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# Systems of parameters and the Cohen-Macaulay property

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# SYSTEMS OF PARAMETERS AND THE COHEN-MACAULAY PROPERTY

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Let $R$ be a commutative, Noetherian, local ring and $M$ a finitely generated $R$-module. Consider the module of homomorphisms $\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ where $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of $M$. When $M = R$ and $R$ is Cohen-Macaulay, Rees showed that this module of homomorphisms is isomorphic to $R/\mathfrak{a}$, and in particular, a free module over $R/\mathfrak{a}$ of rank one. In this work, we study the structure of such modules of homomorphisms for a not necessarily Cohen-Macaulay $R$-module $M$.

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# Contents

# Chapter 1

# Introduction

This thesis makes a contribution to the study of systems of parameters of local rings and modules over local rings. The module of study is well understood in the Cohen-Macaulay case, and so our focus is on the non-Cohen-Macaulay case. In this thesis *rings* are assumed to be commutative. Throughout, a *local ring* is a Noetherian ring with a unique maximal ideal. The notation $(R, \mathfrak{m})$ indicates that $R$ is a local ring whose maximal ideal is $\mathfrak{m}$.

In this chapter, we provide background material. The notion of a Cohen-Macaulay module centers on two key invariants of the module: the dimension and the depth. In Section 1.1 we will focus on the dimension of a module and its systems of parameters. In Section 1.2 we will define the depth of a module and related terms. In Section 1.3 we will define the $I$-torsion functor and state some basic properties of the functor. In Section 1.4 we will define Cohen-Macaulay modules and indicate some relationships among the notions defined in the first two sections when a module is Cohen-Macaulay. In Section 1.5 we discuss the basics of free modules. In Section 1.6 we will discuss the historical background necessary for understanding where the main results fit into the larger picture. Finally, in Section 1.7 we will state the main results.

## 1.1 Dimension and Systems of Parameters

A *prime ideal* of a ring $R$ is a proper ideal $\mathfrak{p}$ such that $R/\mathfrak{p}$ is an integral domain. An equivalent characterization is that $\mathfrak{p}$ is prime if whenever $a, b \in R$ with $ab \in \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. A chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

is said to have *length* $n$. The supremum of the lengths of chains of primes in a ring $R$ is called the *dimension of* $R$, written $\dim R$.

**Example 1.1.** Consider the ring $R = k[\![x,y]\!]/(x^2, xy^m)$ for $m \geq 1$. Since $R/(x) \cong k[\![y]\!]$ and $R/(x,y) \cong k$ are integral domains and $(x) \subsetneq (x,y)$, we know that the dimension of $R$ is at least one. To show that the dimension is at most one, we will use the notion of a system of parameters.

For a ring $R$, recall that the *radical* of an ideal $I \subseteq R$ is the ideal

$$\sqrt{I} := \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

If $\sqrt{I} = \mathfrak{m}$ is a maximal ideal, we say that $I$ is $\mathfrak{m}$-*primary.*

Let $(R, \mathfrak{m})$ be a local ring of dimension $d$. There exist sets of $d$ elements which generate $\mathfrak{m}$-primary ideals, but no ideal generated by fewer than $d$ elements is $\mathfrak{m}$-primary. For a proof of this fact, see [7, Theorem 13.4]. By Krull's Height Theorem [7, Theorem 13.5], the height of an ideal generated by $r$ elements is no more than $r$, so no ideal generated by fewer than $d$ elements is $\mathfrak{m}$-primary. A set of $d$ elements generating an $\mathfrak{m}$-primary ideal is called a *system of parameters.* An ideal generated by a system of parameters is called a *parameter ideal.* We sometimes refer to an element of a system of parameters as a *parameter*.

**Example 1.2.** With $R = k[\![x,y]\!]/(x^2, xy^m)$ as in Example 1.1, consider the ideal $(y)$. Since $x^2 = 0 \in (y)$, both $x$ and $y$ are in $\sqrt{(y)}$ so that $\mathfrak{m} = (x,y) \subseteq \sqrt{(y)}$. Thus $\mathfrak{m} = \sqrt{(y)}$, and $(y)$ is an $\mathfrak{m}$-primary ideal. In Example 1.1 we saw $\dim R \geq 1$. Thus $R$ has dimension one and $y$ is a parameter of $R$.

We now extend the notions of dimension and system of parameters to modules. Let $R$ be a ring and $M$ an $R$-module. The *dimension* of $M$, denoted $\dim_R M$, can be defined to be the dimension of the ring $R/\operatorname{ann}_R(M)$ where

$$\operatorname{ann}_R(M) := \{r \in R \mid rM = 0\}$$

is the *annihilator* of $M$. Since $\operatorname{ann}_R(R) = (0)$ we recover the original definition when the module is the ring itself. More generally, the dimension of the $R$-module $R/I$ is equal to the dimension of $R/I$ as a ring.

**Example 1.3.** With $R = k[\![x,y]\!]/(x^2, xy^m)$ as in Example 1.1 and Example 1.2, consider the $R$-module $M = y^m R$. For $r \in R$, we know $ry^m = 0$ only if $r \in (x)$. As $y^m \in M$ this tells us that $\operatorname{ann}_R(M) \subseteq (x)$. Since $xM = xy^m R = 0$, we see that $\operatorname{ann}_R(M) = \{r \in R \mid r(y^m R) = 0\} = (x)$. We can now compute the dimension of $M$. Since $R/(x) \cong k[\![y]\!]$, we obtain

$$\dim_R M = \dim(R/\operatorname{ann}_R(M)) = \dim(R/(x)) = \dim(k[\![y]\!]) = 1.$$

Next we introduce systems of parameters for modules. Let $R$ be a local ring and $M$ an $R$-module of dimension $d$. There exist sets of $d$ elements $a_1, \ldots, a_d \in R$ such that $M/(a_1, \ldots, a_d)M$ has finite length, that is, such that $M/(a_1, \ldots, a_d)M$ is both an Artinian and a Noetherian $R$-module. However for any $a_1, \dots, a_n \in R$ with $0 \le n < d$, the module $M/(a_1, \dots, a_n)M$ has infinite length. For a proof of this fact, see [7, Theorem 13.4].

A set of $d$ elements $a_1, \dots, a_d \in R$ with the property that $M/(a_1, \dots, a_d)M$ has finite length is called a *system of parameters of* $M$. As in the ring case, an ideal generated by a system of parameters is called a *parameter ideal of* $M$, and we sometimes refer to an element of a system of parameters as a *parameter*. The next proposition says that this definition agrees with the earlier one when $M = R$.

**Proposition 1.4.** *Let* $(R, \mathfrak{m})$ *be a local ring and* $I$ *any ideal of* $R$. *The* $R$*-module* $R/I$ *has finite length if and only if the ideal* $I$ *is* $\mathfrak{m}$*-primary.*

*Proof.* Recall [1, Proposition 1.14]: The radical of an ideal is the intersection of the prime ideals that contain it. Also, if $A \to B$ is a homomorphism of rings and $L$ is a $B$-module, then the length of $L$ as an $A$-module is the same as its length as a $B$-module. Indeed, the $A$-submodules of $L$ are precisely the $B$-submodules of $L$.

As $R$ is Noetherian, so is $R/I$. Thus $R/I$ has finite length if and only if it is Artinian. Since $R$ is Noetherian, we know $R/I$ is Artinian if and only if $R/I$ has dimension zero [1, Theorem 8.5]. Moreover, the ring $R/I$ has dimension zero if and only if the maximal ideal of $R$ is the only prime ideal which contains $I$, which is equivalent to $\mathfrak{m} = \sqrt{I}$ by [1, Proposition 1.14]. $\square$

**Example 1.5.** With $R = k[\![x, y]\!]/(x^2, xy^m)$ and $M = y^m R$ as in Example 1.3, we claim that $\{y\}$ is a system of parameters for $M$, and moreover, $M/yM = (y^m)/(y^{m+1})$ has length one. Indeed, the map

$$R/(x, y) \to (y^m)/(y^{m+1})$$

given by $1 + (x, y) \mapsto y^m + (y^{m+1})$ is $R$-linear and bijective. Thus $M/yM$ is a simple $R$-module, $M/yM$ has length one, and $y$ is a parameter of $M$.

The next proposition is useful for the induction technique used in the proofs of the main results.

**Proposition 1.6.** *Let $R$ be a commutative, local, Noetherian ring and $M$ an $R$-module. If $\{a_1, \ldots, a_i\}$ is part of a system of parameters of $M$, then*

$$\dim_R(M/(a_1, \ldots, a_i)M) = \dim_R(M) - i.$$

*Proof.* Set $\overline{M} = M/a_1M$, $d = \dim_R(M)$, and $\delta = \dim_R(\overline{M})$. We proceed by induction on $i$. Since $a_1$ is a parameter of $M$, we can find elements $x_2, \ldots, x_d \in R$ such that $M/(a_1, x_2, \ldots, x_d)M$ has finite length. Note that

$$\overline{M}/(x_2, \ldots, x_d)\overline{M} \cong M/(a_1, x_2, \ldots, x_d)M,$$

which has finite length. Hence $\dim_R(\overline{M}) \leq \dim_R(M) - 1$. For the other inequality, let $b_2, \ldots, b_\delta$ be a system of parameters of $\overline{M}$. Then

$$M/(a_1, b_2, \ldots, b_\delta)M \cong \overline{M}/(b_2, \ldots, b_\delta)\overline{M}$$

has finite length, and it follows that $\dim_R(M) \leq \dim_R(\overline{M}) + 1$. This completes the proof for the case $i = 1$.

For $i > 1$, note that $\{a_2, \ldots, a_i\}$ is part of a system of parameters of $\overline{M}$. The induction hypothesis then gives

$$\begin{aligned}\dim_R(\overline{M}/(a_2,\ldots,a_i)\overline{M}) &= \dim_R(\overline{M}) - (i-1)\\ &= \dim_R(M) - 1 - (i-1)\\ &= \dim_R(M) - i.\end{aligned}$$

This is the desired result, since $\overline{M}/(a_2,\ldots,a_i)\overline{M} \cong M/(a_1,\ldots,a_i)M$. $\square$

## 1.2 Depth

Depth is another important invariant of a ring and its modules. Roughly speaking, depth measures how many independent elements of the ring behave like indeterminates. An element of a ring $R$ is called a *regular element* on a module $M$ if it is not a zero-divisor on $M$. That is, $r \in R$ is a regular element on $M$ if $rm \neq 0$ for any nonzero $m \in M$.

