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The Hilbert-Samuel function measures the length of powers of a zero-dimensional ideal in a local ring. Samuel showed that over a local ring these lengths agree with a polynomial, called the Hilbert-Samuel polynomial, for sufficiently large powers of the ideal. We examine the coefficients of this polynomial in the case the ideal is generated by a system of parameters, focusing much of our attention on the second Hilbert coefficient. We also consider the Hilbert-Kunz function, which measures the length of Frobenius powers of an ideal in a ring of positive characteristic. In particular, we examine a conjecture of Watanabe and Yoshida comparing the Hilbert-Kunz multiplicity and the length of the ideal and provide a proof in the graded case.

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Contents

Chapter 1

Introduction

The study of Hilbert-Samuel functions began in 1890 with the paper *Über die Theorie der algebraishen Formen* [Hil90] by David Hilbert. In this paper, Hilbert proved that over the polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$ with homogeneous ideal I, the function $H(n) = \dim_{\mathbb{C}} I_n$ agrees with a polynomial for n sufficiently large. Pierre Samuel then extended Hilbert's ideas to $\mathfrak{m}$-primary ideals I in a local ring $(R, \mathfrak{m})$ in his article [Sam51]. In particular, Samuel's work introduced modern multiplicity theory.

Let $(R, \mathfrak{m})$ be a local ring of dimension d, $I \subseteq R$ an $\mathfrak{m}$-primary ideal, and M a finitely generated R-module. The *Hilbert-Samuel function* for I with respect to M is the function $H_{I,M} : \mathbb{Z} \to \mathbb{Z}$ given by $H_{I,M}(n) = \lambda_R(M/I^n M)$. Samuel showed that these functions agree with a polynomial $P_{I,M}(n)$ (called the *Hilbert-Samuel polynomial*) of degree $t = \dim M$ for n sufficiently large. We can always write $P_{I,M}(n)$ in the form

$$P_{I,M}(n) = \sum_{i=0}^{t} (-1)^i e_i(I, M) \binom{n+t-i-1}{t-i}.$$

When $M = R$ we often suppress the M in the notation above. The numbers $e_i(I, M)$ are known as the *Hilbert coefficients* for I. In particular, the number $e(I, M) :=$

$e_0(I, M)$ is known as the *multiplicity* of I with respect to M and has been well-studied. One can compute the multiplicity as the limit

$$e(I, M) = \lim_{n\to\infty} \frac{t!\lambda_R(M/I^n M)}{n^t} \tag{1.1}$$

where $t := \dim M$. The first Hilbert coefficient, $e_1(I, M)$, is also sometimes called the *Chern number* for I with respect to M (see e.g., [Vas08], [MV10]).

One can study the coefficients of the Hilbert-Samuel polynomial to determine what information these coefficients can tell us about the ring or the ideal. For example, Nagata [Nag62] proved the following well-known result concerning the multiplicity of the maximal ideal.

Theorem 1.1 (Nagata). *Suppose $(R, \mathfrak{m})$ is a local unmixed Noetherian ring (i.e., $\dim \widehat{R}/p = \dim \widehat{R}$ for all $p \in \operatorname{Ass}(\widehat{R})$). Then $e(\mathfrak{m}) = 1$ if and only if R is regular.*

The other Hilbert coefficients carry important information as well. When the ring R is Cohen-Macaulay, Northcott [Nor60] and Narita [Nar63] have provided characterizations of $e_1(I)$ and $e_2(I)$, respectively, with the following theorems.

Theorem 1.2 (Northcott). *Suppose $(R, \mathfrak{m})$ is Cohen-Macaulay and I is an $\mathfrak{m}$-primary ideal. Then*

1. *$e_1(I) \geq 0$ with equality if and only if I is a parameter ideal.*

2. *$\lambda_R(R/I) \geq e_0(I) - e_1(I)$.*

Theorem 1.3 (Narita). *Suppose $(R, \mathfrak{m})$ is Cohen-Macaulay and I is an $\mathfrak{m}$-primary ideal. Then $e_2(I) \geq 0$. If $\dim R = 2$, then $e_2(I) = 0$ if and only if I^n has reduction number one for some integer n.*

If R is Cohen-Macaulay and I is a parameter ideal, i.e, $I = (x_1, \ldots, x_d)$ with $d = \dim R$ and $\sqrt{I} = \mathfrak{m}$, it is a straightforward exercise to show that $e(I) = \lambda_R(R/I)$ and $e_i(I) = 0$ for $i = 1, \ldots, d$. However, if R is not Cohen-Macaulay, this is not always the case. In this thesis we will examine the Hilbert coefficients for a parameter ideal in a non-Cohen-Macaulay ring.

Our first main result was inspired by the following theorem of Ghezzi, et al [GGH+10b] which characterizes the Cohen-Macaulayness of an unmixed ring in terms of the first Hilbert coefficient of a parameter ideal. In Section 2.4 we will discuss the case the ring R is unmixed in more detail.

Theorem 1.4 (Ghezzi, et al)**.** *Suppose* $(R, \mathfrak{m})$ *is an unmixed local ring and* q *a parameter ideal. Then* $e_1(q) \leq 0$ *with equality if and only if* R *is Cohen-Macaulay.*

Our main results in Chapter 2 are the following. We will define the postulation number, $n(q)$, in Section 2.1.

Theorem A. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d \geq 2$*. Suppose that* $\operatorname{depth} R \geq d - 1$*. If* q *is a parameter ideal of* R*, then the following hold:*

1. $e_2(q) \leq 0$.
2. $e_2(q) = 0$ *if and only if* $n(q) < 2 - d$ *and* $\operatorname{grade} gr_q(R)_+ \geq d - 1$.
3. $e_2(q) = 0$ *implies* $e_3(q) = e_4(q) = \cdots = e_d(q) = 0$.

Theorem B. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring of dimension* d *and suppose* q *is a parameter ideal for* R *satisfying* $\operatorname{depth} gr_q(R) \geq d - 1$*. Then for* $0 \leq i \leq d + 1$ *and* $n \in \mathbb{Z}$*,*

$$(-1)^i \Delta^{d+1-i}(P_q(n) - H_q(n)) \geq 0.$$

Moreover, $e_i(q) \leq 0$ *for* $i = 1, \ldots, d$*.*

In Chapter 3 we restrict our focus to rings of positive characteristic and consider a characteristic $p > 0$ analogue of the Hilbert-Samuel multiplicity (1.1), known as the Hilbert-Kunz multiplicity of an ideal. Given a local ring $(R, \mathfrak{m})$ of dimension d with maximal ideal $\mathfrak{m}$ of characteristic $p > 0$, an ideal I of R and $q = p^e$ (for some e), we define the e-th *Frobenius power* of the ideal I, denoted $I^{[q]}$, to be the ideal of R generated by the set $\{i^q : i \in I\}$. If I is $\mathfrak{m}$-primary, we define the *Hilbert-Kunz multiplicity* of I by

$$e_{HK}(I, R) := \lim_{q \to \infty} \frac{\lambda_R(R/I^{[q]})}{q^d}. \tag{1.2}$$

As with the Hilbert-Samuel multiplicity, when the ring R is understood, we often suppress the symbol R in the notation above. The idea of studying this limit began with Kunz in [Kun69] where he proved the following:

Theorem 1.5 (Kunz). *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and dimension d. Then the following are equivalent:*

1. *R is a regular local ring.*

2. *R is reduced and flat over $R^p = \{r^p \mid r \in R\}$.*

3. *$\lambda_R(R/\mathfrak{m}^{[q]}) = q^d$ for all $q = p^e$, $e \geq 1$.*

In the same paper, Kunz also showed that for any local ring $(R, \mathfrak{m})$ of characteristic $p > 0$,

$$\lambda_R(R/\mathfrak{m}^{[q]}) \geq q^d \qquad \text{for all } q = p^e, \ e \geq 1.$$

In particular, this shows that for any local ring $(R, \mathfrak{m})$, we have $e_{HK}(\mathfrak{m}) \geq 1$. Monsky [Mon83] then proved that the limit (1.2) exists and is positive for all $\mathfrak{m}$-primary ideals, giving us the Hilbert-Kunz multiplicity.

The Hilbert-Kunz multiplicity can be difficult to compute, and unlike the Hilbert-Samuel multiplicity, $e_{HK}(I)$ is not necessarily an integer. In fact, modulo a conjecture, Monsky ([Mon08a], [Mon08b]) has given examples of non-rational algebraic Hilbert-Kunz multiplicities and even transcendental multiplicities.

In Chapter 3, we consider the following conjecture of Watanabe and Yoshida [WY00].

Conjecture 1.6 (Watanabe-Yoshida)**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of characteristic $p > 0$. Then*

1. *For any $\mathfrak{m}$-primary ideal I, one has $e_{HK}(I, R) \geq \lambda_R(R/I)$.*
2. *For any $\mathfrak{m}$-primary ideal I with $\operatorname{pd}_R(R/I) < \infty$, one has $e_{HK}(I, R) = \lambda_R(R/I)$.*

An example of Miller and Singh [MS00] shows neither part of this conjecture holds in general. We will examine the conjecture in the case that the ring R is graded. Our main result in Chapter 3 gives a positive answer to part (1) of the conjecture in this case.

Theorem C. *Let R be a graded ring of characteristic $p > 0$ and dimension d and I a homogeneous ideal with $\lambda_R(R/I) < \infty$ and $\operatorname{pd}_R(R/I) < \infty$. Then for every $q = p^e$, one has $\lambda_R(R/I^{[q]}) = q^d \lambda_R(R/I)$. In particular, $e_{HK}(I, R) = \lambda_R(R/I)$.*

Also, in Chapter 3 we consider part (1) of Conjecture 1.6 in the context of numerical semigroup rings.

Finally, in Chapter 4 we discuss the following theorem of Hochster and Huneke considering tight closure and the Hilbert-Kunz multiplicity. This is an expository chapter on the theorem and the background needed to prove the theorem.

Theorem 1.7. *(cf. [HH90, Theorem 8.17]) Let (R, m) be a local ring and $J \subseteq I$ m-primary ideals of R.*

1. *If* $I \subseteq J^*$ *then* $e_{HK}(I) = e_{HK}(J)$.

2. *The converse to (1) holds if* R *is equidimensional and either complete or essentially of finite type over a field.*

Chapter 2

The Second Hilbert Coefficient of a Parameter Ideal

2.1 Definitions and Notation

Given a local ring $(R, \mathfrak{m})$, it is a well-known fact that an ideal $I \subseteq R$ is $\mathfrak{m}$-*primary* if and only if $\sqrt{I} = \mathfrak{m}$. If M is an R-module, we write $\lambda_R(M)$ (or simply $\lambda(M)$) for the length of M as an R-module. For an $\mathfrak{m}$-primary ideal I, the *Hilbert-Samuel function* of I with respect to M is defined by $H_{I,M}(n) = \lambda_R(M/I^n M)$ for all integers n. Throughout this thesis we use the convention $I^n = R$ for $n \leq 0$, so that $H_{I,M}(n) = 0$ for $n \leq 0$. Samuel [Sam51] showed that the function $H_{I,M}(n)$ agrees with a polynomial $P_{I,M}(n)$, known as the *Hilbert-Samuel polynomial* for I, of degree $t = \dim M$ for n sufficiently large. Moreover, one can always write $P_{I,M}(n)$ in the form

$$P_{I,M}(n) = \sum_{i=0}^{t} e_i(I, M) \binom{n + t - i - 1}{t - i}$$

where the $e_i(I, M)$'s are known as the *Hilbert coefficients* for I with respect to M. We define the *postulation number* for I with respect to M, denoted $n(I, M)$, to be the largest integer for which $H_{I,M}(n)$ and $P_{I,M}(n)$ disagree. That is,

$$n(I, M) = \min\{j \mid H_{I,M}(n) = P_{I,M}(n)\ \forall n > j\}.$$

Let $\operatorname{Ass} R$ denote the set of associated primes of the ring R and $\operatorname{Assh} R$ the set of associated primes of maximal dimension in R, i.e.,

$$\operatorname{Assh} R = \{p \in \operatorname{Ass} R \mid \dim(R/p) = \dim R\}.$$

We define the *associated graded ring* of an ideal $I \subseteq R$ by $gr_I(R) = \oplus_{n\geq 0} I^n/I^{n+1}$. When R is local, we have that $\cap_{n\geq 0} I^n = 0$ so that for any element $x \in R$, there is a unique integer n so that $x \in I^n \backslash I^{n+1}$. We let x^* denote the image of x in $I^n/I^{n+1} \subseteq gr_I(R)$. For a graded ring $G = \oplus_{n\geq 0} G_n$, we let G_+ denote the ideal $\oplus_{n\geq 1} G_n$.

We say that an element $y \in I$ is *superficial* with respect to a module M if there exists $c \in \mathbb{N}$ such that for all $n \geq c$, $(I^{n+1}M :_M y) \cap I^c M = I^n M$. In particular, if y is also a non-zero-divisor on M, we have $(I^n M :_M y) = I^{n-1}M$ for all n sufficiently large ([SH06, Lemma 8.5.3]). A sequence $y_1, \ldots, y_s \in I$ is said to be a *superficial sequence* for I with respect to M if the image of y_i in $I/(y_1, \ldots, y_{i-1})$ is a superficial element of $I/(y_1, \ldots, y_{i-1})$ with respect to $M/(y_1, \ldots, y_{i-1})M$ for all $i = 1, \ldots, s$.

We provide some preliminary results in the next section and prove the main theorems of this chapter in the following section.

2.2 Some Preliminary Results

When looking at Hilbert-Samuel functions, a common technique is to reduce by a superficial sequence to obtain a ring of smaller dimension. The following Proposition guarantees that when we reduce in this way, the Hilbert coefficients behave nicely.

Proposition 2.2.1. (cf. *[Nag62, 22.6]*) *Let $(R, \mathfrak{m})$ be a Noetherian local ring, I an $\mathfrak{m}$-primary ideal and M a nonzero finitely generated R-module of dimension d. Suppose $y \in I$ is superficial with respect to M. Then $\lambda_R(0 :_M y)$ is finite and*

$$P_{\bar{I},\bar{M}}(n) = P_{I,M}(n) - P_{I,M}(n-1) + \lambda_R(0 :_M y).$$

In particular, we have

$$e_i(\bar{I}, \bar{M}) = \begin{cases} e_i(I, M) & \text{for } i = 0, \dots, d-2 \\ e_{d-1}(I, M) + (-1)^{d-1}\lambda_R(0 :_M y) & \text{for } i = d-1. \end{cases}$$

Proof. We follow the proof of Nagata in [Nag62]. Let c be such that $(I^n M :_M y) \cap I^c M = I^{n-1}M$ for all $n \geq c$. For $n \in \mathbb{Z}$, consider the exact sequence

$$0 \longrightarrow \frac{I^n M :_M y}{I^{n-1}M} \longrightarrow \frac{M}{I^{n-1}M} \xrightarrow{y} \frac{M}{I^n M} \longrightarrow \frac{M}{I^n M + yM} \longrightarrow 0.$$

It follows that

$$\lambda_R\left(\frac{M}{I^n M + yM}\right) = \lambda_R\left(\frac{M}{I^n M}\right) - \lambda_R\left(\frac{M}{I^{n-1}M}\right) + \lambda_R\left(\frac{I^n M :_M y}{I^{n-1}M}\right).$$

For n sufficiently large this gives

$$P_{\bar{I},\bar{M}}(n) = P_{I,M}(n) - P_{I,M}(n-1) + \lambda_R((I^n M :_M y)/I^{n-1}M).$$

So it is enough to show that $\lambda_R((I^nM :_M y)/I^{n-1}M) = \lambda_R(0 :_M y)$ for n sufficiently large. First note that for $n \geq c$

$$\lambda_R\left(\frac{I^nM :_M y}{I^{n-1}M}\right) = \lambda_R\left(\frac{I^nM :_M y}{(I^nM :_M y) \cap I^cM}\right) = \lambda_R\left(\frac{(I^nM :_M y) + I^cM}{I^cM}\right).$$

Note that M/I^cM is a finite length module, and hence artinian. Thus, the descending chain

$$\frac{(I^nM :_M y) + I^cM}{I^cM} \supseteq \frac{(I^{n+1}M :_M y) + I^cM}{I^cM} \supseteq \frac{(I^{n+2}M :_M y) + I^cM}{I^cM} \supseteq \cdots$$

must stabilize, and we have $\lambda_R\left(\frac{(I^nM :_M y)+I^cM}{I^cM}\right)$ is a constant, say C, for n sufficiently large.

We claim that $(I^nM :_M y) \subseteq (0 :_M y) + I^cM$ for n sufficiently large. To see this, note that, using the Artin-Rees lemma, we have

$$y(I^nM :_M y) = I^nM \cap yM = I^{n-k}(I^kM \cap yM) \subseteq yI^{n-k}M$$

for some integer k. So if $x \in (I^nM :_M y)$, for $n \gg 0$, there exists an element $a \in I^{n-k}M \subseteq I^cM$ such that $yx = ya$. Then $x - a \in (0 :_M y)$. This proves the claim.

Now, we have $(0 :_M y) + I^cM \subseteq (I^nM :_M y) + I^cM \subseteq (0 :_M y) + I^cM$ for n sufficiently large. Hence,

$$C = \lambda_R\left(\frac{(I^nM :_M y) + I^cM}{I^cM}\right) = \lambda_R\left(\frac{(0 :_M y) + I^cM}{I^cM}\right) = \lambda_R\left(\frac{0 :_M y}{(0 :_M y) \cap I^cM}\right).$$

Finally, since this C is independent of c (when c is large), we must have the intersection $(0 :_M y) \cap I^cM = 0$. This proves the first statement. The second statement follows directly from the first. $\square$

Note that in the proof above, we show that if $x \in (I^n M :_M y)$ and $c \in \mathbb{Z}$ is such that $(I^n M :_M y) \cap I^c M = I^{n-1}M$ for all $n \geq c$, then for $n \gg 0$, there exists an element $a \in I^c M$ such that $yx = ya$. If y is a non-zero-divisor, this gives that $(I^n M :_M y) \subseteq I^c M$. In particular, this says that for a non-zero-divisor $y \in I$ that is superficial with respect to M, $(I^n M :_M y) = (I^n M :_M y) \cap I^c M = I^{n-1}M$ for $n \gg 0$.

For an element $x \in I^n \backslash I^{n+1}$, recall that x^* denotes the image of x in $gr_I(R)$. If x^* is a non-zero-divisor, the postulation number also behaves nicely when we pass to $I/(x)$. As the proof is short, we include it for completeness.

Lemma 2.2.2. *Let $x \in I \backslash I^2$ be a non-zero-divisor and assume that x^* is a regular element of $gr_I(R)$. Set $\bar{I} = I/(x)$. Then $n(\bar{I}) = n(I) + 1$.*

Proof. Note that from the exact sequence

$$0 = \frac{q^k : x}{q^{k-1}} \to R/q^{k-1} \xrightarrow{y} R/q^k \to R/(q^k, x) \to 0$$

we obtain

$$H_{\bar{I}}(k) = H_I(k) - H_I(k-1) \quad \text{and} \quad P_{\bar{I}}(k) = P_I(k) - P_I(k-1).$$

From this it is clear that $n(\bar{I}) \leq n(I) + 1$. Suppose $n(\bar{I}) < n(I) + 1$. Then we have $H_I(n(I)+1) - H_I(n(I)) = H_{\bar{I}}(n(I)+1) = P_{\bar{I}}(n(I)+1) = P_I(n(I)+1) - P_I(n(I))$. But, since $H_I(n(I)+1) = P_I(n(I)+1)$, we obtain $H_I(n(I)) = P_I(n(I))$, a contradiction. Thus, $n(\bar{I}) = n(I) + 1$. $\square$

We define grade $gr_I(R)_+$ to be the maximal length of a regular sequence for $gr_I(R)$ contained in $gr_I(R)_+$. Then grade $gr_I(R)_+ = \operatorname{depth} gr_I(R)$. The grade of the associated graded ring also behaves nicely with respect to superficial sequences as evidenced by the following lemmas.

