5-2013

Symbolic Powers of Ideals in $k[\mathbf{P}^N]$

Michael Janssen
University of Nebraska-Lincoln, mkjanssen@gmail.com

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by

Michael K. Janssen

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Brian Harbourne

Lincoln, Nebraska

May, 2013

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Michael K. Janssen, Ph. D.

University of Nebraska, 2013

Adviser: Brian Harbourne

Let $I \subseteq k[\mathbf{P}^N]$ be a homogeneous ideal and k an algebraically closed field. Of particular interest over the last several years are ideal containments of symbolic powers of I in ordinary powers of I of the form $I^{(m)} \subseteq I^r$, and which ratios m/r guarantee such containment. A result of [ELS01, HH02] states that, if $I \subseteq k[\mathbf{P}^N]$, where k is an algebraically closed field, then the symbolic power $I^{(Ne)}$ is contained in the ordinary power I^e, and thus, whenever $m/r \geq N$, we have the containment $I^{(m)} \subseteq I^r$. Therefore, for each ideal J, there is a number $a \leq N$ such that $m/r > a$ implies $J^{(m)} \subseteq J^r$. This led Bocci and Harbourne [BH10a] to define the resurgence of I

$$\rho(I) = \sup\left\{m/r \,:\, I^{(m)} \not\subseteq I^r\right\}.$$

In particular, if $m/r > \rho(I)$, then $I^{(m)} \subseteq I^r$. An interesting problem, then, is to compute $\rho(I)$ for various classes of ideals. Much of the work that has been done on this question involves examining ideals of points in $\mathbf{P}^N$. In Chapter 2 we investigate such questions for an ideal defining a certain configuration of points in $\mathbf{P}^2$ using a certain k-vector space basis of $k[\mathbf{P}^2]$ compatible with $I^{(m)}$ and I^r.We are also able to use this approach to verify several conjectures of [HH, BCH11] for our particular class of ideals, and compute some well-known invariants of these ideals, such as $\alpha(I^{(m)})$, $\gamma(I)$, the Castelnuovo-Mumford regularity and the saturation degree.

In Chapter 3, we consider a question raised in [BC11] which is related to the computation of $\gamma(I)$. Bocci and Chiantini classify configurations of points in $\mathbf{P}^2$

based on the difference $t = \alpha(I^{(2)}) - \alpha(I)$, where $I = I(Z)$ and $Z \subseteq \mathbf{P}^2$ is a finite set of points. When $t = 1$, Z is either a set of collinear points or a star configuration of points. We extend that result to configurations of lines in $\mathbf{P}^3$.

ACKNOWLEDGMENTS

First and foremost, I must thank my wife, Laura, for loving and putting up with me for the last several years. I do not wish your experiences on anyone. Thank you for making sure I didn't take myself too seriously. I also want to thank my family: my mom and late father, for supporting and encouraging my education, and my brother, for being a good friend these last several years.

Second, I owe a great debt of gratitude to Brian Harbourne for his patience, graciousness and–did I mention patience? It goes without saying – but I will say it anyway – that I could not have done this without your help. Thanks for making me look forward to our weekly meetings, and for your encouragement and direction for the last four and a half years. You said it best after the defense: it's been fun. Thanks are also due to Mark Walker and Srikanth Iyengar for their helpful comments on this dissertation, and to John Anderson for serving as my outside representative. I also want to thank Susan Cooper for her help, especially early in the research process, and Annika Denkert for her hard work in our collaboration.

I also owe a great deal of thanks to the staff and faculty in the math department. Thank you for supporting graduate students and making the process of getting a Ph.D. more pleasant than it has any right to be. I am certainly biased, but I can't imagine a better environment in which to do this work. Of course, one of the reasons the last several years has been so pleasant is the great colleagues I have been blessed with. In particular, I want to thank those who started with me: Ben Nolting, Courtney Gibbons, Joe Geisbauer, Nathan Corwin, Amanda Croll, and Annika Denkert–thanks for getting me through the first couple of years, and for the conversations since. It's comforting to have people around who are experiencing the same things at the same time and are willing to be candid. I'd also like to thank Lauren Keough for her help keeping me sane in our semester as associate conveners (which was also when I was finishing this research), and for frequently laughing with me, often about things that weren't actually very funny.

I would especially like to thank Zachary Roth, Tom Clark, Eric Eager, and Tanner Auch, for their friendship in and out of the math department. Zach, you've been around since the beginning, and, despite our differing opinions on the necessity of punctuality, have been a friend since Day 1. I've enjoyed our various and ongoing discussions on math, technology, and theology. Tom, getting to know you better the last few years has been a highlight of coming to Avery, and you and Ruth (and Lydia!) have been a joy to Laura and me. Eric, I always enjoyed having you around to distract me with football or any of 13–14 other topics. You have been missed this past year! Tanner, the late nights playing games and Sunday lunches have been a regular source of happiness these last years, and will be missed. I am probably still finding eggs and firemen, so I guess you win.

An under appreciated aspect of one's graduate study is the time spent outside of the department and away from mathematics, and in that part of my life I have been incredibly blessed the past six years. To find a place like Grace Chapel my first week in Lincoln and be welcomed into that community is one of the most genuine examples of grace I've ever experienced. To everyone who has made that a reality, from the many musical folks who have blessed me with their talents and let me tag alongside them, to the community groups we have lived life with the last few years, and everyone in between: thank you.

Contents

Chapter 1

Introduction

This thesis will be primarily concerned with homogeneous ideals I of subschemes of $\mathbf{P}^N$ and their mth symbolic powers, which we denote $I^{(m)}$. Questions involving symbolic powers of ideals have been relevant throughout much of the recent history of commutative algebra and algebraic geometry (e.g., the work of Huneke, such as [Hun81, Hun82, Hun87, HH92, HR98, HH02, HH07]), and are related to orders of vanishing via the following famous result of Zariski and Nagata.

Theorem 1.1 (Zariski, Nagata; see Theorem 3.14 of [Eis99])**.** *Suppose that k is an algebraically closed field and S is a polynomial ring over k. If P is a prime ideal of S, then $P^{(n)}$ is the ideal of forms vanishing to order at least n at every point in the variety corresponding to P.*

(Recall that we say a form F vanishes to order r at a point $p \in \mathbf{P}^N$ if $F \in I(p)^r$, the rth power of the ideal of forms vanishing at p.)

Symbolic powers were first defined in terms of prime ideals, as in Theorem 1.1, but the definition was eventually extended to arbitrary ideals in commutative Noetherian rings.

Definition 1.2. We define the mth symbolic power of an ideal I in a commutative Noetherian ring R to be

$$I^{(m)} := R \cap \left(\cap_{P \in \mathrm{Ass}(I)} (I^m R_P) \right),$$

where $\mathrm{Ass}(I)$ is the set of associated primes of I, $m \in \mathbf{N}$, and R_P denotes the localization of R at the prime ideal P.

Given an ideal I in a commutative Noetherian ring, a natural question to ask is, how do the ordinary and symbolic powers compare? In particular, for which m and r do the containments $I^{(m)} \subseteq I^r$ and $I^r \subseteq I^{(m)}$ hold? It turns out that $I^r \subseteq I^{(m)}$ holds if and only if $r \geq m$ (see [BDH$^+$09, Lemma 8.1.4]). The other direction is a largely open problem, but if I is nontrivial (i.e., $I \neq (0)$, $I \neq R$) and $I^{(m)} \subseteq I^r$, then we must have $m \geq r$. It is the question of for which m and r does the containment $I^{(m)} \subseteq I^r$ hold that we consider in Chapter 2.

In [Swa00], the author states and answers a related question of Schenzel, namely: given an ideal I in a commutative Noetherian local ring R, does there exist an integer k such that $I^{(kn)} \subseteq I^n$ for all $n \geq 1$? Swanson defined the following function:

Definition 1.3. The function $t_I : \mathbf{N} \to \mathbf{N}$ is defined for each $n \geq 1$ to be the smallest integer $t_I(n)$ such that $I^{(t_I(n))} \subseteq I^n$.

Swanson showed that, for a certain class of ideals, $t_I(n)$ is bounded above by kn, where $k \in \mathbf{N}$ depends on I.

Later work in [ELS01] extended the result in a more restrictive setting. Specifically, suppose $Z \subseteq \mathbf{P}^N_{\mathbf{C}}$ is a reduced subscheme with no component of Z having codimension larger than e. Then, if $I = I(Z)$ is the ideal of forms vanishing on

Z, Theorem A of [ELS01] uses the theory of multiplier ideals to demonstrate that $I^{(me)} \subseteq I^m$ for all $m \in \mathbf{N}$.

Hochster and Huneke, using tight closure [HH02], generalized the result of Ein-Lazarsfeld-Smith to ideals in any regular ring containing a field as follows:

Theorem 1.4 (Theorem 1.1, [HH02]). *Let R be a Noetherian ring containing a field. Let h be the largest height of any associated prime of I. Then $I^{(hn+kn)} \subseteq (I^{(k+1)})^n$ for all positive n and nonnegative k. In particular, $I^{(hn)} \subseteq I^n$.*

As a consequence, the authors note, if R has finite dimension d, then we have $I^{(dn)} \subseteq I^n$ for every ideal I.

Nearly every ring in this thesis is a homogeneous coordinate ring of $\mathbf{P}^N$ for some $N \geq 2$ over some algebraically closed field k; in this setting, Theorem 1.4 implies that every nontrivial homogeneous ideal J in $k[\mathbf{P}^N]$ satisifies $J^{(mN)} \subseteq J^m$. The power of Theorem 1.4 is in its generality. Indeed, for no $a < N$ is $J^{(am)} \subseteq J^m$ true for all J and all m [BH10a]. However, for ideals J defining reduced configurations of points in $\mathbf{P}^N$, it is often true that $J^{(am)} \subseteq J^m$ for all m for some $a \in \mathbf{N}$ depending on J. As such ideals are well understood (e.g., all possible Hilbert functions of such ideals are known; see [GMR83]), they have proved to be a fertile ground on which to refine Theorem 1.4.

In the case that I is the ideal of forms vanishing at the points $p_1, p_2, \ldots, p_n \in \mathbf{P}^N$ with multiplicities $m_1, m_2, \ldots, m_n$, respectively, and $I(p_j)$ is the ideal generated by all forms vanishing at p_j, the mth symbolic power of I has a cleaner description:

$$I^{(m)} = \cap_{j=1}^{n} I(p_j)^{m_j m}.$$

This simpler form for $I^{(m)}$ makes working with symbolic powers of points somewhat easier. However, even in this special case, a complete description of which m and r guarantee containment $I^{(m)} \subseteq I^r$ is still difficult. In order to have another way to measure the failure of such containments, Bocci and Harbourne [BH10a] defined the *resurgence* $\rho(I)$, which is an asymptotic measure of the failure of I^r to contain $I^{(m)}$; if $m/r > \rho(I)$, then $I^{(m)} \subseteq I^r$ (see Definition 2.3). Our primary concern in Chapter 2 is, for a particular class of ideals of points in $\mathbf{P}^2$, to completely describe the containments $I^{(m)} \subseteq I^r$ and compute the resurgence. The methods we use to accomplish this goal allow us to verify several conjectures of Harbourne and Huneke for this class of ideals and compute various other invariants of these ideals and their symbolic powers, such as the saturation degree and regularity.

Computing ρ, while in principle easier than completely describing the containments $I^{(m)} \subseteq I^r$, is not a straightforward problem, as there is no general method for doing so. Sometimes, it is enough to give bounds on ρ. Indeed, given a homogeneous ideal $(0) \subsetneq I \subsetneq k[\mathbf{P}^N]$ Bocci and Harbourne [BH10a] give bounds on $\rho(I)$ in terms of the invariants $\alpha(I)$, $\gamma(I)$, and $\operatorname{reg}(I)$. Recall that $\operatorname{reg}(I)$ is the Castelnuovo-Mumford regularity, $\alpha(I)$ is the degree of a nonzero element of I of least degree, and $\gamma(I)$ is the Waldschmidt constant $\gamma(I) := \lim_{m\to\infty} \dfrac{\alpha(I^{(m)})}{m}$ (which always exists; see e.g., [BH10a, Lemma 2.3.1]). Then [BH10a, Theorem 1.2.1] gives, for an ideal $I \subseteq k[\mathbf{P}^N]$ defining a 0-dimensional subscheme (e.g., a set of points),

$$\frac{\alpha(I)}{\gamma(I)} \leq \rho(I) \leq \frac{\operatorname{reg}(I)}{\gamma(I)}.$$

Bocci and Harbourne use the lower bound to verify that there exist sequences of ideals I of points in $\mathbf{P}^N$ for which $\rho(I)$ has limit N, which was not known before. The examples used by Bocci and Harbourne are very special. This raised the question of what other such examples there may be for which $\alpha(I)/\gamma(I)$ is large, and hence for which $\alpha(I)$ is large compared to $\gamma(I)$. This looks to be very hard; Bocci and Chiantini [BC11] take an initial step in this direction by classifying point configurations in the plane for which $\alpha(I^{(2)})$ is as small as possible compared to $\alpha(I)$. More generally, they study the difference to seek to understand the difference $\alpha(I^{(2)}) - \alpha(I)$, where I defines a finite set of points in $\mathbf{P}^2$. In their paper, they classify all configurations of points in $\mathbf{P}^2$ for which the difference is 1 or 2. In Chapter 3, we extend these results and classify certain configurations of lines in $\mathbf{P}^3$ for which the difference is 1.

Chapter 2

Ideals of Almost Collinear Points

In this chapter, we give a more detailed history of the notion of the symbolic power of an ideal of points in $\mathbf{P}^2_k$, where k is an algebraically closed field. We then use a k-vector space basis to provide a complete characterization of the ideal containments $I^{(m)} \subseteq I^r$ and compute the resurgence $\rho(I)$ for a particular class of ideals in $k[\mathbf{P}^2]$. We close by verifying, for this class of ideals, some recent conjectures regarding containments of symbolic powers, and compute several invariants of these ideals.[1]

2.1 Symbolic Powers of Ideals of Points in $\mathbf{P}^2$

The work [BH10a] was, according to the authors, prompted by a question of Huneke:

[1]The work in this chapter was begun as a research project alongside Annika Denkert, and under the direction of Susan Cooper, and eventually culminated in the joint work [DJ12]. We initially were interested in answering these questions for all configurations of points lying on a reducible conic, i.e., a pair of lines. Annika began studying the ideal of a nearly complete intersection (see [DJ12] for more), and I began studying the almost collinear case. While our results are very different (e.g., if I is the ideal of a nearly complete intersection, $\rho(I)$ is constant regardless of the number of points in the configuration, whereas $\rho(I)$ depends on the number of points on the line if I is the ideal of $n+1$ almost collinear points, as we consider in this chapter), the methods we use are similar, and we are often able to verify the same conjectures, e.g., Conjecture 2.25, and so we submitted the paper jointly. In this way, all work presented in this chapter is my original work.

Question 2.1. If I is the radical ideal of a finite set of points in $\mathbf{P}^2$, is it true that $I^{(3)} \subseteq I^2$?

Theorem 1.4 guarantees $I^{(4)} \subseteq I^2$ if I is the ideal of a finite set of points in $\mathbf{P}^2$. Question 2.1 was eventually extended to a conjecture of Harbourne:

Conjecture 2.2 (see Conjecture 8.5.1 of [BDH$^+$09])**.** Let $I \subseteq \mathbf{C}[\mathbf{P}^n]$ be a homogeneous ideal. For $m \geq rn - (n-1)$, we have $I^{(m)} \subseteq I^r$.

The question of the containment $I^{(3)} \subseteq I^2$ turns out to be quite delicate. In fact, this is true if $\mathrm{char}(k) = 2$ (see [BDH$^+$09, Example 8.4.4]). It also holds for ideals of general points (see [BH10a]) and star configurations (see [HH]). However, a recent result of [DST13] uses a set of 12 points coming from the dual of the so-called Hesse configuration to demonstrate that Question 2.1 has a negative answer, which implies Conjecture 2.2 is false. Since this counterexample appeared on the arXiv, additional such examples have been developed by Harbourne and Seceleanu.

In the process of verifying the containment in Question 2.1 in various cases, Bocci and Harbourne develop tools for comparing homogeneous ideals $I^{(m)}$ and I^r more generally. One such tool is the *resurgence*, $\rho(I)$.

Definition 2.3. Let $I \subseteq k[\mathbf{P}^N]$ be a homogeneous ideal. The resurgence, $\rho(I)$, is

$$\rho(I) := \sup \left\{ m/r \,:\, I^{(m)} \not\subseteq I^r \right\}.$$

They also show that the containment result of [ELS01, HH02] (see Theorem 1.4) is optimal in every dimension and codimension in the sense that, for no $a < e$ do we have $I^{(ma)} \subseteq I^m$ for every ideal $I \subseteq k[\mathbf{P}^N]$ with all associated primes of height at most e.

