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On the Betti Number of Differential Modules

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by

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Let $R = k[x_1, \dots, x_d]$ with k a field. A $\mathbb{Z}^d$-graded differential R-module is a $\mathbb{Z}^d$-graded R-module D with a morphism $\delta : D \to D$ such that $\delta^2 = 0$. This dissertation establishes a lower bound on the rank of such a differential module when the underlying R-module is free. We define the Betti number of a differential module and use it to show that when the homology $\ker\delta/\operatorname{im}\delta$ of D is non-zero and finite dimensional over k then there is an inequality $\operatorname{rank}_R D \geq 2^d$. This relates to a problem of Buchsbaum, Eisenbud and Horrocks in algebra and conjectures of Carlsson and Halperin in topology.

Motivated by some steps of this work, further results are proved relating the homotopical Loewy length, derived Loewy length and generalized Loewy length.

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Chapter 1

Introduction

The study of commutative algebra has grown out of the classic study of polynomial equations and their solutions. The principal objects of study are commutative rings and their modules. A basic tool for working with modules is a free resolution, where a module is represented by generators and relations, and those relations are themselves represented by generators and further relations, and so on. Corresponding to these stages of construction, a free resolution is indexed by the integers $\mathbb{Z}$. Alternatively, we can consider a free resolution to be the final product of the construction, so that it is given "all at once" without a corresponding $\mathbb{Z}$ indexing. Viewed in this way a free resolution is a module equipped with a square-zero endomorphism coming from the differentials of the complex. Such an object is known as a differential module.

Differential modules have appeared in several works on chain complexes, but there have not been many studies of differential modules themselves. As described above, they capture much of the structure of a chain complex, but they also allow richer behavior because they are not constrained to a $\mathbb{Z}$ indexing. For example, differential modules simultaneously address problems about free resolutions and problems about free complexes with homology spread among several homological degrees. Their flexibility extends beyond algebraic contexts as well. One such use is in the conjectures of Carlsson and Halperin concerning a lower bound on the rank of DG-modules with non-zero finite length homology [5, 9] (for this connection between differential modules and DG-modules see [1, §5]). This dissertation examines differential modules, particularly $\mathbb{Z}^d$-graded differential modules, with a focus on the problem of finding a lower bound on the rank of a resolving differential module.

To be precise, let k be a field and set $R = k[x_1, \ldots, x_d]$. A *differential R-module* D is an R-module with a square-zero homomorphism $\delta : D \to D$ called the *differential*. The homology of D is defined in the usual way: $H(D) = \ker \delta / \operatorname{im} \delta$. The results of this dissertation are motivated by a conjecture of Avramov, Buchweitz and Iyengar on the rank of a differential module over a local ring ([1, Conjecture 5.3]). Formulated for a $\mathbb{Z}^d$-graded polynomial ring, this reads:

Conjecture 1.0.1. Let $R = k[x_1, \ldots, x_d]$ be a $\mathbb{Z}^d$-graded polynomial ring of dimension d, and F a $\mathbb{Z}^d$-graded differential R-module admitting a finite free flag. If $H(F)$ has non-zero finite length, then

$$\operatorname{rank}_R F \geq 2^d.$$

In this conjecture, a *free flag* on a differential module is a filtration compatible with the differential (see Definition 2.0.8). It provides the appropriate lifting properties for the category of differential modules. From [1, Theorem 5.2] it follows that the conjecture is true when $d \leq 3$.

A consequence of our results is the following result for all dimensions when the differential δ has degree zero:

Theorem. *Let F be a finitely generated $\mathbb{Z}^d$-graded differential R-module with differential $\delta : F \to F$ that is homogeneous of degree zero, such that F is free as an R-module. If $H(F)$ has non-zero finite length then* $\operatorname{rank}_R F \geq 2^d$.

This follows from Theorem 3.4.4 which establishes a lower bound on the rank of a differential module by finding a lower bound on an invariant that corresponds to the role of a Betti number of a module (see Definition 2.0.7).

This result is new even for complexes of R-modules. Given a complex of $\mathbb{Z}^d$-graded free R-modules

$$F = \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \cdots$$

the module $\bigoplus_i F_i$ with differential $\delta = \bigoplus_i \partial^i$ forms a differential module. When $H(F)$ has non-zero finite length as an R-module then we conclude that

$$\sum_i \operatorname{rank}_R F_i \geq 2^d. \tag{1.0.1}$$

This inequality is already known when F is a resolution—i.e. $F_i = 0$ for $i < 0$ and $H_i(F) = 0$ for $i \neq 0$—from the work of Charalambous and Santoni on the Buchsbaum-Eisenbud-Horrocks problem [7, 13]. Recall that for a $\mathbb{Z}$-graded polynomial ring, the Buchsbaum-Eisenbud-Horrocks problem is to show that $\beta_i(M) \geq \binom{d}{i}$ all $\mathbb{Z}$-graded R-modules M with non-zero finite length, where $\beta_i(M)$ is the i-th Betti number of M [4, 10]. Summing the binomial coefficients gives (1.0.1) when F is a free resolution of a non-zero finite length $\mathbb{Z}^d$-graded module M. However, when F is not acyclic it is not clear how to establish (1.0.1) without using differential modules.

Some techniques available for complexes can be directly adapted to the case of differential modules, however there are some subtle difficulties that appear. For example, there may be no way to minimize a resolution in the category of differential modules and this creates an obstruction to applying the usual tools of complexes (see Example 3.2.3, or Theorem 3.2.1 for some positive results). Not many techniques are available for working with differential modules. This work should be seen as a contribution in that direction.

Chapter 2 provides the background in differential modules and $\mathbb{Z}^d$-graded differential modules. We define a notion of a Betti number for differential modules, which serves as an essential notion for establishing a lower bound on the rank of a differential module.

Chapter 3 explores $\mathbb{Z}^d$-graded differential modules with an eye towards lower bounds on the Betti number. We examine several classes of $\mathbb{Z}^d$-graded differential modules, presenting some results and examples, including an example that demonstrates that the Betti number is not always bounded below by 2^d. The main theorem on the lower bound for Betti numbers is proven by using truncation techniques to reduce to a case where differential module adaptations of Santoni's results [13] can be applied.

An important step in finding a lower bound on the Betti number is to take a differential module with finite length homology and replace it by a differential module with finite length. In Chapter 4 we examine this phenomenon for complexes of R-modules and investigate some ways of measuring the size of such a replacement.

Chapter 2

Differential Modules

Throughout, k is a field, $R = k[x_1, \ldots, x_d]$ is the standard $\mathbb{Z}^d$-graded polynomial ring and $\mathfrak{m} = (x_1, \ldots, x_d)$. To be specific, the grading on R is such that the degree $\deg(x_i) \in \mathbb{Z}^d$ of variable x_i is $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 appearing in the i-th coordinate. For $\mathbf{m} \in \mathbb{Z}^d$, we write $\mathbf{m}_i$ to denote the i-th coordinate. Two elements $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ are compared coordinatewise by setting $\mathbf{a} \leq \mathbf{b}$ if $\mathbf{a}_i \leq \mathbf{b}_i$ for all i. This makes $\mathbb{Z}^d$ into a partially ordered group.

Recall that a $\mathbb{Z}^d$-graded module M over R is an R-module that has a decomposition $\bigoplus_{\mathbf{m} \in \mathbb{Z}^d} M_{\mathbf{m}}$ as abelian groups such that multiplication by an element of R of degree $\mathbf{n}$ takes $M_{\mathbf{m}}$ to $M_{\mathbf{m}+\mathbf{n}}$. An R-linear map ϕ between $\mathbb{Z}^d$-graded modules M and N is a *morphism* if $\phi(M_{\mathbf{m}}) \subseteq N_{\mathbf{m}}$. In particular, a complex of $\mathbb{Z}^d$-graded modules is required to have morphisms for its differentials.

