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Annihilators of Local Cohomology Modules

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In many important theorems in the homological theory of commutative local rings, an essential ingredient in the proof is to consider the annihilators of local cohomology modules. We examine these annihilators at various cohomological degrees, in particular at the cohomological dimension and at the height or the grade of the defining ideal. We also investigate the dimension of these annihilators at various degrees and we refine our results by specializing to particular types of rings, for example, Cohen Macaulay rings, unique factorization domains, and rings of small dimension.

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Chapter 1

Introduction

In what follows, all rings are assumed to be commutative, associative and with identity. A *local ring* (R, m) is defined to be a Noetherian ring possessing a unique maximal ideal m. For an R-module M, the *annihilator* of M is defined to be

$$\operatorname{Ann}_R M := \{r \in R \mid rM = 0\}$$

and the *dimension* of M is defined by $\dim M := \dim R/\operatorname{Ann}_R M$; we use "dim" for the Krull dimension. Throughout this dissertation, we define only essential or non-standard terms. We refer the reader to [BH93] for unexplained notation or terminology.

Let R be a Noetherian ring, I an ideal of R, and M an $R-$module. The i^{th} *local cohomology module* of M with respect to I is defined to be

$$\mathrm{H}^i_I(M) = \varinjlim_t \operatorname{Ext}^i_R(R/I^t, M).$$

These modules are nonzero for indices i bounded below by the *grade* of the ideal I and bounded above by the *cohomological dimension* of the ideal I. In other words,

$\operatorname{grade}(I) = \inf\{i \mid H_I^i(R) \neq 0\}$ and $\operatorname{cd}(I) = \sup\{i \mid H_I^i(R) \neq 0\}$. For a summary of the basic properties of local cohomology modules and the main theorems we use throughout this study, see Appendix A.

The theory of local cohomology was first introduced by Grothendieck in 1961 and has become a powerful tool successfully used by many mathematicians to solve problems in both algebraic geometry and commutative algebra. In several instances, the theory has been applied to the study of the various "Homological Conjectures," originally formulated in the 1970s, which still remain at the center of commutative algebra research today. In many of these instances where local cohomology was used, results about the annihilators of local cohomology modules were key to the proofs. Examples of this nature include Roberts' proof of the Intersection Conjecture for characteristic $p > 0$ in [Rob76], Huneke's study of classes of rings which have the uniform Artin-Rees property in [Hun92], and, more recently, Heitmann's proof of the Direct Summand Conjecture in dimension 3 in [Hei02]. These annihilators have also proved useful in other areas of study, including certain vanishing criteria on local cohomology given by Huneke and Koh in [HK91]. The annihilators of local cohomology modules with support in a maximal ideal are well understood (at least in the case that R is the homomorphic image of a Gorenstein ring), but not much is known when the support is in an arbitrary ideal.

For example, it is easily shown (see Corollary A.8 in Appendix A for a proof) that for a Noetherian ring R and ideal $I \subseteq R$,

$$H_I^0(R) \subseteq \operatorname{Ann}_R H_I^i(R)$$

for all $i > 0$. On the other hand, a motivating result for much of this thesis is the following remarkable result:

Proposition 1.1. *[HK91, char p],[Lyu93, char 0] Let R be a regular local ring containing a field and I an ideal. Then $H_I^i(R) \neq 0$ if and only if $\operatorname{Ann}_R H_I^i(R) = 0$.*

Proof. We give here the proof in characteristic $p > 0$, since it is fairly straightforward. The proof in characteristic 0 where R contains $\mathbb{Q}$ follows from Corollary 3.6 in [Lyu93].

Suppose R has characteristic $p > 0$ and $H_I^i(R) \neq 0$. Let F be the Frobenius functor and note $F(H_I^i(R)) = H_I^i(R)$ (since F is exact and commutes with homology). Choose an element $x \in H_I^i(R) \setminus \{0\}$ and let $J = (0 :_R x)$. Then $0 \to R/J \to H_I^i(R)$ defined by $\overline{1} \mapsto x$ is exact. Apply F (e times) to get the exact sequences $0 \to R/J^{[p^e]} \to H_I^i(R)$ for all e. Thus if $y \in \operatorname{Ann}_R H_I^i(R)$, then $y \in \operatorname{Ann}_R R/J^{[p^e]} = J^{[p^e]}$ for all e. By Krull's Intersection Theorem, we have $y \in \bigcap_e J^{[p^e]} = 0$. The other direction is trivial. $\square$

The proposition is certainly not true for arbitrary rings. Theorem 8.1.1 in [BH93] states that

$$\dim R/\operatorname{Ann}_R H_m^i(R) \leq i$$

for a Noetherian local ring R that is the homomorphic image of a Gorenstein local ring. In particular, this says $\operatorname{Ann}_R H_m^i(R) \neq 0$ for $i < \dim R$, and so the proposition does not hold if, for example, R is not Cohen Macaulay. The proposition also remains open for regular local rings of mixed characteristic.

The goal of this dissertation is thus to expand our knowledge of these annihilators. At best, we hope to find a closed form for these annihilators. If a closed form cannot be found, then we would like to establish when these annihilators are "small," that is, either zero or contained within a minimal prime. Our approach to this problem will be two-fold: (1) Examine annihilators of local cohomology modules at particular indices for arbitrary Noetherian rings and (2) Examine annihilators of local cohomology modules of arbitrary index for particular types of rings. In Chapter 2, we examine the annihilators at the index equal to the cohomological dimension of the ideal. The main result of Chapter 2 is the following:

Theorem 2.17. *Let (R, m) be a complete local ring of dimension d and I an ideal of R. Then*

$$\operatorname{Ann}_R H_I^d(R) = \cap\{q \mid q \text{ primary component of } (0), \dim R/q = d, \sqrt{I+q} = m\}.$$

This theorem yields a closed form for these annihilators when the cohomological dimension c coincides with the dimension of the ring. In fact, it generalizes the Hartshorne-Lichtenbaum Vanishing Theorem discussed in Appendix A. Given a ring R and an ideal I of R, we also show the product of all the annihilators of $H_I^i(R)$ is zero. From that we prove that the annihilator of $H_I^c(R)$ is zero whenever the cohomological dimension coincides with the grade of the ideal. In general, we present a conjecture concerning the dimension of $H_I^c(R)$. We prove the conjecture in several cases.

In Chapter 3, we consider these annihilators at another important index, namely the height h of an ideal. Here we establish the following result, where $\operatorname{Min}_R R/I$ is the set of prime ideals of R minimal over I :

Proposition 3.6. *Let (R, m) be local, I an ideal of R, and $p \in \operatorname{Min}_R R/I$. Then*

$$\operatorname{Ann}_R H_I^{\operatorname{ht}(I_p)}(R) \subseteq \cap\{q|q \text{ primary component of } (0), \dim R_p/qR_p = \dim R_p\}.$$

Among other things, this implies the above annihilator is zero when R is a domain. We also establish a bound on the dimension of $H_I^h(R)$ and show, in many cases, this bound is in fact an equality. In this section we also generalize our study of these annihilators to $\operatorname{Ann}_R H_I^h(M)$ for an R−module M.

Lastly, in Chapter 4 we bring the previous results together to examine what happens over domains and other types of rings. The main result of Chapter 4, which utilizes many of the results in the previous chapters, is the following:

Theorem 4.2. *For a complete Cohen Macaulay unique factorization domain R of dimension at most 4, $H_I^i(R) \neq 0$ if and only if $\operatorname{Ann}_R H_I^i(R) = 0$.*

This result generalizes Proposition 1.1 when R has dimension at most 4. We conclude by presenting several questions for future study.

Chapter 2

The Cohomological Dimension

We begin our examination of the annihilators of local cohomology modules at the cohomological dimension of an ideal. Several well-known results (such as Theorem A.17, the Hartshorne-Lichtenbaum Vanishing Theorem) assist our study. Many of the results we present in this chapter are in [Lyn]. Let R be Noetherian and let I be an ideal of R. By examining Corollary A.8, mentioned in the introduction, we see the following:

Remark 2.1. *For $c := \operatorname{cd}(I) > 0$, we have $H_I^0(R) \subseteq \operatorname{Ann}_R H_I^c(R)$ and thus*

$$\dim R/\left(\operatorname{Ann}_R H_I^c(R)\right) \leq \dim R/H_I^0(R).$$

Immediately one could ask when this is an equality and, in fact, we conjecture that it always is.

Conjecture 2.2. *For every local ring R and ideal I of R, if $\operatorname{cd}(I) =: c > 0$, then*

$$\dim\left(R/\operatorname{Ann}_R H_I^c(R)\right) = \dim R/H_I^0(R).$$

In particular, if I contains a non-zero-divisor, then $\dim(R/\operatorname{Ann}_R H_I^c(R)) = \dim R$.

Although the reverse inequality of the expression in Remark 2.1 is not apparent, we do at least have a lower bound on the dimension on the left:

Remark 2.3. *Note that* $0 \neq H_I^c(R) \cong H_I^c(R/\operatorname{Ann}_R H_I^c(R))$, *where* $c = \operatorname{cd}(I)$ *(See Corollary A.11 in Appendix A). Thus*

$$\dim(R/\operatorname{Ann}_R H_I^c(R)) \geq c.$$

When $H_I^c(R)$ is not Artinian, we can do slightly better.

Proposition 2.4. *Let R be a ring, $I \subseteq R$ and $c := \operatorname{cd}(I)$. If $H_I^c(R)$ is not Artinian, then* $\dim(R/\operatorname{Ann}_R H_I^c(R)) \geq c+1$.

Proof. Let $J = \operatorname{Ann}_R H_I^c(R)$. For $J = 0$, we know $\dim R \geq c+1$ since $H_I^c(R)$ is not Artinian by Theorem A.19. Consider the case that $J \neq 0$. Then we see $0 \neq H_I^c(R) \cong H_I^c(R/J)$ by Corollary A.11. Thus $\dim R/J \geq c$ by Remark 2.3. If $\dim R/J = c$, then $H_I^c(R) \cong H_I^c(R/J)$ would be Artinian, a contradiction. Thus $\dim R/J \geq c+1$. □

However, these results are not nearly as strong as Conjecture 2.2. We spend much of this chapter giving evidence for its validity.

Recall that $\operatorname{grade}(I) \leq \operatorname{cd}(I) \leq \dim(R)$ (see Corollary A.6 in Appendix A for proof). We therefore begin our investigation by studying these two extremal values for cohomological dimension and then return to the general case. First, consider the simplest case where there is only one non-zero local cohomology module, that is, $\operatorname{cd}(I) = \operatorname{grade}(I)$.

2.1 The Minimal Case (cd(I) = grade(I))

In order to generalize results that Paul Roberts used to prove the New Intersection Theorem, Peter Schenzel examined the ideals $\operatorname{Ann}_R H_I^i(R)$ in [Sch82]. We generalize one of his results below. This result will allow us to more closely examine the situation of only one non-zero local cohomology module, that is, when the cohomological dimension of an ideal is equal to its grade.

Proposition 2.5. *Let (R, m) be local, M an $R-$module, $I = (x_1, ..., x_t)$ and*

$$F^{\cdot} : 0 \to F^0 \to \cdots \to F^s \to 0$$

a complex of finitely generated free $R-$modules such that $(F^{\cdot} \otimes_R M)_{x_i}$ is exact for all i. Let $a_i = \operatorname{Ann}_R H_I^i(M)$. Then $a_0 \cdots a_i H^i(F^{\cdot} \otimes_R M) = 0$ for all $i \leq s$.

Proof. Our proof requires the theory of spectral sequences and uses the notation in [Wei94]. Let $K^{\cdot}$ be the Čech complex for R with respect to $\underline{x} = x_1, ..., x_t$. Then

$$H^i(K^{\cdot} \otimes_R M) = H^i_{(\underline{x})}(M) = H^i_I(M)$$

by Proposition A.3. Let C be the first quadrant double complex $K^{\cdot} \otimes_R (F^{\cdot} \otimes_R M)$ and consider the following two spectral sequences.