For the remainder, we refer to the following generalization of Huneke's Question 2.1 as the Containment Question:

Question 2.4 (Containment Question)**.** If I is the ideal defining a finite set of points in $\mathbf{P}^N$, for which m and r does the containment $I^{(m)} \subseteq I^r$ hold?

The problem of computing $\rho(I)$ for a given ideal I can be thought of as an asymptotic version of Question 2.4:

Question 2.5 (Asymptotic Containment Question)**.** Given an ideal I defining a finite set of points in $\mathbf{P}^N$, what is $\rho(I)$?

Finding answers to these questions (for a particular class of configurations of points in $\mathbf{P}^2$) is the focus of this chapter. To do so, we develop new methods which rely on computing a k-vector space basis for $k[\mathbf{P}^2]$; see [DJ12] for more.

As previously stated, $I^{(m)} \subseteq I^r$ implies $m \geq r$ when $(0) \neq I \neq (1)$. This fact, together with the result of [ELS01, HH02], implies that $1 \leq \rho(I) \leq N$ for all homogeneous ideals $(0) \neq I \subsetneq k[\mathbf{P}^N]$. (Bocci and Harbourne also connect these questions to Seshadri constants, and then to a famous conjecture of Nagata [Nag61].)

As the resurgence is a limit, it is not clear how, in principle, to evaluate it, and doing so is in general a correspondingly hard problem. However, Bocci and Harbourne find some useful bounds on the resurgence in terms of other familiar invariants.

In particular, Bocci and Harbourne use the following invariants, which we will also make use of (and compute for the class of ideals we consider) in this thesis.

Notation 2.6. Let $(0) \neq I \subsetneq k[\mathbf{P}^N]$ be a nontrivial homogeneous ideal.

- The number $\alpha(I)$ denotes the least degree of a nonzero element of I; that is, if $I = \oplus_{t=0}^{\infty} I_t$, $\alpha(I) = \min\{t : I_t \neq 0\}$.
- The number $\gamma(I) := \lim_{m\to\infty} \dfrac{\alpha(I^{(m)})}{m}$ is the so-called Waldschmidt constant; see [Wal77, Sko77, HH] for more on this invariant.
- The number $\text{reg}(I)$ is the Castelnuovo-Mumford regularity; if I is the ideal of a finite set of points in $\mathbf{P}^N$, $\text{reg}(I)$ is the least $t > 0$ such that $\dim(R/I)_t = \dim(R/I)_{t-1}$, where $\dim(R/I)_j = \dim R_j - \dim I_j$ and I_j (respectively, R_j) is the k-vector space span of homogeneous elements of degree j in I (respectively, R).

Remark 2.7. Bocci and Harbourne show that, for $I \subseteq k[\mathbf{P}^N]$ defining a 0-dimensional subscheme of $\mathbf{P}^N$, $\alpha(I)/\gamma(I) \leq \rho(I) \leq \text{reg}(I)/\gamma(I)$; thus, when $\text{reg}(I) = \alpha(I)$, we have a complete answer to the Asymptotic Containment Question.

Moreover, the assumption $\alpha(I) = \text{reg}(I)$ leads (via Lemmas 2.3.2(a) and 2.3.4 of [BH10a]) to an exact answer to the Containment Question:

Corollary 2.8 (Corollary 1.2 of [BH10b])**.** *Assume $I \subset k[\mathbf{P}^N]$ is the ideal of a 0-dimensional subscheme of $\mathbf{P}^N$ and that $\alpha(I) = reg(I)$. Then $I^{(m)} \subseteq I^r$ if and only if $\alpha(I^{(m)}) \geq r\alpha(I)$.*

In general, it is not often the case that $\alpha(I) = \text{reg}(I)$, and even if equality holds, it can be difficult to compute $\alpha(I^{(m)})$ for $m \gg 0$. It is somewhat easier to compute bounds on $\rho(I)$. For example, in the case in which I defines a star configuration of $\binom{d}{2}$ points in $\mathbf{P}^2$, as in Figure 2.1, [BH10a] finds $\rho(I) = 2(d-1)/d$. (Recall that a *star configuration* of points in $\mathbf{P}^2$ is the set of $\binom{d}{2}$ points obtained by the pairwise intersection of d lines, no three of which pass through any point.)

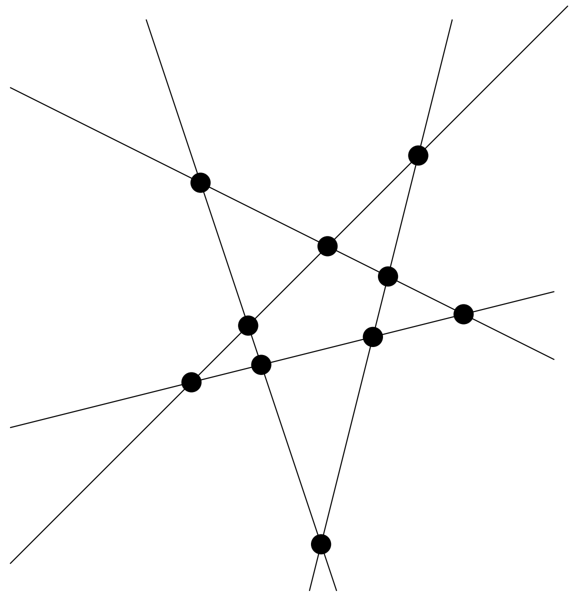

Figure 2.1: A star configuration of points in $\mathbf{P}^2$ with $d = 5$.

Bocci and Harbourne also consider generalizations of star configurations in higher dimensions; if we denote by $S_N(e, s)$ the e-wise intersections of s general hyperplanes in $\mathbf{P}^N$, they obtain the bound $\rho(S_N(e, s)) \geq e(se+1)/s$, with equality if $e = N$.

The simplest situation for which there is a complete solution to Questions 2.4 and 2.5 is that of a complete intersection of points in $\mathbf{P}^2$ (e.g., a single point).

Example 2.9. Suppose $I \subseteq k[\mathbf{P}^2]$ is a complete intersection ideal. (Recall that a complete intersection ideal is one generated by a regular sequence.) Then $I^{(m)} = I^m$ (see [ZS75, Lemma 5, Appendix 6]), whence $\alpha(I^{(m)}) = m\alpha(I)$, $\gamma(I) = \operatorname{reg}(I) = \alpha(I)$, and thus $\rho(I) = 1$.

Bocci and Harbourne [BH10b] also completely solve the Containment Question and the Asymptotic Containment Question in the case that I defines n general points in $\mathbf{P}^2$, $n \leq 9$, and also in the case that I defines any number of points on a smooth conic.

Theorem 2.10 (Theorem 3.4 of [BH10b]). *Assume the points $p_1, p_2, \ldots, p_n \in \mathbf{P}^2$ lie on a smooth conic curve Q'. Let $I = I(Z)$, where $Z = p_1 + p_2 + \cdots + p_n$. Let m and r be positive.*

(a) If n is even or $n = 1$, then $I^{(m)} \subseteq I^r$ if and only if $m \geq r$; in particular, $\rho(I) = 1$.

(b) If $n > 1$ is odd, then $I^{(m)} \subseteq I^r$ if and only if $(n+1)r - 1 \leq nm$; in particular $\rho(I) = (n+1)/n$.

The authors mention that a corresponding theorem for points on a reducible (equivalently, not smooth) conic would depend on the number of points on each of the lines of which the conic is composed. In Section 2.4, we answer this question for one such class of configurations of points on a reducible conic (see also [DJ12]).

2.2 Motivating Our Methods

In this section, we consider an example that motivates not the questions we consider, but our methods for doing so. In particular, we will consider an example which will demonstrate that neither checking α, nor Hilbert functions, nor fixed components can provide complete answers to Questions 2.4 and 2.5, and the bounds they provide are not helpful. We term this particular configuration of points an *almost collinear* configuration of points (see Definition 2.12). An example of 4 almost collinear points is found in Figure 2.2.

Example 2.11. Recall that our goals are to compute $\rho(I)$ and describe exactly for which m and r we have the containment $I^{(m)} \subseteq I^r$. As we will see, the ideal I defining the four points is $I = (x, y) \cap (z, F) = (xz, yz, F)$, where $F = x(x - a_1 y)(x - a_2 y)$. Recall that Theorem 1.4 only gives the bound $\rho(I) \leq 2$; since $I^{(m)} \subseteq I^r$ implies $m \geq r$, we have $1 \leq \rho(I) \leq 2$. Moreover, [BH10a] give the bounds

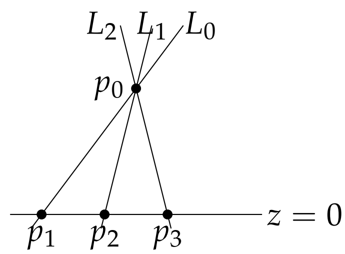

Figure 2.2: Four almost collinear points, where L_0 is defined by the equation $x = 0$, L_1 is defined by the equation $x - a_1 y = 0$, and L_2 is defined by the equation $x - a_2 y = 0$, where $a_1, a_2 \in k$ are nonzero.

$\alpha(I)/\gamma(I) \leq \rho(I) \leq \operatorname{reg}(I)/\gamma(I)$; in the case of 4 almost collinear points, this yields the tighter bound

$$\frac{6}{5} \leq \rho(I) \leq \frac{9}{5},$$

via Lemma 2.30, Proposition 2.31, and Theorem 2.40. While this is certainly an improvement over the Ein-Lazarsfeld-Smith/Hochster-Huneke bound of Theorem 1.4, it still does not allow us to calculate $\rho(I)$ exactly. In fact, our approach will show that $\rho(I) = 9/7$ (see Theorem 2.22).

There is no general approach for computing $\rho(I)$. One can get estimates by computing $\alpha(I^{(m)})$ and $\alpha(I^r)$. Indeed, if $\alpha(I^{(m)}) < \alpha(I^r)$, then $I^{(m)} \not\subseteq I^r$, and thus $m/r \leq \rho(I)$. In fact, $\alpha(I^{(m)}) < \alpha(I^r)$ implies $m\gamma(I) \leq \alpha(I^{(m)}) < \alpha(I^r) = r\alpha(I)$, and thus

$$\frac{m}{r} < \frac{\alpha(I)}{\gamma(I)} \leq \rho(I).$$

(Therefore, the lower bound obtained using $\alpha(I^{(m)}) < \alpha(I^r)$ is never better than $\alpha(I)/\gamma(I) \leq \rho(I)$, and in fact it is where $\alpha(I)/\gamma(I) \leq \rho(I)$ comes from.)

We will see in Lemma 2.30 that $\alpha(I^{(m)}) = \lceil \frac{5m}{3} \rceil$ and $\alpha(I^r) = 2r$. Then $\alpha(I^{(m)}) < \alpha(I^r)$ yields $5m/3 < 2r$, which implies $m/r < 6/5$, so the best lower bound we can get on $\rho(I)$ is $6/5 \leq \rho(I)$, which is no better than we could do using the result of Bocci and Harbourne, as we saw above. [Indeed, taking $m = 3(2s+1)$ and $r = \frac{5(2s+1)+1}{2}$ gives $\alpha(I^{(m)}) = 10s+5$ and $\alpha(I^r) = 10s+6$ (so $\alpha(I^{(m)}) < \alpha(I^r)$) and $\lim_{s\to\infty} \frac{m}{r} = \lim_{s\to\infty} \frac{12s+6}{10s+6} = \frac{6}{5}$, which proves that $6/5$ is indeed a lower bound for $\rho(I)$.]

One might also consider using Hilbert functions to detect failures of containment $I^{(m)} \subseteq I^r$, for if $\dim_k(I^{(m)})_t > \dim_k(I^r)_t$, we must have $I^{(m)} \not\subseteq I^r$. Consider the ideal I of 4 almost collinear points; here we have $I^{(6)} \not\subseteq I^5$. One may then use Macaulay2 [GS] to see the dimensions of $I^{(6)}$ and I^5 in Table 2.1. (Note that, by

Table 2.1: Fixed components of $I^{(6)}$

degree d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\dim(I^{(6)})_d$	0	0	0	0	0	0	0	0	0	0	1	6	16	27	40	54
$\dim(I^5)_d$	0	0	0	0	0	0	0	0	0	0	6	18	31	45	60	76

Theorem 2.40, $\operatorname{reg}(I^5) = 15$, so it's enough to check up to degree 15). However, $I^{(6)} \not\subseteq I^5$, as Theorem 2.24 demonstrates (one may also use Lemmas 2.19 and 2.20 to see that $z^4F^2 \in I^{(6)}$ but $z^4F^2 \notin I^5$). Thus, the Hilbert function is not able to detect every failure of containment.

A final possible approach, and one taken in [BH10b], was to examine the *fixed component* in each degree of the ideal, which Bocci and Harbourne applied to ideals of points on an irreducible conic in $\mathbf{P}^2$. Let f be the form defining the conic; then $q(I_t)$ denotes the largest exponent e such that f^e divides every element of I_t. If $q((I^{(m)})_t) < q((I^r)_t)$, then we cannot have $I^{(m)} \subseteq I^r$. There are ideals I of points in the plane for which this approach again does not detect all failures of containment. In any case, it gives at best a lower bound on $\rho(I)$, and not an exact computation. As a result, we need a different approach to get a complete characterization of the containment $I^{(m)} \subseteq I^r$ and an exact computation of $\rho(I)$.

The approach we use is to first recognize that each ideal I we consider looks a

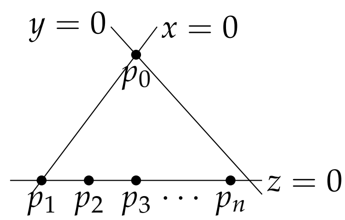

Figure 2.3: $n + 1$ almost collinear points

great deal like a monomial ideal; indeed, two of its generators are monomials, and the third is a form that, while not monomial, can be chosen to be "special" enough with an appropriate choice of coordinates. We then develop a k-basis for the ring $R = k[x, y, z]$ of elements of the form $x^e F^a y^j z^l$, where $0 \leq e < n$ that restricts nicely to k-bases of the ideals $I^{(m)}$ and I^r. In fact, we can provide relatively simple conditions on $i = e + na$, j, and l that easily allow us to decide whether I^r contains $I^{(m)}$, and from there completely answer Questions 2.4 and 2.5.

2.3 Foundation

For the remainder of this chapter, fix $R = k[\mathbf{P}^2] = k[x, y, z]$ and let $Z = p_0 + p_1 + \cdots + p_n$ denote the scheme-theoretic union of $n + 1$ distinct points $p_0, p_1, \ldots, p_n \in \mathbf{P}^2$. The particular configuration of points we are interested in is described in the following definition (and shown in Figure 2.3).

Definition 2.12. Let $Z = p_0 + p_1 + \cdots + p_n$ be a zero-dimensional subscheme of $\mathbf{P}^2$, where $n \geq 3$. We call Z an almost collinear subscheme of $n + 1$ points (or just an almost collinear subscheme, or a set of almost collinear points) if $p_1, p_2, \ldots, p_n$ lie on a line L and p_0 does not lie on L.

Apply a linear change of coordinates so that, as in Figure 2.3, p_0 is defined by the equations $x = 0$ and $y = 0$, p_1 is defined by the equations $x = 0$ and $z = 0$, the line defined by the equation $z = 0$ contains the points $p_1, p_2, \ldots, p_n$, and y vanishes only at p_0. Then $I(p_0) = (x, y)$ and $I(p_1 + p_2 + \cdots + p_n) = (z, F)$, where $F = L_1 L_2 L_3 \cdots L_n$ such that L_i vanishes at p_0 and p_i (so $L_1 = x$ and $L_i = x - l_i y$ if $p_i = (l_i, 1, 0)$ for $i \geq 2$). In particular, F is a form of degree n in the variables x and y only, and x^n is a term in the expansion of F, as the only point on the line defined by $y = 0$ is p_0 (see Figure 2.3). Moreover, $I(Z) = (x, y) \cap (z, F) = (xz, yz, F)$ and $I^{(m)} = (x, y)^{(m)} \cap (z, F)^{(m)} = (x, y)^m \cap (z, F)^m$, since each ideal (x, y) and (z, F) defines a complete intersection (recall Example 2.9). We make the additional assumption that $n \geq 3$; if $n = 1$, the situation described here reduces to a complete intersection. For $n = 2$, see Theorem 2.10.