For $\mathbf{d} \in \mathbb{Z}^d$ the shifted (or twisted) module $M(\mathbf{d})$ is defined to be $M_{\mathbf{d}+\mathbf{m}}$ in degree $\mathbf{m}$ for each $\mathbf{m} \in \mathbb{Z}^d$, with the same R-module structure as M. Given a morphism $\phi : M \to N$ the shifted morphism $M(\mathbf{d}) \to N(\mathbf{d})$ defined by $x \mapsto \phi(x)$ is denoted $\phi(\mathbf{d})$.

We will work with $\mathbb{Z}^d$-graded modules and $\mathbb{Z}^d$-graded differential modules, so definitions will be given in that context for simplicity; see [1, 6] for details concerning arbitrary differential modules.

Definition 2.0.2. A $\mathbb{Z}^d$*-graded differential* R*-module with differential degree* $\mathbf{d} \in \mathbb{Z}^d$ is a $\mathbb{Z}^d$-graded R-module D with a morphism $\delta : D \to D(\mathbf{d})$ such that the composition

$$D(-\mathbf{d}) \xrightarrow{\delta(-\mathbf{d})} D \xrightarrow{\delta} D(\mathbf{d})$$

is zero. We say that δ is the *differential* of D.

When D and E are $\mathbb{Z}^d$-graded differential modules with the same differential degree, define a *morphism* $\phi : D \to E$ to be a morphism of $\mathbb{Z}^d$-graded modules satisfying $\delta^E \circ \phi = \phi \circ \delta^D$. For a fixed differential degree, the category of $\mathbb{Z}^d$-graded differential modules with this notion of a morphism is an abelian category.

The homology of a differential module D is the $\mathbb{Z}^d$-graded R-module

$$H(D) = \ker \delta / \operatorname{im}(\delta(-\mathbf{d})).$$

The $\mathbb{Z}^d$-grading on $H(D)$ is inherited from D by considering $\ker \delta$ and $\operatorname{im}(\delta(-\mathbf{d}))$ as submodules of D with the induced grading. Any $\mathbb{Z}^d$-graded R-module, in particular $H(D)$, will be considered as a differential module with zero differential.

In the usual way, a morphism $\phi : D \to E$ induces a map in homology $H(\phi) : H(D) \to H(E)$. If $H(\phi)$ is an isomorphism we say that ϕ is a *quasi-isomorphism* and write $D \simeq E$ or $\phi : D \xrightarrow{\simeq} E$. Given an exact sequence of differential modules

$$0 \longrightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \longrightarrow 0$$

there is an induced long exact sequence of in homology,

$$\cdots \longrightarrow H(D_1)(i\mathbf{d}) \xrightarrow{H(\alpha)(i\mathbf{d})} H(D_2)(i\mathbf{d}) \xrightarrow{H(\beta)(i\mathbf{d})} H(D_3)(i\mathbf{d}) \xrightarrow{\gamma(i\mathbf{d})} H(D_1)((i+1)\mathbf{d}) \longrightarrow \cdots$$

where i ranges over the integers, and each map is a morphism of $\mathbb{Z}^d$-graded modules (in particular, has degree $\mathbf{0}$). We summarize this sequence by the following diagram

$$\begin{tikzcd} H(D_1) \arrow[rr, "H(\alpha)"] & & H(D_2) \arrow[dl, "H(\beta)"] \\ & H(D_3) \arrow[ul, "\gamma"] & \end{tikzcd} \tag{2.0.1}$$

where the circle indicates that γ is a homomorphism of degree $\mathbf{d}$.

See [6, Chap. IV §1] for a proof.

Bounds on the rank of a differential module will be obtained by comparing the rank and an invariant that we call the Betti number of a differential module. To define the Betti number we will need a notion of a tensor product of differential modules. However adapting the usual definition of a tensor product between complexes fails to produce a differential module when applied to two differential modules. To work around this we recall the construction of a tensor product of a complex and a differential module, along with some of its properties [1, §1].

Definition 2.0.3. For a complex C of $\mathbb{Z}^d$-graded R-modules and a $\mathbb{Z}^d$-graded differential R-module D with differential degree $\mathbf{d}$, define a $\mathbb{Z}^d$-graded differential module $C \boxtimes_R D$ by setting

$$C \boxtimes_R D = \bigoplus_{i \in \mathbb{Z}} (C_i(-i\mathbf{d}) \otimes_R D),$$

with differential defined by

$$\delta^{C \boxtimes_R D}(c \otimes d) = \partial^C(c) \otimes d + (-1)^i c \otimes \delta^D(d),$$

for $c \otimes d \in C_i(-i\mathbf{d}) \otimes_R D$. This makes $C \boxtimes_R D$ into a $\mathbb{Z}^d$-graded differential R-module with differential degree $\mathbf{d}$.

We will need the following facts concerning this product. These results are proved in [1] for arbitrary differential modules, but the proofs hold for $\mathbb{Z}^d$-graded differential modules with the obvious modifications.

Proposition 2.0.4 ([1, 1.9.3])**.** *Let X and Y be $\mathbb{Z}^d$-graded complexes and let D be a $\mathbb{Z}^d$-graded differential module. Then there is a natural isomorphism of $\mathbb{Z}^d$-graded differential modules:*

$$(X \otimes_R Y) \boxtimes_R D = X \boxtimes_R (Y \boxtimes_R D).$$

Proposition 2.0.5 ([1, Proposition 1.10])**.** *Let X and Y be bounded below $\mathbb{Z}^d$-graded complexes of flat R-modules, i.e. $X_i = Y_i = 0$ for sufficiently small i. Then*

1. *the functor $X \boxtimes_R -$ preserves exact sequences and quasi-isomorphisms;*

2. *a quasi-isomorphism $\phi : X \to Y$ induces a quasi-isomorphism*

$$\phi \boxtimes_R D : X \boxtimes_R D \to Y \boxtimes_R D$$

for all $\mathbb{Z}^d$-graded differential R-modules D.

Using this tensor product, we can define a Tor functor between R-modules and differential R-modules, and hence define a Betti number.

Definition 2.0.6. For a $\mathbb{Z}^d$-graded differential R-module D and a $\mathbb{Z}^d$-graded R-module M set

$$\mathrm{Tor}^R(M, D) = H(P \boxtimes_R D)$$

where P is a $\mathbb{Z}^d$-graded free resolution of M. This is well-defined as different choices of free resolution produce quasi-isomorphic differential modules by Proposition 2.0.5.

Definition 2.0.7. We define $\beta^R_{\mathbf{m}}(D)$ to be the *Betti number in degree* $\mathbf{m} \in \mathbb{Z}^d$ of a differential R-module D:

$$\beta^R_{\mathbf{m}}(D) = \operatorname{rank}_k \operatorname{Tor}^R(k, D)_{\mathbf{m}}.$$

Summing over all degrees gives the *Betti number* $\beta^R(D)$:

$$\beta^R(D) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \beta^R_{\mathbf{m}}(D) = \operatorname{rank}_k \operatorname{Tor}^R(k, D).$$

The connection between ranks of differential modules and Betti numbers is provided by free flags, a notion of a free resolution for differential modules [1, §2].

Definition 2.0.8. A *free flag* on a differential module F is a family $\{F^n\}_{n\in\mathbb{Z}}$ of $\mathbb{Z}^d$-graded R-submodules such that

1. $F^n = 0$ for $n < 0$,
2. $F^n \subseteq F^{n+1}$ for all n,
3. $\delta^F(F^n) \subseteq F^{n-1}$ for all n,
4. $\bigcup_{n\in\mathbb{Z}} F^n = F$,
5. F^n/F^{n-1} is a free R-module for all n.

