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The Existence of Solutions for a Nonlinear, Fractional Self-Adjoint Difference Equation

Kevin Ahrendt
University of Nebraska-Lincoln, kahrendt@huskers.unl.edu

THE EXISTENCE OF SOLUTIONS FOR A NONLINEAR, FRACTIONAL SELF-ADJOINT DIFFERENCE EQUATION

by

Kevin Ahrendt

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THE EXISTENCE OF SOLUTIONS FOR A NONLINEAR, FRACTIONAL SELF-ADJOINT DIFFERENCE EQUATION

Kevin Ahrendt, Ph.D.

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Adviser: Allan Peterson

In this work we will explore a fractional self-adjoint difference equation which involves a Caputo fractional difference. In particular, we will develop a Cauchy function for initial value problems and Green's functions for several different types of boundary value problems. We will use the properties of those Green's functions and the Contraction Mapping Theorem to find sufficient conditions for when a nonlinear boundary value problem has a unique solution. We will also investigate the existence of nonnegative solutions for a nonlinear self-adjoint difference that have particular long run behavior.

DEDICATION

This dissertation is dedicated to my parents, Randy and Sandi Ahrendt.

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PREFACE

The history of fractional calculus extends back to 1695 when L'Hopital asked Leibniz about the nature of a one-half derivative. In the 1800s, Liouville gave a strong theoretical foundation for studying fractional derivatives leading to the development of the Riemann-Liouville Definition of a fractional derivative. Caputo later defined the Caputo fractional derivative, which a form of is studied in this work. See [25] for a brief history on fractional calculus.

There are many real world applications for the fractional derivative. In the typical continuous case, fractional calculus can model containment flow in heterogeneous porous media [11] [12], waves in viscoelastic media [1], and waves in complex media like biological tissue [29]. In the discrete case, Atici and Sengul [8] use fractional difference equations to model tumor growth.

The discrete fractional calculus has a domain of a specific time scale. See [13] and [14] for more results on time scales in the general setting. Whole order difference equations are studied in detail in [27]. Work in the discrete fractional calculus was heavily advanced for the delta case in [9] [22] [23]. A broad overview of Discrete Fractional Calculus is given in [17].

The results in Chapter 1 are mostly well known background material. Section 1.4 contains some new results that will be useful in later proofs. Results in Chapter 2 and Section 3.1 contain results where the basic problem has been adjusted to fix mistakes by Ahrendt, et al in [3]. Section 3.2 cites results from [16]. Section 3.3 contains new work. Finally, Chapter 4's results are all new.

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Chapter 1

Introduction to Nabla Fractional Calculus

We will first look at the nabla discrete calculus before going into detail about the fractional case. A full treatment of the nabla discrete calculus is given in [17, Chapter 3]. Closely related is the delta discrete calculus, which appears heavily in ordinary difference equations. For more information on ordinary difference equations, see [27].

For the following results we will let $a \in \mathbb{R}$ be a fixed constant. We will also follow the convention that $b \in \mathbb{R}$ such that $b - a$ is a natural number. Then we define the form of two domains we will be dealing with:

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\} \quad \text{and} \quad \mathbb{N}_a^b := \{a, a+1, \ldots, b-1, b\}.$$

1.1 Basic Results for the Nabla Whole-Order Calculus

1.1.1 Nabla Difference

Definition 1. [17] The *backwards jump operator* $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by

$$\rho(t) := \max\{a, t-1\}.$$

The backwards jump operator can be loosely thought of as the previous point in the domain.

The analog to the derivative in ordinary real-valued calculus is the nabla difference.

Definition 2. [17] Let $f : \mathbb{N}_a \to \mathbb{R}$. Then the *nabla difference of* f is defined by

$$(\nabla f)(t) := f(t) - f(t-1),$$

for $t \in \mathbb{N}_{a+1}$. For convenience, we will use the notation $\nabla f(t) := (\nabla f)(t)$. For $N \in \mathbb{N}_2$, we have that the *Nth order fractional difference* is recursively defined as

$$\nabla^N f(t) := \nabla(\nabla^{N-1} f(t)),$$

for $t \in \mathbb{N}_{a+N}$.

Remark 3. We can reformulate the previous definition in terms of the backwards jump operator. If $f : \mathbb{N}_a \to \mathbb{R}$, then $(\nabla f)(t) := f(t) - f(\rho(t))$ for $t \in \mathbb{N}_{a+1}$.

Remark 4. We treat the difference operator of order 0 as the identity operator, i.e.

$$\nabla^0 f(t) = f(t).$$

With the difference operator in hand, the next theorem shows that the expected results of derivatives in the typical real case have analogs to the nabla discrete case.

Theorem 5 (Properties of the Nabla Difference)**.** *[17] Assume $f, g : \mathbb{N}_a \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Then for $t \in \mathbb{N}_{a+1}$,*

1. $\nabla \alpha = 0$*;*

2. $\nabla \alpha f(t) = \alpha \nabla f(t)$*;*

3. $\nabla(f(t)+g(t)) = \nabla f(t) + \nabla g(t)$;

4. $\nabla\left(f(t)g(t)\right) = f(\rho(t))\nabla g(t) + \nabla f(t)g(t)$;

5. $\nabla \frac{f(t)}{g(t)} = \frac{g(t)\nabla f(t) - f(t)\nabla g(t)}{g(t)g(\rho(t))}$, *if* $g(t) \neq 0$ *for all* $t \in \mathbb{N}_{a+1}$.

We also desire a power rule that matches what we would expect in the typical real case. To do so, we will next define the rising function.

Definition 6. [17] For $t \in \mathbb{N}_a$ and $n \in \mathbb{N}_1$, the *rising function* $t^{\overline{n}}$ is defined as follows:

$$t^{\overline{n}} := t(t+1)(t+2)\cdots(t+n-1).$$

We read $t^{\overline{n}}$ as t to the n rising.

Remark 7. We can reformulate this definition of the rising function using factorial functions when our domain is based at $a = 1$. This form will be useful when we generalize the rising function in the next section. So for $t \in \mathbb{N}_1$ and $n \in \mathbb{N}_1$

$$t^{\overline{n}} := \frac{(t+n-1)!}{(t-1)!}.$$

The rising function as defined gives us a power rule that behaves as expected from the real-valued calculus case.

Theorem 8 (Nabla Power Rule)**.** *[17] For* $n \in \mathbb{N}_1$ *and* $\alpha \in \mathbb{R}$*,*

$$\nabla_t (t+\alpha)^{\overline{n}} = n(t+\alpha)^{\overline{n-1}},$$

for $t \in \mathbb{R}$*.*

1.1.2 Nabla Integral

Just as we have a derivative operator for the nabla calculus, we have an integral operator.

Definition 9. [17] Let $f : \mathbb{N}_a \to \mathbb{R}$ and let $c, d \in \mathbb{N}_a$. Then the *definite nabla integral of f from c to d* is defined by

$$\int_c^d f(s)\nabla s := \begin{cases} \sum_{s=c+1}^d f(s), & c < d, \\ 0, & d \leq c. \end{cases}$$

Remark 10. We can think of the nabla integral as a right-hand Riemann sum from real-valued calculus. Note that $\int_a^b f(s)\nabla s$ does not depend on the value of the function at $t = a$. Since the nabla integral is defined as a sum, we may say *nabla sum* in place of the term *nabla integral*.

Remark 11. We often use a shorthand notation to represent the nabla integral in a similar manner as notating the nabla difference. We say

$$\nabla_a^{-1} f(t) := \int_a^b f(t)\nabla s.$$

Here the 'a' in $\nabla_a^{-1} f(t)$ represents the lower limit of the above integral, in which case we say the integral is based at a. The '-1' represents taking the nabla integral, where the negative sign indicates we are doing the opposite operation of a nabla difference. We can extend this to other integer values where

$$\nabla_a^{-n} f(t) := \int_a^t \int_a^{\tau_1} \int_a^{\tau_2} \cdots \int_a^{\tau_{n-1}} f(\tau_n)\nabla\tau_n \nabla\tau_{n-1} \cdots \nabla\tau_2 \nabla\tau_1.$$

This notation will be used for integration and differentiation of fractional orders.

The next theorem gives properties of the nabla integral, all of which have an analogous result in the real-valued calculus.

Theorem 12 (Properties of the Nabla Integral)**.** *[17] Assume* $f, g : \mathbb{N}_{a+1} \to \mathbb{R}$*,* $b, c, d \in \mathbb{N}_a$ *such that* $b \leq c \leq d$*, and* $\alpha \in \mathbb{R}$*. Then*

1. $\int_b^c \alpha f(t)\nabla t = \alpha \int_b^c f(t)\nabla t$;

2. $\int_b^c (f(t) + g(t))\nabla t = \int_b^c f(t)\nabla t + \int_b^c g(t)\nabla t$;

3. $\int_b^d f(t)\nabla t = \int_b^c f(t)\nabla t + \int_c^d f(t)\nabla t$;

4. $\left|\int_b^c f(t)\nabla(t)\right| \leq \int_b^c |f(t)|\nabla t$;

5. *If* $F(t) := \int_b^t f(s)\nabla s$*, for* $t \in \mathbb{N}_b^c$*, then* $\nabla F(t) = f(t)$*,* $t \in \mathbb{N}_{b+1}^c$;

6. *If* $f(t) \geq g(t)$ *for* $t \in \mathbb{N}_{b+1}^c$*, then* $\int_b^c f(t)\nabla t \geq \int_b^c g(t)\nabla t$.

Definition 13 (Nabla Antidifference)**.** [17] Assume $f : \mathbb{N}_{a+1}^b \to \mathbb{R}$. We say $F : \mathbb{N}_a^b \to \mathbb{R}$ is a *nabla antidifference* of $f(t)$ on $\mathbb{N}_a^b$ provided

$$\nabla F(t) = f(t),$$

for $t \in \mathbb{N}_{a+1}^b$.

Theorem 14 (Fundamental Theorem of Nabla Calculus)**.** *[17] Assume the function* $f : \mathbb{N}_a^b \to \mathbb{R}$ *and let* F *be a nabla antidifference of* f *on* $\mathbb{N}_a^b$*, then*

$$\int_a^b f(t)\nabla t = F(t)\Big|_a^b := F(b) - F(a).$$

The following Leibniz rules are very useful.

Theorem 15 (Leibniz Formulas)**.** *[17] Assume $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. Then, for $t \in \mathbb{N}_{a+1}$,*

$$\nabla\left(\int_a^t f(t,\tau)\nabla\tau\right) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t), \tag{1.1}$$

and

$$\nabla\left(\int_a^t f(t,\tau)\nabla\tau\right) = \int_a^{t-1} \nabla_t f(t,\tau)\nabla\tau + f(t,t). \tag{1.2}$$

In the case where the limits of integration are constant, we can interchange the order of the nabla operator and integration operator.

Theorem 16 (Leibniz Formula with Constant Limits of Integration)**.** *Assume $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$. Then, for $t \in \mathbb{N}_{a+1}$,*

$$\nabla\left(\int_a^b f(t,\tau)\nabla\tau\right) = \int_a^b \nabla_t f(t,\tau)\nabla\tau.$$

1.2 Extending Results to Fractional Values

We want to extend some of the previous results to the fractional case. For example, we can consider a nabla integral of order 1.2 or a nabla difference of order π. To do so, we need to define the Gamma function.

Definition 17. [30] The *gamma function* is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,$$

for $z \in \mathbb{C}\backslash\{\ldots,-2,-1,0\}$.

One of the most useful results of the gamma function that we will frequently use is that it behaves like the factorial function, i.e. $\Gamma(t+1) = t\Gamma(t)$. Furthermore, if $n \in \mathbb{N}_1$, then $\Gamma(n) = (n-1)!$, following the convention that $0! = 1$. For this reason, we say that the Gamma function generalizes the factorial function beyond nonnegative integers.

Since the Gamma function extends the factorial function to values in $\mathbb{C}\backslash\{\ldots,-2,-1,0\}$, it is natural to extend the definition of the rising function as follows, using the form from Remark 7.

Definition 18. [17] Let $t \in \mathbb{R}$ and let $n \in \mathbb{N}_1$. Then the *rising function* is defined by

$$t^{\overline{n}} := (t)(t+1)\cdots(t+n-1) = \frac{\Gamma(t+n)}{\Gamma(t)},$$

where Γ is the gamma function. For $\nu \in \mathbb{R}$, the *generalized rising function* is then defined by

$$t^{\overline{\nu}} := \frac{\Gamma(t+\nu)}{\Gamma(t)}$$

for t and ν such that $t+\nu \notin \{\ldots,-2,-1,0\}$. If t is a non-positive integer and $t+\nu$ is not a non-positive integer then we take by convention $t^{\overline{\nu}} = 0$.

The power rule still holds in this more generalized case.

Theorem 19 (Generalized Power Rules)**.** *[17] For values of t, r, and α so that the values in the following equations make sense as per Definition 18, we have that*

$$\nabla(t+\alpha)^{\overline{r}} = r(t+\alpha)^{\overline{r-1}},$$

and

$$\nabla(\alpha-t)^{\overline{r}} = -r(\alpha-\rho(t))^{\overline{r-1}}.$$

We can also take integrals of the rising functions, getting a generalized power rule for integrals.

Theorem 20 (Power Rules for the Nabla Integral)**.** *[17]*

1. $\int (t-\alpha)^{\overline{r}} \nabla t = \frac{1}{r+1}(t-\alpha)^{\overline{r+1}} + C, \quad r \neq -1;$

2. $\int (\alpha-\rho(t))^{\overline{r}} \nabla t = -\frac{1}{r+1}(\alpha-t)^{\overline{r+1}} + C, \quad r \neq -1.$

1.3 Nabla Taylor Monomials

In this section we will state the definition of the Nabla Taylor monomials in both the whole order case along with several properties.

Definition 21. For $n \in \mathbb{N}_0$, we define the *nabla Taylor monomials* by $H_0(t,a) := 1$ for $t \in \mathbb{N}_a$, and

$$H_n(t,a) = \frac{(t-a)^{\overline{n}}}{n!}, \quad \text{for } t \in \mathbb{N}_{a-n+1}, \quad \text{when } n \in \mathbb{N}_1.$$

Theorem 22 (Properties of Taylor Monomials)**.** *[17] The nabla Taylor monomials satisfy the following properties:*

1. $H_n(t,a) = 0,$ *for* $t \in \mathbb{N}_{a-n+1}^{a}$ *and* $n \in \mathbb{N}_1$*;*

2. $\nabla H_{n+1}(t,a) = H_n(t,a),$ *for* $t \in \mathbb{N}_{a-n+1}$ *and* $n \in \mathbb{N}_0$*;*

3. $\nabla_s H_{n+1}(t,s) = -H_n(t,\rho(s)),$ *for* $t \in \mathbb{N}_s$*;*

4. $\int_a^t H_n(\tau,a)\nabla\tau = H_{n+1}(t,a),$ *for* $t \in \mathbb{N}_a$ *and* $n \in \mathbb{N}_0$*;*

5. $\int_a^t H_n(t,\rho(s))\nabla s = H_{n+1}(t,a),$ *for* $t \in \mathbb{N}_a$ *and* $n \in \mathbb{N}_0$*.*

Theorem 23 (Discrete Whole-Order Taylor's Formula)**.** *[5] Fix* $N \in \mathbb{N}_1$ *and let* $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$*. Then*

$$f(t) = \sum_{k=0}^{N-1} \nabla^k f(a) H_k(t,a) + \int_a^t H_{N-1}(t,\rho(s))\nabla^N f(s)\nabla s,$$

for $t \in \mathbb{N}_a$.

1.4 Fractional Taylor monomials

In this section, we will extend the definition of Taylor monomials to fractional values using the Gamma function. We will also state and prove some properties of the fractional Taylor monomial that will prove useful in later chapters.

Definition 24. [17] Let $\mu \neq -1, -2, -3, \cdots$. Then we define the *$\mu$-th order nabla fractional Taylor monomoial* $H_\mu(t,a)$ by

$$H_\mu(t,a) := \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu+1)},$$

for values of $t \in \mathbb{R}$ such that the right hand side makes sense. By convention, if $t-a$ is a non-positive integer, but $t-a+\mu$ is not a non-positive integer, then $H_\nu(t,a) := 0$.

Theorem 25 (Properties of Fractional Taylor Monomials)**.** *[17] The following hold:*

1. $H_\mu(a,a) = 0$,
2. $\nabla H_\mu(t,a) = H_{\mu-1}(t,a)$,
3. $\nabla H_\mu(t,s) = -H_{\mu-1}(t,\rho(s))$,
4. $\int_a^t H_\mu(s,a)\nabla s = H_{\mu+1}(t,a)$,
5. $\int_a^t H_\mu(t,\rho(s))\nabla s = H_{\mu+1}(t,a)$,
6. $H_{-k}(t,a) = 0$, *for* $k \in \mathbb{N}_1$ *and* $t \in \mathbb{N}_a$,

provided the expressions used are well defined.

Lemma 26. *For* $-1 < \mu < 0$, *we have that* $\lim_{t\to\infty} t^{\overline{\mu}} = 0$.

Proof. We will first consider $\lim_{t\to\infty} \frac{t^{\overline{\mu}}}{\Gamma(\mu+1)}$. Then, for $t \in \mathbb{N}_1$,

$$\begin{aligned}
\frac{t^{\overline{\mu}}}{\Gamma(\mu+1)} &= \frac{\Gamma(t+\mu)}{\Gamma(t)\Gamma(\mu+1)} \\
&= \frac{(\mu+1)^{\overline{t-1}}}{\Gamma(t)} \\
&= \frac{(\mu+1)(\mu+2)\cdots(\mu+t-1)}{(t-1)!} \\
&= \left(\frac{\mu+1}{1}\right)\left(\frac{\mu+2}{2}\right)\cdots\left(\frac{\mu+t-1}{t-1}\right) \\
&= \prod_{n=1}^{t-1} \frac{\mu+n}{n} \\
&= \prod_{n=1}^{t-1} \left(1+\frac{\mu}{n}\right).
\end{aligned}$$

So we then have that $\lim_{t\to\infty} \frac{t^{\overline{\mu}}}{\Gamma(\mu+1)} = \prod_{n=1}^{\infty}\left(1+\frac{\mu}{n}\right)$. Note here that $-1 < \frac{\mu}{n} < 0$ for all $n \in \mathbb{N}_1$, so this infinite product is well defined. But we then have

$$\prod_{n=1}^{\infty}\left(1+\frac{\mu}{n}\right) = \exp\left(\sum_{n=1}^{\infty} \ln\left(1+\frac{\mu}{n}\right)\right).$$

We claim $\sum_{n=1}^{\infty} \ln\left(1+\frac{\mu}{n}\right)$ diverges to $-\infty$. To see this, we will show $\sum_{n=1}^{\infty} -\ln\left(1+\frac{\mu}{n}\right)$ diverges to ∞ by applying the integral test, i.e. we consider the limit

$$\lim_{b\to\infty} \int_1^b -\ln\left(1+\frac{\mu}{x}\right) dx.$$

Note that $-\ln\left(1+\frac{\mu}{x}\right) > 0$ for all $x \in [1,\infty)$, $-\ln\left(1+\frac{\mu}{x}\right)$ is decreasing with respect to x on $[1,\infty)$, and $-\ln\left(1+\frac{\mu}{x}\right)$ is continuous with respect to x on $[1,\infty)$, thus we can apply the integral test. We will integrate $\int_1^b -\ln\left(1+\frac{\mu}{x}\right)dx$ by parts.

$$\int_1^b -\ln\left(1+\frac{\mu}{x}\right)dx = \left(-\ln\left(1+\frac{\mu}{x}\right)x\right)|_{x=1}^b - \int_1^b \frac{\mu x}{\mu x + x^2}dx$$
$$= -\ln\left(1+\frac{\mu}{b}\right)b + \ln(1+\mu) - (\mu \ln|\mu+x|)\,|_{x=1}^b$$
$$= -\ln\left(1+\frac{\mu}{b}\right)b + \ln(1+\mu) - \mu\ln(\mu+b) + \mu\ln(\mu+1).$$

Consider then

$$\lim_{b\to\infty} -\ln\left(1+\frac{\mu}{b}\right)b = \lim_{b\to\infty}\frac{-\ln\left(1+\frac{\mu}{b}\right)}{\frac{1}{b}} = \lim_{b\to\infty}\frac{\frac{\mu}{\mu b+b^2}}{-\frac{1}{b^2}} = \lim_{b\to\infty} -\frac{\mu b}{\mu+b} = -\mu,$$

after applying L'Hopital's rule.

Also,

$$\lim_{b\to\infty} -\mu\ln(\mu+b) = \infty,$$

as $\mu < 0$.

Thus

$$\lim_{b\to\infty} -\ln\left(1+\frac{\mu}{x}\right)dx$$
$$= \lim_{b\to\infty}\left[-\ln\left(1+\frac{\mu}{b}\right)b + \ln(1+\mu) - \mu\ln(\mu+b) + \mu\ln(\mu+1)|\right] = \infty.$$

Therefore, by the integral test, $\sum_{n=1}^{\infty} -\ln\left(1+\frac{\mu}{n}\right)$ diverges to ∞, thus $\sum_{n=1}^{\infty}\ln\left(1+\frac{\mu}{n}\right)$ diverges to $-\infty$. Hence

$$\lim_{t\to\infty}\frac{t^{\overline{\mu}}}{\Gamma(\mu+1)} = \exp\left(\sum_{n=1}^{\infty}\ln\left(1+\frac{\mu}{n}\right)\right) = 0.$$

Finally, since $\lim_{t\to\infty} \frac{t^{\overline{\mu}}}{\Gamma(\mu+1)} = 0$, we get that $\frac{1}{\Gamma(\mu+1)} \lim_{t\to\infty} t^{\overline{\mu}} = 0$, hence

$$\lim_{t\to\infty} t^{\overline{\mu}} = 0.$$

$\square$

Lemma 27. *For $t \in \mathbb{N}_a$, $s \in \mathbb{N}_a^t$, and $\mu > -1$, we have*

$$H_\mu(t,s) \geq 0.$$

Proof. First note if $t = s$, then by convention $H_\mu(t,s) = 0$. So consider $t \in \mathbb{N}_{a+1}$ and $s \in \mathbb{N}_a^{t-1}$. Then

$$H_\mu(t,s) := \frac{(t-s)^{\overline{\mu}}}{\Gamma(\mu+1)} = \frac{\Gamma(t-s+\mu)}{\Gamma(t-s)\Gamma(\mu+1)}.$$

By our assumption on t and s, we have that $t-s \in \mathbb{N}_1$. So $t-s+\mu > 0$ and $t-s > 0$ implying $\Gamma(t-s+\mu) > 0$ and $\Gamma(t-s) > 0$. Finally, $\mu > -1$, so $\mu + 1 > 0$, which implies $\Gamma(\mu+1) > 0$. Hence for $t \in \mathbb{N}_{a+1}$ and $s \in \mathbb{N}_a^{t-1}$, $H_\mu(t,s) > 0$. $\square$

Lemma 28. *For $-1 < \mu < 0$, $t \in \mathbb{N}_{a+2}$, and $s \in \mathbb{N}_{a+2}^t$, we have that*

$$\nabla_s H_\mu(t,\rho(s)) \geq 0.$$

Proof. For $s \in \mathbb{N}_{a+2}^{t}$, consider

$$\begin{aligned}
\nabla_s H_\mu(t,\rho(s)) &= \frac{\nabla_s (t-\rho(s))^{\overline{\mu}}}{\Gamma(\mu+1)} \\
&= \frac{\nabla_s (t+1-s)^{\overline{\mu}}}{\Gamma(\mu+1)} \\
&= -\frac{(\mu)(t+1-\rho(s))^{\overline{\mu-1}}}{\Gamma(\mu+1)} \\
&= -\frac{(\mu)\Gamma(t+1-s+1+\mu-1)}{\Gamma(t+1-s+1)(\mu)\Gamma(\mu)} \\
&= -\frac{\Gamma(t-s+\mu+1)}{\Gamma(t-s+2)\Gamma(\mu)}.
\end{aligned}$$

Since $s \in \mathbb{N}_{a+2}^{t}$, we have that $\Gamma(t-s+\mu+1) > 0$ and $\Gamma(t-s+2) > 0$. Since $-1 < \mu < 0$, $\Gamma(\mu) < 0$. Thus $\nabla_s H_\mu(t,\rho(s)) \geq 0$. □

Lemma 29. *For $\mu \in \mathbb{R}$ such that μ is not a non-positive integer, $s \in \mathbb{N}_a$ and $t \in \mathbb{N}_{s+1}$, we have*

$$H_\mu(t,\rho(s)) = \left(\frac{\mu+1}{t-s}\right) H_{\mu+1}(t,s).$$

Proof. Consider

$$\begin{aligned}
H_\mu(t,\rho(s)) &:= \frac{(t-\rho(s))^{\overline{\mu}}}{\Gamma(\mu+1)} \\
&\overset{\text{Def. 18}}{=} \frac{\Gamma(t-s+1+\mu)}{\Gamma(t-s+1)\Gamma(\mu+1)} \\
&= \frac{\Gamma(t-s+(\mu+1))}{(t-s)\Gamma(t-s)\Gamma(\mu+1)} \cdot \left(\frac{\mu+1}{\mu+1}\right) \\
&= \left(\frac{\mu+1}{t-s}\right) \frac{\Gamma(t-s+(\mu+1))}{\Gamma(t-s)\Gamma(\mu+2)} \\
&= \left(\frac{\mu+1}{t-s}\right) H_{\mu+1}(t,s).
\end{aligned}$$

□

1.5 Basic Results for the Nabla Fractional Calculus

A full discussion of the nabla discrete fractional calculus is in [17, Chapter 3]. Some of the originating results are from [21]. These results have analogs to results in the study of the real fractional calculus. For more information about fractional calculus in the real case, see [30].

Definition 30. [21] Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$. The ν^{th} *order nabla fractional sum of* f is defined by

$$\nabla_a^{-\nu} f(t) := \int_a^t H_{\nu-1}(t, \rho(s)) f(s) \nabla s,$$

for $t \in \mathbb{N}_a$.

We define the Riemann-Liouville nabla fractional difference in terms of a whole order nabla difference and a nabla fractional sum.

