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PRIME IDEALS IN TWO-DIMENSIONAL NOETHERIAN DOMAINS AND FIBER PRODUCTS AND CONNECTED SUMS

by

Ela Celikbas

A DISSERTATION

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PRIME IDEALS IN TWO-DIMENSIONAL NOETHERIAN DOMAINS AND FIBER PRODUCTS AND CONNECTED SUMS

Ela Celikbas, Ph. D.

University of Nebraska, 2012

Advisers: Sylvia Wiegand and Mark Walker

This thesis concerns three topics in commutative algebra:

1) The projective line over the integers (Chapter 2),

2) Prime ideals in two-dimensional quotients of mixed power series-polynomial rings (Chapter 3),

3) Fiber products and connected sums of local rings (Chapter 4),

In the first chapter we introduce basic terminology used in this thesis for all three topics.

In the second chapter we consider the partially ordered set (poset) of prime ideals of the projective line $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ over the integers $\mathbb{Z}$, and we interpret this poset as $\mathrm{Spec}(\mathbb{Z}[x]) \cup \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$ with an appropriate identification.

We have some new results that support Aihua Li and Sylvia Wiegand's conjecture regarding the characterization of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. In particular we show that a possible axiom for $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ proposed by Arnavut, Li and Wiegand holds for some previously unknown cases.

We study the sets of prime ideals of polynomial rings, power series rings and mixed power series-polynomial rings in Chapter 3. Let R be a one-dimensional Noetherian domain and let x and y be indeterminates. We describe the prime spectra of certain two-dimensional quotients of mixed power series/polynomial rings over R, that is, $\mathrm{Spec}(\frac{R[[x]][y]}{Q})$ and $\mathrm{Spec}(\frac{R[y][[x]]}{Q'})$, where Q and Q' are certain height-one prime

ideals of $R[[x]][y]$ and $R[y][[x]]$ respectively.

In the last chapter we describe some ring-theoretic and homological properties of fiber products and connected sums of local rings. For Gorenstein Artin k-algebras R and S where k is a field, the connected sum, $R\#_k S$, is a quotient of the classical fiber product $R \times_k S$. We give basic properties of connected sums over a field and show that certain Gorenstein local k-algebras decompose as connected sums. We generalize structure theorems given by Sally, Elias and Rossi that show two types of Gorenstein local k-algebras are connected sums.

DEDICATION

Lovingly dedicated to my husband, Olgür, my beloved parents, Ali and Leyla, and my dearest sister, Hülya, for their love and support each step of this journey.

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Contents

Chapter 1

Introduction with Basic Commutative Algebra Terms

We discuss prime ideals in Noetherian rings and decomposition of certain Gorenstein rings in this thesis. For this we give a short list of some basic definitions and other relevant terms in commutative algebra.

1.1 Rings and Prime Ideals

A *ring* R is a nonempty set together with two binary operations, addition and multiplication, such that R is an abelian group with respect to addition; multiplication is associative and both right and left distributive over addition, that is, for all a, b, $c \in R$, $a(b+c) = ab+ac$, $(b+c)a = ba+ca$; and there exists a multiplicative identity element 1_R such that $1_R r = r = r1_R$ for all $r \in R$. All rings considered in this thesis are *commutative*; that is, $ab = ba$, for all $a, b \in R$. A *field* is a ring in which $1_R \neq 0_R$ and every nonzero element is invertible; that is, for every $a \in R$, there is an element $b \in R$ with $ab = 1_R$.

An *ideal* in a commutative ring R is a nonempty subset I such that if $a, b \in I$, then $a+b \in I$ and if $r \in R$ and $c \in I$, then $rc \in I$. An ideal P of a commutative ring R is *prime* if $P \neq R$ and if $a, b \in R$ and $ab \in P$ imply $a \in P$ or $b \in P$. The ring R is called an *integral domain* if the ideal (0) is prime. A *maximal ideal* of R is a proper ideal not contained in any other ideal except the whole ring R. If $M \subset R$ is a maximal ideal, then R/M is a field, so M is prime. A ring is called a *local ring* if it has a unique maximal ideal; a ring is *semilocal* if it has only finitely many maximal ideals. A *minimal prime ideal* of R is a prime ideal that does not contain any other prime ideal. The ring R is called *reduced* if, for every nonzero $r \in R$ and every positive integer n, $r^n \neq 0$, that is, R has no nonzero nilpotent elements.

The embedding dimension of a local ring $(R, \mathfrak{m}, k)$, denoted $\operatorname{edim}(R)$, is defined to be the minimal number of generators needed for the maximal ideal $\mathfrak{m}_R$. If I and J are ideals of R, then $(I : J) = \{x \in R \mid xJ \subseteq I\}$. In particular, $\operatorname{ann}(J) = (0 : J)$ denotes the annihilator of J.

For a ring R and a prime ideal P of R, R_P, the *localization* at P, is the set consisting of fractions with denominator not in P; that is, $R_P = \{a/b \mid a \in R,\ b \in R \setminus P\}$. If R is an integral domain, then $R_{(0)}$ consists of all fractions with nonzero denominator and is called the *field of fractions* of R. For example, the field $\mathbb{Q}$ of rational numbers is the field of fractions of the integers $\mathbb{Z}$.

A *Noetherian ring* is a ring that satisfies the ascending chain condition, that is, every strictly ascending chain $I_1 \subset \ldots \subset I_n \subset I_{n+1} \subset \ldots$ of ideals of R is "eventually stationary", that is, the chain has only finitely many terms. If R is a Noetherian ring, then R has only finitely many minimal prime ideals. In this thesis all rings are commutative and Noetherian.

A finite strictly increasing sequence of $n+1$ prime ideals $P_n \subset \ldots \subset P_1 \subset P_0$ of a ring R is called a *chain of primes* of length n. If P is a prime ideal of R, the supremum of the lengths of all chains of primes such that $P = P_0$ is called the *height* of P and is denoted by $\operatorname{ht}(P)$. The *Krull dimension* of a ring R, or simply the *dimension* of R, is denoted by $\dim(R)$ and is defined to be the supremum of the heights of prime ideals in R. For example, the ring of integers has dimension one; the ring $\mathbb{Z}[x]$ of polynomials over the integers has dimension two.

An *Artinian ring* is a ring that satisfies the descending chain condition; that is, for every strictly descending chain $I_1 \supset \ldots \supset I_i \supset I_{i+1} \supset \ldots$ of ideals of R, there exists $k \in \mathbb{N}$ such that $I_k = I_{k+i}$ for all $i \in \mathbb{N}$. A commutative ring R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal, that is, $\dim(R) = 0$, [29, Corollary 8.45].

For a ring R, an *R-module* M is a set with addition and scalar multiplication by elements of R, that satisfies for all $r, s \in R$ and $m, n \in M$:

$$r(sm) = (rs)m \qquad r(m+n) = rm + rn \qquad (r+s)m = rm + sm \qquad 1m = m.$$

A *zerodivisor* on an R-module M is an element $r \in R$ for which there exists $m \in M$ such that $m \neq 0$ but $rm = 0$. An element of R which is not a zerodivisor on M is referred to as a *non-zerodivisor* on M. A sequence $(x_1, \ldots, x_n)$ of elements of M is *M-regular* provided that x_1 is a non-zerodivisor on M and x_i is a non-zerodivisor on $M/(x_1, \cdots, x_{i-1}M)$ for $i = 2, \ldots, n$. The *depth* of an R-module M is defined as $\sup\{n \mid \exists$ an M-regular sequence $(x_1, \ldots, x_n)$ in $M\}$ and is denoted by $\operatorname{depth}_R(M)$ or $\operatorname{depth}(M)$. We denote the length of a module M by $\lambda(M)$.

We write $\mathbb{Z}$ for the ring of integers, $\mathbb{N}$ for the set of natural numbers, $\mathbb{Q}$ for the field of rational numbers, and $\mathbb{R}$ for the field of real numbers. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $|\mathbb{N}| = \aleph_0$.

Let $(R, \mathfrak{m}_R, k)$ denote a commutative local ring for the remaining items for this section. We say R is *Henselian* if for every monic polynomial $f(x) \in R[x]$ satisfying $f(x) \equiv g_0(x)h_0(x)$ modulo $\mathfrak{m}[x]$, where g_0 and h_0 are monic polynomials in $R[x]$ such that $g_0R[x] + h_0R[x] + \mathfrak{m}[x] = R[x]$, there exist monic polynomials $g(x)$ and $h(x)$ in $R[x]$ such that $f(x) = g(x)h(x)$ such that both $g(x) - g_0(x)$ and $h(x) - h_0(x) \in \mathfrak{m}[x]$. In other words, if $f(x)$ factors modulo $\mathfrak{m}[x]$ into two comaximal factors, then this factorization can be lifted back to $R[x]$, [23], [9].

A finitely generated R-module M is called *Cohen-Macaulay* (CM) if $\operatorname{depth}(M) = \dim(M)$ where $\dim(M)$ is defined as $\dim(R/\operatorname{ann}(M))$. A nonzero R-module M is called *maximal Cohen-Macaulay* (MCM) if $\operatorname{depth}(M) = \dim(R)$. We say R is a *Cohen-Macaulay* ring provided $\operatorname{depth}(R) = \dim(R)$.

The local ring R is *Gorenstein Artin* if the k-vector space $\operatorname{ann}_R(\mathfrak{m}_R)$ is one-dimensional; that is, $\dim_k(\operatorname{soc}(R)) = 1$ where $\operatorname{soc}(R) = \operatorname{ann}_R(\mathfrak{m}_R) = \{r \in R \mid r\mathfrak{m}_R = 0\}$. A commutative local ring R is Gorenstein if R is Cohen-Macaulay and $\dim_k(\operatorname{soc}(R)) = 1$.

For an Artinian local ring R, the *Loewy length* of R is $\ell\ell(R) := \max\{i \mid \mathfrak{m}_R^i \neq 0\}$.

1.2 Notation for Partially Ordered Sets

Let U be a partially ordered set, sometimes abbreviated *poset*. A *chain* in U is a totally ordered subset of U. Every poset U we study has a unique minimal element u_0 and every chain in U has finite length.

For $u \in U$, *the height of* u is denoted by $\operatorname{ht}(u)$ and is the length $t \in \mathbb{N}_0$ of a maximal length chain in U of the form $u_0 < u_1 < u_2 \cdots < u_t = u$; *the dimension of* U, $\dim(U)$, is the maximum of $\{\operatorname{ht}(u) \mid u \in U\}$; set $\max(U) = \{\text{maximal elements of } U\}$ and $\min(U) = \{\text{minimal elements of } U\}$. Set $\mathcal{H}_i(U) := \{u \in U \mid \operatorname{ht}(u) = i\}$ for each $i \in \mathbb{N}_0$.

For every pair of elements u, v of U, and every pair of subsets $S \subseteq \mathcal{H}_1(U)$, and $T \subset \mathcal{H}_2(U)$, we define

$$u^{\uparrow} = \{w \in U \mid u < w\}, \qquad v^{\downarrow} = \{w \in U \mid w < v\}, \qquad (u,v)^{\uparrow} = u^{\uparrow} \cap v^{\uparrow},$$

$$S^{\uparrow} = \{t \in U \mid t \in s^{\uparrow}, \text{ for all } s \in S\}, \quad \text{and} \quad L_e(T) = \{x \in U \mid x^{\uparrow} = T\}.$$

For R a commutative ring, the *prime spectrum* of R, denoted by $\operatorname{Spec}(R)$, is the set of all prime ideals of R. $\operatorname{Spec}(R)$ is a partially ordered set, ordered by the inclusion relation on the set of prime ideals of R. We use the same notation for the partially ordered set $\operatorname{Spec}(R)$, such as $P^{\uparrow}$ denotes the prime ideals of R properly containing $P \in \operatorname{Spec}(R)$. Similarly, for a and b elements of R, we define $a^{\uparrow} := \{P \in \operatorname{Spec}(R) \mid a \in P\}$ and $(a,b)^{\uparrow} := \{P \in \operatorname{Spec}(R) \mid a \in P \text{ and } b \in P\}$. We use $\min(R)$ for the set of minimal ideals of R, and $\max(R)$ for the set of maximal ideals of R. Put $V(S) = V_R(S) = \{\mathbf{q} \in \operatorname{Spec}(R) \mid S \subseteq \mathbf{q}\}$, for a subset S of R; for $a \in R$, put $V_R(a) = V_R(\{a\})$. For each $i \in \mathbb{N}_0$, we set $\mathcal{H}_i(R) := \{\mathbf{q} \in \operatorname{Spec}(R) \mid \operatorname{ht}(\mathbf{q}) = i\}$

We illustrate prime spectra using "Spec Graphs" in Chapters 2 and 3. The vertices of a spec graph represent the prime ideals of the spectra and each edge represents an inclusion between the two prime ideals corresponding to the endpoints of the segments.

Chapter 2

The Projective Line over the Integers

The contents of this chapter are contained in the author's paper with Christina Eubanks-Turner: *Projective Line over the Integers*, which appeared in De Gruyter Proceedings in Mathematics, Progress in Commutative Algebra 2.

2.1 Introduction

Let h and k be indeterminates over the integers $\mathbb{Z}$. The *projective line* $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ *over the integers* can be viewed as the partially ordered set under inclusion of all prime ideals of $\mathbb{Z}[h,k]$ that are generated by finite sets of homogeneous polynomials in h and k other than those prime ideals that contain both h and k. For x an indeterminate over $\mathbb{Z}$, the prime spectrum of $\mathbb{Z}[x]$ or $\mathrm{Spec}(\mathbb{Z}[x])$, the partially ordered set of prime ideals of $\mathbb{Z}[x]$ under inclusion, is sometimes called the *affine line* over $\mathbb{Z}$. In this chapter we let $x = h/k$ and we view $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ as the union of its affine pieces $\mathrm{Spec}(\mathbb{Z}[x])$ and $\mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$. In this view of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, the intersection of $\mathrm{Spec}(\mathbb{Z}[x])$ with $\mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$ is identified with $\mathrm{Spec}(\mathbb{Z}[x,\frac{1}{x}])$; cf. Notation 2.2.6(2).

In 1986, Roger Wiegand gave five axioms that characterize the prime spectrum of $\mathbb{Z}[x]$ as a partially ordered set; see [31] and Definition 2.2.3 below. Four of those axioms hold for $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, but $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ fails to satisfy the key fifth axiom of $\mathrm{Spec}(\mathbb{Z}[x])$; see [19]. So far no one has completed a characterization of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, although there have been several related results. In 1994 William Heinzer, David Lantz and Sylvia Wiegand determined those partially ordered sets that occur as the projective line $\mathrm{Proj}(R[h,k])$ when R is a one-dimensional semilocal domain. In 1997, Aihua Li and Sylvia Wiegand described some properties of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. In 2002, Meral Arnavut conjectured that a modified form of the key axiom of $\mathrm{Spec}(\mathbb{Z}[x])$ would complete a characterization of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$; she gave partial results toward her conjecture; see [3] and Axiom 2.4.2 below.

The key axiom for $\mathrm{Spec}(\mathbb{Z}[x])$ stipulates the existence of "radical elements", defined in Definition 2.2.1, for pairs (S,T) of finite subsets of $\mathrm{Spec}(\mathbb{Z}[x])$, where the elements of S have height one and those of T have height two. Radical elements often exist for sets S and T in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, but not always. We expect that the determination of when radical elements exist would lead to a characterization of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

In this chapter we continue the investigation of the projective line over the integers. In the process we give further evidence for Arnavut's conjecture. Among our main results are new cases when radical elements exist, such as Theorem 2.5.5 and Theorem 2.5.8. In Theorem 2.5.5, we show the existence of radical elements when every maximal ideal of $T \cap \mathrm{Spec}(\mathbb{Z}[x])$ has form $(x,p)\mathbb{Z}[x]$, where p is a prime integer; each (x,p) corresponds to exactly one maximal ideal of form $(\frac{1}{x},p)\mathbb{Z}[\frac{1}{x}] \in T \cap \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$; and vice versa. In Theorem 2.5.8, we find radical elements for sets of form

$$S = \{(p_1),\ldots,(p_n),(x),(\tfrac{1}{x}),(x-a),(x-b)\}, \text{ and}$$

$$T = \{(x,p_1),\ldots,(x,p_\ell),(\tfrac{1}{x},p_{\ell+1}),\ldots,(\tfrac{1}{x},p_n)\},$$

where the p_i are prime integers relatively prime to $a, b \in \mathbb{Z}$, under certain conditions. It is difficult to produce prime ideals that are the correct radical elements. For the proof of Theorem 2.5.5, we use Hilbert's Irreducibility Theorem to find radical elements. For the proof of Theorem 2.5.8, we use Euler's theorem. Theorem 2.5.8 is a special case of the conjecture and answers a question in Arnavut's paper [3].

In section 2.2 we restate relevant notation, definitions and previous results of Meral Arnavut, Aihua Li and Sylvia Wiegand from [3], [18], and [19]. In section 2.3 we discuss the *coefficient subset* of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ from [18]. A coefficient subset of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ behaves like the set of all prime ideals of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ generated by prime integers. In section 2.4 we summarize Meral Arnavut's results towards the conjecture. Our new results are in section 2.5; they all support the conjecture.

2.2 Definitions and Previous Results

In this chapter we use the notation for partially ordered sets defined in Section 1.2.

In Definition 2.2.3, we give the five axioms that Roger Wiegand showed characterize $\mathrm{Spec}(\mathbb{Z}[x])$ as a partially ordered set; see [31]. The key axiom is easier to state if we first define "radical element".

Definition 2.2.1. ([18]) Let U be a partially ordered set of dimension two and let S and T be finite subsets of U such that $\emptyset \neq S \subseteq H_1(U)$ and $T \subseteq H_2(U)$. If $w \in H_1(U)$ satisfies (1) and (2), then w is called a *radical element* for (S, T):

(1) $w < t$, for every $t \in T$,

(2) Whenever $m \in U$ is greater than both w and s, for some $s \in S$, then $m \in T$.

(In other words, w is a *radical element* for (S, T) if and only if $\bigcup_{s \in S} (w, s)^{\uparrow} \subseteq T \subset w^{\uparrow}$.)

The following picture illustrates the relations between a radical element and the associated sets S and T in a two-dimensional poset:

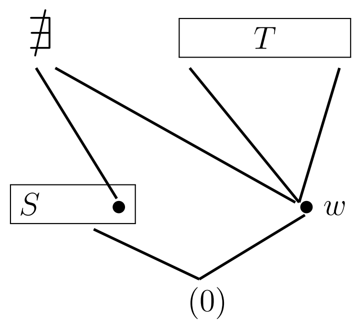

Figure 2.2.1. Radical Element

For convenience we also introduce the following notation that is used later.

Notation 2.2.2. A ht(1,2)-*pair* of a poset U is a pair (S, T) of finite subsets S and T of U such that $\emptyset \neq S \subseteq H_1(U)$ and $T \subseteq H_2(U)$.

Definition 2.2.3. Let U be a partially ordered set. The following five axioms are called the *Countable Integer Polynomial (CZP) Axioms*:

(P1) U is countable and has a unique minimal element.

(P2) U has dimension two.

(P3) For each element u of height-one, $u^{\uparrow}$ is infinite.

(P4) For each pair u, v of distinct elements of height-one, $(u, v)^{\uparrow}$ is finite.

(RW) Every ht(1,2)-pair of U has at least one radical element in U.

Note: Such a set also satisfies Axiom P3′ below, which follows from Axiom RW.

(P3′) For every height-two element t, the set $t^{\downarrow}$ is infinite.

Axiom RW is essential because it distinguishes $\operatorname{Spec}(\mathbb{Z}[x])$ from other similar prime spectra such as $\operatorname{Spec}(\mathbb{Q}[x, y])$ [31]. The following theorem from R. Wiegand shows that the CZP axioms characterize $\operatorname{Spec}(\mathbb{Z}[x])$.

Theorem 2.2.4. ([31]) A partially ordered set U satisfies the CZP axioms of Definition 2.2.3 if and only if U is order isomorphic to $\operatorname{Spec}(\mathbb{Z}[x])$.

Remarks 2.2.5. The first two remarks are from ([18], [19]):

(1) By Theorem 2.2.4, every ht(1,2)-pair of $\operatorname{Spec}(\mathbb{Z}[x])$ has infinitely many radical elements in $\operatorname{Spec}(\mathbb{Z}[x])$.

(2) Since $\operatorname{Spec}(\mathbb{Z}\left[\frac{1}{x}\right]) \cong \operatorname{Spec}(\mathbb{Z}[x])$, every ht(1,2)-pair of $\operatorname{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$ has infinitely many radical elements in $\operatorname{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$.

(3) The following discussion shows how the existence of radical elements is important for showing that two posets U and V that both satisfy axioms for $\operatorname{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ are order isomorphic. Since $\operatorname{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ is a countable set, we would want to define an order-isomorphism φ at each stage between finite subsets F and G of U and V respectively, and then extend φ to U and V. If u_0 and v_0 are the minimal elements of U and V respectively, S is the set of height-one elements of F, T is the set of height-two elements of F, and φ is an order-isomorphism from $F = \{0\} \cup S \cup T$ in U to $G = \{0\} \cup S' \cup T'$ in V, we would try to extend φ so that a radical element for (S, T) goes to a radical element for (S', T'). This is a simplification of the process; actually a height-one set S, and a height-two set T, might be enlarged first and φ defined on enlarged ht(1,2)-pair before defining the map φ on a radical element. The process is described more explicitly in Roger Wiegand's paper [30]. If we knew which pairs had radical elements, we could perhaps obtain such an order-isomorphism.

Notation 2.2.6. As mentioned in the introduction, the projective line over the integers, denoted by $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, where h and k are indeterminates, has two standard interpretations as a partially ordered set. The first interpretation is from algebraic geometry; the second is more ring-theoretic and is used in this paper.

(1) $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ is the set of all prime ideals of $\mathbb{Z}[h,k]$ generated by finite sets of homogeneous polynomials in the variables h and k, but not those prime ideals containing both h and k.

(2) $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]) := \mathrm{Spec}(\mathbb{Z}[x]) \cup \mathrm{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$, where $\mathrm{Spec}(\mathbb{Z}[x]) \cap \mathrm{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$ is identified with $\mathrm{Spec}(\mathbb{Z}[x,\frac{1}{x}])$. In this identification each prime ideal of the form $p\mathbb{Z}[x]$, where p is a prime integer, is considered the same as $p\mathbb{Z}[\frac{1}{x}]$, and $f(x)\mathbb{Z}[x]$ is identified with $x^{-\deg(f)}f(x)\mathbb{Z}[\frac{1}{x}]$, for every irreducible polynomial $f(x)$ of $\mathbb{Z}[x] \setminus x\mathbb{Z}[x]$ with $\deg(f) > 0$.

In particular, in the second view, if $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is irreducible, and $a_n \neq 0$ and $a_0 \neq 0$, then we identify $(f(x)) \in \mathrm{Spec}(\mathbb{Z}[x])$ with $(\frac{1}{x^n}f(x)) \in \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$, written $(f(x)) \sim (\frac{1}{x^n}f(x))$, where

$$\frac{1}{x^n}f(x) = a_0(\tfrac{1}{x})^n + \cdots + a_{n-1}(\tfrac{1}{x}) + a_n.$$

Thus $(x^2+2x+3)\mathbb{Z}[x] \sim (1+\frac{2}{x}+\frac{3}{x^2})\mathbb{Z}[\frac{1}{x}]$. The only elements of $\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$ that are not in $\mathrm{Spec}(\mathbb{Z}[x])$ are the height-one prime $\frac{1}{x}\mathbb{Z}[\frac{1}{x}]$ and the height-two maximals $(p,\frac{1}{x})\mathbb{Z}\left[\frac{1}{x}\right]$, where p is a prime integer. Similarly $x\mathbb{Z}[x]$ is the only height-one element of $\mathrm{Spec}(\mathbb{Z}[x])$ not in $\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$, and $\{(p,x)\mathbb{Z}[x],\ p \text{ is a prime integer}\}$ is the set of all the height-two elements that are in $\mathrm{Spec}(\mathbb{Z}[x])$ but not in $\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{x}\right])$.

Here is an illustration of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ with this interpretation, from [19].

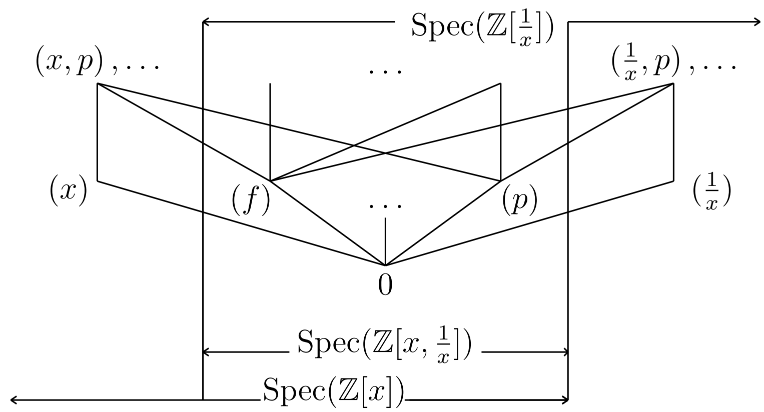

Figure 2.2.6. $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$

The following proposition is useful for finding radical elements. The proof is straightforward and is omitted.

Proposition 2.2.7. ([3]) Let $f(x) = a_n x^n + ... + a_0 \in \mathbb{Z}[x]$, where $a_0, ..., a_n \in \mathbb{Z}$ and $a_n \neq 0$, let $\ell(f)$ denote the leading coefficient a_n of $f(x)$ and let $c(f)$ denote the constant term a_0 of $f(x)$.

(1) If p is a prime integer, then

(a) $f(x) \in (x,p)\mathbb{Z}[x] \iff p \mid c(f)$;

(b) $(f(x)) = (\frac{1}{x^n} f(x)) \subseteq (\frac{1}{x}, p)\mathbb{Z}\left[\frac{1}{x}\right] \iff p \mid \ell(f)$.

(2) If $f(x)$ is an irreducible element of $\mathbb{Z}[x]$ of positive degree in x, then

(c) $\ell(f) = \pm 1 \iff (f, \frac{1}{x})^{\uparrow} = \emptyset \iff (f)^{\uparrow} \subseteq \mathrm{Spec}(\mathbb{Z}[x])$.

(d) $c(f) = \pm 1 \iff (f, x)^{\uparrow} = \emptyset \iff (f)^{\uparrow} \subseteq \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$.

By the following theorem, some adjustment of the CZP axioms of Definition 2.2.3 is necessary in order to describe $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Theorem 2.2.8. ([19]) $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ satisfies Axioms P1-P4 of Definition 2.2.3, but does not satisfy Axiom RW of Definition 2.2.3. Thus $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]) \ncong \mathrm{Spec}(\mathbb{Z}[x])$.

The following example shows that the (RW) axiom fails for $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$:

Example 2.2.9. ([19]) Let $S = \{(\frac{1}{x}), (2), (5)\}$ and $T = \{(x, 2), (\frac{1}{x}, 2), (\frac{1}{x}, 3)\}$ in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. Then the pair (S, T) does not have a radical element in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Proof. Suppose $w \in \mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ is a radical element for (S, T). Then, in order to satisfy Definition 2.2.1.1, $w \subset (x, 2)$ and $w \subset (\frac{1}{x}, 3)$, and so w cannot be generated by a prime integer. Also w cannot be (x) or $(\frac{1}{x})$. Thus $w = (g(x))$, for some irreducible polynomial $g(x) \in \mathbb{Z}[x]$ of positive degree. Write $g(x) = a_nx^n + \cdots + a_1x + a_0$, where $n \geq 1$, $a_0 \neq 0$, and $a_i \in \mathbb{Z}$. Since w is a radical element, $\bigcup_{s\in S}(w, s)^{\uparrow} \subseteq T$.

Case 1: Suppose there exists i, $1 \leq i \leq n$, such that 5 does not divide a_i. Then, modulo 5, g has positive degree. Thus the image $\overline{g}$ of g in $\mathbb{Z}/5\mathbb{Z}$ has at least one irreducible factor $\overline{g_1}$ of positive degree over $\mathbb{Z}/5\mathbb{Z}$, and g_1 can be considered in $\mathbb{Z}[x]$. Now $(5) \in (g_1, 5)$ and $w = (g(x)) \in (g_1, 5)$. But $(g_1, 5) \notin T$, and $(g_1, 5) \in (w, 5)^{\uparrow} \setminus T$. This contradicts Definition 2.2.1(2). Thus Case 1 does not occur.

Case 2: $5 \mid a_i$, for every $i > 0$. Then 5 does not divide a_0 since $g(x)$ is irreducible in $\mathbb{Z}[x]$. Thus $w = (g(x)) \subset (\frac{1}{x}, 5)$, since $5 \mid a_n$, by Proposition 2.2.7(1)(b). Also $(5) \subset (\frac{1}{x}, 5)$; thus $(\frac{1}{x}, 5) \in (w, 5)^{\uparrow}$. But $(\frac{1}{x}, 5) \notin T$, again a contradiction. Therefore (S, T) has no radical element. $\square$

Remark 2.2.10. If there is a radical element w for a ht(1,2)-pair in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, then $w \notin S$. Otherwise, $w^{\uparrow} \subseteq T$ by Definition 2.2.1(1), and this would imply T is infinite by (P3) of Definition 2.2.3, a contradiction.

Our goal in this paper is to determine answers to Questions 2.2.11.

Question 2.2.11. For which ht(1,2)-pairs of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ do radical elements exist? Which pairs have no radical element?

In what follows we obtain partial answers to these questions.

2.3 The Coefficient Subset and Radical Elements of Proj$(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$

In this section we give some more background and describe various ht(1,2)-pairs of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ in order to obtain partial answers to Question 2.2.11. In particular the "coefficient" subset C_0 of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ of prime ideals generated by prime elements of $\mathbb{Z}$ is relevant. It is more feasible that a ht(1,2)-pair (S, T) has a radical element if, for every prime element p of $\mathbb{Z}$ with $(p) \in S$, there is a maximal ideal $M \in T$ so that $p \in M$ Proposition 2.3.5.

First in Proposition 2.3.1 we observe that some ht(1,2)-pairs (S, T) inherit infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ from $\mathrm{Spec}(\mathbb{Z}[x])$ or $\mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$. This is because $\mathrm{Spec}(\mathbb{Z}[x])$ and $\mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$ are CZP Theorem 2.2.4.

Proposition 2.3.1. ([19])

Every ht(1,2)-pair (S, T) of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ has infinitely many radical elements in case (1) or (2) hold:

(1) For every $s \in S$, $s^{\uparrow} \subseteq \mathrm{Spec}(\mathbb{Z}[x])$, and $T \subseteq \mathrm{Spec}(\mathbb{Z}[x])$.

(2) For every $s \in S$, $s^{\uparrow} \subseteq \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$, and $T \subseteq \mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$.

Next we consider subsets of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ like the subsets of prime ideals generated by all prime integers of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$. We consider the existence of radical elements for various ht(1,2)-pairs subject to conditions involving such a "coefficient" subset.

Definition 2.3.2. ([3]) Let U be a poset of dimension two. A subset C of height-one elements is called a *coefficient subset of* U if

(1) For every $p \in C$, $p^{\uparrow}$ is infinite;

(2) For every pair p, q of distinct elements of C, $p \neq q \in C$, $(p,q)^{\uparrow} = \emptyset$;

(3) $\bigcup_{p \in C} p^{\uparrow} = H_2(U)$;

(4) For every $p \in C$ and $u \in H_1(U) \backslash C$, we have $(p,u)^{\uparrow} \neq \emptyset$, and $p^{\uparrow} = \bigcup_{v \in H_1(U) \backslash C} (p,v)^{\uparrow}$.

Definition 2.3.3. Let $A \subseteq H_1(U)$, with $(a,b)^{\uparrow} = \emptyset$ for every $a, b \in A$. A coefficient subset C is said to be *attached to* A if, for every $p \in C$ and every $a \in A$, $|(p,a)^{\uparrow}| = 1$.