We say that a $\mathbb{Z}^d$-graded differential module F with a free flag *resolves* D if there is a quasi-isomorphism $F \xrightarrow{\simeq} D$ in the category of $\mathbb{Z}^d$-graded differential modules.

Many properties of free modules have analogs for differential modules with free flags. We will use the following two.

Proposition 2.0.9. *Let D_1 and D_2 be differential modules and let F be a differential module with a free flag. If $\alpha : D_1 \to D_2$ is a surjective quasi-isomorphism and $\beta : F \to D_2$ is a* *morphism then there is a morphism* $\gamma : F \to D_1$ *such that the following diagram commutes:*

$$\begin{tikzcd} & D_1 \arrow[d, "\alpha", "\simeq"'] \\ F \arrow[ur, dashed, "\gamma"] \arrow[r, "\beta"'] & D_2 \end{tikzcd}$$

Sketch of proof. Let $\{F^n\}_{n\in\mathbb{Z}}$ be a free flag on F. Define $\gamma : F \to D_1$ inductively by defining $\gamma^n : F^n \to D_1$. We can define $\gamma^0 : F^0 \to D_1$ using the usual lifting properties since F^0 is a free R-module. For $n > 0$ we have $F^n = F^{n-1} \oplus (F^n/F^{n-1})$ since F^n/F^{n-1} is free. Assuming that we have $\gamma^{n-1} : F^{n-1} \to D_1$ defined, we can define $\gamma^n : F^n \to D_1$ by using the lifting properties of the free module F^n/F^{n-1} to define a map $F^n/F^{n-1} \to D_1$. The lifting used is important since we need $\delta^{D_1}\gamma = \gamma\delta^F$ and $\alpha\gamma = \beta$. However, any lifting can be modified by adding an appropriate boundary of D_1 so that it has the desired properties. □

Proposition 2.0.10 ([1, Proposition 2.4])**.** *Let F be a $\mathbb{Z}^d$-graded differential module with a free flag. Then the functor $- \boxtimes_R F$ preserves exact sequences and quasi-isomorphisms.*

With differential modules that admit a free flag providing a resolution of a differential module, the Tor functor is balanced, which gives the connection between the rank and Betti number of a differential module.

Lemma 2.0.11. *Let P be a free resolution of a $\mathbb{Z}^d$-graded module M and let F be a free flag resolving a $\mathbb{Z}^d$-graded differential module D. Then $H(P \boxtimes_R D)$ is isomorphic to $H(M \boxtimes_R F)$ as $\mathbb{Z}^d$-graded R-modules.*

Proof. Let $\varepsilon : P \to M$ and $\eta : F \to D$ be $\mathbb{Z}^d$-graded quasi-isomorphisms. Then there are $\mathbb{Z}^d$-graded morphisms

$$P \boxtimes_R D \xleftarrow{P\boxtimes_R \eta} P \boxtimes_R F \xrightarrow{\varepsilon\boxtimes_R F} M \boxtimes_R F.$$

By Proposition 2.0.5 and Proposition 2.0.10 these are quasi-isomorphisms. □

Theorem 2.0.12. *Let F be $\mathbb{Z}^d$-graded differential module admitting a free flag. For all degrees $\mathbf{m} \in \mathbb{Z}^d$ we have $\beta^R_{\mathbf{m}}(F) \leq \operatorname{rank}_k F_{\mathbf{m}}$. Therefore,*

$$\beta^R(F) \leq \operatorname{rank}_R F.$$

Proof. By Lemma 2.0.11,

$$\beta^R_{\mathbf{m}}(F) = \operatorname{rank}_k \operatorname{Tor}^R(k, F)_{\mathbf{m}} = \operatorname{rank}_k H(k \boxtimes_R F)_{\mathbf{m}}.$$

Since k is an R-module, $k \boxtimes_R F = k \otimes_R F$. Since $H(k \otimes_R F)_{\mathbf{m}}$ is a subquotient of $(k \otimes_R F)_{\mathbf{m}}$, we have

$$\operatorname{rank}_k H(k \boxtimes_R F)_{\mathbf{m}} \leq \operatorname{rank}_k (k \otimes_R F)_{\mathbf{m}} = \operatorname{rank}_k F_{\mathbf{m}}.$$

Summing over all degrees gives the inequality for the Betti number,

$$\beta^R(F) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \beta^R_{\mathbf{m}}(F) \leq \sum_{\mathbf{m} \in \mathbb{Z}^d} \operatorname{rank}_k F_{\mathbf{m}} = \operatorname{rank}_k k \otimes_R F = \operatorname{rank}_R F. \qquad \square$$

Remark 2.0.13. When $\delta(F) \subseteq \mathfrak{m}F$ we have $\beta^R_{\mathbf{m}}(F) = \operatorname{rank}_k F_{\mathbf{m}}$ as the differential of $k \boxtimes_R F$ is zero. In general, the inequality can be strict; see Example 3.2.3.

We finish this section by recording a property of the Tor functor for use later.

Lemma 2.0.14. *Consider an exact sequence of $\mathbb{Z}^d$-graded differential R-modules*

$$0 \longrightarrow D_1 \xrightarrow{\alpha} D_2 \xrightarrow{\beta} D_3 \longrightarrow 0\,.$$

For each $\mathbb{Z}^d$-graded R-module M there is an exact commutative diagram of $\mathbb{Z}^d$-graded differ-

ential modules:

$$\begin{tikzcd}
\operatorname{Tor}^R(M, D_1) \arrow[rr, "\operatorname{Tor}(M,\alpha)"] & & \operatorname{Tor}^R(M, D_2) \arrow[dl, "\operatorname{Tor}(M,\beta)"] \\
& \operatorname{Tor}^R(M, D_3) \arrow[ul, "\gamma"] &
\end{tikzcd}$$

Proof. Take a free resolution P of the module M. By Proposition 2.0.5 the sequence of differential modules remains exact after applying $P \boxtimes_R -$:

$$0 \longrightarrow P \boxtimes_R D_1 \xrightarrow{P\boxtimes\alpha} P \boxtimes_R D_2 \xrightarrow{P\boxtimes\beta} P \boxtimes_R D_3 \longrightarrow 0\,.$$

The diagram (2.0.1) coming from this exact sequence is the desired one. $\square$

Chapter 3

Lower Bound on the Betti Number

3.1 Compression

Every complex of R-modules produces a differential module by forming its *compression*. This construction allows results about differential modules to be translated to results about complexes of modules. In fact, the differential modules produced by compressing always have differential degree $\mathbf{0}$ so it is sufficient to restrict to differential modules with differential degree $\mathbf{0}$ if one is interested in establishing results about complexes. Note that not every differential module of differential degree $\mathbf{0}$ arises this way (see Example 3.1.3).

Construction 3.1.1 ([1, 1.3])**.** If C is a complex of $\mathbb{Z}^d$-graded R-modules, then its *compression* is the $\mathbb{Z}^d$-graded differential module

$$C_\Delta = \bigoplus_{i \in \mathbb{Z}} C_i$$

with differential $\delta^{C_\Delta} = \bigoplus_{i \in \mathbb{Z}} \partial_i^C$.

We have $\deg(\delta^{C_\Delta}) = \mathbf{0}$ because the differentials of the complex C are required to have degree zero. By the definition of δ^{C_Δ}, we have $H(C_\Delta) = \bigoplus_{i \in \mathbb{Z}} H_i(C)$.

When the complex C is bounded below and consists of free R-modules then the compression has a free flag. Indeed, suppose $C_i = 0$ for i sufficiently small. Then setting $F^n = \bigoplus_{i \leq n} C_i$ forms a free flag.

Computing the Betti number of a compression is a straight-forward application of Theorem 2.0.12 and Remark 2.0.13.

