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ALGEBRAIC PROPERTIES OF EXT-MODULES OVER COMPLETE INTERSECTIONS

by

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ALGEBRAIC PROPERTIES OF EXT-MODULES OVER COMPLETE INTERSECTIONS

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University of Nebraska, 2014

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We investigate two algebraic properties of Ext-modules over a complete intersection $R = Q/(f_1, \ldots, f_c)$ of codimension c. Given an R-module M, $\mathrm{Ext}^*_R(M, k)$ can be viewed as a graded module over a polynomial ring in c variables with an action given by the Eisenbud operators. We provide an upper bound on the degrees of the generators of this graded module in terms of the regularities of two associated coherent sheaves. In the codimension two case, our bound recovers a bound of Avramov and Buchweitz in terms of the Betti numbers of M. We also provide a description of the differential graded (DG) R-module $\mathbf{R}\operatorname{Hom}^{dg}_R(M, N)$ in terms of well-known DG Q-modules. When $M = N$, this has the structure of a differential graded algebra (DGA) over Q. In the case where $M = Q/I$ with I generated by a Q-regular sequence, we provide explicit generators and relations for the DGA $\mathbf{R}\operatorname{End}^{dg}_R(M)$ using the theory of Clifford algebras. This description generalizes a result of Dyckerhoff, who obtains a similar result in a special case. In the case where $M = k$, our result implies a classical result of Sjödin on the algebraic structure of $\mathrm{Ext}^*_R(k, k)$ over complete intersections.

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Contents

Chapter 1

Introduction

This dissertation concerns algebraic properties of the cohomology modules $\operatorname{Ext}_R^*(M,N)$ over a complete intersection R. There are two primary foci. Viewing $\operatorname{Ext}_R^*(M,k)$ as a cohomologically graded $k[T_1,\dots,T_c]$-module via the Eisenbud operators, we produce an upper bound on the degrees of the generators which generalizes a result of Avramov and Buchweitz [2] in the codimension two case. We also give explicit descriptions of the differential graded R-module $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ and the differential graded R-algebra $\mathbf{R}\operatorname{End}_R^{dg}(M)$, with the latter recovering a result of Sjödin [18] in the case where $M=k$.

Over any commutative ring the modules $\operatorname{Ext}_R^*(M,N)$ provide a measure of how far the left exact functor $\operatorname{Hom}_R(-,N)$ is from being exact. They are a fundamental tool in homological algebra – one needs only to page through any text on the subject to see the myriad applications in which they are used. For example, they provide a characterization of projective and injective modules as well as a classification of extensions of modules.

There are various ways to define $\operatorname{Ext}_R^*(M,N)$. One approach is to consider a projective resolution P of M, apply the functor $\operatorname{Hom}_R(-,N)$ to P, and take cohomology; namely, $\operatorname{Ext}_R^*(M,N) := H^*(\operatorname{Hom}_R(P,N))$. If Q is a projective resolution of N, then one can also define $\operatorname{Ext}_R^*(M,N) = H^*(\operatorname{Hom}_R(P,Q))$. This latter approach is the one we will usually take.

We work almost exclusively over complete intersection rings. Due to their importance throughout this thesis, we describe them here. A commutative Noetherian ring Q is a *local ring* if it has a unique maximal ideal. We usually denote this by (Q, m, k), where m is the maximal ideal of Q and $k = Q/m$ is the *residue field* of Q. A Noetherian commutative ring Q is a *regular local ring* if the minimal number of generators of m equals the Krull dimension of Q. A Q-*regular sequence* is a sequence $f_1, \ldots, f_c \in Q$ such that $(f_1, \ldots, f_c) \neq Q$ and f_j is a non-zero divisor on $Q/(f_1, \ldots, f_{j-1})$ for $1 \leq j \leq c$. A ring R is a *complete intersection of codimension* c if it has the form $R = Q/(f_1, \ldots, f_c)$ for some regular local ring (Q, m, k) and some Q-regular sequence $f_1, \ldots, f_c$. A simple example of a complete intersection is the ring $R = k[[x_1, \ldots, x_n]]/(x_1^{a_1}, \ldots, x_n^{a_n})$, where $k[[x_1, \ldots, x_n]]$ is the regular local ring consisting of formal power series in n variables over a field k and $a_1, \ldots, a_n$ are positive integers.

In chapter 2 we view $\operatorname{Ext}_R^*(M, k)$ as a graded module over the polynomial ring $k[T_1, \ldots, T_c]$ with $|T_j| = 2$ and with the action induced by the Eisenbud operators. It is natural to ask if there is an upper bound on the degrees of the generators of this graded module. For complete intersections of codimension $c = 2$, Avramov and Buchweitz [2] answered this question in the affirmative and gave a bound in terms of the Betti numbers of M. We produce an upper bound which works for any codimension in terms of the Mumford-Castelnuovo regularities of two coherent sheaves on $\mathbb{P}_k^{c-1}$ associated to the module M. Section 2.1 presents the necessary background information for stating the bound. Section 2.2 states and proves the bound, the main result being Theorem 2.11. Section 2.3 shows that in the codimension $c = 2$ case, the regularity bound recovers the bound of Avramov and Buchweitz.

In chapter 3, we consider a result of Sjödin (Theorem 3.1) in [18] which provides generators and relations for the algebra $\operatorname{Ext}_R^*(k, k)$ over a complete intersection. We extend this result to the level of differential graded algebras; namely, we find descriptions of the differential graded R-module $\mathbf{R}\operatorname{Hom}_R^{dg}(M, N)$ and differential graded R-algebra $\mathbf{R}\operatorname{End}_R^{dg}(M)$. Section 3.1 provides the relevant background information of differential graded modules and algebras and sets notation which will be used throughout the chapter. Section 3.2 clarifies our definitions of $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ and $\mathbf{R}\operatorname{End}_R^{dg}(M)$ and shows that they are well-defined up to appropriate equivalences. Section 3.3 gives our most general descriptions of $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ and $\mathbf{R}\operatorname{End}_R^{dg}(M)$ for any finitely generated R-modules M and N. The main results are Theorem 3.19 and Corollary 3.20. In order to give a more explicit description of $\mathbf{R}\operatorname{End}_R^{dg}(M)$ for $M = Q/I$ with I generated by a Q-regular sequence, we utilize the theory of Clifford algebras. The background on this topic is presented in section 3.4, and this is used in section 3.5 to describe $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as a Q-algebra in terms of generators and relations and to describe a differential that makes it a DG module over the Koszul algebra of R. The main results are Theorems 3.29 and 3.32. Section 3.6 provides examples and applications of these descriptions in a few special cases. In particular, we recover a $(\mathbb{Z}/2\mathbb{Z})$-graded differential graded algebra used by Dyckerhoff [11] over hypersurfaces, verify that $\mathbf{R}\operatorname{End}_R^{dg}(M)$ is formal in the quadratic hypersurface case, and describe the Hochschild cohomology of the localization of a polynomial ring with respect to the maximal ideal generated by the variables.

1.1 Conventions

We make a few remarks about notation and conventions that we will follow.

When used to describe rings or modules, the word "graded" will always mean $\mathbb{Z}$-graded. On occasion we will consider $(\mathbb{Z}/2\mathbb{Z})$-graded objects as well, and this will be stated explicitly.

Unless otherwise stated, all graded modules and complexes will be graded cohomologically using superscripts. For example, $L = \bigoplus_{i\in\mathbb{Z}} L^i$. When the use of homological indexing with subscripts is warranted, we will follow the standard convention $L^i = L_{-i}$ for $i \in \mathbb{Z}$. If $x \in L^i$, then we denote the degree of the homogeneous element x by $|x| = i$. If L is a complex, then $L[j]$ is the complex with $L[j]^i = L^{i+j}$ and differential $\partial^i_{L[j]} = (-1)^{|j|}\partial_L^{i+j}$.

Chapter 2

Bounding the Degrees of Generators of $\mathbf{Ext}_R^*(M,k)$ over Complete Intersections

We consider the following theorem of Avramov and Buchweitz from [2]:

Theorem 2.1. *Let (Q, m, k) be a regular local ring, f_1, f_2 a Q-regular sequence, and $R = Q/(f_1, f_2)$ a complete intersection of codimension 2. Let M be a finitely generated R-module and set $g := depth(R) - depth_R(M)$. Let $k[T_1, T_2]$ with $|T_i| = 2$ act on $Ext_R^*(M, k)$ via the Eisenbud operators. Then the graded $k[T_1, T_2]$-module $Ext_R^*(M, k)$ is generated in degrees less than or equal to*

$$\max\{2\beta_g, 2\beta_{g+1} + 1\} + g + 1,$$

where β_i denotes the i^{th} Betti number of M over R.

This proof does not generalize to higher codimensions as it utilizes a decomposition result for graded modules over exterior algebras in two variables. Our goal in this chapter is to find an upper bound for the degrees of the generators of $\mathrm{Ext}_R^*(M, k)$ over a complete intersection $R = Q/(f_1, \dots, f_c)$ of codimension c in terms of the regularities of two coherent sheaves on $\mathbb{P}_k^{c-1}$ associated to M. In the codimension $c = 2$ case, our bound recovers the Avramov-Buchweitz bound in Theorem 2.1.

2.1 Twisted Periodic Complexes and Regularity

Throughout this chapter, let R be a complete intersection of codimension c; namely, $R = Q/(f_1, \dots, f_c)$ with (Q, m, k) a regular local ring and $f_1, \dots, f_c$ a Q-regular sequence. Let M and N be finitely generated R-modules. We will consider $\operatorname{Ext}_R^*(M, N) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}_R^i(M, N)$ as a graded module over $R[T_1, \dots, T_c]$ with an action induced by the Eisenbud operators. These operators are described in [12], but we briefly recall them here.

Let (F, ∂) be a complex of free R-modules. Choose a sequence, not necessarily a complex, of free Q-modules $(\tilde{F}, \tilde{\partial})$ such that $F \cong \tilde{F} \otimes_Q R$ and $\partial = \tilde{\partial} \otimes_Q R$. Note that such a sequence always exists: viewing ∂^j as a matrix with entries in R, choose $\tilde{\partial}^j$ by lifting each entry of this matrix to an element of Q. Since $F \cong \tilde{F} \otimes_Q R$, there exists maps $\tilde{t}_j^n : \tilde{F}^n \to \tilde{F}^{n+2}$ such that $\tilde{\partial}_n^2 = \sum_{j=1}^{c} \tilde{t}_j^n f_j$. Define $t_j^n := \tilde{t}_j^n \otimes_Q R : F^n \to F^{n+2}$, the *Eisenbud operators* on F. Eisenbud proves these maps have the following properties:

(1) The collection of maps $t_j := \{t_j^n\}$ is a morphism of complexes $t_j : F \to F[2]$.

(2) The t_j's are independent of choice of $\tilde{F}$ and $\tilde{t}_j^n$, up to homotopy.

(3) The t_j's are natural up to homotopy, i.e., if $\rho : F \to F'$ is a morphism of complexes of free R-modules and $\{t_j\}$ and $\{t'_j\}$ are Eisenbud operators for F and F', respectively, then $\rho t_j \sim t'_j \rho$.

(4) The t_j's commute up to homotopy.

By considering a free resolution F of M over R, we get induced maps

$$T_j := H(\mathrm{Hom}_R(t_j, N)) : \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^{i+2}_R(M, N)$$

that make $\mathrm{Ext}^*_R(M, N)$ a graded module over the graded ring $R[T_1, \dots, T_c]$ with $|T_j| = 2$. We call this the *action induced by the Eisenbud operators.* Whenever we discuss $\mathrm{Ext}^*_R(M, k)$ as a graded module, it will be via this action.

Gulliksen [13] described a graded $R[T_1, \dots, T_c]$-module structure on $\mathrm{Ext}^*_R(M, N)$, which is shown to agree with the Eisenbud operators up to sign in [3]. With this action Gulliksen was able to prove the following key property of $\mathrm{Ext}^*_R(M, N)$.

Theorem 2.2 (Gulliksen)**.** *Let R be a complete intersection of codimension c and M, N be finitely generated R-modules. Then $\mathrm{Ext}^*_R(M, N)$ is a finitely generated graded $R[T_1, \dots, T_c]$-module with $|T_j| = 2$ and action induced by the Eisenbud operators.*

In order to find an upper bound for the degrees of generators of $\mathrm{Ext}^*_R(M, k)$, we will consider a related object, the stable Ext-modules, which we describe now. The following definitions appear in [10].

Definition 2.3. Given a complete intersection R and any R-module M, a *complete resolution* of M is a morphism of complexes $Y \to P$ with Y_n projective over R for all n and P an R-projective resolution of M, satisfying

(1) $H^i(Y) = 0$ for all $i \in \mathbb{Z}$, i.e., Y is *acyclic*,

(2) $H^i(\mathrm{Hom}_R(Y, W)) = 0$ for all projective R-modules W and all $i \in \mathbb{Z}$,

(3) $Y_n = P_n$ for $n \gg 0$.

If Y is a complete resolution of M, then define the *stable Ext-modules* by

$$\widehat{\operatorname{Ext}}^i_R(M,N) := H^i(\operatorname{Hom}_R(Y,N)).$$

For the remainder of the chapter we restrict to the case where $N = k$, so that $\operatorname{Ext}^*_R(M,k)$ is a finitely generated graded $k[T_1,\dots,T_c]$-module with the action induced by the Eisenbud operators.

From condition (3) in the definition of a complete resolution, we see that there is a natural map $\operatorname{Ext}^*_R(M,k) \to \widehat{\operatorname{Ext}}^*_R(M,k)$ that is an isomorphism in sufficiently large cohomological degrees. If M is a maximal Cohen-Macaulay (MCM) R-module, then we can say a bit more about this natural map. (Recall that an R-module M is MCM if depth $M = \dim R$. See [16] and [5] for more information.)

To do so, we will need a few well known facts about MCM modules over complete intersections; see [15, Theorem 11.5] for proofs. (In fact, these properties hold for any Gorenstein ring, though we will not need this generality.) If M is an MCM R-module, then so is its dual $M^* = \operatorname{Hom}_R(M,R)$. Also, $M^{**} \cong M$ and $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > 0$.

Let P be a free resolution of an MCM R-module M and let F be a free resolution of M^*. Taking the dual of $F \to M^*$ and setting $P_i := F_{-i-1}$ for $i \leq -1$, the above facts applied to the MCM module M^* give an exact sequence $M^{**} \cong M \to P_{\leq -1}$. We splice together the sequences $P \to M$ and $M \to P_{\leq -1}$ using the composition $P_0 \to M \to P_{-1}$ to get an acyclic complex

$$Y = \quad \cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots$$

This Y is a complete resolution of M with $Y_n = P_n$ for all $n \geq 1$. Thus, when M is MCM, the natural map $\operatorname{Ext}^*_R(M,k) \to \widehat{\operatorname{Ext}}^*_R(M,k)$ is an isomorphism in all positive degrees and is surjective in degree 0 with kernel consisting of maps that factor through free R-modules.

The upshot of this natural map is that in order to find an upper bound on the degrees of generators of $\operatorname{Ext}_R^*(M,k)$, it will suffice to find an upper bound of the degrees of generators of $\widehat{\operatorname{Ext}}_R^*(M,k)$. To do this, we use theory developed by Burke and Walker in [8].

Let $\mathbb{P}_k^{c-1} = \operatorname{Proj} k[T_1,\dots,T_c]$. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}_k^{c-1}$, set $\mathcal{F}(1) := \mathcal{F}\otimes\mathcal{O}_{\mathbb{P}_k^{c-1}}(1)$.

Definition 2.4. A *twisted periodic complex of locally free coherent sheaves* $\mathbb{E}$ over $\mathbb{P}_k^{c-1}$ consists of a pair of locally free coherent sheaves $\mathcal{E}^{-1}$, $\mathcal{E}^0$ on $\mathbb{P}_k^{c-1}$ along with morphisms $e^{-1}:\mathcal{E}^{-1}\to\mathcal{E}^0$ and $e^0:\mathcal{E}^0\to\mathcal{E}^{-1}(1)$ such that the compositions $e^0\circ e^{-1}$ and $e^{-1}(1)\circ e^0$ are both 0. Equivalently, $\mathbb{E}=(\mathcal{E},\gamma)$ is a pair consisting of a chain complex

$$\mathcal{E} = (\cdots\to\mathcal{E}^{-2}\to\mathcal{E}^{-1}\to\mathcal{E}^0\to\mathcal{E}^1\to\cdots)$$

of locally free coherent sheaves on $\mathbb{P}_k^{c-1}$ along with a specified isomorphism $\gamma:\mathcal{E}(1)\cong\mathcal{E}[2]$, using the convention that $\mathcal{E}[2]^i=\mathcal{E}^{i+2}$.

