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BOUNDARY VALUE PROBLEMS FOR DISCRETE FRACTIONAL EQUATIONS

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BOUNDARY VALUE PROBLEMS FOR DISCRETE FRACTIONAL EQUATIONS

by

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BOUNDARY VALUE PROBLEMS FOR DISCRETE FRACTIONAL EQUATIONS

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In this dissertation we are interested in proving the existence of solutions for various fractional boundary value problems. Our technique will be to apply certain fixed point theorems. Also comparison theorems for fractional boundary problems and a so-called Liapunov inequality will be given.

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Chapter 1

Introduction

1.1 Brief History of Discrete Fractional Calculus

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders, and their applications appear in numerous diverse fields including engineering, chemistry applied mathematics, economics, biology, control theory and other fields. For example, as mathematical models describing biological phenomena are getting more sophisticated and realistic, the attention needed from specialists is growing at a fast pace. There are several recent areas of specialized research in mathematical biology: Enzyme kinetics, biological tissue analysis, cancer modeling, heart and arterial disease modeling being among the popular ones, see [23]. It is well known that there is a similarity between properties of differential calculus involving the operator $\frac{d}{dx}$ and the properties of discrete calculus involving Δ defined by $\Delta f(x) = f(x+1) - f(x)$ which is known as the forward difference operator. Expectedly, some similar correspondence exists between the operators of continuous fractional and discrete fractional calculus.

A recent interest in discrete fractional calculus has been shown by Atici and Eloe, who in [10] discuss properties of the generalized falling function, a corresponding power rule for fractional delta-operators and the commutativity of fractional sums. They present in [11] additional rules for composing fractional sums and differences but leave many important cases unresolved. It is very important to pay careful attention to various function domains and to the lower limits of summation and differentiation when doing discrete fractional calculus.

The goal of this dissertation is to further develop the theory of fractional calculus on the natural numbers. In Chapter 2 and 3, we present the existence of a positive solution for certain boundary value problems of order ν where $3 < \nu \leq 4$ using two well-known fixed point theorems due to Krasnosel'skii and Banach. Later in the same chapter, we present the existence of multiple positive solutions.

1.2 Gamma Function

Undoubtedly, one of the basic functions of the fractional calculus is the Euler gamma function $\Gamma(x)$, which generalizes the factorial function and allows n to take on non-integer and even complex values. This function and its properties are widely used throughout this dissertation.

1.2.1 Definition of the Gamma Function

The gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,$$

for $z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$.

In figure 1, a graph of the gamma function as a function of the real variable x is given.

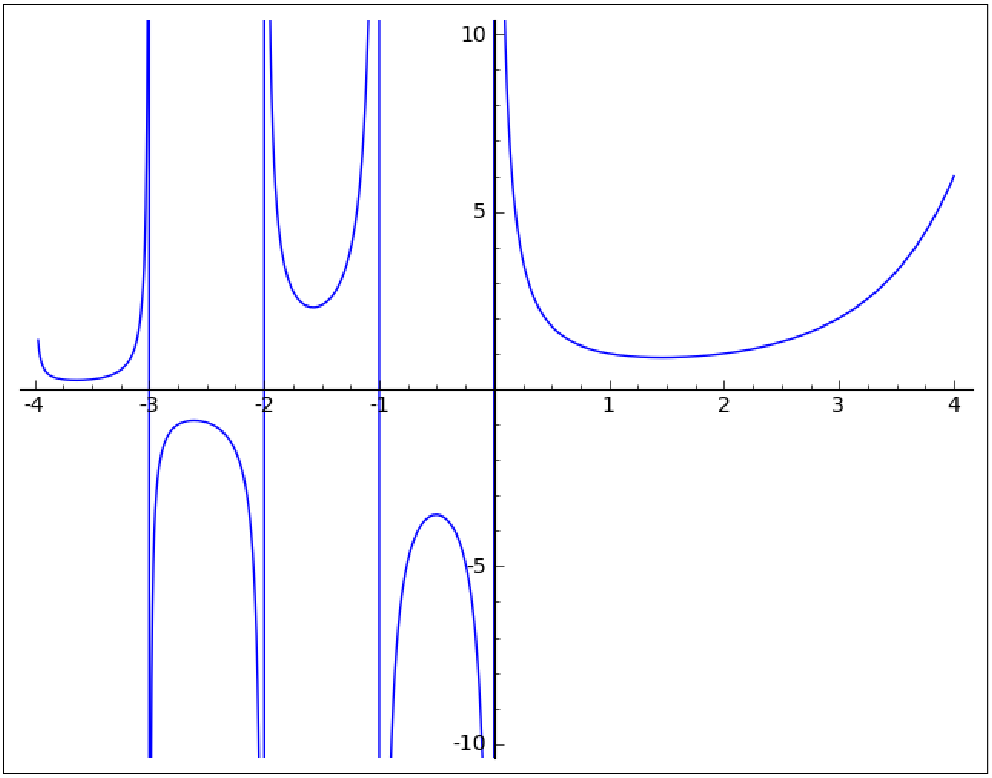

Figure 1.1: Gamma function for real values of x

1.2.2 Some properties of the Gamma Function

Some of the basic properties of the gamma function are

- $\Gamma(x) > 0$ for $x > 0$
- $\Gamma(x+1) = x\Gamma(x)$
- $\Gamma(n+1) = n!$ for $n \in \mathbb{N}_0$
- $\dfrac{\Gamma(x+k)}{\Gamma(x)} = (x+k-1)\cdots(x-1)x$ for $x \in \mathbb{R} \setminus \{-\mathbb{N}_0\}$ and $k \in \mathbb{N}$

Remark 1 • *The generalized falling function is defined by*

$$t^{\underline{r}} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)},$$

for any $t, r \in \mathbb{C}$ *for which the right hand-side is defined. Also, whenever* $t-r+1$ *is a nonpositive integer and the numerator is well defined we make the usual assumption that* $t^{\underline{r}} = 0$.

- *Let* $f : \mathbb{N}_a \to \mathbb{R}$, *then the forward difference operator,* Δ, *is defined by*

$$\Delta f(t) = f(t+1) - f(t), \quad t \in \mathbb{N}_a.$$

- *The following identities hold whenever the generalized falling function is well defined.*

  1. $\nu^{\underline{\nu}} = \Gamma(\nu+1)$.
  2. $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}$.
  3. $t^{\underline{\nu+1}} = (t-\nu)t^{\underline{\nu}}$.

- *For a discrete time scale such as* $\mathbb{N}_a$, $\sigma(s)$ *denotes the next point in the time scale after* s, $\sigma(s) = s+1$.

1.3 Whole-Order Sums

We consider real-valued functions on a shift of the natural numbers

$$f : \mathbb{N}_a \to \mathbb{R}, \text{ where } \mathbb{N}_a := a + \mathbb{N}_0 = \{a, a+1, a+2, \cdots\} \ (a \in \mathbb{R} \ fixed).$$

We define the n^{th}-order sum of f based at a (denoted by $\Delta_a^{-n} f$) by

$$y(t) = (\Delta_a^{-n} f)(t) = \sum_{s=a}^{t-n} \frac{(t-s-1)^{\underline{n-1}}}{(n-1)!} f(s), \quad t \in \mathbb{N}_a.$$

The following are some definitions and facts that are going to be used throughout this dissertation, see [4].

1.4 Fractional-Order Sums and Differences

Definition 2 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then the ν^{th}-order fractional sum of f based at a is given by*

$$(\Delta_a^{-\nu} f)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s), \text{ for } \quad t \in \mathbb{N}_{a+\nu}.$$

Also, we define the trivial sum by $\Delta_a^{-0} f(t) := f(t)$ for $t \in \mathbb{N}_a$.

A useful formula is

$$\Delta_a^{-\nu} f(a+\nu) = f(a),$$

which we will use from time to time.

With the fractional sum in hand, we introduce the fractional differences such as in [4].

Definition 3 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N - 1 < \nu \leq N$. Then we define the ν^{th}-order difference of f based at a, $\Delta_a^{\nu} f$, by*

$$(\Delta_a^{\nu} f)(t) = \Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \quad \text{for } t \in \mathbb{N}_{a+N-\nu}.$$

1.5 The Fractional Power Rule

The next result is a fractional power rule for sums and differences. The proof for this power rule can be found in [4].

Lemma 4 *Let $a \in \mathbb{R}$ and $\mu \geq 0$ be given. Then*

$$\Delta(t-a)^{\underline{\mu}} = \mu(t-a)^{\underline{\mu-1}},$$

for any t for which both sides are well defined. Furthermore, for any $\nu \geq 0$,

$$\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{-\nu}}(t-a)^{\underline{\mu+\nu}},\ for\ t \in \mathbb{N}_{a+\mu+\nu}$$

and

$$\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \mu^{\underline{\nu}}(t-a)^{\underline{\mu-\nu}},\ for\ t \in \mathbb{N}_{a+\mu+N-\nu}.$$

We will later use the power rule to help us solve a ν^{th}-order fractional initial value problem.

Theorem 5 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu \geq 0$ be given with $N-1 < \nu \leq N$. A general solution of the ν^{th}-order fractional difference equation*

$$\Delta_{a+\nu-N}^{\nu} x(t) = 0, \quad t \in \mathbb{N}_a \tag{1.1}$$

is given by

$$x(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}}. \tag{1.2}$$

Also the solution of the ν^{th}-order fractional initial value problem

$$\begin{cases} \Delta_{a+\nu-N}^{\nu} y(t) = f(t), \ \ t \in \mathbb{N}_a \\ \Delta^i y(a+\nu-N) = A_i, \ \ i \in \{0, 1, \cdots, N-1\}, \quad A_i \in \mathbb{R}. \end{cases} \tag{1.3}$$

is given by the variation of constant formula

$$y(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a+\nu-N},$$

where $\alpha_i, 0 \leq i \leq N-1,$ are appropriate constants.

Proof: See Appendix B.

Chapter 2

The Fractional Boundary Value Problem

Given a boundary value problem, its corresponding Green's function is mathematically vital. In this chapter, we are interested in a discrete, nonlinear fractional boundary value problem with right focal boundary conditions. We define an operator A in terms of a certain Green's function in the standard way. This allows us to apply the Krasnosel'skii and Banach fixed point theorems to obtain the existence of positive solutions of our boundary value problem. Also, we use theorems from [8] to obtain multiple positive solutions for the same boundary value problem.

2.1 The Boundary Value Problem

In this section we are concerned with the following fractional boundary value problem and the existence of the related Green's function,

$$\begin{cases} \Delta_{\nu-4}^{\nu} y(t) = f(t, y(t+\nu-2)), \quad t \in \{0, \cdots, b+3\} \\ \Delta^i y(\nu-4) = 0, \quad i = 0, 1 \\ \Delta^j y(b+\nu) = 0, \quad j = 2, 3, \end{cases} \tag{2.1}$$

where

- $3 < \nu \leq 4$
- $b \in \mathbb{N}$
- $f : \{0, \cdots, b+3\} \times \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $y \geq 0$ *and* $f(t, 0) = 0$ for $t \in \{0, \cdots, b+3\}$.

When $\nu = 4$, this boundary value problem has important applications in bending beam problems, see [21] and [22] .

Note the following domains for each function appearing in problem (2.1):

- $D\{\Delta_{\nu-4}^{\nu} y\} = \{0, \cdots, b+3\}$
- $D\{y\} = \{\nu-4, \cdots, b+\nu+3\}$
- $D\{\Delta^i y\} = \{\nu-4, \cdots, b+\nu+3-i\}$, for each $i \in \{0, 1\}$
- $D\{\Delta^j y\} =\{\nu-4, \cdots, b+\nu+3-j\}$, for each $j \in \{2, 3\}$.

In particular, the unknown function in problem (2.1) satisfies

$$y : \{\nu-4, \cdots, b+\nu+3\} \to \mathbb{R}.$$

Similar boundary value problems were studied by

- Goodrich in [3] , where the order of the fractional difference equation is $\nu \in (1, 2]$ requiring only two boundary conditions and the right boundary condition is focal.

- Holm in [4] , where the order of the fractional difference equation is $\nu \in [2, \infty)$, requiring $N := \lceil \nu \rceil$ boundary conditions and a fractional (μ^{th}-order) right boundary condition, where $\mu \in [1, \nu)$. The fractional boundary conditions require more attention but offers great flexibility for border applications

Before deriving the Green's function associated with (2.1), we prove the following theorem about the existence of the Green's function that uses the Variation of Constants Formula for discrete fractional initial value problems.

Definition 6 *We define the Cauchy function $x(t, s)$ of $\Delta^{\nu}_{\nu-4} x = 0$ for $\nu - 4 \leq t \leq b + \nu + 3,\ 0 \leq s \leq b + 3$ by*

$$x(t, s) = \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}.$$

Note that $x(t, s) = 0$ for $0 \leq t \leq s + \nu - 1$. Also for each fixed s, $x(t, s)$, is the solution of the initial value problem

$$\begin{cases} \Delta^{\nu}_{\nu-4} x = 0 \\ x(s + \nu - 3, s) = x(s + \nu - 2, s) = x(s + \nu - 1, s) = 0, \\ x(s + \nu, s) = 1 \end{cases}$$

on $[s + \nu - 3, b + \nu + 3]$.

Theorem 7 (Variation of Constants Formula) *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ and let $x(t, s)$ be the Cauchy function for*

$$Lx = 0, \ where \ \ Lx = \Delta^{\nu}_{\nu-4} x.$$

Then

$$x(t) = \int_a^t x(t,s) f(s) \Delta s$$

is the solution of the initial value problem

$$\begin{cases} Lx = f(t) \\ x(\nu-4) = 0, \quad \Delta x(\nu-4) = 0 \\ \Delta^2 x(\nu-4) = 0, \quad \Delta^3 x(\nu-4) = 0. \end{cases}$$

Next, we prove the following theorem:

Theorem 8 *(Green's Function) Assume that the BVP*

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = 0 \\ y(\nu-4) = \Delta y(\nu-4) = 0 \\ \Delta^2 y(b+\nu) = \Delta^3 y(b+\nu) = 0 \end{cases} \tag{2.2}$$

has only the trivial solution. For each fixed s, let $u(t,s)$ be the unique solution of the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} u(t) = 0 \\ u(\nu-4,s) = \Delta u(\nu-4,s) = 0 \\ \Delta^2 u(b+\nu,s) = -\Delta^2 x(b+\nu,s) \\ \Delta^3 u(b+\nu,s) = -\Delta^3 x(b+\nu,s) \end{cases}$$

where $x(t,s)$ is the Cauchy function for $\Delta^{\nu}_{\nu-4} u = 0$. If $h : \mathbb{N}_0 \longrightarrow \mathbb{R}$ then

$$y(t) = \int_0^{b+3} G(t,s)h(s)\Delta s$$

is the unique solution of the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = h(t) \\ y(\nu-4,s) = \Delta y(\nu-4,s) = 0 \\ \Delta^2 y(b+\nu,s) = \Delta^3 y(b+\nu,s) = 0 \end{cases}$$

where $G(t,s)$ is the Green's function for the problem

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = 0 \\ y(\nu-4) = \Delta y(\nu-4) = 0 \\ \Delta^2 y(b+\nu) = \Delta^3 y(b+\nu) = 0, \end{cases} \tag{2.3}$$

and is given by

$$G(t,s) = \begin{cases} u(t,s), & 0 \le t-\nu+2 \le s \le b+3 \\ v(t,s), & 0 \le s \le t-\nu+1 \le b+3 \end{cases}$$

where $v(t,s) := u(t,s) + x(t,s)$.

Proof: First we show that the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y = 0, \quad t \in \{0, \cdots, b+3\} \\ y(\nu-4) = 0, \quad \Delta y(\nu-4) = 0 \\ \Delta^2 y(b+\nu) = 0, \quad \Delta^3 y(b+\nu) = 0 \end{cases} \tag{2.4}$$

has only the trivial solution. Consider the given BVP, we know from Theorem 5 that the general solution of $\Delta^{\nu}_{\nu-4} y(t) = 0$ is of the form

$$u(t) = At^{\underline{\nu-1}} + Bt^{\underline{\nu-2}} + Ct^{\underline{\nu-3}} + Dt^{\underline{\nu-4}}.$$

From the first boundary condition, we have

$$0 = u(\nu-4) = \sum_{i=0}^{4} C_i(\nu-4)^{\underline{\nu-i}}.$$

Using

$$\begin{aligned}
(\nu-4)^{\underline{\nu-1}} &= \frac{\Gamma(\nu-3)}{\Gamma(-2)} = 0, \\
(\nu-4)^{\underline{\nu-2}} &= \frac{\Gamma(\nu-3)}{\Gamma(-1)} = 0, \\
(\nu-4)^{\underline{\nu-3}} &= \frac{\Gamma(\nu-3)}{\Gamma(0)} = 0, \\
(\nu-4)^{\underline{\nu-4}} &= \frac{\Gamma(\nu-3)}{\Gamma(1)} \neq 0,
\end{aligned}$$

we get that $C = 0$ and a similar argument shows that $D = 0$. Hence

$$u(t) = At^{\underline{\nu-1}} + Bt^{\underline{\nu-2}}.$$

The last two boundary conditions lead to the following system

$$\begin{aligned}
B(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}} + A(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}} &= 0 \\
B(\nu-2)^{\underline{3}}(b+\nu)^{\underline{\nu-5}} + A(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}} &= 0.
\end{aligned}$$

In order to show that $u(t)$ is the trivial solution, we need to show that the above system has only the trivial solution $A = B = 0$. To see this we consider

$$\begin{vmatrix}
(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}} & (\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}} \\
(\nu-2)^{\underline{3}}(b+\nu)^{\underline{\nu-5}} & (\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}
\end{vmatrix}$$

$$= (\nu-2)^{\underline{2}}(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu)^{\underline{\nu-4}} - (\nu-1)^{\underline{2}}(\nu-2)^{\underline{3}}(b+\nu)^{\underline{\nu-3}}(b+\nu)^{\underline{\nu-5}}$$

$$= (\nu-1)^{\underline{2}}(\nu-2)^{\underline{2}}[(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu)^{\underline{\nu-4}} - (\nu-4)(b+\nu)^{\underline{\nu-3}}(b+\nu)^{\underline{\nu-5}}] > 0$$

since $3 < \nu \leq 4$. Thus the BVP (2.4) has only the trivial solution. It then follows by standard arguments that the non-homogenous BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = h(t), \quad t \in \{\nu-4, \cdots, b+\nu+3\} \\ y(\nu-4) = A, \quad \Delta y(\nu-4) = B \\ \Delta^2 y(b+\nu) = C, \quad \Delta^3 y(b+\nu) = D \end{cases}$$

has a unique solution. Since for each fixed s, $u(t,s)$ satisfies a BVP of this form, we get that $u(t,s)$ is uniquely determined. It then follows that $v(t,s)$ and $G(t,s)$ are uniquely determined. Since for each fixed s, $u(t,s)$ is a solution of $\Delta^{\nu}_{\nu-4} x = 0$ and $x(t,s)$ is a solution for $t \geq s+\nu-3$, we have that for each fixed s, $v(t,s) = u(t,s) + x(t,s)$ is also a solution of $\Delta^{\nu}_{\nu-4} x = 0$ for $s+\nu-1 \leq t \leq b+\nu+3$. It follows that $v(t,s)$ satisfies the boundary conditions at $b+\nu$. Now, let $G(t,s)$ be as in the statement of this theorem and consider

$$\begin{aligned} y(t) &= \int_0^{b+3} G(t,s)h(s)\Delta s, \quad t \in \{\nu-4, \cdots, b+\nu+3\} \\ &= \int_0^{t-\nu+2} G(t,s)h(s)\Delta s + \int_{t-\nu+2}^{b+3} G(t,s)h(s)\Delta s \\ &= \int_0^{t-\nu+2} v(t,s)h(s)\Delta s + \int_{t-\nu+2}^{b+3} u(t,s)h(s)\Delta s. \end{aligned}$$

Note that even though $v(t,s)$ is only defined for $0 \leq s \leq t-\nu+1$, the upper limit of integration in the first term above is okay since this delta integral does not depend on the value of the integrand at the upper limit of integration. Hence

$$\begin{aligned} y(t) &= \int_0^{t-\nu+2} [u(t,s)+x(t,s)]h(s)\Delta s + \int_{t-\nu+2}^{b+3} u(t,s)h(s)\Delta s \\ &= \int_0^{b+3} u(t,s)h(s)\Delta s + \int_0^{t-\nu+2} x(t,s)h(s)\Delta s. \end{aligned}$$

Since $x(t,t-\nu+2) = x(t,t-\nu+1) = 0$

$$\begin{aligned} y(t) &= \int_0^{b+3} u(t,s)h(s)\Delta s + \int_0^{t-\nu} x(t,s)h(s)\Delta s \\ &= \int_0^{b+3} u(t,s)h(s)\Delta s + z(t) \end{aligned}$$

where, by the variation of constants formula, z is the solution of the IVP

$$\begin{cases} \Delta^\nu_{\nu-4} z(t) = h(t) \\ z(\nu-4) = z(\nu-3) = z(\nu-2) = z(\nu-1) = 0. \end{cases}$$

It follows that

$$\begin{aligned} \Delta^\nu_{\nu-4} y(t) &= \int_0^{b+3} \Delta^\nu_{\nu-4} u(t,s)h(s)\Delta s + \Delta^\nu_{\nu-4} z(t) \\ &= 0 + \Delta^\nu_{\nu-4} z(t) = h(t). \end{aligned}$$

Hence y is a solution of the non-homogenous equation. It remains to show that y satisfies the boundary conditions. Note that

$$y(\nu-4) = \int_0^{b+3} u(\nu-4,s)h(s)\Delta s + z(\nu-4) = 0$$

and

$$\Delta y(\nu-4)=\int_0^{b+3}\Delta u(\nu-4,s)h(s)\Delta s+\Delta z(\nu-4)=0.$$

Hence y satisfies the first two boundary conditions. Next, we show that the Green's function satisfies the last two boundary conditions. Recall that

$$G(t,s)=\begin{cases} u(t,s), & 0\le t-\nu+2\le s\le b+3\\ v(t,s), & 0\le s\le t-\nu+1\le b+3.\end{cases}$$

Now, consider $v(t,s)=u(t,s)+x(t,s)$, we have

$$\begin{aligned}\Delta^2 v(b+\nu,s) &= \Delta^2 u(b+\nu,s)+\Delta^2 x(b+\nu,s)\\ &= 0 \quad \{\text{from the third boundary condition on u}\}\end{aligned}$$

and

$$\begin{aligned}\Delta^3 v(b+\nu,s) &= \Delta^3 u(b+\nu,s)+\Delta^3 x(b+\nu,s)\\ &= 0 \quad \{\text{ from the fourth boundary condition on u}\}.\end{aligned}$$

Then

$$\begin{aligned}\Delta^2 y(b+\nu) &= \int_0^{b+3}\Delta^2 G(b+\nu,s)h(s)\Delta s\\ &= \int_0^{b+3}\Delta^2 v(b+\nu,s)h(s)\Delta s\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\Delta^3 y(b+\nu) &= \int_0^{b+3} \Delta^3 G(b+\nu,s)h(s)\Delta s \\ &= \int_0^{b+3} \Delta^3 v(b+\nu,s)h(s)\Delta s \\ &= 0\end{aligned}$$

since $v(t,s)$ satisfies the last two boundary conditions and hence $y(t)$ satisfies the last two boundary conditions. Therefore, $y(t)$ is the unique solution of the BVP

$$\begin{cases}\Delta_{\nu-4}^{\nu} y(t) = h(t) \\ y(\nu-4,s) = \Delta y(\nu-4,s) = 0 \\ \Delta^2 y(b+\nu,s) = \Delta^3 y(b+\nu,s) = 0.\end{cases}$$

2.2 The Green's Function

In this section we derive the Green's function associated with (2.1) by solving the corresponding linear boundary value problem

$$\begin{cases}\Delta_{\nu-4}^{\nu} y(t) = 0, \quad t \in \{0,\cdots,b+3\} \\ \Delta^i y(\nu-4) = 0, \quad i = 0,1 \\ \Delta^j y(b+\nu) = 0, \quad j = 2,3.\end{cases}$$

From Theorem 5 we know that for each fixed s

$$u(t,s)=\sum_{i=1}^{4} \alpha_i(s) t^{\underline{\nu-i}}$$

where $\alpha_i \in \mathbb{R}$ and $t \in \{\nu-4,\cdots,b+\nu+3\}$.

