \cdots$$

Remark 2.9. [[4, Remark 6.1.6]] When d is odd, we have a short exact sequence of chain maps of the form

$$0 \longrightarrow A \xrightarrow{\iota} A\langle w\rangle \xrightarrow{\vartheta} A\langle w\rangle \longrightarrow 0$$

where $\vartheta\left(\sum_i a_i w^{(i)}\right) = \sum_i a_i w^{(i-1)}$ is of degree $-d-1$; the resulting long exact sequence in homology has the form

$$\cdots \longrightarrow H_{n-d}(A\langle w\rangle) \xrightarrow{\eth_{n+1}} H_n(A) \xrightarrow{H_n(\iota)} H_n(A\langle w\rangle) \xrightarrow{H_n(\vartheta)} H_{n-d-1}(A\langle w\rangle) \longrightarrow \cdots$$

We note that this differs from the preceding case. In particular, the connecting map $\eth_{n+1}$ in this sequence *does not* take the form of multiplication by $\mathrm{cls}(z)$.

The long exact sequences in homology will be used extensively in Section 2.7.

2.4 The Tate construction and the Cartan construction

In this section, we recall the construction of two families of complexes, due respectively to Tate [25] and Cartan. We adopt the notation of [13, 25].

Let I denote a proper non-zero ideal of R, and $S = R/I$. We fix a generating set $\boldsymbol{f}$ of I. Let E denote the Koszul complex on $\boldsymbol{f}$, i.e., $E = K(\boldsymbol{f}; R)$. We have an identification of E as an extension of R: Let $\boldsymbol{u} = \{u_f : f \in \boldsymbol{f}\}$ denote a set of degree one exterior variables. Then

$$E = R\langle \boldsymbol{u} \,|\, \partial u_f = f \rangle.$$

Construction 2.10. *The Tate construction.* Let $\boldsymbol{z}$ be a set of cycles of degree one such that the homology classes $\{\mathrm{cls}(z) : z \in \boldsymbol{z}\}$ generate $H_1(E)$. Let $\boldsymbol{w} = \{w_z : z \in \boldsymbol{z}\}$ denote a set of degree two divided powers variables. The *Tate construction on* $\boldsymbol{f}$ *and* $\boldsymbol{z}$, denoted $T(\boldsymbol{f};\boldsymbol{z})$ is

$$\begin{aligned} T(\boldsymbol{f};\boldsymbol{z}) &= R\langle \boldsymbol{u}, \boldsymbol{w} \,|\, \partial u_f = f,\, \partial w_z = z \rangle \\ &= E\langle \boldsymbol{w} \,|\, \partial w_z = z \rangle. \end{aligned}$$

Let T denote the Tate construction $T(\boldsymbol{f};\boldsymbol{z})$; we have the equality $H_1(T) = 0$ and isomorphisms $H_0(T) \cong H_0(E) \cong S$.

The Tate construction T has the following explicit presentation. Let W be a graded R-module on the basis $\boldsymbol{w}$, and let $\Gamma^R_* W$ denote the divided powers algebra on W. For integers $j(w) \geq 0$ with $p = \sum_{w \in \boldsymbol{w}} j(w)$, the distinct expressions

$$\prod_{w \in \boldsymbol{w}} w^{(j(w))}$$

form a basis of $\Gamma^R_p W$. This yields the following presentation of the complex:

$$T_n = \bigoplus_{2p+q=n} E_q \otimes_R \Gamma^R_p W,$$

$$\partial_n^T\left(e\otimes\prod_{w\in\boldsymbol{w}} w^{(j(w))}\right)=\partial^E(e)\otimes\prod_{w\in\boldsymbol{w}} w^{(j(w))}$$
$$+(-1)^{|e|}\sum_{w'\in\boldsymbol{w}}\left(z'e\otimes w'^{(j(w')-1)}\prod_{w\neq w'} w^{(j(w))}\right).$$

Note that the differential ∂^T is precisely the extension to all of T (using the Leibniz rule) of the differentials ∂^E and $w_z \mapsto z$.

Remark 2.11. For a local ring $(R, \mathfrak{m})$, we have a uniqueness property of the Tate construction. Let $\boldsymbol{f}$ denote a minimal generating set of I, and let E denote the Koszul complex on $\boldsymbol{f}$. If $\boldsymbol{z}$ is set of degree one cycles whose homology classes form a minimal generating set of $H_1(E)$, then $T(\boldsymbol{f};\boldsymbol{z})$ is unique up to isomorphism (see, for example [6, 1.2]). As such, we will (in this context) simply refer to the *Tate construction on I* without risk of confusion.

Remark 2.12. The explicit presentation of the Tate construction $T = T(\boldsymbol{f};\boldsymbol{z})$ yields a convergent first-quadrant spectral sequence:

$$\left\{d_r^{p,q} : E_{p,q}^r \to E_{p-r,q+r-1}^r\right\}_{r\geq 0}; \qquad E_{pq}^r \implies H_{p+q}(T).$$

The E^0 page is

$$E_{p,q}^0 = E_{q-p} \otimes_R \Gamma_p^R W.$$

Using the notation of [27, Definition 6.5.1] the E^1 and E^2 pages are

$$E_{p,q}^1 = H_q^v(T) = H_{q-p}(E) \otimes_S \Gamma_p^S(S \otimes_R W), \qquad E_{p,q}^2 = H_p^h H_q^v(T).$$

Let H_i denote $H_i(E)$ and let Γ_j denote $\Gamma_j^S(S \otimes_R W)$. For $q \geq 0$, the row E_{*q}^1 of the

E^1 page of the spectral sequence is as follows:

$$0 \longleftarrow H_q \longleftarrow H_{q-1} \otimes_S \Gamma_1 \longleftarrow \cdots \longleftarrow H_1 \otimes_S \Gamma_{q-1} \longleftarrow \Gamma_q \longleftarrow 0 \tag{2.13}$$

This spectral sequence will appear in Section 3.2.3; the realization of the Tate construction as an extension of R will be utilized extensively throughout this work.

We now recall the prototype for the Tate construction: the Cartan construction.

Construction 2.14. *The Cartan construction.* Let B denote a DG R-algebra with differential $\partial^B = 0$. Let $\boldsymbol{y}$ denote a set of generators of B_1, and let $\boldsymbol{w} = \{w_y : y \in \boldsymbol{y}\}$ denote a set of degree two divided powers variables. The *Cartan construction C on B* is the extension

$$C = B\langle \boldsymbol{w} \mid \partial w_y = y \rangle.$$

Remark 2.15. Let C be the Cartan construction on B. Then C is bigraded:

$$C_{p,q} = B_{q-p} \otimes_R \Gamma_p^R W, \qquad C_n = \bigoplus_{p+q=n} C_{p,q}.$$

We have a decomposition of the homology of C:

$$H_n(C) = \bigoplus_{p+q=n} H_p(C_{*,q}), \tag{2.16}$$

Here $C_{*,q}$ is the following *strand* of C:

$$0 \longleftarrow B_q \longleftarrow B_{q-1} \otimes_R \Gamma_1^R W \longleftarrow \cdots \longleftarrow B_1 \otimes_R \Gamma_{q-1}^R W \longleftarrow \Gamma_q^R W \longleftarrow 0. \tag{2.17}$$

We will exploit the similar structure of the the qth row of the E^1 page (2.13) and the strand $C_{*,q}$ (2.17) in the proof of Proposition 3.10.

We will apply the decomposition (2.16) in the proof of Theorem 5.8 in the case where B is a *trivial extension*:

Definition 2.18. The *trivial extension* $k \ltimes U$ is the vector space $k \oplus U$ endowed with the trivial differential $\partial U = 0$ and algebra structure $U^2 = 0$.

We will also be interested in the structure of the Cartan construction in the case where B is an exterior algebra. For such a B, we now describe the structure of the homology of C.