Let $\mathbb{E}$ be a twisted periodic complex and let $\{U_1,\dots,U_c\}$ denote the standard affine cover of $\mathbb{P}_k^{c-1}$. Let $\mathcal{C}^{\bullet\bullet}$ denote the bicomplex formed by applying the Čech construction to $\mathbb{E}$ degreewise:

$$\begin{array}{c}
\Gamma(U_1\cap\cdots\cap U_c,\mathbb{E})\\
\Big\uparrow\\
\vdots\\
\Big\uparrow\\
\displaystyle\bigoplus_{1\le i<j\le c}\Gamma(U_i\cap U_j,\mathbb{E})\\
\Big\uparrow\\
\displaystyle\bigoplus_{i=1}^{c}\Gamma(U_i,\mathbb{E})
\end{array}$$

We define the *hypercohomology* of $\mathbb{E}$ by

$$\mathbb{H}^i(\mathbb{P}_k^{c-1}, \mathbb{E}) := H^i(\mathrm{Tot}(\mathcal{C}^{\bullet\bullet})).$$

Lemma 2.5. *If $\mathbb{E}$ is a twisted periodic complex, then $\mathbb{H}^*(\mathbb{P}_k^{c-1}, \mathbb{E}) := \bigoplus_{i\in\mathbb{Z}} \mathbb{H}^i(\mathbb{P}_k^{c-1}, \mathbb{E})$ is a graded $k[T_1, \dots, T_c]$-module with $|T_j| = 2$.*

Proof. In general, given a Noetherian scheme X and a coherent sheaf $\mathcal{F}$ on X, one has $H^0(X, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$. In our context, any $\varphi \in H^0(\mathbb{P}_k^{c-1}, \mathcal{O}(1)) = kT_1 \oplus \cdots \oplus kT_c$ gives a map $\varphi : \mathcal{O}_{\mathbb{P}_k^{c-1}} \to \mathcal{O}_{\mathbb{P}_k^{c-1}}(1)$, and hence a map of complexes $\mathbb{E} \to \mathbb{E}(1) \cong \mathbb{E}[2]$. In particular, each pair of variables T_j and T_l give maps $T_j, T_l : \mathbb{E} \to \mathbb{E}[2]$ causing the following square to commute:

$$\begin{tikzcd} \mathbb{E} \arrow[r, "T_l"] \arrow[d, "T_j"] & \mathbb{E}[2] \arrow[d, "T_j"] \\ \mathbb{E}[2] \arrow[r, "T_l"] & \mathbb{E}[4] \end{tikzcd}$$

These induce maps $(T_j)_*, (T_l)_* : \mathbb{H}^i(\mathbb{P}_k^{c-1}, \mathbb{E}) \to \mathbb{H}^i(\mathbb{P}_k^{c-1}, \mathbb{E}[2]) \cong \mathbb{H}^{i+2}(\mathbb{P}_k^{c-1}, \mathbb{E})$ in hypercohomology causing the following square to commute:

$$\begin{tikzcd} \mathbb{H}^i(\mathbb{P}_k^{c-1}, \mathbb{E}) \arrow[r, "(T_l)_*"] \arrow[d, "(T_j)_*"] & \mathbb{H}^{i+2}(\mathbb{P}_k^{c-1}, \mathbb{E}) \arrow[d, "(T_j)_*"] \\ \mathbb{H}^{i+2}(\mathbb{P}_k^{c-1}, \mathbb{E}) \arrow[r, "(T_l)_*"] & \mathbb{H}^{i+4}(\mathbb{P}_k^{c-1}, \mathbb{E}) \end{tikzcd}$$

Thus, $\mathbb{H}^*(\mathbb{P}_k^{c-1}, \mathbb{E})$ is a graded $k[T_1, \dots, T_c]$-module. $\square$

We need one last bit of notation. We define *even Ext* and *odd Ext* to be $\operatorname{Ext}_R^{ev}(M,k) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}_R^{2i}(M,k)$ and $\operatorname{Ext}_R^{odd}(M,k) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}_R^{2i+1}(M,k)$, respectively. These are graded $k[T_1,\dots,T_c]$-modules with $|T_j| = 1$ via the action induced by the Eisenbud operators. Note that we are considering these as graded modules with degree i components $\operatorname{Ext}_R^{2i}(M,k)$ and $\operatorname{Ext}_R^{2i+1}(M,k)$, respectively. Gulliksen [13] showed that $\operatorname{Ext}_R^{ev}(M,k)$ and $\operatorname{Ext}_R^{odd}(M,k)$ are both finitely generated.

The following theorem of Burke and Walker shows that to study $\widehat{\operatorname{Ext}}_R^*(M,k)$ it is enough to consider the hypercohomology of a particular twisted periodic complex associated to the R-module M, which they construct in [8].

Theorem 2.6 (Burke-Walker)**.** *Let R be a complete intersection of codimension c and let M be an R-module. There exists a twisted periodic complex $\mathbb{E} = \mathbb{E}(M,k)$ of locally free coherent sheaves on $\mathbb{P}_k^{c-1}$ with the following properties:*

1. *For all i, there is an isomorphism $\widehat{Ext}_R^i(M,k) \cong \mathbb{H}^i(\mathbb{P}_k^{c-1}, \mathbb{E})$.*
2. *The isomorphisms in (1) give an isomorphism of graded $k[T_1,\dots,T_c]$-modules, with $|T_j| = 2$,*
$$\widehat{Ext}_R^*(M,k) \cong \mathbb{H}^*(\mathbb{P}_k^{c-1}, \mathbb{E}),$$
where the action on $\mathbb{H}^*(\mathbb{P}_k^{c-1}, \mathbb{E})$ given by T_j in Lemma 2.5 coincides with the action of the Eisenbud operators on $\widehat{Ext}_R^*(M,k)$.
3. *We have $\mathcal{H}^0(\mathbb{E}) \cong \widetilde{Ext_R^{ev}(M,k)}$ and $\mathcal{H}^1(\mathbb{E}) \cong \widetilde{Ext_R^{odd}(M,k)}$, where $\mathcal{H}^i(\mathbb{E})$ denotes the i^{th} cohomology of the underlying complex of $\mathbb{E}$ and $\widetilde{(-)}$ denotes the coherent sheaf on $\mathbb{P}_k^{c-1}$ associated to the graded module over the standard graded polynomial ring.*

For the remainder of this chapter, $\mathbb{E}$ will always denote the twisted periodic complex which arises from the R-module M and residue field k via Theorem 2.6.

We may now define the necessary invariant which we use to produce our desired upper bound.

Definition 2.7. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}_k^{c-1}$. Then $\mathcal{F}$ is *r-regular* if

$$H^i(\mathbb{P}_k^{c-1}, \mathcal{F}(r-i)) = 0$$

for all $i > 0$. The *(Castelnuovo-Mumford) regularity* of $\mathcal{F}$ is defined to be

$$\operatorname{Reg}(\mathcal{F}) := \inf\{r \mid H^i(\mathbb{P}_k^{c-1}, \mathcal{F}(r-i)) = 0 \text{ for all } i > 0\}.$$

Notice that $\operatorname{Reg}(\mathcal{F}) = -\infty$ is possible. For example, if k is infinite and $\mathcal{F}$ is supported on a finite set of points, then $\mathcal{F} \cong \mathcal{F}(1)$. Hence $H^i(\mathbb{P}_k^{c-1}, \mathcal{F}(r-i)) \cong H^i(\mathbb{P}_k^{c-1}, \mathcal{F}) = 0$ for all $r \in \mathbb{Z}$ and all $i > 0$, so $\operatorname{Reg}(\mathcal{F}) = -\infty$.

We will need the following basic facts about r-regularity. Proofs can be found in [17].

Lemma 2.8. *Let $\mathcal{F}$ be an r-regular coherent sheaf on $\mathbb{P}_k^{c-1}$.*

1. *For $i > 0$, $H^i(\mathbb{P}_k^{c-1}, \mathcal{F}(j)) = 0$ for all $j \geq r - i$.*

2. *If $j \geq r$, then $\mathcal{F}(j)$ is generated by its global sections.*

2.2 An Upper Bound in Codimension c

We want to find an upper bound for the degrees of generators of the graded $k[T_1, \ldots, T_c]$-module $\widehat{\operatorname{Ext}}_R^*(M, k)$. For notational simplicity, we adopt the following notation. For any (possibly non-finitely generated) graded $k[T_1, \ldots, T_c]$-module N, we define

$$\alpha(N) := \inf\{q \mid N \text{ is generated by } N^{\leq q}\}.$$

Notice that $\alpha(N) = -\infty$ when N can be generated in arbitrarily negative degrees. For example, suppose $\mathrm{char}(k) = 0$ and consider the graded $k[T_1, \ldots, T_c]$-module $N := k[X_1, \ldots, X_c]$ with $|X_i| = -2$ and action given by partial differentiation: $T_i \cdot g(X_1, \ldots, X_c) := \dfrac{\partial g}{\partial X_i}$. Then N is generated by $N^{\leq q}$ for all $q \in \mathbb{Z}$. Thus, $\alpha(N) = -\infty$.

This α invariant is well behaved with short exact sequences in the sense of the following lemma.

Lemma 2.9. *Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of graded $k[T_1, \ldots, T_c]$-modules. Then $\alpha(N) \leq \max\{\alpha(N'), \alpha(N'')\}$.*

Proof. Observe that N can be generated by the image of a generating set of N' and a lift of a generating set of N''. If $\max\{\alpha(N'), \alpha(N'')\} = -\infty$, then taking generating sets for N' and N'' of arbitrarily negative degree produces a generating set for N of arbitrarily negative degree, so that $\alpha(N) = -\infty = \max\{\alpha(N'), \alpha(N'')\}$. Otherwise, if $\max\{\alpha(N'), \alpha(N'')\} \in \mathbb{Z}$, then elements in the generating set for N will have degree at most $\max\{\alpha(N'), \alpha(N'')\}$. $\square$

Recall that $\mathcal{H}^0(\mathbb{E}) \cong \widetilde{\mathrm{Ext}_R^{ev}(M,k)}$ and $\mathcal{H}^1(\mathbb{E}) \cong \widetilde{\mathrm{Ext}_R^{odd}(M,k)}$ by Theorem 2.6. For the remainder of the chapter, set $r_0 := \mathrm{Reg}(\mathcal{H}^0(\mathbb{E}))$ and $r_1 := \mathrm{Reg}(\mathcal{H}^1(\mathbb{E}))$. These are the invariants of M which will be used to form our desired bound. We need the following technical lemma. Set $\mathcal{H}^0 = \mathcal{H}^0(\mathbb{E})$ and $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{E})$.

Lemma 2.10. *Set $r^{ev} := \max\{2r_0 - 2, 2r_1\}$ and $r^{odd} := \max\{2r_0 - 1, 2r_1 - 1\}$. Then we have* $\bigoplus_{i \geq \frac{r^{ev}}{2}} \widehat{Ext}_R^{2i}(M,k) \cong \bigoplus_{i \geq \frac{r^{ev}}{2}} H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^0(i))$ *and* $\bigoplus_{i \geq \frac{r^{odd}-1}{2}} \widehat{Ext}_R^{2i+1}(M,k) \cong \bigoplus_{i \geq \frac{r^{odd}-1}{2}} H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^1(i))$ *as graded $k[T_1, \ldots, T_c]$-modules.*

Proof. Let $\mathbb{E}$ be the twisted periodic complex associated to M via Theorem 2.6. Let $\{U_1, \ldots, U_c\}$ be the standard affine cover of $\mathbb{P}_k^{c-1}$ and consider the bicomplex

$$\begin{array}{c}\Gamma(U_1 \cap \cdots \cap U_c, \mathbb{E}) \\ \uparrow \\ \vdots \\ \uparrow \\ \bigoplus_{1 \le i < j \le c} \Gamma(U_i \cap U_j, \mathbb{E}) \\ \uparrow \\ \bigoplus_{i=1}^{c} \Gamma(U_i, \mathbb{E})\end{array}$$

given by the Čech construction, where the lowest nonzero row is in cohomological degree 0. Consider the spectral sequence in which horizontal cohomology is computed first. Consider the edge map $\mathbb{H}^{2i}(\mathbb{P}_k^{c-1}, \mathbb{E}) \to H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^0(i))$ from this spectral sequence. When $i \geq \frac{r^{ev}}{2}$, Lemma 2.8(1) shows that everything lying above $H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^0(i))$ in the spectral sequence will be 0, so that the edge map is an isomorphism. Similarly, when $i \geq \frac{r^{odd}-1}{2}$, the edge map $\mathbb{H}^{2i+1}(\mathbb{P}_k^{c-1}, \mathbb{E}) \to H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^1(i))$ is an isomorphism.

The isomorphism $\widehat{\operatorname{Ext}}_R^*(M,k) \cong \mathbb{H}^*(\mathbb{P}_k^{c-1}, \mathbb{E})$ from Theorem 2.6 completes the proof. $\square$

We now present our main result of the chapter.

Theorem 2.11. *Let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection of codimension c and let M be a finitely generated R-module. Consider the graded $k[T_1, \ldots, T_c]$-module $\widehat{Ext}_R^*(M,k)$ with action given by the Eisenbud operators. Let $\mathbb{E}$ denote the twisted periodic complex associated to M and k in Theorem 2.6 and set $r_0 = Reg(\mathcal{H}^0(\mathbb{E}))$ and $r_1 = Reg(\mathcal{H}^1(\mathbb{E}))$. Then*

$$\alpha\left(\widehat{Ext}_R^*(M,k)\right) \leq \max\{2r_0, 2r_1 + 1\}.$$

Proof. Set $\mathcal{H}^0 = \mathcal{H}^0(\mathbb{E})$ and $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{E})$.

First suppose that at least one of r_0 or r_1 is an integer. For any even $q \geq \max\{2r_0-2, 2r_1\}$, there is a short exact sequence of $k[T_1, \ldots, T_c]$-modules

$$0 \to \bigoplus_{j \geq \frac{q}{2}} H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^0(j)) \to \widehat{\operatorname{Ext}}_R^{ev}(M,k) \to \bigoplus_{i \leq \frac{q}{2}-1} \widehat{\operatorname{Ext}}_R^{2i}(M,k) \to 0$$

by Lemma 2.10. Clearly $\alpha\left(\bigoplus_{i \leq \frac{q}{2}-1} \widehat{\operatorname{Ext}}_R^{2i}(M,k)\right) \leq \frac{q}{2}-1$. Since $r_0 = \operatorname{Reg}(\mathcal{H}^0)$, Lemma 2.8(2) gives us that $\alpha\left(\bigoplus_{j \geq \frac{q}{2}} H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^0(j))\right) = \frac{q}{2}$ as long as $\frac{q}{2} \geq r_0$. So for $q = \max\{2r_0, 2r_1\}$, Lemma 2.9 gives that

$$\begin{aligned}
\alpha\left(\widehat{\operatorname{Ext}}_R^{ev}(M,k)\right) &\leq \max\left\{\alpha\left(\bigoplus_{j \geq \frac{q}{2}} H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^0(j))\right), \alpha\left(\bigoplus_{i \leq \frac{q}{2}-1} \widehat{\operatorname{Ext}}_R^{2i}(M,k)\right)\right\} \\
&\leq \max\left\{\frac{q}{2}, \frac{q}{2}-1\right\} \\
&= \frac{1}{2}\max\{2r_0, 2r_1\}.
\end{aligned}$$

Note that if $r_0 = r_1 = -\infty$, the above argument shows that $\alpha\left(\widehat{\operatorname{Ext}}_R^{ev}(M,k)\right) \leq \frac{q}{2}$ for all even q, hence $\alpha\left(\widehat{\operatorname{Ext}}_R^{ev}(M,k)\right) = -\infty$.

Again, we first suppose that at least one of the regularities r_0 or r_1 is an integer. For any odd $q \geq \max\{2r_0-1, 2r_1-1\}$, there is a short exact sequence of $k[T_1, \ldots, T_c]$-modules

$$0 \to \bigoplus_{j \geq \frac{q-1}{2}} H^0(\mathbb{P}_k^{c-1}, \mathcal{H}^1(j)) \to \widehat{\operatorname{Ext}}_R^{odd}(M,k) \to \bigoplus_{i \leq \frac{q-1}{2}-1} \widehat{\operatorname{Ext}}_R^{2i+1}(M,k) \to 0$$

by Lemma 2.10. Clearly $\alpha\left(\bigoplus_{i \leq \frac{q-1}{2}-1} \widehat{\operatorname{Ext}}_R^{2i+1}(M,k)\right) \leq \frac{q-1}{2}-1$. Since $r_1 = \operatorname{Reg}(\mathcal{H}^1)$, Lemma 2.8(2) gives us that $\alpha\left(\bigoplus_{j\geq\frac{q-1}{2}} H^0(\mathbb{P}_k^{c-1},\mathcal{H}^1(j))\right)=\frac{q-1}{2}$ as long as $\frac{q-1}{2}\geq r_1$. So for $q=\max\{2r_0-1,2r_1+1\}+1$, Lemma 2.9 gives that

$$\begin{aligned}
\alpha\left(\widehat{\operatorname{Ext}}_R^{odd}(M,k)\right) &\leq \max\left\{\alpha\left(\bigoplus_{j\geq\frac{q-1}{2}} H^0(\mathbb{P}_k^{c-1},\mathcal{H}^1(j))\right),\alpha\left(\bigoplus_{i\leq\frac{q-1}{2}-1}\widehat{\operatorname{Ext}}_R^{2i+1}(M,k)\right)\right\}\\
&\leq \max\left\{\frac{q-1}{2},\frac{q-1}{2}-1\right\}\\
&= \frac{1}{2}\max\{2r_0-1,2r_1+1\}.
\end{aligned}$$

Since $\max\{2r_0-1,2r_1+1\}$ is an odd integer, we see that

$$\alpha\left(\widehat{\operatorname{Ext}}_R^{odd}(M,k)\right)\leq\frac{1}{2}\max\{2r_0-2,2r_1\}.$$

Note that if $r_0=r_1=-\infty$, the above argument shows that $\alpha\left(\widehat{\operatorname{Ext}}_R^{odd}(M,k)\right)\leq\frac{q-1}{2}$ for all odd q, hence $\alpha\left(\widehat{\operatorname{Ext}}_R^{odd}(M,k)\right)=-\infty$.

Since $\widehat{\operatorname{Ext}}_R^{*}(M,k)=\widehat{\operatorname{Ext}}_R^{ev}(M,k)\oplus\widehat{\operatorname{Ext}}_R^{odd}(M,k)$ such that $\widehat{\operatorname{Ext}}_R^{2i}(M,k)$ lies in degree $2i$ and $\widehat{\operatorname{Ext}}_R^{2i+1}(M,k)$ lies in degree $2i+1$, combining the above inequalities yields

$$\begin{aligned}
\alpha\left(\widehat{\operatorname{Ext}}_R^{*}(M,k)\right) &\leq \max\left\{2\alpha\left(\widehat{\operatorname{Ext}}_R^{ev}(M,k)\right),2\alpha\left(\widehat{\operatorname{Ext}}_R^{odd}(M,k)\right)+1\right\}\\
&\leq \max\{2r_0,2r_1,2r_0-1,2r_1+1\}\\
&= \max\{2r_0,2r_1+1\}
\end{aligned}$$

□

If M is an MCM R-module, then the natural map $\operatorname{Ext}_R^*(M,k)\to\widehat{\operatorname{Ext}}_R^{*}(M,k)$ is an isomorphism in all positive degrees and a surjection in degree 0. This map and Theorem 2.11 allow us to obtain an upper bound for the degrees of generators of $\operatorname{Ext}_R^*(M,k)$ when M

is an MCM R-module.

Corollary 2.12. *Let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection of codimension c and let M be an MCM R-module. Then*

$$\alpha\left(Ext_R^*(M,k)\right) \leq \max\{2r_0, 2r_1+1, 1\}.$$

The additional 1 in the bound in Corollary 2.12 ensures that the bound is positive, which is necessary since $\operatorname{Ext}_R^i(M,k) = 0$ for all $i < 0$.