From the first boundary condition, we have

$$0 = u(\nu - 4, s) = \sum_{i=1}^{4} \alpha_i(s)(\nu - 4)^{\underline{\nu - i}}.$$

Since

$$\begin{aligned}
(\nu - 4)^{\underline{\nu-1}} &= \frac{\Gamma(\nu - 3)}{\Gamma(-2)} = 0, \\
(\nu - 4)^{\underline{\nu-2}} &= \frac{\Gamma(\nu - 3)}{\Gamma(-1)} = 0, \\
(\nu - 4)^{\underline{\nu-3}} &= \frac{\Gamma(\nu - 3)}{\Gamma(0)} = 0 \\
(\nu - 4)^{\underline{\nu-4}} &= \frac{\Gamma(\nu - 3)}{\Gamma(1)} \neq 0
\end{aligned}$$

we get that $\alpha_4(s) = 0$. Likewise, from the second boundary condition one can show that $\alpha_3(s) = 0$. So we have that

$$u(t, s) = \alpha_2(s)t^{\underline{\nu-2}} + \alpha_1(s)t^{\underline{\nu-1}}.$$

Next, we apply the last two boundary conditions

$$\Delta^j u(b + \nu, s) = -\Delta^j x(b + \nu, s),\ \ j = 2, 3$$

and solve for $\alpha_1(s)$ and $\alpha_2(s)$. Notice the Cauchy function associated with (2.2) is given by the following

$$x(t, s) = \frac{(t - \sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}.$$

So we have the following system of equations:

$$\alpha_2(s)(\nu-2)(\nu-3)(b+\nu)^{\underline{\nu-4}}+\alpha_1(s)(\nu-1)(\nu-2)(b+\nu)^{\underline{\nu-3}} = -\frac{(b+\nu-s-1)^{\underline{\nu-3}}}{\Gamma(\nu-2)}$$

$$\alpha_2(s)(\nu-2)(\nu-3)(\nu-4)(b+\nu)^{\underline{\nu-5}} + \alpha_1(s)(\nu-1)(\nu-2)(\nu-3)(b+\nu)^{\underline{\nu-4}} =$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}}{\Gamma(\nu-3)}.$$

Applying the definition of the falling function and properties of the gamma function, we obtain the following

$$\begin{cases} \alpha_2(s)(\nu-2)^{\underline{2}}(\nu-4)(b+\nu)^{\underline{\nu-4}} + \alpha_1(s)(\nu-1)^{\underline{2}}(\nu-4)(b+\nu)^{\underline{\nu-3}} = \\ \qquad -(\nu-4)\dfrac{(b+\nu-s-1)^{\underline{\nu-3}}}{\Gamma(\nu-2)} \\ -\alpha_2(s)(\nu-2)^{\underline{3}}(b+5)(b+\nu)^{\underline{\nu-5}} - \alpha_1(s)(\nu-1)^{\underline{3}}(b+5)(b+\nu)^{\underline{\nu-4}} = \\ \qquad (b+5)\dfrac{(b+\nu-s-1)^{\underline{\nu-4}}}{\Gamma(\nu-3)}. \end{cases}$$

Adding these two equations we get that

$$\alpha_1(s)[(\nu-1)^{\underline{2}}(\nu-4)(b+\nu)^{\underline{\nu-3}} - (\nu-1)^{\underline{3}}(b+5)(b+\nu)^{\underline{\nu-4}}] =$$

$$(b+5)\frac{(b+\nu-s-1)^{\underline{\nu-4}}}{\Gamma(\nu-3)} - (\nu-4)\frac{(b+\nu-s-1)^{\underline{\nu-3}}}{\Gamma(\nu-2)}$$

$$\Longrightarrow \alpha_1(s)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}[(\nu-4)(b+4) - (\nu-3)(b+5)] =$$

$$(b+5)\frac{(b+\nu-s-1)^{\underline{\nu-4}}}{\Gamma(\nu-3)} - (\nu-4)\frac{(b+\nu-s-1)^{\underline{\nu-3}}}{\Gamma(\nu-2)}$$

$$\implies -\alpha_1(s)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1) =$$

$$\frac{(b+\nu-s-1)^{\underline{\nu-4}}}{\Gamma(\nu-2)}[(b+5)(\nu-3)-(\nu-4)(b-s+3)]$$

$$\implies \alpha_1(s) = -\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.$$

Plugging $\alpha_1(s)$ into one of the equations above, we obtain

$$\alpha_2(s) = \frac{-\dfrac{(b+\nu-s-1)^{\underline{\nu-3}}}{\Gamma(\nu-2)} + \dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+\nu)^{\underline{\nu-3}}}{\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}}{(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}}$$

$$= -\frac{(b+\nu-s-1)^{\underline{\nu-3}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}} + \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$= \frac{(b+\nu-s-1)^{\underline{\nu-4}}[(2\nu+b+s(\nu-4)-3)(b+4)-(b+\nu+1)(b-s+3)]}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.$$

Therefore,

$G : \{\nu-4, \cdots, b+\nu+3\} \times \{0, \cdots, b+3\} \to \mathbb{R}$ given by

$$G(t,s) = \begin{cases} v(t,s), & 0 \le s \le t-\nu+1 \le b+3 \\ u(t,s), & 0 \le t-\nu+2 \le s \le b+3 \end{cases}$$

where

$$u(t,s) = \frac{(b+\nu-s-1)^{\underline{\nu-4}}[(2\nu+b+s(\nu-4)-3)(b+4)-(b-s+3)(b+\nu+1)]}{\Gamma(\nu-1)(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} t^{\underline{\nu-2}}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-1}}$$

and

$$v(t,s)=\frac{(b+\nu-s-1)^{\underline{\nu-4}}[(2\nu+b+s(\nu-4)-3)(b+4)-(b-s+3)(b+\nu+1)]}{\Gamma(\nu-1)(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-2}}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-1}}+\frac{(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}$$

is the Green's function for the BVP (2.2).

Note that if we let $\nu=4$, then we get that the standard Green's function for the BVP

$$\begin{cases}\Delta^4 y=0\\ \Delta^i y(0)=0, \quad i=0,1\\ \Delta^j y(b+4)=0, \quad j=2,3\end{cases}$$

is given by

$$G(t,s)=\begin{cases}\frac{1}{6}[(t-s-1)^{\underline{3}}-t^{\underline{3}}+3(s+1)t^{\underline{2}}], & 0\le s\le t-3\le b+3\\ \\ \frac{-1}{6}t^{\underline{3}}+\frac{(s+1)}{2}t^{\underline{2}}, & 0\le t-2\le s\le b+3.\end{cases}$$

Theorem 9 *Let $G(t,s)$ be the Green's function associated with* (2.1). *Then $G(t,s)$ satisfies the following properties: for each fixed $s\in\{0,\cdots,b+3\}$,*

1. $\min\limits_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s) \geq 0.$

2. $\max\limits_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s) = G(b+\nu+3,s).$

3. $\min\limits_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} G(t,s) \geq \delta \max\limits_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s),$
   where $0<\delta<1$ *is a fixed constant independent of* s *and* $a = \left\lceil \dfrac{b-1}{2} \right\rceil.$

4. $\displaystyle\int_0^{b+3} G(t,s)\Delta s \leq \frac{\Gamma(b+\nu+4)}{(\nu-2)^{\underline{2}}\Gamma(\nu-1)(b+4)} + \frac{(b+\nu+3)^{\underline{2}}(-2b+\nu-11)}{(\nu-1)^{\underline{3}}}$
   $\displaystyle+\frac{\Gamma(b+\nu+4)}{\Gamma(\nu+1)\Gamma(b+4)} - 1.$

Proof:

- For $t \in \{\nu-4,\cdots,s+\nu-3\}$ and fixed $s\in\{0,\ldots,b+3\}$. Consider

$$\Delta u(t,s) = \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(\nu-2)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-3}}$$

$$-\frac{(b-s+3)(b+\nu+1)(b+\nu-s-1)^{\underline{\nu-4}}(\nu-2)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-3}}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(\nu-1)}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-2}} \geq 0$$

$$\Longleftrightarrow \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)}{\Gamma(\nu-2)(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-3}}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-3}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu-2)(\nu-2)(b+\nu)^{\underline{\nu-2}}(b+\nu+1)}t^{\underline{\nu-2}} \geq$$

$$\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)}{\Gamma(\nu-2)(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}t^{\underline{\nu-3}}$$

$$\iff (b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)[(\nu-2)(b+4)t^{\underline{\nu-3}}-(\nu-3)t^{\underline{\nu-2}}]$$

$$\geq (b+\nu+1)(b-s+3)(b+\nu-s-1)^{\underline{\nu-4}}(\nu-2)t^{\underline{\nu-3}}$$

$$\iff (b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)t^{\underline{\nu-3}}[(\nu-2)(b+4)-(\nu-3)(t-\nu+3)]$$

$$\geq (\nu-2)(b+\nu+1)(b-s+3)(b+\nu-s-1)^{\underline{\nu-4}}t^{\underline{\nu-3}}.$$

Since $(b+\nu-s-1)^{\underline{\nu-4}}$ and $t^{\underline{\nu-3}}$ are positive the above inequality is true $\iff$

$$[(2\nu+b+s(\nu-4)-3)(b+4)-(b-s+3)(b+\nu+1)](\nu-2)$$
$$-(2\nu+b+s(\nu-4)-3)(\nu-3)(t-\nu+3)\geq 0. \qquad (2.5)$$

Since $\max\{t\}=s+\nu-3$, we have that if
$[(2\nu+b+s(\nu-4)-3)(b+4)-(b-s+3)(b+\nu+1)](\nu-2)$
$-(2\nu+b+s(\nu-4)-3)(\nu-3)((s+\nu-3)-\nu+3)\geq 0$, then (2.5) is true.

Let $h(s)=[(2\nu+b+s(\nu-4)-3)(b+4)-(b-s+3)(b+\nu+1)](\nu-2)-(2\nu+$

$b+s(\nu-4)-3)s(\nu-3)$. Notice that

$$\begin{aligned}
\Delta h(s) &= (\nu-2)[(\nu-4)(b+4)+b+\nu+1] \\
&- [(\nu-4)(s+1)+2\nu+b+s\nu-4s-3](\nu-3) \\
&= (\nu-2)[\nu b+5\nu-3b-15]-[2\nu s+3\nu-8s-7+b](\nu-3) \\
&= \nu^2 b+5\nu^2-5b\nu-25\nu+6b+30-2\nu^2 s-3\nu^2+14s\nu+16\nu \\
&- \nu b-24s-21+3b \\
&= \nu^2 b+2\nu^2-6b\nu-9\nu+9b-2\nu^2 s+14s\nu-24s+9 \\
&= b(\nu-3)^2-2s(\nu-3)(\nu-4)+2\nu^2-9\nu+9 \\
&> 0.
\end{aligned}$$

Then $h(s)$ is an increasing function of s but

$$\begin{aligned}
h(0) &= [(2\nu+b-3)(b+4)-(b+3)(b+\nu+1)](\nu-2) \\
&= [2\nu b+8\nu+b^2+b-12-b^2-b\nu-4b-3\nu-3](\nu-2) \\
&= (b+5)(\nu-3)(\nu-2)>0.
\end{aligned}$$

Thus, $h(s)\geq 0$ and therefore $\Delta\ u(t,s)\geq 0$ on $[\nu-4,s+\nu-3]$ and $u(t,s)$ is increasing on $[\nu-4,s+\nu-2]$.

- For $t\in\{s+\nu-1,\cdots,b+\nu+2\}$.

  Note that $v(t,s)=u(t,s)+x(t,s)$ and $\Delta v(t,s)=\Delta u(t,s)+\Delta x(t,s)$. But $\Delta u(t,s)>0$ and

$$\begin{aligned}
\Delta x(t,s) &= \frac{(\nu-1)(t-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)} \geq \frac{(\nu-1)(s+\nu-1-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)} \\
&= \frac{(\nu-1)\Gamma(\nu-1)}{\Gamma(\nu)}>0.
\end{aligned}$$

Thus $\Delta v(t,s) > 0$ and $v(t,s)$ is increasing.

Thus, 1 and 2 follow easily.

Next, we prove 3. Let a be defined as in the statement of the theorem. Observe that from earlier in the proof, we know that

$$\min_{t \in \{a+\nu+1,\cdots,b+\nu+1\}} G(t,s) = G(a+\nu+1,s)$$

$$= \begin{cases} \dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(a+\nu+1)^{\underline{\nu-2}} \\ \quad -\dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(b-s+3)(b+\nu+1)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(a+\nu+1)^{\underline{\nu-2}} \\ \quad -\dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(a+\nu+1)^{\underline{\nu-1}} \\ \quad +\dfrac{(a+\nu-s)^{\underline{\nu-1}}}{\Gamma(\nu)}, \\ \qquad\qquad 0 \le s \le a+2 \\ \dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(a+\nu+1)^{\underline{\nu-2}} \\ \quad -\dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(b-s+3)(b+\nu+1)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(a+\nu+1)^{\underline{\nu-2}} \\ \quad -\dfrac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(a+\nu+1)^{\underline{\nu-1}}, \\ \qquad\qquad a+3 \le s \le b+3. \end{cases} \tag{2.6}$$

Moreover, we have

$$\max_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s) = G(b+\nu+3,s)$$

$$= \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(b+\nu+3)^{\underline{\nu-2}}$$

$$- \frac{(b+\nu-s-1)^{\underline{\nu-4}}(b-s+3)(b+\nu+1)}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(b+\nu+3)^{\underline{\nu-2}}$$

$$- \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}(b+\nu+3)^{\underline{\nu-1}} + \frac{(b+\nu+3-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}.$$

Now, let $P(s) = G(b+\nu+3, s)$. Our goal is to show that $P(s)$ is increasing, to do this we show that $\Delta P(s) \geq 0$. We now use the formula

$$\Delta(c-t)^{\underline{\nu}} = -\nu(c-\sigma(t))^{\underline{\nu-1}}.$$

$$\Delta P(s) = -\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-s-2)^{\underline{\nu-4}}[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-1)(b+\nu-s+1)^{\underline{\nu-2}}}{\Gamma(\nu)}$$

$$+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$-\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$= -\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-s-2)^{\underline{\nu-4}}[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$-\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}-\frac{(\nu-1)(b+\nu-s+1)^{\underline{\nu-2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)}$$

$$=-\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(b-s+3)}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{\nu-2}}}{(\nu-1)^{\underline{3}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$+\frac{(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$-\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.$$

Now,

$$-\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{\nu-2}}}{(\nu-1)^{\underline{3}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$+\frac{(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$-\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq 0$$

$$\iff$$

$$-(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}$$
$$+(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}$$
$$+(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}$$
$$-(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{\nu-2}}$$
$$+(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}$$
$$-(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+3)^{\underline{\nu-1}}$$
$$\geq 0$$

$$\iff -(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}$$
$$+(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}$$
$$+(\nu-1)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}$$
$$-(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{3}}(b+\nu-s-2)^{\underline{\nu-5}}$$

$+(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}$

$-(\nu-3)(\nu-4)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+3)^{\underline{\nu-1}} \geq 0.$

For $s \in \{0, \cdots, b+2\}$, $(b+\nu-s-2)^{\underline{\nu-5}} > 0$, we have that the above inequality is true

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)[(2\nu+b+s(\nu-4)-3)(b+4)-$

$(b+\nu+1)(b-s+3)](b+\nu+3)^{\underline{\nu-2}}$

$+(\nu-1)(b-s+3)[(\nu-4)(b+4)+(b+\nu+1)](b+\nu+3)^{\underline{\nu-2}}$

$-(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{3}}+(\nu-3)(\nu-4)(2\nu+b+s(\nu-$

$4)-3)(b+\nu+3)^{\underline{\nu-1}}-(\nu-3)(\nu-4)(b-s+3)(b+\nu+3)^{\underline{\nu-1}}$

≥ 0

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)[(2\nu+b+s(\nu-4)-3)(b+4)$

$-(b+\nu+1)(b-s+3)]\dfrac{(b+\nu+3)^{\underline{3}}}{(b+5)}(b+\nu)^{\underline{\nu-4}}$

$+(\nu-1)(b-s+3)(\nu-3)(b+5)\dfrac{(b+\nu+3)^{\underline{3}}}{(b+5)}(b+\nu)^{\underline{\nu-4}}$

$-(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{3}}$

$+(\nu-3)(\nu-4)[(2\nu+b+s(\nu-4)-3)-(b-s+3)](b+\nu+3)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}$

≥ 0

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)(\nu-3)(b+5)(s+1)\dfrac{(b+\nu+3)^{\underline{3}}}{(b+5)}(b+\nu)^{\underline{\nu-4}}$

$+(\nu-1)(\nu-3)(b+5)(b-s+3)\dfrac{(b+\nu+3)^{\underline{3}}}{(b+5)}(b+\nu)^{\underline{\nu-4}}$

$-(\nu-1)(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-s+1)^{\underline{3}}$

$+(\nu-3)(\nu-4)[(2\nu+b+s(\nu-4)-3)-(b-s+3)](b+\nu+3)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}$

≥ 0

$$\iff -(\nu-1)(\nu-4)(s+1)(b+\nu+3)^{\underline{2}} + (\nu-1)(b-s+3)(b+\nu+3)^{\underline{2}}$$

$$-(\nu-1)(b+\nu-s+1)^{\underline{3}} + (\nu-4)[(2\nu+b+s(\nu-4)-3)-(b-s+3)](b+\nu+3)^{\underline{2}} \geq 0$$

$$\iff$$

$$-(\nu-1)(\nu-4)(s+1)(b+\nu+3)^{\underline{2}} + (\nu-1)(b-s+3)(b+\nu+3)^{\underline{2}}$$

$$-(\nu-1)(b+\nu-s+1)^{\underline{3}} + (\nu-4)(\nu-3)(s+2)(b+\nu+3)^{\underline{2}}$$

$$\geq 0$$

$$\iff$$

$$-(\nu-1)(b+\nu+3)^{\underline{2}}[(\nu-4)(s+1)-(b-s+3)]$$

$$-(\nu-1)(b+\nu-s+1)^{\underline{3}} + (\nu-4)(s+2)(\nu-3)(b+\nu+3)^{\underline{2}} \geq 0$$

$$\iff$$

$$(\nu-4)(b+\nu+3)^{\underline{2}}[(\nu-3)(s+2)-(\nu-1)(s+1)] + (\nu-1)[(b-s+3)(b+\nu+3)^{\underline{2}} - (b+\nu-s+1)^{\underline{3}}] \geq 0$$

$$\iff$$

$$(\nu-4)(b+\nu+3)^{\underline{2}}(\nu-2s-5) + (\nu-1)[(b-s+3)(b+\nu+3)^{\underline{2}} - (b+\nu-s+1)^{\underline{3}}] \geq 0,$$

which is certainly true since

$$(b+\nu+3)(b+\nu+2)(b-s+3) > (b+\nu-s+1)(b+\nu-s)(b+\nu-s-1).$$

Therefore, $P(s)$ is increasing as desired. Which follows that $P(0) < P(s) < P(b+3)$, for all $s \in \{0, \cdots, b+3\}$. So, finding conditions on $0 < \delta < 1$ such that for all $s \in \{0, \cdots, b+3\}$,

$$\min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} G(t,s) \geq \delta \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s)$$

is equivalent to finding conditions on δ such that

$$G(a+\nu+1,s)\geq \delta G(b+\nu+3,s). \tag{2.7}$$

To find such a δ, we consider the following two cases:

Case 1 $s\in\{0,\cdots,a+2\}$: In this case (2.7) becomes,

$$\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(a+\nu-s)^{\underline{\nu-1}}}{\Gamma(\nu)}-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq \delta(\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-s+2)^{\underline{\nu-1}}}{\Gamma(\nu)}-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}).$$

Let

$$Q(s):=\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(a+\nu-s)^{\underline{\nu-1}}}{\Gamma(\nu)}-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.$$

Next we show that $Q(s)$ is increasing function of s on $[0, b+3]$.

$$
\begin{aligned}
&\Delta Q(s)\\
&= -\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&-\frac{(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}-\frac{(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}\geq 0
\end{aligned}
$$

$$
\begin{aligned}
&\iff\\
&-\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}\\
&+\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\\
&+\frac{(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}
\end{aligned}
$$

$$-\frac{(\nu-3)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(a+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}\geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}\geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(\nu-4)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b+\nu)^{\underline{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}\geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(2\nu+b+s(\nu-4)-3)(b+4)$$
$$+(\nu-1)(b-s+3)(\nu-4)(b+4)+(\nu-1)(\nu-4)(b-s+3)(b+\nu+1)$$
$$+(\nu-1)(b-s+3)(b+\nu+1)$$
$$+(\nu-3)(\nu-4)(2\nu+b+s(\nu-4)-3)(a+3)$$

$-(\nu-3)(b-s+3)(\nu-4)(a+3)$

$$-\frac{(\nu-3)(b+\nu)\underline{^{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)\underline{^{\nu-2}}}{(b+\nu-s-2)\underline{^{\nu-5}}(a+\nu+1)\underline{^{\nu-2}}}\geq 0$$

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)(2\nu+b+s(\nu-4)-3)(b+4)$

$+(\nu-1)(b-s+3)(\nu-4)(b+4)+(\nu-1)(\nu-4)(b-s+3)(b+\nu+1)$

$+(\nu-1)(b-s+3)(b+\nu+1)$

$+(\nu-3)(\nu-4)(2\nu+b+s(\nu-4)-3)(a+3)-(\nu-3)(b-s+3)(\nu-4)(a+3)$

$$-\frac{(\nu-3)(b+\nu)\underline{^{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)\underline{^{\nu-2}}}{(b+\nu-s-2)\underline{^{\nu-5}}(a+\nu+1)\underline{^{\nu-2}}}\geq 0$$

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)(\nu-3)(b+5)(s+1)+(b-s+3)(b+5)(\nu-3)(\nu-1)$

$+(\nu-4)(\nu-3)(2\nu+b+s(\nu-4)-3)(a+3)$

$-(b-s+3)(\nu-4)(a+3)(\nu-3)$

$$-\frac{(\nu-3)(b+\nu)\underline{^{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-2)\underline{^{\nu-2}}}{(b+\nu-s-1)\underline{^{\nu-5}}(a+\nu+1)\underline{^{\nu-2}}}\geq 0$$

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)(b+5)(s+1)+(b-s+3)(b+5)(\nu-1)+(\nu-4)(2\nu+b+s(\nu-$

$4)-3)(a+3)-(b-s+3)(\nu-4)(a+3)$

$$-\frac{(b+\nu)\underline{^{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)\underline{^{\nu-2}}}{(b+\nu-s-2)\underline{^{\nu-5}}(a+\nu+1)\underline{^{\nu-2}}}\geq 0$$

$\Longleftrightarrow$

$(\nu-4)[((2\nu+b+s(\nu-4)-3)-(b-s+3))(a+3)-(\nu-1)(b+5)(s+1)]+$

$(b-s+3)(b+5)(\nu-1)-\frac{(b+\nu)\underline{^{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)\underline{^{\nu-2}}}{(b+\nu-s-2)\underline{^{\nu-5}}(a+\nu+1)\underline{^{\nu-2}}}\geq 0$

$\Longleftrightarrow$

$(\nu-4)[(s+2)(\nu-3)(a+3)-(\nu-1)(b+5)(s+1)]+(b-s+3)(b+5)(\nu-$

$$1)-\frac{(b+\nu)^{\underline{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}}{(b+\nu-s-2)^{\underline{\nu-5}}(a+\nu+1)^{\underline{\nu-2}}}\geq 0$$

$\Longleftrightarrow$

$$(\nu-4)[s\nu a-3sa+2a\nu-6a+3s\nu-9s+6\nu-18-5s\nu-sb\nu-\nu b-5\nu+$$
$$5s+sb+b+5]+(b-s+3)(b+5)(\nu-1)$$
$$-\frac{(b+\nu)^{\underline{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}}{(b+\nu-s-2)^{\underline{\nu-5}}(a+\nu+1)^{\underline{\nu-2}}}\geq 0$$

$\Longleftrightarrow$

$$(\nu-4)[s\nu a-sb\nu-3sa+sb-2s\nu-4s-13+\nu-6a+b+2a\nu-\nu b]+(b-s+$$
$$3)(b+5)(\nu-1)$$
$$-\frac{(b+\nu)^{\underline{\nu-4}}(b+\nu+1)(\nu-1)(a+\nu-s-1)^{\underline{\nu-2}}}{(b+\nu-s-2)^{\underline{\nu-5}}(a+\nu+1)^{\underline{\nu-2}}}\geq 0,$$

which is certainly true. Therefore $Q(s)$ is increasing for $s\in\{0,\cdots,b+3\}$.