A sequence $r_1,\ldots,r_n \in R$ is called a *weak regular sequence on $M$* if $r_i$ is a regular element on $M/(r_1,\ldots,r_{i-1})M$ for all $i = 1,\ldots,n$. We take the ideal generated by the empty set to be the zero ideal, so the condition for $i = 1$ is that $r_1$ is a regular element on $M/(0)M = M$. A sequence $r_1,\ldots,r_n \in R$ is a *regular sequence on $M$* if it is a weak regular sequence on $M$ and, in addition, $M/(r_1,\ldots,r_n)M \neq 0$. A *maximal regular sequence* is a regular sequence that cannot be extended to a longer one.

All maximal regular sequences which are contained in a fixed ideal have the same length. See [7, Theorem 16.7] for a proof of this fact. For a local ring $(R,\mathfrak{m})$, the *depth* of an $R$-module $M$ is defined to be the length of any maximal regular sequence contained in $\mathfrak{m}$.

**Example 1.7.** With $R = k[\![x,y]\!]/(x^2, xy^m)$ and $M = y^m R$, as in Example 1.3 and Example 1.5, we claim that $R$ has depth zero and $M$ has depth one.

First suppose $p \in R$ is a non-unit. Then $p = ax + by$ for some $a, b \in R$ and $pxy^{m-1} = ax^2y^{m-1} + bxy^m = 0$ in $R$. Thus the only regular elements of $R$ are units; however, if $p \in R$ is a unit, then $R/(p) = (0)$. Hence, there are no regular sequences on $R$ of length one and $R$ has depth zero.

Next note that the element $y \in R$ is a regular element on $M$. Moreover, $M/yM = (y^m)/(y^{m+1}) \neq 0$, and so $y$ is a regular sequence on $M$. We claim that $y$ is a maximal regular sequence. To see this directly, let $r \in R$. If $r$ is a non-unit we may write $r = ax + by$ for some $a, b \in R$. Then $ry^m = axy^m + by^{m+1} \in (y^{m+1}) = yM$. Thus $r$ is not a regular element on $(y^m)/(y^{m+1}) = M/yM$. If $r \in R$ is a unit, then $(r, y) = R$, so $M/(r, y)M = (0)$. Thus $y$ is a maximal length regular sequence on $M$ and $M$ has depth one.

## 1.3 The $I$-torsion Functor

Let $R$ be a ring, $I \subseteq R$ an ideal, $M$ an $R$-module, and $N \subseteq M$ a submodule. We define $(N :_M I)$ to be the set

$$(N :_M I) := \{m \in M \mid mI \subseteq N\}.$$

This is a submodule of $M$. When $M$ is clear from context, we sometimes write $(N : I)$ instead of $(N :_M I)$. When $I = (a)$ is a principal ideal, we typically write $(N :_M a)$ instead of $(N :_M (a))$.

Let $R$ be a commutative Noetherian ring, $I \subseteq R$ an ideal, and $M$ an $R$-module.

The submodule $\Gamma_I(M)$ of $M$ is defined as

$$\Gamma_I(M) := \bigcup_{i=1}^{\infty} (0 :_M I^i).$$

For any $R$-linear map $f : M \to N$, we define $\Gamma_I(f) : \Gamma_I(M) \to \Gamma_I(N)$ to be the map induced by $f$. This makes $\Gamma_I(-)$ a covariant functor from the category of $R$-modules to itself.

**Example 1.8.** Consider the ring $R = k[\![x, y]\!]/(x^2, xy^m)$ and module $M = y^m R$ from Example 1.7. The ring $R$ is local with maximal ideal $\mathfrak{m} = (x, y)$. For $i \geq m$ it is clear that $\mathfrak{m}^i = (y^i)$ since all other degree $i$ monomials in $x$ and $y$ are zero. It is also easy to see that $(0 :_R \mathfrak{m}^i) = (0 :_R y^i) = (x)$ for $i \geq m$ and $(0 :_M \mathfrak{m}^i) = (0 :_{y^m R} y^i) = (0)$ for $i \geq m$. Thus $\Gamma_{\mathfrak{m}}(R) = (x)$ and $\Gamma_{\mathfrak{m}}(M) = (0)$.

It is straightforward to verify the following properties of the $I$-torsion functor.

**Proposition 1.9.** *Let $R$ be a Noetherian ring, $I \subseteq R$ an ideal, and $M$ a finitely generated $R$-module.*

*(a) The $I$-torsion functor, $\Gamma_I(-)$, is left exact; [5, Exercise 7.2].*

*(b) If $J$ is an ideal of $R$ with $\sqrt{J} = \sqrt{I}$, then $\Gamma_J(M) \cong \Gamma_I(M)$; [5, Proposition 7.3].*

*Now suppose $R$ is a local ring with maximal ideal $\mathfrak{m}$.*

*(c) $M$ has depth zero if and only if $\Gamma_{\mathfrak{m}}(M) \neq 0$; [5, Remark 9.4].*

*(d) $\Gamma_{\mathfrak{m}}(M)$ is Artinian; [5, Exercise 7.7].*

## 1.4 Cohen-Macaulay Modules

Let $R$ be a ring. If $M$ is a nonzero $R$-module, then there is an inequality:

$$\mathrm{depth}_R(M) \leq \dim_R(M);$$

see [3, Proposition 1.2.12]. In the extremal case when the depth and dimension are equal, or when $M = 0$, we say that $M$ is *Cohen-Macaulay*. We say a local ring $R$ is *Cohen-Macaulay* if it is Cohen-Macaulay as a module over itself.

**Example 1.10.** In Example 1.3, Example 1.5 and Example 1.7, we saw that $R = k[\![x,y]\!]/(x^2, xy^m)$ has dimension one but depth zero and $M = y^m R$ has dimension and depth both equal to one. Thus $R$ is not a Cohen-Macaulay ring, but $M$ is a Cohen-Macaulay module.

**Example 1.11.** Consider the ring $R = k[\![x,y]\!]/(x^2)$. It is easy to see that $(x) \subsetneq (x,y)$ is a chain of prime ideals of $R$ and $\sqrt{(y)} = (x,y) = \mathfrak{m}$. Thus $R$ has dimension one and $\{y\}$ is a system of parameters of $R$. To see that $R$ has depth one, we need only produce a regular element of $R$ which is not a unit, since the depth cannot exceed the dimension. It is clear that $y$ is such a regular element and $R$ is a Cohen-Macaulay ring.

In the ring $R = k[\![x,y]\!]/(x^2, xy^m)$, the parameter $y$ is not a regular element as $R$ has depth zero. However, in the ring $R = k[\![x,y]\!]/(x^2)$, the parameter $y$ forms a regular sequence on $R$. More generally, for any Cohen-Macaulay module $M$, the elements $a_1, \ldots, a_i \in R$ are part of a system of parameters of $M$ if and only if $a_1, \ldots, a_i$ is a regular sequence on $M$ [3, Theorem 2.1.2(d)].

## 1.5 Free Modules

Let $R$ be a ring. An $R$-module $F$ is said to be *free* if there is a linearly independent set of generators of $F$. That is, if there is a set of generators, $\{x_\alpha\}_{\alpha \in A}$, that satisfies the property: $\sum_{\alpha \in A} r_\alpha x_\alpha = 0$ if and only if $r_\alpha = 0$ for all $\alpha \in A$. We say a set of linearly independent generators is a *basis* of the free module $F$. If $F$ is a nonzero free $R$-module, then $\operatorname{ann}_R(F) = (0)$. Indeed, let $\{x_\alpha\}_{\alpha \in A}$ be a basis of the free module $F$. If $r \in R$ and $rF = 0$, then in particular $rx_\alpha = 0$ for all $\alpha \in A$. However, by linear independence of the basis, we know that $rx_\alpha = 0$ implies that $r = 0$.

Consider a commutative ring $R$, an ideal $I$, and an $R/I$-module $M$. Note that $M$ is also an $R$-module via the action $r \cdot m := (r+I)m$ for any $r \in R$ and $m \in M$. Clearly $I \subseteq \operatorname{ann}_R(M)$. If $M$ is $R/I$-free then $a \in \operatorname{ann}_R(M)$ also implies $a + I \in \operatorname{ann}_{R/I}(M)$. Equivalently, if $M$ is a free $R/I$-module and $a \in \operatorname{ann}_R M$, then $a + I = 0$ in $R/I$, i.e. $a \in I$. Thus for any $R/I$-module $M$ we know $\operatorname{ann}_R(M) \supseteq I$, and we have equality when $M$ is free.

However, there are non-free $R/I$-modules $M$ such that $\operatorname{ann}_R(M) = I$. For example, for any ring $R$ and proper ideals $I \subsetneq J$ of $R$. The $R/I$-module $M = R/I \oplus R/J$ is not free, but $\operatorname{ann}_R(M) = I$.

## 1.6 Rees' Theorem

The purpose of this section is to provide historical context in order to see how the main results fit into the larger picture. We begin by recalling Rees' Theorem [9, Theorem 2.1], as reformulated in the book by Bruns and Herzog [3, Lemma 1.2.4].

**Theorem 1.12.** *Let $R$ be a ring, $M, N$ be $R$-modules, and $x_1, \ldots, x_n$ elements of*

$\mathrm{ann}_R(N)$ *which form a weak regular sequence on* $M$*. Then*

$$\mathrm{Ext}_R^n(N, M) \cong \mathrm{Hom}_R(N, M/(x_1, \ldots, x_n)M).$$

The following proposition is a direct consequence of Rees' Theorem.

**Proposition 1.13.** *Let* $R$ *be a local ring,* $M$ *a Cohen-Macaulay* $R$*-module of dimension* $d$ *and* $\mathfrak{b} \subseteq \mathfrak{a}$ *parameter ideals of* $M$*. Then*

$$\mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M) \cong M/\mathfrak{a}M.$$

*Proof.* Let $\mathfrak{a}$ and $\mathfrak{b}$ be generated by the systems of parameters $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$ respectively. Since $M$ is Cohen-Macaulay, both $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ are regular sequences on $M$ [3, Theorem 2.1.2(d)]. Applying Theorem 1.12 to $b_1, \ldots, b_d$ and then to $a_1, \ldots, a_d$ yields the following isomorphisms:

$$\begin{aligned} \mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M) &\cong \mathrm{Ext}_R^d(R/\mathfrak{a}, M) \\ &\cong \mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{a}M) \\ &\cong \mathrm{Hom}_{R/\mathfrak{a}}(R/\mathfrak{a}, M/\mathfrak{a}M) \\ &\cong M/\mathfrak{a}M. \end{aligned}$$

This is the desired result. □

When $M = R$ is Cohen-Macaulay and $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals of $R$, Proposition 1.13 says that we have the following isomorphism of $R/\mathfrak{a}$-modules:

$$\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b}) \cong R/\mathfrak{a}.$$

That is, the module of homomorphisms is a free $R/\mathfrak{a}$-module of rank one, and hence is indecomposable. Recently, K. Bahmanpour and R. Naghipour [2, Theorem 2.4] proved the following converse of this statement.