2.3 Recovering the Avramov-Buchweitz Bound

In this section we specialize to the codimension $c = 2$ case; that is, $R = Q/(f_1, f_2)$. We use Theorem 2.11 to recover the bound obtained by Avramov and Buchweitz in Theorem 2.1.

We use the fact that every coherent sheaf $\mathcal{F}$ over $\mathbb{P}_k^1$ decomposes as

$$\mathcal{F} = \bigoplus_{i=1}^{n} \mathcal{O}(e_i) \oplus \mathcal{T},$$

where $e_i \in \mathbb{Z}$ and $\mathcal{T}$ is a *torsion sheaf*, i.e., a coherent sheaf whose support consists of a finite number of points. If $\mathcal{T}$ is a torsion sheaf, then there is a polynomial $P \in k[T_1, T_2]$ of degree d such that multiplication by P induces an isomorphism $\mathcal{T} \cong \mathcal{T}(d)$. If k is an infinite field, then one can find such a P with degree $d = 1$. The above decomposition is useful as we can readily compute the regularities of the various summands.

Lemma 2.13. *Let $e \in \mathbb{Z}$ and let $\mathcal{T}$ be a torsion sheaf on $\mathbb{P}_k^1$. Then $Reg(\mathcal{O}(e)) = -e$ and $Reg(\mathcal{T}) = -\infty$.*

Proof. For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$, we have $H^i(\mathbb{P}^1_k, \mathcal{F}) = 0$ for $i > 1$. Thus,

$$\operatorname{Reg}(\mathcal{F}) = \min\{r \mid H^1(\mathbb{P}^1_k, \mathcal{F}(r-1)) = 0\}.$$

For $\mathcal{F} = \mathcal{O}(e)$, Serre duality gives

$$H^1(\mathbb{P}^1_k, \mathcal{O}(e+r-1)) \cong H^0(\mathbb{P}^1_k, \mathcal{O}(-e-r-1))^*,$$

and this is 0 whenever $-e-r-1 \leq -1$, i.e., whenever $r \geq -e$. So

$$\operatorname{Reg}(\mathcal{O}(e)) = \min\{r \mid r \geq -e\} = -e.$$

For $\mathcal{F} = \mathcal{T}$, taking $r \ll 0$ of the form $r = ld + 1$ yields

$$H^1(\mathbb{P}^1_k, \mathcal{T}(r-1)) = H^1(\mathbb{P}^1_k, \mathcal{T}(ld)) = H^1(\mathbb{P}^1_k, \mathcal{T}) = 0,$$

thus $\operatorname{Reg}(\mathcal{T}) = -\infty$. $\square$

We now apply Theorem 2.11 to the codimension $c = 2$ and recover the Avramov-Buchweitz bound.

Corollary 2.14. *Let $R = Q/(f_1, f_2)$ be a complete intersection of codimension $c = 2$ and let M be a finitely generated R-module. Set $g := depth(R) - depth_R(M)$. Then the graded $k[T_1, T_2]$-module $Ext^*_R(M, k)$ satisfies*

$$\alpha(Ext^*_R(M,k)) \leq \max\{2\beta_g, 2\beta_{g+1} + 1\} + g + 1,$$

where β_i denotes the i^{th} Betti number of M over R.

Proof. As in the original proof in [2], by considering the g^{th} syzygy of M we may assume that $g = 0$. This means that M is an MCM R-module and that we need to show that $\alpha(\operatorname{Ext}^*_R(M,k)) \leq \max\{2\beta_0+1, 2\beta_1+2\}$. By Theorem 2.11, it suffices to show that

$$\max\{2r_0, 2r_1+1\} \leq \max\{2\beta_0+1, 2\beta_1+2\},$$

where $r_0 = \operatorname{Reg}(\mathcal{H}^0)$ and $r_1 = \operatorname{Reg}(\mathcal{H}^1)$. Note that if $r_0 = -\infty = r_1$, the result is trivial. If $r_0 = -\infty \neq r_1$, then our bound is exactly $2r_1+1$ and applying the first argument below will yield the result. Similarly, if $r_0 \neq -\infty = r_1$, then apply only the second argument below. So we may assume that $r_0, r_1 \neq -\infty$.

Write $\mathcal{H}^0 = \bigoplus \mathcal{O}(e_i) \oplus \mathcal{T}_0$ and $\mathcal{H}^1 = \bigoplus \mathcal{O}(f_j) \oplus \mathcal{T}_1$. By Lemma 2.13, we must have $e_i = -r_0$ and $f_j = -r_1$ for some i, j. From the spectral sequence arising from the Čech complex of $\mathbb{E}$, we obtain the short exact sequence

$$0 \to H^1(\mathbb{P}^1_k, \mathcal{H}^1(-1)) \to \widehat{\operatorname{Ext}}^0_R(M,k) \to H^0(\mathbb{P}^1_k, \mathcal{H}^0)) \to 0.$$

Since $\mathcal{O}(-r_1)$ is a direct summand of $\mathcal{H}^1$, we have

$$\begin{aligned}
\hat{\beta}_0 &:= \dim_k \widehat{\operatorname{Ext}}^0_R(M,k) \ \geq\ \dim_k H^1(\mathbb{P}^1_k, \mathcal{H}^1(-1)) \\
&\geq \dim_k H^1(\mathbb{P}^1_k, \mathcal{O}(-r_1-1)) \ =\ \dim_k H^0(\mathbb{P}^1_k, \mathcal{O}(r_1-1)) \\
&= \max\{r_1, 0\},
\end{aligned}$$

so $\hat{\beta}_0 \geq r_1$.

Similarly, we have the short exact sequence

$$0 \to H^1(\mathbb{P}^1_k, \mathcal{H}^0) \to \widehat{\operatorname{Ext}}^1_R(M,k) \to H^0(\mathbb{P}^1_k, \mathcal{H}^1)) \to 0.$$

Since $\mathcal{O}(-r_0)$ is a direct summand of $\mathcal{H}^0$, we have

$$\begin{aligned}\hat{\beta}_1 &:= \dim_k \widehat{\operatorname{Ext}}^1_R(M,k) \;\geq\; \dim_k H^1(\mathbb{P}^1_k, \mathcal{H}^0) \\ &\geq \dim_k H^1(\mathbb{P}^1_k, \mathcal{O}(-r_0)) \;=\; \dim_k H^0(\mathbb{P}^1_k, \mathcal{O}(r_0-2)) \\ &= \max\{r_0-1, 0\},\end{aligned}$$

so $\hat{\beta}_1 \geq r_0 - 1$.

The above inequalities yield $\max\{2r_0, 2r_1+1\} \leq \max\{2\hat{\beta}_0+1, 2\hat{\beta}_1+2\}$. Recall that the map $\operatorname{Ext}^*_R(M,k) \to \widehat{\operatorname{Ext}}^*_R(M,k)$ is an isomorphism in positive degrees and a surjection in degree 0. Thus, $\hat{\beta}_1 = \beta_1$ and $\hat{\beta}_0 \leq \beta_0$, and the result follows. □

Corollary 2.14 suggests a natural question regarding the existence of a module M in the codimension $c = 2$ case where the regularity bound from Theorem 2.11 is strictly smaller than the Avramov-Buchweitz bound. This is the subject of future pursuit.

Chapter 3

A DG Approach to $\mathrm{Ext}_R^*(M, N)$ Over Complete Intersections

Let (Q, m, k) be a regular local ring and let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection. Define the differential graded R-module $\mathbf{R}\operatorname{Hom}_R^{dg}(M, N) := \operatorname{Hom}_R(F, G)$ and differential graded R-algebra $\mathbf{R}\operatorname{End}_R^{dg}(M) := \operatorname{Hom}_R(F, F)$, where F and G are R-free resolutions of the finitely generated R-modules M and N, respectively. The goal of this chapter is to describe these objects as explicitly as possible. To do this, we use a correspondence between R-modules and differential graded modules over the Koszul algebra of R over Q. We obtain a description of $\mathbf{R}\operatorname{Hom}_R^{dg}(M, N)$ in terms of well-known differential graded Q-modules, which has the structure of a differential graded algebra over Q when $M = N$. In the special case where $M = Q/I$ with I generated by a Q-regular sequence, we obtain explicit generators and relations for the DGA $\mathbf{R}\operatorname{End}_R^{dg}(M)$ over Q. In particular, if $M = k$ is the residue field of R, this description allows us to compute cohomology and recover the following result of Sjödin from [18]:

Theorem 3.1 (Sjödin). *Let (Q, m, k) be a regular local ring and let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection with $f_j \in m^2$ for $1 \leq j \leq n$. Suppose m is minimally generated as*

$m = (x_1, \ldots, x_n)$ and write

$$f_j = \sum_{1 \le h \le i \le n} w_{hij} x_h x_i \pmod{m^3}$$

for $w_{hij} \in Q$. Then $Ext^*_R(k,k)$ is generated as a k-algebra by elements $Z_1, \ldots, Z_n$ of degree 1 and $T_1, \ldots, T_c$ of degree 2 subject to the relations

$$Z_i^2 + \sum_{j=1}^{c} \overline{w_{iij}} T_j, \quad [Z_h, Z_i] + \sum_{j=1}^{c} \overline{w_{hij}} T_j, \quad [Z_h, T_i], \quad [T_h, T_i]$$

where $\overline{w_{hij}}$ is the image of w_{hij} in k.

3.1 Differential Graded Algebras and Modules

This section provides a review of the basic concepts and terminology of differential graded algebras and modules found in [1]. We also describe a few basic examples and fix notation which will be used throughout the rest of the chapter.

Definition 3.2. Let Q be a commutative ring. A *differential graded algebra (DGA) over Q* is a graded Q-algebra A equipped with an endomorphism $d_A : A \to A$ of degree 1 satisfying $d_A^2 = 0$ and the Leibniz rule

$$d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b)$$

for all homogeneous elements $a, b \in A$. We say A is *(graded) commutative* if $ab = (-1)^{|a|\cdot|b|} ba$ for all homogeneous $a, b \in A$ and $a^2 = 0$ whenever $|a|$ is odd. A *differential graded (DG) module M over A* is a graded A-module with an endomorphism $d_M : M \to M$ of degree 1 satisfying $d_M^2 = 0$ and the Leibniz rule

$$d_M(am) = d_A(a)m + (-1)^{|a|} a d_M(m)$$

for all homogeneous $a \in A$ and $m \in M$.

We denote the underlying graded algebra of a DGA A by $A^\natural$. Similarly, given a DG A-module M, we denote the underlying graded $A^\natural$-module by $M^\natural$.

Any commutative ring Q concentrated in degree 0 is trivially a DGA with differential $d_Q = 0$. In this case, a DG Q-module is simply a complex of Q-modules. A less trivial example of a DGA is the Koszul algebra $K = \mathrm{Kos}(q_1, \dots, q_r;\ Q)$ over a commutative ring Q, with $\Lambda_Q^i Q^r$ in cohomological degree $-i$ and differential d_K defined by

$$d_K(c_{i_1} \wedge \cdots \wedge c_{i_n}) = \sum_{j=1}^{n} (-1)^{j-1} q_{i_j} c_{i_1} \wedge \cdots \wedge c_{i_{j-1}} \wedge c_{i_{j+1}} \wedge \cdots \wedge c_{i_n},$$

where $c_1, \dots, c_r$ is the canonical basis for Q^r. In this case, a DG K-module can be realized as a complex E of Q-modules equipped with endomorphisms $s_1, \dots, s_r : E \to E$ of degree -1 which satisfy the following properties:

(1) $d_K s_i + s_i d_K = q_i$ for all $1 \le i \le r$

(2) $s_i s_j + s_j s_i = 0$ for all $1 \le i, j \le r$

(3) $s_i^2 = 0$ for all $1 \le i \le r$

We will frequently use the following constructions, so we consider them carefully here.

Example 3.3. Let A be a commutative DGA over Q and let M and N be DG A-modules. We can define DG A-modules $M \otimes_A N$ and $\mathrm{Hom}_A(M, N)$.

We start by letting $(M \otimes_A N)^\natural$ be the quotient of $M \otimes_Q N$ by the submodule spanned by elements of the form $(am) \otimes n - (-1)^{|a|\cdot|m|} m \otimes (an)$. Then $M \otimes_A N$ is a DG A-module with A-action defined by

$$a(m \otimes n) := (am) \otimes n = (-1)^{|a|\cdot|m|} m \otimes (an)$$

and differential defined by

$$d_{M \otimes N}(m \otimes n) := d_M(m) \otimes n + (-1)^{|m|} m \otimes d_N(n).$$

Moreover, if M and N are DGAs over A, then $M \otimes_A N$ has the structure of a DGA over A with multiplication defined on homogeneous elements by

$$(m_1 \otimes n_1)(m_2 \otimes n_2) := (-1)^{|n_1|\cdot|m_2|}(m_1 m_2) \otimes (n_1 n_2).$$

To define the DG A-module $\mathrm{Hom}_A(M, N)$, we start by defining $\mathrm{Hom}_A(M, N)^\natural$ as the collection of Q-linear maps φ of graded modules such that $\varphi(am) = (-1)^{|\varphi|\cdot|a|} a\varphi(m)$ for all homogeneous $a \in A$ and $m \in M$. Then $\mathrm{Hom}_A(M, N)$ is a DG A-module with A-action defined by

$$(a\varphi)(m) := a\varphi(m) = (-1)^{|\varphi|\cdot|a|}\varphi(am)$$

and differential defined by

$$d_{\mathrm{Hom}}(\varphi) := d_N \circ \varphi - (-1)^{|\varphi|} \varphi \circ d_M.$$

If $M = N$, then $\mathrm{Hom}_A(M, M)$ has the structure of a DGA over A with multiplication given by function composition. We will often denote this DGA by $\mathrm{End}_A(M)$ and its differential by d_{End}.

We will frequently use quasi-isomorphisms of DGAs and DG modules. We recall these fundamental notions here. Note that we will discuss a more general notion of quasi-isomorphic DGAs in section 3.2. The following formulation is sufficient for our current needs.

Definition 3.4. Let A and B be DGAs over a commutative ring Q and let M and N be DG A-modules. A map $\varphi : A \to B$ is a *quasi-isomorphism of DGAs* if it is a homomorphism of DGAs which induces an isomorphism in cohomology. A map $\epsilon : M \to N$ is a *quasi-isomorphism of DG A-modules* if it is a homomorphism of DG A-modules which induces an isomorphism in cohomology.

We also recall the following definition given in [1]:

Definition 3.5. Let A be a commutative DGA and M a bounded below DG A-module, i.e., $M^i = 0$ for $i \gg 0$. Then M is *semi-free* if $M^\natural$ is a free $A^\natural$-module.

Semi-free DG A-modules are not free modules in the category of DG A-modules. However, they have the following basic properties, which are sufficient for our needs. Proofs of these properties can be found in [1].

Lemma 3.6. *Let A be a commutative DGA, M a DG A-module, and $\epsilon : N \to N'$ a quasi-isomorphism of DG A-modules.*

1. *If N and N' are semi-free DG A-modules, then the map $M \otimes_A \epsilon : M \otimes_A N \to M \otimes_A N'$ is a quasi-isomorphism.*

2. *If M is a semi-free DG A-module, then $\operatorname{Hom}_A(M, \epsilon) : \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(M, N')$ is a quasi-isomorphism.*

We now set forth notation which will be used throughout the remainder of the chapter. Let $R = Q/(f_1, \ldots, f_c)$ with (Q, m, k) a regular local ring and $f_1, \ldots, f_c$ a Q-regular sequence contained in m^2. Set $V := Q^c$ with canonical basis $e_1, \ldots, e_c$. Let K denote the Koszul algebra $K := \operatorname{Kos}(f_1, \ldots, f_c;\ Q)$, which is a DGA over Q with differential d_K described above. Since $f_1, \ldots, f_c$ is a regular sequence, the canonical surjection $K \twoheadrightarrow R$ is a quasi-isomorphism of DGAs over Q.

Given a graded Q-module N, set $N^* := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_Q(N^{-i}, Q)$ to be the graded dual. Let $T_1, \ldots, T_c$ denote the basis of V^* which is dual to the basis $e_1, \ldots, e_c$ of V, i.e., $T_i(e_j) = \delta_{ij}$.

We will consider the following three DG Q-modules throughout the chapter:

- $\Lambda := \Lambda_Q^\bullet(V)$, the exterior algebra on V, with $|e_i| = -1$ and trivial differential
- $S := \operatorname{Sym}_Q^\bullet(V^*)$, the symmetric algebra on V^*, with $|T_i| = 2$ and trivial differential
- $\Gamma := S^*$, with trivial differential

We make the following observations about these DG Q-modules and will frequently use them without explicitly stating so.

First, $K^\natural = \Lambda^\natural$. Nonzero homogeneous elements of S and Γ always have even degree. Thus, these elements commute in the usual sense. Since S is graded commutative, for every DG S-module M, the dual M^* has a canonical DG S-module structure. Thus, Γ is a DG S-module as it is the dual of the DG S-module S.

Definition 3.7. Let M be a finitely generated R-module. Then P is a *Koszul resolution* of M if it is a degree-wise finitely generated Q-projective resolution of M equipped with the structure of a DG K-module.

Koszul resolutions exist for any finitely generated R-module M and a construction can be found in [2, Section 2.1].

For a Koszul resolution P of M, set $X^\natural = X_P^\natural := \Lambda^\natural \otimes_Q \Gamma^\natural \otimes_Q P^\natural$ and observe that $X^\natural$ is a graded $K^\natural$-module with $K^\natural$-action defined by multiplication on $\Lambda^\natural$. Note that there is a second $K^\natural$-action defined by the action of P, but we will only need the action on the leftmost factor. We wish to equip $X^\natural$ with a differential d_X under which X is quasi-isomorphic to P as DG K-modules. We need the following result of Avramov and Buchweitz from [2], based on the work of Cartan [9].

