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OSCILLATION THEORY OF DYNAMIC EQUATIONS ON TIME SCALES

by

Raegan J. Higgins

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OSCILLATION THEORY OF DYNAMIC EQUATIONS ON TIME SCALES

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University of Nebraska, 2008

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In past years mathematical models of natural occurrences were either entirely continuous or discrete. These models worked well for continuous behavior such as population growth and biological phenomena, and for discrete behavior such as applications of Newton's method and discretization of partial differential equations. However, these models are deficient when the behavior is sometimes continuous and sometimes discrete. The existence of both continuous and discrete behavior created the need for a different type of model. This is the concept behind dynamic equations on time scales. For example, dynamic equations can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

Throughout this work, we will be concerned with certain dynamic equations on time scales. We start with a brief introduction to the time scale calculus and some theory necessary for the new results. The main concern will then be the oscillatory behavior of solutions to certain second order dynamic equations. In Chapter 3, an equation of particular interest is one containing both advanced and delayed arguments. We will use the method of Riccati substitution to prove some oscillation results of the solutions.

In Chapter 4 we again study the oscillatory behavior of a second dynamic equation. However, in this chapter, the equation only has delayed arguments. In addition to using Riccati substitution, we use the method of upper and lower solutions to develop necessary and sufficient conditions for oscillatory solutions. In the final chapter we are

interested in the existence of nonoscillatory solutions of dynamic equations on time scales. The common theme among these results is the use of the Riccati substitution technique and the integration of dynamic inequalities.

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Contents

Chapter 1

Introduction

The theory of time scales is a new area of mathematics that unifies and extends discrete and continuous analysis. The time scale calculus allows us to model situations in which the behavior is both continuous and discrete. For example, it can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a non overlapping population.

In recent years there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on time scales. Already many results concerning second order dynamic equations have been established [3, 7, 15]. In this present work we aim to extend the results of [11] and [16] to dynamic equations on time scales and to improve those of [26]. For oscillation of nonlinear delay dynamic equations, Zhang and Shanliang [26] considered the equation

$$y^{\Delta\Delta}(t) + q(t)f(y(t-\tau)) = 0, \quad t \in \mathbb{T} \tag{1.1}$$

where $\tau \in \mathbb{R}$ and $t - \tau \in \mathbb{T}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and $uf(u) > 0$ for $u \neq 0$. By using comparison theorems, they proved that the oscillation of (1.1) is equivalent to that of the nonlinear dynamic equation

$$y^{\Delta\Delta}(t) + q(t)f(y^{\sigma}(t)) = 0, \quad t \in \mathbb{T} \tag{1.2}$$

where $\sigma(t)$ is the next point in the time scale, and established some sufficient conditions for oscillation by applying the results established in [9] for (1.2) on unbounded above time scales. In Chapter 3 we show that the oscillation of

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0,$$

where $\tau(t)$ is a delay given by a function, τ, of t, is equivalent to that of

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0.$$

on an isolated time scale $\mathbb{T}$ where $\sup \mathbb{T} = \infty$.

In extending the results of [11] to dynamic equations on time scales, we establish oscillation criteria for the second order nonlinear dynamic equation

$$y^{\Delta\Delta} + f\left(t, y^{\sigma}(t), y(\tau(t))\right) = 0 \tag{1.3}$$

with retarded argument in Chapter 4. In order to obtain the results for (1.3), we improve and extend some results of [6] and [17].

In the final chapter, Chapter 5, we are interested in the asymptotic behavior of solutions of dynamic equations on time scales. In [16], the author obtains necessary and sufficient conditions for the existence of a bounded nonoscillatory solution of $y'' + f(t, y)g(y') = 0$ with a prescribed limit at ∞ and necessary and sufficient conditions for a nonoscillatory solution whose derivative has a positive limit at ∞. We extend some of these results to

$$y^{\Delta\Delta} + f(t, y^{\sigma})g(y^{\Delta}) = 0.$$

Chapter 2

Preliminaries

In this chapter we introduce some basic concepts concerning the calculus on time scales. Most of these results will be stated without proof. The proofs can be found in [2] and [5].

2.1 The Calculus on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales.

Definition 2.1.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\},$$

and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}.$$

In the case that $\{s \in \mathbb{T} : s > t\}$ is empty, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t). Similarly, if $\{s \in \mathbb{T} : s < t\}$ is empty, we put $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t).

If $f : \mathbb{T} \to \mathbb{R}$ is a function, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

Points are classified as follows: If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is said to be *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-scattered and left-scattered at the same time are called *isolated*, and points that are both right and left dense are called *dense*.

Definition 2.1.2. The *graininess function*, $\mu : \mathbb{T} \to [0, \infty)$, is defined by

$$\mu(t) := \sigma(t) - t.$$

The *backward graininess function*, $\nu : \mathbb{T} \to [0, \infty)$, is defined by

$$\nu(t) := t - \rho(t).$$

Definition 2.1.3. We also need below the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows: If the maximum, m, of $\mathbb{T}$ is left-scattered, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Throughout this work we make the blanket assumption that a and b are points in $\mathbb{T}$. Often we assume $a \le b$. We then define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \le t \le b\}.$$

2.2 Differentiation

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called delta derivative of f at a point $t \in \mathbb{T}^\kappa$. By convention we will define $\lim_{s \to t} f(s) = f(t)$ if t is an isolated point.

Definition 2.2.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t. Moreover, we say that f is *delta differentiable* (or in short: *differentiable*) on $\mathbb{T}^\kappa$ provided f^Δ exists for all $t \in \mathbb{T}^\kappa$.

Some useful relationships concerning the delta derivative are now given.

Theorem 2.2.2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are functions and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t)$.

Looking at properties *(ii)* and *(iii)* in the above theorem gives us a more intuitive understanding of the derivative that cannot be gained via the definition alone. If $t \in \mathbb{T}$ is right dense, then the delta-derivative behaves much the same way as the usual derivative. It can be viewed as the slope of the tangent line to the function at t, although if t is both right-dense and left-scattered, the limit is a one-sided limit. On the other hand, if t is right-scattered, then $f^{\Delta}(t)$ is the slope of the line segment containing $f(t)$ and $f(\sigma(t))$. In this instance, the behavior of the function to the left of t is irrelevant beyond the requirement that f is continuous at t. Thus, the delta derivative combines the discrete behavior of the forward difference operator and the continuous behavior of the usual derivative.

We next provide the theorem that allows us to find the derivative of sums, products, and quotients of differentiable functions.

Theorem 2.2.3. *Assume f, g are differentiable at $\mathbb{T}^{\kappa}$. Then:*

(i) The sum $f + g : f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) = f(t) g^{\Delta}(t) + f^{\Delta}(t) g(\sigma(t)).$$

(iv) If $g(t)g(\sigma(t)) \neq 0$, then $\dfrac{f}{g}$ is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}.$$

Finally, we present a chain rule which calculates $(f \circ g)^{\Delta}$, where

$$g : \mathbb{T} \to \mathbb{R} \quad \text{and} \quad f : \mathbb{R} \to \mathbb{R}.$$

This chain rule is due to Christian Pötzsche, who derived it first in 1998.

Theorem 2.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$ and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t)$$

holds $t \in \mathbb{T}^{\kappa}$.

2.3 Integration

Of course, the calculus on time scales would not be complete without a concept of integration to complement the derivative. In order to describe functions that are "integrable," we introduce the following concept.

Definition 2.3.1. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this dissertation by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2.3.2. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \quad \forall\, t \in \mathbb{T}^{\kappa}.$$

Then we define the *Cauchy integral* by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \forall\, a, b \in \mathbb{T}.$$

Theorem 2.3.3. *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad \textit{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

The following theorem provides useful properties of delta integrals.

Theorem 2.3.4. *If $a,b,c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f,g \in C_{rd}$, then*

(i) $\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$ *where f, g can be interchanged;*

(vi) If $|f(t)| \leq g(t)$ on $[a,b)$, then

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b g(t)\Delta t;$$

(vii) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$.

The following result provides useful properties of the delta integral.

Theorem 2.3.5. *Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $[a,b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b. \end{cases}$$

Chapter 3

Oscillation Criteria for Functional Dynamic Equations

3.1 Oscillation of Nonlinear Dynamic Equations

We shall consider the second order nonlinear functional dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0 \tag{3.1}$$

and the second order nonlinear dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0 \tag{3.2}$$

on an isolated time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$. We assume $p, q, \tau,$ and f satisfy the following **Condition (*E*)**:

(i) $p \in C_{rd}(\mathbb{T}, (0, \infty))$ satisfies $\displaystyle\int_{t_0}^{\infty} \frac{1}{p(t)}\,\Delta t = \infty, \quad t \in \mathbb{T}.$

(ii) $q \in C_{rd}(\mathbb{T}, \mathbb{R}^{+})$.

(iii) $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ satisfies

$$\lim_{t\to\infty} \tau(t) = \infty \quad \text{and} \quad \exists\, M > 0 \text{ such that } |R(t) - R(\tau(t))| < M \ \forall\, t \in \mathbb{T}$$

where $R(t) = \displaystyle\int_{t_0}^{t} \frac{1}{p(s)}\,\Delta s.$

(iv) $f : \mathbb{R} \to \mathbb{R}$ is continuous, increasing, and

$$f(-u) = -f(u) \text{ for } u \in \mathbb{R} \quad \text{and} \quad uf(u) > 0 \text{ for } u \neq 0.$$

By a solution of (3.1) we mean a nontrivial real-valued function y satisfying (3.1) for $t \geq t_0 \geq a \in \mathbb{T}$, where $a > 0$. A solution y of (3.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (3.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of $(py^\Delta)^\Delta + q(t)f(y(t)) = 0$ which exist on some half line $[t_y, \infty)_\mathbb{T}$ and satisfy $\sup\{|y(t)| : t > t_0\} > 0$ for any $t_0 \geq t_y$.

Definition 3.1.1. A nonempty closed subset K on a Banach space X is called a cone if it possess the following properties:

(i) if $\alpha \in \mathbb{R}^+$ and $x \in K$, then $\alpha x \in K$.

(ii) if $x, y \in K$, then $x + y \in K$.

(iii) if $x \in K \setminus \{0\}$, then $-x \in K$.

Let X be a Banach space and K be a cone with nonempty interior. Then we define a partial ordering $\leq$ on X by

$$x \leq y \quad \text{if and only if} \quad y - x \in K.$$

Our main result, which follows, is an extension of Theorem 2.1 of [26].

Theorem 3.1.2. *Assume (E) holds and* $\dfrac{\mu(t)}{p(t)}$ *is bounded. We further assume* $\tau(t) \leq \sigma(t)$ *for all* t *or* $\tau(t) \geq \sigma(t)$ *for all* t*. Then the oscillation of the second order nonlinear dynamic equation*

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0 \tag{3.2}$$

is equivalent to the oscillation of the second order nonlinear functional dynamic equation

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) = 0. \tag{3.1}$$

We will need the following fixed-point theorem [10].

Theorem 3.1.3. *(Knaster's Fixed-Point Theorem) Let X be a partially ordered Banach space with ordering $\leq$. Let Ω be a subset of X with the following properties: The infimum of Ω belongs to Ω and every nonempty subset of Ω has a supremum which belongs to Ω. If $S : \Omega \to \Omega$ is an increasing mapping, then S has a fixed point in Ω.*

In order to prove Theorem 3.1.2, we will need to begin with the following lemmas.

Lemma 3.1.4. *Assume that (E) holds. A necessary and sufficient condition for equation (3.2) to be oscillatory is that, the inequality*

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) \leq 0, \tag{3.3}$$

has no eventually positive solutions.

Proof. SUFFICIENCY. Assume (3.3) has no eventually positive solutions. Then neither does (3.2), and so $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$ is oscillatory. If y is an eventually negative solution of (3.2), then let $x = -y$. Then x is eventually positive and

$$(px^{\Delta})^{\Delta} + qf(x^{\sigma}) = -(py^{\Delta})^{\Delta} - qf(y^{\sigma}) = -\left[(px^{\Delta})^{\Delta} + qf(x^{\sigma})\right] = 0$$

for $t \geq T$ sufficiently large by Condition (E) (iv). Thus x is an eventually positive solution of (3.3), which is a contradiction. Hence, $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$ is oscillatory.

NECESSITY. Suppose that (3.2) is oscillatory, and by way of contradiction, assume that (3.3) has an eventually positive solution y, namely, there exists $t_0 \in \mathbb{T}$ ($t_0 \geq a$) such that $y(t) > 0$ for $t \geq t_0$. As $\sigma(t) \geq t$ for all t, $\sigma(t) \geq t_0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then $y^{\sigma}(t) > 0$ for $t \geq t_0$. Using this fact along with the sign condition on f in (E), we have $[p(t)y^{\Delta}(t)]^{\Delta} \leq 0$ for $t \geq t_0$, and so $p(t)y^{\Delta}(t)$ is decreasing on $[t_0, \infty)_{\mathbb{T}}$.

We claim that $y^{\Delta}(t) > 0$ for all large t. If not, then for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$, we have $y^{\Delta}(t_1) \leq 0$. It follows that $p(t)y^{\Delta}(t) \leq 0$, $t \in [t_1, \infty)$. Now, if $y^{\Delta}(t_2) < 0$ for some $t_2 \geq t_1$, then

$$\begin{aligned}
y(t) - y(t_2) &= \int_{t_2}^{t} y^{\Delta}(s)\,\Delta s \\
&= \int_{t_2}^{t} \frac{p(s)y^{\Delta}(s)}{p(s)}\,\Delta s \\
&\leq p(t_2)y^{\Delta}(t_2) \int_{t_2}^{t} \frac{\Delta s}{p(s)} \\
&\to -\infty \text{ as t} \to \infty,
\end{aligned}$$

which is a contradiction to our assumption that $y(t) > 0$ for $t \geq t_0$. Hence it follows that $y^{\Delta}(t) \equiv 0$ on $[t_1, \infty)$, and so $(p(t)y^{\Delta}(t))^{\Delta} \equiv 0$ and $q(t)f(y^{\sigma}(t)) > 0$, which is contradictory. Consequently, there exists $T \in \mathbb{T}$ ($T \geq t_0$) such that

$$y(t) > 0, \qquad y^{\Delta}(t) > 0, \qquad \text{and} \qquad (p(t)y^{\Delta}(t))^{\Delta} \leq 0$$

for all $t \geq T$. Since $p(t)y^{\Delta}(t)$ is continuous, the integrals below are well-defined. Integrating $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) \leq 0$ from t to s yields

$$p(s)y^{\Delta}(s) - p(t)y^{\Delta}(t) + \int_{t}^{s} q(u)f(y^{\sigma}(u))\,\Delta u \leq 0, \quad \text{for} \quad s, t \in \mathbb{T} \quad \text{and} \quad s \geq t,$$

i.e.,

$$p(t)y^{\Delta}(t) \geq p(s)y^{\Delta}(s) + \int_{t}^{s} q(u)f(y^{\sigma}(u))\,\Delta u. \tag{3.4}$$

Since $p(t)y^{\Delta}(t) > 0$ is decreasing for $t \geq T$, $\lim_{t\to\infty} p(t)y^{\Delta}(t) = k \geq 0$ exists. Letting $s \to \infty$ in (3.4) we obtain

$$y^{\Delta}(t) \geq \frac{k}{p(t)} + \frac{1}{p(t)} \int_t^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u \geq \frac{1}{p(t)} \int_t^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u. \tag{3.5}$$

Since $\int_t^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u$ exists and is continuous, integrating (3.5) from T to t yields

$$y(t) \geq y(T) + \int_T^t \frac{1}{p(s)} \int_s^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u\,\Delta s, \qquad t \geq T. \tag{3.6}$$

Define X to be the Banach space of all continuous functions on $[a,\infty)_{\mathbb{T}}$ satisfying $\lim_{t\to\infty} x(t) = \infty$, where $\|\cdot\|$ is defined by

$$\|x\| := \max_{t\in[a,\infty)_{\mathbb{T}}} |x(t)| \quad \text{for all} \quad x \in X.$$

Let

$$\Omega := \left\{\omega \in C([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+) : 0 \leq \omega(t) \leq 1 \text{ and } \lim_{t\to\infty} \omega(t) = \infty \text{ for } t \geq t_0\right\},$$

which is endowed with the usual pointwise ordering $\leq$: $\omega_1 \leq \omega_2 \Leftrightarrow \omega_1(t) \leq \omega_2(t)$ for $t \geq t_0$.

One can show that for any nonempty subset N of Ω $\sup N \in \Omega$ and $\inf \Omega \in \Omega$. Define a mapping S on Ω by

$$(S\omega)(t) = \begin{cases} 1, & \text{if } t_0 \leq t \leq T, \\ \frac{1}{y(t)}\left(y(T) + \int_T^t \frac{1}{p(s)} \int_s^{\infty} q(u)f(y^{\sigma}(u)\omega^{\sigma}(u))\,\Delta u\,\Delta s\right), & \text{if } t \geq T. \end{cases}$$

We claim that $S\Omega \subset \Omega$ and S is monotone increasing. For any $\omega \in \Omega$, $(S\omega)(t)$ is certainly continuous and for $t \geq T$,

$$q(t)f(y^{\sigma}(t)\omega^{\sigma}(t)) \leq q(t)f(y^{\sigma}(t))$$

since $0 \leq \omega^{\sigma}(t) \leq 1$ and f is nondecreasing. Therefore, from (3.6), it follows that $0 \leq (S\omega)(t) \leq 1$ for $t \geq T$, and so $S(\omega) \in \Omega$. Moreover, if $\omega_1 \leq \omega_2$, $\omega_1, \omega_2 \in \Omega$, then, since f is nondecreasing, $f(y^{\sigma}(u)\omega_1(u)) \leq f(y^{\sigma}(u)\omega_2(u))$ and so $(S\omega_1)(t) \leq (S\omega_2)(t)$. Therefore, by Knaster's Fixed Point Theorem, there exists $\tilde{\omega} \in \Omega$ such that $S\tilde{\omega} = \tilde{\omega}$. Hence,

$$\tilde{\omega}(t) = \frac{1}{y(t)}\left(y(T) + \int_T^t \frac{1}{p(u)} \int_u^{\infty} q(v)f(y^{\sigma}(v)\tilde{\omega}^{\sigma}(v))\,\Delta v\,\Delta u\right), \text{ for } t \geq T.$$

Observe that
$$\tilde{\omega}(t) \geq \frac{y(T)}{y(t)} > 0 \quad \text{for } t \geq T.$$
Set $z(t) := \tilde{\omega}(t)y(t)$. Then $z(t) > 0$ is continuous and
$$z(t) = y(T) + \int_T^t \frac{1}{p(u)} \int_u^\infty q(v)f(z^\sigma(v))\,\Delta v\,\Delta u, \text{ for } t \geq T.$$
As $z^\Delta(t) = \dfrac{1}{p(t)} \displaystyle\int_t^\infty q(u)f(z^\sigma(u))\,\Delta u$ and $(p(t)z^\Delta(t))^\Delta = -q(t)f(z^\sigma(t))$, $(p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) = 0$ has a positive solution, which is a contradiction to the assumption that all solutions of (3.2) are oscillatory. This completes the proof. □

Lemma 3.1.5. *Assume that (E) holds. Then, every solution of the second order nonlinear functional dynamic equation* $(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) = 0$ *oscillates if and only if the inequality*
$$(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) \leq 0$$
has no eventually positive solutions.

The proof is similar to that of Lemma 3.1.4 and so we omit it.

We can now prove Theorem 3.1.2.

Proof of Theorem 3.1.2. Since $\dfrac{\mu}{p}$ is bounded, there exists $N > 0$ such that $\dfrac{\mu(t)}{p(t)} \leq N$ for all t. Let $K := M + N$, where $M > 0$ is such that
$$|R(t) - R(\tau(t))| < M \ \forall\ t \in \mathbb{T} \quad \text{where } R(t) = \int_{t_0}^t \frac{1}{p(s)}\,\Delta s.$$
SUFFICIENCY. The oscillation of (3.2) implies that of (3.1). Suppose, to the contrary, that y is a nonoscillatory solution of (3.1). We will only consider the case where there exists $t_0 \in \mathbb{T}$ such that $y(t) > 0$ for $t \geq t_0$, since the other case is similar.

From equation (3.1) and Condition (E), there exists $t_1 \in \mathbb{T}$ ($t_1 \geq t_0$) such that
$$y(t) > 0, \qquad (py^\Delta)(t) > 0, \qquad (py^\Delta)^\Delta(t) \leq 0, \qquad y(\tau(t)) > 0, \quad t \geq t_1$$
as in the proof of Lemma 3.1.4. Hence, since $p(t)y^\Delta(t) > 0$ is decreasing for $t \geq t_1$, $\lim_{t\to\infty} p(t)y^\Delta(t) = L \geq 0$ exists. We will distinguish several cases.

(I) Assume $\sigma(t) \leq \tau(t)$ for all t. It follows that $y(\tau(t)) \geq y^\sigma(t) > 0$ as y is increasing. Consequently,
$$(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) \leq (p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) = 0,$$

and so (3.3) has an eventually positive solution. By Lemma 3.1.4, equation (3.2) has a nonoscillatory solution, which is a contradiction.

(II) Suppose next that $\tau(t) \leq \sigma(t)$ for all t.

(a) Assume $L > 0$. Then there exists $t_2 \in \mathbb{T}$ with $t_2 \geq t_1$ such that $p(t)y^\Delta(t) \leq L+1$, for all $t \geq t_2$. Since $\lim\limits_{t\to\infty} \tau(t) = \infty$, there is a $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for $t \geq t_3$. Therefore, if $t \geq t_3$, we have

$$\begin{aligned}
y^\sigma(t) - y(\tau(t)) &= \int_{\tau(t)}^{\sigma(t)} \frac{p(s)y^\Delta(s)}{p(s)}\Delta s \\
&\leq (L+1)\int_{\tau(t)}^{\sigma(t)} \frac{\Delta s}{p(s)} \\
&= (L+1)[R^\sigma(t) - R(t) + R(t) - R(\tau(t))] \\
&\leq (L+1)\left[\left|\int_t^{\sigma(t)} \frac{\Delta s}{p(s)}\right| + |R(t) - R(\tau(t))|\right] \\
&\leq (L+1)\left[\frac{\mu(t)}{p(t)} + M\right].
\end{aligned}$$

Consequently,

$$y(\tau(t)) \geq y^\sigma(t) - (L+1)K, \quad t \geq t_3.$$

Let $z(t) = y(t) - (L+1)K$. Note that for all t large enough,

$$p(t)y^\Delta(t) \geq L.$$

By integrating both sides from t_0 to t we obtain

$$y(t) - y(t_0) \geq L\int_{t_0}^{t} \frac{1}{p(s)}\,\Delta s.$$

By letting $t \to \infty$, we see that $z(t) > 0$ for large enough t. Hence, for all sufficiently large t,

$$z(t) > 0, \quad z^\sigma(t) \leq y(\tau(t)), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) \leq 0.$$

Hence, (3.3) has an eventually positive solution. By Lemma 3.1.4, we have that $(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0$ is nonoscillatory, which is a contradiction.