Lemma 2.2.3. *[HM97, Lemma 2.1] Let $x_1, \ldots, x_k$ be a superficial sequence for I. If* $\operatorname{grade} gr_I(R)_+ \geq k$, *then $x_1^*, \ldots, x_k^*$ is a regular sequence.*

Lemma 2.2.4. *[HM97, Lemma 2.2] Suppose $y_1, \ldots y_k$ is a superficial sequence for an ideal I. Let $\bar{R}$ and $\bar{I}$ denote $R/(y_1, \ldots, y_k)$ and $I/(y_1, \ldots, y_k)$, respectively. If* $\operatorname{grade} gr_{\bar{I}}(\bar{R})_+ \geq 1$, *then* $\operatorname{grade} gr_I(R)_+ \geq k+1$.

We now derive a formula for the d^{th} Hilbert coefficient of a parameter ideal with the following lemma.

Lemma 2.2.5. *Suppose $(R, \mathfrak{m})$ has dimension d. Let I be an $\mathfrak{m}$-primary ideal and $y \in I$ a superficial element. Let $\bar{I} = I/(y)$, $H_{\bar{I}}(k) = \lambda_R(R/(I^k, y))$ and $P_{\bar{I}}(k)$ denote the Hilbert polynomial for $\bar{I}$. Then for $l \gg 0$,*

$$(-1)^d e_d(I) = \sum_{k=1}^{l} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=1}^{l} \lambda_R((I^k : y)/I^{k-1}) + l\lambda_R(0 : y).$$

Furthermore, if y is also a non-zero-divisor on R, we have

$$(-1)^d e_d(I) = \sum_{k=1}^{\infty} (H_{\bar{I}}(k) - P_{\bar{I}}(k)) - \sum_{k=1}^{\infty} \lambda_R((I^k : y)/I^{k-1}).$$

Proof. For $k \in \mathbb{Z}$, consider the exact sequence:

$$0 \to \frac{I^k : y}{I^{k-1}} \to R/I^{k-1} \xrightarrow{y} R/I^k \to R/(I^k, y) \to 0.$$

From this we see that $\lambda_R(R/(y, I^k)) = \lambda_R(R/I^k) - \lambda_R(R/I^{k-1}) + \lambda_R(\frac{I^k : y}{I^{k-1}})$. Subtracting $P_{\bar{I}}(k)$ and summing both sides and we get, for $l \gg 0$,

$$\sum_{k=1}^{l} \left(\lambda(R/(y, I^k) - P_{\bar{I}}(k)\right) = \sum_{k=1}^{l} \left(\lambda(R/I^k) - \lambda(R/I^{k-1}) + \lambda\left(\frac{I^k : y}{I^{k-1}}\right) - P_{\bar{I}}(k)\right)$$

Thus,

$$\begin{aligned}
\sum_{k=1}^{l}(H_{\bar{I}}(k)-P_{\bar{I}}(k)) &= \lambda(R/I^l)-\sum_{k=1}^{l}\sum_{i=0}^{d-1}(-1)^i\binom{k+d-2-i}{d-1-i}e_i(\bar{I})+\sum_{k=1}^{l}\lambda\left(\frac{I^k:y}{I^{k-1}}\right)\\
&= \sum_{i=0}^{d}(-1)^i\binom{l+d-1-i}{d-i}e_i(I)-\sum_{i=0}^{d-1}(-1)^i\binom{l+d-1-i}{d-i}e_i(\bar{I})\\
&\quad+\sum_{k=1}^{l}\lambda\left(\frac{I^k:y}{I^{k-1}}\right)
\end{aligned}$$

where $\lambda(-)=\lambda_R(-)$.

By Proposition 2.2.1, we have $e_i(I)=e_i(\bar{I})$ for $i=0,\ldots,d-2$ and $e_{d-1}(I)=e_{d-1}(\bar{I})-(-1)^{d-1}\lambda_R(0:y)$. Hence,

$$\sum_{k=1}^{l}(H_{\bar{I}}(k)-P_{\bar{I}}(k)) = -l\lambda_R(0:y)+(-1)^d e_d(I)+\sum_{k=1}^{l}\lambda_R((I^k:y)/I^{k-1}).$$

Rearranging, we get

$$(-1)^d e_d(I)=\sum_{k=1}^{l}(H_{\bar{I}}(k)-P_{\bar{I}}(k))-\sum_{k=1}^{l}\lambda_R((I^k:y)/I^{k-1})+l\lambda_R(0:y)$$

and if y is also a non-zero-divisor on R, we have

$$(-1)^d e_d(I)=\sum_{k=1}^{\infty}(H_{\bar{I}}(k)-P_{\bar{I}}(k))-\sum_{k=1}^{\infty}\lambda_R((I^k:y)/I^{k-1})$$

since for $k\gg 0$, $H_{\bar{I}}(k)-P_{\bar{I}}(k)=0$ and $\lambda_R((I^k:y)/I^{k-1})=0$. $\square$

2.3 The first difference function, $\Delta(P_q(n)-H_q(n))$

In this section, we examine the difference function $\Delta(P_q(n)-H_q(n))$ for a parameter ideal q. The techniques we use closely follow those of Marley in [Mar89]. In this section we also prove Theorems A and B.

Definition 2.3.1. *Let* $f : \mathbb{Z} \to \mathbb{Z}$*. The* first difference function*,* $\Delta(f)$*, is defined by* $\Delta(f(n)) = f(n+1) - f(n)$*. We define the* i^{th} difference function *inductively by* $\Delta^i(f) = \Delta(\Delta^{i-1}(f))$*.*

We begin with a proposition describing the Hilbert coefficients for a parameter ideal in a ring of dimension one. The second part of this result is also proved in [GN03, Lemma 2.4(1)]. As the proof is short, we include it here.

Proposition 2.3.2. *[GN03] Suppose* $(R, \mathfrak{m})$ *is a one-dimensional local Noetherian ring and* $q = (x) \subseteq R$ *is a parameter ideal for* R*. Then*

1. $e_0(q) = \lambda_R(R/(H^0_{\mathfrak{m}}(R), x))$*, and*

2. $e_1(q) = -\lambda_R(H^0_{\mathfrak{m}}(R))$*.*

Proof. Consider the short exact sequence

$$0 \to \frac{H^0_{\mathfrak{m}}(R)}{q^n \cap H^0_{\mathfrak{m}}(R)} \to \frac{R}{q^n} \to \frac{R}{(H^0_{\mathfrak{m}}(R), q^n)} \to 0.$$

Using the additivity of length, we have $\lambda_R(R/q^n) = \lambda_R\left(\frac{H^0_{\mathfrak{m}}(R)}{q^n \cap H^0_{\mathfrak{m}}(R)}\right) + \lambda_R\left(\frac{R}{(H^0_{\mathfrak{m}}(R), q^n)}\right)$. Note that for $n \gg 0$, $q^n \cap H^0_{\mathfrak{m}}(R) = 0$ by the Artin-Rees Lemma, so we have $\lambda_R(R/q^n) = \lambda_R(H^0_{\mathfrak{m}}(R)) + \lambda_R(R/(H^0_{\mathfrak{m}}(R), q^n))$ for $n \gg 0$. Now, $R/H^0_{\mathfrak{m}}(R)$ is a Cohen-Macaulay ring and the image of x in $R/H^0_{\mathfrak{m}}(R)$ is a parameter, so we also have $\lambda_R(R/(H^0_{\mathfrak{m}}(R), q^n)) = n\lambda_R(R/(H^0_{\mathfrak{m}}(R), x))$. Putting this together, we have

$$\lambda_R(R/q^n) = n\lambda_R(R/(H^0_{\mathfrak{m}}(R), x)) + \lambda_R(H^0_{\mathfrak{m}}(R)) \qquad \text{for } n \gg 0.$$

It follows that $e_0(q) = \lambda_R(R/(H^0_{\mathfrak{m}}(R), x))$ and $e_1(q) = -\lambda_R(H^0_{\mathfrak{m}}(R))$. □

We now provide another description of the Hilbert coefficients of a parameter ideal in a one-dimensional ring.

Proposition 2.3.3. *Suppose* $(R, \mathfrak{m})$ *is a one-dimensional local Noetherian ring and* $q = (x) \subseteq R$ *is a parameter ideal. Then*

1. *a)* $e_0(q) = \lambda_R(R/\tilde{x})$ *where* $\tilde{x} = ((x^{i+1}) : x^i)$ *for all* $i \gg 0$*, and*

   b) $e_1(q) = \sum_{i=0}^{l-1} (\lambda_R(R/\tilde{x}) - \lambda_R(R/((x^{i+1}) : x^i)))$ *for a fixed integer* l*.*

2. *a)* $P_q(n) - H_q(n) = \sum_{i=n}^{\infty} (\lambda_R(R/((x^{i+1}) : x^i)) - \lambda(R/\tilde{x}))$ *where* $\tilde{x} = ((x^{l+1}) : x^l)$ *for all* $l \gg 0$*, and*

   b) $P_q(n) \geq H_q(n)$ *for all* $n \geq 0$*.*

Proof. Write $q = (x)$. Note that $\lambda_R((x^i)/(x^{i+1})) = \lambda_R(R/((x^{i+1}) : x^i))$ for all i as $((x^{i+1}) : x^i)$ is the kernel of the surjective map $R \to (x^i)/(x^{i+1})$ defined by $1 \mapsto \bar{x^i}$. Then $\lambda(R/q^n) = \sum_{i=0}^{n-1} \lambda_R((x^i)/(x^{i+1})) = \sum_{i=0}^{n-1} \lambda_R(R/((x^{i+1}) : x^i))$. Note

$$((x) : x^0) \subseteq ((x^2) : x) \subseteq ((x^3) : x^2) \subseteq \cdots$$

is an ascending chain, so there must be a point at which it stabilizes. Let

$$l = \min\{i \mid ((x^{n+1}) : x^n) = ((x^{i+1}) : x^i) \text{ for all } n \geq i\}$$

and set $\tilde{x} = ((x^{l+1}) : x^l)$.

For $n \geq l$, we have $\lambda_R(R/(x^n)) = \sum_{i=0}^{l-1} \lambda_R(R/((x^{i+1}) : x^i)) + (n-l)\lambda_R(R/\tilde{x})$. This gives that

$$P_q(n) = \sum_{i=0}^{l-1} \lambda_R(R/((x^{i+1}) : x^i)) + (n-l)\lambda_R(R/\tilde{x}).$$

From this, we see that $e_0(q) = \lambda_R(R/\tilde{x})$ and $e_1(q) = \sum_{i=0}^{l-1} [\lambda_R(R/\tilde{x}) - \lambda_R(R/((x^{i+1} : x^i))]$. This proves (1).

Now if $n \leq l-1$, then $H_q(n) = \sum_{i=0}^{n-1} \lambda_R(R/((x^{i+1}) : x^i)$, and

$$\begin{aligned} P_q(n) - H_q(n) &= \sum_{i=n}^{l-1} \lambda_R(R/((x^{i+1}) : x^i) + (n-l)\lambda_R(R/\tilde{x}) \\ &= \sum_{i=n}^{l-1} \big(\lambda_R(R/((x^{i+1}) : x^i) - \lambda_R(R/\tilde{x})\big) \\ &= \sum_{i=n}^{\infty} \big(\lambda_R(R/((x^{i+1}) : x^i) - \lambda_R(R/\tilde{x})\big), \end{aligned}$$

where the last equality holds since $((x^{i+1}) : x^i) = \tilde{x}$ for all $i \geq l$. This gives $2(a)$.

Note that for all i, we have $((x^{i+1}) : x^i) \subset \tilde{x}$, so $\lambda_R(R/((x^{i+1}) : x^i)) \geq \lambda_R(R/\tilde{x})$ and we have $P_q(n) - H_q(n) \geq 0$. In fact, if $n \geq l$, we have $P_q(n) - H_q(n) = 0$. This gives part $2(b)$ of the proposition. $\square$

This proposition gives us a formula for the postulation number of a parameter ideal in a one-dimensional ring.

Corollary 2.3.4. *Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian ring and $q = (x)$ a parameter ideal. Then*

$$n(q) = \min\{i \mid ((x^{i+1}) : x^i) = ((x^{j+1}) : x^j) \textit{ for all } j \geq i\} - 1.$$

Proof. Let $l = \min\{i \mid ((x^{i+i}) : x^i) = ((x^{j+1}) : x^j) \text{ for all } j \geq i\}$ and $\tilde{x} = ((x^{l+1}) : x^l)$. Then, using part $2(a)$ of Proposition 2.3.3, clearly $n(q) \leq l-1$. If $n(q) < l-1$, then we have $P_q(l) = H_q(l)$ and using $2(a)$ again, this gives $\lambda_R(R/((x^l) : x^{l-1}) = \lambda_R(R/\tilde{x})$. But since $\tilde{x} = ((x^{l+1}) : x^l)$, this says that $((x^l) : x^{l-1}) = ((x^{l+1}) : x^l) = ((x^{i+1}) : x^i)$ for all $i \geq l$, contradicting the minimality of l. Thus, we must have $n(q) = l-1$. $\square$

Corollary 2.3.5. *Let $(R, \mathfrak{m})$ be a one-dimensional local Noetherian ring and $q = (x)$ a parameter ideal. Then*

1. *For $k \in \mathbb{Z}$, if $P_q(k) - H_q(k) = 0$, then $P_q(n) - H_q(n) = 0$ for all $n \geq k$, i.e., $k > n(q)$.*

2. *$\Delta^2(P_q(n) - H_q(n)) = \lambda_R(((x^{n+2}) : x^{n+1})/((x^{n+1}) : x^n))$ for all n.*

3. *$\Delta^2(P_q(n) - H_q(n)) \geq 0$ for all n.*

Proof. For the first statement, suppose $P_q(k) - H_q(k) = 0$. Let $\tilde{x}$ be defined as in Proposition 2.3.3. Then $P_q(k) - H_q(k) = \sum_{i=k}^{\infty} (\lambda_R(R/((x^{i+1}) : x^i) - \lambda_R(R/\tilde{x})) = 0$. As $\lambda_R(R/((x^{i+1}) : x^i)) - \lambda_R(R/\tilde{x}) \geq 0$ for all $i \geq 0$, we must have equality for each $i \geq k$. It follows that $P_q(n) - H_q(n) = 0$ for all $n \geq k$, i.e., $k > n(q)$.

For (2), note by Proposition 2.3.3

$$\Delta(P_q(n) - H_q(n)) \;\;=\;\; \lambda_R(R/\tilde{x}) - \lambda_R(R/((x^{n+1}) : x^n))$$

So,

$$\begin{aligned}
\Delta^2(P_q(n) - H_q(n)) &= \Delta(\Delta(P_q(n) - H_q(n))) \\
&= \Delta(\lambda_R(R/\tilde{x}) - \lambda_R(R/((x^{n+1}) : x^n))) \\
&= -\lambda_R(R/((x^{n+2}) : x^{n+1})) + \lambda_R(R/((x^{n+1}) : x^n)) \\
&= \lambda_R(((x^{n+2}) : x^{n+1})/((x^{n+1}) : x^n)).
\end{aligned}$$

In particular, this says $\Delta(P_q(n) - H_q(n)) \leq 0$ and $\Delta^2(P_q(n) - H_q(n)) \geq 0$ for all n. □

In our next result, we give an explicit formula for $e_1(q)$ over a two-dimensional ring.

Proposition 2.3.6. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension two with parameter ideal q. Suppose $y \in q \backslash \mathfrak{m}q$ is a superficial element for q. Then*

$$e_1(q) = -\lambda_R(0 :_{H^1_{\mathfrak{m}}(R)} y)).$$

Proof. By Propositions 2.2.1 and 2.3.2 , we have

$$e_1(q) = \lambda_R(0 :_R y) - \lambda_R(H^0_{\mathfrak{m}}(R/(y))). \tag{2.1}$$

We will first find formulas for $\lambda_R(0 :_R y)$ and $\lambda_R(H^0_{\mathfrak{m}}(R/(y)))$. Consider the short exact sequence

$$0 \to (0 :_R y) \to R \to R/(0 :_R y) \to 0.$$

Applying local cohomology, we have

$$0 \to H^0_{\mathfrak{m}}(0 :_R y) \to H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(R/(0 :_R y)) \to H^1_{\mathfrak{m}}(0 :_R y) \to \cdots .$$

Note that since $\lambda_R(0 :_R y) < \infty$ (by Proposition 2.2.1) , we have $H^0_{\mathfrak{m}}(0 :_R y) = (0 :_R y)$ and $H^1_{\mathfrak{m}}(0 :_R y) = 0$. So, from the exact sequence

$$0 \to (0 :_R y) \to H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(R/(0 :_R y)) \to 0$$

we have

$$\lambda_R(0 :_R y) = \lambda_R(H^0_{\mathfrak{m}}(R)) - \lambda_R(H^0_{\mathfrak{m}}(R/(0 :_R y))). \tag{2.2}$$

Similarly, we can apply local cohomology to the short exact sequence

$$0 \to R/(0 :_R y) \xrightarrow{\alpha} R \to R/(y) \to 0$$

where $\alpha(\bar{r}) = yr$ to obtain the long exact sequence

$$0 \to H^0_{\mathfrak{m}}(R/(0:_R y)) \to H^0_{\mathfrak{m}}(R) \to H^0_{\mathfrak{m}}(R/(y)) \xrightarrow{g} H^1_{\mathfrak{m}}(R/(0:_R y)) \xrightarrow{f} H^1_{\mathfrak{m}}(R) \to \cdots .$$

Letting $C := \ker(f) \cong \operatorname{im}(g)$, we have

$$\lambda_R(H^0_{\mathfrak{m}}(R/(y)) = \lambda_R(C) + \lambda_R(H^0_{\mathfrak{m}}(R)) - \lambda_R(H^0_{\mathfrak{m}}(R/(0:_R y)). \tag{2.3}$$

Now combining (2.2) and (2.3) with (2.1), we have $e_1(q) = -\lambda_R(C)$.

We claim that $C \cong (0:_{H^1_{\mathfrak{m}}(R)} y)$. First note that as y is superficial for q, y acts as a non-zero-divisor on $R/H^0_{\mathfrak{m}}(R)$. Indeed, suppose $x \in R$ such that $\bar{x}\bar{y} = \bar{0}$ in $R/H^0_{\mathfrak{m}}(R)$. We need to show that $x \in H^0_{\mathfrak{m}}(R)$. There exists $n_1 \in \mathbb{Z}$ such that $xy\mathfrak{m}^{n_1} = (0) \subseteq q^i$ for all $i \geq 0$ and hence $x\mathfrak{m}^n \subseteq (q^i : y)$ for $n \gg 0$ and for all $i \geq 0$. As y is superficial for q, there exists $c \in \mathbb{Z}$ such that $(q^{n+1} : y) \cap q^c = q^n$ for $n \geq c$. Since q is $\mathfrak{m}$-primary, there exists n_2 such that $\mathfrak{m}^{n_2} \subseteq q$. Then note that $\mathfrak{m}^{n_2c} \subseteq q^c$ implies that $x\mathfrak{m}^n \subseteq q^c$ for $n \gg 0$. Thus, for $n \gg 0$, we have $yx\mathfrak{m}^n = 0$ and hence $x\mathfrak{m}^n \subseteq (q^{i+1} : y) \cap q^c = q^i$ for all $i \geq c$. In particular, by Krull's Intersection Theorem, we have $x\mathfrak{m}^n \in \cap_{i\geq c} q^i = 0$. So, $x \in (0 : \mathfrak{m}^n) \subseteq H^0_{\mathfrak{m}}(R)$.