Our solution to the Containment Questions uses a k-vector space basis of R and then restricts it to $I^{(m)}$ and I^r. We can then decide the question of containment $I^{(m)} \subseteq I^r$ by finding a basis element in $I^{(m)}$ which does not lie in I^r, or proving no such element exists. The following lemma demonstrates that this approach is possible.

Lemma 2.13. *Let U and V be subspaces of a vector space W. Let B_W be a basis of W that contains a basis B_U of U and a basis B_V of V. Then $B_U \cap B_V$ is a basis for $U \cap V$.*

Proof. It is enough to show that $B_U \cap B_V$ spans $U \cap V$. Suppose $a \in U \cap V$. We know $a = \sum_{e \in B_W} c_e e$ for $c_e \in k$ (where $c_e = 0$ for all but finitely many e). Since $a \in \operatorname{span}(B_U)$, $c_e \neq 0$ means $e \in B_U$. Similarly, as $a \in \operatorname{span}(B_V)$, $c_e \neq 0$ implies $e \in B_V$. Therefore, if $c_e \neq 0$ we can conclude $e \in B_U \cap B_V$, and thus $a = \sum_{e \in B_W} c_e e = \sum_{e \in B_U \cap B_V} c_e e \in \operatorname{span}(B_U \cap B_V)$. $\square$

Notation 2.14. Given $F \in k[x, y]$ of degree n as above and i a nonnegative integer, use the division algorithm to write $i = an + e$, where $0 \leq e < n$. For each i, we write $H_i := x^e F^a$.

Remark 2.15. Note that $\deg H_i = i$, and, as a polynomial in x, H_i is monic with leading term x^i. Further, $H_{nq} = F^q$, so if $n|b$, $H_a H_b = H_{a+b}$.

Next, we show that x^i is in the span of elements of the form $H_j y^l$, where $j + l = i$.

Lemma 2.16. *Let $i \geq 0$. Then x^i is in the k-span of $H_0 y^i, H_1 y^{i-1}, \ldots, H_{i-1} y, H_i$.*

Proof. This is true for $i = 0$, since $x^0 = 1$. Suppose $i > 0$. Then $x^i - H_i = \sum_{t=0}^{i-1} a_t x^t y^{i-t}$, and by induction on i, each $a_t x^t y^{i-t} \in \operatorname{span}_k \{H_0 y^i, \ldots, H_i y^0\}$, and thus $x^i - H_i \in \operatorname{span}_k \{H_0 y^i, \ldots, H_i y^0\}$. We conclude $x_i \in \operatorname{span}_k \{H_0 y^i, \ldots, H_i y^0\}$. □

We now use this construction to build a k-basis of R, as we exemplify in Example 2.18.

Lemma 2.17. *A k-basis of R is given by $\mathcal{B}_R = \bigcup_{i \geq 0} B_i$, where*

$$B_i = \left\{ H_i y^j z^l \ : \ i = an + e,\ 0 \leq e < n,\ H_i = x^e F^a, \text{ and } j, l \geq 0 \right\}.$$

Proof. By Lemma 2.16, for each $t \geq 0$, x^t is in the span of $H_0 y^t, \ldots, H_t y^0$, hence every monomial $x^t y^s z^l$ is in the span of elements of the form $H_i y^j z^l$ with $i + j = t + s$. Since the monomials of the form $x^t y^s z^l$ span $k[x, y, z]$, so do the elements of the form $H_i y^j z^l$.

The elements $H_i y^j z^l$ are homogeneous and thus the span of those elements of degree d must be the homogeneous component R_d of $R = k[x, y, z]$. There are exactly $\binom{d+2}{2} = \dim_k R_d$ elements of the form $H_i y^j z^l$ of degree d (since the cardinality of the set of those elements of the form $H_i y^j z^l$ is just the number of solutions (i, j, l) to $i + j + l = d$ with $i, j, l \geq 0$). Thus, the elements $H_i y^j z^l$ of degree d

are independent. By homogeneity, any linear dependence among the elements of the form $H_i y^j z^l$ must involve elements of the same degree, hence $\mathcal{B}_R$ is linearly independent, and a k-vector space basis of R. $\square$

We now return to Example 2.11 in the context of the k-vector space basis we have developed for R.

Example 2.18. Suppose $n = 3$. We then have 4 points, arranged as in Figure 2.2. The ring basis is thus

$$\mathcal{B}_R = \bigcup_{i \geq 0} \left\{ H_i y^j z^l \;:\; i = 3a + e,\, 0 \leq e < 3,\, H_i = x^e F^a, \text{ and } j, l \geq 0 \right\},$$

where $F = x(x - a_1 y)(x - a_2 y) = x^3 - (a_1 + a_2)x^2 y + a_1 a_2 x y^2$. In particular, we obtain higher powers of x inductively; for example, $x^3 = H_3 + (a_1 + a_2)H_2 y - a_1 a_2 H_1 y^2$, and we can use this to write

$$\begin{aligned}
x^4 &= H_4 + (a_1 + a_2)x^3 y - a_1 a_2 x^2 y^2 \\
&= H_4 + (a_1 + a_2)(H_3 + (a_1 + a_2)H_2 y - a_1 a_2 H_1 y^2)y - a_1 a_2 H_2 y^2 \\
&= H_4 + (a_1 + a_2)H_3 y + (a_1 + a_2)^2 H_2 y^2 - a_1 a_2 (a_1 + a_2) H_1 y^3 - a_1 a_2 H_2 y^2 \\
&= H_4 + (a_1 + a_2)H_3 y + (a_1 + a_2 - a_1 a_2)H_2 y^2 - a_1 a_2 (a_1 + a_2) H_1 y^3.
\end{aligned}$$

We can continue in this way to write any power of x as a linear combination of elements of the form $H_i y^j z^l$.

The next lemma determines conditions on i, j, l with respect to which elements of the form $H_i y^j z^l$ give a k-basis of the symbolic power $I^{(m)}$.

Lemma 2.19. *Let $m \geq 1$.*

(a) Then $H_i y^j z^l \in I^{(m)}$ if and only if $i, j, l \geq 0$, $i + ln \geq mn$, and $i + j \geq m$.

(b) Moreover, $I^{(m)}$ is the k-vector space span of the elements of the form $H_i y^j z^l$ contained in $I^{(m)}$.

Proof. (a) Suppose $i, j, l \geq 0$, $i + ln \geq mn$, and $i + j \geq m$. Then, since $i, j, l \geq 0$ and $i + j \geq m$, we have $H_i y^j z^l \in (x, y)^m$. Since $i + ln \geq mn$, we have $i/n + l \geq m$, which is equivalent to $\lfloor i/n \rfloor + l \geq m$, which further implies $H_i y^j z^l \in (z, F)^m$. Thus, $H_i y^j z^l \in (x, y)^m \cap (z, F)^m = I^{(m)}$.

Conversely, suppose $H_i y^j z^l \in I^{(m)}$. Since $I^{(m)} = (x, y)^m \cap (z, F)^m$, we know $H_i y^j z^l \in (x, y)^m$, and thus $i + j \geq m$. Also, $H_i y^j z^l \in (z, F)^m = (z, F)^{(m)}$, the order of vanishing of $H_i y^j z^l$ at p_0 must be at least m. Since none of the points $p_1, \cdots, p_n$ are on the lines $x = 0$ or $y = 0$, $H_i y^j z^l \in (z, F)^{(m)}$ if and only if $F^b z^l \in (z, F)^{(m)}$, where $H_i = x^z F^b$. But $F^b z^l \in (z, F)^{(m)}$ if and only if $b + l \geq m$, which holds if and only if $i + ln \geq mn$.

(b) Suppose we show that $(x, y)^m$ is the k-vector space span of the elements of the form $H_i y^j z^l$ contained in $(x, y)^m$, and that $(z, F)^m$ is the k-vector space span of the elements of the form $H_i y^j z^l$ contained in $(z, F)^m$. Then, by Lemma 2.13 and Lemma 2.17, $I^{(m)} = (x, y)^m \cap (z, F)^m$ also is the k-vector space span of the elements of the form $H_i y^j z^l$ contained in $I^{(m)}$. Now, it is an elementary fact that $(x, y)^m$ is the k-span of monomials of the form $x^i y^j z^l$ with $i + j \geq m$, each of which is by Lemma 2.16 in the k-span of elements of the form $H_i y^j z^l$ with $i + j \geq m$, each of which has order of vanishing at p_0 at least m and hence is in $(x, y)^m$. Finally, $(z, F)^m$ is the k-span of elements of the form $x^t F^b y^s z^l$ with $b + l \geq m$. But $x^t y^s$ is in $(x, y)^{t+s}$, and hence $x^t y^s$ is by Lemma 2.16 in the k-span of elements of the form $H_q y^j$ with $q + j = t + s$, so each element $x^t F^b y^s z^l$ with $b + l \geq m$ is in the k-span of elements of the form $H_i y^j z^l$ with $i = q + bn$, $q + j = t + s$ and $b + l \geq m$. But $F^b z^l$ divides each $H_i y^j z^l$, and $F^b z^l \in (z, F)^m$ implies $H_i y^j z^l \in (z, F)^m$. $\square$

We next provide a similar result for I^r, which will eventually allow us to completely answer Questions 2.4 and 2.5.

Lemma 2.20. *Let $r \geq 1$.*

(a) The ideal I^r is the span of the elements of the form $H_i y^j z^l \in I^r$; in addition, if $H_i y^j z^l \in I^r$, then $H_i y^j z^l$ is a product of r elements of I.

(b) Moreover, $H_i y^j z^l \in I^r$ if and only if $i, j, l \geq 0$ and either:

(1) $l < j$ and $i + nl \geq rn$, or

(2) $j \leq l < i + j$ and $i + j + (n-1)l \geq rn$, or

(3) $i + j \leq l$ and $r \leq i + j$.

Before we present the proof of Lemma 2.20, we present the details of (a) in the context of Example 2.18 with $r = 2$.

Example 2.21. Recall from Example 2.18 that $I = (xz, yz, F)$, where $F = x(x - a_1 y)(x - a_2 y) = x^3 - (a_1 + a_2)x^2 y + a_1 a_2 x y^2$. Thus,

$$I^2 = (xyz^2, xyF, yzF, x^2z^2, y^2z^2, F^2) = (H_1 yz^2, H_4 y, H_3 yz, H_2 z^2, y^2 z^2, H_6);$$

that is, I^2 is *generated* by elements which are products of pairs (as $r = 2$) of generators of I. Part (a) of the following proof seeks to show that I^2 is spanned by elements of the ring basis (as described in Lemma 2.17) by taking an arbitrary element of I^2, which is itself a product of basis elements, collecting the powers of x, F, y, and z and rewriting as a linear combination of basis elements which we can check to see are themselves in I^2.

As an example to illustrate how an element of the ordinary power may be written as a linear combination of ring basis elements in I^2, consider $(H_4y)(H_2z^2) \in I^2$ (there is not a unique way to write this as a product of two elements of I, but that causes no trouble). The idea of the proof is to first collect "like factors", e.g., $(H_4y)(H_2z^2) = (xFy)(x^2z^2) = x^3Fyz^2$, and then rewrite the powers of x and F as a linear combination of basis elements. Note that x^3Fyz^2 has a factor of x^3F, which lies in the span of elements of the form $H_uy^vF^B$ such that $u + v = 3$. In fact, $x^3 = H_3 + (a_1 + a_2)H_2y - a_1a_2H_1y^2$, by Example 2.18, and note that each element in the linear combination is a product of $u + v = 3$ linear forms, each of which is in the ideal (x, y). Then we can write $(H_4y)(H_2z^2)$ as a linear combination of basis elements:

$$\begin{aligned} x^3Fyz^2 &= (H_3 + (a_1 + a_2)H_2y - a_1a_2H_1y^2)Fyz^2 \\ &= H_6yz^2 + (a_1 + a_2)H_5y^2z^2 - a_1a_2H_4y^3z^2. \end{aligned}$$

Furthermore, we can write each term in the expansion above as a product of basis elements in I in a nonunique way:

$$\begin{aligned} x^3Fyz^2 &= H_6yz^2 + (a_1 + a_2)H_5y^2z^2 - a_1a_2H_4y^3z^2 \\ &= (H_3y)(H_3z^2) + (a_1 + a_2)(H_4y)(H_1yz^2) - a_1a_2(H_4y)(y^2z^2). \end{aligned}$$

It is straightforward to check that each factor of each term is an element of I, and thus our element can be written as a linear combination of basis elements in I^2, each of which is a product of basis elements in I.

In the following proof, we generalize this and give a method for carrying this process out on an arbitrary element of I^r, which shows both that I^r is the span of elements of the form $H_i y^j z^l \in I^r$ and that every element of I^r is in the span of elements $H_i y^j z^l \in I^r$ which factor as a product of r elements of I, which, by linear independence, shows that every element $H_i y^j z^l \in I^r$ is a product of r elements of I.

Proof of Lemma 2.20. (a) This is true for $r = 1$ by Lemma 2.19(b). Thus I^r is the span of products $H_{i_1} y^{j_1} z^{l_1} \cdots H_{i_r} y^{j_r} z^{l_r}$ of r elements of the form $H_{i_t} y^{j_t} z^{l_t}$, which satisfy $i_t, j_t, l_t \geq 0$, $i_t + l_t n \geq n$ and $i_t + j_t \geq 1$ for $t = 1, \ldots, r$ (i.e., elements of the form $H_{i_t} y^{j_t} z^{l_t} \in I$ for each t).

Write each H_{i_t} as $x^{a_t} F^{b_t}$ where $i_t = b_t n + a_t$ and $0 \leq a_t < n$. Let $B = b_1 + \cdots + b_r$ and let $A = a_1 + \cdots + a_r$. Then $H_{i_1} \cdots H_{i_r} = x^A F^B$ is, by Lemma 2.16, in the span of elements of the form $H_u y^v F^B = H_{u+Bn} y^v$ where $u + v = A$ and $0 \leq u \leq A$.

Since $i_1 + \cdots + i_r = (a_1 + \cdots + a_r) + (b_1 + \cdots + b_r) = A + B$, and since $H_u y^v$ is a product of $u + v = A = a_1 + \cdots + a_r$ linear forms, each of which is in (x, y), we can factor $H_u y^v$ as $G_1 \cdots G_r$ where each G_s is a product of a_s of these linear forms. Thus $H_{u+Bn} y^{v+j_1+\cdots+j_r} z^{l_1+\cdots+l_r} = (G_1 F^{b_1} y^{j_1} z^{l_1}) \cdots (G_r F^{b_r} y^{j_r} z^{l_r})$. Now each $H_{i_t} y^{j_t} z^{l_t}$ satisfies $i_t, j_t, l_t \geq 0$, $i_t + l_t n \geq n$ and $i_t + j_t \geq 1$. Thus $G_t F^{b_t} y^{j_t} z^{l_t}$ satisfies $(a_t + b_t n) + l_t n = i_t + l_t n \geq n$ (thus either $b_t > 0$ or $l_t > 0$ and so $G_t F^{b_t} y^{j_t} z^{l_t}$ vanishes at each point $p_1, \ldots, p_n$) and $(a_t + b_t n) + j_t = i_t + j_t \geq 1$ (so $G_t F^{b_t} y^{j_t} z^{l_t}$ vanishes at p_0) and hence $G_t F^{b_t} y^{j_t} z^{l_t} \in I$. Thus $H_{u+Bn} y^{v+j_1+\cdots+j_r} z^{l_1+\cdots+l_r} \in I^r$.

This shows not only that I^r is the span of the elements of the form $H_i y^j z^l \in I^r$, but also that every element of I^r is in the span of elements $H_i y^j z^l \in I^r$ which factor as a product of r elements of I. But if $H_i y^j z^l \in I^r$, it is in the span only of itself (since elements of this form are linearly independent), so each element $H_i y^j z^l \in I^r$ is itself a product of r elements of I.

(b) Begin with the backward implication, and assume $i, j, l \geq 0$. If $l < j$ and

$i + nl \geq rn$, let $i = bn + a$, where $b = \lfloor i/n \rfloor$. Then $l < j$ implies $F^b(yz)^l$ divides $H_i y^j z^l = x^a F^b y^j z^l$, but $i + nl \geq rn$ implies $b + l \geq r$, so $F^b(yz)^l$ is a product r factors, each of which, being either F or yz, is in I, hence $H_i y^j z^l \in I^r$.