If we filter by columns, we see

$$\begin{aligned} {}^I E_1^{p,q} &= H_v^q(K^p \otimes_R F^{\cdot} \otimes_R M) \\ &= K^p \otimes_R H_v^q(F^{\cdot} \otimes_R M) \text{ since } K^p \text{ is flat for all } p \\ &= \begin{cases} H^q(F^{\cdot} \otimes_R M) & \text{if } p = 0 \\ 0 & \text{if } p > 0 \end{cases}, \end{aligned}$$

since $R_{x_i} \otimes_R H^q(F^\cdot \otimes_R M) = 0$ for all i by assumption. Thus the spectral sequence collapses and we have $H^{p+q}(F^\cdot \otimes_R M) = {}^I E_\infty^{p,q} = H^{p+q}(\mathrm{Tot}(C))$.

Filtering by rows, we have

$$\begin{aligned} {}^{II}E_1^{p,q} &= H_v^q(K^\cdot \otimes_R F^p \otimes_R M) \\ &= H_v^q(K^\cdot \otimes_R M) \otimes_R F^p, \text{ since } F^p \text{ is free and hence flat} \\ &= H_I^q(M) \otimes_R R^{r_p}, \text{ where } r_p = \operatorname{rank} F^p \\ &= (H_I^q(M))^{r_p}. \end{aligned}$$

By definition of a_q, we see $a_q \,{}^{II}E_1^{p,q} = a_q(H_I^q(M))^{r_p} = 0$. Since ${}^{II}E_\infty^{p,q}$ is a subquotient of ${}^{II}E_1^{p,q}$, we also have $a_q \,{}^{II}E_\infty^{p,q} = 0$.

By the classical convergence theorem of spectral sequences [Wei94, Theorem 5.5.1], we have

$${}^{II}E_1^{p,q} \Rightarrow H^{p+q}(\mathrm{Tot}(C)) = H^{p+q}(F^\cdot \otimes_R M).$$

Thus there exists a filtration $\{F^pH^i\}$ with $H^i := H^i(F^\cdot \otimes_R M)$ such that

$$0 = F^{i+1}H^i \subset F^iH^i \subset \cdots \subset F^1H^i \subset F^0H^i = H^i,$$

where $F^pH^i/F^{p+1}H^i = {}^{II}E_\infty^{p,i-p}$ for all p. Since $a_{i-p} \,{}^{II}E_\infty^{p,i-p} = 0$, we see that $a_{i-p}F^pH^i \subset F^{p+1}H^i$ and hence $a_0 \cdots a_i H^i = 0$ for $i \leq s$. $\square$

Corollary 2.6. *Let (R, m) be local, M an $R-$module generated by s elements, $I = (x_1, ..., x_t)$, and $a_i = \mathrm{Ann}_R H_I^i(M)$. Then $(a_0 \cdots a_t)^s \subset \mathrm{Ann}_R M$. In particular, if $M = R$, we have $a_0 \cdots a_t = 0$.*

Proof. Let $\underline{x} = x_1, ..., x_t$, $\underline{x}^{[\ell]} = x_1^\ell, ..., x_t^\ell$ and $F^\cdot = K^\cdot(\underline{x}^{[\ell]}; R)$, the Koszul Complex on $\underline{x}^{[\ell]}$, for $\ell \in \mathbb{N}$. Then

$$H^i(F^\cdot \otimes_R M)_{x_j} \cong H^i(\underline{x}^{[\ell]}; M)_{x_j} = 0$$

for all i (this follows from Lemma 1.1 in [Hun85]). Since $H^i_I(R) = H^i_{I^{[\ell]}}(R)$, Proposition 2.5 gives $a_0 \cdots a_t \subseteq \operatorname{Ann}_R H^t(\underline{x}^{[\ell]}; M)$ for all ℓ and so

$$a_0 \cdots a_t \subseteq \bigcap_\ell \operatorname{Ann}_R \left(H^t(\underline{x}^{[\ell]}; M)\right).$$

Since $H^t(\underline{x}^{[\ell]}; M) = M/(\underline{x}^{[\ell]})M$ (see Section 16 of [Mat89]), we see

$$a_0 \cdots a_t \subseteq \bigcap_\ell \operatorname{Ann}_R \left(M/(\underline{x}^{[\ell]})M\right).$$

Since M is finitely generated, we can find a presentation

$$R^r \xrightarrow{A} R^s \to M \to 0.$$

Tensoring with $R/(\underline{x}^{[\ell]})$ yields

$$(R/(\underline{x}^{[\ell]}))^r \xrightarrow{\overline{A}} (R/(\underline{x}^{[\ell]}))^s \to M/(\underline{x}^{[\ell]})M \to 0.$$

Using the theory of fitting ideals (see Section 20.2 in [Eis04]), we have

$$\left(\operatorname{Ann}_{R/(\underline{x}^{[\ell]})} M/(\underline{x}^{[\ell]})M\right)^s \subset I_s(\overline{A}),$$

where $I_s(\overline{A})$ is the ideal generated by the $s \times s$ minors of $\overline{A}$. Lifting up to R yields

$$\left(\operatorname{Ann}_R M/(\underline{x}^{[\ell]})M\right)^s \subseteq I_s(A) + (\underline{x}^{[\ell]})$$

for all ℓ. By Krull's Intersection Theorem and Proposition 20.7 of [Eis04], we thus have

$$(a_0 \cdots a_t)^s \subseteq \left(\cap_\ell \operatorname{Ann}_R M/(\underline{x}^{[\ell]})M\right)^s \subseteq I_s(A) \subseteq \operatorname{Ann}_R M.$$

□

For R an integral domain, Corollary 2.6 therefore implies one of the annihilators of local cohomology must be zero, though it does not specify which one. If there is only one non-zero local cohomology module, however, we arrive at the following result.

Theorem 2.7. *Let (R, m) be a local ring of dimension d and $I \subseteq R$ an ideal such that $H^i_I(R) = 0$ for all $i \neq g := \operatorname{grade}(I)$. Then $\operatorname{Ann}_R H^g_I(R) = 0$.*

Proof. We have $\operatorname{Ann}_R H^g_I(R) = a_g = a_0 \cdots a_d = 0$ since $a_i = R$ for all $i \neq g$. □

2.1.1 Examples with One Non-Zero Local Cohomology Module

Under what conditions is there only one nonzero local cohomology module? That is, how applicable is Theorem 2.7? To answer that question, we present three corollaries that describe situations where this occurs.

Corollary 2.8. *Let (R, m) be a local ring of characteristic p and I a perfect ideal of grade g. Then $H^i_I(R) = 0$ for all $i \neq g$. Consequently, $\operatorname{Ann}_R H^g_I(R) = 0$.*

Proof. It is clear that $H^i_I(R) = 0$ for all $i < g := \operatorname{grade} I = \operatorname{depth}_I R$. Since the Frobenius functor is exact on finite free resolutions, $\operatorname{pd}_R R/I^{[p^e]} = g$ for all e. Hence $\operatorname{Ext}^i_R(R/I^{[p^e]}, R) = 0$ for all $i > g$. Since $\{I^{[p^e]}\}$ is cofinal with $\{I^t\}$, we therefore see

$$H^i_I(R) = \varinjlim_t \operatorname{Ext}^i_R(R/I^t, R) = \varinjlim_e \operatorname{Ext}^i_R(R/I^{[p^e]}, R) = 0$$

for all $i > g$. By Theorem 2.7, we obtain the desired result. $\square$

Corollary 2.9. *Let (R, m) be local and $I = (x_1, ..., x_t)$ where $\underline{x} := x_1, ..., x_t$ is regular. Then $H_I^i(R) = 0$ for all $i \neq t$. Consequently, $\operatorname{Ann}_R H_I^t(R) = 0$.*

Proof. Recall that $H_I^t(R) \neq 0$ and $H_I^i(R) = 0$ for all $i \neq t$. The result now follows from Theorem 2.7. $\square$

Corollary 2.10. *Let (R, m) be local and I a set theoretic complete intersection of height h. Then $\operatorname{Ann}_R H_I^h(R) = 0$.*

2.1.2 Another Application of Spectral Sequences

In Corollary 2.6, our result on the annihilators $a_i = \operatorname{Ann} H_I^i(M)$ depended on the number of generators of M. By choosing a different spectral sequence argument, however, we get a result on the annihilators of local cohomology that is independent of the number of generators of M.

Proposition 2.11. *Let (R, m) be local, M an $R-$module, and I and J ideals of R with $I \subseteq J$. Then for $a_i = \operatorname{Ann}_R H_I^i(M)$ we have $a_0 \cdots a_i H_J^i(M) = 0$ for all integers $i \geq 0$. In particular, if $\operatorname{Ann}_R H_J^i(M) = 0$ then $a_0 \cdots a_i = 0$.*

Proof. Let $I = (y_1, ..., y_t)$ and $J = (x_1, ..., x_s)$. Let $C^{\cdot}$ be the Čech complex on $y_1, ..., y_t$ and $K^{\cdot}$ be the Čech complex on $x_1, ..., x_s$. Form the first quadrant double complex $D^{\cdot}$ by $C^{\cdot} \otimes_R (K^{\cdot} \otimes_R M)$ and consider the following spectral sequences.

Filter by columns:

$$\begin{aligned} {}^{I}E_1^{p,q} &= H_v^q(C^p \otimes_R K^{\cdot} \otimes_R M) \\ &= C^p \otimes_R H_v^q(K^{\cdot} \otimes_R M) \text{ since } C^p \text{ is flat for all } p \\ &= C^p \otimes_R H_J^q(M) \\ &= \begin{cases} H_J^q(M) & \text{if } p = 0, \\ 0 & \text{if } p > 0 \end{cases} \end{aligned}$$

since $(H_J^q(M))_{y_i} = 0$ for all i. Thus the spectral sequence collapses and

$$H_J^{p+q}(M) = {}^{I}E_\infty^{p,q} = H^{p+q}(\mathrm{Tot}(D^{\cdot})).$$

Filter by rows:

$${}^{II}E_1^{p,q} = H_h^q(C^{\cdot} \otimes_R K^p \otimes_R M) = H_I^q(M) \otimes_R K^p,$$

since K^p is flat. Since $a_q\ {}^{II}E_1^{p,q} = a_q(H_I^q(M) \otimes_R K^p) = 0$ and ${}^{II}E_\infty^{p,q}$ is a subquotient of ${}^{II}E_1^{p,q}$, we have that $a_q\ {}^{II}E_\infty^{p,q} = 0$.

By the classical convergence theorem of spectral sequences, we have

$${}^{II}E_1^{p,q} \Rightarrow H^{p+q}(Tot(D)) = H_J^{p+q}(M).$$

Thus there exists a filtration $\{F^pH^i\}$, for all i, such that

$$0 = F^{n+1}H^i \subset F^iH^i \subset \cdots \subset F^1H^i \subset F^0H^i = H^i$$

where $H^i := H_J^i(M)$ and $F^pH^i/F^{p+1}H^i = {}^{II}E_\infty^{p,i-p}$ for all p. Since $a_{i-p}\ {}^{II}E_\infty^{p,i-p} = 0$, we see

$$a_{i-p}F^pH^i \subset F^{p+1}H^i,$$

and hence $a_0 \cdots a_i H^i_J(M) = 0$ for all i. □

This result is trivial if we take $I = J$ since $a_i H^i_I(M) = 0$ by definition. The power of the proposition comes in finding an ideal J containing I such that more information is known about $H^i_J(M)$. For example, if $I \subseteq (x)$ for some non-zero-divisor $x \in R$, then $a_0 a_1 = 0$. If I contains a non-zero-divisor, then $\operatorname{Ann}_R H^1_I(R) = 0$. More generally, we have the following result.

Corollary 2.12. *With the above notation, suppose* $I \subset (x_1, ..., x_g)$ *where* $x_1, ..., x_g$ *forms a regular sequence. Then* $a_0 \cdots a_g = 0$*. If, in addition,* $\operatorname{grade} I = g$*, then* $a_g = 0$.

Examining the proposition from a slightly different direction, one can also obtain information without knowing the annihilators of $H^i_J(M)$.

Corollary 2.13. *With the above notation, if* $\operatorname{cd}(J) = c$*, then* $(a_0 \cdots a_c)^{c+1} = 0$.