Recall that we have the exact sequence

$$0 \to C \to H^1_{\mathfrak{m}}(R/(0:_R y)) \to H^1_{\mathfrak{m}}(R).$$

Consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H^0_{\mathfrak{m}}(R)/(0:_R y) \arrow[r, "\gamma"] \arrow[d] & H^0_{\mathfrak{m}}(R) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & R/(0:_R y) \arrow[r, "\alpha"] \arrow[d] & R \arrow[r] \arrow[d] & R/(y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R/H^0_{\mathfrak{m}}(R) \arrow[r, "y"] \arrow[d] & R/H^0_{\mathfrak{m}}(R) \arrow[r] \arrow[d] & R/(H^0_{\mathfrak{m}}(R), y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where $K = \ker(\beta) = \mathrm{Coker}(\gamma)$. We can apply local cohomology to obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] & H^1_{\mathfrak{m}}(R/(0:_R y)) \arrow[r] \arrow[d, "\theta"] & H^1_{\mathfrak{m}}(R) \arrow[d, "\tau"] \\
0 \arrow[r] & T \arrow[r] & H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R)) \arrow[r, "y"] & H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R))
\end{tikzcd}$$

where $T = \ker(H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R)) \xrightarrow{y} H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R))$. Note θ and τ are isomorphisms because $H^0_{\mathfrak{m}}(R)/(0:_R y)$ and $H^0_{\mathfrak{m}}(R)$ are both finite-length modules and hence

$$H^i(H^0_{\mathfrak{m}}(R)/(0:_R y)) = 0 \text{ and } H^i_{\mathfrak{m}}(H^0_{\mathfrak{m}}(R)) = 0 \text{ for } i \geq 1.$$

Hence, by the Five Lemma, we have that $C \cong T$.

We now show that $T = (0 :_{H^1_{\mathfrak{m}}(R)} y)$. Note that by applying local cohomology to the exact sequence $0 \to H^0_{\mathfrak{m}}(R) \to R \to R/H^0_{\mathfrak{m}}(R) \to 0$ one can see that $H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R)) \cong H^1_{\mathfrak{m}}(R)$. Furthermore, the kernel of the map $H^1_{\mathfrak{m}}(R) \xrightarrow{y} H^1_{\mathfrak{m}}(R)$ is $(0 :_{H^1_{\mathfrak{m}}(R)} y)$, so $T \cong (0 :_{H^1_{\mathfrak{m}}(R)} y)$. Applying this to the diagram above, we can see that $C \cong (0 :_{H^1_{\mathfrak{m}}(R)} y)$, and hence $e_1(q) = -\lambda_R(0 :_{H^1_{\mathfrak{m}}(R)} y)$. □

Corollary 2.3.7. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension $d \geq 1$ and $q \subseteq R$ a parameter ideal. Then $e_1(q) \leq 0$.*

Proof. Without loss of generality, we may assume that R has infinite residue field by passing to $R[x]_{\mathfrak{m}R[x]}$ if necessary. We will proceed by induction on d. The cases $d = 1$ and $d = 2$ follow from Propositions 2.3.2 and 2.3.6, respectively. Suppose $d > 2$. Then we may choose $y \in q \backslash \mathfrak{m}q$ superficial for q such that $e_1(q) = e_1(q/(y))$. The result now follows by induction as $q/(y)$ is a parameter ideal in the $(d-1)$-dimensional ring $R/(y)$. □

Corollary 2.3.8. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension 2 and $q \subseteq R$ a parameter ideal. Then the following are equivalent:*

1. $e_1(q) = 0$.
2. $H^1_{\mathfrak{m}}(R) = 0$.
3. $R/H^0_{\mathfrak{m}}(R)$ *is Cohen-Macaulay.*

In particular, if R has positive depth, $e_1(q) = 0$ if and only if R is Cohen-Macaulay.

Proof. Note that without loss of generality, we may assume that R has infinite residue field. We will start with proving (1) if and only if (2). Note that $H^1_{\mathfrak{m}}(R)$ is annihilated by a power of $\mathfrak{m}$, and hence, by a power of y. So, by Proposition 2.3.6, $e_1(q) = 0$ if and only if $(0 :_{H^1_{\mathfrak{m}}(R)} y) = 0$ if and only if $H^1_{\mathfrak{m}}(R) = 0$.

For (2) if and only if (3), apply local cohomology to the short exact sequence

$$0 \to H^0_{\mathfrak{m}}(R) \to R \to R/H^0_{\mathfrak{m}}(R) \to 0$$

to obtain the long exact sequence

$$\cdots \to H^1_{\mathfrak{m}}(H^0_{\mathfrak{m}}(R)) \to H^1_{\mathfrak{m}}(R) \to H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R)) \to H^2_{\mathfrak{m}}(H^0_{\mathfrak{m}}(R)) \to \cdots .$$

Note that since $H^0_{\mathfrak{m}}(R)$ is a finite-length module, $H^i_{\mathfrak{m}}(H^0_{\mathfrak{m}}(R)) = 0$ for all $i \geq 1$. So $H^1_{\mathfrak{m}}(R) = 0$ if and only if $H^1_{\mathfrak{m}}(R/H^0_{\mathfrak{m}}(R)) = 0$ if and only if $R/H^0_{\mathfrak{m}}(R)$ is Cohen-Macaulay. The last statement of the Corollary follows immediately. □

We now prove Theorem A, characterizing the second Hilbert coefficient of a parameter ideal in a ring of depth at least $d-1$.

Theorem A. *Suppose $(R, \mathfrak{m})$ is a Noetherian local ring of dimension $d \geq 2$ and* depth $R \geq d-1$. *Let $q \subseteq R$ be a parameter ideal. Then*

1. $e_2(q) \leq 0$.
2. $e_2(q) = 0$ *if and only if* $n(q) < 2-d$ *and* depth $gr_q(R) \geq d-1$.
3. $e_2(q) = 0$ *implies* $e_3(q) = \cdots = e_d(q) = 0$.

Proof. We may assume that R has infinite residue field by passing to $R[x]_{\mathfrak{m}R[x]}$ if necessary. We will proceed by induction on $d = \dim R$. First suppose $d = 2$. Let $q = (y, x)$ where $y \in q \backslash \mathfrak{m}q$ is a superficial non-zero-divisor for R. Let $(\bar{\cdot})$ denote working modulo (y). Now, $\bar{q}$ is a parameter ideal in the one-dimensional ring $\bar{R}$, so by Proposition 2.3.3, $H_{\bar{q}}(k) - P_{\bar{q}}(k) \leq 0$ for all $k \geq 0$. In particular, Lemma 2.2.5 gives

$$\begin{aligned} e_2(q) &= \sum_{k=1}^{\infty}(H_{\bar{q}}(k) - P_{\bar{q}}(k)) - \sum_{k=1}^{\infty}\lambda_R((q^k : y)/q^{k-1}) \\ &\leq 0. \end{aligned}$$

Note that if the left-hand side of the equation above is zero, we must have that $\lambda_R((q^k : y)/q^{k-1}) = 0$ and $P_{\bar{q}}(k) = H_{\bar{q}}(k)$ for all $k \geq 1$. In particular, the condition $\lambda_R((q^k : y)/q^{k-1}) = 0$ for all $k \geq 1$ implies that y^* is a non-zero-divisor in $gr_q(R)$, so $\operatorname{depth} gr_q(R) \geq 1$. Now, since y^* is a non-zero-divisor, Lemma 2.2.2 gives that $n(\bar{q}) = n(q) + 1$; i.e., $n(q) < 0$. This proves the Corollary when $d = 2$.

Now if $\dim R > 2$, then let $y_1, \ldots, y_{d-2} \in q \backslash \mathfrak{m}Q$ be a superficial sequence of non-zero-divisors for R. Then $\bar{q} = q/(y_1, \ldots, y_{d-2})$ is a parameter ideal in the two-dimensional ring $\bar{R} = R/(y_1, \ldots, y_{d-2})$ which has $\operatorname{depth} R \geq 1$. Hence, by induction, we have $e_2(q) = e_2(\bar{q}) \leq 0$.

For (2), first suppose $e_2(q) = 0$. Then by induction $\operatorname{grade} gr_{\bar{q}}(\bar{R})_+ \geq 1$. By Lemma 2.2.4, this implies $\operatorname{grade} gr_q(R)_+ \geq d - 2 + 1 = d - 1$. Finally, this gives $y_1^*, \ldots, y_{d-2}^*$ is a regular sequence by Lemma 2.2.3. Hence, $n(\bar{q}) < 0$ if and only if $n(q) < 2 - d$. This gives the forward implications for (2).

For the backward implication of (2), suppose $n(q) < 2 - d$ (i.e., $H_q(n) = P_q(n)$ for all $n \geq 2 - d$) and $\operatorname{grade} gr_q(R)_+ \geq d - 1$. Then $P_q(n) = 0$ for all $2 - d \leq n \leq 0$. Plugging the values $0, -1, -2, \ldots, 2 - d$ successively into $P_q(n)$, one can see that we get $e_d(q) = e_{d-1}(q) = \cdots = e_2(q) = 0$.

Finally, (3) follows from the proof of (2). $\square$

Corollary 2.3.9. *Suppose $(R, \mathfrak{m})$ is a local Noetherian ring of dimension $d \geq 2$ and* $\operatorname{depth} R \geq d - 1$. *Then for any parameter ideal $q \subseteq R$, we have*

$$\lambda_R(R/q) \leq e_0(q) - e_1(q).$$

Proof. As before, we may assume that $R/\mathfrak{m}$ is infinite. From Proposition 2.3.3, we have that $H_{\bar{q}}(n) \leq P_{\bar{q}}(n)$ for all $n \geq 1$, where $\bar{q} = q/(y_1, \ldots, y_{d-1})$ for $y_1, \ldots, y_{d-1} \in q$ a superficial sequence which is part of a minimal generating set for q. Note that we may also choose $y_1, \ldots, y_{d-1}$ to be a regular sequence as $\operatorname{depth} R \geq d-1$. Now, letting $n = 1$ and using the fact that $e_i(q) = e_i(\bar{q})$ for $i = 0, 1$ since $y_1, \ldots, y_{d-1}$ is a superficial and regular sequence, the result follows. $\square$

In light of the above corollary, one could ask the following question.

Question 2.3.10. *Suppose $(R, \mathfrak{m})$ is a local Noetherian ring of dimension d and $\operatorname{depth} R \geq d-1$. Let $q \subseteq R$ be a parameter ideal. Then does $e_2(q) = 0$ if and only if $\lambda_R(R/q) = e_0(q) - e_1(q)$?*

If $e_2(q) = 0$ above, then by Theorem A one can see that $\lambda_R(R/q) = e_0(q) - e_1(q)$. We believe that the other direction of this question may be related to the questions of Section 2.5. In particular, if we assume additionally that $\operatorname{depth} gr_q(R) \geq d-1$, then the question has an affirmative answer. To see this, let $\underline{y} := y_1, \ldots, y_{d-1} \in q$ be a superficial sequence for q which is also a regular sequence and part of a minimal generating set for q fand let $\bar{\cdot}$ denote working modulo $\underline{y}$. Then

$$\lambda_R(\overline{R}/\bar{q}) = \lambda_R(R/(q, \underline{y})) = \lambda_R(R/q) = e_0(q) - e_1(q) = e_0(\bar{q}) - e_1(\bar{q}) = P_{\bar{q}}(1).$$

Thus, by Corollary 2.3.5, since $\bar{R}$ is one-dimensional, we have $H_{\bar{q}}(n) = P_{\bar{q}}(n)$ for all $n \geq 1$. Moreover, $\underline{y}$ a superficial sequence and $\operatorname{depth} gr_q(R) \geq d-1$ imply that $y_1^*, \ldots, y_{d-1}^*$ is a regular sequence in $gr_q(R)$ (Lemma 2.2.3). Finally, by Lemma 2.2.2 we have $n(\bar{q}) = n(q) + d - 1$, so $n(q) \leq 1-d$. Successively plugging $n = 0, -1, \ldots 1-d$ into $P_q(n) = H_q(n)$, we obtain $e_i(q) = 0$ for $i \geq 2$.

The assumption that $\operatorname{depth} R \geq d-1$ is necessary in Theorem A, as evidenced by the following example.

Example 2.3.11. *Let $R = k[x, y, z, u, v, w]/I$ where I is the intersection of ideals $I = (x+y, z-u, w) \cap (z, u-v, y) \cap (x, u, w)$ and $q = (u-y, z+w, x-v)$. Then R*

is an unmixed ring of dimension three and depth one and q *is a parameter ideal with*

$$P_q(n) = 3\binom{n+2}{3} + 2\binom{n+1}{2} + n.$$

In particular, $e_2(q) = 1 > 0$.

Note that in the example above, one could mod out the ring R by a superficial non-zero-divisor in $q \backslash \mathfrak{m}q$ to obtain an example of a two-dimensional ring $\bar{R}$ of depth zero with parameter ideal $\bar{q}$ satisfying $e_2(\bar{q}) = 1 > 0$.

The upper bound for $e_2(q)$ in Theorem A can be achieved even if R is not Cohen-Macaulay. We also provide an example below with negative second Hilbert coefficient. In both examples, we use the software system Macaulay2 [GS] to compute the Hilbert-Samuel functions.

Example 2.3.12. *Let* $R = k[[x^5, xy^4, x^4y, y^5]] \cong k[[t_1, t_2, t_3, t_4]]/J$ *where* J *is the ideal* $J = (t_2t_3 - t_1t_4, t_2^4 - t_3t_4^3, t_1t_2^3 - t_3^2t_4^2, t_1^2t_2^2 - t_3^3t_4, t_1^3t_2 - t_3^4, t_3^5 - t_1^4t_4)$. *Then* R *is a two-dimensional domain with depth one. The parameter ideal* $q = (x^5, y^5)$ *has Hilbert-Samuel polynomial*

$$P_q(n) = 5\binom{n+1}{2} + 2n,$$

so $e_2(q) = 0$.

Example 2.3.13. *Let* $R = k[x, y, z, t]/((x^2, z^4) \cap (x - y, z + t))$. *Then* R *is a two-dimensional unmixed ring with depth one. The ideal* $q = (x + t + y, z - y)$ *is a parameter ideal with Hilbert-Samuel polynomial*

$$P_Q(n) = 9\binom{n+1}{2} + 2\binom{n}{1} - 1.$$

Hence, $e_2(q) = -1 < 0$*. In this example, we have that* $n(q) = 0$*; that is,* $P_q(0) \neq H_q(0)$ *and* $P_q(n) = H_q(n)$ *for all* $n \geq 1$*. However, we do have that* $\operatorname{depth} gr_q(R) \geq 1$*.*

We now prove the other main result of this section.

Theorem B. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring of dimension* d *and suppose* q *is a parameter ideal for* R *satisfying* $\operatorname{depth} gr_q(R) \geq d-1$*. Then for* $0 \leq i \leq d+1$ *and* $n \in \mathbb{Z}$*,*

$$(-1)^i \Delta^{d+1-i}(P_q(n) - H_q(n)) \geq 0.$$

Proof. We first note that it is enough to prove the result when $i = 0$. Indeed, suppose $g : \mathbb{Z} \to \mathbb{Z}$ satisfies $g(n) = 0$ for all n sufficiently large and $\Delta(g(n)) \geq 0$ for all $n \geq 1-d$. Then we claim $g(n) \leq 0$ for all n. Let N be such that $g(n) = 0$ for all $n \geq N$. Then $\Delta(g(N-1)) = g(N) - g(N-1) \geq 0$ implies $g(N-1) \leq 0$. Inductively, one can show that $g(j) \leq 0$ for all j. In particular, if we set $g(n) = P_q(n) - H_q(n)$, and assume $(-1)^i \Delta^{d+1-i}(g(n)) \geq 0$ for all $n \geq 1-d$, then $(-1)^i \Delta^{d+1-i-1}(g(n)) \leq 0$ gives the theorem for $i+1$. Hence, it is enough to prove

$$\Delta^{d+1}(P_q(n) - H_q(n)) \geq 0 \quad \text{for all } n.$$

We will use induction on the dimension d. Note the case $d = 1$ is proved in Corollary 2.3.5. Suppose $d > 1$. Let $a \in q \backslash \mathfrak{m}q$ such that a^* is a $gr_q(R)$-regular element. Let $\bar{q} = q/(a)$ and $\bar{R} = R/(a)$. Then note that $\operatorname{depth}_{\bar{q}}(\bar{R}) \geq d-2$ and $\bar{q}$ is a parameter ideal for the $(d-1)$-dimensional ring $\bar{R}$. So, by induction,

$$\Delta^d(P_{\bar{q}}(n) - H_{\bar{q}}(n)) \geq 0 \quad \text{for all } n.$$

Now, as a^* is a non-zero-divisor in $gr_q(R)$, we have $H_{\bar{q}}(n) = H_q(n) - H_q(n-1)$ for all n. Similarly, $P_{\bar{q}}(n) = P_q(n) - P_q(n-1)$. Hence,

$$\begin{aligned} \Delta^{d+1}(P_q(n) - H_q(n)) &= \Delta^d(\Delta(P_q(n) - H_q(n)) \\ &= \Delta^d(P_{\bar{q}}(n+1) - H_{\bar{q}}(n+1)) \\ &\geq 0 \quad \text{for all } n. \end{aligned}$$

Thus,

$$\Delta^{d+1}(P_q(n) - H_q(n)) \geq 0 \quad \text{for all } n. \qquad \square$$

Corollary 2.3.14. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d and suppose q is a parameter ideal for R satisfying* $\operatorname{depth} gr_q(R) \geq d-1$. *Suppose $P_q(k) - H_q(k) = 0$ for some k. Then $P_q(n) - H_q(n) = 0$ for all $n \geq k$, i.e., $k > n(q)$.*

Proof. Letting $i = d$ in Theorem B, we have $(-1)^d\Delta(P_q(n) - H_q(n)) \geq 0$ for all n. This gives $(-1)^d(P_q(n+1) - H_q(n+1)) \geq (-1)^d(P_q(n) - H_q(n))$ for all n. In particular, we have

$$0 \geq (-1)^d(P_q(n) - H_q(n)) \geq (-1)^d(P_q(k) - H_q(k)) = 0 \quad \forall\, n \geq k$$

where the first inequality holds because $P_q(N) - H_q(N) = 0$ for $N \gg 0$. Thus, $P_q(n) = H_q(n)$ for all $n \geq k$. $\square$

Remark 2.3.15. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d and suppose q is a parameter ideal for R. For $0 \leq i \leq d-1$, we have the following:*

1. *If $n(q) < i - d$, then $e_j(q) = 0$ for $j \geq i$.*

2. *If* $\operatorname{depth} gr_q(R) \geq d-1$, *the converse to (1) holds.*

Proof. Note (1) follows by using that $P_q(j) = 0$ for $i - d < j < 0$. For (2), suppose depth $gr_q(R) \geq d-1$ and $e_i(q) = 0$ for $j \geq i$. Then $P_q(i-d) = 0 = H_q(i-d)$ and by Corollary 2.3.14, $n(q) < i - d$. □

Question 2.3.16. *Does the converse to part (1) of the remark above hold in general?*

Corollary 2.3.17. *Let* $(R, \mathfrak{m})$ *be a local Noetherian ring of dimension* d *and suppose* q *is a parameter ideal for* R *satisfying* depth $gr_q(R) \geq d-1$. *Then for* $1 \leq i \leq d$

1. $e_i(q) \leq 0$.
2. $(-1)^{j+1}(e_0(q) - e_1(q) + \cdots + (-1)^j e_j(q) - \lambda_R(R/q)) \geq 0$ *for* $j = 1, \ldots, d$.

Proof. Note that it is enough to prove (1) in the case $i = d$, as we can then use our usual argument via reduction by a superficial sequence to obtain $e_i(q) \leq 0$ for $i = 1, \ldots, d-1$. Letting $i = d+1$ in Theorem B, we have

$$(-1)^{d+1}(P_q(n) - H_q(n)) \geq 0 \quad \text{for all } n. \tag{2.4}$$

If $n = 0$, $(-1)^{d+1}((-1)^d(e_d(q) - H_q(0)) \geq 0$ implies $-e_d(q) \geq 0$; that is, $e_d(q) \leq 0$.