If $j \leq l < i + j$ and $i + j + (n-1)l \geq rn$, then $l - j \geq 0$ and $i - (l - j) > 0$. Let $t = \lfloor (i - (l-j))/n \rfloor$ and let $i = bn + a$, where $0 \leq a < n$. Note that $b = \lfloor i/n \rfloor \geq t$; let $G = x^a F^{b-t}$. Then $H_i y^j z^l = x^a F^b (yz)^j z^{l-j} = GF^t(yz)^j z^{l-j}$, but $G \in (x,y)^{a+(b-t)n}$ and $a + (b-t)n = a + bn - nt \geq i - ((i-(l-j))/n)n = l - j$. Thus $H_i y^j z^l = F^t(Gz^{l-j})(yz)^j \in I^t I^{l-j} I^j = I^{t+l}$, but $(i - (l-j)) + nj + n(l-j) = i + j + (n-1)l \geq rn$ implies $(i - (l-j))/n + j + (l-j) \geq r$ and so $t + l \geq r$, whence $H_i y^j z^l \in I^{t+l} \subseteq I^r$.

Finally, if $r \leq i + j \leq l$, then $H_i y^j = G_1 \cdots G_r D$ where each G_t is a linear form in (x,y) and D is a form in $(x,y)^d$ for $d = i + j - r$. Thus $H_i y^j z^l = (G_1 z) \cdots (G_r z)(Dz^{l-r})$, but $(G_1 z) \cdots (G_r z) \in I^r$, hence so is $H_i y^j z^l$.

We now turn to the forward implication, but first a bit of terminology. By *minimal factor of $H_i y^j z^l$ in I* we mean a factor of $H_i y^j z^l$ which is in I but which has no factor of smaller degree which is in I. Minimal factors divisible by z will be called z-factors. Given any $H_i y^j z^l$, note that the minimal factors of $H_i y^j z^l$ in I (if any) are of the form F, yz, xz, and $L_u z$ (where L_u is the linear form vanishing on p_0 and on p_u for some $1 \leq u \leq n$). Let P_s denote a product of s z-factors. Any product $P_s F^t$ which divides $H_i y^j z^l$ satisfies $0 \leq t \leq b$, where $b = \lfloor i/n \rfloor$, and $0 \leq s \leq \min\{l, i + j - nt\}$. It is easy to see that if there are values for s and t satisfying $s + t \geq r$, $0 \leq t \leq b$ and $0 \leq s \leq \min\{l, i + j - nt\}$, then $H_i y^j z^l$ has a factor $P_s F^t \in I^r$ and hence $H_i y^j z^l \in I^r$, while, by part (a), if $H_i y^j z^l \in I^r$ then $H_i y^j z^l$ has a factor $P_s F^t \in I^r$ with $s + t \geq r$ satisfying $0 \leq t \leq b$ and $0 \leq s \leq \min\{l, i + j - nt\}$.

Assume $H_i y^j z^l \in I^r$, and hence there are values for s and t satisfying $s + t \geq r$, $0 \leq t \leq b$ and $0 \leq s \leq \min\{l, i + j - nt\}$. Of course, $i, j, l \geq 0$. Then there are three cases: $l < j$; $j \leq l < i + j$; and $i + j \leq l$.

If $l < j$, then, since $i - nt \geq 0$, we have $\min\{l, i + j - nt\} = l$, so $r \leq t + s \leq b + l \leq i/n + l$, hence $i + ln \geq rn$. This is case (1).

Say $j \leq l < i + j$. If $l \leq i + j - nt$, then $s \leq \min\{l, i + j - nt\} = l$ and $t \leq (i + j - l)/n$, so $r \leq t + s \leq (i + j - l)/n + l$, or, equivalently, $nr \leq i + j + (n-1)l$ as we wanted to show. Suppose instead that $l > i + j - nt$. Let $\delta = l - (i + j - nt)$. Then $s \leq \min\{l, i + j - nt\} = i + j - nt = l - \delta$, so $t = (i + j - l + \delta)/n$ and $r \leq t + s \leq (i + j - l + \delta)/n + l - \delta = (i + j + (n-1)l)/n - \delta(n-1)/n \leq (i + j + (n-1)l)/n$ which again implies $nr \leq i + j + (n-1)l$. This is case (2).

If $i + j \leq l$, then $\min\{l, i + j - nt\} = i + j - nt$, so $r \leq s + t \leq i + j - (n-1)t \leq i + j$. This is case (3). $\square$

With lemmas describing precisely which k-basis elements are in the ideals $I^{(m)}$ and I^r, we now turn to the containment $I^{(m)} \subseteq I^r$ and the computation of $\rho(I)$.

2.4 Containment Questions Answered

We first answer Question 2.5.

Theorem 2.22. *For the ideal I of $n + 1$ almost collinear points,*

$$\rho(I) = \frac{n^2}{n^2 - n + 1}.$$

Proof. Consider $H_i y^j z^l$ where $i = tn^2$, $j = 0$, and $l = tn^2 - tn$, and let $m = tn^2$ and $r = tn^2 - tn + t + 1$. Then $H_i y^j z^l \in I^{(m)}$ for every $t \geq 1$ by Lemma 2.19(a), but $i + j + (n-1)l < rn$ so $I^{(m)} \not\subseteq I^r$ by Lemma 2.20(b)(2), hence $m/r \leq \rho(I)$ for all t. Taking the limit as $t \to \infty$ gives $n^2/(n^2 - n + 1) \leq \rho(I)$.

Now suppose $m/r \geq n^2/(n^2-n+1)$ and hence $m \geq r$. Consider $H_i y^j z^l \in I^{(m)}$. Then $i+j \geq m$ and $i+nl \geq mn$ by Lemma 2.19(a). Now consider cases.

If $l < j$, then $i+nl \geq mn \geq nr$ so $H_i y^j z^l \in I^r$ by Lemma 2.20(b)(1).

If $j \leq l < i+j$, use $i+j \geq m \geq rn^2/(n^2-n+1)$ and $i+nl \geq mn \geq rn^3/(n^2-n+1)$. Arguing by contradiction, suppose that $i+j+(n-1)l < rn$. Then $rn^2 > (n-1)i+i+nj+n(n-1)l = (n-1)(i+nl)+i+nj \geq rn^3(n-1)/(n^2-n+1)+i+nj$ so $rn^2(n^2-n+1) > rn^3(n-1)+(i+nj)(n^2-n+1)$ which simplifies to $rn^2 > (i+nj)(n^2-n+1)$. Using $i+j \geq rn^2/(n^2-n+1)$, this gives $rn^2/(n^2-n+1) > i+nj \geq rn^2/(n^2-n+1)+(n-1)j$, which is impossible. Thus $i+j+(n-1)l \geq rn$ so $H_i y^j z^l \in I^r$ by Lemma 2.20(b)(2).

If $i+j \leq l$, then $i+j \geq m \geq r$ so $H_i y^j z^l \in I^r$ by Lemma 2.20(b)(3).

Thus $m/r \geq n^2/(n^2-n+1)$ implies $I^{(m)} \subseteq I^r$ by Lemma 2.19(b), and so $\rho(I) \leq n^2/(n^2-n+1)$, i.e., $\rho(I) = n^2/(n^2-n+1)$. □

Recall, $\rho(I)$ is the supremum of rationals m/r for which $I^{(m)} \not\subseteq I^r$, and thus $I^{(m)} \subseteq I^r$ whenever $m/r > \rho(I)$ but it may be that there are rationals $m/r \leq \rho(I)$ with $I^{(m)} \subseteq I^r$. We next show that the k-bases found in previous lemmas allow us to completely answer Question 2.4.

The crux of the argument is that containment will fail if and only if we can find $H_i y^j z^l \in I^{(m)} \setminus I^r$. It is known that $I^{(m)} \not\subseteq I^r$ if $m < r$. The constraints we have obtained show that if $m \geq r$, then $i+j \geq m$ and $i+nl \geq mn$ imply $i+j \geq r$ and $i+nl \geq rn$. Thus, we have $H_i y^j z^l \in I^{(m)} \setminus I^r$ if and only if either

1. $m < r$, or

2. $m \geq r$, $i+j \geq m$ and $i+nl \geq mn$ (so $H_i y^j z^l \in I^{(m)}$), and $j \leq l < i+j$, $i+j+(n-1)l \leq rn-1$ (so $H_i y^j z^l \notin I^r$).

If $m \geq r$, we have $H_i y^j z^l \in I^{(m)} \setminus I^r$ if and only if there is a non-negative integer lattice point (i, j, l) satisfying $i + j \geq m$, $i + nl \geq mn$, $j \leq l \leq i + j - 1$ and $i + j + (n-1)l \leq rn - 1$. In fact, we need only concern ourselves with i and l, as the next lemma demonstrates.

Lemma 2.23. *There is such a lattice point (i, j, l) if and only if there is a nonnegative integer lattice point (i', l') satisfying $i' \geq m$, $i' + nl \geq mn$, $l' < i'$ and $i' + (n-1)l' \leq rn - 1$.*

Proof. Given i' and l', just take $i = i'$, $l = l'$, and $j = 0$. Given (i, j, l), take $i' = i + j$ and $l' = l$. $\square$

Therefore, $I^{(m)} \not\subseteq I^r$ if and only if either $m < r$ or there is a nonnegative integer lattice point (i, l) satisfying

$$i \geq m, \quad i + nl \geq mn, \quad l \leq i - 1, \quad \text{and} \quad i + (n-1)l \leq rn - 1. \tag{2.4.A}$$

The following theorem can be thought of as the main result of this section of the thesis. It provides a complete answer to Question 2.4.

Theorem 2.24. *Let I be the ideal of $n+1$ almost collinear points and $m \geq r$ integers. Then $I^{(m)} \subseteq I^r$ if and only if $m > \dfrac{n^2 r - n}{n^2 - n + 1}$.*

Proof. Let P be the point (i, l) where the lines $i + nl = mn$ and $i + (n-1)l = rn - 1$ cross; i.e., $P = (mn - n^2(m-r) - n, n(m-r) + 1)$. Let Q be the point where the lines $l = i - 1$ and $i + nl = mn$ cross; i.e., $Q = (n(m+1)/(n+1), (nm - 1)/(n+1))$. Let U be the point where the lines $m = i$ and $i + nl = mn$ cross; i.e., $U = (m, m(n-1)/n)$. Then (2.4.A) has a solution if and only if the i-coordinate of P is at least as big as the maximum of the i-coordinates of Q and U. Let Q_i and U_i

be these i-coordinates; then $\max(Q_i, U_i) = Q_i$ if $m \leq n$, while $\max(Q_i, U_i) = U_i$ if $m \geq n$.

Thus, assuming $m \geq r$, (2.4.A) has a solution if and only if either $m \leq n$ and $Q_i \leq P_i$, or $m \geq n$ and $U_i \leq P_i$. But $Q_i \leq P_i$ is the same as $n(m+1)/(n+1) \leq mn - n^2(m-r) - n$ or $m \leq r(n+1)/n - (n+2)/n^2 = (rn^2 + rn - n - 2)/n^2$, and $U_i \leq P_i$ is the same as $m \leq mn - n^2(m-r) - n$ or $m \leq (n^2 r - n)/(n^2 - n + 1)$.

Thus, $I^{(m)} \not\subseteq I^r$ holds if and only if either $m < r$, or $m \geq r$ and either $m \leq n$ and $m \leq (rn^2 + rn - n - 2)/n^2$, or $m \geq n$ and $m \leq (n^2 r - n)/(n^2 - n + 1)$. Note, however, that if $1 \leq m < r$, then $r \geq 2$ and so $m \leq (rn^2 + rn - n - 2)/n^2$ holds (since $r \leq (rn^2 + rn - n - 2)/n^2$ if $r \geq 2$ and $n \geq 3$), and also $m \leq (n^2 r - n)/(n^2 - n + 1)$ (since $r \leq (rn^2 + rn - n - 2)/n^2$ if $r \geq 2$ and $n \geq 3$). Thus $m < r$ is subsumed by $m \leq n$ and $m \leq (rn^2 + rn - n - 2)/n^2$, or $m \geq n$ and $m \leq (n^2 r - n)/(n^2 - n + 1)$.

However, we can do even better by ridding ourselves of the need for the two cases $m < n$ and $m \geq n$.

Claim: We have $I^{(m)} \not\subseteq I^r$ if and only if $m \leq \frac{n^2 r - n}{n^2 - n + 1}$.

Proof of Claim. If $m < n$ and $m \leq (rn^2 + rn - n - 2)/n^2$, then routine arithmetic demonstrates $m \leq r$. Now, if $I^{(m)} \not\subseteq I^r$ then we already know that either $m < n$ and $m \leq \frac{rn^2 + rn - n - 2}{n^2}$ or $m \geq n$ and $m \leq \frac{rn^2 - n}{(n^2 - n + 1}$. If $m < n$ and $m \leq \frac{rn^2 + rn - n - 2}{n^2}$, then we now know that $m \leq r$, but $I^{(m)} \not\subset I^r$ implies $r > 1$, and, as we are assuming $n > 1$, it follows that $r \leq \frac{rn^2 - n}{n^2 - n + 1}$, and hence $m \leq (rn^2 - n)/(n^2 - n + 1)$. Conversely, assume $m \leq \frac{rn^2 - n}{n^2 - n + 1}$. If $m \geq n$, then we already know that $I^{(m)} \not\subseteq I^r$, so assume $m < n$. If $m < r$, then $I^{(m)} \not\subseteq I^r$, so we may also assume $r \leq m$. So either $m = r$ or $r + 1 \leq m \leq \frac{rn^2 - n}{n^2 - n + 1}$. If $r + 1 \leq m \leq \frac{rn^2 - n}{n^2 - n + 1}$, then routine arithmetic shows that $n < \frac{n^2 + 1}{n - 1} \leq r \leq m$, which contradicts $m < n$. Thus we must have $m = r < n$. But $m = r = 1$ is impossible since $m = r = 1$

implies $1 \leq \frac{n^2-n}{n^2-n+1}$, which is false, so we must have $1 < m = r < n$. But this implies $m < n$ and more arithmetic demonstrates that $m \leq \frac{rn^2+rn-n-2}{n^2}$ which we have already showed implies $I^{(m)} \not\subseteq I^r$. $\square$

2.5 Application to Conjectures for Ideals of Points

In addition to the questions considered in Section 2.4, we are able to use the basis approach to verify several conjectures of [HH, BCH11] for the case in which I defines $n+1$ almost collinear points. We take a short detour to discuss these conjectures and their origins.

2.5.1 History and Motivation for Conjectures for ideals of points

Recall that, given a nontrivial homogeneous ideal $J \subseteq k[\mathbf{P}^N]$, $\alpha(J)$ denotes the degree of a nonzero element of J of least degree. In the late 1970s, Waldschmidt and Skoda [Wal77, Sko77] used complex analytic techniques to show that, for an ideal I of a finite set of points in $\mathbf{P}^N$,

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)}{N}. \tag{2.5.A}$$

In fact, [HH] shows that this holds for any homogeneous ideal $I \subseteq k[\mathbf{P}^N]$ as an easy corollary of the fact that $I^{(Nm)} \subseteq I^m$ (see [ELS01, HH02] and Theorem 1.4) $(I^{(m)})^N \subseteq I^{(Nm)}$. These containments imply $N\alpha(I^{(m)}) \geq \alpha(I^{(Nm)}) \geq m\alpha(I)$, which gives (2.5.A). Thus, the results of Ein-Lazarsfeld-Smith and Hochster-Huneke lead easily to structural underpinnings for the work of Waldschmidt and Skoda. Later, Chudnovsky [Chu81] improved the bound (2.5.A) for ideals of finite sets of points in $\mathbf{P}^2$:

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)+1}{2}. \tag{2.5.B}$$

In observing that the bound (2.5.A) can be explained as a feature of the structure of symbolic powers, Harbourne and Huneke ask if a similar structure underlies the improved Chudnovsky bound (2.5.B), and make the following conjecture.

Conjecture 2.25 (Conjecture 2.1 of [HH]). Let $I = \cap_i I(p_i)^{m_i} \subseteq k[\mathbf{P}^N]$ be any fat points ideal. Then $I^{(rN)} \subseteq M^{r(N-1)} I^r$ holds for all $r > 0$.

This conjecture easily implies the result of Chudovsky when $N = 2$, and holds for ideals of general points in $\mathbf{P}^2$ (see [HH, Proposition 3.10]) and ideals of almost collinear points (see Proposition 2.35).

Harbourne and Huneke also investigate possible containments if a constant is subtracted off the symbolic exponent and make the following conjecture:

Conjecture 2.26 (Conjecture 4.1.5 of [HH]). Let $I \subseteq k[\mathbf{P}^N]$ be the ideal of a finite set of points $p_i \in \mathbf{P}^N$. Then $I^{(rN-(N-1))} \subseteq M^{(r-1)(N-1)} I^r$ holds for all $r \geq 1$, where M is the irrelevant maximal ideal.

Harbourne and Huneke verify Conjecture 2.26 for ideals $I \subseteq k[\mathbf{P}^N]$ of star configurations and complete intersections. We verify the conjecture for ideals of almost collinear points in Theorem 2.34.