Remark 2.19. Let G be a free R-module on a basis $\boldsymbol{g}$. Set $B = \wedge_*^R G$, and consider B as a DG algebra with differential $\partial^B = 0$; note that $H_*(B) = B$. Let C be the Cartan construction on B. Then $H_p(C_{*,q}) = 0$ for all $(p,q) \neq (0,0)$. Indeed, $\boldsymbol{g}$ is *regular*[2] on B so that [4, Proposition 6.1.7] yields an isomorphism

$$\frac{B}{(\boldsymbol{g})B} \cong H_*(C).$$

In light of this isomorphism and the equalities $(\boldsymbol{g})B = B_{\geq 1}$ and $B_0 = R$ we have $H_n(C) = 0$ for $n > 0$ and $H_0(C) = R$. Now (2.16) yields the desired result.

2.5 Koszul rigidity

One of the motivations of this project is the theory of the *rigidity of the Koszul complex*. Let X be a complex of R-modules, and $\mathcal{C}$ a set of R-modules.

Definition 2.20. We say that X is *rigid with respect to the set* $\mathcal{C}$ if each module A in $\mathcal{C}$ has the following property:

$$H_i(X \otimes_R A) = 0 \implies H_j(X \otimes_R A) = 0 \text{ for all } j \geq i.$$

[2]This is a generalization to DG-algebras of the familiar ring-theoretic property; see [4, §6].

We say that X is *rigid* if X is rigid with respect to the set $\mathcal{C} = \{R\}$. That is, X is rigid if the following implication holds:

$$H_i(X) = 0 \implies H_j(X) = 0 \text{ for all } j \geq i.$$

Under mild hypotheses, the Koszul complex is rigid with respect to the set of finitely generated R-modules:

Theorem 2.21 ([26, Theorem 5.10])**.** *Let $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ be a sequence of elements contained in the Jacobson radical of R, and let M be a finitely generated R-module. If $H_q(K(\boldsymbol{x}; M)) = 0$, then $H_i(K(\boldsymbol{x}; M)) = 0$ for all $i \geq q$.*

Question 2.22. Is the Tate construction rigid? More generally, does there exist a finite set of integers $\mathcal{S}$ and an integer $j = j(\mathcal{S})$ such that

$$H_i(T) = 0 \text{ for all } i \in \mathcal{S} \implies H_i(T) = 0 \text{ for all } i \geq j?$$

The project provides partial answers to this question; see Proposition 3.17, Theorem 4.11, and Theorem 5.8.

2.6 Acyclic closures and deviations

In Chapter 4 we will focus on the case where $(R, \mathfrak{m}, k)$ is a local ring. To this end, we recall two important definitions: acyclic closures and deviations.

Construction 2.23 ([4, Construction 6.3.1]). Suppose that A is a DG algebra such that A_0 is the local ring $(R, \mathfrak{m}, k)$ and each R-module $H_n(A)$ is finitely generated. Let $A \to S$ be a surjective augmentation and set $J = \operatorname{Ker}(R \to S)$. We may construct a semi-free Γ-extension $A \hookrightarrow A\langle X \rangle$ as follows.

(1) Select X_1, a set of variables of degree one, so that $\partial(X_1)$ minimally generates J mod $\partial_1(A_1)$;

(2) For $n \geq 1$, $\{\mathrm{cls}(\partial x) : x \in X_{n+1}\}$ minimally generates $H_n(A\langle X_{\leq n}\rangle)$;

(3) $X = X_{\geq 1}$.

An extension obtained in this manner is called an *acyclic closure* of S over A.

Each package of variables X_i is finite. As such, we may order the variables X so that $|x_i| \leq |x_j|$ whenever $i < j$.

For the local ring $(R, \mathfrak{m}, k)$, an acyclic closure $R\langle X\rangle$ of k over R is a minimal free resolution of k as an R-module; this result was established independently by Gulliksen [14] and Schoeller [24].

The *deviations* ε_n of R measure the number of variables adjoined (in each degree) in the construction of an acyclic closure:

Theorem 2.24 ([4, Theorem 7.1.3])**.** *If $R\langle X\rangle$ is an acyclic closure of k over R, then*

$$\operatorname{card} X_n = \varepsilon_n \qquad \text{for} \ \ n \in \mathbb{Z}.$$

The deviations $\varepsilon_n(R)$ can also be defined in terms of a decomposition of the Poincaré series $P_k^R(t)$; see [4, Remark 7.1.1].

Remark 2.25. We note that the indexing convention of the ε_n used above differs from that of Gulliksen and Levin [13]: ε_n of [13] stands for ε_{n+1} of [4].

Theorem 2.26. *If $\varepsilon_n(R) = 0$ for some $n > 0$, then R is a complete intersection.*

The cases $n = 1$ and 2 are trivial: If $n = 1$ then R is a field; if $n = 2$ then R is regular. The case $n = 3$ is a consequence of a result of Wiebe [28]; see [4, Corollary

7.2.8]. The case $n = 4$ is due to Gulliksen [13, Theorem 3.5.1] and the result was settled in all cases by Halperin [15, Theorem B].

Definition 2.27. Let $R\langle X\rangle$ be an acyclic closure of k over R. A DG algebra Y is said to be a *degree ℓ partial acyclic closure* of k over R if $Y = R\langle X_{\leq \ell}\rangle$.

We make the following observation, which allows certain deviations to be computed from a partial acyclic closure:

Remark 2.28. Let Y be a degree ℓ partial acyclic closure of k over R. Then

$$\operatorname{card} Y_n = \varepsilon_n \text{ for } n \leq \ell \quad \text{and} \quad \operatorname{card} H_\ell(Y) = \varepsilon_{\ell+1}.$$

Consequently, we may restate Theorem 2.26 as follows: *If Y is a degree ℓ partial acyclic closure of k over R and $H_\ell(Y) = 0$, then R is a complete intersection.*

We will apply this observation in Section 4.2 to our goal of detecting the complete intersection property of local rings from the vanishing of the homology of the Tate construction in a single degree.

The result of Theorem 2.26 can also be framed as an answer to the following question:

Question 2.29. Let R be a local ring and ℓ an integer. Does there exists a complex $C = C(\ell)$ such that the implication

$$H_\ell(C) = 0 \implies R \text{ is a complete intersection}$$

holds?

Indeed, Theorem 2.26 yields that the implication holds when $C(\ell)$ is the partial acyclic closure $R\langle X_{\leq \ell}\rangle$. In some cases the Tate construction T gives an affirmative answer; see Section 4.2 and Section 5.2.

2.7 Periodicity and vanishing

We now develop conditions, expressed as a band of vanishing of homology, under which an extension formed by the adjunction of variables of degree two exhibits eventually periodic or eventually vanishing homology. We adopt the notation of [4, §6]. For integers $i \leq j$, let $[i, j]$ denote $\{i, i+1, \ldots, j\}$.

Let A denote a DG R-algebra. Suppose that $\{z_1, \ldots, z_m\}$ is a set degree one cycles of A. Put $A_0 = A$ and for $1 \leq j \leq m$ put $A_j = A_{j-1}\langle w_j \mid \partial w_j = z_j\rangle$.

Lemma 2.30. *Let q and b be integers.*

(1) Suppose $H_i(A_m) = 0$ for all $i \in [q, q+m]$. Then for each j we have $H_i(A_j) = 0$ for all $i \in [q+m-j, q+m]$.

(2) If $H_i(A) = 0$ for all $i > b$, then $H_i(A_1) \cong H_{i+2}(A_1)$ for all $i \geq b$, i.e., $H_*(A_1)$ is periodic of period 2 beginning in degree b.