Proof. Let $b_i = \operatorname{Ann}_R H^i_J(M)$ for all $i \geq 0$. Since $a_0 \cdots a_i \subseteq b_i$ and $b_0 \cdots b_c = 0$ we have

$$a_0(a_0a_1) \cdots (a_0 \cdots a_c) \subseteq b_0 \cdots b_c = 0$$

and thus $(a_0 \cdots a_c)^{c+1} = 0$. □

For example, if R is a domain, then $a_0 \cdots a_c = 0$ whenever $I \subseteq J$ where $\operatorname{cd}(J) = c$.

2.2 The Maximal Case (cd(I) = dim R)

We now consider the situation where $\operatorname{cd}(I) = \dim R$. Unlike the case of only one non-zero local cohomology module, it will not generally be true that $\operatorname{Ann}_R H^d_I(R) = 0$. For example, consider the one-dimensional Cohen Macaulay ring $R = k[x, y]/(xy)$. Here we have

$$H^1_{(\overline{x})}(R) \cong R_x/R \cong k[x, x^{-1}]/k[x]$$

and so $\operatorname{Ann}_R H^1_{(\overline{x})}(R) = (y)$. However, some information can be gained about these annihilators. In particular, we can (and will) give a closed form for $\operatorname{Ann}_R H^d_I(R)$.

First suppose R is complete. By the Cohen Structure Theorem, $R = T/I$ for a regular local ring T and ideal I. Suppose $\operatorname{ht} I = g$ and choose a regular sequence $x_1, ..., x_g \in I$. Define $S := T/(x_1, ..., x_g)$ and $J := I/(x_1, ..., x_g)$. Note S is Gorenstein, $\operatorname{ht} J = 0$ and $R = S/J$. Let $J = q_1 \cap \cdots \cap q_\ell \cap q_{\ell+1} \cap \cdots \cap q_t$ be an irreducible primary decomposition for J where $q_1, ..., q_\ell$ are the primary components such that $\dim S/q_i = \dim S$.

Lemma 2.14. *For S and J as above, $(0 :_S (0 :_S J)) = q_1 \cap \cdots \cap q_\ell$.*

Proof. Let p_i be the associated prime of S/q_i for all i. Now $\operatorname{ht} p_i = 0$ for $i \leq \ell$ and $\operatorname{ht} p_i > 0$ for $i > \ell$. Let $y \in (0 :_S (0 :_S J))$. Localize at p_i for $i \leq \ell$ to get $J_{p_i} = (q_i)_{p_i}$. Then $\frac{y}{1} \in (0 :_{S_{p_i}} (0 :_{S_{p_i}} q_i S_{p_i})) = q_i S_{p_i}$ since S_{p_i} is a zero-dimensional Gorenstein for all $i \leq \ell$. Thus $y \in q_i$ for all $i \leq \ell$ which implies $y \in q_1 \cap \cdots \cap q_\ell$.

Now let $y \in q_1 \cap \cdots \cap q_\ell$. Since $\operatorname{ht}(q_{\ell+1} \cap \cdots \cap q_t) > 0$, there exists a non-zero-divisor $x \in q_{\ell+1} \cap \cdots \cap q_t$. Then $xy \in J$ which implies $xy(0 :_S J) = 0$. Since x is a non-zero-divisor, we have $y(0 :_S J) = 0$ and so $y \in (0 :_S (0 :_S J))$. $\square$

With the above notation, recall that $R = S/J$ is unmixed if and only if we have $\dim S/q_i = \dim S$ for $i = 1, ..., t$.

Remark 2.15. *By Lemma 2.14, R is unmixed if and only if $(0 :_S (0 :_S J)) = J$.*

Relating this back to our study of annihilators, we obtain the following.

Lemma 2.16. *With $R = S/J$ as above, $\operatorname{Ann}_R H^d_m(R) = (0 :_S (0 :_S J))/J$ where $d = \dim R$. Hence, R is unmixed if and only if $\operatorname{Ann}_R H^d_m(R) = 0$.*

Proof. By the Change of Rings principle, $H^d_m(R) \cong H^d_n(S/J)$ where n is the unique maximal ideal of S. Thus it is enough to show $\operatorname{Ann}_S H^d_n(S/J) = (0 :_S (0 :_S J))$. Now, by Matlis Duality,

$$\begin{aligned}
\operatorname{Ann}_S H^d_n(S/J) &= \operatorname{Ann}_S H^d_n(S/J)^\vee \\
&= \operatorname{Ann}_S \operatorname{Hom}_S(S/J, S) \quad \text{(by local duality)} \\
&= \operatorname{Ann}_S(0 :_S J) \\
&= (0 :_S (0 :_S J)).
\end{aligned}$$

□

Now we are ready for the main result of this section, which gives us a closed form for these annihilators when R is complete.

Theorem 2.17. *Let (R, m) be a complete local ring of dimension d and I an ideal of R. Then*

$$\operatorname{Ann}_R H^d_I(R) = \bigcap\{q \mid q \textit{ primary component of } R, \dim R/q = d, \sqrt{I + q} = m\}.$$

Proof. Let $(0) = q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition and let $J = q_1 \cap \cdots \cap q_s$ where $q_1, ..., q_s$ are the primary components with $\sqrt{I + q_i} = m$ and $\dim R/q_i = d$. Note that $\sqrt{I + J} = m$. Also

$$(0 :_R J) \supseteq q_{s+1} \cap \cdots \cap q_t$$

since $(q_{s+1} \cap \cdots \cap q_t)J \subseteq q_1 \cap \cdots \cap q_t = (0)$.

Claim. $H_I^d(R/(0 :_R J)) = 0$

Proof. If not, then there exists a prime p containing $(0 :_R J)$ such that $\dim R/p = d$ and $\sqrt{I+p} = m$ by the Hartshorne-Lichtenbaum Vanishing Theorem (A.17). Since $p \supseteq (0 :_R J)$, we see $p \supseteq q_i$ for some $i \geq s+1$. Therefore $\dim R/q_i = d$ and $p = \sqrt{q_i}$. This implies $\sqrt{I+p} = \sqrt{I+q_i} = m$, contradicting the fact that $i \geq s+1$.

By the Change of Rings principle (A.15),

$$H_I^d(J) = H^d_{IR/\operatorname{Ann}_R J}(J) = H_I^d(R/\operatorname{Ann}_R J) \otimes_R J = 0.$$

By the long exact sequence on local cohomology given by the short exact sequence $0 \to J \to R \to R/J \to 0$, we have $H_I^d(R) \cong H_I^d(R/J) = H^d_{IR/J}(R/J)$. Now R/J is unmixed and since $\sqrt{I+J} = m$, we have $H_I^d(R) \cong H^d_{m/J}(R/J)$. The result now follows from Lemma 2.16. $\square$

To obtain a result for arbitrary local rings R, consider the following remark.

Remark 2.18. *For an arbitrary local ring R, an $R-$module M and an ideal $I \subseteq R$, we have*

$$\operatorname{Ann}_R H_I^i(M) = \left(\operatorname{Ann}_{\hat{R}} H^i_{I\hat{R}}(\hat{M})\right) \cap R$$

where $\hat{R}$ is the m-adic completion of R and $\hat{M}$ is $M \otimes_R \hat{R}$.

Proof. For an arbitrary $R-$module N note that $\operatorname{Ann}_R N = \left(\operatorname{Ann}_R(N \otimes_R \hat{R})\right) \cap R$. (The forward inclusion follows since $rN = 0$ implies $r(N \otimes_R \hat{R}) = 0$ and the reverse inclusion follows from the fact that $N \hookrightarrow N \otimes_R \hat{R}$ is injective.) Thus,

$$\operatorname{Ann}_R H_I^i(M) = \left(\operatorname{Ann}_R(H_I^i(M) \otimes_R \hat{R})\right) \cap R = \left(\operatorname{Ann}_R H_{I\hat{R}}^i(M \otimes_R \hat{R})\right) \cap R.$$

□

Thus for an arbitrary local ring R, we have $\operatorname{Ann}_R H_I^d(R) = \left(\operatorname{Ann}_R H_{I\hat{R}}^d(\hat{R})\right) \cap R$ and so

$$\operatorname{Ann}_R H_I^d(R) = \left(\cap\{q \mid q \text{ primary component of } \hat{R}, \dim \hat{R}/q = d, \sqrt{I\hat{R} + q} = \hat{m}\}\right) \cap R.$$

For the study of the Homological Conjectures, discussed in Chapter 1, it is often crucial to know whether or not a given annihilator is zero. Thus, it is of particular interest to determine when $\operatorname{Ann}_R H_I^d(R) = 0$. By examining when the intersection in Theorem 2.17 is trivial, we obtain the following corollary.

Corollary 2.19. *Let (R, m) be a local ring of dimension d and I an ideal of R with $H_I^d(R) \neq 0$. If R is unmixed and I is $m-$primary, then $\operatorname{Ann}_R H_m^d(R) = 0$.*

If we further suppose R is complete, the converse holds.

Proof. First suppose R is complete. By Theorem 2.17, the annihilator is zero if and only if R is unmixed and $\sqrt{I + q_i} = m$ where $q_1, ..., q_t$ are the primary components of (0). This implies

$$\sqrt{I + q_1 \cap \cdots \cap q_t} = m$$

and thus $\sqrt{I} = m$ since $q_1 \cap \cdots \cap q_t = (0)$. If R is not complete, apply Remark 2.18. □

The assumption that R is complete is essential for the converse. Nagata gives an example of a two-dimensional local domain R in [Nag62] where $\operatorname{Ann}_R H_m^d(R) = 0$, since R is a domain and $\operatorname{ht} m = d$. On the other hand, $\operatorname{Ann}_{\hat{R}} H_{m\hat{R}}^d(\hat{R}) \neq 0$ since $\hat{R}$ is not unmixed.

2.2.1 Ideals with Finite Projective Dimension

For a local ring R of dimension d and an ideal I of R, an interesting question (in part inspired by Proposition 1.1) is whether $\operatorname{Ann}_R H_I^d(R) = 0$ if $H_I^d(R) \neq 0$ and $\operatorname{pd} R/I < \infty$. We show these two conditions imply R is unmixed. To do so, we first recall the New Intersection Theorem, proved by Peskine and Szpiro (for rings essentially of finite type over a field) and Roberts (in complete generality); see [BH93] for the proof for rings essentially of finite type over a field.

New Intersection Theorem. *Let R be a local ring of dimension d and let $F.$ be a complex $0 \to F_k \to \cdots \to F_0 \to 0$ of free modules with $\lambda(H_i(F.)) < \infty$ for all i. Then if $k < d$, the complex $F.$ is exact.*

Corollary 2.20. *Let (R, m) be local and let I, J be ideals of R such that $\sqrt{I+J} = m$. Suppose $\operatorname{pd}_R R/I < \infty$. Then $\operatorname{pd}_R R/I \geq \dim R/J$. In particular,*

$$\operatorname{depth} R \geq \dim R/J + \operatorname{depth} R/I.$$

Proof. Let $t = \operatorname{pd}_R R/I$ and $F. : 0 \to R^{n_t} \to R^{n_{t-1}} \to \cdots \to R^{n_0} \to R/I \to 0$ be a minimal free resolution. Tensor $F.$ with R/J to get

$$\overline{F.} : 0 \to (R/J)^{n_t} \to \cdots \to (R/J)^{n_0} \to 0.$$

Now $H_i(\overline{F.}) = \operatorname{Tor}_i^R(R/I, R/J)$. These modules are finitely generated and have support equal to the maximal ideal. Thus $H_i(\overline{F.})$ has finite length for all i. Since $H_0(\overline{F.}) = R/(I+J) \neq 0$, we see $\overline{F.}$ is not exact. Hence $t \geq \dim R/J$ by the New Intersection Theorem. □

Applying this corollary to our situation, we have the following.

Theorem 2.21. *Let (R, m) be a local ring of dimension d and I and ideal of R with $\operatorname{pd}_R R/I < \infty$ and $H_I^d(R) \neq 0$. If I is $m-$primary, then $\operatorname{Ann}_R H_I^d(R) = 0$.*

If we further assume R is complete, the converse holds.