**Proposition 1.14.** *If $R$ is not Cohen-Macaulay there exist parameter ideals $\mathfrak{b} \subseteq \mathfrak{a}$ such that the $R/\mathfrak{a}$-modules $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ and $R/\mathfrak{a}$ are not isomorphic.*

The focus of this work is to study the structure of the module of homomorphisms $\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ when $M$ is not Cohen-Macaulay and $\mathfrak{b} \subseteq \mathfrak{a}$ are parameter ideals. We focus on showing conditions under which $\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ is decomposable and conditions that imply $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is not a free $R/\mathfrak{a}$-module.

## 1.7 Main Results

In this section we state the main results and compare them to each other. This first result is only for modules of dimension one and depth zero. We show that the module $\operatorname{Hom}_R(R/aR, M/bM)$ is decomposable if the parameter $b$ is chosen to be a multiple of a sufficiently high power of the parameter $a$.

**Theorem 1.15.** *[10, Theorem 3.1] Let $(R, \mathfrak{m})$ be a local ring, and $M$ a nonzero finitely generated $R$-module of dimension one and depth zero. Choose an integer $n$ such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$. For any parameter $a$ of $M$, and any parameter $b$ of $M$ with $b \in (a^{n+1})$, the following $R$-module is decomposable:*

$$\operatorname{Hom}_R(R/aR, M/bM).$$

For higher dimensional modules, we also have a theorem to show that the module $\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ is decomposable; however, this result is weaker since it is not as explicit as the result for modules of dimension one.

**Theorem 1.16.** *[10, Theorem 4.1] Let $R$ be a local ring and $M$ a finitely generated $R$-module of dimension $d$. If $M$ is not Cohen-Macaulay, then, for any system of parameters $\mathbf{a} = a_1, \ldots, a_d$ of $M$, there exist positive integers $n_1, \ldots, n_d$ such that the following $R$-module is decomposable:*

$$\operatorname{Hom}_R(R/(\mathbf{a}), M/(a_1^{n_1}, \ldots, a_d^{n_d})M).$$

For a dimension one ring, we show that $\operatorname{Hom}_R(R/(a), R/(b))$ is decomposable and not a free $R/(a)$-module if the parameter $a$ is chosen to be in a sufficiently high power of the maximal ideal and the parameter $b$ is a multiple of $a^2$.

**Theorem 1.17.** *[10, Theorem 3.3] Let $(R, \mathfrak{m})$ be a local ring of dimension one and depth zero, and $n$ an integer such that $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) = (0)$. For any parameter $a$ in $\mathfrak{m}^n$ and any parameter $b$ in $(a^2)$, the $R/(a)$-module*

$$\operatorname{Hom}_R(R/(a), R/(b))$$

*is decomposable and has a non-free summand.*

For higher dimensional rings, we also have a theorem to show that $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is decomposable and is not $R/\mathfrak{a}$-free; however, as was the case when comparing Theorem 1.15 to Theorem 1.16, this result is weaker as it is less explicit than the result for one dimensional rings.

**Theorem 1.18.** *[10, Theorem 4.2] Let $R$ be a local ring of dimension $d$. If $R$ is not Cohen-Macaulay, then for any system of parameters $a_1, \ldots, a_d$ of $R$, there exist integers $n_1, \ldots, n_d, N_1, \ldots, N_d \in \mathbb{N}$ with $N_i \geq n_i$ for $i = 1, \ldots, d$ such that the*

*$R/(a_1^{n_1}, \ldots, a_d^{n_d})$-module*

$$\operatorname{Hom}_R(R/(a_1^{n_1}, \ldots, a_d^{n_d}), R/(a_1^{N_1}, \ldots, a_d^{N_d}))$$

*is decomposable and has a non-free summand.*

Theorem 1.15 and Theorem 1.17 are for one dimensional rings and modules respectively and are explicit in indicating when $\operatorname{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ is decomposable and when it has a non-free summand. Theorem 1.16 and Theorem 1.18 are similar, but for higher dimensional rings and modules.

In Chapter 2 we will state some preliminary results and prove the main results stated in this section. In Chapter 3 we will provide examples in order to explore some questions about the powers appearing in the main results.

# Chapter 2

# Proof of Results

In this chapter we present proofs of our main results. In Section 2.1 we present some preliminary results. Section 2.2 focuses on our results in dimension one. These results are stronger and more explicit than those in Section 2.3, which focuses on our results in higher dimensions.

## 2.1 Preliminary Results

In this section we include proofs of many of the results due to lack of adequate references. Recall the *support* of an $R$-module, $M$, is defined to be

$$\operatorname{Supp}_R(M) := \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}.$$

An $R$-module $M$ is said to be *faithful* if $\operatorname{ann}_R(M) = 0$. If $M$ is a faithful $R$-module, then $\dim_R M = \dim R$. Indeed, by definition of faithful, $\operatorname{ann}_R(M) = 0$. The next two results are well-known.

**Lemma 2.1.** *Let $R$ be a ring, $I \subseteq R$ an ideal of $R$, and $M$ a finitely generated, faithful $R$-module. Then* $\operatorname{Supp}_R(R/I) = \operatorname{Supp}_R(M/IM)$.

*Proof.* If $\mathfrak{p}$ is a prime ideal of $R$ such that $\mathfrak{p} \notin \operatorname{Supp}_R(R/I)$, then $I_\mathfrak{p} = R_\mathfrak{p}$. Thus $I_\mathfrak{p} M_\mathfrak{p} = R_\mathfrak{p} M_\mathfrak{p} = M_\mathfrak{p}$ so that $\mathfrak{p} \notin \operatorname{Supp}_R(M/IM)$.

For the other inclusion, suppose $\mathfrak{p}$ is a prime ideal of $R$ such that $\mathfrak{p} \notin \operatorname{Supp}_R(M/IM)$. Then $M_\mathfrak{p} = I_\mathfrak{p} M_\mathfrak{p}$. Nakayama's Lemma [7, Theorem 2.2] gives the existence of an element $x \in R_\mathfrak{p}$ such that $xM_\mathfrak{p} = 0$ and $x-1 \in I_\mathfrak{p}$. Say $x-1 = \frac{a}{t}$ with $a \in I$ and $t \in R\setminus\mathfrak{p}$. Rewriting this gives $x = 1+\frac{a}{t}$ or equivalently $xt = t+\frac{a}{1}$. As $xtM_\mathfrak{p} = t(xM_\mathfrak{p}) = 0$ then $(t+\frac{a}{1})M_\mathfrak{p} = 0$. As $M$ is finitely generated, this means that there exists some $s \in R\setminus\mathfrak{p}$ such that $s(t+a)M = 0$. As $M$ is a faithful $R$-module, $s(t+a) = 0 \in \mathfrak{p}$. Since $s \notin \mathfrak{p}$ this implies that $t+a \in \mathfrak{p}$. Hence $a \notin \mathfrak{p}$, since otherwise we would also have $t \in \mathfrak{p}$, a contradiction. We have thus found an element $a \in I \setminus \mathfrak{p}$ and so $(R/I)_\mathfrak{p} = 0$ and hence $\mathfrak{p} \notin \operatorname{Supp}_R(R/I)$. □

**Lemma 2.2.** *Let $(R, \mathfrak{m})$ be a local ring and $M$ a nonzero finitely generated $R$-module. Then* $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$ *if and only if $M$ has finite length.*

*Proof.* First suppose that $M$ has finite length. We'll show $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$ by induction on the length of $M$. When $M$ has length one, $0 \subsetneq M$ is a composition series. In other words, $M$ is a simple $R$-module, so is isomorphic to $R/\mathfrak{m}$. Since $\operatorname{Ass}_R(R/\mathfrak{m}) = \{\mathfrak{m}\}$, we know that $\operatorname{Ass}_R(M) = \{\mathfrak{m}\}$. Since $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$ and their minimal elements coincide [7, Theorem 6.5], we have that $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$ as well. Now suppose that $M$ has length $t$ and that every nonzero $R$-module of length less than $t$ has $\mathfrak{m}$ as the only element of its support. Let

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{t-1} \subsetneq M_t = M$$

be a composition series of $M$. As $\mathfrak{m}$ is the only maximal ideal of $R$, we know that $M_i/M_{i-1} \cong R/\mathfrak{m}$ for each $i = 1, \ldots, t$. Consider the short exact sequence

$$0 \to M_{t-1} \to M \to M/M_{t-1} \to 0.$$

We know that $M_{t-1}$ has length $t-1$ and so $\operatorname{Supp}_R(M_{t-1}) = \{\mathfrak{m}\}$. Also, $M/M_{t-1} \cong R/\mathfrak{m}$ and so $\operatorname{Supp}_R(M/M_t) = \{\mathfrak{m}\}$ as well. If $\mathfrak{p} \in \operatorname{Supp}_R(M)$, either $M_{t-1}$ or $M/M_{t-1}$ is nonzero upon localization at $\mathfrak{p}$ since localization is exact. Since this only holds for $\mathfrak{p} = \mathfrak{m}$, we know that $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$ as claimed.

Now suppose that $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$. This implies $\operatorname{Ass}_R(M) = \{\mathfrak{m}\}$. We wish to show that $M$ has finite length. Since $\mathfrak{m} \in \operatorname{Ass}_R M$, we may choose a nonzero submodule $M_1$ of $M$ with $M_1 \cong R/\mathfrak{m}$. Thus $M$ has finite length submodules, and since $M$ is Noetherian, there exists a submodule $M'$ maximal with respect to having finite length. We wish to show $M' = M$. If not, then $M/M' \neq 0$ so that $\operatorname{Supp}_R(M/M') = \{\mathfrak{m}\}$. As this implies $\mathfrak{m} \in \operatorname{Ass}_R(M/M')$, there is an injection $R/\mathfrak{m} \hookrightarrow M/M'$. We thus have a submodule $M'' \subseteq M$ with $M'' \supsetneq M'$ and $M''/M' \cong R/\mathfrak{m}$. As the length of $M''$ is thus one more than the length of $M'$, we have arrived at a contradiction. Hence $M' = M$ and $M$ has finite length. $\square$

The next result is also well known. We provide a proof for completeness.