Example 2.3.4. The set C_0 of all prime ideals of $\mathbb{Z}[x]$ generated by prime integers is a coefficient subset of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ attached to $\{(x), (\frac{1}{x})\}$. It is also attached to $\{(x), (\frac{1}{x}), (x-1)\}$ or $\{(x), (\frac{1}{x}), (x+1)\}$.

Proposition 2.3.5. ([19]) Let (S,T) be a ht(1,2)-pair and let C be a coefficient subset of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$. Suppose that there exist distinct elements P_0 and P_1 of C such that $P_0 \in S$ and $T \cap P_0^{\uparrow} = \emptyset$, but $T \cap P_1^{\uparrow} \neq \emptyset$. Then

(1) (S,T) has no radical element except possibly P_1,

(2) If $T \nsubseteq P_1^{\uparrow}$, then P_1 is not a radical element by Definition 2.2.1.1,

(3) There exists $Q \in H_1(\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])) \setminus C$ and $t \in P_1^{\uparrow} \cap Q^{\uparrow} \cap T$; thus P_1 is not a radical element for $(S \cup \{Q\}, T)$.

Proof. ([19]) For item 1, let $t \in T \cap P_1^{\uparrow}$. Suppose Q were a radical element for (S,T) and $Q \neq P_1$. If $Q \in C$, then $(Q, P_1)^{\uparrow} = \emptyset$ by (ii) of Definition 2.3.2, and so $t \notin Q^{\uparrow}$, a contradiction to Definition 2.2.1 for Q a radical element. Thus $Q \notin C$, and so there exists $t' \in (P_0, Q)^{\uparrow}$ by (4) of Definition 2.3.2. By hypothesis $t' \in P_0^{\uparrow} \implies t' \notin T$, again contradicting that Q is a radical element. Thus (S, T) has no radical element except possibly P_1.

Item 2 follows directly from Definition 2.2.1.

For item 3, since $P_1^{\uparrow}$ is infinite, there exists $t \in P_1^{\uparrow} \setminus T$. Now by (4) of Definition 2.3.2, there exists $Q \notin C$ with $t \in P_1^{\uparrow} \cap Q^{\uparrow}$. Thus P_1 is not a radical element for the pair $(S \cup \{Q\}, T)$. $\square$

Corollary 2.3.6. ([19]) Let (S, T) be a ht(1,2)-pair in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$. If $T \neq \emptyset$, then there exists a finite subset S' of $H_1(\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}]))$ such that $S \subseteq S'$ and (S', T) has no radical element in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$.

The following results, Proposition 2.3.7 and Theorem 2.3.8, are used later to construct radical elements in various cases.

Proposition 2.3.7. ([16], page 102, exercise 3) Let R be a domain and let y be an indeterminate over R. Suppose (i) $\{a, b\}$ is an R-sequence or (ii) $(a, b) = R$, where $b \neq 0$. Then $(a + by)$ is a prime ideal of $R[y]$.

Theorem 2.3.8. ([17], page 141) **Hilbert's Irreducibility Theorem**. If $f \in \mathbb{Q}[x_1, ..., x_r, x]$ is an irreducible polynomial, then there exist $a_1, ..., a_r \in \mathbb{Q}$ such that $f(a_1, ..., a_r, x)$ remains irreducible in $\mathbb{Q}[x]$.

Meral Arnavut shows that the coefficient subset of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ is unique. She also gives partial results concerning the existence of radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$.

Proposition 2.3.9. ([3]) $C_0 := \{p\mathbb{Z}[x] \mid p \text{ is a prime integer } \}$ is the only coefficient subset of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$.

Proof. We sketch the proof from [3] briefly. If Γ is a coefficient subset of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ such that $\Gamma \neq C_0$, then $\Gamma \cap C_0 = \emptyset$. Let p be a prime integer. Then $(p)^{\uparrow}$ is infinite and $\bigcup_{\gamma\in\Gamma}(\gamma,p)^{\uparrow} = (p)^{\uparrow}$. Hence Γ is infinite. Therefore there exist distinct elements α and β in $\Gamma - C_0 - \{(x), (\frac{1}{x})\}$; say $\alpha = (f(x))$, $\beta = (g(x))$, for two relatively prime irreducible polynomials $f(x)$ and $g(x)$ of $\mathbb{Z}[x]$ of positive degree. By Proposition 2.3.7, $(f + yg)$ is a prime ideal in $\mathbb{Z}[x, y]$, where y is an indeterminate over $\mathbb{Z}[x]$. By Hilbert's Irreducibility Theorem 2.3.8, there exists a prime integer p so that $f + pg$ is irreducible in $\mathbb{Z}[x]$. By Definition 2.3.2(2), no height-two prime ideals contain both f and g. If $(f + pg) \notin \Gamma$, we contradict $(f, g)^{\uparrow} = \emptyset$. Hence $(f + pg) \in \Gamma$. But $(f, p)^{\uparrow} \subseteq (f)^{\uparrow} \cap (f + pg)^{\uparrow} = (f, f + pg)^{\uparrow}$. This contradicts Definition 2.3.2(2). □

Remark 2.3.10. ([3]) Let (S, T) be a ht(1,2)-pair in Proj($\mathbb{Z}$[h, k]). If $T \neq \emptyset$, then (S, T) has at most one radical element in C_0.

Proposition 2.3.11. ([3]) Let (S, T) be a ht(1,2)-pair in Proj($\mathbb{Z}$[h, k]). If (S, T) has a radical element Q in Proj($\mathbb{Z}[h, k]$) then either (1) $S \cap C_0 \subseteq \bigcup_{M\in T}(M^{\downarrow} \cap C_0)$ or (2) $Q \in C_0$. In case (2), if $T = \emptyset$, then $S \subseteq C_0$; if $T \neq \emptyset$, then Q is the only radical element.

Proof. (Sketch from [3]) If (1) fails, there exists $P \in S \cap C_0$ with $T \cap P^{\uparrow} = \emptyset$. Then $Q^{\uparrow} \cap P^{\uparrow} = \emptyset$. Thus, by Definition 2.3.2(4), $Q \in C_0$. Also $\bigcup_{s\in S}(s, Q)^{\uparrow} \subseteq T \subseteq Q^{\uparrow}$. Thus if $T = \emptyset$, then $s \in C_0$, for all $s \in S$. If $T \neq \emptyset$, then T contains an element of form $(f(x), p)$, where p is a prime integer and either $f(x) \in \mathbb{Z}[x]$ has positive degree or $f(x) = \frac{1}{x}$. In either case $(f(x), p) \in Q^{\uparrow}$ implies $(p) = Q$, and so Q is unique. □

Meral Arnavut notes that, if Condition 1 of Proposition 2.3.11(1) is not satisfied, then it is difficult to find radical elements; cf. [3], and Proposition 2.3.5 and Proposition 2.3.11 of this paper.

Proposition 2.3.12. ([3]) Let (S, T) be a ht(1,2)-pair in Proj($\mathbb{Z}$[h, k]) such that

(1) $S \cap C_0 \subseteq \bigcup_{M\in T}(M^{\downarrow} \cap C_0)$, and

(2) $\bigcup_{s\in S}(s,\frac{1}{x})^{\uparrow} \subseteq T$, or (2′) $\bigcup_{s\in S}(s,x)^{\uparrow} \subseteq T$.

Then (S,T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Proof. We give the proof with hypothesis (2); the proof for (2′) is similar (replace $\frac{1}{x}$ by x). Since $(s,\frac{1}{x})^{\uparrow} \subseteq T$, for every $s \in S$, and T is finite, $(\frac{1}{x}) \notin S$. Therefore $S \subseteq \mathrm{Spec}(\mathbb{Z}[x])$. If $T = \emptyset$, then, for every $s \in S$, $s^{\uparrow} \subseteq \mathrm{Spec}(\mathbb{Z}[x])$. Thus, by Proposition 2.3.1, (S,T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[h,k])$ as desired. If $T \neq \emptyset$, let $p_1,\ldots,p_r$ denote the distinct positive prime integers such that

$$\{(p_1),\ldots,(p_r)\} = \bigcup_{M\in T}(M^{\downarrow} \cap C_0).$$

Then, for each $t \in T$, some $p_i \in t$. Let $f_1,\ldots,f_n$ be irreducible polynomials of $\mathbb{Z}[x]$ of positive degree so that $S - C_0 = \{(f_1),\ldots,(f_n)\}$. Let $T' = T - (\frac{1}{x})^{\uparrow}$. Therefore $T' \subseteq \mathrm{Spec}(\mathbb{Z}[x])$ and $S \subseteq \mathrm{Spec}(\mathbb{Z}[x])$. Since $\mathrm{Spec}(\mathbb{Z}[x])$ is CZP, there are infinitely many radical elements for (S,T') in $\mathrm{Spec}(\mathbb{Z}[x])$. By Proposition 2.3.10, (S,T') has at most one radical element in C_0. Thus (S,T') has infinitely many radical elements in $\mathrm{Spec}(\mathbb{Z}[x]) - C_0$. Let P_0 be such a radical element; say $P_0 = (f(x))$, where $f(x)$ is an irreducible polynomial of $\mathbb{Z}[x]$ of positive degree so that $f(x) \notin x\mathbb{Z}[x] \cup f_1\mathbb{Z}[x] \cup \cdots \cup f_n\mathbb{Z}[x] \cup \mathbb{Z}$. Let λ be a positive integer greater than the degree of $f(x)$. Then $f(x)$ and the product $p_1\cdots p_r f_1\cdots f_n x^{\lambda}$ are relatively prime in $\mathbb{Z}[x]$. By Proposition 2.3.7, $(yp_1\cdots p_r\ f_1\cdots f_n x^{\lambda} + f(x))$ is a prime ideal of $\mathbb{Z}[x,y]$, where y is an indeterminate over $\mathbb{Z}[x]$. By Hilbert's Irreducibility Theorem 2.3.8, for each λ, there exists a prime integer p_{λ} such that $g_{\lambda}(x) = p_{\lambda}p_1\cdots p_r f_1\cdots f_n x^{\lambda} + f(x)$ is an irreducible polynomial of $\mathbb{Z}[x]$; thus $w_{\lambda} := (g_{\lambda}(x))$ is a prime ideal of $\mathbb{Z}[x]$. For each $\lambda > \deg(f)$, w_{λ} is a radical element for (S,T) in $\mathrm{Proj}(\mathbb{Z}[h,k])$. Thus (S,T) has infinitely many radical elements. $\square$

2.4 The Conjecture for Proj($\mathbb{Z}$[h, k]) and Previous Partial Results

In Proposition 2.3.12 some conditions are given for a ht(1,2)-pair (S,T) so that there are infinitely many radical elements. Item 2 of Proposition 2.3.12 implies $(\frac{1}{x}) \notin S$ and item $2'$ implies that $(x) \notin S$. In either case, we get infinitely many radical elements in Proj($\mathbb{Z}$[h, k]). If both (x) and $(\frac{1}{x})$ belong to S, it is more difficult to find a radical element. The following conjecture first given by Aihua Li and Sylvia Wiegand, then adjusted by Meral Arnavut, addresses this case; cf. [19], [3].

Proj($\mathbb{Z}$[h, k]) Conjecture 2.4.1. ([3]) Let (S,T) be a ht(1,2)-pair in Proj($\mathbb{Z}$[h, k]). Assume

(1) $S \cap C_0 \subseteq \bigcup_{m \in T}(m^{\downarrow} \cap C_0)$, and

(2) $(x) \in S$, $(\frac{1}{x}) \in S$.

Then there exist infinitely many radical elements for (S,T) in Proj($\mathbb{Z}$[h, k]).

It appears that some axiom regarding the existence of radical elements analogous to Axiom RW is necessary for Proj($\mathbb{Z}$[h, k]). The following axiom was proposed by Li and Wiegand and modified by Arnavut, cf. [19], [3].

Axiom 2.4.2. *Axiom P5.* ([3]) Let U be a poset of dimension two.

(P5a) There exist a unique coefficient subset Γ of U and special elements $u_1, u_2 \in U$ such that $(u_1, u_2)^{\uparrow} = \emptyset$ and Γ is attached to u_1 and u_2. (Thus, for every $\gamma \in \Gamma$, $|(\gamma, u_1)^{\uparrow}| = 1 = |(\gamma, u_2)^{\uparrow}|$.)

(P5b) Let S be a nonempty finite subset of $H_1(U)$ and let T be a nonempty finite subset of $H_2(U)$.

(P5b.1) If $\gamma^{\uparrow} \cap T \neq \emptyset$, for every $\gamma \in S \cap \Gamma$, then there exist infinitely many radical elements for (S, T).

(P5b.2) If there exists an element $\gamma \in S \cap \Gamma$ such that $\gamma^{\uparrow} \cap T = \emptyset$, then there is at most one possible radical element γ_0 for (S, T), and $\gamma_0 \in \Gamma \setminus S$.

Arnavut shows that Conjecture 2.4.1 implies Axiom P5 above for $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ and that $U := \mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]) \setminus C_0$ is CZP; cf. [3]. We believe that this will lead to a complete characterization of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

We give a special case of the Conjecture 2.4.1 when $T = \emptyset$.

Proposition 2.4.3. ([3]) Suppose S is a finite subset of $H_1(\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]))$ of the form

$$S = \{(x), (\tfrac{1}{x}), (f_1), \ldots, (f_n)\},$$

where $f_1, \ldots, f_n$ are monic irreducible polynomials of $\mathbb{Z}[x]$ of positive degree. Then $(S, \emptyset)$ has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Remark 2.4.4. Similarly one can find infinitely many radical elements if $T = \emptyset$ and S is a finite subset of $H_1(\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]))$ such that $S = \{(x), (\frac{1}{x}), (f_1), ..., (f_n)\}$, where $f_1, ..., f_n$ are irreducible polynomials in $\mathbb{Z}[x]$ of positive degree with $c(f_i) = \pm 1$. However we do not know what happens when $T = \emptyset$, $c(f_i) \neq \pm 1$ and $\ell(f_i) \neq \pm 1$. In this case if there is a radical element $(g(x))$ where $g(x)$ is an irreducible polynomial, then $c(g) = \pm 1$, $\ell(g) = \pm 1$ and $(g(x), f_i(x)) = 1$. If we could find such radical elements, the conjecture would hold for $T = \emptyset$. This might help prove the conjecture for the $T \neq \emptyset$ case as well.

Meral Arnavut introduces the following notation and gives some partial results related to the conjecture, recorded here as Theorem 2.4.6, cf. [3].

Notation 2.4.5. Let T be a nonempty finite subset of $H_2(\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]))$.

Let $F := \{p \in \mathbb{Z},\ p \text{ prime} \mid (x,p) \in T\}$ and let $G := \{p \in \mathbb{Z},\ p \text{ prime} \mid (\frac{1}{x},p) \in T\}$. Then $A_1 := F \setminus G$, $A_2 := F \cap G$, and $A_3 := G \setminus F$ are disjoint sets.

Define $a_i := \prod_{p\in A_i} p$, for $i = 1,2,3$. Thus a_1, a_2 and a_3 are pairwise relatively prime integers. For each i, if $A_i = \emptyset$, we set $a_i = 1$. Now let $n \in \mathbb{N}$, and define $f_n(x) \in \mathbb{Z}[x]$ by

$$f_n(x) := \begin{cases} a_3x^n + a_1, & \text{if } F \cap G = \emptyset \ \ (i.e., a_2 = 1) \\ a_2^n a_3 x^2 + a_1a_3x + a_2^n a_1, & \text{if } F \cap G \neq \emptyset \end{cases}$$

Theorem 2.4.6. ([3]) Let (S,T) be an ht(1,2)-pair in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ and let F, G, A_1, A_2, A_3, a_1, a_2, a_3 and f_n be as in as in Notation 2.4.5. Suppose

- $T \subseteq (x)^{\uparrow} \cup (\frac{1}{x})^{\uparrow}$,
- $S \cap C_0 \subseteq \{(p) \mid p \in F \cup G\}$,
- $(x) \in S$, $(\frac{1}{x}) \in S$.

Then:

(1) If $(s,f_n)^{\uparrow} \subseteq T$, for every $s \in S \setminus (C_0 \cup \{(x),(\frac{1}{x})\})$, then (f_n) is a radical element for (S,T) in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

(2) If $S \setminus C_0 = \{(x),(\frac{1}{x})\}$, then (S,T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

(3) If $F \cap G = \emptyset$ and, for every irreducible polynomial $f(x)$ of $\mathbb{Z}[x]$ such that $(f) \in S \setminus (C_0 \cup \{(x),(\frac{1}{x})\})$, (3i) or (3ii) holds, that is,

(3i) $\ell(f)$ is a unit, and a_1 divides every coefficient of $f(x)$ except $\ell(f)$, (3ii) $c(f)$ is a unit, and a_3 divides every coefficient of $f(x)$ except $c(f)$,

then (S,T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[h,k])$.

(4) If $F \cap G = \emptyset$ and $S \setminus C_0 = \{(x), (\frac{1}{x})\} \cup \{(x+\alpha)\}$, for some $\alpha \in \mathbb{Z}$ such that a_1 and α are relatively prime, then (S,T) has a radical element in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Corollary 2.4.7. ([3]) Let

$$S = \{(p_1), ..., (p_n), (x), (\tfrac{1}{x}), (f_1), ..., (f_m)\},$$

$$T = \{(x, p_1), ..., (x, p_\ell), (\tfrac{1}{x}, p_{\ell+1}), ..., (\tfrac{1}{x}, p_n)\},$$

where $0 \le \ell \le n$, $p_1, ..., p_n$ are distinct prime integers.

(1) If $f_i(x) \in \mathbb{Z}[x]$ has the form $x^{d_i} + p_1 ... p_\ell b_i$, for some $d_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ with $1 \le i \le m$, then (S,T) has infinitely many radical elements.

(2) If $f_i(x) \in \mathbb{Z}[x]$ has the form $b_i p_{\ell+1} ... p_n x^{d_i} + 1$, for some $d_i \in \mathbb{N}$ and $b_i \in \mathbb{Z}$ with $1 \le i \le m$, then (S,T) has infinitely many radical elements.

2.5 New Results Supporting the Conjecture

In this section we give some new results that further support Conjecture 2.4.1. We consider various different types of ht(1,2)-pairs in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Theorem 2.5.1. Let (S,T) be an ht(1,2)-pair in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. Suppose

(1) $T \subseteq (x)^{\uparrow}$,

(2) $S \cap C_0 \subseteq \{(p) \mid (x,p) \in T\ ,\ p \in \mathrm{Spec}(\mathbb{Z})\}$,

(3) $S\ \setminus\ C_0 = \{(x), (\frac{1}{x})\} \cup \{(a_1x+1), ..., (a_mx+1)\}$ for some $a_i \in \mathbb{Z}$, $i = 1, ..., m$.

Then (S,T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$.

Proof. Assume that $T \neq \emptyset$. Since $T \subseteq (x)^{\uparrow}$ by (i), we have $F \neq \emptyset$ where $F := \{p \in \mathbb{Z},\ p \text{ prime} \mid (x,p) \in T\}$. Let $\lambda \in \mathbb{N}$ be such that $\lambda \geq 2$. Define

$$g_\lambda(x) := x^{m+\lambda} + b(a_1x+1)\cdots(a_mx+1) \in \mathbb{Z}[x]$$

where the a_i are as in (3) and $b = \prod_{p\in F} p$. We show $w_\lambda = (g_\lambda(x))$ is a radical element for (S,T) in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. By Eisenstein's Criteria, $g_\lambda(x)$ is irreducible in $\mathbb{Z}[x]$. To see that w_λ satisfies Definition 2.2.1, let $t \in T$. Then $t = (x,p)$, for some $p \in F$. But $p \mid c(g_\lambda)$, and so $w_\lambda \subset t$, for every $t \in T$. Let $s \in S$ and let $M \in H_2(\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}]))$ be such that $g_\lambda(x) \in M$ and $s \subset M$. We consider $(w_\lambda, s)^{\uparrow}$ for all possible types of $s \in S$:

(1) Since $(g_\lambda(x), a_ix+1) = (1)$, $(g_\lambda(x), a_ix+1)^{\uparrow} = \emptyset$, for all $i = 1, ..., m$.

(2) Since $(g_\lambda(x), \frac{1}{x}) = (1)$, $(\frac{1}{x}, g_\lambda(x))^{\uparrow} = \emptyset$.

(3) Since $(g_\lambda(x), x) = (x,b)$, $M = (x,p)$, for some $p \in F$, and hence $M \in T$.

(4) Since $(g_\lambda(x), p) = (x^{m+\lambda}, p)$, for $p \in F$ such that $p \mid b$, $M = (x,p)$.
Thus $M \in T$, and so $(p, g_\lambda(x))^{\uparrow} \in T$.

Thus $w_\lambda = (g_\lambda(x))$ is a radical element for (S,T) in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ for each $\lambda \in \mathbb{N}$, and so there are infinitely many radical elements w_λ for (S,T).

If $T = \emptyset$, then take $b = 1$ and define

$$g_\lambda(x) := x^{m+\lambda} + (a_1x+1)\cdots(a_mx+1) \in \mathbb{Z}[x].$$

Similarly $w_\lambda = (g_\lambda(x))$ is a radical element for (S,T) in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$, for $\lambda \in \mathbb{N}$. $\square$

Remark 2.5.2. Similarly there exist infinitely many radical elements for a ht(1,2)-pair (S,T) in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$ satisfying the following:

(1) $T \subseteq (\frac{1}{x})^{\uparrow}$,

(2) $S \cap C_0 \subseteq \{(p) \mid (\frac{1}{x}, p) \in T\ ,\ p \in \mathrm{Spec}(\mathbb{Z})\}$,

(3) $S \setminus C_0 = \{(x), (\frac{1}{x})\} \cup \{(x+a_1), ..., (x+a_m)\}$ for some $a_i \in \mathbb{Z}$, $i = 1, ..., m$.

Proposition 2.5.3. Consider

$$S = \{(p_1), ..., (p_n), (x), (\tfrac{1}{x}), (x+a_1), ..., (x+a_m)\},$$
$$T = \{(x, p_1), (\tfrac{1}{x}, p_2), ..., (\tfrac{1}{x}, p_n), (x+a_1, p_1), ..., (x+a_m, p_1)\}$$

where $p_1, ..., p_n$ are distinct prime integers, $n > 1$, $a_1, \ldots, a_m \in \mathbb{Z}$ and $(a_k, p_1) = 1$ for each $k = 1, ..., m$. Then (S, T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$.

Proof. Let $\lambda \in \mathbb{N}$. Define $h_\lambda(x) = b^\lambda x^\lambda (x+a_1)...(x+a_m) + p_1^\lambda$ where $b = \prod_{i=2}^{n} p_i$. We show that $w_\lambda = (h_\lambda(x))$ is a radical element for (S, T) in $\mathrm{Proj}(\mathbb{Z}[h, k])$. First, by Eisenstein's Criteria for $\mathbb{Z}[\frac{1}{x}]$, $h_\lambda(x)$ is irreducible in $\mathbb{Z}[\frac{1}{x}]$. Also $w_\lambda \subset t$ for all $t \in T$. Let $s \in S$ and $M \in H_2(\mathrm{Proj}\ (\mathbb{Z}[h, k]))$ be such that $h_\lambda(x) \in M$ and $s \subset M$. We consider $(w_\lambda, s)^{\uparrow}$ for all possible types of $s \in S$:

Since $(h_\lambda(x), x+a_k) = (x+a_k, p_1^\lambda)$, $M = (x+a_k, p_1)$ is the only maximal ideal that contains $(h_\lambda(x), x+\alpha_k)$, for $k = 1, ..., m$, that is, $(h_\lambda(x), x+a_k)^{\uparrow} \in T$.

Since $(h_\lambda(x), x) \subseteq (x, p_1^\lambda)$, $M = (x, p_1)$ is the only maximal ideal that contains $(h_\lambda(x), x)$, that is, $(h_\lambda(x), x)^{\uparrow} \in T$.

Since $(h_\lambda(x), \frac{1}{x}) = (b^\lambda, \frac{1}{x})$, for $i = 2, ..., n$, M has form $(\frac{1}{x}, p_i)$ for some i, and the $(\frac{1}{x}, p_i)$ are the only maximal ideals that contain $(h_\lambda(x), \frac{1}{x})$, for $i = 2, ..., n$.

Since $(h_\lambda(x), p_1) = (b^\lambda x^\lambda (x+a_1)...(x+a_m))$, $M = (x, p_1)$ or $M = (x+a_k, p_1) \in T$, for some $k = 1, ..., m$, and these are the only maximal ideals that contain $(h_{\lambda(x)}, p_1)$.

If $s = (p_i)$, for $i = 2, ..., n$, then we get $(h_\lambda(x), p_i) = (p_1^\lambda, p_i) = (1)$ since $p_i \mid b$. Therefore, for each $\lambda \in \mathbb{N}$, w_λ is a radical element for (S, T). $\square$

Proposition 2.5.4. Let $\alpha_1, \ldots, \alpha_m \in \mathbb{Z}$ be such that $\gcd(\alpha_1, \ldots, \alpha_m) \neq 1$. Suppose that $p_1, \ldots, p_k$ are all the prime integers that are factors of any of the α_i and that p_1 divides each of the α_i. Say each $\alpha_i = p_1^{e_{i1}} \cdots p_k^{e_{ik}}$, for some $e_{i\ell} \geq 0$. Then let $n \geq 1$ and choose prime integers $q_1, \ldots, q_n$ distinct from $p_1, \ldots, p_k$. Let

$$B_1 := \{(x, q_j), (x+\alpha_i, q_j)\}_{1 \leq j \leq n}^{1 \leq i \leq m},\ B_2 := \{(x+\alpha_i, p_\ell) \,|\, p_\ell \nmid \alpha_i\}_{1 \leq i \leq m}^{1 \leq \ell \leq k},$$

$$B_3 := \{(x, p_1)\ \ldots\ (x, p_k)\}, \text{ and set}$$

$$S = \{(x), (\tfrac{1}{x}), (x+\alpha_1) \ldots, (x+\alpha_m)\} \cup \{(q_j)\}_{1 \leq j \leq n} \cup \{(p_\ell)\}_{1 \leq \ell \leq k},$$

$$T = B_1 \cup B_2 \cup B_3.$$

Then (S, T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$.

Proof. Let $\lambda \in \mathbb{N}$. Define $a_0 := \prod_{\binom{1 \leq \ell \leq k}{1 \leq j \leq n}} p_\ell \cdot q_j$ and $h_\lambda(x) = x^\lambda (x+\alpha_1) ... (x+\alpha_m) + a_0$.

We show that $w_\lambda = (h_\lambda(x))$ is a radical element for (S, T) in $\mathrm{Proj}(\mathbb{Z}[h, k])$.

Note that $h_\lambda(x)$ is irreducible by Eisenstein in $\mathbb{Z}[x]$, since $\gcd(\alpha_1, \ldots, \alpha_m) \neq 1$ and $p_1 \mid \gcd(\alpha_1, \ldots, \alpha_m)$. Also $w_\lambda \subset t$ for all $t \in T$.

Let $s \in S$ and $t \in H_2(\mathrm{Proj}\ (\mathbb{Z}[h, k]))$ be such that $h_\lambda(x) \in t$ and $s \subseteq t$.

If $s = (q_j)$, $(h_\lambda, q_j) = (x^\lambda (x+\alpha_1) ... (x+\alpha_m), q_j)$, for $j = 1, ..., n$, and so $(h_\lambda, q_j)^\uparrow = \{(x, q_j), (x+\alpha_1, q_j), \ldots, (x+\alpha_m, q_j)\} \subset B_1$. Similarly, if $s = (p_\ell)$, then $(h_\lambda, p_\ell)^\uparrow \subset B_2 \cup B_3$, for $\ell = 1, ..., k$. If $s = (x)$, then $(h_\lambda, x) \subset (x, q_j)$, for all $j, 1 \leq j \leq n$ and also $(h_\lambda, x) \subset (x, p_\ell)$, $1 \leq \ell \leq k$. If $s = (\frac{1}{x})$, then $(h_\lambda, \frac{1}{x}) = (1)$ because h_λ is a monic polynomial of $\mathbb{Z}[x]$. If $s = (x+\alpha_i)$ for some i, $1 \leq i \leq m$, then $(h_\lambda, x+\alpha_i) \subset (x+\alpha_i, q_j) \in B_1$ for some $j, 1 \leq j \leq \ell$. If $p_\ell \nmid \alpha_i$, then $(h_\lambda, x+\alpha_i) \subset (x+\alpha_i, p_\ell) \in B_2$ and if $p_\ell \mid \alpha_i$, $(h_\lambda, x+\alpha_i) \subset (x+\alpha_i, p_\ell) \in B_3$, for each $i, 1 \leq i \leq m$ and $1 \leq \ell \leq k$.

Therefore, in any of the latter cases, $(h_\lambda, x+\alpha_i)^\uparrow \subset T$, for $i, 1 \leq i \leq m$. Thus w_λ is a radical element for (S, T) in $\mathrm{Proj}(\mathbb{Z}[h, k])$. Now, since $\lambda \in \mathbb{N}$, there are infinitely many w_λ in $\mathrm{Proj}(\mathbb{Z}[h, k])$ and so (S, T) has infinitely many radical elements. $\square$

Theorem 2.5.5. There exist infinitely many radical elements for every ht(1,2)-pair in Proj($\mathbb{Z}$[h, k]) of the form $S = \{(x), (\frac{1}{x}), (p_1), ..., (p_n)\}$ and $T = \{(x, p_1), ..., (x, p_n), (\frac{1}{x}, p_1)\ , ..., (\frac{1}{x}, p_n)\}$, where $p_1, \dots, p_n$ are distinct prime integers.

Proof. First consider the subsets

$$S_x := \{(x), (p_1), ..., (p_n)\}, \quad T_x := \{(x, p_1), ..., (x, p_n)\}$$

$$S_{\frac{1}{x}} := \{(\tfrac{1}{x}), (p_1), ..., (p_n)\}, \quad T_{\frac{1}{x}} := \{(\tfrac{1}{x}, p_1), ..., (\tfrac{1}{x}, p_n)\}$$

Then $S_x \cup T_x \subseteq \mathrm{Spec}(\mathbb{Z}[x])$. Thus we see that for every $\lambda \in \mathbb{N}$,

$$f_\lambda(x) := x^\lambda + p_1 \dots p_n \in \mathbb{Z}[x]$$

(f_λ) is a radical element for (S_x, T_x) in $\mathrm{Spec}(\mathbb{Z}[x])$ since $f(x)$ is irreducible by Eisenstein. Similarly for

$$h_\lambda(\frac{1}{x}) := p_1 \dots p_n + \frac{1}{x^\lambda} \in \mathbb{Z}[\frac{1}{x}]$$

$(h(\frac{1}{x}))$ is a radical element for $\{S_{\frac{1}{x}}, T_{\frac{1}{x}}\}$. We identify $h(\frac{1}{x})$ with

$$g_\lambda(x) := x^\lambda h_\lambda(\frac{1}{x}) = x^\lambda p_1 \dots p_n + 1 \in \mathbb{Z}[x].$$

Let y be another indeterminate over $\mathbb{Z}[x]$ and let $k(x) = p_1 ... p_n x^\lambda$. Then

$$f_\lambda(x) g_\lambda(x) = x^{2\lambda} p_1 ... p_n + ((p_1 ... p_n)^2 + 1)x^\lambda + p_1 ... p_n.$$

Since $f_\lambda(x)g_\lambda(x)$ and $k(x)$ are relatively prime elements of $\mathbb{Z}[x]$, $f_\lambda(x)g_\lambda(x) + yk(x)$ is a prime ideal in $\mathbb{Z}[x, y]$ by Proposition 2.3.7. Thus there exists a prime integer q so that $f_\lambda(x)g_\lambda(x) + q(p_1 ... p_n x^\lambda)$ is irreducible in $\mathbb{Z}[x]$ by Hilbert's Irreducibility Theorem 2.3.8.