Proof. By Corollary 2.19, it is enough to show the assumptions imply that R is unmixed. Since $\operatorname{pd}_{\hat{R}} \hat{R}/I\hat{R} = \operatorname{pd}_R R/I < \infty$, $0 \neq H_I^d(R) \otimes \hat{R}$, and $\hat{R}$ unmixed implies R is unmixed, we may assume without loss of generality that R is complete. The Hartshorne-Lichtenbaum Vanishing Theorem (A.17) yields a prime $p \in \operatorname{Spec} R$ such that $\sqrt{I+p} = m$ and $\dim R/p = d$. By the lemma, $\operatorname{depth} R \geq \dim R + \operatorname{depth} R/I$. In particular, this says R is Cohen Macaulay and hence unmixed. □

The question of whether $\operatorname{pd}_R R/I < \infty$ and $H_I^d(R) \neq 0$ imply $\sqrt{I} = m$ appears to be difficult. Utilizing the Hartshorne-Lichtenbaum Vanishing Theorem (A.17), one can show in the complete case that there exists $p \in \operatorname{Spec} R$ such that $\sqrt{I+p} = m$ and $\dim R/p = d$. Lemma 2.20 then gives $\operatorname{depth} R \geq \dim R + \operatorname{depth} R/I$, which implies R is Cohen Macaulay and $\operatorname{depth} R/I = 0$. Looking in a different direction, consider the following conjecture of Peskine and Szpiro. (This conjecture is implied, for example, by the Strong Intersection Theorem.)

An Intersection Conjecture. [PS73] *Let (R, m) be local and M, N $R-$modules with $\operatorname{pd}_R M < \infty$ and $\lambda(M \otimes_R N) < \infty$. Then $\dim M + \dim N \leq \dim R$.*

If we assume this conjecture to be true, we can, in fact, show that I is $m-$primary if $\operatorname{pd}_R R/I < \infty$ and $H_I^d(R) \neq 0$.

Proposition 2.22. *Let (R, m) be a local ring such that the above intersection conjecture holds and $I \subseteq R$ an ideal with $\operatorname{pd}_R R/I < \infty$. Then $H_I^d(R) \neq 0$ if and only if $\operatorname{Ann}_R H_I^d(R) = 0$.*

Proof. By Theorem 2.21, it is enough to show the assumptions imply I is $m-$primary. Since $\sqrt{I\hat{R}} = \hat{m}$ implies $\sqrt{I} = m$, we may again pass to the completion. By the Hartshorne-Lichtenbaum Vanishing Theorem (A.17), there exists $p \in \operatorname{Spec} R$ with $\dim R/p = d$ and $\sqrt{I+p} = m$. Let $M = R/I$ and $N = R/p$ in the above intersection conjecture. Then $\lambda(M \otimes_R N) < \infty$ since $R/I \otimes_R R/p$ is finitely generated with support $\{m\}$. Thus whenever the conjecture holds we have $\dim R/I + \dim R/p \leq \dim R$. This implies $\dim R/I = 0$ since $\dim R/p = \dim R$. Hence $\sqrt{I} = m$. $\square$

2.3 The General Case

We examined the annihilators when the cohomological dimension was as small as possible (that is, equal to the grade of the ideal) and as large as possible (that is, equal to the dimension of the ring). We tie these results back to Conjecture 2.2 with the following proposition:

Proposition 2.23. *Let R be a local ring with $d := \dim(R)$ and let I be an ideal of R with $c := \operatorname{cd}(I) > 0$. If $c \in \{1, \operatorname{grade}(I), d\}$ then Conjecture 2.2 holds.*

Proof. By passing to $R/H_I^0(R)$, we may assume that $H_I^0(R) = 0$ and therefore that I contains a non-zero-divisor. Notice $c = 1$ is then a special case of $c = \operatorname{grade}(I)$. For $c = \operatorname{grade}(I)$, we have $\dim\left(R/\operatorname{Ann}_R H^{\operatorname{grade}(I)}(R)\right) = d$ by Theorem 2.7. If $c = d$, then $\dim\left(R/\operatorname{Ann}_R H_I^d(R)\right) \geq d$ by Remark 2.3. $\square$

Notice that for $c = 0$, we can again apply Theorem 2.7 to get $\operatorname{Ann}_R H_I^0(R) = 0$ and so $\dim\left(R/\operatorname{Ann}_R H_I^0(R)\right) = \dim R$. A more difficult situation arises when we have $cd(I) = \dim R - 1$. To examine this case, we require the following lemma where $\operatorname{Min}(R/J) = \{p \in \operatorname{Spec} R \mid p \text{ is minimal over } J\}$ for an ideal J of R.

Lemma 2.24. *For a Noetherian ring R,*

$$\operatorname{Min}_R R/H_I^0(R) = \{p \in \operatorname{Min}_R R \mid I \not\subseteq p\}.$$

In particular, $\dim R/H_I^0(R) = \max\{\dim R/p \mid p \in \operatorname{Min}_R R$ *and* $I \not\subseteq p\}$.

Proof. Let $J = H_I^0(R) = (0 :_R I^t)$ for $t >> 0$. Suppose $p \in \operatorname{Min}_R R$ and $I \not\subseteq p$. Then, since $J \cdot I^n = 0 \subseteq p$, we see $p \supseteq J$. Therefore $p \in \operatorname{Min}_R R/J$.

Now suppose $p \in \operatorname{Min}_R R/J$. If $I \subseteq p$, then $I_p^\ell \subseteq J_p = (0 : I_p^t)$ for t and ℓ sufficiently large. Then $I_p^{\ell+t} = 0$, which implies $J_p = R_p$, a contradiction. Hence $I \not\subseteq p$, which implies $I_p = R_p$. Thus $J_p = 0$ and so $\operatorname{ht}(p) = \dim R_p = \dim(R/J)_p = 0$, that is, $p \in \operatorname{Min}_R R$. □

Proposition 2.25. *For a local ring R of dimension d at least two and I an ideal of R, if* $\operatorname{cd}(I) = d-1$ *then Conjecture 2.2 holds.*

Proof. By Remark 2.1 and Lemma 2.24, its enough to show

$$\dim\left(R/\operatorname{Ann}_R H_I^{d-1}(R)\right) \geq \max\{\dim R/p \mid p \in \operatorname{Min}_R R \text{ and } I \not\subseteq p\}.$$

We know the left hand side is at least $d-1$ by Remark 2.3. If the right hand side is $d-1$, we are done. Assume $\max\{\dim R/p \mid p \in \operatorname{Min}_R R \text{ and } I \not\subseteq p\} = d$ and further assume by way of contradiction that $\dim R/J = d-1$ for $J := \operatorname{Ann}_R H_I^{d-1}(R)$.

Suppose first that R is complete and unmixed. If J consists solely of zero-divisors, then $J \subseteq p$ for some associated prime p. Since R is unmixed, this would imply $\dim R/J = d$, contradicting our assumption. Thus we let $x \in J$ be a non-zero-divisor with $\dim R/(x) = d-1$. Then $0 \neq H_I^{d-1}(R) \cong H_I^{d-1}(R/(x))$. By the Hartshorne-Lichtenbaum Vanishing Theorem (A.17), there exists a prime ideal $p \supseteq (x)$ such that

$\dim R/p = d-1$ and $\sqrt{I+p} = m$. By Theorem 2.17, $J = \operatorname{Ann}_R H_I^{d-1}(R/(x)) \subseteq q$, where q is the $p-$primary component of (x). By replacing x with x^ℓ, we get that $J \subseteq q_\ell$, where q_ℓ is the $p-$primary component of (x^ℓ). Then $J \subseteq \cap_\ell q_\ell$ and since $q_\ell = (x^\ell)R_p \cap R$ we see $J_p \subseteq \cap_\ell (x^\ell)R_p = (0)$ by Krull's Intersection Theorem. This implies $\dim(R/J)_p = \dim R_p = \operatorname{ht} p$. Of course $\dim R/J = d-1 = \dim R/p$, which implies p is minimal over J, and hence $0 = \dim(R/J)_p = \operatorname{ht}(p)$, a contradiction to the fact that $x \in p$ is a non-zero-divisor. Therefore

$$\dim R/J = d = \max\{\dim R/p \mid p \in \operatorname{Min}_R R, I \not\subseteq p\}.$$

Now suppose R is an arbitrary complete ring and define $U := q_1 \cap \cdots \cap q_\ell$, where q_i are the primary components with $\dim R/q_i = d$. Then $S := R/U$ is unmixed and $\dim U < d$. Furthermore, from the short exact sequence $0 \to U \to R \to S \to 0$ we have

$$H_I^{d-1}(R) \to H_I^{d-1}(S) \to H_I^d(U) = 0$$

where the last term is zero by Corollary A.6 since $\dim U < d$. Thus we have

$$\operatorname{Ann}_R H_I^{d-1}(R) \subseteq \operatorname{Ann}_R H_I^{d-1}(S),$$

which implies

$$\begin{aligned}
\dim_R \left(R/\operatorname{Ann}_R H_I^{d-1}(R)\right) &\geq \dim_R \left(R/\operatorname{Ann}_R H_I^{d-1}(S)\right) \\
&\geq \dim_S \left(S/\operatorname{Ann}_S H_I^{d-1}(S)\right) \\
&\geq \max\{\dim S/p \mid p \in \operatorname{Min}_S S \text{ and } IS \not\subseteq p\}.
\end{aligned}$$

Now, by assumption, $\max\{\dim R/p \mid p \in \operatorname{Min}_R R, I \not\subseteq p\} = d$, which implies there exists $q \in \operatorname{Min}_R R$ with $I \not\subseteq q$ and $\dim R/q = d$. Since $q \supseteq U$, we have $qS \in \operatorname{Min}_S S$

with $IS \not\subseteq qS$ and $\dim S/qS = d$. Thus

$$\dim \left(R/\operatorname{Ann}_R H_I^{d-1}(R)\right) \geq \max\{\dim S/p \mid p \in \operatorname{Min}_S S, IS \not\subseteq p\} = d,$$

and so the result holds for R a complete ring.

For arbitrary R, note that we can reduce to the complete case via the following:

Claim. If $\dim \left(\hat{R}/(\operatorname{Ann}_R H_I^c(R))\hat{R}\right) = \dim \hat{R}/H^0_{I\hat{R}}(\hat{R})$, then

$$\dim \left(R/\operatorname{Ann}_R H_I^c(R)\right) = \dim R/H_I^0(R).$$

Proof. Since $R \to \hat{R}$ is faithfully flat and I is finitely generated, we see $(0 :_{\hat{R}} \hat{I}^s) = (0 :_R I^s)\hat{R}$ for every s and hence $H_I^0(R)\hat{R} = H^0_{I\hat{R}}(\hat{R})$. Also $(\operatorname{Ann}_R H_I^c(R))\hat{R} \subseteq \operatorname{Ann}_{\hat{R}} H^c_{I\hat{R}}(\hat{R})$. Thus

$$\begin{aligned} \dim R/H_I^0(R) &= \dim \hat{R}/H^0_{I\hat{R}}(\hat{R}) \\ &= \dim \left(\hat{R}/\operatorname{Ann}_{\hat{R}} H^c_{I\hat{R}}(\hat{R})\right) \\ &\leq \dim \left(R/\operatorname{Ann}_R H_I^c(R)\right). \end{aligned}$$

By Remark 2.1, we therefore have equality. ∎

By combining Propositions 2.23 and 2.25, we have the following theorem:

Theorem 2.26. *Let R be a local ring with $\dim R \leq 3$, let I be an ideal of R containing a non-zero-divisor, and set $c := \operatorname{cd}(I)$. Then $\dim \left(R/\operatorname{Ann}_R H_I^c(R)\right) = \dim R$. If we further assume R is an integral domain, then $\operatorname{Ann}_R H_I^c(R) = 0$.*

To prove the conjecture in other cases, one could try examining ideals with given properties other than having a specific cohomological dimension. For example, consider the following proposition.