Definition 31. [21] Let $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, and $N := \lceil \nu \rceil$. Then ν^{th} *order Riemann-Liouville nabla fractional difference of* f is defined as

$$\nabla_a^{\nu} f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t),$$

for $t \in \mathbb{N}_{a+N}$.

While we will mostly focus on the Caputo fractional difference given in the upcoming Definition 35, the Riemann-Liouville difference is useful for intermediate steps in certain proofs. In particular, the following composition rules will be useful.

Theorem 32 (Composition Rules)**.** *[2, Theorem 6.1] Let* $\mu > 0$, $\nu > 0$, *and let* $f : \mathbb{N}_a \to \mathbb{R}$. *Define* $N := \lceil \nu \rceil$. *Then*

$$\nabla_a^{-\nu} \nabla_a^{-\mu} f(t) = \nabla_a^{-\nu-\mu} f(t), \quad t \in \mathbb{N}_a,$$

and

$$\nabla_a^{\nu}\nabla_a^{-\mu}f(t) = \nabla_a^{\nu-\mu}f(t), \quad t \in \mathbb{N}_{a+N}.$$

The following specific composition rule is also useful; it enables us to effectively cancel out Riemann-Liouville differences and sums through composition.

Theorem 33. *[17, Corollary 3.122] For* $0 < \mu < 1$ *and* $f : \mathbb{N}_{a+1} \to \mathbb{R}$*, we have that*

$$\nabla_a^{-\mu}\nabla_a^{\mu}f(t) = f(t),$$

for $t \in \mathbb{N}_{a+1}$*.*

It will prove useful later to consider nabla fractional sums consisting of an integral of a function with two variables with constant limits of integration.

Lemma 34. *Let* $c \in \mathbb{N}_a$*,* $d \in \mathbb{N}_c$*, and* $\mu > 0$ *be given. Assume* $f : \mathbb{N}_a \times \mathbb{N}_{c+1} \to \mathbb{R}$*. Then, for* $t \in \mathbb{N}_a$*,*

$$\nabla_a^{-\mu}\left(\int_c^d f(t,\tau)\nabla\tau\right) = \int_c^d \nabla_a^{-\mu}f(t,\tau)\nabla\tau.$$

Proof. For $t \in \mathbb{N}_a$, consider

$$\begin{aligned}
\nabla_a^{-\mu}\left(\int_c^d f(t,\tau)\nabla\tau\right) &= \int_a^t H_{\mu-1}(t,\rho(s))\int_c^d f(s,\tau)\nabla\tau\nabla s \\
&= \sum_{s=a+1}^{t}\sum_{\tau=c+1}^{d} H_{\mu-1}(t,\rho(s))f(s,\tau) \\
&= \sum_{\tau=c+1}^{d}\sum_{s=a+1}^{t} H_{\mu-1}(t,\rho(s))f(s,\tau) \\
&= \int_c^d\int_a^t H_{\mu-1}(t,\rho(s))f(s,\tau)\nabla s\nabla\tau \\
&= \int_c^d \nabla_a^{-\mu}f(t,\tau)\nabla\tau.
\end{aligned}$$

□

The following definition has been adapted from Anastassiou in [4].

Definition 35. Let $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$, $\nu > 0$, $\nu \in \mathbb{R}$, and $N := \lceil \nu \rceil$. The ν^{th} *order Caputo nabla fractional difference* is defined as

$$\nabla_{a^*}^{\nu} f(t) := \nabla_a^{-(N-\nu)}(\nabla^N f(t)),$$

for $t \in \mathbb{N}_{a+1}$. Note that the Caputo difference operator is a linear operator.

Remark 36. Note the difference between the Caputo fractional difference and the Riemann-Liouville fractional difference is the order of the composition of the fractional sum and whole order difference.

The following lemma shows Caputo fractional difference has some particularly nice behavior as a consequence of the order discussed in Remark 36.

Lemma 37. *[3, Lemma 15] Let $0 < \nu < 1$ and let $x : \mathbb{N}_a \to \mathbb{R}$. Then*

$$\nabla_{a*}^{\nu} x(a+1) = \nabla x(a+1).$$

It will be useful later to consider Caputo fractional sums of an integral of a function with two variables with constant limits of integration.

Lemma 38. *Let $c \in \mathbb{N}_a$, $d \in \mathbb{N}_c$, and $\nu > 0$ be given. Assume $f : \mathbb{N}_{a-N+1} \times \mathbb{N}_{c+1} \to \mathbb{R}$ and take $N = \lceil \nu \rceil$. Then for $t \in \mathbb{N}_{a+1}$,*

$$\nabla_{a^*}^{\nu} \left(\int_c^d f(t,\tau) \nabla\tau \right) = \int_c^d \nabla_{a^*}^{\nu} f(t,\tau) \nabla\tau.$$

Proof. For $t \in \mathbb{N}_{a+1}$,

$$\begin{aligned}
\nabla_{a^*}^{\nu}\left(\int_c^d f(t,\tau)\nabla\tau\right) &= \nabla_a^{-(N-\nu)}\nabla^N\left(\int_c^d f(t,\tau)\nabla\tau\right) \\
&= \nabla_a^{-(N-\nu)}\left(\int_c^d \nabla_t^N f(t,\tau)\nabla\tau\right) \\
&= \int_c^d \nabla_a^{-(N-\nu)}\nabla_t^N f(t,\tau)\nabla\tau, && \text{(by Theorem 16)} \\
&= \int_c^d \nabla_{a^*}^{\nu} f(t,\tau)\nabla\tau, && \text{(by Lemma 34).}
\end{aligned}$$

□

Chapter 2

Fractional Self-Adjoint Difference Equations

In the continuous setting, let $\mathbb{D} := \{x : x \text{ and } px' \text{ are continuously differentiable on } \mathbb{R}\}$. Then the second-order formally self-adjoint operator on $\mathbb{D}$ is given by

$$(L_x)(t) := [p(t)x'(t)]' + q(t)x(t),$$

where $p : \mathbb{R} \to (0, \infty)$ and $q : \mathbb{R} \to \mathbb{R}$. This operator has importance in functional analysis (see [24, Chapter 10]) for more details. It also has applications in theoretical physics [6]. Many equivalent results to this chapter in the continuous setting is given in [26, Chapter 5].

The self-adjoint operator studied here for the discrete fractional case follows. Let $\mathcal{D}_a := \{x : \mathbb{N}_a \to \mathbb{R}\}$ and let the fractional self-adjoint operator L_a be defined by

$$(L_a x)(t) := \nabla[p(t)\nabla^{\nu}_{a^*}x(t)] + q(t)x(t-1), \quad t \in \mathbb{N}_{a+2},$$

where $x \in \mathcal{D}_a$, $0 < \nu < 1$, $p : \mathbb{N}_{a+1} \to (0, \infty)$ such that $p(t) \neq \frac{p(a+1)}{H_{-\nu}(t,a)}$ for some $t \in \mathbb{N}_{a+1}$, and $q : \mathbb{N}_{a+2} \to \mathbb{R}$. Note that L_a is a linear operator.

Remark 39 (Domain of x, p, and q in the self-adjoint operator)**.** The operator is defined for $t \in \mathbb{N}_{a+2}$, but $x(t)$ is defined for $t \in \mathbb{N}_a$. This is a result of the Caputo fractional derivative only being defined on $\mathbb{N}_{a+1}$, (see Definition 35), and so then the whole order nabla difference of the Caputo fractional derivative is only defined on $\mathbb{N}_{a+2}$ (see Definition 2).

For the domains of p and q, consider the following expansion of the self-adjoint operator

$$\begin{aligned}
(L_a x)(t) &= \nabla[p(t)\nabla_{a^*}^{\nu} x(t)] + q(t)x(t-1) \\
&= \nabla\left[p(t)\nabla_a^{-(1-\nu)}\nabla x(t)\right] + q(t)x(t-1) \\
&= \nabla\left[p(t)\int_a^t H_{-\nu}(t,\rho(s))\,[x(s)-x(s-1)]\,\nabla s\right] + q(t)x(t-1) \\
&= p(t)\int_a^t H_{-\nu}(t,\rho(s))\,[x(s)-x(s-1)]\,\nabla s \\
&\qquad - p(t-1)\int_a^{t-1} H_{-\nu}(t-1,\rho(s))\,[x(s)-x(s-1)]\,\nabla s + q(t)x(t-1) \\
&= p(t)H_{-\nu}(t,\rho(t))\,[x(t)-x(t-1)] \\
&\qquad + p(t)\int_a^{t-1} H_{-\nu}(t,\rho(s))\,[x(s)-x(s-1)]\,\nabla s \\
&\qquad - p(t-1)\int_a^{t-1} H_{-\nu}(t-1,\rho(s))\,[x(s)-x(s-1)]\,\nabla s + q(t)x(t-1) \\
&= p(t)\,[x(t)-x(t-1)] \\
&\qquad + p(t)\int_a^{t-1} H_{-\nu}(t,\rho(s))\,[x(s)-x(s-1)]\,\nabla s \\
&\qquad - p(t-1)\int_a^{t-1} H_{-\nu}(t-1,\rho(s))\,[x(s)-x(s-1)]\,\nabla s + q(t)x(t-1) \\
&= p(t)x(t) + [q(t)-p(t)]\,x(t-1) \\
&\qquad + p(t)\int_a^{t-1} H_{-\nu}(t,\rho(s))\,[x(s)-x(s-1)]\,\nabla s \\
&\qquad - p(t-1)\int_a^{t-1} H_{-\nu}(t-1,\rho(s))\,[x(s)-x(s-1)]\,\nabla s.
\end{aligned}$$

Since the operator is defined for $t \in \mathbb{N}_{a+2}$, the above expansion shows that $p(t)$ only needs to be defined for $t \in \mathbb{N}_{a+1}$ and $q(t)$ only needs to be defined for $t \in \mathbb{N}_{a+2}$.

Remark 40. The restriction that $p(t) \neq \frac{p(a+1)}{H_{-\nu}(t,a)}$ for some $t \in \mathbb{N}_{a+1}$ will guarantee unique solutions to initial value problems involving the self-adjoint operator (see Theorem 43).

The fractional self-adjoint difference equation behaves similar to a second order difference equation. For instance, a general solution to the homogeneous equation is given by a linear combination of two linearly independent solutions. Note the following theorem relies on the existence and uniqueness of self-adjoint initial value problems, which is given in Theorem 43 in the next section.

Theorem 41 (General Solution of the Homogeneous Equation)**.** *Suppose* $x_1, x_2 : \mathbb{N}_a \to \mathbb{R}$ *are linearly independent solutions to* $L_a x(t) = 0$*. Then a general solution to* $L_a x(t) = 0$ *is given by*

$$x(t) = c_1 x_1(t) + c_2 x_2(t),$$

for $t \in \mathbb{N}_a$*, where* $c_1, c_2 \in \mathbb{R}$ *are arbitrary constants.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions to $L_a x(t) = 0$, and let c_1 and c_2 be arbitrary constants. Then define $x(t) := c_1 x_1(t) + c_2 x_2(t)$. Since L_a is a linear operator, we have that

$$L_a x(t) = c_1 L_a x_1(t) + c_2 L_a x_2(t) = c_1 \cdot 0 + c_2 \cdot 0 = 0,$$

hence $x(t) = c_1 x_1(t) + c_2 x(t)$ solves $L_a x(t) = 0$ for any constants c_1 and c_2.

We now claim that any solution $x(t)$ to $L_a x(t) = 0$ can be written uniquely in the form of $x(t) = c_1 x_1(t) + c_2 x_2(t)$ for suitable constants c_1 and c_2. Since $x_1 : \mathbb{N}_a \to \mathbb{R}$

and $x_2 : \mathbb{N}_a \to \mathbb{R}$, we have that there exists constants α, β, γ, and $\delta \in \mathbb{R}$ such that $x_1(a) = \alpha$, $\nabla x_1(a+1) = \beta$, $x_2(a) = \gamma$, and $\nabla x_2(a+1) = \delta$.

Let $x : \mathbb{N}_a \to \mathbb{R}$ be any arbitrary function that solves $L_a x(t) = 0$. As before, there exists constants A and B such that $x(a) = A$ and $\nabla x(a+1) = B$. Hence $x(t)$ solves the initial value problem

$$\begin{cases} L_a x(t) = 0, \quad t \in \mathbb{N}_{a+2}, \\ x(a) = A, \quad \nabla x(a+1) = B. \end{cases}$$

Consider the vector equation

$$\begin{pmatrix} x_1(a) & x_2(a) \\ \nabla x_1(a+1) & \nabla x_2(a+1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}. \tag{2.1}$$

Note this is equivalent to

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}.$$

We claim that there exists unique $c_1, c_2 \in \mathbb{R}$ that satisfy the above matrix equation. By way of contradiction, suppose not. Then

$$\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} = 0.$$

Then, without lost of generality, there exists a constant $k \in \mathbb{R}$ such that $\alpha = k\gamma$ and $\beta = k\delta$. Relabelling, we get that $x_1(a) = \alpha = k\gamma = kx_2(a)$ and $\nabla x_1(a+1) = \beta = k\delta = k\nabla x_2(a+1)$. But $kx_2(t)$ solve $L_a x(t) = 0$, as L_a is a linear operator, and from before $L_a x_1(t) = 0$. Hence $kx_2(t)$ and $x_1(t)$ satisfy the same difference equation and the same initial conditions, hence by the uniqueness of solutions initial value problems (see Theorem 43) $x_1(t) = kx_2(t)$ for $t \in \mathbb{N}_a$. This implies $x_1(t)$ and $x_2(t)$ are linearly dependent on $\mathbb{N}_a$, which is a contradiction. Hence the vector equation (2.1) has a unique solution, so $x(t)$ and $c_1x_1(t) + c_2x_2(t)$ solve the same initial value problem, hence every solution to $L_a x(t) = 0$ can be expressed uniquely as a linear combination of $x_1(t)$ and $x_2(t)$. □

The proof of the following result is straight forward.

Corollary 42 (General Solution of the Nonhomogeneous Equation)**.** *Suppose* $x_1, x_2 : \mathbb{N}_a \to \mathbb{R}$ *are linearly independent solutions of* $L_a x(t) = 0$ *and* $x_p : \mathbb{N}_a \to \mathbb{R}$ *is a particular solution to* $L_a x(t) = h(t)$ *for some* $h : \mathbb{N}_{a+2} \to \mathbb{R}$*. Then a general solution of* $L_a x(t) = h(t)$ *is given by*

$$x(t) = c_1x_1(t) + c_2x_2(t) + x_p(t),$$

for $t \in \mathbb{N}_a$*, and where* $c_1, c_2 \in \mathbb{R}$ *are arbitrary constants.*

2.1 Self-Adjoint Initial Value Problems

Theorem 43 (Existence and Uniqueness for Self-Adjoint IVPs)**.** *Let* $A, B \in \mathbb{R}$*, and* $h : \mathbb{N}_{a+2} \to \mathbb{R}$ *and assume* $p(t) \neq \frac{p(a+1)}{H_{-\nu}(t,a)}$ *for all* $t \in \mathbb{N}_{a+2}$*. Then the initial value problem*

$$\begin{cases} L_a x(t) = h(t), & t \in \mathbb{N}_{a+2}, \\ x(a) = A, & \nabla x(a+1) = B, \end{cases} \tag{2.2}$$

has a unique solution $x : \mathbb{N}_a \to \mathbb{R}$*.*

Proof. Define $x(t)$ to satisfy the initial conditions in (2.2). It follows that

$$x(a) = A, \quad x(a+1) = A + B,$$

and then for $t \in \mathbb{N}_{a+2}$, define

$$x(t) := \frac{1}{p(t)} \left[h(t) - \left([q(t) - p(t)]\, x(t-1) + p(t) \sum_{s=a+1}^{t-1} H_{-\nu}(t, \rho(s)) \, [x(s) - x(s-1)] \right.\right.$$
$$\left.\left. - p(t-1) \sum_{s=a+1}^{t-1} H_{-\nu}(t-1, \rho(s)) \, [x(s) - x(s-1)] \right) \right]. \tag{2.3}$$

For any fixed $t \in \mathbb{N}_{a+2}$, we have that the coefficient of the $x(a)$ term in (2.3) is given by $-\left[p(t)H_{-\nu}(t,a) - p(t-1)H_{-\nu}(t-1,a)\right] = -\nabla\left[p(t)H_{-\nu}(t,a)\right]$. For $x(t)$, with $t \in \mathbb{N}_{a+2}$, to be uniquely determined by the initial conditions, we need that the coefficient of $x(a)$ is nonzero for some $t \in \mathbb{N}_{a+2}$. So to avoid the situation of $\nabla\left[p(t)H_{-\nu}(t,a)\right] = 0$, we must have $p(t) \neq \frac{C}{H_{-\nu}(t,a)}$, where $C \in \mathbb{R}$ is an arbitrary constant. In particular, we can define $C := p(a+1)$, so then we must have $p(t) \neq \frac{p(a+1)}{H_{-\nu}(t,a)}$ for some $t \in \mathbb{N}_{a+2}$, which is one of our assumptions. Therefore there exists some $t \in \mathbb{N}_{a+2}$ such that $x(t)$ depends on the initial condition $x(a) = A$. See Remark 44 for why there is no issue regarding the coefficient of $x(a+1)$.

Note that when $t = a+2$, $x(a+2)$ depends only on the known functions $p(t)$, $q(t)$, and $h(t)$, as well as $x(a)$ and $x(a+1)$. Hence $x(a+2)$ is uniquely determined from the known functions and the initial conditions. Then for $t = a+3$, we have that $x(a+3)$ depends only on the known functions $p(t)$, $q(t)$, and $h(t)$, as well as $x(a)$, $x(a+1)$, and $x(a+2)$. Hence $x(a+3)$ is uniquely determined by the known functions and the previous values of x. Continuing on in this fashion, we get that $x(t)$ is uniquely determined by $p(t)$, $q(t)$, $h(t)$, and $x(s)$ for $s = a, a+1, \ldots, t-1$. Thus for $t \in \mathbb{N}_a$, $x(t)$ is uniquely defined by the initial conditions and the recursion equation. (2.3).

Consider the following rearrangement of (2.3) using integral notation instead of summation notation,

$$\begin{aligned} p(t)x(t) &+ [q(t) - p(t)]\, x(t-1) \\ &\quad + p(t) \int_a^{t-1} H_{-\nu}(t, \rho(s))\, [x(s) - x(s-1)]\, \nabla s \\ &\quad - p(t-1) \int_a^{t-1} H_{-\nu}(t-1, \rho(s))\, [x(s) - x(s-1)]\, \nabla s \\ &= h(t). \end{aligned} \tag{2.4}$$

Here the left hand side of the above equation is simply the expanded form of the self-adjoint operator given in Remark 39, hence $x(t)$ as defined above satisfies the initial conditions and the self-adjoint equation

$$L_a x(t) = \nabla[p(t) \nabla_{a^*}^{\nu} x(t)] + q(t)x(t) = h(t).$$

Therefore $x(t)$ uniquely solves (2.2). □

Remark 44. For $t \in \mathbb{N}_{a+3}$, we have that the coefficient of $x(a+1)$ is given by

$$\begin{aligned} p(t)\, [H_{-\nu}(t,a) - H_{-\nu}(t,a+1)] - p(t-1)\, [H_{-\nu}(t-1,a) - H_{-\nu}(t-1,a+1)] \\ = \nabla\, [p(t)\, (H_{-\nu}(t,a) - H_{-\nu}(t,a+1))] \\ = \nabla\, [p(t)\, (H_{-\nu}(t,a) - H_{-\nu}(t-1,a))] \\ = \nabla\, [p(t) H_{-\nu-1}(t,a)]\, . \end{aligned}$$

The coefficient of $x(a+1)$ needs to be nonzero for some $t \in \mathbb{N}_{a+3}$, which requires $p(t) \neq \frac{C}{H_{-\nu-1}(t,a)}$ for all $t \in \mathbb{N}_{a+3}$ and for some constant $C \in \mathbb{R}$. In particular, if we take $C = p(a+1)$, we require $p(t) \neq \frac{p(a+1)}{H_{-\nu-1}(t,a)}$ for all $t \in \mathbb{N}_{a+3}$. Recall that $p : \mathbb{N}_{a+1} \to (0,\infty)$, so $p(a+1) > 0$. However, $H_{-\nu-1}(t,a) < 0$ for $t \in \mathbb{N}_{a+2}$, thus the condition that p is a positive valued function already avoids the possible issue of the coefficient of $x(a+1)$ being zero for all $t \in \mathbb{N}_{a+2}$.

Definition 45. The *Cauchy function for $L_a x(t) = 0$* is the function $x(t,s)$, where $x : \mathbb{N}_a \times \mathbb{N}_{a+2} \to \mathbb{R}$, which for any fixed $s \in \mathbb{N}_{a+2}$, satisfies the initial value problem

$$\begin{cases} L_{\rho(s)} x(t,s) = 0, \quad t \in \mathbb{N}_{s+1} \\ x(\rho(s), s) = 0, \\ \nabla x(s,s) = \frac{1}{p(s)}. \end{cases} \tag{2.5}$$

Theorem 46 (Variation of Constants)**.** *Let $h : \mathbb{N}_{a+2} \to \mathbb{R}$. Then the solution to the initial value problem*

$$\begin{cases} L_a x(t) = h(t), \quad t \in \mathbb{N}_{a+2}, \\ x(a) = 0 \\ \nabla x(a+1) = 0, \end{cases} \tag{2.6}$$

is given by

$$x(t) = \int_{a+1}^{t} x(t,s)h(s)\nabla s,$$

where $x(t,s)$ is the Cauchy function for the homogeneous equation.

Proof. Define $x(t) := \int_{a+1}^{t} x(t,s)h(s)\nabla s$, where $x(t,s)$ is the Cauchy function for $L_a x(t) = 0$. Note that $x(a) = \int_{a+1}^{a} x(a,s)h(s)\nabla s = 0$ by convention and $x(a+1) = \int_{a+1}^{a+1} x(a+1,s)h(s)\nabla s = 0$. Hence $\nabla x(a+1) = x(a+1) - x(a) = 0$, thus the initial conditions are satisfied.

Now consider

$$\begin{aligned}\nabla\left[p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s)\right] &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{\rho(s)}\nabla_t x(t,s)\right]\\ &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{\rho(s)}\left(x(t,s)-x(t-1,s)\right)\right]\\ &= \nabla\left[p(t)\int_{\rho(s)}^{t} H_{-\nu}(t,\rho(\tau))\left(x(\tau,s)-x(\tau-1,s)\right)\nabla\tau\right].\end{aligned} \tag{2.7}$$

Using Leibniz's Formula (1.1), we have

$$\begin{aligned}\nabla\left[p(t)\nabla^{\nu}_{a^*}x(t)\right] &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{a}\nabla x(t)\right]\\ &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{a}\nabla\int_{a+1}^{t} x(t,s)h(s)\nabla s\right]\\ &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{a}\left(\int_{a+1}^{t}\nabla_t x(t,s)h(s)\nabla s + x(\rho(t),t)h(t)\right)\right]\\ &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{a}\int_{a+1}^{t}\nabla_t x(t,s)h(s)\nabla s\right],\end{aligned}$$

where we used the first initial condition in the definition of the Cauchy function. Continuing this expansion, we get

$$\begin{aligned}\nabla\left[p(t)\nabla^{\nu}_{a^*}x(t)\right] &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{a}\int_{a+1}^{t}\nabla_t x(t,s)h(s)\nabla s\right]\\ &= \nabla\left[p(t)\nabla^{-(1-\nu)}_{a}\int_{a+1}^{t}\left(x(t,s)-x(t-1,s)\right)h(s)\nabla s\right]\\ &= \nabla\left[p(t)\int_{a}^{t} H_{-\nu}(t,\rho(\tau))\int_{a+1}^{\tau}\left(x(\tau,s)-x(\tau-1,s)\right)h(s)\nabla s\nabla\tau\right]\\ &= \nabla\left[p(t)\sum_{\tau=a+1}^{t}\sum_{s=a+2}^{\tau} H_{-\nu}(t,\rho(\tau))\left(x(\tau,s)-x(\tau-1,s)\right)h(s)\right],\end{aligned}$$

using the sum definition of the nabla integral. By interchanging the order of summation, applying Leibniz's formula (1.2), using both initial conditions in the definition of the Cauchy function, and using (2.7), we get

$$
\begin{aligned}
\nabla\left[p(t)\nabla^{\nu}_{a^*}x(t)\right] &= \nabla\left[p(t)\sum_{s=a+2}^{t}h(s)\sum_{\tau=s}^{t}H_{-\nu}(t,\rho(\tau))\left(x(\tau,s)-x(\tau-1,s)\right)\right] \\
&= \sum_{s=a+2}^{t-1}\left(h(s)\nabla_t\left[p(t)\sum_{\tau=s}^{t}H_{-\nu}(t,\rho(\tau))\left(x(\tau,s)-x(\tau-1,s)\right)\right]\right) \\
&\quad + h(t)p(t)H_{-\nu}(t,\rho(t))\left(x(t,t)-x(t-1,t)\right) \\
&= \int_{a+1}^{t-1}\Big(h(s) \\
&\quad \cdot\nabla_t\left[p(t)\int_{\rho(s)}^{t}H_{-\nu}(t,\rho(\tau))\left(x(\tau,s)-x(\tau-1,s)\right)\nabla\tau\right]\Big)\nabla s \\
&\quad + h(t)p(t)\left(\frac{1}{p(t)}-0\right) \\
&= h(t)+\int_{a+1}^{t-1}h(s)\left(\nabla\left[p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s)\right]\right)\nabla s.
\end{aligned}
$$

So finally,

$$
\begin{aligned}
L_a x(t) &= \nabla\left[p(t)\nabla^{\nu}_{a^*}x(t)\right]+q(t)x(t-1) \\
&= h(t)+\int_{a+1}^{t-1}h(s)\left(\nabla\left[p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s)\right]\right)\nabla s+q(t)\int_{a+1}^{t-1}x(t-1,s)h(s)\nabla s \\
&= h(t)+\int_{a+1}^{t-1}h(s)\left(\nabla\left[p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s)\right]+q(t)x(t-1,s)\right)\nabla s \\
&= h(t)+\int_{a+1}^{t-1}h(s)L_{\rho(s)}x(t,s)\nabla s \\
&= h(t).
\end{aligned}
$$

Hence $x(t)=\int_{a+1}^{t}x(t,s)h(s)$ solves the initial value problem (2.6). □

Theorem 47 (Variation of Constants with Non-Zero Initial Conditions)**.** *Let* h :

$\mathbb{N}_{a+2} \to \mathbb{R}$. *Then the solution to the initial value problem*

$$\begin{cases} L_a x(t) = h(t), \quad t \in \mathbb{N}_{a+2}, \\ x(a) = A, \\ \nabla x(a+1) = B, \end{cases}$$

where $A, B \in \mathbb{R}$ *are arbitrary constants, is given by*

$$x(t) = x_0(t) + \int_{a+1}^{t} x(t,s)h(s)\nabla s,$$

where $x_0(t)$ *solves the initial value problem*

$$\begin{cases} L_a x_0(t) = 0, \quad t \in \mathbb{N}_{a+2}, \\ x_0(a) = A, \\ \nabla x_0(a+1) = B. \end{cases}$$

Proof. This proof follows from the linearity of the self-adjoint operator. □

2.1.1 Cauchy Function Examples

Example 48. Find the Cauchy function for $\nabla\left[p(t)\nabla^{\nu}_{a^*}x(t)\right] = 0$.