**Proposition 2.3.** *Let $a_1, \ldots, a_d$ be elements of $R$ and $\overline{a_1}, \ldots, \overline{a_d}$ their images in $R/\operatorname{ann}_R(M)$. Then $\{a_1, \ldots, a_d\}$ is a system of parameters of $M$ if and only if $\{\overline{a_1}, \ldots, \overline{a_d}\}$ is a system of parameters of the ring $R/\operatorname{ann}_R(M)$.*

*Proof.* Note $\{a_1, \ldots, a_d\}$ is a system of parameters of $M$ as an $R$-module if and only if $\{\overline{a_1}, \ldots, \overline{a_d}\}$ is a system of parameters of $M$ as an $R/\operatorname{ann}_R(M)$-module. We may thus assume $\operatorname{ann}_R(M) = 0$. The desired result is thus that $\{a_1, \ldots, a_d\}$ is a system of parameters of $M$ if and only if $\{a_1, \ldots, a_d\}$ is a system of parameters of $R$.

Suppose $\{a_1, \ldots, a_d\}$ is a system of parameters of $M$ and write $\mathfrak{a} = (a_1, \ldots, a_d)$, so $M/\mathfrak{a}M$ has finite length as an $R$-module. By Lemma 2.2 we know $\operatorname{Supp}_R(M/\mathfrak{a}M) = \{\mathfrak{m}\}$. Lemma 2.1 then tells us that $\operatorname{Supp}_R(R/\mathfrak{a}) = \{\mathfrak{m}\}$. By Lemma 2.2 we have that $R/\mathfrak{a}$ has finite length and $\mathfrak{a} = (a_1, \ldots, a_d)$ is a parameter ideal of $R$.

For the other direction, suppose $\{a_1, \ldots, a_d\}$ is a system of parameters of $R$ and write $\mathfrak{a} = (a_1, \ldots, a_d)$ so that $R/\mathfrak{a}$ has finite length. As the length of $R/\mathfrak{a}$ as an $R$-module is the same as its length as an $R/\mathfrak{a}$-module, we know that $R/\mathfrak{a}$ is both an Artinian and Noetherian ring. (Of course, all Artinian rings are Noetherian [1, Theorem 8.5].) As $M/\mathfrak{a}M$ is a finitely generated $R/\mathfrak{a}$-module, it must be both Artinian and Noetherian as well, and hence have finite length as an $R/\mathfrak{a}$-module. The length of $M/\mathfrak{a}M$ as an $R$-module is the same as its length as an $R/\mathfrak{a}$-module, and hence $M/\mathfrak{a}M$ has finite length as an $R$-module as well. Thus $\mathfrak{a} = (a_1, \ldots, a_d)$ is a parameter ideal of $M$. □

The integer $n$ appearing in the next result plays a key role in the main results.

**Lemma 2.4.** *Let $(R, \mathfrak{m})$ be a local ring and $M$ a finitely generated $R$-module. There exists an integer $n$ such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$.*

*Proof.* Since $\Gamma_{\mathfrak{m}}(M)$ is Artinian (see Proposition 1.9), the descending chain of submodules

$$(\mathfrak{m}M \cap \Gamma_{\mathfrak{m}}(M)) \supseteq (\mathfrak{m}^2 M \cap \Gamma_{\mathfrak{m}}(M)) \supseteq \cdots$$

must stabilize. That is, there is some $n \in \mathbb{N}$ such that

$$\mathfrak{m}^{n+i} M \cap \Gamma_{\mathfrak{m}}(M) = \mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M)$$

for all integers $i \geq 0$. Thus

$$\begin{aligned}\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) &= \bigcap_{i \geq n} (\mathfrak{m}^i M \cap \Gamma_{\mathfrak{m}}(M)) \\ &\subseteq \bigcap_{i \geq 1} \mathfrak{m}^i M \\ &= (0).\end{aligned}$$

The last equality is by the Krull Intersection Theorem [7, Theorem 8.10]. $\square$

*Remark* 2.5. In fact, for any finite-length submodule $L \subseteq M$, we have $L \subseteq \Gamma_{\mathfrak{m}}(M)$, and hence $\mathfrak{m}^n M \cap L = (0)$ where $n$ is the integer of Lemma 2.4.

The next result is in the spirit of [8, Prop 4.7.13]. We include a proof in order to obtain specific bounds on the powers of $a$ in this special case.

**Proposition 2.6.** *Let $R$ be any commutative ring, $M$ an $R$-module, and $a, b \in R$. Then, for arbitrary positive integers $p \leq q \leq r$, we have an equality*

$$(ba^r M : a^p) = a^{r-q}(ba^q M : a^p) + (0 :_M a^p).$$

*Proof.* First let $x \in (ba^r M : a^p)$. Then $a^p x = ba^r y$ for some $y$ in $M$. Now

$$a^p(x - ba^{r-p}y) = 0,$$

so that $x - ba^{r-p}y \in (0 :_M a^p)$. Additionally,

$$ba^{r-p}y = a^{r-q} \cdot ba^{q-p}y \in a^{r-q}(ba^q M : a^p).$$

We now have

$$x = ba^{r-p}y + (x - ba^{r-p}y) \in a^{r-q}(ba^qM : a^p) + (0 :_M a^p).$$

For the other inclusion, it is clear that $(0 :_M a^p) \subseteq (ba^rM : a^p)$, so it suffices to prove $a^{r-q}(ba^qM : a^p) \subseteq (ba^rM : a^p)$. To that end, let $a^{r-q}y \in a^{r-q}(ba^qM : a^p)$ with $y \in (ba^qM : a^p)$. We can write $a^py = ba^qw$ for some $w$ in $M$. Thus we may rewrite $a^p \cdot a^{r-q}y$ as follows:

$$\begin{aligned} a^pa^{r-q}y &= a^{r-q}a^py \\ &= a^{r-q}ba^qw \\ &= ba^rw, \end{aligned}$$

which is in $ba^rM$. Thus $a^{r-q}(ba^qM : a^p) + (0 :_M a^p) \subseteq (ba^rM : a^p)$ as desired. $\square$

The next result will be applied in Section 2.3 in the situation where $I = (a_1, \ldots, a_d)$ is a parameter ideal and $J$ is of the form $(a_1^{n_1}, a_2, \ldots, a_d)$ for a positive integer $n_1$.

**Lemma 2.7.** *Let $R$ be a Noetherian ring, $J \subseteq I$ proper ideals of $R$ with $\sqrt{I} = \sqrt{J}$, and $N$ an $R$-module. If $\operatorname{Hom}_R(R/J, N)$ is decomposable, then so is $\operatorname{Hom}_R(R/I, N)$.*

*Proof.* Suppose that $\operatorname{Hom}_R(R/J, N) = X \oplus Y$ where $X$ and $Y$ are nonzero $R$-modules. Since Hom and $\otimes$ are adjoint functors, there are isomorphisms

$$\begin{aligned} \operatorname{Hom}_R(R/I, N) &\cong \operatorname{Hom}_R((R/I) \otimes_R (R/J), N) \\ &\cong \operatorname{Hom}_R(R/I, \operatorname{Hom}_R(R/J, N)) \\ &\cong \operatorname{Hom}_R(R/I, X \oplus Y) \\ &\cong \operatorname{Hom}_R(R/I, X) \oplus \operatorname{Hom}_R(R/I, Y). \end{aligned}$$

By symmetry, it suffices to show that $\operatorname{Hom}_R(R/I, X) \neq 0$. It is clear that $JX = (0)$ since $X \subseteq \operatorname{Hom}_R(R/J, N)$. Choose $\mathfrak{p} \in \operatorname{Ass}_R X$ and note that $J \subseteq \mathfrak{p}$. Indeed, we know $JX = (0)$ and any $\mathfrak{p} \in \operatorname{Ass}_R(X)$ has the form $\mathfrak{p} = \operatorname{ann}(x)$ for some $x \in X$. Since $Jx = (0)$, we obtain that $J \subseteq \mathfrak{p}$. Since $\sqrt{J} = \sqrt{I}$, one has $I \subseteq \mathfrak{p}$, so there are maps

$$R/I \twoheadrightarrow R/\mathfrak{p} \hookrightarrow X.$$

The composition of these maps is nonzero, and so $\operatorname{Hom}_R(R/I, X) \neq (0)$ as desired. □

*Remarks* 2.8.

1. The hypothesis that $J \subseteq I$ is necessary. For any pair of ideals $I, J$, if we let $N = R/I$, then

   $$\operatorname{Hom}_R(R/I, N) = \operatorname{Hom}_R(R/I, R/I) \cong R/I$$

   is indecomposable. However, it is possible that $\operatorname{Hom}_R(R/J, N)$ is decomposable. For instance, let $k$ be a field and consider the ring $R = k[\![x, y]\!]/(x^2, xy)$ along with the ideals $I = (y^2)$, $J = (y)$ of $R$. Here $\sqrt{I} = \sqrt{J} = (x, y)$ but $J \not\subseteq I$. In Example 3.2 we will show

   $$\operatorname{Hom}_R(R/J, R/I) = \operatorname{Hom}_R(R/(y), R/(y^2)) \cong \frac{(y)}{(y^2)} \oplus \frac{(x, y^2)}{(y^2)}.$$

2. The hypothesis that $\sqrt{I} = \sqrt{J}$ is also necessary. For example, let $k$ be a field, $R = k[\![x, y, z]\!]/(x^2, xyz)$, $N = R/(y^2)$, $I = (y, z)$, and $J = (y)$. We have $J \subseteq I$, but $\sqrt{J} = (x, y) \subsetneq (x, y, z) = \sqrt{I}$. Using this notation we obtain

   $$\operatorname{Hom}_R(R/J, N) \cong \frac{(y^2) : y}{(y^2)} = \frac{(y)}{(y^2)} \oplus \frac{(xz, y^2)}{(y^2)}$$

decomposes but

$$\operatorname{Hom}_R(R/I, N) \cong \frac{(y^2) : (y, z)}{(y^2)} = \frac{(xy, y^2)}{(y^2)}$$

is cyclic, and hence indecomposable.

This next result will be used in Section 2.3 in an induction argument.

**Lemma 2.9.** *Let $R$ be a local ring and $M$ an $R$-module of dimension $d \geq 2$. If $M$ is not Cohen-Macaulay, then for any system of parameters $a_1, \ldots, a_d$ of $M$, there exist positive integers $i$ and $s$ such that $M/a_i^s M$ is not Cohen-Macaulay.*

*Proof.* If some $a_i$ is $M$-regular, then $M/a_iM$ is not Cohen-Macaulay, so we may assume that each $a_i$ is a zero-divisor on $M$. Suppose, by way of contradiction, that $M/a_1^sM$ is Cohen-Macaulay for each $s \geq 1$. Then $a_2, \ldots, a_d$ is a regular sequence on $M/a_1^sM$ for all integers $s \geq 1$. In particular $a_2$ is $M/a_1^sM$-regular for all integers $s \geq 1$. We claim this implies $a_2$ is $M$-regular. Indeed, suppose $a_2m = 0$ for some $m \in M$. Then $a_2\overline{m} = 0$ in $M/a_1^sM$ for all integers $s \geq 1$, so that $m \in a_1^sM$ for all integers $s \geq 1$. By the Krull Intersection Theorem [7, Theorem 8.10], we have $m = 0$ and hence $a_2$ is $M$-regular which gives the desired contradiction. □

The next example, noticed by Ryan Karr, shows that even when every element in a of the parameters is a zero-divisor, $M$ may have positive depth.