However, Conjecture 2.26 turns out to be false in general (see [DST13]).

Harbourne and Huneke also give conditions on an ideal $I \subseteq k[\mathbf{P}^2]$ of points in $\mathbf{P}^2$ for there to be an m such that $I^{(mt)} = (I^{(m)})^t$ for all $t \geq 1$ (see [HH, Proposition 3.5]). Unfortunately, the conditions they require do not hold for an ideal defining $n+1$ almost collinear points in $\mathbf{P}^2$, but nonetheless we will prove that $I^{(nt)} =$

$(I^{(n)})^t$ for all $t \geq 1$ (see Proposition 2.32). A consequence of this equality is that the symbolic power algebra $\oplus_m I^{(m)}$ is Noetherian (see [HH]).

In this thesis, we are also able to affirmatively answer a few questions of Harbourne and Huneke, namely:

Question 2.27 (Question 4.2.3 in [HH])**.** Is it true for all positive integers m and t that $I^{(t(m+N-1))} \subseteq M^{t(N-1)}(I^{(m)})^t$?

(See Theorem 2.33 for a proof in the almost collinear case.) Moreover, an affirmative answer to Question 2.27 implies an affirmative answer to Conjecture 2.25, which we note in Corollary 2.35.

Further, an affirmative answer to Question 2.27 in turn implies a positive answer to the following question:

Question 2.28 (Question 4.2.1 in [HH])**.** Let I be the radical ideal for a finite set of points in $\mathbf{P}^N$. Is it true for all $m \geq 1$ that

$$\frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \leq \gamma(I)?$$

Harbourne and Huneke also ask two similar questions which, for ideals in $\mathbf{P}^2$, are identical to the two just mentioned, and so are omitted here.

More such questions were asked in [BCH11] (an inquisitive reader is referred to that paper, as it very helpfully organizes the various conjectures and their implications). One in particular that is considered in this thesis is their Conjecture 3.9, which implies Conjecture 2.26 in much the same way that Question 2.27 implies Conjecture 2.25.

Conjecture 2.29 (Conjecture 3.9 of [BCH11])**.** Let $I \subseteq k[\mathbf{P}^N]$ be the radical ideal of a finite set of points in $\mathbf{P}^N$. Then $I^{(t(m+N-1)-N+1)} \subseteq M^{(t-1)(N-1)}(I^{(m)})^t$.

This conjecture has been verified in the following cases: if $I^{(r)} = I^r$ for all $r \geq 1$; if $N = 2$ and I is the ideal of $n \geq 5$, n odd, points on a smooth conic with $\mathrm{char}(k) = 0$; and when I is the radical ideal of a finite set of $n = s^2$ points in $\mathbf{P}^2$, $s \geq 3$ (see [BCH11]). As a demonstration of the power and utility of the basis approach, we are able to use it to verify this conjecture (and, with $m = 1$, Conjecture 2.25, also) in Theorem 2.34 (and Corollary 2.36) for ideals of $n+1$ amost collinear points.

2.5.2 Conjectures and Applications for ideals of points in the almost collinear case

In this section we consider proofs of the conjectures and questions found in Section 2.5.1. We also demonstrate that the basis approach can be used to compute the saturation degree of I^m and the regularity of ordinary and symbolic powers, when I defines $n+1$ almost collinear points in $\mathbf{P}^2$. We will make some use of divisors on blow-ups of $\mathbf{P}^2$, so let us recall a few basic facts and definitions about divisors and intersection theory. Let $p_0, p_1, \dots, p_n \in \mathbf{P}^2$ be a set of $n+1$ almost collinear points, with $n \geq 3$. Let $\pi : X \to \mathbf{P}^2$ denote the blow-up of the $n+1$ almost collinear points, and let $E_i = \pi^{-1}(p_i)$ and L denote the class of a line. Then $L, E_0, E_1, \dots, E_n$ form a basis of the divisor class group $\mathrm{Cl}(X)$, whose intersection form is defined by $-L^2 = E_i^2 = -1$, and $E_i \cdot E_j = 0 = E_i \cdot L$, when $i \neq j$. Finally, recall that a divisor D is nef if $D \cdot C \geq 0$ for every effective divisor C.

First, we compute $\alpha(I^{(m)})$ when I defines $n+1$ almost collinear points.

Lemma 2.30. *Let $n \geq 3$ and $I = I(Z) \subseteq k[\mathbf{P}^2]$ be the ideal of forms vanishing on $n+1$ almost collinear points. Then $\alpha(I^{(m)}) = \lceil m(2n-1)/n \rceil$ and $\alpha(I^r) = 2r$.*

Proof. Suppose L_0 is the line containing the n collinear points $p_1, \cdots, p_n$, and suppose L_i is a line containing p_0 and p_i for each $1 \leq i \leq n$. Set $a = \lceil m(2n-1)/n \rceil$, and write $m = bn + r$, where $0 \leq r < n$ (thus, $b = \lfloor m/n \rfloor$). Then $a = \lceil 2m - (bn+r)/n \rceil = \lceil 2m - b - r/n \rceil = 2m - b$. Finally, $L_0^{m-b} L_1^{b+1} \cdots L_r^{b+1} L_{r+1}^{b} \cdots L_n^{b}$ has degree $2m - b$ and vanishes at each p_j to order at least m, which demonstrates that $\alpha(I^{(m)}) \leq \lceil 2m(n-1)/n \rceil$.

For a lower bound, we consider the blow-up $X \xrightarrow{\pi} \mathbf{P}^2$ on the $n+1$ points. Set $D = nL - (n-1)E_0 - E'$, where $E' = E_1 + E_2 + \cdots + E_n$ and $E = E_0 + E'$. Then D is nef, as it meets each $\tilde{L}_i = L - E_0 - E_i$ (in this case, $\tilde{L}_i$ is the proper transform of the L_i passing through p_0 and p_i) and E_0 nonnegatively, and $D = \tilde{L}_1 + \tilde{L}_2 + \cdots + \tilde{L}_n + E_0$. Thus, $D \cdot (\alpha L - mE) \geq 0$, where $\alpha = \alpha(I^{(m)})$, and since $D \cdot (\alpha L - mE) = n\alpha - m(n-1) - mn \geq 0$ we see $\alpha \geq (2n-1)m/n$, and thus $\alpha \geq \lceil m(2n-1)/n \rceil$, which proves $\alpha(I^{(m)}) = \lceil m(2n-1)/n \rceil$.

As for the ordinary power, $\alpha(I) = 2$ immediately gives $\alpha(I^r) = 2r$. $\square$

This computation of $\alpha(I^{(m)})$ allows us to easily compute Waldschmidt's constant, $\gamma(I)$. Recall that $\gamma(I) = \lim_{m\to\infty} \dfrac{\alpha(I^{(m)})}{m}$. By [Wal77], $\gamma(I)$ always exists, so we may use a convenient subsequence to compute the limit.

Proposition 2.31. *For I, the ideal of $n+1$ almost collinear points, $\gamma(I) = 2 - \dfrac{1}{n}$.*

Proof. Set $\alpha_j = \alpha(I^{(j)})$. Then $\gamma(I) = \lim_{m\to\infty} \dfrac{\alpha_m}{m} = \lim_{t\to\infty} \dfrac{\alpha(I^{(tn)})}{tn} = \lim_{t\to\infty} \dfrac{\lceil \frac{tn(2n-1)}{n} \rceil}{tn} = \lim_{t\to\infty} \dfrac{t(2n-1)}{tn} = 2 - \dfrac{1}{n}$. $\square$

Recall that $I(p_0) = (x, y)$. Therefore, if P is a degree d form in x and y only, P vanishes to order d at p_0, as $I(p_0)^{(d)} = I(p_0)^d = (x, y)^d$, since a single point is a complete intersection.

Proposition 2.32. *Given I as above, $I^{(nt)} = (I^{(n)})^t$ for all $t \geq 1$. Moreover, we have $n = \min\left\{e \,:\, I^{(et)} = (I^{(e)})^t \ \forall t \geq 1\right\}$.*

Proof. To prove the first statement, we use induction on t. When $t = 1$, the statement is clear. For $t > 1$, write $(I^{(n)})^t = I^{(n)}(I^{(n)})^{t-1} = I^{(n)}I^{(n(t-1))}$. Thus we need only show $I^{(n)}I^{(n(t-1))} = I^{(nt)}$. The forward containment is obvious, so consider the reverse.

Consider $H_i y^j z^l \in I^{(nt)}$. By Lemma 2.19, $i + ln \geq n^2 t$ and $i + j \geq nt$, where the first inequality is equivalent to $\lfloor i/n \rfloor + l \geq nt$. Assume $\lfloor i/n \rfloor \geq 1$ and $l \geq n-1$. Then $H_i y^j z^l = x^e F^{\lfloor i/n \rfloor} z^l = (Fz^{n-1})\left(x^e F^{\lfloor i/n \rfloor - 1} z^{l-n+1}\right)$. Notice that $Fz^{n-1} \in I^{(n)}$; we claim $x^e F^{\lfloor i/n \rfloor - 1} z^{l-n+1} = H_{i-n} y^j z^{l-n+1} \in I^{(n(t-1))}$. We have by hypothesis that $i + j \geq nt$ and $i + ln \geq n^2 t$. The latter inequality is equivalent to $(i-n) + (l - (n-1))n \geq (t-1)n^2$, and subtracting n from both sides of the former shows that $(i-n) + j \geq n(t-1)$; this proves the claim in the case of $\lfloor i/n \rfloor \geq 1$ and $l \geq n-1$.

Suppose now that $\lfloor i/n \rfloor = 0$. This means that $0 \leq i < n$, and thus $H_i = x^i$, so $H_i y^j z^l = x^i y^j z^l$. We have by hypothesis that $i + j \geq nt$ and $l = \lfloor i/n \rfloor + l \geq nt$. Therefore, we can factor $x^i y^j z^l = (yz)^n (x^i y^{j-n} z^{l-n})$, where $(yz)^n \in I^{(n)}$ and $x^i y^{j-n} z^{l-n} \in I^{(n(t-1))}$.

Finally, suppose $l \leq n-2$. Write $l = n - 2 - \delta$, where $0 \leq \delta \leq n-2$. We know $i + ln \geq n^2 t$, so $i \geq n^2 t - ln = n^2 t - (n - 2 - \delta)n = n^2(t-1) + (\delta+2)n$, and thus $b = \lfloor i/n \rfloor \geq n(t-1) + \delta + 2$. Set $\varepsilon = b - n(t-1) - \delta - 2$. Using these constraints on i, j, l, we can write $H_i y^j z^l = x^e F^b y^j z^l = (x^e y^j F^{n(t-1)}) F^{\varepsilon} F^{\delta+2} z^l$. Since F vanishes at each point, $F^{n(t-1)}$ vanishes to order $n(t-1)$ at each point, and thus $x^e y^j F^{n(t-1)} \in I^{(n(t-1))}$. As $l = n - 2 - \delta$, we see $l + \delta + 2 = n$; therefore, $F^{\delta+2} z^l \in (z, F)^n$. Additionally, $\delta + 2 \geq 1$, so $F \in (x, y)^n$ and thus $F^{\delta+2} z^l \in (x, y)^n \cap (z, F)^n = I^{(n)}$. Therefore, when $H_i y^j z^l \in I^{(nt)}$ and $l \leq n-2$, we conclude $H_i y^j z^l \in$

$I^{(n(t-1))}I^{(n)}$, which completes the proof of the first statement.

To see the second statement, recall the computation of $\alpha(I^{(m)})$ from Lemma 2.30, and assume $e < n$. We know that $\alpha(I^{(et)}) = \lceil et(2n-1)/n \rceil = \lceil t(2e - e/n) \rceil$ and $\alpha((I^{(e)})^t) = t\lceil e(2n-1)/n \rceil = t\lceil 2e - e/n \rceil = 2et$, so that when $t \geq n/e$ we have $\alpha(I^{(et)}) < \alpha((I^{(e)})^t)$. Thus, the ideals cannot be equal for all $t \geq 1$. □

As a result of Proposition 2.32, we can conclude that the symbolic power algebra $\oplus I^{(m)}$ is Noetherian. This is a homogeneous version of Theorem 1.3 in [Sch88]. We now verify Question 2.27 (and thus Conjecture 2.25) for $n+1$ almost collinear points in $\mathbf{P}^2$.

Theorem 2.33. *If I is the ideal of $n+1$ almost collinear in $\mathbf{P}^2$ and $M = (x, y, z) \subseteq k[\mathbf{P}^2]$, then $I^{(t(m+1))} \subseteq M^t(I^{(m)})^t$ for all $t, m \geq 1$.*

Proof. We wish to factor a basis element $H_i y^j z^l \in I^{(t(m+1))}$ into a product of a form of degree t and a product of t forms, each of which vanishes to order m on the set of $n+1$ points. Then

$$i + nl \geq nt(m+1) \tag{2.5.C}$$

and

$$i + j \geq t(m+1). \tag{2.5.D}$$

If $l = 0$, then (2.5.C) becomes $i \geq nt(m+1)$, which means H_i has a factor of F^{tm+t}. It is clear that $F^t \in M^t$, as $\deg F^t = nt \geq t$, and $F^m \in I^{(m)}$, as F vanishes at each of the $n+1$ points. Thus, $H_i y^j z^l \in M^t(I^{(m)})^t$.

Now assume $l \geq 1$.

If $1 \leq l < t$, then $l + \xi = t$ for some $\xi \geq 1$. We see that (2.5.C) becomes $i + n(t-\xi) \geq nt(m+1)$, and thus $i \geq n(tm+\xi)$. Then $F^{tm+\xi}$ is a factor of H_i;

as before, $F^{tm} \in (I^{(m)})^t$, and, as $l+\xi = t$, $\deg z^l F^\xi = l + n\xi \geq l + \xi = t$, so $z^l F^\xi \in M^t$, whence $H_i y^j z^l \in M^t(I^{(m)})^t$.

If $l \geq mt$, write $l = mt + \xi$. Recall that $F = L_1 L_2 \cdots L_n$, where L_i is a linear form vanishing at p_0 and p_i, for $1 \leq i \leq n$. Set $i = nb + e$, where $0 \leq e < n$. Thus, we can factor $H_i y^j z^l = x^e F^b y^j z^l = x^e L_1^b L_2^b \cdots L_n^b y^j z^{mt} z^{l-mt}$. If $j \geq t$, then $y^t \in M^t$, so write $H_i y^j z^l = y^t(x^e L_1^b L_2^b \cdots L_n^b y^{j-t} z^{mt} z^{l-mt})$. As z^m vanishes to order m at the n collinear points, we may group the linear factors of $H_i y^{j-t}$ as $G_1 G_2 \cdots G_t G$, where $\deg G_d = m$ and $\deg G = i + j - t - mt \geq 0$; then $G_d z^m \in I^{(m)}$ for every d, $1 \leq d \leq t$, so $H_i y^j z^l \in M^t(I^{(m)})^t$. If, on the other hand, $j < t$, set $\delta = t - j$. Then (2.5.D) becomes $i \geq tm + \delta$, so we again factor $H_i = G_1 G_2 \cdots G_t G$, where $\deg G_d = m$, and $\deg G = i - tm \geq \delta$. Since $\delta + j = t$, $G y^j \in M^t$, and, as G_d vanishes to order m at p_0, $G_d z^m \in I^{(m)}$ for every d, $1 \leq d \leq t$. Thus, $H_i y^j z^l \in M^t(I^{(m)})^t$.

Finally, suppose l satisfies $st \leq l < (s+1)t$, where $1 \leq s \leq m-1$, and write $l = (s+1)t - \xi$, $1 \leq \xi \leq t$. Then (2.5.C) becomes $i \geq n(t(m-s)+\xi)$, so H_i has a factor of $F^{t(m-s)+\xi}$. Notice that $F^{m-s}z^s$ vanishes to order m at each of the n collinear points. We consider two cases: $j \geq t$ and $j < t$.

Case 1: Assume $j \geq t$. Then it is obvious that $y^t \in M^t$. If $n(m-s) \geq m$, then F^{m-s} vanishes to order m at p_0, and $F^{m-s}z^s \in I^{(m)}$,which proves that $H_i y^j z^l \in M^t(I^{(m)})^t$ if $n(m-s) \geq m$.