For fixed $s \in \mathbb{N}_{a+1}$, we consider the initial value problem

$$\begin{cases} \nabla\left[p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s)\right] = 0, \quad t \in \mathbb{N}_{s+1}, \\ x(\rho(s), s) = 0, \\ \nabla x(s,s) = \frac{1}{p(s)}. \end{cases}$$

Integrating the above difference equation on both sides from s to t and using the

Fundamental Theorem of Nabla Calculus yields

$$p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s) - p(s)\nabla^{\nu}_{\rho(s)^*}x(s,s) = 0.$$

By Lemma 37 and the second initial condition in the definition of the Cauchy function, this is equivalent to

$$p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s) - p(s)\nabla x(s,s) = p(t)\nabla^{\nu}_{\rho(s)^*}x(t,s) - p(s)\frac{1}{p(s)} = 0,$$

hence

$$\nabla^{\nu}_{\rho(s)^*}x(t,s) = \nabla^{-(1-\nu)}_{\rho(s)}\nabla x(t,s) = \frac{1}{p(t)}.$$

By composing both sides with the operator $\nabla^{1-\nu}_{\rho(s)}$, we get that

$$\nabla x(t,s) = \nabla^{1-\nu}_{\rho(s)}\frac{1}{p(t)}.$$

Using Theorem 32, we get

$$\nabla x(t,s) = \nabla\nabla^{-\nu}_{\rho(s)}\frac{1}{p(t)} = \nabla\int_{\rho(s)}^{t} H_{\nu-1}(t,\rho(\tau))\frac{1}{p(\tau)}\nabla\tau,$$

and so by integrating from $\rho(s)$ to t and using the Fundamental Theorem of Nabla Calculus,

$$x(t,s) - x(\rho(s),s) = \int_{\rho(s)}^{t} H_{\nu-1}(t,\rho(\tau))\frac{1}{p(\tau)}\nabla\tau - \int_{\rho(s)}^{\rho(s)} H_{\nu-1}(\rho(s),\rho(\tau))\frac{1}{p(\tau)}\nabla\tau.$$

Then by the first initial condition in the definition of the Cauchy function and by convention on nabla integrals, we get

$$x(t,s) = \int_{\rho(s)}^{t} H_{\nu-1}(t,\rho(\tau))\frac{1}{p(\tau)}\nabla\tau = \nabla_{\rho(s)}^{-\nu}\frac{1}{p(t)}.$$

Example 49. Find the Cauchy function for $\nabla\nabla_{a^*}^{\nu}x(t) = 0$. Note this is a specific case of Example 48 where $p(t) \equiv 1$. Hence

$$\begin{aligned} x(t,s) &= \nabla_{\rho(s)}^{-\nu}1 \\ &= \int_{\rho(s)}^{t} H_{\nu-1}(t,\rho(\tau))\nabla\tau \\ &= -H_{\nu}(t,\tau)\big|_{\tau=\rho(s)}^{t} \\ &= H_{\nu}(t,\rho(s)). \end{aligned}$$

Hence the Cauchy function for $\nabla\nabla_{a^*}^{\nu}x(t) = 0$ is given by $x(t,s) = H_{\nu}(t,\rho(s))$.

2.2 Self-Adjoint Boundary Value Problems

In this section we develop techniques to solve boundary value problems for the fractional self-adjoint operator involving the Caputo difference. See Brackins [15] for a similar development using the Riemann-Liouville definition of a fractional difference. In the continuous setting, [28] has some work on boundary value problems involving fractional derivatives. Some work in the delta case on fractional boundary value problems is given in [19]. In [7], they develop Green's functions in the delta case.

We are interested in the homogeneous self-adjoint boundary value problem

$$\begin{cases} L_a x(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\ \alpha x(a) - \beta \nabla x(a+1) = 0, \\ \gamma x(b) + \delta \nabla x(b) = 0, \end{cases} \tag{2.8}$$

and the corresponding nonhomogeneous self-adjoint boundary value problem

$$\begin{cases} L_a x(t) = h(t), \quad t \in \mathbb{N}_{a+2}^b, \\ \alpha x(a) - \beta \nabla x(a+1) = A, \\ \gamma x(b) + \delta \nabla x(b) = B, \end{cases} \tag{2.9}$$

where $0 < \nu < 1$; $b - a \in \mathbb{N}_2$; α, β, γ, δ, A, and B are real-valued constants such that $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$; and $h : \mathbb{N}_{a+2}^b \to \mathbb{R}$. Note that the difference equation is satisfied only for $t \in \mathbb{N}_{a+2}^b$, but solutions to these boundary value problems are defined on $\mathbb{N}_a^b$.

Theorem 50. *Assume* (2.8) *has only the trivial solution. Then* (2.9) *has a unique solution.*

Proof. Let $x_1, x_2 : \mathbb{N}_a \to \mathbb{R}$ be two linearly independent solutions of $L_a x(t) = 0$. Then by Theorem 41, a general solution is given by $x(t) = c_1 x_1(t) + c_2 x_2(t)$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Note that $x(t)$ satisfies the boundary conditions in (2.8) if it satisfies the following system of equations

$$\begin{cases} \alpha \left[c_1 x_1(a) + c_2 x_2(a)\right] - \beta \nabla \left[c_1 x_1(a+1) + c_2 x_2(a+1)\right] = 0, \\ \gamma \left[c_1 x_1(b) + c_2 x_2(b)\right] + \delta \nabla \left[c_1 x_1(b) + c_2 x_2(b)\right] = 0, \end{cases}$$

if and only if it satisfies the following equivalent system of equations

$$\begin{cases} c_1\left[\alpha x_1(a) - \beta\nabla x_1(a+1)\right] + c_2\left[\alpha x_2(a) - \beta\nabla x_2(a+1)\right] = 0, \\ c_1\left[\gamma x_1(b) + \delta\nabla x_1(b)\right] + c_2\left[\gamma x_2(b) + \delta\nabla x_2(b)\right] = 0, \end{cases}$$

which is equivalent to the following vector equation

$$\begin{pmatrix} \alpha x_1(a) - \beta\nabla x_1(a+1) & \alpha x_2(a) - \beta\nabla x_2(a+1) \\ \gamma x_1(b) + \delta\nabla x_1(b) & \gamma x_2(b) + \delta\nabla x_2(b) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But $x(t)$ is the trivial solution if and only if $c_1 = c_2 = 0$ if and only if

$$D := \begin{vmatrix} \alpha x_1(a) - \beta\nabla x_1(a+1) & \alpha x_2(a) - \beta\nabla x_2(a+1) \\ \gamma x_1(b) + \delta\nabla x_1(b) & \gamma x_2(b) + \delta\nabla x_2(b) \end{vmatrix} \neq 0.$$

Hence if (2.8) has only the trivial solution, then $D \neq 0$.

Now consider (2.9). By Corollary 42, a general solution to $L_a x(t) = h(t)$ is given by $x(t) = a_1 x_1(t) + a_2 x_2(t) + x_p(t)$, where $a_1, a_2 \in \mathbb{R}$ are arbitrary constants and $x_p(t) : \mathbb{N}_a \to \mathbb{R}$ is a particular solution to $L_a x(t) = h(t)$. To satisfy the boundary conditions in (2.9), $x(t)$ must satisfy the following system of equations

$$\begin{cases} \alpha\left[a_1 x_1(a) + a_2 x_2(a)\right] - \beta\nabla\left[a_1 x_1(a+1) + a_2 x_2(a+1)\right] \\ \qquad = A - \alpha x_p(a) + \beta\nabla x_p(a+1), \\ \gamma\left[a_1 x_1(b) + a_2 x_2(b)\right] + \delta\nabla\left[a_1 x_1(b) + a_2 x_2(b)\right] \\ \qquad = B - \gamma x_p(b) - \delta\nabla x_p(b), \end{cases}$$

if and only if it satisfies the following equivalent system of equations

$$\begin{cases} a_1\left[\alpha x_1(a)-\beta\nabla x_1(a+1)\right]+a_2\left[\alpha x_2(a)-\beta\nabla x_2(a+1)\right] \\ \qquad = A-\alpha x_p(a)+\beta\nabla x_p(a+1), \\ a_1\left[\gamma x_1(b)+\delta\nabla x_1(b)\right]+a_2\left[\gamma x_2(b)+\delta\nabla x_2(b)\right] \\ \qquad = B-\gamma x_p(b)-\delta\nabla x_p(b), \end{cases}$$

which is equivalent to the following vector equation

$$\begin{pmatrix} \alpha x_1(a)-\beta\nabla x_1(a+1) & \alpha x_2(a)-\beta\nabla x_2(a+1) \\ \gamma x_1(b)+\delta\nabla x_1(b) & \gamma x_2(b)+\delta\nabla x_2(b) \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} A-\alpha x_p(a)+\beta\nabla x_p(a+1) \\ B-\gamma x_p(b)-\delta\nabla x_p(b) \end{pmatrix}.$$

But from before, $D \neq 0$, hence there exists unique $a_1, a_2 \in \mathbb{R}$ such that the above matrix equation is satisfied. Hence $x(t) = a_1x_1(t) + a_2x_2(t) + x_p(t)$ uniquely solves the nonhomogeneous boundary value problem (2.9). □

Theorem 51. *Let* $r := \alpha\gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b} + \alpha\delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b} + \frac{\beta\gamma}{p(a+1)}$. *Then the boundary value problem*

$$\begin{cases} \nabla\left[p(t)\nabla_{a^*}^{\nu}x(t)\right]=0, \quad t\in\mathbb{N}_{a+2}^{b}, \\ \alpha x(a)-\beta\nabla x(a+1)=0, \\ \gamma x(b)+\delta\nabla x(b)=0, \end{cases} \tag{2.10}$$

has only the trivial solution if and only if $r \neq 0$.

Proof. Note that $x_1(t) = 1$ and $x_2(t) = \nabla_a^{-\nu}\frac{1}{p(t)}$ are two linearly independent solutions to $\nabla\left[p(t)\nabla_{a^*}^{\nu}x(t)\right]=0$, so by Theorem 41 a general solution to $\nabla\left[p(t)\nabla_{a^*}^{\nu}x(t)\right]=0$ is given by $x(t)=c_1+c_2\nabla_a^{-\nu}\frac{1}{p(t)}$, where $c_1, c_2\in\mathbb{R}$ are arbitrary constants. Consider the left boundary condition

$$\begin{aligned}\alpha x(a)-\beta\nabla x(a+1)&=\alpha\left[c_1+c_2\nabla_a^{-\nu}\frac{1}{p(t)}\right]|_{t=a}-\beta\left[c_1\nabla 1+c_2\nabla\nabla_a^{-\nu}\frac{1}{p(t)}\right]|_{t=a+1}\\&=c_1\alpha-c_2\beta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=a+1}\\&=c_1\alpha-c_2\beta\int_a^{a+1}H_{\nu-2}(a+1,\rho(s))\frac{1}{p(s)}\nabla s\\&=c_1\alpha-c_2\frac{\beta}{p(a+1)}\\&=0.\end{aligned}$$

Hence c_1 and c_2 satisfy $c_1\alpha-c_2\frac{\beta}{p(a+1)}=0$. Now consider the right boundary condition

$$\begin{aligned}\gamma x(b)+\delta\nabla x(b)&=\gamma\left[c_1+c_2\nabla_a^{-\nu}\frac{1}{p(t)}\right]|_{t=b}+\delta\left[c_1\nabla 1+c_2\nabla\nabla_a^{-\nu}\frac{1}{p(t)}\right]|_{t=b}\\&=c_1\gamma+c_2\gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b}+c_2\delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b}\\&=0.\end{aligned}$$

Hence c_1 and c_2 satisfy $c_1\gamma+c_2\left[\gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b}+\delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b}\right]=0$.

Using these boundary conditions, c_1 and c_2 must satisfy the following vector equation

$$\begin{pmatrix}\alpha & -\frac{\beta}{p(a+1)}\\ \gamma & \gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b}+\delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

Note then the determinant of the previous matrix is given by

$$\begin{vmatrix} \alpha & -\frac{\beta}{p(a+1)} \\ \gamma & \gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b} + \delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b} \end{vmatrix} = \alpha\gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b} + \alpha\delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b}$$

$$+\frac{\beta\gamma}{p(a+1)}$$

$$= r,$$

thus the boundary value problem (2.10) has only the trivial solution if and only if $r \neq 0$. □

Example 52 (Non-unique BVP Solution)**.** The following boundary value problem does not have a unique solution.

$$\begin{cases} \nabla[\nabla_{a*}^{\nu}x(t)] = h(t), \quad t \in \mathbb{N}_{a+2}^{b}, \\ \nabla x(a+1) = A, \\ \nabla x(b) = B. \end{cases}$$

To see this, note this BVP is a specific case of (2.10) where $\alpha = 0$, $\beta = -1$, $\gamma = 0$, $\delta = 1$, and $p(t) \equiv 1$. Then

$$r := \alpha\gamma\left(\nabla_a^{-\nu}\frac{1}{p(t)}\right)|_{t=b} + \alpha\delta\left(\nabla_a^{1-\nu}\frac{1}{p(t)}\right)|_{t=b} + \frac{\beta\gamma}{p(a+1)} = 0.$$

Hence the above boundary value problem does not have a unique solution by Theorem 51.

Definition 53 (Green's Function)**.** Assume that (2.8) has only the trivial solution. We define the Green's function $G(t,s)$ where $G : \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b \to \mathbb{R}$ for the homogeneous boundary value problem (2.8) by

$$G(t,s) := \begin{cases} u(t,s), & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{\max\{t+1,a+2\}}^b, \\ v(t,s), & t \in \mathbb{N}_{a+1}^b \text{ and } s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}, \end{cases}$$

where, for each fixed $s \in \mathbb{N}_{a+2}^b$, $u(t,s)$ solves the boundary value problem

$$\begin{cases} L_a u(t,s) = 0, \quad t \in \mathbb{N}_{a+2}^b \\ \alpha u(a,s) - \beta \left(\nabla_t u(t,s)\right)|_{t=a+1} = 0, \\ \gamma u(b,s) + \delta \left(\nabla_t u(t,s)\right)|_{t=b} = -\left[\gamma x(b,s) + \delta \left(\nabla_t x(t,s)\right)|_{t=b}\right], \end{cases}$$

and $v(t,s) := u(t,s) + x(t,s)$, where $x(t,s)$ is the Cauchy function for $L_a x(t) = 0$.

Remark 54. Note some care is needed when specifying domains for $u(t,s)$ and $v(t,s)$ while respecting the domains for the t and s components of the Green's function. For example, in the case of $u(t,s)$, if $t = a$, then $t+1 = a+1$, but $t+1 = a+1 \notin \mathbb{N}_{a+2}^b$, which is the domain for s. Hence we use $\max\{t+1, a+2\}$ for the lower bound on the domain of s. Also for the case of $u(t,s)$, when $t = b$, we would have $s \in \mathbb{N}_{b+1}^b$, which is an impossible situation. Hence we restrict the domain of t to be in $\mathbb{N}_a^{b-1}$. Similarly, the upper bound for the domain of s in the definition of $v(t,s)$ is $\min\{t+1, b\}$ and the lower bound for the domain of t is $a+1$.

Remark 55. When dealing with the Green's function defined piecewise, there is an overlap of the domains for $u(t,s)$ and $v(t,s)$ when $s = t+1$. The Green's function is still well defined, as $u(t,s) = v(t,s)$ when $s = t+1$. To see this, note

$$v(t,t+1) := u(t,t+1) + x(t,t+1) = u(t,t+1) + 0 = u(t,t+1),$$

where $x(t,s)$ is the Cauchy function for $L_a x(t) = 0$.

Theorem 56 (Green's Function Theorem)**.** *Assume* (2.8) *has only the trivial solution. Then the solution to the nonhomogeneous boundary value problem* (2.9)*, with* $A = B = 0$*, is given by*

$$x(t) = \int_{a+1}^{b} G(t,s)h(s)\nabla s,$$

where $G(t,s)$ *is the Green's function for the homogeneous boundary value problem* (2.8)*.*

Proof. Assume (2.8) has only the trivial solution and consider, for $t \in \mathbb{N}_{a+1}^{b}$,

$$\begin{aligned} x(t) &= \int_{a+1}^{b} G(t,s)h(s)\nabla s \\ &= \int_{a+1}^{t} v(t,s)h(s)\nabla s + \int_{t}^{b} u(t,s)h(s)\nabla s \\ &= \int_{a+1}^{t} [u(t,s) + x(t,s)]h(s)\nabla s + \int_{t}^{b} u(t,s)h(s)\nabla s \\ &= \int_{a+1}^{b} u(t,s)h(s)\nabla s + \int_{a+1}^{t} x(t,s)h(s)\nabla s. \end{aligned} \tag{2.11}$$

When $t = a$, we have

$$\begin{aligned} x(a) &= \int_{a+1}^{b} G(a,s)h(s)\nabla s \\ &= \int_{a+1}^{b} u(a,s)h(s)\nabla s + 0 \\ &= \int_{a+1}^{b} u(a,s)h(s)\nabla s + \int_{a+1}^{a} x(a,s)h(s)\nabla s \\ &= \left(\int_{a+1}^{b} u(t,s)h(s)\nabla s + \int_{a+1}^{t} x(t,s)h(s)\nabla s \right) |_{t=a}. \end{aligned} \tag{2.12}$$

Hence for $t \in \mathbb{N}_a^b$,

$$x(t) = \int_{a+1}^{b} u(t,s)h(s)\nabla s + \int_{a+1}^{t} x(t,s)h(s)\nabla s.$$

Note that by the variation of constants formula given in Theorem 46,
$z(t) := \int_{a+1}^{t} x(t,s)h(s)\nabla s$ solves the initial value problem

$$\begin{cases} L_a z(t) = h(t), \quad t \in \mathbb{N}_{a+2} \\ z(a) = 0, \\ \nabla z(a+1) = 0. \end{cases}$$

Thus $x(t) = \int_{a+1}^{b} u(t,s)h(s)\nabla s + z(t)$. Using Theorem 16 and Lemma 38, composing both sides of the previous equation with the operator L_a gives

$$\begin{aligned} L_a x(t) &= L_a \int_{a+1}^{b} u(t,s)h(s)\nabla s + L_a z(t) \\ &= L_a \int_{a+1}^{t} u(t,s)h(s)\nabla s + L_a \int_{t}^{b} u(t,s)h(s)\nabla s + h(t) \\ &= \int_{a+1}^{b} L_a u(t,s)h(s)\nabla s + h(t) \\ &= \int_{a+1}^{b} 0 \cdot h(s)\nabla s + h(t) \\ &= h(t). \end{aligned}$$

Hence $x(t)$ satisfies the difference equation in the nonhomogeneous boundary value problem (2.9).

Now consider the left boundary condition

$$\begin{aligned}
\alpha x(a) - \beta \nabla x(a+1) &= \alpha \left[\int_{a+1}^{b} u(a,s)h(s)\nabla s + z(a) \right] \\
&\quad - \beta \left(\nabla \left[\int_{a+1}^{b} u(t,s)h(s)\nabla s + z(t) \right] \right) |_{t=a+1} \\
&= \int_{a+1}^{b} \alpha u(a,s)h(s)\nabla s + \alpha z(a) \\
&\quad - \int_{a+1}^{b} \beta \left(\nabla_t u(t,s) \right) |_{t=a+1} h(s)\nabla s - \beta \left(\nabla z(t) \right) |_{t=a+1} \\
&= \int_{a+1}^{b} \left[\alpha u(a,s) - \beta \left(\nabla_t u(t,s) \right) |_{t=a+1} \right] h(s)\nabla s \\
&\quad + \left[\alpha z(a) - \beta \nabla z(a+1) \right] \\
&= \int_{a+1}^{b} 0 \cdot h(s)\nabla s + \alpha \cdot 0 - \beta \cdot 0 \\
&= 0.
\end{aligned}$$

Hence the left boundary condition in (2.9) is satisfied.

For the right boundary condition, consider

$$
\begin{aligned}
\gamma x(b)+\delta\nabla x(b) &= \gamma\left[\int_{a+1}^{b} u(b,s)h(s)\nabla s + z(b)\right] \\
&\quad + \delta\left(\nabla\left[\int_{a+1}^{b} u(t,s)h(s)\nabla s + z(t)\right]\right)|_{t=b} \\
&= \int_{a+1}^{b} \gamma u(b,s)h(s)\nabla s + \gamma z(b) \\
&\quad + \int_{a+1}^{b} \delta\left(\nabla_t u(t,s)\right)|_{t=b} h(s)\nabla s + \delta\left(\nabla z(t)\right)|_{t=b} \\
&= \int_{a+1}^{b} \left[\gamma u(b,s) + \delta\left(\nabla_t u(t,s)\right)_{t=b}\right] h(s)\nabla s \\
&\quad + \gamma z(b) + \delta\left(\nabla z(t)\right)|_{t=b} \\
&= \int_{a+1}^{b} \left[\gamma u(b,s) + \delta\left(\nabla_t u(t,s)\right)|_{t=b}\right] h(s)\nabla s \\
&\quad + \gamma\int_{a+1}^{b} x(b,s)h(s)\nabla s + \delta\left(\nabla_t \int_{a+1}^{b} x(t,s)h(s)\nabla s\right)|_{t=b} \\
&= \int_{a+1}^{b} \left[\gamma u(b,s) + \delta\left(\nabla_t u(t,s)\right)|_{t=b}\right] h(s)\nabla s \\
&\quad + \int_{a+1}^{b} \left[\gamma x(b,s) + \delta\left(\nabla_t x(t,s)\right)|_{t=b}\right] h(s)\nabla s \\
&= \int_{a+1}^{b} -\left[\gamma x(b,s) + \delta\left(\nabla_t x(t,s)\right)|_{t=b}\right] h(s)\nabla s \\
&\quad + \int_{a+1}^{b} \left[\gamma x(b,s) + \delta\left(\nabla_t x(t,s)\right)|_{t=b}\right] h(s)\nabla s \\
&= 0.
\end{aligned}
$$

Thus the right boundary condition in (2.9) is satisfied. Therefore

$$x(t) = \int_{a+1}^{b} G(t,s)h(s)\nabla s$$

solves the nonhomogeneous boundary value problem (2.9) with $A = B = 0$. □

Remark 57. If at the start of the proof we instead consider $t \in \mathbb{N}_a^{b-1}$, we would split the Green's function into two integrals at the point $t+1$ instead of t as in (2.11) and considered the $t = b$ case separately in place of (2.12). This would result in the same end result as a consequence of Remark 55.

Corollary 58. *Assume that* (2.8) *has only the trivial solution. Then the solution to the nonhomogeneous boundary value problem* (2.9)*, with arbitrary* $A, B \in \mathbb{R}$*, is given by*

$$x(t) = w(t) + \int_{a+1}^{b} G(t,s)h(s)\nabla s,$$

where $G(t,s)$ *is the Green's function for the homogeneous boundary value problem* (2.8)*, and* $w(t)$ *solves the boundary value problem*

$$\begin{cases} L_a w(t) = 0, \quad \mathbb{N}_{a+2}^{b}, \\ \alpha w(a) - \beta \nabla w(a+1) = A, \\ \gamma w(b) + \delta \nabla w(b) = B. \end{cases}$$

Proof. The proof follows immediately from Theorem 56 and the linearity of the operator L_a. $\square$

Chapter 3

Green's Functions for Specific Boundary Value Problems

In this chapter, we will continue to look at the fractional self-adjoint difference equation

$$\nabla[p(t)\nabla^{\nu}_{a^*}x(t)] + q(t)x(t-1) = h(t), \quad t \in \mathbb{N}^{b}_{a+2},$$

with various boundary conditions.

3.1 A Conjugate Boundary Value Problem

In this section, we investigate the Green's function for a conjugate boundary value problem, called as such because the boundary conditions only depend on x at a and b. Similar results hold in the continuous setting (see [26]). Also, Brackins [15] investigates a conjugate, fractional, self-adjoint boundary value problem involving the Riemann-Liouville fractional difference.

In particular, we consider the Green's function for the conjugate, fractional, self-adjoint boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = 0, \\ x(b) = 0. \end{cases} \tag{3.1}$$

3.1.1 Conjugate Green's Function

Here we determine the Green's function for the conjugate, fractional, self-adjoint boundary value problem (3.1).