Proposition 3.8 (Avramov-Buchweitz)**.** *Consider the graded K-module $L^\natural := \Lambda^\natural \otimes_Q \Gamma^\natural \otimes_Q \Lambda^\natural$ with K-action defined by multiplication on the leftmost factor of $\Lambda^\natural$. Define $d_L : L^\natural \to L^\natural$ by*

$$d_L(\lambda_1 \otimes \gamma \otimes \lambda_2) := d_K(\lambda_1) \otimes \gamma \otimes \lambda_2 + (-1)^{|\lambda_1|}\lambda_1 \otimes \gamma \otimes d_K(\lambda_2) - \delta_L(\lambda_1 \otimes \gamma \otimes \lambda_2),$$

where $\delta_L(\lambda_1 \otimes \gamma \otimes \lambda_2) := \sum_{j=1}^{c} e_j\lambda_1 \otimes T_j\gamma \otimes \lambda_2 - (-1)^{|\lambda_1|}\lambda_1 \otimes T_j\gamma \otimes e_j\lambda_2$. Then L equipped with d_L is a DG K-module which is quasi-isomorphic to K via the map $\epsilon : L \to K$ defined by

$$\epsilon(\lambda_1 \otimes \gamma \otimes \lambda_2) := \lambda_1 \pi_0(\gamma)\lambda_2,$$

where π_0 denotes projection onto the degree 0 component of Γ.

To obtain the desired differential on $X^\natural$, we tensor L with the Koszul resolution P.

Lemma 3.9. *For $X^\natural = \Lambda^\natural \otimes_Q \Gamma^\natural \otimes_Q P^\natural$, define an endomorphism $d_X : X^\natural \to X^\natural$ by*

$$d_X(\lambda \otimes \gamma \otimes p) := d_K(\lambda) \otimes \gamma \otimes p + (-1)^{|\lambda|}\lambda \otimes \gamma \otimes d_P(p) - \delta_X(\lambda \otimes \gamma \otimes p),$$

where $\delta_X(\lambda \otimes \gamma \otimes p) := \sum_{j=1}^{c} e_j\lambda \otimes T_j\gamma \otimes p - (-1)^{|\lambda|}\lambda \otimes T_j\gamma \otimes e_j p$. Then $X^\natural$ equipped with d_X is a DG K-module which is quasi-isomorphic to P via the map $\eta : X \to P$ defined by

$$\eta(\lambda \otimes \gamma \otimes p) := \lambda \pi_0(\gamma)p,$$

where π_0 denotes projection onto the degree 0 component of Γ.

Proof. We consider $L^\natural \otimes_{K^\natural} P^\natural$ as a $K^\natural$-module via the action on L. There is a canonical isomorphism of $K^\natural$-modules $\mu : L^\natural \otimes_{K^\natural} P^\natural \to X^\natural$ given by

$$\mu(\lambda_1 \otimes \gamma \otimes \lambda_2 \otimes p) = \lambda_1 \otimes \gamma \otimes \lambda_2 p.$$

Observe that $L^\natural \otimes_{K^\natural} P^\natural$ has an induced differential as in Example 3.3, given by

$$d_{L\otimes_K P}(\lambda_1 \otimes \gamma \otimes \lambda_2 \otimes p) = d_L(\lambda_1 \otimes \gamma \otimes \lambda_2) \otimes p + (-1)^{|\lambda_1|+|\lambda_2|}\lambda_1 \otimes \gamma \otimes \lambda_2 \otimes d_P(p).$$

We will show that $d_X\mu = \mu d_{L\otimes_K P}$, i.e., that d_X corresponds with the canonical differential on $L \otimes_K P$ under the isomorphism μ.

To this end, we observe that

$$\begin{aligned}
\mu d_{L\otimes_K P}(\lambda_1 \otimes \gamma \otimes \lambda_2 \otimes p) &= d_K(\lambda_1) \otimes \gamma \otimes \lambda_2 p + (-1)^{|\lambda_1|}\lambda_1 \otimes \gamma \otimes d_K(\lambda_2)p \\
&\quad - \sum_{j=1}^{c} e_j\lambda_1 \otimes T_j\gamma \otimes \lambda_2 p - (-1)^{|\lambda_1|}\lambda_1 \otimes T_j\gamma \otimes e_j\lambda_2 p \\
&\quad + (-1)^{|\lambda_1|+|\lambda_2|}\lambda_1 \otimes \gamma \otimes \lambda_2 d_P(p) \\
&= d_K(\lambda_1) \otimes \gamma \otimes \lambda_2 p + (-1)^{|\lambda_1|}\lambda_1 \otimes \gamma \otimes d_P(\lambda_2 p) \\
&\quad - \sum_{j=1}^{c} e_j\lambda_1 \otimes T_j\gamma \otimes \lambda_2 p - (-1)^{|\lambda_1|}\lambda_1 \otimes T_j\gamma \otimes e_j\lambda_2 p \\
&= d_X(\lambda_1 \otimes \gamma \otimes \lambda_2 p) \\
&= d_X\mu(\lambda_1 \otimes \gamma \otimes \lambda_2 \otimes p)
\end{aligned}$$

This verifies that d_X is a differential on $X^\natural$ and that $X \cong L \otimes_K P$ as DG K-modules.

The DG K-module $L = \Lambda \otimes_Q \Gamma \otimes_Q \Lambda$ is a semi-free DG K-module, as $\Gamma^\natural$ and $\Lambda^\natural$ are both bounded below free Q-modules, so that $L^\natural = \Lambda^\natural \otimes_Q \Gamma^\natural \otimes_Q \Lambda^\natural$ is a bounded below free $K^\natural$-module. Since $\epsilon : L \xrightarrow{\sim} K$ is a quasi-isomorphism of semi-free DG K-modules by

Proposition 3.8,

$$X \cong L \otimes_K P \xrightarrow{\epsilon \otimes P} K \otimes_K P \cong P$$

is also a quasi-isomorphism of DG K-modules, by Lemma 3.6(1). Using the definition of ϵ in Proposition 3.8, we see that $\eta = \epsilon \otimes P$. $\square$

Let M and N be finitely generated R-modules with Koszul resolutions P and P', respectively. Define $X^\natural := \Lambda^\natural \otimes_Q \Gamma^\natural \otimes_Q P^\natural$ and $(X')^\natural := \Lambda^\natural \otimes_Q \Gamma^\natural \otimes_Q (P')^\natural$, and equip them with differentials d_X and $d_{X'}$ as defined in Lemma 3.9. Note that X and X' are semi-free DG K-modules, since $\Gamma^\natural$ and $P^\natural$ are both bounded below free Q-modules.

3.2 Defining $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ and $\mathbf{R}\operatorname{End}_R^{dg}(M)$

In this section we let (Q, m, k) be a regular local ring and let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection. Let M and N be finitely generated R-modules. At the beginning of the chapter we informally introduced $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ and $\mathbf{R}\operatorname{End}_R^{dg}(M)$. We recall these definitions here.

Definition 3.10. Let R be a complete intersection and let M and N be finitely generated R-modules. Let F and G be free resolutions of M and N, respectively, over R. Define $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N) := \operatorname{Hom}_R(F,G)$ and $\mathbf{R}\operatorname{End}_R^{dg}(M) := \operatorname{Hom}_R(F,F)$.

Observe that $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ is a DG R-module, or equivalently a complex of R-modules, and that $\mathbf{R}\operatorname{End}_R^{dg}(M)$ is a DGA over R. The goal of this section is to show that these definitions of $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ and $\mathbf{R}\operatorname{End}_R^{dg}(M)$ are independent of choices of free resolutions up to homotopy equivalence and quasi-isomorphisms of DGAs over R, respectively. This is well-known but worth mentioning explicitly. We also describe a correspondence between R-modules and DG K-modules, where K is the Koszul algebra $K = \operatorname{Kos}(f_1, \ldots, f_c;\ Q)$.

The definition of $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ requires choices of free resolutions M and N. The next Lemma shows that our definition is independent of these choices up to homotopy equivalence.

Lemma 3.11. *The DG R-module* $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$ *is well-defined up to homotopy equivalence.*

Proof. Let F, F' be R-free resolutions of M and G, G' be R-free resolutions of N. Since any two free resolutions are homotopy equivalent, there exists chain maps $\alpha : F \to F'$ and $\alpha' : F' \to F$ such that $\alpha'\alpha \sim \mathrm{id}_F$ and $\alpha\alpha' \sim \mathrm{id}_{F'}$. Similarly, there are chain maps $\beta : G \to G'$ and $\beta' : G' \to G$ such that $\beta'\beta \sim \mathrm{id}_G$ and $\beta\beta' \sim \mathrm{id}_{G'}$. We define maps $g : \operatorname{Hom}_R(F,G) \to \operatorname{Hom}_R(F',G')$ and $h : \operatorname{Hom}_R(F',G') \to \operatorname{Hom}_R(F,G)$ by $g(\theta) := \beta\theta\alpha'$ and $h(\psi) := \beta'\psi\alpha$, respectively. Since $hg(\theta) = \beta'\beta\theta\alpha'\alpha$ with $\beta'\beta \sim \mathrm{id}_G$ and $\alpha'\alpha \sim \mathrm{id}_F$, we have that $hg \sim \mathrm{id}_{\operatorname{Hom}_R(F,G)}$. Similarly, since $gh(\psi) = \beta\beta'\psi\alpha\alpha'$ with $\beta\beta' \sim \mathrm{id}_{G'}$ and $\alpha\alpha' \sim \mathrm{id}_{F'}$, we have that $gh \sim \mathrm{id}_{\operatorname{Hom}_R(F',G')}$. Thus, $\operatorname{Hom}_R(F,G)$ and $\operatorname{Hom}_R(F',G')$ are homotopy equivalent. $\square$

We now turn our attention to $\mathbf{R}\operatorname{End}_R^{dg}(M)$. Unlike $\mathbf{R}\operatorname{Hom}_R^{dg}(M,N)$, this has the structure of a DGA over R. The definition still involves the choice of free resolution, but we show that it is independent of this choice up to quasi-isomorphism of DGAs over R. We now need the more general notion alluded to in the comment preceding Definition 3.4.

Definition 3.12. Let A and B be DGAs over R. Then A and B are *quasi-isomorphic* (as DGAs) if there exists a diagram

$$A = A_0 \xrightarrow{\alpha_0} A_1 \xleftarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} A_{n-1} \xleftarrow{\alpha_{n-1}} A_n = B$$

such that each A_i is a DGA over R and each α_i is a homomorphism of DGAs over R which induces an isomorphism on cohomology.

Note that α_0 or α_{n-1} could be the identity map, so similar diagrams with the maps on the ends reversed would also yield valid quasi-isomorphisms of DGAs.

Lemma 3.13. *The DGA* $\mathbf{R}\operatorname{End}_R^{dg}(M)$ *is well defined up to quasi-isomorphisms of DGAs over* R*.*

Proof. Let F and F' be any R-free resolutions of M and let G be the minimal free resolution of M. By minimality of G, there are chain maps $\alpha : F \to G$ and $\beta : G \to F$ such that $\alpha\beta = \mathrm{id}_G$. Define $g : \operatorname{Hom}_R(G, G) \to \operatorname{Hom}_R(F, F)$ by $g(\theta) := \beta\theta\alpha$. The proof of Lemma 3.11 shows that g is a homotopy equivalence, and hence a quasi-isomorphism of DG R-modules. For any $\theta, \theta' \in \operatorname{Hom}_R(G, G)$, observe that

$$g(\theta\theta') = \beta\theta\theta'\alpha = \beta\theta\alpha\beta\theta'\alpha = g(\theta)g(\theta'),$$

since $\alpha\beta = \mathrm{id}_G$. Thus, g is a quasi-isomorphism of DGAs. Applying the above argument to F' results in a quasi-isomorphism of DGAs $h : \operatorname{Hom}_R(G, G) \to \operatorname{Hom}_R(F', F')$. This produces a diagram

$$\operatorname{Hom}_R(F, F) \xleftarrow{g} \operatorname{Hom}_R(G, G) \xrightarrow{h} \operatorname{Hom}_R(F', F').$$

So $\operatorname{Hom}_R(F, F)$ and $\operatorname{Hom}_R(F', F')$ are quasi-isomorphic as DGAs. $\square$

We now explore the correspondence between R-modules and DG K-modules up to quasi-isomorphism. Any R-module becomes a DG K-module by restriction of scalars along the quasi-isomorphism $K \xrightarrow{\sim} R$. Conversely, given a DG K-module E we obtain an R-module $E \otimes_K^{\mathbf{L}} R$.

Let P and P' be Koszul resolutions for M and N, respectively, and let $X = X_P$ and $X' = X_{P'}$ be the semi-free DG K-modules defined in Lemma 3.9. Since $K \xrightarrow{\sim} R$ and X' is semi-free, we get a quasi-isomorphism $X' \xrightarrow{\sim} X' \otimes_K R$ of DG K-modules by Lemma 3.6(1).

By Lemma 3.6(2) and hom-tensor adjointness, we get a quasi-isomorphism

$$\phi : \operatorname{Hom}_K(X, X') \xrightarrow{\sim} \operatorname{Hom}_K(X, X' \otimes_K R) \cong \operatorname{Hom}_R(X \otimes_K R, X' \otimes_K R)$$

of DG K-modules. For $\tau \in \operatorname{Hom}_K(X, X')$, we see that $\phi(\tau)(x \otimes r) = \tau(x) \otimes r$. Note that

$$X \otimes_K R = \Lambda \otimes_Q \Gamma \otimes_Q P \otimes_K R \cong \Gamma \otimes_Q P \otimes_Q R,$$

which is a free resolution of M over R by [2, Theorem 2.4]. Thus, $X \otimes_K R$ and $X' \otimes_K R$ are R-free resolutions of M and N, respectively, so we have a quasi-isomorphism of DG K-modules

$$\phi : \operatorname{Hom}_K(X, X') \xrightarrow{\sim} \mathbf{R}\operatorname{Hom}_R^{dg}(M, N).$$

When $M = N$, we may choose $P = P'$ and hence $X = X'$. Thus, we have a quasi-isomorphism of DG K-modules

$$\phi : \operatorname{Hom}_K(X, X) \xrightarrow{\sim} \operatorname{Hom}_R(X \otimes_K R, X \otimes_K R) = \mathbf{R}\operatorname{End}_R^{dg}(M).$$

For $\tau, \tau' \in \operatorname{Hom}_K(X, X)$, we have

$$\phi(\tau\tau')(x \otimes r) = (\tau\tau'(x) \otimes r) = \phi(\tau)(\tau'(x) \otimes r) = \phi(\tau)\phi(\tau')(x \otimes r),$$

so ϕ preserves compositions and thus is a quasi-isomorphism of DGAs over Q. In particular, $H^*(\operatorname{Hom}_K(X, X')) = \operatorname{Ext}_R^*(M, N)$ and $H^*(\operatorname{Hom}_K(X, X)) = \operatorname{Ext}_R^*(M, M)$. Henceforth, we will set out to explicitly describe the DG K-module $\operatorname{Hom}_K(X, X')$ and the DGA $\operatorname{Hom}_K(X, X)$ over Q, up to quasi-isomorphism.

3.3 The DG Module for $\mathbf{Ext}_R^*(M, N)$

Fix finitely generated R-modules M and N with Koszul resolutions P and P', respectively. Let $X = X_P$ and $X' = X_{P'}$ be the DG K-modules defined in Lemma 3.9.

The goal of this section is to define a DG K-module structure on the graded Q-module $S \otimes_Q \operatorname{Hom}_Q(P, P')$ and define a quasi-isomorphism $\widetilde{\Psi} : S \otimes_Q \operatorname{Hom}_Q(P, P') \to \operatorname{Hom}_K(X, X')$ of DG K-modules. This will give an explicit model of $\mathbf{R} \operatorname{Hom}_R^{dg}(M, N)$ up to quasi-isomorphism of DG K-modules.

We first show that $S \otimes_Q \operatorname{Hom}_Q(P, P')$ is isomorphic to $\operatorname{Hom}_K(X, P')$ as graded Q-modules. To do this, we need the following canonical isomorphisms of modules.

Remark 3.14.

1. Let A be a commutative ring and let L, M, and N be graded A-modules. If L and M are projective and L is degree-wise finitely generated, then there is a canonical isomorphism of graded A-modules

$$\psi : L \otimes_A \operatorname{Hom}_A(M, N) \to \operatorname{Hom}_A(L^* \otimes_A M, N)$$

defined by $\psi(l \otimes \chi)(\xi \otimes m) := l(\xi)\chi(m)$, where $l(\xi)$ is the natural action of $l \in L$ on $\xi \in L^* = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_A(L^{-i}, A)$.

2. If A and B are graded commutative rings, $A \to B$ a graded ring homomorphism, M a graded A-module, and N a graded B-module, then there is a canonical isomorphism of graded modules $\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_B(B \otimes_A M, N)$ defined by $\phi \mapsto \widetilde{\phi}$, where

$$\widetilde{\phi}(b \otimes m) := (-1)^{|\phi| \cdot |b|} b\phi(m).$$

To avoid being pedantic and simplify notation, we omit the $\natural$ notation in the statements and proofs of the following three lemmas.

Lemma 3.15. *There is an isomorphism of graded Q-modules*

$$\Psi : S \otimes_Q \mathrm{Hom}_Q(P, P') \to \mathrm{Hom}_K(X, P')$$

given by the following: for each $\alpha \otimes \beta \in S \otimes_Q \mathrm{Hom}_Q(P, P')$, we define

$$\Psi(\alpha \otimes \beta)(\lambda \otimes \gamma \otimes p) := (-1)^{|\beta|\cdot|\lambda|}\lambda\alpha(\gamma)\beta(p).$$

Proof. Since S and P are Q-projective, S is degree-wise finitely generated, and $\Gamma = S^*$, Remark 3.14(1) gives an isomorphism of graded Q-modules

$$\Psi_1 : S \otimes_Q \mathrm{Hom}_Q(P, P') \to \mathrm{Hom}_Q(\Gamma \otimes_Q P, P')$$

defined by

$$\Psi_1(\alpha \otimes \beta)(\gamma \otimes p) := \alpha(\gamma)\beta(p).$$

Since $K^\natural = \Lambda^\natural$, Remark 3.14(2) gives an isomorphism of graded Q-modules

$$\Psi_2 : \mathrm{Hom}_Q(\Gamma \otimes_Q P, P') \to \mathrm{Hom}_K(X, P')$$

defined by

$$\Psi_2(\phi)(\lambda \otimes \gamma \otimes p) := (-1)^{|\phi|\cdot|\lambda|}\lambda\phi(\gamma \otimes p).$$

Notice that $|\Psi_1(\alpha \otimes \beta)| = |\alpha| + |\beta| \equiv |\beta| \pmod 2$ since $|\alpha|$ is even. Thus, $(-1)^{|\Psi_1(\alpha\otimes\beta)|} = (-1)^{|\beta|}$. Therefore, $\Psi = \Psi_2\Psi_1$ and is an isomorphism of graded Q-modules. $\square$

We now define a K-action and differential on $S \otimes_Q \mathrm{Hom}_Q(P, P)$ under which Ψ is an isomorphism of DG K-modules. We use Ψ and the canonical DG K-module structure on $\mathrm{Hom}_K(X, P')$ to define these.