**Example 2.10.** Consider the ring $R = k[\![x, y, z]\!]/(x^2, xyz)$. It is clear that the rings $R/(x, y, z) \cong k$, $R/(x, y) \cong k[\![z]\!]$, and $R/(x) \cong k[\![y, z]\!]$ are all domains and so the dimension of $R$ is at least 2. Since $x$ is nilpotent, we have $x \in \sqrt{(y, z)}$ so that $\sqrt{(y, z)} = (x, y, z) = \mathfrak{m}$. Hence the dimension of $R$ is 2 and $\{y, z\}$ is a system of parameters of $R$. Both $y$ and $z$ are zero-divisors in $R$ since $xz \cdot y = xy \cdot z = 0$ and $xz, xy \neq 0$ in $R$.

To see that $R$ has positive depth, note that $\operatorname{Ass}_R R = \{(x), (x, y), (x, z)\}$. As $y - z \notin (x) \cup (x, y) \cup (x, z)$ we know that it is a regular element on $R$ [7, Theorem 6.1(ii)]. We saw in Example 1.7 that $R/(y - z) \cong k[\![x, y]\!]/(x^2, xy^2)$ has depth zero. Hence $y - z$ is a maximal length regular sequence on $R$ and $R$ has depth one. Thus we have found a ring of positive depth where the system of parameters consists only of zero-divisors.

Note that both $R/(y)$ and $R/(z)$ are Cohen-Macaulay rings of dimension one in this example, but that $R/(y^n)$ and $R/(z^n)$ have dimension one and depth zero for all integers $n \geq 1$.

The next example shows that for an arbitrary parameter $a$ of a non-Cohen-Macaulay module, $M$, it is possible that $M/a^s M$ is Cohen-Macaulay for all $s$. Hence, we cannot strengthen Lemma 2.9 to say that for any parameter $a$ of a non-Cohen-Macaulay module $M$, there is an integer $s$ such that $M/a^s M$ is non-Cohen-Macaulay.

**Example 2.11.** Consider the ring $R = k[\![x, y, z]\!]/(x^2, xy)$. It is clear that the rings $R/(x, y, z) \cong k$, $R/(x, y) \cong k[\![z]\!]$, and $R/(x) \cong k[\![y, z]\!]$ are all domains and so the dimension of $R$ is at least two. Since $x$ is nilpotent, $x \in \sqrt{I}$ for every ideal $I$ of $R$. Thus, $\sqrt{(y, z)} = (x, y, z)$ is the maximal ideal of $R$. Hence $R$ has dimension two and $\{y, z\}$ is a system of parameters of $R$. As $z$ is a regular element of $R$ which is in the maximal ideal, $R$ must have at least depth one. However, $R/(z) \cong k[\![x, y]\!]/(x^2, xy)$ has depth zero; see Example 1.7. Thus $z$ is a maximal length regular sequence and $R$ has depth one.

Next, we consider the quotients $S_n := R/(y^n) = k[\![x, y, z]\!]/(x^2, xy, y^n)$ for $n \in \mathbb{N}$. We claim these quotients are all Cohen-Macaulay rings of dimension one. Indeed, $S_n/(x, y, z)S_n \cong k$ and $S_n/(x, y)S_n \cong k[\![z]\!]$ are both domains, so $S_n$ has dimension at least one for all $n$. Since $\sqrt{(y)S_n} = (x, y, z)S_n$ for all $n \in \mathbb{N}$ we have that $\{y\}$ is a system of parameters for $S_n$. Thus $\dim S_n = 1$ for all $n$. To see that $S_n$ has depth one, we simply need to note that $z$ is a regular element in the maximal ideal of $S_n$.

## 2.2 Dimension One

We start the proofs of the main results with those for modules of dimension one and depth zero since we are able to obtain stronger bounds in this case. We show $\operatorname{Hom}_R(R/aR, M/bM)$ decomposes if the parameter $b$ is chosen to be in a sufficiently high power of the ideal generated by an arbitrary parameter $a$. Recall:

**Theorem 1.15.** *[10, Theorem 3.1] Let $(R, \mathfrak{m})$ be a local ring, and $M$ a nonzero finitely generated $R$-module of dimension one and depth zero. Choose an integer $n$ such that $\mathfrak{m}^n M \cap \Gamma_{\mathfrak{m}}(M) = (0)$. For any parameter $a$ of $M$, and any parameter $b$ of $M$ with $b \in (a^{n+1})$, the following $R$-module is decomposable:*

$$\operatorname{Hom}_R(R/aR, M/bM).$$

*Remark* 2.12. The integer $n$ in the statement exists by Lemma 2.4. Note that $n \geq 1$ because $\Gamma_{\mathfrak{m}}(M) \neq (0)$; see Proposition 1.9.

*Proof of Theorem 1.15.* Set $S := R/\operatorname{ann}_R(M)$, and let $(\bar{\ })$ denote the image in $S$. Then $\bar{a}$ and $\bar{b}$ are parameters of $M$ as an $S$-module. Moreover there is an $R$-module isomorphism

$$\operatorname{Hom}_S(S/\bar{a}S, M/\bar{b}M) \cong \operatorname{Hom}_R(R/aR, M/bM).$$

By replacing $R$ with $S$, we may thus assume that $M$ is a faithful $R$-module.

Write $b = ca^{n+1}$. Since $M$ is faithful, we have $\sqrt{(a)} = \mathfrak{m}$ (see Proposition 2.3) and so

$$(0 :_M a) \subseteq \Gamma_{(a)}(M) = \Gamma_{\mathfrak{m}}(M).$$

Thus we know

$$(0 :_M a) \cap ca^n M \subseteq \Gamma_{\mathfrak{m}}(M) \cap \mathfrak{m}^n M = (0). \tag{2.1}$$

By Proposition 2.6, with $p = q = 1$ and $r = n + 1$, we have

$$(ca^{n+1}M : a) = a^n(caM : a) + (0 :_M a). \tag{2.2}$$

We now claim that

$$a^n(caM : a) = ca^n M.$$

Indeed, it is clear that $ca^n M \subseteq a^n(caM : a)$. For the reverse inclusion, let $x$ be in $a^n(caM : a)$ and write $x = a^n m$ for some $m \in (caM : a)$. We have $am = cam'$ for some $m' \in M$. Then

$$x = a^n m = a^{n-1} \cdot am = ca^n m' \in ca^n M.$$

Equation (2.2) is thus equivalent to

$$(bM : a) = ca^n M + (0 :_M a). \tag{2.3}$$

Next we claim that

$$ca^{n+1}M = ca^n M \cap \left[(0 :_M a) + ca^{n+1}M\right]. \tag{2.4}$$

It is clear that $ca^{n+1}M \subseteq ca^n M \cap [(0 :_M a) + ca^{n+1}M]$. For the other inclusion, let $x$

be in $ca^nM \cap [(0 :_M a) + ca^{n+1}M]$, and write

$$x = ca^n y = z + ca^{n+1}w$$

for some $y, w \in M$ and $z \in (0 :_M a)$. Then

$$\begin{aligned} z &= ca^n y - ca^{n+1}w \\ &\in (0 :_M a) \cap ca^nM = (0) && \text{by (2.1).} \end{aligned}$$

Equation (2.4) follows. Now there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_R(R/aR, M/bM) &\cong \frac{(bM : a)}{bM} \\ &\cong \frac{ca^nM + (0 :_M a)}{ca^{n+1}M} && \text{by (2.3)} \\ &\cong \frac{ca^nM}{ca^{n+1}M} \oplus \frac{(0 :_M a) + ca^{n+1}M}{ca^{n+1}M} && \text{by (2.4).} \end{aligned}$$

All that remains to prove is that both summands are nonzero.

If the summand on the left were zero, then $ca^nM = (a) \cdot ca^nM$ so that $ca^nM = (0)$ by Nakayama's Lemma. This would be a contradiction as $ca^{n+1} = b$ is a parameter of $M$.

If the summand on the right were zero, then

$$(0 :_M a) \subseteq ca^{n+1}M.$$

By Equation (2.1) we would then have

$$(0 :_M a) = (0 :_M a) \cap ca^{n+1}M = (0).$$

This would also give a contradiction as $\operatorname{depth}_R M = 0$. Thus $\operatorname{Hom}_R(R/aR, M/bM)$ is decomposable, as desired. □

When $R$ is a Cohen-Macaulay ring, we know from Proposition 1.13 that the $R/\mathfrak{a}$-module $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b}) \cong R/\mathfrak{a}$ is not only indecomposable, but also free. When $R$ is one-dimensional and not Cohen-Macaulay, we can prove that in addition to being decomposable, this module will be non-free if the parameters are chosen to be in sufficiently high powers of the maximal ideal. In comparing this to Theorem 1.15 we see that the requirements for showing $\operatorname{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is not a free $R/\mathfrak{a}$-module are greater than those required to show that it is decomposable since the integer $n$ in the two theorems is the same when $M = R$. Recall:

**Theorem 1.17.** *[10, Theorem 3.3] Let $(R, \mathfrak{m})$ be a local ring of dimension one and depth zero, and $n$ an integer such that $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) = (0)$. For any parameter $a$ in $\mathfrak{m}^n$ and any parameter $b$ in $(a^2)$, the $R/(a)$-module*

$$\operatorname{Hom}_R(R/(a), R/(b))$$

*is decomposable and has a non-free summand.*

*Remark* 2.13. Again, the integer $n$ in the statement exists by Lemma 2.4 and must be positive since $\Gamma_{\mathfrak{m}}(R) \neq (0)$; see Proposition 1.9.

*Proof of Theorem 1.17.* We will first prove that the module decomposes. Both the proof of this fact and the decomposition obtained are similar to those found in the proof of Theorem 1.15. Write $I = \Gamma_{\mathfrak{m}}(R)$. For any $x \in \mathfrak{m}^n$, we know $(x) \cap I = (0)$ and hence $xI = (0)$. If $x \in \mathfrak{m}^n$ is also a parameter, then we know $\sqrt{(x)} = \mathfrak{m}$ and so $\Gamma_{(x)}(R) = I$, and $(0 : x) = I$ as well. To see that $(0 : x) = I$, note that $\sqrt{(x)} = \mathfrak{m}$

and $xI = 0$ imply

$$I \subseteq (0 : x) \subseteq \Gamma_{(x)}(R) = I,$$

whence $(0 : x) = I$.