Suppose now that $n(m-s) < m$. Set $\delta = j - t$; then $\delta \geq 0$. Since $i + j \geq t(m+1)$, we know $i + t + \delta \geq t(m+1)$, whence $i + \delta \geq tm$. This means that we can factor $H_i y^\delta$ into a product of t factors, each vanishing at p_0 to order m; say $H_i y^\delta = G_1 G_2 \cdots G_t \cdot G$, where $\deg G_d = m$ for $1 \leq d \leq t$, and $\deg G = i + \delta - mt$. Our aim is to do this in such a way so that each G_d, when multiplied by a particular power of z, will vanish to order m at each of the $n+1$ points. Note that $i \geq n(t(m-s)+\xi)$. Then $F^{t(m-s)+\xi}$ divides H_i, and so H_i has t factors of F^{m-s}. Define

$G_1 = F^{m-s}Q_1$, where Q_1 is a product of $m - n(m-s)$ linear factors of $H_i y^\delta / F^{t(m-s)}$. Now recursively define, for each d satisfying $2 \le d \le t$, $G_d = F^{m-s}Q_d$, where Q_d is a product of $m - n(m-s)$ linear factors of $\frac{H_i y^\delta}{F^{t(m-s)}Q_1Q_2\cdots Q_{d-1}}$ (note that this is possible, as $F^{t(m-s)}|H_i$, $i + \delta \ge tm$, each G_d has degree m, and the factors Q_d are distinct and chosen so that $Q_d|H_i y^\delta$). Then G_d vanishes to order m at p_0 by construction. Notice that $G_d z^s \in I^{(m)}$ as $\deg G_d = m$ (and thus vanishes to order m at p_0), and $F^{m-s}z^s$ vanishes to order m at $p_1, p_2, \ldots, p_n$. Therefore, $H_i y^j z^l = z^t G z^{t-\xi} \prod_{d=1}^{t} G_d z^s \in M^t (I^{(m)})^t$.

Case 2: Now suppose $j < t$, and set $\delta = t - j$. Recall that $l = (s+1)t - \xi$, $1 \le \xi \le t$ and $1 \le s \le m-1$ and (2.5.C) becomes $i \ge n(t(m-s) + \xi)$, so H_i has a factor of $F^{t(m-s)+\xi}$. It is clear that $F^{m-s}z^s$ vanishes to order m at $p_1, p_2, \ldots, p_n$, as both F and z vanish once at each of the n points. Moreover, if $n(m-s) \ge m$, F^{m-s} vanishes to order m at p_0, so $F^{t(m-s)}z^{st} \in (I^{(m)})^t$. Since $F^\xi z^{l-st} = F^\xi z^{t-\xi}$ has degree $n\xi + t - \xi \ge t$ (since $n \ge 3$), $F^\xi z^{t-\xi} \in M^t$, and thus $H_i y^j z^l \in M^t (I^{(m)})^t$ if $n(m-s) \ge m$.

Suppose instead that $n(m-s) < m$ and recall that (2.5.C) becomes $i \ge n(t(m-s) + \xi)$ and (2.5.D) becomes $i \ge mt + \delta$. Define $A_1 = F^{m-s}B_1$, where B_1 is a product of $m - n(m-s)$ linear factors of $H_i/F^{t(m-s)}$. Recursively, for d satisfying $1 < d \le t$, define $A_d = F^{m-s}B_d$, where B_d is a product of $m - n(m-s)$ linear factors of $\dfrac{H_i}{F^{t(m-s)}B_1B_2\cdots B_{d-1}}$. (Note that we can do this, as $\deg H_i = i$ and $\deg A_1A_2\cdots A_t = mt$; since $F^{t(m-s)}|H_i$ and $F^{t(m-s)}|A_1\cdots A_t$, and the other factors of $A_1\cdots A_t$ are linear factors of H_i [enough linear factors exist, since $i \ge mt$], it follows that $A_1\cdots A_t|H_i$.) Then A_d is a form in x and y only of degree $n(m-s) + m - n(m-s) = m$, so A_d vanishes to order m at p_0. Moreover, $A_d z^s$ vanishes to order m at each of the $n+1$ points, so $A_d z^s \in I^{(m)}$. Let A be the form satisfying $H_i = (A_1A_2\cdots A_t)A$; then $\deg A = i - mt \ge \delta$, so $\deg Ay^j \ge \delta + j = t$, so

$Ay^j \in M^t$. Thus, $Ay^j \in M^t$ and $A_1 A_2 \cdots A_t z^l \in (I^{(m)})^t$, so $H_i y^j z^l \in M^t (I^{(m)})^t$. $\square$

We now use a similar method to verify Conjecture 2.29 for $n+1$ almost collinear points in $\mathbf{P}^2$.

Theorem 2.34. *Let I be the ideal of $n+1$ almost collinear points and $M = (x, y, z)$ the irrelevant maximal ideal; recall that $n \geq 3$. Then $I^{(t(m+1)-1)} \subseteq M^{t-1}(I^{(m)})^t$ for all $m \geq 1$ and for all $t \geq 1$.*

Proof. Consider $H_i y^j z^l \in I^{(t(m+1)-1)}$. Then i, j, l satisfy

$$i + nl \geq n(t(m+1) - 1) \tag{2.5.E}$$

and

$$i + j \geq t(m+1) - 1. \tag{2.5.F}$$

Suppose first that $l = 0$. Then (2.5.E) becomes $i \geq n(t(m+1) - 1)$, so H_i has a factor of $F^{t(m+1)-1} = F^{t-1}F^{mt} \in M^{t-1}(I^{(m)})^t$.

Now suppose that $1 \leq l < t$. Then there exists $\xi > 0$ such that $l + \xi = t$. Then (2.5.E) becomes $i + n(t - \xi) \geq n(t(m+1) - 1)$, so $i \geq n(tm + \xi - 1)$. Thus, H_i has a factor of $F^{tm+\xi-1}$; since $\xi \geq 1$ and $F^{\xi-1}z^l = F^{\xi-1}z^{t-\xi}$ has degree at least $\xi - 1 + t - \xi = t - 1$, we have $F^{\xi-1}z^l \in M^{t-1}$. Also, it is clear that $F^{tm} \in (I^{(m)})^t$, whence $H_i y^j z^l \in M^{t-1}(I^{(m)})^{t-1}$.

Now suppose $l \geq mt$. Recall that $F = L_1 L_2 \cdots L_n$, where L_i is a linear form vanishing at p_0 and p_i for $1 \leq i \leq n$. Set $i = nb + e$, $0 \leq e < n$. Factor $H_i y^j z^l = x^e F^b y^j z^l = x^e L_1^b L_2^b \cdots L_n^b z^{mt} z^{l-mt}$. If $j \geq t-1$, then $y^j \in M^t$, so we write $H_i y^j z^l = y^{t-1}(x^e L_1^b \cdots L_n^b y^{j-t+1} z^{mt} z^{l-mt})$ and consider the factors in parentheses. Notice that $i + j - t + 1 \geq tm + t - 1 - t + 1 = tm$, so we may factor $H_i y^{j-t+1} = G_1 \cdots G_t G$, where $\deg G_d = m$ for d satisfying $1 \leq d \leq t$ and

$\deg G = i + j - t + 1 - mt \geq 0$. Then G_d vanishes to order m at p_0 and thus $G_d z^m \in I^{(m)}$, whence $H_i y^j z^l \in M^{t-1}(I^{(m)})^t$.

On the other hand, if $0 \leq j < t-1$, set $\delta = t - 1 + j$; then (2.5.F) becomes $i \geq tm + \delta$. Factor H_i as $G_1 G_2 \cdots G_t G$, where $\deg G_d = m$ and $\deg G = i - tm \geq \delta$. Then $\deg y^j G = j + \delta = t - 1$, so $y^j G \in M^{t-1}$. For the same reasons as before, $G_d z^m \in I^{(m)}$ for each d, and thus $H_i y^j z^l \in M^{t-1}(I^{(m)})^t$.

Now assume $st \leq l < (s+1)t$, where $1 \leq s \leq m-1$ and write $l = (s+1)t - \xi$, where $1 \leq \xi \leq t$. Then (2.5.E) becomes

$$i \geq n(t(m-s) + \xi - 1) \tag{2.5.G}$$

This means that H_i has a factor of the form $F^{t(m-s)+\xi-1}$, which we will use repeatedly throughout the remainder of the proof. Moreover, $z^l = z^{st} z^{t-\xi}$, and note $t - \xi \geq 0$ by definition. Next, a fact.

Fact: $F^{m-s} z^s$ vanishes to order m at $p_1, p_2, \ldots, p_n$ (the n collinear points).

The reason for this is that z^s vanishes to order s at the n collinear points, and, as F vanishes once at each of the n points, F^{m-s} vanishes to order $m - s$ at the points; thus the product $F^{m-s} z^s$ vanishes to order $m - s + s = m$ at each of the n points.

Case 1: Assume $j \geq t - 1$. As $y^{t-1} \in M^{t-1}$, it remains to be seen that $H_i y^{j-t+1} z^l \in (I^{(m)})^t$. If $n(m-s) \geq m$, then F^{m-s} vanishes to order m at p_0, so $F^{m-s} z^s \in I^{(m)}$ by the Fact. Thus, $H_i y^j z^l \in M^{t-1}(I^{(m)})^t$.

On the other hand, if $n(m-s) < m$, set $\delta = j - t + 1$ and note $\delta \geq 0$. Then (2.5.F) becomes $i + \delta \geq tm$. By (2.5.G), H_i has a factor of $F^{t(m-s)+\xi-1}$. Since $\xi \geq 1$, H_i has t factors of F^{m-s}. Define $G_1 = F^{m-s} E_1$, where E_1 is a collection of $m - n(m-s) > 0$ linear factors of $\dfrac{H_i y^\delta}{F^{t(m-s)}}$. Now, for each d satisfying $1 < d \leq t$, recursively define $G_d = F^{m-s} E_d$, where E_d is a product of $m - n(m-s) > 0$ linear factors of $\frac{H_i y^\delta}{F^{t(m-s)}E_1 \cdots E_{d-1}}$. Note that this is possible, as $\deg F^{t(m-s)}E_1E_2\cdots E_t = nt(m-s) + t(m-n(m-s)) = tm \leq i+\delta$ and the E_d's are chosen to be factors of $H_i y^\delta$. Then $G_d z^s = F^{m-s}E_d z^s \in I^{(m)}$, so $H_i y^\delta z^l \in (I^{(m)})^t$.

Case 2: Assume $j < t-1$, and set $\delta = t-1-j$. Recall that $l = (s+1)t - \xi$, where $1 \leq \xi \leq t$ and $1 \leq s \leq m-1$. Then (2.5.G) implies H_i has a factor of the form $F^{t(m-s)+\xi-1}$. By the Fact above, $F^{m-s}z^s$ vanishes to order m at each of the collinear points. If $\deg F^{m-s} = n(m-s) \geq m$, then $F^{m-s}z^s \in I^{(m)}$, so $F^{t(m-s)}z^{ts} \in (I^{(m)})^t$. Multiplying the remaining powers of F and z together yields $F^{\xi-1}z^{t-1}$, which as degree at least $t-1$ and hence $F^{\xi-1}z^{t-1} \in M^{t-1}$. Therefore, $H_i y^j z^l \in M^{t-1}(I^{(m)})^t$.

If, on the other hand, $n(m-s) < m$, then (2.5.F) becomes $i \geq mt+\delta$. Again, H_i has a factor of $F^{t(m-s)+\xi-1}$, where $\xi \geq 1$. Define $A_1 = F^{m-s}B_1$, where B_1 is a product of $m - n(m-s) > 0$ linear factors of $\frac{H_i}{F^{t(m-s)}}$. For d satisfying $1 < d \leq t$, define $A_d = F^{m-s}B_d$, where B_d is a product of $m-n(m-s) > 0$ linear factors of $\frac{H_i}{F^{t(m-s)}B_1B_2\cdots B_{d-1}}$. Choose B so that $H_i = (F^{t(m-s)}B_1B_2\cdots B_t)B$. Note that $\deg A_d = m$ for each d satisfying $1 \leq d \leq m$. As before, this is all possible by the way the B_d's are chosen, and the fact that $\deg H_i = i \geq mt+\delta = nt(m-s) + t(m-n(m-s)) + \delta = \deg(F^{t(m-s)}A_1A_2\cdots A_t) + \delta$. As $\deg A_d = m$ and $F^{m-s}|A_d$, $A_d z^s \in I^{(m)}$ for each d; moreover, $\deg B \geq \delta$, so $\deg By^j \geq \delta + j = t-1$, whence $By^j \in M^{t-1}$. Thus, $H_i y^j z^l \in M^{t-1}(I^{(m)})^t$, which completes the proof. □

We now turn to proving Conjecture 2.25.

Corollary 2.35. *For the ideal of $n+1$ almost collinear points, $I^{(2r)} \subseteq M^r I^r$, where $M = (x,y,z)$ is the irrelevant maximal ideal.*

Proof. Apply Theorem 2.33 with $m = 1$. □

The next proposition provides an affirmative answer to Conjecture 2.26.

Corollary 2.36. *For the ideal of $n+1$ almost collinear points, $I^{(2r-1)} \subseteq M^{r-1}I^r$, where $M = (x, y, z)$ is the irrelevant maximal ideal.*

Proof. Apply Theorem 2.34 with $m = 1$. □

Corollary 2.37. *Given I as defined above, $M = (x, y, z)$ the irrelevant maximal ideal, and $m \geq 1$, $I^{(m)} \subseteq M^{\lfloor m/2 \rfloor} I^{\lceil m/2 \rceil}$.*

Proof. If m is even, see Corollary 2.35; if m is odd, see Corollary 2.36. □

Corollary 2.38. *Given I and M as above, $I^{(nt)} \subseteq \left(M^{\lfloor n/2 \rfloor} I^{\lceil n/2 \rceil}\right)^t$ for all $t \geq 1$.*

Proof. Apply Proposition 2.32 and Corollary 2.37. □

Additionally, the basis approach can be used to compute the saturation degree and regularity of the ideal I^m. Recall that, if I is an ideal of points, the saturation degree of I^m is the least degree t so that $(I^{(m)})_d = (I^m)_d$ for all $d \geq t$.

Theorem 2.39. *Given the ideal I of $n+1$ almost collinear points, satdeg$(I^m) = n(m-1)+2$.*

Proof. Recall that, when I is an ideal of points, $\mathrm{sat}(I^m) = I^{(m)}$. Additionally, $(I^m)_t \subseteq (I^{(m)})_t$ for every $t \geq 1$, as $I^m \subseteq I^{(m)}$. Thus, it's enough to show that $(I^{(m)})_t \subseteq (I^m)_t$ for every degree $t \geq n(m-1)+2$ and that this is the least such degree for which containment holds.

We show that $\mathrm{satdeg}(I^m) \leq n(m-1)+2$ in two steps: first we see that $(I^{(m)})_t \subseteq (I^m)_t$ if (1) $t \geq mn$ and then (2) if $n(m-1)+2 \leq t \leq nm-1$. Let $t \geq mn$ and $H_i y^j z^l \in (I^{(m)})_t$. Then $i+j+l = t \geq mn$, $i+j \geq m$, and $i+ln \geq mn$. It's enough to see that $i+j+(n-1)l \geq mn$; this is clear, though, as $i+j+(n-1)l =$

$i + j + l + (n-2)l \geq mn + (n-2)l \geq mn$, so $H_i y^j z^l \in (I^m)_t$. This shows that $\mathrm{satdeg}(I^m) \leq nm$.

Now suppose $t = n(m-1) + 2 + \delta$, where $0 \leq \delta \leq n-3$, i.e., $n(m-1)+2 \leq t \leq nm-1$. Suppose $H_i y^j z^l \in (I^{(m)})_t$, so that $i + j + l = t$. As before, we wish to show that $i + j + (n-1)l \geq nm$. We see that $i + j + (n-1)l = i + j + l + (n-2)l \geq n(m-1) + 2 + (n-2)l + \delta = nm - (n-2) + (n-2)l + \delta$. It's enough to see that $(n-2)l + \delta \geq n-2$. Notice, though, that we must have $l \geq 1$, for if $l = 0$, $mn > t \geq i = i + ln \geq mn$, with the last inequality coming from the fact that $H_i y^j z^l \in (I^{(m)})_t$. This is an obvious contradiction, so $l \geq 1$, and thus $(n-2)l + \delta \geq n-2$. Therefore, $\mathrm{satdeg}(I^m) \leq n(m-1)+2$.