Lemma 3.16. *The graded Q-module $S \otimes_Q \operatorname{Hom}_Q(P, P')$ has a K-module structure with K-action defined by*

$$e_j(\alpha \otimes \beta) := \alpha \otimes (e_j\beta),$$

where $(e_j\beta)(p) := e_j\beta(p)$ is given by the K-action on P'. Under this action,

$$\Psi : S \otimes_Q \operatorname{Hom}_Q(P, P') \to \operatorname{Hom}_K(X, P')$$

is an isomorphism of graded K-modules.

Proof. By Lemma 3.15, the Q-module $S \otimes_Q \operatorname{Hom}_Q(P, P')$ is isomorphic to $\operatorname{Hom}_K(X, P')$, which has a K-module structure given by the action

$$(e_j\phi)(\lambda \otimes \gamma \otimes p) := e_j\phi(\lambda \otimes \gamma \otimes p) = (-1)^{|\phi|}\phi(e_j\lambda \otimes \gamma \otimes p).$$

We will show that the desired K-action on $S \otimes_Q \operatorname{Hom}_Q(P, P')$ corresponds with this K-action on $\operatorname{Hom}_K(X, P')$ under the isomorphism Ψ; that is, $e_j\Psi(\alpha \otimes \beta) = \Psi(e_j(\alpha \otimes \beta))$. We have that

$$\begin{aligned}
(e_j\Psi(\alpha \otimes \beta))(\lambda \otimes \gamma \otimes p) &= e_j(-1)^{|\beta|\cdot|\lambda|}\lambda\alpha(\gamma)\beta(p) \\
&= (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\alpha(\gamma)e_j\beta(p) \\
&= (-1)^{(|\beta|+1)\cdot|\lambda|}\lambda\alpha(\gamma)e_j\beta(p) \\
&= \Psi(\alpha \otimes (e_j\beta))(\lambda \otimes \gamma \otimes p) \\
&= \Psi(e_j(\alpha \otimes \beta))(\lambda \otimes \gamma \otimes p)
\end{aligned}$$

Thus, Ψ is an isomorphism of K-modules. $\square$

Lemma 3.17. *Define an endomorphism d on the graded K-module $S \otimes_Q \operatorname{Hom}_Q(P, P')$ by*

$$d(\alpha \otimes \beta) := \alpha \otimes d_{\operatorname{Hom}}(\beta) + \sum_{j=1}^{c} \alpha T_j \otimes [e_j, \beta],$$

where $[e_j, \beta] = e_j\beta - (-1)^{|\beta|}\beta e_j$. Then $S \otimes_Q \operatorname{Hom}_Q(P, P')$ equipped with d is a DG K-module and

$$\Psi : S \otimes_Q \operatorname{Hom}_Q(P, P') \to \operatorname{Hom}_K(X, P')$$

is an isomorphism of DG K-modules.

Remark 3.18. We are abusing the notation $[e_j, \beta]$ here. In the term $e_j\beta$ the e_j denotes the K-action on the DG K-module P', whereas in the term βe_j the e_j denotes the K-action on the DG K-module P. If $P = P'$, then $[e_j, \beta]$ would be the genuine commutator.

Proof. The canonical differential on $\operatorname{Hom}_K(X, P')$ is given by

$$d_{\operatorname{Hom}}(\phi) = d_{P'}\phi - (-1)^{|\phi|}\phi d_X.$$

It suffices to compute $d_{\operatorname{Hom}}(\Psi(\alpha \otimes \beta))$ and show that this corresponds with $d(\alpha \otimes \beta)$ under the isomorphism Ψ.

To this end, we have

$$\begin{aligned} d_{P'}\Psi(\alpha \otimes \beta)(\lambda \otimes \gamma \otimes p) &= d_{P'}\left((-1)^{|\beta|\cdot|\lambda|}\lambda\alpha(\gamma)\beta(p)\right) \\ &= (-1)^{|\beta|\cdot|\lambda|}\left(d_K(\lambda)\alpha(\gamma)\beta(p) + (-1)^{|\lambda|}\lambda\alpha(\gamma)d_{P'}(\beta(p))\right) \\ &= (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\alpha(\gamma)\beta(p) + (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\alpha(\gamma)d_{P'}(\beta(p)) \end{aligned}$$

and

$$\begin{aligned}
\Psi(\alpha\otimes\beta)d_X(\lambda\otimes\gamma\otimes p) &= \Psi(\alpha\otimes\beta)\left(d_K(\lambda)\otimes\gamma\otimes p\right) + (-1)^{|\lambda|}\Psi(\alpha\otimes\beta)\left(\lambda\otimes\gamma\otimes d_P(p)\right) \\
&- \Psi(\alpha\otimes\beta)\left(\sum_{j=1}^{c} e_j\lambda\otimes T_j\gamma\otimes p - (-1)^{|\lambda|}\lambda\otimes T_j\gamma\otimes e_jp\right) \\
&= (-1)^{|\beta|(|\lambda|+1)}d_K(\lambda)\alpha(\gamma)\beta(p) + (-1)^{|\lambda|+|\beta|\cdot|\lambda|}\lambda\alpha(\gamma)\beta(d_P(p)) \\
&\quad -\sum_{j=1}^{c}(-1)^{|\beta|(|\lambda|-1)}e_j\lambda\alpha(T_j\gamma)\beta(p) - (-1)^{|\lambda|+|\beta|\cdot|\lambda|}\lambda\alpha(T_j\gamma)\beta(e_jp) \\
&= (-1)^{|\beta|\cdot|\lambda|+|\beta|}d_K(\lambda)\alpha(\gamma)\cdot\beta(p) + (-1)^{|\lambda|+|\beta|\cdot|\lambda|}\lambda\alpha(\gamma)\cdot\beta(d_P(p)) \\
&-\sum_{j=1}^{c}(-1)^{|\beta|(|\lambda|-1)+|\lambda|}\lambda\alpha(T_j\gamma)e_j\beta(p) - (-1)^{|\lambda|+|\beta|\cdot|\lambda|}\lambda\alpha(T_j\gamma)\beta(e_jp)
\end{aligned}$$

Combining these, we have

$$\begin{aligned}
d_{\mathrm{Hom}}(\Psi(\alpha\otimes\beta))(\lambda\otimes\gamma\otimes p) &= d_{P'}\Psi(\alpha\otimes\beta)(\lambda\otimes\gamma\otimes p) - (-1)^{|\beta|}\Psi(\alpha\otimes\beta)d_X(\lambda\otimes\gamma\otimes p) \\
&= (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\alpha(\gamma)\beta(p) - (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\alpha(\gamma)\beta(p) + \\
&\quad (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\alpha(\gamma)d_{P'}(\beta(p)) - (-1)^{|\beta|+|\lambda|+|\beta|\cdot|\lambda|}\lambda\alpha(\gamma)\beta(d_P(p)) + \\
&\sum_{j=1}^{c}(-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\alpha(T_j\gamma)e_j\beta(p) - (-1)^{|\beta|+|\lambda|+|\beta|\cdot|\lambda|}\lambda\alpha(T_j\gamma)\beta(e_jp) \\
&= (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\left[\lambda\alpha(\gamma)d_{P'}(\beta(p)) - (-1)^{|\beta|}\lambda\alpha(\gamma)\beta(d_P(p))\right] \\
&\quad + (-1)^{|\beta|\cdot|\lambda|-|\lambda|}\sum_{j=1}^{c}\lambda\alpha(T_j\gamma)e_j\beta(p) - (-1)^{|\beta|}\lambda\alpha(T_j\gamma)\beta(e_jp) \\
&= (-1)^{(|\beta|+1)\cdot|\lambda|}\lambda\alpha(\gamma)\left[d_{P'}(\beta(p)) - (-1)^{|\beta|}\lambda\alpha(\gamma)\cdot\beta(d_P(p))\right] \\
&\quad + (-1)^{(|\beta|-1)\cdot|\lambda|}\sum_{j=1}^{c}\lambda\alpha(T_j\gamma)\left[e_j\beta(p) - (-1)^{|\beta|}\beta(e_jp)\right]
\end{aligned}$$

$$= \Psi\left(\alpha \otimes d_{\mathrm{Hom}}(\beta) + \sum_{j=1}^{c} \alpha T_j \otimes [e_j, \beta]\right)$$
$$= \Psi(d(\alpha \otimes \beta))$$

Thus $d_{\mathrm{Hom}}\Psi(\alpha \otimes \beta) = \Psi(d(\alpha \otimes \beta))$, and so d is indeed a differential on $S \otimes_Q \mathrm{Hom}_Q(P, P')$ which makes Ψ an isomorphism of DG K-modules. $\square$

Lemma 3.9 gives us a quasi-isomorphism $\eta' : X' \to P'$ given by

$$\eta'(\lambda \otimes \gamma \otimes p') = \lambda \pi_0(\gamma) p'.$$

Since $X = \Lambda \otimes_Q \Gamma \otimes_Q P$ is a semi-free DG K-module, the induced map

$$\widetilde{\eta}' := \mathrm{Hom}_K(X, \eta') : \mathrm{Hom}_K(X, X') \to \mathrm{Hom}_K(X, P')$$

defined by $\widetilde{\eta}'(\phi) := \eta'\phi$ is a quasi-isomorphism by Lemma 3.6(2).

Define a K-module map $\widetilde{\Psi} : S \otimes_Q \mathrm{Hom}_Q(P, P') \to \mathrm{Hom}_K(X, X')$ by

$$\widetilde{\Psi}(\alpha \otimes \beta)(\lambda \otimes \gamma \otimes p) := (-1)^{|\beta|\cdot|\lambda|} \lambda \otimes \alpha\gamma \otimes \beta(p),$$

where $\alpha\gamma$ is the natural action of S on its dual $\Gamma = S^*$. Since

$$\begin{aligned}
\widetilde{\eta}'\widetilde{\Psi}(\alpha \otimes \beta)(\lambda \otimes \gamma \otimes p) &= (-1)^{|\beta|\cdot|\lambda|} \eta'(\lambda \otimes \alpha\gamma \otimes \beta(p)) \\
&= (-1)^{|\beta|\cdot|\lambda|} \lambda \pi_0(\alpha\gamma) \beta(p) \\
&= (-1)^{|\beta|\cdot|\lambda|} \lambda \alpha(\gamma) \beta(p) \\
&= \Psi(\alpha \otimes \beta),
\end{aligned}$$

we see that the following triangle of graded K-modules commutes:

$$\begin{tikzcd} S \otimes_Q \mathrm{Hom}_Q(P, P') \arrow[d, "\widetilde{\Psi}"'] \arrow[dr, "\Psi"] & \\ \mathrm{Hom}_K(X, X') \arrow[r, "\widetilde{\eta}'"'] & \mathrm{Hom}_K(X, P') \end{tikzcd}$$

Our last step is to show that the differential d on $S \otimes_Q \mathrm{Hom}_Q(P, P')$ defined in Lemma 3.17 corresponds to the canonical differential d_{Hom} on $\mathrm{Hom}_K(X, X')$ under $\widetilde{\Psi}$.

Theorem 3.19. *Let $R = Q/(f_1, \dots, f_c)$ be a complete intersection, M and N be finitely generated R-modules, and P and P' be Koszul resolutions of M and N, respectively. If X and X' are the DG K-modules defined in Lemma 3.9, then the map*

$$\widetilde{\Psi} : S \otimes_Q \mathrm{Hom}_Q(P, P') \to \mathrm{Hom}_K(X, X')$$

is a quasi-isomorphism of DG K-modules. Hence, there is a quasi-isomorphism

$$S \otimes_Q \mathrm{Hom}_Q(P, P') \to \mathbf{R}\,\mathrm{Hom}_R^{dg}(M, N)$$

of DG K-modules and an isomorphism of R-modules $H^*(S \otimes_Q \mathrm{Hom}_Q(P, P')) \cong Ext_R^*(M, N)$.

Proof. It remains to verify that $d_{\mathrm{Hom}}\widetilde{\Psi}(\alpha \otimes \beta) = \widetilde{\Psi}(d(\alpha \otimes \beta))$. Recall that the canonical differential on $\mathrm{Hom}_K(X, X')$ is given by

$$d_{\mathrm{Hom}}(\phi) := d_{X'}\phi - (-1)^{|\phi|}\phi d_X.$$

We have that

$$\begin{aligned}
d_{X'}\widetilde{\Psi}(\alpha\otimes\beta)(\lambda\otimes\gamma\otimes p) &= (-1)^{|\beta|\cdot|\lambda|}d_{X'}(\lambda\otimes\alpha\gamma\otimes\beta(p)) \\
&= (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\otimes\alpha\gamma\otimes\beta(p) + (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\otimes\alpha\gamma\otimes d_{P'}(\beta(p)) \\
&\quad - (-1)^{|\beta|\cdot|\lambda|}\sum_{j=1}^{c} e_j\lambda\otimes T_j\alpha\gamma\otimes\beta(p) - (-1)^{|\lambda|}\lambda\otimes T_j\alpha\gamma\otimes e_j\beta(p) \\
&= (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\otimes\alpha\gamma\otimes\beta(p) + (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\otimes\alpha\gamma\otimes d_{P'}(\beta(p)) \\
&\quad - (-1)^{|\beta|\cdot|\lambda|}\sum_{j=1}^{c} e_j\lambda\otimes \alpha T_j\gamma\otimes\beta(p) - (-1)^{|\lambda|}\lambda\otimes \alpha T_j\gamma\otimes e_j\beta(p)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\Psi}(\alpha\otimes\beta)d_X(\lambda\otimes\gamma\otimes p) &= \widetilde{\Psi}(\alpha\otimes\beta)(d_K(\lambda)\otimes\gamma\otimes p) + (-1)^{|la|}\widetilde{\Psi}(\alpha\otimes\beta)(\lambda\otimes\gamma\otimes d_P(p)) \\
&\quad - \sum_{j=1}^{c}\widetilde{\Psi}(\alpha\otimes\beta)(e_j\lambda\otimes T_j\gamma\otimes p) - (-1)^{|\lambda|}\widetilde{\Psi}(\alpha\otimes\beta)(\lambda\otimes T_j\gamma\otimes e_j p) \\
&= (-1)^{|\beta|(|\lambda|+1)}d_K(\lambda)\otimes\alpha\gamma\otimes\beta(p) + (-1)^{|\lambda|+|\beta|\cdot|\lambda|}\lambda\otimes\alpha\gamma\otimes\beta(d_P(p)) \\
&\quad - \sum_{j=1}^{c}(-1)^{|\beta|(|\lambda|-1)}e_j\lambda\otimes\alpha T_j\gamma\otimes\beta(p) - (-1)^{|\lambda|+|\beta|\cdot|\lambda|}\lambda\otimes\alpha T_j\gamma\otimes\beta(e_j p)
\end{aligned}$$

Combining these and observing that $|\widetilde{\Psi}(\alpha\otimes\beta)| = |\alpha| + |\beta| \equiv |\beta| \pmod 2$, we get

$$
\begin{aligned}
d_{\mathrm{Hom}}\widetilde{\Psi}(\alpha\otimes\beta)(\lambda\otimes\gamma\otimes p) &= d_{X'}\widetilde{\Psi}(\alpha\otimes\beta)(\lambda\otimes\gamma\otimes p) - (-1)^{|\beta|}\widetilde{\Psi}(\alpha\otimes\beta)d_X(\lambda\otimes\gamma\otimes p) \\
&= (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\otimes\alpha\gamma\otimes\beta(p) - (-1)^{|\beta|\cdot|\lambda|}d_K(\lambda)\otimes\alpha\gamma\otimes\beta(p) \\
&\quad + (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\lambda\otimes\alpha\gamma\otimes d_{P'}(\beta(p)) - (-1)^{|\beta|+|\lambda|+|\beta|\cdot|\lambda|}\lambda\otimes\alpha\gamma\otimes\beta(d_P(p)) \\
&\quad - (-1)^{|\beta|\cdot|\lambda|}\sum_{j=1}^{c} e_j\lambda\otimes\alpha T_j\gamma\otimes\beta(p) + (-1)^{|\beta|\cdot|\lambda|}e_j\lambda\otimes\alpha T_j\gamma\otimes\beta(p) \\
&\quad + (-1)^{|\beta|\cdot|\lambda|+|\lambda|}\sum_{j=1}^{c}\lambda\otimes\alpha T_j\gamma\otimes e_j\beta(p) - (-1)^{|\beta|+|\lambda|+|\beta|\cdot|\lambda|}\lambda\otimes\alpha T_j\gamma\otimes\beta(e_jp) \\
&= (-1)^{(|\beta|+1)\cdot|\lambda|}\lambda\otimes\alpha\gamma\otimes\left[d_{P'}\beta - (-1)^{|\beta|}\beta d_P\right](p) \\
&\quad + (-1)^{(|\beta|-1)\cdot|\lambda|}\sum_{j=1}^{c}\lambda\otimes\alpha T_j\gamma\otimes\left[e_j\beta - (-1)^{|\beta|}\beta e_j\right](p) \\
&= (-1)^{(|\beta|+1)|\lambda|}\lambda\otimes\alpha\gamma\otimes d_{\mathrm{Hom}}(\beta) + (-1)^{(|\beta|-1)|\lambda|}\sum_{j=1}^{c}\lambda\otimes\alpha T_j\gamma\otimes[e_j,\beta](p) \\
&= \widetilde{\Psi}\left(\alpha\otimes d_{\mathrm{Hom}}(\beta) + \sum_{j=1}^{c}\alpha T_j\otimes[e_j,\beta]\right)(\lambda\otimes\gamma\otimes p) \\
&= \widetilde{\Psi}(d(\alpha\otimes\beta))(\lambda\otimes\gamma\otimes p)
\end{aligned}
$$

Thus $d_{\mathrm{Hom}}\widetilde{\Psi}(\alpha\otimes\beta) = \widetilde{\Psi}(d(\alpha\otimes\beta))$ and $\widetilde{\Psi}$ is a map of DG K-modules.