For (2), we will first prove the case $j = \dim R = d$. Indeed, letting $n = 1$ in equation (2.4), we see

$$(-1)^{d+1}(e_0(q) - e_1(q) + \cdots + (-1)^d e_d - \lambda_R(R/q)) \geq 0.$$

Now, let $a_1, \ldots a_{d-j} \in q \backslash q^2$ be part of a minimal generating set for q such that $a_1^*, \ldots, a_{d-j}^*$ is a $gr_q(R)$-regular sequence. Then, setting $\bar{R} = R/(a_1, \ldots a_{d-j})$ and $\bar{q} = q/(a_1, \ldots a_{d-j})$ we have $\bar{q}$ is a parameter ideal in the j-dimensional ring $\bar{R}$, and depth $gr_{\bar{q}}(\bar{R}) \geq j-1$. Finally, $\lambda_R(R/q) = \lambda_{\bar{R}}(\bar{R}/\bar{q})$ and since $a_1, \ldots a_{d-j}$ defines a superficial regular sequence in R, we have $e_i(\bar{q}) = e_i(q)$ for all $i = 0, \ldots, j$. It follows that

$$(-1)^{j+1}(e_0(q) - e_1(q) + \cdots + (-1)^j e_j(q) - \lambda_R(R/q)) \geq 0. \qquad \square$$

Corollary 2.3.18. *Let $(R, \mathfrak{m})$ be a local Noetherian ring of dimension d and suppose q is a parameter ideal for R satisfying $\operatorname{depth} gr_q(R) \geq d-1$. Suppose $e_i(q) = 0$ for some $1 \leq i \leq d-1$. Then $e_j(q) = 0$ for $i \leq j \leq d$.*

Proof. Note that it is enough to prove that $e_{i+1}(q) = 0$. Reducing by a superficial sequence if necessary, we may assume that $i = d-1$. Since $e_0(q) > 0$, we must have that $d > 1$, so by assumption, $\operatorname{depth} gr_q(R) > 0$. Let $a \in q$ be such that $a^* \in gr_q(R)$ is a non-zero-divisor. Then $e_{d-1}(\bar{q}) = e_{d-1}(q) = 0$ implies that $P_{\bar{q}}(0) = 0 = H_{\bar{q}}(0)$. Now, by Corollary 2.3.14, $n(\bar{q}) \leq -1$. As $n(\bar{q}) = n(q) + 1$, this gives $n(q) \leq -2$, and in particular, $(-1)^d e_d(q) = P_q(0) = H_q(0) = 0$. $\square$

Question 2.3.19. *Is there an example of a parameter ideal with $e_i = 0$ and $e_{i+1} \neq 0$?*

2.4 The Unmixed Case

In this section we provide an alternative proof of Theorem A in the case R is unmixed. We restate the theorem below:

Theorem Â. *Let $(R, \mathfrak{m})$ be an unmixed Noetherian local ring of dimension $d \geq 2$. Suppose that $\operatorname{depth} R \geq d-1$. If q is a parameter ideal of R, then the following hold:*

1. $e_2(q) \leq 0$
2. $e_2(q) = 0$ *if and only if* $n(q) < 2 - d$ *and* $\operatorname{grade} gr_q(R)_+ \geq d-1$
3. $e_2(q) = 0$ *implies* $e_3(q) = e_4(q) = \cdots = e_d(q) = 0$.

Although this theorem is less general than Theorem A, the methods used to prove Theorem Â are quite different than the methods used in the previous section. We include the proof here in the hopes that the proof techniques will be useful in proving more results concerning other Hilbert coefficients of parameter ideals in unmixed rings.

To prove Theorem Â, we want to show that if $R = S/I$ with S a Gorenstein ring of the same dimension as R and $q \subseteq R$ is a parameter ideal of R, then there is a parameter ideal $Q \in S$ so that $QR = q$. The following lemma will allow us to find such a Q. We use the same proof technique here as Lemma 3.1 in [GHV09], where they prove the lemma in the case S is Cohen-Macaulay and I is prime.

Lemma 2.4.1. *Let $(S, \mathfrak{n})$ be a local Noetherian ring of dimension d with infinite residue field and let $R = S/I$ with $\dim R = \dim S$. Suppose $x_1, \ldots, x_d \in S$ such that $\dim(S/(I, x_1, \ldots, x_i)) = d - i$ for $i = 1, \ldots, d$. Then there exist $a_1, \ldots, a_d \in S$ such that $\dim(S/(a_1, \ldots, a_i)) = d - i$ and $x_i - a_i \in I$ for $i = 1, \ldots, d$.*

Proof. Let $I = (c_1, \ldots, c_n)$ and let $p_1, \ldots, p_l$ be the dimension d (ie. $\dim S/p = d$) primes of S not containing I. We will show that there exists $\alpha \in S \backslash \mathfrak{n}$ such that $x_1 + c_1\alpha + c_2\alpha^2 + \cdots + c_n\alpha^n \notin p_i$ for $i = 1, \ldots, l$. Suppose not. Then since $S/\mathfrak{n}$ is infinite, there exist $\alpha_1, \ldots, \alpha_{n+1} \in S \backslash \mathfrak{n}$ and a fixed k between 1 and l such that $\alpha_i + \mathfrak{n} \neq \alpha_j + \mathfrak{n}$ whenever $i \neq j$ and $x_1 + c_1\alpha_i + \cdots c_n\alpha_i^n \in p_k$ for $i = 1, \ldots, n+1$. Let A be the Vandermonde matrix determined by the α_i. We have

$$A \begin{bmatrix} x_1 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^n \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_{n+1} & \alpha_{n+1}^2 & \cdots & \alpha_{n+1}^n \end{bmatrix} \begin{bmatrix} x_1 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{n+1} \end{bmatrix}$$

where $g_1, \dots, g_{n+1} \in p_k$. Note A is invertible as $\det A = \prod_{i<j}(\alpha_j - \alpha_i) \notin \mathfrak{n}$ so $\det A$ is a unit in S. Thus we have $c_1, \dots, c_n \in p_k$ so that $I \subseteq p_k$, a contradiction. Now let $a_1 = x_1 + c_1\alpha + c_2\alpha^2 + \cdots + c_n\alpha^n$. Then $a_1 \notin p$ for any dimension d prime in S not containing I. In fact, we have $a_1 \notin p$ for any dimension d prime of S. To see this, suppose $p \supseteq I$ is prime with $\dim S/p = d$. Then note that as $x_1 - a_1 \in I$, by assumption we have $\dim S/(I, a_1) = d - 1$. If $a_1 \in p$, then $\dim(S/p) \leq d - 1$, contradicting our assumption that $\dim S/p = d$. Thus, we must have that $a_1 \notin p$ for any prime $p \subseteq S$ with $\dim S/p = d$, and hence $\dim S/(a_1) = d - 1$.

Now let $q_1, \dots, q_s$ be the dimension $d-1$ primes of (a_1) not containing I. Using a similar argument to that above, we can find $\beta \in S\backslash\mathfrak{n}$ such that the element $a_2 = x_2 + c_1\beta + \cdots + c_n\beta^n$ is not in p_i for $i = 1, \dots, s$. Note again that if p is a dimension $d-1$ prime over (a_1) such that $p \supseteq I$, then we have $S/(I, a_1, a_2) = S/(I, x_1, x_2)$ so by assumption $\dim(S/(I, a_1, a_2)) = d - 2$. If p contains a_2, we have $\dim(S/p) \leq d - 2$, a contradiction. Thus a_2 is not contained in any dimension $d-1$ prime over (a_1). We can continue in this way to obtain $a_1, \dots, a_d$ such that $\dim(S/(a_1, \dots, a_i)) = d - i$ for $1 \leq i \leq d$. $\square$

When we consider the Hilbert coefficients of an ideal in an unmixed ring, the following results will allow us to assume that the ring has infinite residue field.

Lemma 2.4.2. *Suppose $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a faithfully flat local ring extension and $I \subseteq R$ is an ideal with $\sqrt{I} = \mathfrak{m}$. Then* $\operatorname{depth} S/IS = \operatorname{depth} S/\mathfrak{m}S$.

Proof. Note S/IS and $S/\mathfrak{m}S$ are S/IS-modules, so $\operatorname{depth}_{S/IS} S/IS = \operatorname{depth}_S S/IS$ and $\operatorname{depth}_{S/IS} S/\mathfrak{m}S = \operatorname{depth}_S S/\mathfrak{m}S$. Now, $R \to S$ flat implies $R/I \to S/IS$ is flat. Hence, tensoring with R/I, it's enough to show $\operatorname{depth} S = \operatorname{depth} S/\mathfrak{m}S$ when $\dim R = 0$ and $R \to S$ is flat. We claim that $\operatorname{depth} S/\mathfrak{m}^n S = \operatorname{depth} S/\mathfrak{m}S$ for all

$n \geq 1$. Then, since $\dim R = 0$, $\mathfrak{m}^n = 0$ for n sufficiently large and so we'll have $\operatorname{depth} S = \operatorname{depth} S/\mathfrak{m}S$.

To prove the claim, we proceed by induction. The result clearly holds for $n = 1$, so assume $n > 1$ and $\operatorname{depth} S/\mathfrak{m}^n S = \operatorname{depth} S/\mathfrak{m}S$. Consider the exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^{n+1} \to R/\mathfrak{m}^n \to 0.$$

Note that $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong (R/\mathfrak{m})^t$ for some t. Now, since S is flat over R, we have

$$0 \to (S/\mathfrak{m}S)^t \to S/\mathfrak{m}^{n+1} \to S/\mathfrak{m}^n \to 0$$

is exact. By induction, we have $\operatorname{depth}(S/\mathfrak{m}S)^t = \operatorname{depth} S/\mathfrak{m}^n$. It follows from the exact sequence above that $\operatorname{depth} S/\mathfrak{m}^{n+1} = \operatorname{depth} S/\mathfrak{m}$. $\square$

Theorem 2.4.3. *Let R be a local ring which is the homomorphic image of a Cohen-Macaulay ring such that* $\operatorname{Ass} R = \operatorname{Assh} R$. *Then* $\operatorname{Ass} \widehat{R} = \operatorname{Assh} \widehat{R}$.

Proof. Write $R = S/I$. Then $\widehat{R} = \widehat{S}/I\widehat{S}$. Let $Q \in \operatorname{Ass} \widehat{S}/I\widehat{S}$ and consider $Q \cap S = p$. Now, $Q \in \operatorname{Ass} \widehat{S}/I\widehat{S}$ means that p consists of zero-divisors on S/I, so $p \subseteq q \in \operatorname{Ass} S/I$ for some q. However, $\dim R/q = \dim R$ by assumption, so we must have $q = p$ is minimal over I.

Now, $S_p \to \widehat{S}_Q$ is a flat local ring extension. So, $\operatorname{depth} \widehat{S}_Q = \operatorname{depth} S_p + \operatorname{depth} \widehat{S}_Q/p\widehat{S}_Q$ by [BH93, 1.2.16]. Note that as $p \in \operatorname{Min}(S/I)$, $\sqrt{I_p} = pS_p$, so applying Lemma 2.4.2 to the extension $S_p \to \widehat{S}_Q$, we have $\operatorname{depth} \widehat{S}_Q/p\widehat{S}_Q = \operatorname{depth} \widehat{S}_Q/I\widehat{S}_Q = 0$ as $Q \in \operatorname{Ass}(\widehat{S}/I\widehat{S})$. Thus, $\operatorname{depth} \widehat{S}_Q = \operatorname{depth} S_p$. Now, as S is Cohen-Macaulay, $\widehat{S}_Q$ and S_p are Cohen-Macaulay, so this gives $\dim \widehat{S}_Q = \dim S_p$. Finally, as S and $\widehat{S}$ are Cohen-Macaulay, we have $\dim S_p + \dim S/p = \dim S = \dim \widehat{S} = \dim \widehat{S}_Q + \dim \widehat{S}/Q$.

This implies

$$\dim S/I = \dim S/p = \dim \widehat{S}/Q,$$

i.e., $\dim \widehat{R} = \dim \widehat{R}/q$ for any $q \in \operatorname{Ass} \widehat{R}$. $\square$

Note that given an unmixed ring $(R, \mathfrak{m})$ and an $\mathfrak{m}$-primary ideal I, we may pass to the $\mathfrak{m}$-adic completion $\widehat{R}$ so that $e_i(I) = e_i(I\widehat{R})$ for each i. Now $\widehat{R}$ satisfies $\operatorname{Ass} \widehat{R} = \operatorname{Assh} \widehat{R}$. Passing to $T := \widehat{R}[x]_{\widehat{\mathfrak{m}}\widehat{R}[x]}$, we obtain a ring with infinite residue field such that $\operatorname{Ass} T = \operatorname{Assh} T$ and we still have $e_i(I) = e_i(IT)$ for $i = 0, \dots, d$. Note that since $\widehat{R}$ is the homomorphic image of a Cohen-Macaulay ring (by the Cohen Structure Theorem), T is also. Thus, we may apply Theorem 2.4.3, to obtain that T is a local unmixed ring with infinite residue field.

We now prove the main theorem in the case $\dim R = 2$.

Proposition 2.4.4. *Suppose $(R, \mathfrak{m})$ is an unmixed ring of dimension two. Then $e_2(q) \leq 0$ for all parameter ideals q of R. Moreover, equality holds if and only if* $\operatorname{grade} gr_q(R)_+ \geq 1$ *and $P_q(n) = H_q(n)$ $\forall n \geq 0$.*

Proof. As above, we may assume that R has infinite residue field. By passing to the $\mathfrak{m}$-adic completion $\widehat{R}$, we may also assume R is complete and is the homomorphic image of a two-dimensional Gorenstein local ring $(S, \mathfrak{n})$. Furthermore, note that as R is unmixed, $\operatorname{depth} R > 0$. Let q be a parameter ideal for R. Then there exists a parameter ideal $Q \subseteq S$ such that $QR = q$ by Lemma 2.4.1. Since $gr_Q(R) \cong gr_q(R)$, we have $e_i(Q, R) = e_i(q)$ for $i = 0, 1, 2$. As R is unmixed and S is Gorenstein, we may assume R is a first syzygy of S [EG85, Theorem 3.5]. So we have

$$0 \to R \to S^n \to C \to 0.$$

Let $y \in Q$ be a superficial element with respect to R such that y is part of a minimal generating set for Q and $y \notin p$ for all $p \in \mathrm{Ass}_S(C)\backslash\{\mathfrak{n}\}$ and all $p \in \mathrm{Ass}_S(R)$. Tensor the sequence above with $S/(y)$ to obtain

$$0 \to T = \mathrm{Tor}_1^S(S/yS, C) \to R/yR \xrightarrow{\varphi} S^n/yS^n \to C/yC \to 0.$$

Let R' denote R/yR and S' denote the image of φ. We consider the short exact sequence

$$0 \to T \to R' \to S' \to 0.$$

Note $\lambda(T) < \infty$ as $T_p = 0$ for all $p \neq \mathfrak{n}$ by the choice of y.

Let $Q = (y, z)$. Then we can apply the Snake Lemma to

$$\begin{tikzcd}
0 \arrow[r] & T \cap z^n R' \arrow[r] \arrow[d] & z^n R' \arrow[r] \arrow[d] & z^n S' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & R' \arrow[r] & S' \arrow[r] & 0
\end{tikzcd}$$

to obtain

$$\lambda(R'/z^n R') = \lambda(T/(T \cap z^n R')) + \lambda(S'/z^n S')$$

for all n. Now, for $n \gg 0$, we have $T \cap z^n R' = (0)$ since by the Artin-Rees Lemma, there exists a k such that $T \cap z^n R' \subseteq z^{n-k}(z^k R' \cap T) \subseteq z^{n-k} T$ and $z^{n-k} T = 0$ for $n \gg 0$ as $\lambda_R(T) < \infty$. Furthermore, $\lambda(S'/z^n S') = n\lambda(S'/zS')$ for all n since z is regular on $S/(y)$ and hence on $S' \subseteq S^n/yS^n$. From this we get that $e_0(\bar{Q}) = \lambda(S'/zS')$ and $e_1(\bar{Q}) = -\lambda(T)$. Since y is a superficial non-zero-divisor for R, we have that

$e_0(\bar{Q}) = e_0(Q)$ and $e_1(\bar{Q}) = e_1(Q)$. Now by Lemma 2.2.5,

$$
\begin{aligned}
e_2(q) &= \sum_{k=1}^{\infty} \big(H_{\bar{Q}}(k) - P_{\bar{Q}}(k)\big) - \sum_{k=1}^{\infty} \lambda_R((Q^k : y)/Q^{k-1}) \\
&= \sum_{k=1}^{\infty} \big(\lambda(T/(T \cap z^k R')) + \lambda(S'/z^k S') - (k\lambda(S'/zS') + \lambda(T))\big) \\
&\qquad\quad - \sum_{k=1}^{\infty} \lambda_R((Q^k : y)/Q^{k-1}) \\
&= \sum_{k=1}^{\infty} -\lambda(T \cap z^k R') - \sum_{k=1}^{\infty} \lambda_R((Q^k : y)/Q^{k-1}) \\
&\leq 0.
\end{aligned}
$$

This proves the first part of the proposition.

For the second statement, suppose $e_2(q) = 0$. Note that in the above equation we have shown that $H_{\bar{Q}}(k) - P_{\bar{Q}}(k) = -\lambda(T \cap z^k R') \leq 0$ for all $k \geq 1$. If the left hand side of the above equation is 0, we must have $Q^k : y = Q^{k-1}$ for all $k \geq 1$, i.e., $y*$ is a non-zero-divisor for $gr_Q(R) \cong gr_q(R)$. Furthermore, since $H_{\bar{Q}}(n) - P_{\bar{Q}}(n) \leq 0$ for all $n \geq 1$, we must have $H_{\bar{Q}}(n) = P_{\bar{Q}}(n)$ for all $n \geq 1$. Finally, note that since y^* is a non-zero-divisor in $gr_Q(R)$, $H_{\bar{Q}}(n) = P_{\bar{Q}}(n)$ for all $n \geq 1$ if and only if $H_Q(n) = P_Q(n)$ for all $n \geq 0$ by Lemma 2.2.2. For the other direction, note that $e_2(Q) = P_Q(0) = H_Q(0) = 0$. □

To prove the main theorem of this section we want to be able to choose a superficial sequence $y_1, \ldots, y_i \in q$ which is part of a minimal generating set for q so that $R/(y_1, \ldots, y_i)$ remains unmixed. The following theorems and propositions will allow us to make such a choice.

Theorem 2.4.5. *[BH93, Theorem 2.1.15] Let R be a Cohen-Macaulay local ring, and p a prime ideal. Then $\dim \widehat{R}/q = \dim R/p$ for all $q \in \operatorname{Ass}(\widehat{R}/p\widehat{R})$. In particular, $p\widehat{R}$*

is an unmixed ideal.

Proof. This follows from Theorem 2.4.3, since $\operatorname{Ass} R/p = \operatorname{Assh} R/p$. □

The following propositions can be found in [GN01]. The proofs here are those of [GN01], but we include more detail.

Proposition 2.4.6. *[GN01, Lemma 3.2] Suppose R is a homomorphic image of a Cohen-Macaulay local ring satisfying $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{m\}$. Then*

$$\mathfrak{F} := \{p \in \operatorname{Spec} R \mid \text{ height}(p) > 1 = \operatorname{depth} R_p, p \neq \mathfrak{m}\}$$

is a finite set.

Proof. [GN01] Let $\widehat{\mathfrak{F}} := \{P \in \operatorname{Spec} \widehat{R} \mid \text{ height}_{\widehat{R}} P > 1 = \operatorname{depth} \widehat{R}_P, P \neq \mathfrak{m}\widehat{R}\}$. Let $p \in \mathfrak{F}$ and $P \in \operatorname{Min}(\widehat{R}/p\widehat{R})$. Then note that $\operatorname{height}_{\widehat{R}} P > 1, \operatorname{depth} \widehat{R}_P = 1$, and $p = P \cap R$. To see this, first note that R/p is a domain, so P contracts to the zero ideal in R/p as P consists of zero-divisors in $\widehat{R}/p\widehat{R}$; that is, $P \cap R = p$. Now, as in the proof of Theorem 2.4.3, we have $\operatorname{depth} \widehat{R}_P = \operatorname{depth} R_p + \operatorname{depth} \widehat{R}_P/p\widehat{R}_P = 1 + 0$ as $p \in \mathfrak{F}$ and $P \in \operatorname{Ass}(\widehat{R}/p\widehat{R})$. Finally, [Mat89, Theorem 15.1] gives $\operatorname{height} P = \operatorname{height} p + \dim \widehat{R}_P/p\widehat{R}_P > 1$. Hence, $P \in \widehat{\mathfrak{F}}$, so $\mathfrak{F} \subseteq \{P \cap R \mid P \in \widehat{\mathfrak{F}}\}$ and it's enough to prove that $\widehat{\mathfrak{F}}$ is a finite set.