(b) Assume $L = 0$. Since both $y^\Delta(t)$ and $y(t)$ are positive, there exists $\epsilon_0 > 0$ and $t_2 \geq t_1$ such that $y(t) > M\epsilon_0$ for all $t \geq t_2$. Corresponding to this ϵ_0, there exists $t_3 \geq t_1$ such that $p(t)y^\Delta(t) \leq \epsilon_0$ for all $t \geq t_3$. Now, if

$t \geq T := \max\{t_2, t_3\}$, we have

$$\begin{aligned} y^\sigma(t) - y(\tau(t)) &= \int_{\tau(t)}^{\sigma(t)} \frac{p(s)y^\Delta(s)}{p(s)}\,\Delta s \\ &\leq \epsilon_0 \int_{\tau(t)}^{\sigma(t)} \frac{\Delta s}{p(s)} \\ &\leq \epsilon_0 \left[\left| \int_t^{\sigma(t)} \frac{\Delta s}{p(s)} \right| + |R(t) - R(\tau(t))| \right] \\ &\leq \epsilon_0 \left[\frac{\mu(t)}{p(t)} + M \right]. \end{aligned}$$

Consequently,

$$y(\tau(t)) \geq y^\sigma(t) - \epsilon_0 K, \quad t \geq T.$$

Again, we set $z(t) := y(t) - \epsilon_0 K$. Then for sufficiently large t

$$z(t) > 0, \quad z^\sigma(t) \leq y(\tau(t)), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) \leq 0.$$

Hence, (3.3) has an eventually positive solution. Again by Lemma 3.1.4, $(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0$ is nonoscillatory, which is a contradiction.

NECESSITY. The oscillation of (3.1) implies that of equation (3.2). Suppose that there is a nonoscillatory solution $y(t)$ of (3.2) and without loss of generality, we assume there exists $t_1 \in \mathbb{T}$ such that

$$y(t) > 0, \quad p(t)y^\Delta(t) > 0, \quad \text{and} \quad (p(t)y^\Delta(t))^\Delta \leq 0, \quad t \geq t_1.$$

Since $p(t)y^\Delta(t) > 0$ is decreasing for $t \geq t_1$, $\lim\limits_{t\to\infty} p(t)y^\Delta(t) = L \geq 0$ exists. We distinguish several cases.

(I) Assume $\tau(t) \leq t$ for all t. As y is increasing, $y^\sigma(t) \geq y(\tau(t))$. Furthermore, as f is increasing, we have

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) \leq (p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0.$$

So $y(t)$ is an eventually positive solution of $(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) \leq 0$. By Lemma 3.1.5, equation (3.1) is nonoscillatory, which is a contradiction.

(II) Suppose $\tau(t) \geq \sigma(t)$ for all t.

(a) Assume $L > 0$. It follows that there exists $t_2 \in \mathbb{T}$ with $t_2 \geq t_1$ such that $p(t)y^\Delta(t) \leq L+1$ for all $t \geq t_2$. Since $\lim\limits_{t\to\infty} \tau(t) = \infty$, there is a $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for $t \geq t_3$. Therefore, if $t \geq t_3$, we have

$$\begin{aligned} y(\tau(t)) - y^\sigma(t) &= \int_{\sigma(t)}^{\tau(t)} \frac{p(s)y^\Delta(s)}{p(s)}\,\Delta s \\ &\leq (L+1)[R(\tau(t)) - R(\sigma(t))] \\ &\leq (L+1)[R(\tau(t)) - R(t) + R(t) - R(\sigma(t))] \\ &\leq (L+1)[M+N], \end{aligned}$$

which leads to

$$y^\sigma(t) \geq y(\tau(t)) - (L+1)K, \quad t \geq t_3.$$

Let $z(t) := y(t) - (L+1)K$. Then for sufficiently large t, we have

$$z(t) > 0, \quad z(\tau(t)) \leq y^\sigma(t), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z(\tau(t))) \leq 0.$$

This leads to a contradiction as in part (I) above.

(b) Assume $L = 0$. Since $y^\Delta(t) > 0$ and $y(t) > 0$, there is an $\epsilon_0 > 0$ and a $t_2 \geq t_1$ such that $y(t) > M\epsilon_0$ for all $t \geq t_2$. Corresponding to this ϵ_0, there exists $t_3 \geq t_1$ such that $p(t)y^\Delta(t) \leq \epsilon_0$ for all $t \geq t_3$. Now, if $t \geq T := \max\{t_2, t_3\}$, we have

$$\begin{aligned} y(\tau(t)) - y^\sigma(t) &= \int_{\sigma(t)}^{\tau(t)} \frac{p(s)y^\Delta(s)}{p(s)}\,\Delta s \\ &\leq \epsilon_0 \int_{\sigma(t)}^{\tau(t)} \frac{\Delta s}{p(s)} \\ &\leq \epsilon_0 \left[M + \frac{\mu(t)}{p(t)}\right], \end{aligned}$$

and so $y^\sigma(t) \geq y(\tau(t)) - \epsilon_0 K$ for $t \geq T$. Now set $z(t) := y(t) - \epsilon_0 K$. Then for sufficiently large t

$$z(t) > 0, \quad z(\tau(t)) \leq y^\sigma(t), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) \leq 0,$$

which again leads to a contradiction.

This completes the proof. □

Remark 3.1.6. Under the assumptions Theorem 3.1.2 we see that the functional τ in equation (3.1) has no influence on its oscillation.

As a corollary to Theorem 3.1.2 we have the following:

Corollary 3.1.7. *Let $\mathbb{T} = \mathbb{Z}$ and $\tau : \mathbb{Z} \to \mathbb{Z}$. Assume $q : \mathbb{N}_0 \to \mathbb{N}_0$ is continuous and f, τ, and p satisfy (E). Then, the oscillation of the two equations*

$$\Delta(p(t)\Delta y(t)) + q(t)f(y(t+1)) = 0$$

and

$$\Delta(p(t)\Delta y(t)) + q(t)f(y(\tau(t))) = 0$$

is equivalent.

Proof. Since $\mu(t) = 1 \ \forall \ t \in \mathbb{Z}$ and $y^\sigma(t) = y(t+1)$, the result follows from Theorem 3.1.2. $\square$

Remark 3.1.8. One can prove analogous results when considering

$$(p(t)y^\Delta(t))^\Delta + q_1(t)f_1(y(\tau_1(t))) + q_2(t)f_2(y(\tau_2(t))) = 0$$

and

$$(p(t)y^\Delta(t))^\Delta + q_1(t)f_1(y^\sigma(t)) + q_2(t)f_2(y^\sigma(t)) = 0$$

and their corresponding inequalities.

Let $r \in \mathcal{R}$ and assume that $p \cdot r$ is a differentiable function. Assume that
(E_1) There exists $M > 0$ such that $r(t)e_r(t, t_0) \leq M$ for all large t.
(E_2) Condition (E) holds, $\dfrac{\mu(t)}{p(t)}$ is bounded, and $|f(u)| \geq K|u|$ for $u \neq 0$ for some $K > 0$, and define the auxiliary functions

$$\begin{aligned}
H_1(t) &= H_1(t, t_0) := 1 + \frac{\mu(t)}{p(t)\int_{t_0}^{t} \frac{\Delta s}{p(s)}}, \\
H_2(t) &= H_2(t, t_0) := \frac{1 + \mu(t)r(t)}{p(t)e_r(t, t_0)}, \\
H_3(t) &= H_3(t, t_0) := e_r(\sigma(t), t_0)\left[Kp(t) + \frac{1}{2}r^\Delta(t) + \frac{r^2(t)}{4H_1(t)}\right], \\
H_4(t) &= H_4(t, t_0) := r(t) - \frac{r(t)(1 + \mu(t)r(t))}{H_1(t)},
\end{aligned}$$

for $t > t_0$, for some $t_0 \in \mathbb{T}$.

By combining Theorem 3.1.2 and Theorem 3.1 in [9], we obtain the following result:

Theorem 3.1.9. *Assume that (E_1) and (E_2) hold. Furthermore, assume that there exists $r \in \mathcal{R}^+$ such that $p \cdot r$ is differentiable and such that for any $t_0 \geq a$ there exists* *a $t_1 > t_0$ so that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} H(s)\Delta s = \infty,$$

where

$$H(t) = H(t,t_0) = H_3(t) - \frac{(H_4(t))^2 H_1(t)}{4H_2(t)},$$

for $t > t_0$. Then equation (3.1) is oscillatory on $[a,\infty)_{\mathbb{T}}$.

We end this section with comparing $(p(t)y^{\Delta})^{\Delta} + q(t)f(y(\tau(t)))$ to

$$(p(t)y^{\Delta}(t))^{\Delta} + \tilde{q}(t)g(y(\tilde{\tau}(t))) = 0, \tag{3.7}$$

on a time scale $\mathbb{T}$ where $\tilde{q}, g,$ and $\tilde{\tau}$ satisfy condition (E) and $\dfrac{\mu}{p}$ is bounded.

From Theorem 3.1.2 we see that the oscillation of (3.7) is equivalent to that of

$$(p(t)y^{\Delta}(t))^{\Delta} + \tilde{q}(t)g(y^{\sigma}(t)) = 0. \tag{3.8}$$

We get the following result.

Theorem 3.1.10. *Assume condition (E) holds and $\dfrac{\mu}{p}$ is bounded on $\mathbb{T}$. Further assume that $\tilde{q}(t) \le q(t)$ for all large t and $|g(u)| \le |f(u)|$ for $|u| > 0$. Then, the oscillation of equation (3.7) implies that of equation (3.1).*

Proof. Otherwise, without loss of generality, we assume that (3.1) has an eventually positive solution. From Theorem 3.1.2, equation (3.2) also has an eventually positive solution $y(t)$. Then

$$(p(t)y^{\Delta}(t))^{\Delta} + \tilde{q}(t)g(y^{\sigma}(t))) \le (p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t))) = 0,$$

which implies (3.8) has an eventually positive solution. So, equation (3.7) also has an eventually positive solution, which is a contradiction. $\square$

3.2 Oscillation of a Linear Dynamic Equation

In this section we give two theorems about the oscillatory behavior of

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)y^{\sigma}(t) = 0 \tag{3.9}$$

on a time scale $\mathbb{T}$ where $\sup \mathbb{T} = \infty$, $p \in C_{rd}(\mathbb{T},(0,\infty))$ and $q \in C_{rd}(\mathbb{T},\mathbb{R})$. These are Theorems 3.2.2 and 3.2.7.

We impose the following condition

$$\int_a^\infty \frac{1}{p(s)}\,\Delta s = \infty \quad \text{and} \quad \int_a^\infty q(s)\,\Delta s < \infty \quad \text{for some} \quad a \in \mathbb{T}. \tag{E$_3$}$$

To prove our main result, we need the following lemma.

Lemma 3.2.1. *[14] Assume*

$$\liminf_{t\to\infty} \int_T^t q(s)\,\Delta \ge 0 \qquad \textit{and } \not\equiv 0 \tag{E$_4$}$$

for all large T, and

$$\int_\tau^\infty \frac{1}{p(s)}\,\Delta s = \infty. \tag{E$_5$}$$

If y is a solution of (3.9) such that $y(t) > 0$, for $t \in [T,\infty)_{\mathbb{T}}$, then there exists $S \in [T,\infty)_{\mathbb{T}}$ such that $y^\Delta(t) > 0$ for $t \in [S,\infty)_{\mathbb{T}}$.

Proof. The proof is by contradiction. We consider two cases:

(a) Suppose that $y^\Delta(t) < 0$ for $t \in [T,\infty)_{\mathbb{T}}$. Define $Q(t,T) = \int_T^t q(s)\Delta s$. We may assume, by condition (E$_4$), that T is such that $Q(t,T) \ge 0$ for $t \in [T,\infty)_{\mathbb{T}}$. Indeed, if no such T exists, then for $T \in [\tau,\infty)_{\mathbb{T}}$ fixed but arbitrary, we define

$$T_1 = T_1(T) := \sup\left\{t > T : \int_T^t q(s)\Delta s < 0\right\}.$$

If $T_1 = \infty$, then choosing $t_n \to \infty$ such that $Q(t_n,T) < 0$ for all n, we obtain a contradiction to (E$_4$). Hence, we must have T_1 is finite, which implies that $Q(t,T_1) \ge 0$ for $t \in [T_1,\infty)_{\mathbb{T}}$. Now an integration by parts gives (with $T_1 = T$)

$$\begin{aligned}
\int_T^t q(s)y^\sigma(s)\Delta s &= \int_T^t Q^\Delta(s,T)y^\sigma(s)\Delta s \\
&= Q(t,T)y(t) - \int_T^t Q(s,T)y^\Delta(s)\Delta s \\
&\ge 0.
\end{aligned}$$

Integrating (3.9) we have, from this last estimate,

$$y^\Delta(t) \le \frac{p(T)y^\Delta(T)}{p(t)} \tag{3.10}$$

for $t \in [T,\infty)_{\mathbb{T}}$. Integrating (3.10) for $t \ge T$ we see that $y(t) \to -\infty$ by (E$_5$), a contradiction. Therefore, $y^\Delta(t) < 0$ cannot hold for all large t.

(b) Next, if $y^\Delta(t) \not> 0$ eventually, then for every (large) $T \in [\tau, \infty)_\mathbb{T}$ there exists T_0 in $[T, \infty)_\mathbb{T}$ such that $y^\Delta(T_0) \le 0$ and we may suppose that $\liminf_{t\to\infty} \int_{T_0}^t q(s)\Delta s \ge 0$. Since $y(t) > 0$ for $t \in [T, \infty)$, the function $z(t) := \dfrac{p(t)y^\Delta(t)}{y(t)}$ satisfies the Riccati equation

$$z^\Delta(t) + q(t) + \frac{z^2(t)}{p(t) + \mu(t)z(t)} = 0$$

for $t \in [T, \infty)_\mathbb{T}$ with $p(t) + \mu(t)z(t) > 0$. Integrating the Riccati equation from T_0 to t gives

$$z(t) = z(T_0) - \int_{T_0}^t q(s)\Delta s - \int_{T_0}^t \frac{z^2(s)}{p(s) + \mu(s)z(s)}\Delta s.$$

Therefore it follows that $\limsup_{t\to\infty} z(t) < 0$, using the facts that $z(T_0) \le 0$, $z(t)$ is eventually nontrivial, and (E_4) holds. Hence there exists $T_2 \in [T, \infty)_\mathbb{T}$ such that $z(t) < 0$ for $t \in [T_2, \infty)_\mathbb{T}$ and so $y^\Delta(t) < 0$ for $t \in [T_2, \infty)_\mathbb{T}$, a contradiction to part (a). The proof is complete.

□

Before we state Theorem 3.2.2, we need the following defintions.

$$\begin{aligned}
A_0(t) &= \int_t^\infty q(s)\,\Delta s, \\
A_1(t) &= A_0(t) + \int_t^\infty \frac{A_0^2(s)}{p(s) + \mu(s)A_0(s)}\,\Delta s, \\
&\vdots \\
A_n(t) &= A_0(t) + \int_t^\infty \frac{A_{n-1}^2(s)}{p(s) + \mu(s)A_0(s)}\,\Delta s,
\end{aligned}$$

if the integrals on the right-hand side exist.

Our first result is a generalization of Theorem 3.1 of [26].

Theorem 3.2.2. *Assume (E_3) and (E_4) hold, and one of the following two conditions holds:*

(i) there exists some positive integer m such that A_n is well defined for $n = 0, 1, 2, \ldots,$ $m-1$, and

$$\lim_{t\to\infty} \int_a^t \frac{A_{m-1}^2(s)}{p(s) + \mu(s)A_{m-1}(s)}\,\Delta s = \infty.$$

(ii) A_n is well defined for $n = 0, 1, 2, \ldots,$ and there exists $t^* \in \mathbb{T}$ $(t^* \geq t_0)$ such that

$$\lim_{n\to\infty} A_n(t^*) = \infty$$

for all n.

Then the second order dynamic equation

$$(p(t)y^\Delta(t))^\Delta + q(t)y^\sigma(t) = 0 \tag{3.9}$$

is oscillatory.

Proof. If not, without loss of generality, we assume (3.9) has an eventually positive solution $y(t)$. From Lemma 3.2.1, we get that there exists $t_1 \in \mathbb{T}$ $(t_1 \geq t_0)$ such that

$$y(t) > 0 \quad \text{and} \quad y^\Delta(t) > 0 \quad \text{for all} \quad t \geq t_1.$$

Define the function z by

$$z(t) = \frac{p(t)y^\Delta(t)}{y(t)} \quad \text{for} \quad t \geq t_1. \tag{3.11}$$

Then $z(t) > 0$ and

$$p(t) + \mu(t)z(t) = p(t) + \mu(t)\frac{p(t)y^\Delta(t)}{y(t)} = \frac{p(t)y(t) + p(t)\mu(t)y^\Delta(t)}{y(t)} > 0,$$

for $t \geq t_1$. From (3.11) we get that

$$\begin{aligned}
z^\Delta(t) &= \frac{(p(t)y^\Delta(t))^\Delta y(t) - (p(t)y^\Delta(t))y^\Delta(t)}{y(t)y^\sigma(t)} \\
&= \frac{(p(t)y^\Delta(t))^\Delta}{y^\sigma(t)} - \frac{p(t)(y^\Delta(t))^2}{y(t)y^\sigma(t)} \cdot \frac{p(t)}{p(t)} \cdot \frac{y^2(t)}{y^2(t)} \\
&= -q(t) - \left(\frac{p(t)y^\Delta(t)}{y(t)}\right)^2 \frac{y(t)}{p(t)y^\sigma(t)} \\
&= -q(t) - z^2(t)\frac{y(t)}{p(t)(y(t) + \mu(t)y^\Delta(t))} \\
&= -q(t) - z^2(t) \cdot \frac{y(t)}{y(t)} \left(\frac{1}{p(t)(1 + \mu(t)\frac{y^\Delta(t)}{y(t)})}\right) \\
&= -q(t) - \frac{z^2(t)}{p(t) + \mu(t)z(t)}
\end{aligned}$$

for $t \geq t_1$. Hence, z is a solution of the Riccati equation

$$z^{\Delta}(t) = -q(t) - \frac{z^2(t)}{p(t) + \mu(t)z(t)}, \quad t \geq t_1. \tag{3.12}$$

Integrating both sides of (3.12) from t_1 to t we get

$$z(t) - z(t_1) + \int_{t_1}^{t} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\, \Delta s = -\int_{t_1}^{t} q(s)\, \Delta s, \quad t \geq t_1.$$

Then, as $z(t) > 0$,

$$\int_{t_1}^{t} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\, \Delta s \leq z(t_1) - \int_{t_1}^{t} q(s)\, \Delta s \leq z(t_1), \quad t \geq t_1.$$

Letting $t \to \infty$ we have that

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\, \Delta s < \infty.$$

Integrating (3.12) from t to s we obtain

$$\begin{aligned} z(t) &= z(s) + \int_{t}^{s} q(\tau)\, \Delta\tau + \int_{t}^{s} \frac{z^2(\tau)}{p(\tau) + \mu(\tau)z(\tau)}\, \Delta\tau \\ &> \int_{t}^{s} q(\tau)\, \Delta\tau + \int_{t}^{s} \frac{z^2(\tau)}{p(\tau) + \mu(\tau)z(\tau)}\, \Delta\tau. \end{aligned}$$

for s, $t \in \mathbb{T}$ and $s \geq t \geq t_1$. Letting $s \to \infty$ we have

$$z(t) \geq \int_{t}^{\infty} q(s)\, \Delta s + \int_{t}^{\infty} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\, \Delta s, \quad t \geq t_1. \tag{3.13}$$

Assume Condition (i) holds and $m = 1$. From (3.13) we obtain that $z(t) \geq A_0(t)$ for all $t \geq t_1$.

Observe that $F(u) = \dfrac{u^2}{c_1 + c_2 u}$ is increasing for $u > 0$, where c_1, $c_2 \geq 0$ are constants. It follows that

$$\int_{t}^{\infty} \frac{A_0^2(s)}{p(s) + \mu(s)A_0(s)}\, \Delta s \leq \int_{t}^{\infty} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\, \Delta s < \infty.$$

This contradicts (i). If $m > 1$, we have

$$z(t) \geq \int_{t}^{\infty} q(s)\, \Delta s + \int_{t}^{\infty} \frac{A_0^2(s)}{p(s) + \mu(s)A_0(s)}\, \Delta s = A_1(t), \quad \text{for } t \geq t_1.$$

Repeating the above procedure, we get that $z(t) \geq A_{m-1}(t)$ for all $t \geq t_1$, and

$$\int_t^\infty \frac{A_{m-1}^2(s)}{p(s) + \mu(s)A_{m-1}(s)} \Delta s \leq \int_t^\infty \frac{z^2(s)}{p(s) + \mu(s)z(s)} \Delta s < \infty,$$

which contradicts Condition (i).

Assume that Condition (ii) holds. Similar to the above proof, we obtain $A_n(t) \leq z(t)$ for $n = 0, 1, 2, \ldots$. Then, as $y(t) > 0$,

$$\lim_{n\to\infty} A_n(t^*) \leq z(t^*) < \infty,$$

which gives a contradiction to Condition (ii). The proof is complete. □

Remark 3.2.3. If $\mathbb{T} = \mathbb{R}$ and $p(t) = 1$ for all t, then Theorem 3.2.2 is the same as Yan's result for second order linear differential equations [25].

To prove the next result, we need the following lemmas:

Lemma 3.2.4. *[2, Theorem 4.61] Assume $a \in \mathbb{T}$, $p > 0$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If $(py^\Delta)^\Delta(t) + q(t)y^\sigma(t) = 0$ has a positive solution on $[a, \omega)$, then there is a positive solution u, called a recessive solution at ω, such that for any second linearly independent solution v, called a dominant solution at ω,*

$$\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0, \quad \int_a^\omega \frac{\Delta t}{p(t)u(t)u^\sigma(t)} = \infty, \quad \textit{and} \quad \int_b^\omega \frac{\Delta t}{p(t)v(t)v^\sigma(t)} < \infty,$$

where $b < \omega$ is sufficiently close. Furthermore

$$\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)}$$

for $t < \omega$ sufficiently close.

Lemma 3.2.5. *[2, Theorem 4.55] Assume z is a solution of the the Riccati equation*

$$Rz = 0, \quad \textit{where} \quad Rz(t) := z^\Delta(t) + q(t) + \frac{z^2(t)}{p(t) + \mu(t)z(t)}$$

on $[a, \sigma^2(b)]_\mathbb{T}$ with $p(t) + \mu(t)z(t) > 0$ on $[a, \sigma^2(b)]_\mathbb{T}$. Let u be a continuous function on $[a, \sigma^2(b)]_\mathbb{T}$ whose derivative is piecewise right-dense continuous with

$u(a) = u(\sigma^2(b)) = 0$. *Then we have for all* $t \in [a, \sigma^2(b)]_{\mathbb{T}}$,

$$(zu^2)^\Delta(t) = p(t)[u^\Delta(t)]^2 - q(t)u^2(\sigma(t)) - \left\{ \frac{z(t)u^\sigma(t)}{\sqrt{p(t)+\mu(t)z(t)}} - \sqrt{p(t)+\mu(t)z(t)}\,u^\Delta(t) \right\}^2.$$

Using the previous lemmas, we have the following theorem which was proven for differential equations by Kelley and Peterson in [21].

Theorem 3.2.6. *Assume* $I = [a, \infty)_{\mathbb{T}}$. *If* $\int_a^\infty \frac{\Delta t}{p(t)} = \infty$ *and there is a* $t_0 \geq a$ *and a* $u \in C^1_{rd}[t_0, \infty)$ *such that* $u(t) > 0$ *on* $[t_0, \infty)_{\mathbb{T}}$ *and*

$$\int_{t_0}^\infty \{q(t)[u^\sigma(t)]^2 - p(t)[u^\Delta(t)]^2\}\Delta t = \infty,$$

then the second-order dynamic equation

$$(p(t)y^\Delta(t))^\Delta + q(t)y^\sigma(t) = 0 \tag{3.9}$$

is oscillatory on I.