Proposition 2.27. *Let (R, m) be local, unmixed, and let I and J be ideals of R such that $\sqrt{I+J} = m$, $\dim R/J = c$, and $H^i_I(R) = 0$ for all $i > c$ (e.g., $\mu(I) = c$). Then $\dim(R/\operatorname{Ann}_R H^c_I(R)) = \dim R$.*

Proof. First notice $H^c_I(R) \otimes_R R/J = H^c_m(R/J) \neq 0$ and so $\operatorname{cd}(I) = c$. Define q_t to be the $Q-$primary component of J^t for some prime $Q \supseteq J$ with $\dim R/Q = c$. Then for all t we have

$$A := \operatorname{Ann}_R H^c_I(R) \subseteq \operatorname{Ann}_R H^c_m(R/J^t) \subseteq q_t$$

by Theorem 2.17. Thus $A_Q \subseteq \cap_t (q_t)_Q = \cap_t (J^t)_Q = (0)_Q$ by Krull's Intersection Theorem. So A is contained in some associated prime of R. Since R is unmixed, we have $\dim R/A = \dim R$. $\square$

As an example, let $R := k[\![x_1, ..., x_t, u_1, ..., u_t]\!]$ and $f := \sum_i u_i x_i$. Then we have $S := R/(f)$ is an unmixed domain because f is irreducible. Take $I = (\overline{x}_1, ..., \overline{x}_t)$ and $J = (\overline{u}_1, ..., \overline{u}_t)$. Then $\dim R/J = t$ and $H^i_I(R) = 0$ for all $i > t$ by Proposition A.5 since $t = \operatorname{ara}(I)$, the *arithmetic rank* of I. Since R is an integral domain, Proposition 2.27 gives $\operatorname{Ann}_S H^t_I(S) = 0$.

Moving in a slightly different direction (still assuming $\operatorname{cd}(I)$ is arbitrary, but imposing other conditions on I), we have the following characterization of the annihilator when I is generated by c elements.

Proposition 2.28. *Let R be a ring and $I = (x_1, ..., x_c)$. Then $r \in \operatorname{Ann}_R H^c_I(R)$ if and only if for every k there exists ℓ such that $r(x_1 \cdots x_c)^s \in (x_1^{\ell+k}, ..., x_c^{\ell+k})$; that is,*

$$\operatorname{Ann}_R H^c_I(R) = \bigcap_k \bigcup_\ell \left((x_1^{\ell+k}, ..., x_c^{\ell+k}) : x_1^\ell \cdots x_c^\ell\right).$$

In particular, $H^c_I(R) = 0$ if and only if for every $k \geq 1$ there exists ℓ such that $(x_1 \cdots x_c)^\ell \in (x_1^{\ell+k}, ..., x_c^{\ell+k})$.

Proof. For ease of notation, define $\underline{x} := (x_1, ..., x_c)$ and $\underline{x}^k := (x_1^k, ..., x_c^k)$. Recall

$$H_I^c(R) = \varinjlim \left(R/(\underline{x}) \xrightarrow{x_1 \cdots x_c} R/(\underline{x}^2) \xrightarrow{x_1 \cdots x_c} R/(\underline{x}^3) \to \cdots \right)$$

(because $H_I^i(R) = \varinjlim_t H^i(\underline{x}^t; R)$ and $H^c(\underline{x}^t; R) = R/(\underline{x}^t)$). For each k, let

$$\psi_k : R/(\underline{x}^k) \to \varinjlim_t H^c(\underline{x}^t; R) = H_I^c(R)$$

be the canonical map. Then every element of $H_I^c(R)$ is equal to $\psi_k(u)$ for some $u \in R/(\underline{x}^k)$. Let $r \in \operatorname{Ann}_R H_I^c(R)$ and fix k. Then $0 = r\psi_k(1) = \psi_k(\overline{r})$. Hence there exists $\ell >> 0$ such that $(x_1 \cdots x_c)^\ell \overline{r} = 0$ in $R/(\underline{x}^{k+\ell})$; that is, $(x_1 \cdots x_c)^\ell r \in (\underline{x}^{k+\ell})$. Thus

$$r \in ((\underline{x}^{k+\ell}) : x_1^\ell \cdots x_c^\ell).$$

Since this holds for all k, the forward direction of the statement holds. For the other direction, suppose for every j there exists ℓ such that

$$r(x_1 \cdots x_c)^\ell \in (x_1^{\ell+j}, ..., x_c^{\ell+j}).$$

Then $r(x_1 \cdots x_c)^\ell \cdot \overline{1} = 0$ in $R/(\underline{x}^{\ell+j})$ and hence $r\psi_j(\overline{1}) = 0$ for all j. Thus, $r\psi_j(\overline{u}) = 0$ for all $\overline{u} \in R/(\underline{x}^j)$, which implies $r \in \operatorname{Ann}_R H_I^c(R)$. $\square$

As an example, suppose $I = (x, y)$. Then we have

$$\operatorname{Ann}_R H_I^2(R) = \bigcap_k \bigcup_n \left((x^{n+k}, y^{n+k}) : x^n y^n \right).$$

While this does give a closed form for $\operatorname{Ann}_R H_I^2(R)$, it is not clear from this formula what the dimension of $\operatorname{Ann} H_I^2(R)$ is. Thus further work is needed to understand what happens for this particular scenario which is the simplest case currently unknown for Conjecture 2.2.

Chapter 3

The Height

For this chapter, we continue our study of local cohomology of rings by examining when the index equals the height of an ideal. For Cohen Macaulay rings, the height coincides with the grade of the ideal and so it gives the first non-vanishing local cohomology module. Much of the work we do over rings can be generalized to modules and we therefore spend the latter half of this chapter focusing on local cohomology of modules.

3.1 Local Cohomology of Rings at the Height

To start, we consider the results of the previous chapter with respect to the maximal ideal m, where we were able to find a closed form for the annihilator. To do so, we first introduce some convenient notation.

Let (R, m) be a Noetherian local ring and I an ideal of R. Define

$$U(I) := q_1 \cap \cdots \cap q_t,$$

where q_i are the isolated primary components of I; that is, $\dim R/q_i = \dim R/I$ for all i. For the zero ideal, we write $U(R)$ for $U((0))$. If R is complete and unmixed, then $U(R) = 0$. Hochster and Huneke gave a characterization of $U(R)$ in [HH94] which we now extend to $U(I)$:

Proposition 3.1. *For a Noetherian local ring R,*

$$U(I) = \{r \in R | \dim (R/(I : r)) < \dim R/I\}.$$

Proof. Let $I = q_1 \cap \cdots \cap q_s \cap \cdots \cap q_\ell$ be a primary decomposition of I such that $U(I) = q_1 \cap \cdots \cap q_s$ and each q_i is p_i-primary for p_i a prime ideal of R and $i = 1, ..., \ell$. By properties of colon ideals, we have for $r \in R$ that $(I : r) = \bigcap_{i=1}^{\ell}(q_i : r)$ and hence

$$\sqrt{(I : r)} = \bigcap_{i=1}^{\ell} \sqrt{(q_i : r)}.$$

Note that

$$\sqrt{(q_i : r)} = \begin{cases} p_i, & \text{if } r \notin q_i, \\ R, & \text{if } r \in q_i. \end{cases}$$

Hence $\sqrt{(I : r)} = \bigcap_{i=1}^{t} p_{i_j}$ where $\{q_{i_j}\}_{j=1}^{t}$ is the complete set of primary components q with $r \notin q$. Then

$$\dim (R/(I : r)) = \dim \left(R/\sqrt{(I : r)}\right) = \max_j \dim R/p_{i_j}.$$

Thus $\dim (R/(I : r)) = \dim R/I$ if and only if $r \notin q_i$ for some $i \leq s$. In other words, $\dim (R/(I : r)) < \dim R$ if and only if $r \in q_i$ for all $i \leq s$. Therefore

$$U(R) = \{r \in R | \dim (R/(I : r)) < \dim R/I\}.$$

□

Recall that Theorem 2.17 showed $\operatorname{Ann}_{\hat{R}} H^d_{\hat{m}}(\hat{R}) = U(\hat{R})$, and hence Proposition 3.1 gives us another way to characterize this annihilator. A natural question that arises is what happens in more general settings. Observe the following.

Proposition 3.2. *Let R be a Noetherian local ring and let I be an ideal of R. Then*

1. *$U(I) = U(I\hat{R}) \cap R$.*
2. *If R is catenary and p is a prime ideal containing a prime ideal $q \supseteq I$ with $\dim R/q = \dim R/I$, then $U(I)_p = U(I_p)$. If $p = q$, then $U(I)_p = I_p$.*

Proof. 1. Let $r \in R$; note that $(I :_R r)\hat{R} = (I :_{\hat{R}} r)$. Thus

$$\dim R/(I :_R r) = \dim \hat{R}/(I :_R r)\hat{R} = \dim \hat{R}/(I :_{\hat{R}} r).$$

Since $\dim \hat{R}/I\hat{R} = \dim R/I$, the result follows from Proposition 3.1.

2. Since R is catenary, we know $\dim R/Q = \dim R/p + \operatorname{ht}(p/Q)$ for any prime $Q \subseteq p$. Since $\dim R/I = \dim R/q$ and q is minimal over I, we also have that $\dim R/I = \dim R/p + \operatorname{ht}(p/I)$.

Let $I = q_1 \cap \cdots \cap q_\ell \cap \cdots \cap q_r$ where $\dim R/q_i = \dim R/I$ for $1 < i \leq \ell$ and $\dim R/q_i < \dim R/I$ for $\ell < i \leq r$. Then $I_p = (q_1)_p \cap \cdots \cap (q_r)_p$. For $1 < i \leq \ell$, if $q_i \subseteq p$ we have

$$\begin{aligned} \dim R_p/(q_i)_p = \operatorname{ht}(p/q_i) &= \dim R/q_i - \dim R/p \\ &= \dim R/I - \dim R/p = \operatorname{ht}(p/I) = \dim R_p/I_p. \end{aligned}$$

Similarly, for $\ell < i \leq r$, if $q_i \subseteq p$ we have $\dim R_p/(q_i)_p < \dim R_p/I_p$. Thus

$$U(I_p) = (q_1)_p \cap \cdots \cap (q_\ell)_p = U(I)_p.$$

If we further have that $p = q$, that is, $\dim R/p = \dim R/I$, then I_p is p_p-primary, and so $U(I_p) = I_p$.

□

In particular, by part 1 of Proposition 3.2, $U(R) = U(\hat{R}) \cap R$. By combining this with Theorem 2.17, we arrive at the following.

Corollary 3.3. *For a Noetherian local ring (R, m) of dimension d, we have*

$$\operatorname{Ann}_R H^d_m(R) = (\operatorname{Ann}_{\hat{R}} H^d_{\hat{m}}(\hat{R})) \cap R = U(\hat{R}) \cap R = U(R).$$

Next we apply this equation to the annihilators of local cohomology modules over an arbitrary ideal I.

Proposition 3.4. *For (R, m) local, I an ideal of R, and $p \in \operatorname{Min}_R R/I$, we have*

$$\dim\left(R/\operatorname{Ann}_R H_I^{\operatorname{ht}(I_p)}(R)\right) \geq \dim R_p + \dim R/p.$$

Proof. Let $J := \operatorname{Ann}_R H_I^{\operatorname{ht}(I_p)}(R)$. Then

$$J_p \subseteq \operatorname{Ann}_{R_p}\left(H_{I_p}^{\operatorname{ht}(I_p)}(R_p)\right) = U(R_p)$$

by Corollary 3.3. Thus J_p is contained in some minimal prime of maximal dimension of R_p. By lifting to R, we see $J \subseteq q$ where $q \subseteq p$ for q prime with $\dim R_p/q_p = \dim R_p$. Then

$$\dim R/J \geq \dim R/q \geq \dim(R/q)_p + \dim R/p = \dim R_p + \dim R/p.$$

□

This bound is tight. As an example, consider $R = k[\![x, y, z]\!]/((x) \cap (y, z))$ and $p = (\overline{y}, \overline{z})$. Here $H_p^0(R) = (\overline{x})$ and so $\operatorname{Ann}_R H_p^0(R) = p$. Since $\dim R_p = 0$, this shows

$$\dim\left(R/\operatorname{Ann}_R H_p^0(R)\right) = \dim R/p = \dim R_p + \dim R/p.$$

We can say more if we place restrictions on the ring. In particular, the statement of Conjecture 2.2 is valid for the height of the ideal in a Cohen Macaulay ring:

Corollary 3.5. *If (R, m) is equidimensional and catenary (e.g. Cohen Macaulay), then $\dim\left(R/\operatorname{Ann}_R H_I^h(R)\right) = \dim R$ where $h = \operatorname{ht} I$.*

It would be nice to have a closed form for $\operatorname{Ann}_R H_I^h(R)$ as in the case for $\operatorname{Ann}_R H_I^d(R)$. Although we not found one, we can say the following:

Proposition 3.6. *Let (R, m) be local, I an ideal of R, and $p \in \operatorname{Min}_R R/I$. Then*

$$J := \operatorname{Ann}_R H_I^{\operatorname{ht}(I_p)}(R) \subseteq \bigcap \{q | q \text{ primary component of } (0) \text{ and } \dim R_p/qR_p = \dim R_p\}.$$

Proof. Recall from the proof of Proposition 3.4 that $J_p \subseteq U(R_p)$. Now lift to R. $\square$

This leads us to determine the height of these annihilators.