Proof. For (1), by induction we may assume $m = 1$. Then $H_i(A_1) = 0$ for $i \in \{q, q+1\}$. The equality $H_{q+1}(A) = 0$ now follows from immediately from the following portion of exact sequence in homology associated with Tate's exact homology triangle [4, Remark 6.1.6]:

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{q+1}(A_1) \arrow[dll, rounded corners] \\
H_{q+2}(A) \arrow[r] & H_{q+2}(A_1) \arrow[r] & H_q(A_1) \arrow[dll, rounded corners] \\
H_{q+1}(A) \arrow[r] & H_{q+1}(A_1) \arrow[r] & \cdots
\end{tikzcd}$$

The result of (2) also follows from [4, Remark 6.1.6]: For each $i \geq b$ we have an isomorphism $H_{i+2}(A_1) \cong H_i(A_1)$, so that $H_*(A_1)$ is eventually periodic of period 2. The extremal case occurs when $i = b$, namely $H_{b+2}(A_1) \cong H_b(A_1)$, so that the periodicity begins in the desired position. $\square$

Proposition 2.31. *If $H_i(A) = 0$ for all $i > b$, $q \geq b + 1 - m$, and $H_i(A_m) = 0$ for all $i \in [q, q + m]$, then $H_i(A_m) = 0$ for all $i \geq b + 1 - m$.*

Proof. We proceed by induction on m. Consider the case $m = 1$. By Lemma 2.30 (2), the vanishing of $H_i(A)$ for $i > b$ yields that $H_*(A_1)$ is periodic of period 2 beginning in degree b. By hypothesis, $H_i(A_1) = 0$ for $i \in \{q, q + 1\}$. As $q \geq b$, we have that one representative from each of the two isomorphism classes of $H_{\geq b}(A_1)$ vanishes, so that $H_i(A_1) = 0$ for all $i \geq b$.

Suppose now that for each $1 \leq a < m$ the statement holds for the adjunction of a variables of degree two, and that $H_i(A_m) = 0$ for all $i \in [q, q + m]$. By Lemma 2.30 (1), $H_i(A_{m-1}) = 0$ for all $i \in [q + 1, q + m]$. By induction, we have that $H_i(A_{m-1}) = 0$ for all $i \geq b + 1 - (m - 1)$, so that Lemma 2.30 (2) yields that $H_*(A_m)$ is periodic of period 2 starting in degree $b + 1 - m$. As $q \geq b + 1 - m$, we have that (at least) one representative from each of the two isomorphism classes of $H_{\geq b+1-m}(A_m)$ vanishes, which completes the proof. $\square$

For a non-negative integer m, let $[m]$ denote the set $[0, m] = \{0, 1, \ldots, m\}$. We now state a more general version of Proposition 2.31.

Fix a sequence $d_1, \ldots, d_n$ of strictly increasing odd integers, and let B be the extension

$$B = A\langle w_{ij} \mid \partial w_{ij} = z_{ij},\ 1 \leq i \leq n, 1 \leq j \leq m_i \rangle,$$

where $|z_{ij}| = d_i$. For $1 \le q \le n$, put

$$\mathcal{S}_q = \bigcup_{s=1}^{d_q+1} \left\{ -\left(\sum_{j=1}^{q} d_j t_j \right) + s \,:\, (t_1, \dots, t_q) \in [m_1 - 1] \times [m_2] \times \cdots \times [m_q] \right\},$$

and set

$$\mu = \sum_{j=1}^{n} m_j d_j.$$

Proposition 2.32. *Suppose $H_i(A) = 0$ for all $i > b$, $k \ge b - \mu$, and $H_{k+i}(B) = 0$ for all $i \in \mathcal{S}_n$. Then $H_i(B) = 0$ for all $i \ge b + 1 - \mu$.*

The following lemma highlights two situations in which the adjunction of a variable to annihilate a non-zero homology class preserves the vanishing of homology in a higher degree. Here, $(R, \mathfrak{m}, k)$ is a local ring.

Lemma 2.33. *Let A be a DG R-algebra and assume that $H_0(A) = k$. Let i be an integer, and suppose that $H_i(A) \neq 0$. Let z be a cycle representing a non-zero homology class in $H_i(A)$ and set $B = A\langle w \,|\, \partial w = z \rangle$.*

(1) If $i \ge 2$ is even and $H_1(A) = 0 = H_{i+2}(A)$, then $H_1(B) = 0 = H_{i+2}(B)$.

(2) If $H_{i+1}(A) = 0$, then $H_{i+1}(B) = 0$.

Proof. For (1), the equality $H_1(B) = 0$ is clear, and the equality $H_{i+2}(B) = 0$ follows immediately from a portion of the exact sequence from [4, Remark 6.1.5]:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{i+2}(A) & \longrightarrow & H_{i+2}(B) & \xrightarrow{H_{i+1}(\vartheta)} & H_1(A) & \longrightarrow \cdots \\ & \parallel & & & & \parallel & \\ & 0 & & & & 0 & \end{array}$$

For (2), let ζ denote $\mathrm{cls}(z)$. Suppose first that i is even.

We consider the following portion of exact sequence in homology of [4, Remark 6.1.5]:

$$\begin{array}{ccccccccc}\cdots \longrightarrow & H_{i+1}(A) & \longrightarrow & H_{i+1}(B) & \xrightarrow{H_i(\vartheta)} & H_0(A) & \xrightarrow{\cdot\zeta} & H_i(A) & \longrightarrow \cdots \\ & \shortparallel & & & & \shortparallel & & & \\ & 0 & & & & k & & & \end{array}$$

Multiplication by ζ is injective on $H_0(A)$, so that $H_{i+1}(B) = 0$, as desired.

In the case where i is odd, the relevant portion of the exact sequence in homology from [4, Remark 6.1.6] is the following:

$$\begin{array}{ccccccccccc} H_{i+1}(A) & \longrightarrow & H_{i+1}(B) & \xrightarrow{H_{i+1}(\vartheta)} & H_0(B) & \xrightarrow{\eth_{i+1}} & H_i(A) & \xrightarrow{H_i(\iota)} & H_i(B) & \longrightarrow & H_{-1}(B) \\ \shortparallel & & & & \shortparallel & & & & & & \shortparallel \\ 0 & & & & k & & & & & & 0 \end{array}$$

By construction $H_i(\iota)(\zeta) = 0$, and so $H_i(\iota)$ is not injective. Thus $\eth_{i+1}$ is not the zero map and so $\eth_{i+1}$ is injective. Thus $H_{i+1}(B) = 0$. $\square$

Chapter 3

Quasi-complete intersection ideals

Let $(R, \mathfrak{m})$ be a commutative, Noetherian, local ring and let I be a proper, non-zero ideal of R. Fix a generating set $\boldsymbol{f}$ of I, and let E be the Koszul complex on $\boldsymbol{f}$.

Definition 3.1. We say that I is a *complete intersection* ideal if I can be generated by a regular sequence. As R is local, this condition is tantamount to the following (equivalent) conditions:

(1) $H_1(E) = 0$.

(2) $H_i(E) = 0$ for all $i > 0$.

Here, we see a situation in which the structure of $H_1(E)$ can determine the structure of $H_*(E)$, and this structure can be detected from an ideal-theoretic perspective.

Suppose now that R is a commutative (not necessarily local) ring and set $S = R/I$. There is a canonical homomorphism of graded S-algebras:

$$\begin{aligned} \lambda_*^S : \wedge_*^S H_1(E) &\to H_*(E), \\ \mathrm{cls}(z_1) \wedge \cdots \wedge \mathrm{cls}(z_m) &\mapsto \mathrm{cls}(z_1 \wedge \cdots \wedge z_m). \end{aligned}$$

Definition 3.2. We say that I is a *quasi-complete intersection* if $H_1(E)$ is free over S and λ_*^S is an isomorphism.

Remark 3.3. As Avramov, Henriques, and Şega [5, §1] note, the existence of *some* isomorphism of graded S-algebras

$$\lambda : H_*(E) \xrightarrow{\cong} \wedge_*^S H_1(E)$$

guarantees the quasi-complete intersection property. Indeed, if λ is such an isomorphism, then the composition

$$\wedge_*^S H_1(E) \xrightarrow{\wedge_*^S(\lambda_1)} \wedge_*^S H_1(E) \xrightarrow{\lambda^{-1}} H_*(E)$$

is an isomorphism and is equal to λ_*^S, as both the composition and λ_*^S restrict to $\mathrm{id}^{H_1(E)}$.