Since $\widetilde{\eta}'\widetilde{\Psi} = \Psi$ with Ψ and $\widetilde{\eta}'$ an isomorphism and quasi-isomorphism of DG K-modules, respectively, we have that $\widetilde{\Psi}$ is also a quasi-isomorphism of DG K-modules.

The last claim follows from the quasi-isomorphism $\mathrm{Hom}_K(X,X') \xrightarrow{\sim} \mathbf{R}\,\mathrm{Hom}_R^{dg}(M,N)$ from section 3.2. $\square$

We now specialize to the case when $M = N$. We may choose $P = P'$ and hence $X = X'$. In this case both $S\otimes_Q\mathrm{Hom}_Q(P,P)$ and $\mathrm{Hom}_K(X,X)$ have canonical multiplicative structures under which they are DGAs over Q.

Corollary 3.20. *If, in addition to the hypotheses of Theorem 3.19, $M = N$ and $P = P'$, then $S \otimes_Q \operatorname{End}_Q(P)$ is a DGA over Q and the map*

$$\widetilde{\Psi} : S \otimes_Q \operatorname{End}_Q(P) \to \operatorname{Hom}_K(X, X)$$

is a quasi-isomorphism of DGAs. Moreover, there is a quasi-isomorphism

$$S \otimes_Q \operatorname{End}_Q(P) \to \mathbf{R}\operatorname{End}_R^{dg}(M)$$

of DGAs over Q and an isomorphism of graded R-algebras $H^*(S \otimes_Q \operatorname{End}_Q(P)) \cong \operatorname{Ext}_R^*(M, M)$.

Proof. To see that $S \otimes_Q \operatorname{End}_Q(P)$ is a DGA over Q, we must show that the differential defined in Lemma 3.17 satisfies the Leibniz rule. Note that since $|\alpha|$ is even for all homogeneous $\alpha \in S$, multiplication on $S \otimes_Q \operatorname{End}_Q(P)$ is given by

$$(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = (-1)^{|\beta_1| \cdot |\alpha_2|}(\alpha_1 \alpha_2) \otimes (\beta_1 \circ \beta_2) = (\alpha_1 \alpha_2) \otimes (\beta_1 \circ \beta_2).$$

Thus, we have

$$\begin{aligned}
d(\alpha_1\alpha_2 \otimes \beta_1 \circ \beta_2) &= \alpha_1\alpha_2 \otimes d_{\mathrm{End}}(\beta_1 \circ \beta_2) + \sum_{j=1}^{c} \alpha_1\alpha_2 T_j \otimes [e_j, \beta_1 \circ \beta_2] \\
&= \alpha_1\alpha_2 \otimes \big(d_{\mathrm{End}}(\beta_1) \circ \beta_2 + (-1)^{|\beta_1|}\beta_1 \circ d_{\mathrm{End}}(\beta_2)\big) \\
&\quad + \sum_{j=1}^{c} \alpha_1\alpha_2 T_j \otimes \big([e_j, \beta_1] \circ \beta_2 + (-1)^{|\beta_1|}\beta_1 \circ [e_j, \beta_2]\big) \\
&= \alpha_1\alpha_2 \otimes d_{\mathrm{End}}(\beta_1) \circ \beta_2 + \sum_{j=1}^{c} \alpha_1\alpha_2 T_j \otimes [e_j, \beta_1] \circ \beta_2 \\
&\quad + (-1)^{|\beta_1|}\left(\alpha_1\alpha_2 \otimes \beta_1 \circ d_{\mathrm{End}}(\beta_2) + \sum_{j=1}^{c} \alpha_1\alpha_2 T_j \otimes \beta_1 \circ [e_j, \beta_2]\right) \\
&= d(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) + (-1)^{|\beta_1|}(\alpha_1 \otimes \beta_1) d(\alpha_2 \otimes \beta_2).
\end{aligned}$$

So d satisfies the Leibniz rule and $S \otimes_Q \mathrm{End}_Q(P)$ is a DGA over Q.

To see that $\widetilde{\Psi}$ is a quasi-isomorphism of DGAs, we need to check that $\widetilde{\Psi}$ preserves multiplication. Thus, we have

$$\begin{aligned}
(\widetilde{\Psi}(\alpha_1 \otimes \beta_1) \circ \widetilde{\Psi}(\alpha_2 \otimes \beta_2))(\lambda \otimes \gamma \otimes p) &= (-1)^{|\beta_2|\cdot|\lambda|}\widetilde{\Psi}(\alpha_1 \otimes \beta_1)(\lambda \otimes \alpha_2\gamma \otimes \beta_2(p)) \\
&= (-1)^{|\beta_2|\cdot|\lambda|}(-1)^{|\beta_1|\cdot|\lambda|}\lambda \otimes \alpha_1\alpha_2\gamma \otimes \beta_1(\beta_2(p)) \\
&= (-1)^{(|\beta_1|+|\beta_2|)\cdot|\lambda|}\lambda \otimes (\alpha_1\alpha_2)\gamma \otimes (\beta_1 \circ \beta_2)(p) \\
&= \widetilde{\Psi}((\alpha_1\alpha_2) \otimes (\beta_1 \circ \beta_2))(\lambda \otimes \gamma \otimes p) \\
&= \widetilde{\Psi}((\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2))(\lambda \otimes \gamma \otimes p)
\end{aligned}$$

The corollary then follows immediately from Theorem 3.19. $\square$

Suppose M is a finitely generated R-module which, when viewed as a Q-module, has the form $M = Q/I$ with I generated by a regular sequence. For the Koszul resolution of M, we may choose P so that $P^\natural = \Lambda_Q^\bullet(Q^n)$. The remainder of this chapter focuses on obtaining a more explicit description of $S \otimes_Q \mathrm{End}_Q(\Lambda_Q^\bullet(Q^n))$. We will do this by realizing $\mathrm{End}_Q(\Lambda_Q^\bullet(Q^n))^\natural$ as a Clifford algebra.

3.4 Clifford Algebras

In this section we present an abbreviated exposition of the theory of Clifford algebras; a more thorough treatment can be found in [14]. Throughout this section let Q be any commutative ring and let M be a Q-module.

Definition 3.21. A *quadratic form* on M is a map $q : M \to Q$ such that

(1) $q(am) = a^2q(m)$ for all $a \in Q$, $m \in M$,

(2) $b_q(m, n) := q(m + n) - q(m) - q(n)$ is Q-bilinear.

The main example we will need is that of the hyperbolic space of a projective module.

Example 3.22. Let W be a finitely generated projective Q-module and define a Q-module $H(W) := W \oplus W^*$. Define a map $q_H : H(W) \to Q$ by $q_H((w, \varphi)) := \varphi(w)$ for $w \in W$ and $\varphi \in W^*$. Then q_H is a quadratic form on $H(W)$. This is called the *hyperbolic space* of W.

We will define the Clifford algebra as a quotient of the tensor algebra. Recall that the tensor algebra of M over Q is

$$T_Q(M) := Q \oplus T^1(M) \oplus T^2(M) \oplus \cdots \oplus T^i(M) \oplus \cdots ,$$

where $T^i(M) = M \otimes_Q M \otimes_Q \cdots \otimes_Q M$ (i factors), with multiplication given by concatenation.

Definition 3.23. Let q be a quadratic form on a Q-module M. The *Clifford algebra* associated to the pair (M, q) is defined as

$$C_Q(M, q) := T_Q(M)/I(q),$$

where $T_Q(M)$ is the tensor algebra of M over Q and $I(q)$ is the two-sided ideal of $T_Q(M)$ generated by all elements of the form $m \otimes m - q(m)$ for $m \in M$.

Note that $I(q)$ has the property that every term of every generator has even degree. Thus, the decomposition $T_Q(M) = T^{\mathrm{ev}}(M) \oplus T^{\mathrm{odd}}(M)$, where $T^{\mathrm{ev}}(M) := Q \oplus T^2(M) \oplus T^4(M) \oplus \cdots$ and $T^{\mathrm{odd}}(M) := T^1(M) \oplus T^3(M) \oplus \cdots$, induces a canonical $(\mathbb{Z}/2\mathbb{Z})$-grading on the Clifford algebra $C_Q(M, q)$. In general, a Clifford algebra is not $\mathbb{Z}$-graded. An important exception is illustrated in the following example.

Example 3.24. Let W be a finitely generated projective Q-module. Then the Clifford algebra of the hyperbolic space $H(W)$ has a canonical $\mathbb{Z}$-grading. Set $|w| = -1$ and $|\varphi| = 1$

for all $w \in W$ and $\varphi \in W^*$. Since $q_H(w,\varphi) = \varphi(w)$, $I(q)$ is generated by elements of the form

$$(w,\varphi) \otimes (w,\varphi) - \varphi(w) = w \otimes \varphi + \varphi \otimes w - \varphi(w),$$

which are all homogeneous of degree 0. Thus $I(q)$ is a homogeneous ideal of $T_Q(H(W))$, and hence $C_Q(H(W), q_H)$ is $\mathbb{Z}$-graded.

We end this section with two classical results involving Clifford algebras. Theorem 3.25 states that the Clifford algebra associated to a free module is itself a free module. We will need this fact in the next section. Theorem 3.26 allows us to realize $\mathrm{End}_Q(\Lambda_Q^\bullet(W))$ as the Clifford algebra of the pair $(H(W), q_H)$ from Example 3.22. (Recall that the goal is to describe $\mathrm{End}_Q(\Lambda_Q^\bullet(Q^n))$ more explicitly.) Proofs of both theorems can be found in [14].

Theorem 3.25 (Poincaré-Birkhoff-Witt)**.** *If M is a free Q-module with basis $\{b_1, \ldots, b_r\}$ and q is a quadratic form on M, then $C_Q(M,q)$ is also a free Q-module with basis given by $\{b_{i_1} b_{i_2} \cdots b_{i_l} \mid 1 \le i_1 < i_2 < \cdots < i_l \le r,\ l \ge 0\}$. In particular, if $\mathrm{rank}_Q M = r$, then $\mathrm{rank}_Q C_Q(M,q) = 2^r$.*

Theorem 3.26. *For any commutative ring Q and finitely generated projective Q-module W, consider the $(\mathbb{Z}/2\mathbb{Z})$-grading on $\mathrm{End}_Q(\Lambda_Q^\bullet(W))$ which is induced by the decomposition $\Lambda_Q^\bullet(W) = \Lambda_Q^{ev}(W) \oplus \Lambda_Q^{odd}(W)$ into even and odd components. Then there is an isomorphism*

$$\rho : C_Q(H(W), q_H) \to \mathrm{End}_Q(\Lambda_Q^\bullet(W))$$

of $(\mathbb{Z}/2\mathbb{Z})$-graded Q-modules given by $\rho((w,\varphi)) = \lambda_w + c_\varphi$, where λ_w denotes left multiplication by $w \in W$ and c_φ denotes contraction by $\varphi \in W^*$, extending the map $\varphi : W \to Q$ to all of $\Lambda_Q^\bullet(W)$ via the Leibniz rule.

Remark 3.27. Since $\operatorname{End}_Q(\Lambda_Q^\bullet(W))$ is a $\mathbb{Z}$-graded Q-module with $|\lambda_w| = -1$ and $|c_\varphi| = 1$, we make $C_Q(H(W), q_H)$ into a $\mathbb{Z}$-graded Q-module by setting $|w| = -1$ and $|\varphi| = 1$ for all $w \in W$ and $\varphi \in W^*$. Note that this agrees with the $\mathbb{Z}$-grading from example 3.24. This makes ρ an isomorphism of $\mathbb{Z}$-graded Q-modules, and hence we have an isomorphism

$$S \otimes \rho : S \otimes_Q C_Q(H(W), q_H) \to S \otimes_Q \operatorname{End}_Q(\Lambda_Q^\bullet(W))$$

of graded Q-modules.

3.5 Modules Defined by Regular Sequences

In this section we return to the situation where (Q, m, k) is a regular local ring, $R = Q/(f_1, \dots, f_c)$ is a complete intersection of codimension c, and $K = \operatorname{Kos}(f_1, \dots, f_c; Q)$ is the Koszul algebra. Let $I = (x_1, \dots, x_n)$ be an ideal of Q generated by a Q-regular sequence $x_1, \dots, x_n$ with $f_j \in I$ for $1 \leq j \leq c$. Let $M := Q/I$ regarded as an R-module. In this case, we will obtain explicit generators and relations for the DG K-module $S \otimes_Q \operatorname{End}_Q(P)$ by viewing $\operatorname{End}_Q(P)$ as a Clifford algebra.

We remark here that Tate first introduced the idea of killing cycles in arbitrary DGAs to commutative algebra in [19]. These ideas permeate throughout this chapter.

Since $f_j \in I = (x_1, \dots, x_n)$, we can choose $v_{ij} \in Q$ with $f_j = \sum_{i=1}^{n} v_{ij} x_i$. We define $v_j := \begin{pmatrix} v_{1j} \\ \vdots \\ v_{nj} \end{pmatrix} \in Q^n$. We choose a Koszul resolution P of M over Q such that $P^\natural = \Lambda_Q^\bullet(W)$, where $W := Q^n$ has canonical basis $y_1, \dots, y_n$ with $|y_i| = -1$. The K-action on P is given by $e_j p := \lambda_{v_j} p$ and the differential is given by the contraction $c_{\underline{x}}$, where $\underline{x} : W \xrightarrow{(x_1, \dots, x_n)} Q$. With this explicit DG K-module structure on P, the differential on $S \otimes_Q \operatorname{End}_Q(\Lambda_Q^\bullet(W))$

defined in Lemma 3.17 can be written as

$$d(\alpha \otimes \beta) = \alpha \otimes [c_{\underline{x}}, \beta] + \sum_{j=1}^{c} \alpha T_j \otimes [\lambda_{v_j}, \beta].$$

Lemmas 3.16 and 3.17 give a DG K-module structure for $S \otimes_Q \operatorname{End}_Q(\Lambda_Q^{\bullet}(W))$. We use this structure along with the graded Q-module isomorphism $S \otimes \rho$ from Remark 3.27 to make $S \otimes_Q C_Q(H(W), q_H)$ a DG K-module.

Lemma 3.28. *Let $\alpha \otimes (w, \varphi) \in S \otimes_Q C_Q(H(W), q_H)$. Define a K-action on $S \otimes_Q C_Q(H(W), q_H)$ by $e_j(\alpha \otimes (w, \varphi)) := \alpha \otimes (v_j(w, \varphi))$ and an endomorphism d_C of $S \otimes_Q C_Q(H(W), q_H)$ by*

$$d_C(\alpha \otimes (w, \varphi)) := \alpha \otimes [\underline{x}, w] + \sum_{j=1}^{c} \alpha T_j \otimes [v_j, \varphi].$$

Then $S \otimes_Q C_Q(H(W), q_H)$ equipped with d_C is a DG K-module and

$$S \otimes \rho : S \otimes_Q C_Q(H(W), q_H) \to S \otimes_Q \operatorname{End}_Q(\Lambda_Q^{\bullet}(W))$$

is an isomorphism of DG K-modules. Moreover, $S \otimes \rho$ is an isomorphism of DGAs over Q.

Proof. The given K-action corresponds with the one in Lemma 3.16 under $S \otimes \rho$. Indeed,

$$\begin{aligned}
(S \otimes \rho)(e_j(\alpha \otimes (w, \varphi))) &= (S \otimes \rho)(\alpha \otimes v_j(w, \varphi)) \\
&= \alpha \otimes \lambda_{v_j}(\lambda_w + c_\varphi) \\
&= e_j(\alpha \otimes (\lambda_w + c_\varphi)) \\
&= e_j(S \otimes \rho)(\alpha \otimes (w, \varphi)).
\end{aligned}$$

To see that $S \otimes \rho$ is an isomorphism of DG K-modules, it remains to show that d_C corresponds with the differential d defined in Lemma 3.17. We show that $(S \otimes \rho) d_C = d(S \otimes \rho)$.

Indeed,

$$\begin{aligned}(S\otimes\rho)d_C(\alpha\otimes(w,\varphi)) &= (S\otimes\rho)\left(\alpha\otimes[\underline{x},w]+\sum_{j=1}^{c}T_j\alpha\otimes[v_j,\varphi]\right)\\ &= \alpha\otimes[c_x,\lambda_w]+\sum_{j=1}^{c}T_j\alpha\otimes[\lambda_{v_j},c_\varphi]\\ &= d(\alpha\otimes(\lambda_w+c_\varphi))\\ &= d(S\otimes\rho)(\alpha\otimes(w,\varphi))\end{aligned}$$

To see that $S\otimes\rho$ is an isomorphism of DGAs over Q, we show that $S\otimes\rho$ preserves multiplication. Note that in $S\otimes_Q C_Q(H(W),q_H)$, multiplication is defined by the rule

$$(\alpha_1\otimes(w_1,\varphi_1))(\alpha_2\otimes(w_2,\varphi_2)) := \alpha_1\alpha_2\otimes(w_1,\varphi_1)(w_2,\varphi_2)$$

since $|\alpha_2|$ is even. To this end, we have

$$\begin{aligned}(S\otimes\rho)((\alpha_1\otimes(w_1,\varphi_1))(\alpha_2\otimes(w_2,\varphi_2))) &= (S\otimes\rho)(\alpha_1\alpha_2\otimes(w_1,\varphi_1)(w_2,\varphi_2))\\ &= \alpha_1\alpha_2\otimes(\lambda_{w_1}+c_{\varphi_1})(\lambda_{w_2}+c_{\varphi_2})\\ &= (\alpha_1\otimes(\lambda_{w_1}+c_{\varphi_1}))(\alpha_2\otimes(\lambda_{w_2}+c_{\varphi_2}))\\ &= ((S\otimes\rho)(\alpha_1\otimes(w_1,\varphi_1)))((S\otimes\rho)(\alpha_2\otimes(w_2,\varphi_2)))\end{aligned}$$

□

We now view $S\otimes_Q C_Q(H(W),q_H)$ as $C_S(H(W_S),q_H)$ via extension of scalars, where $W_S := W\otimes_Q S\cong S^n$. Note that $C_S(H(W_S),q_H)$ is a DG K-module with differential defined by

$$d(w,\varphi)=[\underline{x},w]+\sum_{j=1}^{c}[v_j,\varphi]T_j$$

where $(w, \varphi) \in H(W_S)$.