Lemma 3.1.2. *Let C be a bounded below complex of free modules that is minimal in the sense that $\partial_n^C(C_n) \subseteq \mathfrak{m}C_{n-1}$. Then*

$$\beta(C_\Delta) = \sum_i \operatorname{rank}_R C_i.$$

When C is a minimal free resolution of a module M we have

$$\beta(C_\Delta) = \sum_i \beta_i(M),$$

where $\beta_i(M)$ is the usual Betti number of M.

Proof. Since C is a bounded below complex of free modules, C_Δ has a free flag. We have

$$\delta(C_\Delta) = \bigoplus_{i \in \mathbb{Z}} \partial_i(C_i) \subseteq \bigoplus_{i \in \mathbb{Z}} \mathfrak{m}C_{i-1} = \mathfrak{m}C_\Delta,$$

so by Remark 2.0.13 we have

$$\beta(C_\Delta) = \operatorname{rank}_R C_\Delta = \sum_i \operatorname{rank}_R C_i.$$

When C is a minimal free resolution of M we have $\operatorname{rank}_R C_i = \beta_i(M)$, which completes the proof. $\square$

Obviously differential modules with non-zero differential degree do not come from compressing a complex, but the following shows that there are also differential modules with differential degree zero that are not compressions of a complex.

Example 3.1.3. Let $R = k[x, y]$ and let $F = R(0,0) \oplus R(-1,0) \oplus R(0,-1) \oplus R(-1,-1)$. Viewing F as column vectors, define a differential δ by left-multiplication by the matrix

$$\begin{bmatrix} 0 & x & y & xy \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a differential module with $\deg \delta = \mathbf{0}$. Represented diagrammatically this has the form of a Koszul complex on x, y modified by adding an additional map:

$$\begin{tikzcd} R(-1,-1) \arrow[r, "\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]"'] \arrow[rr, bend left, "xy"] & R(-1,0) \oplus R(0,-1) \arrow[r, "{[x\ y]}"'] & R(0,0) \arrow[r] & 0 \end{tikzcd}.$$

Reading the diagram from right to left produces a free flag:

$$\begin{aligned} 0 \subset R(0,0) \subset R(0,0) \oplus R(-1,0) \oplus R(0,-1) \subset \\ R(0,0) \oplus R(-1,0) \oplus R(0,-1) \oplus R(-1,-1) = F. \end{aligned}$$

To calculate $H(F)$, consider the first differential submodule of the flag $F^0 = R(0,0)$. It is straight-forward to see that

$$\begin{aligned} H(F^0) &= R(0,0) \\ H(F/F^0) &= (R(-1,0) \oplus R(0,-1))/R(-y \oplus x). \end{aligned}$$

From the short exact sequence

$$0 \longrightarrow F^0 \longrightarrow F \longrightarrow F/F^0 \longrightarrow 0$$

we have the long exact sequence

$$\cdots \longrightarrow H(F/F^0) \xrightarrow{\alpha} H(F^0) \xrightarrow{\beta} H(F) \longrightarrow H(F/F^0) \xrightarrow{\alpha} H(F^0) \longrightarrow \cdots$$

where the map α is given by the matrix $\begin{bmatrix} x & y \end{bmatrix}$. Since α is injective, β must be a surjection, giving

$$H(F) = H(F^0)/\operatorname{im}\alpha = R/(x,y) = k.$$

To compute the Betti number, note that $\delta(F) \subseteq \mathfrak{m}F$, so we have $\beta^R(F) = \operatorname{rank}_R F = 4$ by Remark 2.0.13.

3.2 Non-positive differential degree

Every differential R-module with a free flag is free as an R-module, but not conversely (see Example 3.2.4). Even when a differential module admits a free flag there may be no way to "minimize," unlike finite free complexes that can be decomposed into an acyclic complex and a minimal complex C with $\partial(C) \subseteq \mathfrak{m}C$ (see Example 3.2.3). Restricting to the case of a differential module D with $\deg \delta^D \leq \mathbf{0}$ we can avoid both of these difficulties.

Theorem 3.2.1. *Let F be a finitely generated $\mathbb{Z}^d$-graded differential R-module with $\deg \delta^F \leq$ $\mathbf{0}$ that is free as an R-module. Then F has a free flag and a submodule F' that is a direct summand in the category of $\mathbb{Z}^d$-graded differential R-modules such that*

1. *F' has a free flag,*

2. $\delta(F') \subseteq \mathfrak{m}F'$,

3. $H(F') = H(F)$.

Remark 3.2.2. The hypothesis that $\deg \delta \leq \mathbf{0}$ is necessary. See Examples 3.2.3 and 3.2.4.

Proof. We induce on $\operatorname{rank}_R F$: if $\operatorname{rank}_R F = 1$ then the differential of F is multiplication by an element of R. Since R is a domain, this element must be zero; hence $F^0 = F$ is a free flag. As $\delta(F^0) = 0$ we conclude that $\delta(F) \subseteq \mathfrak{m}F$ as well.

Now suppose $\operatorname{rank}_R F > 1$. If $\delta(F) \not\subseteq \mathfrak{m}F$ then there is some homogeneous basis element e with $\delta(e) \notin \mathfrak{m}F$. We first show that $\overline{e}, \delta(\overline{e}) \in F/\mathfrak{m}F$ are linearly independent over k. Suppose that there is a linear relation $\delta(\overline{e}) = a\overline{e}$ with $a \in k$. Since $\delta^2 = 0$, we have $0 = a\delta(\overline{e}) = a^2\overline{e}$, a contradiction.

So e and $\delta(e)$ are linearly independent. By Nakayama's lemma we can take $\{e, \delta(e)\}$ to be part of a basis of F. Let $G = Re \oplus R\delta(e)$. Then G is a differential sub-module. So we have an exact sequence of differential modules:

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0\,. \tag{3.2.1}$$

Since $H(G) = 0$, the long exact sequence in homology coming from (3.2.1) shows that $H(F/G) = H(F)$. The module F/G is free since G is generated by basis elements of F. So by induction F/G has a free flag $\{G^n\}_{n\in\mathbb{Z}}$. By Proposition 2.0.9, F/G is a direct summand of F as differential modules. Setting

$$\begin{aligned} F^0 &= R\delta^F(e), \\ F^1 &= R\delta^F(e) \oplus Re, \\ F^n &= R\delta^F(e) \oplus Re \oplus G^{n-2}, \quad n \geq 2 \end{aligned}$$

gives a free flag on F. The induction hypothesis also shows that F/G has a direct summand F' with a free flag such that $\delta(F') \subseteq \mathfrak{m}F'$ and such that $H(F') = H(F/G) = H(F)$. This completes the proof when $\delta(F) \not\subseteq \mathfrak{m}F$.

Now suppose that $\delta(F) \subseteq \mathfrak{m}F$. In this case it suffices to show that F has a free flag. Let $e_1, \ldots, e_n$ be a homogeneous basis for F, and let $\mathbf{n}$ be a minimal element of $\{\deg(e_1), \ldots, \deg(e_n)\}$ under the partial order on $\mathbb{Z}^d$. Set

$$G = \bigoplus_{\deg(e_i)=\mathbf{n}} Re_i.$$

Then $\delta^F(G) \subseteq G$ since $\deg(\delta^F(e_i)) \leq \deg(e_i)$ for all i as the degree of δ^F is non-positive in each coordinate. So G is a differential sub-module.

We claim that $\delta^F|_G = 0$. When $\deg \delta^F < \mathbf{0}$, we have $\delta^F|_G = 0$ as $\deg(\delta^F(e_i)) < \deg(e_i)$ and all the generators e_i of G have the same degree. When $\deg \delta^F = \mathbf{0}$ the matrix representing $\delta^F|_G$ has entries in k since all generators of G are in the same degree. So $\delta^F|_G = 0$, otherwise there would be an element of $\delta^F(G)$ that is not in $\mathfrak{m}F$, contrary to assumption.