We show that $w_\lambda := (r_\lambda(x)) = (f_\lambda(x)g_\lambda(x) + qk(x))$ is a radical element for (S, T) for all $\lambda \in \mathbb{N}$. First observe

$$r_\lambda(x) = p_1...p_n x^{2\lambda} + ((p_1...p_n)^2 + 1 + qp_1...p_n)x^\lambda + p_1...p_n.$$

It is easy to see that $w_\lambda \subset t$ for every $t \in T$, since $p_i \mid \ell(r)$ and $p_i \mid c(r)$, for $i = 1, ..., n$. Also $(r_\lambda, x) \subseteq (p_i, x)$ for all $i = 1, ..., n$. Similarly $(r_\lambda, \frac{1}{x}) \subseteq (\frac{1}{x}, p_i)$, for all $i = 1, ..., n$. Moreover $(r_\lambda, p_i) = (x^\lambda, p_i)$ and so (x, p_i) is the only maximal element that contains $(r_\lambda, p_i)^\uparrow$, for $i = 1, ..., n$. Thus $(r_\lambda, p_i)^\uparrow \in T$, for $i = 1, ..., n$. Therefore w_λ is a radical element for each $\lambda \in \mathbb{N}$. $\square$

Example 2.5.6. There are infinitely many radical elements for every ht(1,2)-pair in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$ of the form

$$S = \{(x), (\tfrac{1}{x}), (2), (3), (5)\}$$

$$T = \{(x, 2), (x, 3), (x, 5), (\tfrac{1}{x}, 2), (\tfrac{1}{x}, 3), (\tfrac{1}{x}, 5)\}.$$

First consider the following subsets as in the previous proof of Theorem 2.5.5:

$$S_x = \{(x), (2), (3), (5)\}, \quad T_x = \{(x, 2), (x, 3), (x, 5)\}$$
$$S_{\frac{1}{x}} = \{(\tfrac{1}{x}), (2), (3), (5)\}, \quad T_{\frac{1}{x}} = \{(\tfrac{1}{x}, 2), (\tfrac{1}{x}, 3), (\tfrac{1}{x}, 5)\}.$$

Then $S_x \cup T_x \subseteq \mathrm{Spec}\mathbb{Z}[x]$ and for every $\lambda \in \mathbb{N}$, $f_\lambda(x) := x^\lambda + 30$ in $\mathbb{Z}[x]$ generates a radical element for (S_x, T_x) in $\mathrm{Spec}(\mathbb{Z}[\frac{1}{x}])$. Similarly $h_\lambda(\frac{1}{x}) := 30 + \frac{1}{x^\lambda} \in \mathbb{Z}[\frac{1}{x}]$ is a radical element for $(S_{\frac{1}{x}}, T_{\frac{1}{x}})$. We identify $h_\lambda(\frac{1}{x})$ with $g_\lambda(x) := 30x^\lambda + 1 \in \mathbb{Z}[x]$. Let y be another indeterminate over $\mathbb{Z}[x]$. Since $f_\lambda(x)g_\lambda(x)$ and $30x^\lambda$ are relatively prime elements of $\mathbb{Z}[x]$, $(f_\lambda(x)g_\lambda(x)+y(30x^\lambda))$ is a prime ideal of $\mathbb{Z}[x, y]$ by Proposition 2.3.7. There exists a prime integer q so that $f_\lambda(x)g_\lambda(x) + q(30x^\lambda)$ is irreducible in $\mathbb{Z}[x]$ by

Hilbert's Irreducibility Theorem 2.3.8. Therefore $w = (f_\lambda(x)g_\lambda(x)+30qx^\lambda)$ is a radical element for each $\lambda \in \mathbb{N}$.

In [3], Arnavut raises some questions about particular ht(1,2)-pairs in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. We consider one such unanswered question below.

Question 2.5.7. Does (S,T) have a radical element if

$$S = \{(p_1), ..., (p_n), (x), (\tfrac{1}{x}), (x-a), (x-b)\},$$

$$T = \{(x,p_1), ..., (x,p_\ell), (\tfrac{1}{x}, p_{\ell+1}), ..., (\tfrac{1}{x}, p_n)\},$$

where $0 \le \ell \le n$, $\gcd(ab, p_1...p_\ell) = 1$, and the p_i are distinct prime integers for $i = 1, ..., n$?

Theorem 2.5.8 answers Question 2.5.7 in a special case.

Theorem 2.5.8. Assume a and b are relatively prime integers and let S and T be the following subsets of $\mathrm{Proj}(\mathbb{Z}[h,k])$:

$$S := \{(p_1), \ldots, (p_n), (x), (\tfrac{1}{x}), (x-a), (x-b)\},$$

$$T := \{(x,p_1), \ldots, (x,p_\ell), (\tfrac{1}{x}, p_{\ell+1}), \ldots, (\tfrac{1}{x}, p_n)\},$$

where $0 \le \ell \le n$, $\gcd(ab, p_1...p_\ell) = 1$, and the p_i are distinct prime integers for $i = 1, ..., n$. Suppose also that $\mathbf{pq}$ divides $(1-\mathbf{p}^t)(b^2+ab+a^2)+\mathbf{q}a^3b^3$ and $(1-\mathbf{p}^t+\mathbf{q}b^2a^2)(b+a)$ where $\mathbf{p} = p_1 \ldots p_\ell$, $\mathbf{q} = p_{\ell+1} \ldots p_n$, $t = \mathrm{lcm}(\phi(a^2), \phi(b^2))$, and ϕ is the Euler phi function. Then (S,T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[h,k])$.

Proof. Consider the polynomial $g(x;u,v,w)$ of the form

$$g(x;u,v,w) = \mathbf{q}x^4 + (\mathbf{pq}u)x^3 + (\mathbf{pq}v)x^2 + (\mathbf{pq}w)x + (\mathbf{p})^t$$

where $t = \operatorname{lcm}(\phi(a^2), \phi(b^2))$.

We show there exist infinitely many triples u, v, and $w \in \mathbb{Z}$ such that $(g(x; u, v, w))$ is a radical element for (S, T) in $\operatorname{Proj}(\mathbb{Z}[h, k])$.

First, by Euler's theorem, $(\mathbf{p})^{\phi(a^2)} \equiv 1 \pmod{a^2}$ and $(\mathbf{p})^{\phi(b^2)} \equiv 1 \pmod{b^2}$, since $\gcd(a, \mathbf{p}) = 1$ and $\gcd(b, \mathbf{p}) = 1$. Thus $\mathbf{p}^t - 1 \equiv 0 \pmod{a^2 b^2}$, that is, $a^2 b^2$ divides $\mathbf{p}^t - 1$.

To find u, v, and $w \in \mathbb{Z}$, we solve the system of linear equations $g(a; u, v, w) = 1$ and $g(b; u, v, w) = 1$; that is,

$$\mathbf{q}a^4 + \mathbf{pq}a^3 u + \mathbf{pq}a^2 v + \mathbf{pq}aw + \mathbf{p}^t = 1, \text{ and}$$

$$\mathbf{q}b^4 + \mathbf{pq}b^3 u + \mathbf{pq}b^2 v + \mathbf{pq}bw + \mathbf{p}^t = 1.$$

This becomes:

$$u + \frac{v}{a} + \frac{w}{a^2} = \frac{1 - \mathbf{p}^t - \mathbf{q}a^4}{\mathbf{pq}a^3} \tag{2.5.1}$$

$$u + \frac{v}{b} + \frac{w}{b^2} = \frac{1 - \mathbf{p}^t - \mathbf{q}b^4}{\mathbf{pq}b^3} \tag{2.5.2}$$

By subtracting (5.2) from (5.1), we get

$$v\left(\frac{b-a}{ab}\right) + w\left(\frac{b^2 - a^2}{a^2 b^2}\right) = \frac{(1-p^t)(b^3 - a^3) + q(b-a)a^3 b^3}{pqa^3 b^3}\cdot \tag{2.5.3}$$

After simplifying (5.3), we deduce

$$v + w\left(\frac{a+b}{ab}\right) = \frac{(1-p^t)(b^2 + ab + a^2) + qa^3 b^3}{pqa^2 b^2}\cdot \tag{2.5.4}$$

Therefore, for every $w = abk$ where $k \in \mathbb{Z}$, we get

$$v = \frac{(1-p^t)(b^2+ab+a^2)+qa^3b^3}{pqa^2b^2} - (a+b)k. \tag{2.5.5}$$

Similarly, by eliminating v, and letting $w = abk$ for $k \in \mathbb{Z}$, we get

$$u = -\frac{(1-\mathbf{p}^t+\mathbf{q}b^2a^2)(b+a)}{\mathbf{pq}a^2b^2} - k. \tag{2.5.6}$$

Note that $\mathbf{pq}$ divides $(1-\mathbf{p}^t)(b^2+ab+a^2)+\mathbf{q}a^3b^3$ and $(1-\mathbf{p}^t+\mathbf{q}b^2a^2)(b+a)$. Moreover a^2b^2 divides $1-p^t$. Hence u and v are integers in (5.5) and (5.6).

Now we claim that for every triple of integers u, v and w that we have found above, the polynomial $g(x;u,v,w) := \mathbf{q}x^4+(\mathbf{pq}u)x^3+(\mathbf{pq}v)x^2+(\mathbf{pq}w)x+(\mathbf{p})^t \in \mathbb{Z}[x]$ generates a radical element for (S,T). First $g(x;u,v,w)$ is irreducible by Eisenstein's Criteria in $\mathbb{Z}[\frac{1}{x}]$. Since $c(g) = \mathbf{p}^t$ and $\ell(g) = \mathbf{q}$, we have $(g(x;u,v,w)) \subseteq z$, $\forall\ z \in T$. Consider $(g(x;u,v,w),s)^{\uparrow}$ for each $s \in S$:

For $s = (x)$, $(g(x;u,v,w),x) = (\mathbf{p}^t,x) \subseteq (p_i,x) \in T$, where $i = 1,\ldots,\ell$.

For $s = (\frac{1}{x})$, we have $(g(x;u,v,w),\frac{1}{x}) = (\mathbf{q},\frac{1}{x})$. The only maximal ideals containing $(\mathbf{q},\frac{1}{x})$ are $(p_j,\frac{1}{x}) \in T$, for $j = \ell+1,\ldots,n$.

For $s = (p_i)$, where $i = 1,\ldots,\ell$, we get $(g(x;u,v,w),p_i) = (\mathbf{q}x^4,p_i)$. The only maximal ideals containing $(\mathbf{q}x^4,p_i)$ are (x,p_i), since $(\mathbf{q},p_i) = (1)$.

For $s = (p_j)$, where $j = \ell+1,\ldots,n$, we have $(g(x;u,v,w),p_j) = (\mathbf{p}^t,p_j) = (1)$.

For $s = (x-a)$, we have $(g(x;u,v,w),x-a) = 1$ since $g(a;u,v,w) = (1)$.

Similarly, for $s = (x-b)$, we get $(g(x;u,v,w),x-b) = (1)$ since $g(b;u,v,w) = 1$.

Therefore we conclude that $(g(x;u,v,w))$ is a radical element for (S,T), for all u,v and $w \in \mathbb{Z}$ as chosen in the proof. Thus there are infinitely many radical elements for this (S,T)-pair in $\mathrm{Proj}(\mathbb{Z}[\mathrm{h},\mathrm{k}])$. $\square$

Example 2.5.9. For $S = \{(2), (3), (x), (\frac{1}{x}), (x-5), (x-7)\}$ and $T = \{(x, 2), (\frac{1}{x}, 3)\}$, the polynomial $g(x; u, v, w) := 3x^4 + 6ux^3 + 6vx^2 + 6wx + 2^{420} \in \mathbb{Z}[x]$ generates a radical element for (S, T) for $w = 0$,

$$u = \frac{(1 - 2^{420} + 3 \cdot 5^2 \cdot 7^2)(5 + 7)}{2 \cdot 3 \cdot 5^2 \cdot 7^2} \in \mathbb{Z} \text{ , and}$$
$$v = \frac{(1 - 2^{420})(5^2 + 35 + 7^2) + 3 \cdot 5^3 \cdot 7^3}{2 \cdot 3 \cdot 5^2 \cdot 7^2} \in \mathbb{Z} \text{ .}$$

Note that u and v are integers since $5^2 \cdot 7^2$ divides $1 - 2^{420}$, and also $2 \cdot 3 = 6$ divides the numerators $(1 - 2^{420} + 3 \cdot 5^2 \cdot 7^2)(5 + 7)$ and $(1 - 2^{420})(5^2 + 35 + 7^2) + 3 \cdot 5^3 \cdot 7^3$.

Also, if $w = 5 \cdot 7 \cdot k = 35k$, for $k \in \mathbb{Z}$, then we get different integers u and v, that is, $g(x; u, v, w)$ generates a different radical element for every $k \in \mathbb{Z}$. Therefore (S, T) has infinitely many radical elements in $\mathrm{Proj}(\mathbb{Z}[h, k])$.

2.6 Summary and Questions

There is still much to be done for the characterization of $\mathrm{Proj}(\mathbb{Z}[\mathrm{h}, \mathrm{k}])$. In particular, the determination of which (S, T)-pairs have radical elements appears to be very challenging. In the future we hope to address some of the following questions:

Question 2.6.1. (1) In the setting of Theorem 4.2.7 with

$$S = \{(p_1), ..., (p_n), (x), (\frac{1}{x}), (f_1), ..., (f_m)\},$$
$$T = \{(x, p_1), ..., (x, p_l), (\frac{1}{x}, p_{l+1}), ..., (\frac{1}{x}, p_n)\},$$

where $0 \leq l \leq n$, $p_1, ..., p_n$ are distinct prime integers, is there a radical element for (S, T) if

(i) The leading coefficient of $f_1 ... f_m$ is not a unit and $p_{l+1} ... p_n$ does not divide the leading coefficient of f_i, for some i?

(ii) The constant coefficient of $f_1 ... f_m$ is not a unit and $p_1 ... p_l$ does not divide the constant coefficient of f_i, for some i?

(iii) $\gcd(p_1 ... p_n, \ell(f_1 ... f_m)) = 1$ and $gcd(p_1 ... p_n, c(f_1 ... f_m)) = 1$?

(2) Does the (S, T)-pair in Theorem 2.5.8 have a radical element if we remove some assumptions?

(3) Let $u_1, ..., u_n, v_1, ..., v_m \in H_1(\text{Proj } (\mathbb{Z}[h, k])) - C_0$, and let $P \in C_0$. Does there exist a $Q \in C_0$ such that $|\bigcup_{i=1}^{n}(u_i, P)^{\uparrow}| = |\bigcup_{j=1}^{m}(v_j, P)^{\uparrow}|$?

(4) What happens if we change T?

Chapter 3

Prime Ideals in Quotients of Mixed Power Series/Polynomials

This chapter contains work in progress with Christina Eubanks-Turner and Sylvia Wiegand.

3.1 Introduction

Over the past sixty years many algebraists have studied Kaplansky's question, posed in 1950: "Which partially ordered sets occur as the prime spectrum of a Noetherian ring?", [13], [14], [21], [22]. His question is still open and difficult, even when restricted to two-dimensional Noetherian domains. Some progress has been made in describing Spec(R), the partially ordered set of prime ideals of R, for certain two-dimensional polynomial rings R and power series rings, [11], [12], [28], [30], [31]. For example, as we mention in the previous chapter, in his 1986 article, Roger Wiegand gives axioms characterizing Spec($\mathbb{Z}[x]$), the prime spectrum of the ring of polynomials in one variable x over $\mathbb{Z}$ [31]. In 1989 William Heinzer and Sylvia Wiegand characterized the prime spectrum of $R[y]$ for R a one-dimensional countable semilocal Noetherian domain and y an indeterminate, [11]. Chandni Shah extended their characterization to the uncountable case, [28].[i] By "characterizing" a prime spectrum, we mean giving a set of axioms such that the prime spectrum satisfies the given axioms and any two partially ordered sets satisfying the axioms are order-isomorphic. In 1996 Aihua Li and Sylvia Wiegand proved that, if $f, g_1, \ldots, g_m \in \mathbb{Z}[y]$ and f is nonzero, then the prime spectrum of $\mathbb{Z}[y][\frac{g_1}{f}, \ldots, \frac{g_m}{f}]$ is order-isomorphic to the prime spectrum of $\mathbb{Z}[y]$, [19]. Two years later, in 1998, Serpil Saydam and Sylvia Wiegand showed that, if D is a ring of algebraic integers in a field K that is a finite extension of the rational numbers, $f, g_1, \ldots, g_m \in \mathbb{Z}[y]$, f is nonzero and x is an indeterminate, then $\operatorname{Spec}(D[x, \frac{g_1}{f}, \ldots, \frac{g_m}{f}])$ is order isomorphic to $\operatorname{Spec}(\mathbb{Z}[x])$, [27]. In 2006, William Heinzer, Christel Rotthaus, and Sylvia Wiegand described the prime spectrum of $R[[x]]$, the power series ring in an indeterminate x over a one-dimensional integral domain R [10]. [ii]

A *birational extension* of an integral domain is an overring contained in its field of fractions. In their 2011 article, Christina Eubanks-Turner, Melissa Luckas, and Serpil Saydam study prime spectra of birational extensions of $R[[x]]$ of form $B = R[[x]][g/f]$, where R is a one-dimensional Noetherian domain with infinitely many maximal ideals and f and $g \in R[[x]]$ are such that $f \neq 0$, and either $\{f, g\}$ is a $R[[x]]$-sequence or $(f, g) = R[[x]]$ [8]. They characterize $\operatorname{Spec}(B)$ when R is a countable Dedekind domain. If y is another indeterminate, then $R[[x]][g/f]$ is isomorphic to $\dfrac{R[[x]][y]}{(fy - g)}$, the ring of the three-dimensional *mixed power series/polynomial ring* $R[[x]][y]$ modulo the height-one prime ideal $(fy - g)$ of $R[[x]][y]$.

The primary goal of this chapter is to describe the prime spectra of two-dimensional

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[i]The generalization by Chandni Shah needed a small cardinality fix [32].

[ii]Their description became a characterization with the cardinality fix of [32].

quotients of $R[[x]][y]$ and $R[y][[x]]$, that is, $\operatorname{Spec}(\frac{R[[x]][y]}{Q})$ and $\operatorname{Spec}(\frac{R[y][[x]]}{Q'})$, for certain height-one prime ideals Q of $R[[x]][y]$ and Q' of $R[y][[x]]$, where R is a one-dimensional Noetherian domain. We give examples of spectra that arise when R is the ring $\mathbb{Z}$ of integers and Q and Q' are particular prime ideals of their respective rings; see Example 3.6.1 and Example 3.6.2. Although the rings $R[y][[x]]$ and $R[[x]][y]$ are similar for R a one-dimensional Noetherian domain, their prime spectra are different. For example, $xy + 1$ is an element of both $R[[x]][y]$ and $R[y][[x]]$; $xy + 1$ generates a height-one prime ideal in $R[[x]][y]$, but is a unit of $R[y][[x]]$. We also study the maximal ideals of $R[[x]][y]$ and $R[y][[x]]$. For example we show that there are no height-one maximal ideals in $R[y][[x]]$ or in $R[[x]][y]$ and, in the case when R has infinitely many maximal ideals, there are no height-two maximal ideals in $R[y][[x]]$; see Proposition 3.3.3 and Proposition 3.3.5(1).

If R is a one-dimensional Noetherian domain and Q and Q' are certain height-one prime ideals of $R[[x]][y]$ and $R[y][[x]]$, respectively, then the dimensions of $\frac{R[[x]][y]}{Q}$ and $\frac{R[y][[x]]}{Q'}$ are usually two; see section 5. In certain exceptional cases for Q and Q' these dimensions are both 1 and the prime spectrum resembles a fan; see Definition 3.5.1 and Theorem 3.5.2. We give a set of axioms in Definition 3.5.4 that are satisfied by the two-dimensional image rings of mixed power series/polynomial rings; see Theorem 3.5.5. In the two-dimensional case, there are finitely many nonmaximal j-primes in these mixed power series/polynomial rings. Generally we avoid letting Q or Q' be the ideal generated by x unless R is semilocal. In case Q or Q' is (x), the prime spectrum we seek is order-isomorphic to $R[y]$. When $Q = (x)R[[x]][y]$ and R is semilocal or $R = \mathbb{Z}$, $\operatorname{Spec}(R[y])$ has been characterized, [11], [28], [32], [31]. However, if $R = \mathbb{Q}[z]$, a polynomial ring over the rational numbers in another indeterminate z, $\operatorname{Spec}(\mathbb{Q}[z, y])$ is unknown.

We have some partial results concerning which partially ordered sets satisfy the axioms of Definition 3.5.4; see Proposition 3.7.1. For certain height-one prime ideals of Q and Q' in the case where $R[[x]][y]/Q$ and $R[y][[x]]/Q'$ are two-dimensional, we can compute the cardinality of the set of height-one maximal ideals. As we point out in Corollary 3.5.6, most of the spectra of $R[[x]][y]/Q$ and $R[y][[x]]/Q'$are determined by the spectrum of $R[y]/I$. We give examples of some spectra that arise when $R = \mathbb{Z}$.

3.2 Notation and Background

In this section we give more notation, we describe previous results, and we list basic facts and remarks about prime spectra of polynomial rings and power series rings. We use the notation for partially ordered sets from Section 1.2.

In Remark 3.2.1 we establish that the rings we study in this chapter are well behaved.

Remarks 3.2.1. (1) If a ring A is Cohen-Macaulay and x_i and y_j are indeterminates over A for $1 \leq i \leq n$, $1 \leq j \leq m$, and $n, m \in \mathbb{N}_0$, then the *mixed polynomial/power series rings* $A[[\{x_i\}_{i=1}^n]][\{y_j\}_{j=1}^m]$ and $A[\{y_j\}_{j=1}^m][[\{x_i\}_{i=1}^n]]$ are Cohen-Macaulay [20, Theorem 17.7]. Thus they are *catenary*; that is, for every inclusion of prime ideals $P \subseteq Q$, any two maximal chains of prime ideals from P to Q have the same length [20, Theorem 17.9].
(2) If R is a Noetherian integral domain of dimension one, then R is Cohen-Macaulay [20, Exercise 17.1, p. 139]. Thus every mixed polynomial/power series that is a finite extension of a one-dimensional Noetherian domain R is catenary by (1).

Lemma 3.2.2 is useful for counting prime ideals in our rings.

Lemma 3.2.2. [32, Lemma 4.2], [8, Lemma 3.6] Let T be a Noetherian domain, let y be an indeterminate and let I be a proper ideal of T. Let $\beta = |T|$ and $\rho = |T/I|$. Then (1) $|T[y]| = |(T/I)[y]| = \rho \cdot \aleph_0 = \beta \cdot \aleph_0$, and (2) $|T[[y]]| = \beta^{\aleph_0} = \rho^{\aleph_0}$.

Theorem 3.2.3 gives information concerning the relative heights of prime ideals of the polynomial ring $A[y]$ or the power series ring $A[[x]]$ and their contractions to A when A is Noetherian.

Theorem 3.2.3. [20, Theorem 15.1] Let $\varphi : A \to B$ be a homomorphism of Noetherian rings, let P be a prime ideal of B, and set $\mathfrak{p} = P \cap A$ (identified with $\varphi^{-1}(P)$). Then:

(i) $\operatorname{ht} P \leq \operatorname{ht} \mathfrak{p} + \dim(B_P/\mathfrak{p}B_P)$;

(ii) If φ is flat, or more generally if the going-down theorem holds between A and B, then equality holds in item i.

3.2.1 Results, Basic Facts about $\operatorname{Spec}(A[y])$

In Remarks 3.2.4 and several results following it we describe prime ideals in a polynomial ring over a Noetherian domain A.

Remarks 3.2.4. Let A be a Noetherian domain of dimension d with field of fractions K and let y be an indeterminate over K.

(1) If I is a nonzero ideal of $A[y]$ such that $I \cap A = (0)$, then $I = h(y)K[y] \cap A[y]$, for some $h(y) \in A[y]$ of degree ≥ 1. This follows since $K[y] = (A \setminus \{0\})^{-1}A[y]$ is a principal ideal domain (PID). Thus the set of prime ideals P of $A[y]$ such that $P \cap A = (0)$ is in one-to-one order-preserving correspondence with the set of height-one prime ideals of $K[y]$, via $P \mapsto PK[y] \mapsto PK[y] \cap A[y]$. If P is a prime ideal of $R[y]$ such that $P \cap A = (0)$, then $\operatorname{ht}(P) = 1$.

(2) If $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}A[y]$ is a prime ideal of $A[y]$ and $\mathrm{ht}(\mathfrak{p}A[y]) = \mathrm{ht}(\mathfrak{p})$, [24, Proposition 10].

(3) If P is a prime ideal of $A[y]$, then $\mathrm{ht}(P \cap A) \leq \mathrm{ht}(P) \leq \mathrm{ht}(P \cap A) + 1$ Theorem 3.2.3.

(4) If M is a prime ideal of $A[y]$ of height $d+1$, then M is a maximal ideal of $A[y]$, the ideal $\mathfrak{m} = M \cap A$ is a maximal ideal of A of height d, and $M = (\mathfrak{m}, h(y))A[y]$, where $\overline{h(y)}$ is irreducible in $\overline{A[y]} = A[y]/(\mathfrak{m}[y]) \cong (A/\mathfrak{m})[y]$; This follows from item (3) and [16, Theorem 28, p. 17].

(5) If $P \in \mathrm{Spec}(A[y])$ with $\mathrm{ht}(P) = 1$ and $(0) \neq \mathfrak{p} = P \cap A$, then $\mathrm{ht}(\mathfrak{p}) = 1$ and $\mathfrak{p}A[y] = P$. This follows from item (2) and item (3).

(6) If $\mathfrak{q} \in \mathrm{Spec}(A)$ and $b \in A$ is such that $(1 + yb, \mathfrak{q})A[y] = A[y]$, then $b \in \mathfrak{q}$.

To see this, write $(1 + yb)f(y) + g(y) = 1$, where $f(y) \in A[y]$ and $g(y) \in \mathfrak{q}A[y]$. Then $f(y) \notin \mathfrak{q}A[y]$ and

$$f(y) + ybf(y) - 1 \in \mathfrak{q}A[y] \tag{3.2.4.0}$$

Write $f(y) = a_0 + a_1y + a_2y^2 + \cdots a_ny^n + y^{n+1}h(y)$, where $a_n \notin \mathfrak{q}$ and $h(y) \in \mathfrak{q}A[y]$. Looking at the coefficient of y^{n+1} in Equation 3.2.4.0, we conclude that $a_nb \in \mathfrak{q}A[y] \cap A = \mathfrak{q}$. Since $a_n \notin \mathfrak{q}$, we have $b \in \mathfrak{q}$, as desired.

Lemma 3.2.5. Let A be a Noetherian domain. If $Q \in \mathrm{Spec}(A[y])$ is a height-one maximal ideal, then

(1) $Q \cap A = (0)$,

(2) $\dim(A) = 1$ and $|\max(A)| < \infty$; say $\max(A) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_t\}$, and

(3) Q contains an element of form $h(y) = yg(y) + 1$, where $g(y) \in (\cap_{i=1}^{t} \mathfrak{m}_i)[y]$.

Moreover if A is one-dimensional and semi-local with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ and $Q \in \operatorname{Spec}(A[y])$ is minimal over an element of form $h(y) = yg(y) + 1$, where $0 \neq g(y) \in (\cap_{i=1}^{t} \mathfrak{m}_i)[y]$, then Q is a height-one maximal ideal of $A[y]$.

Proof. For item 1, by Remarks 3.2.4(5), if $Q \cap A \neq (0)$, then $Q = (Q \cap A)A[y] \subsetneq (Q \cap A, y)A[y]$, a contradiction to Q maximal. Thus item 1 holds.

For item 2, we refer to [16, Theorems 24, p. 15 and 146, p. 107], where a ring A such that a maximal ideal of $A[Y]$ intersects A in (0) is called a G-domain.

For item 3, if $(\cap_{i=1}^{t} \mathfrak{m}_i)[y] \subseteq Q$, then $\mathfrak{m}_i[y] \subseteq Q$, for some $i = 1, \ldots, t$, and so $\mathfrak{m}_i[y] = Q$, since $\operatorname{ht}(\mathfrak{m}_i[y]) \geq 1 = \operatorname{ht}(Q)$ by Remark 3.2.4(2). This yields a contradiction since $(\mathfrak{m}_i[y], y)$ properly contains $\mathbf{m}_i[y]$. Thus $((\cap_{i=1}^{t} \mathfrak{m}_i)[y], Q) = A[y]$, and so there exist $h(y) \in Q$, $s(y) \in (\cap_{i=1}^{t} \mathfrak{m}_i)[y]$, and $r_1(y)$, $r_2(y) \in A[y]$ such that $h(y)r_1(y) + s(y)r_2(y) = 1$. Let $f(y) := h(y)r_1(y)$. Then $f(y) \in Q$ since $h(y) \in Q$. Let $^{-}$ denote the image in $A/((\cap_{i=1}^{t} \mathfrak{m}_i)[y])$. Then $\overline{f(y)} + \overline{s(y)r_2(y)} = \overline{1}$, and so $\overline{f(y)} = \overline{1}$. Therefore $f(y) = yg(y) + a_0 \in Q$ where $g(y) \in (\cap_{i=1}^{t} \mathfrak{m}_i)[y]$ and $a_0 = 1 + b$, where $b \in \cap_{i=1}^{t} \mathfrak{m}_i$; thus a_0 is a unit of A. Now replacing f by $a_0^{-1} f$ yields the result.

For the moreover statement, if $Q \cap A = \mathfrak{m}_i$ for some i, then $1 \in (\mathfrak{m}_i, f(y)) \subseteq Q$, a contradiction. Thus the statement holds. $\square$

Theorem 3.2.6 was proved by Heinzer and S. Wiegand in the countable case, then for other cardinalities by Shah and R. Wiegand and S. Wiegand. The theorem has been slightly adjusted here using the fact that $|(R/\mathfrak{m})[y]| = |R[y]|$, for every $\mathfrak{m} \in \max R$ by Lemma 3.2.2.

Theorem 3.2.6. [11, Theorem 2.7], [28, Theorem 2.4], [32, Theorem 3.1] Let R be a one-dimensional Noetherian domain with exactly n maximal ideals, $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, let

y be an indeterminate and let $\alpha = |R[y]|$. Then there exist exactly two possibilities for $U = \mathrm{Spec}(R[y])$ up to cardinality, depending upon whether or not R is Henselian. [iii]

- In case R is **not Henselian**, U satisfies Axioms $I - VI$ below:

(I) $|U| = |R|$ and U has a unique minimal element $u_0 = (0)$.

(II) $|\mathcal{H}_1(U) \cap \max(U)| = \alpha$.

(III) $\dim(U) = 2$, $|\mathcal{H}_2(U)| = \alpha$.

(IV) There exist exactly n height-one elements $u_1, \ldots, u_n$ such that $|u_i^{\uparrow}|$ is infinite. Furthermore for $1 \leq i \leq n$

(i) $u_1^{\uparrow} \cup \cdots \cup u_n^{\uparrow} = |\mathcal{H}_2(U)|$; (ii) $u_i^{\uparrow} \cap u_j^{\uparrow} = \emptyset$ if $i \neq j$; (iii) $|u_i^{\uparrow}| = \alpha$.

(V) If $\mathrm{ht}(v) = 1$ and $v \neq u_i$ for all i with $1 \leq i \leq n$, then $|v^{\uparrow}| < \infty$.

(VI) For every nonempty finite subset of T of $\mathcal{H}_2(U)$, $|L_e(T)| = \alpha$,

- In case R is **Henselian**, then $n = 1$, and U satisfies Axioms $I - V$ and VI':

(VI') For every finite nonempty subset $T \subseteq \mathcal{H}_2(U)$, $L_e(T) = \emptyset$ if $|T| > 1$, and $|L_e(T)| = \alpha$ if $T = 1$.

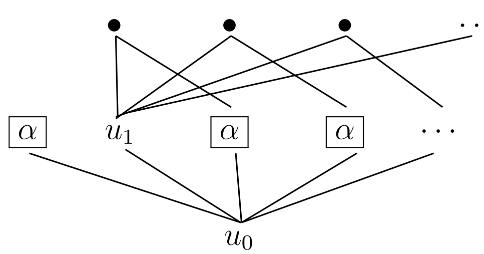


Diagram 3.2.6.h: $\mathrm{Spec}(R[y])$, R Henselian

[iii] For the definition of Henselian; see Section 1.1.