We first claim that $\operatorname{Ass} \widehat{R} \subseteq \operatorname{Assh} \widehat{R} \cup \{\mathfrak{m}\widehat{R}\}$. To see this, suppose that $P \in \operatorname{Ass} \widehat{R} \backslash \{m\widehat{R}\}$. Then there exists $p \in \operatorname{Ass} R$ with $p \neq \mathfrak{m}$ such that $P \in \operatorname{Ass}_{\widehat{R}} \widehat{R}/p\widehat{R}$. Indeed, let $p = P \cap R$. Then the extension $R_p \to \widehat{R}_P$ is a faithfully flat extension and so $\operatorname{depth} \widehat{R}_P = \operatorname{depth} R_p + \operatorname{depth} \widehat{R}_P/p\widehat{R}_P$. As $P \in \operatorname{Ass} \widehat{R}$, the left-hand side of the equation is zero. Thus, $\operatorname{depth} \widehat{R}_P/p\widehat{R}_P = 0 = \operatorname{depth} R_p$. This gives that $p \in \operatorname{Ass} R$ and $P \in \operatorname{Ass}_{\widehat{R}} \widehat{R}/p\widehat{R}$.

By assumption, we have $p \in \operatorname{Assh} R$, so $\dim(\widehat{R}/p\widehat{R}) = \dim(R/p) = d$. Now, by Theorem 2.4.5, we have $\dim \widehat{R}/P = \dim R/p = d$, so $P \in \operatorname{Assh} \widehat{R}$. This gives the claim. We may now assume that R is a complete local ring.

Let K_R be the canonical module for R. (Recall that for a complete local ring R we can define $K_R := \operatorname{Hom}_R(H^d_{\mathfrak{m}}(R), E_R(k))$. See [BH93, Remark 3.5.10] for more.) Let $\varphi : R \to S = \operatorname{Hom}_R(K_R, K_R)$ be the canonical map given by $\varphi(r)(x) = rx$ for $r \in R$ and $x \in K_R$. Then setting $U = \operatorname{Ker}\varphi$ and $C = \operatorname{Coker}\varphi$, we have an exact sequence $0 \to U \to R \to S \to C \to 0$. First note that $U_p = 0$ for $p \in \operatorname{Assh} R$. To see this note that $(K_R)_p = K_{R_p}$ and $S_p = \operatorname{Hom}_{R_p}(K_{R_p}, K_{R_p}) \cong R_p$ since $p \in \operatorname{Assh} R$. Localize at $p \in \mathfrak{F}$ to obtain the short exact sequence $0 \to R_p \to S_p \to C_p \to 0$. By the choice of p, we have $\operatorname{depth} R_p = 1$ and $\operatorname{depth}_{R_p} S_p \geq 2$ by [HH90, Remark 2.2(f)]. By the Depth Lemma, this implies $\operatorname{depth} C_p = 0$ and hence $p \in \operatorname{Ass}_R(C)$. Thus, $\mathfrak{F} \subseteq \operatorname{Ass}_R(C)$, a finite set. $\square$

Proposition 2.4.7. *[GN01, Lemma 3.3] Suppose R is a homomorphic image of a Cohen-Macaulay local ring satisfying $\operatorname{Ass} R \subseteq \operatorname{Assh} R \cup \{m\}$. Let q be a parameter ideal. Set $d = \dim R$. Then there exists a system $a_1, a_2, \ldots, a_d$ of generators of q such that $\operatorname{Ass}(R/q_i) \subseteq \operatorname{Assh}(R/q_i) \cup \{\mathfrak{m}\}$ for all $0 \leq i \leq d$ where $q_i = (a_1, a_2, \ldots, a_i)$.*

Proof. [GN01] Note if $d = 0$, the result holds. So assume $d > 0$. It is enough to prove the proposition in the case $i = 1$. Let $\mathfrak{F}$ be as in Proposition 2.4.6. Choose $a_1 \in q$ so that

$$a_1 \in q \setminus \left(\mathfrak{m}q \bigcup \left(\bigcup_{p \in \operatorname{Assh} R} p \right) \bigcup \left(\bigcup_{p \in \mathfrak{F}} p \right) \right).$$

Let $p \in \operatorname{Ass}(R/(a_1))$ with $p \neq \mathfrak{m}$. We want to show that $\dim R/p = \dim R/(a_1)$. Note that since p is an associated prime for $R/(a_1)$, $\operatorname{depth}(R_p/a_1R_p) = 0$. However, since $p \notin \operatorname{Ass}(R)$ (by the choice of a_1), we have $\operatorname{depth} R_p > 0$. Thus, we must have $\operatorname{depth} R_p = 1$. This gives $\operatorname{height} p = 1$ as $p \notin \mathfrak{F}$. Note by assumption, R is catenary and equidimensional. So we have $\dim R/p = \dim R - \operatorname{height} p = d - 1 = \dim R/(a_1)$. Thus, $p \in \operatorname{Assh}(R/(a_1))$. $\square$

Note that we can choose the element a_1 above to be a superficial element of q. To see this, we consider the following from [SH06].

Proposition 2.4.8. (cf. *[SH06, Proposition 8.5.7]*) *Let R be a Noetherian ring with infinite residue field and I an ideal. Then there exists an integer c such that for all $n \geq c$, $(I^n :_R x) \cap I^c = I^{n-1}$. Furthermore, there exists a non-empty Zariski-open set U of I/mI such that whenever $r \in I$ with image in I/mI in U, then r is superficial for I.*

Corollary 2.4.9. *[SH06, Corollary 8.5.9] Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field. Let I be an ideal of R and $P_1, \ldots, P_r$ ideals in R not containing I. Then for any finitely generated R-module M there exists a superficial element for I with respect to M that is not contained in any P_i.*

If q is a parameter ideal of R and $\operatorname{depth} R \geq 1$, then note that $q \not\subseteq p$ for any $p \in \mathfrak{F} \cup \operatorname{Ass} R$. Hence, by Corollary 2.4.9, we may choose a_1 in Proposition 2.4.7 to be a superficial non-zero-divisor.

We are now ready to prove Theorem Â.

Theorem Â. *Let $(R, \mathfrak{m})$ be an unmixed Noetherian ring of dimension $d \geq 2$. Suppose that* $\operatorname{depth} R \geq d - 1$. *If q is a parameter ideal of R, then the following hold:*

1. $e_2(q) \leq 0$.

2. $e_2(q) = 0$ *if and only if* $n(q) < 2 - d$ *and* $\operatorname{grade} gr_q(R)_+ \geq d - 1$.

3. $e_2(q) = 0$ *implies* $e_3(q) = e_4(q) = \cdots = e_d(q) = 0$.

Proof. We will proceed by induction on $d := \dim R$. As in Proposition 2.4.4, we may assume that R is complete and has infinite residue field. Note that the case $d = 2$ is given by Proposition 2.4.4. So suppose $d = \dim R > 2$. By Proposition 2.4.7 we may choose a superficial non-zero-divisor $a \in q$ which is a minimal generator for q such that $\operatorname{Ass}(R/aR) \subseteq \operatorname{Assh}(R/aR) \cup \{\mathfrak{m}\}$. Since $\operatorname{depth} R \geq d - 1 > 1$, we have $\operatorname{depth}(R/aR) \geq 1$, so $\mathfrak{m} \notin \operatorname{Ass}(R/aR)$. Thus, R/aR is unmixed. Let $\bar{q}$ be the image of q in R/aR. Then, by induction, we have $e_2(q) = e_2(\bar{q}) \leq 0$. Then $e_2(\bar{q}) = 0$. This gives us (1). The proof of (2) and (3) is identical to the proof in Theorem A. $\square$

2.5 When does the associated graded ring have positive depth?

Our work in the previous sections led us to the following question:

Question 2.5.1. *Let $q \subseteq R$ be a parameter ideal in a local Noetherian ring R. Is* $\operatorname{depth} gr_q(R) = \operatorname{depth} R$*?*

In this section we examine this question, and present some situations in which the question has an affirmative answer.

Definition 2.5.2. *Let $(R, \mathfrak{m})$ be a d-dimensional ring and $I \subseteq R$ an ideal. Let $I = I(p_1) \cap \cdots \cap I(p_n)$ be a primary decomposition for I, where $I(p_i)$ is p_i-primary. We define the* unmixed part *of I to be*

$$U(I) := \bigcap_{p \in \operatorname{Assh} R} I(p).$$

Lemma 2.5.3. *Suppose $(R, \mathfrak{m})$ is a one-dimensional local Noetherian ring. Then $H^0_{\mathfrak{m}}(R) = U((0))$.*

Proof. Let $(0) = p_1 \cap p_2 \cap \cdots \cap p_l \cap p'$ be a primary decomposition of (0) where p' is the $\mathfrak{m}$-primary component. Then for $n \gg 0$, we have

$$H^0_{\mathfrak{m}}(R) = (0 : \mathfrak{m}^n) = (p_1 : \mathfrak{m}^n) \cap \cdots \cap (p_l : \mathfrak{m}^n) \cap (p' : \mathfrak{m}^n) = p_1 \cap \cdots \cap p_l = U((0)).$$

□

Lemma 2.5.4. *Suppose* $(R, \mathfrak{m})$ *is a two-dimensional Noetherian ring and* $q = (x, y)$ *is a parameter ideal satisfying* $yH^0_{\mathfrak{m}}(R/(x)) = 0$ *(or* $xH^0_{\mathfrak{m}}(R/(y)) = 0$*). Then*

$$U((x)) \cap q^n = xq^{n-1}$$

(or $U((y)) \cap q^n = yq^{n-1}$*) for all integers* $n > 0$.

Proof. Let $u \in U((x)) \cap q^n$. Then $u \in q^n = xq^{n-1} + yq^{n-1}$. So we may write $u = x\alpha + y\beta$ where $\alpha, \beta \in q^{n-1}$ and hence $y\beta = u - x\alpha \in U((x))$. Note y is $R/U((x))$-regular since (x, y) is a system of parameters. This gives $\beta \in U((x))$.

Suppose $n = 1$. Then the assumption that $yH^0_{\mathfrak{m}}(R/(x)) = 0$ implies that $yU((x)) \subseteq (x)$ as $H^0_{\mathfrak{m}}(R/(x)) = U((x))/(x)$. Hence, $\beta \in U((x))$ implies $y\beta \in (x)$ and we have $u = x\alpha + y\beta \in (x)$. This gives $U((x)) \cap q = (x)R$.

Now suppose $n > 1$ and assume $U((x)) \cap q^{n-1} = (x)q^{n-2}$. Now, note that $\beta \in U((x)) \cap q^{n-1} = (x)q^{n-2}$ gives $y\beta \in (x)q^{n-1}$. Hence, $u = x\alpha + y\beta \in (x)q^{n-1}$ and we have $U((x)) \cap q^n = (x)q^{n-1}$. □

Corollary 2.5.5. *Suppose* $(R, \mathfrak{m})$ *is a two-dimensional Noetherian ring and* $q = (x, y)$ *is a parameter ideal with* x *a non-zero-divisor satisfying* $yH^0_{\mathfrak{m}}(R/(x)) = 0$. *Then* $\operatorname{depth} gr_q(R) > 0$.

Proof. By the lemma above, $x(q^n : x) = (x) \cap q^n \subseteq U((x)) \cap q^n = xq^{n-1}$ for all

$n \geq 1$. As x is a non-zero-divisor, we have $(q^n : x) = q^{n-1}$ for all $n \geq 1$, i.e., x^* is a non-zero-divisor in $gr_q(R)$. □

Definition 2.5.6. *Let M be a finitely generated R module. Then we say a system of parameters $\boldsymbol{x} := x_1, \ldots, x_d$ is a* standard system of parameters *if*

$$(\boldsymbol{x})H^i_{\mathfrak{m}}(M/(x_1, \ldots, x_j)M) = 0$$

holds for all non-negative integers i, j with $i + j < d$.

We say M is Buchsbaum *if every system of parameters $\boldsymbol{x}$ for M forms a weak M-sequence, that is,*

$$(x_1, \ldots, x_{i-1})M : x_i = (x_1, \ldots, x_{i-1})M : \mathfrak{m}$$

holds for all $1 \leq i \leq d$.

M is quasi-Buchsbaum *if at least one (and hence every) system of parameters $\boldsymbol{x}$ for M contained in $\mathfrak{m}^2$ forms a weak M-sequence.*

We remark that M is quasi-Buchsbaum if and only if $\mathfrak{m}H^i_{\mathfrak{m}}(M) = 0$ for all $0 \leq i < d$.

Suppose $(R, \mathfrak{m})$ is a two-dimensional ring of positive depth and $q = (x, y)$ is a parameter ideal. Then by the above corollary we have depth $gr_q(R) > 0$ if q is also a standard parameter ideal. Goto [Got83] has also shown that if the ring R is Buchsbaum, depth $gr_q(R) > 0$.

Lemma 2.5.7. *[Suz87, Theorem 3.6] Let R be a quasi-Buchsbaum ring and $a \in \mathfrak{m}^2$ part of a system of parameters. Then $R/(a)$ is quasi-Buchsbaum.*

This gives us the following:

Corollary 2.5.8. *Suppose R is a quasi-Buchsbaum ring of depth at least $d-1$ and $q \subseteq \mathfrak{m}^2$ is a parameter ideal. Then the following hold:*

1. grade $gr_q(R)_+ \geq d-1$.

2. $e_i(q) \leq 0$ *for* $1 \leq i \leq d$.

3. *For* $2 \leq i \leq d$*, we have*

   a) $e_i(q) = 0$ *if and only if* $H_q(n) = P_q(n)\ \forall n \geq i-d$*, and*

   b) $e_i(q) = 0$ *implies* $e_{i+1}(q) = e_{i+2}(q) = \cdots = e_d(q) = 0$.

Proof. Let $q = x_1, \ldots, x_d$ where $x_1, \ldots, x_{d-2}$ is a superficial sequence and $x_1, \ldots, x_{d-1}$ is a regular sequence. Then, by Lemma 2.5.7, we have $\bar{R} = R/(x_1, \ldots x_{d-2})$ is a two-dimensional quasi-Buchsbaum ring. Moreover, depth $\bar{R} \geq 1$. Note, again by Lemma 2.5.7, $R/(x_1, \ldots, x_{d-1})$ is quasi-Buchsbaum and hence $(x_d)H^0_{\mathfrak{m}}(R/(x_1, \ldots, x_{d-1})) = 0$. By Corollary 2.5.5, we have grade $gr_{\bar{q}}(\bar{R})_+ > 0$. This implies grade $gr_q(R)_+ \geq 1 + d - 2 = d - 1$ by Lemma 2.2.4, proving (1). Now, (2) follows from Corollary 2.3.17. Combining the results of Remark 2.3.15 and Corollary 2.3.18, we obtain (3). □

Chapter 3

A Conjecture of Watanabe and Yoshida

3.1 Definitions and Notation

In this chapter, we examine a conjecture of Watanabe and Yoshida and provide a proof of this conjecture in the case we have a graded ring R. We begin with some notation. Throughout this chapter all rings are assumed to be commutative Noetherian of prime characteristic $p > 0$ unless otherwise noted. All graded rings $R = \oplus_{i \geq 0} R_i$ are finitely generated over the Artinian local ring R_0. Let (R, m) be a local ring of dimension d with maximal ideal $\mathfrak{m}$. For an ideal I of R and $q = p^e$ (for some e), we let $I^{[q]}$ denote the ideal of R generated by the set $\{i^q : i \in I\}$. Given an $\mathfrak{m}$-primary ideal I of R, let $\lambda_R(R/I)$ denote the length of R/I as an R-module. One then defines the Hilbert - Kunz multiplicity of I by

$$e_{HK}(I, R) := \lim_{q \to \infty} \frac{\lambda_R(R/I^{[q]})}{q^d}. \tag{3.1}$$

One can also define the Hilbert-Kunz multiplicity using the Frobenius functor. We define R^F to be the R-bimodule with additive group R and multiplication given by $a \cdot r \circ b = arb^p$ for $a, b \in R$ and $r \in R^F$. The *Frobenius functor* F is then given by $F(M) = R^F \otimes_R M$. It can be shown that $F(R/I) = R/I^{[p]}$, and hence $e_{HK}(I) = \lim_{e\to\infty} \dfrac{\lambda_R(F^e(R/I))}{p^{ed}}$.

Many theorems that hold for the Hilbert-Samuel multiplicity of an ideal have an analogous statement for the Hilbert-Kunz multiplicity. For example, Watanabe and Yoshida [WY00], and later Huneke-Yao [HY02], proved that an unmixed Noetherian local ring of characteristic p is regular if and only if $e_{HK}(\mathfrak{m}) = 1$. This provided a Hilbert-Kunz analogue to the famous theorem of Nagata (Theorem 1.1) that an unmixed Noetherian local ring is regular if and only if the Hilbert-Samuel multiplicity $e(\mathfrak{m}) = 1$. Another example is given by a theorem of Rees [Ree61] which states that in a formally equidimensional Noetherian ring with $\mathfrak{m}$-primary ideals $I \subseteq J$, $J \subseteq \bar{I}$ if and only if $e(I) = e(J)$, where $\bar{I}$ denotes the integral closure of the ideal I. In the appendix, we treat the Hilbert-Kunz analogue to this theorem, due to Hochster and Huneke [HH90].

In [WY00], Watanabe and Yoshida made the following conjecture:

Conjecture 3.1.1. *Let (R, m) be a Cohen-Macaulay local ring of characteristic $p > 0$. Then*

1. *For any $\mathfrak{m}$-primary ideal I, one has $e_{HK}(I, R) \geq \lambda_R(R/I)$.*

2. *For any $\mathfrak{m}$-primary ideal I with $\operatorname{pd}_R(R/I) < \infty$, one has $e_{HK}(I, R) = \lambda_R(R/I)$.*

Note that the Hilbert-Samuel analogue of part (1) of the conjecture can be shown to hold using a reduction of the ideal I. Dutta [Dut83] has shown Conjecture 3.1.1 holds when R is a complete intersection ring (but not necessarily graded). Note part

(2) of Conjecture 3.1.1 is also easily shown to hold over a regular local ring due to the flatness of the Frobenius. However, neither part of the conjecture holds for *all* Cohen-Macaulay rings. In particular, Miller and Singh [MS00] have constructed a module M of finite projective dimension over a Gorenstein ring R of dimension 5 with

$$220 = \lim_{n\to\infty} \frac{\lambda_R(F_R^n(M))}{p^{5n}} < \lambda_R(M) = 222.$$

From the proof of Theorem 6.4 in [Kur04], there exist $\mathfrak{m}$-primary ideals $J, I_1, \ldots, I_t$ of finite projective dimension such that $I_1, \ldots, I_t$ are parameter ideals and

1. $\lambda_R(M) = \lambda_R(R/J) - \sum_{i=1}^{t} \lambda_R(R/I_i)$, and
2. $\lim_{n\to\infty} \frac{\lambda_R(F^n(M))}{p^{5n}} = e_{HK}(J, R) - \sum_{i=1}^{t} \lambda_R(R/I_i)$.

Thus, for the ideal $J \subset R$, we have $e_{HK}(J, R) < \lambda_R(R/J)$, contradicting both parts of the conjecture above.

What we are able to prove, using basic properties of Poincaré series and a result of Avramov and Buchweitz [AB93], is the following:

Theorem 3.1.2. *Let R be a graded ring of characteristic $p > 0$ and dimension d, and I a homogeneous ideal with $\lambda(R/I) < \infty$ and $\mathrm{pd}(R/I) < \infty$. Then for every $q = p^e$, one has $\lambda(R/I^{[q]}) = q^d \lambda(R/I)$. In particular, $e_{HK}(I, R) = \lambda(R/I)$.*

Theorem 3.1.2 also follows from a conjecture of Szpiro [Szp82, Conjecture C2]. Szpiro sketches a proof of this conjecture in the graded case, using different methods than what we employ here.