3.1 Principal quasi-complete intersection ideals

In this section, we discuss an ideal-theoretic characterization of *principal* quasi-complete intersection ideals. For such a principal ideal $I = (f)$, let E denote the Koszul complex $K(f;R)$. Then E has the following form:

$$E: \quad 0 \longrightarrow R \xrightarrow{\cdot f} R \longrightarrow 0$$

As such, $H_1(E) = \{r \in R : rf = 0\} = (0 :_R f)$ and $H_i(E) = 0$ for $i \geq 2$. With these observations, we note that in order for a principal ideal (f) to be a quasi-complete intersection, it must be the case that $H_1(E)$ is a free S-module and $\operatorname{rank}_S H_1(E) \leq 1$. In particular, $I = (f)$ is a quasi-complete intersection ideal if and only if one of the following conditions holds:

(1) $(0:_R f) = 0$.

(2) $(0:_R f) \cong R/(f)$.

An element satisfying the former condition is *regular*; an element satisfying the latter is called an *exact zerodivisor*. When f is an exact zero divisor, there exists $g \in R$ with $(0:_R f) = (g)$ and $(0:_R g) = (f)$, so that (f, g) is an *exact pair* in the sense of Kiełpiński, Simson, and Tyc [19, Definition 1.1]. Collectively, an element which is either regular or an exact zerodivisor is called an *exact element*.

As an analog to a regular sequence, we say that a sequence $\boldsymbol{f} = f_1, \ldots, f_n$ is an *exact sequence* if

(1) f_i is exact on $R/(f_1, \ldots, f_{i-1})$ for $i = 1, \ldots, n$, and

(2) $R \neq (\boldsymbol{f})$.

Recall the ideal-theoretic characterization of complete intersection ideals as ideals which can be generated by a regular sequence (Definition 3.1). The following result gives one portion of an ideal-theoretic characterization of quasi-complete intersection ideals:

Theorem 3.4. *[[5, Theorem 3.7]] Suppose $f_1, \ldots, f_n$ is a sequence of exact elements in R and put $I = (f_1, \ldots, f_n)$. Then I is a quasi-complete intersection ideal, and*

$$\mathrm{grade}_R(S) = \mathrm{card}\{i \in [1, n] \,:\, f_i \textit{ is regular on } R/(f_1, \ldots, f_{i-1})\}.$$

The result that an exact ideal is necessarily a quasi-complete intersection also follows from [19, Theorem 1.8]. Indeed, if $f_1, \ldots, f_n$ is a sequence of exact elements in R then there exists $g_1, \ldots, g_n \in R$ such that the $(\boldsymbol{f}, \boldsymbol{g}) = ((f_1, g_1), \ldots, (f_n, g_n))$ is a sequence of exact pairs (see [19, Definition 1.1]). Let $R(f_i, g_i)$ denote the complex

$$0 \longleftarrow R \xleftarrow{\cdot f_i} R \xleftarrow{\cdot g_i} R \xleftarrow{\cdot f_i} \cdots$$

Then [19, Theorem 1.8] yields that the complex $U = \bigotimes_{i=1}^{n} R(f_i, g_i)$ satisfies $H_i(U) = 0$ for $i > 0$. Noting that U is precisely $T(\boldsymbol{f}; \boldsymbol{g})$, the result of Blanco, Majadas, and Rodicio [10, Theorem 1] shows that $I = (\boldsymbol{f})$ is a quasi-complete intersection.

However, this result does not extend to a complete ideal-theoretic characterization of quasi-complete intersection ideals. That is, there exists a quasi-complete intersection ideal which cannot be generated by a sequence of exact elements. Kustin, Şega, and Vraciu [20] construct such an ideal:

Example 3.5 ([20, Example 4.1])**.** Let $X = \{X_1, \ldots, X_5\}$ be a set of indeterminates, and let J be the ideal of $\mathbb{Z}[X]$ generated by the following eight elements:

$$X_1^2 - X_2X_3, \quad X_2^2 - X_3X_5, \quad X_3^3 - X_1X_4, \quad X_4^2, \quad X_5^2, \quad X_3X_4, \quad X_2X_5, \quad X_4X_5.$$

Let A denote the ring $\mathbb{Z}[X]/J$; we denote the image of the variable X_i in A by x_i. Let f_1 and f_2 be the elements

$$f_1 = x_1 + x_2 + x_4 \quad \text{and} \quad f_2 = x_2 + x_3 + x_5.$$

Fix a field L and set $B = L \otimes_{\mathbb{Z}} A$. Let I denote the ideal $(f_1, f_2)B$.

By [20, Proposition 4.3(3)], I is a quasi-complete intersection ideal. Moreover, [20, Theorem 4.5] yields that I is neither principal nor properly contains a non-zero quasi-complete intersection ideal. Consequently Theorem 3.4 yields that I contains no exact elements; in particular, I cannot be generated by a sequence of exact elements.

3.2 Homological characterizations

In this section, we present various characterizations of the quasi-complete intersection property. The following characterization is due to Blanco, Majadas, and Rodicio. Let $\boldsymbol{z}$ be a set of degree one cycles whose homology classes generate $H_1(E)$ and let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$.

Theorem 3.6 ([10, Theorem 1]). *The following conditions are equivalent:*

(1) I is a quasi-complete intersection and $\boldsymbol{z}$ represents a basis for the free S-module $H_1(E)$.

(2) $H_i(T) = 0$ for all $i > 0$.

3.2.1 Two-generated quasi-complete intersection ideals

Section 3.1 describes a homological characterization of principal quasi-complete intersection ideals. A result of Kustin, Şega, and Vraciu ([20, Lemma 1.7]) provide (in the local case) an analogous classification for two-generated quasi-complete intersection ideals in terms of the vanishing of the homology of the Tate construction and a double annihilator condition.

In this section $(R, \mathfrak{m})$ is a local (Noetherian) ring and $\boldsymbol{f} = \{f_1, \ldots, f_n\}$ is a minimal generating set for the ideal I. The size of a minimal generating set of an R-module M is denoted $\nu_R(M)$.

The following construction provides the framework for the double annihilator condition.

Construction 3.7 ([20, 1.4]). Fix a basis $v_1, \ldots, v_n$ of E_1 with $\partial^E(v_i) = f_i$. Suppose that $\boldsymbol{z} = \{z_1, \ldots, z_n\}$ is a set of degree one cycles of E such that the homology classes $\{\mathrm{cls}(z_i)\}$ minimally generate $H_1(E)$. Then there exist $\{a_{ij} : i, j \in [1, n]\} \subset R$ such that

$$z_i = \sum_{j=1}^{n} a_{ij} v_j.$$

Let A denote the matrix (a_{ij}) and set $\Delta = \det A$. Then the map $\lambda_n^S : \wedge_n^S H_1(E) \to H_n(E)$ is given by

$$\operatorname{cls}(z_1) \wedge \cdots \wedge \operatorname{cls}(z_n) \mapsto \Delta v_1 \cdots v_n.$$

The equality $\nu_R(H_1(E)) = n$ holds whenever I is a quasi-complete intersection with $\operatorname{depth}(I, R) = 0$; see [5, 1.2].

Lemma 3.8 ([20, Lemma 1.7]). *Suppose $\nu_R(I) = 2$ and $\operatorname{depth}(I, R) = 0$. Then the following statements are equivalent:*

(1) I is a quasi-complete intersection.

(2) $H_2(T) = 0$, $(0 :_R I) = (\Delta)$, and $(0 :_R \Delta) = I$, where Δ is as defined in Construction 3.7.

3.2.2 Initial band of vanishing of $H_*(T)$

We now present the new results which characterize quasi-complete intersection ideals as those for which a finite portion of the homology of the Tate construction vanishes. We make the following assumptions: I is a non-zero proper ideal of R. Fix a generating set $\boldsymbol{f}$ of I and let E denote the Koszul complex $K(\boldsymbol{f}; R)$. Let $\boldsymbol{z}$ denote a set of degree one cycles such that the homology classes $\{\operatorname{cls}(z) : z \in \boldsymbol{z}\}$ generate $H_1(E)$.

Our first characterization is in terms of an initial band of vanishing of the homology of the Tate construction.

Theorem 3.9. *Suppose that $I = (\boldsymbol{f})$ is a proper, non-zero ideal of R, and set $b = \max\{i : H_i(E) \neq 0\}$. Suppose $\boldsymbol{z}$ is a set of cycles whose homology classes* *generate $H_1(E)$. Let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$. The following conditions are equivalent:*

(1) I is a quasi-complete intersection ideal and $\boldsymbol{z}$ represents a basis of the S-module $H_1(E)$.