Assume now that $t = n(m-1)+1$. Notice that $H_{n(m-1)}z = zF^{m-1} \in (I^{(m)})_t$, as $i + j = n(m-1) \geq m$ and $i + ln = n(m-1) + n \geq mn$; however, $i + j + (n-1)l = n(m-1) + 0 + n - 1 = nm - n + n - 1 = nm - 1 < mn$ (and $0 = j \leq l = 1 < i + j = n(m-1)$), so $zF \notin (I^m)_t$. Thus, $\mathrm{satdeg}(I^m) > n(m-1)+1$, and we conclude $\mathrm{satdeg}(I^m) = n(m-1)+2$. $\square$

Recall now that, since I cuts out a 0-dimensional subscheme Z, $\mathrm{reg}(I)$ is the least degree t such that the Hilbert function of I in degree d, which we denote $H(R/I,d)$, equals the Hilbert polynomial of I in degree d, denoted $\mathrm{HP}(R/I,d)$, for every $d \geq t-1$. Moreover, since $H(R/I,d) - \mathrm{HP}(R/I,d) = h^1(\mathbf{P}^N, \mathcal{I}(Z)(d))$ for all d, where $\mathcal{I}$ is the sheafification of I, we see that $\mathrm{reg}(I)$ is the least t such that $h^1(\mathbf{P}^N, \mathcal{I}(Z)(t-1)) = 0$.

By a result of [GGP95], $\mathrm{reg}(I^m) = \max\left\{\mathrm{satdeg}(I^m), \mathrm{reg}(I^{(m)})\right\}$. Thus, it's enough to compute $\mathrm{reg}(I^{(m)})$.

Theorem 2.40. *Given the ideal I of $n+1$ almost collinear points, $\mathrm{reg}(I^m) = \mathrm{reg}(I^{(m)}) = mn$.*

Proof. Let $\pi : X \to \mathbf{P}^2$ denote the blow-up of the 0-dimensional subscheme $Z = p_0 + p_1 + \cdots + p_n$, and let $E_i = \pi^{-1}(p_i)$. Furthermore, let L denote the class of a line, with $-L^2 = E_i^2 = -1$, and $E_i \cdot E_j = 0 = E_i \cdot L$, when $i \neq j$.

Define a divisor $F_t = tL - mE_0 - m(E_1 + \cdots + E_n)$, and let $\delta = \min\{t : F_t \text{ is nef}\}$. By [Har86, Theorem I.6.3], $\delta \leq \operatorname{reg}(I^{(m)}) \leq \delta + 1$. We claim that $\delta = mn$.

If we intersect F_t with the class of the line through the n points, $L - E_1 - E_2 - \cdots - E_n$, and we get $F_t \cdot (L - E_1 - \cdots - E_n) = t - mn$; for F_t to be nef, we need $t - mn \geq 0$, so $t \geq mn$. Thus, $\delta \geq mn$. Now, it is a fact that, if G is nef, any multiple of G will be nef, so it's enough to show that $F = nL - E_0 - E_1 - \cdots - E_n$ is nef, and then mF will be, also. Additionally, we use the fact that if $C = C_1 + \cdots + C_r$, where each C_i is irreducible, then C is nef if and only if $C \cdot C_i \geq 0$ for all i.

So, write $F = (L - E_1 - \cdots - E_n) + (L - E_0) + (n-2)L$; as F meets these divisors nonnegatively, F is nef, and thus $mF = mnL - mE_0 - m(E_1 + \cdots + En)$.

Therefore, $\delta = mn$, so $mn \leq \operatorname{reg}(I^{(m)}) \leq mn + 1$.

Now, let $Z = p_0 + p_1 + \cdots + p_n$, whence $I^{(m)} = I(mZ)$, and recall

$$\operatorname{reg}(I^{(m)}) = \min\left\{t : h^1(\mathbf{P}^2, \mathcal{I}(mZ)(t-1)) = 0\right\}.$$

By [Har86, Proposition I.6.3], $\delta \leq \operatorname{reg}(I^{(m)}) \leq \delta + 1$. By [Har86, Theorem I.4.1], $h^1(\mathbf{P}^2, \mathcal{I}(Z)(\delta)) = 0$, so to decide if $\operatorname{reg}(I^{(m)})$ is δ or $\delta + 1$ we need to check $h^1(\mathbf{P}^2, \mathcal{I}(Z)(\delta - 1))$; if it's 0, then $\operatorname{reg}(I^{(m)}) = \delta$, and if not, $\operatorname{reg}(I^{(m)} = \delta + 1$.

Again, we work on the blow-up, and use the fact $h^1(X, \mathcal{O}_X(tL - m(E_0 + \cdots + E_n))) = h^1(\mathbf{P}^2, \mathcal{I}(mZ)(t))$. Let Λ be the proper transform of the line through $p_1, \ldots, p_n$. Thus, $\mathcal{O}_X(\Lambda) = \mathcal{O}_X(L - E_1 - \cdots - E_n)$. We get a short exact sequence, where $A = (mn-2)L - (m-1)(E_1 + \cdots + E_n) - mE_0$, $B = (mn-1)L - m(E_0 +$

$\cdots + E_n)$, and $C = -1$:

$$0 \to \mathcal{O}_X(A) \to \mathcal{O}_X(B) \to \mathcal{O}_\Lambda(C) \to 0.$$

We get a long exact sequence of cohomology, and get the following table of dimensions:

Table 2.2: Cohomology Dimensions

	$\mathcal{O}_X(A)$	$\mathcal{O}_X(B)$	$\mathcal{O}_\Lambda(C)$
h^0	–	–	0
h^1	$*$	$**$	0
h^2	–	–	–
$\vdots$			

We will show $**$ is 0 by showing that $*$ is, which we do by showing that A is effective and nef. It is clear that A is effective, as $A = (m-1)\Lambda + m(L - E_0) + ((mn-2) - (2m-1))L$. Since A is effective and can be written as $A = (m-1)\Lambda + D$, where D is nef, A is nef if $A \cdot \Lambda \geq 0$, . We see that $A \cdot \Lambda = m + (m(n-2) - 1) + (m-1)(1-n) = n - 2 \geq 1$ when $n \geq 3$, as it is in our case, whence A is effective and nef. Thus, $h^1(X, \mathcal{O}_X(A)(\delta - 1)) = 0$, and therefore $\operatorname{reg}(I^m) = \max\left\{\operatorname{satdeg}(I^m), \operatorname{reg}(I^{(m)})\right\} = \max\{n(m-1)+2, mn\} = mn$.

As $mn > n(m-1) + 2$ for $n \geq 3$, the result follows. $\square$

Chapter 3

Fattening of Subschemes of $\mathbf{P}^N$

In this section, we follow the lead of [BC11] and show how differences in the invariant α can be used to classify certain classes of subschemes of $\mathbf{P}^3$. Specifically, we will seek to classify arithmetically Cohen-Macaulay codimension 2 subschemes of $\mathbf{P}^3$ in the manner Bocci and Chiantini classified points in $\mathbf{P}^2$ (the reasons for our additional assumptions will become clear later). The first section of this chapter will seek to motivate our consideration of the invariant α by relating it to the Hilbert function and γ, following the work of [BC11, DST12]. The second section will contain our results classifying arithmetically Cohen-Macaulay codimension 2 subschemes of $\mathbf{P}^3$.

3.1 The Importance of α

Much is known about finite sets of reduced points $Z \subseteq \mathbf{P}^2$. In particular, [GMR83] classified all possible Hilbert functions of finite sets of reduced points (their classification is more general, and concerns points in projective space of any dimension N). However, not much is known about the double scheme, $2Z$. Recall:

Definition 3.1. Let $Z \subseteq \mathbf{P}^N$ be a reduced subscheme defined by $I = I(Z)$. The double scheme (often called the double point scheme if Z is a set of reduced points, or the fattening) is the subscheme of $\mathbf{P}^N$ defined by $I^{(2)}$.

In fact, knowing the Hilbert function of Z does not determine the Hilbert function of $2Z$. In [GMS06], possible Hilbert functions of $2Z$ are given for a fixed Hilbert function of Z. Additionally, in [GHM12], Hilbert functions of the double scheme $2Z$ are given if $Z \subseteq \mathbf{P}^2$ is a finite set of 8 or fewer points or a set of points on a conic. Now, it is not difficult to see that the number α is the degree in which the Hilbert function of the ideal first deviates from that of the ring. Indeed, recall that the Hilbert function of a homogeneous ideal $I \subseteq R = k[\mathbf{P}^N]$ in degree t is $H(R/I, t) = \dim_k(R_t) - \dim_k(I_t)$. If $t < \alpha(I)$, $\dim_k(I_t) = 0$, hence $H(R/I, t) = \dim_k(R_t) = \binom{t+N}{N}$.

Thus, Bocci and Chiantini, rather than compute α (or even Hilbert functions) of various planar point configurations in $\mathbf{P}^2$ (or their symbolic powers), chose to study the difference $t := \alpha(2Z) - \alpha(Z)$. This path was chosen as a result of their further desire to understand the Waldschmidt constant

$$\gamma(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

Understanding $\alpha(I^{(m)})$ for all $m \geq 1$ is a difficult task, while computing the difference t is a more tractable task (though, as we will see, it grows difficult quickly). Recall as well, from Remark 2.7, that Waldschmidt's constant, $\gamma(I)$, together with $\alpha(I)$ and the regularity of I, bounds the resurgence $\rho(I)$: $\alpha(I)/\gamma(I) \leq \rho(I) \leq \operatorname{reg}(I)/\gamma(I)$. Thus, the difference t is related to both γ and ρ.

An important first observation about t is that $t \geq 1$ always; indeed, if k has characteristic 0, let F be a form of minimal degree $\alpha(2Z)$ vanishing to order at least

2 at each point of Z. Then the partial derivatives of F vanish on Z, and the degree of the partial derivatives is less than the degree of F. If k has characteristic $p > 0$, then it may happen that every partial derivative of F is identically 0. In that case, F is the pth power of some form G, which vanishes at each point of Z, and thus $t \geq 1$.

We follow the lead of [BC11] and say that a subscheme Z has type $(d-t, d)$ if $\alpha(Z) = d - t$ and $\alpha(2Z) = d$.

Bocci and Chiantini examine cases when t is small; specifically, they consider $t = 1, 2$. When $t = 1$, they are able to use Bézout's Theorem to find:

Theorem 3.2 (Example 3.1, Proposition 3.2, and Theorem 3.3 of [BC11])**.** *Let $Z \subseteq \mathbf{P}^2$ be a finite set of points. Then $t = 1$ if and only if either Z is a set of collinear points and $\alpha(Z) = 1$ or Z is a star configuration of points and $\alpha(Z) = d - 1$.*

That $\alpha(Z) = 1$ and $\alpha(2Z) = 2$ when Z is a set of collinear points is clear. A star configuration of points in $\mathbf{P}^2$ is the finite subset Z of $\binom{d}{2}$ points of pairwise intersection of d lines, where $d \geq 3$. See Figure 3.1 for a star configuration Z when $d = 4$. Let F be the product of the four linear forms corresponding to the lines, and G be the form F divided by one of the linear forms. Then it is clear that F vanishes to order 2 at each of the six points and G vanishes to order 1 at each point; Bocci and Chiantini show that F and G are forms of minimal degree vanishing to order 2 and 1, respectively, which means that Z has type $(3, 4)$.

When $t = 2$, Bocci and Chiantini also obtain classification results, though these are much more complicated, occupying approximately 75% of [BC11]. The situation can be roughly described as follows: either $\alpha(2Z) = 4$ and Z lies in a conic, or $\alpha(2Z) > 4$ and Z lies in the nodes of the union of rational curves.

There are several possible avenues for generalizing these results; the first we

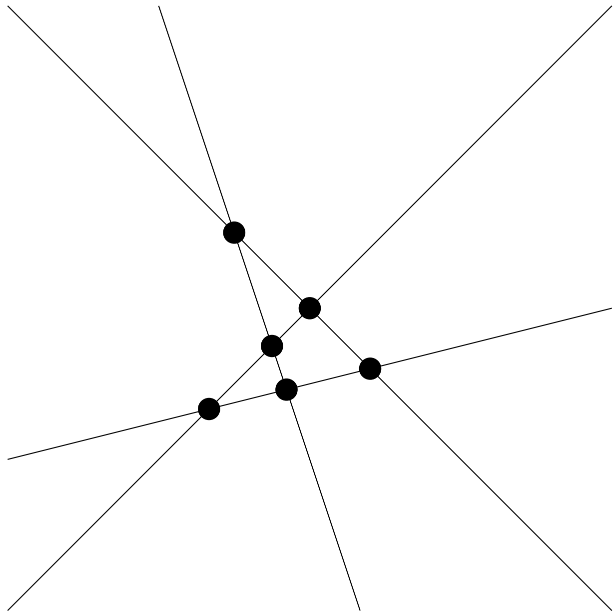

Figure 3.1: A star configuration formed by the pairwise intersection of 4 lines in $\mathbf{P}^2$.

consider is to look at higher symbolic powers. We borrow the following notation from [DST12]:

Notation 3.3. Let $Z \subseteq \mathbf{P}^2$ be a finite fixed set of arbitrary points. Then we use the notation $\alpha_{m,n}(Z) := \alpha(I^{(m)}) - \alpha(I^{(n)})$ for $m > n$.

In [DST12], Dumnicki et al. obtain stronger results by requiring the successive differences $\alpha_{m+1,m}$ to be constant as m increases.

They then prove:

Theorem 3.4 (Theorems 3.1 and 4.14 of [DST12]). *If*

$$\alpha_{2,1}(Z) = \alpha_{3,2}(Z) = \cdots = \alpha_{t+1,t}(Z) = d,$$

then

1. *for $d = 1$ and $t \geq 2$ the set Z is contained in a line, i.e., $\alpha(Z) = 1$;*

2. *for $d = 2$ and $t \geq 4$ the set Z is contained in a conic, i.e., $\alpha(Z) = 2$.*

Moreover, both results are sharp, i.e., there are examples showing that one cannot relax the assumptions on t.

The authors believe that such a result should be true for cubics as well.

A second avenue for generalizing the results of Bocci and Chiantini is to consider subschemes of higher dimensional projective spaces, and this is the direction we will take in the remainder of this thesis. However, rather than look at point configurations, we will examine configurations of lines in $\mathbf{P}^3$. With some additional reasonable assumptions, we are able to use Bocci and Chiantini's results to describe configurations of lines in $\mathbf{P}^3$ for which $t = 1$.

3.2 Lines in $\mathbf{P}^3$

Throughout the remainder, let $S = k[\mathbf{P}^3] = k[x, y, z, w]$ and $R = k[\mathbf{P}^2] = k[x, y, z]$ be the homogeneous coordinate rings of $\mathbf{P}^3$ and $\mathbf{P}^2$, respectively.

Broadly speaking, the two types of configurations of lines in $\mathbf{P}^3$ we will discuss are the coplanar configurations and the pseudo-star configurations.

Definition 3.5. A pseudo-star configuration (or pseudostar) of lines in $\mathbf{P}^3$ is a finite collection of lines formed by the pairwise intersection of hyperplanes such that no three of the hyperplanes meet in a line.

There is a growing body of literature on the study of star configurations (see [GHM12] and the references therein). Indeed, star configurations were one of the first examples studied in [BH10a] in which the resurgence was introduced. The easiest examples, of course, are star configurations of points in $\mathbf{P}^2$, but star configurations can be defined in any codimension in any projective space. As defined in [GHM12], a star configuration of lines in $\mathbf{P}^3$ is a collection of lines formed by the pairwise intersections of hyperplanes which meet properly, meaning that the intersection of j hyperplanes is empty or has codimension j. For the case of the pseudostars, we replace the requirement that the planes meet properly with the requirement that no three of the planes meet in a line; therefore, it may be that in a pseudostar in $\mathbf{P}^3$, more than three planes meet in a single point.

The easiest example of a pseudostar in $\mathbf{P}^3$ is a star configuration of lines.

Another easy example of a pseudostar in $\mathbf{P}^3$ is a projective cone over a star configuration of points in $\mathbf{P}^2$:

Example 3.6. Suppose $I \subseteq R$ defines a star configuration Z of points in $\mathbf{P}^2$. The projective cone over Z is a subscheme of $\mathbf{P}^3$ defined by the extension IS of I to S. This is an example of a pseudostar.

In this chapter, we are primarily concerned with configurations of lines in $\mathbf{P}^3$ which are arithmetically Cohen-Macaulay. Recall that a subscheme $X \subseteq \mathbf{P}^n$ is arithmetically Cohen-Macaulay (ACM) if the homogeneous coordinate ring $k[\mathbf{P}^n]/I(X)$ of the subscheme is Cohen-Macaulay. Several familiar linear configurations are ACM.