3.2 Proof of Theorem 3.1.2

Let R be a graded ring and M a nonzero finitely generated graded R-module. Let $P_M(t) = \sum_{i\in\mathbb{Z}} \lambda_{R_0}(M_i)\, t^i$ denote the *Hilbert series* for M. Note that if $\lambda_R(M) < \infty$, we have $\lambda_R(M) = P_M(1)$. Further note $P_{M(-k)}(t) = t^k P_M(t)$ for any $k \in \mathbb{Z}$, where if $M = \oplus_{i\in\mathbb{Z}} M_i$, $M(-k)$ denotes the graded R-module with $M(-k)_i = M_{i-k}$. We recall the following proposition concerning the Hilbert series for M.

Proposition 3.2.1. (cf. *[Smo72, Theorem 5.5]*) *Let R be a graded Noetherian ring and M a nonzero finitely generated graded R-module of dimension l. Then there exist positive integers $s_1, \ldots, s_l$ and a polynomial $p(t) \in \mathbb{Z}[t, t^{-1}]$ with $p(1) \neq 0$ such that*

$$P_M(t) = \frac{p(t)}{\prod_{i=1}^{l}(1 - t^{s_i})}.$$

Proof. We use the proof from [Mar]. We will proceed by induction on l. If $l = 0$, then $M_n = 0$ for all but finitely many n and thus $p(t) = P_M(t) \in \mathbb{Z}[t, t^{-1}]$. Note $p(1) = \lambda_R(M) \neq 0$. Now suppose $l > 0$. Let $x \in R_+$ be a superficial element for M and let $s_l = \deg x$. Consider the exact sequence

$$0 \to (0 :_M x)_n \to M_n \xrightarrow{x} M_{n+s_l} \to (M/xM)_{n+s_l} \to 0.$$

As length is additive on exact sequences, we get

$$\lambda_R(M_{n+s_l}) - \lambda_R(M_n) = \lambda_R((M/xM)_{n+s_l}) + \lambda_R((0 :_M x)_n) \tag{3.2}$$

and so multiplying by t^{n+s_l} we have

$$\lambda_R(M_{n+s_l})t^{n+s_l} - \lambda_R(M_n)t^{n+s_l} = \lambda_R((M/xM)_{n+s_l})t^{n+s_l} + \lambda_R((0 :_M x)_n)t^{n+s_l}.$$

Summing over all n, we get

$$P_M(t) - t^{s_l} P_M(t) = P_{M/xM}(t) - t^{s_l} P_{(0:_M x)}(t).$$

Now, since x was superficial, $\dim M/xM = l - 1$ and since $(0 :_M x)$ has finite length, $\dim(0 :_M x) = 0$ or $(0 :_M x) = 0$. Therefore, by induction, there exist $p_1(t), p_2(t) \in \mathbb{Z}[t, t^{-1}]$ with $p_1(t) \neq 0$ and positive integers $s_1, \ldots, s_{l-1}$ such that

$$(1 - t^{s_l}) P_M(t) = \frac{p_1(t)}{\prod_{i=1}^{l-1}(1 - t^{s_i})} - t^{s_l} p_2(t).$$

Thus,

$$P_M(t) = \frac{p_1(t) - t^{s_l} \prod_{i=1}^{l-1}(1 - t^{s_i}) p_2(t)}{\prod_{i=1}^{l}(1 - t^{s_i})}.$$

Let $p(t) = p_1(t) - t^{s_l} \prod_{i=1}^{l-1}(1 - t^{s_i}) p_2(t) \in \mathbb{Z}[t, t^{-1}]$. It remains to show that $p(1) \neq 0$. If $l > 1$, then $p(1) = p_1(1) \neq 0$. If $d = 1$, then

$$p(1) = p_1(1) - p_2(1) = \lambda_R(M/xM) - \lambda_R((0 :_M x)).$$

Suppose that $\lambda_R(M/xM) = \lambda_R((0 :_M x))$. Then by equation (3.2), we have,

$$\begin{aligned}
\lambda_R(M_n) + \cdots \lambda_R(M_{n+s_l}) &= \sum_{i=-\infty}^{n} \lambda_R(M_{i+s_l}) - \lambda_R(M_i) \\
&= \sum_{i=-\infty}^{n} \lambda_R((M/xM)_{i+s_l}) - \sum_{i=-\infty}^{n} (\lambda_R((0 :_m x)_i) \\
&= \lambda_R(M/xM) - \lambda_R((0 :_M x)) \\
&= 0
\end{aligned}$$

Hence, $\lambda_R(M_n) = 0$ for n sufficiently large, and thus $\dim M = 0$, a contradiction. $\square$

The following result is a special case of a result in [AB93]. Since the proof is not difficult, we include it here for completeness.

Proposition 3.2.2. *[AB93, Lemma 7] Let M be a finitely generated graded R-module with finite length and finite projective dimension. Let $\mathfrak{m}$ be the maximal ideal of R_0 and let $K = R/(\mathfrak{m} + R_+) = R_0/\mathfrak{m}$. Set $\chi_M^R(t) := \sum_i (-1)^i P_{\mathrm{Tor}_i^R(M,K)}(t)$. Then one has $P_M(t) = \chi_M^R(t) P_R(t)$.*

Proof. [AB93] Let $F. : 0 \to F_s \to \cdots \to F_1 \to F_0 \to 0$ be a minimal free resolution for M where $F_i \cong \oplus_{j=0}^{r_i} R(-j)^{b_{ij}}$ with $b_{ij} \in \mathbb{N}$. Then evaluating the Hilbert series for M using the resolution, one gets

$$P_M(t) = \sum_{i,j \in \mathbb{Z}} (-1)^i b_{ij} P_R(t) t^j. \tag{3.3}$$

Note this sum is well-defined as there are only finitely many nonzero b_{ij}. Moreover, $\chi_M^R(t)$ is also well-defined as $\mathrm{Tor}_i^R(M,K)$ is finitely generated graded and $\mathrm{Tor}_i^R(M,K) = 0$ for $i > \mathrm{pd}_R M$. Now, tensoring $F.$ with K, we obtain the complex

$$\cdots \to \oplus K(-j)^{b_{sj}} \to \cdots \to \oplus K(-j)^{b_{0j}} \to 0$$

where each map is the zero map. Note that for $n \in \mathbb{Z}$, $\mathrm{Tor}_i^R(M,K)_n = H_i(F. \otimes K)_n$, so

$$\dim_K \mathrm{Tor}_i^R(M,K)_n = \dim_K \left(\left(\oplus_{j \in \mathbb{Z}} K(-j)^{b_{ij}} \right)_n \right) = \sum_{j \in \mathbb{Z}} b_{ij} \dim_K K_{n-j} = b_{in} < \infty$$

(as $K_i = 0$ for $i \neq 0$). Thus, $P_{\mathrm{Tor}_i^R(M,K)}(t) = \sum_{j \in \mathbb{Z}} b_{ij} t^j$.

Now, by the invariance of Euler-Poincare characteristics, we have

$$\begin{aligned}\chi_M^R(t) &= \sum_{i\in\mathbb{Z}}(-1)^i P_{\mathrm{Tor}_i^R(M,K)}(t)\\ &= \sum_{i,j\in\mathbb{Z}}(-1)^i b_{ij}t^j\\ &= \frac{P_M(t)}{P_R(t)}. \qquad \text{(by equation (3.3))}\end{aligned}$$

This gives $P_R(t)\,\chi_M^R(t) = P_M(t)$. Note also that we have $\chi_M^R(t)\in\mathbb{Z}[t,t^{-1}]$. □

Note if we can show $\lambda_R(F^e(R/I)) = p^{ed}\lambda_R(R/I)$ for all e sufficiently large, then $e_{HK}(I,R) = \lambda_R(R/I)$ since $e_{HK}(I)$ is obtained by taking a limit of increasing powers of the Frobenius applied to R/I. We now prove the main theorem of this chapter.

Theorem C. *Let R be a graded ring of characteristic p and let M be a finitely generated graded R-module with $\lambda_R(M)<\infty$ and $\mathrm{pd}_R(M)<\infty$. Then $\lambda_R(F^e(M)) = q^d\lambda_R(M)$ for all $q=p^e$. In particular, one has $e_{HK}(I,R)=\lambda_R(R/I)$ for all zero-dimensional homogeneous ideals I of finite projective dimension.*

Proof. Let $G.$ be a minimal graded free resolution for M, with $G_i = \bigoplus_{j=0}^{r_i} R(-j)^{b_{ij}}$ and $b_{ij}\in\mathbb{N}$. Let $F^e(-)$ denote the Frobenius functor and $q=p^e$. Note that $L. = F^e(G.)$ is a minimal graded free resolution for $F^e(M)$ by [PS73, Theorem 1.7] where each twist by j in $G.$ is multiplied by a factor of q and the b_{ij} remain the same. That is, $L_i = \bigoplus_{j=1}^{r_i} R(-jq)^{b_{ij}}$. Now, by the lemma above, we have $P_M(t) = \chi_M^R(t)\,P_R(t)$ and $P_{F^e(M)}(t) = \chi_{F^e(M)}^R(t)\,P_R(t)$.

By Proposition 3.2.1, we can write $P_R(t) = \dfrac{p(t)}{\prod_{i=1}^d(1-t^{s_i})}$ where $d=\dim R$ and the $s_i\in\mathbb{N}$ are nonzero. Each term in the denominator can be factored as $1-t^{s_i} = (1-t)g_i(t)$ with $g_i(t)\in\mathbb{Z}[t]$ and $g_i(1)\neq 0$. Letting $g(t)=\prod_i g_i(t)$, we can rewrite $P_R(t) = \dfrac{p(t)}{(1-t)^d g(t)}$ where $p(1)/g(1)$ is a nonzero rational number.

Since $\lambda_R(M) < \infty$, $P_M(t) \in \mathbb{Z}[t, t^{-1}]$. So, we have

$$P_M(t) = \chi_M^R(t)\, P_R(t) = \chi_M^R(t)\, \frac{p(t)}{(1-t)^d\, g(t)} \in \mathbb{Z}[t, t^{-1}].$$

Since $p(1) \neq 0$, we must have $(1-t)^d$ divides $\chi_M^R(t)$ in $\mathbb{Z}[t, t^{-1}]$; say $\chi_M^R(t) = \tilde{\chi}_M^R(t) \cdot (1-t)^d$, for some $\tilde{\chi}_M^R(t) \in \mathbb{Z}[t, t^{-1}]$. So,

$$P_M(t) = \tilde{\chi}_M^R(t)\, \frac{p(t)}{g(t)}. \tag{3.4}$$

In the proof of Proposition 3.2.2, we see that $\chi_M^R(t) = \sum_{i,j\in\mathbb{Z}} (-1)^i\, b_{ij} t^j$. Applying this to the resolution $L.$ of $F^e(M)$, we also have

$$\chi_{F^e(M)}^R(t) = \sum_{i,j\in\mathbb{Z}} (-1)^i\, b_{ij} t^{qj} = \chi_M^R(t^q) = \tilde{\chi}_M^R(t^q) \cdot (1-t^q)^d.$$

Thus,

$$\begin{aligned} P_{F^e(M)}(t) &= \chi_{F^e(M)}^R(t)\, \frac{p(t)}{(1-t)^d\, g(t)} \\ &= \tilde{\chi}_M^R(t^q)\, (1-t^q)^d\, \frac{p(t)}{(1-t)^d\, g(t)} \\ &= \tilde{\chi}_M^R(t^q) \left(1 + t + \cdots + t^{q-1}\right)^d \frac{p(t)}{g(t)}. \end{aligned}$$

Letting $t = 1$, we get

$$\begin{aligned} \lambda_R(F^e(M)) &= P_{F^e(M)}(1) = \tilde{\chi}_M^R(1^q) \left(1 + 1 + \cdots + 1^{q-1}\right)^d \frac{p(1)}{g(1)} \\ &= \tilde{\chi}_M^R(1)\, (q)^d\, \frac{p(1)}{g(1)} \\ &= q^d P_M(1) \qquad \text{(by (3.4))} \\ &= q^d \lambda_R(M). \end{aligned}$$

Finally, if I is a zero-dimensional homogeneous ideal and $q = p^e$, we have

$$e_{HK}(I,R) = \lim_{q\to\infty} \lambda_R\left(F^e(R/I)\right)/q^d = \lambda_R\left(R/I\right). \qquad \square$$

3.3 Numerical Semigroup Rings

We would like to determine whether the first part of the conjecture of Watanabe and Yoshida holds in a graded ring of positive characteristic. In the hopes of making progress in this direction, we will consider the first part of Watanabe and Yoshida's conjecture in the case that the ring R is a numerical semigroup ring of characteristic $p > 0$. Recall the conjecture:

Conjecture 3.3.1. *[WY00] Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of characteristic $p > 0$. Then for any $\mathfrak{m}$-primary ideal I, one has $e_{HK}(I,R) \geq \lambda_R(R/I)$.*

In considering this conjecture, one could ask the stronger question:

Question 3.3.2. *Under what conditions is $\lambda_R(R/I^{[p^e]}) = p^e\lambda_R(R/I)$ for all $e \geq 1$?*

In this section, we consider the above question for numerical semigroup rings. Let $S \subseteq \mathbb{N}$. We say $S =< a_1, \ldots, a_n >$ is a *numerical semigroup* if $\mathbb{N}\backslash S$ is a finite set or equivalently, if $\gcd(a_1, \ldots, a_n) = 1$. The *Frobenius number* of a numerical semigroup S is the largest natural number not in S, denoted $f(S)$. For $n \in S$ we define the *Apéry set* of n in S to be $\mathrm{Ap}(S,n) = \{s \in S \mid s - n \notin S\}$. In particular, $\mathrm{Ap}(S,n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$ where $w(i)$ is the least element of S congruent to i modulo n for all $0 \leq i \leq n-1$. Let k be a field. Then if S is a numerical semigroup, the ring $R = k[t^s : s \in S]$ is a *numerical semigroup ring*. We will show that if R is a numerical semigroup ring and I a homogeneous ideal of R then $\lambda_R(F^e(R/I)) \geq p^e\lambda_R(R/I)$ for all $e \gg 0$.

We begin with a claim.

Claim 3.3.3. *Let $R = k[t^{a_1}, \dots, t^{a_n}]$ be a numerical semigroup ring and $I = (t^s)$ an ideal of R. Then $\lambda_R(R/I) = s$. In particular, $\lambda_R(F^e(R/I)) = p^e s = p^e \lambda_R(R/I)$.*

Proof. Let $S =< a_1, \dots, a_n >$ and $\mathrm{Ap}(S, s) = \{\alpha_0, \dots, \alpha_{s-1}\}$ where $\alpha_j \in S$ and $\alpha_j \equiv j \pmod{s}$ for $0 \le j \le s-1$. Let $\overline{t^\alpha}$ denote the image of t^α in R/I for any $t^\alpha \in R$. We claim that $\{\overline{t^a} : a \in \mathrm{Ap}(S, s)\}$ is a k-basis for R/I. First note that if $g \in S \setminus \mathrm{Ap}(S, s)$, then $g \equiv \alpha_j \pmod{s}$ for some j. So $g - \alpha_j = ns$ for some $n \in \mathbb{N} \setminus \{0\}$ and $t^g = t^s \cdot t^{(n-1)s} \cdot t^{\alpha_j} \in (t^s) = I$. From the definition of $\mathrm{Ap}(S, s)$, we see that the elements $t^{\alpha_j} \not\equiv t^{\alpha_l} \pmod{I}$ for $j \ne l$. So $\{\overline{t^a} : a \in \mathrm{Ap}(S, s)\}$ is a k-basis for R/I. Since $\mathrm{Ap}(S, s)$ has s elements, we have $s = \dim_k(R/I) = \lambda_R(R/I)$. The final statement of the claim is clear since $(t^s)^{[p^e]} = (t^{sp^e})$. $\square$

Theorem 3.3.4. *Let $R = k[t^{a_1}, \dots, t^{a_n}]$ be a numerical semigroup ring and $I = (t^{s_1}, \dots, t^{s_m})$ a homogeneous ideal with $s_1 = \min\{s_i : 1 \le i \le m\}$. Let $S =< a_1, \dots, a_n >$. Then for $p^e > f(S)$, $\lambda_R(F^e(R/I)) \ge p^e \lambda_R(R/I)$. In fact, $e_{HK}(I) = s_1$.*

Proof. Let $g = f(S)$ and note that $t^{g+i} \in R$ for all $i \ge 1$. We will show that for $p^e > g$, $I^{[p^e]} = (t^{s_1})^{[p^e]}$. The containment $(t^{s_1})^{[p^e]} \subseteq I^{[p^e]}$ is clear. For the other containment, let e be such that $p^e > g$. Then for $1 \le i \le m$, we have $t^{s_i p^e} = t^{(s_i - s_1)p^e} \cdot t^{s_1 p^e} \in (t^{s_1})^{[p^e]}$. (Note $t^{(s_i - s_1)p^e} \in R$ as $(s_i - s_1)p^e > g$.) Thus, $I^{[p^e]} \subseteq (t^{s_1})^{[p^e]}$ and it follows that $I^{[p^e]} = (t^{s_1})^{[p^e]}$ for all $p^e > g$. In this case,

$$
\begin{aligned}
\lambda_R\left(F^e(R/I)\right) &= \lambda_R\left(R/I^{[p^e]}\right) \\
&= \lambda_R\left(R/(t^{s_1})^{[p^e]}\right) \\
&= p^e \lambda_R\left(R/(t^{s_1})\right) \quad \text{(by Claim 3.3.3)} \\
&\ge p^e \lambda_R\left(R/I\right)
\end{aligned}
$$

Finally, note that

$$\begin{aligned} e_{HK}(I) &= \lim_{e\to\infty} \lambda_R(R/I^{[p^e]}))/p^e \\ &= \lim_{e\to\infty} \lambda_R\left(R/(t^{s_1})^{[p^e]}\right)/p^e \\ &= \lambda_R(R/(t^{s_1})) = s_1. \end{aligned}$$

□

Remark 3.3.5. *Note that in the statement of Theorem 3.3.4 above, the hypothesis* $p^e > f(S)$ *could be replaced by* $\min\{s_i - s_1 : i \neq 1\} > f(S)/p^e$*. Then for such an* e*, we'll have* $\lambda_R(F^e(R/I)) = s_1 p^e \geq p^e \lambda_R(R/I)$*. In particular, if* $f(S)/p < 1$ *(i.e.,* $f(S) < p$*), then we have* $\lambda_R(F^e(R/I)) = s_1 p^e \geq p^e \lambda_R(R/I)$ *for all* $e \geq 1$*.*

Chapter 4

A Theorem of Hochster and Huneke

This final chapter is an expository chapter on a theorem of Hochster and Huneke ([HH90, Theorem 8.17]) which characterizes the Hilbert-Kunz multiplicity of two ideals by their tight closures. We follow closely the treatment given in [HH90], but expand on some of the details. In preparing this we found one implication in line 12 of page 81 of the proof in [HH90] which we were unable to justify. However, we were able to use an argument suggested by Neil Epstein to work around this implication. This argument can be found toward the end of the proof of Theorem 4.2.4.

4.1 Background

We begin with some notation. Throughout this section all rings are assumed to be of prime characteristic p and (except in obvious exceptional cases) Noetherian as well. For a ring R we let R° denote the elements of R which are not in any minimal prime of R. For an ideal I of R and $q = p^e$ (for some e), we let $I^{[q]}$ denote the ideal of R

generated by the set $\{i^q : i \in I\}$. The *tight closure* of I, denoted by I^*, is defined to be the set of all elements $x \in R$ such that there exists $c \in R^\circ$ with $cx^q \in I^{[q]}$ for all $q = p^e$ sufficiently large.

Let $(R, \mathfrak{m})$ be a local ring of dimension d. Given an $\mathfrak{m}$-primary ideal I of R, we let $\lambda(R/I)$ denote the length of R/I as an R-module. We define the *Hilbert-Kunz multiplicity*, $e_{HK}(I, M)$, of I with respect to an R-module M as in Chapter 3. If $M = R$, we will suppress the M.