Let $a \in \mathfrak{m}^n$ and $b \in (a^2)$ be parameters and write $b = ca^2$. Applying Proposition 2.6 with $p = q = 1$ and $r = 2$, we obtain the equality

$$((ca^2) : a) = a((ca) : a) + (0 : a). \tag{2.5}$$

We now note that $a((ca) : a) = (ca)$. We may thus rewrite Equation (2.5) as

$$((b) : a) = (ca) + I. \tag{2.6}$$

Next we wish to show that

$$(ca^2) = (ca) \cap \left[I + (ca^2)\right]. \tag{2.7}$$

The inclusion $\subseteq$ is clear. For the other inclusion, let $x \in (ca) \cap [I + (ca^2)]$ and write $x = rca = s + r'ca^2$ for some $r, r' \in R$ and $s \in I$. Then

$$\begin{aligned} s &= rca - r'ca^2 \\ &\in (a) \cap I \\ &\subseteq \mathfrak{m}^n \cap I = (0). \end{aligned}$$

Thus $s = 0$ and $x = r'ca^2 \in (ca^2)$. This gives the existence of the following isomorphisms of $R/(a)$-modules:

$$\begin{aligned}
\operatorname{Hom}_R(R/(a), R/(b)) &\cong \frac{((b):a)}{(b)} \\
&\cong \frac{(ca)+I}{(ca^2)} && \text{by (2.6)} \\
&\cong \frac{(ca)}{(ca^2)} \oplus \frac{I+(ca^2)}{(ca^2)} && \text{by (2.7).}
\end{aligned}$$

Next we show that both summands are nonzero.

If the summand on the left were zero, then Nakayama's Lemma would imply that $ca = 0$, a contradiction as $ca^2 = b$ is a parameter and hence nonzero.

If the summand on the right were zero, then $I \subseteq (ca^2)$ so that

$$\Gamma_{\mathfrak{m}}(R) = I = I \cap (ca^2) \subseteq I \cap (a) = (0),$$

a contradiction as the depth of $R$ is zero (see Proposition 1.9).

We now show that the summand on the left, that is, $(ca)/(ca^2)$, is not a free $R/(a)$-module. To that end, recall that $I \cap (a) = (0)$, but $I \neq (0)$, so we can choose an element $y \in I \setminus (a)$. We know $ya = 0$ since $aI = (0)$. In particular, $yca \in (ca^2)$. Thus $y + (a)$ is a nonzero element of

$$\operatorname{ann}_{R/(a)} \left( \frac{(ca)}{(ca^2)} \right),$$

and hence $(ca)/(ca^2)$ is not free as an $R/(a)$-module. □

## 2.3 Higher Dimensions

In higher dimensions, we can also prove a decomposition theorem. Recall:

**Theorem 1.16.** *[10, Theorem 4.1] Let $R$ be a local ring and $M$ a finitely generated $R$-module of dimension $d$. If $M$ is not Cohen-Macaulay, then, for any system of parameters $\mathbf{a} = a_1, \ldots, a_d$ of $M$, there exist positive integers $n_1, \ldots, n_d$ such that the following $R$-module is decomposable:*

$$\operatorname{Hom}_R(R/(\mathbf{a}), M/(a_1^{n_1}, \ldots, a_d^{n_d})M).$$

*Proof.* As in the proof of Theorem 1.15, we may reduce to the case that $M$ is a faithful module. We proceed by induction on $d$, the case $d = 1$ being covered by Theorem 1.15.

Assume, now, that $d \geq 2$. By Lemma 2.9, we can find some positive integer $i \leq d$ and a positive integer $n_i$ such that $M/a_i^{n_i}M$ is not Cohen-Macaulay. We may harmlessly assume $i = 1$. Set

$$\overline{R} := R/(a_1^{n_1}), \qquad \overline{M} := M/a_1^{n_1}M, \qquad \text{and} \qquad \bar{\mathfrak{a}} := (\overline{a_2}, \ldots, \overline{a_d}).$$

Then $\bar{\mathfrak{a}}$ is a parameter ideal of $\overline{M}$. Since $\overline{M}$ has dimension $d-1$ and is not Cohen-Macaulay, by induction there are positive integers $n_2, \ldots, n_d$ such that the $R$-module

$$U := \operatorname{Hom}_{\overline{R}}(\overline{R}/\bar{\mathfrak{a}}, \overline{M}/(\overline{a_2}^{n_2}, \ldots, \overline{a_d}^{n_d})\overline{M})$$

is decomposable. Since there is an isomorphism

$$U \cong \operatorname{Hom}_R(R/(a_1^{n_1}, a_2, \ldots, a_d), M/(a_1^{n_1}, a_2^{n_2}, \ldots, a_d^{n_d})M),$$

we apply Lemma 2.7, with $N = M/(a_1^{n_1}, \ldots, a_d^{n_d})M$, $I = (a_1, \ldots, a_d)$, and $J = (a_1^{n_1}, a_2, \ldots, a_d)$, to obtain the existence of the desired decomposition. $\square$

Recall the result below, which is a version of Theorem 1.17 for rings of arbitrary dimension.

**Theorem 1.18.** *[10, Theorem 4.2] Let $R$ be a local ring of dimension $d$. If $R$ is not Cohen-Macaulay, then for any system of parameters $a_1, \ldots, a_d$ of $R$, there exist integers $n_1, \ldots, n_d, N_1, \ldots, N_d \in \mathbb{N}$ with $N_i \geq n_i$ for $i = 1, \ldots, d$ such that the $R/(a_1^{n_1}, \ldots, a_d^{n_d})$-module*

$$\operatorname{Hom}_R(R/(a_1^{n_1}, \ldots, a_d^{n_d}), R/(a_1^{N_1}, \ldots, a_d^{N_d}))$$

*is decomposable and has a non-free summand.*

*Proof.* We proceed by induction on $d$.

For $d = 1$, we choose $n$ such that $\mathfrak{m}^n \cap \Gamma_{\mathfrak{m}}(R) = (0)$ and set $n_1$ and $N_1$ to be $n$ and $2n$, respectively. Theorem 1.17 then gives the desired decomposition and existence of a non-free summand.

Now suppose that $d \geq 2$. By Lemma 2.9, we can find integers $i$ and $n_i$ such that $R/(a_i^{n_i})$ is not Cohen-Macaulay. We may harmlessly assume $i = 1$. Set $S := R/(a_1^{n_1})$ let $(^-)$ denote the image in $S$. Then $\overline{a_2}, \ldots, \overline{a_d}$ is a system of parameters of $S$ and, by induction, there exist integers $n_2, \ldots, n_d, N_2, \ldots, N_d$ such that the $S/(\overline{a_2}^{n_2}, \ldots, \overline{a_d}^{n_d})$-module

$$U := \operatorname{Hom}_S(S/(\overline{a_2}^{n_2}, \ldots, \overline{a_d}^{n_d}), S/(\overline{a_2}^{N_2}, \ldots, \overline{a_d}^{N_d}))$$

is decomposable and has a non-free summand. Note that

$$S/(\overline{a_2}^{n_2}, \ldots, \overline{a_d}^{n_d}) \cong R/(a_1^{n_1}, \ldots, a_d^{n_d})$$

and

$$S/(\overline{a_1}^{N_2}, \ldots, \overline{a_d}^{N_d}) \cong R/(a_1^{n_1}, a_2^{N_2}, \ldots, a_d^{N_d}).$$

Setting $N_1 = n_1$ we then have

$$U \cong \mathrm{Hom}_R(R/(a_1^{n_1}, \ldots, a_d^{n_d}), R/(a_1^{N_1}, \ldots, a_d^{N_d})),$$

and this is decomposable and has a non-free summand. $\square$

# Chapter 3

# Examples

Choosing $b = a^t$ in Theorem 1.15 gives $\mathrm{Hom}_R(R/aR, M/a^tM)$ is decomposable for all integers $t \geq n+1$. Similarly, if $a$ is a parameter of $R$, Theorem 1.17 says that $\mathrm{Hom}_R(R/(a^t), R/(a^T))$ is decomposable and has a non-free summand for $t \geq n$ and $T \geq 2t$. Thus, these theorems provide lower bounds sufficient to show that the module is decomposable and not free. One natural question to ask is whether or not these bounds are optimal, that is, are there smaller values of $t$ and $T$ that will also cause these modules to be decomposable and have non-free summands.

Theorem 1.16 and Theorem 1.18, which are for higher dimensional rings and modules, say there exist integers $\eta_i$, $n_i$, and $N_i$ such that both

$$A := \mathrm{Hom}_R(R/\mathfrak{a}, M/(a_1^{\eta_1}, \ldots, a_d^{\eta_d})M)$$

and

$$B := \mathrm{Hom}_R(R/(a_1^{n_1}, \ldots, a_d^{n_d}), R/(a_1^{N_1}, \ldots, a_d^{N_d}))$$

are decomposable and $B$ has a non-free summand. These statements are not explicit about the powers which will work; however, the proofs indicate a method for iteratively choosing the integers to be sufficiently large so that the modules will decompose and $B$ will have a non-free summand. For $d \geq 2$, examples seem to indicate that

$$\mathrm{Hom}_R(R/\mathfrak{a}, R/(a_1^{n_1}, \ldots, a_d^{n_d}))$$

decomposes for all $n_i \geq N$ with $N$ chosen to be sufficiently large. However, Example 3.6 shows that Theorem 1.15 is not strong enough to use the induction technique in Theorem 1.16 to prove the existence of such an integer $N$. So the question remains: Can we find bounds on the powers of the parameters that guarantee the modules $A$ and $B$ above are decomposable and $B$ has a non-free summand?

The purpose of this chapter is to explore this question by way of examples. In particular, we focus on the structure of the $R/\mathfrak{a}$-module $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ for concrete examples of $R$, $\mathfrak{a}$, and $\mathfrak{b}$.

Let us take $M = R$ in Theorem 1.16. Our first example shows that sometimes

$$\mathrm{Hom}_R(R/(a_1, \ldots, a_d), R/(a_1^{n_1}, \ldots, a_d^{n_d}) \cong R/(a_1, \ldots, a_d)$$

even when $R$ is not Cohen-Macaulay and at least one of the $n_i$'s is greater than one.