Since $\delta^F|_G = 0$ we get $\delta^F(F^0) = 0$ by setting $F^0 = G$. As F^0 is generated by basis elements of F, the quotient F/F^0 is a free R-module, so the induction hypothesis produces a free flag $\{G^n\}_{n\in\mathbb{Z}}$ for F/F^0. Setting $F^n = F^0 \oplus G^{n-1}$ for $n \geq 0$ and $F^n = 0$ for $n < 0$ gives a free flag on F. $\square$

The next example illustrates several difficulties in dealing with differential modules with non-zero differential degree. It provides an obstruction to extending Theorem 3.4.4 to differential modules with $\deg \delta > \mathbf{0}$. Furthermore, by [1, Theorem 5.2], a differential module over $k[x, y]$ with a free flag must have rank at least 4. So this also shows that Lemma 3.2.1 cannot be extended to differential modules with $\deg \delta > \mathbf{0}$ as no summand can have a free flag.

Example 3.2.3. Let $R = k[x,y]$ and let $F = R(0,0) \oplus R(0,1) \oplus R(1,0) \oplus R(1,1)$ have differential given by the matrix,

$$\delta = \begin{bmatrix} 0 & x & y & 1 \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is a differential module with differential degree $(1,1)$. As a diagram it is

$$\begin{tikzcd} R(1,1) \arrow[r, "{\left[\begin{smallmatrix} -y \\ x \end{smallmatrix}\right]}"'] \arrow[rr, bend left, "1"] & R(0,1) \oplus R(1,0) \arrow[r, "{[x\ y]}"'] & R(0,0) \arrow[r] & 0\,. \end{tikzcd} \tag{3.2.2}$$

As in Example 3.1.3, reading the diagram from right to left gives a free flag. The same computation from Example 3.1.3 shows that $H(F) = k$. As F has a free flag, we can compute $\beta^R(F)$ by $\operatorname{rank}_k H(k \boxtimes_R F)$. Applying $k \boxtimes_R -$ to (3.2.2) we have the vector space k^4 (suppressing the grading) with differential given by the diagram:

$$\begin{tikzcd} k \arrow[r, "0"'] \arrow[rr, bend left, "1"] & k^2 \arrow[r, "0"'] & k \arrow[r] & 0\,. \end{tikzcd}$$

The homology is k^2, so $\beta^R(F) = 2$.

This final example shows that a differential module that is free as an R-module need not have a free flag; thus Lemma 3.2.1 cannot be strengthened to apply to differential modules with $\deg \delta > \mathbf{0}$.

Example 3.2.4 ([1, Example 5.6])**.** Let F be as in Example 3.2.3. Let e be the basis element in degree $(-1,-1)$ and set $G = Re \oplus R\delta^F(e)$. Then calculation shows that F/G is the differential module $D = R(0,1) \oplus R(1,0)$ with

$$\delta = \begin{bmatrix} xy & -y^2 \\ x^2 & -xy \end{bmatrix}.$$

This is a differential module with $\deg \delta = (1,1)$. Since $H(G) = 0$, an exact sequence argument shows that the map $F \to F/G$ is a quasi-isomorphism; hence $H(D) = H(F) = k$. As F admits a free flag, it is a resolution of D. So we have $\beta^R(D) = \beta^R(F) = 2$.

The differential module D itself cannot have a free flag since $\operatorname{rank}_R D = 2 < 4$, as noted before Example 3.2.3.

3.3 High-low decompositions

The main tool, Theorem 3.3.6, we use for finding a bound on the Betti number comes from an inequality of Santoni [13] formulated to apply to differential modules. The essential idea is to use information about the "top" and "bottom" degree parts to derive information about the entire module. The meaning of "top" and "bottom" is made precise by a *high-low decomposition*, Definition 3.3.5.

Let y be an indeterminate over $R = k[x_1, \ldots, x_d]$ with $\deg y = (0, \ldots, 0, 1) \in \mathbb{Z}^{d+1}$, so that $R[y]$ is a $\mathbb{Z}^{d+1}$-graded ring. In this section we will be concerned with comparing $\mathbb{Z}^{d+1}$-graded differential modules over $R[y]$ with $\mathbb{Z}^d$-graded differential modules over R. Via the inclusion $R \hookrightarrow R[y]$, any $\mathbb{Z}^{d+1}$-graded differential module over $R[y]$ can be considered as a $\mathbb{Z}^d$-graded differential module over R, with the action of R fixing the $(d+1)$-th coordinate of the $\mathbb{Z}^{d+1}$-grading. The following result allows this change of rings to be applied to the Tor functor.

Lemma 3.3.1. *Let M be a $\mathbb{Z}^{d+1}$-graded $R[y]$-module and D a $\mathbb{Z}^{d+1}$-graded differential $R[y]$-*

module. View $R[y] \boxtimes_R D$ *as a* $R[y]$*-module via the action* $r(s \otimes d) = (rs) \otimes d$*. Then*

$$\mathrm{Tor}^{R[y]}(M, R[y] \boxtimes_R D) \cong \mathrm{Tor}^R(M, D).$$

Proof. Let P be a $\mathbb{Z}^{d+1}$-graded free resolution of M over $R[y]$. Then using Proposition 2.0.4 one gets:

$$\begin{aligned} \mathrm{Tor}^{R[y]}(M, R[y] \boxtimes_R D) &= H(P \boxtimes_{R[y]} (R[y] \boxtimes_R D)) \\ &\cong H((P \otimes_{R[y]} R[y]) \boxtimes_R D) \\ &\cong H(P \boxtimes_R D) \\ &= \mathrm{Tor}^R(M, D). \end{aligned}$$

□

Let $\mathcal{C}$ be a class of $\mathbb{Z}^{d+1}$-graded differential $R[y]$-modules which is closed under taking submodules and quotients. Take λ to be a superadditive function from $\mathcal{C}$ to an ordered commutative monoid such that $\lambda(C) \geq 0$ for all $C \in \mathcal{C}$. Recall that λ is superadditive if an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of differential modules in $\mathcal{C}$ gives an inequality $\lambda(B) \geq \lambda(A) + \lambda(C)$.

Example 3.3.2. For our purposes, $\mathcal{C}$ will be the collection of $\mathbb{Z}^{d+1}$-graded differential $R[y]$-modules with non-zero homology in finitely many degrees and λ will be the length of the differential module.

Lemma 3.3.3. *Let* B *be a* $\mathbb{Z}^{d+1}$*-graded differential* $R[y]$*-module and suppose we have the* *following commutative diagrams in $\mathcal{C}$:*

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, two heads, "\psi_A"'] & B \arrow[d, "\psi_B"] \\ A'' \arrow[r, hook, "\iota''"'] & B'' \end{tikzcd} \qquad \begin{tikzcd} B' \arrow[r, two heads, "\varepsilon'"] \arrow[d, "\phi_B"'] & C' \arrow[d, hook, "\phi_C"] \\ B \arrow[r, "\varepsilon"'] & C \end{tikzcd}$$

Then for each $\mathbf{m} \in \mathbb{Z}^{d+1}$ the following inequalities hold:

$$\lambda(\operatorname{im}\iota)_{\mathbf{m}} \geq \lambda(\operatorname{im}\iota'')_{\mathbf{m}} \qquad \textit{and} \qquad \lambda(\operatorname{im}\varepsilon)_{\mathbf{m}} \geq \lambda(\operatorname{im}\varepsilon')_{\mathbf{m}}.$$

Furthermore, if $\varepsilon\iota = 0$ then

$$\lambda(B_{\mathbf{m}}) \geq \lambda(\operatorname{im}\iota'')_{\mathbf{m}} + \lambda(\operatorname{im}\varepsilon')_{\mathbf{m}}.$$

Proof. For the first inequality, there is a surjection $\operatorname{im}\iota \twoheadrightarrow \operatorname{im}\psi_B\iota$, so

$$\lambda(\operatorname{im}\iota)_{\mathbf{m}} \geq \lambda(\operatorname{im}\psi_B\iota)_{\mathbf{m}} = \lambda(\operatorname{im}\iota''\psi_A)_{\mathbf{m}}.$$

Because ψ_A is surjective there is also a surjection $\operatorname{im}\iota''\psi_A \twoheadrightarrow \operatorname{im}\iota''$. This gives the desired inequality, $\lambda(\operatorname{im}\iota)_{\mathbf{m}} \geq \lambda(\operatorname{im}\iota'')_{\mathbf{m}}$.