Now, our goal is to show that δ in (2.7) is less than 1. Since $P(s)$ is increasing and positive and $Q(s)$ is increasing we have that if we choose δ so that $Q(0)\geq\delta P(a+2)$ it follows that $Q(s)\geq\delta P(s)$ for $s\in\{0,\cdots,a+2\}$. Now we show that we can choose $0<\delta<1$ so that $Q(0)\geq\delta P(a+2)$,

$$\frac{(b+\nu-1)^{\underline{\nu-4}}[(2\nu+b-3)(b+4)-(b+\nu+1)(b+3)](a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(a+\nu)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}}-\frac{(b+\nu-1)^{\underline{\nu-4}}(2\nu+b-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq\delta(\frac{(b+\nu-a-3)^{\underline{\nu-4}}(2\nu+b+(a+2)(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-a-3)^{\underline{\nu-4}}(b+\nu+1)(b-a+1)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-a)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}}-\frac{(b+\nu-a-3)^{\underline{\nu-4}}(2\nu+b+(a+2)(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}})$$

$$\Longleftrightarrow$$

$$\frac{(b+\nu-1)^{\underline{\nu-4}}(b+5)(\nu-3)(\nu-1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}\frac{(a+3)^{\underline{2}}}{(a+\nu+1)}}{\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(\nu-1)^{\underline{3}}}$$

$$-\frac{(\nu-3)(a+3)(b+\nu-1)^{\underline{\nu-4}}(2\nu+b-3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq \delta(\frac{(b+\nu-a-3)^{\underline{\nu-4}}(\nu-3)(\nu-1)(b+5)(a+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-3)(b+\nu-a)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-a-3)^{\underline{\nu-4}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$-\frac{(\nu-3)(b+\nu-a-3)^{\underline{\nu-4}}([a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}})$$

$$\Longleftrightarrow$$

$$(b+\nu-1)^{\underline{\nu-4}}(b+5)(\nu-3)(\nu-1)(a+\nu+1)^{\underline{\nu-2}}+$$

$$(\nu-3)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}\frac{(a+3)^{\underline{2}}}{(a+\nu+1)}(a+\nu+1)^{\underline{\nu-2}}$$

$$-(\nu-3)(a+3)(b+\nu-1)^{\underline{\nu-4}}(2\nu+b-3)(a+\nu+1)^{\underline{\nu-2}}$$

$$\geq \delta((b+\nu-a-3)^{\underline{\nu-4}}(\nu-3)(\nu-1)(b+5)(a+3)(b+\nu+3)^{\underline{\nu-2}}$$

$$+(\nu-3)(b+\nu-a)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}(b+\nu-a-3)^{\underline{\nu-4}}$$

$$-(\nu-3)(b+\nu-a-3)^{\underline{\nu-4}}[a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{\nu-1}})$$

$$\Longleftrightarrow$$

$$(b+\nu-1)^{\underline{\nu-4}}(b+5)(\nu-3)(\nu-1)(a+\nu+1)^{\underline{\nu-2}}+$$

$$(\nu-3)(b+\nu+1)\frac{(a+3)^{\underline{2}}(b+\nu)}{(b+4)(a+\nu+1)}(b+\nu-1)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}$$

$$-(\nu-3)(a+3)(b+\nu-1)^{\underline{\nu-4}}(2\nu+b-3)(a+\nu+1)^{\underline{\nu-2}}$$

$$\geq \delta((b+\nu-a-3)^{\underline{\nu-4}}(\nu-3)(\nu-1)(b+5)(a+3)\frac{(b+\nu+3)^{\underline{4}}}{(b+5)^{\underline{2}}}(b+\nu-1)^{\underline{\nu-4}}$$

$$+(\nu-3)\frac{(b+\nu-a)^{\underline{3}}(b+\nu)}{(b+4)}(b+\nu+1)(b+\nu-1)^{\underline{\nu-4}}(b+\nu-a-3)^{\underline{\nu-4}}$$

$$-(\nu-3)(b+\nu-a-3)^{\underline{\nu-4}}[a(\nu-4)+4\nu+b-11]\frac{(b+\nu+3)^{\underline{4}}}{(b+4)}(b+\nu-1)^{\underline{\nu-4}})$$

$$\Longleftrightarrow$$

$$(a+\nu+1)^{\underline{\nu-2}}[(b+5)(\nu-1)+(b+\nu+1)\frac{(a+3)^{\underline{2}}(b+\nu)}{(b+4)(a+\nu+1)}-(a+3)(2\nu+b-3)]$$

$$\geq \delta((b+\nu-a-3)^{\underline{\nu-4}}[(\nu-1)(b+5)(a+3)\frac{(b+\nu+3)^{\underline{4}}}{(b+5)^{\underline{2}}}$$

$$+\frac{(b+\nu-a)^{\underline{3}}(b+\nu)}{(b+4)}(b+\nu+1)\ -[a(\nu-4)+4\nu+b-11]\frac{(b+\nu+3)^{\underline{4}}}{(b+4)}])$$

$$\Longleftrightarrow$$

$$(a+\nu+1)^{\underline{\nu-2}}[(b+5)^{\underline{2}}(\nu-1)+(b+\nu+1)\frac{(a+3)^{\underline{2}}(b+\nu)}{(a+\nu+1)}-(a+3)(b+4)(2\nu+b-3)]$$

$$\geq\ \delta((b+\nu-a-3)^{\underline{\nu-4}}[(\nu-1)(a+3)(b+\nu+3)^{\underline{4}}+(b+\nu-a)^{\underline{3}}(b+\nu)(b+\nu+1)-[a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{4}}])$$

$$\Longleftrightarrow$$

$$(a+\nu+1)^{\underline{\nu-2}}[(b+5)^{\underline{2}}(\nu-1)(a+\nu+1)+(b+\nu+1)(a+3)^{\underline{2}}(b+\nu)$$

$$-(a+3)(b+4)(a+\nu+1)(2\nu+b-3)]$$

$\geq \delta((b+\nu-a-3)^{\underline{\nu-4}}(a+\nu+1)[(\nu-1)(a+3)(b+\nu+3)^{\underline{4}}+(b+\nu-a)^{\underline{3}}(b+\nu)(b+\nu+1)-[a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{4}}])$.

Consider the LHS

$$
\begin{aligned}
& (a+\nu+1)^{\underline{\nu-2}}[(b+5)^{\underline{2}}(\nu-1)(a+\nu+1)+(b+\nu+1)(a+3)^{\underline{2}}(b+\nu) \\
- & (a+3)(b+4)(a+\nu+1)(2\nu+b-3)].
\end{aligned}
$$

Since the expression

$(b+5)^{\underline{2}}(\nu-1)(a+\nu+1)+(b+\nu+1)(a+3)^{\underline{2}}(b+\nu)-(a+3)(b+4)(a+\nu+1)(2\nu+b-3)$

is increasing in ν for $4 \geq \nu \geq 3$ we get that

$$
\begin{aligned}
& (a+\nu+1)^{\underline{\nu-2}}[(b+5)^{\underline{2}}(\nu-1)(a+\nu+1)+(b+\nu+1)(a+3)^{\underline{2}}(b+\nu) \\
- & (a+3)(b+4)(a+\nu+1)(2\nu+b-3)] \\
\geq & (a+4)[2(b+5)^{\underline{2}}(a+4)+(b+4)^{\underline{2}}(a+3)^{\underline{2}} \\
- & (a+3)(a+4)(b+4)^{\underline{2}}] \\
= & (a+4)(b+4)[2(b+5)(a+4)+(b+3)(a+3)^{\underline{2}} \\
- & (a+3)(a+4)(b+3)] \\
= & (a+4)(b+4)[2ab+8b+10a+40-2ab-6b-6a-18] \\
= & (a+4)(b+4)[2b+4a+22].
\end{aligned}
$$

Next, consider

$(b+\nu-a-3)^{\underline{\nu-4}}(a+\nu+1)[(\nu-1)(a+3)(b+\nu+3)^{\underline{4}}+(b+\nu-a)^{\underline{3}}(b+\nu)(b+\nu+1)$
$-\ [a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{4}}]$. Since the expression

$(\nu-1)(a+3)(b+\nu+3)^{\underline{4}}+(b+\nu-a)^{\underline{3}}(b+\nu)(b+\nu+1)$- $[a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{4}}$

is increasing function in ν for $4 \geq \nu \geq 3$ we get that

$(b+\nu-a-3)^{\underline{\nu-4}}(a+\nu+1)[(\nu-1)(a+3)(b+\nu+3)^{\underline{4}}+(b+\nu-a)^{\underline{3}}(b+\nu)(b+\nu+1)- \ [a(\nu-4)+4\nu+b-11](b+\nu+3)^{\underline{4}}]$

$\leq (a+5)[3(a+3)(b+7)^{\underline{4}}+(b-a+4)^{\underline{3}}(b+5)^{\underline{2}}-(b+5)(b+7)^{\underline{4}}]$

$= (a+5)(b+5)^{\underline{2}}[3(a+3)(b+7)^{\underline{2}}+(b-a+4)^{\underline{3}}-(b+7)^{\underline{3}}]$

$= (a+5)(b+5)^{\underline{2}}[(b+7)^{\underline{2}}(3a-b+4)+(b-a+4)^{\underline{3}}]$.

Thus

$$\delta = \frac{Q(0)}{P(a+2)} \geq \frac{(a+4)(b+4)[2b+4a+22]}{(a+5)(b+5)^{\underline{2}}[(b+7)^{\underline{2}}(3a-b+4)+(b-a+4)^{\underline{3}}]} = \delta^*.$$

Now, by our choice of a, we have

$(a+4)(b+4)[2b+4a+22] < (a+5)(b+5)^{\underline{2}}[(b+7)^{\underline{2}}(3a-b+4)+(b-a+4)^{\underline{3}}]$.

Therefore, the condition on $0<\delta^*<1$ is

$$0 < \delta^* \leq \frac{(a+4)(b+4)[2b+4a+22]}{(a+5)(b+5)^{\underline{2}}[(b+7)^{\underline{2}}(3a-b+4)+(b-a+4)^{\underline{3}}]}.$$

will satisfy

$$Q(s) \geq \delta^* P(s), \quad \text{for } s \in \{0, \cdots, a+2\}.$$

Case 2 $s \in \{a+3, \cdots, b+3\}$: In this case (2.7) becomes,

$$\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq \hat{\delta}(\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-s+2)^{\underline{\nu-1}}}{\Gamma(\nu)}-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}).$$

We want to find $\hat{\delta}$ so that

$$R(s) \geq \hat{\delta}P(s) \quad for \quad s \in \{a+3, \cdots, b+3\}$$

where $P(s)$ is as previously defined and $R(s)$ is given by

$$R(s) = \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.$$

Since $P(s)$ is increasing and positive we have that if we choose $\hat{\delta}$ so that $R(s) \geq \hat{\delta}P(b+3)$ it follows that $R(s) \geq \hat{\delta}P(s)$ for $s \in \{a+3, \cdots, b+3\}$. We proceed to use an argument similar to the previous case. So

$$\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq \hat{\delta}(\frac{(b+\nu-(b+3)-1)^{\underline{\nu-4}}(2\nu+b+(b+3)(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-(b+3)-1)^{\underline{\nu-4}}(b+\nu+1)(b-(b+3)+3)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-(b+3)+2)^{\underline{\nu-1}}}{\Gamma(\nu)}$$

$$-\frac{(b+\nu-(b+3)-1)^{\underline{\nu-4}}(2\nu+b+(b+3)(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}})$$

$$\iff$$

$$\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq \hat{\delta}(\frac{(\nu-4)^{\underline{\nu-4}}(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)^{\underline{\nu-1}}}{\Gamma(\nu)}-\frac{(\nu-4)^{\underline{\nu-4}}(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}})$$

$\Longleftrightarrow$

$$
\begin{aligned}
& \frac{(\nu-1)(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
- & \frac{(\nu-1)(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
- & \frac{(\nu-3)(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
\geq & \hat{\delta}(\frac{(\nu-1)(\nu-4)^{\underline{\nu-4}}(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
+ & 1-\frac{(\nu-3)(\nu-4)^{\underline{\nu-4}}(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}})
\end{aligned}
\tag{2.8}
$$

$\Longleftrightarrow$

$$
\begin{aligned}
& \frac{(\nu-1)(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
& -\frac{(\nu-1)(b+\nu-s-1)^{\underline{\nu-4}}(b+\nu+1)(b-s+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
& -\frac{(\nu-3)(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
& \geq \hat{\delta}(\frac{(\nu-1)\Gamma(\nu-3)(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
& -\frac{(\nu-3)\Gamma(\nu-3)(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
& +\frac{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}).
\end{aligned}
$$

Next we show that $R(s)$ is increasing on $[a+3,\cdots,b+3]$. Consider

$$\Delta R(s) =$$

$$-\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \geq 0$$

$$\Longleftrightarrow$$

$$-\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$+\frac{(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$+\frac{(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-3)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(a+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b+\nu-s-2)^{\underline{\nu-4}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b+\nu-s-2)^{\underline{\nu-4}}(\nu-4)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$\geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(\nu-4)(b+4)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(b-s+3)(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-1)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(b+\nu+1)(a+\nu+1)^{\underline{\nu-2}}$$
$$+(\nu-3)(\nu-4)(b+\nu-s-2)^{\underline{\nu-5}}(2\nu+b+s(\nu-4)-3)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b-s+3)(b+\nu-s-2)^{\underline{\nu-5}}(\nu-4)(a+3)(a+\nu+1)^{\underline{\nu-2}}$$
$$\geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(2\nu+b+s(\nu-4)-3)(b+4)$$
$$+(\nu-1)(b-s+3)(\nu-4)(b+4)+(\nu-1)(\nu-4)(b-s+3)(b+\nu+1)$$
$$+(\nu-1)(b-s+3)(b+\nu+1)$$
$$+(\nu-3)(\nu-4)(2\nu+b+s(\nu-4)-3)(a+3)$$
$$-(\nu-3)(b-s+3)(\nu-4)(a+3)\geq 0$$

$\iff$

$$-(\nu-1)(\nu-4)(\nu-3)(b+5)(s+1)+(b-s+3)(b+5)(\nu-3)(\nu-1)$$
$$+(\nu-4)(\nu-3)(2\nu+b+s(\nu-4)-3)(a+3)$$

$-(b-s+3)(\nu-4)(a+3)(\nu-3)$

≥ 0

$\Longleftrightarrow$

$-(\nu-1)(\nu-4)(b+5)(s+1)+(b-s+3)(b+5)(\nu-1)+(\nu-4)(2\nu+b+s(\nu-4)-3)(a+3)-(b-s+3)(\nu-4)(a+3)\geq 0$

$\Longleftrightarrow$

$(\nu-4)[(s+2)(\nu-3)(a+3)-(\nu-1)(b+5)(s+1)]+(b-s+3)(b+5)(\nu-1)\geq 0$

$\Longleftrightarrow$

$(\nu-4)[s\nu a-sb\nu-3sa+sb-2s\nu-4s-13+\nu-6a+b+2a\nu-\nu b]+(b-s+3)(b+5)(\nu-1)\geq 0,$

which is certainly true since both terms are nonnegative. Therefore $R(s)$ is increasing for $s\in\{a+3,\cdots,b+3\}$. Since $R(s)$ is an increasing we have that if we choose $\hat{\delta}$ so that $R(a+3)\geq\hat{\delta}P(b+3)$ it follows that $R(s)\geq\hat{\delta}P(s)$ for $s\in\{a+3,\cdots,b+3\}$. Now we show that we can choose $0<\hat{\delta}<1$ so that $R(a+3)\geq\hat{\delta}P(b+3)$. Since $R(a+3)\geq\hat{\delta}P(b+3)$ we have from (2.8) with replacing s by $a+3$ that

$$\frac{(\nu-1)(b+\nu-a-4)^{\underline{\nu-4}}(2\nu+b+(a+3)(\nu-4)-3)(b+4)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-1)(b+\nu-a-4)^{\underline{\nu-4}}(b+\nu+1)(b-(a+3)+3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-3)(b+\nu-a-4)^{\underline{\nu-4}}(2\nu+b+(a+3)(\nu-4)-3)(a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq\hat{\delta}(\frac{(\nu-1)\Gamma(\nu-3)(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-3)\Gamma(\nu-3)(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$+\frac{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}).$$

Combining the first two terms and simplifying the first term on the left hand side we get that the above inequality is true

$\iff$

$$\frac{(\nu-1)(b+\nu-a-4)^{\underline{\nu-4}}(a+4)(b+5)(\nu-3)(a+\nu+1)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-3)(b+\nu-a-4)^{\underline{\nu-4}}[5(\nu-3)+a(\nu-4)+b](a+\nu+1)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$\geq \hat{\delta}(\frac{(\nu-1)\Gamma(\nu-3)(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)}$$

$$-\frac{(\nu-3)(\nu-4)^{\underline{\nu-4}}(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}$$

$$+\frac{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}{\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}})$$

$\iff$

$$(\nu-1)(b+\nu-a-4)^{\underline{\nu-4}}(a+4)(b+5)(\nu-3)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(\nu-3)(b+\nu-a-4)^{\underline{\nu-4}}[5(\nu-3)+a(\nu-4)+b](a+\nu+1)^{\underline{\nu-1}}$$
$$\geq \hat{\delta}((\nu-1)\Gamma(\nu-3)(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}$$
$$-\ \Gamma(\nu-3)(\nu-3)(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}}+\Gamma(\nu-2)(\nu-1)^{\underline{3}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}})$$

$\iff$

$$(\nu-1)(b+\nu-a-4)^{\underline{\nu-4}}(a+4)(b+5)(a+\nu+1)^{\underline{\nu-2}}$$
$$-(b+\nu-a-4)^{\underline{\nu-4}}[5(\nu-3)+a(\nu-4)+b](a+\nu+1)^{\underline{\nu-1}}$$

$\geq \hat{\delta}((\nu-1)\Gamma(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}} - \Gamma(\nu-3)(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}} + \Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}})$

$\Longleftrightarrow$

$(b+\nu-a-4)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}[(a+4)(\nu-1)(b+5)-(a+3)[5(\nu-3)+a(\nu-4)+b]$
$\geq \hat{\delta}((\nu-1)\Gamma(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}} - \Gamma(\nu-3)(\nu-3)(b+5)(b+\nu+3)^{\underline{\nu-1}} + \Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}})$

$\Longleftrightarrow$

$(b+\nu-a-4)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}[(a+4)(\nu-1)(b+5)-(a+3)[5(\nu-3)+a(\nu-4)+b]$
$\geq \hat{\delta}((\nu-1)\Gamma(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}} - \Gamma(\nu-3)(\nu-3)(b+5)^2(b+\nu+3)^{\underline{\nu-2}} + \Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}})$

$\Longleftrightarrow$

$(b+\nu-a-4)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}[(a+4)(\nu-1)(b+5)-(a+3)[5(\nu-3)+a(\nu-4)+b]$

$\geq \hat{\delta}((b+\nu+3)^{\underline{\nu-2}}[(\nu-1)\Gamma(\nu-3)(b+5)^{\underline{2}} - \Gamma(\nu-3)(\nu-3)(b+5)^2] + \Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}})$.

Consider the LHS

$(b+\nu-a-4)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}[(a+4)(\nu-1)(b+5)-(a+3)(5(\nu-3)+a(\nu-4)+b)]$.

Since the expression $(a+4)(\nu-1)(b+5)-(a+3)(5(\nu-3)+a(\nu-4)+b)$ is increasing in ν and $3 < \nu \leq 4$ we get that

$(b+\nu-a-4)^{\underline{\nu-4}}(a+\nu+1)^{\underline{\nu-2}}[(a+4)(\nu-1)(b+5)-(a+3)(5(\nu-3)+a(\nu-4)+b)]$
$\geq (b-a-1)^{\underline{-1}}(a+4)[2(a+4)(b+5)-(a+3)(b-a)]$
$= (b-a-1)^{\underline{-1}}(a+4)[ab+13a+a^2+5b+40]$.