Proof. We prove this theorem by contradiction. So assume (3.9) is nonoscillatory on I. Lemma 3.2.4, there is a dominant solution y at ∞ such that for $t_1 \geq a$, sufficiently large,

$$\int_{t_1}^\infty \frac{\Delta t}{p(t)y(t)y^\sigma(t)} < \infty,$$

and we may assume $y(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Let t_0 and u be as in the statement of this theorem. Let $T = \max\{t_0, t_1\}$; then let

$$z(t) := \frac{p(t)y^\Delta(t)}{y(t)}, \quad t \geq T.$$

It follows that

$$\begin{aligned} z^\Delta(t) &= \frac{(p(t)y^\Delta(t))^\Delta y(t) - p(t)(y^\Delta(t))^2}{y(t)y^\sigma(t)} \\ &= -\frac{q(t)y^\sigma(t)}{y^\sigma(t)} - \left(\frac{p(t)y^\Delta(t)}{y(t)}\right)^2 \frac{y(t)}{p(t)y^\sigma(t)} \\ &= -q(t) - z^2(t)\frac{y(t)}{p(t)[y(t)+\mu(t)y^\Delta(t)]} \\ &= -q(t) - \frac{z^2(t)}{p(t)+\mu(t)z(t)} \end{aligned}$$

and

$$p(t) + \mu(t)z(t) > 0 \quad \text{for all } t \geq T.$$

Then by Lemma 3.2.5, we have for $t \geq T$

$$\begin{aligned}
&(zu^2)^\Delta(t) \\
&= p(t)[u^\Delta(t)]^2 - q(t)u^2(\sigma(t)) - \left\{ \frac{z(t)u(\sigma(t))}{\sqrt{p(t)+\mu(t)z(t)}} - \sqrt{p(t)+\mu(t)z(t)}\,u^\Delta(t) \right\}^2 \\
&\leq p(t)[u^\Delta(t)]^2 - q(t)u^2(\sigma(t)).
\end{aligned}$$

Integrating from T to t, we obtain

$$z(t)u^2(t) \leq z(T)u^2(T) - \int_T^t \left\{ q(t)u^2(\sigma(t)) - p(t)[u^\Delta(t)]^2 \right\} \Delta t$$

which implies

$$\lim_{t\to\infty} z(t)u^2(t) = -\infty.$$

However, then there is a $T_1 \geq T$ such that for $t \geq T_1$

$$z(t) = \frac{p(t)y^\Delta(t)}{y(t)} < 0.$$

This implies that $y^\Delta(t) < 0$ for $t \geq T_1$, and hence y is decreasing on $[T_1, \infty)_{\mathbb{T}}$. However,

$$\begin{aligned}
\int_{T_1}^\infty \frac{1}{p(s)}\,\Delta s &= y(T_1)y^\sigma(T_1)\int_{T_1}^\infty \frac{1}{p(s)y(T_1)y^\sigma(T_1)}\,\Delta s \\
&\leq y(T_1)y^\sigma(T_1)\int_{T_1}^\infty \frac{1}{p(s)y(s)y^\sigma(s)}\,\Delta s \\
&< \infty,
\end{aligned}$$

which is a contradiction. □

We conclude this section with an example that shows how Theorem 3.2.6 can be used to obtain oscillation criteria.

Example 3.2.7. If $a > 0$ and

$$\int_a^\infty \sigma^\alpha(t)q(t)\,\Delta t = \infty,$$

where $0 < \alpha < 1$, then $y^{\Delta\Delta} + q(t)y^\sigma = 0$ is oscillatory on $[a, \infty)_{\mathbb{T}}$.

We will show that this follows from Theorem 3.2.6. In the Pötzsche Chain Rule [2, Theorem 1.90], let $g(t) = t$ and $f(t) = t^{\frac{\alpha}{2}}$, for $0 < \alpha < 1$. Then with

$u(t) = (f \circ g)(t) = t^{\frac{\alpha}{2}}$, we have

$$\begin{aligned} u^{\Delta}(t) = (f \circ g)^{\Delta}(t) &= \left\{ \int_0^1 \frac{\alpha}{2}[t + h\mu(t) \cdot 1]^{\frac{\alpha-2}{2}}\, dh \right\} \cdot 1 \\ &= \frac{\alpha}{2} \int_0^1 (t + h\mu(t))^{\frac{\alpha-2}{2}}\, dh \\ &\leq \frac{\alpha}{2} \int_0^1 t^{\frac{\alpha-2}{2}}\, dh \\ &= \frac{\alpha}{2} t^{\frac{\alpha-2}{2}} \end{aligned}$$

since $\alpha - 2 < 0$. Therefore, it follows that $(u^{\Delta}(t))^2 \leq \dfrac{\alpha^2}{4} t^{\alpha-2}$ for all t. Hence,

$$\int_a^{\infty} \left\{ q(t)[u^{\sigma}(t)]^2 - p(t)[u^{\Delta}(t)]^2 \right\} \Delta t \geq \int_a^{\infty} \left\{ q(t)\sigma^{\alpha}(t) - \frac{\alpha^2}{4} t^{\alpha-2} \right\} \Delta t = \infty$$

since $0 < \alpha < 1$ implies

$$\int_a^{\infty} t^{\alpha-2}\, \Delta t < \infty.$$

Thus $y^{\Delta\Delta} + q(t)y^{\sigma} = 0$ is oscillatory on $[a, \infty)_{\mathbb{T}}$ by Theorem 3.2.6.

3.3 Oscillation of a Nonlinear Dynamic Equation with Advanced and Delayed Arguments

In this section we establish several oscillation results (Theorems 3.3.5-3.3.10) for the second order nonlinear functional dynamic equation

$$y^{\Delta\Delta} + f\big(t, y^{\sigma}(t), y^{\tau_1}(t), y^{\tau_2}(t), y^{\xi_1}(t), y^{\xi_2}(t)\big) = 0 \tag{3.14}$$

on a time scale $[t_0, \infty)_{\mathbb{T}}$ where $f \in C(\mathbb{T} \times \mathbb{R}^5, \mathbb{R})$. We shall assume

$$\tau_i(t) \leq t \leq \sigma(t) \leq \xi_i(t)$$

for all $t \in \mathbb{T}$ and $\tau_i,\ \xi_i \in C_{rd}(\mathbb{T}, \mathbb{T})$ for $i = 1, 2$. We also assume

$$\lim_{t \to \infty} \tau_i(t) = \infty = \lim_{t \to \infty} \xi_i(t)$$

for $i = 1, 2$. Here we use the notation $y^{\tau}(t) = y(\tau(t))$ and $y^{\xi}(t) = y(\xi(t))$. Our goal is to establish some new oscillation and nonoscillation results for this equation. We apply results from the theory of lower and upper solutions for related dynamic equations along with some additional estimates on the positive solutions.

Concerning the function $f = f(t, u, v_1, v_2, w_1, w_2)$, we will always assume that f satisfies the following **Condition (*A*)**:

$$f(t, u, v_1, v_2, w_1, w_2) = -f(t, -u, -v_1, -v_2, -w_1, -w_2)$$

and

$$f(t, u, v_1, v_2, w_1, w_2) > 0 \quad \text{if} \quad u, v_1, v_2, w_1, w_2 > 0, t \in \mathbb{T}.$$

We begin with the following preliminary lemmas.

Lemma 3.3.1. *Let $y \in C^2_{rd}[t_0, \infty)_{\mathbb{T}}$ satisfy $y(t) > 0,\ y^{\Delta}(t) > 0,\ y^{\Delta\Delta}(t) \leq 0$ for $t \geq T \geq t_0$. Then for each $0 < k < 1$ there exists $T_k \geq T \geq t_0$ such that the following hold:*

(i)

$$y^{\tau}(t) := y(\tau(t)) \geq k y^{\sigma}(t) \frac{\tau(t)}{\sigma(t)}, \ \ t \geq T_k,$$

and

(ii)

$$y^{\xi}(t) := y(\xi(t)) \leq y^{\sigma}(t) \frac{\xi(t)}{k\sigma(t)}, \ \ t \geq T_k.$$

Proof. (i) For $t > T \geq t_0$ we have

$$y^{\sigma}(t) - y^{\tau}(t) = \int_{\tau(t)}^{\sigma(t)} y^{\Delta}(s)\Delta s \leq y^{\Delta}(\tau(t))(\sigma(t) - \tau(t))$$

as y is decreasing, and so

$$y^{\sigma}(t) \leq y^{\tau}(t) + y^{\Delta}(\tau(t))(\sigma(t) - \tau(t)). \tag{3.15}$$

Also we have

$$y^{\tau}(t) - y(T) = \int_{T}^{\tau(t)} y^{\Delta}(s)\Delta s \geq y^{\Delta}(\tau(t))(\tau(t) - T) \tag{3.16}$$

and hence

$$\frac{y^{\tau}(t)}{y^{\Delta}(\tau(t))} \geq \frac{y(T)}{y^{\Delta}(\tau(t))} + (\tau(t) - T) \tag{3.17}$$

which implies

$$\frac{y^{\Delta}(\tau(t))}{y^{\tau}(t)} \leq \frac{1}{(\tau(t) - T) + \dfrac{y(T)}{y^{\Delta}(\tau(t))}} < \frac{1}{\tau(t) - T}. \tag{3.18}$$

Therefore, (3.15) and (3.18) imply

$$\begin{aligned}
\frac{y^\sigma(t)}{y^\tau(t)} &\le 1 + \frac{y^\Delta(\tau(t))}{y^\tau(t)}(\sigma(t) - \tau(t)) \\
&\le 1 + \frac{\sigma(t) - \tau(t)}{\tau(t) - T} \\
&= \frac{\sigma(t) - T}{\tau(t) - T}.
\end{aligned} \tag{3.19}$$

Now given any $0 < k < 1$, there exists T_k such that

$$\frac{\sigma(t) - T}{\tau(t) - T} < \frac{1}{k}\frac{\sigma(t)}{\tau(t)}, \quad t \ge T_k. \tag{3.20}$$

Consequently, we have from (3.19) and (3.20)

$$y^\tau(t) \ge k y^\sigma(t)\frac{\tau(t)}{\sigma(t)}, \quad t \ge T_k$$

and this completes the proof of (i).

The proof of (ii) is similar. We have for $T < t \le \sigma(t) \le \xi(t)$

$$y^\xi(t) - y^\sigma(t) = \int_{\sigma(t)}^{\xi(t)} y^\Delta(s)\Delta s \le y^\Delta(\sigma(t))(\xi(t) - \sigma(t))$$

and so we have

$$\frac{y^\xi(t)}{y^\sigma(t)} \le 1 + \frac{y^\Delta(\sigma(t))}{y^\sigma(t)}(\xi(t) - \sigma(t)). \tag{3.21}$$

Also we have

$$y^\sigma(t) \ge y(T) + y^\Delta(\sigma(t))(\sigma(t) - T)$$

so that

$$\frac{y^\sigma(t)}{y^\Delta(\sigma(t))} \ge \frac{y(T)}{y^\Delta(\sigma(t))} + (\sigma(t) - T) \ge k\sigma(t), \quad t \ge T_k,\ 0 < k < 1.$$

Hence, from (3.21) we have

$$\frac{y^\xi(t)}{y^\sigma(t)} \le 1 + \frac{\xi(t) - \sigma(t)}{k\sigma(t)} = \frac{(k-1)\sigma(t) + \xi(t)}{k\sigma(t)} \le \frac{\xi(t)}{k\sigma(t)}, \quad t \ge T_k.$$

This completes the proof of the lemma. □

We continue with the following result for the case when $f(t, u, v_1, v_2, w_1, w_2)$

satisfies the following **Condition $(\boldsymbol{B})$**:

For each fixed $t \in \mathbb{T}$, f is nonincreasing in $w_1, w_2 > 0$ for fixed $u, v_1, v_2 > 0$, f is nondecreasing in $v_1, v_2 > 0$ for fixed $u, w_1, w_2 > 0$, and f is nondecreasing in $u > 0$ for fixed $v_1, v_2, w_1, w_2 > 0$.

We introduce the functions $g_i(t)$, $h_i(t)$ defined by

$$g_i(t) := \frac{\tau_i(t)}{\sigma(t)}, \quad h_i(t) := \frac{\xi_i(t)}{\sigma(t)}, \tag{3.22}$$

where $i = 1, 2$. In order to prove our main results, we need a method for studying boundary value problems (BVP). Namely we will define functions called upper and lower solutions that, not only imply the existence of a BVP but also provide bounds on the location of the solution. Consider the second-order equation

$$y^{\Delta\Delta} = f(t, y^\sigma) \tag{3.23}$$

where f is continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$.

Definition 3.3.2. [2, Definition 6.53] We say that $\alpha \in C^2_{rd}$ is a *lower solution* of (3.23) on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided

$$\alpha^{\Delta\Delta}(t) \geq f(t, \alpha^\sigma(t)) \quad \text{for all} \quad t \in [a, b]_{\mathbb{T}}.$$

Similarly, $\beta \in C^2_{rd}$ is called an *upper solution* of (3.23) on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided

$$\beta^{\Delta\Delta}(t) \leq f(t, \beta^\sigma(t)) \quad \text{for all} \quad t \in [a, b]_{\mathbb{T}}.$$

Theorem 3.3.3. *[2, Theorem 6.54] Let f be continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution α and an upper solution β of* (3.23) *with*

$$\alpha(a) \leq A \leq \beta(a) \quad \text{and} \quad \alpha(\sigma^2(b)) \leq B \leq \beta(\sigma^2(b))$$

such that

$$\alpha(t) \leq \beta(t) \quad \text{for all} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}}.$$

Then the BVP

$$y^{\Delta\Delta} = f(t, y^\sigma) \text{ on } [a, b]_{\mathbb{T}}, \quad y(a) = A, \quad y(\sigma^2(b)) = B$$

has a solution y with

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \text{for all} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}}.$$

The following is a generalization of Theorem 7.4 of [20].

Theorem 3.3.4. *Let f be continuous on $[a,b]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution α and an upper solution β of (3.23) with $\alpha(t) \leq \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. Then for any $\alpha(a) \leq c \leq \beta(a)$ the BVP*

$$y^{\Delta\Delta} = f(t, y^{\sigma}), \quad y(a) = c \tag{3.24}$$

has a solution y with

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \textit{for all} \quad t \in [a,\infty)_{\mathbb{T}}.$$

Proof. It follows from Theorem 3.3.3 that for each $n \geq 1$ there is a solution $y_n(t)$ of $[a, a+n]_{\mathbb{T}}$ with $y_n(a) = c$, $y_n(a+n) = \beta(a+n)$ and $\alpha(t) \leq y_n(t) \leq \beta(t)$ on $[a, a+n]_{\mathbb{T}}$. Thus, for any fixed $n \geq 1$, $y_m(t)$ is a solution on $[a, a+n]_{\mathbb{T}}$ satisfying $\alpha(t) \leq y_m(t) \leq \beta(t)$ for all $m \geq n$. Hence, for $m \geq n$, the sequence $y_m(t)$ is pointwise bounded on $[a, a+n]_{\mathbb{T}}$.

We claim that $\{y_m(t)\}$ is equicontinuous on $[a, a+n]_{\mathbb{T}}$ for any fixed $n \geq 1$. Since f is continuous and $y_m(t) \leq \beta(t)$ for all $t \in [a, a+n]_{\mathbb{T}}$, there is constant $K > 0$ such that $|y_m^{\Delta\Delta}(t)| = |f(t, y_m^{\sigma}(t))| \leq K$ for all $t \in [a, a+n]_{\mathbb{T}}$. It follows that

$$\begin{aligned}
y_m^{\Delta}(t) - y_m^{\Delta}(a) &= \int_a^t y_m^{\Delta\Delta}(s)\,\Delta s \\
&\leq \int_a^t K\,\Delta s \\
&= K(t-a) \\
&\leq K(a+n-a) \\
&= Kn
\end{aligned}$$

which gives that

$$|y_m^{\Delta}(t)| \leq |y_m^{\Delta}(a)| + |Kn| =: L.$$

Consequently,

$$|y_m(t) - y_m(s)| = |\int_s^t y_m^{\Delta}\,\Delta s| \leq L|t-s| < \epsilon$$

for all $t, s \in [a, a+n]_{\mathbb{T}}$ provided $|t-s| < \delta = \dfrac{\epsilon}{L}$. Hence the claim holds.

So by Ascoli-Arzela and a standard diagonalization argument, $\{y_m(t)\}$ contains a subsequence which converges uniformly on all compact subintervals $[a, a+n]_{\mathbb{T}}$ of $[a,\infty)_{\mathbb{T}}$ to a solution $y(t)$, which is the desired solution of the (3.24) that satisfies $\alpha(t) \leq y(t) \leq \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. $\square$

In the results that follow by

$$\int^{\infty} tf\,(t, u, v_1 v_2, w_1, w_2)\,\Delta t = \infty$$

we mean

$$\int_T^{\infty} tf\left(t, u, v_1 v_2, w_1, w_2\right) \Delta t < \infty$$

for some T sufficiently large.

Our first main result in this section is:

Theorem 3.3.5. *Assume conditions (A) and (B) hold. Then all bounded solutions of* $y^{\Delta\Delta} + f\big(t, y^{\sigma}(t), y^{\tau_1}(t), y^{\tau_2}(t), y^{\xi_1}(t), y^{\xi_2}(t)\big) = 0$ *are oscillatory in case*

$$\left| \int^{\infty} tf\left(t, \alpha, \alpha k g_1(t), \alpha k g_2(t), \frac{\alpha}{k^2} h_1(t), \frac{\alpha}{k^2} h_2(t)\right) \Delta t \right| = \infty \tag{3.25}$$

for all $\alpha \neq 0$ *and for some* $k \in (0,1)$, *where* $g_i(t)$, $h_i(t)$ *for* $i = 1, 2$ *are given by* (3.22).

Proof. If not, let $u(t)$ be a bounded nonoscillatory solution which we may assume satisfies

$$u(t) > 0, \quad u_i^{\tau}(t) > 0, \quad t \geq T \geq t_0, \quad i = 1, 2.$$

Consequently, $u^{\Delta\Delta}(t) = -f(t, u^{\sigma}(t), u^{\tau_1}(t), u^{\tau_2}(t), u^{\xi_1}(t), u^{\xi_2}(t)) < 0$ for $t \geq T$ and so $u^{\Delta}(t)$ is decreasing for $t \geq T$. It follows that $u^{\Delta}(t) > 0$ for $t \geq T$. Indeed, if $u^{\Delta}(t_1) \leq 0$ for some $t_1 \geq T$, then $u^{\Delta}(t) \leq 0$ for all $t \geq t_1$. Now if $u^{\Delta}(t_2) < 0$ for some $t_2 \geq t_1$, then

$$u(t) - u(t_2) = \int_{t_2}^{t} u^{\Delta}(s) \Delta s \leq u^{\Delta}(t_2)(t - t_2) \to -\infty$$

as $t \to \infty$, which is a contradiction to our assumption that $u(t) > 0$ for $t \geq T \geq t_0$. Also, if $u^{\Delta}(t_1) = 0$, then $u^{\Delta}(t) \equiv 0$ on $[t_1, \infty)_{\mathbb{T}}$, and so

$$u^{\Delta\Delta}(t) \equiv 0 = -f(t, u^{\sigma}(t), u^{\tau_1}(t), u^{\tau_2}(t), u^{\xi_1}(t), u^{\xi_2}(t)),$$

which is again a contradiction. Hence, we conclude that for all $t \geq T$

$$u(t) > 0, \quad u^{\Delta}(t) > 0, \quad u^{\tau_i}(t) > 0$$

for $i = 1, 2$. From Lemma 3.3.1, given $0 < k < 1$, there exists $T_k^1 \geq T$ such that

$$u^{\tau_1}(t) \geq k g_1(t) u^{\sigma}(t) \text{ and } u^{\xi_1}(t) \leq \frac{1}{k} h_1(t) u^{\sigma}(t)$$

for $t \geq T_k^1$ and there exists $T_k^2 \geq T$ such that

$$u^{\tau_2}(t) \geq k g_2(t) u^{\sigma}(t) \text{ and } u^{\xi_2}(t) \leq \frac{1}{k} h_2(t) u^{\sigma}(t)$$

for $t \geq T_k^2$. By the monotonicity assumption on f we have

$$\begin{aligned} 0 &= u^{\Delta\Delta}(t) + f\left(t, u^\sigma(t), u^{\tau_1}(t), u^{\tau_2}(t), u^{\xi_1}(t), u^{\xi_2}(t)\right) \\ &\geq u^{\Delta\Delta}(t) + f\left(t, u^\sigma(t), kg_1(t)u^\sigma(t), kg_2(t)u^\sigma(t), \frac{1}{k}h_1(t)u^\sigma(t), \frac{1}{k}h_2(t)u^\sigma(t)\right) \end{aligned} \tag{3.26}$$

for $t \geq T_k := \max\{T_k^1, T_k^2\}$. Now, if we set

$$F(t, u^\sigma(t)) := f\left(t, u^\sigma(t), kg_1(t)u^\sigma(t), kg_2(t)u^\sigma(t), \frac{1}{k}h_1(t)u^\sigma(t), \frac{1}{k}h_2(t)u^\sigma(t)\right),$$

then (3.26) shows that $\beta(t) := u(t)$ is an upper solution for the dynamic equation $u^{\Delta\Delta} + F(t, u^\sigma(t)) = 0$. Also, the constant function $\alpha(t) := u(T_k)$ satisfies the inequality $\alpha^{\Delta\Delta}(t) + F(t, \alpha^\sigma(t)) \geq 0$, and so $\alpha(t)$ is a lower solution. Therefore, by Theorem 3.3.4, the BVP

$$y^{\Delta\Delta} + F(t, y^\sigma(t)) = 0, \quad y(T_k) = u(T_k)$$

has a solution $y(t)$ with

$$u(T_k) \leq y(t) \leq u(t), \quad t \geq T_k.$$

It follows that $y(t) > 0$ and $y^{\Delta\Delta}(t) \leq 0$. Therefore $y^\Delta(t) > 0$. Now, since $y(t)$ is bounded, we have that $\lim_{t\to\infty} y(t) := L > 0$ exists. Integration for $T_k < s < \tilde{T}$ implies

$$y^\Delta(\tilde{T}) - y^\Delta(s) + \int_s^{\tilde{T}} F(r, y^\sigma(r))\Delta r = 0.$$

Letting $\tilde{T} \to \infty$ we obtain

$$y^\Delta(s) \geq \int_s^\infty F(r, y^\sigma(r))\Delta r,$$

and so integrating again for $T_k < \tilde{t} < t$, we obtain

$$\begin{aligned} y(t) \;\; - \;\; y(\tilde{t}) &= \int_{\tilde{t}}^{t} y^{\Delta}(s)\Delta s \\ &\geq \int_{\tilde{t}}^{t}\int_{s}^{\infty} F(r, y^{\sigma}(r))\Delta r \Delta s \\ &= \int_{\tilde{t}}^{t}\int_{\tilde{t}}^{r} F(r, y^{\sigma}(r))\Delta s \Delta r + \int_{t}^{\infty}\int_{\tilde{t}}^{t} F(r, y^{\sigma}(r))\Delta s \Delta r \\ &= \int_{\tilde{t}}^{t} (r-\tilde{t})F(r, y^{\sigma}(r))\Delta r + \int_{t}^{\infty} (t-\tilde{t})F(r, y^{\sigma}(r))\Delta r \\ &\geq \int_{\tilde{t}}^{t} (r-\tilde{t})F(r, y^{\sigma}(r))\Delta r. \end{aligned} \tag{3.27}$$

From (3.27) we have

$$y(t) \geq y(\tilde{t}) + \int_{\tilde{t}}^{t} (r-\tilde{t})F(r, y^{\sigma}(r))\Delta r > \int_{\tilde{t}}^{t} (r-\tilde{t})F(r, y^{\sigma}(r))\Delta r.$$

Since $y(t) \leq u(t) \leq M$ for some $M > 0$ and $\int_{\tilde{t}}^{t} (r-\tilde{t})F(r, y^{\sigma}(r))\Delta r$ is an increasing function of t, it follows that

$$\int_{\tilde{t}}^{\infty} (r-\tilde{t})F(r, y^{\sigma}(r))\Delta r < \infty.$$

Since $2r \geq \tilde{t}$ for r sufficiently large, it follows that $\int_{\tilde{t}}^{\infty} rF(r, y^{\sigma}(r))\Delta r < \infty$. For the same $0 < k < 1$ as in the first part of the proof, we may assume that we have $y(t) \geq kL$ for $t \geq \tilde{T}_k \geq T_k$. Since $y(t) \leq y^{\sigma}(t) \leq L$, using the monotonicity of f,

$$\begin{aligned} f&\left(t, y^{\sigma}(t), kg_1(t)y^{\sigma}(t), kg_2(t)y^{\sigma}(t), \frac{1}{k}h_1(t)y^{\sigma}(t), \frac{1}{k}h_2(t)y^{\sigma}(t)\right) \\ &\geq f\left(t, y^{\sigma}(t), kg_1(t)y^{\sigma}(t), kg_2(t)y^{\sigma}(t), \frac{L}{k}h_1(t), \frac{L}{k}h_2(t)\right) \\ &\geq f\left(t, kL, k^2g_1(t)L, k^2g_2(t)L, \frac{L}{k}h_1(t), \frac{L}{k}h_2(t)\right). \end{aligned}$$

Therefore, with $\alpha := kL$, it follows that

$$\int_{\tilde{T}_k}^{\infty} rf\left(r, \alpha, \alpha kg_1(r), \alpha kg_2(r), \frac{\alpha}{k^2}h_1(r), \frac{\alpha}{k^2}h_2(r)\right)\Delta r < \infty,$$

a contradiction to our assumption (3.25).