In [HH90], Hochster and Huneke give a proof of the following result:

Theorem 4.1.1. *(cf. [HH90, Theorem 8.17]) Let (R, m) be a local ring and $J \subseteq I$ $\mathfrak{m}$-primary ideals of R.*

1. *If $I \subseteq J^*$ then $e_{HK}(I) = e_{HK}(J)$.*

2. *The converse to (1) holds if R is equidimensional and either complete or essentially of finite type over a field.*

We note that this theorem provides an analogue to the following result of Rees [Ree61] concerning Samuel multiplicity and integral closure. For an ideal of a ring R, let $\overline{I}$ denote the integral closure of I. If R is local of dimension d and I is $\mathfrak{m}$-primary, then the Samuel multiplicity of I is defined by

$$e(I) := d! \lim_{n\to\infty} \frac{\lambda(R/I^n)}{n^d}.$$

Theorem 4.1.2. *(cf. [Ree61, Theorem 3.2] Let (R, m) be a local ring (not necessarily of positive characteristic) and $J \subseteq I$ $\mathfrak{m}$-primary ideals of R.*

1. *If $I \subseteq \overline{J}$ then $e(I) = e(J)$.*

2. *If R is formally equidimensional then the converse to (1) holds.*

Note that the converses of part (1) of both Theorem 4.1.1 and 4.1.2 hold under essentially the same conditions.

We begin with a discussion of qth roots. Assume R is reduced and let $\operatorname{Ass}_R R = \operatorname{Min}_R R = \{p_1, \ldots, p_n\}$. For each i let $k_i = R_{p_i} = Q(R/p_i)$. By the Chinese Remainder Theorem, the total quotient ring $Q = Q(R)$ is isomorphic to $k_1 \times \cdots \times k_n$. For each i let $\overline{k_i}$ denote a fixed algebraic closure of k_i and let $\overline{Q}$ denote $\overline{k_1} \times \cdots \times \overline{k_n}$. Clearly, R embeds in $\overline{Q}$ in a natural way. For $q = p^e$ let $R^{1/q} = \{u \in \overline{Q} \mid u^q \in R\}$. Then $R^{1/q}$ is an integral extension of R but in general is not finite. For each $r \in R$ there exists a unique $u \in R^{1/q}$, denoted $r^{1/q}$, such that $u^q = r$. For, if $u^q = r = w^q$ for $u, w \in \overline{Q}$, then $0 = u^q - w^q = (u-w)^q$; since $R^{1/q} \subseteq \overline{Q}$ is reduced, we have $u = w$. Note that the map $\varphi : R \longrightarrow R^{1/q}$ given by $\varphi(r) = r^{1/q}$ is a ring isomorphism, so $R^{1/q}$ is a Noetherian ring. If R is local with maximal ideal $\mathfrak{m}$ then the maximal ideal of $R^{1/q}$ is $m^{1/q} = \{x^{1/q} \mid x \in m\}$.

Note that the inclusion map $R \hookrightarrow R^{1/q}$, where $q = p^e$, is essentially the same as the the e^{th} iteration of the Frobenius map $f^e : R \to R$ defined by $f^e(r) = r^{p^e}$. To be precise, let S be the ring R, but viewed as an R-module via f^e. Then the map $\rho : S \longrightarrow R^{1/q}$ given by $\rho(s) = s^{1/q}$ is easily seen to be an isomorphism of R-algebras (i.e., a ring isomorphism which is R-linear). Hence, for example, by a result of Kunz [Kun69] R is regular if and only if $R^{1/q}$ is a flat R-module for some (equivalently, every) q.

Lemma 4.1.3. *Let $A \subseteq R$ be rings, with A a Noetherian domain and $A \subseteq R$ a finite integral extension. Then R is torsion-free as an A-module if and only if* $\dim R/p = \dim R$ *for all $p \in \operatorname{Ass}_R R$. In the case R is torsion-free over A, we have $Q(A) \subseteq R_W = Q(R)$, where $W = A \backslash \{0\}$ and $Q(-)$ denotes the total quotient ring.*

Proof. For the reverse implication, suppose $a \cdot r = 0$,with $a \in A\backslash\{0\}, r \in R\backslash\{0\}$.

Then $a \in P$ for some $P \in \operatorname{Ass}_R R$ as $a \in A \subseteq R$ is a zero-divisor. Now $A/(P \cap A) \subseteq R/P$ is a finite integral extension, and $\dim R = \dim R/P = \dim A/(P \cap A) = \dim A$. But A is a domain, so we must have $P \cap A = 0$. We have $a \in P \cap A$, so we must have $a = 0$, giving a contradiction. Thus, R must be a torsion-free A-module.

Conversely, let $P \in \operatorname{Ass}_R R$. Since R is torsion-free, $P \cap A = (0)$. This implies $\dim R/P = \dim A = \dim R$.

In the case R is torsion-free over A, let $W' = \{s \in R \mid s \text{ a non-zero-divisor}\}$. As R is torsion-free over A, $W \subseteq W'$, so $Q(A) \subseteq R_W \subseteq R_{W'}$ and $Q(R) = R_{W'} = (R_W)_{W'} = Q(R_W)$. But as $Q(A) \subseteq R_W$ is integral, $\dim R_W = 0$. Hence, $Q(R_W) = R_W$. □

Lemma 4.1.4. *Let $A \subseteq R$ be rings. Consider the ring homomorphism $\varphi : R \otimes_A A^{1/q} \to R\left[A^{1/q}\right]$ given by $\varphi\left(r \otimes a^{1/q}\right) = ra^{1/q}$. Then φ is onto and* $\ker \varphi$ *is nilpotent.*

Proof. Clearly φ is onto. Note, if $\Sigma r_i \otimes a_i^{1/q} \in \ker \varphi$, then we have $\Sigma r_i a_i^{1/q} = 0$. Taking q^{th} powers, we get $\Sigma r_i^q a_i = 0$, and so, $\Sigma r_i^q a_i \otimes 1 = 0$. We can move the a_i's to the other side of the tensor product to obtain $0 = \Sigma r_i^q \otimes a_i = \left(\Sigma r_i \otimes a_i^{1/q}\right)^q$. Hence, $\Sigma r_i \otimes a_i^{1/q}$ is nilpotent. □

Example 4.1.5. *Note that $R \otimes_A A^{1/p}$ need not be reduced, even if R and A are fields. For example, let k be an imperfect field (i.e., $k = \mathbb{F}_p(t)$, where t is an indeterminate) and $A = k$ and $R = k^{1/p}$. Choose $a \in k \backslash k^p$. Then $\beta = a^{1/p} \otimes 1 - 1 \otimes a^{1/p}$ is nonzero as $\{a^{1/p} \otimes 1, 1 \otimes a^{1/p}\}$ is part of a k-basis for $k^{1/p} \otimes_k k^{1/p}$ since $a^{1/p}$ is part of a k-basis for $k^{1/p}$. However, $\beta^p = 0$.*

We want to determine when the map φ of Lemma 4.1.4 is an isomorphism, or equivalently, assuming R and A are reduced, under what conditions $R \otimes_A A^{1/q}$ is reduced. We begin this exploration with some remarks concerning separability.

Definition 4.1.6. *Let $A \subseteq S$ be a finite ring extension where A is a Noetherian domain and S is reduced and torsion-free over A. Then $Q(A) \subseteq Q(S) \cong k_1 \times \cdots \times k_n$ where each k_i is a finite field extension of $Q(A)$. An element $s \in S$ is* separable *over A if each component of its image in $k_1 \times \cdots \times k_n$ is separable over $Q(A)$. We say S is* separable over *A if each $s \in S$ is separable over A, or equivalently, each k_i is separable over $Q(A)$.*

Remark 4.1.7. *([Mat89], page 199) If E/F is a finite separable field extension, then for every field extension L of F, $E \otimes_F L$ is reduced.*

Lemma 4.1.8. *Let $A \subseteq S$ be a finite separable ring extension with A a Noetherian domain and S reduced and torsion-free over A. Let B be a reduced A-algebra. Then*

1. *If B is flat, then $S \otimes_A B$ is reduced.*

2. *If S is flat and B is torsion-free, then $S \otimes_A B$ is reduced.*

Proof. We first prove part (1). Note that as S, B are reduced, S and B both inject into a product of fields, say, $S \hookrightarrow k_1 \times \cdots \times k_n$ and $B \hookrightarrow l_1 \times \cdots \times l_m$. By hypothesis $Q(A) \subseteq k_i$ is separable for all i. We want to show that $S \otimes_A B$ injects into a reduced ring.

Claim: $S \otimes_A B \hookrightarrow (k_1 \times \cdots \times k_n) \otimes_{Q(A)} (l_1 \times \cdots l_m) \cong \prod_{i,j} \left(k_i \otimes_{Q(A)} l_j\right)$.

Proof of claim: Note $0 \to S \to k_1 \times \cdots \times k_n$ is exact. Applying $- \otimes_A B$, we obtain

$$0 \;\; \to \;\; S \otimes_A B \to (k_1 \times \cdots \times k_n) \otimes_A B \cong (k_1 \otimes_A B) \times \cdots \times (k_n \otimes_A B) \quad \text{is exact,}$$

with the injection coming from the fact that B is flat. It suffices to show that if a field k is separable over $Q(A)$ and $B \hookrightarrow l_1 \times \cdots \times l_m$, then $k \otimes_A B \hookrightarrow k \otimes_A (l_1 \times \cdots \times l_m)$.

Now, we have $0 \to B \to l_1 \times \cdots \times l_m$ is exact. Applying $Q(A) \otimes_A -$ and then $k \otimes_{Q(A)} -$, we obtain

$$\begin{aligned} 0 &\to Q(A) \otimes_A B \to Q(A) \otimes_A (l_1 \times \cdots \times l_m) \cong l_1 \times \cdots \times l_m \quad \text{is exact and} \\ 0 &\to k \otimes_{Q(A)} (Q(A) \otimes_A B) \to k \otimes_{Q(A)} (l_1 \times \cdots \times l_m) \quad \text{is exact,} \end{aligned}$$

where the isomorphism in the first line comes from the fact that $l_i \supseteq Q(A)$ for all i. But, $k \otimes_{Q(A)} (Q(A) \otimes_A B) \cong k \otimes_A B$ and $k \otimes_{Q(A)} (l_1 \times \cdots \times l_m) = \prod_i \left(k \otimes_{Q(A)} l_i\right)$ is reduced by separability. Thus, we have $k \otimes_A B \hookrightarrow \prod_i \left(k \otimes_{Q(A)} l_i\right)$, and thus, $k \otimes_A B$ is reduced, implying that $S \otimes_A B$ is reduced.

Now, to prove part (2), suppose S is flat and B is torsion-free. Using the notation above, since S is flat over A, we have

$$0 \to S \otimes_A B \to (S \otimes_A l_1) \times \cdots \times (S \otimes_A l_m)$$

is exact. Therefore, it suffices to show that $S \otimes_A l_i$ is reduced for all i. Since B is torsion-free over A, $Q(A) \subseteq l_i$ for all i, and so $Q(A) \otimes_A l_i \cong l_i$ for all i.

Hence, $k_j \otimes_A l_i \cong k_j \otimes_{Q(A)} l_i$ is reduced for any j, since k_j is separable over $Q(A)$. Since l_i is flat over A,

$$0 \to S \otimes_A l_i \to (k_1 \times \cdots \times k_n) \otimes_A l_i \cong (k_1 \otimes_A l_i) \times \cdots \times (k_n \otimes_A l_i)$$

is exact. Therefore, $S \otimes_A l_i$ is reduced.

$\square$

Remark 4.1.9. *Suppose we have $A \subseteq R$, a finite extension with A a Noetherian domain and R reduced and torsion-free over A. Then $A^{1/q} \subseteq R\left[A^{1/q}\right]$ is a finite* *extension and $R\left[A^{1/q}\right]$ is reduced and torsion-free over $A^{1/q}$. Furthermore, the extension $A^{1/q} \subseteq R\left[A^{1/q}\right]$ is separable for $q \gg 0$.*

Proof. Using the notation from the beginning of this section, we have $R \subseteq Q(R) \cong k_1 \times \cdots \times k_n$ where k_i is a field for all i. As R is torsion-free over A, $Q(A) \subseteq Q(R)$ by Lemma 4.1.3. Let $\overline{k_i}$ denote the algebraic closure of k_i. Since R/p_i is a finite extension of A, we have each $\overline{k_i}$ is an algebraic extension of $Q(A)$. Note $R\left[A^{1/q}\right] \subseteq \overline{k_1} \times \cdots \times \overline{k_n}$, a product of fields, so $R\left[A^{1/q}\right]$ is reduced for any q. Now, suppose $a^{1/q}\left(\Sigma r_i a_i^{1/q}\right) = 0$. Then, taking q^{th} powers, we have $a\left(\Sigma r_i^q a_i\right) = 0$. If $a \neq 0$, then $\Sigma r_i^q a_i = 0$ since R is torsion-free over A. Taking q^{th} roots, we have $\Sigma r_i a_i^{1/q} = 0$. Thus, $R\left[A^{1/q}\right]$ is torsion-free over $A^{1/q}$.

To see that $A^{1/q} \subseteq R\left[A^{1/q}\right]$ is a finite extension, note that if $R = A\alpha_1 + \cdots + A\alpha_n$, with $\alpha_i \in R$, then $R[A^{1/q}] = A^{1/q}[R] = A^{1/q}\alpha_1 + \cdots + A^{1/q}\alpha_n$. Thus, $R[A^{1/q}]$ is a finitely generated $A^{1/q}$-module.

Finally, to see that the extension $A^{1/q} \subseteq R\left[A^{1/q}\right]$ is separable for $q \gg 0$, write $R = A\alpha_1 + \cdots + A\alpha_n$. Recall that for a field F of characteristic $p > 0$ and an element α in the algebraic closure of F, α^{p^n} is separable over F for all $n \gg 0$. So, for each i, there exists a $q_i = p^{e_i}$ such that $\alpha_i^{q_i}$ is separable over A, and therefore α_i is separable over A^{1/q_i}. Now, let $q = \max_i\{q_i\}$. Then we have R, and hence $R[A^{1/q}]$, is separable over $A^{1/q}$. □

Proposition 4.1.10. *Let $A \subseteq R$ be a finite ring extension where A is a regular local ring and R is reduced and torsion-free over A. Then there exists a power q' of p such that for all $q \geq q'$, $S = R[A^{1/q}]$ is reduced, torsion-free, and separable over $A^{1/q}$. Furthermore, for all $Q \geq q \geq q'$, $S \otimes_{A^{1/q}} A^{1/Q} \cong S[A^{1/Q}]$.*

Proof. Note by Remark 4.1.9, there is a q' such that $A^{1/q} \subseteq S = R[A^{1/q}]$ is a finite separable extension and S is reduced and torsion-free over $A^{1/q}$ for all $q \geq q'$. As

A is regular, for $Q \geq q \geq q'$, we have $A^{1/Q}$ is flat over $A^{1/q}$. So by Lemma 4.1.8, $S \otimes_{A^{1/q}} A^{1/Q}$ is reduced and thus, by Lemma 4.1.4, we have $S \otimes_{A^{1/q}} A^{1/Q} \cong S[A^{1/Q}]$. □

Definition 4.1.11. *Let R be a ring and S an R-algebra. We say that S is* smooth *over* R, *sometimes denoted as S/R* smooth, *if given an R-algebra T, an ideal N of T satisfying $N^2 = 0$, and an R-algebra homomorphism $u : S \to T/N$, then there exists an R-algebra homomorphism $v : S \to T$ lifting u; i.e., $u = \pi v$ where $\pi : T \to T/N$ is the natural surjection.*

We refer the reader to [Mat89] and [Mat70] for a detailed treatment of smoothness. We summarize some of the important properties of smoothness in the following remark:

Remark 4.1.12. *We list below several properties of smoothness:*

1. *([Mat89], 28.2) If A is an R algebra, then S/R smooth implies $S \otimes_R A$ is smooth over A. In particular, localization and taking quotients preserve smoothness.*

2. *([Mat89], 28.1) T/S and S/R smooth imply T/R is smooth.*

3. *([Mat70], 28.E) If W is a multiplicatively closed subset of R, then R_W/R is smooth.*

4. *([Mat89], 28.9; [Gro64], 19.7.1) If S/R is smooth and S, R are Noetherian, then S/R is flat.*

5. *([Mat70], 28.I, 28.L) Let R be a field and S a finite extension field of R. Then S/R is smooth if and only if S/R is separable.*

6. *([Mat70], 29.E; [Mat89] corollary to theorem 30.5) If S is a finitely generated R-algebra then $\{p \in \operatorname{Spec} R \mid S_p/R_p$ is smooth$\}$ is an open set. In particular, if*

S_p/R_p is smooth for some $p \in \operatorname{Spec} R$ then there exists an $f \in R\backslash p$ such that S_f/R_f is smooth.

7. *([Mat70], 28.K) If k is a field and A is a local ring which is smooth over k, then A is a regular local ring.*

8. *([Mat70], section 28, example 1) Let A be a ring. Then $A[x]$ is smooth over A.*

Proposition 4.1.13. *Suppose S is smooth over R, where R and S are Noetherian rings. If R is reduced, so is S.*

Proof. Let $P \in \operatorname{Ass}_S S$. Note that it suffices to show that S_P is a field. Let $Q = P \cap R$ be the contraction of P to R. Since P consists of zero-divisors on S, so does Q. On the other hand, since S is flat over R, any non-zero-divisor on R is a non-zero-divisor on S (i.e., S is torsion-free over R). Hence, Q must consist of zero-divisors on R, so there exists $Q' \in \operatorname{Ass}_R R$ such that $Q \subseteq Q'$. But, as R is reduced, the associated primes are minimal, and so Q must be a minimal prime of R. Furthermore, as R is reduced, R_Q is a field. Now, we have S_Q is smooth over R_Q, and $PS_Q \in \operatorname{Ass}_{S_Q} S_Q$. Thus, we have reduced to the case that R is a field.

Now, S_P is smooth over S, and so by transitivity, S_P is smooth over R. By (7) in the remarks above, S_P is a regular local ring of depth zero, i.e., S_P is a field. □

Proposition 4.1.14. *Suppose $R \subseteq S$ is a finite extension where R is a Noetherian domain and S is reduced, torsion-free, and separable over R. Then there exists $d \in R\backslash\{0\}$ such that S_d is smooth over R_d.*

Proof. We have $Q(R) \subseteq Q(S) \cong k_1 \times \cdots \times k_n$, where k_i is a finite separable field extension of $Q(R)$ for each i. By part (5) of Remark 4.1.12, each k_i is smooth over $Q(R)$. Thus, $Q(S)$ is smooth over $Q(R)$. Since $Q(S) \cong S_W$ where $W = R\backslash\{0\}$ (see

Lemma 4.1.3), we have S_W is smooth over R_W. The result now follows by part (6) of Remark 4.1.12. $\square$

Theorem 4.1.15. *Suppose S/R is smooth, finite, and R is a Noetherian domain of characteristic p. Then for any $q = p^e$, $S^{1/q} = S\left[R^{1/q}\right]$.*

Proof. Note that since S/R is smooth and R is a domain we have S is reduced (Proposition 4.1.13) and torsion-free over A (Remark 4.1.12, part (4)). Also, since $S \cong S^{1/q}$ as rings, and this isomorphism restricts to $R \cong R^{1/q}$, we get that $S^{1/q}/R^{1/q}$ is smooth and finite. Thus, it is enough to prove the theorem for $e = 1$, since $S^{1/p} = S\left[R^{1/p}\right]$ implies $S^{1/p^2} = S^{1/p}\left[R^{1/p}\right] = S\left[R^{1/p^2}\right]$.

Case 1: R is a field. Then we have $R \hookrightarrow S \cong k_1 \times \cdots \times k_n$ with each k_i/R a finite algebraic separable field extension. Note $S^{1/p} = k_1^{1/p} \times \cdots \times k_n^{1/p}$. It is enough to show that $k_i^{1/p} = k_i\left[R^{1/p}\right]$, because then $S^{1/p} = k_1^{1/p} \times \cdots \times k_n^{1/p} = k_1\left[R^{1/p}\right] \times \cdots \times k_n\left[R^{1/p}\right] = (k_1 \times \cdots \times k_n)\left[R^{1/p}\right] = S\left[R^{1/p}\right]$.

Claim: If L/K is a finite separable algebraic field extension of characteristic p, then $L^{1/p} = L\left[K^{1/p}\right]$.