Corollary 3.7. *With the notation as in Proposition 3.6, $\operatorname{ht} J = 0$. In particular, if R is an integral domain, then*

$$\operatorname{Ann}_R H_I^{\operatorname{ht}(I_p)}(R) = 0$$

for every prime $p \in \operatorname{Min}_R R/I$.

Notice that the dimension of R/J need not be equal to the dimension of R. Consider $R = k[\![x,y,z]\!]/((x) \cap (y,z))$ and $p = (\overline{y}, \overline{z})$ again. Here

$$\dim \left(R/ \operatorname{Ann}_R H_p^0(R)\right) = 1 \neq 2 = \dim R.$$

This does, however, lead us to a bound on the dimension.

Corollary 3.8. *For (R,m) local, I an ideal of R, $\operatorname{ht}(I) = h$, and $J = \operatorname{Ann}_R H_I^h(R)$, we have*

$$\dim R/J \geq \min\{\dim R/p | p \in \operatorname{Min}_R R\}.$$

The above example shows that equality can occur for this bound. If we further assume the ring R is equidimensional then this implies $\dim R/J = \dim R$.

3.1.1 An Interesting Situation

Theorem 6.1 in [HKM09] says for a Noetherian ring R containing a field of characteristic 0 that $H^3_{I_2(A)}(R) \cong H^6_{I_1(A)}(R)$ for a 2×3 matrix A with entries in R. Here, $I_n(A)$ denotes the ideal generated by the $n \times n$ minors of A. We combine this with the Hilbert-Burch theorem below:

Hilbert-Burch Theorem. *Let (R,m) be a Noetherian local ring and $I \subseteq R$ an ideal such that $\operatorname{pd}_R R/I = 2$. Then the minimal resolution of R/I has the form $0 \to R^t \xrightarrow{A} R^{t+1} \to R \to R/I \to 0$ where $t+1 = \mu_R(I)$. Moreover, $I = xI_t(A)$ for some non-zero-divisor x.*

Thus we obtain the following result.

Theorem 3.9. *Let (R,m) be a Noetherian local ring containing a field of characteristic 0. Suppose $I \subset R$ is perfect of grade 2 and $\mu(I) = 3$. If $H_I^3(R) \neq 0$, then* $\dim (R/ \operatorname{Ann}_R H_I^3(R)) \geq 6$.

Proof. Since I is perfect, we have $\operatorname{pd}_R R/I = \operatorname{grade} I = 2$. By the Hilbert-Burch Theorem, there exists a 2×3 matrix A such that $I = I_2(A)$. By Theorem 6.1 in [HKM09], $H^3_I(R) \cong H^6_{I_1(A)}(R)$. Hence $\dim(R/\operatorname{Ann}_R H^3_I(R)) = \dim\left(R/\operatorname{Ann}_R H^6_{I_1(A)}(R)\right) \geq 6$ by Remark 2.3. □

For a ring of characteristic p, every ideal with these hypotheses would yield $H^3_I(R) = 0$ (because $H^3_I(R) \cong \varinjlim_e \operatorname{Ext}^3_R(R/I^{[p^e]}, R) = 0$ since $\operatorname{pd}_R R/I^{[p^e]} = 2$ for all e since the Frobenius functor is exact on acyclic complexes of free $R-$modules).

Corollary 3.10. *Let R be a complete Noetherian ring containing a field of characteristic 0 and $\dim R = 6$, and I a perfect ideal of R of grade 2 and $\mu(I) = 3$. Then*

$$\operatorname{Ann}_R H^3_I(R) = \cap\{q | q \text{ isolated primary component}, \dim R/q = 6, \sqrt{J+q} = m\},$$

where $J = I_1(A)$ and A is as given in Hilbert-Burch.

3.2 Local Cohomology of Modules at the Height

A natural question that arises in this investigation is whether we can generalize the results in the previous section to modules.

For the remainder of this section, define $\overline{R} := R/\operatorname{Ann}_R M$. As a generalization of Corollary 3.3, we obtain the following.

Proposition 3.11. *Let (R, m) be local and M a finitely generated $R-$module such that $\dim_R M = d$. Then $\operatorname{Ann}_R H^d_m(M) = U(\operatorname{Ann}_R M)$.*

Proof. By Proposition 3.1, we have that $U(\operatorname{Ann}_R M) = U((\operatorname{Ann}_R M)\hat{R}) \cap R$. Since $\operatorname{Ann}_R H^d_m(M) = \operatorname{Ann}_{\hat{R}} H^d_{\hat{m}}(M\hat{R}) \cap R$, it is enough to show

$$\operatorname{Ann}_{\hat{R}} H^d_{\hat{m}}(M\hat{R}) = U((\operatorname{Ann}_R M)\hat{R}).$$

Thus we may assume R is complete. Then by the Cohen Structure Theorem, there exists a Gorenstein local ring (S, n) with $\dim S = \dim R$ and an ideal $T \subseteq S$ with $R = S/T$. By the Change of Rings Principle, we have $H^d_m(M) \cong H^d_n(M)$ and thus it is enough to show $\operatorname{Ann}_S H^d_n(M) = U(\operatorname{Ann}_S M)$. Therefore we may assume R is a complete Gorenstein ring.

Let $I := \operatorname{Ann}_R H^d_m(M)$ and $J := U(\operatorname{Ann}_R M)$. By Corollary 3.3 applied to $\overline{R}$, we see $J \cdot H^d_m(\overline{R}) = 0$. Since $H^d_m(M) \cong H^d_m(\overline{R}) \otimes_R M$, we see $J \cdot H^d_m(M) = 0$ and hence $J \subseteq I$. Thus we need only show $I/J = 0$, or equivalently, $(I/J)_p = 0$ for all $p \in \operatorname{Ass} R/J$. Since R/J is unmixed (by definition) and $\dim M = d$, we see

$$\operatorname{Ass} R/J = \{p \in \operatorname{Spec} R | \dim R/p = d, p \supseteq \operatorname{Ann}_R M\}.$$

Now, for all $p \in \operatorname{Ass} R/J$, we have

$$J_p = U(\operatorname{Ann}_R M)_p = U(\operatorname{Ann}_{R_p} M_p) = \operatorname{Ann}_{R_p}(M_p)$$

by Proposition 3.2 applied to $I = \operatorname{Ann}_R M$.

Since R is Gorenstein, local duality gives

$$I = \operatorname{Ann}_R H^d_m(M)^\vee = \operatorname{Ann}_R \operatorname{Hom}_R(M, R).$$

Note that R_p is the injective hull of $k(p) = R_p/pR_p$, and so

$$I_p = \operatorname{Ann}_R(\operatorname{Hom}_{R_p}(M_p, R_p)) = \operatorname{Ann}_{R_p} M_p^\vee = \operatorname{Ann}_{R_p} M_p = J_p.$$

Thus $(I/J)_p = I_p/J_p = 0$ for all $p \in \operatorname{Ass} R/J$ and hence $I/J = 0$. $\square$

In particular, this shows for a finitely generated faithful $R-$module M with $\dim_R M = d$ that $\operatorname{Ann}_R H^d_m(M) = U(R)$. Continuing our generalization, we use these results to examine what happens at the height of our ideal.

Proposition 3.12. *Let (R, m) be local, I an ideal of R, and $p \in \operatorname{Min}_R R/I$. Let M be a finitely generated $R-$module with $\dim_R M = d$ and $J = \operatorname{Ann}_R \left(H_I^{\operatorname{ht}(I\overline{R}_p)}(M) \right)$. Then*

$$J \subseteq \cap\{q|q \textit{ is a primary component in } R \textit{ of } \operatorname{Ann}_R M, \dim R_p/qR_p = \dim \overline{R}_p\}$$

Proof. By Proposition 3.11, we see $J\overline{R}_p \subseteq \operatorname{Ann}_{\overline{R}_p} \left(H^{\operatorname{ht}(I\overline{R}_p)}_{I\overline{R}_p}(M\overline{R}_p) \right) = U(\overline{R}_p)$. Thus

$$J\overline{R} \subseteq \cap\{\overline{q}|\overline{q} \text{ is a primary component in } \overline{R} \text{ of } (0), \dim \overline{R}_p/q\overline{R}_p = \dim \overline{R}_p\},$$

which implies

$$J \subseteq \cap\{q|q \text{ is a primary component in } R \text{ of } \operatorname{Ann}_R M, \dim \overline{R}_p/q\overline{R}_p = \dim \overline{R}_p\},$$

since if $q_1 \cap \cdots \cap q_t$ is an irredundant primary decomposition in R of $\operatorname{Ann}_R M$, then $\overline{q_1} \cap \cdots \cap \overline{q_t}$ is an irredundant primary decomposition in $\overline{R}$ of (0). $\square$

If M is faithful, this shows $\operatorname{ht} J = 0$ as in the case $M = R$. If R is equidimensional and M faithful, then $\dim R/J = \dim R$.

Corollary 3.13. *For (R,m) local, I an ideal of R, $p \in \operatorname{Min}_R R/I$, M a finitely generated $R-$module, and $J = \operatorname{Ann}_R H^{\operatorname{ht}(I\overline{R}_p)}(M)$, we have*

$$\dim R/J \geq \dim \overline{R}_p + \dim \overline{R}/p\overline{R}.$$

Proof. First note

$$\overline{J} = \left(\operatorname{Ann}_R \left(H_I^{\operatorname{ht}(I\overline{R}_p)}(M)\right)\right)\overline{R} \subseteq \operatorname{Ann}_{\overline{R}} \left(H_{I\overline{R}}^{\operatorname{ht}(I\overline{R}_p)}(M\overline{R})\right)$$

and so

$$\operatorname{Ann}_R \left(H_I^{\operatorname{ht}(I\overline{R}_p)}(M)\right) \subseteq \operatorname{Ann}_R \left(H_{I\overline{R}}^{\operatorname{ht}(I\overline{R}_p)}(M\overline{R})\right).$$

Then

$$\dim R/J \geq \dim \overline{R}/\overline{J} \geq \dim \overline{R}/\operatorname{Ann}_{\overline{R}} H_{\overline{I}}^{\operatorname{ht}(I\overline{R}_p)}(\overline{M}).$$

Thus we may assume M is faithful, which implies $\operatorname{Ann}_R H_m^d(M) = U(R)$. The result now follows exactly from the proof for $M = R$. $\square$

Chapter 4

Generalizations

We have examined local cohomology modules over an arbitrary ring at a few specific indices. However, one can also consider the local cohomology modules from another viewpoint. In this chapter, we focus on local cohomology modules of varying indices over specific classes of rings.

4.1 Domains

Recall in Corollary 2.6 we showed the product of annihilators of local cohomology modules is zero for certain ideals. Over an integral domain this implies that at least one of the annihilators is zero. On the other hand, S. Goto gave an example in [Got84] that showed there exist Noetherian local domains (R, m) such that $m \subseteq \operatorname{Ann}_R H^i_m(R)$ for every $1 \leq i \leq \dim R - 1$ with equality for at least two i. Thus more hypotheses are required for a domain to say when these annihilators are zero or even what the dimension of these annihilators are. We therefore restrict our attention to that of a Cohen Macaulay domain in the results that follow.

For a complete local domain R of dimension d, the a corollary of the Hartshorne-

Lichtenbaum Vanishing Theorem (Corollary A.18) states $\sqrt{I} = m$ if and only $H^d_I(R) \neq 0$. In the case that $\sqrt{I} = m$, we know $H^i_I(R) = 0$ for all $i < d$ (that is, there is only one nonzero local cohomology module) and so

$$\operatorname{Ann}_R H^i_I(R) = \begin{cases} 0, & \text{if } i = d, \\ R, & \text{if } i < d. \end{cases}$$

Thus we are left with the case that $\sqrt{I} \neq m$, in which case $\operatorname{cd}(I) < d$.