**Example 3.1.** Let $R = k[\![x, y]\!]/(x^2, xy^2)$. This ring has dimension one and depth zero as was shown in Examples 1.2 and 1.7. Consider the parameter $y$ of $R$. Let $r \in R$ and write $r$ using coset notation:

$$r = r_0 + r_1x + r_2y + r_3xy + r_4y^2 + r_5y^3 + \cdots + (x^2, xy^2)$$

with with $r_i \in k$. Looking at

$$ry = r_0y + r_1xy + r_2y^2 + r_3xy^2 + r_4y^3 + r_5y^4 + \cdots + (x^2, xy^2)$$

we see that $ry \in (y^2) \subset R$ if and only if $r_0 = r_1 = 0$. This happens exactly when $r \in (y) \subset R$. Thus

$$\operatorname{Hom}_R(R/(y), R/(y^2)) \cong \frac{(y^2) :_R y}{(y^2)} = \frac{(y)}{(y^2)} \cong R/(y).$$

The second isomorphism is given by $ry + (y^2) \leftarrow\!\shortmid r + (y)$.

The next example shows that the bound in Theorem 1.15 is close to optimal.

**Example 3.2.** Let $R = k[\![x, y]\!]/(x^2, xy^m)$. This ring has dimension one and depth zero as was shown in Example 1.2 and Example 1.7. Consider the parameter $y$ of $R = k[\![x, y]\!]/(x^2, xy^m)$ and set

$$U_t := \operatorname{Hom}_R(R/(y), R/(y^t)) \cong \frac{(y^t) : y}{(y^t)}.$$

We will show $U_t$ is cyclic (and hence indecomposable) for $t \leq m$ and decomposable for $t > m$. For any $r \in R$ we may write $r$ in coset notation:

$$r = \sum_{i=0}^{\infty} r_i y^i + \sum_{i=0}^{m-1} r_i' x y^i + (x^2, xy^m)$$

with $r_i, r_i' \in k$. In this notation

$$ry = \sum_{i=0}^{\infty} r_i y^{i+1} + \sum_{i=0}^{m-2} r_i' x y^{i+1} + (x^2, xy^m). \tag{3.1}$$

Let $t \leq m$. We know that $ry \in (y^t) \subset R$ if the only nonzero terms of the sums in (3.1) are those with $i + 1 < t$. So $ry \in (y^t) \subset R$ if and only if $r_i = r_i' = 0$ for $i < t - 1$.

This gives that $r \in ((y^t) :_R y)$ if and only if

$$r = \sum_{i=t-1}^{\infty} r_i y^i + \sum_{i=t-1}^{m-1} r_i' x y^i + (x^2, xy^m).$$

That is, $r \in ((y^t) :_R y)$ if and only if $r \in (y^{t-1}) \subset R$. Hence $U_t \cong (y^{t-1})/(y^t)$ is cyclic.

Now let $t > m$. Since $xy^m = 0$ we now have that $ry \in (y^t)$ if and only if the only nonzero terms in (3.1) are those with in the first sum with $i + 1 < t$. This gives that $ry \in (y^t)$ if and only if

$$r = \sum_{i=t-1}^{\infty} r_i y^i + r_{m-1}' x y^{m-1} + (x^2, xy^m).$$

That is, $r \in ((y^t) :_R y)$ if and only if $r \in (y^{t-1}, xy^{m-1})$. Since $t > m$ we know $xy^{t-1} = 0$ in $R$. As $k[\![x, y]\!]$ has unique factorization, we have $(y^{t-1}) \cap (xy^{m-1}, y^t) = (y^t)$ in $R$. Thus

$$U_t \cong \frac{(y^{t-1}, xy^{m-1})}{(y^t)} \cong \frac{(y^{t-1})}{(y^t)} \oplus \frac{(xy^{m-1}, y^t)}{(y^t)}$$

is decomposable for all $t > m$.

Note that $\mathfrak{m}^i = (y^i)$ for $i \geq m + 1$. For $i \geq m + 1$ and $r \in R$ we have that $ry^i = 0$ if and only if $r \in (x)$ and so

$$\Gamma_{\mathfrak{m}}(R) = \bigcup_{i=1}^{\infty} (0 : \mathfrak{m}^i) = \bigcup_{i=m+1}^{\infty} (x) = (x).$$

Moreover $\mathfrak{m}^i \cap \Gamma_{\mathfrak{m}}(R) = (0)$ precisely when $i \geq m + 1$ and so Theorem 1.15 gives only that $U_t$ decomposes for $t \geq m + 2$. Since our computations showed that $U_t$ actually decomposes for $t \geq m + 1$, the bound obtained in Theorem 1.15 is, at worst, one away from a tight bound.

Before we present the next example, we recall the following well-known result:

**Proposition 3.3.** *[6, Proposition 3.1, Theorem 3.7] Let $R$ be a ring and $M$ an $R$-module of finite length. The non-units of the non-commutative ring* $\mathrm{End}_R(M)$ *form a two-sided ideal if and only if $M$ is indecomposable.*

In particular, this means that for parameter ideals $\mathfrak{a}, \mathfrak{b}$ of an $R$-module, $M$, the module $U := \mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ is indecomposable if and only if the non-units of $\mathrm{End}_R(U)$ form a two-sided ideal.

The next example shows that it is possible that the module $\mathrm{Hom}_R(R/\mathfrak{a}, R/\mathfrak{b})$ is neither cyclic nor decomposable. It also shows that the bound in Theorem 1.15 may be quite far from optimal.

**Example 3.4.** Consider the parameter $y^2$ of $R = k[\![x, y]\!]/(x^2, xy^m)$ for $m \geq 3$. Set

$$U_t := \mathrm{Hom}_R(R/(y^2), R/(y^t)).$$

We claim that $U_t$ is

$$\begin{cases} \text{cyclic,} & \text{if } t < m+1, \\ \text{indecomposable, but not cyclic,} & \text{if } t = m+1 \\ \text{decomposable,} & \text{if } t > m+1. \end{cases} \tag{3.2}$$

For small values of $t$, this is easily seen to be cyclic (and hence indecomposable). If $t = 0$, then

$$U_t = \mathrm{Hom}_R(R/(y^2), R) \cong (0) : y^2 = (xy^{m-2})$$

is cyclic. When $t = 1$, we have that

$$U_t = \mathrm{Hom}_R(R/(y^2), R/(y)) \cong \frac{(y) : y^2}{(y)} = R/(y)$$

is also cyclic. When $t = 2$, then

$$U_t = \mathrm{Hom}_R(R/(y^2), R/(y^2)) \cong \frac{(y^2) : y^2}{(y^2)} = R/(y^2)$$

is also cyclic. For what follows, we'll focus on the case that $t \geq 3$ and the representation of $U_t$ as $\dfrac{(y^t) : y^2}{(y^t)}$. We will also use the following notation. Let $r \in R$ and write $r$ in coset notation:

$$r = \sum_{i=0}^{\infty} a_i y^i + \sum_{i=0}^{m-1} b_i x y^i + (x^2, xy^m). \tag{3.3}$$

In this notation

$$ry^2 = \sum_{i=0}^{\infty} a_i y^{i+2} + \sum_{i=0}^{m-3} b_i x y^{i+2} + (x^2, xy^m). \tag{3.4}$$

Suppose that $3 \leq t < m + 1$. We show that $U_t$ is cyclic for all such values of $t$. We have that $ry^2 \in (y^t)$ if and only if the only nonzero terms of (3.4) are those in either sum with $i + 2 \geq t$. This means that $a_i = b_i = 0$ for $i = 0, 1, \ldots, t - 1$. As $t - 2 \leq m - 2$ this gives that $ry^2 \in (y^t)$ if and only if

$$r = \sum_{i=t-2}^{\infty} a_i y^i + \sum_{i=t-2}^{m-2} b_i x y^i + (x^2, xy^m).$$

That is, $r \in ((y^t) : y^2)$ if and only if $r \in (y^{t-2})$. Thus, whenever $3 \leq t < m + 1$, the module $U_t \cong (y^{t-2})/(y^t)$ is cyclic.

Next consider the case that $t > m + 1$. We again have that $ry^2 \in (y^t)$ if and only if the only nonzero terms of (3.4) are those in either sum with $i + 2 \geq t$. Since $t - 2 > m - 2$ we know that the only nonzero summands appear in the first sum. This means that $a_i = 0$ for $i = 0, 1, \ldots, t - 1$ and $b_i = 0$ for $i = 0, \ldots, m - 3$. Thus

$ry^2 \in (y^t)$ if and only if

$$r = \sum_{i=t-2}^{\infty} a_i y^i + \sum_{i=m-2}^{m-1} b_i x y^i + (x^2, xy^m).$$

That is, $r \in ((y^t) : y^2)$ if and only if $r \in (y^{t-2}, xy^{m-2})$. Moreover, we claim that

$$(y^{t-2}) \cap (xy^{m-2}, y^t) = (y^t).$$

Indeed, we have that $t \geq m+2$ and so writing $f \in (y^{t-2}) \cap (xy^{m-2}, y^t)$ as

$$f = ay^{t-2} = bxy^{m-2} + cy^t$$

with $a, b, c \in R$ we see that $bxy^{m-2}$ must be of the form $b'xy^{t-2} = 0$, since $R$ is a quotient of the unique factorization domain $k[\![x, y]\!]$. Hence $f \in (y^t)$. The other inclusion is clear. This gives the desired decomposition for $t > m+1$:

$$U_t \cong \frac{(y^t) : y^2}{(y^t)} = \frac{(xy^{m-2}, y^{t-2})}{(y^t)} \cong \frac{(xy^{m-2}, y^t)}{(y^t)} \oplus \frac{(y^{t-2})}{(y^t)}.$$

We finally consider the case where $t = m+1$. For this case we have that $ry^2$ is in $(y^t) = (y^{m+1})$ if and only if the only nonzero terms of (3.4) are those with $i + 2 \geq m+1$, or equivalently $i \geq m-1$. Since $m-1 > m-3$ all of these nonzero terms are in the first sum of (3.4). Thus $ry^2 \in (y^t)$ if and only if $a_i = 0$ for $i < m-1$ and $b_i = 0$ for $i < m-2$. This gives $r \in ((y^t) : y^2)$ if and only if $r \in (xy^{m-2}, y^{m-1})$ so that

$$U_t = \operatorname{Hom}_R(R/(y^2), R/(y^t)) \cong \frac{(y^t) : y^2}{(y^t)} = \frac{(xy^{m-2}, y^{m-1})}{(y^t)}.$$