This completes the proof. $\square$

The next result shows that a converse of Theorem 3.3.5 is true under an additional assumption.

Theorem 3.3.6. *Assume f satisfies conditions (A) and (B) and that*

$$\liminf_{t\to\infty} g_i(t) := m_i > 0 \quad \textit{and} \quad \limsup_{t\to\infty} h_i(t) := M_i < \infty, \tag{3.28}$$

where $g_i(t)$, $h_i(t), 1 \le i \le 2$, are defined in (3.22). Also, assume that $\sigma(t)/t$ is bounded. Then, if

$$y^{\Delta\Delta} + f\left(t, y^{\sigma}(t), y^{\tau_1}(t), y^{\tau_2}(t), y^{\xi_1}(t), y^{\xi_2}(t)\right) = 0 \tag{3.14}$$

has a bounded nonoscillatory solution, it follows that

$$\left|\int^{\infty} \sigma(t) f\left(t, \alpha, \alpha k\tilde{m}, \alpha k\tilde{m}, \frac{\alpha\tilde{M}}{k^2}, \frac{\alpha\tilde{M}}{k^2}\right)\Delta t\right| < \infty \tag{3.29}$$

for some $\alpha \neq 0$ and for any $0 < k < 1$, where $\tilde{m}$ and $\tilde{M}$ satisfy $\tilde{m} < m_i$ and $\tilde{M} > M_i$ for $i = 1, 2$.

Proof. Note that for any β

$$\left|\int^{\infty} \sigma(t) f(t, \beta, \ldots, \beta)\Delta t\right| < \infty$$

if, and only if,

$$\left|\int^{\infty} t f(t, \beta, \ldots, \beta)\Delta t\right| < \infty$$

since $\sigma(t)/t$ is bounded on $\mathbb{T}$.

Assume (3.28) holds. Then for $\epsilon > 0$ with $\epsilon < \min\{m_1,\ m_2\}$, there exists $t_i \ge t_0$ such that $g_i(t) \ge m_i - \epsilon := \tilde{m}_i$ provided $t \ge t_i$ and there exists $T_i \ge t_0$ such that $h_i(t) \le M_i + \epsilon := \tilde{M}_i$ provided $t \ge T_i$, $i = 1, 2$. It follows that for $\alpha > 0$

$$\alpha g_i(t) \ge \alpha\tilde{m}_i \ge \alpha\tilde{m}$$

where $\tilde{m} := \min\{\tilde{m}_1,\ \tilde{m}_2\} \le \tilde{m}_i < m_i$, and

$$\frac{1}{\alpha} h_i(t) \le \frac{1}{\alpha}\tilde{M}_i \le \frac{1}{\alpha}\tilde{M}$$

where $\tilde{M} := \max\{\tilde{M}_1,\ \tilde{M}_2\} \ge \tilde{M}_i > M_i$, $i = 1, 2$. Assume (3.14) has a bounded nonoscillatory solution. Then by Theorem 3.3.5,

$$\left|\int^{\infty} tf\left(t, \alpha, \alpha k g_1(t), \alpha k g_2(t), \frac{\alpha}{k^2} h_1(t), \frac{\alpha}{k^2} h_2(t)\right) \Delta t\right| < \infty$$

for some $\alpha \neq 0$ and for all $0 < k < 1$. By the monotonicity assumption of f, we have

$$\int^{\infty} tf\left(t, \alpha, \alpha k\tilde{m}, \alpha k\tilde{m}, \frac{\alpha\tilde{M}}{k^2}, \frac{\alpha\tilde{M}}{k^2}\right) \Delta t < \infty,$$

which proves the result. □

The previous result says that condition (3.28) is sufficient in order to replace the auxiliary functions $g_i(t),\ h_i(t)$ for $i = 1, 2$, given by (3.22) with upper bounds. Our next result gives a sufficient condition for

$$y^{\Delta\Delta} + f\big(t, y^{\sigma}(t), y^{\tau_1}(t), y^{\tau_2}(t), y^{\xi_1}(t), y^{\xi_2}(t)\big) = 0 \tag{3.14}$$

to have bounded nonoscillatory solutions.

Theorem 3.3.7. *Assume f satisfies conditions (A) and (B). If*

$$\left|\int^{\infty} \sigma(t) f\left(t, \alpha, \alpha, \alpha, \frac{\alpha}{2}, \frac{\alpha}{2}\right) \Delta t\right| < \infty, \tag{3.30}$$

for all $\alpha \neq 0$, then (3.14) *has a bounded nonoscillatory solution.*

Proof. If (3.30) holds, assume to be specific that $\alpha > 0$ and let $0 < \beta < \alpha$. Choose $T \geq t_1 \in \mathbb{T}$ such that $\tau_1(t), \tau_2(t) \geq t_1$ for $t \geq T$ and such that

$$\int_T^{\infty} \sigma(t) f\left(t, \alpha, \alpha, \alpha, \frac{\alpha}{2}, \frac{\alpha}{2}\right) \Delta t < \frac{\beta}{2}.$$

Define $y_0(t) \equiv \beta$ for $t \geq t_0$ and

$$y_{n+1}(t) = \begin{cases} \beta - \int_T^{\infty} (\sigma(s) - T) f(s, y_n^{\sigma}(s), y_n^{\tau_1}(s), y_n^{\tau_2}(s), y_n^{\xi_1}(s), y_n^{\xi_2}(s)) \Delta s, & t < T, \\ \beta - \int_t^{\infty} (\sigma(s) - t) f(s, y_n^{\sigma}(s), y_n^{\tau_1}(s), y_n^{\tau_2}(s), y_n^{\xi_1}(s), y_n^{\xi_2}(s)) \Delta s, & t \geq T \end{cases}$$

Observe $t_1 \leq \tau_i(t) \leq t \leq \sigma(t) \leq \xi_i(t)$ for all $t \geq T$ and $i = 1, 2$. We claim that

$$\frac{\beta}{2} \leq y_n(t) \leq \beta \quad t \geq T \quad \text{and} \quad \text{all } n \geq 0. \tag{3.31}$$

By construction the claim holds for $y_0(t)$. Notice that when $\tau_i(t) < T \leq t$ for any $i = 1, 2$, $y_n(\tau_i(t)) < \beta$ as $y^{\Delta}(t) \equiv 0$ for all $t \in \mathbb{T}$ less than T. Assume the inequality holds for $y_m(t)$, $1 \le m \le n$. Then for $t \ge T$

$$\begin{aligned}
y_{m+1}(t) &= \beta - \int_t^\infty (\sigma(s) - t) f(s, y_m^\sigma(s), y_m^{\tau_1}(s), y_m^{\tau_2}(s), y_m^{\xi_1}(s), y_m^{\xi_2}(s))\Delta s \\
&\ge \beta - \int_t^\infty \sigma(s) f(s, y_m^\sigma(s), y_m^{\tau_1}(s), y_m^{\tau_2}(s), y_m^{\xi_1}(s), y_m^{\xi_2}(s))\Delta s \\
&\ge \beta - \int_t^\infty \sigma(s) f\left(s, \alpha, \alpha, \alpha, \frac{\alpha}{2}, \frac{\alpha}{2}\right)\Delta s \\
&> \beta - \frac{\beta}{2} \\
&= \frac{\beta}{2}.
\end{aligned}$$

Furthermore, since $s \ge T$, we have $y_m^\sigma(s), y_m^{\tau_1}(s), y_m^{\tau_2}(s), y_m^{\xi_1}(s), y_m^{\xi_2}(s)$ are all positive. Hence by condition (A)

$$(\sigma(s) - t) f(s, y_m^\sigma(s), y_m^{\tau_1}(s), y_m^{\tau_2}(s), y_m^{\xi_1}(s), y_m^{\xi_2}(s)) \ge 0$$

for $s \ge t \ge T$. Consequently, $y_{m+1}(t) \le \beta$ for $t \ge T$. Therefore, by induction, (3.31) holds.

It remains to show that the set $\{y_n(t)\}_{n=0}^\infty$ is equicontinuous. To do this, we show that $\left\{y_n^\Delta(t)\right\}_{n=0}^\infty$ is uniformly bounded. It follows that

$$\begin{aligned}
|y_n^\Delta(t)| = \Big| 0 - \Big[&\int_t^\infty -f(s, y_n^\sigma(s), y_n^{\tau_1}(s), y_n^{\tau_2}(s), y_n^{\xi_1}(s), y_n^{\xi_2}(s))\Delta s \\
&+ (\sigma(t) - \sigma(t)) f(t, y_n^\sigma(t), y_n^{\tau_1}(t), y_n^{\tau_2}(t), y_n^{\xi_1}(t), y_n^{\xi_2}(t))\Big]\Big| \\
= &\left| \int_t^\infty f(s, y_n^\sigma(s), y_n^{\tau_1}(s), y_n^{\tau_2}(s), y_n^{\xi_1}(s), y_n^{\xi_2}(s))\Delta s \right| \\
\le &\left| \int_t^\infty f(s, \beta, \beta, \beta, \beta/2, \beta/2)\Delta s \right| \\
\le &\int_T^\infty |\sigma(s) f(s, \beta, \beta, \beta, \beta/2, \beta/2)|\,\Delta s \\
< &\frac{\beta}{2}.
\end{aligned}$$

Therefore, the Ascoli-Arzela theorem along with a standard diagonalization argument yields a subsequence of $\{y_n(t)\}_{n=0}^\infty$ which converges uniformly on compact subintervals of $[T, \infty)_{\mathbb{T}}$ to a solution $y(t)$ of (3.14) satisfying $\beta/2 \le y(t) < \beta$, $t \ge T$. This proves the theorem. □

We next introduce the following condition which replaces the nonincreasing assumption for the function f in the w_1, w_2 variables by assuming f is nondecreasing in w_1, w_2 for $w_1, w_2 > 0$ and for fixed $t \in \mathbb{T}$ and $u, v_1, v_2 > 0$.

The function f is said to satisfy **Condition** $(\tilde{B})$ if for each fixed $t \in \mathbb{T}$, f is nondecreasing in $w_1, w_2 > 0$ for fixed $u, v_1, v_2 > 0$, f is nondecreasing in $v_1, v_2 > 0$ for fixed $u, w_1, w_2 > 0$, and f is nondecreasing in $u > 0$ for fixed $v_1, v_2, w_1, w_2 > 0$.

Of course, if f is independent of $w_1, w_2 > 0$, then conditions (B) and $(\tilde{B})$ coincide.

Now we have the following:

Theorem 3.3.8. *Assume conditions* (A) *and* $(\tilde{B})$ *hold. Then all solutions of*

$$y^{\Delta\Delta} + f\left(t, y^{\sigma}(t), y^{\tau_1}(t), y^{\tau_2}(t), y^{\xi_1}(t), y^{\xi_2}(t)\right) = 0 \tag{3.14}$$

are oscillatory in case

$$\left| \int^{\infty} t f(t, \alpha, \alpha k g_1(t), \alpha k g_2(t), \alpha, \alpha) \Delta t \right| = \infty \tag{3.32}$$

for all $\alpha \neq 0$ *and for some* $k \in (0,1)$*, where* $g_i(t)$ *for* $i = 1, 2$ *is given by* (3.22).

Proof. If u is an eventually positive solution of (3.14), as in the proof of Theorem 3.3.5, we conclude that for all $t \geq t_1 \geq t_0$

$$u(t) > 0, \quad u^{\Delta}(t), \quad u^{\tau_1}(t) > 0, \quad u^{\tau_2}(t) > 0.$$

Hence, by Lemma 3.3.1 given $0 < k < 1$, there exists $T_k = \max\{T_1,\ T_2\} \geq t_1$ so that $u^{\tau_i}(t) \geq k g_i(t) u^{\sigma}(t)$ for all $t \geq T_k$ and $i = 1, 2$. Furthermore, since $u^{\Delta}(t) > 0$ for $t \geq t_1$ and $i = 1, 2$, $u^{\xi_i}(t) \geq u^{\sigma}(t)$. Therefore, by the monotonicity assumption on f from $(\tilde{B})$, it follows that

$$\begin{aligned} 0 &= u^{\Delta\Delta}(t) + f\left(t, u^{\sigma}(t), u^{\tau_1}(t), u^{\tau_2}(t), u^{\xi_1}(t), u^{\xi_2}(t)\right) \\ &\geq u^{\Delta\Delta}(t) + f(t, u^{\sigma}(t), k g_1(t) u^{\sigma}(t), k g_2(t) u^{\sigma}(t), u^{\sigma}(t), u^{\sigma}(t)). \end{aligned}$$

If we set $\tilde{F}(t, u^{\sigma}) := f(t, u^{\sigma}(t), k g_1(t) u^{\sigma}(t), k g_2(t) u^{\sigma}(t), u^{\sigma}(t), u^{\sigma}(t))$, then the remainder of the proof is similar to that of Theorem 3.3.5. $\square$

If we replace assumption (B) by $(\tilde{B})$ in Theorem 3.3.6, then we may give a necessary and sufficient condition for the existence of a bounded nonoscillatory solution. In this case, we only need to assume the first part of (3.28).

Theorem 3.3.9. *Assume* f *satisfies* (A) *and* $(\tilde{B})$ *and that*

$$\liminf_{t \to \infty} g_i(t) := m_i > 0, \tag{3.33}$$

where $g_i(t) := \dfrac{\tau_i(t)}{\sigma(t)}$, $1 \leq i \leq 2$. *Assume further that* $\sigma(t)/t$ *is bounded. Then equation*

(3.14) *has a bounded nonoscillatory solution, if and only if,*

$$\left| \int^{\infty} \sigma(t) f(t, \alpha, \alpha, \alpha, \alpha, \alpha) \Delta t \right| < \infty \tag{3.34}$$

for some $\alpha \neq 0$.

Proof. Assume (3.14) has a bounded nonoscillatory solution. By condition (3.33), we have for any $\epsilon > 0$ with $\epsilon < \min\{m_i | 1 \leq i \leq 2\}$ and any $\delta > 0$, there exists $t_i \geq t_0$ such that

$$\delta g_i(t) \geq \delta(m_i - \epsilon) = \delta \tilde{m}_i$$

where $\tilde{m}_i < m_i, i = 1, 2$. By definition of $g_i(t)$, we have $0 \leq m_i \leq g_i(t) \leq 1$ for $t \geq t_i, i = 1, 2$. Hence

$$\delta m \leq \delta \tilde{m}_i < \delta m_i \leq \delta$$

for $i = 1, 2$, where $m := \min\{\tilde{m}_1, \tilde{m}_2\}$. Then by Theorem 3.3.8 and the monotonicity of f, we have

$$\begin{aligned} \int^{\infty} t f(t, \delta m, \delta m, \delta m, \delta m, \delta m) \Delta t &\leq \int^{\infty} t f(t, \delta m, \delta m, \delta m, \delta, \delta) \Delta t \\ &\leq \int^{\infty} t f(t, \delta m, \delta \tilde{m}_1, \delta \tilde{m}_2, \delta, \delta) \Delta t \\ &< \int^{\infty} t f(t, \delta m, \delta m_1, \delta m_2, \delta, \delta) \Delta t \\ &\leq \int^{\infty} t f(t, \delta, \delta g_1(t), \delta g_2(t), \delta, \delta) \Delta t \\ &< \infty. \end{aligned}$$

By letting $\alpha = \delta m$, we obtain (3.34) is necessary for the existence of a bounded nonoscillatory solution.

Conversely, if (3.34) holds, assume to be specific that $\alpha > 0$ and let $0 < \beta < \alpha$. Choose $T \geq t_1 \geq t_0$ such that $\tau_1(t), \tau_2(t) \geq t_1$ for $t \geq T$ such that

$$\int_T^{\infty} t f(t, \alpha, \alpha, \alpha, \alpha, \alpha) \Delta t < \frac{\beta}{2}.$$

If we define $y_0(t) \equiv \beta$ for $t \geq t_0$ and

$$y_{n+1}(t) = \begin{cases} \beta - \int_T^{\infty} (\sigma(s) - T) f(s, y_n^{\sigma}(s), y_n^{\tau_1}(s), y_n^{\tau_2}(s), y_n^{\xi_1}(s), y_n^{\xi_2}(s)) \Delta s, & t < T, \\ \beta - \int_t^{\infty} (\sigma(s) - t) f(s, y_n^{\sigma}(s), y_n^{\tau_1}(s), y_n^{\tau_2}(s), y_n^{\xi_1}(s), y_n^{\xi_2}(s)) \Delta s, & t \geq T, \end{cases}$$

then the remainder of the proof is similar to that of Theorem 3.3.7. $\square$

To extend Theorems 3.3.5 and 3.3.6 to unbounded solutions, we introduce the class Φ of functions ϕ such that $\phi(u)$ denotes a continuous nondecreasing function of

u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty.$$

We will say that $f(t,u,v_1,v_2,w_1,w_2)$ satisfies **Condition *(H)*** provided for some $\phi \in \Phi$ there exists $c \neq 0$ and $0 < \alpha < 1$ such that for all $t \geq T$

$$\inf_{|u|\geq c} \frac{f(t,u,\alpha g_1(t)u,\alpha g_2(t)u,\frac{1}{\alpha}h_1(t)u,\frac{1}{\alpha}h_2(t)u)}{\phi(u)} \geq k\left|f\left(t,c,\alpha g_1(t)c,\alpha g_2(t)c,\frac{1}{\alpha}h_1(t)c,\frac{1}{\alpha}h_2(t)c\right)\right|$$

for some positive constant k.

We may now prove the following result:

Theorem 3.3.10. *Suppose $\phi \in \Phi$. Assume f satisfies conditions (A), (B), and (H). Then all solutions of*

$$y^{\Delta\Delta} + f\left(t, y^{\sigma}(t), y^{\tau_1}(t), y^{\tau_2}(t), y^{\xi_1}(t), y^{\xi_2}(t)\right) = 0 \tag{3.14}$$

are oscillatory in case

$$\left|\int^{\infty} tf\left(t,\alpha,\alpha k g_1(t),\alpha k g_2(t),\frac{\alpha}{k^2}h_1(t),\frac{\alpha}{k^2}h_2(t)\right)\Delta t\right| = \infty \tag{3.25}$$

holds for all $\alpha \neq 0$, where k is the constant appearing in condition (H).

Proof. If (3.25) holds for all $\alpha \neq 0$, assume $u(t)$ be a nonoscillatory solution of (3.14) with

$$u(t) > 0, \quad u(\tau_1(t)) > 0, \quad u(\tau_2(t)) > 0 \quad \text{for } t \geq T.$$

As in the proof of Theorem 3.3.5, given $0 < \alpha < 1$ from condition (H) there exists $T_\alpha \geq T$ such that

$$u^{\Delta\Delta} + f\left(t,u^{\sigma}(t),\alpha g_1(t)u^{\sigma}(t),\alpha g_2(t)u^{\sigma}(t)\frac{1}{\alpha}h_1(t)u^{\sigma}(t),\frac{1}{\alpha}h_2(t)u^{\sigma}(t)\right) \leq 0, \quad t \geq T_\alpha. \tag{3.35}$$

Hence we obtain a solution $y(t)$ of

$$y^{\Delta\Delta} + f\left(t,y^{\sigma}(t),\alpha g_1(t)y^{\sigma}(t),\alpha g_2(t)y^{\sigma}(t),\frac{1}{\alpha}h_1(t)y^{\sigma}(t),\frac{1}{\alpha}h_2(t)y^{\sigma}(t)\right) = 0 \tag{3.36}$$

with $0 < u(T_\alpha) \leq y(t) \leq u(t), t \geq T_\alpha$. We next define the continuously differentiable real-valued function

$$G(u) := \int_{u_0}^{u} \frac{ds}{\phi(s)},$$

where $u_0 := y(T_\alpha) > 0$. Observe that $G'(u) = 1/\phi(u)$. By the Pötzche Chain Rule,

$$(G(y(t))^{\Delta} = \left(\int_0^1 \frac{dh}{\phi(y_h(t))}\right) y^{\Delta}(t) \geq \left(\int_0^1 \frac{dh}{\phi(y^\sigma(t))}\right) y^{\Delta}(t) = \frac{y^{\Delta}(t)}{\phi(y^\sigma(t))}$$

where $y_h(t) := y(t) + h\mu(t)y^{\Delta}(t) \leq y^\sigma(t)$. Since ϕ is nondecreasing, we have $\dfrac{1}{\phi(y_h(t))} \geq \dfrac{1}{\phi(y^\sigma(t))}$. Consequently,

$$(G(y(t)))^{\Delta} \geq \frac{y^{\Delta}(t)}{\phi(y^\sigma(t)))} \tag{3.37}$$

Furthermore, since $y(t) > 0$ and $y^{\Delta}(t)$ is nonincreasing, $\lim_{t\to\infty} y^{\Delta}(t) = L_1$ with $0 \leq L_1 < \infty$. Now integrating (3.36) for $t \geq \tilde{T} \geq T_\alpha$ gives

$$\begin{aligned} 0 &= y^{\Delta}(t) - y^{\Delta}(\tilde{T}) \\ &\quad + \int_{\tilde{T}}^{t} f\left(s, y^\sigma(s), \alpha g_1(s)y^\sigma(s), \alpha g_2(s)y^\sigma(s), \frac{1}{\alpha}h_1(s)y^\sigma(s), \frac{1}{\alpha}h_2(s)y^\sigma(s)\right)\Delta s \end{aligned}$$

and letting $t \to \infty$ in the above, we obtain

$$\begin{aligned} &y^{\Delta}(\tilde{T}) \\ &\quad = L_1 + \int_{\tilde{T}}^{\infty} f\left(s, y^\sigma(s), \alpha g_1(s)y^\sigma(s), \alpha g_2(s)y^\sigma(s), \frac{1}{\alpha}h_1(s)y^\sigma(s), \frac{1}{\alpha}h_2(s)y^\sigma(s)\right)\Delta s \\ &\quad \geq \int_{\tilde{T}}^{\infty} f\left(s, y^\sigma(s), \alpha g_1(s)y^\sigma(s), \alpha g_2(s)y^\sigma(s), \frac{1}{\alpha}h_1(s)y^\sigma(s), \frac{1}{\alpha}h_2(s)y^\sigma(s)\right)\Delta s \\ &\quad > \int_{\tilde{T}}^{t} f\left(s, y^\sigma(s), \alpha g_1(s)y^\sigma(s), \alpha g_2(s)y^\sigma(s), \frac{1}{\alpha}h_1(s)y^\sigma(s), \frac{1}{\alpha}h_2(s)y^\sigma(s)\right)\Delta s. \end{aligned}$$