Note here that this is a specific case of the general homogeneous boundary value problem (2.8) where $q(t) \equiv 0$ and $p(t) \equiv 1$, with $\alpha = 1$, $\beta = 0$, $\gamma = 1$, and $\delta = 0$. Since

$$\begin{aligned} r &:= \alpha\gamma \left(\nabla_a^{-\nu} \frac{1}{p(t)} \right) |_{t=b} + \alpha\delta \left(\nabla_a^{1-\nu} \frac{1}{p(t)} \right) |_{t=b} + \frac{\beta\gamma}{p(a+1)} \\ &= (\nabla_a^{\nu} 1)|_{t=b} + 0 + 0 \\ &= \int_a^b H_{\nu-1}(b, \rho(s)) \cdot 1 \nabla s \\ &= H_\nu(b, a) \\ &\neq 0, \end{aligned}$$

by Theorem 51, (3.1) has only the trivial solution, thus we apply Definition 53, i.e. we want to, for each fixed $s \in \mathbb{N}_{a+2}^b$, solve the boundary value problem

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} u(t, s) = 0, \quad t \in \mathbb{N}_{a+2}^b \\ u(a, s) = 0, \\ u(b, s) = -x(b, s) = -H_\nu(b, \rho(s)), \end{cases} \tag{3.2}$$

where $x(t, s) = H_\nu(t, \rho(s))$ is the Cauchy function for $\nabla\nabla_{a^*}^{\nu} x(t) = 0$ found in Example 49.

As in the proof of Theorem 51, $x_1(t) = 1$ and $x_2(t) = \nabla_a^{-\nu} 1$ are two linearly independent solutions to $L_a x(t) = 0$. Hence a general solution to (3.2) is given by

$$u(t,s) = c_1(s) + c_2(s)\nabla_a^{-\nu}1.$$

Since $\left(\nabla_a^{-\nu}1\right)(t) = H_\nu(t,a)$, we have that $u(t,s) = c_1(s) + c_2(s)H_\nu(t,a)$. Applying the first boundary condition, we get that

$$u(a,s) = c_1(s) + c_2(s)H_\nu(a,a) = c_1(s) = 0.$$

Thus $u(t,s) = c_2(s)H_\nu(t,a)$. Applying the second boundary condition gives

$$u(b,s) = c_2(s)H_\nu(b,a) = -H_\nu(b,\rho(s)).$$

Hence $c_2(s) = -\frac{H_\nu(b,\rho(s))}{H_\nu(b,a)}$, and therefore

$$u(t,s) = -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)},$$

which from Definition 53 implies

$$v(t,s) = -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} + H_\nu(t,\rho(s)).$$

Therefore the Green's function for (3.1) is given by

$$G(t,s) = \begin{cases} -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)}, & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{\max\{t+1,a+2\}}^{b}, \\ -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} + H_\nu(t,\rho(s)), & t \in \mathbb{N}_{a+1}^{b} \text{ and } s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}. \end{cases}$$

3.1.2 Conjugate Green's Function Properties

Lemma 59. *The Green's function for the boundary value problem* (3.6) *satisfies*

$$\nabla_t G(t,s) \leq 0, \tag{3.3}$$

for $t \in \mathbb{N}_{a+1}^{b-1}$ *and* $s \in \mathbb{N}_{t+1}^{b}$, *and*

$$\nabla_t G(t,s) \geq 0, \tag{3.4}$$

for $t \in \mathbb{N}_{a+2}^{b}$ *and* $s \in \mathbb{N}_{a+2}^{t}$.

Proof of (3.3)*:* Let $t \in \mathbb{N}_{a+1}^{b-1}$ and $s \in \mathbb{N}_{t+1}^{b}$. Then $G(t,s) = u(t,s)$. Thus consider

$$\begin{aligned}
\nabla_t G(t,s) &= \nabla_t u(t,s) \\
&= \nabla_t \left(-\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} \right) \\
&= -\frac{H_\nu(b,\rho(s))H_{\nu-1}(t,a)}{H_\nu(b,a)}.
\end{aligned}$$

Since $0 < \nu < 1$, we have $\nu > -1$ and $\nu - 1 > -1$. So by Lemma 27, we have $H_\nu(b,\rho(s)) > 0$, $H_{\nu-1}(t,a) > 0$, and $H_\nu(b,a) > 0$. Hence, for $t \in \mathbb{N}_{a+1}^{b-1}$ and $s \in \mathbb{N}_{t+1}^{b}$, $\nabla_t G(t,s) \leq 0$. □

Proof of (3.4)*:* Let $t \in \mathbb{N}_{a+2}^{b}$ and $s \in \mathbb{N}_{a+2}^{t}$. Then $G(t,s) = v(t,s)$, so

$$\begin{aligned}
\nabla_t G(t,s) &= \nabla_t v(t,s) \\
&= -\nabla_t \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} + \nabla_t H_\nu(t,\rho(s)) \\
&= -\frac{H_\nu(b,\rho(s))H_{\nu-1}(t,a)}{H_\nu(b,a)} + H_{\nu-1}(t,\rho(s))
\end{aligned}$$

This is nonnegative if and only if

$$\frac{H_\nu(b,\rho(s))H_{\nu-1}(t,a)}{H_\nu(b,a)} \le H_{\nu-1}(t,\rho(s)). \tag{3.5}$$

Since $s \in \mathbb{N}_{a+2}^t$ and $0 < \nu < 1$, we have that $t - \rho(s) > 0$ implying $H_{\nu-1}(t,\rho(s)) > 0$, so (3.5) is equivalent to

$$\frac{H_\nu(b,\rho(s))H_{\nu-1}(t,a)}{H_\nu(b,a)H_{\nu-1}(t,\rho(s))} \le 1$$

Breaking this down, consider

$$\begin{aligned}\frac{H_\nu(b,\rho(s))}{H_\nu(b,a)} &= \frac{(b-\rho(s))^{\overline{\nu}}}{\Gamma(\nu+1)} \cdot \frac{\Gamma(\nu+1)}{(b-a)^{\overline{\nu}}} \\ &= \frac{\Gamma\left(b-\rho(s)+\nu\right)}{\Gamma\left(b-\rho(s)\right)} \cdot \frac{\Gamma(b-a)}{\Gamma(b-a+\nu)} \\ &= \frac{\Gamma\left(b-\rho(s)+\nu\right)}{\Gamma(b-a+\nu)} \cdot \frac{\Gamma(b-a)}{\Gamma\left(b-\rho(s)\right)}.\end{aligned}$$

Using the property that $\Gamma(x+1) = x\Gamma(x)$, we can rewrite the previous equality as

$$\begin{aligned}&\frac{\Gamma\left(b-\rho(s)+\nu\right)}{\Gamma(b-a+\nu)}\frac{\Gamma(b-a)}{\Gamma\left(b-\rho(s)\right)} \\ &\qquad = \frac{\Gamma(b-\rho(s)+\nu)}{(b-(a+1)+\nu)\Gamma(b-(a+1)+\nu)} \cdot \frac{(b-(a+1))\Gamma(b-(a+1))}{\Gamma(b-\rho(s))} \\ &\qquad = \frac{\Gamma(b-\rho(s)+\nu)}{(b-(a+1)+\nu)(b-(a+2)+\nu)\Gamma(b-(a+2)+\nu)} \\ &\qquad\qquad \cdot \frac{(b-(a+1))(b-(a+2))\Gamma(b-(a+2))}{\Gamma(b-\rho(s))} \\ &\qquad = \frac{\Gamma(b-\rho(s)+\nu)}{(b-(a+1)+\nu)(b-(a+2)+\nu)\cdots(b-\rho(s)+\nu)\Gamma(b-\rho(s)+\nu)} \\ &\qquad\qquad \cdot \frac{(b-(a+1))(b-(a+2))\cdots(b-\rho(s))\Gamma(b-\rho(s))}{\Gamma(b-\rho(s))} \\ &\qquad = \frac{(b-(a+1))(b-(a+2))\cdots(b-\rho(s))}{(b-(a+1)+\nu)(b-(a+2)+\nu)\cdots(b-\rho(s)+\nu)}.\end{aligned}$$

Note that this is shown in general using the $\cdots$ notation for multiplication of decreasing factors, but may not make sense for small s. However this is not an issue, because $s \in \mathbb{N}_{a+2}^{t}$, the above Gamma property expansions will have at least one term, i.e. if, at its worst, $s = a + 2$, then

$$\frac{H_\nu(b, \rho(s))}{H_\nu(b, a)} = \frac{(b - (a + 1))}{(b - (a + 1) + \nu)}.$$

For large s in the domain, the above expansions using shorthand for decreasing factors will make sense as written.

Since $0 < \nu < 1$, we get that $\frac{b-(a+1)}{b-(a+1)+\nu} \leq 1$, $\frac{b-(a+2)}{b-(a+2)+\nu} \leq 1$, $\ldots$, $\frac{b-\rho(s)}{b-\rho(s)+\nu} \leq 1$, hence $\frac{H_\nu(b,\rho(s))}{H_\nu(b,a)} \leq 1$.

By a similar expansion with the same notation issue noted, we get

$$\frac{H_{\nu-1}(t, a)}{H_{\nu-1}(t, \rho(s))} = \frac{(t - a + \nu)(t - a + \nu + 1) \cdots (t - \rho(s) + \nu - 1)}{(t - a + 1)(t - a + 2) \cdots (t - \rho(s))}.$$

Again, since $0 < \nu < 1$, we have that $\frac{t-a+\nu}{t-a+1} \leq 1$, $\frac{t-a+\nu+1}{t-a+2} \leq 1$, $\ldots$, $\frac{t-\rho(s)+\nu-1}{t-\rho(s)} \leq 1$, thus $\frac{H_{\nu-1}(t,a)}{H_{\nu-1}(t,\rho(s))} \leq 1$.

Combining these two results, we get that $\frac{H_\nu(b,\rho(s))H_{\nu-1}(t,a)}{H_\nu(b,a)H_{\nu-1}(t,\rho(s))} \leq 1$, i.e. (3.5) is true. Hence $\nabla_t G(t, s) \geq 0$ for $t \in \mathbb{N}_{a+2}^{b}$ and $s \in \mathbb{N}_{a+2}^{t}$. $\square$

Remark 60. For Lemma 59, we note that again care must be taken when specifying the domain of t and s.

If $t \in \mathbb{N}_{a}^{b-1}$ and $s \in \mathbb{N}_{\max\{t+1,a+2\}}^{b}$, then $G(t, s) = u(t, s)$. But consider $\nabla_t u(t, s) = u(t, s) - u(t-1, s)$. The term $u(t-1, s)$ is defined for $t \in \mathbb{N}_{a+1}^{b}$ and $s \in \mathbb{N}_{\max\{t,a+2\}}^{b}$. So for $u(t, s)$ and $u(t-1, s)$ to be both well defined, we must have $t \in \mathbb{N}_{a}^{b-1} \cap \mathbb{N}_{a+1}^{b} = \mathbb{N}_{a+1}^{b-1}$ and $s \in \mathbb{N}_{\max\{t+1,a+2\}}^{b} \cap \mathbb{N}_{\max\{t,a+2\}}^{b} = \mathbb{N}_{\max\{t+1,a+2\}}^{b}$. Since $t \in \mathbb{N}_{a+1}^{b-1}$, $\max\{t+1, a+2\} = t + 1$. Hence the domain for $\nabla_t u(t, s)$ is $t \in \mathbb{N}_{a+1}^{b-1}$ and $s \in \mathbb{N}_{t+1}^{b}$.

In the other case, if $t \in \mathbb{N}_{a+1}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}$, then $G(t,s) = v(t,s)$. But $\nabla_t v(t,s) = v(t,s) - v(t-1,s)$. The term $v(t-1,s)$ is defined for $t \in \mathbb{N}_{a+2}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{t,b\}}$. Therefore, for $v(t,s)$ and $v(t-1,s)$ to be both well defined, we must have $t \in \mathbb{N}_{a+1}^{b} \cap \mathbb{N}_{a+2}^{b} = \mathbb{N}_{a+2}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}} \cap \mathbb{N}_{a+2}^{\min\{t,b\}} = \mathbb{N}_{a+2}^{\min\{t,b\}}$. But since $t \in \mathbb{N}_{a+2}^{b}$, $\min\{t,b\} = t$. Therefore the domain for $\nabla_t v(t,s)$ is $t \in \mathbb{N}_{a+2}^{b}$ and $s \in \mathbb{N}_{a+2}^{t}$.

Theorem 61. *The Green's function for the boundary value problem*

$$\begin{cases} \nabla\nabla_{a*}^{\nu} x(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ x(a) = 0, \\ x(b) = 0, \end{cases} \tag{3.6}$$

where $b-a \in \mathbb{N}_2$ and $0 < \nu < 1$, given by

$$G(t,s) = \begin{cases} -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)}, & t \in \mathbb{N}_{a}^{b-1} \text{ and } s \in \mathbb{N}_{\max\{t+1,a+2\}}^{b}, \\ -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} + H_\nu(t,\rho(s)), & t \in \mathbb{N}_{a+1}^{b} \text{ and } s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}. \end{cases}$$

satisfies the inequalities

1. $G(t,s) \leq 0, \quad$ *for* $t \in \mathbb{N}_a^b$ *and* $s \in \mathbb{N}_{a+2}^{b}$,
2. $G(t,s) \geq -\frac{(b-a)^2}{4(b-a)^{\overline{\nu}}\Gamma(\nu+1)}, \quad$ *for* $t \in \mathbb{N}_a^b$ *and* $s \in \mathbb{N}_{a+2}^{b}$,
3. $\int_{a+1}^{b} |G(t,s)|\nabla s \leq \frac{(b-a)^2}{4\Gamma(\nu+2)}, \quad$ *for* $t \in \mathbb{N}_a^b$,

First we prove a lemma that will simplify the proof of Theorem 61.

Lemma 62. *Let $t \in \mathbb{N}_1$ and $0 < \nu < 1$. Then*

$$t^{\overline{\nu}} \leq t.$$

Proof. Since $t \in \mathbb{N}_1$, we have that by properties of the Gamma function

$$t^{\overline{\nu}} = \frac{\Gamma(t+\nu)}{\Gamma(t)} \leq \frac{\Gamma(t+1)}{\Gamma(t)} = \frac{t\Gamma(t)}{\Gamma(t)} = t.$$

□

Proof of Theorem 61 part (1): Let $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{t+1,a+2\}}^b$. Then $G(t,s) = u(t,s)$. From (3.3), we know that $u(t,s)$ is nonincreasing in t. thus the maximum of $u(t,s)$ will occur when $t = a$. So

$$u(a,s) = -\frac{H_\nu(b,\rho(s))H_\nu(a,a)}{H_\nu(b,a)} = 0,$$

hence $u(t,s) \leq 0$ when $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{t+1,a+2\}}^b$.

Let $t \in \mathbb{N}_{a+1}^b$ and $s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}$, so then $G(t,s) = v(t,s)$. By (3.4), $v(t,s)$ is a nondecreasing function in t. Thus the maximum of $v(t,s)$ will occur when $t = b$. Consider

$$v(b,s) = -\frac{H_\nu(b,\rho(s))H_\nu(b,a)}{H_\nu(b,a)} + H_\nu(b,\rho(s)) = -H_\nu(b,\rho(s)) + H_\nu(b,\rho(s)) = 0.$$

Thus $v(t,s) \leq 0$ for all $t \in \mathbb{N}_{a+1}^b$ and $s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}$, thus showing $G(t,s) \leq 0$ for $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^b$. □

Proof of Theorem 61 part (2): By Lemma 59, we have that the minimum of the

Green's function occurs when $t = \rho(s)$. So

$$\begin{aligned} G(t,s) &\geq G(\rho(s),s) \\ &= -\frac{H_\nu(b,\rho(s))H_\nu(\rho(s),a)}{H_\nu(b,a)} \\ &= -\frac{(b-\rho(s))^{\overline{\nu}}}{\Gamma(\nu+1)}\frac{\Gamma(\nu+1)}{(b-a)^{\overline{\nu}}}\frac{(\rho(s)-a)^{\overline{\nu}}}{\Gamma(\nu+1)}. \end{aligned}$$

But $s \in \mathbb{N}_{a+2}^{b}$, so $b-\rho(s) \in \mathbb{N}_1$ and $\rho(s)-a \in \mathbb{N}_1$, thus by Lemma 62, we have

$$G(t,s) \geq -\frac{(b-\rho(s))(\rho(s)-a)}{(b-a)^{\overline{\nu}}\Gamma(\nu+1)} = -\frac{(b-s+1)(s-1-a)}{(b-a)^{\overline{\nu}}\Gamma(\nu+1)}.$$

But $-(b-s+1)(s-1-a)$ is a parabola opening upwards with roots of $s = a+1$ and $s = b+1$ which has its minimum value at $s = \frac{(a+1)+(b+1)}{2} = \frac{a+b}{2}+1$, so

$$\begin{aligned} G(t,s) &\geq -\frac{\left(b-\left(\frac{a+b}{2}+1\right)+1\right)\left(\left(\frac{a+b}{2}+1\right)-1-a\right)}{(b-a)^{\overline{\nu}}\Gamma(\nu+1)} \\ &= -\frac{\left(\frac{b}{2}-\frac{a}{2}\right)\left(\frac{b}{2}-\frac{a}{2}\right)}{(b-a)^{\overline{\nu}}\Gamma(\nu+1)} \\ &= -\frac{(b-a)^2}{4(b-a)^{\overline{\nu}}\Gamma(\nu+1)}. \end{aligned}$$

□

Proof of Theorem 61 part (3): Part (1) shows that $G(t,s) \leq 0$, hence

$$\begin{aligned}\int_{a+1}^{b} |G(t,s)|\nabla s &= -\int_{a+1}^{b} G(t,s)\nabla s \\ &= -\int_{a+1}^{t} v(t,s)\nabla s - \int_{t}^{b} u(t,s)\nabla s \\ &= -\int_{a+1}^{t} \left[u(t,s) + x(t,s)\right]\nabla s - \int_{t}^{b} u(t,s)\nabla s \\ &= -\int_{a+1}^{b} u(t,s)\nabla s - \int_{a+1}^{t} x(t,s)\nabla s \\ &= \int_{a+1}^{b} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)}\nabla s - \int_{a+1}^{t} H_\nu(t,\rho(s)\nabla s.\end{aligned}$$

Applying Theorem 25 to integrate yields

$$\begin{aligned}\int_{a+1}^{b} |G(t,s)|\nabla s &= \frac{H_{\nu+1}(b,a+1)H_\nu(t,a)}{H_\nu(b,a)} - H_{\nu+1}(t,a+1) \\ &= H_\nu(t,a)\frac{(b-a-1)^{\overline{\nu+1}}}{\Gamma(\nu+2)}\frac{\Gamma(\nu+1)}{(b-a)^{\overline{\nu}}} - \frac{(t-a-1)^{\overline{\nu+1}}}{\Gamma(\nu+2)} \\ &= \frac{H_\nu(t,a)}{\nu+1}\frac{\Gamma(b-a-1+\nu+1)}{\Gamma(b-a-1)}\frac{\Gamma(b-a)}{\Gamma(b-a+\nu)} \\ &\quad - \frac{\Gamma(t-a-1+\nu+1)}{\Gamma(t-a-1)\Gamma(\nu+2)} \\ &= \frac{H_\nu(t,a)}{\nu+1}(b-a-1) - \frac{\Gamma(t-a+\nu)(t-a-1)}{\Gamma(t-a)(\nu+1)\Gamma(\nu+1)} \\ &= \frac{(t-a)^{\overline{\nu}}}{(\nu+1)\Gamma(\nu+1)}(b-a-1) - \frac{(t-a)^{\overline{\nu}}}{(\nu+1)\Gamma(\nu+1)}(t-a-1) \\ &= \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}\left[(b-a-1)-(t-a-1)\right] \\ &= \frac{(t-a)^{\overline{\nu}}(b-t)}{\Gamma(\nu+2)}.\end{aligned}$$

When $t = a$ we get that $\int_{a+1}^{b} |G(t,s)|\nabla s = 0$, so let $t \in \mathbb{N}_{a+1}^{b}$. Then, since $0 < \nu < 1$ and $t-a \in \mathbb{N}_1$, we have that by Lemma 62 $(t-a)^{\overline{\nu}} \leq (t-a)$. Thus $\int_{a+1}^{b} |G(t,s)|\nabla s \leq \frac{(t-a)(b-t)}{\Gamma(\nu+2)}$ for $t \in \mathbb{N}_{a+1}^{b}$. Note that the numerator is a parabola opening downwards

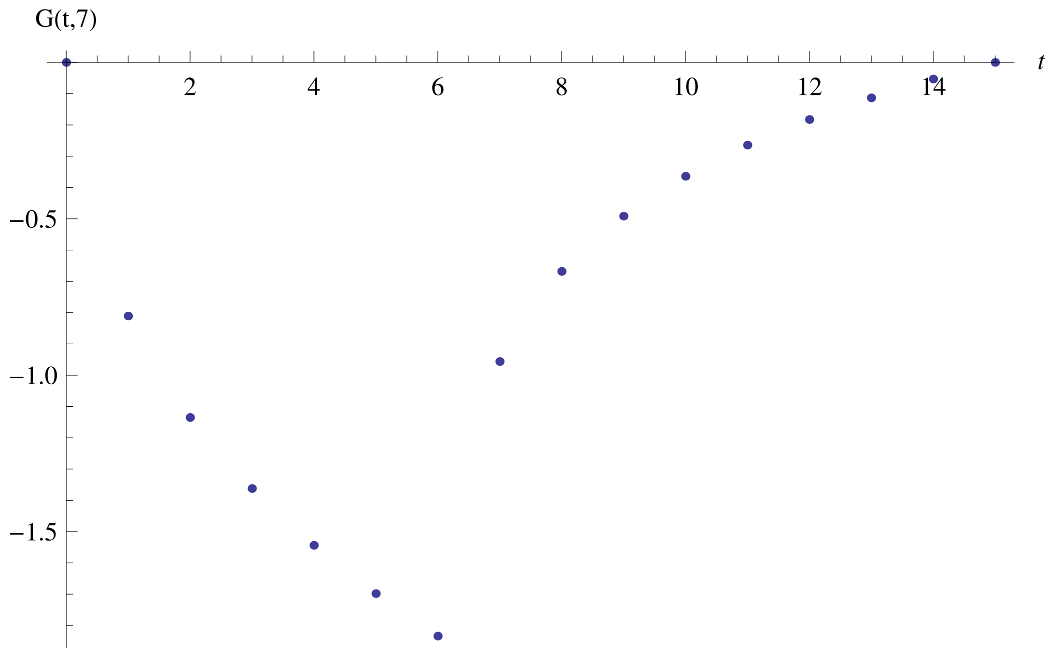

Figure 3.1: Conjugate Green's Function as a function of t where $b = 15$, $a = 0$, $\nu = 0.4$ and fixed $s = 7$.

with its maximum value at $t = \frac{a+b}{2}$. Hence

$$\frac{(t-a)(b-t)}{\Gamma(\nu+2)} \leq \frac{(\frac{a+b}{2}-a)(b-\frac{a+b}{2})}{\Gamma(\nu+2)} = \frac{(b-a)^2}{4\Gamma(\nu+2)}.$$

Therefore $\int_{a+1}^{b} |G(t,s)|\nabla s \leq \frac{(b-a)^2}{4\Gamma(\nu+2)}$, for all $t \in \mathbb{N}_a^b$. □

3.1.3 Graph of Conjugate Green's Function

Example 63. Consider the Green's function for (3.1) where $b = 15$, $a = 0$, and $\nu = 0.4$. Fixing $s = 7$, the graph of the Green's function is given in Figure 3.1.

Figure 3.1 illustrates the properties of the Green's function previously proven. First we see that $G(t,7) \leq 0$ for $t \in \mathbb{N}_0^{15}$ and that the minimum occurs when $t = \rho(s) = 6$ as per Theorem 61. Also, if $t \leq \rho(s) = 6$, then $\nabla_t G(t,7) \leq 0$ as in (3.3), and if $t \geq s$, then $\nabla_t G(t,7) \geq 0$ as in (3.4).

3.2 A Right Focal Boundary Value Problem

The results in this short section are from St. Goar in [16]. The results are included because it represents a specific and important case for the more general three point boundary value problem given in Section 3.3. Note similar results for the delta case is given in [18].

We consider the fractional, homogeneous, self-adjoint, right focal boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = 0, \\ \nabla x(b) = 0, \end{cases} \tag{3.7}$$

and the corresponding fractional, nonhomogeneous, self-adjoint, right focal boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = A, \\ \nabla x(b) = B, \end{cases} \tag{3.8}$$

where $0 < \nu < 1$, $b - a \in \mathbb{N}_2$, and $h(t) : \mathbb{N}_{a+2} \to \mathbb{R}$.

3.2.1 Right Focal Green's Function

Definition 64. [16, Theorem 5.10] The Green's function for (3.7) is given by

$$G(t,s) = \begin{cases} -\frac{H_{\nu-1}(b,\rho(s))H_{\nu}(t,a)}{H_{\nu-1}(b,a)}, & t \in \mathbb{N}^{b-1}_{a} \text{ and } s \in \mathbb{N}^{b}_{\max\{t+1,a+2\}}, \\ -\frac{H_{\nu-1}(b,\rho(s))H_{\nu}(t,a)}{H_{\nu-1}(b,a)} + H_{\nu}(t,\rho(s)), & t \in \mathbb{N}^{b}_{a+1} \text{ and } s \in \mathbb{N}^{\min\{t+1,b\}}_{a+2}. \end{cases}$$

3.2.2 Right Focal Green's Function Properties

Theorem 65. *[16, Theorem 5.11] The Green's function for the boundary value problem* (3.7) *given by*

$$G(t,s) = \begin{cases} -\frac{H_{\nu-1}(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)}, & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{\max\{t+1,a+2\}}^b, \\ -\frac{H_{\nu-1}(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} + H_\nu(t,\rho(s)), & t \in \mathbb{N}_{a+1}^b \text{ and } s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}. \end{cases}$$

satisfies the inequalities

1. $G(t,s) \leq 0$, *for* $t \in \mathbb{N}_a^b$ *and* $s \in \mathbb{N}_{a+2}^b$,

2. $G(t,s) \geq -\frac{b-a+\nu-1}{\nu}$, *for* $t \in \mathbb{N}_a^b$ *and* $s \in \mathbb{N}_{a+2}^b$,

3. $\int_{a+1}^b |G(t,s)|\nabla s \leq \frac{(b-a)(b-a-1)}{\nu\Gamma(2+\nu)}$, *for* $t \in \mathbb{N}_a^b$.

3.2.3 Graph of Right Focal Green's Function

Example 66. Consider the Green's function for (3.7) where $b = 15$, $a = 0$, and $\nu = 0.4$. Fixing $s = 7$, the graph of the Green's function is given in Figure 3.2.