We now claim that this module is indecomposable. To that end, note that the set $\{y^{m-1}, y^m, xy^{m-2}, xy^{m-1}\}$ forms a $k$-basis for $U_t$. We will show the non-units of $\operatorname{End}_R(U_t)$ form a two-sided ideal. Let $\phi \in \operatorname{End}_R(U_t)$ and write

$$\phi(y^{m-1}) = ay^{m-1} + by^m + cxy^{m-2} + dxy^{m-1}$$

and

$$\phi(xy^{m-2}) = ey^{m-1} + fy^m + gxy^{m-2} + hxy^{m-1}$$

where $a, b, c, d, e, f, g, h \in k$. Since $\phi$ is $R$-linear, we know that $y\phi(xy^{m-2}) = \phi(xy^{m-1}) = x\phi(y^{m-1})$. Hence,

$$ey^m + gxy^{m-1} = y\phi(xy^{m-2}) = x\phi(y^{m-1}) = axy^{m-1}$$

which implies $e = 0$ and $g = a$. In turn, this gives

$$\phi(y^m) = y\phi(y^{m-1}) = ay^m + cxy^{m-1}.$$

We may thus represent $\phi$ as a matrix in $M_4(k)$ using $y^{m-1}, y^m, xy^{m-2}, xy^{m-1}$ as the ordering of the basis:

$$\phi = \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & f & 0 \\ c & 0 & a & 0 \\ d & c & h & a \end{pmatrix}.$$

Hence $\operatorname{End}_R(U_t)$ is the set of all matrices of the form of $\phi$. Note that the determinant of such a matrix is $a^4$ and so $\phi \in \operatorname{End}_R(U_t)$ is a unit if and only if the diagonal entry in the matrix representation of $\phi$ is nonzero. Let $I \subseteq \operatorname{End}_R(U_t)$ be the set of all non-units of $\mathrm{End}_R(U_t)$. That is, $I$ is the set of matrices in $M_4(k)$ of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & f & 0 \\ c & 0 & 0 & 0 \\ d & c & h & 0 \end{pmatrix}$$

where $b, c, d, f, h \in k$. The sum of any two matrices in $I$ is clearly in $I$. Also, letting $X \in I$ and $Y \in \mathrm{End}_R(U_t)$ we have

$$XY = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & f & 0 \\ c & 0 & 0 & 0 \\ d & c & h & 0 \end{pmatrix} \cdot \begin{pmatrix} a' & 0 & 0 & 0 \\ b' & a' & f' & 0 \\ c' & 0 & a' & 0 \\ d' & c' & h' & a' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a'b + fc' & 0 & a'f & 0 \\ a'c & 0 & 0 & 0 \\ a'd + b'c + c'h & a'c & f'c + a'h & 0 \end{pmatrix}$$

and

$$YX = \begin{pmatrix} a' & 0 & 0 & 0 \\ b' & a' & f' & 0 \\ c' & 0 & a' & 0 \\ d' & c' & h' & a' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & f & 0 \\ c & 0 & 0 & 0 \\ d & c & h & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a'b + f'c & 0 & a'f & 0 \\ a'c & 0 & 0 & 0 \\ bc' + ch' + a'd & a'c & f'c + a'h & 0 \end{pmatrix}$$

which are both elements of $I$. Thus, $I$ is a two sided ideal of $\mathrm{End}_R(U_t)$ so that $U_t$ is an indecomposable $R$-module by Proposition 3.3.

This completes the proof of the assertions in (3.2). We showed in Example 3.2 that $\mathfrak{m}^i \cap \Gamma_{\mathfrak{m}}(R) = (0)$ if and only if $i \geq m+1$. Theorem 1.15 thus only predicts that $U_t$ decomposes for $t \geq 2m+4$ and so in this case the bound obtained in Theorem 1.15 is especially poor for large values of $m$.

*Remark* 3.5. A more efficient way to compute $\mathrm{End}_R(U_{m+1})$ for a fixed value of $m$ and field $k$ in Example 3.4 is using Macaulay2 [4]. The following code, generated with the help of Frank Moore, determines the form of an arbitrary element of $\mathrm{End}_R(U_t)$ given a module $U_t$ which is an ideal in a quotient of a polynomial ring $R$:

```
linearMaps = method()
linearMaps (Module, Symbol) := (Ut, T) -> (
   --- We first set up the objects we need to work with.
   R := ring Ut;
   gensUt := flatten entries gens Ut;
   dimenUt := numgens source basis Ut;
   numgensUt := #gensUt;
   varList := toList(T_(1,1)..T_(numgensUt,dimenUt));
   B := QQ[varList,MonomialOrder=>Lex];
   rGens := gens R;
   idealR := ideal R;
   Q := B[rGens];
   S := Q/sub(idealR,Q);
   --- This builds the k-matrix representing an R-linear map.
   basisUt := sub((gens Ut) * (matrix basis Ut),S);
   M1 := transpose sub(genericMatrix(B,dimenUt,numgensUt),S);
   M2 := (transpose basisUt);
   genMatr := M1 * M2;
   M3 := transpose genMatr * sub(matrix basis Ut,S);
   Temp1 := coefficients(M3, Monomials=>flatten entries basisUt);
   myMatrix := last Temp1;
```

```
    --- Now we find the relationships between the coefficients
    --- coming from the fact that some elements of the basis
    --- can be obtained in more than one way.
    syzMatr := transpose genMatr * syz sub(matrix {gensUt},S);
    Temp2 := flatten entries basisUt;
    Temp3 := coefficients(syzMatr, Monomials => Temp2);
    myEquations := select(flatten entries last Temp3, f -> f != 0);
    --- Finally, we apply the relations to the k-matrix
    --- to obtain an arbitrary element of End_R(Ut).
    C := B/apply(myEquations,f -> sub(f,B));
    QQQ := C[rGens];
    SS := QQQ/sub(idealR, QQQ);
    sub(myMatrix,C)
)
```

With this algorithm in Macaulay2, we are prepared to compute an arbitrary element of $\mathrm{End}_R(U_t)$. In order to do this for the example $R = \mathbb{Q}[\![x, y]\!]/(x^2, xy^m)$ and $U_t = \mathrm{Hom}_R(R/(y^2), R/(y^t))$ with $m = 3$ and $t = 4$ we use the following input. The first two input lines define the ring $R$ and module $U_t$.

```
i1 : R = QQ[x,y]/ideal(x^2,x*y^3);
i2 : Ut = Hom(coker matrix{{y^2}},coker matrix{{y^4}});
i3 : M = linearMaps(Ut,T)
```

The output for the third line is

```
o3 = | T_(2,3) 0       0       0       |
     | T_(1,2) T_(2,3) T_(2,2) 0       |
```

```
| T_(1,3) 0       T_(2,3) 0       |
| T_(1,4) T_(1,3) T_(2,4) T_(2,3) |
```

which is an arbitrary element of $\mathrm{End}_R(U_t)$. By making the following identification of the coefficients used in Example 3.4 and by Macaulay2:

```
T_(1,1) = a = T_(2,3)                T_(2,1) = e = 0
T_(1,2) = b                          T_(2,2) = f
T_(1,3) = c                          T_(2,4) = h
T_(1,4) = d
```

we see that this is the same as the matrix as the one painstakingly computed by hand in Example 3.4.

Theorem 1.15 and Theorem 1.17, which give bounds on the powers needed to make the module $\mathrm{Hom}_R(R/\mathfrak{a}, M/\mathfrak{b}M)$ decompose and be non-free, apply only in dimension one. However, examples seem to indicate that the $R/\mathfrak{a}$-module

$$\mathrm{Hom}_R(R/\mathfrak{a}, R/(a_1^{n_1}, \ldots, a_d^{n_d}))$$

is neither free nor indecomposable if the $n_i$ are large enough. One such example is explained below.

**Example 3.6.** Consider the ring $R = k[\![x, y, z]\!]/(x^2, xyz)$. We saw in Example 2.10 that this ring has dimension two and depth one. Note that $\{y, z\}$ is a system of parameters of $R$. If $n_1 \geq 2$, then $S_{n_1} := R/(y^{n_1})$ is not Cohen-Macaulay. Indeed, as $y^{n_1}$ is a parameter, $S_{n_1}$ has dimension one. The depth of $S_{n_1}$ is zero since the nonzero element $xy^{n_1-1}$ is in the socle. Letting $\mathfrak{m}$ be the maximal ideal of $S_{n_1}$ we have $\mathfrak{m}^i \cap \Gamma_\mathfrak{m}(S_{n_1}) = 0$ if and only if $i \geq n_1 + 2$. By symmetry, the same holds for the ring

$T_{n_2} := R/(z^{n_2})$. Thus Theorem 1.15 gives that $U_{n_1,n_2} := \mathrm{Hom}_R(R/(y,z), R/(y^{n_1}, z^{n_2})$ decomposes for all $n_1, n_2 \geq 2$ with $|n_1 - n_2| > 2$. However, direct computation shows that $U_{n_1,n_2}$ actually decomposes as

$$U_{n_1,n_2} \cong \frac{(xy^{n_1-1}, y^{n_1}, z^{n_2})}{(y^{n_1}, z^{n_2})} \oplus \frac{(xz^{n_2-1}, y^{n_1}, z^{n_2})}{(y^{n_1}, z^{n_2})} \oplus \frac{(y^{n_1-1}z^{n_2-1}, y^{n_1}, z^{n_2})}{(y^{n_1}, z^{n_2})}$$

for all $n_1, n_2 \geq 2$. See Figure 3.1 for a visual representation of this.

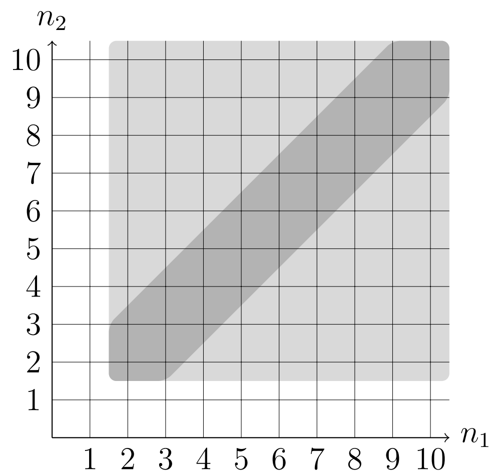


Figure 3.1: In this figure, a lattice point $(n_1, n_2)$ corresponds to the module $\mathrm{Hom}_R(R/(y,z), R/(y^{n_1}, z^{n_2}))$ from Example 3.6. The modules corresponding to lattice points in the light grey regions are known to decompose due to Theorem 1.15. The modules corresponding to lattice points in the middle dark grey region are known to decompose by direct computation. The modules corresponding to lattice points where $n_1 = 1$ or $n_2 = 1$ are indecomposable since $R/(y)$ and $R/(z)$ are both Cohen-Macaulay rings.

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