(2) $H_i(T) = 0$ for all $i > 0$.

(3) $H_i(T) = 0$ for $i = 2, \ldots, b+2$.

With the goal of establishing Theorem 3.9, we first prove a preliminary result which connects the properties of the map λ_*^S to the low-degree vanishing of $H_*(T)$. Recall the spectral sequence of Remark 2.12 associated with the Tate construction; the map $d_1^{1,1}$ of the spectral sequence of is given by

$$d_1^{1,1} : S \otimes_R W \to H_1(E), \quad s \otimes_R w_i \mapsto s\,\mathrm{cls}(z_i).$$

The construction of T yields that $d_1^{1,1}$ is surjective, $S \otimes_R W$ free over S, and $H_1(T) = 0$. In addition, $\lambda_1^S : \wedge_1^S H_1(E) \to H_1(E)$ is the identity map.

Proposition 3.10. *Let $k \geq 1$ be an integer. The following statements are equivalent:*

(1) $H_i(T) = 0$ for $i = 2, \ldots, k+1$.

(2) $d_1^{1,1}$ is an isomorphism, λ_i^S is an isomorphism for $i = 2, \ldots, k$, and λ_{k+1}^S is surjective.

Proof. (1) $\implies$ (2): We first establish that $d_1^{1,1} : S \otimes_R W \to H_1(E)$ is injective (and is thus an isomorphism of S-modules). Recall that the terms $E_{p,q}^0$ are non-zero only for (p, q) satisfying $q \geq p \geq 0$. Thus $E_{2,1}^0 = 0$, and so $E_{1,1}^2 = \operatorname{Ker} d_1^{1,1}$. Moreover,

$E^2_{1,1} = E^\infty_{1,1}$ is (isomorphic to) a subquotient of $H_2(T)$, so that $d_1^{1,1}$ is injective, as desired.

We now focus on the properties of the maps λ_i described in (2). In addition, we show that the following condition holds:

(3) $E^2_{0,k+1} = 0$ and $E^2_{p,q} = 0$ for all $(p,q) \neq (0,0)$ with $0 \leq p \leq q \leq p+k-1$.

We will establish (2) and (3) by induction on k.

Suppose $k = 1$. We begin with (3) and show that $E^2_{q,q} = 0$ for all $q \geq 1$. Let C be the Cartan construction on the free S-module $H_1(E)$ and let D^n denote the strand $C_{*,n}$ (see Construction 2.14). Let Γ_i denote $\Gamma_i^S(W \otimes_R S)$. For each $q \geq 1$, we have morphisms relating the row E^1_{*q} of the E^1 page of the spectral sequence to the strand D^q:

$$\begin{tikzcd}
E^1_{*q}: & \cdots & H_2(E) \otimes_S \Gamma_{q-2} \arrow[l] & H_1(E) \otimes_S \Gamma_{q-1} \arrow[l] & \Gamma_q \arrow[l] & 0 \arrow[l] \\
D^q: & \cdots & \wedge_2^S H_1(E) \otimes_S \Gamma_{q-2} \arrow[l] \arrow[u, "\lambda_2^S \otimes \Gamma_{q-2}"] & H_1(E) \otimes_S \Gamma_{q-1} \arrow[l] \arrow[u, "\lambda_1^S \otimes \Gamma_{q-1}", "="'] & \Gamma_q \arrow[l] \arrow[u, "="'] & 0 \arrow[l]
\end{tikzcd} \tag{3.11}$$

From this diagram we conclude that $E^2_{q,q} = H_q(D^q)$. But Remark 2.19 yields that $H_q(D^q) = 0$ for all $q \geq 1$, and so $E^2_{q,q} = 0$ for all $q \geq 1$, as desired. It remains to show that $E^2_{0,2} = 0$. We have $E^2_{0,2} = E^\infty_{0,2}$; this is (isomorphic to) a subquotient of $H_2(T)$, so that $E^2_{0,2} = 0$.

For (2) note that λ_1^S is the identity map on $H_1(E)$; we now show that λ_2^S is surjective. Note that $E^1_{-1,2} = 0$, and so $E^2_{0,2} = \operatorname{Coker} d_1^{1,2}$. But $E^2_{0,2} = 0$, and so $d_1^{1,2}$ is surjective. We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E^1_{*,2}: & 0 & H_2(E) \arrow[l] & H_1(E)\otimes_S \Gamma_1 \arrow[l, "d_1^{1,2}"'] \\
D^2_*: & 0 & \wedge^S_2 H_1(E) \arrow[l] \arrow[u, "\lambda_2^S"'] & \wedge^S_1 H_1(E)\otimes_S \Gamma_1 \arrow[l] \arrow[u, "\cong", "\lambda_1^S\otimes\Gamma_1"']
\end{tikzcd}$$

Thus λ_2^S is surjective, as desired.

Suppose now that $k \geq 2$. By construction and by induction $E^2_{1,k} = E^\infty_{1,k}$; this is (isomorphic to) a subquotient of $H_{k+1}(T)$, and so $E^2_{1,k} = 0$. Similarly, $E^2_{0,k} = 0$. These equalities yield the following commutative diagram with exact rows:

$$\begin{tikzcd}
E^1_{*k}: & 0 & H_k(E) \arrow[l] & H_{k-1}(E)\otimes_S \Gamma_1 \arrow[l] & H_{k-2}(E)\otimes_S \Gamma_2 \arrow[l] \\
D^k: & 0 & \wedge^S_k H_1(E) \arrow[l] \arrow[u, "\lambda_k^S"'] & \wedge^S_{k-1} H_1(E)\otimes_S \Gamma_1 \arrow[l] \arrow[u, "\cong", "\lambda_{k-1}^S\otimes\Gamma_1"'] & \wedge^S_{k-2} H_1(E)\otimes_S \Gamma_{k-2} \arrow[l] \arrow[u, "\cong", "\lambda_{k-2}^S\otimes\Gamma_2"']
\end{tikzcd} \tag{3.12}$$

An application of the four lemma now gives that λ_k^S is an isomorphism.

To establish (3) it remains to show that $E^2_{pq} = 0$ for all $(p, q) = (q - k + 1, q)$, where $q \geq k + 1$. For each such q, we have the following commutative diagram:

$$\begin{tikzcd}
E^1_{*q}: & H_k(E)\otimes_S \Gamma_{q-k} & H_{k-1}(E)\otimes_S \Gamma_{q-k+1} \arrow[l] & H_{k-2}(E)\otimes \Gamma_{q-k+2} \arrow[l] \\
D^q: & \wedge^S_k H_1(E)\otimes_S \Gamma_{q-k} \arrow[u, "\cong", "\lambda_k^S\otimes\Gamma_{q-k}"'] & \wedge^S_{k-1} H_1(E)\otimes_S \Gamma_{q-k+1} \arrow[l] \arrow[u, "\cong", "\lambda_{k-1}^S\otimes\Gamma_{q-k+1}"'] & \wedge^S_{k-2} H_1(E)\otimes \Gamma_{q-k+2} \arrow[l] \arrow[u, "\cong", "\lambda_{k-2}^S\otimes\Gamma_{q-k+2}"']
\end{tikzcd} \tag{3.13}$$

Hence we have an isomorphism $E^2_{q-k+1,q} \cong H_{q-k+1}(D^q)$. Noting that $q-k+1 \geq 2$, Remark 2.19 yields $H_{q-k+1}(D^q) = 0$, and so $E^2_{q-k+1,q} = 0$ for each $q \geq k + 1$.