Next consider

$(b+\nu+3)^{\underline{\nu-2}}\Gamma(\nu-3)[(\nu-1)(b+5)^{\underline{2}} - (\nu-3)(b+5)^2] + \Gamma(\nu-2)(\nu-1)^{\underline{2}}(b+\nu+1)(b+\nu)^{\underline{\nu-4}}$

$\leq (b+7)^{2}(b+5)3(b+4)+6(b+5)$

$= 3(b+5)[(b+7)^{2}(b+4)+2].$

$$\hat{\delta} = \frac{R(a+3)}{P(b+3)} \geq \frac{(b-a-1)^{-1}(a+4)[ab+13a+a^2+5b+40]}{3(b+5)[(b+7)^{2}(b+4)+2]} = \bar{\delta}.$$

Now by our choice of a, we get

$(b-a-1)^{-1}(a+4)[ab+13a+a^2+5b+40] < 3(b+5)[(b+7)^{2}(b+4)+2.$

Therefore, the condition on $0 < \bar{\delta} < 1$ is

$$0 < \bar{\delta} \leq \frac{(b-a-1)^{-1}(a+4)[ab+13a+a^2+5b+40]}{3(b+5)[(b+7)^{2}(b+4)+2]}$$

will satisfy

$$R(s) \geq \bar{\delta}P(s), \quad for \quad s \in \{a+3, \cdots, b+3\}.$$

Thus, combining both cases above ,we shown that any choice of $0 < \delta < 1$ satisfying

$$\delta \leq \min\{M, N\}$$

where

$$M := \frac{(a+4)[ab+13a+a^2+5b+40]}{3(b-a)(b+5)[(b+7)^{2}(b+4)+2]}$$

$$N := \frac{(a+4)(b+4)[2b+4a+22]}{(a+5)(b+5)^{2}[(b+7)^{2}(3a-b+4)+(b-a+4)^{3}}$$

the Green's function $G(t,s)$ will satisfy

$$\min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} G(t,s) \geq \delta \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s).$$

Next we prove 4. Consider

$$\begin{aligned}
\int_0^{b+3} G(t,s)\Delta s &= \int_0^{t-\nu+2} v(t,s)\Delta s + \int_{t-\nu+2}^{b+3} u(t,s)\Delta s \\
&= \int_0^{t-\nu+2} [u(t,s)+x(t,s)]\Delta s + \int_{t-\nu+2}^{b+3} u(t,s)\Delta s \\
&= \int_0^{b+3} u(t,s)\Delta s + \int_0^{t-\nu+2} x(t,s)\Delta s \\
&= \int_0^{b+3} u(t,s)\Delta s + \int_0^{t-\nu} x(t,s)\Delta s.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^{t-\nu} x(t,s)\Delta s &= \int_0^{t-\nu} \frac{(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}\Delta s \\
&= -\frac{(t-s)^{\underline{\nu}}}{\nu\Gamma(\nu)}\Big|_{s=0}^{s=t-\nu} \\
&= -\frac{\nu^{\underline{\nu}}}{\Gamma(\nu+1)} + \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} = \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} - 1.
\end{aligned}$$

Also

$$\begin{aligned}
\int_0^{b+3} u(t,s)\Delta s &= \beta_1(t)\int_0^{b+3} (\nu-3)(b+5)(s+1)(b+\nu-s-1)^{\underline{\nu-4}}\Delta s \\
&+ \beta_2(t)\int_0^{b+3} (2\nu+b+s(\nu-4)-3)(b+\nu-s-1)^{\underline{\nu-4}}\Delta s,
\end{aligned}$$

where

$$\begin{aligned}\beta_1(t) &= \frac{t^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}\\ \beta_2(t) &= \frac{t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.\end{aligned}$$

Note that

$$\begin{aligned}&\int_0^{b+3}(b+\nu-s-1)^{\underline{\nu-4}}[(\nu-3)(b+5)(s+1))]\Delta s\\ =\;& -[(s+1)(b+5)](b+\nu-s)^{\underline{\nu-3}}|_{s=0}^{s=b+3} - \frac{(b+5)(b+\nu-s)^{\underline{\nu-2}}}{(\nu-2)}|_{s=0}^{s=b+3}\\ =\;& -(b+5)[(b+4)(\nu-3)^{\underline{\nu-3}} - (b+\nu)^{\underline{\nu-3}} + \frac{(\nu-3)^{\underline{\nu-2}}}{(\nu-2)} - \frac{(b+\nu)^{\underline{\nu-2}}}{(\nu-2)}]\\ =\;& -(b+5)[(b+4)\Gamma(\nu-2) - (b+\nu)^{\underline{\nu-3}} - \frac{(b+3)(b+\nu)^{\underline{\nu-3}}}{(\nu-2)}]\\ =\;& -(b+5)[(b+4)\Gamma(\nu-2) - \frac{(b+\nu)^{\underline{\nu-3}}(b+\nu+1)}{(\nu-2)}]\\ =\;& (b+5)[\frac{(b+\nu)^{\underline{\nu-3}}(b+\nu+1)}{(\nu-2)} - (b+4)\Gamma(\nu-2)].\end{aligned}$$

Next we look at

$$\begin{aligned}
&\int_0^{b+3}(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)\Delta s \\
= & -\frac{(2\nu+b+s(\nu-4)-3)(b+\nu-s)^{\underline{\nu-3}}}{(\nu-3)}\Big|_{s=0}^{s=b+3}+\int_0^{b+3}\frac{(\nu-4)(b+\nu-s-1)^{\underline{\nu-3}}}{(\nu-3)}\Delta s \\
= & -\frac{1}{(\nu-3)}[\Gamma(\nu-2)(b+5)(\nu-3)-(b+2\nu-3)(b+\nu)^{\underline{\nu-3}} \\
+ & \frac{(\nu-4)(\nu-3)^{\underline{\nu-2}}}{(\nu-2)}-\frac{(\nu-4)(b+\nu)^{\underline{\nu-2}}}{(\nu-2)}] \\
= & -\frac{1}{(\nu-3)}[\Gamma(\nu-2)(\nu-3)(b+5)-(b+2\nu-3)(b+\nu)^{\underline{\nu-3}} \\
- & \frac{(\nu-4)(b+3)(b+\nu)^{\underline{\nu-3}}}{(\nu-2)}] \\
= & -\Gamma(\nu-2)(b+5)+\frac{2(b+\nu)^{\underline{\nu-3}}(\nu b+\nu^2-2\nu-3b-3)}{(\nu-2)^{\underline{2}}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^{b+3}u(t,s)\Delta s &= \beta_1(t)[-(b+5)\Gamma(\nu-2)(b+4)+\frac{(b+5)(b+\nu+1)(b+\nu)^{\underline{\nu-3}}}{(\nu-2)}] \\
&+ \beta_2(t)[-\Gamma(\nu-2)(b+5)+\frac{2(\nu b+\nu^2-2\nu-3b-3)(b+\nu)^{\underline{\nu-3}}}{(\nu-2)^{\underline{2}}}].
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^{b+3} G(t,s)\Delta s &= \frac{t^{\underline{\nu-2}}\left[\frac{(b+5)(b+\nu+1)(b+\nu)^{\underline{\nu-3}}}{(\nu-2)} - (b+5)^{\underline{2}}\Gamma(\nu-2)\right]}{(\nu-2)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&- \frac{t^{\underline{\nu-1}}\left[\frac{2(b+\nu)^{\underline{\nu-3}}(\nu b+\nu^2-2\nu-3b-3)}{(\nu-2)^{\underline{2}}} - \Gamma(\nu-2)(b+5)\right]}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&+ \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} - 1 \\
&= \frac{t^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-3}}(b+5)(b+\nu+1)(\nu-1)}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} + \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} \\
&- \frac{t^{\underline{\nu-2}}2(\nu b+\nu^2-2\nu-3b-3)(t-\nu+2)(b+\nu)^{\underline{\nu-3}}}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} - 1 \\
&+ \frac{t^{\underline{\nu-2}}\Gamma(\nu-1)(b+5)[(t-\nu+2)(\nu-3)-(\nu-1)(b+4)]}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.
\end{aligned}$$

Since the third term is 0 for $t = \{\nu-4, \nu-3, \nu-2\}$ and it is less than zero for $\nu - 1 \leq t \leq b+\nu+3$ we get that

$$\begin{aligned}
\int_0^{b+3} G(t,s)\Delta s &= \frac{t^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-3}}(b+5)(b+\nu+1)(\nu-1)}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} + \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} - 1 \\
&+ \frac{t^{\underline{\nu-2}}\Gamma(\nu-1)(b+5)[(t-\nu+2)(\nu-3)-(\nu-1)(b+4)]}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_0^{b+3} G(t,s)\Delta s &\leq \frac{(b+5)(b+\nu+3)^{\underline{\nu-2}}(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-3}}}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&+ \frac{(b+5)(b+\nu+3)^{\underline{\nu-2}}(-2b+\nu-11)\Gamma(\nu-1)}{(\nu-1)^{\underline{3}}\Gamma(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&+ \frac{\Gamma(b+\nu+4)}{\Gamma(\nu+1)\Gamma(b+4)} - 1 \\
&= \frac{\Gamma(b+\nu+4)[\frac{\Gamma(b+\nu+2)(\nu-1)}{\Gamma(b+4)} + \Gamma(\nu-1)(-2b+\nu-11)]}{\Gamma(\nu-1)(\nu-1)^{\underline{3}}\Gamma(b+\nu+2)} \\
&+ \frac{\Gamma(b+\nu+4)}{\Gamma(\nu+1)\Gamma(b+4)} - 1 \\
&= \frac{\Gamma(b+\nu+4)}{(\nu-2)^{\underline{2}}\Gamma(\nu-1)\Gamma(b+4)} + \frac{\Gamma(b+\nu+4)(-2b+\nu-11)}{(\nu-1)^{\underline{3}}\Gamma(b+\nu+2)} \\
&+ \frac{\Gamma(b+\nu+4)}{\Gamma(\nu+1)\Gamma(b+4)} - 1 \\
&= \frac{\Gamma(b+\nu+4)}{(\nu-2)^{\underline{2}}\Gamma(\nu-1)\Gamma(b+4)} + \frac{(b+\nu+3)^{\underline{2}}(-2b+\nu-11)}{(\nu-1)^{\underline{3}}} \\
&+ \frac{\Gamma(b+\nu+4)}{\Gamma(\nu+1)\Gamma(b+4)} - 1.
\end{aligned}$$

This completes the proof of Theorem 9.

Theorem 10 *(Comparison Theorem) Assume that u and v satisfy*

$$\begin{aligned}
u(\nu-4) &\geq v(\nu-4) \\
\Delta u(\nu-4) &\geq \Delta v(\nu-4) \\
\Delta^2 u(b+\nu) &\geq \Delta^2 v(b+\nu) \\
\Delta^3 u(b+\nu) &\geq \Delta^3 v(b+\nu).
\end{aligned}$$

In addition, suppose that $Lu(t) \geq Lv(t)$ *for* $t \in \{0, \cdots, b+3\}$*, where* $Ly := \Delta^{\nu}_{\nu-4} y$*. Then*

$$u(t) \geq v(t), \quad \text{for} \quad t \in \{\nu - 4, \cdots, b + \nu + 3\}.$$

Proof: Put $w(t) = u(t) - v(t)$ and let $h(t)$ be defined by $h(t + \nu - 2) = Lw(t) = Lu(t) - Lv(t) \geq 0$, $t \in \{0, \cdots, b+3\}$. Then it follows that w is a solution of the BVP

$$\begin{aligned}
Lw(t) &= h(t + \nu - 2) \\
w(\nu - 4) &= C_1 := u(\nu - 4) - v(\nu - 4) \geq 0, \\
\Delta w(\nu - 4) &= C_2 := \Delta u(\nu - 4) - \Delta v(\nu - 4) \geq 0, \\
\Delta^2 w(b + \nu) &= C_3 := \Delta^2 u(b + \nu) - \Delta^2 v(b + \nu) \geq 0, \\
\Delta^3 w(b + \nu) &= C_4 := \Delta^3 u(b + \nu) - \Delta^3 v(b + \nu) \leq 0.
\end{aligned}$$

Then $w(t)$ is given by the formula

$$w(t) = \phi(t) + \int_0^{b+3} G(t, s) h(s) \Delta s, \quad t \in \{\nu - 4, \cdots, b + \nu + 3\},$$

where ϕ is the solution of the BVP

$$\begin{aligned}
L\phi &= 0 \\
\phi(\nu - 4) &= C_1, \quad \Delta\phi(\nu - 4) = C_2, \\
\Delta^2 \phi(b + \nu) &= C_3, \quad \Delta^3 \phi(b + \nu) = C_4
\end{aligned}$$

and $G(t,s)$ is the Green's function for the BVP (2.3). Moreover, Theorem 9 shows that $G(t,s) \geq 0$ on its domain. So as $h(t) \geq 0$, it follows that

$$w(t) = \phi(t) + \int_0^{b+3} G(t,s)h(s)\Delta s \geq 0, \quad t \in \{\nu-4, \cdots, b+\nu+3\},$$

provided

$$\phi(t) \geq 0, \quad \text{on} \quad \{\nu-4, \cdots, b+\nu+3\}.$$

To see this, first note that

$$\phi(t) = a_1 t^{\underline{\nu-1}} + a_2 t^{\underline{\nu-2}} + a_3 t^{\underline{\nu-3}} + a_4 t^{\underline{\nu-4}}.$$

From the first boundary condition on ϕ, we get that $a_4 = \frac{C_1}{\Gamma(\nu-3)}$. From the second boundary condition on ϕ, we get $a_3 = \frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}$. Now we use the last two boundary conditions on ϕ to find a_2 and a_1. Using the boundary conditions at $b+\nu$ we get the following system

$$C_3 = a_1(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}} + a_2(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}} + \left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)(\nu-3)^{\underline{2}}(b+\nu)^{\underline{\nu-5}} + \frac{C_1(\nu-4)^{\underline{2}}(b+\nu)^{\underline{\nu-6}}}{\Gamma(\nu-3)}$$

$$C_4 = a_1(\nu-1)^{\underline{3}}(b+\nu)^{\underline{\nu-4}} + a_2(\nu-2)^{\underline{3}}(b+\nu)^{\underline{\nu-5}} + \left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)(\nu-3)^{\underline{3}}(b+\nu)^{\underline{\nu-6}} + \frac{C_1(\nu-4)^{\underline{3}}(b+\nu)^{\underline{\nu-7}}}{\Gamma(\nu-3)}.$$

Multiplying the first equation by $(\nu-3)$ and the second equation by $-(b+4)$ we get $(\nu-3)C_3 - (b+4)C_4 + \left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)[(\nu-3)^{\underline{3}}(b+4)(b+\nu)^{\underline{\nu-6}} - (\nu-3)(\nu-3)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}] + \frac{C_1}{\Gamma(\nu-3)}[(\nu-4)^{\underline{3}}(b+4)(b+\nu)^{\underline{\nu-7}} - (\nu-3)(\nu-4)^{\underline{2}}(b+\nu)^{\underline{\nu-6}}] =$

$$a_2[(\nu-2)^{\underline{2}}(\nu-3)(b+\nu)^{\underline{\nu-4}} - (b+4)(\nu-2)^{\underline{3}}(b+\nu)^{\underline{\nu-5}}]$$

$\Longrightarrow$

$$(\nu-3)C_3 - (b+4)C_4 + \left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)(b+\nu)^{\underline{\nu-6}}(\nu-3)^{\underline{2}}[(\nu-5)(b+4) - (\nu-3)(b+6)] + \frac{C_1(\nu-4)^{\underline{2}}(b+\nu)^{\underline{\nu-7}}}{\Gamma(\nu-3)}[(\nu-6)(b+4)-(\nu-3)(b+7)] = a_2(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}[(\nu-3)(b+5)-(b+4)(\nu-4)]$$

$\Longrightarrow$

$$(\nu-3)C_3 - (b+4)C_4 - \left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)(b+\nu)^{\underline{\nu-6}}(\nu-3)^{\underline{2}}(2(b+\nu+1))$$

$$-\frac{C_1(\nu-4)^{\underline{2}}(b+\nu)^{\underline{\nu-7}}}{\Gamma(\nu-3)}(3(b+\nu+1)) = a_2(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}(b+\nu+1)$$

$\Longrightarrow$

$$\frac{(\nu-3)C_3-(b+4)C_4}{(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}(b+\nu+1)} - 2\left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)\frac{(\nu-3)^{\underline{2}}(b+\nu)^{\underline{\nu-6}}}{(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}}$$

$$-\frac{3C_1(\nu-4)^{\underline{2}}(b+\nu)^{\underline{\nu-7}}}{\Gamma(\nu-3)(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}} = a_2$$

$\Longrightarrow$

$$\frac{[(\nu-3)C_3-(b+4)C_4]\Gamma(b+6)}{(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)} - 2\left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)\frac{(\nu-4)}{(\nu-2)(b+6)}$$

$$-\frac{3C_1(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-2)^{\underline{2}}(b+7)^{\underline{2}}} = a_2.$$

Next we find a_1.

$$a_1 = \frac{C_3}{(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}}} - \frac{a_2(\nu-2)^{\underline{2}}(b+\nu)^{\underline{\nu-4}}}{(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}}}$$

$$-\left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)\frac{(\nu-3)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}}{(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}}} - \frac{C_1(\nu-4)^{\underline{2}}(b+\nu)^{\underline{\nu-6}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}}}$$

$\Longrightarrow$

$$a_1 = \frac{C_3}{(\nu-1)^{\underline{2}}(b+\nu)^{\underline{\nu-3}}} - \frac{a_2(\nu-3)}{(\nu-1)(b+4)}$$

$$-\left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)\frac{(\nu-3)^{\underline{2}}}{(\nu-1)^{\underline{2}}(b+5)^{\underline{2}}} - \frac{C_1(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}(b+6)^{\underline{3}}}$$

$\Longrightarrow$

$$a_1 = \frac{C_3\Gamma(b+4)}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+1)} - \frac{(\nu-3)[(\nu-3)C_3-(b+4)C_4]\Gamma(b+6)}{(\nu-1)(b+4)(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)}$$

$$+\frac{2(\nu-3)^{\underline{2}}}{(\nu-1)^{\underline{2}}(b+4)(b+6)}\left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)$$

$$+\frac{3(\nu-3)(\nu-4)^{\underline{2}}C_1}{\Gamma(\nu-3)(\nu-1)(b+4)(\nu-2)^{\underline{2}}(b+7)^{\underline{2}}} - \left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)\frac{(\nu-3)^{\underline{2}}}{(\nu-1)^{\underline{2}}(b+5)^{\underline{2}}}$$

$$-\frac{C_1(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}(b+6)^{\underline{3}}}$$

$$a_1 = \frac{C_3\Gamma(b+4)}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+1)} - \frac{[(\nu-3)C_3-(b+4)C_4]\Gamma(b+6)}{(\nu-1)^{\underline{2}}(b+4)\Gamma(b+\nu+2)}$$

$$+\frac{(\nu-3)^{\underline{2}}}{(\nu-1)^{\underline{2}}}\left(\frac{C_2}{\Gamma(\nu-2)} - \frac{C_1(\nu-4)}{\Gamma(\nu-2)}\right)\left(\frac{2}{(b+4)(b+6)} - \frac{1}{(b+5)^{\underline{2}}}\right)$$

$$+\frac{C_1(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}}\left(\frac{3}{(b+4)(b+7)^{\underline{2}}} - \frac{1}{(b+6)^{\underline{3}}}\right).$$

Hence

$$\phi(t) = a_1 t^{\underline{\nu-1}} + a_2 t^{\underline{\nu-2}} + a_3 t^{\underline{\nu-3}} + a_4 t^{\underline{\nu-4}}$$

where a_1, a_2, a_3 and a_4 are given above.

$$\begin{aligned}
\phi(t) &= C_1\left(\frac{2(\nu-4)^2}{(\nu-2)\Gamma(\nu-2)(b+6)} - \frac{3(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-2)^{\underline{2}}(b+7)^{\underline{2}}}\right)t^{\underline{\nu-2}} \\
&- C_1\left(\frac{(\nu-3)^{\underline{2}}(\nu-4)}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)}\left(\frac{2}{(b+4)(b+6)} - \frac{1}{(b+5)^{\underline{2}}}\right)\right)t^{\underline{\nu-1}} \\
&+ C_1\frac{(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}}\left(\frac{3}{(b+7)^{\underline{2}}(b+4)} - \frac{1}{(b+6)^{\underline{3}}}\right)t^{\underline{\nu-1}} \\
&+ C_1\left(-\frac{(\nu-4)t^{\underline{\nu-3}}}{\Gamma(\nu-2)} + \frac{t^{\underline{\nu-4}}}{\Gamma(\nu-3)}\right) \\
&+ C_2\left(\frac{(\nu-3)^{\underline{2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)}\left(\frac{2}{(b+4)(b+6)} - \frac{1}{(b+5)^{\underline{2}}}\right)t^{\underline{\nu-1}}\right) \\
&+ C_2\left(-\frac{2(\nu-4)t^{\underline{\nu-2}}}{(\nu-2)(b+6)\Gamma(\nu-2)} + \frac{t^{\underline{\nu-3}}}{\Gamma(\nu-2)}\right) \\
&+ C_3\left(\frac{\Gamma(b+4)}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+1)} - \frac{(\nu-3)\Gamma(b+6)}{(\nu-1)^{\underline{2}}(b+4)\Gamma(b+\nu+2)}\right)t^{\underline{\nu-1}} \\
&+ C_3\frac{(\nu-3)\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)} \\
&+ C_4\left(\frac{\Gamma(b+6)t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+2)} - \frac{(b+4)\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)}\right).
\end{aligned}$$

Next, we show that $\phi(t) \geq 0$:

First we show that the terms involving C_4 are nonnegative:

$$C_4\left(\frac{\Gamma(b+6)t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+2)} - \frac{(b+4)\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)}\right)$$

$$\begin{aligned}
&= \frac{C_4\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-1)^{\underline{3}}\Gamma(b+\nu+2)}[(\nu-3)(t-\nu+2)-(\nu-1)(b+4)] \\
&\geq \frac{C_4\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-1)^{\underline{3}}\Gamma(b+\nu+2)}(\nu-3)(b+5)-(b+4)(\nu-1) \geq 0.
\end{aligned}$$

Now we show that the coefficient of C_3 is nonnegative:

$$\left(\frac{\Gamma(b+4)t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+1)} - \frac{(\nu-3)\Gamma(b+6)t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}(b+4)\Gamma(b+\nu+2)}\right) + \frac{(\nu-3)\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)}$$

$$= \frac{\Gamma(b+4)(b+4)(4-\nu)t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(b+\nu+2)} + \frac{(\nu-3)\Gamma(b+6)t^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(b+\nu+2)} \geq 0.$$

Next we show that the coefficient of C_2 is nonnegative:

$$\frac{2(\nu-3)^{\underline{2}}t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+4)(b+6)} - \frac{2(\nu-4)t^{\underline{\nu-2}}}{(\nu-2)(b+6)\Gamma(\nu-2)}$$

$$-\frac{(\nu-3)^{\underline{2}}t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+5)^{\underline{2}}} + \frac{t^{\underline{\nu-3}}}{\Gamma(\nu-2)}$$

$$= \frac{2(\nu-4)t^{\underline{\nu-2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+4)(b+6)}[(\nu-3)(t-\nu+2)-(b+4)(\nu-1)]$$

$$-\frac{(\nu-3)^{\underline{2}}t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+5)^{\underline{2}}} + \frac{t^{\underline{\nu-3}}}{\Gamma(\nu-2)}.$$

But $(\nu-3)(t-\nu+2)-(\nu-1)(b+4) \leq (\nu-3)(b+5)-(\nu-1)(b+4) \leq 0$. Hence

$$\frac{2(\nu-4)t^{\underline{\nu-2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+4)(b+6)}[(\nu-3)(t-\nu+2)-(b+4)(\nu-1)]$$

$$-\frac{(\nu-3)^{\underline{2}}t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+5)^{\underline{2}}} + \frac{t^{\underline{\nu-3}}}{\Gamma(\nu-2)} \geq 0.$$

Finally, we show that the coefficient of C_1 is nonnegative:

$$\frac{2(\nu-4)^2t^{\underline{\nu-2}}}{(\nu-2)\Gamma(\nu-2)(b+6)} - \frac{(\nu-3)^{\underline{2}}(\nu-4)t^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+6)^{\underline{2}}} - \frac{3(\nu-4)^{\underline{2}}t^{\underline{\nu-2}}}{\Gamma(\nu-3)(\nu-2)^{\underline{2}}(b+7)^{\underline{2}}}$$

$$-\frac{(\nu-4)t^{\underline{\nu-3}}}{\Gamma(\nu-2)} + \frac{(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}(b+4)(b+6)}\left(\frac{3}{(b+7)} - \frac{1}{(b+5)}\right)t^{\underline{\nu-1}} + \frac{t^{\underline{\nu-4}}}{\Gamma(\nu-3)}$$

$$= \frac{(\nu-4)^2t^{\underline{\nu-2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+6)^{\underline{2}}}[2(\nu-1)(b+5)-(\nu-3)(t-\nu+2)]$$

$$-\frac{(\nu-4)t^{\underline{\nu-3}}}{(\nu-2)^{\underline{2}}\Gamma(\nu-3)(b+7)^{\underline{2}}}[3(\nu-5)(t-\nu+3)+(\nu-2)(b+7)^{\underline{2}}] + \frac{2(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}(b+7)^{\underline{3}}}t^{\underline{\nu-1}}$$

$$+\frac{t^{\underline{\nu-4}}}{\Gamma(\nu-3)}.$$

Note that

$2(\nu-1)(b+5)-(\nu-3)(t-\nu+2) \geq 2(\nu-1)(b+5)-(\nu-3)(b+5) = (b+5)(\nu+1) \geq 0$

and

$$\begin{aligned} 3(\nu-5)(t-\nu+3)+(\nu-2)(b+7)^{\underline{2}} &\geq 3(\nu-5)(b+6)+(\nu-2)(b+7)^{\underline{2}} \\ &= (b+6)(10\nu-29+b(\nu-2)) \geq 0. \end{aligned}$$

Thus

$$\frac{(\nu-4)^2 t^{\underline{\nu-2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+6)^{\underline{2}}}[2(\nu-1)(b+5)-(\nu-3)(t-\nu+2)]$$

$$-\frac{(\nu-4)t^{\underline{\nu-3}}}{(\nu-2)^{\underline{2}}\Gamma(\nu-3)(b+7)^{\underline{2}}}[3(\nu-5)(t-\nu+3)+(\nu-2)(b+7)^{\underline{2}}]+\frac{2(\nu-4)^{\underline{2}}}{\Gamma(\nu-3)(\nu-1)^{\underline{2}}(b+7)^{\underline{3}}}t^{\underline{\nu-1}}$$

$+\frac{t^{\underline{\nu-4}}}{\Gamma(\nu-3)} \geq 0$. Hence $\phi(t) \geq 0$ and therefore

$$u(t) \geq v(t), \quad \text{for} \quad t \in \{\nu-4, \cdots, b+\nu+3\}.$$

2.3 Positive Solutions of a Nonlinear Boundary Value Problem

In this section, we use the properties of the Green's function corresponding to the nonlinear boundary value problem (2.1). Recall that the problem is

$$\begin{cases} \Delta_{\nu-4}^{\nu} y(t) = f(t, y(t+\nu-2)), \quad t \in \{0, \cdots, b+3\} \\ \Delta^i y(\nu-4) = 0, \quad i = 0, 1 \\ \Delta^j y(b+\nu) = 0, \quad j = 2, 3 \end{cases}$$

where

- $3 < \nu \leq 4$
- $b \in \mathbb{N}$
- $f : \{0, \cdots, b+3\} \times \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $y \geq 0$ and $f(t, 0) = 0$ for $t \in \{0, \cdots, b+3\}$.