For the second inequality, there is an inclusion $\operatorname{im}\varepsilon' \hookrightarrow \operatorname{im}\varepsilon$ since ϕ_C is injective. By superadditivity, $\lambda(\operatorname{im}\varepsilon)_{\mathbf{m}} \geq \lambda(\operatorname{im}\varepsilon')_{\mathbf{m}}$.

For the final inequality, note that $\varepsilon\iota = 0$ implies that $\operatorname{im}\iota \subseteq \ker\varepsilon$. The exact sequence

$$0 \longrightarrow \ker\varepsilon \longrightarrow B \longrightarrow \operatorname{im}\varepsilon \longrightarrow 0\,,$$

then implies $\lambda(B_{\mathbf{m}}) \geq \lambda(\operatorname{im}\varepsilon)_{\mathbf{m}} + \lambda(\operatorname{im}\iota)_{\mathbf{m}} \geq \lambda(\operatorname{im}\varepsilon')_{\mathbf{m}} + \lambda(\operatorname{im}\iota'')_{\mathbf{m}}$ using the first two inequalities. $\square$

Lemma 3.3.4. *Let D be a $\mathbb{Z}^{d+1}$-graded differential $R[y]$-module. Viewing $R[y] \boxtimes_R D$ as a $R[y]$-module via the action $r(s \otimes d) = (rs) \otimes d$, there is a sequence of $\mathbb{Z}^{d+1}$-graded differential $R[y]$-modules*

$$0 \longrightarrow (R[y] \boxtimes_R D)(-\deg y) \xrightarrow{\sigma} R[y] \boxtimes_R D \xrightarrow{\varepsilon} D \longrightarrow 0\,,$$

with $\sigma(1 \otimes d) = y \otimes d - 1 \otimes yd$ and $\varepsilon(a \otimes d) = ad$. This sequence is exact and functorial in D. The map σ is given by multiplication by y if and only if $yD = 0$.

Proof. It is straight-forward to check that σ and ε are morphisms and that the sequence is exact and functorial. Evidently σ is multiplication by y when $yD = 0$. The exactness of the sequence shows that the converse holds. $\square$

The following definition and theorem are differential module versions of Santoni's results for R-modules [13].

Definition 3.3.5. A $\mathbb{Z}^{d+1}$-graded differential $R[y]$-module D admits a *high-low decomposition* if there are non-zero $\mathbb{Z}^{d+1}$-graded differential $R[y]$-modules D_h and D_ℓ each annihilated by y, and there are morphisms of differential $R[y]$-modules $D_h \hookrightarrow D$ and $D \twoheadrightarrow D_\ell$ that split in the category of $\mathbb{Z}^d$-graded differential R-modules.

Theorem 3.3.6. *Let K be a $\mathbb{Z}^{d+1}$-graded $R[y]$-module such that $yK = 0$, and assume $\mathcal{C}$ is closed under $\operatorname{Tor}^{R[y]}(K, -)$. Let $D \in \mathcal{C}$ be a $\mathbb{Z}^{d+1}$-graded differential module with differential degree $\mathbf{d}$ which admits a high-low decomposition. Then for all $\mathbf{m} \in \mathbb{Z}^{d+1}$*

$$\lambda(\operatorname{Tor}^{R[y]}(K, D)_{\mathbf{m}}) \geq \lambda(\operatorname{Tor}^{R}(K, D_\ell)_{\mathbf{m}}) + \lambda(\operatorname{Tor}^{R}(K, D_h)_{\mathbf{m}+\mathbf{d}-\deg y}).$$

Proof. Applying the functoriality of Lemma 3.3.4 to the high-low decomposition $D_h \hookrightarrow D$

and $D \twoheadrightarrow D_\ell$ gives two exact commutative diagrams:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (R[y] \boxtimes_R D_h)(-\deg y) \arrow[r, "\sigma'"] \arrow[d] & & R[y] \boxtimes_R D_h \arrow[r, "\varepsilon'"] \arrow[d] & D_h \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (R[y] \boxtimes_R D)(-\deg y) \arrow[rr, "\sigma"] & & R[y] \boxtimes_R D \arrow[r, "\varepsilon"] & D \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & (R[y] \boxtimes_R D)(-\deg y) \arrow[r, "\sigma"] \arrow[d] & R[y] \boxtimes_R D \arrow[r, "\varepsilon"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (R[y] \boxtimes_R D_\ell)(-\deg y) \arrow[r, "\sigma''"] \arrow[d] & R[y] \boxtimes_R D_\ell \arrow[r, "\varepsilon''"] \arrow[d] & D_\ell \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In both diagrams the first two columns are split exact over $R[y]$ due to the high-low decomposition. Because D_h and D_ℓ are annihilated by y, Lemma 3.3.4 implies that σ' and σ'' are multiplication by y. The $R[y]$-action on $\operatorname{Tor}^{R[y]}(K, -)$ is via K and $yK = 0$, so after applying $\operatorname{Tor}^{R[y]}(K, -)$ and using Lemma 3.3.1 the maps σ' and σ'' become zero, leaving

$$\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & \\
 & 0 \arrow[r] & \operatorname{Tor}^R(K, D_h) \arrow[r, "\varepsilon'"] \arrow[d] & \operatorname{Tor}^{R[y]}(K, D_h) \arrow[r, "\gamma'"] \arrow[d] \arrow[dr, phantom, "(\dagger)"] & \operatorname{Tor}^R(K, D_h)(\mathbf{d} - \deg y) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[rr, "\sigma"] & & \operatorname{Tor}^R(K, D) \arrow[r, "\varepsilon"] & \operatorname{Tor}^{R[y]}(K, D) \arrow[r, "\gamma"] & \operatorname{Tor}^R(K, D)(\mathbf{d} - \deg y) \arrow[r, "\sigma(\mathbf{d})"] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdots \arrow[r, "\sigma"] & \mathrm{Tor}^R(K,D) \arrow[r, "\varepsilon"] \arrow[d] \arrow[dr, phantom, "(\ddagger)"] & \mathrm{Tor}^{R[y]}(K,D) \arrow[r, "\gamma"] \arrow[d] & \mathrm{Tor}^R(K,D)(\mathbf{d}-\deg y) \arrow[r, "\sigma(\mathbf{d})"] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{Tor}^R(K,D_\ell) \arrow[r, "\varepsilon''"] \arrow[d] & \mathrm{Tor}^{R[y]}(K,D_\ell) \arrow[r, "\gamma''"] & \mathrm{Tor}^R(K,D_\ell)(\mathbf{d}-\deg y) \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 &
\end{tikzcd}$$

Lemma 3.3.3 on the commutative squares (†) and (‡) completes the proof. □

3.4 Lower bound on the Betti number

In order to apply the results for high-low decompositions we need to establish some results on the existence of high-low decompositions D_h and D_ℓ with $H(D_h) \neq 0$ and $H(D_\ell) \neq 0$.

Recall that $\mathbf{m}_i$ denotes the i-th coordinate of a d-tuple $\mathbf{m} \in \mathbb{Z}^d$.