For (2), it remains to show that λ^S_{k+1} is surjective. As $E^1_{-1,k+1} = 0$, we have $\operatorname{Coker} d_1^{1,k+1} = E^2_{0,k+1}$. But $E^2_{0,k+1} = E^\infty_{0,k+1}$; this is (isomorphic to) a subquotient of

$H_{k+1}(T)$, so that $E^2_{0,k+1} = 0$. This in turn yields that $d_1^{1,k+1}$ is surjective. From this, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
E^1_{*,k+1}: & 0 & H_{k+1}(E) \arrow[l] & H_k(E) \otimes_S \Gamma_1 \arrow[l, "d_1^{1,k+1}"'] \\
D^{k+1}_*: & 0 & \wedge^S_{k+1} H_1(E) \arrow[l] \arrow[u, "\lambda^S_{k+1}"'] & \wedge^S_k H_1(E) \otimes_S \Gamma_1 \arrow[l] \arrow[u, "\cong", "\lambda^S_k \otimes \Gamma_1"']
\end{tikzcd} \tag{3.14}$$

From this, we conclude that λ^S_{k+1} is surjective.

(2) $\Longrightarrow$ (1): As above, let C denote the Cartan construction on the free S-module $H_1(E)$ and consider its strands D^n. First, we have that $E^2_{p,q} = 0$ for $(p,q) = (0,k)$ and for all $(p,q) \neq (0,0)$ with $0 \leq p \leq q \leq p+k-1$. Indeed, by utilizing commutative diagrams analogous to (3.11), (3.12), and (3.13), we have that $E^2_{p,q}$ is isomorphic to $H_p(D^q)$ for $(p,q) = (0,k)$ and for all $(p,q) \neq (0,0)$ with $0 \leq p \leq q \leq p+k-1$. Consequently, Remark 2.19 yields that $E^2_{p,q} = 0$ for all such (p,q). Second, noting that λ^S_{k+1} is surjective, we see from a diagram analogous to (3.14) that $E^2_{0,k+1} = 0$ as well.

In particular, this vanishing of E^2 in this region yields that $E^\infty_{p,q} = E^2_{p,q} = 0$ for all (p,q) satisfying $0 < p+q \leq k+1$. For each such (p,q) we have a finite filtration

$$0 = F_{-1}H_{p+q} \subseteq F_0 H_{p+q} \subseteq \cdots \subseteq F_{p+q}H_{p+q} = H_{p+q}(T). \tag{3.15}$$

Each quotient of consecutive terms has the form

$$\frac{F_p H_{p+q}}{F_{p-1} H_{p+q}} \cong E^\infty_{p,q} = 0.$$

We therefore conclude that each containment in (3.15) is an equality, and hence $H_{p+q}(T) = 0$ for all $0 < p+q \leq k+1$, so that $H_i(T) = 0$ for all $i = 2, \ldots, k+1$. $\square$

With Proposition 3.10 in hand, we now provide the proof of our first characterization, Theorem 3.9. Note that the result of Blanco, Majadas, and Rodicio ([10, Theorem 1]) establishes the equivalence (1) $\Longleftrightarrow$ (2).

Proof of Theorem 3.9. (1) $\Longrightarrow$ (2) was established by Tate [25], and (2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (1): By Proposition 3.10, $H_1(E)$ is free as an S-module via $d_1^{1,1} : S \otimes_R W \to H_1(E)$, so that $H_1(E)$ has the desired basis. Moreover, λ_i^S is an isomorphism for $i = 1, 2, \ldots, b+1$. As $H_{b+1}(E) = 0$, we have that $\wedge_{b+1}^S H_1(E) = 0$, and so $\operatorname{rank}_S H_1(E) \leq b$. Then for each $i > b+1$ we have the equality $\wedge_i^S H_1(E) = 0$ and λ_i^S is an isomorphism (of zero modules). $\square$

Remark 3.16. When R is Noetherian, the integer b can be computed as follows: For $I = (f_1, \ldots, f_c)$ and $I \neq I^2$, [21, Theorem 16.8] yields that $b = c - \operatorname{depth}(I, R)$, where $\operatorname{depth}(I, R)$ denotes the length of a maximal R-sequence contained in I.

3.2.3 General band of vanishing of $H_*(T)$

Utilizing the vanishing and periodicity results of Section 2.7, we now make an observation about the eventual vanishing of the homology of the Tate construction.

Proposition 3.17. *Suppose $\boldsymbol{z} = \{z_1, \ldots, z_m\}$ is a set of cycles of degree one whose homology classes generate $H_1(E)$. Let T be Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$. The following conditions are equivalent:*

(1) $H_i(T) = 0$ *for all* $i \geq b - m + 1$.

(2) $H_i(T) = 0$ *for all* $i \gg 0$.

(3) There exists an integer $q \geq b - m + 1$ *such that* $H_i(T) = 0$ *for all* $i \in [q, q+m]$.

Proof. The implications (1) $\implies$ (2) $\implies$ (3) are immediate.

(3) $\implies$ (1). Note that $T = E\langle w_1, \ldots, w_m \,|\, \partial w_i = z_i \rangle$, where $|z_i| = 1$; Proposition 2.31 now yields that $H_i(T) = 0$ for all $i \geq b - m + 1$. $\square$

Remark 3.18. Proposition 3.17 shows that the Tate construction exhibits Koszul-like rigidity (see Section 2.5), in that a band of vanishing (of sufficient width) of $H_*(T)$ starting in degree q guarantees that $H_i(T) = 0$ for (at least) $i \geq q$. This gives a partial answer to Question 2.22.

Remark 3.19. If I is a quasi-complete intersection ideal, then $\operatorname{rank}_S H_1(E) = b$. Indeed, we have the following equalities:

$$\begin{aligned}
\operatorname{rank}_S H_1(E) &= \max\{i \,:\, \wedge_i^S H_1(E) \neq 0\} \\
&= \max\{i \,:\, H_i(E) \neq 0\} \\
&= b.
\end{aligned}$$

The following characterization is Proposition 3.17 in the case $b = m$:

Proposition 3.20. *Suppose $\boldsymbol{z} = \{z_1, \ldots, z_b\}$ is a set of cycles whose homology classes generate $H_1(E)$. Let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$. The following conditions are equivalent:*

(1) I is a quasi-complete intersection.

(2) $H_i(T) = 0$ for all $i \gg 0$.

(3) There exists an integer $q \geq 1$ such that $H_i(T) = 0$ for all $i \in [q, q + b]$.

Chapter 4

Complete intersections

Let $(R, \mathfrak{m}, k)$ be a local (Noetherian) ring.

Definition 4.1. We say that R is a *complete intersection* if the $\mathfrak{m}$-adic completion $\widehat{R}$ can be written as a quotient of a (complete) regular local ring by a regular sequence.

Remark 4.2. Heitmann and Jorgensen [16, Corollary 1.6] show that if R is a complete intersection which is a one-dimensional integral domain, then itself has a presentation $R = Q/J$ where Q is a regular local ring and J is generated by a Q-regular sequence. However, this property does not hold in general. Indeed, Heitmann and Jorgensen [16, Theorem 2.25] construct a three-dimensional complete intersection domain which is not itself the quotient of a regular local ring by a regular sequence.

Let E denote the Koszul complex on a minimal generating set of $\mathfrak{m}$, and let T denote the Tate construction on $\mathfrak{m}$ (see Remark 2.11). The following result of Assmus characterizes (local) complete intersection rings in terms of the homological structure of E and T:

Theorem 4.3 ([1, Theorem 2.7])**.** *The following conditions are equivalent:*

(1) R is a complete intersection.

(2) $H_*(E)$ *is the exterior algebra on* $H_1(E)$.

(3) $H_2(E) = H_1(E)^2$.

(4) $H_2(T) = 0$

In light of Remark 3.3, the equivalence (1) $\iff$ (2) of Theorem 4.3 can be restated as follows:

$$R \text{ is a complete intersection} \iff \mathfrak{m} \text{ is a quasi-complete intersection.}$$

In this chapter, we continue the theme of Chapter 3 and discuss the quasi-complete intersection property of $\mathfrak{m}$. These results will come in two flavors:

(I) *We improve upon the result of Proposition 3.20.*

The quasi-complete intersection property of $\mathfrak{m}$ (and hence the complete intersection property of R) can be detected from a small band of vanishing of $H_*(T)$; see Theorem 4.9.

(II) *We generalize the implication (4)* $\implies$ *(1) of Theorem 4.3.*

In general, the complete intersection property of R can be detected from the vanishing of $H_3(T)$ or $H_4(T)$; see Theorem 4.11. (The case of the vanishing of $H_*(T)$ in a single degree will be further explored in Chapter 5).