Note that $y_1, \dots, y_n$ is a basis for W_S. Let $Y_1, \dots, Y_n$ be the basis of W_S^* dual to $y_1, \dots, y_n$; that is, $Y_i(y_j) = \delta_{ij}$. Then $H(W_S)$ is a free S-module with basis $y_1, \dots, y_n, Y_1, \dots, Y_n$. Observe that

$$\begin{aligned} d(y_i) &= [\underline{x}, y_i] + \sum_{j=1}^{c} [v_j, y_i] T_j \;=\; \underline{x}(y_i) \;=\; x_i \\ d(Y_i) &= [\underline{x}, Y_i] + \sum_{j=1}^{c} [v_j, Y_i] T_j \;=\; \sum_{j=1}^{c} Y_i(v_j) T_j \;=\; \sum_{j=1}^{c} v_{ij} T_j \end{aligned}$$

We now use this Clifford algebra with differential to define an explicit DGA over Q which is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$. Let $\mathcal{A}$ be the DGA over Q with:

- generators $T_1, \dots, T_c, y_1, \dots, y_n, Y_1, \dots, Y_n$ with $|T_j| = 2$, $|y_i| = -1$, and $|Y_i| = 1$,
- relations y_i^2, Y_i^2, $[T_h, T_i]$, $[y_h, y_i]$, $[Y_h, Y_i]$, $[T_h, y_i]$, $[T_h, Y_i]$, and $[y_h, Y_i] - \delta_{hi}$, where $[a, b] = ab - (-1)^{|a|\cdot|b|} ba$ and δ_{hi} is the Kronecker delta,
- a differential defined on the generators by $d_{\mathcal{A}}(T_j) = 0$, $d_{\mathcal{A}}(y_i) = x_i$, $d_{\mathcal{A}}(Y_i) = \sum_{j=1}^{c} v_{ij} T_j$, extended to all of $\mathcal{A}$ via the Leibnitz rule.

Theorem 3.29. *Let $R = Q/(f_1, \dots, f_c)$ be a complete intersection and let $I = (x_1, \dots, x_n)$ be an ideal of Q generated by a Q-regular sequence $x_1, \dots, x_n$ with $f_j \in I$ for $1 \le j \le c$. Let $M = Q/I$ regarded as an R-module. Then $\mathcal{A}$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DGAs over Q. Moreover, if we define a K-action on $\mathcal{A}$ by $e_j 1_{\mathcal{A}} := \sum_{i=1}^{n} v_{ij} y_i$, then $\mathcal{A}$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DG K-modules as well.*

Remark 3.30. Note that $\mathcal{A}$ can be viewed as a DG S-module, in which case the above generating set and list of relations can be reduced. We will not take this viewpoint as our goal is to realize $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as a DGA over Q, up to quasi-isomorphism.

Proof. First note that there is a quasi-isomorphism $C_S(H(W_S), q_H) \to \mathbf{R}\operatorname{End}_R^{dg}(M)$ of DGAs over Q by Lemma 3.28 and Corollary 3.20. It suffices to show that the DGA $C_S(H(W_S), q_H)$ is precisely $\mathcal{A}$.

The fact that $C_S(H(W_S), q_H)$ has the same generating set and differential as $\mathcal{A}$ has been established in the previous discussion. It remains to verify the relations. Since $|T_j| = 2$ for $1 \leq j \leq c$, we get the relations $[T_h, T_i]$, $[T_h, y_i]$, and $[T_h, Y_i]$. For the remaining relations, recall that the quadratic form q_H applied to a pair $(w, \varphi) \in H(W_S)$ is $q_H(w, \varphi) = \varphi(w)$. In particular, $q_H(w) = 0 = q_H(\varphi)$. Thus, $w^2 = q_H(w) = 0$ and $\varphi^2 = q_H(\varphi) = 0$ for all $w \in W_S$ and $\varphi \in W_S^*$ by the defining condition of a Clifford algebra. Since $y_i \in W_S$ and $Y_i \in W_S^*$, we immediately obtain the relations y_i^2 and Y_i^2. For any $y_h, y_i \in W_S$ and $Y_h, Y_i \in W_S^*$, we obtain the relations $[y_h, y_i] = (y_h + y_i)^2 = 0$ and $[Y_h, Y_i] = (Y_h + Y_i)^2 = 0$. Lastly, we have $[y_h, Y_i] = (y_h + Y_i)^2 = Y_i(y_h) = \delta_{ij}$.

The claim that $\mathcal{A}$ and $\mathbf{R}\operatorname{End}_R^{dg}(M)$ are quasi-isomorphic as DG K-modules then follows from Lemma 3.28, Corollary 3.20, and the observation that $v_j = \displaystyle\sum_{i=1}^{n} v_{ij} y_i$. $\square$

The generators and relations for $\mathcal{A}$ are not ideal for computing cohomology, and in particular for recovering Sjödin's Theorem in the case where $M = k$. In order to obtain more useful generators and relations for this purpose, we make an additional assumption. In addition to M being an R-module of the form $M = Q/I$ with I generated by a Q-regular sequence, we now assume that $f_j \in I^2$ for each $1 \leq j \leq c$. Note that $M = k$ is such a module by our assumption that $f_j \in m^2$.

In this case, since $f_j \in I^2 = (x_1, \ldots, x_n)^2$, we can choose $v_{ij} \in I$ and $w_{hij} \in Q$ such that

$$f_j = \sum_{i=1}^{n} v_{ij} x_i \qquad \text{and} \qquad v_{ij} = \sum_{h=1}^{n} w_{hij} x_h.$$

Observe that we have

$$f_j = \sum_{1 \le h,i \le n} w_{hij} x_h x_i$$

for each $1 \le j \le c$. By starting with this linear combination of the $x_h x_i$'s, we see that the v_{ij}'s and w_{hij}'s can be chosen so that $w_{hij} = 0$ whenever $h > i$. Thus, we can write

$$f_j = \sum_{1 \le h \le i \le n} w_{hij} x_h x_i \tag{3.31}$$

for each $1 \le j \le c$. Define $w_{ij} := \begin{pmatrix} w_{1ij} \\ \vdots \\ w_{nij} \end{pmatrix} \in Q^n$.

Recall that $H(W_S)$ is a free S-module with a basis $y_1, \ldots, y_n, Y_1, \ldots, Y_n$. For $1 \le i \le n$, define elements

$$X_i := Y_i - \sum_{j=1}^{c} w_{ij} T_j \in H(W_S).$$

Since $\sum_{j=1}^{c} w_{ij} T_j \in W_S$, the set $y_1, \ldots, y_n, X_1, \ldots, X_n$ is also a basis for $H(W_S)$. Viewing $X_i \in C_S(H(W_S), q_H)$, we have

$$\begin{aligned} d(X_i) &= d(Y_i) - d\left(\sum_{j=1}^{c} w_{ij} T_j\right) = \sum_{j=1}^{c} v_{ij} T_j - \underline{x}\left(\sum_{j=1}^{c} w_{ij} T_j\right) \\ &= \sum_{j=1}^{c} v_{ij} T_j - \sum_{j=1}^{c} \underline{x}(w_{ij} T_j) = \sum_{j=1}^{c} v_{ij} T_j - \sum_{j=1}^{c} v_{ij} T_j \\ &= 0 \end{aligned}$$

So $y_1, \ldots, y_n, X_1, \ldots, X_n$ is a basis for the free S-module $H(W_S)$ and the differential on $C_S(H(W_S), q_H)$ is defined by $d(y_i) = x_i$ and $d(X_i) = 0$ for $1 \le i \le n$.

Let $\mathcal{A}'$ be the DGA over Q with:

- generators $T_1, \ldots, T_c, y_1, \ldots, y_n, X_1, \ldots, X_n$ with $|T_j| = 2$, $|y_i| = -1$, and $|X_i| = 1$,
- relations $y_i^2,\ X_i^2 + \sum_{j=1}^{c} w_{iij} T_j,\ [T_h, T_i],\ [y_h, y_i],\ [X_h, X_i] + \sum_{j=1}^{c} w_{hij} T_j,\ [T_h, y_i],\ [T_h, X_i]$, and $[y_h, X_i] - \delta_{hi}$, where $[a, b] = ab - (-1)^{|a|\cdot|b|} ba$, δ_{hi} is the Kronecker delta, the w_{hij}'s are chosen as in equation 3.31, and $h < i$ in the relation for $[X_h, X_i]$
- a differential defined on the generators by $d_{\mathcal{A}'}(T_j) = 0,\ d_{\mathcal{A}'}(y_i) = x_i,\ d_{\mathcal{A}'}(X_i) = 0$, extended to all of $\mathcal{A}'$ via the Leibnitz rule.

Theorem 3.32. *Let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection and let $I = (x_1, \ldots, x_n)$ be an ideal of Q generated by a Q-regular sequence $x_1, \ldots, x_n$ with $f_j \in I^2$ for $1 \le j \le c$. Let $M = Q/I$ regarded as an R-module. Then $\mathcal{A}'$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DGAs over Q. Moreover, with the K-action on $\mathcal{A}'$ defined by $e_j 1_{\mathcal{A}'} := \sum_{i=1}^{n} v_{ij} y_i$, $\mathcal{A}'$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DG K-modules as well.*

Proof. Everything follows as in Theorem 3.29 except for the relations involving the X_i's, which we verify here:

$$\begin{aligned}
X_i^2 &= Y_i^2 - Y_i \sum_{j=1}^{c} w_{ij} T_j - \sum_{j=1}^{c} w_{ij} T_j Y_i - \left(\sum_{j=1}^{c} w_{ij} T_j \right)^2 \ = \ - \left[\sum_{j=1}^{c} w_{ij} T_j, Y_i \right] \\
&= -Y_i \sum_{j=1}^{c} w_{ij} T_j \ = \ - \sum_{j=1}^{c} w_{iij} T_j \\
[X_h, X_i] &= [Y_h, Y_i] - \left[Y_h, \sum_{j=1}^{c} w_{ij} T_j \right] - \left[\sum_{j=1}^{c} w_{hj} T_j, Y_i \right] + \left[\sum_{j=1}^{c} w_{hj} T_j, \sum_{j=1}^{c} w_{ij} T_j \right] \\
&= -(Y_h \sum_{j=1}^{c} w_{ij} T_j + Y_i \sum_{j=1}^{c} w_{hj} T_j) \ = \ - \sum_{j=1}^{c} (w_{hij} + w_{ihj}) T_j = - \sum_{j=1}^{c} w_{hij} T_j \\
[y_h, X_i] &= [y_h, Y_i] - \left[y_h, \sum_{j=1}^{c} w_{ij} T_j \right] \ = \ Y_i(y_h) \ = \ \delta_{ih}
\end{aligned}$$

$\square$

Taking $M = k$, this description of $\mathcal{A}'$ recovers Sjödin's description of $\mathrm{Ext}^*_R(k,k)$ upon taking cohomology. In fact, we get a bit more than this, as illustrated in the following corollary.

Corollary 3.33. *Let $R = Q/(f_1, \ldots, f_c)$ be a complete intersection and let $I = (x_1, \ldots, x_n)$ be an ideal of Q generated by a Q-regular sequence $x_1, \ldots, x_n$ with $f_j \in I^2$ for $1 \leq j \leq c$. Let $w_{hij} \in Q$ be as in equation 3.31. Then $Ext^*_R(Q/I, Q/I)$ is generated as a (Q/I)-algebra by elements $Z_1, \ldots, Z_n$ of degree 1 and $T_1, \ldots, T_c$ of degree 2 subject to the relations*

$$Z_i^2 + \sum_{j=1}^{c} \overline{w_{iij}} T_j, \quad [Z_h, Z_i] + \sum_{j=1}^{c} \overline{w_{hij}} T_j, \quad [Z_h, T_i], \quad [T_h, T_i],$$

where $\overline{w_{hij}}$ is the image of w_{hij} in Q/I. In particular, when $I = m$, we obtain Sjödin's description of $Ext^*_R(k,k)$ as a k-algebra (Theorem 3.1).

Proof. Since $H(W_S) \cong S^{2n}$ with basis $y_1, \ldots, y_n, X_1, \ldots, X_n$, the Poincaré-Birkhoff-Witt theorem (Theorem 3.25) says that $\mathcal{A}' \cong C_S(H(W_S), q_H)$ is a free S-module of rank 2^{2n} with basis consisting of elements of the form $y_{i_1} \cdots y_{i_k} X_{j_1} \cdots X_{j_l}$ with $1 \leq i_1 < \cdots < i_k \leq n$, $1 \leq j_1 < \cdots < j_l \leq n$, $0 \leq k, l \leq n$. Thus, there is an isomorphism of free S-modules, and hence of Q-modules,

$$\mathcal{A}' \to \Lambda^\bullet_S(H(W_S))$$

which maps $y_{i_1} \cdots y_{i_k} X_{j_1} \cdots X_{j_l}$ to $y_{i_1} \wedge \cdots \wedge y_{i_k} \wedge X_{j_1} \wedge \cdots \wedge X_{j_l}$. We make $\Lambda^\bullet_S(H(W_S))$ into a DG K-module by setting $|y_i| = -1$, $|X_i| = 1$, $d(y_i) = x_i$, and $d(X_i) = 0$. Then the above Q-module isomorphism is an isomorphism of DG K-modules by Theorem 3.32. We define $W'_S := \mathrm{Span}\langle X_1, \ldots, X_n \rangle$, so that $H(W_S) = W_S \oplus W'_S$. This decomposition gives an isomorphism

$$\Lambda^\bullet_S(H(W_S)) \to \Lambda^\bullet_S(W_S) \otimes_S \Lambda^\bullet_S(W'_S)$$

which maps $y_{i_1}\wedge\cdots\wedge y_{i_k}\wedge X_{j_1}\wedge\cdots\wedge X_{j_l}$ to $(y_{i_1}\wedge\cdots\wedge y_{i_k})\otimes(X_{j_1}\wedge\cdots\wedge X_{j_l})$. This is an isomorphism of DG K-modules by carrying the grading and differential from $\Lambda_S^\bullet(H(W_S))$ through the map. Observe that $\Lambda_S^\bullet(W_S) = \mathrm{Kos}(x_1,\dots,x_n\,;S) \xrightarrow{\sim} S/(x_1,\dots,x_n) \cong (Q/I)[T_1,\dots,T_c]$ while $\Lambda_S^\bullet(W_S')$ has trivial differential. So we have a quasi-isomorphism

$$\Lambda_S^\bullet(W_S)\otimes_S\Lambda_S^\bullet(W_S') \xrightarrow{\sim} (Q/I)[T_1,\dots,T_c]\otimes_S\Lambda_S^\bullet(W_S') \xrightarrow{\sim} \Lambda_{Q/I}((Q/I)^n)[T_1,\dots,T_c],$$

where $(Q/I)^n$ has basis $\overline{X_1},\dots,\overline{X_n}$, with $\overline{X_i}$ the image of X_i in Q/I. Composing the above maps gives a quasi-isomorphism $\epsilon : \mathcal{A}' \xrightarrow{\sim} \Lambda_{Q/I}((Q/I)^n)[T_1,\dots,T_c]$. By tracking the generators $T_1,\dots,T_c,y_1,\dots,y_n,X_1,\dots,X_n$ of $\mathcal{A}'$ from Theorem 3.32 through ϵ, we see that $\epsilon(T_j)=T_j$, $\epsilon(y_i)=0$, and $\epsilon(X_i)=\overline{X_i}$. Set $Z_i=\overline{X_i}$ for $1\le i\le n$. Hence,

$$\mathrm{Ext}_R^*(Q/I,Q/I) = H^*(\mathcal{A}') = \Lambda_{Q/I}((Q/I)^n)[T_1,\dots,T_c]$$

is generated as a (Q/I)-algebra by elements $Z_1,\dots,Z_n$ of degree 1 and $T_1,\dots,T_c$ of degree 2 subject to the relations from Theorem 3.32; namely,

$$Z_i^2+\sum_{j=1}^c \overline{w_{iij}}T_j,\quad [Z_h,Z_i]+\sum_{j=1}^c \overline{w_{hij}}T_j,\quad [Z_h,T_i],\quad [T_h,T_i].$$

□

3.6 Examples

We now look at Theorems 3.29 and 3.32 in a few special cases and use this to recover some known results related to the work of Dyckerhoff. We also use Theorem 3.29 to compute Hochschild cohomology of the localization of a polynomial ring.

3.6.1 Hypersurface Case

A *hypersurface* is a complete intersection of codimension $c = 1$; specifically, $R = Q/(f)$, with f a non-zero divisor in a regular local ring Q. The notation in Theorems 3.29 and 3.32 can be simplified in the hypersurface case. Since hypersurfaces are an important class of complete intersections, we present the results of these theorems in this special case.

As in the setup of Theorem 3.29, let $I = (x_1, \dots, x_n)$ be an ideal of Q generated by a Q-regular sequence $x_1, \dots, x_n$ with $f \in I$. Let $M = Q/I$ regarded as an R-module. Then there exist $v_i \in Q$ with $f = \sum_{i=1}^{n} v_i x_i$. If $f \in I^2$ as in Theorem 3.32, then as in equation 3.31 we have elements $w_{hi} \in Q$ such that $f = \sum_{1 \le h \le i \le n} w_{hi} x_h x_i$. We then define $X_i = Y_i - w_{ii} T \in H(W_S)$.

We now state the theorems in the hypersurface case as a formal remark.

Remark 3.34. Let $R = Q/(f)$ be a hypersurface and let I and M be as above.

(1) Let $\mathcal{A}$ be the DGA over Q with:

- generators $T, y_1, \dots, y_n, Y_1, \dots, Y_n$ with $|T| = 2$, $|y_i| = -1$, and $|Y_i| = 1$,
- relations $y_i^2,\ Y_i^2,\ [y_h, y_i],\ [Y_h, Y_i],\ [T, y_i],\ [T, Y_i],\ [y_h, Y_i] - \delta_{hi}$,
- a differential defined on the generators by $d_{\mathcal{A}}(T) = 0,\ d_{\mathcal{A}}(y_i) = x_i,\ d_{\mathcal{A}}(Y_i) = v_i T$, extended to all of $\mathcal{A}$ via the Leibniz rule.