For a domain R, we know $H^0_I(R) = 0$. For a Cohen Macaulay domain, we further have $\operatorname{Ann}_R H^1_I(R) = 0$ if and only if $H^1_I(R) \neq 0$ (by Proposition 3.7). We have the analogous statement for $\operatorname{Ann}_R H^2_I(R)$ if we add an additional hypothesis, but the proof requires a bit more work.

Lemma 4.1. *Let R be a Cohen Macaulay unique factorization domain and I an ideal of R. If $H^2_I(R) \neq 0$, then $\operatorname{Ann}_R H^2_I(R) = 0$.*

Proof. If $H^1_I(R) = 0$, then $\operatorname{ht}(I) = \operatorname{grade}(I) = 2$ and the result follows from Proposition 3.7. So assume $H^1_I(R) \neq 0$ (and hence $\operatorname{ht}(I) = 1$). Without loss of generality, assume $I = \sqrt{I}$. We claim $I = (f) \cap J$ where $\operatorname{ht}(J) \geq 2$ and $\operatorname{ht}(J, f) \geq \operatorname{ht} J + 1$. To see this, first express $I = \sqrt{I}$ as an (irredundant) intersection of primes:

$$I = p_1 \cap \cdots \cap p_\ell \cap q_1 \cap \cdots \cap q_s$$

where $\operatorname{ht}(p_i) = 1$ and $\operatorname{ht}(q_i) \geq 2$. As R is a unique factorization domain, each p_i is principal, say $p_i = (f_i)$. Then $p_1 \cap \cdots \cap p_\ell = (f)$ where $f := f_1 \cap \cdots \cap f_\ell$. If $f \in q_i$ for some i, then $p_j \subseteq q_i$ for some i, a contradiction. Let $J = q_1 \cap \cdots \cap q_s$. Then $\operatorname{ht}(J) \geq 2$ and $\operatorname{ht}(J, f) \geq \operatorname{ht} J + 1$.

The Mayer-Vietoris Sequence for local cohomology thus gives

$$0 \to H^2_J(R) \to H^2_I(R) \to H^3_{(J,f)}(R) \to H^3_J(R) \to H^3_I(R) \to H^4_{(J,f)}(R).$$

If $\operatorname{ht}(J) > 2$, then $\operatorname{ht}(J,f) > 3$ and so $H^2_J(R) = H^3_{(J,f)}(R) = 0$. Thus $H^2_I(R) = 0$, a contradiction. Therefore $\operatorname{ht}(J) = 2$. Then $\operatorname{Ann}_R H^2_I(R) \subseteq \operatorname{Ann}_R H^2_J(R) = 0$ by Proposition 3.7. $\square$

We combine this lemma with the results of our previous sections. This leads us to the following result.

Theorem 4.2. *For a complete Cohen Macaulay unique factorization domain R of dimension at most 4, $H^i_I(R) \neq 0$ if and only if $\operatorname{Ann}_R H^i_I(R) = 0$.*

Proof. If $\sqrt{I} = m$, we are done by Corollary 2.19. For ideals with $\sqrt{I} \neq m$, we have $c < d$ by Corollary A.18 since R is a complete domain. Note that we have already proved the following:

- $a_g = 0$ for $g := \operatorname{grade}(I)$
- $a_c = 0$ if $c := \operatorname{cd}(I) \in \{1, 2, d-1, d\}$

The grade case results from Proposition 3.7 (since $\operatorname{grade}(I) = \operatorname{ht}(I)$ in a Cohen Macaulay ring). The $\operatorname{cd}(I) = 2$ case follows from Lemma 4.1 and the $\operatorname{cd}(I) = 1, d-1$, and d cases are proved in Chapter 2.

These results cover every situation except for one: namely, R has dimension 4 and I is an ideal with $H^i_I(R) \neq 0$ for $i = 1, 2$, and 3. Let $a_i = \operatorname{Ann}_R H^i_I(R)$. Then $a_1 = a_3 = 0$ by the above cases and $a_2 = 0$ by Lemma 4.1. $\square$

From here, we can generalize Proposition 1.1 for rings of dimension at most 4:

Corollary 4.3. *For a regular local ring R of dimension at most 4, $H_I^i(R) \neq 0$ if and only if $\operatorname{Ann}_R H_I^i(R) = 0$.*

Proof. In the proof of Theorem 4.2, we used that R was complete to say $H_I^d(R) = 0$ if and only if $\sqrt{I} = m$. For an arbitrary regular local ring R, this fact remains true since $H_I^d(R) = 0$ if and only if $H_I^d(R) \otimes_R \hat{R} = 0$. Also $\sqrt{I} = m$ if and only if $\sqrt{I\hat{R}} = \hat{m}$. Since a regular local ring is a Cohen Macaulay unique factorization domain, the result follows from Theorem 4.2. □

Recall that Proposition 1.1 said that $H_I^i(R) \neq 0$ if and only if $\operatorname{Ann}_R H_I^i(R) = 0$ for a regular local ring R containing a field. Thus Corollary 4.3 provides evidence that $H_I^i(R) \neq 0$ if and only if $\operatorname{Ann}_R H_I^i(R) = 0$ for arbitrary regular local rings.

4.2 Continuing Research

To continue this study of annihilators of local cohomology modules, there are numerous directions one might take and questions to consider.

Question 1. *For a local ring of dimension d and an ideal I of cohomological dimension d with $\operatorname{pd}_R R/I < \infty$, must I be $m-$primary?*

Theorem 2.21 showed given the above conditions, that I $m-$primary implies $\operatorname{Ann}_R H_I^d(R) = 0$. We presented one situation in which this occurs (that is, we assume certain intersection conjectures), but it is unknown in general whether I is $m-$primary.

Question 2. *Under what hypotheses on R is $\dim\left(R/\operatorname{Ann}_R H_I^c(R)\right) = \dim R/H_I^0(R)$ where I is an ideal of R with $c := \operatorname{cd}(I)$.*

The first open case of Conjecture 2.2 occurs at $c = 2$. If R is a Cohen Macaulay unique factorization domain, the conjecture holds for $c = 2$, cf. Lemma 4.1. However, the conjecture remains open in general. I am currently working with notions such as the ideal transform to discover more information about this case.

Question 3. *For R a Cohen Macaulay ring, what is $\dim(R/\operatorname{Ann} H^i_I(R))$ when $\operatorname{ht}(I) < i \leq \operatorname{cd}(I)$?*

Corollary 3.7 showed this dimension to be d for $i = \operatorname{ht}(I)$. Conjecture 2.2 applied to a Cohen Macaulay ring and ideal I with $\operatorname{ht}(I) > 0$ leads us to believe the dimension is again d for $i = \operatorname{cd}(I)$. A natural question then is whether these annihilators have dimension d for $i > \operatorname{ht}(I)$.

Question 4. *For a complete Cohen Macaulay unique factorization domain R, is $\operatorname{Ann}_R H^i_I(R) = 0$ if and only if $H^i_I(R) \neq 0$?*

Theorem 4.2, which proved the question for rings of dimension at most four, followed from reduction to the case where the index was either the grade or the cohomological dimension and there the result was known from previous cases. We hope that we can again reduce to a known result when rings have higher dimension and thus answer this question.

Question 5. *Does Question 4 hold for a Cohen Macaulay normal domain?*

A natural generalization for Theorem 4.2 is to relax the restrictions on R to a Cohen Macaulay normal ring. Of course, even if we strengthen our ring to an arbitrary regular local ring, there is little known about these annihilators (for dimension at least five).

Question 6. *Can we utilize characteristic p techniques to determine more specific properties on the annihilators discussed above?*

In [HK91], Huneke and Koh used properties of characteristic p rings to establish results on the vanishing of local cohomology modules. It could prove useful to explore these and other techniques applied to the annihilators of local cohomology to identify further properties in the case of rings of prime characteristic.

Appendix A

Summary of Basic Results on Local Cohomology

In this appendix, we summarize topics of local cohomology that are used throughout this thesis and provide proofs for the lesser-known results. For a more in-depth treatment of the subject, we refer the reader to [BS98].

A.1 Definitions and basic results

Throughout this section, let R be a Noetherian ring, I an ideal of R, and M an $R-$module. In Chapter 1, we define the i^{th} *local cohomology module* of M with respect to I to be $\mathrm{H}^i_I(M) = \varinjlim_t \mathrm{Ext}^i_R(R/I^t, M)$. While this is a convenient definition, it is neither the only definition nor the standard definition given in an introduction to the subject.

A.1.1 Local Cohomology as a Right Derived Functor

Definition. Define $\Gamma_I(M)$ to be the additive left exact covariant functor

$$\Gamma_I(M) := \cup_{t\geq 1}^{\infty}(0 :_M I^t) = \{m \in M | I^t m = 0 \text{ for some } t\},$$

and if $f : M \to N$ then $\Gamma_I(f)$ is the restriction map $\Gamma_I(M) \to \Gamma_I(N)$. The i^{th} *local cohomology* of M with respect to I is $H_I^i(M) := R^i\Gamma_I(M)$, where R^iF is the right derived functor of a given covariant left exact functor F.

Using this definition, we can quickly deduce the following.

Proposition A.1. *Let R be Noetherian. Then for any ideal I of R we have $\Gamma_I = \Gamma_{\sqrt{I}}$. In particular, $H_I^i(M) = H_{\sqrt{I}}^i(M)$ for all $i \geq 0$ and for all $R-$modules M.*

Proof. Since R is Noetherian, $\sqrt{I}$ is finitely generated. Thus there exists t such that $(\sqrt{I})^t \subseteq I$. Let $x \in \Gamma_{\sqrt{I}}(M)$. Then there exists k such that $(\sqrt{I})^k x = 0$, which implies $I^k x \subseteq (\sqrt{I})^k x = 0$. Therefore $x \in \Gamma_I(M)$.

Let $x \in \Gamma_I(M)$. Then there exists k such that $I^k x = 0$. Since $(\sqrt{I})^t \subseteq I$, we see $(\sqrt{I})^{kt} \subseteq I^k$ and so $(\sqrt{I})^{kt} x = 0$. Therefore $x \in \Gamma_{\sqrt{I}}(M)$. □

Let $\{I_t\}, \{J_t\}$ be two decreasing chains of ideals. We say the chains are *cofinal* if for all t there exists k such that $J_k \subseteq I_t$, and for all s there exists ℓ such that $I_\ell \subseteq J_s$. Using our original definition for local cohomology, we obtain the following.

Proposition A.2. *If $\{I_t\}$ is a descending chain of ideals cofinal with $\{I^t\}$ then $H_I^i(M) = \varinjlim_t \operatorname{Ext}_R^i(R/I_t, M)$.*

Proof. See [I+07, Remark 7.9]. □

A.1.2 Local Cohomology in terms of Čech Cohomology

In practice, computing local cohomology modules is a difficult task. A useful way to compute local cohomology, which we have used in many of the examples presented throughout this work, is by using the Čech complex.

Definition. Let $\underline{x} = x_1, ..., x_t \in R$. The *Čech complex* on R with respect to $x_1, ..., x_t$ is given by

$$\begin{aligned} C^{\cdot}(x_1; R) &:= 0 \to R \to R_{x_1} \to 0 \text{ where } r \mapsto \tfrac{r}{1} \\ C^{\cdot}(x_1, ..., x_t; R) &:= C^{\cdot}(x_1, ..., x_{t-1}; R) \otimes_R C^{\cdot}(x_t; R) \\ &= \otimes_{i=1}^{t} C^{\cdot}(x_i; R). \end{aligned}$$

For an $R-$module M, let $C^{\cdot}(\underline{x}; M) := C^{\cdot}(\underline{x}; R) \otimes_R M$. The i^{th} *Čech cohomology* of M is $H^i_{\underline{x}}(M) := H^i(C^{\cdot}(\underline{x}; M))$.

It turns out that local cohomology and Čech cohomology are naturally isomorphic, provided the ring is Noetherian.