4.1 General band of vanishing of $H_*(T)$

In this section, we present the results of this chapter which fall into the first flavor, namely those that improve upon Proposition 3.20. The size of a minimal generating set of an R-module M is denoted $\nu_R(M)$.

Remark 4.4. If R is complete, then the *Cohen Structure Theorem* yields that R has a *minimal regular presentation*. I.e., there exists a (complete) regular local ring $(Q, \mathfrak{n})$ such that $R = Q/J$ and $J \subset \mathfrak{n}^2$. Then

$$\nu_Q(J) = \varepsilon_2(R) = \nu_R(H_1(E)),$$

where the first equality is due to [4, Corollary 7.1.5] and the second is [4, Theorem 7.1.3].

The following construction describes a convenient (minimal) generating set of $H_1(E)$:

Construction 4.5 ([1, pp 196-197]). Suppose that R is complete and fix a minimal presentation $R = Q/J$. Let $\boldsymbol{x} = x_1, \ldots, x_n$ denote a minimal generating set of $\mathfrak{m}$, and $a_1, \ldots, a_m$ denote a minimal generating set of J. As $J \subset \mathfrak{n}^2$, we have an equality $\operatorname{embdim} Q = \operatorname{embdim} R$, so that we may select a minimal generating set $t_1, \ldots, t_n$ of $\mathfrak{n}$ such that the image in R of t_j is x_j. For each $i \in \{1, \ldots, m\}$ write

$$a_i = \sum_{j=1}^{n} c_{ij} t_j.$$

For each $i \in \{1, \ldots, m\}$ define the element z_i of $E = R\langle u_1, \ldots, u_n \,|\, \partial u_j = x_j \rangle$ by

$$z_i = \sum_{j=1}^{n} c'_{ij} u_j,$$

where c'_{ij} is the image in R of c_{ij}. Then $\boldsymbol{z} = \{z_1, \ldots, z_n\}$ is a set of degree one cycles of E representing a minimal generating set of $H_1(E)$.

In the next construction, we introduce an intermediate complete intersection ring $(Q', \mathfrak{n}')$ and describe a relationship between the Tate constructions on $\mathfrak{n}'$ and $\mathfrak{m}$. As usual, we set $b = \max\{i : H_i(E) \neq 0\}$.

Construction 4.6. Suppose that R is complete (with minimal regular presentation $R = Q/J$) and that $\operatorname{rank}_k H_1(E) = b$, so that $b = \nu_Q(J)$. Select a maximal Q-sequence $a_1, \ldots, a_h$ in J so that the images $\{a_i{}'\}$ in $J/\mathfrak{n}J$ are linearly independent over $Q/\mathfrak{n}$; we may extend the sequence to a minimal generating set $a_1, \ldots, a_b$ of J. Put $J' = (a_1, \ldots, a_h)$ and let $(Q', \mathfrak{n}')$ denote $(Q/J', \mathfrak{n}/J')$.

Let K' denote the Koszul complex on a minimal generating set $\boldsymbol{x'}$ of $\mathfrak{n}'$. As before, $h = \nu_{Q'}(H_1(K'))$. Apply Construction 4.5 to find a set of degree one cycles $\boldsymbol{z'} = \{z'_1, \ldots, z'_h\}$ whose homology classes form a minimal generating set for $H_1(K')$. For $i = 1, \ldots, h$, set

$$z_i = z'_i \otimes_{Q'} R',$$

the image of z'_i in $E = K' \otimes_{Q'} R$. The set of cycles $\{z_1, \ldots, z_h\}$ is the first h cycles given by Construction 4.5, and so the set may be extended to a set of cycles $\boldsymbol{z} = z_1, \ldots, z_b$ whose homology classes form a minimal generating set of $H_1(E)$.

Let F' denote the Tate construction on $\boldsymbol{x'}$ and $\boldsymbol{z'}$, so that

$$F' = K'\langle w_1, \ldots, w_h \,|\, \partial w_i = z'_i\rangle.$$

Set

$$F = F' \otimes_{Q'} R = E\langle w_1, \ldots, w_h \,|\, \partial w_i = z_i\rangle.$$

Then

$$T = F\langle w_{h+1}, \ldots, w_b \,|\, \partial w_i = z_i\rangle,$$

where T is the Tate construction on $\boldsymbol{x}$ and $\boldsymbol{z}$.

In the setup of the previous construction, we will be interested in showing that the natural surjection $\pi : Q' \to R$ is an isomorphism. The following result of Auslander and Buchsbaum will be useful for this effort.

Proposition 4.7 ([2, Proposition 6.2])**.** *Let M be a finitely-generated R-module and*

$$0 \longrightarrow X_p \longrightarrow X_{p-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

an exact sequence where the X_i are free R-modules. The following conditions are equivalent:

(1) $(0 :_R M) \neq 0$.

(2) $\sum_{i=0}^{p}(-1)^i \operatorname{rank}_R X_i = 0$.

(3) $(0 :_R M)$ *contains a non-zerodivisor on R.*

Remark 4.8. The previous result relates $\operatorname{Ker}(\pi : Q' \to R)$ to $\operatorname{pd}_{Q'} R$. Indeed, Construction 4.6 yields that $(0 :_{Q'} R)$ (which is precisely $\operatorname{Ker}\pi$) contains only zerodivisors. Thus, we have the implication

$$\operatorname{pd}_{Q'} R < \infty \implies (0 :_{Q'} R) = 0.$$

We now provide the improvement of Proposition 3.20. As usual, we set $b = \max\{i : H_i(E) \neq 0\}$ and T is the Tate construction on $\mathfrak{m}$.

Theorem 4.9. *Suppose there exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i = [q, q+b-1]$. Then R is a complete intersection.*

Proof. Here we follow the strategy of Gulliksen [12]. Without loss of generality, we may assume that R is complete. Recall the notation of Construction 4.6. Let $\pi : Q' \to R$ be the natural surjection. We will show that $\operatorname{Ker}\pi = 0$; by Remark 4.8 it will be enough to show that $\operatorname{pd}_{Q'} R < \infty$.

As Q' is a complete intersection, Theorem 4.3 yields F' is a Q'-free resolution of k, so that $\operatorname{Tor}_i^{Q'}(R,k) = H_i(F)$. By hypothesis, there exists an integer $q \geq 2$ such that $H_i(T) = 0$ for $i = [q, q+b-1]$. Noting that we have obtained T from F by adjoining at most $b-1$ variables of degree two, Lemma 2.30 (1) yields that $H_{q+b-1}(F) = 0$. This implies that $\operatorname{Tor}_{q+b-1}^{Q'}(R,k) = 0$ for some $q \geq 2$. Hence, $\operatorname{pd}_{Q'} R < \infty$, completing the proof. □

Let $R\langle X\rangle$ denote an acyclic closure of k over R and order the variables X such that $|x_i| \leq |x_j|$ for $i < j$; see Construction 2.23. Fix an integer p and let Y denote the extension $R\langle x_i : i \leq p\rangle$. (An example of such an extension is a partial acyclic closure $R\langle X_{\leq n}\rangle$.) We note that the following result appears implicitly in work of Gulliksen [12]:

Proposition 4.10. *Let F be as in Construction 4.6 and suppose that $F \subseteq Y$. If $H_i(Y) = 0$ for all $i \gg 0$, then R is a complete intersection.*

Proof. The DG-algebra Y satisfies the conditions of [12, Lemma 1]. Now $H_i(Y) = 0$ for all $i \gg 0$ and Y is obtained from F by an adjunction of (finitely many) variables, so a repeated application of [12, Lemma 2] yields $H_i(F) = 0$ for all $i \gg 0$. But $H_i(F) = \operatorname{Tor}_i^{Q'}(R,k)$, so that $\operatorname{pd}_{Q'} R < \infty$. Consequently, Remark 4.8 yields that R is a complete intersection. □

In particular, Gulliksen [12] implicitly contains a proof of the local case of the equivalence (1) $\Longleftrightarrow$ (2) of Proposition 3.20: *The eventual vanishing of $H_*(T)$ is equivalent to the complete intersection property of R (i.e, the quasi-complete intersection property of $\mathfrak{m}$).*

4.2 Vanishing of $H_*(T)$ in a single low degree

We now show that certain vanishing of $H_*(T)$ in a single degree detects the complete intersection property of R.