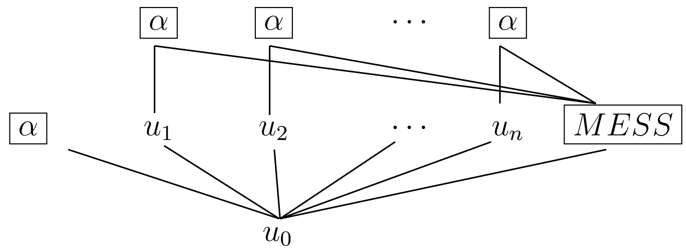


Diagram 3.2.6.nh: Spec($R[y]$), R non-Henselian.

The relations satisfied by the MESS box are too complicated to show, but they are described in Axiom VI.

Notes 3.2.7. If R is a one-dimensional Noetherian domain with infinitely many maximal ideals, and y is an indeterminate, then Spec($R[y]$) is not known in general. The following cases are known:

(1) Spec($\mathbb{Z}[y]$) has a characterization of five axioms, [31, Theorem 2] .

(2) Let k be an algebraic extension of a finite field and let z be another indeterminate. Then Spec($k[z, y]$) is order-isomorphic to Spec($\mathbb{Z}[y]$); see [31, Theorem 2].

(3) Let D be an order in an algebraic number field. Then Spec($D[y]$) is order-isomorphic to Spec($\mathbb{Z}[y]$), [31, Theorem 1].

(4) In case D is an order in an algebraic number field, z is another indeterminate, and $Q = (fz - g)$, where f, g is an $R[y]$-sequence or $f \neq 0$ and $(g, f) = R[y]$, then Spec($D[y][z]/Q$) $\cong$ Spec($\mathbb{Z}[y]$), [27].

The prime spectrum of $\mathbb{Q}[z, y]$, where $\mathbb{Q}$ is the rational numbers is unknown but it is known that it is *not* order-isomorphic to Spec($\mathbb{Z}[y]$), [31].

3.2.2 Results, Basic Facts about $\mathrm{Spec}(A[[x]])$

Next we describe prime ideals in power series rings over a Noetherian domain A.

Remarks 3.2.8. Let A be a Noetherian domain and x an indeterminate.

(1) Every maximal ideal of $A[[x]]$ has the form $(\mathfrak{m}, x)A[[x]]$ where $\mathfrak{m}$ is a maximal ideal of A, [23, Theorem 15.1] (Nagata). Thus x is in every maximal ideal of $A[[x]]$.

(2) If $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}A[[x]] \in \mathrm{Spec}(A[[x]])$ and $\mathrm{ht}(\mathfrak{p}A[[x]]) = \mathrm{ht}(\mathfrak{p})$, [5, Theorem 4] or [4, Theorem 4].

(3) If P is a prime ideal of $A[[x]]$, then $\mathrm{ht}(P \cap A) \leq \mathrm{ht}(P) \leq \mathrm{ht}(P \cap A) + 1$ Theorem 3.2.3.

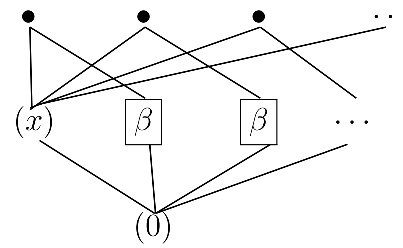
The following characterization of $\mathrm{Spec}(R[[x]])$ for R a one-dimensional Noetherian domain is due to Heinzer, Rotthaus, S. Wiegand [10] and R. Wiegand and S. Wiegand [32].

Theorem 3.2.9. [10, Theorem 3.4], [32, Theorem 4.3] Let R be a one-dimensional Noetherian domain and let x be an indeterminate. Let $\beta = |R[[x]]|$ and let $\alpha = |\max(R)|$. Then $U := \mathrm{Spec}(R[[x]])$ satisfies axioms I, III, IV, and V of Theorem 3.2.6, and U satisfies (II^*) and (VI^*) below. The unique nonmaximal element of IV is $u_x = (x)R[[x]]$. Each maximal ideal of $R[[x]]$ has form $(\mathfrak{m}, x)R[[x]]$, where $\mathfrak{m} \in \max(R)$.

(II^*) $\mathcal{H}_1(U) \cap \max(U) = \emptyset$.

(VI^*) $L_e(T) = \beta$ if $|T| = 1$.

Thus $\mathrm{Spec}(R[[x]])$ is shown in the following diagram:



$(\#\{\text{bullets}\} = \alpha)$

Diagram 3.2.9.0: Spec($R[[x]]$)

In Diagram 3.2.9.0, the cardinality of the set of bullets equals the cardinality of $\max(R)$ since the set of height-two maximal ideals of $R[[x]]$ is in one-to-one correspondence with the set of maximal ideals of the coefficient ring R. The boxed β beneath each maximal ideal of $R[[x]]$ means that there are exactly β prime ideals in that position (beneath that maximal ideal and no other). Every two posets described by Diagram 3.2.9.0 are order-isomorphic.

3.2.3 Mixed Power Series/Polynomial Rings

Here we give some properties of prime spectra for mixed power series/polynomial rings.

Remarks 3.2.10. Let A be a Noetherian domain and x an indeterminate.

(1) The inclusion map $\varphi : A[[x]][y] \hookrightarrow A[y][[x]]$ is flat since $A[y][[x]]$ is the (x)-adic completion of $A[[x]][y]$. In general φ is not faithfully flat; for example, the proper ideal $\mathcal{I} = (xy-1)A[[x]][y]$ of $A[[x]][y]$ satisfies $\mathcal{I}A[y][[x]] = A[y][[x]]$.

(2) In case $\mathcal{M}$ is a maximal ideal of $A[y][[x]]$, we see that

$$\varphi_{\mathcal{M}} : A[[x]][y]_{\mathcal{M}\cap A[[x]][y]} \hookrightarrow A[y][[x]]_{\mathcal{M}}$$

is faithfully flat. Thus, for every proper ideal $\mathcal{I}$ of $A[[x]][y]_{\mathcal{M}\cap A[[x]][y]}$, we have

$$\mathcal{I}A[y][[x]]_{\mathcal{M}} \cap A[[x]][y]_{\mathcal{M}\cap A[[x]][y]} = \mathcal{I}.$$

The proper ideals of $A[[x]][y]_{\mathcal{M}\cap A[[x]][y]}$ are in one-to-one correspondence with the ideals of $A[[x]][y]$ contained in $\mathcal{M} \cap A[[x]][y]$ and the ideals of $A[y][[x]]_{\mathcal{M}}$ are in one-to-one correspondence with the ideals of $A[y][[x]]$ contained in $\mathcal{M}$. Thus prime ideals of $A[y][[x]]$ contained in $\mathcal{M}$ intersect down to prime ideals of $A[[x]][y]$ contained in $\mathcal{M} \cap A[[x]][y]$ via

$$\mathcal{P} \mapsto \mathcal{P} \cap A[[x]][y] \in \operatorname{Spec}(A[[x]][y]) \cap (\mathcal{M} \cap A[[x]][y])^{\downarrow},$$

for $\mathcal{P} \in \operatorname{Spec}(A[y][[x]]) \cap (\mathcal{M})^{\downarrow}$, and, for $\mathcal{I}$ an ideal of $A[[x]][y]$ contained in $\mathcal{M} \cap A[[x]][y]$,

$$\mathcal{I} = \mathcal{I}A[y][[x]] \cap A[[x]][y].$$

(3) An ideal $\mathcal{M}$ is a maximal ideal of $A[y][[x]]$ if and only if $\mathcal{M} = (M, x)A[y][[x]]$, for some maximal ideal M of $A[y]$, by Remarks 3.2.8(1). This implies that $\mathcal{M} \cap A[[x]][y] = (M, x)A[[x]][y]$ is a maximal ideal of $A[[x]][y]$ by Remarks 3.2.8(1).

Conversely if $\mathcal{M}$ is a maximal ideal of $A[[x]][y]$ of height $d+2$, then by Remarks 3.2.4.(4), $\mathcal{M} \cap A[[x]]$ has height $d+1$ and $x \in \mathcal{M}$. Thus, $\mathcal{M}/(x)$ is a maximal ideal of $A[[x]][y]/(x) \cong A[y]$ and has height $d+1$. By the correspondence between ideals of $A[[x]][y]$ containing x and ideals of $A[y]$, we see that $\mathcal{M}/(x)$ corresponds to a maximal ideal M of $A[y]$, and since (M, x) (considered in $A[[x]][y]$) also corresponds to $\mathcal{M}$, we take $\mathcal{M} = (M, x)$. The upshot of this is that a maximal ideal of $A[[x]][y]$ of height $d+2$ has form (M, x), where $M \in \max(A[y])$.

Proposition 3.2.11. Let A be a Noetherian domain, let x, y be indeterminates and let Q be a height-one prime ideal of $A[[x]][y]$ such that $Q \cap A[y] \neq (0)$. Then

$$Q = (Q \cap A[y])A[[x]][y] = (Q \cap A[y])A[y][[x]] \cap A[[x]][y].$$

Proof. Since $Q \cap A[y]$ is a prime ideal of $A[y]$ and $\mathrm{ht}(Q \cap A[y]) = 1$ by Remarks 3.2.4(5), we have $(Q \cap A[y])A[y][[x]]$ is a prime ideal of $A[y][[x]]$ of height one by Remarks 3.2.8(2). Also $Q \cap A[y]$ is contained in a maximal ideal N of $A[y]$. Then $\mathcal{M} = (N, x)A[y][[x]]$ is a maximal ideal of $A[y][[x]]$ and $(Q \cap A[y])A[y][[x]] \subseteq \mathcal{M}$ and so $(Q \cap A[y])A[[x]][y] = ((Q \cap A[y])A[y][[x]]) \cap A[[x]][y]$ is a prime ideal of $A[[x]][y]$ using Remarks 3.2.10(2). Since $(0) \neq (Q \cap A[y])A[[x]][y] \subseteq Q$ and $\mathrm{ht}(Q) = 1$, we have $Q = (Q \cap A[y])A[[x]][y]$, as desired. $\square$

Proposition 3.2.12. Let A be a Noetherian domain, let x and y be indeterminates and let Q be a prime ideal of $A[[x]][y]$ such that $(Q, x)A[[x]][y] \neq A[[x]][y]$. Then

$$Q = (QA[y][[x]]) \cap A[[x]][y].$$

Proof. It suffices to show that $(Q, x) \neq A[[x]][y] \implies QA[y][[x]] \subseteq \mathcal{N}$, for some maximal ideal $\mathcal{N}$ of $A[y][[x]]$, by Remarks 3.2.10(2). Let $(Q, x) = (I, x)$, where I is an ideal of $A[y]$. Then I is a proper ideal of $A[y]$ since $(Q, x)A[[x]][y] \neq A[[x]][y]$. Therefore there exists a maximal ideal N of $A[y]$ containing I. It follows that $\mathcal{N} = (N, x)$ is a maximal ideal of $A[y][[x]]$ that contains $(I, x)A[y][[x]]$. Since $QA[y][[x]] \subseteq N$, we are done. $\square$

Proposition 3.2.13. Let $e, n \in \mathbb{N}_0$, let A be a Cohen-Macaulay integral domain of dimension e, and let $x, y_1, \ldots, y_n$ be indeterminates.

(1) An ideal I of $A[[x]][y_1, \dots, y_n]$ is a maximal ideal of height $n+e+1$ if and only if $I = (\mathbf{m}, x)A[[x]][y_1, \dots, y_n]$ for some maximal ideal $\mathbf{m}$ of $A[y_1, \dots, y_n]$ of height $n + e$; see [8, Proposition 3.2].

(2) If $\mathbf{m}$ is a maximal ideal of $A[y_1, \dots, y_n]$ of height t, then $(\mathbf{m}, x)A[[x]][y_1, \dots, y_n]$ is a maximal ideal of $A[[x]][y_1, \dots, y_n]$ of height $t + 1$; see [8, Proposition 3.2]

(3) Every maximal ideal of $A[y_1, \dots, y_n][[x]]$ has form $(\mathbf{n}, x)$ where $\mathbf{n}$ is a maximal ideal of $A[y_1, \dots, y_n]$; see Remarks 3.2.8(1).

In case $n = 0$, the first item of the next proposition is given by Heinzer, Rotthaus, and S.Wiegand in [10]. It is extended to $R[[x]][y_1, \dots, y_n]$ for $n \geq 1$ in [8].

Proposition 3.2.14. Let $e, n \in \mathbb{N}_0$, let R be an e-dimensional Noetherian domain, and let $x, y_1, \dots, y_n$ be indeterminates.

(1) [8, Proposition 3.11] Let Q be a height-$(n+e)$ prime ideal in $R[[x]][y_1, \dots, y_n]$. If $x \notin Q$, then Q is contained in a unique maximal ideal of $R[[x]][y_1, \dots, y_n]$.

(2) [10, Proposition 2.4] Let P be a prime ideal of height $e+n$ in $R[y_1, \dots, y_n][[x]]$. If $x \notin P$, then P is contained in a unique maximal ideal of $R[y_1, \dots, y_n][[x]]$.

3.2.4 Counting Intermediate Prime Ideals

For A either $R[[x]][y_1, \dots, y_n]$ or $R[y_1, \dots, y_n][[x]]$ and a nonmaximal prime ideal Q of A with $\mathrm{ht}(Q) = \dim(A) - 2$, we are interested in the number of prime ideals between Q and a maximal ideal of height equal to the dimension of A.

We observe that, for certain prime ideals Q of A with $\mathrm{ht}(Q) = \dim(A) - 2$, there are no maximal ideals of maximal height that contain Q.

Remark 3.2.15. Let e and n be nonnegative integers with $n+e \geq 2$, let R be a Cohen-Macaulay e-dimensional integral domain and let $x, y_1, \ldots, y_n$ be indeterminates. Let Q be a prime ideal of $R[[x]][y_1, \ldots, y_n]$ or of $R[y_1, \ldots, y_n][[x]]$ with $\operatorname{ht}(Q) = n+e-1$. In any of the following three cases, Q is not contained in a maximal ideal of height $n+e+1$.

(1) $(Q, x) = (1)$,

(2) Every height-$(n+e)$ prime ideal containing (Q, x) is a maximal ideal, or

(3) $(Q, \mathfrak{m}) = (1)$ for every $\mathfrak{m} \in \max(R)$,

To see this, we observe that, for $A = R[[x]][y_1, \ldots, y_n]$ or $A = R[y_1, \ldots, y_n][[x]]$, every maximal ideal $\mathcal{M}$ of A having maximal height $n+e+1$ has form $\mathcal{M} = (N, x)A$, where N is a maximal ideal of $R[[y_1, \ldots, y_n]$ of height $n+e$, by Proposition 3.2.8(1) or by Remarks 3.2.10(3). Furthermore, $\operatorname{ht}(N \cap R) = e$, by repeated use of Remark 3.2.4(4), and so $\mathfrak{m} \subseteq N$, for some $\mathfrak{m} \in \max R$ of height e. In cases 1 or 3, if Q were contained in a maximal ideal $\mathcal{M}$ of either ring, where $\operatorname{ht}(\mathcal{M}) = n+e+1$, then $Q \cup \{x\} \subseteq \mathcal{M}$ or $Q \cup \mathfrak{m} \subseteq \mathcal{M}$ would imply that $1 \in M$, a contradiction. In case 2, no maximal ideal containing (Q, x) is contained in a larger maximal ideal.

When R is a one-dimensional Noetherian domain and $n = 1$, we use Proposition 3.2.16, an adjustment of [8, Proposition 3.8, Remark 3.9], to find the cardinality of the set of all height-two prime ideals of $R[[x]][y]$ or $R[y][[x]]$ that are properly between a height-one prime ideal Q and a height-three maximal ideal P such that $x \notin Q$, $(Q, x) \neq (1)$, and (Q, x) is not a maximal ideal.

Proposition 3.2.16. Let e and n be nonnegative integers with $n+e \geq 2$, let R be a Cohen-Macaulay e-dimensional integral domain and let $x, y_1, \ldots, y_n$ be indeterminates. Let A be either $R[[x]][y_1, \ldots, y_n]]$ or $R[y_1, \ldots, y_n][[x]]$. Let $Q \subseteq P$ be prime ideals of A with $\operatorname{ht}(Q) = n+e-1$ and $\operatorname{ht}(P) = n+e+1$. Then $Q^{\uparrow} \cap P^{\downarrow}$ contains

$|R[[x]]|$ height-$(n+e)$ prime ideals in either of the following cases:

(1) $A = R[[x]][y_1, \ldots, y_n], x \notin Q$, and $\mathfrak{m} \nsubseteq Q$, for every $\mathfrak{m} \in \max(R)$, or

(2) $A = R[y_1, \ldots, y_n][[x]]$ and $x \notin Q$.

Proof. In either case for A, the prime ideal P has form $(\mathbf{n}, x)A$, where $\mathbf{n}$ is a maximal ideal of $R[y_1, \ldots, y_n]$ of height $n+e$, by Proposition 3.2.8(1) or by Remarks 3.2.10(3), as in the proof of Remarks 3.2.15. By repeated use of Remark 3.2.4(4), there exists $\mathfrak{m} \in \max(R)$ with $\mathfrak{m} = P \cap R$. Thus $\mathrm{ht}(\mathbf{n}) = n + e$, and we have

$$A/P \cong (A/(xA)/(P/(xA)) \cong R[y_1, \ldots, y_n]/\mathbf{n}; \quad \text{and} \quad R/\mathfrak{m} \hookrightarrow R[y_1, \ldots, y_n]/\mathbf{n}.$$

Let $\beta := |(\mathcal{H}_{n+e}(A)) \cap (Q^{\uparrow} \cap P^{\downarrow})|$, let $\gamma := |R/\mathfrak{m}|$ and let $\gamma_1 := |R[y_1, \ldots, y_n]/\mathbf{n}|$. Then $\gamma^{\aleph_0} = {\gamma_1}^{\aleph_0}$ since $\gamma_1 = \gamma \cdot \aleph_0$, and $|A| = |R[[x]]| = \gamma^{\aleph_0}$, by Lemma 3.2.2. Since A is Noetherian implies every ideal of A is finitely generated, we have $\beta \leq \gamma^{\aleph_0}$.

Let $N_1, \ldots, N_m$ be all the minimal prime ideals of A containing (Q, x) and contained in P; that is, $(Q, x) \subseteq N_i \subseteq P$ for each i. Since $\mathrm{ht}(Q) = n + e - 1$ and $Q \subset (Q, x)$, Krull's Principal Ideal Theorem and the catenary condition of Remarks 3.2.1 imply $\mathrm{ht}(N_i) = n + e$, for each i. Since $\mathrm{ht}(P) = n + e + 1$, and the N_i have height $n + e$, we see $(\mathbf{n}, x) = P \nsubseteq N_1 \cup \ldots \cup N_m$.

For item 1 with $A = R[[x]][y_1, \ldots, y_n]$, since $\mathfrak{m} \nsubseteq Q$, we have $\mathfrak{m} \nsubseteq N_i$ for each i and so $\mathfrak{m} \nsubseteq N_1 \cup \ldots \cup N_m$. Let $a \in \mathfrak{m} \setminus (N_1 \cup \ldots \cup N_m)$ and let C be a complete set of γ distinct coset representatives of $R/\mathfrak{m}$.

For item 2, with $A = R[y_1, \ldots, y_n][[x]]$, since $x \in N_i$, for every i, we have $\mathbf{n} \nsubseteq N_1 \cup \ldots \cup N_m$. Let $a \in \mathbf{n} \setminus (N_1 \cup \ldots \cup N_m)$ and let C be a complete set of γ_1 distinct coset representatives of $R[y_1, \ldots, y_n]/\mathbf{n}$.

Set $H = \{a + \sum_{i=1}^{\infty} w_i x^i \big| w_i \in C\}$. Then $H \subseteq (\mathbf{n}, x)A = P$, but $a \in \mathbf{n} \backslash (N_1 \cup \ldots \cup N_m)$ and so $h \notin (Q, x)$ for every $h \in H$.

Claim: Let $\mathbf{p} \in \mathcal{H}_{n+e}(A)$ be strictly between Q and P; that is, $Q \subsetneq \mathbf{p} \subsetneq P = (\mathbf{n}, x)$. Then $\mathbf{p}$ contains at most one element of H.

Proof. If $\mathbf{p}$ contains two distinct elements h_1 and h_2 of H, then, for h_1 and h_2 as given below with $w_i, v_i \in C$, for every $i \in \mathbb{N}$, we have

$$\begin{aligned} h_1 \quad &:= \quad a + \sum_{i=1}^{\infty} w_i x^i \in \mathbf{p} \cap H \quad \text{ and } h_2 := a + \sum_{i=1}^{\infty} v_i x^i \in \mathbf{p} \cap H; \\ h_1 \neq h_2 \quad &\Rightarrow \quad h_1 - h_2 = \sum_{i=1}^{\infty} w_i x^i - \sum_{i=1}^{\infty} v_i x^i = \sum_{i=1}^{\infty} (w_i - v_i) x^i \in \mathbf{p} \\ &\Rightarrow \quad h_1 - h_2 = x^t((w_t - v_t) + (w_{t+1} - v_{t+1})x + \ldots) \in \mathbf{p}, \end{aligned}$$

where t is the smallest positive integer so that $w_t \neq v_t$.

Since $\mathbf{p}$ is prime, $x \in \mathbf{p}$ or $(w_t - v_t) + (w_{t+1} - v_{t+1})x + \ldots \in \mathbf{p}$. If $x \in \mathbf{p}$, then $(Q, x) \subseteq \mathbf{p} \subseteq (\mathbf{n}, x)$, and so, since $\text{ht}(\mathbf{p}) = n + e$, we have $\mathbf{p} = N_i$, for some i. Then $h_1 \in N_i$ and $x \in N_i$ would imply $a \in N_i$, a contradiction to the choice of a. Thus $x \notin \mathbf{p}$. On the other hand, if $(w_t - v_t) + (w_{t+1} - v_{t+1})x + \ldots \in \mathbf{p}$, then, in case $A = R[[x]][y_1, \ldots, y_n]$, we have $(w_t - v_t) + (w_{t+1} - v_{t+1})x + \ldots \in (\mathfrak{m}, x)$, and so $w_t - v_t \in \mathfrak{m}$, a contradiction to w_t and v_t in distinct cosets of $\mathfrak{m}$.

In case $A = R[y_1, \ldots, y_n][[x]]$, we have $(w_t - v_t) + (w_{t+1} - v_{t+1})x + \ldots \in (\mathbf{n}, x)$, and so $w_t - v_t \in \mathbf{n}$, a contradiction to w_t and v_t in distinct cosets of $\mathbf{n}$.

Therefore the claim holds. $\square$

We return to the proof of Proposition 3.2.16. By Remarks 3.2.1, A is catenary. Since A is also Noetherian, Krull's Principal Ideal Theorem implies that there is at least one prime ideal of height $n + e$ between P and (Q, h) for each $h \in H$.

Since distinct elements of H yield distinct prime ideals in $Q^{\uparrow} \cap P^{\downarrow}$, there are at least $\gamma^{\aleph_0}$ height-$(n+e)$ prime ideals in $Q^{\uparrow} \cap P^{\downarrow}$. Thus $\gamma^{\aleph_0} \leq \beta$. Now we have $|Q^{\uparrow} \cap P^{\downarrow}| = \beta = |R[[x]]|$, and so the proposition holds. □

3.3 Maximal Ideals and j-spectra

In this section, we study the set of maximal ideals of three-dimensional mixed polynomial/power series rings, with emphasis on the numbers of various types that arise. We use the following setting:

Setting 3.3.1. Let R be a one-dimensional Noetherian domain with field of fractions K and let x and y be indeterminates over K. Let A be either $R[y][[x]]$ or $R[[x]][y]$.

We begin with the maximal ideals of maximal height, that is, height three.

Proposition 3.3.2. Assume Setting 3.3.1 and let $\mathcal{M}$ be a height-three maximal ideal of A. Then $\mathcal{M} = (\mathfrak{m}, x, h(y))A$, for some $\mathfrak{m} \in \max(R)$ and some $\overline{h(y)}$ irreducible in $\overline{R[y]} = R[y]/(\mathfrak{m}[y]) \cong (R/\mathfrak{m})[y]$. Conversely, the ideals $(\mathfrak{m}, x, h(y))A$ are maximal and have height three, for every $\mathfrak{m} \in \max(R)$ and $\overline{h(y)}$ irreducible in $\overline{R[y]} = R[y]/(\mathfrak{m}[y]) \cong (R/\mathfrak{m})[y]$. Thus there are $|R[y]| = |R| \cdot \aleph_0$ height-three maximal ideals of A.

Proof. For $R[y][[x]]$, $\mathcal{M} = (M, x)$, where $M \in \max(R[y])$ and $\mathrm{ht}(M) = 2$, by Remark 3.2.8(1). By [8, Proposition 3.4], such a maximal ideal M of $R[y]$ has the form $(\mathfrak{m}, h(y))R[y]$, where $\mathfrak{m} \in \max(R)$ and $\overline{h(y)}$ is irreducible in $\overline{R[y]} \cong (R/\mathfrak{m})[y]$. Thus every maximal height-three ideal of $R[y][[x]]$ is generated by $\mathfrak{m}, x$ and $h(y) \in R[y]$ as desired.

For $R[[x]][y]$, $\mathcal{M} = (M, x)$, where $M \in \max(R[y])$ and $\mathrm{ht}(M) = 2$, by Remarks 3.2.10(3). As in the paragraph above, this implies that $\mathcal{M}$ has the desired form, and so the result holds.

For the converse, the ideal of A generated by $\mathfrak{m}, x$ and $h(y) \in R[y]$, where $\overline{h(y)}$ is irreducible in $\overline{R[y]} = R[y]/(\mathfrak{m}[y]) \cong (R/\mathfrak{m})[y]$, are in one-to-one correspondence with the ideals of $R[y]$ generated by $\mathfrak{m}$ and $h(y) \in R[y]$ via the natural map $\pi_x : A \to R[y]$ with kernel (x), and these are maximal ideals of $R[y]$.

Let γ denote the cardinality of the set of all maximal ideals of form $(\mathfrak{m}, x, h(y))$ in A. Then by the correspondence above, γ equals the cardinality of the set of all maximal ideals of form $(\mathfrak{m}, h(y))$ in $R[y]$. Since every ideal is finitely generated, $\gamma \le |R[y]| = |R| \cdot \aleph_0$. Furthermore γ is at least as big as the number of maximal ideals of form $(\mathfrak{m}, h(y))$ in $R[y]$ for a fixed $\mathfrak{m} \in \max(R)$. Since each maximal ideal $(\mathfrak{m}, h(y))$ in $R[y]$ corresponds to a height-one maximal ideal of $k[y]$, where $k = R/\mathfrak{m}$; each is generated by an irreducible element of the PID $k[y]$. Thus γ is at least the cardinality of a complete set of nonassociate irreducible elements of $k[y]$. There are (at least) $|k[y]| = |R/\mathfrak{m}[y]|$ of these.[iv] Hence, using Lemma 3.2.2, $\gamma \ge |R/\mathfrak{m}[y]| = |R[y]|$, and so we have $\gamma = |R| \cdot \aleph_0$, as desired. □

Proposition 3.3.3. There are no height-one maximal ideals in $R[y][[x]]$ or in $R[[x]][y]$.

Proof. If $\mathcal{M}$ is a maximal ideal of $R[y][[x]]$, then by Remark 3.2.8(1), $\mathcal{M} = (M, x)$, where $M \in \max(R[y])$. Thus $\mathrm{ht}(M) \ge 1$, but $x \notin \mathcal{M}$, and so $\mathrm{ht}(\mathcal{M}) \ge 2$, as desired for $R[y][[x]]$.

Since $\dim(R[[x]]) = 2$, Lemma 3.2.5(2) implies that $R[[x]][y]$ has no height-one maximal ideals. □

3.3.1 Height-two Maximal Ideals

We consider the maximal ideals of height-two in $R[[x]][y]$ and $R[y][[x]]$. First we

[iv] It is enough to prove this for a finite field k, since always $\{(y-\alpha)\}_{\alpha \in k}$ consists of nonassociate irreducible elements and has cardinality $|k|$. Thus if k is infinite, then $|\{$ nonassociate irreducible elements $\}| \ge |k| = |k| \cdot \aleph_0 = |k[y]|$, using Lemma 3.2.2. If k is finite, it is straightforward to show that $|\{$ nonassociate irreducible elements $\}| = \aleph_0 = |k[y]|$.

prove a lemma adjusted from [10, Proposition 2.4].

Lemma 3.3.4. Let R be a one-dimensional Noetherian domain, let x be an indeterminate, and let Q be a height-one prime ideal of $R[[x]]$. If $x \notin Q$, then $D = R[[x]]/Q$ is a one-dimensional local domain with maximal ideal $\mathbf{m}_D$ that is complete with respect to the (x)-adic topology, and so D is Henselian.

Proof. Since $x \notin Q$, Q is not maximal by Remarks 3.2.8(1). Thus $D = R[[x]]/Q$ has dimension one. By [20, Theorem 8.7], D is complete with respect to the xD-adic topology and every maximal ideal of D is a minimal prime of the principal ideal xD. Therefore D is a complete semilocal ring. Since D is an integral domain, it is local by [20, Theorem 8.15]. Therefore D is a Henselian local domain with maximal ideal $\mathbf{m}_D$, [20, Theorem 8.3]. $\square$

Proposition 3.3.5. Let R be a one-dimensional Noetherian domain and let x and y be indeterminates.

(1) If R has infinitely many maximal ideals, then $R[y][[x]]$ has no height-two maximal ideals.

(2) If $\mathcal{M}$ is a height-two maximal ideal of $R[[x]][y]$, then (i) $\operatorname{ht}(\mathcal{M} \cap R[[x]]) = 1$ and (ii) $\operatorname{ht}(\mathcal{M} \cap R[y]) \leq 1$.

(3) If $\mathcal{M}$ is a height-two maximal ideal of $R[[x]][y]$ and $x \notin \mathcal{M}$, then $\mathcal{M}$ contains an element $1 + xyg(x, y)$, for some $0 \neq g(x, y) \in R[[x]][y]$. If $P := \mathcal{M} \cap R[[x]]$, then $D := R[[x]]/P$ is a one-dimensional Henselian Noetherian local domain and $\mathcal{M} = (P, Q)R[[x]][y]$, where Q is the preimage in $R[[x]][y]$ of a height-one maximal ideal of $(R[[x]]/P)[y]$ under the natural homomorphism $\pi_P : R[[x]] \to D$ with kernel P, extended to $R[[x]][y] \to D[y]$ by defining $\pi_P(y) = y$.

(4) If R has infinitely many maximal ideals and $\mathcal{M}$ is a height-two maximal ideal of $R[[x]][y]$, then $x \notin \mathcal{M}$.

(5) If R is semilocal, then there are one-to-one correspondences among the set of height-two maximal ideals of $R[y][[x]]$, the set of height-one maximal ideals of $R[y]$ and the set of height-two maximal ideals of $R[[x]][y]$ that contain x:

$$\begin{aligned}(\max(R[y][[x]])) \cap (\mathcal{H}_2(R[y][[x]]) &\leftrightarrow (\max(R[y])) \cap \mathcal{H}_1(R[y]) \\ &\leftrightarrow (V_{R[[x]][y]}(x)) \cap (\max(R[[x]][y]) \cap (\mathcal{H}_2(R[[x]][y]))\end{aligned}$$

via $\mathcal{N} \to \mathcal{N} \cap R[y] = N \to \mathcal{N}' = (N, x)R[[x]][y]$.

Thus a height-two maximal ideal $\mathcal{M}$ of $R[y][[x]]$ has form $\mathcal{M} = (M, x)R[y][[x]]$, where M is a height-one maximal ideal of $R[y]$, and $(M, x)R[[x]][y]$ is also a maximal ideal of $R[[x]][y]$ that contains x.

(6) If $Q \in \operatorname{Spec}(R[y][[x]])$ is minimal over $(yg(y) + 1)R[y][[x]]$, for some $0 \neq g(y) \in \mathcal{J}(R)(R[y])$, then Q is a height-one prime ideal of $R[y][[x]]$, R is semilocal and the only maximal ideal of $R[y][[x]]$ containing Q is (Q, x).