Definition 3.4.1. Let D be a $\mathbb{Z}^d$-graded differential module and let $1 \leq i \leq d$. We say that D is *bounded in the i-th direction* if there are $a, b \in \mathbb{Z}$ such that $\mathbf{m}_i \notin [a,b]$ implies $D_{\mathbf{m}} = 0$.

Remark 3.4.2. When D is finitely generated the condition that D is bounded in the i-th direction for all i is equivalent to the condition that $\operatorname{rank}_k D < \infty$.

Lemma 3.4.3. *Let D be a $\mathbb{Z}^d$-graded differential module with $H(D) \neq 0$. Fix an index $1 \leq i \leq d$ and suppose that $(\deg \delta^D)_i = 0$. If $H(D)$ is bounded in the i-th direction then there is a $\mathbb{Z}^d$-graded differential module D' that is quasi-isomorphic to D such that D' has a high-low decomposition D'_h and D'_ℓ with $H(D'_h)$ and $H(D'_\ell)$ both non-zero.*

Proof. Let $a \in \mathbb{Z}$ be the largest integer such that $H(D)_{\mathbf{m}} = 0$ whenever $\mathbf{m}_i < a$. Such an integer exists because $H(D)$ is non-zero and bounded in the i-th direction. Set

$$E = \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}^d \\ a \leq \mathbf{m}_i}} D_{\mathbf{m}}.$$

This is an R-submodule. Since $(\deg \delta^D)_i = 0$ it is closed under δ^D as well. So E is a differential submodule of D. By the definition of E, we have

$$D/E = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ \mathbf{m}_i<a}} D_{\mathbf{m}}.$$

Let z be a cycle in $(D/E)_{\mathbf{m}}$. If $\mathbf{m}_i \geq a$ then $z = 0$ as $(D/E)_{\mathbf{m}} = 0$. If $\mathbf{m}_i < a$ then $z \in (D/E)_{\mathbf{m}} = D_{\mathbf{m}}$ so there is a $z' \in D$ with $\delta^D(z') = z$ as $H(D)_{\mathbf{m}} = 0$. So $\delta^{D/E}(z'+E) = z$. Therefore $H(D/E)_{\mathbf{m}} = 0$ for all $\mathbf{m} \in \mathbb{Z}^d$, and so $H(D/E) = 0$. From a short exact sequence we conclude that $E \simeq D$.

Let $b \in \mathbb{Z}$ be the smallest integer such that $H(E)_{\mathbf{m}} = 0$ when $\mathbf{m}_i > b$. Again, such an integer exists because $H(E) \cong H(D)$ is non-zero and bounded in the i-th direction. Set

$$E' = \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ b+1\leq\mathbf{m}_i}} E_{\mathbf{m}}.$$

Then E' is a differential submodule of E with $H(E') = 0$ by the definition of b. Set $D' = E/E'$. From a short exact sequence we conclude that $H(E/E') \cong H(E)$ so that $D' = E/E' \simeq E \simeq D$.

By construction, $D'_{\mathbf{m}} = 0$ for $\mathbf{m}_i < a$ and for $\mathbf{m}_i > b$. Also, by the definitions of a and b, there are $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^d$ with $\mathbf{n}_i = a$ and $\mathbf{n}'_i = b$ such that $H(D')_{\mathbf{n}} \neq 0$ and $H(D')_{\mathbf{n}'} \neq 0$; hence $D'_{\mathbf{n}} \neq 0$ and $D'_{\mathbf{n}'} \neq 0$ as well.

Set

$$D'_\ell := \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ \mathbf{m}_i=a}} D'_{\mathbf{m}} \qquad \text{and} \qquad D'_h := \bigoplus_{\substack{\mathbf{m}\in\mathbb{Z}^d \\ \mathbf{m}_i=b}} D'_{\mathbf{m}}.$$

Then D'_ℓ and D'_h are both non-zero and annihilated by x_i. The two morphisms $D'_h \hookrightarrow D'$ and $D' \twoheadrightarrow D'_\ell$ split in the category of differential modules because $(\deg \delta^D)_i = 0$. So

D'_ℓ and D'_h form a high-low decomposition. As noted above $H(D'_\ell)$ and $H(D'_h)$ are both non-zero, so D' is the desired differential module. $\square$

The proof of the following theorem uses Theorem 3.3.6 inductively, after first using Lemma 3.4.3 to find a differential module with a high-low decomposition.

Note that $H(D)$ is not required to be finitely generated in the following theorem. If $H(D)$ is finitely generated then the hypothesis on $H(D)$ is equivalent to $0 < \operatorname{rank}_k H(D) < \infty$; see Remark 3.4.2.

Theorem 3.4.4. *If D is a $\mathbb{Z}^d$-graded differential module with $\deg \delta^D = \mathbf{0}$ and such that $H(D) \neq 0$ is bounded in the i-th direction for all i, then*

$$\beta^R(D) \geq 2^d.$$

Proof. Use induction on d. For $d = 0$, so that $R = k$, we have

$$\operatorname{Tor}^k(k, D) = H(k \boxtimes_k D) \cong H(D) \neq 0.$$

So $\beta^k(D) \geq 1$.

Now suppose $d > 1$. Then $H(D)$ is bounded in the d-th direction by assumption. By Proposition 2.0.10 the Betti number is preserved under quasi-isomorphisms, so Lemma 3.4.3 allows us to assume that D has a high-low decomposition D_h and D_ℓ with $H(D_h) \neq 0$ and $H(D_\ell) \neq 0$. By definition of a high-low decomposition, $H(D_h)$ and $H(D_\ell)$ are submodules of $H(D)$ since the splitting happens in the category of differential modules. In particular, $H(D_h)$ and $H(D_\ell)$ are bounded in the i-th direction for all i.

So induction hypothesis applies to D_h and D_ℓ. From Theorem 3.3.6 we have:

$$\begin{aligned}\beta^R(D) &\geq \beta^{k[x_1,\dots,x_{d-1}]}(D_\ell) + \beta^{k[x_1,\dots,x_{d-1}]}(D_h)\\ &\geq 2^{d-1} + 2^{d-1}\\ &= 2^d.\end{aligned}$$ □

Remark 3.4.5. Example 3.2.3 shows that Theorem 3.4.4 cannot be extended to differential modules D with $\deg \delta^D > \mathbf{0}$.

Via Theorem 2.0.12 this result provides an affirmative answer to Conjecture 1.0.1 when $\deg \delta = \mathbf{0}$.

Corollary 3.4.6. *If F is a finitely generated $\mathbb{Z}^d$-graded differential module that is free as an R-module such that $\deg \delta^F = \mathbf{0}$ and such that $H(F)$ has non-zero finite length then*

$$\operatorname{rank}_R F \geq 2^d.$$

Proof. By Lemma 3.2.1, F has a free flag. So Theorem 2.0.12 implies that $\beta^R(F) \leq \operatorname{rank}_R F$. Applying Theorem 3.4.4 gives the desired inequality. □

Chapter 4

Derived Loewy Length

4.1 Notions of Loewy length

Lemma 3.4.3 can be seen as a means of taking a differential module with finite length homology and replacing it by a new differential module that has finite length. In fact, a similar result is true for chain complexes (see Theorem 4.1.1 below). In this chapter we will take this as motivation for investigating and comparing ways to measure the size of such replacements.

The following is essentially contained in [12].