Now multiplying by $\Big(\phi(y^\sigma(\tilde{T}))\Big)^{-1}$, we obtain

$$\begin{aligned}
&\frac{y^\Delta(\tilde{T})}{\phi(y^\sigma(\tilde{T}))}\\
&\geq \frac{1}{\phi(y^\sigma(\tilde{T}))}\int_{\tilde{T}}^{t} f\left(s, y^\sigma(s), \alpha g_1(s)y^\sigma(s), \alpha g_2(s)y^\sigma(s), \frac{1}{\alpha}h_1(s)y^\sigma(s), \frac{1}{\alpha}h_2(s)y^\sigma(s)\right)\\
&\geq \int_{\tilde{T}}^{t} \frac{f\left(s, y^\sigma(s), \alpha g_1(s)y^\sigma(s), \alpha g_2(s)y^\sigma(s), \frac{1}{\alpha}h_1(s)y^\sigma(s), \frac{1}{\alpha}h_2(s)y^\sigma(s)\right)}{\phi(y^\sigma(s))}\Delta s \qquad (3.38)\\
&\geq \int_{\tilde{T}}^{t} kf\left(s, c, \alpha g_1(s)c, \alpha g_2(s)c, \frac{1}{\alpha}h_1(s)c, \frac{1}{\alpha}h_2(s)c\right)\Delta s
\end{aligned}$$

for sufficiently large $\tilde{T}$ (by condition (H)) where $c := u(T_\alpha) > 0$. Observe that since $y^\Delta(t) > 0$, we have $\lim_{t\to\infty} y(t) = L_2$ with $0 < L_2 < \infty$ and so

$$\lim_{t\to\infty} G(y(t)) = \lim_{t\to\infty}\int_{u_0}^{y(t)} \frac{du}{\phi(u)} = \int_{u_0}^{L_2}\frac{du}{\phi(u)} = L < \infty. \qquad (3.39)$$

Therefore as $t \to \infty$ we have

$$\int_{T_\alpha}^{\infty} (G(y(s)))^\Delta = \lim_{t\to\infty}[G(y(t)) - G(y(T_\alpha))] < \infty.$$

We integrate (3.38) for $t \geq \tilde{T}$ and using (3.37) to obtain

$$\begin{aligned}
&\textstyle\int_{\tilde{T}}^{t}(G(y(s)))^\Delta\Delta s\\
&\geq \textstyle\int_{\tilde{T}}^{t}\frac{y^\Delta(s)}{\phi(y^\sigma(s))}\Delta s\\
&\geq \textstyle\int_{\tilde{T}}^{t}\int_{\tilde{T}}^{s} kf\left(r, c, \alpha g_1(r)c, \alpha g_2(r)c, \frac{1}{\alpha}h_1(r)c, \frac{1}{\alpha}h_2(r)c\right)\Delta r\Delta s\\
&= k\textstyle\int_{\tilde{T}}^{t}(s-\tilde{T})f\left(s, c, \alpha g_1(s)c, \alpha g_2(s)c, \frac{1}{\alpha}h_1(s)c, \frac{1}{\alpha}h_2(s)c\right)\Delta s. \qquad (3.40)
\end{aligned}$$

However, the left side of (3.40) is bounded as $t \to \infty$ whereas the right side is unbounded by assumption (3.25). This contradiction shows that all solutions of (3.14) are oscillatory. $\square$

We would like to illustrate some of the results above by means of several examples. We first consider the linear case when the equation contains an advanced argument. In the first example, we need the following lemma. Using this lemma is often referred to as the Riccati substitution technique. (See Theorem 4.42 of [2]).

Lemma 3.3.11. *The linear equation*

$$Ly \equiv y^{\Delta\Delta} + q(t)y^\sigma = 0$$

is nonoscillatory if and only if there is a function z satisfying the Riccati dynamic inequality

$$z^\Delta + q(t) + S(z)(t) \leq 0 \tag{3.41}$$

with $1+\mu(t)z(t) > 0$ for large t, where

$$S(z) := \frac{z^2}{1+\mu(t)z}.$$

Example 3.3.12. Consider the linear functional dynamic equation

$$y^{\Delta\Delta} + p(t)y^\sigma(t) + q(t)y^\tau(t) + r(t)y^\sigma(t) = 0 \tag{3.42}$$

where $p(t), q(t), r(t) > 0$ for $t \geq t_0 > 0$ and are rd-continuous. If we set

$$Q(t) := p(t) + q(t)\frac{\tau(t)}{\sigma(t)} + r(t)$$

then (3.42) is oscillatory in case

$$y^{\Delta\Delta} + \lambda Q(t)y^\sigma = 0 \tag{3.43}$$

is oscillatory for some $0 < \lambda < 1$. To see this, suppose that $u(t)$ is a nonoscillatory solution of (3.42) with $u(t) > 0$ for $t \geq T$. Since $u^{\Delta\Delta}(t) \leq 0$ for all t, we have $u^\Delta(t) > 0$ for $t \geq T$. Then by Lemma 3.3.1, for $\lambda < k < 1$ there is a $T_k \geq T$ such that

$$u^{\Delta\Delta}(t) + \left(p(t) + kq(t)\frac{\tau(t)}{\sigma(t)} + r(t)\right)u^\sigma(t) \leq 0,\ t \geq T_k.$$

Then with $z(t) := \dfrac{u^\Delta(t)}{u(t)}$, we see that $z(t)$ satisfies the Riccati dynamic inequality (3.41) with $q(t)$ replaced by $p(t)+kq(t)\frac{\tau(t)}{\sigma(t)}+r(t)$. By Lemma 3.3.11, this means that the equation

$$y^{\Delta\Delta} + \left(p(t) + kq(t)\frac{\tau(t)}{\sigma(t)} + r(t)\right)y^\sigma(t) = 0$$

is nonoscillatory and so by the Sturm-Picone Comparison Theorem [14, Lemma 6], (3.43) is also nonoscillatory. This contradiction shows that (3.42) is oscillatory. If we apply a specific oscillation criterion, we conclude that (3.42) is oscillatory if

$$\liminf\ t\int_t^\infty \left(p(s) + q(s)\frac{\tau(s)}{\sigma(s)} + r(s)\right)\Delta s > \frac{1}{4}$$

(see Example 3.4 in [13]).

Example 3.3.13. Let $f(t,u,v) := p(t)u^{\gamma_1} + q(t)v^{\gamma_2}$, where $\gamma_1, \gamma_2 > 0$ and are the quotients of odd positive integers. We assume also that $p(t), q(t) > 0$ for all large t and are rd-continuous. Observe condition (A) holds so that with $g(t) = \dfrac{\tau(t)}{\sigma(t)}$ we conclude from Theorem 3.3.5 that all bounded solutions of

$$y^{\Delta\Delta} + p(t)(y^{\sigma}(t))^{\gamma_1} + q(t)(y^{\tau}(t))^{\gamma_2} = 0 \tag{3.44}$$

are oscillatory if

$$\int^{\infty} t\left(p(t) + q(t)(g(t))^{\gamma_2}\right)\Delta t = \infty.$$

If $\gamma_1, \gamma_2 > 1$, then with $\phi(u) = u^{\gamma}$, where $1 < \gamma < min\{\gamma_1, \gamma_2\}$, it is not difficult to show that $f(t,u,v) = p(t)u^{\gamma_1} + q(t)v^{\gamma_2}$ satisfies condition (H). Therefore, from Theorem 3.3.10, we conclude that all solutions of (3.44) are oscillatory provided

$$\int^{\infty} t\left(p(t)\alpha^{\gamma_1} + q(t)(\alpha g(t))^{\gamma_2}\right)\Delta t = \infty \tag{3.45}$$

for all $\alpha \neq 0$. Moreover, (3.45) holds for all $\alpha \neq 0$ if and only if

$$\int^{\infty} tp(t)\Delta t + \int^{\infty} tq(t)(g(t))^{\gamma_2}\Delta t = \infty. \tag{3.46}$$

Conversely, if $\liminf_{t\to\infty} g(t) := m_1 > 0$, then (3.46) is necessary for all solutions of (3.44) to be oscillatory.

As an illustration of the situation when f involves an advanced argument, we consider

Example 3.3.14. Suppose that

$$f(t,u,v,w) := \frac{p(t)u^{\gamma_1} + q(t)v^{\gamma_2}}{1 + r(t)w^2}$$

where $p(t), q(t) > 0,\ r(t) \geq 0$ are rd-continuous and $\gamma_1, \gamma_2 > 0$. From Theorem 3.3.5, we conclude that all bounded solutions of

$$y^{\Delta\Delta} + \frac{p(t)(y^{\sigma}(t))^{\gamma_1} + q(t)(y^{\tau}(t))^{\gamma_2}}{1 + r(t)(y^{\xi}(t))^2} = 0 \tag{3.47}$$

are oscillatory in case

$$k^4 \int^{\infty} \frac{t\left[p(t)\alpha^{\gamma_1} + q(t)(\alpha g(t))^{\gamma_2}\right]}{k^4 + r(t)(\alpha h(t))^2}\Delta t = \infty$$

for all $\alpha \neq 0$ and some $k \in (0,1)$. Moreover, (3.47) has a bounded nonoscillatory solution iff

$$\int^{\infty} tf(t,\alpha,\alpha,\alpha)\Delta t < \infty$$

for some $\alpha \neq 0$.

The results in the last two examples may be regarded as extensions of some oscillation criteria due to Atkinson [1].

Chapter 4

Oscillation Criteria for Nonlinear Delay Dynamic Equations

4.1 Oscillation of a Dynamic Equation with a Single Delay

It is the purpose of this section to establish oscillation criteria for second order nonlinear dynamic equations with a single retarded argument. Specifically, we consider the equation

$$y^{\Delta\Delta} + f\big(t, y^{\sigma}(t), y(\tau(t))\big) = 0, \tag{4.1}$$

where $f \in C([0,\infty)_{\mathbb{T}} \times \mathbb{R}^2), \tau \in C([0,\infty)_{\mathbb{T}}, [0,\infty)_{\mathbb{T}})$, and

$$0 < \tau(t) \le t, \quad \lim_{t\to\infty} \tau(t) = \infty. \tag{4.2}$$

We shall restrict our attention to solutions of (4.1) which exist on the time scale $[T,\infty)_{\mathbb{T}}$. Our main results include Theorems 4.1.2 and 4.1.11.

We begin this section by introducing the auxiliary functions $H(t,t_0)$ and $\eta_i(t,t_0)$ defined by

$$H(t,t_0) = t - t_0 \quad \text{and} \quad \eta_i(t,t_0) := \frac{H(\tau_i(t),t_0)}{H(\sigma(t),t_0)}, \quad 1 \le i \le n.$$

From Erbe and Peterson [8], we have the following lemma.

Lemma 4.1.1. *[8, Lemma 1.2] Let $y(t)$ be a solution of*

$$\big(p(t)y^{\Delta}(t)\big)^{\Delta} + \sum_{i=1}^{n} q_i(t)y(\tau_i(t)) = 0$$

which satisfies

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad \big(p(t)y^{\Delta}(t)\big)^{\Delta} \leq 0$$

for all $\tau_i(t) \geq T \geq t_0$*. Then for each* $1 \leq i \leq n$ *we have*

$$y(\tau_i(t)) > \eta_i(t,T)y^{\sigma}(t), \quad \tau_i(t) > T.$$

We continue with a general result for the case when $y^{\Delta\Delta} + f\big(t, y^{\sigma}(t), y(\tau(t))\big) = 0$ is linear of the form

$$y^{\Delta\Delta} + q(t)y(\tau(t)) = 0, \tag{4.3}$$

where $q \in C[0,\infty)_{\mathbb{T}}$.

Theorem 4.1.2. *Suppose* $q(t) \geq 0$ *for* $t > 0$*. Assume that the equation*

$$y^{\Delta\Delta} + \lambda\frac{\tau(t)}{\sigma(t)}q(t)y^{\sigma}(t) = 0, \tag{4.4}$$

is oscillatory on $(0,\infty)_{\mathbb{T}}$ *for some* $0 < \lambda < 1$*. Then all solutions of (4.3) are oscillatory.*

Proof. Suppose, to the contrary, that (4.3) has a nonoscillatory solution and without loss of generality, we assume there is an eventually positive solution u. That is, there exists $T_0 \in [0,\infty)_{\mathbb{T}}$ such that $u(t) > 0$ for $t \geq T_0$. Since $\tau(t) \to \infty$ as $t \to \infty$, there exists $T \in [0,\infty)_{\mathbb{T}}$ such that $u(t) > 0$ and $u(\tau(t)) > 0$ for $t \geq T$. Hence $u^{\Delta}(t)$ decreases to a limit which must be nonnegative. In fact, we must have $u^{\Delta}(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ for if $u^{\Delta}(t_0) = 0$ for some $t_0 > T$, then $u^{\Delta}(t) \equiv 0$ on $[t_0,\infty)_{\mathbb{T}}$. Consequently, from (4.3) we would have $q(t) \equiv 0$ on $[t_0,\infty)_{\mathbb{T}}$, since $u(\tau(t)) > 0$ on $[T,\infty)_{\mathbb{T}}$, contradicting the fact that (4.4) is oscillatory. So we have

$$u(t) > 0, \quad u^{\Delta}(t) > 0, \quad u^{\Delta\Delta}(t) \leq 0 \quad \text{on} \quad [T,\infty)_{\mathbb{T}}.$$

From Lemma 4.1.1 we have $u(\tau(t)) > \eta(t,T)u^{\sigma}(t),\ \tau(t) > T$. So for any $0 < k < 1$ there is a $T_k \geq T$ such that

$$u(\tau(t)) \geq \frac{\tau(t) - T}{\sigma(t) - T} \geq k\frac{\tau(t)}{\sigma(t)}u^{\sigma}(t), \quad t \geq T_k.$$

It follows that

$$u^{\Delta\Delta}(t) + k\frac{\tau(t)}{\sigma(t)}q(t)u^{\sigma}(t) \leq 0, \quad t \geq T_k. \tag{4.5}$$

Let $z(t) = \dfrac{u^{\Delta}(t)}{u(t)}$ and $Q(t) = k\dfrac{\tau(t)}{\sigma(t)}q(t)$. Also, let

$$S[z] = \frac{z^2}{1 + \mu(t)z}.$$

Then $p(t)+\mu(t)z(t) = 1+\mu(t)\dfrac{u^\Delta(t)}{u(t)} > 0$ for $t \geq T$ and

$$\begin{aligned} z^\Delta + Q + S(z) &= \frac{uu^{\Delta\Delta}-(u^\Delta)^2}{uu^\sigma} + Q + \left(\frac{u^\Delta}{u}\right)^2 \frac{1}{1+\mu\frac{u^\Delta}{u}} \\ &= \frac{uu^{\Delta\Delta}-(u^\Delta)^2}{uu^\sigma} + Q + \left(\frac{u^\Delta}{u}\right)^2 \frac{u}{u+\mu u^\Delta} \\ &= \frac{uu^{\Delta\Delta}-(u^\Delta)^2}{uu^\sigma} + Q + \frac{(u^\Delta)^2}{uu^\sigma} \\ &= \frac{u^{\Delta\Delta}}{u^\sigma} + Q \\ &\leq 0. \end{aligned}$$

Hence, by Lemma 1.1 of [8], $u^{\Delta\Delta} + Qu^\sigma = 0$ is nonoscillatory. Choosing $0 < k < 1$ such that $k > \lambda$, we have $Q(t) > \lambda\dfrac{\tau(t)}{\sigma(t)}q(t) =: R(t)$. Thus, by the Sturm-Picone Comparison Theorem [14, Lemma 6], $u^{\Delta\Delta} + R(t)u^\sigma(t) = 0$ is nonoscillatory. This contradiction proves the theorem. □

As a corollary to Theorem 4.1.2, we have

Corollary 4.1.3. *All solutions of*

$$y^{\Delta\Delta} + q(t)y(\tau(t)) = 0, \tag{4.3}$$

are oscillatory in case either of the following holds:

(i) $\displaystyle\int^\infty (\sigma(t))^{\alpha-1}\tau(t)q(t)\,\Delta t = \infty$ *for some* $0<\alpha<1$

(ii) $\displaystyle\liminf_{t\to\infty} t\int_t^\infty \frac{\tau(t)}{\sigma(t)}q(t)\,\Delta t > \frac{1}{4}$ *and* $\mu(t)$ *is bounded.*

Proof. If (i) holds, then for any $\lambda > 0$, $\displaystyle\int^\infty \sigma^\alpha(t)\lambda\frac{\tau(t)}{\sigma(t)}q(t)\,\Delta t = \infty$. By Example 3.2.7, $y^{\Delta\Delta} + \lambda\dfrac{\tau(t)}{\sigma(t)}q(t)y^\sigma(t) = 0$ is oscillatory since $0<\alpha<1$. Hence, by Theorem 4.1.2, all solutions of $y^{\Delta\Delta}+q(t)y(\tau(t)) = 0$, equation (4.3), are oscillatory. Next assume (ii) holds. Then by Theorem 3.1 of [23], $y^{\Delta\Delta} + \dfrac{\tau(t)}{\sigma(t)}q(t)y^\sigma(t) = 0$ is oscillatory. Since $\mu(t)$ is bounded, we have that $y^{\Delta\Delta} + \dfrac{\tau(t)}{\sigma(t)}q(t)y(\tau(t)) = 0$ is oscillatory by Theorem 3.1.2 with $f(t) = t$. Since $0 < \tau(t) \leq t \leq \sigma(t)$, by the Sturm-Picone Comparison Theorem [14, Lemma 6], (4.3) is oscillatory. □

To prove another corollary of Theorem 4.1.2, we need the following results.

Lemma 4.1.4. *If y is an eventually positive solution of*

$$y^{\Delta\Delta} + \lambda \frac{\tau(t)}{\sigma(t)} q(t) y^{\sigma}(t) = 0, \tag{4.4}$$

then there is a $t_0 \in [a, \infty)_{\mathbb{T}}$ such that

$$y(t) \geq y(\tau(t)) > 0, \quad y^{\Delta}(t) > 0, \quad \textit{and} \quad y^{\Delta\Delta}(t) < 0 \tag{4.6}$$

for all $t \geq t_0 \geq a$.

Lemma 4.1.5. *[4] Let $a \in \mathbb{T}$. If $\mathbb{T}$ is a time scale that is unbounded above, then*

$$\int_a^{\infty} \frac{\Delta t}{t} = \infty.$$

Lemma 4.1.6. *If*

$$\int^{\infty} \tau(t) q(t) \Delta t = \infty, \tag{4.7}$$

then every bounded solution of equation (4.4) is oscillatory on $[a, \infty)_{\mathbb{T}}$.

Proof. Suppose that there exists an eventually positive and bounded solution y of (4.4). Then there exists $t_0 \in \mathbb{T}$ such that (4.6) holds, and without loss of generality, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$0 < \alpha < y(t) < \beta \quad \text{for all} \quad t \geq t_0.$$

Let $Y(t) = t y^{\Delta}(t)$. Then

$$\begin{aligned}
Y(t) &= Y(t_0) + \int_{t_0}^{t} Y^{\Delta}(s)\, \Delta s \\
&= Y(t_0) + \int_{t_0}^{t} \left\{ y^{\Delta}(s) + \sigma(s) y^{\Delta\Delta}(s) \right\} \Delta s \\
&= Y(t_0) + \int_{t_0}^{t} \left\{ y^{\Delta}(s) - \sigma(s) \lambda \frac{g(s)}{\sigma(s)} q(s) y^{\sigma}(s) \right\} \Delta s \\
&= Y(t_0) + y(t) - y(t_0) - \lambda \int_{t_0}^{t} g(s) q(s) y^{\sigma}(s)\, \Delta s \\
&\leq Y(t_0) + \beta - y(t_0) - \lambda \int_{t_0}^{t} g(s) q(s) y^{\sigma}(s)\, \Delta s \\
&\leq Y(t_0) + \beta - y(t_0) - \lambda \alpha \int_{t_0}^{t} g(s) q(s)\, \Delta s \\
&\to -\infty \text{ as } t \to \infty,
\end{aligned}$$

i.e., there is a constant $M > 0$ such that

$$y^{\Delta}(t) \leq -\frac{M}{t} \quad \text{for } t \geq T$$

for some $T \geq t_0$, and this implies that $\lim_{t\to\infty} y(t) = -\infty$ by Lemma 4.1.5, contradicting $y(t) > 0$ for all $t \geq t_0$. Thus every bounded solution of $y^{\Delta\Delta} + \lambda \frac{\tau(t)}{\sigma(t)} q(t) y^{\sigma}(t) = 0$ is oscillatory. □

In addition to the above lemmas, we should recall the following about upper and lower solutions. Consider the second-order equation

$$y^{\Delta\Delta} = f(t, y^{\sigma}) \tag{4.8}$$

where f is continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$.

Definition 4.1.7. [2, Definition 6.53] We say that $\alpha \in C^2_{rd}$ is a *lower solution* of (4.8) on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided

$$\alpha^{\Delta\Delta}(t) \geq f(t, \alpha^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]_{\mathbb{T}}.$$

Similarly, $\beta \in C^2_{rd}$ is called an *upper solution* of (4.8) on $[a, \sigma^2(b)]_{\mathbb{T}}$ provided

$$\beta^{\Delta\Delta}(t) \leq f(t, \beta^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]_{\mathbb{T}}.$$

Theorem 4.1.8. *[2, Theorem 6.54] Let f be continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution α and an upper solution β of (4.8) with*

$$\alpha(a) \leq A \leq \beta(a) \quad \text{and} \quad \alpha(\sigma^2(b)) \leq B \leq \beta(\sigma^2(b))$$

such that

$$\alpha(t) \leq \beta(t) \quad \text{for all} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}}.$$

Then the BVP

$$y^{\Delta\Delta} = f(t, y^{\sigma}) \text{ on } [a, b]_{\mathbb{T}}, \quad y(a) = A, \quad y(\sigma^2(b)) = B$$

has a solution y with

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \text{for all} \quad t \in [a, \sigma^2(b)]_{\mathbb{T}}.$$

Recall the following generalization of Theorem 7.4 of [20].

Theorem 4.1.9. *Let f be continuous on $[a, b]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution α and an upper solution β of (4.8) with $\alpha(t) \leq \beta(t)$ for all $t \in [a, \infty)_{\mathbb{T}}$. Then* *for any* $\alpha(a) \le c \le \beta(a)$ *the BVP*

$$y^{\Delta\Delta} = f(t, y^\sigma) \quad y(a) = c \tag{4.9}$$

has a solution y *with*

$$\alpha(t) \le y(t) \le \beta(t) \quad \textit{for all} \quad t \in [a, \infty)_{\mathbb{T}}.$$

Now we can state and prove another corollary of Theorem 4.1.2.

Corollary 4.1.10. *All bounded solutions of the linear second-order dynamic equation* $y^{\Delta\Delta} + q(t)y(\tau(t)) = 0$ *are oscillatory in case*

$$\int^\infty \tau(t)q(t)\Delta t = \infty \tag{4.7}$$

holds.