If $\mathcal{M}$ is a height-two maximal ideal of $R[[x]][y]$ and $x \in \mathcal{M}$, then R is semilocal and $\mathcal{M} = (M, x)$, where M is a height-one maximal ideal of $R[y]$,

(7) If R is semilocal,

a) Every height-two maximal ideal of $R[y][[x]]$ containing x contains an element of form $yg(y) + 1$, for some $0 \neq g(y) \in \mathcal{J}(R)(R[y])$, where $\mathcal{J}(R) = \cap_{\mathfrak{m} \in \max R}\, \mathfrak{m}$, the Jacobson radical of R.

b) There are $|R|$ height-two maximal ideals of $R[y][[x]]$.

Proof. For item 1, if R has infinitely many maximal ideals, then $R[y]$ has no height-one maximal ideals by Lemma 3.2.5(2), and so every maximal ideal of $R[y]$ has height two. Since every maximal ideal of $R[y][[x]]$ has form (M, x) where M is a maximal ideal of $R[y]$, every maximal ideal of $R[y][[x]]$ has height three.

For item 2, part i, if $\mathcal{M}$ is a height-two maximal of $R[[x]][y]$, and $\operatorname{ht}(\mathcal{M} \cap R[[x]]) = 2$, then $\mathcal{M} \cap R[[x]]$ is a height-two maximal ideal of $R[[x]]$, and so the ideal $(\mathcal{M} \cap R[[x]])R[[x]][y]$ is prime, has height two, and thus equals $\mathcal{M}$, but

$$(\mathcal{M} \cap R[[x]])R[[x]][y] \subsetneq (\mathcal{M} \cap R[[x]], y)R[[x]][y] \neq R[[x]][y],$$

a contradiction to $\mathcal{M}$ maximal. Also, if $(\mathcal{M} \cap R[[x]]) = (0)$, then $\operatorname{ht}(\mathcal{M}) \leq 1$, by Remarks 3.2.4(3). Therefore $\operatorname{ht}(\mathcal{M} \cap R[[x]]) = 1$. Thus item 2.i holds.

Similarly, for item 2, part ii, if $\operatorname{ht}(\mathcal{M} \cap R[y]) = 2$, then

$$\mathcal{M} = (\mathcal{M} \cap R[y])R[[x]][y] \subsetneq (\mathcal{M} \cap R[y], x)R[[x]][y],$$

a contradiction to $\mathcal{M}$ maximal. Thus $\operatorname{ht}(\mathcal{M} \cap R[y]) \leq 1$

For item 3, suppose $y \in \mathcal{M}$. This implies that $\mathcal{M}/(y)$ is a maximal ideal of $R[[x]][y]/(y) \cong R[[x]]$. Thus $\mathcal{M} = (M, y)$, where M is a maximal ideal of $R[[x]]$, and so, by Theorem 3.2.9, M has height two. But then $\mathcal{M} = (M, y)$ has height three, a contradiction. Thus we may assume that $y \notin \mathcal{M}$. Hence $xy \notin \mathcal{M}$ and so $1 \in (\mathcal{M}, xy)$. We write $1 = f(x, y) - xyg(x, y)$, where $f(x, y) \in \mathcal{M}$ and $g(x, y) \in R[[x]][y]$. Then $f(x, y) = 1 + xyg(x, y)$, as desired for the first statement. (Since $f(x, y) \neq 1$, $g(x, y \neq 0$.) For the second statement we use Lemma 3.3.4.

For items 4 and 5, we see that $x \in \mathcal{M} \implies \mathcal{M}/(x)$ is a maximal ideal of $R[[x]][y]/(x) \cong R[y]$. Thus $\mathcal{M} = (M, x)$, where M is a maximal ideal of $R[y]$ and $\operatorname{ht}(M) = 1$. By Lemma 3.2.5(2), $\max R$ is finite. Thus item 4 holds.

For item 5, if $\mathcal{N}$ is a maximal ideal of $R[y][[x]]$, then $\mathcal{N} = (N, x)$, where N is a maximal ideal of $R[y]$ by Remark 3.2.8(1), and then $\mathrm{ht}(N) = 1$. Using the canonical map $\pi'_x : R[y][[x]] \to R[y]$ with kernel (x), yields the first part of the correspondence. The analogous canonical map $\pi_x : R[[x]][y] \to R[y]$ yields the second part.

For item 6, the first statement, we have that $\mathrm{ht}(Q) \leq 1$ by Krull's Principal Ideal Theorem. Since $Q \neq (0)$, $\mathrm{ht}(Q) = 1$. By the definition of Q, we see that $Q \cap R[y] \neq (0)$ and $Q = (Q \cap R[y])R[y][[x]]$ by Proposition 3.2.11. If $Q \cap R \neq (0)$, then $\mathrm{ht}(Q) \geq 1$, a contradiction. Thus $Q \cap R = (0)$, and so, by Lemma 3.2.5(3), $Q \cap R[y]$ is a height-one maximal ideal of $R[y]$. By Lemma 3.2.5(2), R is semilocal. By Remark 3.2.8(1), every maximal ideal $\mathcal{M}$ of $R[y][[x]]$ has form (M, x), where M is some maximal ideal of $R[y]$. Then $M \subseteq Q \cap R[y]$, and so $Q \cap R[y] = M$, $\mathcal{M} = (Q, x)$, and this is the only maximal ideal containing Q.

For the second statement, since $x \in \mathcal{M}$, the image $\mathcal{M}/(x)$ is a maximal ideal of $R[[x]][y]/xR[[x]][y] \cong R[y]$ and so corresponds to a height-one maximal ideal M of $R[y]$. This implies that $\mathcal{M} = (M, x)R[[x]][y]$, and that R is semilocal by Lemma 3.2.5(2).

For part a of item 7, we have from item 5 that a height-two maximal ideal $\mathcal{N}$ of $R[y][[x]]$ has form $\mathcal{N} = (N, x)$, where N is a height-one maximal ideal of $R[y]$. By Lemma 3.2.5(3), N contains an element $yg(y) + 1$, where $0 \neq g(y)$ has coefficients in $\mathcal{J}(R) = \cap_{\mathfrak{m} \in \max R} \mathfrak{m}$ and every prime ideal Q of $R[y]$ minimal over such an element of $R[y]$ is a height-one maximal ideal of $R[y]$.

For part b of item 7, let $a \in \mathcal{J}$ with $a \neq 0$, and, for each $b \in R$, define the polynomial $h_b(y) = ay(y - b) + 1$. The ring R is infinite, because every finite integral domain is a field. We claim that if $b \neq c \in R$, then at most one of $h_b(y)$ and $h_c(y)$ is an element of a height-one prime ideal Q of $R[y]$. This is because such a prime ideal Q is a height-one maximal ideal of Q and has 0 intersection with R, whereas if both were contained in Q, then

$$0 \neq ay(b-c) = h_b(y) - h_c(y) \in Q \implies 0 \neq b - c \in R \cap Q,$$

a contradiction. Thus the number of such Q is at least $|R|$. On the other hand $R[y]$ is Noetherian and every prime ideal is finitely generated and so the number of ideals of $R[y]$ is at most $|R[y]| \cdot \aleph_0 = |R| \cdot \aleph_0 \cdot \aleph_0 = |R|$, since R is infinite. Thus part b holds. $\square$

Proposition 3.3.6. For every height-one prime ideal Q of $R[[x]]$ such that $x \notin Q$, there exist $|R[[x]]|$ height-two maximal ideals $\mathcal{N}$ of $R[[x]][y]$ containing Q, and $|R|$ height-three maximal ideals $\mathcal{M}$ of $R[[x]][y]$ that contain Q.

Proof. Since $Q \neq (x)$, Theorem 3.2.9 implies there exists a unique $\mathfrak{m} \in \max(R)$ with $Q \subsetneq (\mathfrak{m}, x)R[[x]]$.

For the second statement of Proposition 3.3.6, we see that every maximal ideal of the form $\mathcal{M} = (\mathfrak{m}, x, h(y))$, where the image $\overline{h(y)}$ in $\overline{R[[x]][y]} = R[[x]][y]/(\mathfrak{m}, x)$ is an irreducible polynomial, is a height-three maximal ideal of $R[[x]][y]$ that contains Q and there are $|R| = |R[y]|$ of them, as shown in the proof of Proposition 3.3.2. Every $\mathfrak{n} \in \max(R)$ with $\mathfrak{n} \neq \mathfrak{m}$ is comaximal with Q by Theorem 3.2.9. Thus every height-three maximal ideal $\mathcal{N} = (\mathfrak{n}, x, h(y))$ with $\mathfrak{n} \in \max(R)$ and $\mathfrak{n} \neq \mathfrak{m}$ does not contain Q. Thus the $\mathcal{M} = (\mathfrak{m}, x, h(y))$ as above are the only maximal height-three ideals that contain Q. Thus the second statement holds.

For the first statement of Proposition 3.3.6, by Lemma 3.3.4, $D = R[[x]]/Q$ is a one-dimensional Henselian Noetherian local domain. The unique maximal ideal of D is $\mathfrak{m}_D = \pi_Q(\mathfrak{m}, x)$, where $\pi_Q : R[[x]][y] \to D[y]$ with $\ker(\pi_Q) = Q$.

As in the proof of Proposition 3.2.16, let $\mathcal{C}$ be a complete set of distinct coset representatives of $R/\mathfrak{m}$ and let $0 \neq a \in \mathfrak{m}$. Let

$$\mathcal{H} = \{h(x,y) = 1 + y(a + \sum_{i=1}^{\infty} c_i x^i)\}_{c_i \in \mathcal{C}} \subseteq R[[x]][y].$$

For convenience, let $h_c = a + \sum_{i=1}^{\infty} c_i x^i$ denote the y coefficent of $h \in \mathcal{H}$. Notice that if $h_c \in Q$, then $1 = -h_c y + (1 + yh_c)$, and so in this case Q is comaximal in $R[[x]][y]$ with the element $h = 1 + yh_c$.

Claim 3.3.7. (1) If $(h, Q)R[[x]][y] = R[[x]][y]$, then $h_c \in Q$.

(2) There exists at most one $h \in \mathcal{H}$ with $h_c \in Q$.

(3) For $h \neq \ell \in \mathcal{H}$, and $h, \ell \notin Q$, $(Q, h, \ell) = R[[x]][y]$.

Proof. For item 1, we use Remark 3.2.4(6). For items 2 and 3, assume that h and ℓ are distinct elements of $\mathcal{H}$ with $\ell_c = a + \sum_{i=1}^{\infty} d_i x^i$ and $h_c = a + \sum_{i=1}^{\infty} c_i x^i$. Let i be the smallest coefficient so that $c_i \neq d_i$. Then $h - \ell = y(h_c - \ell_c)$ and so

$$h - \ell = x^i y(c_i - d_i + \sum_{j=i+1}^{\infty} (c_j - d_j)x^{j-i}) = x^i yu; \quad u = c_i - d_i + \sum_{j=i+1}^{\infty} (c_j - d_j)x^{j-i},$$

for some $i \in \mathbb{N}$. Since c_i and d_i are distinct cosets of $\mathfrak{m}$ in R, we have $c_i - d_i \notin \mathfrak{m}$ and u is not an element of $(\mathfrak{m}, x)R[[x]]$.

For item 2, suppose that $\ell_c, h_c \in Q$. Then $x \notin Q$, $y \notin Q$ and $\ell_c - h_c \in Q$ imply $u \in Q$. However $Q \subseteq (x, \mathfrak{m})$, a contradiction to the above argument. This proves item 2.

For item 3, suppose a maximal ideal $\mathcal{M}$ contains $Q \cup \{h\} \cup \{\ell\}$. Then $x^i yu \in \mathcal{M}$, and so $x \in \mathcal{M}$, $y \in \mathcal{M}$ or $u \in \mathcal{M}$. If $x \in \mathcal{M}$, then $\mathcal{M} \cap R[[x]]$ is a prime ideal that contains Q and x, and so $(\mathfrak{m}, x) \subseteq \mathcal{M}$. However, also $h \in \mathcal{M} \implies ay + 1 \in \mathcal{M}$, whereas $a \in \mathfrak{m} \implies ay + 1$ is comaximal with $(\mathfrak{m}, x)$ a contradiction, and so $x \notin \mathcal{M}$.

Suppose that $y \in \mathcal{M}$. Then $h = 1 + yh_c \in \mathcal{M} \implies 1 \in \mathcal{M}$, a contradiction. Finally, if $u \in \mathcal{M}$, we saw above that $u \in R[[x]]$, but $u \notin (\mathfrak{m}, x)$. By Theorem 3.2.9, u is in no maximal ideal of $R[[x]]$ that contains Q and thus u is comaximal with Q, again a contradiction. This proves item 3, and so the claim is proved. □

For each $h \in \mathcal{H}$, *except*, if such an h exists, the one h such that $(h, Q) = (1)$, let $\mathcal{N}_h$ be a prime ideal minimal over $(Q, h(x, y))R[[x]][y]$. Since Q has height one and $R[[x]][y]$ is catenary by Remarks 3.2.1, we have $\operatorname{ht}(\mathcal{N}_h) = 2$.

Every height-three maximal ideal containing Q has form $(x, \mathfrak{n}, h(y))$ for some $h(y) \in R[y]$ and $\mathfrak{n} \in \max(R)$, using Proposition 3.3.2. By the observation, from Theorem 3.2.9, that every maximal ideal of R other than $\mathfrak{m}$ is comaximal with Q, the only possible height-three maximal containing $\mathcal{N}_h$ is $(x, \mathfrak{m}, h(y))$ for some $h(y) \in R[y]$. Since each $h_c \in (\mathfrak{m}, x)$, we see that $h \notin (x, \mathfrak{m})$. Thus each $\mathcal{N}_h$ is a height-two maximal ideal and so there are at least $|\mathcal{H}| = |R/\mathfrak{m}|^{\aleph_0} = |R[[x]]|$ height-two maximal ideals containing Q, using Lemma 3.2.2. Since $R[[x]][y]$ is Noetherian implies every ideal is finitely generated, we have that the number of height-two maximal ideals containing Q is less than or equal to $|R[[x]][y]| = |R[[x]]|$, and we are done. □

3.3.2 j-primes of $R[[x]][y]$ and $R[y][[x]]$

We start with the definition of a j-prime ideal and j-spectrum of a commutative ring.

Definition 3.3.8. Let A be a commutative ring.

- A *j-prime* of A is a prime ideal of A that is an intersection of maximal ideals of A;
- The j-spectrum of A is $j\text{-Spec}(A) := \{j\text{-primes} \in \operatorname{Spec}(A)\}$.

For U a partially ordered set, we say that $u \in U$ is a *j-element* if u is an intersection of maximal elements of U.

Note 3.3.9. A maximal ideal of a ring A, respectively a maximal element of a partially ordered set U, is considered to be a j-prime, respectively a j-element. The non-maximal j-primes provide crucial information for the determination of prime spectra for our rings.

We use Setting and Notation 3.3.10 in the remainder of this subsection.

Setting and Notation 3.3.10. Let R be a one-dimensional domain and let x and y be indeterminates. Let A be either $R[[x]][y]$ or $R[y][[x]]$ and let Q be a height-one prime ideal of A such that $x \notin Q$ and $(Q, x)A \neq A$. Set $B := A/Q$. Let I be a nonzero ideal of $R[y]$ such that $(I, x)A = (Q, x)A$.

Note 3.3.11. The ideal I from Setting and Notation 3.3.10 is a height-one ideal of $R[y]$; that is, every prime P of $R[y]$ minimal over I has height one.

Proof. If $I = (0)$, then $(I, x) = (x) \neq (Q, x)$, since $Q \neq (0)$ and $(x) \not\supseteq Q$. Thus $I \neq (0)$. Thus $\text{ht}(P) \geq 1$. Since $\text{ht}(Q) = 1$ and $x \notin Q$, $\text{ht}(I, x) = \text{ht}(Q, x) \leq 2$ by Krull's principal ideal theorem because A is catenary 3.2.1. By the same reasoning, since $x \notin P$, $\text{ht}(P, x) > \text{ht}(P)$ and so $\text{ht}(P) = 1$. □

We show in this subsection that the j-primes of A that contain Q also contain x. It follows that each j-prime of A corresponds to a minimal prime ideal of $R[y]/I$ and vice-versa. We begin to demonstrate this correspondence with the following remarks.

Remarks 3.3.12. With Setting and Notation 3.3.10, consider the following canonical surjections:

(1) $\pi : A \longrightarrow B = A/Q$ with $\ker(\pi) = Q$,

(2) $\pi_x : A \longrightarrow R[y]$ with $\ker(\pi_x) = (x)$.

(i) Since Q is a height-one prime ideal of A, B is a Noetherian integral domain with $\dim(B) \leq 2$.

(ii) π and π_x induce natural order-preserving maps π^{-1} and π_x^{-1} on the prime spectra;

$\pi^{-1} : \mathrm{Spec}(B) \to \mathrm{Spec}(A); \quad \pi_x^{-1} : \mathrm{Spec}(R[y]) \cong \mathrm{Spec}(A/xA) \to \mathrm{Spec}(A)$

$$\mathrm{Spec}(B) \cong V_A(Q) \quad \text{and} \quad \mathrm{Spec}(R[y]) \cong V_A(x),$$

$$V_{R[y]}(I) \cong V_A(x, I) = V_A(x, Q) \cong V_B(x).$$

(iii) Since A is catenary, the correspondences in Remark 3.3.12(ii) above imply that, for each $n \leq 2$, the height-n prime ideals of A can be identified with the height-$(n+1)$ prime ideals of A containing Q ; and the height-n prime ideals of $R[y]$ can be identified with height-$(n+1)$ prime ideals of A containing x.

(iv) For a commutative ring C and a height-one prime ideal Q of $C[[x]]$, the ideal $(Q, x)C[[x]]$ is a proper ideal of $C[[x]]$, [8]. To see this, if $(Q, x)C[[x]] = C[[x]]$, then there exists $b(x) \in C[[x]]$ such that $x(b(x)) - 1 \in Q$, a contradiction since $bx - 1$ is a unit of $C[[x]]$. Thus $(Q, x) \neq (1)$.

Proposition 3.3.13. Assume Setting 3.3.10. By Remarks 3.3.12,

(1) $\mathrm{Spec}(B/xB) \cong \mathrm{Spec}(R[y]) \cap (V_{R[y]}I) \cong \mathrm{Spec}(R[y]/I)$.

(2) j-$\mathrm{Spec}(B) \setminus \{(0)\} \setminus \{\text{height-one maximal elements}\} \cong \mathrm{Spec}(B/xB)$; $\mathrm{Spec}(B/xB) = j$-$\mathrm{Spec}(B/xB)$, and $\mathrm{Spec}(R[y]/I) = j$-$\mathrm{Spec}(R[y]/I)$.

(3) The height-one prime ideals of B that contain x correspond to the height-one prime ideals of $R[y]$ that contain I.

Proof. Item 1 follows from Remarks 3.3.12(ii). For item 2, suppose that P is a nonzero nonmaximal height-one j-prime ideal of B. Then P is an intersection of height-two maximal ideals of B. Every maximal ideal M of height two of $B = A/Q$ contains x since every height-three maximal ideal of A contains x by Proposition 3.3.2. Thus $x \in P$ and the correspondences in Remarks 3.3.12(ii) carry P over to a unique element of $\operatorname{Spec}(B/xB)$ that is the intersection of maximal ideals there, and then, as in item 1, to a unique element of j-$\operatorname{Spec}(R[y]/I)$. Furthermore, if M is a height-two maximal ideal of B, then $x \in M$, and item 1 shows that M corresponds to a unique maximal element of j-$\operatorname{Spec}(R[y]/I)$. For every element of j-$\operatorname{Spec}(R[y]/I)$, the steps are reversible by Remarks 3.3.12 and we have the desired isomorphisms. Note that $R[y][[x]]$ has no height-one maximal ideals if $\max(R) = \infty$ by Lemma 3.2.5(2).

For item 3, we observe that the nonmaximal height-one j-primes are the minimal elements of $V_B(x)$ by item 2. $\square$

In the next section we describe $\operatorname{Spec}(R[y]/I)$, where I is an ideal of $R[y]$ such that $(I, x) = (Q, x)$ and $(I, x) = (Q', x)$, as in Setting 3.3.10.

3.4 Prime Spectra for Images of $R[y]$

In this section we give a description and examples of $\operatorname{Spec}(R[y]/I)$, where R is a one-dimensional Noetherian domain, y is an indeterminate, and I is a height-one ideal of $R[y]$.

In particular we show that $\operatorname{Spec}(R[y]/I)$ satisfies the following definition.

Definition 3.4.1. Let $\ell \in \mathbb{N}_0$ and let $\gamma_1, \ldots, \gamma_\ell$ be cardinal numbers. Let F be a finite partially ordered set of dimension at most one with ℓ minimal elements such that every height-one element of F is greater than at least two height-zero elements of F. A partially ordered set U of dimension 0 or 1 has *image polynomial type* $(\ell; (\gamma_1, \ldots, \gamma_\ell); F)$, abbreviated as (IPT), if there exists an order-isomorphism $\varphi : F \to U$ such that, if $P_1, P_2, \ldots, P_\ell$ are the minimal elements of F:

(1) $|U| = |F| + \gamma_1 + \ldots + \gamma_\ell$.

(2) $\min(U) = \mathcal{H}_0(U) = \{\varphi(P_1), \ldots, \varphi(P_\ell)\}$.

(3) $\mathcal{H}_1(U) = \bigcup \varphi(P_i)^\uparrow = \varphi(F \setminus \{P_1, \ldots, P_\ell\}) \cup \bigcup_{i=1}^t T_i$, where $T_i := \varphi(P_i)^\uparrow \setminus (\cup_{j \neq i} \varphi(P_j)^\uparrow)$, that is, $L_e(T_i) = \{\varphi(P_i)\}$; $|T_i| = \gamma_i$.

(4) $\{\varphi(P_1), \ldots, \varphi(P_\ell)\} \supseteq \{u \in U \mid |u^\uparrow| = \infty,\ \mathrm{ht}(u) = 0\}$, the set of nonmaximal j-elements of U.

(5) For every $i \neq j$, $\varphi(P_i)^\uparrow \cap \varphi(P_j)^\uparrow = \varphi(P_i^\uparrow \cap P_j^\uparrow) \subseteq \varphi(F)$.

Notes 3.4.2. (i) The axioms are somewhat redundant for more clarity.
(ii) If $\gamma_i = \gamma$ for every i with $1 \leq i \leq \ell$, we abbreviate the type of the image polynomial poset U to $(\ell; \gamma; F)$.

We record our setting for the rest of this section:

Setting and Notation 3.4.3. Let R be a one-dimensional Noetherian domain, let y be an indeterminate, and let I be an ideal of $R[y]$ of height one. We identify $\mathrm{Spec}(R[y]/I)$ with $V_{R[y]}(I)$ and consider three categories of minimal elements of $V_{R[y]}(I)$, namely:

Define $V_0(I) := \{\mathfrak{m}R[y] \mid I \subseteq \mathfrak{m}R[y], \mathfrak{m} \in \max R\}$;

$V_1(I) := \mathcal{H}_1(R[y]) \cap \max(R[y]) \cap V_{R[y]}(I)$;

$V_2(I) := \mathcal{H}_1(R[y]) \cap V_{R[y]}(I) \setminus V_0(I) \setminus V_1(I)$.

Let $V_0(I) := \{q_1, \ldots, q_t\}$; $V_1(I) = \{q_{t+1}, \ldots, q_m\}$; $V_2(I) = \{q_{m+1}, \ldots, q_\ell\}$.

Then $V_0(I) \cup V_1(I) \cup V_2(I) = \{q_1, \dots, q_\ell\}$ is the set of minimal elements of $V_{R[y]}(I)$ and they correspond to the minimal elements of $\operatorname{Spec}(R[y]/I)$.

Let $F = \{q_1, \dots, q_\ell\} \cup \{q_i^\uparrow \cap q_j^\uparrow\}_{1 \le i < j \le \ell}$, a subset of $V_{R[y]}(I)$. For each i with $1 \le i \le \ell$, let $T_i = q_i^\uparrow \setminus (\bigcup_{j \ne i} q_j^\uparrow)$, the height-two maximal ideals of $R[y]$ that contain q_i but none of the other q_js. Then $T_i = \emptyset$ if $q_i \in V_1(I)$.

Let $\gamma_i = |T_i|$ for each i with $1 \le i \le \ell$.

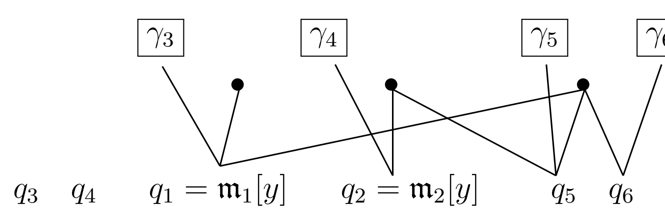
Thus $\operatorname{Spec}(R[y]/I)$ corresponds to $F \cup \bigcup \{T_i\}_{i=1}^{\ell}$.

Proposition 3.4.4. With Setting and Notation 3.4.3, $\operatorname{Spec}(R[y]/I)$ is a partially ordered set of image polynomial type $(\ell; (\gamma_1, \dots, \gamma_\ell); F)$, for $\ell, \{\gamma_i\}_{i=1}^{\ell}, F$ as in (3.4.3).

Proof. Let $q_1, \dots, q_\ell$, F and T_i be as defined in Setting and Notation 3.4.3. Then $\operatorname{Spec}(R[y]/I) = F \cup \bigcup \{T_i\}_{i=1}^{\ell}$ satisfies the axioms of Definition 3.4.1; that is, $\operatorname{Spec}(R[y]/I)$ has image polynomial type $(\ell; (\gamma_1, \dots, \gamma_\ell); F)$. $\square$

Remarks and Pictures 3.4.5.

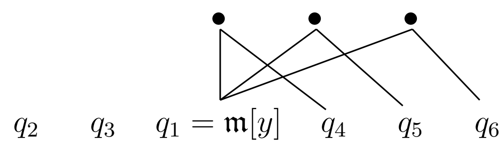
(1) If R is semilocal, with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$, then $V_1(I) \ne \emptyset$ by Lemma 3.2.5(2). Here is a possible picture of $\operatorname{Spec}(R[y]/I)$ in this case:



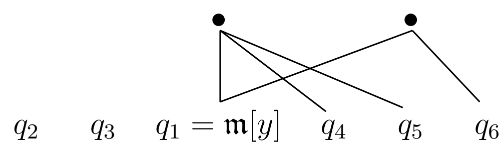
R semilocal, $s = 2$, $\ell = 6$

(2) If R is Henselian, then R is local, say with maximal ideal $\mathfrak{m}$ and $R[y]$ has the special property that each nonmaximal height-one prime is contained in a unique maximal ideal.

Here are two possible pictures of $\mathrm{Spec}(R[y]/I)$ in this case:



R Henselian, $\ell = 6$



R Henselian, $\ell = 6$

(3) If $\max(R)$ is infinite, then $R[y]$ has no maximal ideals of height one and so $V_1(I) = \emptyset$ by Lemma 3.2.5(2). A possible picture of $\mathrm{Spec}(R[y]/I)$, for $R = \mathbb{Z}$, is in Example 3.6.1.

3.5 Putting It All Together

In this section we give our main results for the spectra of homomorphic images of three-dimensional polynomial/power series rings, over a one-dimensional Noetherian domain, as described in the introduction.

For R a one-dimensional Noetherian domain and x and y indeterminates over R, we give a partial description of prime spectra of homomorphic images of the mixed power series/polynomial rings $R[[x]][y]$ and $R[y][[x]]$ modulo a height-one prime ideal. In some cases, we can determine these spectra more precisely.

Definition 3.5.4 of this section contains a general set of axioms that are satisfied by the two-dimensional image rings of these mixed power series/polynomial rings. These axioms hold for the partially ordered sets of prime ideals of image rings of the form $B = R[[x]][y]/Q$ and $B' = R[y][[x]]/Q'$, where Q and Q' are height-one prime ideals of $R[[x]][y]$ and $R[y][[x]]$ respectively, except for three special cases for Q: Namely, (Q, x) is a maximal ideal, $(Q, x) = (1)$ and $(Q, \mathfrak{m}) = (1)$ for every $\mathfrak{m} \in \max(R)$; and the analogous cases for Q'. First we take care of the special cases for Q and Q'.

3.5.1 Special Cases

Definition 3.5.1. A partially ordered set U is a *fan* if U is one-dimensional with a unique minimal element. This includes a partially ordered set U with exactly two elements, one of height zero and another one of height one above the minimal element.

Theorem 3.5.2. Let R be a one-dimensional Noetherian domain and let x and y be indeterminates over R. Let $A = R[[x]][y]$ or $R[y][[x]]$, let Q be a height-one prime ideal of A, and let $B = A/Q$. In any of the following three cases, $\operatorname{Spec}(B)$ is a fan.

(i) Every height-two prime ideal of A containing $(Q, x)A$ is maximal.

(ii) $(Q, x) = (1)$.

(iii) $(Q, \mathfrak{m}) = (1)$, for every $\mathfrak{m} \in \max R$.

Moreover, if $A = R[y][[x]]$, then $\operatorname{Spec}(B)$ is a fan with two elements.

Proof. In each of these cases, Q is not contained in any height-three prime ideal of either ring A by Remark 3.2.15. Since no maximal ideals of A have height one by Proposition 3.3.3, every maximal ideal of A containing Q has height two. It follows that $\dim(B) = 1$. Since B is an integral domain, $\operatorname{Spec}(B)$ is a fan.

To see that there is just one maximal ideal in A/Q in case $A = R[[x]][y]$, we use Proposition 3.3.5(7). $\square$

We believe that $\operatorname{Spec}(B)$ is a fan with $|R[[x]]|$ elements if $A = R[[x]][y]$ in Theorem 3.5.2. This cardinality argument is in progress.

3.5.2 The General Case of Dimension Two

Except for the special cases of Theorem 3.5.2, the prime spectra of homomorphic images of mixed power series/polynomial rings $R[[x]][y]$ and $R[y][[x]]$ by height-one prime ideals are two dimensional and satisfy the axioms of Definition 3.5.4, if R is a one-dimensional Noetherian domain and x and y are indeterminates over R; see Theorem 3.5.5.

We use the following setting and notation for the rest of this section.

Setting and Notation 3.5.3. Let R be a one-dimensional Noetherian domain and let x and y be indeterminates over R and let $A = R[[x]][y]$ or $R[y][[x]]$. Let Q be a height-one prime ideal of A such that $x \notin Q$, no prime ideal of height two containing $(Q, x)A$ is maximal, the ideal $(Q, x)A$ is not all of A, and, for some $\mathfrak{m} \in \max R$, the ideal $(Q, \mathfrak{m})A \neq A$. Set $B := A/Q$, and let I be the height-one ideal of $R[y]$ such that $(I, x)A = (Q, x)A$. We refer also to Setting and Notation 3.4.3; in particular, let the set F, the sets T_i and the cardinalities γ_i, for $1 \leq i \leq \ell$, be as defined there.

The axioms of Definition 3.5.4 are intentionally redundant, in order to explain the situation in more detail.

Definition 3.5.4. Let $\ell \in \mathbb{N}_0$ and let $\epsilon, \beta, \gamma_1, \ldots, \gamma_\ell$ be cardinal numbers with $\epsilon, \gamma_i \leq \beta$, for each γ_i. Let F be a finite partially ordered set of dimension at most one with ℓ minimal elements such that every non-minimal maximal element of F is greater than at least two minimal elements of F. A partially ordered set U of dimension 1 or 2 is *image polynomial power series of type* $(\epsilon; \beta; \ell; (\gamma_1, \ldots, \gamma_\ell); F)$, abbreviated as (IPPS), if there exists an order-isomorphism $\varphi : F \to U$ such that, if $P_1, P_2, \ldots, P_\ell$ are the minimal elements of F:

(1) $|U| = \beta$ and U has a unique minimal element u_0.

(2) $|\{\mathcal{H}_1(U) \cap \max(U)\}| = \epsilon; \quad \{\varphi(P_1), \ldots, \varphi(P_\ell)\} \subseteq \mathcal{H}_1(U)$.