Define the Banach space $(\mathbb{B}, ||\cdot||)$ by

$$\mathbb{B} := \begin{cases} y : \{\nu-4, \cdots, b+\nu+3\} \to \mathbb{R} \\ \Delta^i y(\nu-4) = 0, \quad i = 0, 1 \\ \Delta^j y(b+\nu) = 0, \quad j = 2, 3 \end{cases}$$

together with the norm

$$||y||_{\mathbb{B}} = ||y|| := \max_{t \in \{\nu-4, \cdots, b+\nu+3\}} |y(t)|.$$

Define the completely continuous operator $A : \mathbb{B} \to \mathbb{B}$ by

$$Ay(t) := \sum_{s=0}^{b+3} G(t, s) f(s, y(s+\nu-2)), \quad t \in \{\nu-4, \cdots, b+\nu+3\}.$$

According to the work that has been done in [4] and Theorem 5, we conclude that every fixed point of the operator A is a solution of (2.1). In the following we will apply Krasnosel'skiis's theorem to establish the existence of fixed points for A, and hence of positive solutions of (2.1).

2.3.1 Krasnosel'skiis's Theorem

The following theorem is attributed to the Soviet mathematician Mark Krasnosel'skiis (1920–1997), who worked primarily in nonlinear functional analysis in the Ukraine and Russia.

Theorem 11 *Let $\mathbb{B}$ be a Banach space, $K \subseteq \mathbb{B}$ be a cone and let Ω_1 and Ω_2 be open sets contained in $\mathbb{B}$ such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subseteq \Omega_2$. Suppose that A: $K \cap (\bar{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator. Then A has at least one fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$ if either one of the following hold:*

- $||Ay|| \leq ||y||$ *for* $y \in K \cap \partial\Omega_1$ *and* $||Ay|| \geq ||y||$ *for* $y \in K \cap \partial\Omega_2$.
- $||Ay|| \geq ||y||$ *for* $y \in K \cap \partial\Omega_1$ *and* $||Ay|| \leq ||y||$ *for* $y \in K \cap \partial\Omega_2$.

To apply Krasnosel'skiis's Theorem to (2.1), we define

$$a := \left\lceil \frac{b-1}{2} \right\rceil$$

$$\eta := \left(\sum_{s=0}^{b+3} G(b+\nu+3, s) \right)^{-1}$$

$$\mu := \left(\sum_{s=a+3}^{b+3} G(b+\nu+3, s) \right)^{-1}$$

Also, we make the following assumptions on the nonlinearity f.

Assume

(H_1) There exists an $R_1 > 0$ such that for each $0 \leq y \leq R_1$ and $t \in \{0, \cdots, b+3\}$,

$$f(t, y) \leq \eta R_1.$$

(H_2) There exists an $R_2 > 0$ such that for each $\delta R_2 \leq y \leq R_2$ and $t \in \{a+3, \cdots, b+3\}$,

$$f(t, y) \geq \mu R_2.$$

(H_3) Either $R_1 < \delta R_2$ or $\mu R_2 \leq \eta R_1$.

Note that $\eta < \mu$ and $0 < \delta < 1$ (where δ is the constant found in Theorem 9). With this in mind, we make assumption (H_3) to avoid the case where assumptions (H_1) and (H_2) contradict each other.

Theorem 12 *Assume that* (H_1), (H_2) *and* (H_3) *hold. Then the BVP* (2.1) *has at least one positive solution* y_1 *with* $\min\{R_1, R_2\} \leq ||y_1|| \leq \max\{R_1, R_2\}$.

Proof: Assume f satisfies assumptions (H_1), (H_2) and (H_3). Define the cone $K \subseteq \mathbb{B}$ by

$$K := \begin{cases} y \in \mathbb{B} : y(t) \geq 0 \quad \text{on} \quad \{\nu - 4, \cdots, b + \nu + 3\} \\ \text{and} \\ \min\limits_{t \in \{a+\nu+1, \cdots, b+\nu+1\}} y(t) \geq \delta ||y||. \end{cases}$$

First we show that A maps K into K. Since $y \in K$, $t \in \{\nu - 4, \cdots, b+\nu+3\}$, $G(t, s)$ is nonnegative everywhere and f is nonnegative for $y \geq 0$, we have

$$Ay(t) = \sum_{s=0}^{b+3} G(t,s)f(s, y(s+\nu-2)) \geq 0, \quad t \in \{\nu-4, \cdots, b+\nu+3\}.$$

Furthermore,

$$\begin{aligned}
& \min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} Ay(t) \\
= & \sum_{s=0}^{b+3} \left(\min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} G(t,s) \right) f(s, y(s+\nu-2)) \\
\geq & \sum_{s=0}^{b+3} \left(\delta \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} G(t,s) \right) f(s, y(s+\nu-2)) \\
= & \delta \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s)f(s, y(s+\nu-2)) \\
= & \delta ||Ay||.
\end{aligned}$$

Thus, A maps K into K.

Now, define the open subsets $\Omega_R := \{y \in \mathbb{B} : ||y|| < R\}$.

Observe that

$$y \in \partial\Omega_{R_1} \Longrightarrow ||y|| = R_1 \text{ and } z \in \partial\Omega_{R_2} \Longrightarrow ||z|| = R_2.$$

where R_1 and R_2 are defined in (H_1) and (H_2).

According to (H_3), we have two cases to consider:

Case 1 $R_1 < \delta R_2$:

If $R_1 < \delta R_2$, we may apply Krasnosel'skiis's Theorem to show the existence of a positive solution to BVP (2.1):

- Suppose that $y \in K \cap \partial\Omega_{R_1}$. Then

$$
\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s)f(s,y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)) \\
&\leq \eta R_1 \left(\sum_{s=0}^{b+3} G(b+\nu+3,s) \right) \\
&= R_1 \\
&= ||y||.
\end{aligned}
$$

- Suppose that $y \in K \cap \partial\Omega_{R_2}$. Then

$$
\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s)f(s,y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)).
\end{aligned}
$$

Now, since $y \in K \cap \partial\Omega_{R_2}$, we have that

$$\min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} y(t) \geq \delta||y||,$$

and hence

$$\delta R_2 \leq y(t) \leq R_2, \text{ for } \quad t \in \{a+\nu+1,\cdots,b+\nu+1\}.$$

Therefore, assumption (H_2) implies that

$$f(t-\nu+2, y(t)) \geq \mu R_2 \text{ , for } t \in \{a+\nu+1, \cdots, b+\nu+1\}$$

$$\Longrightarrow f(s, y(s+\nu-2)) \geq \mu R_2, \text{ for } s \in \{a+3, \cdots, b+3\}.$$

Hence

$$\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s) f(s, y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\
&\geq \mu R_2 \sum_{s=a+3}^{b+3} G(b+\nu+3, s) \\
&= R_2 \\
&= ||y||.
\end{aligned}$$

Since we have proved that

- $||Ay|| \leq ||y||$, for all $y \in K \cap \partial\Omega_{R_1}$,
- $||Ay|| \geq ||y||$, for all $y \in K \cap \partial\Omega_{R_2}$.

Krasnosel'skiis's Theorem part 1 with $\Omega_1 := \Omega_{R_1}$ and $\Omega_2 := \Omega_{R_2}$ implies that A

has a fixed point $y_1 \in K \cap (\bar{\Omega}_{R_2} \setminus \Omega_{R_1})$. It follows that y_1 is a solution to the BVP (2.1) with $R_1 \leq ||y_1|| \leq R_2$.

Case 2 $\mu R_2 \leq \eta R_1$:

If $\mu R_2 \leq \eta R_1$, we apply Krasnosel'skiis's Theorem part 2 to show the existence of a positive solution to the BVP (2.1):

- Suppose $y \in K \cap \partial\Omega_{R_1}$. Then

$$\begin{aligned} ||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s) f(s, y(s+\nu-2)) \\ &= \sum_{s=0}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\ &\leq \eta R_1 \sum_{s=0}^{b+3} G(b+\nu+3, s), \quad \text{by } (H_1) \\ &= R_1 \\ &= ||y||. \end{aligned}$$

- Suppose that $y \in K \cap \partial\Omega_{R_2}$. Then

$$\min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} y(t) \geq \delta ||y||,$$

and hence

$$\delta R_2 \leq y(t) \leq R_2, \text{for} \quad t \in \{a+\nu+1, \cdots, b+\nu+1\}.$$

Therefore, assumption (H_2) implies that

$$f(t-\nu+2, y(t)) \geq \mu R_2 \text{ , for } t \in \{a+\nu+1, \cdots, b+\nu+1\}$$

$$\Longrightarrow f(s, y(s+\nu-2)) \geq \mu R_2, \text{ for } s \in \{a+3, \cdots, b+3\}.$$

Hence,

$$\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s) f(s, y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\
&\geq \mu R_2 \sum_{s=a+3}^{b+3} G(b+\nu+3, s) \\
&= R_2 \\
&= ||y||.
\end{aligned}$$

Since we have proved that

- $||Ay|| \geq ||y||$, for all $y \in K \cap \partial\Omega_{R_2}$,
- $||Ay|| \leq ||y||$, for all $y \in K \cap \partial\Omega_{R_1}$.

Krasnosel'skiis's Theorem part 2 with $\Omega_1 := \Omega_{R_2}$ and $\Omega_2 := \Omega_{R_1}$ implies that A has a fixed point $y_1 \in K \cap (\bar{\Omega}_{R_1} \setminus \Omega_{R_2})$. It follows that y_1 is a solution to the BVP (2.1) with $R_2 \leq ||y_1|| \leq R_1$.

Since R_1 and R_2 from assumptions (H_1) and (H_2) are positive, we see that in either case, y_1 is a nontrivial solution to the BVP (2.1) with

$$\min\{R_1, R_2\} \leq ||y_1|| \leq \max\{R_1, R_2\}.$$

2.3.2 Banach's Theorem

The Banach fixed-point Theorem (also known as the contraction mapping theorem) is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The theorem is named after Stefan Banach (1892-1945), and was first stated by him in 1922.

Theorem 13 *Let (X, d) be a complete metric space and suppose $f : X \to X$ is a contraction mapping, where $0 \leq L < 1$ is a constant such that*

$$|f(x) - f(y)| \leq L|x - y|, \text{ for all } x, y \in X.$$

Then

- *f has a unique fixed point $y \in X$.*
- *$\lim_{n\to\infty} f^n(x) = y$, for all $x \in X$.*
- *$d(f^n(x), y) \leq \dfrac{L^n}{1-L} d(x, f(x))$, for all $x \in X$ and $n \in \mathbb{N}$.*

Here we use the notation $f^n(x) = (fofo \cdots of)(x)$ n times.

Since $(\mathbb{B}, || \cdot ||)$ is a complete metric space (with $d(y_1, y_2) := ||y_1 - y_2||$), we may apply the Banach Contraction Theorem to prove the existence of a unique solution to the BVP (2.1), provided the nonlinearity f satisfies a uniform Lipschitz condition.

Theorem 14 *The nonlinear boundary value problem* (2.1) *has a unique solution provided there exists a* $0 \leq \gamma < \eta$ *such that* f *satisfies the uniform Lipschitz condition*

$$|f(t, y_1) - f(t, y_2)| \leq \gamma |y_1 - y_2|$$

for all $t \in \{0, \cdots, b+3\}$, $y_1, y_2 \in \mathbb{R}$ *and* $\eta := \left(\sum_{s=0}^{b+3} G(b+\nu+3, s) \right)^{-1}$ *is the fixed constant involving the Green's function.*

Proof: Assume that there exists a $0 \leq \gamma < \eta$ such that f satisfies the uniform Lipschitz condition above. Then given any $y_1, y_2 \in \mathbb{R}$, we may estimate

$$\begin{aligned}
|Ay_1(t) - Ay_2(t)| &= |\sum_{s=0}^{b+3} G(t,s) f(s, y_1(s+\nu-2)) - \sum_{s=0}^{b+3} G(t,s) f(s, y_2(s+\nu-2))| \\
&= |\sum_{s=0}^{b+3} G(t,s)(f(s, y_1(s+\nu-2)) - f(s, y_2(s+\nu-2)))| \\
&\leq \sum_{s=0}^{b+3} G(t,s)|f(s, y_1(s+\nu-2)) - f(s, y_2(s+\nu-2))| \\
&\leq \sum_{s=0}^{b+3} G(b+\nu+3, s)\gamma |y_1(s+\nu-2) - y_2(s+\nu-2)| \\
&\leq \sum_{s=0}^{b+3} G(b+\nu+3, s)\gamma ||y_1 - y_2|| \\
&= \gamma ||y_1 - y_2|| \sum_{s=0}^{b+3} G(b+\nu+3, s) \\
&= \frac{\gamma}{\eta} ||y_1 - y_2||,
\end{aligned}$$

for $t \in \{\nu-4, \cdots, b+\nu+3\}$. Hence

$$||Ay_1 - Ay_2|| \leq \frac{\gamma}{\eta}||y_1 - y_2||.$$

Then A is a contraction mapping with Lipschitz constant $\frac{\gamma}{\eta} \in [0,1)$. Banach's Contraction Mapping Theorem implies A has a unique fixed point $y_1 \in \mathbb{B}$, and y_1 is the unique solution to the BVP (2.1). Furthermore, we obtain this unique solution by calculating for any $y \in \mathbb{B}$ the function

$$\lim_{n\to\infty} A^n y.$$

Moreover, for any chosen initial function $y \in \mathbb{B}$, Banach's Theorem part (3) allows us to calculate the error of estimation at any step $n \in \mathbb{N}$ using the formula

$$||A^n y - y_1|| \leq \frac{\gamma^n}{\eta^n - \gamma^n}||Ay - y||.$$

Remark 15 *We claim that*

$$0 \leq \frac{\Gamma(\nu-1)}{L} < \eta,$$

where

$$\begin{aligned} L \;:=\; & \sum_{s=0}^{b+3} \frac{\Gamma(b+\nu-s)(b+\nu+3)^2(b+4)(2\nu+b+s(\nu-4)-3)}{(\nu-3)\Gamma(b-s+4)(b+5)} \\ + \; & \sum_{s=0}^{b+3} \frac{\Gamma(b+\nu-s+3)}{(\nu-1)\Gamma(b-s+4)}. \end{aligned}$$

To see this consider

$$
\begin{aligned}
\frac{1}{\eta} &= \sum_{s=0}^{b+3} G(b+\nu+3,s) \\
&= \sum_{s=0}^{b+3} \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
&- \frac{(b+\nu-s-1)^{\underline{\nu-4}}(b-s+3)(b+\nu+1)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
&- \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{\Gamma(\nu-2)(\nu-1)^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
&+ \frac{(b+\nu-s+2)^{\underline{\nu-1}}}{\Gamma(\nu)} \\
&< \sum_{s=0}^{b+3} \frac{(b+\nu-s-1)^{\underline{\nu-4}}(2\nu+b+s(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{\Gamma(\nu-2)(\nu-2)^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-4}}(b+\nu+1)} \\
&+ \frac{(b+\nu-s+2)^{\underline{\nu-1}}}{\Gamma(\nu)} \\
&= \sum_{s=0}^{b+3} \frac{\dfrac{\Gamma(b+\nu-s)\Gamma(b+\nu+4)}{\Gamma(b-s+4)\Gamma(b+6)}(2\nu+b+s(\nu-4)-3)(b+4)}{\Gamma(\nu-2)\Gamma(\nu-1)(b+\nu+1)\dfrac{\Gamma(b+\nu+1)}{\Gamma(b+5)}} \\
&+ \sum_{s=0}^{b+3} \frac{\dfrac{\Gamma(b+\nu-s+3)}{\Gamma(b-s+4)}}{(\nu-1)\Gamma(\nu-1)} \\
&= \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{b+3} \frac{\Gamma(b+\nu-s)(b+\nu+3)^{\underline{2}}(b+4)(2\nu+b+s(\nu-4)-3)}{\Gamma(\nu-2)\Gamma(b-s+4)(b+5)} \\
&+ \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{b+3} \frac{\Gamma(b+\nu-s+3)}{(\nu-1)\Gamma(b-s+4)}.
\end{aligned}
$$

Example 16 *Consider the following boundary value problem*

$$\begin{cases} \Delta^{\pi}_{\pi-4} y = (y(t+\pi-2)(t+1))^2, \quad t \in \{0, \cdots, 15\} \\ y(\pi-4) = \Delta y(\pi-4) = 0, \\ \Delta^2 y(12+\pi) = \Delta^3 y(12+\pi) = 0. \end{cases} \tag{2.9}$$

Here, we have the nonlinear problem (2.1) *with*

$\nu = \pi,\ f(t, y) = y^2(t+1)^2,$
$b = 12,\ a = 6$

and

$$D\{y\} = \{\pi - 4, \cdots, \pi + 15\}.$$

Let us apply Krasnosel'skiis's Theorem to show that (2.9) *has at least one positive solution. To do this, we must find constants* $R_1, R_2 > 0$ *such that the assumptions* $(H_1), (H_2)$ *and* (H_3) *are satisfied. first, we calculate the following three constants from Theorem 9:*

$$\begin{aligned} \delta &:= \frac{(a+4)(b+4)[2b+4a+22]}{(a+5)(b+5)^{\underline{2}}[(b+7)^{\underline{2}}(3a-b+4)+(b-a+4)^{\underline{3}}]} \approx 0.0009041. \\ \eta &:= \left(\sum_{s=0}^{15} G(b+\nu+3, s)\right)^{-1} \approx 0.000705. \\ \mu &:= \left(\sum_{s=9}^{15} G(b+\nu+3, s)\right)^{-1} \approx 0.00196. \end{aligned}$$

- *We want to find* $R_1 > 0$ *such that*

$$\begin{aligned} 0 &\le f(t,y) \le \eta R_1, \quad for \quad 0 \le y \le R_1 \quad and \quad t \in \{0, \cdots, 15\} \\ \Longleftrightarrow 0 &\le (t+1)^2 y^2 \le \eta R_1 \\ \Longleftarrow 0 &\le {R_1}^2 16^2 \le \eta R_1 \\ \Longleftrightarrow 0 &\le R_1 \le \frac{\eta}{(16)^2} \approx 0.000002753. \end{aligned}$$

Then (H_1) *holds with* $R_1 := 0.000002753$.

- *We want to find* R_2 *such that*

$$\begin{aligned} f(t,y) &\ge \mu R_2, \quad for \quad \delta R_2 \le y \le R_2 \; and \quad t \in \{a+3, \cdots, b+3\} \\ f(t,y) &\ge \mu R_2, \quad for \quad \delta R_2 \le y \le R_2 \; and \quad t \in \{9, \cdots, 15\} \\ \Longleftrightarrow (t+1)^2 y^2 &\ge \mu R_2. \end{aligned}$$

Hence if we choose R_2 *so that*

$$(\delta R_2 10)^2 \ge \mu R_2,$$

then

$$(t+1)^2 y^2 \ge \mu R_2,$$

and hence we can take $R_2 = 23.981$. *Then* (H_2) *holds with* $R_2 := 23.981$.

- (H_3) *also holds, since* $R_1 \approx 0.000002753 < 0.02168 \approx \delta R_2$.

Therefore, Krasnosel'skiis's Theorem implies that (2.9) *has a positive solution* y_1 *on* $\{\pi - 4, \cdots, \pi + 15\}$. *Furthermore,*

$$R_1 \leq ||y_1|| \leq R_2.$$

Example 17 *Consider the fractional boundary value problem*

$$\begin{cases} \Delta^{\pi}_{\pi-4} y = \dfrac{y(t+\pi-2)}{(t+12)^3}, \quad t \in \{0, \cdots, 15\} \\ y(\pi - 4) = \Delta y(\pi - 4) = 0, \\ \Delta^2 y(12+\pi) = \Delta^3 y(12+\pi) = 0. \end{cases} \tag{2.10}$$

The BVP (2.10) *is of the form* (2.9), *where*

$\nu = \pi,\ f(t,y) = \dfrac{y}{(t+12)^3},$
$b = 12,\ a = 6$

In this, problem , we apply Banach Theorem to show that (2.10) *has a unique solution on* $\{\pi - 4, \cdots, \pi + 15\}$. *Note that for* $t \in \{0, \cdots, 15\}$ *and* $y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= \left| \frac{y_1}{(t+12)^3} - \frac{y_2}{(t+12)^3} \right| \\ &= \frac{1}{(t+12)^3} |y_1 - y_2| \\ &\leq \frac{1}{12^3} |y_1 - y_2| \\ &= \frac{1}{1728} |y_1 - y_2|. \end{aligned}$$

So our Lipschitz constant is $\gamma = \dfrac{1}{1728}$. *Since*

$$0 \leq \gamma = \frac{1}{1728} \approx 0.0005787 < \eta \approx 0.000705,$$

Banach's Theorem implies the existence of a unique solution of (2.10) *on* $\{\pi - 4, \cdots, \pi + 15\}$.

2.4 Multiple Positive Solutions of a Nonlinear Boundary Value Problem

In this section, we give some results about the existence of two positive solutions of a nonlinear boundary value problem. We shall first state the following theorems that can be found in [8] on fixed point index and multiple fixed points. Recall that the operator A and the cone K are defined as in Section 2.3.

Theorem 18 *Let K be a retract of the real Banach space $\mathbb{B}$. Then, for every relatively bounded open subset Ω of K and every completely continuous operator $A : \bar{\Omega} \to K$ which has no fixed points on $\partial\Omega$, there exists an integer $i(A, \Omega, K)$ satisfying:*

- *Normality: $i(A, \Omega, K) = 1$ if $Ay = y_0 \in \Omega$ for any $y \in \bar{\Omega}$.*
- *Additivity: $i(A, \Omega, K) = i(A, \Omega_1, K) + i(A, \Omega_2, K)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that A has no fixed points on $\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)$.*
- *Homotopy invariance: $i(H(t, \cdot), \Omega, K)$ is independent of t whenever $H : [0, 1] \times \bar{\Omega} \to K$ is completely continuous and $H(t, y) = y$ for any $(t, y) \in [0, 1] \times \partial\Omega$.*
- *Permanence: $i(A, \Omega, K) = i(A, \Omega \cap Y)$ if Y is a retract of K and $A(\bar{\Omega}) \subset Y$.*

$i(A, \Omega, K)$ is called the fixed point index of A on Ω with respect to K and is uniquely defined.

Theorem 19 *Let $\mathbb{B}$ be a Banach space and $K \subset \mathbb{B}$ be a cone. For $p > 0$ assume that $A : K_p \to K$ is compact such that $Ay \neq y$ for every $y \in \partial K_p := \{y \in K : ||y|| = p\}$. Then*

- *If* $||Ay|| \geq ||y||$, *for every* $y \in \partial K_p$, $i(A, K_p, K) = 0$.

- *If* $||Ay|| \leq ||y||$, *for every* $y \in \partial K_p$, $i(A, K_p, K) = 1$.