Lemma 3.7. *Any collection of coplanar lines in* $\mathbf{P}^3$ *is ACM.*

Proof. If $I \subseteq S$ is the ideal of coplanar lines, then I is a complete intersection ideal, and thus S/I is Cohen-Macaulay. $\square$

Lemma 3.8. *Let* $\mathbb{L}$ *denote a finite union of lines in* $\mathbf{P}^3$*. If* $\mathbb{L}$ *is a star configuration of lines in* $\mathbf{P}^3$ *or a projective cone over a star configuration of points in* $\mathbf{P}^2$*,* $\mathbb{L}$ *and* $2\mathbb{L}$ *are ACM.*

Proof. If $\mathbb{L}$ is a star configuration of lines in $\mathbf{P}^3$, then $\mathbb{L}$ and $2\mathbb{L}$ are ACM by [GHM12, Proposition 2.9 and Theorem 3.1], respectively. Suppose $\mathbb{L}$ is a projective cone over a star configuration Z in $\mathbf{P}^2$. Then $I(\mathbb{L}) = I(Z)S$, and $(R/I(Z))[w] \cong S/I(Z)S = S/I(\mathbb{L})$. Since $R/I(Z)$ is Cohen-Macaulay, so is $(R/I(Z))[w]$, and hence also $S/I(\mathbb{L})$. Therefore $\mathbb{L}$ is ACM. A similar argument can be carried out for $I(2Z) = (I(Z))^{(2)}$. □

Proposition 3.9. *Pseudostars and their symbolic squares are ACM.*

Proof. The reduced case was proved, though not explicitly, in [GHM12, Proposition 2.9] (but see [GHM12, Remark 2.13]). The symbolic square case can be found in the first part of the proof of [GHM12, Theorem 3.2], as the assumption that the hyperplanes meet properly can be relaxed to the assumption that no three hyperplanes contain a line. □

Proposition 3.10 (Corollary 1.3.8 of [Mig98]). *Let $\mathbb{X} \subseteq \mathbf{P}^N$ be an arithmetically Cohen-Macaulay scheme of dimension at least 1, and suppose $H \subseteq \mathbf{P}^N$ is a general hyperplane. Let $\mathbb{X} \cap H$ denote the general hyperplane section of $\mathbb{X}$, $S = k[\mathbf{P}^N]$, and $R = S/I(H) \cong k[\mathbf{P}^{N-1}]$. Then the Hilbert function of $R/I(\mathbb{X} \cap H)$ is given by*

$$H(R/I(\mathbb{X} \cap H), t) = H(S/I(X), t) - H(S/I(X), t-1).$$

A useful corollary of Proposition 3.10 is the following.

Corollary 3.11. *Suppose $\mathbb{X} \subseteq \mathbf{P}^N$ is an arithmetically Cohen-Macaulay scheme of dimension at least 1, and $H \subseteq \mathbf{P}^N$ is a general hyperplane. If $\mathbb{X} \cap H$ denotes the general hyperplane section of $\mathbb{X}$, then $\alpha(\mathbb{X}) = \alpha(\mathbb{X} \cap H)$.*

Proof. This follows immediately from Proposition 3.10 and the definitions of the Hilbert function and α. □

Corollary 3.12. *Let $\mathbb{L}$ be a pseudostar in $\mathbf{P}^3$ formed by the pairwise intersection of d planes, no three of which contain any line. Then $\alpha(\mathbb{L}) = d - 1$ and $\alpha(2\mathbb{L}) = d$.*

Proof. Apply Corollary 3.11. □

The following proposition shows that if a general hyperplane intersects three or more lines in $\mathbf{P}^3$ in collinear points, the lines must lie in a plane. We make use of the notion of the dual space of $\mathbf{P}^3$, which we denote $(\mathbf{P}^3)^*$. Recall the dual relationship: a point $(a, b, c, d) \in \mathbf{P}^3$ corresponds to a hyperplane $ax + by + cz + dw = 0$ in $(\mathbf{P}^3)^*$.

Proposition 3.13. *A general hyperplane intersects $d \geq 3$ non-coplanar lines in $\mathbf{P}^3$ in d non-collinear points.*

Proof. Without loss of generality, let $\mathbb{L} = \ell_1 \cup \ell_2 \cup \ell_3$ be a collection of three non-coplanar lines in $\mathbf{P}^3$. Our goal is to show explicitly that the set of all planes which either contain one of the lines, or meet the lines in fewer than 3 distinct points, or meet the lines in exactly 3 collinear points, is contained in a proper closed subset of $(\mathbf{P}^3)^*$. This will ensure that the collection of planes which meet the non-coplanar lines in non-collinear points forms a nonempty open subset of $(\mathbf{P}^3)^*$.

First, note that the set $\Gamma_p \subset (\mathbf{P}^3)^*$ of all planes containing a given point p is a proper closed subset of the set of all planes. Thus, the set Γ_ℓ of planes that contain a given line ℓ is also a proper closed subset of the set of all planes, since $\Gamma_\ell = \Gamma_p \cap \Gamma_q$, where p and q are any two distinct points of ℓ. Thus, a general plane meets any finite set of lines in a finite set of points, as the union of Γ_{ℓ_i}, $1 \leq i \leq n$ will again be a proper closed subset.

If the number of points in which a plane meets $\mathbb{L}$ is less than 3, it must be that one of the points is a point at which at least two lines intersect. There are at most 3 such intersection points, and as mentioned above the planes containing any given point forms a proper closed subset of $(\mathbf{P}^3)^*$. Since the union of finitely many proper closed subsets is a proper closed subset, the set of all planes which contain one of the 3 lines or which meet the d lines in fewer than 3 points is contained in a proper closed subset of $(\mathbf{P}^3)^*$. It is now enough to show that the set of all planes which meet the 3 lines in exactly 3 collinear points is also contained in a proper closed subset C of $(\mathbf{P}^3)^*$.

We consider two cases: either two lines intersect, or all three lines are skew.

If the lines ℓ_1 and ℓ_2 intersect, $\ell_1 \cup \ell_2$ determine a plane H'. By assumption, there is no hyperplane containing all three lines, so ℓ_3 and H' meet in a point $p' \in H'$. Then, if any plane H meets $\ell_1 \cup \ell_2 \cup \ell_3$ in collinear points on a line L, it must be that $L = H \cap H'$, and thus L contains p. Indeed, if H'' meets ℓ_3 in a point other than p', we have $H'' \cap H' = L'$, a line which meets ℓ_1 and ℓ_2 in collinear points, and $H'' \cap \ell_3 = p''$, where p'' does not lie on L' as it does not lie in the plane H' formed by $\ell_1 \cup \ell_2$.

We now turn to the case in which the three lines ℓ_1, ℓ_2, ℓ_3 are skew (equivalently, we assume no two of the lines are coplanar). We consider planes meeting $\mathbb{L}$ in collinear points which contain none of the lines. Without loss of generality, suppose two of the lines are coordinate lines, i.e., $I(\ell_1) = (x, w)$, $I(\ell_2) = (y, z)$ and $I(\ell_3) = (x - y, z - w)$. Let H be a hyperplane defined by $\alpha x + \beta y + \gamma z + \delta w = 0$. Then $H \cap \ell_1 = (0, \gamma, -\beta, 0)$, $H \cap \ell_2 = (-\delta, 0, 0, \alpha)$, and $H \cap \ell_3 = (-(+\delta), -(\gamma + \delta), \alpha + \beta, \alpha + \beta) \in \mathbf{P}^3$.

Notice that the line passing through $H \cap \ell_1$ and $H \cap \ell_2$ is defined by the planes $\beta y + \gamma z = 0$ and $\alpha x + \delta w = 0$. If we evaluate the equation for either plane at $H \cap \ell_3$ we get $\alpha\gamma - \delta\beta = 0$. This is a quadric surface Q in $(\mathbf{P}^3)^*$, which is a proper closed subset.

Therefore, the union of all planes which meets 3 noncoplanar lines in collinear points is a proper closed subset of $(\mathbf{P}^3)^*$. $\square$

Another way to say this is:

Corollary 3.14. *If $d \geqslant 3$ lines in $\mathbf{P}^3$ intersect a general hyperplane H in collinear points, then the lines are coplanar.*

We set the following notation.

Notation 3.15. Let $H_1, H_2, \ldots, H_d \subset \mathbf{P}^3$ be hyperplanes, no three of which contain any line. Set $\ell_{ij} = H_i \cap H_j$ for all $i < j$, and put $\mathbb{L} = \bigcup_{1 \leq i < j \leq d} \ell_{ij}$.

We now come to the main result of this chapter, which describes an extension of Bocci and Chiantini's $t = 1$ result for points in $\mathbf{P}^2$. In $\mathbf{P}^2$ every codimension 2 subscheme is ACM, as all finite sets of points in any $\mathbf{P}^N$ are ACM. In higher dimensions, not every codimension 2 subscheme in $\mathbf{P}^N$ is ACM. However, the natural generalization of their result seems to be for ACM codimension 2 subschemes (but see Question 3.20).

We follow the lead of [BC11] and say that $\mathbb{L}$ has type $(d-1, d)$ if $\alpha(\mathbb{L}) = d - 1$ and $\alpha(2\mathbb{L}) = d$.

Theorem 3.16. *Let $\mathbb{L}$ be a union of lines $\ell_1, \ell_2, \ldots, \ell_s$.*

(a) If $\mathbb{L}$ is ACM of type $(d-1, d)$ for some $d > 1$, then $\mathbb{L}$ is either a pseudostar or coplanar.

(b) If $\mathbb{L}$ is either a pseudostar or coplanar, then $\mathbb{L}$ has type $(d-1, d)$ for some $d > 1$.

Remark 3.17. We only make the ACM assumption for the part of the theorem in which it is used, so as to preserve as much generality as possible.

Proof of Theorem 3.16. Note that we may assume we have $s \geq 4$ lines and treat the cases in which $1 \leq s \leq 3$ in an ad hoc fashion.

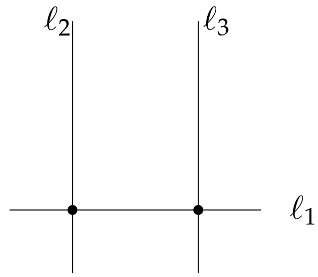

Figure 3.2: Three lines ℓ_1, ℓ_2, ℓ_3 in $\mathbf{P}^3$ such that $\ell_2 \cap \ell_3 = \emptyset$.

Fewer than 4 lines in (a): Indeed, if $s = 1$, we have a single line, which is coplanar, so (a) holds.

For $s = 2$, either the lines meet, in which case they are coplanar, or the lines are skew. If the lines ℓ_1, ℓ_2 are skew, then, without loss of generality, we may take $I(\ell_1) = (x, y)$ and $I(\ell_2) = (z, w)$, so $I(\mathbb{L}) = (x, y) \cap (z, w)$, $\alpha(\mathbb{L}) = 2$, and $\alpha(2\mathbb{L}) = \alpha((x, y)^2 \cap (z, w)^2) = 4$ so $\mathbb{L}$ has type $(d-2, d)$. In either case, if $s = 2$, (a) holds.

If $s = 3$, we have three possible ACM configurations. If the lines meet in a single point, they are either coplanar or a pseudostar. If the lines intersect pairwise, they are coplanar. The last case involves lines ℓ_1, ℓ_2, ℓ_3 such that ℓ_2 and ℓ_3 do not meet, but $\ell_2 \cap \ell_1 \neq \emptyset$ and $\ell_3 \cap \ell_1 \neq \emptyset$, as in Figure 3.2. In this case, we can, after an appropriate change of coordinates, assume $I(\ell_1) = (x, z)$, $I(\ell_2) = (y, z)$, and $I(\ell_3) = (x, w)$. One can easily verify that $\mathbb{L} = \ell_1 \cup \ell_2 \cup \ell_3$ is of type $(2, 4)$. Thus, (a) is satisfied for $1 \leq s \leq 3$, and so we make the assumption in the proof of (a) that $s \geq 4$.

Fewer than 4 lines in (b): Note that if $s = 1$ then $\mathbb{L}$ consists of a single coplanar line of type $(1, 2)$. Similarly, if $s = 2$, the only possibility to consider is that $\mathbb{L}$ is coplanar of type $(1, 2)$. When $s = 3$, $\mathbb{L}$ may be coplanar or a pseudostar. If $\mathbb{L}$ is a pseudostar of 3 lines, note that $\mathbb{L}$ is actually the projective cone over a star configuration of 3 points in $\mathbf{P}^2$ formed by the pairwise intersection of 3 hyperplanes. It is easily verified that this configuration of lines has type $(2,3)$. Thus, (b) is satisfied for $1 \leq s \leq 3$, so we may assume $s \geq 4$ in (b) also (or, as we shall refer to it, $s-1 \geq 3$).

We now consider (a) and assume $s \geq 4$.

Suppose $\mathbb{L}$ has type $(d-1,d)$ for some $d \geq 2$, and let H denote a general hyperplane. As $\mathbb{L}$ is ACM, we can apply Proposition 3.10 to $\mathbb{L}$ to see that $\alpha(\mathbb{L}) = \alpha(\mathbb{L} \cap H) = d-1$, and since $d = \alpha(2\mathbb{L}) \geq \alpha(2(\mathbb{L} \cap H)) > \alpha(\mathbb{L} \cap H) = d-1$ (see [BC11]), the general hyperplane sections $\mathbb{L} \cap H$ must have type $(d-1,d)$ in $H \cong \mathbf{P}^2$. By [BC11], this means that the general hyperplane sections $\mathbb{L} \cap H$ of $\mathbb{L}$ are either a set of collinear points or a star of points in $\mathbf{P}^2$.

If $\mathbb{L} \cap H$ is a set of collinear points, we must have that $\mathbb{L}$ is a set of coplanar lines (see Proposition 3.13). Otherwise, by Proposition 3.13 (since $s \geq 4$ and thus $s-1 \geq 3$) we have d (non-disjoint) collections of $d-1$ collinear points (in fact, we have $\binom{d}{2}$ points total, since $\mathbb{L} \cap H$ is a star in $H \cong \mathbf{P}^2$). Each of the $\binom{d}{2}$ points is the hyperplane section of exactly one of the ℓ_{ij}'s, so we must have $s = \binom{d}{2}$ lines ℓ_{ij}, with d (non-disjoint) collections of $d-1 \geq 3$ coplanar lines. Moreover, since we have d hyperplanes meeting in $\binom{d}{2}$ lines, it must be that no three hyperplanes meet in a line, or else we would have strictly fewer than $\binom{d}{2}$ lines, and thus strictly fewer than $\binom{d}{2}$ hyperplane sections. Thus, $\mathbb{L}$ forms a pseudostar.

We now turn to (b) and again assume $s \geq 4$. If $\mathbb{L}$ lies in a plane, then $\mathbb{L}$ has type $(1,2)$. Assume now that $\mathbb{L}$ is a pseudostar. Then a general hyperplane H meets each H_i in a line L_i; as H is general, L_i meets each ℓ_{ij}, $j \neq i$ in distinct points $p_{ij} \in H \cong \mathbf{P}^2$. The points p_{ij}, $j \neq i$, form a star configuration of points in $H \cong \mathbf{P}^2$, as each line L_i contains $s-1 \geq 3$ points p_{ij}, $j \neq i$, each point p_{ij} lies on exactly two lines, L_i and L_j, and we have exactly $\binom{d}{2}$ points. By Proposition 3.9, $\mathbb{L}$ and $2\mathbb{L}$ are

ACM, so Proposition 3.10 applies to the general hyperplane sections of $\mathbb{L}$ and $2\mathbb{L}$ to give that $\alpha(\mathbb{L}) = d - 1$ and $\alpha(2\mathbb{L}) = d$. $\square$

3.3 Future Work

It seems as that pairing this approach with an inductive argument may generalize Theorem 3.16 to ACM codimension 2 subschemes of $\mathbf{P}^N$, $N > 3$, but this has not yet been explored.

There are several other avenues for future work.

We made heavy use of the assumption that the lines in question in $\mathbf{P}^3$ are arithmetically Cohen-Macaulay (ACM). A natural question, then, is:

Question 3.18. Which configurations of lines in $\mathbf{P}^3$ are ACM?

Also:

Question 3.19. Does there exist an ACM configuration of lines in $\mathbf{P}^3$ which is not a pseudostar or a collection of coplanar lines?

Similarly,

Question 3.20. Does there exist a configuration of lines of type $(d-1, d)$ which is not ACM?

An early example that I tried to understand was the example of three skew lines in $\mathbf{P}^3$, as such an arrangement has three general points as its general hyperplane section, which is also an example of a star configuration. However, it can be shown that three skew lines has type $(2, 4)$. Moreover, three skew lines are not ACM, as they are not connected.

As every finite set of points in $\mathbf{P}^N$ (for any $N \geq 1$) is ACM, another natural question to ask is:

Question 3.21. Which configurations of points in $\mathbf{P}^3$ have type $(d-1, d)$?

In [BC11], the authors also classify configurations of points in $\mathbf{P}^2$ which have type $(d-2, d)$. Thus, we ask:

Question 3.22. Which arrangements of lines in $\mathbf{P}^3$ have type $(d-2, d)$? Which arrangements of points in $\mathbf{P}^3$ have type $(d-2, d)$?

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