Proof. Let $u(t)$ be a bounded nonoscillatory solution of $u^{\Delta\Delta} + q(t)u(\tau(t)) = 0$ with $u(t) > 0$ and $u(\tau(t)) > 0$ for $t \ge T$. Since $u^{\Delta\Delta}(t) \le 0$ for all t, we have $u^\Delta(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. As in the proof of Theorem 4.1.2, for any $0 < k < 1$ there exists a $T_k \ge T$ such that

$$u^{\Delta\Delta}(t) + k\frac{\tau(t)}{\sigma(t)}q(t)u^\sigma(t) \le 0$$

for $t \ge T \ge T_k$. Let $\alpha(t) = u(T)$ and $\beta(t) = u(t)$. Then

$$f(t, \alpha^\sigma(t)) = -\lambda\frac{\tau(t)}{\sigma(t)}q(t)u^\sigma(T) \le 0 = \alpha^{\Delta\Delta}(t)$$

and

$$f(t, \beta^\sigma(t)) = -\lambda\frac{\tau(t)}{\sigma(t)}q(t)u^\sigma(t) \ge \beta^{\Delta\Delta}(t) \qquad \text{with } k = \lambda.$$

So α, β are lower and upper solutions, respectively, of $y^{\Delta\Delta} + \lambda\frac{\tau(t)}{\sigma(t)}q(t)y^\sigma(t) = 0$, equation (4.4). As u is increasing, $\beta(t) \ge \alpha(t)$ on $[T_k, \infty)_{\mathbb{T}}$. Then by Theorem 4.1.9, there is a solution $y(t)$ of (4.4) satisfying $u(T) \le y(t) \le u(t)$ on $[T_k, \infty)_{\mathbb{T}}$. As u is bounded, y is a bounded nonoscillatory solution of (4.4). This is a contradiction to Lemma 4.1.6 and proves the theorem.

□

We end this section with a general result for the case when

$$y^{\Delta\Delta} + f(t, y^\sigma(t), y(\tau(t))) = 0$$

is linear of the form

$$(p(t)y^{\Delta})^{\Delta} + q(t)y(\tau(t)) = 0, \tag{4.10}$$

where $p : \mathbb{T} \to (0, \infty)$ is rd-continuous and satisfies $\displaystyle\int^{\infty} \frac{1}{p(t)}\,\Delta t = \infty$ and $q \in C_{rd}[0, \infty)_{\mathbb{T}}$.

Theorem 4.1.11. *Suppose $q(t) \geq 0$ for $t > 0$. Assume that the equation*

$$(p(t)y^{\Delta})^{\Delta} + \lambda\frac{\tau(t)}{\sigma(t)}q(t)y^{\sigma}(t) = 0$$

is oscillatory on $(0, \infty)_{\mathbb{T}}$ for some $0 < \lambda < 1$. Then all solutions of (4.10) are oscillatory.

Proof. If not, we may assume that $u(t)$ is a nonoscillatory solution of (4.10) with $u(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. As $\tau(t) \to \infty$ as $t \to \infty$, we may also assume that $u(\tau(t)) > 0$ for $t \geq T$. Consequently,

$$(p(t)u^{\Delta}(t))^{\Delta} = -q(t)u(\tau(t)) \leq 0 \quad \text{for } t \geq T.$$

Integrating twice from T to t gives

$$u(t) \leq u(T) + p(T)u^{\Delta}(T)\int_T^t \frac{1}{p(s)}\,\Delta s \quad t \geq T.$$

As shown in Theorem 3.1.2, it follows that $p(t)u^{\Delta}(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. So we have

$$u(t) > 0, \quad p(t)u^{\Delta}(t) > 0, \quad (p(t)u^{\Delta}(t))^{\Delta} \leq 0, \quad u(\tau(t)) > 0 \quad \text{on} \quad [T, \infty)_{\mathbb{T}}.$$

From Lemma 4.1.1 we have $u(\tau(t)) > \eta(t, T)u^{\sigma}(t),\ \tau(t) > T$. So for any $0 < k < 1$ there is a $T_k \geq T$ such that

$$u(\tau(t)) \geq \frac{\tau(t) - T}{\sigma(t) - T}u^{\sigma}(t) \geq k\frac{\tau(t)}{\sigma(t)}u^{\sigma}(t), \quad t \geq T_k.$$

Since $(pu^{\Delta})^{\Delta} + q(t)u(\tau(t)) = 0$, we have

$$(p(t)u^{\Delta}(t))^{\Delta} + k\frac{\tau(t)}{\sigma(t)}q(t)u^{\sigma}(t) \leq 0, \quad t \geq T_k.$$

Let $z(t) = \dfrac{p(t)u^{\Delta}(t)}{u(t)}$ and $Q(t) = k\dfrac{\tau(t)}{\sigma(t)}q(t)$. Then we have

$$p(t) + \mu(t)z(t) = p(t) + \mu(t)\frac{u^{\Delta}(t)}{u(t)} > 0 \quad \text{for} \quad t \geq T$$

and

$$
\begin{aligned}
z^\Delta + Q + S(z) &= \frac{u(pu^\Delta)^\Delta - p(u^\Delta)^2}{uu^\sigma} + Q + \left(\frac{pu^\Delta}{u}\right)^2 \frac{1}{p + \mu\frac{pu^\Delta}{u}} \\
&= \frac{u(pu^\Delta)^\Delta - p(u^\Delta)^2}{uu^\sigma} + Q + \left(\frac{pu^\Delta}{u}\right)^2 \frac{u}{p(u + \mu u^\Delta)} \\
&= \frac{u(pu^\Delta)^\Delta - p(u^\Delta)^2}{uu^\sigma} + Q + \frac{p(u^\Delta)^2}{uu^\sigma} \\
&= \frac{(pu^\Delta)^\Delta}{u^\sigma} + Q \\
&\leq 0.
\end{aligned}
$$

Hence, by Lemma 1.1 of [8], $u^{\Delta\Delta} + Qu^\sigma = 0$ is nonoscillatory. Choosing $\lambda < k < 1$, we have $Q(t) > \lambda\dfrac{\tau(t)}{\sigma(t)}q(t) =: R(t)$. Thus, by the Sturm-Picone Comparison Theorem [14, Lemma 6], $u^{\Delta\Delta} + R(t)u^\sigma(t) = 0$ is nonoscillatory. This contradiction proves the theorem. $\square$

4.2 Oscillation of a Dynamic Equation with Several Delays

In this section we establish several oscillation results for the nonlinear delay dynamic equation

$$y^{\Delta\Delta} + f(t, y^\sigma(t), y(\tau_1(t)), \ldots, y(\tau_n(t))) = 0 \tag{4.11}$$

where for some positive integer n, $f \in C\left([0,\infty)_\mathbb{T} \times \mathbb{R}^{n+1}\right), \tau_i \in C\left([0,\infty)_\mathbb{T}, \mathbb{T}^+\right)$ for $1 \leq i \leq n$, and

$$0 < \tau_i(t) \leq t, \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \quad 1 \leq i \leq n. \tag{4.12}$$

Results of particular importance are Theorems 4.2.2 and 4.2.4. We also provide examples to further establish the importance of Theorem 4.2.2.

Throughout this section we shall assume $f(t, u, v_1, \ldots, v_n)$ satisfies

$$f(t, u, v_1, \ldots, v_n) = -f(t, -u, -v_1, \ldots, -v_n) \tag{4.13}$$

and

$$f(t, u, v_1, \ldots, v_n) > 0 \quad \text{for } u, v_1, \ldots, v_n > 0 \quad \text{and all} \quad t \geq 0. \tag{4.14}$$

Further, we assume that for each fixed t and $u > 0$, $f(t, u, v_1, \ldots, v_n)$ is nondecreasing in v_i for $v_i > 0$, $1 \leq i \leq n$ and that for each fixed t and $v_i > 0$, $1 \leq i \leq n$,

$f(t, u, v_1, \ldots, v_n)$ is nondecreasing in u for $u > 0$.

Our first result is a generalization of Theorem 3 of [24].

Theorem 4.2.1. *Let $f(t, y)$ be a continuous function of the variables $t \geq t_0$ and $|y| < \infty$. Assume that for all $t > 0$ and $y \neq 0$, $yf(t, y) > 0$, and for each fixed t, $f(t, y)$ is nondecreasing in y for $y > 0$. Then a necessary condition for*

$$y^{\Delta\Delta} + f(t, y^\sigma) = 0, \quad t \geq t_0 \tag{4.15}$$

to have a bounded nonoscillatory solution is that

$$\int^\infty t f(t, c) \Delta t < \infty \tag{4.16}$$

for some constant $c > 0$.

Proof. Suppose $y(t)$ is a bounded eventually positive solution of (4.15). So there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that $y(t) > 0$ for $t \geq T$. As $f(t, y) > 0$ for all $y > 0$, $y^{\Delta\Delta}$ is eventually negative. So y^Δ is decreasing and tends to a limit L that is positive, zero, negative, or $-\infty$. If $L < 0$ or if $L = -\infty$, y would be eventually negative. Hence $\lim_{t\to\infty} y^\Delta(t) = L$ with $0 \leq L < \infty$. In fact, $L = 0$, since if $L > 0$, then $y(t)$ would be unbounded. Integrating (4.15) from s to t, we obtain

$$y^\Delta(s) = y^\Delta(t) + \int_s^t f(u, y^\sigma(u)) \Delta u.$$

Letting $t \to \infty$ gives

$$y^\Delta(s) = \int_s^\infty f(\tau, y^\sigma(\tau)) \Delta\tau \tag{4.17}$$

since $0 = L < \infty$.

As $y(t)$ is nondecreasing and bounded, it increases to a finite limit $l > 0$. By the monotonicity of f, we have for any $\epsilon > 0$ with $l - \epsilon > 0$ there exists T_ϵ such that

$$f(t, l) \geq f(t, y^\sigma(t)) \geq f(t, l - \epsilon), \quad t \geq T_\epsilon \tag{4.18}$$

for all $t \geq T$. Integrating (4.17) from T to t, we have

$$\begin{aligned}
y(t) - y(T) &\geq \int_T^t \int_s^\infty f(u, y^\sigma(u)) \Delta u \Delta s \\
&= \int_T^t \int_T^u f(u, y^\sigma(u)) \Delta s \Delta u + \int_t^\infty \int_T^t f(u, y^\sigma(u)) \Delta s \Delta u \\
&= \int_T^t (u - T) f(u, y^\sigma(u)) \Delta u + \int_t^\infty (t - T) f(u, y^\sigma(u)) \Delta u \\
&\geq \int_T^t (t - T) f(u, y^\sigma(u)) \Delta u \\
&\geq \int_T^t (t - T) f(u, l - \epsilon) \Delta u
\end{aligned}$$

by (4.18). So

$$y(t) \geq y(T) + \int_T^t (u - T) f(u, l - \epsilon) \Delta u > \int_T^t (u - T) f(u, l - \epsilon) \Delta u. \tag{4.19}$$

Therefore, $l \geq y(t) \geq \int_T^t (u - T) f(u, l - \epsilon) \Delta u$. Since $l - \epsilon > 0$, letting $t \to \infty$ in (4.19) implies

$$\int_T^\infty (u - T) f(u, l - \epsilon) \Delta t < \infty.$$

Hence (4.16) is necessary. $\square$

Recall the auxiliary functions $H(t, t_0)$ and $\eta_i(t, t_0)$ defined by

$$H(t, t_0) = t - t_0 \quad \text{and} \quad \eta_i(t, t_0) := \frac{H(\tau_i(t), t_0)}{H(\sigma(t), t_0)}, \quad 1 \leq i \leq n.$$

The following two results are extensions of Erbe [11].

Theorem 4.2.2. *All bounded solutions of the nonlinear dynamic equation*

$$y^{\Delta\Delta} + f\left(t, y^\sigma(t), y(\tau_1(t)), \ldots, y(\tau_n(t))\right) = 0 \tag{4.11}$$

are oscillatory in case

$$\left| \int^\infty t f(t, \alpha, \alpha\eta_1(t, t_0), \ldots, \alpha\eta_n(t, t_0)) \Delta t \right| = \infty \tag{4.20}$$

for all $\alpha \neq 0$ and $t_0 \in \mathbb{T}$.

Proof. Assume not and let $u(t)$ be a bounded nonoscillatory solution of (4.11) which we may assume satisfies

$$u(t) > 0, \quad u^{\Delta}(t) > 0, \quad u^{\Delta\Delta}(t) \leq 0, \quad u(\tau_i(t)) > 0, \quad t \geq T \geq t_0, \quad 1 \leq i \leq n.$$

By Lemma 4.1.1, we know that for each $1 \leq i \leq n$,

$$u(\tau_i(t)) > \eta_i(t, T)u^{\sigma}(t), \quad \tau_i(t) > T.$$

Then by the monotonicity of f we have

$$u^{\Delta\Delta} + f(t, u^{\sigma}(t), \eta_1(t, T)u^{\sigma}(t), \ldots, \eta_n(t, T)u^{\sigma}(t))$$
$$\leq u^{\Delta\Delta} + f(t, u^{\sigma}(t), u(\tau_1(t)), \ldots, u(\tau_n(t))) = 0.$$

Define

$$F(t, w) := f(t, w, \eta_1(t, T)w, \ldots, \eta_n(t, T)w).$$

Then it follows that $F(t, u^{\sigma}(t)) = f(t, u^{\sigma}(t), \eta_1(t, T)u^{\sigma}(t), \ldots, \eta_n(t, T)u^{\sigma}(t))$. Applying Theorem 4.1.9 with $\alpha(t) \equiv u(T) \leq u(t) \equiv \beta(t)$, we obtain the existence of a solution $y(t)$ of

$$y^{\Delta\Delta} + F(t, y^{\sigma}) = 0$$

with $u(T) \leq y(t) \leq u(t)$ on $[T, \infty)_{\mathbb{T}}$. However, by Theorem 4.2.1 it follows that

$$\left| \int^{\infty} tF(t, c)\Delta t \right| < \infty$$

for some constant $c \neq 0$. Since $\lim_{t\to\infty} \dfrac{\eta_i(t, T)}{\eta_i(t, t_0)} = 1$, given any $0 < k < 1$, there exists T_k such that $\eta_i(t, T) \geq k\eta_i(t, t_0)$, $t \geq T_k, 1 \leq i \leq n$. Hence

$$\left| \int^{\infty} tf(t, ck, ck\eta_1(t, t_0), \ldots, ck\eta_n(t, t_0))\Delta t \right| < \infty$$

by the monotonicity of f and the fact that $0 < k < 1$. It follows that with $\tilde{c} = ck$

$$\left| \int^{\infty} tf(t, \tilde{c}, \tilde{c}\eta_1(t, t_0), \ldots, \tilde{c}\eta_n(t, t_0))\Delta t \right| < \infty,$$

which contradicts (4.20). This completes the proof. $\square$

The next theorem shows that the converse of Theorem 4.2.2 is true under additional assumptions.

Theorem 4.2.3. *Assume for each $i \in \{1, \ldots, n\}$ there exists $\rho_i > 0$ such that*

$$\liminf_{t\to\infty} \eta_i(t, t_0) \geq \rho_i \quad \textit{for} \quad t_0 \in \mathbb{T} \tag{4.21}$$

and assume $\sigma(t)/t$ is bounded on $\mathbb{T}$. Then the nonlinear dynamic equation

$$y^{\Delta\Delta} + f(t, y^{\sigma}(t), y(\tau_1(t)), \ldots, y(\tau_n(t))) = 0 \tag{4.11}$$

has a bounded nonoscillatory solution if, and only if,

$$\left| \int^{\infty} \sigma(t) f(t, \alpha, \alpha, \ldots, \alpha) \Delta t \right| < \infty \tag{4.22}$$

for some $\alpha \neq 0$.

Proof. Note that for any β

$$\left| \int^{\infty} \sigma(t) f(t, \beta, \ldots, \beta) \Delta t \right| < \infty$$

if, and only if,

$$\left| \int^{\infty} t f(t, \beta, \ldots, \beta) \Delta t \right| < \infty$$

since $\sigma(t)/t$ is bounded on $\mathbb{T}$.

We prove first the necessity of (4.22). By assumption, given any $\epsilon > 0$ with $\epsilon < \frac{1}{2} \min\{\rho_i | 1 \leq i \leq n\}$ there exists $T_i \geq t_0$ such that $\eta_i(t, t_0) \geq \rho_i - \epsilon$ for $t \geq T_i$ and $1 \leq i \leq n$. Assume (4.11) has a bounded nonoscillatory solution. Then by Theorem 4.2.2, $\left| \int^{\infty} t f(t, \alpha, \alpha\eta_1, \ldots, \alpha\eta_n) \Delta t \right| < \infty$ for some $\alpha \neq 0$. From (4.21), we have

$$\alpha\eta_i(t, t_0) \geq \alpha(\rho_i - \epsilon) =: \alpha\tilde{\rho}_i, \quad t \geq T_i$$

where $0 < \tilde{\rho}_i < \rho_i$. Hence, by the monotonicity of f, we have

$$\left| \int^{\infty} t f(t, \alpha, \alpha\tilde{\rho}_1, \ldots, \alpha\tilde{\rho}_n) \Delta t \right| < \infty.$$

Let $\tilde{\rho} := \min\{\tilde{\rho}_i | 1 \leq i \leq n\}$. Observe that $\eta_i(t, t_0) \leq 1$ implies that $0 < \tilde{\rho}_i \leq 1$ for $t \geq T_i$ and for all i. Hence $\alpha\tilde{\rho} \leq \alpha\tilde{\rho}_i \leq \alpha$ for all $1 \leq i \leq n$. Thus

$$\left| \int^{\infty} t f(t, \alpha\tilde{\rho}, \alpha\tilde{\rho}, \ldots, \alpha\tilde{\rho}) \Delta t \right| < \infty,$$

which shows that (4.22) is necessary.

Suppose (4.22) holds. Assume to be specific that $\alpha > 0$ and let $0 < \beta < \alpha$. Choose $T \geq t_1 \geq t_0$ so that $\tau_i(t) \geq t_1$ if $t \geq T$ for all $i = 1, \ldots, n$ and so that

$$\int_T^{\infty} \sigma(s) f(s, \beta, \beta, \ldots, \beta) \Delta s < \frac{\beta}{2}.$$

Define $y_0(t) \equiv \beta$ for $t \geq t_0$ and

$$y_{m+1}(t) = \begin{cases} \beta - \int_T^\infty (\sigma(s) - T) f(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s)))\Delta s, & t < T, \\ \beta - \int_t^\infty (\sigma(s) - t) f(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s)))\Delta s, & t \geq T. \end{cases}$$

Observe $t_1 \leq \tau_i(t) \leq t$ for all $t \geq T$ and $i = 1, \ldots, n$. We claim that

$$\frac{\beta}{2} \leq y_m(t) \leq \beta \quad t \geq T \quad \text{and} \quad \text{all } m \geq 0. \tag{4.23}$$

By construction the claim holds for $y_0(t)$. Notice that when $\tau_i(t) < T \leq t$ for any $i = 1, \ldots, n$, $y_m(\tau_i(t)) < \beta$ as $y_m^\Delta(t) \equiv 0$ for all $t \in \mathbb{T}$ less than T. Assume the inequality holds for $y_k(t)$, $1 \leq k \leq m$. Then for $t \geq T$

$$\begin{aligned} y_{m+1}(t) &= \beta - \int_t^\infty (\sigma(s) - t) f(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s)))\Delta s \\ &\geq \beta - \int_t^\infty \sigma(s) f(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s)))\Delta s \\ &> \beta - \int_t^\infty \sigma(s) f\left(s, \beta, \beta, \ldots, \beta\right) \Delta s \\ &> \beta - \frac{\beta}{2} \\ &= \frac{\beta}{2}. \end{aligned}$$

Furthermore, since $s \geq T$, we have $y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s))$ are all positive. Hence by (4.14)

$$(\sigma(s) - t) f\big(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s))\big) \geq 0$$

for $s \geq t \geq T$. Consequently, $y_{m+1}(t) \leq \beta$ for $t \geq T$. Therefore, by induction, the claim holds.

Furthermore, for $t \geq T$,

$$\begin{aligned} y_{m+1}^\Delta(t) &= 0 - (\sigma(t) - \sigma(t)) f(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s))) \\ &\quad - \int_t^\infty (-1) f(s, y_m^\sigma(s), y_m(\tau_1(s)), \ldots, y_m(\tau_n(s)))\Delta s \\ &\leq \int_t^\infty f(s, \beta, \beta, \ldots, \beta)\Delta s, \end{aligned}$$

and so $\left\{y_m^\Delta(t)\right\}_{m=0}^\infty$ is uniformly bounded for $t \geq T$. Hence, the set $\{y_m(t)\}_{m=0}^\infty$ is uniformly bounded and equicontinuous. Therefore, the Ascoli-Arzela theorem along with a standard diagonalization argument yields a subsequence of $\{y_m(t)\}_{m=0}^\infty$ which converges uniformly on compact subintervals of $[T, \infty)$ to a solution $y(t)$ of (4.11) satisfying $\beta/2 \le y(t) < \beta$, $t \ge T$. This proves the theorem as $\beta > 0$. $\square$

We shall note later (see Remark 4.2.7) that the converse of Theorem 4.2.2 is not true. To extend Theorems 4.2.2 and 4.2.3 to unbounded solutions, we introduce the class Φ of functions ϕ such that $\phi(u)$ is a nondecreasing continuous function of u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty.$$

We will say that $f(t, u, v_1, \dots, v_n)$ satisfies **Condition (C)** provided for some $\phi \in \Phi$ there exists $c \neq 0$ and $0 < \alpha < 1$ such that

$$\liminf_{|u|\to\infty} \frac{f(t, u, \alpha\eta_1(t, t_0), \dots, \alpha\eta_n(t, t_0))}{\phi(u)} \ge k|f(t, c, \alpha\eta_1(t, t_0)c, \dots, \alpha\eta_n(t, t_0)c)| \quad (4.24)$$

for some positive constant k and all $t \ge T$.

We continue with a generalization of Theorem 4 of [24].

Theorem 4.2.4. *Suppose $\phi \in \Phi$. Let $f(t, y)$ be a continuous function of the variables $t \ge t_0$ and $|y| < \infty$ such that for all $t > 0$ $yf(t, y) > 0$, $y \neq 0$ and satisfies with respect to $\phi(y)$ the following conditions: there is a $c \neq 0$ such that*

$$\liminf_{|y|\to\infty} \frac{f(t, y)}{\phi(y)} \ge k|f(t, c)|, \quad (4.25)$$

for some positive constant k, and for all $t \ge T$,

$$\lim_{|y|\to\infty} \left| \int^{y} \frac{1}{\phi(u)} du \right| < \infty. \quad (4.26)$$

Assume also that $\mu(t)/t$ is bounded on $\mathbb{T}$. Then a necessary and sufficient condition for the second-order dynamic equation

$$y^{\Delta\Delta} + f(t, y^{\sigma}) = 0 \quad (4.15)$$

to be oscillatory is that

$$\left| \int^{\infty} \sigma(t) f(t, c) \Delta t \right| = \infty, \quad (4.27)$$

for all $c \neq 0$.

Proof. Note that if $\mu(t)/t$ is bounded, then (4.27) holds if and only if $\left|\int^{\infty} t f(t, c)\Delta t\right| = \infty$ for all $c \neq 0$.

Assume (4.15) is oscillatory and $\left|\int^{\infty} t f(t, c)\Delta t\right| < \infty$ for some $c \neq 0$. Hence, by Theorem 4.2.1, $y^{\Delta\Delta}(t) + f(t, y^{\sigma}(t)) = 0$ has a bounded nonoscillatory solution. So condition (4.27) is necessary.