Theorem 20 *Let* $\Omega_{r_1}, \Omega_{r_2}$ *and* Ω_{r_3} *be three open sets in the Banach space* $\mathbb{B}$ *such that* $0 \in \Omega_{r_1}, \bar{\Omega}_{r_1} \subset \Omega_{r_2}$ *and* $\bar{\Omega}_{r_2} \subset \Omega_{r_3}$. *Let* $A : K \cap (\bar{\Omega} r_3 \setminus \Omega r_1) \to K$ *be a completely continuous operator. Suppose that*

- $||Ay|| \geq ||y||$, *for every* $y \in K \cap \partial\Omega_{r_1}$;

- $||Ay|| \leq ||y||$ *and* $Ay \neq y$, *for every* $y \in K \cap \partial\Omega_{r_2}$;

- $||Ay|| \geq ||y||$, *for every* $y \in K \cap \partial\Omega_{r_3}$.

Then A *has at least two fixed points in* $K \cap (\bar{\Omega} r_3 \setminus \Omega r_1)$.

With the assumptions on the nonlinearity f in Section 2.3 , we assume the following
(H_4) $\displaystyle\lim_{y\to 0^+} \min_{t\in\{\nu-4,\cdots,b+\nu+3\}} \frac{f(t,y)}{y} = \lim_{y\to\infty} \min_{t\in\{\nu-4,\cdots,b+\nu+3\}} \frac{f(t,y)}{y} = \infty$ (sub-superlinear).

(H_5) $\displaystyle\lim_{y\to 0^+} \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \frac{f(t,y)}{y} = \lim_{y\to\infty} \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \frac{f(t,y)}{y} = 0$ (super-sublinear).

With the above assumptions on f we prove the following theorems.

Theorem 21 *Assume that* (H_1) *and* (H_4) *hold. Then the BVP* (2.1) *has at least two positive solutions* y_1 *and* y_2 *with* $0 < ||y_1|| < R_1 < ||y_2||$.

Proof: Choose $M > 0$ such that $\delta M \displaystyle\sum_{s=a+3}^{b+3} G(b+\nu+3, s) > 1$. By the first part of (H_4), we can choose $r > 0$ such that $r < R_1$ and $f(t, y) \geq My$ for $t \in \{\nu-4, \cdots, b+\nu+3\}$

and $0 \le y \le r$. Then if $y \in K \cap \partial\Omega_r$, we have

$$\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s)f(s,y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)) \\
&\ge \sum_{s=0}^{b+3} G(b+\nu+3,s)My(s+\nu-2) \\
&\ge M\delta \sum_{s=a+3}^{b+3} G(b+\nu+3,s)||y|| \\
&= M\delta||y|| \sum_{s=a+3}^{b+3} G(b+\nu+3,s) \\
&\ge ||y||.
\end{aligned}$$

Then

$$i(A,\Omega_r,K) = 0$$

by Theorem 19. By the second part of (H_4), there exists $p > 0$ such that $f(t,y) \ge My$ for $y \ge p$. So choose $R > \max\{R_1, \frac{p}{\delta}\}$, where R_1 is the constant guaranteed by (H_1). Then for $y \in K \cap \partial\Omega_R$, we have

$$\min_{t\in\{a+\nu+1,\cdots,b+\nu+1\}} y(t) \ge \delta||y|| \ge \delta R \ge p.$$

and

$$\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s)f(s,y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3,s)My(s+\nu-2) \\
&\geq M\delta \sum_{s=a+3}^{b+3} G(b+\nu+3,s)||y|| \\
&\geq ||y||.
\end{aligned}$$

Hence

$$i(A,\Omega_R,K) = 0.$$

For $y \in K \cap \partial\Omega_{R_1}$, we have

$$\begin{aligned}
||Ay|| &= \max_{t\in\{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s)f(s,y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3,s)f(s,y(s+\nu-2)) \\
&\leq \eta \sum_{s=0}^{b+3} G(b+\nu+3,s)y(s+\nu-2), \quad by\ (H_1) \\
&\leq \eta \sum_{s=0}^{b+3} G(b+\nu+3,s)R_1 \\
&= \eta \sum_{s=0}^{b+3} G(b+\nu+3,s)||y|| \\
&= ||y||.
\end{aligned}$$

Hence by Theorem 19, we have

$$i(A, \Omega_{R_1}, K) = 1.$$

Since $Ay \neq y$ for every $y \in K \cap \partial\Omega_{R_1}$Theorem 20 (with r_1, r_2 and r_3 replaced by r, R_1 and R respectively) implies that there exist a fixed point $y_1 \in \bar{\Omega}_R \setminus \Omega_{R_1}$ and another fixed point $y_2 \in \bar{\Omega}_{R_1} \setminus \Omega_r$. It follows that the BVP (2.1) has at least two positive solutions y_1 and y_2 with $||y_1|| \leq R_1 \leq ||y_2||$.

Theorem 22 *Assume that* (H_2) *and* (H_5) *hold. Then the BVP* (2.1) *has at least two positive solutions* y_1 *and* y_2 *with* $0 < ||y_1|| < R_2 < ||y_2||$.

Proof: Using both parts of (H_5) and the continuity of $f(t, y)$ with respect to y, we have for any $\epsilon > 0$ there exists $M > 0$ such that $f(t, y) \leq M + \epsilon y$, for every $y > 0$ and $t \in \{\nu - 4, \cdots, b + \nu + 3\}$. For $y \in K$

$$\begin{aligned} ||Ay|| &= \max_{t \in \{\nu-4,\cdots,b+\nu+3\}} \sum_{s=0}^{b+3} G(t, s) f(s, y(s + \nu - 2)) \\ &= \sum_{s=0}^{b+3} G(b + \nu + 3, s) f(s, y(s + \nu - 2)) \\ &\leq \sum_{s=0}^{b+3} G(b + \nu + 3, s)(M + \epsilon y(s + \nu - 2)). \end{aligned}$$

Then, if we choose $R > \max\{R_2, \dfrac{ML}{1-\epsilon L}\}$ (where R_2 is the constant guaranteed in (H_2)) large enough and $\epsilon > 0$ small enough, we obtain

$$\begin{aligned} & \sum_{s=0}^{b+3} G(b+\nu+3, s)(M + \epsilon y(s+\nu-2)) \\ \leq & \sum_{s=0}^{b+3} G(b+\nu+3, s)(M + \epsilon R) \\ = & \; LM + \epsilon LR \\ < & \; R = ||y|| \end{aligned}$$

with $y \in K \cap \partial\Omega_R$, where $L = \sum_{s=0}^{b+3} G(b+\nu+3, s)$. Thus

$$i(A, \Omega_R, K) = 1.$$

Again by the first part of (H_5), if $0 < \epsilon < \dfrac{1}{L}$ is given, then we can choose $r > 0$ such that

$$f(t, y) \leq \epsilon y, \qquad \text{for every} \quad 0 \leq y \leq r.$$

So, if we further assume $0 < r < R_2$ and $y \in K \cap \partial\Omega_r$, then

$$\begin{aligned} ||Ay|| & = \max_{t \in \{\nu-4, \cdots, b+\nu+3\}} \sum_{s=0}^{b+3} G(t, s) f(s, y(s+\nu-2)) \\ & = \sum_{s=0}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\ & \leq \sum_{s=0}^{b+3} G(b+\nu+3, s) \epsilon y(s+\nu-2) \\ & \leq L\epsilon r \\ & \leq ||y||. \end{aligned}$$

Hence

$$i(A, \Omega_r, K) = 1.$$

On the other hand, for $y \in K \cap \partial\Omega_{R_2}$, we have

$$\min_{t \in \{a+\nu+1, \cdots, b+\nu+1\}} y(t) \geq \delta ||y|| = \delta R_2.$$

Then

$$\begin{aligned}
||Ay|| &= \max_{t \in \{\nu-4, \cdots, b+\nu+3\}} \sum_{s=0}^{b+3} G(t,s) f(s, y(s+\nu-2)) \\
&= \sum_{s=0}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3, s) f(s, y(s+\nu-2)) \\
&\geq \sum_{s=a+3}^{b+3} G(b+\nu+3, s) \mu R_2 \\
&= R_2 \\
&= ||y||.
\end{aligned}$$

Thus

$$i(A, \Omega_{R_2}, K) = 0.$$

Hence Theorem 20 implies that there exist a fixed point $y_1 \in \bar{\Omega}_R \setminus \Omega_{R_2}$ and another fixed point $y_2 \in \bar{\Omega}_{R_2} \setminus \Omega_r$. It follows that the BVP (2.1) has at least two positive solutions y_1 and y_2 with $0 < ||y_1|| \leq R_2 \leq ||y_2||$.

2.5 Liapunov Inequality

Theorem 23 *(Liapunov Inequality) If*

$$\int_{\nu-4}^{b+\nu+3} |q(t)|\Delta t \leq \frac{(\nu-1)^{\underline{3}}}{(b+\nu+3)^{\underline{2}}(2b+11-\nu)+(\nu-1)^{\underline{3}}}, \tag{2.11}$$

then the BVP

$$\begin{cases} \Delta_{\nu-4}^{\nu} y + q(t)y(t+\nu-4) = 0 \\ y(\nu-4) = \Delta y(\nu-4) = 0 \\ \Delta^2 y(b+\nu) = \Delta^3 y(b+\nu) = 0 \end{cases} \tag{2.12}$$

has only the trivial solution.

Proof: Assume that the inequality (2.11) holds but the BVP (2.12) has a nontrivial solution. Then $y(t)$ is a nontrivial solution of the BVP

$$\begin{aligned} \Delta_{\nu-4}^{\nu} y &= -q(t)y(t+\nu-4), \quad t \in \{0, \cdots, b+7\} \\ y(\nu-4) &= \Delta y(\nu-4) = 0 \\ \Delta^2 y(b+\nu) &= \Delta^3 y(b+\nu) = 0. \end{aligned}$$

Since $y(t)$ is a solution of the above BVP

$$y(t) = \int_0^{b+3} G(t,s)[-q(s)y(s+\nu-4)]\Delta s$$

where $G(t,s)$ is the Green's function for the BVP (2.3). Pick $t_0 \in \{\nu-4, \cdots, b+\nu+3\}$ such that

$$|y(t_0)| = \max\{|y(t)| : \nu - 4 \leq t \leq b + \nu + 3\}.$$

consider

$$\begin{aligned} ||y|| = |y(t_0)| &= |\int_0^{b+3} G(t_0, s)[-q(s)y(s+\nu-4)\Delta s| \\ &\leq \int_0^{b+3} G(t_0, s)|q(s)||y(s+\nu-4)|\Delta s \\ &\leq \int_0^{b+3} G(t_0, s)|q(s)|||y||\Delta s \\ &\leq \int_0^{b+3} G(b+\nu+3, s)|q(s)|||y||\Delta s. \end{aligned}$$

If $q(t) \equiv 0$ on $\{\nu-4, \cdots, b+\nu+3\}$, then by Theorem 8 the BVP (2.12) has only the trivial solution. So assume that $q(t) \not\equiv 0$ on $\{\nu-4, \cdots, b+\nu+3\}$. In the proof of Theorem 9 we proved that $P(s) := G(b+\nu+3, s)$ is increasing with maximum value $J := P(b+3)$. Since $G(t,s) \leq G(b+\nu+3, b+3)$ and $q(t) \not\equiv 0$, we have that

$$||y|| = |y(t_0)| \quad < \quad ||y|| \int_0^{b+3} J|q(s)|\Delta s \tag{2.13}$$

where by Theorem 9,

$$
\begin{aligned}
J \quad &:= \quad P(b+3) = G(b+\nu+3, b+3) \\
&= \quad \frac{(b+\nu-(b+3)-1)^{\underline{\nu-4}}(2\nu+b+(b+3)(\nu-4)-3)(b+4)(b+\nu+3)^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&- \quad \frac{(b+\nu-(b+3)-1)^{\underline{\nu-4}}(b-(b+3)+3)(b+\nu+1)(b+\nu+3)^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&- \quad \frac{(b+\nu-(b+3)-1)^{\underline{\nu-4}}(2\nu+b+(b+3)(\nu-4)-3)(b+\nu+3)^{\underline{\nu-1}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&- \quad \frac{(b+\nu-(b+3)+2)^{\underline{\nu-1}}}{\Gamma(\nu)} \\
&= \quad \frac{\Gamma(\nu-3)(\nu-3)(b+5)^{\underline{2}}(b+\nu+3)^{\underline{\nu-2}}}{(\nu-2)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} - \frac{\Gamma(\nu-3)(\nu-3)(b+5)^{2}(b+\nu+3)^{\underline{\nu-2}}}{(\nu-1)^{\underline{2}}\Gamma(\nu-2)(b+\nu+1)(b+\nu)^{\underline{\nu-4}}} \\
&+ \quad 1 \\
&= \quad \frac{(\nu-1)(b+4)(b+\nu+3)^{\underline{2}}}{(\nu-1)^{\underline{3}}} - \frac{(\nu-3)(b+5)(b+\nu+3)^{\underline{2}}}{(\nu-1)^{\underline{3}}} + 1 \\
&= \quad \frac{(b+\nu+3)^{\underline{2}}(2b+11-\nu)}{(\nu-1)^{\underline{3}}} + 1 = \frac{(\nu-1)^{\underline{3}} + (b+\nu+3)^{\underline{2}}(2b+11-\nu)}{(\nu-1)^{\underline{3}}}.
\end{aligned}
$$

Dividing both sides of (2.13) by $||y||$, we get the inequality

$$1 < \left(\frac{(b+\nu+3)^{\underline{2}}(2b+11-\nu) + (\nu-1)^{\underline{3}}}{(\nu-1)^{\underline{3}}} \right) \int_0^{b+3} |q(s)|\Delta s.$$

Hence

$$\int_{\nu-4}^{b+\nu+3} |q(t)|\Delta t > \left(\frac{(\nu-1)^{\underline{3}}}{(b+\nu+3)^{\underline{2}}(2b+11-\nu) + (\nu-1)^{\underline{3}}} \right)$$

which contradicts (2.11).

Chapter 3

The Conjugate Boundary Value Problem

In this chapter we consider a fourth order boundary value problem with $(2,2)$ conjugate boundary conditions.

3.1 The Boundary Value Problem

In this section we are concerned with the nonlinear fractional boundary value problem

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = f(t, y(t+\nu-2)), \quad t \in \{0, \cdots, b+1\} \\ \Delta^{i} y(\nu-4) = 0, \quad i = 0, 1 \\ \Delta^{j} y(b+\nu) = 0, \quad j = 0, 1, \end{cases} \tag{3.1}$$

where

- $3 < \nu \leq 4$
- $b \in \mathbb{N}$
- $f : \{0, \cdots, b+1\} \times \mathbb{R} \to \mathbb{R}$ is continuous and nonnegative for $y \geq 0$ and $f(t, 0) = 0$ for $t \in \{0, \cdots, b+1\}$.

Note the following domains for each function appearing in problem (3.1):

- $D\{\Delta^{\nu}_{\nu-4} y\} = \{0, \cdots, b+1\}$
- $D\{y\} = \{\nu - 4, \cdots, b + \nu + 1\}$
- $D\{\Delta^i y\} = \{\nu - 4, \cdots, b + \nu + 1 - i\}$, for each $i \in \{0, 1\}$
- $D\{\Delta^j y\} = \{\nu - 4, \cdots, b + \nu + 1 - j\}$, for each $j \in \{0, 1\}$.

In particular, the unknown function in problem (3.1) satisfies

$$y : \{\nu - 4, \cdots, b + \nu + 1\} \to \mathbb{R}.$$

Using Theorem 5 and methods in Chapter 2, we prove the following theorem.

Theorem 24 *(Green's Function) For each fixed s, let $u(t, s)$ be the unique solution of the BVP*

$$\begin{cases} \Delta^{\nu}_{\nu-4} u(t) = 0 \\ u(\nu - 4, s) = \Delta u(\nu - 4, s) = 0 \\ u(b + \nu, s) = -x(b + \nu, s) \\ \Delta u(b + \nu, s) = -\Delta x(b + \nu, s) \end{cases} \tag{3.2}$$

where $x(t,s)$ *is the Cauchy function for* $\Delta^{\nu}_{\nu-4}u = 0$.

If $h : \mathbb{N}_0 \to \mathbb{R}$*, then*

$$y(t) = \int_0^{b+1} G(t,s)h(s)\Delta s$$

is the unique solution of the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4}y(t) = h(t), \quad t \in \{0, \cdots, b+1\} \\ y(\nu-4,s) = \Delta y(\nu-4,s) = 0 \\ y(b+\nu,s) = \Delta y(b+\nu,s) = 0 \end{cases}$$

where $G(t,s)$ *is the Green's function for the problem*

$$\begin{cases} \Delta^{\nu}_{\nu-4}y(t) = 0 \\ y(\nu-4) = \Delta y(\nu-4) = 0 \\ y(b+\nu) = \Delta y(b+\nu) = 0 \end{cases} \tag{3.3}$$

and is given by

$$G(t,s) = \begin{cases} v(t,s), \quad 0 \le s \le t-\nu+1 \le b+1 \\ u(t,s), \quad 0 \le t-\nu+2 \le s \le b+1 \end{cases}$$

where $v(t,s) := u(t,s) + x(t,s)$.

Proof: Similar to the proof of Theorem 8. See Appendix A.

3.2 The Derivation of Green's Function

Following the same method as in Chapter 2, we derive the Green's function for the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = 0, \quad t \in \{0, \cdots, b+1\} \\ \Delta^{i} y(\nu - 4) = 0, \quad i = 0, 1 \\ \Delta^{j} y(b+\nu) = 0, \quad j = 0, 1. \end{cases}$$

The Green's function for the homogeneous problem corresponding to (3.1) (see appendix A) is given by the following

$$G(t,s) = \begin{cases} v(t,s), \quad 0 \le s \le t - \nu + 1 \le b+1 \\ u(t,s), \quad 0 \le t - \nu + 2 \le s \le b+1 \end{cases} \tag{3.4}$$

where

$$u(t,s) = \frac{(b+\nu-s-1)^{\underline{\nu-2}}(\nu-1)(b+3)(s+1)}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} t^{\underline{\nu-2}}$$

$$- \frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} t^{\underline{\nu-1}}$$

and

$$v(t,s) = \frac{(b+\nu-s-1)^{\underline{\nu-2}}(\nu-1)(b+3)(s+1)}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} t^{\underline{\nu-2}}$$

$$- \frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} t^{\underline{\nu-1}} + \frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)}.$$

Note that if $\nu = 4$, then we get that the standard Green's function for the BVP

$$\begin{cases} \Delta^4 y = 0 \\ \Delta^i y(0) = 0, \quad i = 0, 1 \\ \Delta^j y(b+4) = 0, \quad j = 0, 1 \end{cases}$$

is given by

$$G(t,s) = \begin{cases} \frac{(b-s+3)^{\underline{2}}(s+1)}{2(b+5)^{\underline{2}}} t^{\underline{2}} - \frac{(b-s+3)^{\underline{2}}(b+2s+7)}{6(b+5)^{\underline{3}}} t^{\underline{3}} + \frac{(t-s-1)^{\underline{3}}}{6}, \\ \qquad\qquad 0 \le s+3 \le t \le b+4, \\ \frac{(b-s+3)^{\underline{2}}(s+1)}{2(b+5)^{\underline{2}}} t^{\underline{2}} - \frac{(b-s+3)^{\underline{2}}(b+2s+7)}{6(b+5)^{\underline{3}}} t^{\underline{3}}, \\ \qquad\qquad 0 \le t \le s+2 \le b+3. \end{cases}$$

Theorem 25 *Let $G(t,s)$ be the Green's function for (3.3). Then $G(t,s)$ satisfies the following properties: for each fixed $s \in \{0, \cdots, b+1\}$,*

1. $G(t,s) \ge 0$ *for* $t \in \{\nu - 4, \cdots, b + \nu + 1\}$.
2. $\displaystyle\int_0^{b+1} G(t,s)\Delta s \le \frac{2\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}$.

Proof: Similar to the proof of Theorem 9. See Appendix A.

Theorem 26 *(Comparison Theorem) Assume that u and v satisfy*

$$\begin{aligned} u(\nu-4) &= v(\nu-4), \quad \Delta u(\nu-4) \geq \Delta v(\nu-4), \\ u(b+\nu) &\geq v(b+\nu), \quad \Delta u(b+\nu) \geq \Delta v(b+\nu). \end{aligned}$$

In addition, suppose that $Lu(t) \geq Lv(t)$ *for* $t \in \{0, \cdots, b+1\}$, *where* $Ly := \Delta^{\nu}_{\nu-4} y$. *Then*

$$u(t) \geq v(t), \quad \textit{for} \quad t \in \{\nu-4, \cdots, b+\nu-1\}.$$

Proof: Similar to the proof of Theorem 10. See appendix A

3.3 Existence of a Unique Solution of a Nonlinear Boundary Value Problem

In this section we use the Contraction Mapping Theorem to establish the existence of a unique solution of the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y = f(t, y(t+\nu-2)), \quad t \in \{0, \cdots, b+1\} \\ y(\nu-4) = A, \quad \Delta y(\nu-4) = B \\ y(b+\nu) = C, \quad \Delta y(b+\nu) = D. \end{cases} \tag{3.5}$$

Theorem 27 *Assume* $f : \{0, \cdots, b+1\} \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfy a uniform Lipschitz condition with respect to the second variable on* $\{0, \cdots, b+1\} \times \mathbb{R}$ *with*

Lipschitz constant K. If

$$2\Gamma(b+\nu+2) < \frac{\Gamma(\nu+1)\Gamma(b+2)}{K}, \tag{3.6}$$

then the BVP (3.6) *has a unique solution.*

Proof: Let $\mathbb{B}$ be the Banach space of continuous functions on $\{0, \cdots, b+1\}$ with the norm

$$||y|| := max\{|y(t)| : t \in \{\nu-4, \cdots, b+\nu+1\}.$$

Note that y is a solution of the BVP (3.6) *iff y is a solution of the integral equation*

$$y(t) = z(t) + \int_0^{b+1} G(t,s) f(s, y(s+\nu-2))\Delta s, \quad t \in \{\nu-4, \cdots, b+\nu+1\},$$

where z is the solution of the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} z = 0 \\ z(\nu-4) = A, \quad \Delta z(\nu-4) = B \\ z(b+\nu) = C, \quad \Delta z(b+\nu) = D \end{cases}$$

and the $G(t,s)$ is the Green's function given for the BVP (3.3).

Define $T : \mathbb{B} \to \mathbb{B}$ by

$$Ty(t) = z(t) + \int_0^{b+1} G(t,s) f(s, y(s+\nu-2))\Delta s,$$

for $t \in \{\nu-4, \cdots, b+\nu+1\}$. Hence the BVP (3.6) *has a unique solution iff the operator T has a unique fixed point.*

We will use the Contraction Mapping Theorem to show that T has a unique fixed point in $\mathbb{B}$. Let $y_1, y_2 \in \mathbb{B}$ and consider

$$\begin{aligned}
|Ty_1(t) - Ty_2(t)| &= |z(t) + \int_0^{b+1} G(t,s) f(s, y_1(s+\nu-2))\Delta s \\
&- z(t) - \int_0^{b+1} G(t,s) f(s, y_2(s+\nu-2))\Delta s| \\
&= |\int_0^{b+1} G(t,s) f(s, y_1(s+\nu-2))\Delta s \\
&- \int_0^{b+1} G(t,s) f(s, y_2(s+\nu-2))|\Delta s \\
&\leq \int_0^{b+1} G(t,s)|f(s, y_1(s+\nu-2)) - f(s, y_2(s+\nu-2))|\Delta s \\
&\leq \int_0^{b+1} G(t,s) K|y_1(s+\nu-2) - y_2(s+\nu-2)|\Delta s \\
&\leq K \int_0^{b+1} G(t,s)\Delta s ||y_1 - y_2|| \\
&\leq KL||y_1 - y_2||,
\end{aligned}$$

where

$$L := \frac{2\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}.$$

for $t \in \{\nu-4, \cdots, b+\nu+1\}$, where for the last inequality we used Theorem 25. It follows that

$$||Ty_1 - Ty_2|| \leq KL||y_1 - y_2||,$$

where from (3.7) we have $KL < 1$. Hence T is a contraction mapping on $\mathbb{B}$, and by the Contraction Mapping Theorem we get the desired conclusion.