(3) $\mathcal{H}_2(U) = \bigcup \varphi(P_i)^\uparrow = (\varphi(F) \setminus \{\varphi(P_1), \ldots, \varphi(P_\ell)\}) \cup \{T_i\}_{i=1}^{\ell}$, where each $T_i = \varphi(P_i)^\uparrow \setminus (\cup_{j \neq i} \varphi(P_j)^\uparrow)$ and $|T_i| = \gamma_i$.

(4) $\{\varphi(P_1), \ldots, \varphi(P_\ell)\}$ contains the set $\{u \in U \mid |u^\uparrow| = \infty,\ \mathrm{ht}(u) = 1\}$ of nonmaximal j-elements of U.

(5) For every $u \in \mathcal{H}_1(U) \setminus \varphi(F)$, there exists a unique maximal element in U that is greater than or equal to u.

(6) For every $1 \leq i < j \leq \ell$, $\varphi(P_i)^\uparrow \cap \varphi(P_j)^\uparrow = \varphi(P_i^\uparrow \cap P_j^\uparrow) \subseteq \varphi(F)$.

(7) For every finite nonempty subset $T \subseteq \mathcal{H}_2(U) \setminus F$, $L_e(T) = \emptyset$ if $|T| > 1$ and $|L_e(T)| = \beta$ if $|T| = 1$.

Main Theorem 3.5.5. Let R be a one-dimensional Noetherian domain, let x and y be indeterminates, let A be $R[[x]][y]$ or $R[y][[x]]$, let Q be a height-one prime ideal of A with $Q \neq (x)$; $(Q, x) \neq (1)$; no height-two prime ideal containing $(Q, \mathfrak{m})$ is maximal; and $(Q, \mathfrak{m}) \neq (1)$, for every $\mathfrak{m} \in \max R$, set $\beta = \big|R[[x]]\big|$ and let $B := A/Q$. Then $\mathrm{Spec}(B)$ is image polynomial power series of type $(\epsilon; \beta; \ell; (\gamma_1, \ldots, \gamma_\ell); F)$, for F, ℓ and γ_i as in Settings 3.5.3 and 3.4.3 and some cardinal number ϵ.

Proof. To determine the type, we need F, ϵ, ℓ and the γ_i. We assign ϵ to be the number of height-one maximal ideals of B for the generality of this theorem.

To identify the other parts of the type and check the axioms in Definition 3.5.4, we proceed as follows: Let I be the height-one prime ideal of $R[y]$ such that $(I, x)A =$

$(Q,x)A$, let $\ell = |\min(R[y]/I)|$, let $\{q_1,\ldots,q_\ell\}$ be the minimal elements of $V_{R[y]}(I)$ that correspond to $\min(R[y]/I)$ and let F be as in Notation 3.4.3. For each i with $1 \le i \le \ell$, let $\varphi(q_i) = P_i := \pi(q_i,x)A \in \operatorname{Spec}(B)$. By Remarks 3.3.12, $V_{R[y]}(I) \cong V_B(x)$. Thus the P_i corresponds to q_i via $\pi(\pi_x^{-1}(q_i)) = \pi(q_i,x)A = P_i$, where $\pi : A \to A/Q$ and $\pi_x^{-1} : A \to R[y]$ are the canonical surjections in Remarks 3.3.12. Similarly let each $T_{i,B}$ be the set of prime ideals of B containing only P_i (of the elements of $\varphi(F)$—these correspond to the height-two prime ideals of $R[y]$ such that the only element of F contained in them is q_i.

Now let $F_B = \{P_1,\ldots,P_\ell\} \cup \{P_i^{\uparrow} \cap P_j^{\uparrow}\}_{i\neq j}$, and let each $\gamma_{i,B}$ be the cardinality of $P_i^{\uparrow} \setminus (\cup_{j\neq i} P_j^{\uparrow})$. Thus F_B corresponds to the set F of Setting 3.4.3 and γ_i is as defined there.

We show that $\operatorname{Spec}(A/Q)$ satisfies the axioms in Definition 3.5.4.

Since $Q \neq (x)$, $(Q,x) \neq (1)$, $(Q,\mathfrak{m}) \neq (1)$ and no height-two prime ideal containing $(Q,\mathfrak{m})$ is maximal for all $\mathfrak{m} \in \max(R)$, we have A has β height-two prime ideals containing Q and contained in each height-three maximal ideal by Proposition 3.2.16. Thus $\operatorname{Spec}(A/Q)$ has at least β elements. Since $|A| = \beta$ and A is Noetherian, $|\operatorname{Spec}(A)| = \beta$. Since A/Q is an integral domain, axiom 1 holds.

By Proposition 3.4.4, $\operatorname{Spec}(R[y]/I)$ is a partially ordered set of image polynomial type $(\ell;(\gamma_1,\ldots,\gamma_\ell);F)$, that is, $\operatorname{Spec}(R[y]/I)$ satisfies the axioms in Definition 3.4.1. By the correspondence, $\operatorname{Spec}(A/Q)$ also satisfies axioms 3, 4, and 6 in Definition 3.5.4, and also $|P_i^{\uparrow}| = |\varphi(q_i)^{\uparrow}| = \gamma_i$.

Every height-two prime ideal N of A with $x \notin N$ is contained in a unique maximal ideal of A by Proposition 3.2.14. Thus, by the correspondence, for every height-one prime ideal $\overline{N}$ of $\operatorname{Spec}(A/Q) \setminus \varphi(F)$, there exists a unique maximal ideal in A/Q that contains $\overline{N}$, that is, axiom 5 holds.

Now let T be a finite nonempty subset of height-two prime ideals of A/Q not in

$\varphi(F)$. By Proposition 3.2.16, there are β height-two prime ideals between any single height-three maximal ideal of A and Q. Then, by Proposition 3.2.14, we see that all but finitely many of these height-two prime ideals are only in the height-three maximal ideal. Thus $|L_e(T)| = \beta$ if $|T| = 1$. If $|T| > 1$, then $|L_e(T)| = \emptyset$ since the elements of T that are not in $\varphi(F)$ are the height-two elements above exactly one minimal element of $\varphi(F)$. This proves axiom 7. $\square$

The following corollary follows from the proof of Theorem 3.5.5.

Corollary 3.5.6. Let R be a countable one-dimensional Noetherian domain, let x and y be indeterminates, let $A = R[[x]][y]$ or $R[y][[x]]$, let Q be a height-one prime ideal of A and let $B = A/Q$. Assume $|\max(R)|$ is infinite. Then $\operatorname{Spec}(B) \setminus \{$the set of height-one maximal ideals$\}$ is determined by $\operatorname{Spec}(R[y]/I)$, where I is a height-one prime ideal of $R[y]$ such that $(I, x)A = (Q, x)A$.

3.6 Examples over the Ring of Integers

Example 3.6.1. For $\alpha = (2y-1)\cdot 3\cdot(y+1)\cdot y\cdot(y(y+1)+6)\cdot 2\cdot(3y+1)$, what is $\operatorname{Spec}(\mathbb{Z}[[x]][y]/(x-\alpha))$? First we consider $\operatorname{Spec}(\mathbb{Z}[y]/(\alpha))$. Note that $(3, y+1), (3, y), (5, y+2), (2, y+1)$ are the only maximal ideals of $\mathbb{Z}[y]$ that contain two or more of the height-one prime ideals minimal over α. This is because

(a) the sets $\{y+1, y\}$ and $\{2, 3\}$ are certainly comaximal in $\mathbb{Z}[y]$.

(b) For p a prime element of $\mathbb{Z}$ with $p > 3$, the set $\{\overline{y+1}, \overline{y}, \overline{y(y+1)+6}\}$, where $^{-}$ denotes image in $(\mathbb{Z}/p)[y]$, is comaximal.

(c) The set $\{\overline{2y-1}, \overline{3y+1}, \overline{y+1}, \overline{y}\}$ is comaximal in $(\mathbb{Z}/p\mathbb{Z})[y]$ for $p > 5$. To see this, first, $\overline{2(p-1)/2} = \overline{1} \implies \overline{2y-1} \equiv \overline{y-(k-1)/2}$, which is comaximal with all the other elements given, mod p. Secondly, the inverse of $\overline{3}$ in $\mathbb{Z}/p\mathbb{Z}$ is $\overline{k}$, where

$3 \le k \le p-1$, since $3 \cdot 1 \not\equiv 1$ and $3 \cdot 2 \not\equiv 1 \pmod p$. Since $\overline{3y+1} = \overline{y+k}$ in $\mathbb{Z}/p\mathbb{Z}$, we see that it is comaximal in $\mathbb{Z}/p\mathbb{Z}$ with the other elements of the set, if $p > 5$.

(d) For $p = 5$,

$$\overline{2y-1} = \overline{2y+4} = \overline{2(y+2)} = \overline{y+2} = \overline{3(y+2)} = \overline{(3y+6)} = \overline{(3y+1)}$$
$$\implies (5, 2y-1) = (5, 3y+1).$$

Thus $\mathrm{Spec}(Z[y]/(\alpha))$ looks like the diagram below:

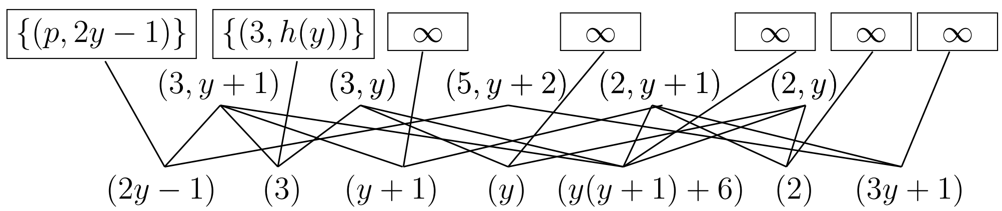


Diagram 3.6.1.0: $\mathrm{Spec}(\mathbb{Z}[y]/(\alpha))$

That is, there is a countably infinite clump of height-one prime ideals above $(2y-1)$, one for each prime integer $p \ge 7$. There is an infinite clump of height-one prime ideals above (3), one for each maximal ideal $\overline{(h(y))}$ in $(\mathbb{Z}/3\mathbb{Z})[y]$, where $(3, h(y))$ is a maximal ideal of $\mathbb{Z}[y]$ and $(3, h(y))$ is not already represented among the other height-one prime ideals listed. The other boxes labeled "∞" show similar sets of height-one prime ideals.

From Diagram 3.6.1.0 we see that j-$\mathrm{Spec}(\mathbb{Z}[[x]][y]/(x-\alpha))$ looks like this diagram:

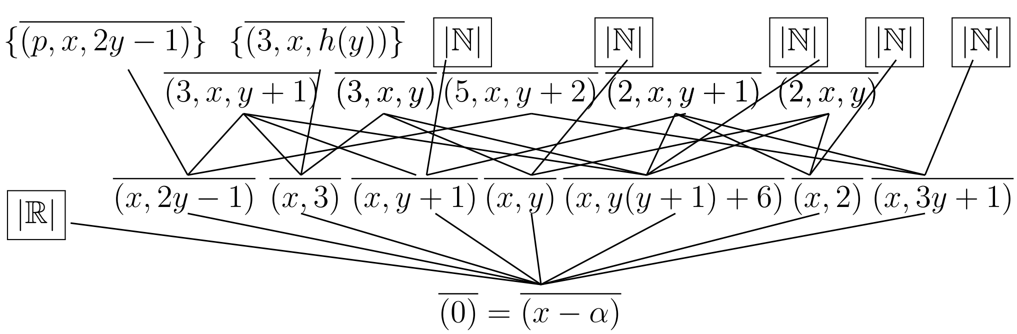


Diagram 3.6.1.1: $\mathrm{Spec}(\mathbb{Z}[[x]][y]/(x-\alpha))$

Example 3.6.2. Consider $Q' = (x + 455y^2 + 322y + 56)$ in $\mathbb{Z}[y][[x]]$. What is $\mathrm{Spec}(\mathbb{Z}[y][[x]]/Q')$? Let $B' = \mathbb{Z}[y][[x]]/Q'$ First note that the element $x + 455y^2 + 322y + 56$ is irreducible in $\mathbb{Z}[y][[x]]$ since it has degree one in x. Thus Q' is a prime ideal of $\mathbb{Z}[y][[x]]$. The minimal prime ideals of $(Q', x)\mathbb{Z}[y][[x]]$ are $(x, 13y+4)$, $(x, 5y+2)$, $(x, 7)$ and so these correspond to the prime ideals $u_1 := (x, 5y+2)B'$, $u_2 := (x, 13y+4)B'$, $u_3 := (x, 7)B'$. Thus the spec graph below shows the relations for B'.

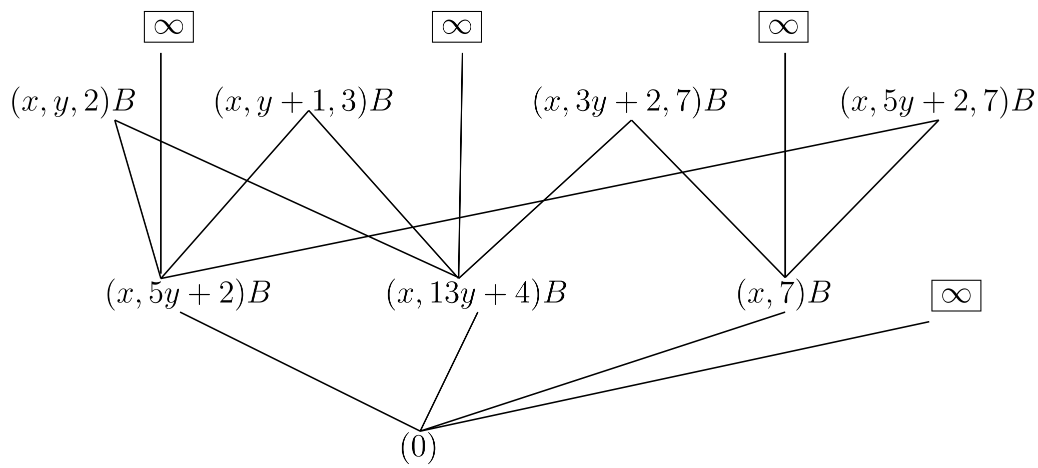


Diagram 3.6.2.1

3.7 Work in Progress

In future work we hope to give more details about which ϵ, γ_i and F can occur in the types of $\mathrm{Spec}(A/Q)$, for Q a height-one prime ideal of $A = R[[x]][y]$ or $R[y][[x]]$, where R is a one-dimensional Noetherian domain and x and y are indeterminates.

We have the following result for $A = R[[x]][y]$ and Q a height-one prime ideal of $R[[x]]$. The proof follows from Proposition 3.3.6.

Proposition 3.7.1. If $Q \in \mathrm{Spec}(R[[x]])$ has height one, then $\mathrm{Spec}(R[[x]][y]/QR[[x]][y])$

has $|R[[x]]|$ height-one maximal ideals. That is, $\operatorname{Spec}(R[[x]][y]/QR[[x]][y])$ is image polynomial-power series of type $(|R[[x]]; |R[[x]]; \ell; (\gamma_1, \ldots, \gamma_\ell; F)$, where ℓ, F and each γ_i are as found in Settings 3.5.3 and 3.4.3.

Chapter 4

Fiber Products and Connected Sums of Local Rings

The contents of this chapter are work in progress with H. Ananthnarayan and Z. Yang:

4.1 Introduction

We start this chapter by discussing the fiber product of local rings R and S over another local ring T, denoted $R \times_T S$. Our main goal is to analyze basic homological properties of fiber product rings; some of which were given in [2]. We present several examples to illustrate the set of zero-divisors, reducedness and Cohen-Macaulayness of such rings. These examples complement the existing literature and provide motivation for further study of fiber product rings.

If $T = k$, R and S are Artinian, neither of which is isomorphic to k, then the fiber product ring $R \times_k S$ cannot be Gorenstein; see Proposition 4.2.12. Therefore H. Ananthnarayan, L. Avramov and F. Moore, using fiber products, introduced and studied the *connected sum* of R and S over T, denoted $R\#_T S$, and defined to be quotients of the fiber product $R \times_T S$; see [2]. This construction produces Gorenstein local rings under mild conditions. For example, if R and S are Gorenstein Artin k-algebras, then it follows from [2, Theorem 2.8] that $R\#_k S$ is also Gorenstein. If R and S are Gorenstein Artin k-algebras, and if $R\#_k S$ is a nongraded Gorenstein ring, then we prove in Proposition 4.3.12 that the associated graded ring of $R\#_k S$ is a fiber product.

J. Sally, in 1979, characterized stretched Gorenstein local rings when the characteristic of k is different from two [26]. J. Elias and M. E. Rossi, three decades later, proved a similar structure theorem for short Gorenstein local rings for the case where k is an algebraically closed field of characteristic zero [7]. In case Q is a Gorenstein k-algebra that is either stretched or short, the structure theorems of Sally and Elias – Rossi imply that Q is a connected sum Theorem 4.3.2. Our main result of the last section of this chapter, Theorem 4.3.24, generalizes these structure theorems; it shows that Gorenstein local k-algebras whose associated graded rings have certain structures decompose as connected sums.

4.2 Fiber Products

We start with the definition of the fiber product of local rings and an example.

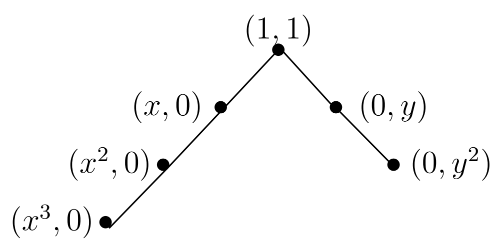
Definition 4.2.1. Let R, S and T be commutative rings with ring maps $\varepsilon_R : R \to T$, and $\varepsilon_S : S \to T$. The *fiber product* of R and S over T, denoted $R \times_T S$, is defined as $R \times_T S = \{(r, s) \in R \times S \mid \varepsilon_R(r) = \varepsilon_S(s)\}$.

The fiber product $P := R \times_T S$ is a subring of $R \times S$ and it is the pullback of ε_R and ε_S, i.e., the following diagram commutes for the natural projection maps

$p_R : R \times_T S \to R$ and $p_S : R \times_T S \to S$:

$$\begin{tikzcd} & R \arrow[dr, "\varepsilon_R"] & \\ P \arrow[ur, "p_R"] \arrow[dr, "p_S"'] & & T \\ & S \arrow[ur, "\varepsilon_S"'] & \end{tikzcd} \tag{4.2.1}$$

Example 4.2.2. Let $R = k[X]/(X^4)$, $S = k[Y]/(Y^3)$ and $T = k$, where k is a field. Then $R \times_k S \cong k[X,Y]/(X^4, Y^3, XY)$. We denote the respective images of X and Y in R and S by x and y. The set $\{(1,1), (x,0), (x^2,0), (x^3,0), (0,y), (0,y^2)\}$ is a k-basis for $R \times_k S$ and we draw the following picture to represent $R \times_k S$:



$$R \times_k S = k[X,Y]/(XY, X^4, Y^3)$$

Here vertices represent the k-basis elements and horizontal rows represent the degrees of the monomials of $R \times_k S$. Multiplying by x takes it into the next row to the left and multiplying by y takes it into the next row to the right.

In 1985 Ogoma explored when the fiber product of Noetherian rings is again Noetherian. His main result is as follows:

Theorem 4.2.3. [25, Theorem 2.1] Let R and S be Noetherian rings. Set $C = \varepsilon_R(R) \cap \varepsilon_S(S)$ where $\varepsilon_R : R \to T$ and $\varepsilon_S : S \to T$ are the ring maps in Definition 4.2.1. Then the fiber product $R \times_T S$ is Noetherian if and only if

(1) C is Noetherian;

(2) I/I^2 and J/J^2 are finite C-modules where $\operatorname{Ker}(\varepsilon_R) = I$ and $\operatorname{Ker}(\varepsilon_S) = J$.

Setup 4.2.4. We assume that R, S and T are Noetherian rings with surjective maps $\varepsilon_R : R \to T$, and $\varepsilon_S : S \to T$. We set $I = \operatorname{Ker}(\varepsilon_R)$, $J = \operatorname{Ker}(\varepsilon_S)$, and $P = R \times_T S$.

We list some basic properties of the fiber product $P = R \times_T S$ also [1] and [2].

Remarks 4.2.5. With notation as in Setup 4.2.4,

(1) $R \times_T S \subseteq R \times_k S \subseteq R \times S$.

(2) For $(r, s) \in R \times_T S$, we have

(i) $r \in I$ if and only if $s \in J$.

(ii) $(r, 0) \in P$ and $(0, s) \in P$ if and only if $r \in I$ and $s \in J$. Hence $(I, 0)$ and $(0, J)$ are ideals of P which we identify with I and J in P, respectively. With this identification, we have $I \cap J = 0$ in P.

(iii) r is a unit in R if and only if s is a unit in S if and only if (r, s) is a unit in $R \times_T S$.

(3) For ideals $\mathbf{p} \subseteq I$ and $\mathbf{q} \subseteq J$, the natural projection maps $R \times_T S$ to R and S induce the isomorphisms $R/\mathbf{p} \cong P/(\mathbf{p}, J)$, and $S/\mathbf{q} \cong P/(I, \mathbf{q})$. In particular, $R \cong P/J$, $S \cong P/I$ and $T \cong P/I + J$. If $S = T$, then $R \times_T T \cong R$.

(4) We have the following exact sequences of P-modules

$$0 \longrightarrow I \oplus J \longrightarrow R \oplus S \xrightarrow{\varepsilon_R \oplus \varepsilon_S} T \oplus T \longrightarrow 0 \tag{4.2.2}$$

$$0 \longrightarrow R \times_T S \xrightarrow{\eta} R \oplus S \xrightarrow{(\varepsilon_R, -\varepsilon_S)} T \longrightarrow 0 \tag{4.2.3}$$

Also 4.2.3 yields a relation between the lengths of P-modules:

$$\lambda(R \times_T S) + \lambda(T) = \lambda(R) + \lambda(S) \tag{4.2.4}$$

Let $(R, \mathfrak{m}_R, k)$, $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ be local rings.

(5) $P = R \times_T S$ is local with unique maximal ideal:

$$\mathfrak{m}_P = \mathfrak{m}_R \times_T \mathfrak{m}_S = \{(x, y) \in \mathfrak{m}_R \times \mathfrak{m}_S : \varepsilon_R(x) = \varepsilon_S(y)\}.$$

(6) For $(r, s) \in P$, $(r, s) \in \mathfrak{m}_P = \mathfrak{m}_R \times_T \mathfrak{m}_S$ if and only if $r \in \mathfrak{m}_R$ and $s \in \mathfrak{m}_S$.

(7) If $(0 :_R I) \subseteq I$ and $(0 :_S J) \subseteq J$, then $(0 :_{R \times_T S} (I + J)) = \{(r, s) \mid r \in (0 :_R I)\}$, and $(0 :_{R \times_T S} (I + J)) = \{(r, s) \mid s \in (0 :_S J)\}$.

In particular, taking $I = \mathfrak{m}_R$ and $J = \mathfrak{m}_S$, we get

$$\mathrm{soc}(R \times_k S) = \{(r, s) \mid r \in \mathrm{soc}(R), s \in \mathrm{soc}(S)\}.$$

Proposition 4.2.6. [1, Proposition 4.3] Let $Z \xrightarrow{f_R} R \xrightarrow{\varepsilon_R} T$ and $Z \xrightarrow{f_S} S \xrightarrow{\varepsilon_S} T$ be such that $\varepsilon_R f_R = \varepsilon_S f_S$. Then there is a ring homomorphism $\phi : Z \to R \times_T S$ defined by $\phi(z) = (f_R(z), f_S(z))$. Furthermore,

(i) If $\mathrm{Ker}(f_R) \cap \mathrm{Ker}(f_S) = 0$, then ϕ is injective.

(ii) If $\mathrm{Ker}(\varepsilon_R f_R) = \mathrm{Ker}(f_R) + \mathrm{Ker}(f_S)$ (or $\mathrm{Ker}(\varepsilon_S f_S) = \mathrm{Ker}(f_R) + \mathrm{Ker}(f_S)$), then ϕ is surjective.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f_R"] \arrow[dr, dotted, "\phi"] \arrow[ddr, bend right, "f_S"'] & & \\
& R \times_T S \arrow[r, "p_R"'] \arrow[d, "p_S"] & R \arrow[d, "\varepsilon_R"'] \\
& S \arrow[r, "\varepsilon_S"] & T
\end{tikzcd}$$

Corollary 4.2.7. [1, Corollary 4.4] Let $(A, \mathfrak{m}, k)$ be a local ring, $\mathfrak{p}$ and $\mathfrak{q}$ be ideals in A. Then

$$A/(\mathfrak{p} \cap \mathfrak{q}) \cong A/\mathfrak{p} \times_{A/(\mathfrak{p}+\mathfrak{q})} A/\mathfrak{q}.$$

In particular, if $\mathfrak{p} \cap \mathfrak{q} = 0$, then $A \cong A/\mathfrak{p} \times_{A/(\mathfrak{p}+\mathfrak{q})} A/\mathfrak{q}$.

Proof. [1, Corollary 4.4] Set $Z = A/(\mathfrak{p}\cap\mathfrak{q})$, $R = A/\mathfrak{p}$, $S = A/\mathfrak{q}$ and $T = A/(\mathfrak{p}+\mathfrak{q})$ in Proposition 4.2.6. Note that $\mathrm{Ker}(f_R) = \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{q})$, $\mathrm{Ker}(f_S) = \mathfrak{q}/(\mathfrak{p} \cap \mathfrak{q})$ and $\mathrm{Ker}(\varepsilon_R f_R) = \mathrm{Ker}(\varepsilon_S f_S) = (\mathfrak{p} + \mathfrak{q})/(\mathfrak{p} \cap \mathfrak{q})$. This completes the proof. □

As a consequence, we have a nice presentation for the fiber products of quotients of polynomial rings over a field k.

Theorem 4.2.8. [1, Theorem 4.19] Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of the polynomial rings $k[X_1, \ldots, X_m]$ and $k[Y_1, \ldots, Y_n]$ over a field k, respectively. If $R = k[X_1, \ldots, X_m]/\mathcal{I}$ and $S = k[Y_1, \ldots, Y_n]/\mathcal{J}$, then

$$R \times_k S \cong k[\underline{X}, \underline{Y}]/(\mathcal{I}, \mathcal{J}, X_iY_j : 1 \leq i \leq m, 1 \leq j \leq n).$$

Corollary 4.2.9. [1, Corollary 4.20] If R and S are graded quotients of polynomials over k, then $R \times_k S$ is also graded.

Before working on the homological properties of the fiber products, we give more examples.

Examples 4.2.10. Let k be a field.

(1) If $R = k[X, Y]$, $S = k[Z, W]$ and $T = k$, then, by Theorem 4.2.8, we have $R \times_k S \cong k[X, Y, Z, W]/(XZ, XW, YZ, YW)$.

(2) Let $R = k[X,Y]/(XY^2, X^3 - Y^2)$, $S = k[Z,W]/(ZW, Z^2 - W^2)$ and $T = k$. Then $R \times_k S = k[X,Y,Z,W]/(XZ, XW, YZ, YW, ZW, XY^2, Z^2 - W^2, X^3 - Y^2)$ by Theorem 4.2.8.

(3) Let $R = k[X]$, $S = k[Y]$ and $T = k[Z]/(Z^n)$ where $n \in \mathbb{N}$ and $n \geq 2$. Then we have $R \times_T S \cong k[X,Y]/(X^nY - Y^2)$ if we take $A = k[X,Y]/(X^nY - Y^2)$, $\mathfrak{p} = (X^n - Y)$ and $\mathfrak{q} = (Y)$ in Corollary 4.2.7.

4.2.1 Homological Properties of Fiber Product Rings

In this section we study some homological properties of fiber product rings. We start by giving some information about the numerical invariants of the fiber products which are also listed in [2]. Assume $(R, \mathfrak{m}_R, k)$, $(S, \mathfrak{m}_S, k)$ and $(T, \mathfrak{m}_T, k)$ are local rings.

Remarks 4.2.11. [2, Lemma 1.5] With notation as in Setup 4.2.4, the following inequalities and equalities hold:

(1) $\operatorname{edim}(R \times_T S) \geq \operatorname{edim}(R) + \operatorname{edim}(S) - \operatorname{edim}(T)$.

(2) $\dim(R \times_T S) = \max\{\dim(R), \dim(S)\} \geq \min\{\dim(R), \dim(S)\} \geq \dim(T)$.

(3) $\operatorname{depth}(R \times_T S) \geq \min\{\operatorname{depth}(R), \operatorname{depth}(S), \operatorname{depth}(T) + 1\}$ and

$\operatorname{depth}(T) \geq \min\{\operatorname{depth}(R), \operatorname{depth}(S), \operatorname{depth}(R \times_T S) - 1\}$.

Proposition 4.2.12. [2, Proposition 1.7] Assume that T is Cohen-Macaulay and set $d = \dim(T)$. The ring $P := R \times_T S$ is Cohen-Macaulay of dimension d if and only if R and S are Cohen-Macaulay of dimension d. If P is Cohen-Macaulay of dimension

d, then the following inequalities hold:

$$\begin{aligned}\operatorname{type}(R)+\operatorname{type}(S) &\geq \operatorname{type}(R\times_T S)\\ &\geq \max\{\operatorname{type}(R)+\operatorname{type}(S)-\operatorname{type}(T),\operatorname{type}_R(I)+\operatorname{type}_S(J)\}.\end{aligned}$$

If, in addition, I and J are non-zero, then $R\times_T S$ is not Gorenstein.

Remark 4.2.13. If $\dim(R\times_T S)\neq\dim(T)$, then the Cohen-Macaulayness of $R\times_T S$ does not imply that R and S are Cohen-Macaulay.

The next example shows that if $R\times_T S$ is Gorenstein, then R and S are not necessarily Cohen-Macaulay, in general.

Example 4.2.14. Let $R=k[X,Y]/(Y^2)$, $S=k[X,Y]/(X^2,XY)$ and $T=k[X,Y]/(X^2,XY,Y^2)$. Then, by Corollary 4.2.7, we have $P=k[X,Y]/(X^2Y)\cong R\times_T S$ which is Gorenstein. Note that $\dim(P)=1$ and $\dim(T)=0$. R and T are Cohen-Macaulay, but S is not Cohen-Macaulay.

Proposition 4.2.15. Assume R and S are Cohen-Macaulay of dimension d. Then

$$\begin{aligned}R\times_T S \text{ is Cohen-Macaulay} &\iff I \text{ is MCM } R\text{-module.}\\ &\iff J \text{ is MCM } S\text{-module.}\\ &\iff \operatorname{depth}(T)\geq d-1.\end{aligned}$$

Proof. It follows from Lemma 4.2.11 that $\dim(P)=\max\{\dim(R),\dim(S)\}$=d and

$$\operatorname{depth}(P)\geq\min\{\operatorname{depth}(R),\operatorname{depth}(S),\operatorname{depth}(T)+1\}$$

$$\operatorname{depth}(T)\geq\min\{\operatorname{depth}(R),\operatorname{depth}(S),\operatorname{depth}(P)-1\}$$

Thus either $\operatorname{depth}(P) = d$ and $\operatorname{depth}(T) \geq d - 1$, or $\operatorname{depth}(P) = \operatorname{depth}(T) + 1$. If $\operatorname{depth}(P) = d$, then $\operatorname{depth}(T) \geq d - 1$. If $\operatorname{depth}(P) = t \leq d$, then $\operatorname{depth}(T) \geq t - 1$. On the other hand, $\operatorname{depth}(P) = t \geq \min\{d, \operatorname{depth}(T) + 1\}$, i.e., $t \geq \operatorname{depth}(T) + 1$. ($t \geq \operatorname{depth}(T) + 1 \geq (t - 1) + 1 = t$). Similarly, either $\operatorname{depth}(I) = d$ and $\operatorname{depth}(T) \geq d - 1$ or $\operatorname{depth}(I) = \operatorname{depth}(T) + 1$. Therefore

$$\operatorname{depth}(P) = \min\{d, \operatorname{depth}(T) + 1\} = \operatorname{depth}_R(I) = \operatorname{depth}_S(J).$$

In particular,

$$\begin{aligned} \operatorname{depth}(P) = d &\iff \operatorname{depth}(T) \geq d - 1 \\ &\iff \operatorname{depth}_R(I) = d \\ &\iff \operatorname{depth}_S(J) = d. \end{aligned}$$

This proves the claim. □

Recall that the *type* of a finitely generated module M over a local ring R is defined as $\operatorname{type}(M) = \dim_k \operatorname{Ext}^{\operatorname{depth}(M)}(k, M)$; see [6, Definition 1.2.15].