Conversely, assume (4.27) holds and let $y(t)$ be a solution of (4.15) where we assume y is eventually positive. As in the proof of Theorem 4.2.1, $y^{\Delta}(t)$ is decreasing and hence must tend to a nonnegative limit. Since (4.27) holds, from Theorem 4.2.1, we see that $y(t)$ cannot be bounded. So we assume that $\lim_{t\to\infty} y(t) = \infty$.

We next define the continuously differentiable real-valued function

$$G(u) := \int_{u_0}^{u} \frac{ds}{\phi(s)}.$$

Observe that $G'(u) = 1/\phi(u)$. By the Pötzsche Chain Rule [2, Theorem 1.90],

$$(G(y(t))^{\Delta} = \left(\int_0^1 \frac{dh}{\phi(y_h(t))}\right) y^{\Delta}(t) \geq \left(\int_0^1 \frac{dh}{\phi(y^{\sigma}(t)}\right) y^{\Delta}(t) = \frac{y^{\Delta}(t)}{\phi(y^{\sigma}(t))},$$

where $y_h(t) := y(t) + h\mu(t)y^{\Delta}(t) \leq y^{\sigma}(t)$. Since ϕ is nondecreasing we have that $\dfrac{1}{\phi(y_h(t))} \geq \dfrac{1}{\phi(y^{\sigma}(t))}$. Now integrating (4.15) for $t \geq T$ gives

$$0 = y^{\Delta}(t) - y^{\Delta}(T) + \int_T^t f(s, y^{\sigma}(s))\Delta s,$$

and since $y^{\Delta}(t) > 0$ for all large t, we obtain

$$y^{\Delta}(T) = y^{\Delta}(t) + \int_T^t f(s, y^{\sigma}(s))\Delta s > \int_T^t f(s, y^{\sigma}(s))\Delta s.$$

Now multiplying by $(\phi(y^{\sigma}(T)))^{-1}$, we obtain

$$\begin{aligned} \frac{y^{\Delta}(T)}{\phi(y^{\sigma}(T))} &\geq \frac{1}{\phi(y^{\sigma}(T)} \int_T^t f\left(s, y^{\sigma}(s)\right)\Delta s \\ &\geq \int_T^t \frac{f(s, y^{\sigma}(s))}{\phi(y^{\sigma}(s))}\Delta s \qquad (4.28) \\ &\geq \int_T^t \frac{k}{2} f(s, c)\Delta s \end{aligned}$$

for sufficiently large T. Since $\lim_{t\to\infty} y(t) = \infty$, we have

$$\lim_{t\to\infty} G(y(t)) = \lim_{t\to\infty} \int_T^{y(t)} \frac{du}{\phi(u)} = \int_T^{\infty} \frac{du}{\phi(u)} < \infty.$$

We integrate (4.28) for $t \geq T$ to obtain

$$\begin{aligned}
\int_T^t (G(y(s)))^{\Delta} \Delta s &\geq \int_T^t \frac{y^{\Delta}(s)}{\phi(y^{\sigma}(s))} \Delta s \\
&\geq \int_T^t \int_T^s \frac{k}{2} f\,(r, c)\,\Delta r \Delta s \qquad (4.29)\\
&= \frac{k}{2} \int_T^t (s - T) f\,(s, c)\,\Delta s.
\end{aligned}$$

However, by letting $t \to \infty$ in (4.29), the left side is bounded whereas the right side is unbounded by assumption (4.27), a contradiction. □

We may now prove the following result:

Theorem 4.2.5. *Assume f satisfies condition (C) and that $\sigma(t)/t$ is bounded. Then all solutions of $y^{\Delta\Delta} + f(t, y^{\sigma}(t), y(\tau_1(t)), \ldots, y(\tau_n(t))) = 0$ are oscillatory in case*

$$\left| \int^{\infty} \sigma(t) t f(t, \alpha, \alpha\eta_1(t, t_0), \ldots, \alpha\eta_n(t, t_0)) \Delta t \right| = \infty \qquad (4.30)$$

holds for all $\alpha \neq 0$. In addition, if $\liminf_{t\to\infty} \eta_i(t, t_0) \geq \rho_i$ (ie, inequality (4.21) holds), then (4.30) is also necessary.

Proof. Assume (4.30) holds for all $\alpha \neq 0$ and let $u(t)$ be a nonoscillatory solution of $u^{\Delta\Delta} + f(t, u^{\sigma}(t), u(\tau_1(t)), \ldots, u(\tau_n(t))) = 0$ which we may assume satisfies

$$u(t) > 0, \quad u^{\Delta}(t) > 0, \quad u^{\Delta\Delta}(t) \leq 0, \quad u(\tau_i(t)) > 0, \quad t \geq T \geq t_0, \quad 1 \leq i \leq n.$$

By Lemma 4.1.1 we know that for each $1 \leq i \leq n$

$$u(\tau_i(t)) > \eta_i(t, T) u^{\sigma}(t), \quad \tau_i(t) > T.$$

By the monotonicity of f we have

$$u^{\Delta\Delta} + f(t, u^{\sigma}(t), \eta_1(t, T) u^{\sigma}(t), \ldots, \eta_n(t, T) u^{\sigma}(t)) \leq 0.$$

As in the proof of Theorem 4.2.2, we obtain the existence of a solution $y(t)$ of $y^{\Delta\Delta} + F(t, y^{\sigma}) = 0$ with $0 < u(T) \leq y(t) \leq u(t)$, $t \geq T$. Now by Theorem 4.2.4, it follows that

$$\left| \int^{\infty} \sigma(t) f(t, c, c\eta_1(t, T), \ldots, c\eta_n(t, T)) \Delta t \right| < \infty,$$

for some $c \neq 0$, which is a contradiction.

Conversely, assume (4.21) holds and (4.30) does not for some $\alpha \neq 0$. It follows that for any $\epsilon > 0$ with $\epsilon < \frac{1}{2} \min \{\tilde{\rho}_i | 1 \leq i \leq n\}$ there exists $T_i \geq t_0$ such that

$\eta_i(t,t_0) \geq \rho_i - \epsilon$ for $t \geq T_i$ and $1 \leq i \leq n$. Let $\tilde{\rho}_i := \rho_i - \epsilon$, $1 \leq i \leq n$ and $\tilde{\rho} := \min\{\tilde{\rho}_i | 1 \leq i \leq n\}$. Then

$$\alpha\eta_i(t,t_0) \geq \alpha(\rho_i - \epsilon) = \alpha\tilde{\rho}_i \geq \alpha\tilde{\rho}$$

for $t \geq T$ for $t \geq T_i$. Then by the monotonicity of f and the fact that $\eta_i \leq 1$ for $t \geq T$, we have

$$\left| \int^{\infty} \sigma(t) f(t, \alpha\tilde{\rho}, \ldots, \alpha\tilde{\rho}) \Delta t \right| < \infty,$$

which gives (4.22). Therefore, by Theorem 4.2.3,

$$y^{\Delta\Delta} + f(t, y^{\sigma}(t), y(\tau_1(t)), \ldots, y(\tau_n(t))) = 0$$

has a bounded nonoscillatory solution. □

As corollaries to these results, we obtain and extend the results of Gollwitzer [17] for the equation

$$y^{\Delta\Delta}(t) + q(t)(y(\tau(t)))^{\gamma} = 0 \tag{4.31}$$

where $0 < \tau(t) \leq t$ and $q(t)$ is continuous and eventually nonnegative on $[T,\infty)_{\mathbb{T}}$, and $\gamma > 1$ is the quotient of odd integers.

Corollary 4.2.6. *All solutions of* (4.31) *are oscillatory provided*

$$\int^{\infty} t^{1-\gamma} q(t)(\tau(t))^{\gamma} \Delta t = \infty \tag{4.32}$$

and $\mu(t)/t$ is bounded.

Proof. Assume (4.32) holds. Define $\phi(u) := u^{\gamma}$. Then $u\phi(u) > 0$ for $u \neq 0$ and by Theorem 2.6 of Erbe and Hilger [12], $\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty$ and $u\phi(u) > 0$ for $u \neq 0$. Let $f(t,u,v) := q(t)v^{\gamma}(t)$ and let $c = 1$ and $0 < \alpha < 1$. Then

$$\begin{aligned} \frac{f(t,u,\alpha\eta(t)u)}{\phi(u)} &= \frac{\alpha^{\gamma} q(t) \left(\frac{\tau(t)}{t}\right)^{\gamma} u^{\gamma}}{u^{\gamma}} \\ &= \alpha^{\gamma} q(t) \left(\frac{\tau(t)}{t}\right)^{\gamma} \\ &= k|f(t,c,\alpha\eta(t)c)| \end{aligned}$$

for $k = 1$ and all $t \geq T$. Furthermore

$$\left| \int^{\infty} t(f,t,\alpha,\alpha\eta(t)) \Delta t \right| = \left| \int^{\infty} t^{1-\gamma} q(t) \alpha^{\gamma} (\tau(t))^{\gamma} \Delta t \right| = \infty.$$

Hence, by Theorem 4.2.5, equation (4.31) is oscillatory. $\square$

Remark 4.2.7. Theorem 4.2.2 shows that

$$\int^{\infty} tq(t)\Delta t = \infty \tag{4.33}$$

is a necessary condition for all solutions of (4.31) to oscillate, in case $\gamma > 1$, with just the assumptions that $0 < \tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. However, (4.33) is no longer sufficient as the following examples demonstrate.

Example 4.2.8. Let $\mathbb{T} = [1, \infty)_{\mathbb{R}}$. Consider equation (4.31) with

$$q(t) = \beta(1-\beta)t^{\alpha} \quad \text{and} \quad \tau(t) = t^{\delta},$$

where $\alpha = \beta(1-\gamma\delta) - 2$ with $0 < \beta, \delta < 1$ and $\gamma\delta < 1$, and γ is the quotient of odd integers. For this example, $y(t) = t^{\beta}$ is a nonoscillatory solution but $\int^{\infty} tq(t)\Delta t = \infty$.

We have

$$\begin{aligned} y''(t) + q(t)(y(\tau(t)))^{\gamma} &= \beta(\beta-1)\left[t^{\beta-2} - t^{\alpha+\beta\gamma\delta}\right] \\ &= \beta(\beta-1)\left[t^{\beta-2} - t^{\beta(1-\gamma\delta)-2+\beta\gamma\delta}\right] \\ &= \beta(\beta-1)\left[t^{\beta-2} - t^{\beta-2}\right] \\ &= 0. \end{aligned}$$

However,

$$\begin{aligned} \int^{\infty} tq(t)dt &= \int_T^{\infty} t\beta(1-\beta)t^{\alpha}dt, \quad \text{for } T \text{ sufficiently large} \\ &= \beta(1-\beta)\lim_{R\to\infty}\int_T^R t^{1+\alpha}dt \\ &= \frac{\beta(1-\beta)}{2+\alpha}\lim_{R\to\infty}\left[R^{2+\alpha} - T^{2+\alpha}\right] \\ &= \infty \end{aligned}$$

since $2 + \alpha > 0$.

Example 4.2.9. For $q > 1$, let $\mathbb{T} = q^{\mathbb{N}_0}$. We want to find a function $Q : \mathbb{T} \to \mathbb{R}$ and a function $y : \mathbb{T} \to \mathbb{R}$ such that $y(t)$ is a nonoscillatory solution $y^{\Delta\Delta} + Q(t)y^{\gamma}(\tau(t)) = 0$ and $\int^{\infty} tQ(t)\Delta t = \infty$, where $\tau(t) : \mathbb{T} \to \mathbb{T}$ is such that $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$ and $\gamma > 1$ is a quotient of odd positive integers.

Let $y(t) = t^{\beta}$ with $\beta < 1$. Since $\sigma(t) = tq > t$, we have

$$y^{\Delta}(t) = t^{\beta-1}\frac{q^{\beta}-1}{q-1}.$$

After simplifying, we obtain

$$y^{\Delta\Delta}(t) = \frac{q^\beta - 1}{(q-1)^2} t^{\beta-2}[q^{\beta-1} - 1].$$

It follows that

$$Q(t) = \frac{q^\beta - 1}{(q-1)^2} \cdot \frac{t^{\beta-2}[1 - q^{\beta-1}]}{y^\gamma(\tau(t))} =: C_q \frac{t^{\beta-2}}{y^\gamma(\tau(t))}.$$

Now choose $\tau(t) = q^{\left\lfloor \frac{k}{2} \right\rfloor}$, where $\lfloor\cdot\rfloor$ is the greatest integer function. Then $\tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$. Consequently

$$Q(t) = \begin{cases} C_q t^{\frac{\beta(2-\gamma)-4}{2}}, & \text{if } k \text{ is even,} \\ C_q t^{\frac{\beta(2-\gamma)-4}{2}} q^{\frac{\gamma\beta}{2}}, & \text{if } k \text{ is odd.} \end{cases} \tag{4.34}$$

Now

$$\begin{aligned} \int^\infty tQ(t)dt &= C_q \lim_{R\to\infty} \begin{cases} \int_T^R t^{\frac{\beta(2-\gamma)}{2}} dt, & \text{if } k \text{ is even,} \\ \int_T^R t^{\frac{\beta(2-\gamma)}{2}} q^{\frac{\gamma\beta}{2}} dt, & \text{if } k \text{ is odd} \end{cases} \quad \text{for } T \text{ sufficiently large} \\ &= C_q(q-1) \lim_{n\to\infty} \begin{cases} \displaystyle\sum_{k=m}^{n-1} q^{k\frac{\beta(2-\gamma)}{2}}, & \text{if } k \text{ is even,} \\ q^{\frac{\gamma\beta}{2}} \displaystyle\sum_{k=m}^{n-1} q^{k\frac{\beta(2-\gamma)}{2}}, & \text{if } k \text{ is odd} \end{cases} \\ &= \infty \end{aligned}$$

if $q^{\frac{\beta(2-\gamma)}{2}} > 1$. That means $\beta(2-\gamma)$ must be nonnegative. If it were negative, then $y^{\Delta\Delta}(t)$ would be positive, which cannot happen as shown in the proof of Theorem 4.2.2. Hence, in order for the dynamic equation $y^{\Delta\Delta} + Q(t)y^\gamma(\tau(t)) = 0$ to have a nonoscillatory solution $y(t) = t^\beta$ and for $\int^\infty tQ(t)\Delta t = \infty$, we must choose $0 \leq \beta < 1$ and $1 < \gamma \leq 2$.

In order to prove another extension of a second theorem of Gollwitzer [17], we need the following result.

Lemma 4.2.10. *Let $y(t)$ be a positive solution of* (4.31) *defined on $[t_0, \infty)_{\mathbb{T}}$ for some $t_0 > 0$ that satisfies $y^\Delta(t) > 0$ and $y^{\Delta\Delta}(t) \leq 0$ on $[t_0, \infty)_{\mathbb{T}}$. If*

$$\int^\infty (\tau(t))^\gamma q(t)\Delta t = \infty, \tag{4.35}$$

where $0 < \gamma < 1$ is the quotient of positive integers, then there exists a $t_1 \geq t_0$ such *that*

$$\frac{y(t)}{y^{\Delta}(t)} \geq t$$

for $t \geq t_1$ *and* $\dfrac{y(t)}{t}$ *is decreasing on* $[t_1, \infty)_{\mathbb{T}}$.

Proof. Let $y(t)$ be as in the statement of the lemma and assume (4.35) holds. Let $Y(t) := y(t) - ty^{\Delta}(t)$. Then $Y^{\Delta}(t) = -\sigma(t)y^{\Delta\Delta}(t) \geq 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$. This implies that $Y(t)$ is increasing on $[t_0, \infty)_{\mathbb{T}}$. We claim there is a $t_1 \in [t_0, \infty)$ such that $Y(t) \geq 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume not, then $Y(t) < 0$ on $[t_0, \infty)_{\mathbb{T}}$. Therefore

$$\left(\frac{y(t)}{t}\right)^{\Delta} = \frac{ty^{\Delta}(t) - y(t)}{t\sigma(t)} = -\frac{Y(t)}{t\sigma(t)} > 0, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

which implies that $\dfrac{y(t)}{t}$ is increasing on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_2 \in [t_0, \infty)_{\mathbb{T}}$ such that $\tau(t) \geq \tau(t_0)$ for $t \geq t_2$. Then $y(\tau(t))/\tau(t) \geq y(\tau(t_1))/\tau(t_1) =: d > 0$, which gives $y(\tau(t)) \geq d\tau(t)$ for $t \geq t_2$. Now by integrating both sides of (4.31) from t_2 to t, we obtain

$$y^{\Delta}(t) - y^{\Delta}(t_2) + \int_{t_2}^{t} q(s)(y(\tau(s)))^{\gamma}\Delta s = 0.$$

This implies that

$$y^{\Delta}(t_2) \geq \int_{t_2}^{t} q(s)(y(\tau(s)))^{\gamma}\Delta s \geq d^{\gamma}\int_{t_2}^{t} q(s)\tau^{\gamma}(s)\Delta s,$$

which contradicts (4.35). Hence there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $Y(t) \geq 0$ on $[t_1, \infty)_{\mathbb{T}}$. Moreover,

$$\left(\frac{y(t)}{t}\right)^{\Delta} = \frac{ty^{\Delta}(t) - y(t)}{t\sigma(t)} = -\frac{Y(t)}{t\sigma(t)} \leq 0, \quad t \in [t_0, \infty)_{\mathbb{T}},$$

and we have that $\dfrac{y(t)}{t}$ is decreasing on $[t_1, \infty)_{\mathbb{T}}$. This completes the proof of the lemma. □

We end this section with an extension of a third theorem of Gollwitzer [17] for equation

$$y^{\Delta\Delta}(t) + q(t)(y(\tau(t)))^{\gamma} = 0. \tag{4.31}$$

Theorem 4.2.11. *Let* $0 < \gamma < 1$ *in equation* (4.31). *Then all solutions of* (4.31) *are oscillatory if, and only if,* (4.35) *holds.*

Proof. We first show that condition (4.35) is sufficient. Assume $y(t)$ is a positive solution and $q(t) \geq 0$ for sufficiently large t. Since (4.35) holds we see that for sufficiently large t, $y^{\Delta\Delta}(t) \leq 0$, $y^{\Delta}(t) > 0$, and $y(t) > 0$. Let c be fixed and sufficiently large. If we multiply both members of (4.31) by $\left(y^{\Delta}(\tau(t))\right)^{-\gamma}$ and use Lemma 4.2.10, we obtain

$$[y^{\Delta}(\tau(t))]^{-\gamma}y^{\Delta\Delta}(t) + q(t)(\tau(t))^{\gamma} \leq 0.$$

Since $y^{\Delta}(t)$ is nonincreasing and positive, we see that $y^{\Delta}(\tau(t)) \geq y^{\Delta}(t)$ for $t \geq c$ and hence

$$[y^{\Delta}(t)]^{-\gamma}y^{\Delta\Delta}(t) + q(t)(\tau(t))^{\gamma} \leq 0.$$

Integrating we obtain

$$\int_c^t [y^{\Delta}(s)]^{-\gamma}y^{\Delta\Delta}(s)\Delta s + \int_c^t q(s)L^{\gamma}(\tau(s))^{\gamma}\Delta s \leq 0.$$

It remains to show that the first term is bounded below. Once we obtain this, we will reach a contradiction since (4.35) holds. To this end, consider $[(y^{\Delta})^{-\gamma+1}]^{\Delta}$. Using the Pötzsche Chain Rule [2, Theorem 1.90], we have

$$[(y^{\Delta}(t))^{-\gamma+1}]^{\Delta} = (1-\gamma)\left(\int_0^1 [(1-h)y^{\Delta}(t) + hy^{\Delta}(\sigma(t))]^{-\gamma}dh\right)y^{\Delta\Delta}(t).$$

As $y^{\Delta\Delta}(t) \leq 0$ for all sufficiently large t, we have $y^{\Delta}(\sigma(t)) \leq y^{\Delta}(t)$. Consequently,

$$(1-h)y^{\Delta}(t) + hy^{\Delta}(\sigma(t)) \leq (1-h)y^{\Delta}(t) + hy^{\Delta}(t) = y^{\Delta}(t).$$

Now as $0 < \gamma < 1$ and $y^{\Delta\Delta}(t) \leq 0$, we have

$$[(y^{\Delta}(t))^{-\gamma+1}]^{\Delta} \leq (1-\gamma)y^{\Delta\Delta}(t)\left(y^{\Delta}(t)\right)^{-\gamma}.$$

Consequently,

$$\begin{aligned}
\int_c^t \left(y^{\Delta}(s)\right)^{-\gamma} y^{\Delta\Delta}(s)\Delta s &\geq \frac{1}{1-\gamma}\int_c^t [(y^{\Delta}(s))^{-\gamma+1}]^{\Delta}\Delta s \\
&= \frac{1}{1-\gamma}\left[(y^{\Delta}(t))^{-\gamma+1} - (y^{\Delta}(c))^{-\gamma+1}\right] \\
&\geq 0 \\
&> -\infty
\end{aligned}$$

as $t \to \infty$. Hence equation $y^{\Delta\Delta}(t) + q(t)(y(\tau(t)))^{\gamma} = 0$ is oscillatory.

Conversely, suppose that $\int^{\infty}(\tau(s))^{\gamma}q(s)\Delta s < \infty$. It is sufficient to construct a nonoscillatory solution on some half-line $[t_0, \infty)_{\mathbb{T}}$. Choose t_0 so large that

$$\int_{t_0}^{\infty}(\tau(s))^{\gamma}q(s)\Delta s < \frac{1}{2}.$$

Consider the solution $y(t)$ which is defined by the initial data

$$y^{\Delta}(t_0) = 1, \quad y(t) = 0, \; t \geq t_0.$$

Then $y(t) > 0$ for some $t > t_0$. We claim that this solution does not vanish on $[t_0, \infty)_{\mathbb{T}}$. If $y(t_1) = 0$ for some $t_1 > t_0$, then there exits $t_2 \in (t_0, t_1)_{\mathbb{T}}$ such that $y^{\Delta}(t_2) \leq 0$. However, this will contradict the following statement: the function $y^{\Delta}(t)$ can never vanish on $(t_0, t_1)_{\mathbb{T}}$. Since $y^{\Delta\Delta}(t) \leq 0$ on $(t_0, t_1)_{\mathbb{T}}$, we see, after two integrations, that

$$y(t) \leq (t - t_0), \quad t_0 \leq t \leq t_1.$$

From $y^{\Delta\Delta}(t) + q(t)(y(\tau(t)))^{\gamma} = 0$, we have

$$y^{\Delta}(t) = 1 - \int_{t_0}^{t} q(s)(y(\tau(s)))^{\gamma} \Delta s \geq 1 - \int_{t_0}^{t} q(s)(\tau(s) - t_0)^{\gamma} \Delta s \geq \frac{1}{2}.$$

Hence $y^{\Delta}(t)$ never vanishes and the proof is complete. $\square$

Chapter 5

Asymptotic Behavior for Functional Dynamic Equations

5.1 Asymptotic Behavior of Dynamic Equations

In this section we are concerned with the asymptotic behavior of the solutions of the following second order nonlinear dynamic equation:

$$y^{\Delta\Delta} + f(t, y^\sigma)g(x^\Delta) = 0 \tag{5.1}$$

where $\sup \mathbb{T} = \infty$. Our goal is to establish conditions for existence of solutions of (5.1). The conditions are given in Theorems 5.1.1, 5.1.4, and 5.1.5.

We shall assume the following conditions hold:

(A_0) $f, f_y : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ are continuous in y and rd-continuous in t and $g : \mathbb{R} \to \mathbb{R}$ is continuous.