Theorem 4.11. *If $H_i(T) = 0$ for $i = 3$ or 4, then R is a complete intersection.*

Proof. We may assume that R is not a complete intersection, so that $H_2(T) \neq 0$.

If $H_3(T) \neq 0$, then we apply Lemma 2.33 (2) and adjoin variables of degree three to obtain a partial acyclic closure B of k with $H_i(B) = 0$ for $i \in \{1, 2, 3\}$. This yields $\varepsilon_4(R) = 0$, so that by a result of Gulliksen [13, Theorem 3.5.1], R is a complete intersection, a contradiction.[1]

Suppose now that $H_4(T) = 0$. We adjoin variables of degrees 3 and 4; applying Lemma 2.33 (1) and (2), we obtain a partial acyclic closure V of k with $H_i(V) = 0$ for $i = 1, 2, 3, 4$, so that $\varepsilon_5(R) = 0$. Now Halperin [15, Theorem B] gives that R is a complete intersection, a contradiction. $\square$

The preceding result and the earlier work of Assmus (Theorem 4.3) suggest the following question:

Question 4.12. Does the implication

$$H_i(T) = 0 \implies R \text{ is a complete intersection}$$

hold for every $i \geq 0$?

This question will be further explored in Chapter 5.

[1] Recall (from Remark 2.25) that the indexing convention of the ε_n differs from that of Gulliksen and Levin [13]. In particular, ε_3 of [13] stands for ε_4 of [4], so that the theorem of Gulliksen applies.

Chapter 5

Rigidity of the Tate construction

In this chapter, we present a class of rings for which Question 4.12 has an affirmative answer. We begin by presenting the background necessary to introduce this class.

5.1 Golod rings

In this section, $(Q, \mathfrak{n}, k)$ and $(R, \mathfrak{m}, k)$ are local rings. Let $\varphi : Q \to R$ denote a homomorphism of local rings which induces the identity on k. Let M denote a finitely generated R-module.

Recall the *Poincaré series* of M over R:

$$P_M^R(t) = \sum_{n=0}^{\infty} \beta_n^R(t) t^n \in Z[\![t]\!].$$

The following result relates the Betti numbers of M over R and Q.

Proposition 5.1 ([4, Proposition 3.3.2])**.** *Then there is a coefficientwise inequality of formal power series*

$$P_M^R(t) \preccurlyeq \frac{P_M^Q(t)}{1 - t(P_R^Q(t) - 1)}. \tag{5.2}$$

Assume further that $Q \to R$ is a minimal regular presentation. Let K^R denote the Koszul complex on a minimal generating set of $\mathfrak{m}$, and set $K^M = K^R \otimes_R M$. Recall the *codepth* of R is defined as

$$\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R.$$

Then Proposition 5.1 becomes:

Proposition 5.3 ([4, Proposition 4.1.4])**.** *There is a coefficientwise inequality of formal power series*

$$P_M^R(t) \preccurlyeq \frac{\displaystyle\sum_{i=0}^{\operatorname{edim} R - \operatorname{depth} M} \operatorname{rank}_k H_i(K^M) t^i}{1 - \displaystyle\sum_{j=1}^{\operatorname{codepth} R} \operatorname{rank}_k H_j(K^R) t^{t+1}}.$$

A ring for which $P_k^R(t)$ has the fastest growth allowed by Proposition 5.3 is called a *Golod ring*; (5.3) takes the form

$$P_k^R(t) = \frac{(1+t)^{\operatorname{edim} R}}{1 - \displaystyle\sum_{j=1}^{\operatorname{codepth} R} \operatorname{rank}_k H_j(K^R) t^{t+1}}. \tag{5.4}$$

Remark 5.5. As noted in [4, p 47], the Golod property of local rings does not fit in the heirarchy

$$\text{regular} \implies \text{complete intersection} \implies \text{Gorenstein} \implies \text{Cohen-Macaulay.}$$

In particular, a Golod ring which is Gorenstein is necessarilly a hypersurface, but a Golod ring need not be Cohen-Macaulay. A further disparty is that the Golod property is not stable under localization; see [4, Example 5.2.6].

5.2 Golod homomorphisms

Definition 5.6 ([4, §3.3])**.** A surjective homomorphism $\varphi : Q \to R$ is called a *Golod homomorphism* if equality holds in (5.2) for $M = k$.

Remark 5.7. Suppose that $\psi : Q \to R$ is a minimal regular presentation. Then R is Golod precisely when ψ is a Golod homomorphism.

Theorem 5.8. *Suppose that there exists a complete intersection ring Q and a Golod homomorphism $\varphi : Q \to \widehat{R}$. If $H_i(T) = 0$ for some $i \geq 5$, then R is a complete intersection.*

Proof. Without loss of generality, we may assume that R is complete. By [8, Proposition 5.13] we may further assume that $\operatorname{depth}_Q(R) = 0$. We endeavor to show that $\operatorname{Ker} \varphi = 0$. By Remark 4.8 it is enough to show that $\operatorname{pd}_Q R < \infty$.

Let F' denote the Tate construction on $\mathfrak{n}$, and put $F = R \otimes_Q F'$. As Q is a complete intersection, we have that F' is a Q-free resolution of k. Let A denote the trivial extension $k \ltimes H_{\geq 1}(F)$; see Definition 2.18. Then [3, Theorem 2.3] yields that F and A are equivalent as DG-algebras.

Let $\boldsymbol{y}$ be a set of cycles of degree one whose homology classes form a minimal generating set of $H_1(F)$, and let C denote the Cartan construction on A and $\boldsymbol{y}$. Then [13, Proposition 1.3.5] yields the equivalence $T \simeq C$. Thus, there exists an integer $i \geq 5$ with $H_i(C) = 0$.

Remark 2.15 yields that C has a direct sum decomposition

$$C = \bigoplus_{j \geq 0} D^j,$$

where D^j is the strand C_{*j}:

$$0 \leftarrow H_j(F) \xleftarrow{\partial_1^{D_j}} H_{j-1}(F) \otimes \Gamma_1^k W \xleftarrow{\partial_2^{D_j}} \cdots \leftarrow H_1(F) \otimes \Gamma_{j-1}^k W \xleftarrow{\partial_j^{D_j}} \Gamma_j^k W \leftarrow 0$$

Consequently we have the following decomposition of the homology of C:

$$H_k(C) = \bigoplus_{i \geq 0} H_i(D^{k-i}) = \bigoplus_{i=0}^{k} H_i(D^{k-i}). \tag{5.9}$$

The equivalence $F \simeq A$ yields that $[H_{\geq 1}(F)]^2 = 0$, and so the differential $\partial_i^{D_j}$ is zero for each i in $[1, j-1]$. In light of (5.9), this yields that $H_0(D^k) = H_k(F)$ for each $k \geq 2$, so that $H_k(C)$ contains $H_k(F)$ as a summand for each $k \geq 2$. As such, $H_i(F) = 0$, and so $\operatorname{Tor}_i^Q(R, k) = 0$. Therefore, $\operatorname{pd}_Q R < \infty$, and hence R is a complete intersection. $\square$

Remark 5.10. The hypotheses of Theorem 5.8 are satisfied in the following situations:

(1) R is a Golod ring,

(2) R is Gorenstein and $\operatorname{embdim} R = 4$; see [18, Theorem B],

(3) $\operatorname{codepth} R \leq 3$; see [8, Proposition 6.1],

(4) $\mathfrak{m}$ has a *Conca generator* (i.e., there exists $x \in \mathfrak{m}$ such that $x^2 = 0$ and $\mathfrak{m}^2 = x\mathfrak{m}$); see [7, Theorem 1.4].

(5) R is a *compressed Gorenstein ring* of socle degree s and embedding dimension e for $2 \leq s \neq 3$ and $e > 1$; see [23, Theorem 5.1].

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