Proposition 4.2.16. Assume R and S are Cohen-Macaulay of dimension d. If $P = R \times_T S$ is Gorenstein, then $\operatorname{type}(T) = 1$.

Proof. It follows from Proposition 4.2.15 that $\operatorname{depth}(T) \geq d - 1$ since P is Cohen-Macaulay. Since P is Gorenstein, T cannot be Cohen-Macaulay of dimension d by Proposition 4.2.12. Therefore $\operatorname{depth}(T) = d - 1$. Consider the short exact sequence of P-modules: $0 \to P \to R \oplus S \to T \to 0$. Applying $\operatorname{Hom}_P(-, k)$, we see that the sequence $\operatorname{Ext}^{d-1}(R \oplus S, k) \to \operatorname{Ext}^{d-1}(T, k) \to \operatorname{Ext}^d(P, k)$ is exact. Since $\operatorname{depth}(R) = \operatorname{depth}(S) = d$, $\operatorname{Ext}^{d-1}(R \oplus S, k) = 0$. Thus $\operatorname{Ext}^{d-1}(T, k) \hookrightarrow \operatorname{Ext}^d(P, k)$.

Recall that P is Gorenstein ring. Hence $\operatorname{type}(P) = 1$ by [**?**, Theorem 3.2.10], i.e., $\dim_k \operatorname{Ext}^d(P, k) = 1$. Moreover $\operatorname{Ext}^{d-1}(T, k) \neq 0$ as $\operatorname{depth}(T) = d - 1$. This implies that $\dim_k \operatorname{Ext}^{d-1}(T, k) = 1$, i.e., $\operatorname{type}(T) = 1$. □

4.2.2 When are Fiber Product Rings Reduced?

In this section we analyze when fiber product rings are reduced. Recall that a commutative ring R is called *reduced* if it has no non-zero nilpotent elements, i.e., if for every nonzero $r \in R$ and every positive integer n, $r^n \neq 0$, equivalently, if $x \in R$ and $x^2 = 0$, then $x = 0$. The nilpotent elements of R form an ideal of R, called the *nilradical* of R. Therefore R is reduced if and only if its nilradical is the zero ideal. For an ideal $\mathcal{I}$ of R, the radical of $\mathcal{I}$, is denoted by $\sqrt{\mathcal{I}}$ and is defined as $\sqrt{\mathcal{I}} = \{r \in R \mid r^n \in \mathcal{I} \text{ for some } n > 0\}$. Also $\sqrt{\mathcal{I}}$ is the intersection of all prime ideals containing $\mathcal{I}$. Moreover, $R/\mathcal{I}$ is reduced if and only if $\mathcal{I} = \sqrt{\mathcal{I}}$.

Throughout this section, we assume Setup 4.2.4.

Remarks 4.2.17. Assume Setup 4.2.4.

(1) For ideals I and J in $P = R \times_T S$, we have $I \cap J = (0)$. By Corollary 4.2.7, $P/(I \cap J) \cong P \cong P/J \times_{P/(I+J)} P/I$. Thus

$$\begin{aligned}
P \text{ is reduced} &\iff (0) = I \cap J = \sqrt{I \cap J};\\
R \text{ is reduced} &\iff J = \sqrt{J};\\
T \text{ is reduced} &\iff I + J = \sqrt{I + J};\\
S \text{ is reduced} &\iff I = \sqrt{I}.
\end{aligned}$$

(2) The following statements are equivalent:

(i) R and S are domains.

(ii) I and J are prime ideals of $R \times_T S$.

(iii) $R \times_T S$ is a reduced ring and I and J are prime ideals of $R \times_T S$.

Proof. It is clear that (iii) implies (ii), and (ii) implies (iii) since $I \cap J = (0)$. Moreover the equivalence of (i) and (ii) follows from the fact that $R \cong P/J$ and $S \cong P/I$ by Remark 4.2.5(3). □

The next proposition gives criteria for fiber product rings to be reduced.

Proposition 4.2.18. Let R, S and T be local rings given as in Setup 4.2.4. Then

(i) If R and S are reduced, then the fiber product $R \times_T S$ is reduced.

(ii) If $R \times_T S$ and T are reduced, then R and S are reduced.

Proof. (i) By Remark 4.2.5(3), $R \cong P/J$ and $S \cong P/I$. Since R and S are reduced, $\sqrt{J} = J$ and $\sqrt{I} = I$. Then it follows that $R \times_T S$ is reduced since $0 = I \cap J = \sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$.

(ii) Since $R \times_T S$ and T are reduced, $(0) = I \cap J = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ and $I + J = \sqrt{I + J}$ by Remark 4.2.17. Thus $I \oplus J = I + J = \sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$. Note that $\sqrt{J} \subseteq \sqrt{I} + \sqrt{J}$ implies that $\sqrt{J} \subseteq \sqrt{\sqrt{I} + \sqrt{J}} = I + J$. Similarly, $\sqrt{I} \subseteq I + J$. Therefore $\sqrt{I} + \sqrt{J} = I + J$. Now we show $J = \sqrt{J}$. Clearly $J \subseteq \sqrt{J}$. Let $x \in \sqrt{J}$. Then $x \in \sqrt{I} + \sqrt{J} = I + J$. Write $x = a + b$ where $a \in J$ and $b \in I$. Thus $x - a = b \in I \subseteq \sqrt{I}$. Since $x - a \in \sqrt{J}$, $b \in \sqrt{J} \cap \sqrt{I} = (0)$, that is, $x - a = b = 0$. Hence $x = a \in I$. Thus $J = \sqrt{J}$. Similarly $I = \sqrt{I}$. Therefore R and S are reduced.

□

The following example shows that Proposition 4.2.18(ii) requires T to be reduced.

Example 4.2.19. Let $A = k[X,Y,Z,W]$ where k is a field, X,Y,Z and W are indeterminates. Consider the ideals $\mathfrak{p} = (Z) \cap (X,Y^2) = (XZ, ZY^2)$ and $\mathfrak{q} = (X) \cap (Z,W^2) = (XZ, XW^2)$ in A. Then $\mathfrak{p} \cap \mathfrak{q} = (XZ)$ and $\mathfrak{p} + \mathfrak{q} = (XZ, W^2X, Y^2Z)$. By Corollary 4.2.7, we have $R \times_T S = A/(\mathfrak{p} \cap \mathfrak{q}) = k[X,Y,Z,W]/(XZ)$ where $R = A/\mathfrak{p} = k[X,Y,Z,W]/(XZ,ZY^2)$, $S = A/\mathfrak{q} = k[X,Y,Z,W]/(XZ,XW^2)$ and $T = A/(\mathfrak{p}+\mathfrak{q}) = k[X,Y,Z,W]/(XZ,W^2X,Y^2Z)$. Here R, S and T are not reduced since $\mathfrak{p} \neq \sqrt{\mathfrak{p}} = (XZ, YZ)$, $\mathfrak{q} \neq \sqrt{\mathfrak{q}} = (XZ, XW)$ and $\mathfrak{p}+\mathfrak{q} \neq \sqrt{\mathfrak{p}+\mathfrak{q}} = (XZ, XW, YZ)$. But $R \times_T S$ is reduced since $\mathfrak{p} \cap \mathfrak{q} = \sqrt{\mathfrak{p} \cap \mathfrak{q}}$.

The next example shows that Proposition 4.2.18(i) does not imply that T is reduced.

Example 4.2.20. Let $A = k[X,Y]$ where k is a field, X,Y are indeterminates. Consider the ideals $\mathfrak{p} = (X - Y^2)$, $\mathfrak{q} = (X)$ in A. Then $\mathfrak{p}+\mathfrak{q} = (X,Y^2)$, $\mathfrak{p} \cap \mathfrak{q} = (X(X-Y^2))$ in A. By Corollary 4.2.7, we have $R \times_T S = A/(\mathfrak{p} \cap \mathfrak{q}) = k[X,Y]/(X^2 - XY^2)$ where $R = A/\mathfrak{p} = k[X,Y]/(X-Y^2)$, $S = A/\mathfrak{q} = k[X,Y]/(X) \cong k[Y]$ and $T = A/(\mathfrak{p}+\mathfrak{q}) = k[X,Y]/(X,Y^2) = k[Y]/(Y^2)$. Note that in this example $R \times_T S$, R, S are reduced, but T is not.

Remark 4.2.21. Let $(R, \mathbf{m}_R, k)$, $(S, \mathbf{m}_S, k)$ and $(T, \mathbf{m}_T, k)$ be local rings given as in Setup 4.2.4. If $P = R \times_T S$ is reduced, but R and S are not, then we can rearrange P so that $P \cong R' \times_{T'} S'$ where R' and S' are reduced:

By Remarks 4.2.5(3), we have $R = P/J$, $S = P/I$ and $T = P/(I+J)$. Since R and S are not reduced, $I \neq \sqrt{I}$ and $J \neq \sqrt{J}$. Since P is reduced, $0 = I \cap J = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. Let $\mathfrak{p}' = \sqrt{J}$, $\mathfrak{q}' = \sqrt{I}$. Thus, by Corollary 4.2.7, $P \cong R' \times_{T'} S'$ where $R' = P/\mathfrak{p}'$, $S' = P/\mathfrak{q}'$ and $T' = P/(\mathfrak{p}' + \mathfrak{q}')$.

4.2.3 Zero-Divisors of the Fiber Product

A *zerodivisor* on an R-module M is an element $r \in R$ for which there exists $m \in M$ such that $m \neq 0$ but $rm = 0$. By $\mathsf{Z}(\_)$, we denote the set of zero-divisors of a module. In this section we investigate the set of zero-divisors of the fiber product, $P := R \times_T S$, i.e., $\mathsf{Z}(R \times_T S)$.

We consider the following subsets of P:

$$\mathcal{Z}_1 = \{(0,j) \in P \mid j \in J\}$$

$$\mathcal{Z}_2 = \{(i,0) \in P \mid i \in I\}$$

$$\mathcal{Z}_3 = \{(x,y) \in P \mid x \in \mathsf{Z}(R) \setminus \{0\}, y \in \mathsf{Z}(S) \setminus \{0\}\}$$

$$\mathcal{Z}_4 = \{(x,y) \in P \mid x \notin \mathsf{Z}(R),\ \exists\, 0 \neq k \in J \ \text{ such that } \ ky = 0\}$$

$$\mathcal{Z}_5 = \{(x,y) \in P \mid y \notin \mathsf{Z}(S),\ \exists\, 0 \neq \ell \in I \text{ such that } \ell x = 0\}$$

Remark 4.2.22. Clearly $\mathsf{Z}(R \times_T S) \subseteq \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. However this inclusion is not an equality in general: Although $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$ is contained in $\mathsf{Z}(R \times_T S)$, the following example shows that $\mathcal{Z}_3$ may not be contained in $\mathsf{Z}(R \times_T S)$.

Example 4.2.23. Let $R = k[X,Y,Z]/(X^2, XY, XZ)$, $S = k[U,V]/(U^2, UV)$ and $T = k[\epsilon, \delta]/(\epsilon^2, \epsilon\delta, \delta^2)$. Consider the maps $\varepsilon_R : R \to T$ via $X \mapsto \epsilon$, $Y \mapsto \epsilon$, $Z \mapsto \delta$ and $\varepsilon_S : S \to T$ via $U \mapsto \delta$, $V \mapsto \epsilon$. Let x, y, z, u, v denote the respective images of X, Y, Z, U, V in R and S. Note that $\operatorname{Ker}(\varepsilon_R) = (y^2, yz, z^2)$ and $\operatorname{Ker}(\varepsilon_S) = (v^2)$. Here $y \in \mathsf{Z}(R)$ and $v \in \mathsf{Z}(S)$. Also $(y,v) \in P$ since $\varepsilon_R(y) = \varepsilon_S(v)$. Then $(y,v) \in \mathcal{Z}_3$. Suppose $(y,v)(a,b) = (0,0)$ for some $(a,b) \neq (0,0)$ in $R \times_T S$. Then $ya = 0$ in R and $vb = 0$ in S. This implies that $a \in (x)$ and $b \in (u)$. Then $(a,b) = (x,0)$ or $(0,u)$ or (x,u). However, none of these options belong to $R \times_T S$. Therefore $(y,v) \notin \mathsf{Z}(R \times_T S)$.

Question 4.2.24. Let $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}_S, k)$ be local rings given as in Setup 4.2.4. If $T = k$, then is $\mathsf{Z}(R \times_k S) = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$?

4.3 Connected Sums

4.3.1 Connected Sums of Local Rings over a Field

If $(R, \mathfrak{m}_R, k)$ and $(S, \mathfrak{m}, k)$ are local rings with $R \neq k \neq S$, then $P = R \times_k S$ is a local ring and $\mathrm{soc}(P) = \mathrm{soc}(R) \oplus \mathrm{soc}(S)$ by Remarks 4.2.5(5) and Remarks 4.2.5(7). As a consequence of this, when R and S are Artinian local rings with $R \neq k \neq S$, $P = R \times_k S$ is not Gorenstein. However Ananthnarayan, Avramov and Moore construct a suitable quotient which is a Gorenstein ring; see [2].

Definition 4.3.1. [2] Let R and S be Gorenstein Artin local rings with $R \neq k \neq S$. Let $\mathrm{soc}(R) = (\delta_R)$ and $\mathrm{soc}(S) = (\delta_S)$. Identifying δ_R with $(\delta_R, 0)$ and δ_S with $(0, \delta_S)$, we define a *connected sum* of R and S over k, denoted $R\#_k S$, as $R\#_k S = (R \times_k S)/(\delta_R - u\delta_S)$, where u is a unit in S.

Since connected sums are quotients of fiber products, we have the following presentation of connected sums of Gorenstein Artin quotients of polynomial rings over a field k.

Theorem 4.3.2. [1, Theorem 4.22] Let $\mathcal{I}$ and $\mathcal{J}$ be ideals of the polynomial rings $k[X_1, \ldots, X_m]$ and $k[Y_1, \ldots, Y_n]$ over a field k, respectively. If $R = k[X_1, \ldots, X_m]/\mathcal{I}$ and $S = k[Y_1, \ldots, Y_n]/\mathcal{J}$, then

$$\begin{aligned} R\#_k S &\cong (R \times_k S)/(\delta_R - u\delta_S) \\ &\cong k[\underline{X}, \underline{Y}]/(\mathcal{I}, \mathcal{J}, \Delta_R - u\Delta_S, X_i Y_j : 1 \leq i \leq m, 1 \leq j \leq n), \end{aligned}$$

where u is a unit in S, $\Delta_R \in k[\underline{X}]$ and $\Delta_S \in k[\underline{Y}]$ are such that their respective images in R and S are δ_R and δ_S.

The following example shows that connected sums over R and S over k depend on the unit u chosen.

Example 4.3.3. [2, Example 3.1] Let $R = \mathbb{Q}[Y]/(Y^3)$ and $S = \mathbb{Q}[Z]/(Z^3)$. Let y and z denote the respective images of Y and Z in R and S. Then $\mathrm{soc}(R) = (y^2)$ and $\mathrm{soc}(S) = (z^2)$. The connected sums $Q_1 = (R \times_k S)/(y^2 - z^2)$ and $Q_2 = (R \times_k S)/(y^2 - pz^2)$ are not isomorphic where p is a prime number not congruent to 3 modulo 4. For a proof of this fact; see [2].

Corollary 4.3.4. [1, Corollary 4.23] Let R and S be graded Artinian local quotients of polynomial rings over k such that $\ell\ell(R) = \ell\ell(S)$. Then $R\#_k S$ is also graded.

Examples 4.3.5. Let k be a field.

(1) Let $R = k[X]/(X^4)$, $S = k[Y]/(Y^3)$ and $T = k$. By Example 4.2.2, we have $R \times_k S = k[X,Y]/(XY, X^4, Y^3)$. Then $R\#_k S \cong (R \times_k S)/(X^3 - uY^2) \cong k[X,Y]/(XY, X^3 - uY^2)$, where $\mathrm{soc}(R) = (X^3)$, $\mathrm{soc}(S) = (Y^2)$, and u is a unit.

(2) Let $R = k[X]/(X^4)$, $S = k[Y,Z]/(YZ, Y^2 - Z^2)$, and $T = k$. By Theorem 4.2.8, we have $R \times_k S = k[X,Y,Z]/(XY, XZ, YZ, X^4, Y^2 - Z^2)$. Then it follows that $R\#_k S = k[X,Y,Z]/(XY, XZ, YZ, X^3 - uY^2, Y^2 - Z^2)$, where u is a unit.

The following theorem is a special case of [2, Theorem 2.8].

Theorem 4.3.6. [2, Theorem 2.8] Let R and S be Gorenstein Artin local rings with $R \neq k \neq S$. Then a connected sum of R and S over k is also Gorenstein.

Definition 4.3.7. Let $(R, \mathfrak{m}_R, k)$ be a Noetherian local ring.

(1) The *graded ring associated to the maximal ideal* $\mathfrak{m}_R$ of R, denoted $\mathrm{gr}_{\mathfrak{m}_R}(R)$ (or simply $\mathrm{gr}(R)$), is defined as $\mathrm{gr}(R) \cong \oplus_{i=0}^{\infty} \mathfrak{m}_R^i/\mathfrak{m}_R^{i+1}$.

(2) If $R = \oplus_{i \geq 0} R_i$ is a finitely generated graded k-algebra, where $R_0 = k$ and R_i consist of the elements in R of degree i, we define the *Hilbert function* of R as $H_R(i) = \dim_k(R_i)$ for $i \geq 0$. If R is not graded, we define $H_R(i) = H_{\mathrm{gr}(R)}(i)$.

Remark 4.3.8. Let us now list some notation and facts about associated graded rings needed for the rest of this chapter.

(a) Note that for any $n \geq 0$, a minimal generating set of $\mathrm{gr}(R)_n = \oplus_{i=n}^{\infty} \mathfrak{m}_R^i / \mathfrak{m}_R^{i+1}$, the nth power of the maximal ideal $\mathrm{gr}(R)_1$ of $\mathrm{gr}(R)$, lifts to a minimal generating set of $\mathfrak{m}_R^n$.

(b) Let $x \in R$ be such that $x \in \mathfrak{m}_R^i \setminus \mathfrak{m}_R^{i+1}$. We define $x^* \in \mathrm{gr}(R)$, called the *initial form* of x, to be the element of degree i that is the image of x in $\mathfrak{m}^i / \mathfrak{m}^{i+1}$. Let $I \subseteq R$ be an ideal. We define I^* to be the ideal in $\mathrm{gr}(R)$ defined by $\langle x^* : x \in I \rangle$. Note that if $A \cong R/I$, then $\mathrm{gr}(A) \cong \mathrm{gr}(R)/I^*$.

4.3.2 Properties of Connected Sums

We prove some basic properties of connected sums in this section. We begin with the following remarks.

Remarks 4.3.9. Let the notation be as in Definition 4.3.1.

(1) If $(\delta_S) = \mathrm{soc}(S)$, then $(u\delta_S) = \mathrm{soc}(S)$ for any unit $u \in S$. Hence, one can write $R \#_k S \cong (R \times_k S)/(\delta_R - \delta'_S)$, where $(\delta_R) = \mathrm{soc}(R)$ and $(\delta'_S) = \mathrm{soc}(S)$.

(2) Since $0 \neq \delta_R - u\delta_S \in \mathrm{soc}(R \times_k S)$, we have $\lambda(R \#_k S) = \lambda(R \times_k S) - 1$.

(3) As a consequence of item 2, we have a relation between the Hilbert functions of R, S and $R \#_k S$: $H_{R \#_k S} = H_R + H_S - H_k$.

Proposition 4.3.10. Let R and S be Gorenstein Artin local rings with $R \neq k \neq S$. Let $Q \cong R\#_k S$. Then $Q/\operatorname{soc}(Q) \cong R/\operatorname{soc}(R) \times_k S/\operatorname{soc}(S)$.

Proof. Let $P = R \times_k S$, $(\delta_R) = \operatorname{soc}(R)$ and $(\delta_S) = \operatorname{soc}(S)$. We know that $\operatorname{soc}(P) = \operatorname{soc}(R) \oplus \operatorname{soc}(S)$. Let $\pi : P \twoheadrightarrow Q$ be the natural surjection. Since $Q \cong P/(\delta_R - u\delta_S)$ for some unit $u \in S$, and $\lambda(Q) = \lambda(P) - 1$, $\pi(\delta_R) \neq 0$ in Q. Hence $\pi(\delta_R) \in \operatorname{soc}(Q)$ and $\dim_k(\operatorname{soc}(Q)) = 1$ force $\operatorname{soc}(Q) = (\pi(\delta_R))$. Thus $Q/\operatorname{soc}(Q) \cong (R \times_k S)/(\delta_R - u\delta_S, \delta_R) \cong R/(\delta_R) \times_k S/(\delta_S)$. $\square$

Lemma 4.3.11. If both R and S are Artinian k-algebras, then

$$\operatorname{gr}(R \times_k S) \cong \operatorname{gr}(R) \times_k \operatorname{gr}(S).$$

Proof. Let $P = R \times_k S$. By Remarks 4.2.5.(3), we have $R \cong P/J$, $S \cong P/I$, and $k = P/(I + J)$. Thus if $I = (y_1, \ldots, y_m)$ and $J = (z_1, \ldots, z_n)$, we see that $\mathfrak{m}_P = (y_1, \ldots, y_m, z_1, \ldots, z_n)$ is the maximal ideal of P. Hence $\mathfrak{m}_P^* = (y_1^*, \ldots, y_m^*, z_1^*, \ldots, z_n^*)$ is the maximal ideal $\mathfrak{m}_{\operatorname{gr}(P)}$ of $\operatorname{gr}(P)$. Thus $(y_1^*, \ldots, y_m^*) \subseteq I^*$ and $(z_1^*, \ldots, z_n^*) \subseteq J^*$ forces $I^* + J^* = \mathfrak{m}_{\operatorname{gr}(P)}$. Since $\lambda(P) = \lambda(R) + \lambda(S) - 1$ and $\lambda(\operatorname{gr}(P)) = \lambda(P)$, we have $\lambda(\operatorname{gr}(P)) = \lambda(\operatorname{gr}(R) \times_k \operatorname{gr}(S))$.

Now, by Remark 4.3.8(b), $R = P/J$ and $S = P/I$ implies that $\operatorname{gr}(R) \cong \operatorname{gr}(P)/J^*$ and $\operatorname{gr}(S) \cong \operatorname{gr}(P)/I^*$. In particular, the natural projection $\operatorname{gr}(P) \twoheadrightarrow k$ factors through the surjective maps $\operatorname{gr}(P) \twoheadrightarrow \operatorname{gr}(R)$ and $\operatorname{gr}(P) \twoheadrightarrow \operatorname{gr}(S)$. Hence $\operatorname{gr}(P)$ maps onto $\operatorname{gr}(R) \times_k \operatorname{gr}(S)$. Since $\lambda(\operatorname{gr}(P)) = \lambda(\operatorname{gr}(R) \times_k \operatorname{gr}(S))$, we get the desired isomorphism. $\square$

Proposition 4.3.12. Let R and S be Gorenstein Artin k-algebras with $ll(R) \neq ll(S)$. Then the associated ring of $R\#_{\mathsf{k}}S$ is a fiber product.

Moreover, if $ll(R)$ and $ll(S)$ are at least 3, Q is not a standard graded k-algebra.

Proof. Let $P = R \times_k S$ and $Q = R\#_k S$. Let $\operatorname{soc}(R) = (\delta_R)$ and $\operatorname{soc}(S) = (\delta_S)$. Since $Q \cong P/(\delta_R - u\delta_S)$ for some unit u in S, by Remark 4.3.8(b), we have $\operatorname{gr}(Q) \cong \operatorname{gr}(P)/(\delta_R - u\delta_S)^*$. Without loss of generality, we may assume that $ll(R) > ll(S)$. Hence $(\delta_R - u\delta_S)^* = (u\delta_S)^*$. Thus we see that $\operatorname{gr}(Q) \cong (\operatorname{gr}(R) \times_k \operatorname{gr}(S))/(u\delta_S)^* \cong \operatorname{gr}(R) \times_k \operatorname{gr}(S/\delta_S)$.

Finally, if $ll(R) > ll(S) \geq 3$, then $\operatorname{gr}(R) \neq k \neq \operatorname{gr}(S/\delta_S)$. Hence $\operatorname{gr}(Q)$ is not Gorenstein, by Remark 4.2.1(d). Thus $Q \ncong \operatorname{gr}(Q)$, hence Q is not standard graded.

□

The following example illustrates the situation in Proposition 4.3.12.

Example 4.3.13. Let $Q = k[X,Y]/(X^2, XY - Y^3)$. Then Q is a nongraded connected sum, i.e., for $U = X - Y^2$ and $V = Y$, $Q \cong k[U,V]/(U^3,V) \#_k k[U,V]/(U,V^5)$ $\cong k[X]/(X^3)\#_k k[X,Y]/(X - Y^2, Y^5)$. Also we have $\operatorname{gr}(Q) \cong k[X,Y]/(XY, X^2, Y^5) \cong k[X]/(X^3) \times_k k[Y]/(Y^5)$.

Propositions 4.3.10 and 4.3.12 lead to the following questions:

Questions 4.3.14. Let R, S and Q be Gorenstein Artin k-algebras.

(1) If $Q/\operatorname{soc}(Q) \cong R/\operatorname{soc}(R) \times_k S/\operatorname{soc}(S)$, then is $Q \cong R\#_k S$?

(2) If $\operatorname{gr}(Q)$ is a fiber product, can we decompose Q as a connected sum?

One can see from Remarks 4.2.5.(3) that if $P = R \times_k S$, then R and S can be identified with appropriate quotients of P. On the other hand, if $Q = R\#_k S$, in general, it is not clear how one can recover R and S from Q. The following proposition shows that one can do so when Q is a k-algebra.

Proposition 4.3.15. Let R and S be Gorenstein Artin k-algebras. If $Q = R\#_k S$ with $Q \cong k[Y_1, \ldots, Y_m, Z_1, \ldots, Z_n]/I_Q$, $R \cong k[Y_1, \ldots, Y_m]/I_R$, $S \cong k[Z_1, \ldots, Z_n]/I_S$, then $I_R = I_Q \cap k[\underline{Y}]$ and $I_S = I_Q \cap k[\underline{Z}]$.

Proof. By Theorem 4.3.2, we see that $I_Q = I_R + I_S + (\Delta_R - \Delta_S) + (Y_i Z_j : 1 \leq i \leq m, 1 \leq j \leq n)$, where $\mathrm{soc}(R) = (\delta_R)$, $\mathrm{soc}(S) = (\delta_S)$ and Δ_R and Δ_S are the respective preimages of δ_R and δ_S in $k[\underline{Y}]$ and $\mathsf{k}[\underline{Z}]$. Hence it is clear that $I_R \subseteq I_Q \cap \mathsf{k}[\underline{Y}]$ and $I_S \subseteq I_Q \cap k[\underline{Z}]$. To complete the proof, it is enough to show that $I_Q \cap k[\underline{Y}] \subseteq I_R$, since $I_Q \cap k[\underline{Z}] \subseteq I_S$ will follow by symmetry.

Let $F(\underline{Y}) \in I_Q \cap k[\underline{Y}]$. We can write $F = F_1(\underline{Y}) + F_2(\underline{Z}) + \sum F_{ij} Y_i Z_j + (\Delta_R - \Delta_S)G$, where $F_1 \in I_R$, $F_2 \in I_S$ and $F_{ij}, G \in k[\underline{Y}, \underline{Z}]$. Write $G = G_1(\underline{Y}) + G_2(\underline{Z}) + \sum G_{ij} Y_i Z_j$, where $G_1 \in k[\underline{Y}]$, $G_2 \in k[\underline{Z}]$ and $G_{ij} \in k[\underline{Y}, \underline{Z}]$. Now,

$$F_2(\underline{Z}) + \sum F_{ij} Y_i Z_j + (G - G_1(\underline{Y}))\Delta_R - G\Delta_S = F - F_1 - \Delta_R G_1(\underline{Y}) \in k[\underline{Y}].$$

Since every monomial in $F_2(\underline{Z}) + \sum F_{ij} Y_i Z_j + (G - G_1(\underline{Y}))\Delta_R - G\Delta_S$ is a multiple of some Z_j, the sum must be zero. Thus $F = F_1(\underline{Y}) + \Delta_R \cdot G_1(\underline{Y})$. Hence we need to prove $\Delta_R \cdot G_1(\underline{Y}) \in I_R$.

Write $G_1 = c + H$, where $H \in (\underline{Y})$ and $c \in k$ is a constant. Note that $H\Delta_R \in I_R$ since $Y_i \cdot \Delta_R \in I_R$ for each i, hence the proof is complete if we prove $c = 0$.

Note that $c\Delta_R + \Delta_R H(\underline{Y}) = F - F_1 \in I_Q$. Hence $H\Delta_R \in I_R \subseteq I_Q$ forces $c\Delta_R \in I_Q$. Since δ_R generates $\mathrm{soc}(R)$, $\Delta_R \notin I_R$, and hence not in I_Q. Therefore c cannot be a unit, forcing $c = 0$, as desired. $\square$

A question that comes up naturally at this juncture is whether the converse of the above statement is true, i.e., if $Q = k[\underline{Y}, \underline{Z}]/I_Q$ is Gorenstein Artin, is $Q \simeq R\#_k S$, where $R = k[\underline{Y}]/I_Q \cap k[\underline{Y}]$ and $S = k[\underline{Z}]/I_Q \cap k[\underline{Z}]$?

The following show that R and S defined as above are not necessarily Gorenstein when Q is Gorenstein Artin k-algebra. However, we have a positive answer in the situation of Theorem 4.3.17.

Example 4.3.16. Let $Q = \dfrac{k[x, y, z]}{(xy - z^3, x^3, y^3)}$. Then, for $I_Q = (xy - z^3, x^3, y^3)$, we have $I_Q \cap k[y, z] = (y^3, y^2z^3, yz^6, z^9)$. Then $S = k[y, z]/I_Q \cap k[y, z]$ which is not Gorenstein.

Theorem 4.3.17. Let $Q = k[Y_1, \ldots, Y_m, Z_1, \ldots, Z_n]/I_Q$ be a Gorenstein Artin local ring. Let $R = k[\underline{Y}]/I_R$ and $S = k[\underline{Z}]/I_S$ where $I_R = I_Q \cap k[\underline{Y}]$ and $I_S = I_Q \cap k[\underline{Z}]$. Suppose $Y_i \cdot Z_j \in I_Q$ for $1 \leq i \leq m$, $1 \leq j \leq n$. Then

(a) R and S are Gorenstein Artin and

(b) $Q \cong R \#_k S$.

Proof. Note that the inclusions $k[\underline{Y}]$, $k[\underline{Z}] \hookrightarrow k[\underline{Y}, \underline{Z}]$ induce inclusions $R \hookrightarrow Q$ and $S \hookrightarrow Q$. Let y and z denote the respective images of Y and Z in the quotient rings Q, R and S.

(a) Let $f \in \operatorname{soc}(R)$. Then $Y_i \cdot F \in I_R \subseteq I_Q$ for each i, where $F \in k[\underline{Y}]$ is a preimage in $k[\underline{Y}]$ of f. Moreover, since $Y_iZ_j \in I_Q$ for each i and j, $Z_jF \in I_Q$. Hence $f \in \operatorname{soc}(Q)$. Therefore $0 \neq \operatorname{soc}(R) \subseteq \operatorname{soc}(Q)$ which is a one-dimensional k-vector space. Thus $\dim_k(\operatorname{soc}(R)) = 1$, i.e., R is Gorenstein Artin.

We can show that S is also a Gorenstein Artin local ring by a similar argument.