Example 28 *Consider the fractional boundary value problem*

$$\begin{cases} \Delta^{\pi}_{\pi-4} y = \dfrac{y(t+\pi-2)}{(e^t+35)^2}, \quad t \in \{0, \cdots, 8\} \\ y(\pi-4) = \Delta y(\pi-4) = 0, \\ y(7+\pi) = \Delta y(7+\pi) = 0. \end{cases} \tag{3.7}$$

The BVP (3.8) *is of the form* (3.6), *where*

$\nu = \pi,\ f(t,y) = \dfrac{y}{(e^t+35)^2},$
$b = 7$

In this, problem , we apply Theorem 27 *to show that for* K *sufficiently small* (3.8) *has a unique solution on* $\{\pi-4, \cdots, \pi+8\}$. *Note that for* $t \in \{0, \cdots, 8\}$ *and* $y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} |f(t,y_1) - f(t,y_2)| &= \left| \frac{y_1}{(e^t+35)^2} - \frac{y_2}{(e^t+35)^2} \right| \\ &= \frac{1}{(e^t+35)^2} |y_1 - y_2| \\ &\leq \frac{1}{36^2} |y_1 - y_2| \\ &= \frac{1}{1296} |y_1 - y_2|. \end{aligned}$$

So our Lipschitz constant is $K = \dfrac{1}{1296}$. *Since*

$$0 \leq K = \frac{1}{1296} \approx 0.00071 < 0.000853 = \frac{\Gamma(\pi+1)\Gamma(9)}{2\Gamma(9+\pi)}$$

Theorem 27 *implies the existence of a unique solution of* (3.8) *on* $\{\pi-4, \cdots, \pi+8\}$.

Appendix A

Proofs

A.1 Existence of The Green's Function

A.1.1 Proof of Theorem 24

first we show that the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = h(t) \\ y(\nu-4) = A, \quad \Delta y(\nu-4) = B \\ y(b+\nu) = C, \quad \Delta y(b+\nu) = D \end{cases}$$

has a unique solution. To show this BVP has a unique solution, it suffices to show that

$$\begin{cases} \Delta^{\nu}_{\nu-4} y = 0 \\ y(\nu-4) = \Delta y(\nu-4) = 0 \\ y(b+\nu) = \Delta y(b+\nu) = 0 \end{cases} \tag{A.1}$$

has only the trivial solution. Consider the given BVP, we know from Theorem 5 that the general solution of $\Delta^{\nu}_{\nu-4}y(t) = 0$ is of the form

$$u(t) = \alpha_1 t^{\underline{\nu-1}} + \alpha_2 t^{\underline{\nu-2}} + \alpha_3 t^{\underline{\nu-3}} + \alpha_4 t^{\underline{\nu-4}}$$

where $\alpha_i \in \mathbb{R}$ and $t \in \{\nu-4, \cdots, b+\nu+1\}$.

From the first boundary condition, we have

$$0 = u(\nu-4) = \sum_{i=1}^{4} \alpha_i (\nu-4)^{\underline{\nu-i}}.$$

Using

$$\begin{aligned}
(\nu-4)^{\underline{\nu-1}} &= \frac{\Gamma(\nu-3)}{\Gamma(-2)} = 0, \\
(\nu-4)^{\underline{\nu-2}} &= \frac{\Gamma(\nu-3)}{\Gamma(-1)} = 0, \\
(\nu-4)^{\underline{\nu-3}} &= \frac{\Gamma(\nu-3)}{\Gamma(0)} = 0, \\
(\nu-4)^{\underline{\nu-4}} &= \frac{\Gamma(\nu-3)}{\Gamma(1)} \neq 0,
\end{aligned}$$

we get that $\alpha_4 = 0$. Similarly, one can show that $\alpha_3 = 0$ from the second boundary condition. Hence

$$u(t) = \alpha_1 t^{\underline{\nu-1}} + \alpha_2 t^{\underline{\nu-2}}.$$

The last two boundary conditions lead to the following system

$$\begin{aligned}
\alpha_1 (b+\nu)^{\underline{\nu-1}} + \alpha_2 (b+\nu)^{\underline{\nu-2}} &= 0 \\
\alpha_1 (\nu-1)(b+\nu)^{\underline{\nu-2}} + \alpha_2 (\nu-2)(b+\nu)^{\underline{\nu-3}} &= 0.
\end{aligned}$$

In order to show that $u(t)$ is the trivial solution, we need to show that the above system has only the trivial solution $\alpha_1 = \alpha_2 = 0$. To see this we consider

$$\begin{vmatrix} (b+\nu)^{\underline{\nu-1}} & (b+\nu)^{\underline{\nu-2}} \\ (\nu-1)(b+\nu)^{\underline{\nu-2}} & (\nu-2)(b+\nu)^{\underline{\nu-3}} \end{vmatrix}$$

$$\begin{aligned} &= (\nu-2)(b+\nu)^{\underline{\nu-1}}(b+\nu)^{\underline{\nu-3}} - (\nu-1)(b+\nu)^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-2}} \\ &= (\nu-2)(b+3)^{\underline{2}}((b+\nu)^{\underline{\nu-3}})^2 - (\nu-1)(b+3)^2((b+\nu)^{\underline{\nu-3}})^2 \\ &= (b+3)((b+\nu)^{\underline{\nu-3}})^2[(\nu-2)(b+2) - (\nu-1)(b+3)] \\ &= (b+3)((b+\nu)^{\underline{\nu-3}})^2[-(\nu+b+1)] \neq 0, \end{aligned}$$

since $3 < \nu \leq 4$. Thus the BVP (3.4) has only the trivial solution. It then follows by standard arguments that the non-homogenous BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = h(t), \quad t \in \{\nu-4, \cdots, b+\nu+1\} \\ y(\nu-4) = A, \quad \Delta y(\nu-4) = B \\ y(b+\nu) = C, \quad \Delta y(b+\nu) = D \end{cases}$$

has unique solution. Since for each fixed s, $u(t,s)$ satisfies a BVP of this form, we get that $u(t,s)$ is uniquely determined. It follows that $v(t,s)$ and $G(t,s)$ are uniquely determined. Since for each fixed s, $u(t,s)$ is a solution of $\Delta^{\nu}_{\nu-4} x = 0$ and $x(t,s)$ is a solution for $t \geq s+\nu-3$, we have that for each fixed s, $v(t,s) = u(t,s) + x(t,s)$ is also a solution of $\Delta^{\nu}_{\nu-4} x = 0$ for $s+\nu-1 \leq t \leq b+\nu+1$. It follows that $v(t,s)$ satisfies the boundary conditions at $b+\nu$.

Now, let $G(t,s)$ be as in the statement of this theorem and consider

$$\begin{aligned} y(t) &= \int_0^{b+1} G(t,s)h(s)\Delta s, \quad t \in \{\nu-4, \cdots, b+\nu+1\} \\ &= \int_0^{t-\nu+2} G(t,s)h(s)\Delta s + \int_{t-\nu+2}^{b+1} G(t,s)h(s)\Delta s \\ &= \int_0^{t-\nu+2} v(t,s)h(s)\Delta s + \int_{t-\nu+2}^{b+1} u(t,s)h(s)\Delta s. \end{aligned}$$

Note that even though $v(t,s)$ is only defined for $0 \le s \le t-\nu+1$, the upper limit of integration in the first term is okay since the delta integral does not depend on the value of the integrand at the upper limit of integration. Hence

$$\begin{aligned} y(t) &= \int_0^{t-\nu+2} [u(t,s)+x(t,s)]h(s)\Delta s + \int_{t-\nu+2}^{b+1} u(t,s)h(s)\Delta s \\ &= \int_0^{b+1} u(t,s)h(s)\Delta s + \int_0^{t-\nu+2} x(t,s)h(s)\Delta s. \end{aligned}$$

Since $x(t,t-\nu+2) = x(t,t-\nu+1) = 0$

$$\begin{aligned} y(t) &= \int_0^{b+1} u(t,s)h(s)\Delta s + \int_0^{t-\nu} x(t,s)h(s)\Delta s \\ &= \int_0^{b+1} u(t,s)h(s)\Delta s + q(t) \end{aligned}$$

where, by the variation of constants formula, q is the solution of the IVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} q(t) = h(t) \\ q(\nu-4) = q(\nu-3) = q(\nu-2) = q(\nu-1) = 0. \end{cases}$$

It follows that

$$\begin{aligned} \Delta^{\nu}_{\nu-4} y(t) &= \int_0^{b+1} \Delta^{\nu}_{\nu-4} u(t,s) h(s) \Delta s + \Delta^{\nu}_{\nu-4} q(t) \\ &= 0 + \Delta^{\nu}_{\nu-4} q(t) = h(t). \end{aligned}$$

Hence y is a solution of the non-homogenous equation. It remains to show that y satisfies the boundary conditions. Note that

$$y(\nu-4) = \int_0^{b+1} u(\nu-4,s) h(s) \Delta s + q(\nu-4) = 0$$

and

$$\Delta y(\nu-4) = \int_0^{b+1} \Delta u(\nu-4,s) h(s) \Delta s + \Delta q(\nu-4) = 0.$$

Hence y satisfies the first two boundary conditions. Next, we show that the Green's function satisfies the last two boundary conditions. Recall that

$$G(t,s) = \begin{cases} u(t,s), & 0 \le t-\nu+2 \le s \le b+1 \\ v(t,s), & 0 \le s \le t-\nu+1 \le b+1. \end{cases}$$

Now, consider $v(t,s) = u(t,s) + x(t,s)$, we have

$$\begin{aligned} v(b+\nu, s) &= u(b+\nu, s) + x(b+\nu, s) \\ &= 0 \quad \{\text{from the third boundary condition on u}\} \end{aligned}$$

and

$$\begin{aligned} \Delta v(b+\nu, s) &= \Delta u(b+\nu, s) + \Delta x(b+\nu, s) \\ &= 0 \quad \{\text{from the fourth boundary condition on u}\}. \end{aligned}$$

Then

$$\begin{aligned} y(b+\nu) &= \int_0^{b+1} G(b+\nu, s) h(s) \Delta s \\ &= \int_0^{b+1} v(b+\nu, s) h(s) \Delta s \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \Delta y(b+\nu) &= \int_0^{b+1} \Delta G(b+\nu, s) h(s) \Delta s \\ &= \int_0^{b+1} \Delta v(b+\nu, s) h(s) \Delta s \\ &= 0 \end{aligned}$$

since $v(t,s)$ satisfies the last two boundary conditions and hence $y(t)$ satisfies the last two boundary conditions. Therefore, $y(t)$ is the unique solution of the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = h(t) \\ y(\nu-4,s) = \Delta y(\nu-4,s) = 0 \\ y(b+\nu,s) = \Delta y(b+\nu,s) = 0. \end{cases}$$

A.2 The Derivation of Green's Function

Following the same method as in Chapter 2, we derive the Green's function for the BVP

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = 0, \quad t \in \{0, \cdots, b+1\} \\ \Delta^{i} y(\nu-4) = 0, \quad i = 0, 1 \\ \Delta^{j} y(b+\nu) = 0, \quad j = 0, 1. \end{cases}$$

Since for each fixed s, $u(t,s)$ solves the fractional difference equation $\Delta^{\nu}_{\nu-4} u = 0$ we have that

$$u(t,s) = \sum_{i=1}^{4} \alpha_i(s) t^{\underline{\nu-i}}$$

for $t \in \{\nu-4, \cdots, b+\nu+1\}$.

From the first boundary condition, we have

$$0 = u(\nu-4,s) = \sum_{i=1}^{4} \alpha_i(s)(\nu-4)^{\underline{\nu-i}}.$$

Since

$$\begin{aligned}(\nu-4)^{\underline{\nu-1}} &= \frac{\Gamma(\nu-3)}{\Gamma(-2)}=0,\\ (\nu-4)^{\underline{\nu-2}} &= \frac{\Gamma(\nu-3)}{\Gamma(-1)}=0,\\ (\nu-4)^{\underline{\nu-3}} &= \frac{\Gamma(\nu-3)}{\Gamma(0)}=0,\\ (\nu-4)^{\underline{\nu-4}} &= \frac{\Gamma(\nu-3)}{\Gamma(1)}\neq 0,\end{aligned}$$

we get that $\alpha_4 = 0$. Similarly, one can show that $\alpha_3 = 0$ from the second boundary condition. So we have that

$$u(t,s)=\alpha_1(s)t^{\underline{\nu-1}}+\alpha_2(s)t^{\underline{\nu-2}}.$$

Next, we apply the last two boundary conditions and the fact that the Cauchy function is given by

$$x(t,s)=\frac{(t-\sigma(s))^{\underline{\nu-1}}}{\Gamma(\nu)},$$

to obtain the following system of equations

$$\begin{cases}\alpha_2(s)(b+\nu)^{\underline{\nu-2}}+\alpha_1(s)(b+\nu)^{\underline{\nu-1}}=-\dfrac{(b+\nu-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}\\ \alpha_2(s)(\nu-2)(b+\nu)^{\underline{\nu-3}}+\alpha_1(s)(\nu-1)(b+\nu)^{\underline{\nu-2}}=-\dfrac{(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu-1)}.\end{cases}$$

Multiplying the first equation by $(\nu-2)$ and the second equation by $-(b+3)$ and adding these two equations we get

$$\alpha_1(s)[(\nu-2)(b+\nu)^{\underline{\nu-1}}-(b+3)(\nu-1)(b+\nu)^{\underline{\nu-2}}]=\frac{(b+3)(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu-1)}-\frac{(\nu-2)(b+\nu-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}$$

$\Longrightarrow$

$$\alpha_1(s)[(\nu-2)(b+2)(b+\nu)^{\underline{\nu-2}} - (b+3)(\nu-1)(b+\nu)^{\underline{\nu-2}}] = \frac{(b+3)(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} - \frac{(\nu-2)(b+\nu-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}$$

$\Longrightarrow$

$$\alpha_1(s)(b+\nu)^{\underline{\nu-2}}[(\nu-2)(b+2) - (b+3)(\nu-1)] = \frac{(b+3)(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu-1)} - \frac{(\nu-2)(b-s+1)(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}$$

$\Longrightarrow$

$$-\alpha_1(s)(b+\nu)^{\underline{\nu-2}}(b+\nu+1) = \frac{(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}[(b+3)(\nu-1) - (b-s+1)(\nu-2)]$$

$\Longrightarrow$

$$-\alpha_1(s)(b+\nu)^{\underline{\nu-2}}(b+\nu+1) = \frac{(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}(2\nu+b+s(\nu-2)-1)$$

$\Longrightarrow$

$$\alpha_1(s) = -\frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}.$$

So

$$\alpha_2(s) = \frac{-\alpha_1(s)(b+\nu)^{\underline{\nu-1}} - \dfrac{(b+\nu-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}}{(b+\nu)^{\underline{\nu-2}}}$$

$\Longrightarrow$

$$\alpha_2(s) = \frac{\dfrac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)(b+2)(b+\nu)^{\underline{\nu-2}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} - \dfrac{(b+\nu-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}}{(b+\nu)^{\underline{\nu-2}}}$$

$\Longrightarrow$

$$\alpha_2(s) = \frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)(b+2) - (b+\nu+1)(b+\nu-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}$$

$\Longrightarrow$

$$\alpha_2(s) = \frac{(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b+2) - (b+\nu+1)(b-s+1)]}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}.$$

Therefore,

$$u(t,s) = -\frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} t^{\underline{\nu-1}} +$$

$$\frac{(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b+2) \,-\, (b+\nu+1)(b-s+1)]}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} t^{\underline{\nu-2}}$$

and $v(t,s) = u(t,s) + x(t,s)$.

A.2.1 Proof of Theorem 25

Proof: First we prove 1.

- For $t = \nu - 4$

  $G(\nu-4, s) = u(\nu-4, s) = 0.$

- For $t = \nu - 3$

  $G(\nu-3, s) = u(\nu-3, s) = 0.$

- For $t \in \{\nu-2, \cdots, s+\nu-2\}$, we have

  $G(t,s) = u(t,s)$
  $$= \frac{(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b+2) - (b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}$$

  $$-\frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)t^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \geq 0$$

  $\iff$

  $(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b+2) - (b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}$

  $-(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)t^{\underline{\nu-1}} \geq 0$

$\iff$

$[(2\nu + b + s(\nu-2) - 1)(b+2) - (b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}$

$-(2\nu + b + s(\nu-2) - 1)t^{\underline{\nu-1}} \geq 0$

$\iff$

$[(2\nu + b + s(\nu-2) - 1)(b+2) - (b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}$

$-(2\nu + b + s(\nu-2) - 1)(t-\nu+2)t^{\underline{\nu-2}} \geq 0$

$\iff$

$(2\nu + b + s(\nu-2) - 1)[(b+2) - (t-\nu+2)] - (b-s+1)(b+\nu+1) \geq 0$

$\iff$

$(2\nu + b + s(\nu-2) - 1)(b-t+\nu) - (b-s+1)(b+\nu+1) \geq 0$

$\Longleftarrow$

$(2\nu + b + s(\nu-2) - 1)(b-s+2) - (b-s+1)(b+\nu+1) \geq 0$, which is true since the first term is greater than the second term.

- For $t \in \{s+\nu-1, \cdots, b+\nu+1\}$, we have

$G(t,s) = v(t,s) =$

$$\frac{(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu + b + s(\nu-2) - 1)(b+2) - (b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu + b + s(\nu-2) - 1)t^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} + \frac{(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)} \geq 0$$

$\iff$

$$\frac{(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu + b + s(\nu-2) - 1)(b+2) - (b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}$$

$$-\frac{(b+\nu-s-1)^{\underline{\nu-2}}(2\nu + b + s(\nu-2) - 1)t^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}$$

$$+\frac{(b+\nu+1)(b+\nu)^{\underline{\nu-2}}(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \geq 0$$

$$\Longleftrightarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b+2)-(b-s+1)(b+\nu+1)]t^{\underline{\nu-2}}$$
$$-(b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)t^{\underline{\nu-1}}$$
$$+(b+\nu+1)(b+\nu)^{\underline{\nu-2}}(t-s-1)^{\underline{\nu-1}} \geq 0$$

$$\Longleftrightarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}t^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b+2)-(b-s+1)(b+\nu+1)-$$
$$(2\nu+b+s(\nu-2)-1)(t-\nu+2)]$$
$$+(b+\nu+1)(b+\nu)^{\underline{\nu-2}}(t-s-1)^{\underline{\nu-1}} \geq 0$$

$$\Longleftrightarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b-t+\nu)-(b-s+1)(b+\nu+1)]$$
$$+\frac{(b+\nu+1)(b+\nu)^{\underline{\nu-2}}(t-s-1)^{\underline{\nu-1}}}{t^{\underline{\nu-2}}} \geq 0$$

$$\Longleftarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)(b-(b+\nu-1)+\nu)-(b-s+1)(b+\nu+1)]$$
$$+\frac{(b+\nu+1)(b+\nu)^{\underline{\nu-2}}((s+\nu)-s-1)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-2}}} \geq 0$$

$$\Longleftrightarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)-(b-s+1)(b+\nu+1)]$$
$$+\frac{(b+\nu+1)(b+\nu)^{\underline{\nu-2}}(\nu-1)^{\underline{\nu-1}}}{(b+\nu-1)^{\underline{\nu-2}}} \geq 0$$

$$\Longleftrightarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)-(b-s+1)(b+\nu+1)]$$
$$+\frac{(b+\nu+1)(b+\nu)^{\underline{\nu-2}}\Gamma(\nu)}{(b+\nu-1)^{\underline{\nu-2}}} \geq 0$$

$$\Longleftrightarrow$$

$$(b+\nu-s-1)^{\underline{\nu-2}}[(2\nu+b+s(\nu-2)-1)-(b-s+1)(b+\nu+1)]$$

$$+\frac{(b+\nu+1)(b+\nu)^{\underline{\nu-2}}\Gamma(\nu)\Gamma(b+2)}{(b+\nu)} \geq 0,$$

since the first and the third term are positive and greater than the second term.

Thus $G(t,s) \geq 0$ for $t \in \{\nu-4, \cdots, b+\nu+1\}$.

Next we prove 2. Consider

$$\begin{aligned}
\int_0^{b+1} G(t,s)\Delta s &= \int_0^{t-\nu+2} v(t,s)\Delta s + \int_{t-\nu+2}^{b+1} u(t,s)\Delta s \\
&= \int_0^{t-\nu+2} [u(t,s)+x(t,s)]\Delta s + \int_{t-\nu+2}^{b+1} u(t,s)\Delta s \\
&= \int_0^{b+1} u(t,s)\Delta s + \int_0^{t-\nu+2} x(t,s)\Delta s \\
&= \int_0^{b+1} u(t,s)\Delta s + \int_0^{t-\nu} x(t,s)\Delta s.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^{t-\nu} x(t,s)\Delta s &= \int_0^{t-\nu} \frac{(t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}\Delta s \\
&= -\frac{(t-s)^{\underline{\nu}}}{\nu\Gamma(\nu)}\Big|_{s=0}^{s=t-\nu} \\
&= -\frac{\nu^{\underline{\nu}}}{\Gamma(\nu+1)} + \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} = \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} - 1.
\end{aligned}$$

Also

$$\begin{aligned}
\int_0^{b+1} u(t,s)\Delta s &= \alpha_1(t)\int_0^{b+1} (\nu-1)(b+3)(s+1)(b+\nu-s-1)^{\underline{\nu-2}}\Delta s \\
&+ \alpha_2(t)\int_0^{b+1} (2\nu+b+s(\nu-2)-1)(b+\nu-s-1)^{\underline{\nu-2}}\Delta s,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_1(t) &= \frac{t^{\underline{\nu-2}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \\
\alpha_2(t) &= \frac{t^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}}.
\end{aligned}$$

Note that

$$
\begin{aligned}
&\int_0^{b+1} (b+\nu-s-1)^{\underline{\nu-2}}[(\nu-1)(b+3)(s+1))]\Delta s \\
&= -[(s+1)(b+3)](b+\nu-s)^{\underline{\nu-1}}|_{s=0}^{s=b+1} - \frac{(b+3)(b+\nu-s)^{\underline{\nu}}}{\nu}|_{s=0}^{s=b+1} \\
&= -(b+3)[(b+2)(\nu-1)^{\underline{\nu-1}} - (b+\nu)^{\underline{\nu-1}} + \frac{(\nu-1)^{\underline{\nu}}}{\nu} - \frac{(b+\nu)^{\underline{\nu}}}{\nu}] \\
&= -(b+3)[(b+2)\Gamma(\nu) - (b+\nu)^{\underline{\nu-1}} - \frac{(b+1)(b+\nu)^{\underline{\nu-1}}}{\nu}] \\
&= -(b+3)[(b+2)\Gamma(\nu) - \frac{(b+\nu)^{\underline{\nu-1}}(b+\nu+1)}{\nu}] \\
&= (b+3)[\frac{(b+\nu)^{\underline{\nu-1}}(b+\nu+1)}{\nu} - (b+2)\Gamma(\nu)].
\end{aligned}
$$