3.3 A Three Point Boundary Value Problem

In this section we consider a particular self-adjoint, three point boundary value problem. See Goodrich [20] for similar work using the delta fractional difference.

In particular, we want to consider the fractional, homogeneous, self-adjoint, three

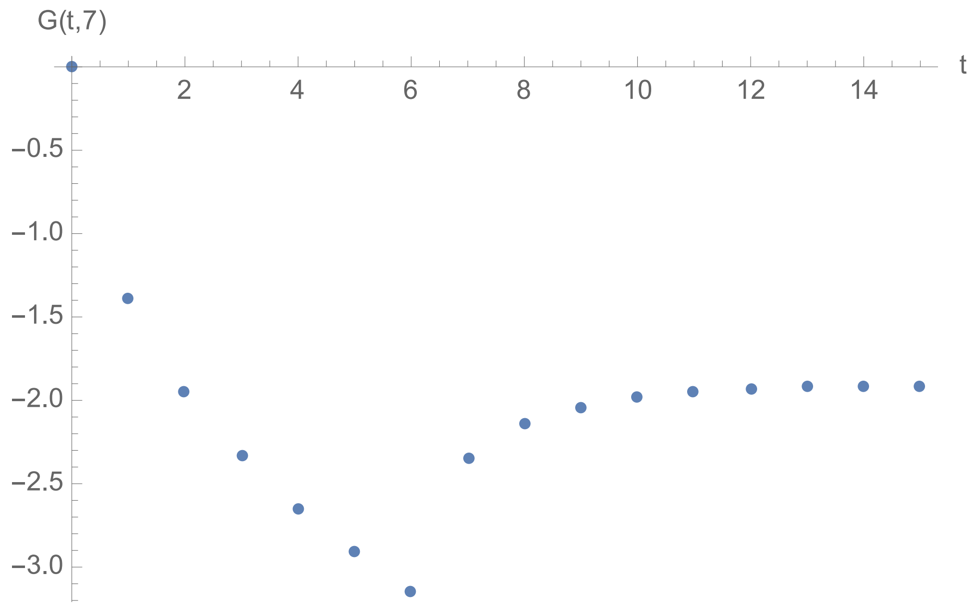

Figure 3.2: Right Focal Green's Function as a function of t where $b = 15$, $a = 0$, $\nu = 0.4$ and fixed $s = 7$.

point boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = 0, \\ x(b) - \alpha x(a+k) = 0, \end{cases} \tag{3.9}$$

and the corresponding fractional, nonhomogeneous, self-adjoint, three point boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = 0, \\ x(b) - \alpha x(a+k) = 0, \end{cases} \tag{3.10}$$

where $0 < \nu < 1$, $b - a \in \mathbb{N}_2$, $h : \mathbb{N}^{b}_{a+2} \to \mathbb{R}$, $0 \le \alpha \le 1$, and $k \in \mathbb{N}^{(b-a)-1}_{1}$.

Note when $\alpha = 0$, this corresponds to the fractional self-adjoint boundary value problem studied in Section 3.1, and when $\alpha = 1$ and $k = b - a - 1$, this corresponds to the fractional self-adjoint boundary value problem studied in Section 3.2.

3.3.1 Three Point Green's Function

We are concerned with finding a Green's function for (3.9)

Using the Cauchy function from Example 49, we have a general solution to (3.10) is given by

$$x(t) = c_1 + c_2 H_\nu(t, a) + \int_{a+1}^{t} H_\nu(t, \rho(s))h(s)\nabla s,$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

The first boundary condition yields

$$x(a) = 0 = c_1 + c_2 H_\nu(a, a) + \int_{a+1}^{a} H_\nu(a, \rho(s))h(s)\nabla s = c_1,$$

i.e. $x(t) = c_2 H_\nu(t, a) + \int_{a+1}^{t} H_\nu(t, \rho(s))h(s)\nabla s$, for some $c_2 \in \mathbb{R}$. Note that the second boundary condition is equivalent to $\alpha x(a + k) - x(b) = 0$, so we consider

$$\begin{aligned} \alpha x(a+k) - x(b) &= \alpha \left[c_2 H_\nu(a+k, a) + \int_{a+1}^{a+k} H_\nu(a+k, \rho(s))h(s)\nabla s \right] \\ &\quad - \left[c_2 H_\nu(b, a) + \int_{a+1}^{b} H_\nu(b, \rho(s))h(s)\nabla s \right] \\ &= c_2 \left[\alpha H_\nu(a+k, a) - H_\nu(b, a) \right] \\ &\quad + \int_{a+1}^{a+k} \alpha H_\nu(a+k, \rho(s))h(s)\nabla s - \int_{a+1}^{b} H_\nu(b, \rho(s))h(s)\nabla s \\ &= 0, \end{aligned}$$

so we get that

$$c_2 = \frac{\int_{a+1}^{b} H_\nu(b,\rho(s))h(s)\nabla s - \int_{a+1}^{a+k} \alpha H_\nu(a+k,\rho(s))h(s)\nabla s}{\alpha H_\nu(a+k,a) - H_\nu(b,a)}.$$

For convenience, define $\Omega := \alpha H_\nu(a+k,a) - H_\nu(b,a)$. Thus the solution to (3.10) is given by

$$x(t) = \int_{a+1}^{b} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s - \int_{a+1}^{a+k} \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s + \int_{a+1}^{t} H_\nu(t,\rho(s))h(s)\nabla s. \tag{3.11}$$

From this, we can deduce the Green's function for (3.9). Consider the case where we let $t \in \mathbb{N}_a^b$ such that $a+k \leq t$. Then (3.11) is equivalent to

$$\begin{aligned} x(t) = &\Bigg(\int_{a+1}^{a+k} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s + \int_{a+k}^{t} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s \\ &+ \int_{t}^{b} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s\Bigg) \\ &- \int_{a+1}^{a+k} \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s \\ &+ \int_{a+1}^{a+k} H_\nu(t,\rho(s))h(s)\nabla s + \int_{a+k}^{t} H_\nu(t,\rho(s))h(s)\nabla s \\ = &\int_{a+1}^{a+k} \left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s))\right) h(s)\nabla s \\ &+ \int_{a+k}^{t} \left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s))\right) h(s)\nabla s \\ &+ \int_{t}^{b} \left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega}\right) h(s)\nabla s. \end{aligned} \tag{3.12}$$

Now let $t \in \mathbb{N}_a^b$ such that $t \leq a+k$. Then (3.11) is equivalent to

$$\begin{aligned}
x(t) =& \Bigg(\int_{a+1}^{t} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s + \int_{t}^{a+k} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s \\
&+ \int_{a+k}^{b} \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s\Bigg) \\
&- \Bigg(\int_{a+1}^{t} \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s \\
&+ \int_{t}^{a+k} \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} h(s)\nabla s\Bigg) \\
&+ \int_{a+1}^{t} H_\nu(t,\rho(s))h(s)\nabla s \\
=& \int_{a+1}^{t} \left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s))\right) h(s)\nabla s \\
&+ \int_{t}^{a+k} \left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega}\right) h(s)\nabla s \\
&+ \int_{a+k}^{b} \left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega}\right) h(s)\nabla s.
\end{aligned} \tag{3.13}$$

Using the convention that $\int_a^t f(s)\nabla s = 0$ for $t \leq a$, we can combine the two cases for $t \in \mathbb{N}_a^b$ as

$$x(t) = \int_{a+1}^{b} G(t,s)h(s)\nabla s,$$

where

$$G(t,s) = \begin{cases} g_1(t,s), & t \in \mathbb{N}_{a+1}^b \text{ and } s \in \mathbb{N}_{a+2}^{\min\{a+k,t\}}, \\ g_2(t,s), & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{t+1}^{a+k}, \\ g_3(t,s), & t \in \mathbb{N}_{a+1}^b \text{ and } s \in \mathbb{N}_{a+k+1}^{t}, \\ g_4(t,s), & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{\max\{a+k,t\}+1}^{b}, \end{cases}$$

where

$$
\begin{aligned}
g_1(t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s)) \\
g_2(t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} \\
g_3(t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s)) \\
g_4(t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega},
\end{aligned}
$$

recalling $\Omega := \alpha H_\nu(a+k,a) - H_\nu(b,a)$. Therefore, $G(t,s)$, given by (3.14), is the Green's function for the homogeneous, self-adjoint, three point boundary value problem (3.9).

Remark 67. Note that when $t = \rho(s) = s-1$, we have that

$$
\begin{aligned}
g_1(t,\rho(s)) = g_1(t,t+1) &= \frac{H_\nu(b,t)H_\nu(t,a)}{\Omega} - \alpha\frac{H_\nu(a+k,t)H_\nu(t,a)}{\Omega} + H_\nu(t,t) \\
&= \frac{H_\nu(b,t)H_\nu(t,a)}{\Omega} - \alpha\frac{H_\nu(a+k,t)H_\nu(t,a)}{\Omega} \\
&= g_2(t,t+1).
\end{aligned}
$$

Hence $g_1(t,s) = g_2(t,s)$ when $t = \rho(s)$.

Further, in the case where $t = \rho(s) = s-1$, we have that

$$
\begin{aligned}
g_3(t,t+1) &= \frac{H_\nu(b,t)H_\nu(t,a)}{\Omega} + H_\nu(t,t) \\
&= \frac{H_\nu(b,t)H_\nu(t,a)}{\Omega} \\
&= g_4(t,t+1).
\end{aligned}
$$

Hence $g_3(t,s) = g_4(t,s)$ when $t = \rho(s)$. Thus in our piecewise definition of the Green's function, the pieces agree on the boundary line $t = \rho(s)$.

Remark 68. We now consider the case where $s = a + k + 1$. Then we have

$$\begin{aligned} g_1(t, a+k+1) &= \frac{H_\nu(b, a+k)H_\nu(t, a)}{\Omega} - \alpha\frac{H_\nu(a+k, a+k)H_\nu(t, a)}{\Omega} + H_\nu(t, a+k) \\ &= \frac{H_\nu(b, a+k)H_\nu(t, a)}{\Omega} + H_\nu(t, a+k) \\ &= g_3(t, a+k+1). \end{aligned}$$

Hence $g_1(t, s) = g_3(t, s)$ when $s = a + k + 1$. Still in the case where $s = a + k + 1$, consider

$$\begin{aligned} g_2(t, a+k+1) &= \frac{H_\nu(b, a+k)H_\nu(t, a)}{\Omega} - \alpha\frac{H_\nu(a+k, a+k)H_\nu(t, a)}{\Omega} \\ &= \frac{H_\nu(b, a+k)H_\nu(t, a)}{\Omega} \\ &= g_4(t, a+k+1). \end{aligned}$$

Hence $g_2(t, s) = g_4(t, s)$ when $s = a + k + 1$. So we have in the piecewise definition of the Green's function, the pieces agree on the boundary line $s = a + k + 1$.

With Remark 67 and Remark 68 in hand, we can rewrite the Green's function for (3.9).

Definition 69 (Three Point BVP Green's Function)**.** The Green's function for the three point homogeneous boundary value problem (3.9) is given by

$$G(t, s) = \begin{cases} g_1(t, s), & t \in \mathbb{N}_{a+1}^{b} \text{ and } s \in \mathbb{N}_{a+2}^{\min\{a+k, t\}+1}, \\ g_2(t, s), & t \in \mathbb{N}_{a}^{a+k} \text{ and } s \in \mathbb{N}_{\max\{a+2, t+1\}}^{a+k+1}, \\ g_3(t, s), & t \in \mathbb{N}_{a+k}^{b} \text{ and } s \in \mathbb{N}_{a+k+1}^{\min\{b, t+1\}}, \\ g_4(t, s), & t \in \mathbb{N}_{a}^{b-1} \text{ and } s \in \mathbb{N}_{\max\{a+k, t\}+1}^{b}. \end{cases} \tag{3.14}$$

3.3.2 Special Cases of the Three Point Boundary Value Problem

Recall from Section 3.1, the Green's function for the conjugate boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = 0, \\ x(b) = 0. \end{cases}$$

is given by

$$C(t,s) = \begin{cases} -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)}, & t \in \mathbb{N}^{b-1}_{a} \text{ and } s \in \mathbb{N}^{b}_{\max\{t+1,a+2\}}, \\ -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} + H_\nu(t,\rho(s)), & t \in \mathbb{N}^{b}_{a+1} \text{ and } s \in \mathbb{N}^{\min\{t+1,b\}}_{a+2}. \end{cases}$$

Remark 70 (Conjugate BVP Special Case)**.** Let $\alpha = 0$, and note that in this case the three point boundary value problem reduces to the conjugate boundary value problem. So consider the Green's function (3.14) with $\alpha = 0$. We have that $g_1(t,s) = g_3(t,s)$ and $g_2(t,s) = g_4(t,s)$. Further, $g_1(t,s) = -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} + H_\nu(t,\rho(s))$ and $g_2(t,s) = -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)}$. Thus when $\alpha = 0$, we see that $C(t,s) = G(t,s)$.

Recall from Section 3.2, the Green's function for the right focal boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^{b}_{a+2}, \\ x(a) = 0, \\ \nabla x(b) = 0, \end{cases}$$

is given by

$$R(t,s) = \begin{cases} -\frac{H_{\nu-1}(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)}, & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{\max\{t+1,a+2\}}^{b}, \\ -\frac{H_{\nu-1}(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} + H_\nu(t,\rho(s)), & t \in \mathbb{N}_{a+1}^{b} \text{ and } s \in \mathbb{N}_{a+2}^{\min\{t+1,b\}}. \end{cases}$$

Remark 71 (Right Focal BVP Special Case)**.** When $\alpha = 1$ and $k = b - a - 1$, we have that the three point boundary value problem (3.10) is equivalent to the right focal boundary value problem. To verify that the Green's functions match in this case, first consider, with $\alpha = 1$ and $k = b - a - 1$,

$$\Omega = H_\nu(b-1,a) - H_\nu(b,a) = -\nabla_t H_\nu(t,a)|_{t=b} = -H_{\nu-1}(b,a).$$

When $\alpha = 1$ and $k = b - a - 1$, the domain for $g_2(t,s)$ is $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{a+2,t+1\}}^{b}$. Let $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{a+2,t+1\}}^{b}$ and consider

$$\begin{aligned} g_2(t,s) &= -\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} + \frac{H_\nu(b-1,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} \\ &= -\frac{H_\nu(t,a)}{H_{\nu-1}(b,a)}\left[H_\nu(b,\rho(s)) - H_\nu(b-1,\rho(s))\right] \\ &= -\frac{H_{\nu-1}(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)}, \end{aligned}$$

which matches the appropriate piecewise portion of the Green's function for the right focal boundary value problem with domain $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{a+2,t+1\}}^{b}$.

Similarly, when $\alpha = 1$ and $k = b - a - 1$, the domain for $g_1(t,s)$ is $t \in \mathbb{N}_{a+1}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{b,t+1\}}$. So let $t \in \mathbb{N}_{a+1}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{b,t+1\}}$ and consider

$$\begin{aligned} g_1(t,s) &= g_2(t,s) + H_\nu(t,\rho(s)) \\ &= -\frac{H_{\nu-1}(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)} + H_\nu(t,\rho(s)), \end{aligned}$$

which again matches the appropriate piecewise portion of the Green's function with domain $t \in \mathbb{N}_{a+1}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{b,t+1\}}$.

Since $k = b-a-1$ in this special case, we have the $g_3(t,s)$ and $g_4(t,s)$ components of the Green's function occur when $s = b$. But by Remark 68 we have $g_1(t,s) = g_3(t,s)$ and $g_2(t,s) = g_4(t,s)$ when $s = b = a+k+1$

Therefore in the case of $\alpha = 1$ and $k = b-a-1$, we have $G(t,s) = R(t,s)$.

3.3.3 Three Point Green's Function Properties

In this subsection we explore some properties of the Green's function for the boundary value problem (3.9). The general strategy for the proofs of these results is to eliminate the terms with α by considering continuous derivatives with respect to α and then use results already proven for the conjugate boundary value problem case. To start, we show the constant term Ω is negative.

Lemma 72. *For $k \in \mathbb{N}_1^{(b-a)-1}$ and $0 \leq \alpha \leq 1$,*

$$\Omega := \Omega(\alpha) = \alpha H_\nu(a+k,a) - H_\nu(b,a) < 0.$$

Proof. Consider

$$\Omega'(\alpha) = H_\nu(a+k,a) = \frac{(a+k-a)^{\overline{\nu}}}{\Gamma(\nu)} = \frac{\Gamma(k+\nu)}{\Gamma(k)\Gamma(\nu)}.$$

Since $k \in \mathbb{N}_a^{(b-a)-1}$ and $\nu > 0$, we have that $\Gamma(k+\nu) > 0$, $\Gamma(k) > 0$, and $\Gamma(\nu) > 0$. Hence $\Omega'(\alpha) > 0$, i.e. Ω is an increasing function in α. Therefore

$$\max_{\alpha \in [0,1]} \Omega(\alpha) = \Omega(1) = 1 \cdot H_\nu(a+k,a) - H_\nu(b,a).$$

Therefore, $\Omega < 0$, for $\alpha \in [0,1]$, if and only if

$$H_\nu(a+k,a) < H_\nu(b,a),$$

if and only if

$$k^{\overline{\nu}} < (b-a)^{\overline{\nu}},$$

which is true because $k \in \mathbb{N}_1^{(b-a)-1}$. Therefore $\Omega < 0$ as claimed. □

Lemma 73. *If*

$$f(\alpha) := \frac{H_\nu(b,\rho(s))H_\nu(t,a) - \alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\alpha H_\nu(a+k,a) - H_\nu(b,a)},$$

then

$$f'(\alpha) \leq 0,$$

for $k \in \mathbb{N}_1^{(b-a)-1}$, $t \in \mathbb{N}_a^b$, *and* $s \in \mathbb{N}_{a+2}^b$.

Proof. By the quotient rule,

$$\begin{aligned} f'(\alpha) = &\frac{[\alpha H_\nu(a+k,a) - H_\nu(b,a)]\left(-H_\nu(a+k,\rho(s))H_\nu(t,a)\right)}{\left(\alpha H_\nu(a+k,a) - H_\nu(b,a)\right)^2} \\ &- \frac{[H_\nu(b,\rho(s))H_\nu(t,a) - \alpha H_\nu(a+k,\rho(s))H_\nu(t,a)]\, H_\nu(a+k,a)}{\left(\alpha H_\nu(a+k,a) - H_\nu(b,a)\right)^2} \\ = &\frac{H_\nu(b,a)H_\nu(a+k,\rho(s))H_\nu(t,a) - H_\nu(b,\rho(s))H_\nu(t,a)H_\nu(a+k,a)}{\left(\alpha H_\nu(a+k,a) - H_\nu(b,a)\right)^2}. \end{aligned}$$

Then $f'(\alpha) \leq 0$ if and only if

$$H_\nu(b,a)H_\nu(a+k,\rho(s))H_\nu(t,a) \leq H_\nu(b,\rho(s))H_\nu(a+k,a)H_\nu(t,a). \tag{3.15}$$

Note if $t = a$ then (3.15) holds trivially, so we only need to consider $t \in \mathbb{N}_{a+1}^b$. If

$t \in \mathbb{N}_{a+1}^{b}$, then $H_\nu(t,a) > 0$, so (3.15) is equivalent to

$$H_\nu(b,a)H_\nu(a+k,\rho(s)) \leq H_\nu(b,\rho(s))H_\nu(a+k,a). \tag{3.16}$$

Since $k \in \mathbb{N}_1^{(b-a)-1}$ and $s \in \mathbb{N}_{a+2}^{b}$, we have that $H_\nu(a+k,a) > 0$ and $H_\nu(b,\rho(s)) > 0$ respectively. Hence $H_\nu(b,\rho(s))H_\nu(a+k,a) > 0$. But if $k = 1$ and $s \in \mathbb{N}_{a+2}^{b}$, then $H_\nu(a+k,\rho(s)) = 0$ by convention, so (3.16) holds. Hence we only need to consider $k \in \mathbb{N}_2^{(b-a)-1}$. Again, by convention, $H_\nu(a+k,\rho(s)) = 0$ for $s \in \mathbb{N}_{a+k+1}^{b}$. Hence if $s \in \mathbb{N}_{a+k+1}^{b}$, we have that (3.16) holds. Hence we only need to consider $s \in \mathbb{N}_{a+2}^{a+k}$.

So for $k \in \mathbb{N}_2^{(b-a)-1}$ and $s \in \mathbb{N}_{a+2}^{a+k}$, (3.16) is true if and only if

$$\frac{H_\nu(b,a)H_\nu(a+k,\rho(s))}{H_\nu(b,\rho(s))H_\nu(a+k,a)} \leq 1. \tag{3.17}$$

Consider

$$\begin{aligned}
\frac{H_\nu(a+k,\rho(s))}{H_\nu(a+k,a)} &= \frac{(a+k-\rho(s))^{\overline{\nu}}}{(a+k-a)^{\overline{\nu}}} \\
&= \frac{\Gamma(a+k-\rho(s)+\nu)}{\Gamma(a+k-\rho(s))}\frac{\Gamma(a+k-a)}{\Gamma(a+k-a+\nu)} \\
&= \frac{\Gamma(a+k-\rho(s)+\nu)}{\Gamma(a+k-a+\nu)}\frac{\Gamma(a+k-a)}{\Gamma(a+k-\rho(s))} \\
&= \left(\frac{\Gamma(a+k-\rho(s)+\nu)}{\Gamma(a+k-\rho(s)+\nu+\rho(s)-a)}\right)\left(\frac{\Gamma(a+k-\rho(s)+\rho(s)-a)}{\Gamma(a+k-\rho(s))}\right) \\
&= \frac{1}{(a+k-\rho(s)+\nu)^{\overline{\rho(s)-a}}}(a+k-\rho(s))^{\overline{\rho(s)-a}} \\
&= \frac{(a+k-\rho\left(s\right))\cdots(k-2)(k-1)}{(a+k-\rho(s)+\nu)\cdots(k-2+\nu)(k-1+\nu)} \\
&= \left(\frac{a+k-\rho(s)}{a+k-\rho(s)+\nu}\right)\cdots\left(\frac{k-2}{k-2+\nu}\right)\left(\frac{k-1}{k-1+\nu}\right).
\end{aligned}$$

Note that there are exactly $\rho(s)-a$ decreasing factors. Since $s \in \mathbb{N}_{a+2}^{a+k}$ and $k \in \mathbb{N}_2^{(b-a)-1}$, there will always be at least one factor in this expansion.

In a similar manner, we can expand

$$\frac{H_\nu(b,a)}{H_\nu(b,\rho(s))} = \left(\frac{b-\rho(s)+\nu}{b-\rho(s)}\right)\cdots\left(\frac{b-(a+2)+\nu}{b-(a+2)}\right)\left(\frac{b-(a+1)+\nu}{b-(a+1)}\right),$$

where there are exactly $\rho(s)-a$ factors. Again, since $s \in \mathbb{N}_{a+2}^{a+k}$ and $k \in \mathbb{N}_2^{(b-a)-1}$, we have there will always be at least one factor in this expansion.

Therefore

$$\begin{aligned}
&\frac{H_\nu(b,a)H_\nu(a+k,\rho(s))}{H_\nu(b,\rho(s))H_\nu(a+k,a)} \\
&= \left(\frac{a+k-\rho(s)}{a+k-\rho(s)+\nu}\right)\cdots\left(\frac{k-2}{k-2+\nu}\right)\left(\frac{k-1}{k-1+\nu}\right) \\
&\quad \left(\frac{b-\rho(s)+\nu}{b-\rho(s)}\right)\cdots\left(\frac{b-(a+2)+\nu}{b-(a+2)}\right)\left(\frac{b-(a+1)+\nu}{b-(a+1)}\right) \\
&= \left[\left(\frac{a+k-\rho(s)}{a+k-\rho(s)+\nu}\right)\left(\frac{b-\rho(s)+\nu}{b-\rho(s)}\right)\right]\cdots \\
&\quad \cdot\left[\left(\frac{k-2}{k-2+\nu}\right)\left(\frac{b-(a+2)+\nu}{b-(a+2)}\right)\right] \\
&\quad \cdot\left[\left(\frac{k-1}{k-1+\nu}\right)\left(\frac{b-(a+1)+\nu}{b-(a+1)}\right)\right],
\end{aligned} \tag{3.18}$$

where there are $\rho(s)-a$ grouped factors in square brackets.

Note

$$\left[\left(\frac{k-1}{k-1+\nu}\right)\left(\frac{b-(a+1)+\nu}{b-(a+1)}\right)\right] \leq 1 \tag{3.19}$$

if and only if

$$(k-1)(b-(a+1)+\nu) \leq (k-1+\nu)(b-(a+1))$$

if and only if

$$(k-1)(b-(a+1)) + \nu(k-1) \leq (k-1)(b-(a+1)) + \nu(b-(a+1))$$

if and only if

$$(k-1) \leq (b-a)-1,$$

but $k \in \mathbb{N}_2^{(b-a)-1}$, hence (3.19) holds. We can similarly show the other $\rho(s)-a-1$ grouped factors in (3.18) are less than or equal to 1. Hence (3.17) holds, so we have $f'(\alpha) \leq 0$. □

Theorem 74. *The Green's function for* (3.9) *given by* (3.14) *satisfies*

$$G(t,s) \leq 0,$$

for $t \in \mathbb{N}_a^b$ *and* $s \in \mathbb{N}_{a+2}^b$.

Proof. We will start with the easier cases. First, let $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{a+k,t\}+1}^b$, i.e.

$$G(t,s) = g_4(t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega}.$$

By the domain on s and t, we have that $H_\nu(b,\rho(s)) \geq 0$ and $H_\nu(t,a) \geq 0$. Also, by Lemma 72, $\Omega < 0$. Hence $g_4(t,s) \leq 0$ for $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{a+k,t\}+1}^b$.