Proposition A.3. *Assume R is Noetherian and $\underline{x}$ and M are as above. Then $H^i_{\underline{x}}(M) = H^i_{(\underline{x})}(M)$, that is, the Čech cohomology and local cohomology for M are naturally isomorphic.*

Proof. See [BS98, Theorem 5.1.19]. □

One particularly useful outcome of this natural isomorphism is the following.

Proposition A.4. *Let R be Noetherian, M be an $R-$module, and $I = (x_1, ..., x_t)$ an ideal. Then $H^t_I(M) \cong M_{x_1 \cdots x_t} / \sum_{i=1}^{t} M_{x_1 \cdots \hat{x}_i \cdots x_t}$.*

It also follows from Proposition A.3 that local cohomology vanishes above a certain invariant.

Definition. Let I be an ideal in a Noetherian ring R. The *arithmetic rank* of I is

$$\operatorname{ara}(I) = \inf\{t|\text{ there exists } x_1, ..., x_t \in R \text{ with } \sqrt{(x_1, ..., x_t)} = \sqrt{I}\}.$$

We present this invariant only to use the proposition below, thus refer the reader to [I+07, Lecture 9] for more properties of the arithmetic rank.

Proposition A.5. *For R Noetherian and I an ideal of R, we have $H^i_I(R) = 0$ for all $i > \operatorname{ara}(I)$.*

Corollary A.6. *For R a Noetherian local ring and I an ideal of R, we have*

$$\operatorname{grade}(I) \leq \operatorname{cd}(I) \leq \dim(R).$$

Proof. The first inequality is immediate from the definitions of grade and cohomological dimension. The second inequality follows from the fact that every ideal in a local ring can be generated up to radical by $\dim R$ elements (that is, $\operatorname{ara}(I) \leq d$) and that, by definition of cohomological dimension, $\operatorname{cd}(I) \leq \operatorname{ara}(I)$. $\square$

A.2 Long Exact Sequences

There are three standard long exact sequences on local cohomology that we use repeatedly throughout this dissertation.

Theorem A.7. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of $R-$modules. Then there exists a natural long exact sequence*

$$\begin{aligned} 0 &\to H^0_I(L) \to H^0_I(M) \to H^0_I(M) \to \cdots \\ &\to H^i_I(L) \to H^i_I(M) \to H^i_I(N) \to \cdots \end{aligned}$$

Proof. See [BS98, 1.2.2(iv)]. □

There are several results that come from this long exact sequence.

Corollary A.8. *For $i > 0$, we have $H_I^0(R) \subseteq \operatorname{Ann}_R H_I^i(R)$.*

Proof. Note that $0 \to H_I^0(R) \to R \to R/H_I^0(R) \to 0$ is exact and that, by definition, $H_I^0(R) = H_I^0(H_I^0(R))$. Thus applying the above long exact sequence gives us that $H_I^i(R/H_I^0(R)) \cong H_I^i(R)$ for all $i > 0$. □

Corollary A.9. *Let (R, m) be Noetherian, M an $R-$module, and $I \subseteq R$ an ideal with $c := \operatorname{cd}(I)$. Then $H_I^i(M) = 0$ for all $i > c$.*

Proof. Define $s := \sup\{n \in \mathbb{Z} | H_I^n(M) \neq 0$ for some $R-$module $M\}$. Then for $i > s$ we see $H_I^i(M) = 0$ for all $R-$modules M. So it is enough to show $s = \operatorname{cd}(I)$. Choose M such that $H_I^s(M) \neq 0$ and choose a surjective homomorphisms $\phi : F \twoheadrightarrow M$ where F is a free $R-$module. Let $K = \ker \phi$. Then $0 \to K \to F \to M \to 0$ is exact and we can apply the long exact sequence on local cohomology:

$$H_I^s(K) \to H_I^s(F) \to H_I^s(M) \to H_I^{s+1}(K) = 0$$

Note $H_I^{s+1}(K) = 0$ by how we defined s. This implies $H_I^s(F) \neq 0$ (else $H_I^s(M) = 0$). Since local cohomology and direct sums commute (this follows from the analogous statement for Čech Cohomology), we must have $H_I^s(R) \neq 0$. Since $H_I^i(R) = 0$ for all $i > s$, we see $s = \operatorname{cd}(I)$. □

Corollary A.10. *Let (R, m) be Noetherian, I an ideal of R and $c = \operatorname{cd}(I)$. Then $H_I^c(-)$ is right exact.*

Proof. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of $R-$modules. Let $K = \ker\beta$ and consider the following commutative diagram where the top row is exact:

$$\begin{tikzcd} K \arrow[r, "\gamma"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\ A \arrow[u, "\alpha'"] \arrow[ur, "\alpha"'] & & & \end{tikzcd}$$

Note that α' is surjective. Applying the long exact sequence on local cohomology and using the previous corollary, we get the exact sequence

$$H^c_I(K) \xrightarrow{\overline{\gamma}} H^c_I(B) \xrightarrow{\overline{\beta}} H^c_I(C) \to H^{c+1}_I(K) = 0$$

By functorial properties, this gives us the following commutative diagram

$$\begin{tikzcd} H^c_I(K) \arrow[r, "\overline{\gamma}"] & H^c_I(B) \arrow[r, "\overline{\beta}"] & H^c_I(C) \arrow[r] & 0 \\ H^c_I(A) \arrow[u, "\overline{\alpha'}"] \arrow[ur, "\overline{\alpha}"'] & & & \end{tikzcd}$$

where now $\overline{\alpha'}$ is onto (since α' was onto, apply the long exact sequence on local cohomology to the sequence $0 \to \ker(\alpha') \to A \xrightarrow{\alpha'} K \to 0$ and use the fact that $H^{c+1}_I(\ker(\alpha')) = 0$). Thus, in order to show $H^c_I(-)$ is right exact, it is enough to show $\operatorname{im}\overline{\alpha} = \ker\overline{\beta}$. This follows from a minimal amount of diagram chasing using the facts that $\overline{\alpha} = \overline{\gamma}\overline{\alpha'}$, that the top row is exact, and that $\overline{\alpha'}$ is onto. $\square$

Corollary A.11. *Let* $c = \operatorname{cd}(I)$. *Then* $H^c_I(R) \otimes_R M = H^c_I(M)$.

Proof. First note that the result holds for free modules since local cohomology commutes with direct sums. Now find a presentation (not necessarily finite) for M :

$$F \to G \to M \to 0.$$

By right exactness, we may apply $- \otimes_R H_I^c(R)$ to obtain

$$F \otimes_R H_I^c(R) \to G \otimes_R H_I^c(R) \to M \otimes_R H_I^c(R) \to 0$$

or apply $H_I^c(-)$ to obtain

$$H_I^c(F) \to H_I^c(G) \to H_I^c(M) \to 0.$$

These two sequences give us the following diagram

$$\begin{tikzcd}
F \otimes_R H_I^c(R) \arrow[r] \arrow[d, "\cong"] & G \otimes_R H_I^c(R) \arrow[r] \arrow[d, "\cong"] & M \otimes_R H_I^c(R) \arrow[r] & 0 \arrow[d, "\cong"] \\
H_I^c(F) \arrow[r] & H_I^c(G) \arrow[r] & H_I^c(M) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, we obtain the desired result. □

The other two long exact sequences given below are also frequently used in studying specific examples in local cohomology.

Theorem A.12 (Mayer-Vietoris sequence). *Let R be a Noetherian ring, $I, J \subseteq R$, M an $R-$module. Then there exists a natural long exact sequence*

$$\begin{aligned} 0 \ &\to H^0_{I+J}(M) \to H^0_I(M) \oplus H^0_J(M) \to H^0_{I\cap J}(M) \to \cdots \\ &\to H^i_{I+J}(M) \to H^i_I(M) \oplus H^i_J(M) \to H^i_{I\cap J}(M) \to \cdots . \end{aligned}$$

Proof. See [BS98, 3.2.3]. □

Lemma A.13. *Let R be a Noetherian ring, I an ideal, $x \in R$, and M an $R-$module. Then there exists a long exact sequence*

$$\cdots \to H^i_{(I,x)}(M) \to H^i_I(M) \to H^i_{I_x}(M_x) \to H^{i+1}_{(I,x)}(M) \to \cdots .$$

Proof. See [BS98, Proposition 8.1.2]. □

A.3 Other Results

We saw in the previous section that local cohomology at the cohomological dimension commutes with tensor products. This commutative property also holds in other contexts, such as the following.

Proposition A.14. *Let S be a flat $R-$algebra with R, S Noetherian. Let I be an ideal of R and M an $R-$module. Then $H^i_I(M) \otimes_R S \cong H^i_{IS}(M \otimes_R S)$ for all $i \geq 0$.*

Proof. See [BS98, Theorem 4.3.2]. □

In particular, local cohomology commutes with $- \otimes_R \hat{R}$.

Theorem A.15 (**Change of Rings Principle**)**.** *Let S be an $R-$algebra, where R and S are Noetherian. Let I be an ideal of R and M an $S-$module. Then for all i $H^i_I(M) \cong H^i_{IS}(M)$ where we consider M as an $R-$module on the left hand side and as an $S-$module on the right hand side.*

Proof. See [BS98, Theorem 4.2.1]. □

Theorem A.16 (**Local Duality**)**.** *Let (R, m) be a complete Cohen Macaulay local ring of dimension d. Then for all finitely generated $R-$modules M,*

$$\operatorname{Ext}^{d-i}_R(M, \omega_R) \cong H^i_m(M)^\vee \quad \textit{and} \quad \operatorname{Ext}^{d-i}_R(M, \omega_R)^\vee \cong H^i_m(M)$$

for all i where $(-)^\vee = \operatorname{Hom}_R(-, E_R(R/m))$.

Proof. See [BS98, Theorem 11.2.8]. □

Theorem A.17 (**Hartshorne-Lichtenbaum Vanishing Theorem**). *Let (R, m) be a local ring of dimension d and I an ideal of R. The following are equivalent*

1. $H_I^d(R) = 0$
2. $\dim \hat{R}/(I\hat{R} + p) > 0$ *for all* $p \in \operatorname{Spec} \hat{R}$ *such that* $\dim \hat{R}/p = d$.

Proof. See [BS98, Theorem 8.2.1]. □

Corollary A.18. *Let (R, m) be a complete local domain of dimension d and I an ideal of R. Then $H_I^d(R) \neq 0$ if and only if $\sqrt{I} = m$.*

Theorem A.19. *Let (R, m) be a local ring, I an ideal of R and M a finite $R-$module of dimension t. Then $H_I^t(M)$ is Artinian.*

Proof. Since $H_I^t(M)$ is Artinian if and only if $H_I^t(M) \otimes_R \hat{R} \cong H_{I\hat{R}}^t(\hat{M})$ is Artinian, it suffices to prove the theorem in the case that R is a complete local ring. We may certainly suppose $H_I^t(M) \neq 0$. Then by the Hartshorne-Lichtenbaum Vanishing Theorem (A.17), there exists p such that $\dim R/p = d$ and $\sqrt{I + p} = m$. Let $\{p_1, ..., p_s\}$ be the set of such primes. Then $\sqrt{I + p_1 \cap \cdots \cap p_s} = m$. Consider the Mayer-Vietoris Sequence:

$$\cdots \to H^d_{I+p_1\cap\cdots\cap p_s}(R) \to H_I^d(R) \oplus H^d_{p_1\cap\cdots\cap p_s}(R) \to H^d_{I\cap p_1\cap\cdots\cap p_s} \to 0.$$

If we can show the last term is zero, then the result will hold since we know that $H^d_{I+p_1\cap\cdots\cap p_s}(R) = H^d_m(R)$ is Artinian. So suppose, by way of contradiction, that $H^d_{I\cap p_1\cap\cdots\cap p_s} \neq 0$. Then we can again apply the Hartshorne-Lichtenbaum Vanishing

Theorem (A.17) to find a prime ideal q with $\dim R/q = d$ and $\sqrt{q + I \cap p_1 \cap \cdots \cap p_s} = m$. Then $\sqrt{q+I} = m$, which implies $q = p_j$ for some j and so

$$\sqrt{q} = \sqrt{q + I \cap p_1 \cap \cdots \cap p_s} = m,$$

a contradiction. $\square$

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