Proof of claim: Note L/K separable implies $L^{1/p}/K^{1/p}$ is separable. Now, $K^{1/p} \subseteq L\left(K^{1/p}\right) \subseteq L^{1/p}$ and $L^{1/p}/K^{1/p}$ separable imply $L^{1/p}/L\left(K^{1/p}\right)$ is separable. Let $\alpha \in L^{1/p}$. Then $\alpha^p \in L \subseteq L\left(K^{1/p}\right)$. Let $\mathrm{Min}(\alpha, L\left(K^{1/p}\right))$ denote the minimal polynomial of α over the field $L\left(K^{1/p}\right)$. Then we have $\mathrm{Min}\left(\alpha, L\left(K^{1/p}\right)\right) \mid (x^p - \alpha^p)$ implies $\mathrm{Min}\left(\alpha, L\left(K^{1/p}\right)\right) = x - \alpha$ by the separability of $L^{1/p}/L\left(K^{1/p}\right)$. Thus, $\alpha \in L\left(K^{1/p}\right)$. This gives that $L^{1/p} \subseteq L\left(K^{1/p}\right)$. The other containment is clear, giving $L^{1/p} = L\left(K^{1/p}\right) = L\left[K^{1/p}\right]$. $\square_{claim}$

General Case: We want to show that $S^{1/p} = S\left[R^{1/p}\right]$. Since S/R is smooth, $Q(S) = S_{(0)}$ is smooth over $R_{(0)}$; i.e., S is separable over R, by Remark 4.1.12, part (5). Since S is flat over R and $R^{1/p}$ is torsion-free over R, $S \otimes_R R^{1/p}$ is reduced by part (2) of Lemma 4.1.8. Hence, by Lemma 4.1.4, $S \otimes_R R^{1/p} \cong S[R^{1/p}]$.

Define $\varphi : S \otimes_R R^{1/p} \to S^{1/p}$ by $\varphi\left(s \otimes r^{1/p}\right) = sr^{1/p}$. We claim that φ is an isomorphism. Note that it is enough to show that φ is an isomorphism locally at any maximal ideal of R. Thus, we may assume that (R, m) is local. Now, S/R is finite and flat, which implies $S \otimes_R R^{1/p}/R^{1/p}$ is finite and flat. Recall that if T is a finitely generated flat module over a Noetherian ring then T is free. Therefore, we have $S \otimes_R R^{1/p}$ is a free $R^{1/p}$-module. Similarly, $S^{1/p}$ is a finitely generated free $R^{1/p}$-module.

Over a local ring, a map of finitely generated free modules is an isomorphism if and only if it is an isomorphism after tensoring with the residue field. Now, we have $S \otimes_R R^{1/p} \xrightarrow{\varphi} S^{1/p}$ is a map of free $R^{1/p}$-modules. Tensoring with $R/mR \left(\cong R^{1/p}/m^{1/p}\right)$ gives the map $S/mS \otimes_{R/m} (R/m)^{1/p} \xrightarrow{\bar{\varphi}} (S/mS)^{1/p}$. Since S/mS is smooth over R/m, this is an isomorphism by the first case. Thus, φ is an isomorphism. $\square$

Now, suppose we have $R \subseteq S$ with R a domain and S finite, torsion-free, and separable over R. By Proposition 4.1.14, there exists a $d \in R/\{0\}$ such that S_d/R_d is smooth. The previous theorem then implies that $\left(S^{1/q}/S\left[R^{1/q}\right]\right)_d = 0$ for any $q = p^e$, so that, given q, there is a power l of d such that $d^l S^{1/q} \subseteq S\left[R^{1/q}\right]$. Using the following lemma, one can find a single power of d that will work for all q.

Proposition 4.1.16. *(see 6.4 in [HH90]) Let $A \subseteq R$ be a finite ring extension where A is a regular local ring and R is reduced, torsion-free, and separable over A. Then there exists $c \in A \backslash \{0\}$ such that $cR^{1/q} \subseteq R[A^{1/q}]$ for all q.*

Proof. By Proposition 4.1.14, there exists $d \in A\backslash\{0\}$ such that R_d is smooth over A_d. In the discussion above, we showed that there exists a power $b = d^l$ of d such that $bR^{1/p} \subseteq R\left[A^{1/p}\right]$. Let $h = 1 + 1/p + \cdots + 1/p^e$.

Claim: $b^h R^{1/pq} \subseteq R\left[A^{1/pq}\right]$ for all $q = p^e$.

Proof: We proceed by induction on e. If $e = 0$, then $h = 1$ and $b^1 R^{1/p} \subseteq R\left[A^{1/p}\right]$. If $e > 0$, take q^{th} roots to obtain $b^{1/q} R^{1/pq} \subseteq R^{1/q}\left[A^{1/pq}\right]$. Now, with $h' = h - 1/q$, we have

$$b^h R^{1/pq} = b^{h'} b^{1/q} R^{1/pq} \subseteq b^{h'} R^{1/q}\left[A^{1/pq}\right] \subseteq \left[R\left[A^{1/q}\right]\right]\left[A^{1/pq}\right] = R[A^{1/pq}]$$

where the last containment comes from the induction hypothesis. $\square_{claim}$

Now, since b^h divides b^2 (in $A^{1/q}$) for all h, setting $c = b^2$, we have that $cR^{1/q} \subseteq R\left[A^{1/q}\right]$ for all q, as required.

$\square$

4.2 The Proof

Before we begin the proof of the main result of this chapter, we recall some definitions and a lemma.

Definition 4.2.1. *Let R be a Noetherian ring of characteristic $p > 0$ and I an ideal of R. We say $x \in I^*$, the* tight closure *of I, if there exists a $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$. We say $c \in R^\circ$ is a q'*-weak test element *if there exists q' such that for all $I \subseteq R$ and all $x \in I^*$, we have $cx^q \in I^{[q]}$ for all $q \geq q'$. The element c is a* locally stable q'-weak test element *if its image in every local ring of R is also a q'-weak test element. Finally, c is a* completely stable q'-weak test element *if it is locally stable and its image in the completion of each local ring of R is a q'-weak test element.*

Remark 4.2.2. *(see [HH90], 6.18 and 6.19) Let R be a reduced, equidimensional local ring of characteristic $p > 0$ and suppose that either R is complete or essentially of finite type over a field. Then R has a completely stable q'-weak test element.*

Lemma 4.2.3. *(see [HH90], 8.16) Let J be an ideal in a reduced Noetherian ring R which has a q'-weak test element $c \in R^\circ$. Suppose that $x \in R$ such that $x \notin J^*$. Then for any $d \in R^\circ$, $dx^q \notin J^{[q]*}$ for all $q \gg 0$. In particular, for any $d \in R^\circ$, $dx^q \notin J^{[q]}R^{1/q}$ for all $q \gg 0$.*

Proof. We will use the contrapositive to prove the lemma. Suppose that there exists $d \in R^\circ$ such that $dx^q \in J^{[q]*}$ for all $q \gg 0$. We will show that $x \in J^*$ by showing $c^{q'+1}d^{q'}x^{qq'} \in J^{[qq']}$ for all $q \gg 0$. Note if $dx^q \in J^{[q]*}$, then we have $cd^{Qq'}x^{qQq'} \in J^{[qQq']}$ for all Q, as c is a weak q'-test element. So, $(cdx^q)^{Qq'} \in \left(J^{[q]}\right)^{[Qq']}$. Hence, for all Q, we have $1 \cdot (cdx^q)^{Qq'} \in \left(J^{[q]}\right)^{[Qq']}$, which shows that $cdx^q \in J^{[q]*}$ for all $q \gg 0$. But since c is a q'-weak test element, we have $c\,(cdx^q)^{q'} \in J^{[q][q']}$ for all $q \gg 0$, i.e., $c^{q'+1}d^{q'}x^{qq'} \in J^{[qq']}$ for all $q \gg 0$, which shows that $x \in J^*$.

To prove the last statement of the lemma, note that if $v \in J^{[q]}R^{1/Q}$, then $v^Q \in J^{[q][Q]}$ and as above, $1 \cdot v^{Qq''} \in J^{[qQq'']}$ for all q''. Hence, $v \in J^{[q]*}$. In particular, $dx^q \notin J^{[q]*}$ for $q \gg 0$ implies $dx^q \notin J^{[q]}R^{1/q}$ for all $q \gg 0$. $\square$

We are now prepared to prove the main result. The proof here follows closely that of Theorem 8.17 in [HH90], with some details expanded.

Theorem 4.2.4. *[HH90] Suppose (R, m) is a local ring and let $J \subseteq I$ be $\mathfrak{m}$-primary ideals of R.*

1. *If $I \subseteq J^*$ then $e_{HK}(I) = e_{HK}(J)$.*
2. *The converse to (1) holds if R is analytically unramified, formally equidimensional, and has a completely stable q_1-weak test element c.*

Proof. (1) Let $I = (x_1, \ldots, x_n)$. Then since $I \subseteq J^*$, for each x_i there exists $a_i \in R^\circ$ such that $a_i x_i^q \in J^{[q]}$ for all $q \gg 0$. Let $a = a_1 \cdots a_n \in R^\circ$. Then since $ax_i^q \in J^{[q]}$

for all $q \gg 0$, we have $aI^{[q]} \subseteq J^{[q]}$. As J is $\mathfrak{m}$-primary, there exists an n such that $m^n I \subset J$. Set $K = m^n$. Let b be a bound on the number of generators for I. Then $I^{[q]}/J^{[q]}$ has at most b generators and is annihilated by $K^{[q]} + aR$, so that $I^{[q]}/J^{[q]}$ is a homomorphic image of $\left(R/(K^{[q]} + aR)\right)^b$. Thus, $\lambda(I^{[q]}/J^{[q]}) \leq b\lambda\left(R/(K^{[q]} + aR)\right)$. Let $S = R/aR$. Since $a \in R^{\circ}$, $\dim S \leq d-1$. So we have $\lambda\left(I^{[q]}/J^{[q]}\right) \leq b\lambda\left(S/K^{[q]}S\right)$. By a result of Monsky, we have $\lambda(S/K^{[q]}S) \leq e_{HK}(KS)q^{d-1} + Cq^{d-2}$ for some constant C. Therefore,

$$
\begin{aligned}
0 \ \leq \ e_{HK}(J) - e_{HK}(I) \ &= \ \lim_{q\to\infty} \left[\lambda\left(R/J^{[q]}\right)/q^d - \lambda\left(R/I^{[q]}\right)/q^d\right] \\
&= \ \lim_{q\to\infty} \frac{1}{q^d}\lambda\left(I^{[q]}/J^{[q]}\right) \\
&\leq \ \lim_{q\to\infty} \frac{1}{q^d}\left[e_{HK}(KS)q^{d-1} + Cq^{d-2}\right] \\
&= \ 0.
\end{aligned}
$$

(2) Suppose $e_{HK}(I) = e_{HK}(J)$ and $I \not\subseteq J^*$, i.e., there exists $x \in I \backslash J^*$, with $x \neq 0$. Then, as above, we have that $\lim_{q\to\infty} \frac{1}{q^d}\lambda(I^{[q]}/J^{[q]}) = 0$. Our goal is to show that there exists a constant $\gamma > 0$ such that $\lambda\left(I^{[q]}/J^{[q]}\right) \geq \gamma q^d$ for $q \gg 0$ to obtain a contradiction.

Choose $q \geq q_1$ such that $cx^q \notin J^{[q]}$. Since $J^{[q]}\widehat{R} \cap R = J^{[q]}$, we have $cx^q \notin J^{[q]}\widehat{R}$. As $(J\widehat{R})^{[q]} = J^{[q]}\widehat{R}$ and c is a completely stable q_1-weak test element, we have $x \notin (J\widehat{R})^*$. So we may assume $(R, \mathfrak{m})$ is complete, local, reduced and equidimensional. By the Cohen Structure theorem, R is module finite over a complete regular local domain $A = k\left[[x_1, \ldots, x_d]\right]$ and as R is equidimensional, by Lemma 4.1.3, we have R is also torsion-free over A.

By Proposition 4.1.10, we can choose q'' such that $S = R\left[A^{1/q''}\right]$ is separable over

$A^{1/q''}$ and by Lemma 4.1.16 we can choose $d \in A^\circ$ such that $dS^{1/q} \subseteq S\left[A^{1/qq''}\right]$ for any q. Since $x \in I$ is not in J^*, by Lemma 4.2.3 we can choose q' such that $dx^q \notin J^{[q]}R^{1/q}$ for any $q \geq q'$.

Let $K_Q \subseteq A$ be the ideal of elements $a \in A$ such that $ax^Q \in J^{[Q]}$. Note the map $A/K_Q \to I^{[Q]}/J^{[Q]}$ given by $\bar{a} \mapsto \overline{ax^Q}$ is an injection. Since A and R have the same residue class field, $\lambda\left(I^{[Q]}/J^{[Q]}\right) \geq \lambda\left(A/K_Q\right)$, and so it is enough to show there exists $\gamma > 0$ such that $\lambda\left(A/K_Q\right) \geq \gamma Q^d$ for $Q \gg 0$.

Now assume $Q \geq q'q''$ and write $Q = qq'q''$. Then $a \in K_Q$ implies that $ax^Q \in J^{[Q]}$. Taking $(1/q)^{th}$ powers, we have $a^{1/q}x^{q'q''} \in J^{[q'q'']}R^{1/q} \subseteq J^{[q'q'']}S^{1/q}$. This implies $da^{1/q}x^{q'q''} \in J^{[q'q'']}\left(dS^{1/q}\right) \subseteq J^{[q'q'']}S\left[A^{1/qq''}\right]$.

Note that as A is regular, $A^{1/qq''}$ is $A^{1/q''}$- flat and so we have $S \otimes_{A^{1/q''}} A^{1/qq''} \cong S\left[A^{1/qq''}\right]$ is S-flat, with the isomorphism coming from Proposition 4.1.10. Therefore,

$$a^{1/q} \in \left(J^{[q'q'']}S[A^{1/qq''}] :_{S[A^{1/qq''}]} dx^{q'q''}\right) \cong \left(J^{[q'q'']}S :_S dx^{q'q''}\right) S\left[A^{1/qq''}\right].$$

Now, $\left(J^{[q'q'']}S :_S dx^{q'q''}\right) \subseteq \left(J^{[q'q'']}R^{1/q''} :_{R^{1/q''}} dx^{q'q''}\right)$ as $S = R\left[A^{1/q''}\right] \subseteq R^{1/q''}$. By choice of q', $dx^{\tilde{q}} \notin J^{[\tilde{q}]}R^{1/\tilde{q}}$ for any $\tilde{q} \geq q'$. Therefore, $\left(J^{[q'q'']}R^{1/q''} :_{R^{1/q''}} dx^{q'q''}\right) \neq R^{1/q''}$. Thus, $(J^{[q'q'']}S :_S dx^{q'q''}) \subseteq \mathfrak{m}^{1/q''}$, and so $a^{1/q} \in \mathfrak{m}^{1/q''}S[A^{1/qq''}] \subseteq \mathfrak{m}^{1/q''}R^{1/qq''}$. Taking qq'' powers, we get $a^{q''} \in \mathfrak{m}^{[q]}R$. So if $Q \geq q'q''$ and $a \in K_Q$, then $a^{q''} \in \mathfrak{m}^{[Q/q'q'']}R \cap A$.

Let $\mathfrak{m}_A$ denote the maximal ideal of A. Note we can find $D = p^n$ such that $\mathfrak{m}^D \subseteq \mathfrak{m}_A R$ and then $a^{q''} \in \mathfrak{m}^{[Q/q'q'']}R \cap A \subseteq \mathfrak{m}^{Q/q'q''}R \cap A \subseteq \mathfrak{m}_A^{Q/q'q''D}R \cap A \subseteq \mathfrak{m}_A^{(Q/q'q''D)-t}$ for some constant t and $Q \gg 0$ (the last inclusion coming from the Artin - Rees Lemma). Now for large Q we'll have

$$\mathfrak{m}_A^{(Q/q'q''D)-t} \subseteq \mathfrak{m}_A^{Q/B} \qquad \text{where } B = q'q''Dp.$$

Thus, for $Q \gg 0$, $a^{q''} \in \mathfrak{m}_A^{Q/B}$. Let $d = \dim A$. Then note that $\mathfrak{m}_A^{dq} \subseteq \mathfrak{m}_A^{[q]}$ for any $q = p^n$. Let $p^e \geq d$. Then we have for large Q, $a^{q''} \in \mathfrak{m}_A^{Qp^e/Bp^e} \subseteq \mathfrak{m}_A^{Qd/Bp^e} \subseteq \mathfrak{m}_A^{[Q/Bp^e]}$. Now let $H = \mathfrak{m}_A^{[Q/Bp^e q'']}$. For large Q, $A = \left(H^{[q'']} :_A a^{q''}\right) = (H :_A a)^{[q'']}$ (the last equality coming from the fact that A is regular and so the Frobenius is flat). So, we have $A = (H :_A a)$ and hence $a \in H = \mathfrak{m}_A^{[Q/Bp^e q'']}$ for large Q. As a was an arbitrary element of K_Q, we have for large Q, $K_Q \subseteq \mathfrak{m}_A^{[Q/B']} \subseteq \mathfrak{m}_A^{Q/B'}$ where $B' = Bp^e q''$, and so $\lambda\left(A/K_Q\right) \geq \lambda(A/\mathfrak{m}_A^{Q/B'})$. Let $0 < \gamma < \frac{1}{d!(B')^d}$. Then as A is regular we have

$$\begin{aligned}
\lambda(A/K_Q) \;\geq\; \lambda(A/\mathfrak{m}_A^{Q/B'}) \;&=\; \binom{Q/B' + d - 1}{d} \\
&=\; \frac{Q^d}{d!(B')^d} + \text{lower terms} \\
&\geq\; \gamma Q^d \quad \text{for } Q \gg 0.
\end{aligned}$$

$\square$

The argument in the last paragraph of the proof above allowing us to conclude that $K_Q \subseteq \mathfrak{m}_A^{Q/B'}$ was suggested to us by Neil Epstein and allowed us to avoid the implication on page 81, line 12 of the original proof in [HH90].

Corollary 4.2.5. *Let $(R, \mathfrak{m})$ be a local ring and $J \subseteq I$ $\mathfrak{m}$-primary ideals of R.*

1. *If $I \subseteq J^*$ then $e_{HK}(I) = e_{HK}(J)$.*

2. *The converse to (1) holds if R is equidimensional and either complete or essentially of finite type over a field.*

Proof. Note that part (1) follows from the previous theorem. To prove part (2), suppose that $e_{HK}(I) = e_{HK}(J)$ and R is equidimensional and either complete or essentially of finite type over a field. We first note that that we may reduce to the case R is a domain by going modulo a minimal prime. To see this, recall the associativity formula for Hilbert-Kunz multiplicity says that for any ideal I, $e_{HK}(I) = \sum_{P \in \mathrm{Assh}(R)} e_{HK}(I, R/P)\lambda_{R_P}(R_P)$ where $\mathrm{Assh}(R) = \{P \in \mathrm{Ass}(R) \mid \dim R/P = \dim R\}$. Let $\mathrm{Min}(R) = \{P_1, \ldots, P_n\}$ and I_i and J_i denote the images of I and J respectively in R/P_i. As R is equidimensional, $\mathrm{Assh}(R) = \mathrm{Min}(R)$ and so we have $e_{HK}(I) = \sum_{i=1}^{n} e_{HK}(I_i)\lambda(R_{P_i}) = \sum_{i=1}^{n} e_{HK}(J_i)\lambda(R_{P_i}) = e_{HK}(J)$. Since $J \subseteq I$, we have $e_{HK}(I_i) \leq e_{HK}(J_i)$ for all i. The equality of the Hilbert-Kunz multiplicities of I and J forces $e_{HK}(I_i) = e_{HK}(J_i)$ for all i. Furthermore, an element $x \in R$ is in the tight closure of J if and only if its image is in J_i^* for all i. (See [BH93], Proposition 10.1.2.) Thus, we may reduce to the case R is a domain.

By Remark 4.2.2, R has a completely stable test element. If R is a complete domain, we are done by Theorem 4.2.4. If R is a domain which is essentially of finite type over a field, then the completion of R is analytically unramified (see [Mat70], page 251, Lemma 2) and formally equidimensional (see [Mat89], Theorem 31.6 (iii)). The result once again follows from Theorem 4.2.4. $\square$

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