Next we look at

$$
\begin{aligned}
&\int_0^{b+1} (b+\nu-s-1)^{\underline{\nu-2}}(2\nu+b+s(\nu-2)-1)\Delta s \\
&= -\frac{(2\nu+b+s(\nu-2)-1)(b+\nu-s)^{\underline{\nu-1}}}{(\nu-1)}|_{s=0}^{s=b+1} + \int_0^{b+1} \frac{(\nu-2)(b+\nu-s-1)^{\underline{\nu-1}}}{(\nu-1)}\Delta s \\
&= -\frac{1}{(\nu-1)}[\Gamma(\nu)(b+3)(\nu-1) - (b+2\nu-1)(b+\nu)^{\underline{\nu-1}} + \frac{(\nu-2)(\nu-1)^{\underline{\nu}}}{\nu} \\
&- \frac{(\nu-2)(b+\nu)^{\underline{\nu}}}{\nu}] \\
&= -\frac{1}{(\nu-1)}[\Gamma(\nu)(\nu-1)(b+3) - (b+2\nu-1)(b+\nu)^{\underline{\nu-1}} - \frac{(\nu-2)(b+1)(b+\nu)^{\underline{\nu-1}}}{\nu}] \\
&= -\Gamma(\nu)(b+3) + \frac{(b+\nu)^{\underline{\nu-1}}(2\nu b+2\nu^2-2b-2)}{\nu^{\underline{2}}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^{b+1} u(t,s)\Delta s &= \alpha_1(t)[(b+3)(-\Gamma(\nu)(b+2) + \frac{(b+\nu+1)(b+\nu)^{\underline{\nu-1}}}{\nu}] \\
&+ \alpha_2(t)[-\Gamma(\nu)(b+3) + \frac{(2\nu b+2\nu^2-2b-2)(b+\nu)^{\underline{\nu-1}}}{\nu^{\underline{2}}}].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^{b+1} G(t,s)\Delta s &= \frac{t^{\underline{\nu-2}}[\frac{(b+3)(b+\nu+1)(b+\nu)^{\underline{\nu-1}}}{\nu} - (b+3)^{\underline{2}}\Gamma(\nu)]}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \\
&- \frac{t^{\underline{\nu-1}}[\frac{(b+\nu)^{\underline{\nu-1}}(2\nu b+2\nu^2-2b-2)}{\nu^{\underline{2}}} - \Gamma(\nu)(b+3)]}{\Gamma(\nu)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \\
&+ \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} - 1 \\
&= \frac{t^{\underline{\nu-2}}(b+\nu)^{\underline{\nu-1}}(b+3)(b+\nu+1)(\nu-1)}{\Gamma(\nu+1)(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \\
&- \frac{t^{\underline{\nu-2}}(2\nu b+2\nu^2-2b-2)(t-\nu+2)(b+\nu)^{\underline{\nu-2}}}{\Gamma(\nu+1)(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \\
&+ \frac{t^{\underline{\nu-2}}(\nu-1)\Gamma(\nu+1)(b+3)(t-\nu+2-b-2)}{\Gamma(\nu+1)(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} + \frac{t^{\underline{\nu}}}{\Gamma(\nu+1)} - 1 \\
&\leq \frac{(b+3)(b+\nu+1)^{\underline{\nu-2}}[(\nu-1)(b+\nu+1)(b+\nu)^{\underline{\nu-1}} + (\nu-1)\Gamma(\nu+1)}{(\nu-1)\Gamma(\nu+1)(b+\nu+1)(b+\nu)^{\underline{\nu-2}}} \\
&+ \frac{\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)} - 1 \\
&= \frac{\frac{\Gamma(b+\nu+2)(\nu-1)}{\Gamma(b+2)} + \Gamma(\nu+1)(\nu-1)}{\Gamma(\nu+1)(\nu-1)} + \frac{\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)} - 1 \\
&= \frac{2\Gamma(b+\nu+2)}{\Gamma(\nu+1)\Gamma(b+2)}.
\end{aligned}
$$

A.2.2 Proof of Theorem 26

Put $w(t) = u(t) - v(t)$ and let $h(t)$ be defined by $h(t+\nu-2) = Lw(t) = Lu(t) - Lv(t) \geq 0$, $t \in \{0, \cdots, b+1\}$. Then it follows that w is a solution of the BVP

$$\begin{aligned}
Lw(t) &= h(t+\nu-2) \\
w(\nu-4) &= C_1 := u(\nu-4) - v(\nu-4) = 0, \\
\Delta w(\nu-4) &= C_2 := \Delta u(\nu-4) - \Delta v(\nu-4) \geq 0, \\
w(b+\nu) &= C_3 := u(b+\nu) - v(b+\nu) \geq 0, \\
\Delta w(b+\nu) &= C_4 := \Delta u(b+\nu) - \Delta v(b+\nu) \leq 0.
\end{aligned}$$

Then $w(t)$ is given by the formula

$$w(t) = \phi(t) + \int_0^{b+1} G(t,s)h(s)\Delta s, \quad t \in \{\nu-4, \cdots, b+\nu-1\}.$$

where ϕ is the solution of the BVP

$$\begin{aligned}
L\phi &= 0 \\
\phi(\nu-4) &= 0, \quad \Delta\phi(\nu-4) = C_2, \\
\phi(b+\nu) &= C_3, \quad \Delta\phi(b+\nu) = C_4
\end{aligned}$$

and $G(t,s)$ is the Green's function for the BVP (3.3). Moreover, Theorem 25 shows that $G(t,s) \geq 0$ on its domain. So as $h(t) \geq 0$, it follows that

$$w(t) = \phi(t) + \int_0^{b+1} G(t,s)h(s)\Delta s \geq 0, \quad t \in \{\nu-4, \cdots, b+\nu-1\},$$

provided

$$\phi(t) \geq 0 \quad \text{on} \quad \{\nu-4, \cdots, b+\nu-1\}.$$

To see this, first note that

$$\phi(t) = a_1 t^{\underline{\nu-1}} + a_2 t^{\underline{\nu-2}} + a_3 t^{\underline{\nu-3}} + a_4 t^{\underline{\nu-4}}.$$

From the first boundary condition on ϕ, we get that $a_4 = 0$. From the second boundary condition on ϕ, we get $a_3 = \dfrac{C_2}{\Gamma(\nu-2)}$. Now we use the last two boundary conditions on ϕ to find a_2 and a_1. Using the boundary conditions at $b+\nu$ we get the following system

$$C_3 = a_1(b+\nu)^{\underline{\nu-1}} + a_2(b+\nu)^{\underline{\nu-2}} + \left(\frac{C_2}{\Gamma(\nu-2)}\right)(b+\nu)^{\underline{\nu-3}}$$

$$C_4 = a_1(\nu-1)(b+\nu)^{\underline{\nu-2}} + a_2(\nu-2)(b+\nu)^{\underline{\nu-3}} + \left(\frac{C_2}{\Gamma(\nu-2)}\right)(\nu-3)(b+\nu)^{\underline{\nu-4}}.$$

Multiplying the first equation by $(\nu-1)$ and the second equation by $-(b+2)$, we get the following

$$a_2[(\nu-1)(b+\nu)^{\underline{\nu-2}} - (\nu-2)(b+2)(b+\nu)^{\underline{\nu-3}}] = (\nu-1)C_3 - (b+2)C_4$$
$$- \left(\frac{C_2}{\Gamma(\nu-2)}\right)[(\nu-1)(b+\nu)^{\underline{\nu-3}} - (\nu-3)(b+2)(b+\nu)^{\underline{\nu-4}}]$$

$$\Longrightarrow$$

$$a_2[(\nu-1)(b+\nu)^{\underline{\nu-2}} - (\nu-2)(b+2)(b+\nu)^{\underline{\nu-3}}] = (\nu-1)C_3 - (b+2)C_4$$
$$- \left(\frac{C_2}{\Gamma(\nu-2)}\right)(b+\nu)^{\underline{\nu-4}}[(\nu-1)(b+4) - (\nu-3)(b+2)]$$

$$\Longrightarrow$$

$$a_2(b+\nu+1)(b+\nu)^{\underline{\nu-3}} = (\nu-1)C_3 - (b+2)C_4$$
$$-\left(\frac{C_2}{\Gamma(\nu-2)}\right)(b+\nu)^{\underline{\nu-4}}2(b+\nu+1)$$
$$\Longrightarrow$$
$$a_2(b+\nu+1)(b+5)^{\underline{2}}(b+\nu)^{\underline{\nu-5}} = (\nu-1)C_3 - (b+2)C_4$$
$$-\left(\frac{C_2}{\Gamma(\nu-2)}\right)(b+5)(b+\nu)^{\underline{\nu-5}}2(b+\nu+1)$$
$$\Longrightarrow$$
$$a_2 = \frac{(\nu-1)C_3-(b+2)C_4}{(b+\nu+1)(b+5)^{\underline{2}}(b+\nu)^{\underline{\nu-5}}} - \left(\frac{C_2}{\Gamma(\nu-2)}\right)\frac{2}{(b+4)}.$$

Next we find a_1:

$$a_1 = \frac{C_3}{(b+\nu)^{\underline{\nu-1}}} - \frac{a_2(b+\nu)^{\underline{\nu-2}}}{(b+\nu)^{\underline{\nu-1}}} - \left(\frac{C_2}{\Gamma(\nu-2)}\right)\frac{(b+\nu)^{\underline{\nu-3}}}{(b+\nu)^{\underline{\nu-1}}}$$
$$\Longrightarrow$$
$$a_1 = \frac{C_3\Gamma(b+2)}{\Gamma(b+\nu+1)} - \frac{a_2}{(b+2)} - \left(\frac{C_2}{\Gamma(\nu-2)}\right)\frac{1}{(b+3)^{\underline{2}}}$$
$$\Longrightarrow$$
$$a_1 = \frac{C_3\Gamma(b+2)}{\Gamma(b+\nu+1)} - \frac{[(\nu-1)C_3-(b+2)C_4]\Gamma(b+4)}{\Gamma(b+\nu+2)(b+2)}$$
$$+\left(\frac{C_2}{\Gamma(\nu-2)}\right)\left[\frac{2}{(b+4)(b+2)} - \frac{1}{(b+3)^{\underline{2}}}\right]$$
$$\Longrightarrow$$
$$a_1 = \frac{C_3\Gamma(b+2)}{\Gamma(b+\nu+1)} - \frac{[(\nu-1)C_3-(b+2)C_4]\Gamma(b+4)}{\Gamma(b+\nu+2)(b+2)}$$
$$+\left(\frac{C_2}{\Gamma(\nu-2)}\right)\frac{1}{(b+4)^{\underline{2}}}.$$

Hence

$$\phi(t) = a_1 t^{\underline{\nu-1}} + a_2 t^{\underline{\nu-2}} + a_3 t^{\underline{\nu-3}} + a_4 t^{\underline{\nu-4}},$$

where a_1, a_2, a_3 and a_4 are given above. Furthermore,

$$\begin{aligned}
\phi(t) &= C_2\left(\frac{t^{\underline{\nu-1}}}{(b+4)^{\underline{2}}\Gamma(\nu-2)} - \frac{2t^{\underline{\nu-2}}}{(b+4)\Gamma(\nu-2)} + \frac{t^{\underline{\nu-3}}}{\Gamma(\nu-2)}\right)\\
&+ C_3\left[\left(\frac{\Gamma(b+2)}{\Gamma(b+\nu+1)} - \frac{(\nu-1)\Gamma(b+4)}{\Gamma(b+\nu+2)(b+2)}\right)t^{\underline{\nu-1}} + \frac{(\nu-1)\Gamma(b+4)t^{\underline{\nu-2}}}{\Gamma(b+\nu+2)}\right]\\
&+ C_4\left[\frac{\Gamma(b+4)t^{\underline{\nu-1}}}{\Gamma(b+\nu+2)} - \frac{(b+2)\Gamma(b+4)t^{\underline{\nu-2}}}{\Gamma(b+\nu+2)}\right].
\end{aligned}$$

Hence to complete the proof we just need to show that the coefficients of C_2 and C_3 are nonnegative and the coefficient of C_4 is nonpositive on $\{\nu-4, \cdots, b+\nu-1\}$. First we show that the coefficient of C_4 is nonpositive:

$$\left[\frac{\Gamma(b+4)t^{\underline{\nu-1}}}{\Gamma(b+\nu+2)} - \frac{(b+2)\Gamma(b+4)t^{\underline{\nu-2}}}{\Gamma(b+\nu+2)}\right]$$

$$\begin{aligned}
&= \frac{\Gamma(b+4)t^{\underline{\nu-2}}}{\Gamma(b+\nu+2)}[(t-\nu+2)-(b+2)]\\
&\leq \frac{\Gamma(b+4)t^{\underline{\nu-2}}}{\Gamma(b+\nu+2)}[(b+1)-(b+2)] \leq 0.
\end{aligned}$$

Next we show that the coefficient of C_3 is nonnegative:

$$\frac{\Gamma(b+2)t^{\underline{\nu-1}}}{\Gamma(b+\nu+1)} - \frac{(\nu-1)\Gamma(b+4)t^{\underline{\nu-1}}}{(b+2)\Gamma(b+\nu+2)} + \frac{(\nu-1)\Gamma(b+4)t^{\underline{\nu-2}}}{\Gamma(b+\nu+2)}$$

$$= \frac{\Gamma(b+2)t^{\underline{\nu-1}}}{\Gamma(b+\nu+1)} + \frac{(\nu-1)\Gamma(b+4)t^{\underline{\nu-2}}}{(b+2)\Gamma(b+\nu+2)}[(b+2)-(t-\nu+2)]$$

$$\geq \frac{\Gamma(b+2)t^{\underline{\nu-1}}}{\Gamma(b+\nu+1)} + \frac{(\nu-1)\Gamma(b+4)t^{\underline{\nu-2}}}{(b+2)\Gamma(b+\nu+2)}[(b+2)-(b+1)] \geq 0.$$

Finally we show that the coefficient of C_2 is nonnegative:

$$\frac{t^{\underline{\nu-1}}}{(b+4)^{\underline{2}}\Gamma(\nu-2)} - \frac{2t^{\underline{\nu-2}}}{(b+4)\Gamma(\nu-2)} + \frac{t^{\underline{\nu-3}}}{\Gamma(\nu-2)}$$

$$= \frac{t^{\underline{\nu-3}}}{(b+4)^{\underline{2}}\Gamma(\nu-2)}[(t-\nu+3)^{\underline{2}} - 2(t-\nu+3)(b+3) + (b+4)^{\underline{2}}]$$

$$\geq \frac{t^{\underline{\nu-3}}}{(b+4)^{\underline{2}}\Gamma(\nu-2)}[(b+2)^{\underline{2}} - 2(b+3)^{\underline{2}} + (b+4)^{\underline{2}}]$$

$$= \frac{t^{\underline{\nu-3}}}{(b+4)^{\underline{2}}\Gamma(\nu-2)}[-(b+2)(b+5) + (b+4)^{\underline{2}}] \geq 0.$$

Therefore,

$$u(t) \geq v(t), \quad \text{for} \quad t \in \{\nu-4, \cdots, b+\nu-1\}.$$

Appendix B

Definitions and Theorems

We provide some definitions and theorems that have been used in this dissertation without motivation or proof.

Theorem 29 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, with $N-1 < \nu \leq N$. The following two definitions for the fractional difference $\Delta_a^\nu f(t) : \mathbb{N}_{a+N-\nu} \to \mathbb{R}$ are equivalent:*

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \tag{B.1}$$

$$\Delta_a^\nu f(t) := \begin{cases} \dfrac{1}{\Gamma(-\nu)} \displaystyle\sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & N-1 < \nu < N \\ \Delta^N f(t), & \nu = N. \end{cases} \tag{B.2}$$

Definition 30 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then*

- *the ν^{th}-order fractional sum of f is given by*

$$\Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s), \quad t \in \mathbb{N}_{a+\nu}.$$

- *the ν^{th}-order fractional difference of f is given by*

$$\Delta_a^\nu f(t) := \begin{cases} \dfrac{1}{\Gamma(-\nu)} \displaystyle\sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & \nu \notin \mathbb{N}, \quad t \in \mathbb{N}_{a+N-\nu}. \\ \Delta^N f(t), \quad \nu = N \in \mathbb{N}, & t \in \mathbb{N}_{a+N-\nu}. \end{cases}$$

Lemma 31 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given. For any $k \in \mathbb{N}_0$ and $\mu > 0$ with $M-1 < \mu \leq M$, we have*

$$\Delta^k \Delta_a^{-\mu} f(t) = \Delta_a^{k-\mu} f(t), \quad for \quad t \in \mathbb{N}_{a+\mu}. \tag{B.3}$$

$$\Delta^k \Delta_a^{\mu} f(t) = \Delta_a^{k+\mu} f(t), \quad for \quad t \in \mathbb{N}_{a+M-\mu}. \tag{B.4}$$

Theorem 32 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given, and suppose $\nu, \mu > 0$ with $N-1 < \nu \leq N$. Then*

$$\Delta_{a+\mu}^{\nu} \Delta_a^{-\mu} f(t) = \Delta_a^{\nu-\mu} f(t), \quad for \ t \in \mathbb{N}_{a+\mu+N-\nu}.$$

Theorem 33 *Let $f : \mathbb{N}_a \to \mathbb{R}$ be given and suppose $k \in \mathbb{N}_0$ and $\nu > 0$. Then for $t \in \mathbb{N}_{a+\nu}$,*

$$\Delta_a^{-\nu} \Delta^k f(t) = \Delta_a^{k-\nu} f(t) - \sum_{j=0}^{k-1} \frac{\Delta^j f(a)}{\Gamma(\nu-k+j+1)} (t-a)^{\underline{\nu-k+j}}.$$

Moreover, if $\mu > 0$ with $M-1 < \mu \leq M$, then for $t \in \mathbb{N}_{a+M-\mu+\nu}$,

$\Delta_{a+M-\mu}^{-\nu} \Delta_a^{\mu} f(t) =$

$$\Delta_a^{\mu-\nu} f(t) - \sum_{j=0}^{M-1} \frac{\Delta_a^{j-(M-\mu)} f(a+M-\mu)}{\Gamma(\nu-M+j+1)} (t-a+M+\mu)^{\underline{\nu-M+j}}.$$

Here we give a proof of Theorem 5 that was stated in Chapter 1. Recall the following theorem

Theorem 34 *let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given with $N-1 < \nu \leq N$. Consider the ν^{th}-order fractional difference equation*

$$\Delta^{\nu}_{a+\nu-N} y(t) = f(t), \quad t \in \mathbb{N}_a \tag{B.5}$$

and the corresponding ν^{th}-order fractional initial value problem

$$\begin{cases} \Delta^{\nu}_{a+\nu-N} = f(t), \quad t \in \mathbb{N}_a \\ \Delta^{i} y(a+\nu-N) = A_i, \quad i \in \{0, \cdots, N-1\}, \quad A_i \in \mathbb{R}. \end{cases} \tag{B.6}$$

The general solution to $(B.6)$ is

$$y(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t), \quad t \in \{\mathbb{N}_{a+\nu-N}\}, \tag{B.7}$$

where α_i, $0 \leq i \leq N-1$, are real constants.

Proof: Let f and ν be given as in the statement of the theorem. For arbitrary but fixed constants $\alpha_i, 0 \leq i \leq N-1 \in \mathbb{R}$, define $y : \mathbb{N}_{a+\nu-N} \to \mathbb{R}$ by

$$y(t) = \sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t).$$

Here, we extend the usual domain of the fractional sum $\Delta_a^{-\nu} f$ from $\mathbb{N}_{a+\nu}$ to the largest set $\mathbb{N}_{a+\nu-N}$ by including the N zeros of $\Delta_a^{-\nu} f$ discussed in [4] Section 1.1.3. We will show that any function y of the above form is a solution to $(B.6)$ and that every solution to $(B.6)$ must be of this form. Observe that for $t \in \{\mathbb{N}_a\}$,

$$\begin{aligned}\Delta^{\nu}_{a+\nu-N} y(t) &= \Delta^{\nu}_{a+\nu-N}[\sum_{i=0}^{N-1} \alpha_i (t-a)^{\underline{i+\nu-N}} + \Delta_a^{-\nu} f(t)] \\ &= \sum_{i=0}^{N-1} \alpha_i \Delta^{\nu}_{a+\nu-N} (t-a)^{\underline{i+\nu-N}} + \Delta^{\nu}_{a+\nu-N} \Delta_a^{-\nu} f(t).\end{aligned}$$

At this point, we would like to apply power rule from Section 1.5 within the summation and Theorem 31 on the second term, but neither may be applied directly due to the incorrect lower limit on the operator $\Delta^{\nu}_{a+\nu-N}$. However, in this case, we obtain the correct lower limit by throwing away the zero terms involved. Writing each term out by definition, we have

$$\begin{aligned}\Delta^{\nu}_{a+\nu-N} (t-a)^{\underline{i+\nu-N}} &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+\nu-N}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} (s-a)^{\underline{i+\nu-N}} \\ &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+\nu-N+i}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} (s-a)^{\underline{i+\nu-N}} \\ &= \Delta^{\nu}_{a+i+\nu-N} (t-a)^{\underline{i+\nu-N}}, \quad for \quad i \in \{0, \cdots, N-1\}\end{aligned}$$

and

$$\begin{aligned}\Delta^{\nu}_{a+\nu-N} \Delta_a^{-\nu} f(t) &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+\nu-N}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} \Delta_a^{-\nu} f(s) \\ &= \frac{1}{\Gamma(-\nu)} \sum_{s=a+\nu}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} \Delta_a^{-\nu} f(s) \\ &= \Delta^{\nu}_{a+\nu} \Delta_a^{-\nu} f(t).\end{aligned}$$

Therefore, applying the power rule and Theorem 31, we have

$$\begin{aligned}\Delta^{\nu}_{a+\nu-N}y(t) &= \sum_{i=0}^{N-1}\alpha_i\Delta^{\nu}_{a+i+\nu-N}(t-a)^{\underline{i+\nu-N}}+\Delta^{\nu}_{a+\nu}\Delta^{-\nu}_{a}f(t)\\ &= \sum_{i=0}^{N-1}\alpha_i\frac{\Gamma(i+\nu-N+1)}{\Gamma(i-N+1)}(t-a)^{\underline{i-N}}+\Delta^{\nu-\nu}_{a}f(t)\\ &= f(t).\end{aligned}$$

Next, we show that every solution of $(B.6)$ has the correct form. Suppose that $z:\mathbb{N}_{a+\nu-N}\to\mathbb{R}$ is a solution to $(B.6)$. Then we apply Theorem 32 to directly solve $(B.6)$ for z:

$$\begin{aligned}&\Delta^{\nu}_{a+\nu-N}z(t)=f(t), \quad for \quad t\in\mathbb{N}_a\\ \Longrightarrow\;&\Delta^{-\nu}_{a}\Delta^{\nu}_{a+\nu-N}z(t)=\Delta^{-\nu}_{a}, for \quad t\in\mathbb{N}_{a+\nu-N}\\ \Longrightarrow\;&\Delta-a+\nu-N^0z(t)-\sum_{i=0}^{N-1}\frac{\Delta^{i-(N-\nu)}_{a+\nu-N}z(a)}{\Gamma(\nu-N+i+1)}(t-a)^{\underline{\nu-N+i}}=\Delta^{-\nu}_{a}f(t)\\ \Longrightarrow\;&z(t)=\sum_{i=0}^{N-1}\frac{\Delta^{i-(N-\nu)}_{a+\nu-N}z(a)}{\Gamma(\nu-N+i+1)}(t-a)^{\underline{\nu-N+i}}+\Delta^{-\nu}_{a}f(t).\end{aligned}$$

Since z has the same form as $(B.3)$, we have shown that

$$y(t)=\sum_{i=0}^{N-1}\alpha_i(t-a)^{\underline{i+\nu-n}}+\Delta^{-\nu}_{a}f(t), \quad t\in\mathbb{N}_{a+\nu-N}$$

is the general solution of $(B.6)$.

Appendix C

Future Work

The domain $\mathbb{N}_a$ that we considered in the earlier chapters is an example of a time scale (nonempty closed subset of the reals).

We could consider the same type of problems where our functions are defined on a time scale. In particular the q-time scale

$$q^{\mathbb{N}_0} := \{1, q, q^2, q^3, \cdots\}, \quad \text{where } q > 1 \text{ is fixed.}$$

which has important applications in quantum theory and asymptotic methods.

Further work that can be developed is to consider the periodic boundary value problem

$$\begin{cases} \Delta^{\nu}_{\nu-4} y(t) = f(t) \\ y(\nu-4) = y(b+\nu) \\ \Delta y(\nu-4) = \Delta y(b+\nu) \\ \Delta^2 y(\nu-4) = \Delta^3 y(b+\nu) \\ \Delta^3 y(\nu-4) = \Delta^3 y(b+\nu). \end{cases}$$

for $3 < \nu \leq 4$. Also, we may be able to prove some properties about the Green's function corresponding to the above BVP that lead to the existence of multiple positive solutions. We would also like to work with some initial value problems of ν^{th}-order (with $3 < \nu \leq 4$) and use Laplace Transformation to solve these types of IVPs.

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