(A_1) $f(t, 0) = 0,\ t \in [0, \infty)_\mathbb{T}$.

(A_2) $f_y(t, y) \geq 0$ and is nondecreasing in y for $t \in [0, \infty)_\mathbb{T}$ and $y \geq 0$.

(A_3) $g(v) > 0$ for all $v \in \mathbb{R}$.

We shall study (5.1) by considering the equation

$$y^{\Delta\Delta} + f_y(t, \alpha)y = 0, \tag{5.2}$$

where α is some real constant depending on the solutions of (5.1). To do this we should recall the results from Section 4.1 on upper and lower solutions.

We now establish necessary and sufficient conditions for the existence of certain types of solutions of (5.1).

Theorem 5.1.1. *Assume (A_0)-(A_3) hold and let $\alpha_0 > 0$. Furthermore, assume $\sigma(t)/t$ is bounded. Then the following statements are equivalent:*

(i) For each $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of $u^{\Delta\Delta}(t) + f(t, u^\sigma)g(u^\Delta) = 0$ satisfying $\lim_{t\to\infty} u_\alpha(t) = \alpha$.

(ii) $\int^{\infty} \sigma(t) f_y(t, \alpha)\Delta t < \infty \quad$ *for* $\quad 0 < \alpha < \alpha_0.$

Proof. Assume $\int^{\infty} \sigma(t) f_y(t, \alpha_1)\Delta t = \infty$ for some $0 < \alpha_1 < \alpha_0$ and let $\alpha_1 < \beta < \alpha_0$. Let $u_\beta(t)$ be the corresponding solution of $u_\beta^{\Delta\Delta} + f(t, u_\beta^\sigma)g(u_\beta^\Delta) = 0$ with $\lim_{t\to\infty} u_\beta(t) = \beta$. Choose $\delta > 0$ such that $\alpha_1 + \delta < \beta$ and let $T \geq 0$ be such that $u_\beta^\sigma(t) \geq \alpha_1 + \delta$ for all $t \geq T$. Then for $t \geq T$

$$u_\beta^{\Delta\Delta} = -f(t, u_\beta^\sigma)g(u_\beta^\Delta) \leq 0.$$

Hence $u_\beta^\Delta > 0$ and decreases to a limit, and this limit must be zero since u_β is bounded. Therefore, $u_\beta(t) \leq \beta$ for $t \geq T$. By applying the Mean Value Theorem, we obtain

$$\frac{f(t, u_\beta^\sigma(t)) - f(t, \alpha_1)}{u_\beta^\sigma(t) - \alpha_1} = f_y(t, \eta(t)) \quad \text{for some } \eta(t) \in (\alpha_1, u(\beta^\sigma(t))).$$

Now by the monotonicity of f_y (condition (A_2)), we have

$$\begin{aligned} f_y(t, \alpha_1) &\leq f_y(t, \eta(t)) \\ &\leq \frac{f(t, u_\beta^\sigma(t)) - f(t, \alpha_1)}{u_\beta^\sigma(t)) - \alpha_1} \\ &\leq \frac{u_\beta^\sigma(t)}{u_\beta^\sigma(t) - \alpha_1} \frac{f(t, u_\beta^\sigma(t)}{u_\beta^\sigma(t)} \\ &\leq \frac{\beta}{\delta} \frac{f(t, u_\beta^\sigma(t))}{u_\beta^\sigma(t)} \end{aligned}$$

for $t \geq T$. Since $\lim_{t\to\infty} u^\Delta(t) = 0$, there exists $T_1 \geq T$ such that $g(u_\beta^\Delta(t)) \geq \dfrac{g(0)}{2} > 0$ for all $t \geq T_1$. Hence, for $t \geq T_1$, we have

$$\begin{aligned} u_\beta^{\Delta\Delta}(t) &= -f(t, u_\beta^\sigma(t))g(u_\beta^\Delta(t)) \\ &\leq -\frac{f_y(t, \alpha_1)}{\beta} \delta u_\beta^\sigma(t) \frac{g(0)}{2} \\ &= -k f_y(t, \alpha_1) u_\beta^\sigma(t) \end{aligned}$$

where $k = g(0)\dfrac{\delta}{2\beta}$. Also, $\alpha_1^{\Delta\Delta} = 0 \geq -k f_y(t, \alpha_1)\alpha_1$. Hence, by Theorem 3.3.4, there is a solution $z(t)$ of $z^{\Delta\Delta} + k f_y(t, \alpha_1) z^\sigma = 0$ with $0 < \alpha_1 \leq z(t) \leq u_\beta(t) \leq \beta$ on $[T, \infty)_{\mathbb{T}}$. By Theorem 4.2.1, it follows that

$$\int^{\infty} kct f_y(t, \alpha_1)\Delta t < \infty$$

for some $c > 0$. Since $\sigma(t)/t$ is bounded, we have

$$\int^{\infty} \sigma(t) f_y(t, \alpha_1)\Delta t < \infty,$$

which is the desired contradiction.

Conversely, let $0 < \alpha < \alpha_0$ be such that

$$\int^{\infty} \sigma(t) f_y(t, \alpha)\Delta t < \infty$$

and let

$$M = \max\{g(v) : 0 \le v \le \alpha\}.$$

Choose $T \ge 0$ such that

$$\int_T^{\infty} (\sigma(s) - T) f_y(s, \alpha)\Delta s < \frac{1}{M} \quad \text{and} \quad \int_T^{\infty} f_y(s, \alpha)\Delta s < \frac{1}{M}.$$

We shall now define a sequence of functions on $[T, \infty)_{\mathbb{T}}$ in the following manner:

Let $y_0(t) = \alpha,\ t \ge T$. Now for $t \ge T$

$$\begin{aligned}
0 &\le \int_t^{\infty} (\sigma(s) - t) f(s, \alpha) g(0)\Delta s \\
&= \int_t^{\infty} (\sigma(s) - t)[f(s, \alpha) - f(s, 0)] g(0)\Delta s \\
&= \int_t^{\infty} (\sigma(s) - t)\alpha f_y(s, \eta(s)) g(0)\Delta s, \quad \eta(s) \in (0, \alpha) \\
&\le \int_t^{\infty} (\sigma(s) - t)\alpha f_y(s, \alpha) g(0)\Delta s \\
&\le \alpha M \int_t^{\infty} (\sigma(s) - t) f_y(s, \alpha)\Delta s \\
&\le \alpha M \int_T^{\infty} (\sigma(s) - T) f_y(s, \alpha)\Delta s \\
&< \alpha.
\end{aligned}$$

By defining $y_1(t) := \alpha - \int_t^{\infty} (\sigma(s) - t) f(s, \alpha) g(0)\Delta s,\ t \ge T$, we have $0 \le y_1(t) < \alpha$.

Differentiating y_1, we obtain

$$\begin{aligned} y_1^{\Delta}(t) &= 0 - \left[\int_t^{\infty} -f(s,\alpha)g(0)\Delta s + (\sigma(t)-\sigma(t))f(t,\alpha)g(0)\right] \\ &= \int_t^{\infty} f(s,\alpha)g(0)\Delta s \\ &\leq M\int_t^{\infty} f(s,\alpha)\Delta s \\ &= M\int_t^{\infty} [f(s,\alpha)-f(s,0)]\Delta s \\ &= \alpha M\int_t^{\infty} f_y(s,\eta(s))\Delta s, \quad \eta(s)\in(0,\alpha) \\ &\leq \alpha M\int_T^{\infty} f_y(s,\alpha)\Delta s \\ &< \alpha. \end{aligned}$$

So $0 \leq y_1^{\Delta}(t) < \alpha$ for $t \geq T$. Proceeding inductively, we define for all $m \geq 1$

$$y_{m+1}(t) := \alpha - \int_t^{\infty} (\sigma(s)-t)f(s,y_m^{\sigma}(s))g(y_m^{\Delta}(s))\Delta s, \quad t \geq T, \tag{5.3}$$

and obtain $0 \leq y_m(t),\ y_m^{\Delta}(t) \leq \alpha$ for all $m \geq 1$. Hence the sequence $\{y_m(t)\}_{m=0}^{\infty}$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem along with a standard diagonalization argument yields a uniformly convergent subsequence $\{y_{m_k}(t)\}$ on compact subintervals of $[T,\infty)_{\mathbb{T}}$. Let

$$u_\alpha(t) := \lim_{k\to\infty} y_{m_k}(t),$$

for $t \in [T,\infty)$. It follows that

$$\lim_{k\to\infty} f(t,y_{m_k}(t))g(y_{m_k}^{\Delta}(t)) = f(t,u_\alpha(t))g(u_\alpha^{\Delta}(t))$$

uniformly on compact subintervals of $[T,\infty)_{\mathbb{T}}$. Replacing m in equation 5.3 by m_k and letting $k \to \infty$, we get

$$u_\alpha(t) = \alpha - \int_t^{\infty} (\sigma(s)-t)f(s,u_\alpha^{\sigma}(s))g(u_\alpha^{\Delta}(s))\Delta s$$

on $[T,\infty)_{\mathbb{T}}$. It follows that $u_\alpha(t)$ is a solution of $u_\alpha^{\Delta\Delta}(t) + f(t,u_\alpha^{\sigma})(t)g(u_\alpha^{\Delta}) = 0$. As $\lim_{t\to\infty} u_\alpha(t) = \alpha$, the proof is complete. □

Remark 5.1.2. If $f(t,y) = -f(t,-y)$ and $g(v) > 0$ and is continuous for $v \in \mathbb{R}$, then we see that $\int^{\infty} \sigma(t)f_y(t,\alpha)\Delta t < \infty$ for $0 < |\alpha| < \alpha_0$ if and only if for each

$0 < |\alpha| < \alpha_0$ there is a solution $u_\alpha(t)$ of $u^{\Delta\Delta} + f(t, u^\sigma)g(u^\Delta) = 0$ with $\lim_{t\to\infty} u_\alpha(t) = \alpha$.

Corollary 5.1.3. *$\int^\infty \sigma(t) f_y(t, \alpha)\Delta t < \infty$ for all $\alpha > 0$ if and only if there is a solution $u_\alpha(t)$ of $u^{\Delta\Delta} + f(t, u)g(u^\Delta) = 0$ with $\lim_{t\to\infty} u_\alpha(t) = \alpha$ for all $\alpha > 0$.*

In [22] it is shown that $y'' + a(t)y^{2n+1} = 0$, $n \geq 0$, where $a(t) \geq 0$ for $t \geq 0$ and $g(v) = 1$ for all v, has solutions for which

$$\lim_{t\to\infty} \frac{y(t)}{t} = \alpha > 0$$

if and only if

$$\int^\infty t^{2n+1} a(t)dt < \infty.$$

We will show that an analogous result is true for the dynamic equation (5.1) provided $f(t, y)$ satisfies the following additional condition.

(A_4) There exist real numbers $c > 0$ and $\lambda > 0$ such that $\liminf_{v\to\infty} \dfrac{f(t, v)}{v f_v(t, cv)} \geq \lambda > 0$, for all sufficiently large t.

We first establish the following result.

Theorem 5.1.4. *Assume (A_0)-(A_3) hold and let there exist a real number $\beta > 0$ with*

$$\int^\infty \sigma(t) f_y(t, \beta\sigma(t))\Delta t < \infty.$$

Then there exist solutions to $y^{\Delta\Delta} + f(t, y^\sigma(t))g(y^\Delta) = 0$, say $y(t)$, such that $\lim_{t\to\infty} \dfrac{y(t)}{t}$ exists and is positive.

Proof. Let $T > 0$ be such that

$$\int_T^\infty \sigma(t) f_y(t, \beta\sigma(t))\Delta t < \frac{1}{2M},$$

where $M = \max\{g(v) : 0 \leq v \leq \beta\}$. We define a solution of $u^{\Delta\Delta} + f(t, u^\sigma)g(u^\Delta) = 0$ by

$$u(T) = 0, \quad u^\Delta(T) = \beta,$$

and we assert that the solution satisfies $u^\Delta(t) \geq \dfrac{\beta}{2}$ for $t \geq T$. Observe that $u(t) > 0$ and $u^\Delta(t) > 0$ for some $t > T$. Assume, for the sake of contradiction, that there is a $\delta > 0$ with $\delta < \dfrac{\beta}{2}$ and a $t_1 > T$ with $u^\Delta(t_1) = \delta$ and $u(t) > 0$ on $(T, t_1]_{\mathbb{T}}$. Then for

$T \le t \le t_1$ we have

$$u^{\Delta}(T) = u^{\Delta}(t) + \int_T^t f(s, u^{\sigma}(s))g(u^{\Delta}(s))\Delta s. \tag{5.4}$$

Since $u^{\Delta\Delta}(t) \le 0$ on $(T, t_1]_{\mathbb{T}}$ and $u(t)$ is decreasing on $(T, t_1]_{\mathbb{T}}$, we have

$$u^{\Delta}(t) \le \beta \text{ on } (T, t_1)_{\mathbb{T}} \quad \text{and} \quad u(t) \le \beta(t - T) \text{ on } (T, t_1)_{\mathbb{T}}.$$

Applying the Mean Value Theorem to (5.4) and the monotonicity of f_y we have

$$\begin{aligned}
\beta = u^{\Delta}(T) &= u^{\Delta}(t) + \int_T^t f(s, u^{\sigma}(s))g(u^{\Delta}(s))\Delta s \\
&\le u^{\Delta}(t) + M \int_T^t f(s, u^{\sigma}(s))\Delta s \\
&= u^{\Delta}(t) + M \int_T^t [f(s, u^{\sigma}(s)) - f(s, 0)]\Delta s \\
&= u^{\Delta}(t) + M \int_T^t u^{\sigma}(s) f_y(s, \eta(s))\Delta s, \quad 0 < \eta(s) < u^{\sigma}(s) \\
&\le u^{\Delta}(t) + M\beta \int_T^t (\sigma(s) - T) f_y(s, u^{\sigma}(s))\Delta s \\
&\le u^{\Delta}(t) + M\beta \int_T^t \sigma(s) f_y(s, \beta\sigma(s))\Delta s \\
&< u^{\Delta}(t) + M\beta \frac{1}{2M} \\
&= u^{\Delta}(t) + \frac{\beta}{2}.
\end{aligned}$$

Hence, $u^{\Delta}(t_1) > \frac{\beta}{2}$, a contradiction. Thus, $u^{\Delta}(t) \ge \frac{\beta}{2}$ on $[T, \infty)_{\mathbb{T}}$ and $\lim_{t\to\infty} u^{\Delta}(t)$ exists and is positive. By L'Hôpital's Rule [2, Theorem 1.120] , we have $\lim_{t\to\infty} \frac{u(t)}{t}$ exists and is positive. □

If we assume condition (A_4), then we may establish the converse of Theorem 5.1.4.

Theorem 5.1.5. *Assume conditions (A_0)-(A_4) hold. Then* (5.1) *has a solution, say $y(t)$, such that $\lim_{t\to\infty} \frac{y(t)}{t}$ exists and is positive if and only if*

$$\int^{\infty} \sigma(t) f_y(t, \beta\sigma(t))\Delta t < \infty \text{ for some } \beta > 0.$$

Proof. Let $\alpha > 0$ and let $y(t)$ be solution of (5.1) with

$$\lim_{t\to\infty} \frac{y(t)}{t} = \alpha.$$

Let $T \geq 0$ be such that $y(t) \geq \alpha t/2$ for $t \geq T$ and let

$$m := \min\{g(v) : 0 \leq v \leq y^{\Delta}(T)\}.$$

By condition (A_4), there is a $T_1 \geq T$ such that

$$f(t, y^{\sigma}(t)) \geq \lambda y^{\sigma}(t) f_y(t, cy^{\sigma}(t)) \geq \lambda \frac{\alpha\sigma(t)}{2} f_y\left(t, \frac{\alpha\sigma(t)}{2}\right) = k\sigma(t) f_y\left(t, \frac{\alpha\sigma(t)}{2}\right)$$

for $t \geq T_1$, where $k = \dfrac{\lambda\alpha}{2}$. Since $0 < y^{\Delta}(t) \leq y^{\Delta}(T)$ for $t \geq T$, we have

$$f(t, y^{\sigma}(t)) g(y^{\Delta}(t)) \geq mk\sigma(t) f_y\left(t, \frac{c\alpha\sigma(t)}{2}\right), \quad t \geq T_1.$$

Therefore,

$$\begin{aligned} y^{\Delta}(T_1) &= y^{\Delta}(t) + \int_{T_1}^{t} f(s, y^{\sigma}(s)) g(y^{\Delta}(s)) \Delta s \\ &\geq y^{\Delta}(t) + \int_{T_1}^{t} mk\sigma(s) f_y\left(s, \frac{c\alpha\sigma(s)}{2}\right). \end{aligned}$$

Since $\lim_{t\to\infty} y^{\Delta}(t) \geq 0$,

$$\int_{T_1}^{\infty} \sigma(s) f_y\left(s, \frac{c\alpha\sigma(s)}{2}\right) < \infty,$$

and this proves the theorem. □

5.2 Asymptotic Behavior of a Dynamic Equation with a Single Delay

In this section we establish a result similar to Theorem 5.1.1 for the second order nonlinear functional equation

$$y^{\Delta\Delta} + f(t, y^{\tau}) g(y^{\Delta}) = 0 \tag{5.5}$$

where $0 < \tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$ with $\sup \mathbb{T} = \infty$. We assume conditions (A_0)-(A_3) hold as well.

We can prove a result similar to .

Theorem 5.2.1. *Assume (A_0)-(A_3) hold and let $\alpha_0 > 0$. If for each $0 < \alpha < \alpha_0$ there is a solution $u_\alpha(t)$ of $u^{\Delta\Delta}(t) + f(t, u^\tau)g(u^\Delta) = 0$ satisfying $\lim_{t\to\infty} u_\alpha(t) = \alpha$, then $\int^\infty \tau(t) f_y(t, \alpha)\Delta t < \infty$ for $0 < \alpha < \alpha_0$.*

Proof. Assume $\int^\infty \tau(t) f_y(t, \alpha_1)\Delta t = \infty$ for some $0 < \alpha_1 < \alpha_0$ and let $\alpha_1 < \beta < \alpha_0$. Let $u_\beta(t)$ be the corresponding solution of $u_\beta^{\Delta\Delta} + f(t, u_\beta^\tau)g(u_\beta^\Delta) = 0$ with $\lim_{t\to\infty} u_\beta(t) = \beta$. Choose $\delta > 0$ such that $\alpha_1 + \delta < \beta$ and let $T \geq 0$ be such that $u_\beta^\tau(t) \geq \alpha_1 + \delta$ for all $t \geq T$. Then by condition (A_2), for $t \geq T$

$$u_\beta^{\Delta\Delta} = -f(t, u_\beta^\tau)g(u_\beta^\Delta) \leq 0.$$

Therefore $u_\beta^\Delta > 0$ is decreasing and is concave down. It follows that $u_\beta(t)$ increases to β. Consequently, $u_\beta^\Delta > 0$ and decreases to zero. So we have

$$u_\beta(t) > 0, \quad u_\beta^\Delta(t) > 0, \quad \text{and} \quad u_\beta^{\Delta\Delta}(t) \leq 0 \quad \text{on } [T, \infty)_{\mathbb{T}}.$$

Then by Lemma 4.1.1 we have $u(\tau(t)) > \eta(t, T)u^\sigma(t),\ \tau(t) > T$. So for any $0 < k < 1$ there is a $T_k \geq T$ such that

$$u(\tau(t)) \geq \frac{\tau(t) - T}{\sigma(t) - T} u^\sigma(t) \geq k\frac{\tau(t)}{\sigma(t)} u^\sigma(t), \quad t \geq T_k.$$

Define

$$F(t, u) := f\left(t, ku(t)\frac{\tau(t)}{\sigma(t)}\right).$$

Since f is increasing in the second variable, we have

$$f(t, u_\beta(\tau(t))) \geq f\left(t, ku_\beta^\sigma(t)\frac{\tau(t)}{\sigma(t)}\right) = F(t, u_\beta^\sigma(t)).$$

By applying the Mean Value Theorem, we obtain

$$\frac{F(t, u_\beta^\sigma(t)) - F(t, \alpha_1)}{u_\beta^\sigma(t) - \alpha_1} = \frac{\partial F}{\partial u}(t, \eta(t)) = k\frac{\tau(t)}{\sigma(t)} f_u(t, \eta(t)) \quad \text{for some } \eta(t) \in (\alpha_1, \beta^\sigma(t)).$$

Now by the monotonicity of f_u (condition (A$_2$)), we have

$$\begin{aligned} k\frac{\tau(t)}{\sigma(t)}f_u(t,\eta(t)) &= \frac{F(t,u_\beta^\sigma(t))-F(t,\alpha_1)}{u_\beta^\sigma(t)-\alpha_1} \\ &\leq \frac{u_\beta^\sigma(t)}{u_\beta^\sigma(t)-\alpha_1}\frac{F(t,u_\beta^\sigma(t))}{u_\beta^\sigma(t)} \\ &\leq \frac{u_\beta^\sigma(t)}{u_\beta^\tau(t)-\alpha_1}\frac{F(t,u_\beta^\sigma(t))}{u_\beta^\sigma(t)} \\ &\leq \frac{\beta}{\delta}\frac{F(t,u_\beta^\sigma(t))}{u_\beta^\sigma(t)} \end{aligned}$$

for $t \geq T$. Since $\lim_{t\to\infty} u^\Delta(t) = 0$, there exists $T_1 \geq T$ such that $g(u_\beta^\Delta(t)) \geq \dfrac{g(0)}{2} > 0$ for all $t \geq T_1$. Hence, for $t \geq T_1$, we have

$$\begin{aligned} u_\beta^{\Delta\Delta}(t) &= -f(t,u_\beta^\tau(t))g(u_\beta^\Delta(t)) \\ &= -F(t,u_\beta^\sigma(t))g(u_\beta^\Delta(t)) \\ &\leq -F(t,u_\beta^\sigma(t))\frac{g(0)}{2} \\ &\leq -k\frac{\tau(t)}{\sigma(t)}f_u(t,\alpha_1)\frac{\delta}{\beta}u_\beta^\sigma(t)\frac{g(0)}{2} \\ &= -m\frac{\tau(t)}{\sigma(t)}f_u(t,\alpha_1)u_\beta^\sigma(t) \end{aligned}$$

where $m = g(0)\dfrac{k\delta}{2\beta}$. Also, $\alpha_1^{\Delta\Delta} = 0 \geq -m\dfrac{\tau(t)}{\sigma(t)}f_u(t,\alpha_1)\alpha_1$. Hence, by Theorem 3.3.4, there is a solution $z(t)$ of $z^{\Delta\Delta} + m\dfrac{\tau(t)}{\sigma(t)}f_z(t,\alpha_1)z^\sigma = 0$ with $0 < \alpha_1 \leq z(t) \leq u_\beta(t) \leq \beta$ on $[T,\infty)_{\mathbb{T}}$. By Theorem 4.2.1, it follows that

$$\int^\infty tcm\frac{\tau(t)}{\sigma(t)}f_z(t,\alpha_1)\Delta t < \infty$$

for some $c > 0$. Since $\tau(t) \leq t \leq \sigma(t)$ and $mc > 0$, we have

$$\int^\infty \tau(t)f_z(t,\alpha_1)\Delta t < \infty,$$

which is the desired contradiction. $\square$

In the future, we would like to develop similar results to those of Section 5.1 for the second order nonlinear functional equation

$$y^{\Delta\Delta} + f(t, y^{\tau})g(y^{\Delta}) = 0 \tag{5.5}$$

where $0 < \tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$ with $\sup \mathbb{T} = \infty$.

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