Now let $t \in \mathbb{N}_a^{a+k}$ and $s \in \mathbb{N}_{t+1}^{a+k+1}$, i.e.

$$G(t,s) = g_2(t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega}.$$

Then $g_2(t,s) \leq 0$ if and only if

$$\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} \leq \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega}.$$

This inequality is immediately true when $t = a$ and $s = a+k+1$, so consider $t \in \mathbb{N}_{a+1}^{a+k}$ and $s \in \mathbb{N}_{t+1}^{a+k}$. Then for this new domain, we have $H_\nu(t,a) > 0$ and

$H_\nu(a+k,\rho(s)) > 0$. Also, note that if $\alpha = 0$, we have by the previous case that $g_2(t,s) \leq 0$, hence we consider $\alpha \in (0,1]$. Since $\Omega < 0$ by Lemma 72, $g_2(t,s) \leq 0$ if and only if

$$\frac{H_\nu(b,\rho(s))}{\alpha H_\nu(a+k,\rho(s))} \geq 1.$$

Since $\alpha \in (0,1]$, hence

$$\frac{H_\nu(b,\rho(s))}{\alpha H_\nu(a+k,\rho(s))} \geq \frac{H_\nu(b,\rho(s))}{H_\nu(a+k,\rho(s))} = \frac{(b-\rho(s))^{\overline{\nu}}}{(a+k-\rho(s))^{\overline{\nu}}}, \tag{3.20}$$

but $a+k \in \mathbb{N}_{a+1}^{b-1}$ and $s \in \mathbb{N}_{t+1}^{a+k}$, so inequality (3.20) is true. Therefore $g_2(t,s) \leq 0$.

Consider the case when $t \in \mathbb{N}_{a+k}^{b}$ and $s \in \mathbb{N}_{a+k+1}^{t+1}$, i.e.

$$G(t,s) = g_3(t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s)).$$

In the proof of Lemma 72 we see Ω is an increasing function in α, hence $\frac{1}{\Omega}$ is a decreasing function in α. So

$$\begin{aligned} g_3(t,s) &= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s)) \\ &= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\alpha H_\nu(a+k,a) - H_\nu(b,a)} + H_\nu(t,\rho(s)) \\ &\leq \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{-H_\nu(b,a)} + H_\nu(t,\rho(s)). \end{aligned}$$

But by the proof of Theorem 61 part 1, we see that $\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{-H_\nu(b,a)} + H_\nu(t,\rho(s) \leq 0$, hence $g_3(t,s) \leq 0$.

Finally we consider the case when $t \in \mathbb{N}_{a+1}^{b}$ and $s \in \mathbb{N}_{a+2}^{\min\{a+k,t\}+1}$, i.e

$$G(t,s) = g_1(t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} + H_\nu(t,\rho(s)).$$

Note that $g_1(t,s) \leq 0$ if and only if

$$\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} \leq -H_\nu(t,\rho(s)). \tag{3.21}$$

By Lemma 73, we have the left hand side of (3.21) is decreasing in α, hence

$$\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega} \leq \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{-H_\nu(b,a)}.$$

We have in the proof of Theorem 61 part (1) that

$$\frac{-H_\nu(b,\rho(s))H_\nu(t,a)}{H_\nu(b,a)} \leq -H_\nu(t,\rho(s)),$$

so (3.21) holds, hence $g_1(t,s) \leq 0$. □

Definition 75. Define the function $F : [0,1] \times \mathbb{N}_1^{(b-a)-1} \times \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b \to \mathbb{R}$ as

$$F(\alpha,k,t,s) := \begin{cases} f_1(\alpha,k,t,s), & t \in \mathbb{N}_{a+1}^b \text{ and } s \in \mathbb{N}_{a+2}^{\min\{a+k,t\}+1}, \\ f_2(\alpha,k,t,s), & t \in \mathbb{N}_a^{a+k} \text{ and } s \in \mathbb{N}_{\max\{a+2,t+1\}}^{a+k+1}, \\ f_3(\alpha,k,t,s), & t \in \mathbb{N}_{a+k}^b \text{ and } s \in \mathbb{N}_{a+k+1}^{\min\{b,t+1\}}, \\ f_4(\alpha,k,t,s), & t \in \mathbb{N}_a^{b-1} \text{ and } s \in \mathbb{N}_{\max\{a+k,t\}+1}^b, \end{cases}$$

where

$$
\begin{aligned}
f_1(\alpha,k,t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} + H_\nu(t,\rho(s)) \\
f_2(\alpha,k,t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} \\
f_3(\alpha,k,t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} + H_\nu(t,\rho(s)) \\
f_4(\alpha,k,t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)}, \\
\Omega(\alpha,k) &:= \alpha H_\nu(a+k,a) - H_\nu(b,a).
\end{aligned}
$$

Lemma 76. *The function $F(\alpha,k,t,s)$ satisfies*

$$\frac{\partial}{\partial\alpha}F(\alpha,k,t,s) \leq 0,$$

for all $k \in \mathbb{N}_1^{(b-a)-1}$, $t \in \mathbb{N}_a^b$, and $s \in \mathbb{N}_{a+2}^b$.

Proof. We will show each component of $F(\alpha,k,t,s)$ is non-increasing in α. Let $k \in \mathbb{N}_1^{(b-a)-1}$ be fixed but arbitrary.

Suppose $t \in \mathbb{N}_{a+1}^b$ and $s \in \mathbb{N}_{a+2}^{\min\{a+k,t\}+1}$. Then $F(\alpha,k,t,s) = f_1(\alpha,k,t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} + H_\nu(t,\rho(s))$. Note from Lemma 73, $\frac{\partial}{\partial\alpha}f_1(\alpha,k,t,s) \leq 0$.

Suppose $t \in \mathbb{N}_a^{a+k}$ and $s \in \mathbb{N}_{\max\{a+2,t+1\}}^{a+k+1}$. Then $F(\alpha,k,t,s) = f_2(\alpha,k,t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} - \frac{\alpha H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)}$. Again by Lemma 73, $\frac{\partial}{\partial\alpha}f_2(\alpha,k,t,s) \leq 0$.

Suppose $t \in \mathbb{N}_{a+k}^b$ and $s \in \mathbb{N}_{a+k+1}^{\min\{b,t+1\}}$. Then $F(\alpha,k,t,s) = f_3(\alpha,k,t,s) = \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)} + H_\nu(t,\rho(s))$. By the proof Lemma 72, we have that $\Omega(\alpha,k)$ is a non-decreasing function in α, hence $\frac{1}{\Omega(\alpha,k)}$ is a non-increasing function in α. This implies $\frac{\partial}{\partial\alpha}f_3(\alpha,k,t,s) \leq 0$.

Finally, suppose $t \in \mathbb{N}_a^{b-1}$ and $s \in \mathbb{N}_{\max\{a+k,t\}+1}^b$. Then $F(\alpha,k,t,s) = f_4(\alpha,k,t,s) =$

$\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(\alpha,k)}$. Again, by the proof Lemma 72, we have that $\Omega(\alpha, k)$ is a non-decreasing function in α, hence $\frac{1}{\Omega(\alpha,k)}$ is a non-increasing function in α. This implies $\frac{\partial}{\partial\alpha} f_4(\alpha, k, t, s) \leq 0$. □

Lemma 77. *For fixed $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^b$, the function $F(1, k, t, s)$ satisfies*

1. $\nabla_k F(1, k, t, s) \leq 0$, *for* $k \in \mathbb{N}_2^{\rho(s)-a}$,
2. $\nabla_k F(1, k, t, s) \geq 0$, *for* $k \in \mathbb{N}_{s-a}^{(b-a)-1}$.

Proof. Fix $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^b$. For $k \in \mathbb{N}_2^{\rho(s)-a}$, consider

$$\begin{aligned}\nabla_k \frac{H_\nu(b, \rho(s))H_\nu(t, a)}{\Omega(1, k)} &= \nabla_k \frac{H_\nu(b, \rho(s))H_\nu(t, a)}{H_\nu(a + k, a) - H_\nu(b, a)} \\ &= -\frac{H_\nu(b, \rho(s))H_\nu(t, a)H_{\nu-1}(a + k, a)}{(H_\nu(a + k, a) - H_\nu(b, a))\,(H_\nu(a + k - 1, a) - H_\nu(b, a))}.\end{aligned}$$

Since $t \in \mathbb{N}_a^b$, $s \in \mathbb{N}_{a+2}^b$, $k \in \mathbb{N}_2^{\rho(s)-a}$, $\nu > -1$, and $\nu - 1 > -1$, we have by Lemma 27 that $H_\nu(b, \rho(s)) > 0$, $H_\nu(t, a) \geq 0$, and $H_{\nu-1}(a + k, a) > 0$. Also, by Lemma 72, $H_\nu(a + k, a) - H_\nu(b, a) < 0$ and $H_\nu(a + k - 1, a) - H_\nu(b, a) < 0$. Therefore

$$\nabla_k \frac{H_\nu(b, \rho(s))H_\nu(t, a)}{\Omega(1, k)} \leq 0,$$

for $k \in \mathbb{N}_2^{\rho(s)-a}$. Since $\nabla_k f_3(1, k, t, s) = \nabla_k f_4(1, k, t, s) = \nabla_k \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,k)}$, we have that $\nabla_k f_3(1, k, t, s) \leq 0$ and $\nabla_k f_4(1, k, t, s) \leq 0$ for $k \in \mathbb{N}_2^{\rho(s)-a}$. But note when $k \in \mathbb{N}_1^{\rho(s)-a}$, $F(1, k, t, s) = f_3(1, k, t, s)$ or $F(1, k, t, s) = f_4(1, k, t, s)$, hence we have shown $\nabla_k F(1, k, t, s) \leq 0$ for $k \in \mathbb{N}_2^{\rho(s)-a}$.

If $b - a > 2$, then consider the case where we fix $t \in \mathbb{N}_a^b$ and $s \in \mathbb{N}_{a+2}^{b-1}$. For

$k \in \mathbb{N}_{s-a}^{(b-a)-1}$, consider

$$\begin{aligned}
&\nabla_k \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,k)} - \frac{H_\nu(a+k,\rho(s))H_\nu(t,a)}{\Omega(1,k)} \\
&\quad = \nabla_k \frac{H_\nu(b,\rho(s))H_\nu(t,a) - H_\nu(a+k,\rho(s))H_\nu(t,a)}{H_\nu(a+k,a) - H_\nu(b,a)} \\
&\quad = \frac{[H_\nu(a+k,a) - H_\nu(b,a)]\left(-H_{\nu-1}(a+k,\rho(s))H_\nu(t,a)\right)}{\left(H_\nu(a+k,a) - H_\nu(b,a)\right)\left(H_\nu(a+k-1,a) - H_\nu(b,a)\right)} \\
&\qquad - \frac{[H_\nu(b,\rho(s))H_\nu(t,a) - H_\nu(a+k,\rho(s))H_\nu(t,a)]\, H_{\nu-1}(a+k,a)}{\left(H_\nu(a+k,a) - H_\nu(b,a)\right)\left(H_\nu(a+k-1,a) - H_\nu(b,a)\right)}.
\end{aligned} \tag{3.22}$$

Note then that (3.22) is nonnegative if and only if

$$\begin{aligned}
\frac{H_\nu(t,a)}{\Omega(1,k)\Omega(1,k-1)} &\left[H_{\nu-1}\left(a+k,\rho(s)\right)\left(H_\nu\left(b,a\right) - H_\nu\left(a+k,a\right)\right)\right. \\
&\qquad \left. - H_{\nu-1}\left(a+k,a\right)\left(H_\nu\left(b,\rho(s)\right) - H_\nu\left(a+k,\rho(s)\right)\right)\right] \\
&\geq 0.
\end{aligned} \tag{3.23}$$

By Lemma 27 and Lemma 72, we have that $\frac{H_\nu(t,a)}{\Omega(1,k)\Omega(1,k-1)} \geq 0$. Hence (3.23) is true if and only if

$$\begin{aligned}
H_{\nu-1}(a+k,\rho(s))&[H_\nu(b,a) - H_\nu(a+k,a)] \\
&\geq H_{\nu-1}(a+k,a)\left[H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))\right],
\end{aligned}$$

which is true if and only if

$$\frac{H_\nu(b,a) - H_\nu(a+k,a)}{H_{\nu-1}(a+k,a)} \geq \frac{H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))}{H_{\nu-1}(a+k,\rho(s))}, \tag{3.24}$$

as $H_{\nu-1}(a+k,a) > 0$ and $H_{\nu-1}(a+k,\rho(s)) > 0$ since $s \in \mathbb{N}_{a+2}^{b}$ is fixed and $k \in \mathbb{N}_{s-a}^{(b-a)-1}$.

Note that (3.24) holds if we can show the right hand side is non-increasing in s, i.e.

$$\nabla_s \frac{H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))}{H_{\nu-1}(a+k,\rho(s))} \leq 0. \tag{3.25}$$

Using the quotient rule, we get

$$\begin{aligned}
&\nabla_s \frac{H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))}{H_{\nu-1}(a+k,\rho(s))} \\
&= \frac{H_{\nu-1}(a+k,\rho(s))\left[-H_{\nu-1}(b,\rho(s)-1) + H_{\nu-1}(a+k,\rho(s)-1)\right]}{H_{\nu-1}(a+k,\rho(s))H_{\nu-1}(a+k,\rho(s)-1)} \\
&\quad - \frac{\left[H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))\right]\left(-H_{\nu-2}(a+k,\rho(s)-1)\right)}{H_{\nu-1}(a+k,\rho(s))H_{\nu-1}(a+k,\rho(s)-1)} \\
&\leq 0,
\end{aligned}$$

if and only if

$$\begin{aligned}
H_{\nu-1}(a+k,\rho(s))&\left[-H_{\nu-1}(b,\rho(s)-1) + H_{\nu-1}(a+k,\rho(s)-1)\right] \\
&- \left[H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))\right]\left(-H_{\nu-2}(a+k,\rho(s)-1)\right) \\
&\leq 0.
\end{aligned} \tag{3.26}$$

Applying Lemma 29 to all the monomials with $\rho(s)-1$ in the second variable, we get

(3.26) holds if and only

$$
\begin{aligned}
H_{\nu-1}(a+k,\rho(s)) &\left[-\left(\frac{\nu}{b-\rho(s)}\right) H_\nu(b,\rho(s)) + \left(\frac{\nu}{a+k-\rho(s)}\right) H_\nu(a+k,\rho(s))\right] \\
&\quad - [H_\nu(b,\rho(s)) - H_\nu(a+k,\rho(s))] \\
&\quad \cdot \left(-\left(\frac{\nu-1}{a+k-\rho(s)}\right) H_{\nu-1}(a+k,\rho(s))\right) \\
&= H_{\nu-1}(a+k,\rho(s)) \\
&\quad \cdot \Bigg[-\left(\frac{\nu}{b-\rho(s)}\right) H_\nu(b,\rho(s)) + \left(\frac{\nu}{a+k-\rho(s)}\right) H_\nu(a+k,\rho(s)) \\
&\quad + \left(\frac{\nu-1}{a+k-\rho(s)}\right) H_\nu(b,\rho(s)) \\
&\quad - \left(\frac{\nu-1}{a+k-\rho(s)}\right) H_\nu(a+k,\rho(s))\Bigg] \\
&\le 0.
\end{aligned}
$$

Since $H_{\nu-1}(a+k,\rho(s)) \ge 0$ by Lemma 27, we have this is true if and only if

$$
\begin{aligned}
\Bigg[-\left(\frac{\nu}{b-\rho(s)}\right) H_\nu(b,\rho(s)) &+ \left(\frac{\nu}{a+k-\rho(s)}\right) H_\nu(a+k,\rho(s)) \\
&+ \left(\frac{\nu-1}{a+k-\rho(s)}\right) H_\nu(b,\rho(s)) \\
&- \left(\frac{\nu-1}{a+k-\rho(s)}\right) H_\nu(a+k,\rho(s))\Bigg] \\
&\le 0.
\end{aligned}
$$

Since $\Gamma(\nu+1)>0$, this is true if and only if

$$\begin{aligned}\Bigg[-\left(\frac{\nu}{b-\rho(s)}\right)(b-\rho(s))^{\overline{\nu}}+\left(\frac{\nu}{a+k-\rho(s)}\right)(a+k-\rho(s))^{\overline{\nu}}&\\ +\left(\frac{\nu-1}{a+k-\rho(s)}\right)(b-\rho(s))^{\overline{\nu}}&\\ -\left(\frac{\nu-1}{a+k-\rho(s)}\right)(a+k-\rho(s))^{\overline{\nu}}\Bigg]&\\ \leq 0.&\end{aligned} \tag{3.27}$$

Define $q:=b-\rho(s)$ and $r:=a+k-\rho(s)$. Then (3.27) is equivalent to

$$\begin{aligned}&-\left(\frac{\nu}{q}\right)q^{\overline{\nu}}+\left(\frac{\nu}{r}\right)r^{\overline{\nu}}+\left(\frac{\nu-1}{r}\right)q^{\overline{\nu}}-\left(\frac{\nu-1}{r}\right)r^{\overline{\nu}}\\ &=-\left(\frac{\nu}{q}\right)q^{\overline{\nu}}+\left(\frac{1}{r}\right)r^{\overline{\nu}}+\left(\frac{\nu-1}{r}\right)q^{\overline{\nu}}\\ &\leq 0,\end{aligned}$$

which is true if and only

$$-\nu r q^{\overline{\nu}}+q r^{\overline{\nu}}+q(\nu-1)q^{\overline{\nu}}\leq 0,$$

if and only if

$$-\nu r q^{\overline{\nu}}+q r^{\overline{\nu}}\leq q(1-\nu)q^{\overline{\nu}},$$

if and only if

$$-\nu\frac{r}{q}+\frac{r^{\overline{\nu}}}{q^{\overline{\nu}}}\leq 1-\nu. \tag{3.28}$$

To show (3.28), we will find the maximum of the left hand side. We want to show

$$\nabla_r\left(-\nu\frac{r}{q}+\frac{r^{\overline{\nu}}}{q^{\overline{\nu}}}\right)=\frac{\nu r^{\overline{\nu-1}}}{q^{\overline{\nu}}}-\frac{\nu}{q}=\nu\left(\frac{r^{\overline{\nu-1}}}{q^{\overline{\nu}}}-\frac{1}{q}\right)\leq 0,\quad \text{for } r\in\mathbb{N}_{q+1}. \tag{3.29}$$

As $\nu > 0$, this is true if and only if

$$\frac{r^{\overline{\nu-1}}}{q^{\overline{\nu}}} - \frac{1}{q} \leq 0,$$

if and only if

$$r^{\overline{\nu-1}} \leq \frac{q^{\overline{\nu}}}{q} = \frac{\Gamma(q+\nu)}{\Gamma(q)q} = \frac{\Gamma(q+1+\nu-1)}{\Gamma(q+1)} = (q+1)^{\overline{\nu-1}}.$$

But note $\nabla_r r^{\overline{\nu-1}} = (\nu-1)\frac{\Gamma(r+\nu-2)}{\Gamma(r)} \leq 0$ for $r \in \mathbb{N}_{q+2}$. Hence for $r \in \mathbb{N}_{q+1}$,

$$r^{\overline{\nu-1}} \leq (q+1)^{\overline{\nu-1}},$$

i.e. (3.29) holds.

We now want to show

$$\nabla_r \left(-\nu\frac{r}{q} + \frac{r^{\overline{\nu}}}{q^{\overline{\nu}}}\right) = \nu\left(\frac{r^{\overline{\nu-1}}}{q^{\overline{\nu}}} - \frac{1}{q}\right) \geq 0, \quad \text{for } r \in \mathbb{N}_2^q. \tag{3.30}$$

Again, as $\nu > 0$, this is true if and only if

$$r^{\overline{\nu-1}} \geq (q+1)^{\overline{\nu-1}},$$

but still $\nabla_r r^{\nu-1} \leq 0$ for $r \in \mathbb{N}_3^q$, hence

$$r^{\overline{\nu-1}} \geq (q+1)^{\overline{\nu-1}},$$

i.e. (3.30) holds.

Since (3.29) and (3.30) hold, we have that $\max_{r\in\mathbb{N}_2}\{-\nu\frac{r}{q} + \frac{r^{\overline{\nu}}}{q^{\overline{\nu}}}\} = -\nu\frac{q}{q} + \frac{q^{\overline{\nu}}}{q^{\overline{\nu}}} = 1-\nu$, therefore (3.28) is true. Thus it is shown that (3.22) is nonnegative. Further, $\nabla_k f_1(1,k,t,s)$ and $\nabla_k f_2(1,k,t,s)$ are equivalent to (3.22). Finally, note that for $k \in \mathbb{N}_{\rho(s)-a}^{(b-a)-1}$, $F(1,k,t,s) = f_1(1,k,t,s)$ or $F(1,k,t,s) = f_2(1,k,t,s)$, hence $\nabla_k F(1,k,t,s) \geq 0$ for $k \in \mathbb{N}_{s-a}^{(b-a)-1}$. □

Theorem 78. *The Green's function for* (3.9) *given by* (3.14) *satisfies*

$$\int_{a+1}^{b} |G(t,s)| \nabla s \leq \frac{\left(b + \frac{(b-a+1)^2}{\nu} - a\right)^2}{4\Gamma(\nu+2)},$$

for $t \in \mathbb{N}_a^b$.

Proof. By Theorem 74, we have that $G(t,s) \leq 0$. Combining this with Lemma 76, we have $F(1,k,t,s) \leq G(t,s) \leq 0$. Finally, note Lemma 77 implies $F(1,\rho(s)-a,t,s) \leq F(1,k,t,s)$. Hence

$$F(1,\rho(s)-a,t,s) \leq G(t,s) \leq 0,$$

implying

$$|G(t,s)| \leq |F(1,\rho(s)-a,t,s)|.$$

Note when $\alpha = 1$ and $k = \rho(s) - a$, we have that

$$\begin{aligned} f_1(\alpha,k,t,s) = f_1(1,\rho(s)-a,t,s) &:= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)} \\ &\quad - \frac{H_\nu(a+\rho(s)-a,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)} \\ &\quad + H_\nu(t,\rho(s)) \\ &= \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)} + H_\nu(t,\rho(s)) \\ &= f_3(1,\rho(s)-a,t,s). \end{aligned}$$

Similarly,

$$f_2(\alpha,k,t,s) = f_2(1,\rho(s)-a,t,s) = f_4(1,\rho(s)-a,t,s).$$

Therefore,

$$F(1,\rho(s)-a,t,s)=\begin{cases}\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)}, & t\in\mathbb{N}_a^{b-1} \text{ and } s\in\mathbb{N}_{\max\{t+1,a+2\}}^{b}\\ \frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)}+H_\nu(t,\rho(s)), & t\in\mathbb{N}_{a+1}^{b} \text{ and } s\in\mathbb{N}_{a+2}^{\min\{t+1,b\}}.\end{cases}$$

Thus

$$\begin{aligned}\int_{a+1}^{b}|G(t,s)|\nabla s &\le \int_{a+1}^{b}|F(1,\rho(s)-a,t,s)|\nabla s\\ &= -\int_{a+1}^{b}F(1,\rho(s)-a,t,s)\nabla s\\ &= -\int_{a+1}^{t}\left(\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)}+H_\nu(t,\rho(s))\right)\nabla s\\ &\quad -\int_{t}^{b}\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{\Omega(1,\rho(s)-a)}\nabla s\\ &= \int_{a+1}^{b}\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{-\Omega(1,\rho(s)-a)}\nabla s-\int_{a+1}^{t}H_\nu(t,\rho(s))\nabla s.\end{aligned}$$

Note that $-\Omega(1,\rho(s)-a)=H_\nu(b,a)-H_\nu(a+\rho(s)-a,a)$ is a decreasing function in s by Lemma 27 and since $s\in\mathbb{N}_1^{(b-a)-1}$, we have

$$\begin{aligned}-\Omega(1,\rho(s)-a) &:= H_\nu(b,a)-H_\nu(a+\rho(s)-a,a)\\ &\ge H_\nu(b,a)-H_\nu(b-a-1,a)\\ &= H_{\nu-1}(b,a).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{a+1}^{b}|G(t,s)|\nabla s &\le \int_{a+1}^{b}\frac{H_\nu(b,\rho(s))H_\nu(t,a)}{H_{\nu-1}(b,a)}\nabla s-\int_{a+1}^{t}H_\nu(t,\rho(s))\nabla s\\ &= \frac{H_{\nu+1}(b,a+1)H_\nu(t,a)}{H_{\nu-1}(b,a)}-H_{\nu+1}(t,a+1).\end{aligned}$$

Consider

$$\begin{aligned}
\frac{H_{\nu+1}(b,a+1)}{H_{\nu-1}(b,a)} &= \frac{\Gamma(b-a-1+\nu+1)}{\Gamma(b-a-1)\Gamma(\nu+2)}\frac{\Gamma(b-a)\Gamma(\nu)}{\Gamma(b-a+\nu-1)} \\
&= \frac{(b-a+\nu-1)\Gamma(b-a+\nu-1)}{\Gamma(b-a-1)(\nu+1)(\nu)\Gamma(\nu)}\frac{(b-a-1)\Gamma(b-a-1)\Gamma(\nu)}{\Gamma(b-a+\nu-1)} \\
&= \frac{(b-a+\nu-1)(b-a-1)}{(\nu+1)\nu},
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{H_{\nu+1}(b,a+1)H_\nu(t,a)}{H_{\nu-1}(b,a)} - H_{\nu+1}(t,a+1) \\
&\quad = \frac{(b-a+\nu-1)(b-a-1)}{(\nu+1)\nu}H_\nu(t,a) - H_{\nu+1}(t,a+1) \\
&\quad = \frac{(b-a+\nu-1)(b-a-1)}{(\nu+1)\nu}\frac{\Gamma(t-a+\nu)}{\Gamma(t-a)\Gamma(\nu+1)} \\
&\qquad - \frac{\Gamma(t-a-1+\nu+1)}{\Gamma(t-a-1)\Gamma(\nu+2)} \\
&\quad = \frac{(b-a+\nu-1)(b-a-1)}{\nu}\frac{\Gamma(t-a+\nu)}{\Gamma(t-a)\Gamma(\nu+2)} \\
&\qquad - \frac{\Gamma(t-a+\nu)(t-a-1)}{\Gamma(t-a)\Gamma(\nu+2)} \\
&\quad = \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+2)}\left(\frac{(b-a+\nu-1)(b-a-1)}{\nu} - t-a-1\right).
\end{aligned}$$

Since $0<\nu<1$, by Lemma 62, we have that

$$\begin{aligned}
&\frac{H_{\nu+1}(b,a+1)H_\nu(t,a)}{H_{\nu-1}(b,a)} - H_{\nu+1}(t,a+1) \\
&\quad \le \frac{(t-a)}{\Gamma(\nu+2)}\left(\frac{(b-a+\nu-1)(b-a-1)}{\nu} - t-a-1\right) \\
&\quad = \frac{(t-a)}{\Gamma(\nu+2)}\left(\frac{(b-a-1)^2+\nu(b-a-1)}{\nu} - \frac{\nu(t-a-1)}{\nu}\right) \\
&\quad = \frac{(t-a)}{\Gamma(\nu+2)}\left(\frac{(b-a-1)^2}{\nu} + b - t\right).
\end{aligned}$$

Note that this is a parabola in t opening downwards with roots of $t=a$ and $t=$

$b + \frac{(b-a+1)^2}{\nu}$, hence it attains its maximum value at $t = \frac{b}{2} + \frac{(b-a+1)^2}{2\nu} + \frac{a}{2}$. Therefore

$$\frac{(t-a)}{\Gamma(\nu+2)}\left(\frac{(b-a-1)^2}{\nu} + b - t\right) \leq \frac{\left(\frac{b}{2} + \frac{(b-a+1)^2}{2\nu} - \frac{a}{2}\right)}{\Gamma(\nu+2)}\left(\frac{b}{2} + \frac{(b-a+1)^2}{2\nu} - \frac{a}{2}\right)$$
$$= \frac{\left(b + \frac{(b-a+1)^2}{\nu} - a\right)^2}{4\Gamma(\nu+2)}.$$

Hence

$$\int_{a+1}^{b} |G(t,s)|\nabla s \leq \frac{\left(b + \frac{(b-a+1)^2}{\nu} - a\right)^2}{4\Gamma(\nu+2)}.$$

□

3.3.4 Graphs of Three Point Green's Functions

Example 79. Consider the Green's function for (3.10) where $b = 15$, $a = 0$, and $\nu = 0.4$. Fixing $s = 7$, graphs of the Green's function for various α and k values are given in Figures 3.3, 3.4, 3.5, and 3.6.