Then $\mathcal{A}$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DGAs over Q. Moreover, if we define a K-action on $\mathcal{A}$ by $e1_{\mathcal{A}} := \sum_{i=1}^{n} v_i y_i$, then $\mathcal{A}$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DG K-modules as well.

(2) Suppose $f \in I^2$. Let $\mathcal{A}'$ be the DGA over Q with:

- generators $T, y_1, \dots, y_n, X_1, \dots, X_n$ with $|T| = 2$, $|y_i| = -1$, and $|X_i| = 1$,
- relations $y_i^2,\ X_i^2 + w_{ii} T,\ [y_h, y_i],\ [X_h, X_i] + w_{hi} T,\ [T, y_i],\ [T, X_i],\ [y_h, X_i] - \delta_{hi}$,

- a differential defined on the generators by $d_{\mathcal{A}'}(T) = 0,\ d_{\mathcal{A}'}(y_i) = x_i,\ d_{\mathcal{A}'}(X_i) = 0$, extended to all of $\mathcal{A}'$ via the Leibnitz rule.

Then $\mathcal{A}'$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DGAs over Q. Moreover, with the K-action on $\mathcal{A}'$ defined by $e1_{\mathcal{A}'} := \sum_{i=1}^{n} v_i y_i$, $\mathcal{A}'$ is quasi-isomorphic to $\mathbf{R}\operatorname{End}_R^{dg}(M)$ as DG K-modules as well.

This hypersurface version can recover a result of Dyckerhoff [11], who is interested in computing the DGA of endomorphisms of a certain $(\mathbb{Z}/2\mathbb{Z})$-graded module which we now define. To do so, we consider two categories associated to the hypersurface $R = Q/(f)$, namely the homotopy category of matrix factorizations of (Q, f) and the singularity category of R.

A *matrix factorization* for (Q, f) is a free $(\mathbb{Z}/2\mathbb{Z})$-graded Q-module $E = E_0 \oplus E_1$ of finite rank equipped with a Q-linear endomorphism d of E of degree 1 (i.e., $d(E_i) \subseteq E_{i+1}$ with subscripts taken modulo 2) satisfying $d^2 = f \cdot \mathrm{id}_E$. A *homomorphism* of matrix factorizations from E to F is a $(\mathbb{Z}/2\mathbb{Z})$-graded Q-linear map $\alpha : E \to F$ of degree 0 such that $d_F \circ \alpha = \alpha \circ d_E$. Two homomorphisms $\alpha, \beta : E \to F$ are *homotopic* if there exists a homomorphism $h : E \to F$ of degree 1 such that $d_F \circ h + h \circ d_E = \alpha - \beta$. Note that this is an equivalence relation and is preserved by composition. The *homotopy category of matrix factorizations*, denoted $[MF(Q, f)]$, is the category whose objects are matrix factorizations of (Q, f) and whose morphisms are the equivalence classes of homomorphisms from E to F formed by identifying homotopic homomorphisms.

Let $D^b(R)$ denote the bounded derived category of finitely generated R-modules. A complex in $D^b(R)$ is *perfect* if it is isomorphic in $D^b(R)$ to a bounded complex of finitely generated free R-modules. Let $\mathrm{Perf}(R)$ denote the full subcategory of perfect complexes in $D^b(R)$, which is known to form a thick subcategory. So we can define the Verdier quotient $D_{\mathrm{sg}}(R) := D^b(R)/\mathrm{Perf}(R)$, called the *singularity category* of R. Observe that a finitely generated R-module M can be viewed as an object in $D_{\mathrm{sg}}(R)$ by considering it as a complex concentrated in degree 0.

Buchweitz [6] and Eisenbud [12] proved that there is an equivalence of categories

$$[MF(Q,f)] \to D_{\mathrm{sg}}(R)$$

given by $(E,d) \mapsto \operatorname{coker}\left(E_1 \xrightarrow{d} E_0\right)$. This equivalence allows us to associate to every finitely generated R-module M a matrix factorization M^{stab}.

Dyckerhoff gives a description for $\operatorname{End}(k^{\mathrm{stab}})$, the $(\mathbb{Z}/2\mathbb{Z})$-graded DGA of endomorphisms of k^{stab}. We recover this algebra using the description of $\mathcal{A}$ in Remark 3.34(1) by setting $T = 1$. Note that since $|T| = 2$, this action does indeed produce a $(\mathbb{Z}/2\mathbb{Z})$-graded module.

Proposition 3.35. *Let $R = Q/(f)$ be a hypersurface and let $\mathcal{A}$ be the DGA described in Remark 3.34(1). The $(\mathbb{Z}/2\mathbb{Z})$-graded DGA $\operatorname{End}(k^{stab})$ is recovered by setting $T = 1$ and viewing the resulting object as a $(\mathbb{Z}/2\mathbb{Z})$-graded module in the natural way. In other words,*

$$\operatorname{End}(k^{stab}) \cong \mathcal{A} \otimes_{Q[T]} Q[T]/(T-1).$$

Proof. Using the generators and relations of $\mathcal{A}$ given in Remark 3.34(1), setting $T = 1$ yields the $(\mathbb{Z}/2\mathbb{Z})$-graded algebra generated by elements

$$\{y_1, \ldots, y_n, Y_1, \ldots, Y_n\}$$

of degree 1 subject to the relations

$$\{y_i^2,\ Y_i^2,\ [y_h, y_i],\ [Y_h, Y_i],\ [Y_h, y_i] - \delta_{hi}\}.$$

and equipped with the differential d defined by $d(y_i) = x_i$ and $d(Y_i) = v_i$. These are precisely the generators, relations and differential for $\mathrm{End}(k^{\mathrm{stab}})$ given by Dyckerhoff in [11, Section 4.1]. $\square$

3.6.2 Homogeneous Quadratic Case

In this section we return to the general codimension c case, though we only consider homogeneous quadratic complete intersections. We begin with the definition.

Definition 3.36. A *(homogeneous) quadratic complete intersection* is a complete intersection $R = Q/(f_1, \ldots, f_c)$ such that

(1) (Q, m, k) is a regular local k-algebra for which the composition $k \subseteq Q \twoheadrightarrow Q/m = k$ is an isomorphism,

(2) m is generated by a Q-regular sequence $x_1, \ldots, x_n$, and

(3) there exist elements $w_{hij} \in k \subseteq Q$ such that, for all $1 \leq j \leq c$,

$$f_j = \sum_{1 \leq h \leq i \leq n} w_{hij} x_h x_i.$$

The key point in (3) is that the w_{hij}'s are actually in k. For example, we may take Q to be either $k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ or $k[[x_1, \ldots, x_n]]$ with each f_j a nonzero quadratic form in $k[x_1, \ldots, x_n]$.

Our goal in this section is to show that the DGA $\mathcal{A}'$ described in Theorem 3.32 is formal when R is a homogeneous quadratic complete intersection. Recall that a DGA is *formal* if it is quasi-isomorphic to its cohomology viewed as a DGA with trivial differential. This is a special case of a result of Beilinson, Ginzburg, and Soergel for Koszul algebras in [4]. It was also shown by Dyckerhoff [11, Section 5.5] in the case of a hypersurface $R = Q/(f)$ with $Q = k[[x_1, \ldots, x_n]]$ and f a nonzero quadratic form.

Proposition 3.37. *Let $R = Q/(f_1, \ldots, f_c)$ be a quadratic complete intersection and let M be a finitely generated R-module. Let $\mathcal{A}'$ be the DGA described in Theorem 3.32. Then $\mathcal{A}'$ is formal.*

Proof. We define a DGA $\mathcal{B}$ over k with generators

$$\{T_1, \ldots, T_c, X_1, \ldots, X_n\}$$

with $|T_j| = 2$ and $|X_i| = 1$, subject to the relations

$$\left\{ X_i^2 + \sum_{j=1}^{c} w_{iij} T_j,\ [T_h, T_i],\ [X_h, X_i] + \sum_{j=1}^{c} w_{hij} T_j,\ [T_h, X_i] \right\}$$

with $w_{hij} \in k$ chosen via equation 3.31, and with trivial differential. Since $k \subseteq Q$, we see that $\mathcal{B} \subseteq \mathcal{A}'$. In fact, this containment is a quasi-isomorphism. Recalling that $S = Q[T_1, \ldots, T_c]$, observe that

$$\mathcal{B} \cong \Lambda_S(S^n) \cong \Lambda_k(k^n)[T_1, \ldots, T_n] \cong H^*(\mathcal{A}'),$$

where S^n and k^n each have the basis $X_1, \ldots, X_n$. Thus, $\mathcal{A}'$ is formal. $\square$

3.6.3 Hochschild Cohomology

We now look at the Hochschild cohomology of the localization of a polynomial ring with respect to the maximal ideal generated by the variables. This is an application of Theorem 3.29 involving a module $M = Q/I \neq k$. A related computation is given by Buchweitz and Roberts in [7].

Before considering the example, we recall the notion of Hochschild cohomology.

Definition 3.38. Let A be a commutative ring and let B be a flat commutative A-algebra.

Then the *Hochschild cohomology of B over A* is defined by

$$HH^*(B/A) := \operatorname{Ext}^*_{B\otimes_A B}(B,B),$$

where B is viewed as a $(B\otimes_A B)$-module via multiplication.

Consider the local ring $B = k[z_1,\dots,z_n]_{m_B}$, where $m_B = (z_1,\dots,z_n)$. Let $g_1,\dots,g_c$ be a B-regular sequence contained in m_B and set $A = k[u_1,\dots,u_c]_{(u_1,\dots,u_c)}$. We show that there is a flat k-algebra homomorphism $\varphi : A \to B$ defined by $\varphi(u_j) = g_j$. Indeed, consider the k-algebra homomorphism $\tilde{\varphi} : k[u_1,\dots,u_c] \to B$ defined by $\tilde{\varphi}(u_j) = g_j$. If we have $h(u_1,\dots,u_c) \in k[u_1,\dots,u_c]$ with $h(0,\dots,0) \neq 0$, then $\tilde{\varphi}(h) = h(f_1,\dots,f_c) \notin m_B$. Thus $\tilde{\varphi}(h)$ is invertible, so that φ is a well-defined k-algebra homomorphism. Viewing B as an A-algebra via φ, we see that B is flat, since for $i>0$ we have

$$\operatorname{Tor}^i_A(k,B) = H^i(\operatorname{Kos}(u_1,\dots,u_c;A)\otimes_A B) \cong H^i(\operatorname{Kos}(g_1,\dots,g_c;B)) = 0$$

as $g_1,\dots,g_c$ form a B-regular sequence.

Note that $\mathrm{HH}^*(B/A) = \operatorname{Ext}^*_{B\otimes_A B}(B,B)$ is a $(B\otimes_A B)$-module. If $\mu : B\otimes_A B \to B$ denotes the canonical multiplication map on $B\otimes_A B$, then observe that $\ker\mu$ acts trivially on B, and so $HH^*(B/A)$ is actually a B-module via μ.

When considering $B\otimes_A B$, we will let $z_1,\dots,z_n$ denote the variables for the first factor of B and let $w_1,\dots,w_n$ denote the variables for the second factor of B. Though $B\otimes_A B$ is not a local ring, $m := (z_1,\dots,z_n,w_1,\dots,w_n)$ is a maximal ideal with residue field k. Since $\mu^{-1}(m_B) = m$, if $\alpha \in B\otimes_A B$ with $\alpha \notin m$, then $\mu(\alpha) \notin m_B$ and is therefore invertible. Hence, localizing the Hochschild cohomology modules does not change the modules, i.e.,

$\mathrm{HH}^*(B/A)_m \cong \mathrm{HH}^*(B/A)$. In particular, we have

$$HH^*(B/A) \cong \mathrm{Ext}^*_{(B\otimes_A B)_m}(B, B).$$

Define $Q := (B \otimes_k B)_m = k[z_1, \ldots, z_n, w_1, \ldots, w_n]_{(z_1,\ldots,z_n,w_1,\ldots,w_n)}$, where we now view $m = (z_1, \ldots, z_n, w_1, \ldots, w_n)$ as a maximal ideal of $B \otimes_k B$. For each $1 \leq j \leq c$, set $f_j := g_j(z_1, \ldots, z_n) - g_j(w_1, \ldots, w_n) \in Q$ and $R := Q/(f_1, \ldots, f_c)$. Observe that Q is a regular local ring, $f_1, \ldots, f_c$ is a Q-regular sequence, and R is a complete intersection. Furthermore, we see that $(B \otimes_A B)_m \cong R$ by realizing it as a quotient of $Q = (B \otimes_k B)_m$ by identifying the actions of u_j through φ on each factor of B. Thus,

$$(B \otimes_A B)_m \cong (B \otimes_k B)_m/(g_1 \otimes 1 - 1 \otimes g_1, \ldots, g_c \otimes 1 - 1 \otimes g_c) = Q/(f_1, \ldots, f_c) = R.$$

Set $x_i := z_i - w_i \in Q$ and $I := (x_1, \ldots, x_n) \subseteq Q$. Observe that $x_1, \ldots x_n$ form a Q-regular sequence. For each $1 \leq j \leq c$, we have $f_j \in I$. Note that we can realize the $(B \otimes_A B)_m$-module $M := B$ as a quotient of $Q = (B \otimes_k B)_m$ by identifying the actions of z_i on the first factor of Q with the action of w_i on the second factor. So

$$M = B \cong (B \otimes_k B)_m/(z_1 - w_1, \ldots, z_n - w_n) = Q/I.$$

Therefore, $HH^*(B/A) \cong \mathrm{Ext}^*_{(B\otimes_A B)_m}(B, B) \cong \mathrm{Ext}^*_R(M, M)$ where $R = Q/(f_1, \ldots, f_c)$ is a complete intersection, $M = Q/I$ with each $f_j \in I$, and I is generated by the regular sequence $x_1, \ldots, x_n$. We may now apply Theorem 3.29 to obtain the following Proposition.

Proposition 3.39. *Let $B = k[z_1, \ldots, z_n]_{(z_1,\ldots,z_n)}$ and let $g_1, \ldots, g_c$ be a B-regular sequence. Let $A = k[u_1, \ldots, u_c]_{(u_1,\ldots,u_c)}$ and consider B as a flat A-algebra via $\varphi : A \to B$ defined by $\varphi(u_i) = g_i$. Choose elements $v_{ij} \in Q = (B \otimes_k B)_{(z_1,\ldots,z_n,w_1,\ldots,w_n)}$ which satisfy the equation*

$g_j(z_1, \ldots, z_n) - g_j(w_1, \ldots, w_n) = \sum_{i=1}^{n} v_{ij}(z_i - w_i)$. *Then* $HH^*(B/A)$ *is quasi-isomorphic to the cohomology of the DGA over* Q *with generators*

$$\{T_1, \ldots, T_c, y_1, \ldots, y_n, Y_1, \ldots, Y_n\}$$

with $|T_j| = 2$, $|y_i| = -1$, *and* $|Y_i| = 1$, *subject to the relations*

$$\{y_i^2,\ Y_i^2,\ [T_h, T_i],\ [y_h, y_i],\ [Y_h, Y_i],\ [T_h, y_i],\ [T_h, Y_i],\ [y_h, Y_i] - \delta_{hi}\},$$

with differential given by $d(T_j) = 0$, $d(y_i) = z_i - w_i$, *and* $d(Y_i) = \sum_{j=1}^{c} v_{ij} T_j$.

To give an explicit description of $\mathrm{HH}^*(B/A)$, we need to find $v_{ij} \in Q$ which satisfy $f_j = \sum_{i=1}^{n} v_{ij} x_i$. Such v_{ij}'s must satisfy a nice property which we now illustrate.

For notational simplicity, we let $Z := (z_1, \ldots, z_n)$ and $W := (w_1, \ldots, w_n)$. Given any $h(Z, W) \in Q = k[z_1, \ldots, z_n, w_1, \ldots, w_n]_{(z_1, \ldots, z_n, w_1, \ldots, w_n)}$, let $\overline{h}(Z)$ denote the rational function in $k[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)}$ obtained by setting $w_i = z_i$. We claim the v_{ij}'s satisfy $\overline{v_{ij}} = \dfrac{\partial g_j(Z)}{\partial z_i}$. Recall that $f_j(Z, W) = g_j(Z) - g_j(W)$ and $x_i = z_i - w_i$. So for $1 \le h \le n$ and $1 \le j \le c$, there exist $v_{hj}(Z, W) \in Q$ with

$$f_j(Z, W) = \sum_{h=1}^{n} v_{hj}(Z, W)(z_h - w_h).$$

Taking the partial derivative with respect to z_i on both sides, we obtain

$$\frac{\partial g_j(Z)}{\partial z_i} = \frac{\partial f_j(Z, W)}{\partial z_i} = \sum_{h=1}^{n} \frac{\partial v_{hj}(Z, W)}{\partial z_i}(z_h - w_h) + v_{ij}(Z, W).$$

Setting $w_i = z_i$, we obtain $\dfrac{\partial g_j(Z)}{\partial z_i} = \overline{v_{ij}}(Z)$.

We end by verifying this calculation with an explicit example when $c = 1$. Let $B =$

$k[z_1, z_2]_{(z_1,z_2)}$ and let $g = z_1^2 + 2z_2^3$. We then have $Q = k[z_1, z_2, w_1, w_2]_{(z_1,z_2,w_1,w_2)}$ as well as $f = z_1^2 + 2z_2^3 - w_1^2 - 2w_2^3$. We need to write $f = v_1(z_1 - w_1) + v_2(z_2 - w_2)$ (where $v_1 = v_{11}$ and $v_2 = v_{21}$). We may choose $v_1 = z_1 + w_1$ and $v_2 = 2(z_2^2 + z_2 w_2 + w_2^2)$. Note that

$$\overline{v_1} = 2z_1 = \frac{\partial g}{\partial z_1} \qquad \text{and} \qquad \overline{v_2} = 6z_2^2 = \frac{\partial g}{\partial z_2},$$

so these satisfy the above property.

Our choices for v_1 and v_2 are not unique. For example, we could have also chosen $v_1 = z_1 + w_1 + (z_2 - w_2)$ and $v_2 = 2(z_2^2 + z_2 w_2 + w_2^2) - (z_1 - w_1)$. These satisfy our defining equation as well as $\overline{v_i} = \dfrac{\partial g}{\partial z_i}$ for $i = 1, 2$.

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