(b) Let $\operatorname{soc}(R) = (\delta_R)$, $\operatorname{soc}(S) = (\delta_S)$, Δ_R and Δ_S be the respective preimages of δ_R and δ_S in $k[\underline{Y}]$ and $k[\underline{Z}]$. We will show that $I_Q = I_R + I_S + (Y_i \cdot Z_j : 1 \leq i \leq m, 1 \leq j \leq n) + (\Delta_R - u\Delta_S)$, for some unit $u \in k$.

From the hypothesis, to prove

$$I_R + I_S + (Y_i \cdot Z_j : 1 \leq i \leq m, 1 \leq j \leq n) + (\Delta_R - u\Delta_S) \subseteq I_Q,$$

we only need to prove $\Delta_R - u\Delta_S \in I_Q$ for some unit $u \in k$. From the proof of (a), we note that $0 \neq \delta_R \in \operatorname{soc}(Q)$ and $0 \neq \delta_S \in \operatorname{soc}(Q)$. Since Q is Gorenstein, $\delta_R = u\delta_S$ in Q, i.e., there is a unit $u \in k$ such that $\Delta_R - u\Delta_S \in I_Q$.

In order to prove the reverse inclusion, consider $F \in I_Q$. Write $F = F_1(\underline{Y}) + F_2(\underline{Z}) + \sum F_{ij}Y_iZ_j$ where $F_1 \in k[\underline{Y}]$, $F_2 \in k[\underline{Z}]$ and $F_{ij} \in k[\underline{Y}, \underline{Z}]$. Since $Y_i \cdot Z_j \in I_Q$ for $1 \leq i \leq m$, $1 \leq j \leq n$, $F_1(\underline{Y}) + F_2(\underline{Z}) \in I_Q$. Furthermore, the same reason forces Y_iF_1, $Z_jF_2 \in I_Q$ for each i and j. In particular, $F_1 \in (\Delta_R) + I_R$ and $F_2 \in (\Delta_S) + I_S$. Note that $F_1 \in I_R \iff F_2 \in I_S$, and the proof is complete if this happens.

Let f_1 and f_2 be the respective images of F_1 and F_2 in R and S. Suppose $F_1 \notin I_R$, $F_2 \notin I_S$. Then $f_1 \in \operatorname{soc}(R)$ and $f_2 \in \operatorname{soc}(S)$ imply $f_1 = u_R\delta_R$ in R and $f_2 = u_S\delta_S$ in S for some units $u_R, u_S \in k$. Since $\delta_R = u\delta_S$ in Q, $f_1 = -f_2$ in Q forces $u_S = -uu_R$. Thus $F_1 - u_R\Delta_R = G_1 \in I_R$, $F_2 + uu_R\Delta_S = G_2 \in I_S$. Thus $F = F_1 + F_2 + \sum F_{ij}Y_iZ_j = G_1 + G_2 + \sum F_{ij}Y_iZ_j + u_R(\Delta_R - u\Delta_S)$, as desired. $\square$

4.3.3 Decomposing a Gorenstein Artin Ring as a Connected Sum

In this section we explore the connections between associated graded rings and connected sums. In particular, we study conditions on the associated graded ring of an Artinian Gorenstein ring which force it to be a connected sum.

We start with the definition of short and stretched Gorenstein rings.

Definition 4.3.18. Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin local ring with Hilbert function H_Q.

(i) We say that Q is a *short Gorenstein ring* if $H_Q = (1, h, n, 1)$, i.e., if $\mathfrak{m}_Q^3 = \operatorname{soc}(Q)$.

(ii) We say Q is a *stretched Gorenstein ring* if $H_Q = (1, h, 1, \ldots, 1)$, i.e., $\mathfrak{m}_Q^2$ is principal and $\mathfrak{m}_Q^3 \neq 0$.

Example 4.3.19. Let $Q = k[x, y, z]/(xy, xz, yz, x^3 - y^2, y^2 - z^2)$. Then Q is both stretched and short Gorenstein k-algebra since $H_Q = (1, 3, 1, 1)$.

In her paper([26]) on stretched Gorenstein rings, Sally proved the following structure theorem for stretched Gorenstein local rings $(Q, \mathfrak{m}_Q, k)$ when $\mathrm{char}(k) \neq 2$.

Theorem 4.3.20. [26, Theorem 1.1, Corollary 1.2] Let $(Q, \mathfrak{m}_Q, k)$ be a stretched local Gorenstein Artin ring of length $h+s$, embedding dimension h and $\mathfrak{m}_Q^s = \mathrm{soc}(Q)$ with $s > 2$ and $h > 1$. Let $Q = S/I$, where $(S, \mathbf{n})$ is a regular local ring of dimension h and the characteristic of $S/\mathbf{n}$ is not 2. Then

$$Q \cong S/(\{Z_i Z_j \mid i \neq j, Z_i Y, Y^t - U_i Z_i^2 : 1 \leq i, j \leq h-1\}),$$

where $\mathbf{n} = (Y, Z_1, \ldots, Z_{h-1})$ and the U_i are units in S.

In [7], Elias and Rossi proved a similar structure theorem for short Gorenstein local rings $(Q, \mathfrak{m}_Q, k)$ when $\mathrm{char}(k) = 0$ and k is algebraically closed. The next theorem is a special case of their theorem in the k-algebra case.

Theorem 4.3.21. [7, Theorem 4.1] Let $(Q, \mathfrak{m}_Q, k)$ be a short local Gorenstein Artin k-algebra with Hilbert function $H_Q = (1, h, n, 1)$. Then $Q \cong R \#_k S$ where R is a graded Gorenstein k-algebra with $H_R = (1, n, n, 1)$ and S is a Gorenstein Artin k-algebra $\mathfrak{m}_S^3 = 0$.

If Q is a Gorenstein Artin k-algebra, then in either short Gorenstein or the stretched Gorenstein case, Q is a connected sum by Theorem 4.3.2. Theorem 4.3.24 at the end of this section generalizes these two results of Sally and Elias–Rossi.

Next we give a Gorenstein ring construction due to A. Iarrobino, [15]. He studied a filtration of ideals of an associated graded ring, $G = \mathrm{gr}(Q)$, of a Gorenstein Artin local ring Q and showed that there is a graded Gorenstein quotient of $G = \mathrm{gr}(Q)$.

Definition and Theorem 4.3.22. [15] Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin local ring with associated graded ring $G = \mathrm{gr}_{\mathfrak{m}_Q}(Q)$ and $\mathfrak{m}_Q^s = \mathrm{soc}(Q)$. Consider a filtration of ideals $0 \subseteq C(s-2) \subseteq \cdots \subseteq C(1) \subseteq C(0) = G$, where the i^{th} graded piece of $C(a)$ is given by

$$C(a)_i = \frac{(0 :_Q \mathfrak{m}_Q^{s-a-i+1}) \cap \mathfrak{m}_Q^i}{(0 :_Q \mathfrak{m}_Q^{s-a-i+1}) \cap \mathfrak{m}_Q^{i+1}}.$$

Then $Q(0) := G/C(1)$ is a graded Gorenstein quotient of G with $\deg(\mathrm{soc}(Q(0))) = s$. Furthermore, since $C(1)_i = 0$ for $i \geq s-1$ by definition, $H_{Q(0)}(i) = H_G(i)$ for $i \geq s-1$.

Remark 4.3.23. By the above discussion, using the fact that the Hilbert function of a graded Gorenstein k-algebra is palindromic, we see the following:

(i) If Q is a short Gorenstein ring with $H_Q = (1, h, n, 1)$, then $H_{Q(0)} = (1, n, n, 1)$.

(ii) If Q is a stretched Gorenstein ring with $H_Q = (1, h, 1, \ldots, 1)$, then $H_{Q(0)} = (1, 1, 1, \ldots, 1)$.

Thus if Q is either a short or a stretched Gorenstein ring, we see that there is a surjective map $\pi : G = \mathrm{gr}_{\mathfrak{m}}(Q) \longrightarrow\!\!\!\!\rightarrow Q(0)$ such that $\mathrm{Ker}(\pi)_i = 0$ for $i \geq 2$.

This observation leads us to the following theorem.

Theorem 4.3.24. Let $(Q, \mathbf{m}_Q, k)$ be Gorenstein Artin k-algebra. Let $\pi : G = \mathrm{gr}(Q) \twoheadrightarrow A$ be a surjective map where A is a graded Gorenstein with $\deg(\mathrm{soc}(A)) = s \geq 3$. Assume $\ker(\pi)_i = 0$, $i \geq 2$. Then $Q \cong R \#_k S$ where R is a Gorenstein ring such that $\mathrm{gr}(R) = A$ and S is a Gorenstein ring with $\mathfrak{m}_S^3 = 0$.

Remarks 4.3.25. We first make a few observations. Let the setup be as in the hypothesis of the Theorem 4.3.24. Note that the induced map $\pi : \mathfrak{m}_G^i \longrightarrow \mathfrak{m}_A^i$ is an isomorphism for $i \geq 2$.

(a) $(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 = \mathfrak{m}_Q^{s-1}$.

Proof. Let $w \in (0 :_Q \mathfrak{m}_Q^2)$. Then $\pi(w^*) \in (0 :_A \mathfrak{m}_A^2)$. Since A is graded Gorenstein, we have $(0 :_A \mathfrak{m}_A^2) = \mathfrak{m}_A^{s-1}$ and hence $\pi(w^*) \in \mathfrak{m}_A^{s-1}$. Suppose further $w \in \mathfrak{m}_Q^2$. Then $\deg(w^*) \geq 2$ in G. Since $\pi : G_i \longrightarrow A_i$ is an isomorphism for $i \geq 2$, $\pi(w^*) \in \mathfrak{m}_A^{s-1} = A_{s-1}$ forces $w^* \in \mathfrak{m}_G^{s-1}$, i.e., $w \in \mathfrak{m}_Q^{s-1}$. Thus $(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 \subseteq \mathfrak{m}_Q^{s-1}$. The other inclusion is clear since $\mathfrak{m}_Q^s = \operatorname{soc}(Q)$.

(b) $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$ is annihilated by $\mathfrak{m}_Q$ and $\dim_k((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \operatorname{edim}(Q) - \operatorname{edim}(A)$.

Proof. By (a),

$$\frac{(0 :_Q \mathfrak{m}_Q^2)}{\mathfrak{m}_Q^{s-1}} \cong \frac{(0 :_Q \mathfrak{m}_Q^2)}{(0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2} \cong \frac{(0 :_Q \mathfrak{m}_Q^2) + \mathbf{m}_Q^2}{\mathbf{m}_Q^2}$$

is annihilated by $\mathfrak{m}_Q$.

Let $n = \lambda(\ker(\pi)) = \operatorname{edim}(Q) - \operatorname{edim}(A)$. Note that since $(0 :_Q \mathfrak{m}_Q^2)$ is the canonical module of $Q/\mathfrak{m}_Q^2$, $\lambda(0 :_Q \mathfrak{m}_Q^2) = \lambda(Q/\mathfrak{m}_Q^2)$. Also, $\ker(\pi)_i = 0$ for $i \geq 2$ gives $\lambda(\mathfrak{m}_Q^i) = \lambda(\mathfrak{m}_G^i) = \lambda(\mathfrak{m}_A^i)$ for $i \geq 2$. Hence $\lambda((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda(\mathfrak{m}_A^{s-1}) = \lambda(Q/\mathfrak{m}_Q^2) - \lambda(A/\mathfrak{m}_A^2)$, where the last equality follows from $\lambda(\mathfrak{m}_A^{s-1}) = \lambda(A/\mathfrak{m}_A^2)$, which holds since A is a graded Gorenstein ring.

Thus $\dim_{\mathsf{k}}((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \lambda((0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}) = \operatorname{edim}(Q) - \operatorname{edim}(A) = n$.

(c) If $w \in (0 :_Q \mathfrak{m}_Q^2)\backslash\mathfrak{m}_Q^{s-1}$, then $w \in \mathfrak{m}_Q\backslash\mathfrak{m}_Q^2$, $w{\cdot}\mathfrak{m}_Q = \operatorname{soc}(Q)$ and $w^* \in \operatorname{soc}(G)\backslash\mathfrak{m}_G^2$.

*P*roof. If $w \in \mathfrak{m}_Q^2$, then $w \in \mathfrak{m}_Q^2 \cap (0 :_Q \mathfrak{m}_Q^2) = \mathfrak{m}_Q^{s-1}$ by (a). Hence $w \notin \mathfrak{m}_Q^2$ and so $w^* \notin \mathfrak{m}_G^2$. Now $w \cdot \mathfrak{m}_Q^2 = 0$ implies that $w\mathfrak{m}_Q \subseteq \operatorname{soc}(Q)$. Since Q is Gorenstein and $s \geq 3$, $\operatorname{soc}(Q) \subseteq w\mathfrak{m}_Q$ proving $w\mathfrak{m}_Q = \operatorname{soc}(Q) = \mathfrak{m}_Q^s$. In particular, since $s \geq 3$, $w^* \in \operatorname{soc}(G) \setminus \mathfrak{m}_G^2$.

We first prove the following proposition:

Proposition 4.3.26. Let $(Q, \mathfrak{m}_Q, k)$ be a Gorenstein Artin k-algebra. Let $\pi : G = \operatorname{gr}(Q) \twoheadrightarrow A$ be a surjective map where A is a graded Gorenstein with $\deg(\operatorname{soc}(A)) = s \geq 3$. Assume $\ker(\pi)_i = 0$, $i \geq 2$. Then there is a minimal generating set $\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$ of $\mathfrak{m}_Q$ such that

(i) $y_i z_j = 0$ in Q for $1 \leq i \leq m$, $1 \leq j \leq n$. Futhermore

(ii) $(z_1, \ldots, z_n) \subseteq (0 :_Q \mathfrak{m}_Q^2)$ and hence for every $1 \leq i, j \leq n$, $z_i z_j \in \operatorname{soc}(Q)$,

(iii) $(z_1, \ldots, z_n) \cap (0 :_Q (z_1, \ldots, z_n)) = \operatorname{soc}(Q)$ and

(iv) $\mathfrak{m}_Q^i = (y_1, \ldots, y_m)^i$ for $i \geq 2$.

Proof. By Remark 4.3.25(b), we can choose elements $z_1, \ldots, z_n \in (0 :_Q \mathfrak{m}_Q^2) \setminus \mathfrak{m}_Q^{s-1}$ such that their images form a basis for $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$. By Remark 4.3.25(c), $z_i \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$ for each i.

(ii) Notice that $z_1, \ldots, z_n$ is a part of a minimal generating set of $\mathfrak{m}_Q$, i.e., $z_1, \ldots, z_n$ are linearly independent modulo $\mathfrak{m}_Q^2$. Indeed, suppose $a_1 z_1 + a_2 z_2 + \ldots + a_n z_n = 0$ (mod $\mathfrak{m}_Q^2$). Thus $\sum_{i=1}^n a_i z_i \in \mathfrak{m}_Q^2 \cap 0 :_Q \mathfrak{m}_Q^2 = \mathfrak{m}_Q^{s-1}$ by Remark 4.3.25(a), and hence $\sum a_i \overline{z_i} = 0$ in $(0 :_Q \mathfrak{m}_Q^2)/\mathfrak{m}_Q^{s-1}$. Since $\{\overline{z_1}, \ldots, \overline{z_n}\}$ is linearly independent, $a_i \equiv 0$ (mod $\mathfrak{m}_Q$) for all i. Thus $a_i \in \mathfrak{m}_Q$ for all i, proving that $z_1, \ldots, z_n$ are linearly independent modulo $\mathfrak{m}_Q^2$.

Finally, since $z_i \mathfrak{m}_Q^2 = 0$, $z_i \mathbf{m}_Q \subseteq \operatorname{soc}(Q)$ for each i. Hence for every $1 \leq i, j \leq n$, $z_i z_j \in \operatorname{soc}(Q)$.

(iii) If $w \in (z_1, \ldots, z_n) \subseteq (0 :_Q \mathfrak{m}_Q^2)$, then $w \cdot \mathfrak{m}_Q^{s-1} = 0$. Hence, if $w \in (z_1, \ldots, z_n) \cap (0 :_Q (z_1, \ldots, z_n))$, then $w \in (0 :_Q ((z_1, \ldots, z_n)) + \mathfrak{m}_Q^{s-1}) = (0 :_Q (0 :_Q \mathfrak{m}_Q^2)) = \mathfrak{m}_Q^2$ since Q is Gorenstein Artin. Thus $w \in (0 :_Q \mathfrak{m}_Q^2) \cap \mathfrak{m}_Q^2 = \mathfrak{m}_Q^{s-1}$ by Remark 4.3.25(a). Write $w = \sum_{i=1}^n a_i z_i$. Since $w \in \mathfrak{m}_Q^{s-1}$ and $z_1, \ldots, z_n$ are linearly independent modulo $\mathfrak{m}_Q^{s-1}$, $a_i \in \mathfrak{m}_Q$. Thus, $w \in (z_1, \ldots, z_n) \cdot \mathfrak{m}_Q = \mathfrak{m}_Q^s = \operatorname{soc}(Q)$. Therefore $(z_1, \ldots, z_n) \cap (0 :_Q (z_1, \ldots, z_n)) \subseteq \operatorname{soc}(Q)$. Since Q is Gorenstein, the other inclusion is clear and the equality follows, proving (iii).

(i) Since $n = \lambda(\ker(\pi)) = \operatorname{edim}(Q) - \operatorname{edim}(A)$, we can find elements $y_1, \ldots, y_m \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$ that extend $z_1, \ldots, z_n$ to a minimal generating set $\{y_1, \ldots, y_m, z_1, \ldots, z_n\}$ of $\mathbf{m}_Q$, where $m = \operatorname{edim}(A)$. Now $\mathfrak{m}_Q \cdot (z_1, \cdots, z_n) = \operatorname{soc}(Q)$ implies that $\lambda(z_1, \cdots, z_n) = \lambda((z_1, \cdots, z_n)/\mathfrak{m}_Q(z_1, \cdots, z_n)) + 1 = n + 1$. Hence

$$
\begin{aligned}
\lambda(0 :_Q (z_1, \cdots, z_n)/\mathfrak{m}_Q^2) &= \lambda(0 :_Q (z_1, \cdots, z_n)) - \lambda(\mathfrak{m}_Q^2) \\
&= \lambda(Q/(z_1, \cdots, z_n)) - \lambda(\mathfrak{m}_Q^2) \\
&= \lambda(Q/\mathfrak{m}_Q^2) - \lambda((z_1, \cdots, z_n)) \\
&= 1 + \lambda(\mathfrak{m}_Q/\mathfrak{m}_Q^2) - (n+1) \\
&= \operatorname{edim}(Q) - n = \operatorname{edim}(A) = m.
\end{aligned}
$$

Let $x_1, \ldots, x_m \in 0 :_Q (z_1, \ldots, z_n) \setminus \mathfrak{m}_Q^2$ be elements whose images form a basis for $(0 :_Q (z_1, \ldots, z_n)/\mathfrak{m}_Q^2)$. Then $x_i \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$, $i = 1, \ldots, m$, are such that $x_1, \ldots, x_m$ are linearly independent modulo $\mathfrak{m}_Q^2$.

Fix i, $1 \leq i \leq m$. Write $x_i = \sum_{j=1}^m a_{ij} y_j + \sum_{k=1}^n b_{ik} z_k$. If $a_{ij} \in \mathfrak{m}_Q$ for every j, then $\sum_{j=1}^m a_{ij} y_j \in \mathfrak{m}_Q^2 \subseteq 0 :_Q (z_1, \ldots, z_n)$. Hence $x_i - \sum_{j=1}^m a_{ij} y_j = \sum_{k=1}^n b_{ik} z_k \in 0 :_Q (z_1, \ldots, z_n) \cap (z_1, \ldots, z_n) = \mathfrak{m}_Q^s$ by Claim 3. Hence $x_i \in \mathfrak{m}_Q^2$, a contradiction.

Thus for each i, there is a j such that $a_{ij} \notin \mathfrak{m}_Q$. Without loss of generality, suppose that a_{11} is a unit. Replace x_i by $x_i - a_{i1} a_{11}^{-1} x_1$ for $i \geq 2$ to assume $a_{i1} = 0$

for $i \geq 2$. By the same argument as above, we first assume that a_{22} is a unit, and hence by replacing x_i by $x_i - a_{i2}a_{22}^{-1}x_2$ for $j \geq 3$, can assume that $a_{i2} = 0$ for $i \geq 3$. Continuing thus, we can assume that a_{ii} is a unit for all i, $1 \leq i \leq m$, and $a_{ij} = 0$ for $i > j$.

We now show that $(x_{m-i}, \ldots, x_m, z_1, \ldots, z_n) = (y_{m-i}, \ldots, y_m, z_1, \ldots, z_n)$ by induction on $m - i$. Let $i = 0$. Since a_{mm} is a unit and $a_{mj} = 0$ for $j < m$, $y_m = a_{mm}^{-1}x_m - (\sum_{k=1}^{n} a_{mm}^{-1}b_{mk}z_k)$. This proves the statement for the base case of the induction.

Suppose $(x_{m-i}, \ldots, x_m, z_1, \ldots, z_n) = (y_{m-i}, \ldots, y_m, z_1, \ldots, z_n)$ for $0 \leq i < m - 1$. As before, we can show that $y_{m-i+1} \in (x_{m-i+1}, y_{m-i}, \ldots, y_m, z_1, \ldots, z_n)$ since $a_{m-i+1,m-i+1} \notin \mathfrak{m}_Q$ and $a_{m-i+1,j} = 0$ for $j < m + i - 1$. Then induction shows that $y_{m-i+1} \in (x_{m-i+1}, \ldots, x_m, z_1, \ldots, z_n)$. Replacing the y_i's by x_i's, we can choose y_i such that $y_i \in 0 :_Q (z_1, \ldots, z_n)$, i.e., $y_i z_j = 0$ for $1 \leq i \leq m$, $1 \leq j \leq n$.

(iv) Next we show that $\mathfrak{m}_Q^i = (y_1, \ldots, y_m)^i$ for $i \geq 2$. Since $z_i \in 0 :_Q \mathfrak{m}_Q^2$, $\pi(z_i^*) \in 0 :_A \mathfrak{m}_A^2 = \mathfrak{m}_A^{s-1}$. But $z_i \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$ implies that $\deg(z_i^*) = 1$ in G. Hence either $\deg(\pi(z_i^*)) = 1$ or $\pi(z_i^*) = 0$ in A. Since $\pi(z_i^*) \in \mathfrak{m}_A^{s-1}$, $\deg(\pi(z_i^*)) \neq 1$, forcing $\pi(z_i^*) = 0$ in A. Counting lengths, we see that $\ker(\pi) = (z_1^*, \ldots, z_n^*)$.

Now $\mathfrak{m}_A = \pi(\mathfrak{m}_G) = \pi(y_1^*, \ldots, y_m^*, z_1^*, \ldots, z_n^*)$. Since $\pi(z_i^*) = 0$, we have $\mathfrak{m}_A = (\pi(y_1^*), \ldots, \pi(y_m^*))$. Thus $\mathfrak{m}_A^i = (\pi(y_1^*), \ldots, \pi(y_m^*))^i$ for each i. Since $\pi : \mathfrak{m}_G^i \longrightarrow \mathfrak{m}_A^i$ is an isomorphism for $i \geq 2$, we have $\mathfrak{m}_G^i = (y_1^*, \ldots, y_m^*)^i$ and hence $\mathfrak{m}_Q^i = (y_1, \ldots, y_m)^i$ for $i \geq 2$. □

As a consequence, one can prove the following lemma, which we use in the proof of Theorem 4.3.24.

Lemma 4.3.27. Let the notation be as in Proposition 4.3.26. Write $Q \cong \tilde{Q}/I_Q$

where $\tilde{Q} = k[Y_1, \ldots, Y_m, Z_1, \ldots, Z_n]$ and $I_Q \subseteq (\underline{Y}, \underline{Z})^2$. Then

$$I_Q^* = I_R^* + (Y_i^* Z_j^*, Z_j^* Z_k^* : 1 \leq i \leq m, 1 \leq j, k \leq n)$$

where $I_R = k[Y_1, \ldots, Y_m] \cap I_Q$.

Proof. By the above Proposition, $\mathbf{m}_Q = (y_1, \ldots, y_m, z_1, \ldots, z_n)$ where $y_i z_j = 0$, $1 \leq i \leq m, 1 \leq j \leq n$. Hence we can write $Q \simeq \tilde{Q}/I_Q$ where $\tilde{Q} = k[Y_1, \ldots, Y_m, Z_1, \ldots, Z_n]$, $I_Q \subseteq (\underline{Y}, \underline{Z})^2$ and $Y_i Z_j \in I_Q$ for each i and j. Hence for $1 \leq i \leq n$, $1 \leq j \leq m$, $Y_i^* Z_j^* \in I_Q^*$. Also, since $I_R \subseteq I_Q, I_R^* \subseteq I_Q^*$.

Let $\delta \in (y_1, \ldots, y_m)^s$ generate $\mathrm{soc}(Q)$ and $\Delta \in (Y_1, \ldots, Y_m)^s$ be its preimage in $\tilde{Q}$. By Proposition 4.3.26(ii), $z_i z_j = g_{ij}\delta$ for some g_{ij} in Q. Let $G_{ij} \in \tilde{Q}$ be a lift of g_{ij}. One can see that since $\Delta \in (Y_1, \ldots, Y_m)^s$ and $s \geq 3$, $Z_i^* Z_j^* = (Z_i Z_j - G_{ij}\Delta)^* \in I_Q^*$ for $1 \leq i, j \leq m$. Thus we have proved $I_R^* + (Y_i^* Z_j^*, Z_j^* Z_k^* : 1 \leq i \leq m, 1 \leq j, k \leq n) \subseteq I_Q^*$.

For the other inclusion, consider $F \in I_Q$. Write $F = F_1(\underline{Y}) + F_2(\underline{Z}) + F_3$, where $F_1 \in (\underline{Y})k[\underline{Y}]$, $F_2 \in (\underline{Z})k[\underline{Z}]$ and $F_3 \in (Y_i Z_j : 1 \leq i \leq m, 1 \leq j \leq n) \subseteq I_Q$. Since the set of the monomials appearing in each F_i is disjoint from the set of the monomials appearing in the other two, $F^* \subseteq (F_1^*, F_2^*, F_3^*)$.

Now, $F_3^* \subseteq (Y_i^* Z_j^* : 1 \leq i \leq m, 1 \leq j \leq n)$. Moreover, since $F \in I_Q \subseteq (\underline{Y}, \underline{Z})^2$, we see that $F_2 \in (\underline{Z})^2 k[\underline{Z}]$ and hence $F_2^* \subseteq (Z_j^* Z_k^* : 1 \leq j, k \leq n)$. If $\deg(F_1)^* > \min\{\deg(F_2^*), \deg(F_3^*)\}$, then $F^* \in (F_2^*, F_3^*) \subseteq (Y_i^* Z_j^*, Z_j^* Z_k^* : 1 \leq i \leq n, 1 \leq j, k \leq m)$ and we are done.

Thus, in order to prove the lemma, it is enough to prove the following claim.

*C*laim: Suppose $\deg(F_1)^* \leq \min\{\deg(F_2^*), \deg(F_3^*)\}$. Then $F_1^* \in I_R^*$.

Proof of Claim. Since $F_2 \in (\underline{Z})^2\mathsf{k}[\underline{Z}]$, there are $F_{ij} \in \mathsf{k}[\underline{Z}]$ such that

$$F_2 = \sum_{1\le i,j\le n} F_{ij}Z_iZ_j = \sum_{1\le i,j\le n} F_{ij}(Z_iZ_j - G_{ij}\Delta) + \sum_{1\le i,j\le n} F_{ij}G_{ij}\Delta.$$

We see that $F_2 = F_4 + F_5$, where $F_4 = \sum_{1\le i,j\le n} F_{ij}(Z_iZ_j - G_{ij}\Delta) \in I_Q$ and $F_5 = \sum_{1\le i,j\le n} F_{ij}G_{ij}\Delta$. Note that $\deg F_5^* > \deg(F_2^*)$ (since $\deg(\Delta^*) = s \ge 3 > \deg(Z_iZ_j)^*$).

If for some k, Z_k divides $F_{ij}G_{ij}$, then $(\underline{Y})(\underline{Z}) \subseteq I_Q$, forces $F_{ij}G_{ij}\Delta \in I_Q$. Hence we can rewrite $F_5 = F_5' + F_5''$, where

$$F_5' = \sum_{Z_k \text{ divides } F_{ij}G_{ij} \text{ for some } k} F_{ij}G_{ij}\Delta \in I_Q \text{ and } F_5'' = \sum_{\text{No } Z_k \text{ divides } F_{ij}G_{ij}} F_{ij}G_{ij}\Delta,$$

i.e., $F_5'' \in \mathsf{k}[Y_1,\dots,Y_n]$. Thus $F = F_1+F_3+F_4+F_5'+F_5''$, where $F_1, F_5'' \in \mathsf{k}[Y_1,\dots,Y_n]$, $F_3, F_4, F_5' \in I_Q$ and $\deg(F_5''^*) > \deg(F_2^*) \ge \deg(F_1^*)$. In other words, $F_1 + F_5'' \in I_Q \cap \mathsf{k}[Y_1,\dots,Y_n] = I_R$ and $(F_1 + F_5'')^* = F_1^*$. This proves the claim and hence the lemma. $\square$

*P*roof of Theorem 4.3.24. As observed in the proof of the above lemma, we can write $Q \cong \tilde{Q}/I_Q$ where $\tilde{Q} = k[Y_1,\dots,Y_m,Z_1,\dots,Z_n]$, $I_Q \subseteq (\underline{Y},\underline{Z})^2$ and $Y_iZ_j \in I_Q$ for each i and j. Let $I_R = k[Y_1,\dots,Y_m] \cap I_Q$ and $I_S = k[Z_1,\dots,Z_n] \cap I_Q$.

For $1 \le i \le n$, $1 \le j \le m$, $Y_iZ_j \in I_Q$. Therefore, by Theorem 4.3.17, $Q \cong R\#_{\mathsf{k}}S$, where $R = \mathsf{k}[Y_1,\dots,Y_m]/I_R$ and $S = k[Z_1,\dots,Z_n]/I_S$.

Let us first prove that $\mathbf{m}_S^3 = 0$. Notice that by Proposition 4.3.26(ii),

$$(Y_1,\dots,Y_m,Z_1,\dots Z_n)^2 \cdot Z_j \subseteq I_Q$$

for each j. Therefore $(Z_1,\dots,Z_n)^3 \subseteq I_S$ proving $\mathbf{m}_S^3 = 0$.

Now we want to prove that $\mathrm{gr}(R) \cong A$. Since $G \cong \mathrm{gr}(\tilde{Q})/I_Q^*$, we have

$$A \cong G/(z_1^*, \ldots, z_n^*) \cong \mathrm{gr}(\tilde{Q})/(I_Q^* + (Z_1^*, \ldots, Z_n^*)).$$

Now, by the above lemma,

$$I_Q^* = I_R^* + (Y_i^* Z_j^*, Z_j^* Z_k^* : 1 \leq i \leq n, 1 \leq j, k \leq m).$$

Hence we get

$$\begin{aligned}
\mathrm{gr}(R) \cong k[Y_1^*, \ldots, Y_m^*]/I_R^* &\cong \mathrm{gr}(\tilde{Q})/(I_R^* + (Z_1^*, \ldots, Z_n^*)) \\
&\cong \mathrm{gr}(\tilde{Q})/(I_Q^* + (Z_1^*, \ldots, Z_n^*)) \\
&\cong A.
\end{aligned}$$

□

Theorem 4.3.24 yields to the following question.

Question 4.3.28. Let Q be a Gorenstein Artin k-algebra. Assume $\mathrm{gr}(Q) \cong A \times_k B$ where either

(1) A is a graded Gorenstein Artin k-algebra and $B \cong k[Z]/(Z^t)$ with $t \leq s-2$, or

(2) $A \cong k[Y]/(Y^s)$ and B is a Teter ring with $\mathfrak{m}_B^{s-2} = 0$.

Is $Q \cong R \#_k S$ for some Gorenstein Artin rings R and S?

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