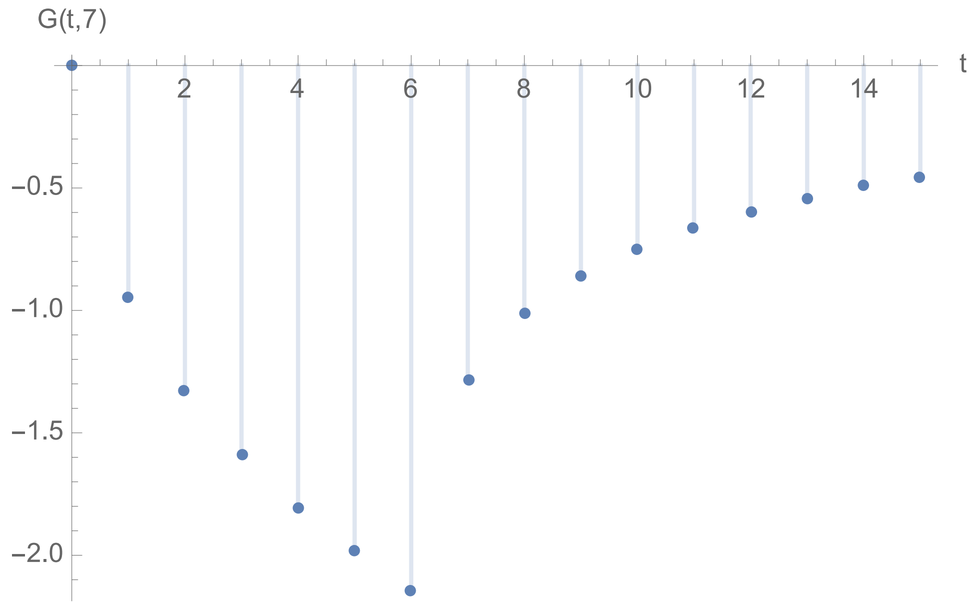

Figure 3.3: Right Focal Green's Function as a function of t where $b = 15$, $a = 0$, $\nu = 0.4$, $\alpha = 0.25$, $k = 4$, and fixed $s = 7$.

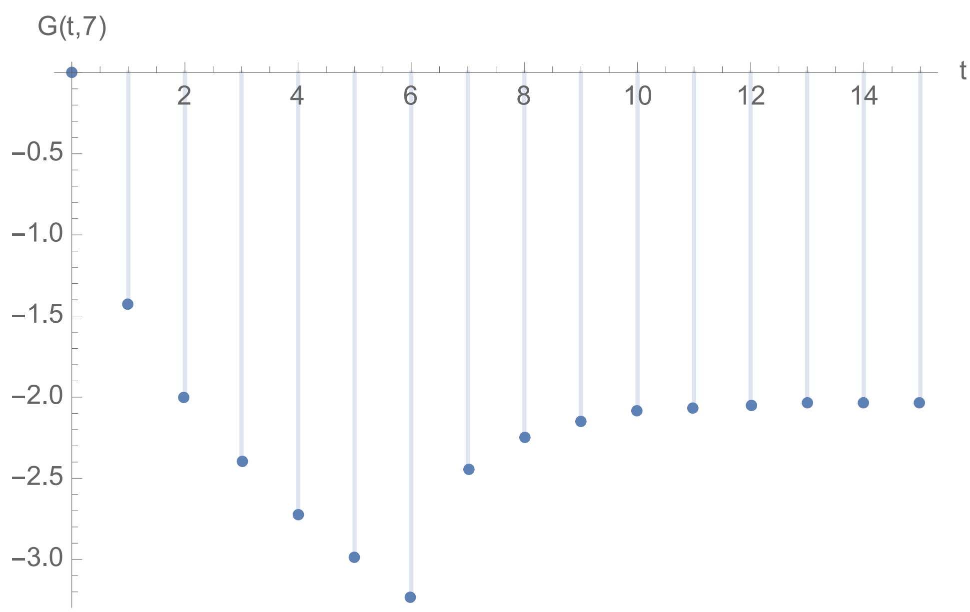

Figure 3.4: Right Focal Green's Function as a function of t where $b = 15$, $a = 0$, $\nu = 0.4$, $\alpha = 0.75$, $k = 4$, and fixed $s = 7$.

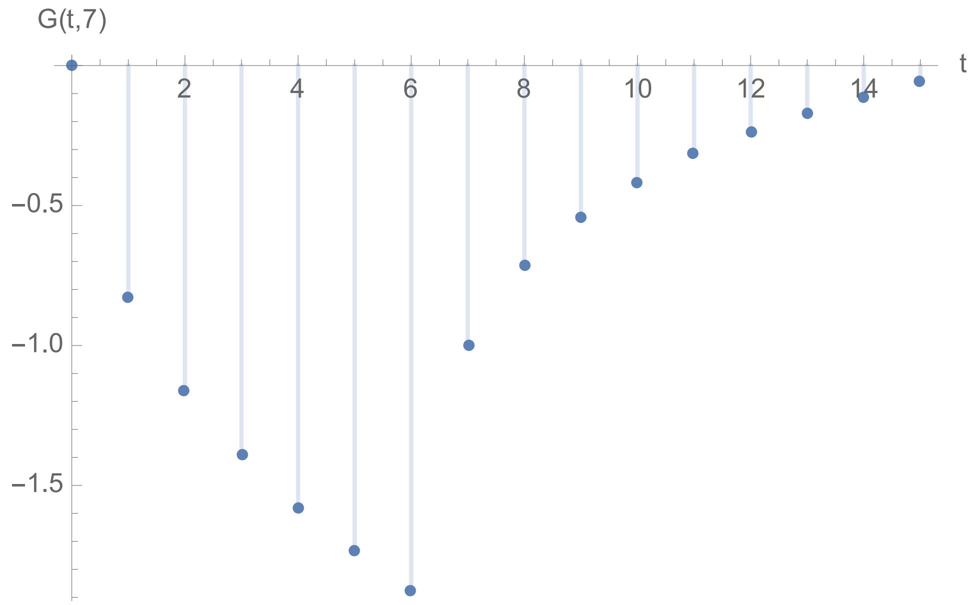

Figure 3.5: Right Focal Green's Function as a function of t where $b = 15$, $a = 0$, $\nu = 0.4$, $\alpha = 0.25$, $k = 12$, and fixed $s = 7$.

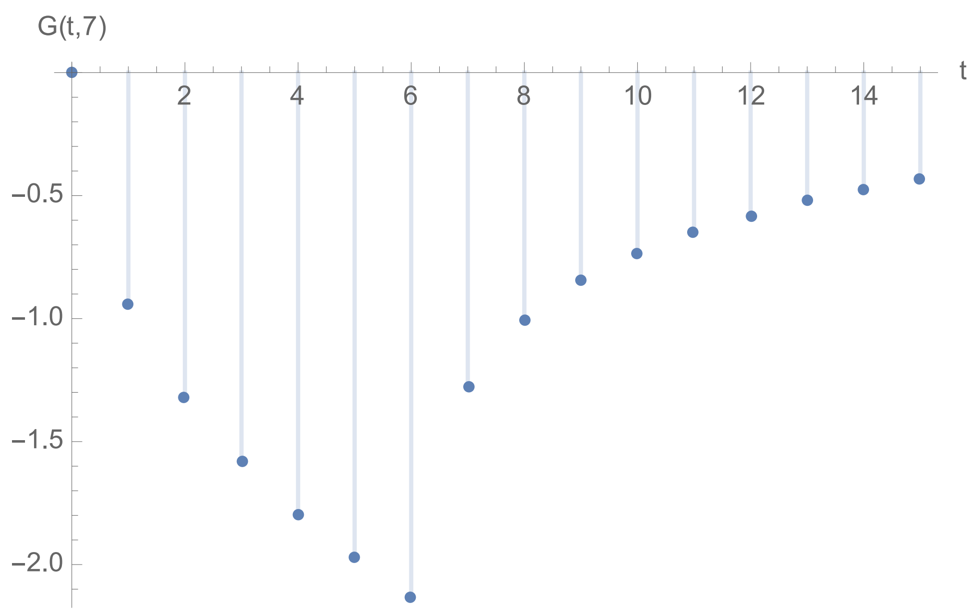

Figure 3.6: Right Focal Green's Function as a function of t where $b = 15$, $a = 0$, $\nu = 0.4$, $\alpha = 0.75$, $k = 12$, and fixed $s = 7$.

Chapter 4

Applications of the Contraction Mapping Theorem to Fractional Self-Adjoint Difference Equations

Definition 80 (Contraction Mapping)**.** Let (X, d) be a metric space. Then a map $T : X \to X$ is called a *contraction mapping* on X if there exists $q \in [0, 1)$ such that $d(T(x), T(y)) \leq qd(x, y)$ for all x, y in X.

Theorem 81. *[24] Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x_0 in X (i.e. $T(x_0) = x_0$).*

4.1 Long Run Behavior of Equations with Generalized Forcing Terms

Here we will focus on the long run behavior of solutions to the self-adjoint nonlinear fractional difference equation

$$\nabla[p(t)\nabla^{\nu}_{a^*}x(t)] = F(t, x(t-1)), \quad \text{for } t \in \mathbb{N}_{a+2}, \tag{4.1}$$

where $p : \mathbb{N}_{a+1} \to (0, \infty)$, $0 < \nu < 1$, and $F : \mathbb{N}_{a+2} \times \mathbb{R} \to [0, \infty)$.

Remark 82. In a similar methodology of the proof for Theorem 43, we can prove that that the initial value problem

$$\begin{cases} \nabla[p(t)\nabla^{\nu}_{a^*}x(t)] = F(t, x(t-1)), \quad t \in \mathbb{N}_{a+2}, \\ x(a) = 0, \\ \nabla x(a+1) = 0, \end{cases}$$

has a unique solution. Note that in this section, we specify only one initial condition, $\nabla x(a+1) = 0$. These results however also specify a boundary condition at infinity, namely the long run behavior of a solution.

4.1.1 Long Run Behavior Theorem

This subsection's main result is Theorem 87. In order to prove it, we build up an equivalence of solutions between the forced fractional difference equation (4.1) and an integral equation in Theorem 83. Using Theorem 83 and a Lipschitz condition on the nonlinear forcing term, the Contraction Mapping Theorem is applied in Theorem 87 to guarantee the existence of a unique solution with long run behavior approaching any nonnegative real number. See [10] for a similar result in the delta discrete fractional setting.

Theorem 83. *Let* $p : \mathbb{N}_{a+1} \to (0, \infty)$, $0 < \nu < 1$, *and* $F : \mathbb{N}_{a+2} \times \mathbb{R} \to [0, \infty)$. *Define* $\zeta := \{x : \mathbb{N}_a \to [0, \infty) | x(t)$ *is bounded on* $\mathbb{N}_a$ *and* $\nabla x(a+1) = 0\}$. *Furthermore, assume for all* $x \in \zeta$,

$$\int_a^{\infty} \left| \int_a^k \frac{H_{\nu-2}(k, \rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s \right| \nabla k < \infty. \tag{4.2}$$

Then, $x(t)$ is a solution to the integral equation

$$x(t) = L - \int_t^\infty \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k, \tag{4.3}$$

if and only if the forced fractional self-adjoint difference equation

$$\nabla[p(t)\nabla_{a^*}^{\nu} x(t)] = F(t, x(t-1)), \quad t \in \mathbb{N}_{a+2} \tag{4.4}$$

has solution $x \in \zeta$ with $\lim_{t\to\infty} x(t) = L$.

Before this result is proven, we need two lemmas to simplify the proof. This first lemma gives us a way to rewrite the integral equation given in Theorem 83.

Lemma 84. *Let $p : \mathbb{N}_{a+1} \to (0,\infty)$, $0 < \nu < 1$, $F : \mathbb{N}_{a+2} \times \mathbb{R} \to [0,\infty)$, and $x : \mathbb{N}_a \to \mathbb{R}$. Assume (4.2) holds. then*

$$\begin{aligned} -\int_t^\infty \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} &\int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k \\ &= \int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s. \end{aligned} \tag{4.5}$$

Proof. Assume (4.2) and define $z(s) := \frac{1}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau$. Then consider

$$\begin{aligned} \int_t^\infty \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} &\int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k \\ &= \int_t^\infty \int_a^k H_{\nu-2}(k,\rho(s)) z(s)\nabla s\nabla k \\ &= \sum_{k=t+1}^{\infty} \sum_{s=a+1}^{k} H_{\nu-2}(k,\rho(s)) z(s) \\ &= \sum_{k=t+1}^{\infty} \sum_{s=a+1}^{t} H_{\nu-2}(k,\rho(s)) z(s) + \sum_{k=t+1}^{\infty} \sum_{s=t+1}^{k} H_{\nu-2}(k,\rho(s)) z(s). \end{aligned}$$

Since this summation is absolutely convergent by assumption, we can interchange the order of summation to get

$$\begin{aligned}
&\sum_{k=t+1}^{\infty}\sum_{s=a+1}^{t} H_{\nu-2}(k,\rho(s))z(s) + \sum_{k=t+1}^{\infty}\sum_{s=t+1}^{k} H_{\nu-2}(k,\rho(s))z(s) \\
&\qquad = \sum_{s=a+1}^{t}\sum_{k=t+1}^{\infty} H_{\nu-2}(k,\rho(s))z(s) + \sum_{s=t+1}^{\infty}\sum_{k=s}^{\infty} H_{\nu-2}(k,\rho(s))z(s).
\end{aligned}$$

Rewriting some of the inner summations in integral form gives

$$\begin{aligned}
&\sum_{s=a+1}^{t}\sum_{k=t+1}^{\infty} H_{\nu-2}(k,\rho(s))z(s) + \sum_{s=t+1}^{\infty}\sum_{k=s}^{\infty} H_{\nu-2}(k,\rho(s))z(s) \\
&\qquad = \sum_{s=a+1}^{t}\int_{t}^{\infty} H_{\nu-2}(k,\rho(s))z(s)\nabla k + \sum_{s=t+1}^{\infty}\int_{\rho(s)}^{\infty} H_{\nu-2}(k,\rho(s))z(s)\nabla k.
\end{aligned}$$

Evaluating these integrals using Theorem 25 and applying Lemma 26, we get that

$$\begin{aligned}
&\sum_{s=a+1}^{t}\int_{t}^{\infty} H_{\nu-2}(k,\rho(s))z(s)\nabla k + \sum_{s=t+1}^{\infty}\int_{\rho(s)}^{\infty} H_{\nu-2}(k,\rho(s))z(s)\nabla k \\
&\qquad = \sum_{s=a+1}^{t}\left(H_{\nu-1}(k,\rho(s))z(s)\big|_{k=t}^{k\to\infty}\right) + \sum_{s=t+1}^{\infty}\left(H_{\nu-1}(k,\rho(s))z(s)\big|_{k=\rho(s)}^{k\to\infty}\right) \\
&\qquad = \sum_{s=a+1}^{t}\left(0 - H_{\nu-1}(t,\rho(s))\right)z(s) + \sum_{s=t+1}^{\infty}(0-0)\,z(s) \\
&\qquad = -\sum_{s=a+1}^{t} H_{\nu-1}(t,\rho(s))z(s) \\
&\qquad = -\int_{a}^{t}\frac{H_{\nu-1}(t,\rho(s))}{p(s)}\int_{a+1}^{s} F(\tau,x(\tau-1))\nabla\tau\nabla s.
\end{aligned}$$

Therefore

$$-\int_t^\infty \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k$$
$$= \int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s.$$

□

This next lemma helps show that the integral equation in Theorem 83 is nonnegative.

Lemma 85. *Let* $p : \mathbb{N}_{a+1} \to (0,\infty)$*,* $0 < \nu < 1$*,* $F : \mathbb{N}_{a+2} \times \mathbb{R} \to [0,\infty)$*, and* $x : \mathbb{N}_a \to \mathbb{R}$*. Then*

$$\int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s \geq 0,$$

for all $t \in \mathbb{N}_a$*.*

Proof. Note that by our assumptions on $p(t)$ and $F(t, x(t))$ being nonnegative, we have that

$$\frac{1}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau \geq 0, \tag{4.6}$$

for all $s \in \mathbb{N}_{a+2}$. Note for $s = a+1$, we have that $\frac{1}{p(s)}\int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau = 0$, hence (4.6) holds for $s \in \mathbb{N}_{a+1}$.

Since $\nu - 1 > -1$, we have by Lemma 27 that $H_{\nu-1}(t,\rho(s)) \geq 0$ for $t \in \mathbb{N}_{a+1}$ and $s \in \mathbb{N}_{a+1}^t$. Finally, if $t = a$, we have $\int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s = 0$. Hence

$$\int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s \geq 0,$$

for all $t \in \mathbb{N}_a$. □

Proof of Theorem 83: Assume that x satisfies (4.3) and that (4.2) holds. Then, taking the nabla difference of (4.3), we get that

$$\begin{aligned}\nabla x(t) &= \int_a^t \frac{H_{\nu-2}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s \\ &= \nabla_a^{1-\nu} \frac{1}{p(t)} \int_{a+1}^t F(\tau, x(\tau-1))\nabla\tau.\end{aligned}$$

Note here that $\nabla x(a+1) = 0$. Composing both sides of the previous equation with the operator $\nabla_a^{-(1-\nu)}$ yields

$$\nabla_a^{-(1-\nu)}\nabla x(t) = \nabla_{a^*}^{\nu} x(t) = \frac{1}{p(t)} \int_{a+1}^t F(\tau, x(\tau-1))\nabla\tau.$$

Rearranging and taking the nabla difference, we get that

$$\nabla\left[p(t)\nabla_{a^*}^{\nu} x(t)\right] = F(t, x(t-1)),$$

hence $x(t)$ satisfies (4.4).

Since by assumption (4.2) holds, we get that there exists $N \in \mathbb{N}_a$ such that

$$\left|\int_t^\infty \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k\right| \leq 1$$

for all $t \in \mathbb{N}_N$. Thus $|x(t)| \leq \max\{|x(a)|, |x(a+1)|, \ldots, |x(N-1)|, L+1\}$, hence $x(t)$ is bounded on $\mathbb{N}_a$. Also, by Lemma 84 and Lemma 85 we have that $x(t) \geq 0$. Therefore $x \in \zeta$. Finally, since (4.2) holds, $\lim_{t\to\infty} x(t) = L$. Hence the forward direction is proven.

Now assume $x \in \zeta$ where $\lim_{t\to\infty} x(t) = L$ is a solution to (4.4). Then

$$\nabla\left[p(t)\nabla_{a^*}^{\nu} x(t)\right] = F(t, x(t-1)).$$

By integrating from $a+1$ to t, we get

$$p(t)\nabla^{\nu}_{a^*}x(t) - p(a+1)\nabla^{\nu}_{a^*}x(a+1) = \int_{a+1}^{t} F(\tau, x(\tau-1))\nabla\tau.$$

By Lemma 37 and since $x \in \zeta$, we have that $\nabla^{\nu}_{a^*}x(a+1) = \nabla x(a+1) = 0$. Hence

$$p(t)\nabla^{\nu}_{a^*}x(t) = \int_{a+1}^{t} F(\tau, x(\tau-1))\nabla\tau.$$

Rearranging and rewriting yields

$$\nabla^{\nu}_{a^*}x(t) = \nabla_a^{-(1-\nu)}\nabla x(t) = \frac{1}{p(t)}\int_{a+1}^{t} F(\tau, x(\tau-1))\nabla\tau.$$

Now composing both sides with the operator $\nabla_a^{1-\nu}$ gives

$$\begin{aligned}\nabla_a^{1-\nu}\nabla_a^{-(1-\nu)}\nabla x(t) = \nabla x(t) &= \nabla_a^{1-\nu}\frac{1}{p(t)}\int_{a+1}^{t} F(\tau, x(\tau-1))\nabla\tau \\ &= \int_a^t \frac{H_{\nu-2}(t,\rho(s))}{p(s)}\int_{a+1}^{s} F(\tau, x(\tau-1))\nabla\tau\nabla s.\end{aligned}$$

Finally, by integrating from t to infinity, we get that

$$\lim_{b\to\infty} x(b) - x(t) = \int_t^{\infty}\int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)}\int_{a+1}^{s} F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k.$$

But $\lim_{b\to\infty} x(b) = L$ by assumption, hence

$$x(t) = L - \int_t^{\infty}\int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)}\int_{a+1}^{s} F(\tau, x(\tau-1))\nabla\tau\nabla s\nabla k,$$

i.e. $x(t)$ satisfies (4.3). $\square$

Lemma 86. *For $0 < \nu < 1$ and for any fixed $t \in \mathbb{N}_a$, we have that*

$$\max_{s \in \mathbb{N}_{a+1}^t} |H_{\nu-1}(t, \rho(s))| = 1.$$

Proof. First note that when $t = a$, we have by convention that $H_{\nu-1}(t, \rho(s)) = 0$ for $s \in \mathbb{N}_{a+1}$, so let $t \in \mathbb{N}_{a+1}$ be fixed but arbitrary. Since $\nu - 1 > -1$, we have by Lemma 27 that $H_{\nu-1}(t, \rho(s)) \geq 0$. Thus $|H_{\nu-1}(t, \rho(s))| = H_{\nu-1}(t, \rho(s))$, for $s \in \mathbb{N}_{a+1}^t$. By Lemma 28, we have that $\nabla_s H_{\nu-1}(t, \rho(s)) \geq 0$. Therefore

$$\max_{s \in \mathbb{N}_{a+1}^t} |H_{\nu-1}(t, \rho(s))| = H_{\nu-1}(t, \rho(t)) = 1.$$

□

Theorem 87. *Assume $p : \mathbb{N}_{a+1} \to (0, \infty)$, $0 < \nu < 1$, and $F : \mathbb{N}_{a+2} \times \mathbb{R} \to [0, \infty)$ satisfies a Lipschitz condition with respect to its second variable in $\mathbb{N}_{a+2} \times \mathbb{R}$, i.e. there exists $M > 0$ such that, if $u, v \in \mathbb{R}$, then for each fixed $t \in \mathbb{N}_{a+2}$,*

$$|F(t, u) - F(t, v)| \leq M|u - v|.$$

Let $\zeta := \{x : \mathbb{N}_a \to [0, \infty) : x(t)$ is bounded on $\mathbb{N}_a$, and $\nabla x(a + 1) = 0\}$ and $\|\cdot\|$ be the supremum norm, noting that $(\zeta, \|\cdot\|)$ is a complete metric space. If

1. *the series $\sum_{k=a+1}^{\infty} \left| \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau - 1)) \nabla\tau \nabla s \right|$ converges for all $x \in \zeta$, and*

2. *$\beta := M \int_a^{\infty} \frac{s-(a+1)}{p(s)} \nabla s < 1$,*

then there exists a unique nonnegative solution of the fractional difference equation (4.4) with $\lim_{t\to\infty} x(t) = L$ for any $L \geq 0$.

Proof. Let $(\zeta, \|\cdot\|)$ be the complete metric space noted above, $L \geq 0$ be fixed but arbitrary, and define the operator T as

$$Tx(t) := L - \int_t^\infty \int_a^k \frac{H_{\nu-2}(k, \rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau - 1))\nabla\tau\nabla s\nabla k.$$

We want to show that T is a contraction mapping in ζ, so first we show $T : \zeta \to \zeta$. Note that in summation notation, Tx is equivalent to

$$Tx(t) = L - \sum_{k=t+1}^{\infty} \int_a^k \frac{H_{\nu-2}(k, \rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau - 1))\nabla\tau\nabla s,$$

and so by the convergence given in Condition 1, we have that $Tx(t)$ is well defined for $t \in \mathbb{N}_a$. Also by the absolute convergence given in Condition 1, there exists $N \in \mathbb{N}_a$ such that

$$\left| \sum_{k=t+1}^{\infty} \int_a^k \frac{H_{\nu-2}(k, \rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau - 1))\nabla\tau\nabla s \right| \leq 1,$$

for all $t \in \mathbb{N}_N$. This implies

$$|Tx(t)| \leq \max\{|Tx(a)|, |Tx(a + 1)|, \ldots, |Tx(N - 1)|, L + 1\},$$

for all $t \in \mathbb{N}_a$, hence Tx is bounded for all $t \in \mathbb{N}_a$. We also have by Lemma 84 and Lemma 85 that $Tx(t) \geq 0$ for all $t \in \mathbb{N}_a$. Finally,

$$\nabla Tx(t) = \int_a^t \frac{H_{\nu-2}(t, \rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau - 1))\nabla\tau\nabla s,$$

so

$$\begin{aligned}
\nabla Tx(a+1) &= \int_a^{a+1} \frac{H_{\nu-2}(a+1,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s \\
&= \sum_{s=a+1}^{a+1} \frac{H_{\nu-2}(a+1,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau \\
&= \frac{H_{\nu-2}(a+1,a)}{p(a+1)} \int_{a+1}^{a+1} F(\tau, x(\tau-1))\nabla\tau \\
&= 0.
\end{aligned}$$

Hence $Tx \in \zeta$, thus $T : \zeta \to \zeta$.

We now show that Tx is contraction mapping. To see this, let $x, y \in \zeta$ and $t \in \mathbb{N}_a$ be fixed but arbitrary. By Lemma 84

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \Bigg| \left(L + \int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x(\tau-1))\nabla\tau\nabla s \right) \\
&\quad - \left(L + \int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, y(\tau-1))\nabla\tau\nabla s \right) \Bigg| \\
&= \left| \int_a^t \frac{H_{\nu-1}(t,\rho(s))}{p(s)} \int_{a+1}^s \left(F(\tau, x(\tau-1)) - F(\tau, y(\tau-1))\right) \nabla\tau\nabla s \right| \\
&\leq \int_a^t \frac{|H_{\nu-1}(t,\rho(s))|}{p(s)} \int_{a+1}^s \left|F(\tau, x(\tau-1)) - F(\tau, y(\tau-1))\right| \nabla\tau\nabla s.
\end{aligned}$$

Using the Lipschitz condition assumption, we get that

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_a^t \frac{|H_{\nu-1}(t,\rho(s))|}{p(s)} \int_{a+1}^s M\left|x(\tau-1) - y(\tau-1)\right| \nabla\tau\nabla s \\
&\leq M \int_a^t \frac{|H_{\nu-1}(t,\rho(s))|}{p(s)} \int_{a+1}^s \|x-y\| \nabla\tau\nabla s \\
&= M\|x-y\| \int_a^t \frac{|H_{\nu-1}(t,\rho(s))|}{p(s)} (s-(a+1))\nabla s.
\end{aligned}$$

By Lemma 86,

$$\begin{aligned}|Tx(t)-Ty(t)| &\leq M\,\|x-y\| \int_a^t \frac{1}{p(s)}\left(s-(a+1)\right)\nabla s\\ &\leq M\,\|x-y\| \int_a^\infty \frac{s-(a+1)}{p(s)}\nabla s\\ &< \beta\,\|x-y\|\,,\end{aligned}$$

using Condition 2 to simplify. Since Tx and Ty are bounded and the previous inequality is true for all $t \in \mathbb{N}_a$, we have that

$$\|Tx - Ty\| \leq \beta\,\|x-y\|\,,$$

but $\beta < 1$ by assumption, therefore T is a contraction mapping on ζ. Thus there exists a unique fixed point $x_0 \in \zeta$, i.e. $x_0 = Tx_0$. This implies

$$x_0(t) = L - \int_t^\infty \int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} \int_{a+1}^s F(\tau, x_0(\tau-1))\nabla\tau\nabla s\nabla k,$$

so by Theorem 83, $x_0(t)$ solves the forced fractional self-adjoint difference equation

$$\nabla\left[p(t)\nabla^\nu_{a^*}x(t)\right] = F(t, x(t-1)), \quad t \in \mathbb{N}_{a+2},$$

with $\lim_{t\to\infty} x_0(t) = L$. □

4.1.2 Example

Example 88. Let $a = 0$, $\nu = 0.6$, and $p(t) := t^3$. Let $M \in \mathbb{R}$ such that $0 < M < \frac{1}{\left(\frac{\pi^2}{6}-\zeta(3)\right)} \approx 2.258$, where ζ is the Riemann zeta function. Finally, define $F(t,x) := M(1+\sin(x))$. We will apply the previous theorem. To show that F is uniformly

Lipschitz with respect to its second variable, consider the known identity for $u, v \in \mathbb{R}$ and fixed $t \in \mathbb{N}_{a+2}$

$$|F(t,u)-F(t,v)| = |(M+M\sin(u))-(M+M\sin(v))| = M|\sin(u)-\sin(v)| \leq M|u-v|.$$

To show the first hypothesis holds, consider

$$\begin{aligned}
&\sum_{k=a+1}^{\infty}\left|\int_a^k \frac{H_{\nu-2}(k,\rho(s))}{p(s)} \int_{a+1}^{s} F(\tau, x(\tau-1))\nabla\tau\nabla s\right| \\
&\qquad = \sum_{k=1}^{\infty}\left|\int_0^k \frac{H_{-1.4}(k,\rho(s))}{s^3}\int_1^s M\left(1+\sin\left(x\left(\tau-1\right)\right)\right)\nabla\tau\nabla s\right| \\
&\qquad \leq \sum_{k=1}^{\infty}\int_0^k \left|\frac{H_{-1.4}(k,\rho(s))}{s^3}\right| \int_1^s |1+\sin(x(\tau-1))|\nabla\tau\nabla s \\
&\qquad \leq 2M\sum_{k=1}^{\infty}\int_0^k \left|\frac{H_{-1.4}(k,\rho(s))(s-1)}{s^3}\right|\nabla s \\
&\qquad = 2M\sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{s-1}{s^3}\left|H_{-1.4}(k,\rho(s))\right|.
\end{aligned}$$

Interchanging the summations, we get

$$2M\sum_{k=1}^{\infty}\sum_{s=1}^{k}\frac{|H_{-1.4}(k,\rho(s))|}{s^2} = 2M\sum_{s=1}^{\infty}\frac{s-1}{s^3}\sum_{k=s}^{\infty}\left|H_{-1.4}(k,\rho(s))\right|.$$

Note $H_{-1.4}(k,\rho(s)) = 1$ when $k = s$ and $H_{-1.4}(k,\rho(s)) \leq 0$ when $k \in \mathbb{N}_{s+1}^{\infty}$, hence eliminating the absolute value yields

$$\begin{aligned} 2M\sum_{s=1}^{\infty}\frac{s-1}{s^3}\sum_{k=s}^{\infty}|H_{-1.4}(k,\rho(s))| &= 2M\sum_{s=1}^{\infty}\frac{s-1}{s^3}\left[1-\sum_{k=s+1}^{\infty}H_{-1.4}(k,\rho(s))\right] \\ &= 2M\sum_{s=1}^{\infty}\frac{s-1}{s^3}\left[1-\int_s^{\infty}H_{-1.4}(k,\rho(s))\nabla k\right] \\ &= 2M\sum_{s=1}^{\infty}\frac{s-1}{s^3}\left[1-\left(\lim_{b\to\infty}H_{-0.4}(k,\rho(s))|_{k=s}^{k=b}\right)\right]. \end{aligned}$$

By Lemma 26, $\lim_{b\to\infty} H_{-0.4}(b,s)=0$ for fixed $s\in\mathbb{N}_1^{\infty}$. Therefore

$$\sum_{k=a+1}^{\infty}\left|\int_a^k\frac{H_{\nu-2}(k,\rho(s))}{p(s)}\int_{a+1}^s F(\tau,x(\tau-1))\nabla\tau\nabla s\right|\le 4M\sum_{s=1}^{\infty}\frac{s-1}{s^3}<\infty,$$

hence the first hypothesis of Theorem 87 holds.

Finally, we show the second hypothesis holds, i.e. we consider

$$\beta := M\int_a^{\infty}\frac{s-(a+1)}{p(s)}\nabla s = M\sum_{s=1}^{\infty}\frac{s-1}{s^3}=M\sum_{s=1}^{\infty}\frac{s-1}{s^3}=M\left(\frac{\pi^2}{6}-\zeta(3)\right)<1.$$

Thus Theorem 87 applies, so for any fixed $L\ge 0$, there exists a unique nonnegative $x(t)$ that solves the nonlinear self-adjoint difference equation

$$\nabla\left[t^3\nabla_{0^*}^{0.6}x(t)\right]=M\left(1+\sin\left(x\left(t-1\right)\right)\right),$$

where $\lim_{t\to\infty}x(t)=L$.

4.2 Unique Solutions to Nonlinear Boundary Value Problems

In this section, we will look at a theorem that uses the Contraction Mapping Theorem which give sufficient conditions for unique solutions to the specific boundary value problems studied in Chapter 3.

We will consider the nonlinear conjugate boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = F(t, x(t-1)), \quad t \in \mathbb{N}^b_{a+2}, \\ x(a) = A, \\ x(b) = B, \end{cases} \tag{4.7}$$

the nonlinear right-focal boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = F(t, x(t-1)), \quad t \in \mathbb{N}^b_{a+2}, \\ x(a) = A, \\ \nabla x(b) = B, \end{cases} \tag{4.8}$$

and the nonlinear three point boundary value problem

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = F(t, x(t-1)), \quad t \in \mathbb{N}^b_{a+2}, \\ x(a) = 0, \\ x(b) - \alpha x(a+k) = 0, \end{cases} \tag{4.9}$$

where $0 < \nu < 1$, $b - a \in \mathbb{N}_2$, $A, B \in \mathbb{R}$, $F : \mathbb{N}^b_{a+2} \times \mathbb{R} \to \mathbb{R}$, $0 \le \alpha \le 1$, and $k \in \mathbb{N}_1^{(b-a)-1}$.

4.2.1 Unique Solutions to a Nonlinear Conjugate BVP

We will consider the following nonlinear, self-adjoint, conjugate boundary value problem given in (4.7)

Theorem 89. *Assume $F : \mathbb{N}^b_{a+2} \times \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition with respect to its second variable in $\mathbb{N}^b_{a+2} \times \mathbb{R}$, i.e. there exists a constant $K > 0$ such that, if*

$u, v \in \mathbb{R}$ and $t \in \mathbb{N}_{a+2}^{b}$,

$$|F(t,u) - F(t,v)| \leq K\,|u - v|\,.$$

If $b - a < \frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}}$, then the nonlinear self-adjoint boundary value problem (4.7) has a unique solution.

Proof. If x is a solution to (4.7), then it is a solution to the linear self-adjoint boundary value problem

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} x(t) = h(t) := F(t, x(t-1)), \quad t \in \mathbb{N}_{a+2}^{b}, \\ x(a) = A, \\ x(b) = B. \end{cases} \tag{4.10}$$

By Corollary 58, the boundary value problem (4.10) has solution $x(t)$ if and only if $x(t)$ is a solution to the integral equation

$$x(t) = w(t) + \int_{a+1}^{b} G(t,s)h(s)\nabla s = w(t) + \int_{a+1}^{b} G(t,s)f(s, x(s-1))\nabla s, \tag{4.11}$$

where $w : \mathbb{N}_a^b \to \mathbb{R}$ is the unique solution to

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} w(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ w(a) = A, \\ w(b) = B, \end{cases}$$

and $G(t,s)$ is the Green's function for

$$\begin{cases} \nabla\nabla^{\nu}_{a^*}x(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ x(a) = 0, \\ x(b) = 0. \end{cases}$$

Define $\zeta := \{x : \mathbb{N}^b_a \to \mathbb{R}\}$ and $\|x\| := \max_{t\in\mathbb{N}^b_a} |x(t)|$, noting that $(\zeta, \|\cdot\|)$ is a complete metric space. Define the operator T on $(\zeta, \|\cdot\|)$ as

$$Tx(t) := w(t) + \int_{a+1}^{b} G(t,s)f(s, x(s-1))\nabla s,$$

where $w(t)$ and $G(t,s)$ are given as above. Note that $T : \zeta \to \zeta$. We claim T is a contraction mapping, so for a fixed $t \in \mathbb{N}^b_a$, consider for arbitrary $x, y \in \zeta$

$$\begin{aligned} |Tx(t) - Ty(t)| &= \left| \int_{a+1}^{b} G(t,s)f(s,x(s-1))\nabla s - \int_{a+1}^{b} G(t,s)f(s,y(s-1))\nabla s \right| \\ &= \left| \int_{a+1}^{b} G(t,s)\left[f(s,x(s-1)) - f(s,y(s-1))\right]\nabla s \right| \\ &\leq \int_{a+1}^{b} |G(t,s)||f(s,x(s-1)) - f(s,y(s-1))|\nabla s. \end{aligned}$$

Using the Lipschitz condition,

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \int_{a+1}^{b} |G(t,s)|K|x(s-1) - y(s-1)|\nabla s \\ &\leq K \int_{a+1}^{b} |G(t,s)| \, \|x - y\| \, \nabla s \\ &\leq K \, \|x - y\| \int_{a+1}^{b} |G(t,s)|\nabla s. \end{aligned}$$

Then, by the bound on the Green's function given in Theorem 61, we get

$$|Tx(t) - Ty(t)| \leq K \|x - y\| \frac{(b-a)^2}{4\Gamma(\nu+2)} = \alpha \|x - y\|, \tag{4.12}$$

where $\alpha := K\frac{(b-a)^2}{4\Gamma(\nu+2)}$. Note that this is true for all $t \in \mathbb{N}_a^b$, so

$$\|Tx - Ty\| \leq \alpha \|x - y\|,$$

but, by assumption, $b - a < \frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}}$, so

$$\begin{aligned} \alpha &< \frac{K}{4\Gamma(\nu+2)} \left(\frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}} \right)^2 \\ &= \frac{K}{4\Gamma(\nu+2)} \left(\frac{4\Gamma(\nu+2)}{K} \right) \\ &= 1, \end{aligned}$$

hence T is a contraction mapping on ζ. So by the Contraction Mapping Theorem, there exists a unique fixed point $x_0 \in \zeta$ such that

$$x_0(t) = Tx_0(t) = w(t) + \int_{a+1}^{b} G(t,s) f(s, x_0(s)) \nabla s,$$

hence $x_0(t)$ is the unique solution to the nonlinear self-adjoint boundary value problem (4.7). □

4.2.1.1 Nonlinear Conjugate BVP Example

Example 90. Consider the following boundary value problem.

$$\begin{cases} \nabla\nabla_{0*}^{0.6}x(t) = K\sqrt{x^2(t-1)+5} + h(t), \quad t \in \mathbb{N}_2^5, \\ x(0) = A, \\ x(5) = B, \end{cases}$$

where K is a constant such that $0 < K < \frac{4}{25}\Gamma(2.6) \approx 0.22874$ and $h : \mathbb{N}_2^5 \to \mathbb{R}$.

Note that this is a specific case of (4.7) where $a = 0$, $b = 5$, $\nu = 0.6$, and $F(t,x) = K\sqrt{x^2+5} + h(t)$. Then note $F_x(t,x) = \frac{Kx}{\sqrt{x^2+5}} \leq K$ for all $x \in \mathbb{R}$, hence our Lipschitz constant for $F(t,x)$ with respect to the second variable is K. But now we have

$$\frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}} = \frac{2\sqrt{\Gamma(0.6+2)}}{\sqrt{K}} > \frac{2\sqrt{\Gamma(2.6)}}{\sqrt{\frac{4}{25}\Gamma(2.6)}} = 5 = b - a.$$

Hence, by Theorem 89, we have that the above boundary value problem has a unique solution.

4.2.2 Unique Solutions to a Nonlinear Right Focal BVP

We will consider the following nonlinear, self-adjoint, conjugate boundary value problem given in (4.8)

Theorem 91. *Assume $F : \mathbb{N}_{a+2}^b \times \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition with respect to its second variable in $\mathbb{N}_{a+2}^b \times \mathbb{R}$, i.e. there exists a constant $K > 0$ such that, if $u, v \in \mathbb{R}$ and $t \in \mathbb{N}_{a+2}^b$,*

$$|F(t,u) - F(t,v)| \leq K\,|u-v|\,.$$

If $(b-a)(b-a-1) < \frac{\nu\Gamma(2+\nu)}{K}$*, then the nonlinear self-adjoint boundary value problem* (4.8) *has a unique solution.*

Proof. The proof will follow nearly identically to Theorem 89. In this case, we have that $w : \mathbb{N}_a^b \to \mathbb{R}$ will solve the boundary value problem

$$\begin{cases} \nabla\nabla_{a^*}^{\nu} w(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\ w(a) = A, \\ \nabla w(b) = B. \end{cases}$$

In place of (4.12), we would have, as a result of Theorem 65, that

$$|Tx(t) - Ty(t)| \leq K \|x - y\| \frac{(b-a)(b-a-1)}{\nu\Gamma(2+\nu)} = \alpha \|x - y\|$$

where $\alpha := K\frac{(b-a)(b-a-1)}{\nu\Gamma(2+\nu)}$. But by assumption $(b-a)(b-a-1) < \frac{\nu\Gamma(2+\nu)}{K}$, so

$$\alpha := K\frac{(b-a)(b-a-1)}{\nu\Gamma(2+\nu)} < K\frac{\nu\Gamma(2+\nu)}{K\nu\Gamma(2+\nu)} = 1,$$

hence we get T is a contraction mapping on ζ, and the proof finishes in similar manner to Theorem 89. □

4.2.2.1 Nonlinear Right Focal BVP Example

Example 92. Consider the following boundary value problem.

$$\begin{cases} \nabla\nabla_{0^*}^{0.6} x(t) = K\sqrt{x^2(t-1) + 5} + h(t), \quad t \in \mathbb{N}_2^5, \\ x(0) = A, \\ \nabla x(5) = B, \end{cases}$$

where K is a constant such that $0 < K < \frac{0.6\Gamma(2.6)}{20} \approx 0.04289$ and $h : \mathbb{N}_2^5 \to \mathbb{R}$.

Note that this is a specific case of (4.8) where $a = 0$, $b = 5$, $\nu = 0.6$, and $F(t, x) = K\sqrt{x^2+5} + h(t)$. Then note $F_x(t, x) = \frac{Kx}{\sqrt{x^2+5}} \leq K$ for all $x \in \mathbb{R}$, hence our Lipschitz constant for $F(t, x)$ with respect to the second variable is K. But now we have

$$\frac{\nu\Gamma(2+\nu)}{K} = \frac{0.6\Gamma(2.6)}{K} > 5 \cdot 4 = (b-a)(b-a-1).$$

Hence, by Theorem 91, we have that the above boundary value problem has a unique solution.

4.2.3 Unique Solutions to a Nonlinear Three-Point BVP

We will consider the following nonlinear, self-adjoint, conjugate boundary value problem given in (4.9)

Theorem 93. *Assume $F : \mathbb{N}_{a+2}^b \times \mathbb{R} \to \mathbb{R}$ satisfies a Lipschitz condition with respect to its second variable in $\mathbb{N}_{a+2}^b \times \mathbb{R}$, i.e. there exists a constant $K > 0$ such that, if $u, v \in \mathbb{R}$ and $t \in \mathbb{N}_{a+2}^b$,*

$$|F(t, u) - F(t, v)| \leq K\,|u - v|\,.$$

If $(b + \frac{(b-a+1)^2}{\nu} - a) < \frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}}$, then the nonlinear self-adjoint boundary value problem (4.9) *has a unique solution.*

Proof. The proof will follow nearly identically to Theorem 89. In this case, we have that $w \equiv 0$.

In place of (4.12), we would have, as a result of Theorem 78, that

$$|Tx(t) - Ty(t)| \leq K\,\|x - y\|\,\frac{(b + \frac{(b-a+1)^2}{\nu} - a)^2}{4\Gamma(\nu+2)} = \alpha\,\|x - y\|$$

where $\alpha := K\frac{(b+\frac{(b-a+1)^2}{\nu}-a)^2}{4\Gamma(\nu+2)}$. But by assumption $(b+\frac{(b-a+1)^2}{\nu}-a) < \frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}}$, so

$$\alpha := K\frac{(b+\frac{(b-a+1)^2}{\nu}-a)^2}{4\Gamma(\nu+2)} < K\frac{4\Gamma(\nu+2)}{K4\Gamma(\nu+2)} = 1,$$

hence we get T is a contraction mapping on ζ, and the proof finishes in similar manner to Theorem 89. □

4.2.3.1 Nonlinear Right Focal BVP Example

Example 94. Consider the following boundary value problem.

$$\begin{cases} \nabla\nabla_{0*}^{0.6}x(t) = K\sqrt{x^2(t-1)+5}+h(t), \quad t\in\mathbb{N}_2^5, \\ x(0)=0, \\ x(5)-\alpha x(3) = B, \end{cases}$$

where K is a constant such that $0 < K < \frac{4}{65^2}\Gamma(2.6) \approx 0.00135$, $0 \leq \alpha \leq 1$, and $h : \mathbb{N}_2^5 \to \mathbb{R}$.

Note that this is a specific case of (4.9) where $a = 0$, $k = 3$, $b = 5$, $\nu = 0.6$, and $F(t,x) = K\sqrt{x^2+5}+h(t)$. Then note $F_x(t,x) = \frac{Kx}{\sqrt{x^2+5}} \leq K$ for all $x \in \mathbb{R}$, hence our Lipschitz constant for $F(t,x)$ with respect to the second variable is K. But now we have

$$\frac{2\sqrt{\Gamma(\nu+2)}}{\sqrt{K}} = 2\frac{\sqrt{\Gamma(2.6)}}{\sqrt{K}} > 2\frac{\sqrt{\Gamma(2.6)}}{\sqrt{\frac{4}{65^2}\Gamma(2.6)}} = 65$$
$$= \left(5+\frac{6^2}{0.6}-0\right)^2 = \left(b+\frac{(b-a+1)^2}{\nu}-a\right)^2.$$

Hence, by Theorem 93, we have that the above boundary value problem has a unique solution.

Bibliography

[1] *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press, 2010.

[2] K. Ahrendt, L. Castle, and K. Yochman. Laplace transforms for the nabla-difference operator and a fractional variation of parameters formula. *Communications in Applied Analysis*, 2011.

[3] K. Ahrendt, L. DeWolf, L. Mazurowski, K. Mitchell, T. Rolling, and D. Veconi. Initial and boundary value problems for the caputo fractional self-adjoint difference equations. *Enlightenment in Pure and Applied Mathematics*, 2, 2016.

[4] G.A. Anastassiou. Nabla discrete fractional calculus and nabla inequalities. *Mathematical and Computer Modelling*, 2010.

[5] D.R. Anderson. Taylor polynomials for nabla dynamic equations on time scales. *PanAmerican Mathematical Journal*, 12, 2002.

[6] George B. Arfken and Hans J. Weber. *Mathematical Methods for Physicists*. Academic Press, 6 edition, 2005.

[7] Ferhan M. Atici and Paul W. Eloe. Two-point boundary value problems for finite fractional difference equations. *Journal of Difference Equations and Applications*, 17:445–456, 2011.

[8] Ferhan M. Atici and Sevgi Sengul. Modeling with fractional difference equations. *J. Math. Anal. Appl.*, 2010.

[9] F. M. Attici and P. W. Eloe. A transform method in discrete fractional calculus. *International Journal of Difference Equations*, 2:165–176, 2007.

[10] Pushp R. Awasthi. *Boundary Value Problems for Discrete Fractional Equations*. PhD thesis, University of Nebraska-Lincoln, 2013.

[11] D. Benson, R. Shumer, S. Wheatcraft, and S. Meerschaert. Fractional dispersion, levy motion, and the made tracer tests. *Transport Porous Media*, 42:211–240, 2001.

[12] D. Benson, S. Wheatcraft, and M. Meerschaert. Application of a fractional advection-dispersion equation. *Water Resources*, 36:1403–1412, 200.

[13] Martin Bohner and Allan Peterson. *Dynamic Equations on Time Scales*. Birkhäuser, 2001.

[14] Martin Bohner and Allan Peterson. *Advances in Dynamic Equations on Time Scales*. Birkhäuser, 2003.

[15] A. Brackins. *Boundary Value Problems of Nabla Fractional Difference Equations*. PhD thesis, University of Nebraska-Lincoln, 2014.

[16] Julia St. Goar. *A Caputo Boundary Value Problem in Nabla Fractional Calculus*. PhD thesis, University of Nebraska-Lincoln, 2016.

[17] C. Goodrich and A. C. Peterson. *Discrete Fractional Calculus*. Springer International Publishing, 2015.

[18] C. S. Goodrich. Solutions to a discrete right-focal fractional problem. *International Journal of Difference Equations*, 5:195–216, 2010.

[19] C. S. Goodrich. On a fractional boundary value problem with fractional boundary conditions. *Applied Mathematics Letters*, 25:1101–1105, 2012.

[20] Christopher S. Goodrich. *An Analysis of Nonlocal Boundary Value Problems of Fractional and Integer Order*. PhD thesis, University of Nebraska-Lincoln, 2012.

[21] J. Hein, S. McCarthy, N. Gaswick, B. McKain, and K. Spear. Laplace transforms for the nabla difference operator. *PanAmerican Mathematical Journal*, 21(3):79–96, 2011.

[22] Michael Holm. Sum and difference compositions in discrete fracctional calculus. *CUBO Mathematical Journal*, 13, 2011.

[23] Michael Holm. *The theory of discrete fractional calculus: Development and application*. PhD thesis, University of Nebraska-Lincoln, 2011.

[24] John Hunter and Bruno Nachtergaele. *Applied Analysis*. World Scientific Publishing, 2001.

[25] Francesco Mainardi J. A. Tenreiro Machado, Virginia Kriyakova. A poster about the recent history of fractional calculus. *Frtactional Calculus and Applied Analysis*, 13(3), 2010.

[26] W.G. Kelley and A.C. Peterson. *The Theory of Differential Equations*. Springer, second edition, 2010.

[27] W.G. Kelley and A.C. Peterson. *Difference Equations*. Academic Press, second edition, 2011.

[28] S. Hamani M. Benchohra and S. K. Ntouyas. Boundary value problems for differential equations with fractional order and nonlocal conditions. *Nonlinear Analysis: Theory Methods and Applications*, 71:2391–3296, 2009.

[29] S. P. Näsholm and S. Holm. On a fractional zener elastic wave equation. *Fractional Calculus and Applied Analysis*, 16(1):26–50, 2013.

[30] I. Podlubny. *Fractional Differential Equations.* Academic Press, 1999.