Theorem 4.1.1. *Let R be a noetherian ring and let C be a complex of finitely generated R-modules such that $H(C)$ has finite length. Then there is a complex C' such that C' has finite length and C' is isomorphic to C in the derived category.*

Proof. If $H(C) = 0$ then take C' to be the zero complex. Otherwise there are $a, b \in \mathbb{Z}$ with $H_a(C) \neq 0$ and $H_b(C) \neq 0$ and $H_i(C) = 0$ for $i < a$ or $i > b$. Let $C_{\subseteq a}$ denote the soft truncation above a, and let $C_{\supseteq b}$ denote the soft truncation below b. Truncating in succession we get the complex $(C_{\subseteq a})_{\supseteq b}$:

$$0 \longrightarrow C_a/\partial(C_{a+1}) \longrightarrow C_{a-1} \longrightarrow \cdots \longrightarrow C_{b+1} \longrightarrow \ker \partial_b \longrightarrow 0\ .$$

Note that there is a quasi-isomorphism from C to $C_{\subseteq a}$ and a quasi-isomorphism from $(C_{\subseteq a})_{\supseteq b}$ to $C_{\subseteq a}$. So in the derived category the complex $(C_{\subseteq a})_{\supseteq b}$ is isomorphic to C.

So we may assume that $C_i = 0$ for $i < a$ or $i > b$. Proceed by induction on $a - b$. If $a - b = 0$, then $H_a(C) = C_a$ and $H_a(C)$ has finite length by hypothesis. Now suppose $a - b > 0$. Let $E(H_a(C))$ denote the injective envelope of $H_a(C)$. Since $H_a(C)$ injects into C_a we have the following diagram

$$\begin{tikzcd} H_a(C) \arrow[r] \arrow[d] & C_a\ . \arrow[dl, dashed, "\phi"] \\ E(H_a(C)) & \end{tikzcd}$$

Let $C'_a = \operatorname{im}\phi$. Then C'_a is artinian as it is the submodule of the injective hull of the finite length module $H_a(C)$. Since C'_a is finitely generated, it is also noetherian, and hence has finite length. Note that the restriction of ϕ to $H_a(C)$ must be injective by commutativity of the diagram.

Using the push-out $C'_a \amalg_{C_a} C_{a-1}$, build the following diagram:

$$\begin{tikzcd} C_a \arrow[r, "\partial_a"] \arrow[d, "\phi"] & C_{a-1} \arrow[r] \arrow[d] & C_{a-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & C_b \arrow[d, equal] \\ C'_a \arrow[r, "\partial'_a"] & C'_a \amalg_{C_a} C_{a-1} \arrow[r, "\partial'_{a-1}"] & C_{a-2} \arrow[r] & \cdots \arrow[r] & C_b \end{tikzcd}$$

where $\partial'_a(m) = (m, 0)$ and $\partial'_{a-1}(m, n) = \partial_{a-1}(n)$.

We claim that this is a quasi-isomorphism between the rows. An element of $C'_a \amalg_{C_a} C_{a-1}$ is zero if and only if can be written in the form $(\phi(m), \partial_a(m))$. So m is in the kernel of ∂'_a if and only if there is an $m' \in C_a$ with $\phi(m') = m$ and $\partial_a(m') = 0$. Thus every element $m \in \ker \partial'_a$ is in $\phi(H_a(C))$. Since ϕ is injective when restricted to $H_a(C)$, we get an isomorphism $H_a(C) \cong \ker \partial'_a$, so the homology of the two rows agrees here.

In degree $a-1$, a homology class $[z]$ maps to the class $[(0,z)]$. We can write a cycle (m,n) of $C'_a \amalg_{C_a} C_{a-1}$ in the form $(0, n + \partial_a(m'))$ with $\phi(m') = m$. Since $\partial_a(n) = 0$, this shows that map in degree $a-1$ homology is surjective. If the cycle $(0,z)$ is a boundary

$$\partial'_a(m) = (m, 0) = (0, \partial_a(m'))$$

with $\phi(m') = m$, then $(0, z - \partial_a(m'))$ is zero. In particular there is an $n' \in C_a$ such that $\partial_a(n) = z - \partial_a(m')$. Thus z is a boundary, and we conclude that the map is a quasi-isomorphism.

Now note that the cycles $\ker \partial'_{a-1}$ have finite length as there is an exact sequence

$$0 \longrightarrow \partial'_a(C'_a) \longrightarrow \ker \partial'_{a-1} \longrightarrow H_{a-1}(C) \longrightarrow 0$$

and both $\partial'_a(C'_a)$ and $H_{a-1}(C)$ have finite length. So the induction hypothesis applies to the complex

$$C'_a \amalg_{C_a} C_{a-1} \xrightarrow{\partial'_{a-1}} C_{a-2} \longrightarrow \cdots \longrightarrow C_b\ .$$

Splicing the quasi-isomorphic finite length complex with C'_a via the map $\partial'_a : C'_a \to \ker \partial'_{a-1}$ gives the desired finite length complex C'. $\square$

In this chapter we will be concerned with measuring the sizes of possible replacements. Let $(R, \mathfrak{m})$ be a commutative local ring with maximal ideal $\mathfrak{m}$. Denote the derived category of R by $\mathcal{D}(R)$.

Definition 4.1.2. Let M be an R-module that is not necessarily finitely generated. The

Loewy length $\ell\ell_R(M)$ of M is

$$\ell\ell_R(M) = \inf\{n \geq 0 \mid \mathfrak{m}^n M = 0\}.$$

For a complex C of R-modules, define $\ell\ell_R(C)$ to be the Loewy length of the underlying module.

Subscripts are omitted when the ring is understood.

When M is finitely generated then the Loewy length is finite if and only if the ordinary length is finite, in the sense of a composition series. We are interested in extending this invariant to the derived category $\mathcal{D}(R)$. One option is to allow replacements by objects that are isomorphic in the derived category. This gives the homotopical Loewy length as defined by Avramov, Iyengar and Miller [2].

Definition 4.1.3. Let C be a complex of R-modules. Define the *homotopical Loewy length* $\mathrm{h}\ell\ell_R(C)$ to be

$$\mathrm{h}\ell\ell_R(C) = \inf\{\ell\ell(V) \mid C \simeq V\}.$$

Another option is to allow replacements by objects that "contain" the object of interest. Allowing this extra flexibility appears to give an invariant with better homological properties.

Definition 4.1.4. Let C be a complex of R-modules. Define the *derived Loewy length* $\mathrm{d}\ell\ell_R(C)$ to be

$$\mathrm{d}\ell\ell_R(C) = \inf\{\ell\ell(V) \mid C \text{ is a retract of } V \text{ in } \mathcal{D}(R)\}.$$

Obviously

$$\ell\ell(H(C)) \leq \mathrm{d}\ell\ell(C) \leq \mathrm{h}\ell\ell(C) \leq \ell\ell(C),$$

but not much is known about the precise value of $\mathrm{h}\ell\ell(C)$ or $\mathrm{d}\ell\ell(C)$.

We will focus attention on the Koszul complex of a minimal generating set for $\mathfrak{m}$. Recall that the Koszul complex $K(x_1, \ldots, x_n)$ on a sequence of elements $x_1, \ldots, x_n$ in R is defined to be

$$K(x_1, \ldots, x_n) = K(x_1) \otimes_R \cdots \otimes_R K(x_n),$$

where $K(x_i)$ is the complex

$$0 \longrightarrow R \xrightarrow{x_i} R \longrightarrow 0\,.$$

Throughout, K^R will denote the Koszul complex of a minimal generating set for $\mathfrak{m}$. Different choices for minimal generating sets give isomorphic Koszul complexes, however the derived Loewy length is independent of the choice even if we allow non-minimal generating sets.

Proposition 4.1.5. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be sequences of elements of R that generate the same ideal. Then*

$$\mathrm{d}\ell\ell(K(x_1, \ldots, x_n)) = \mathrm{d}\ell\ell(K(y_1, \ldots, y_m)).$$

Proof. It suffices to show that

$$\mathrm{d}\ell\ell(K(x_1, \ldots, x_n)) = \mathrm{d}\ell\ell(K(x_1, \ldots, x_n, y))$$

for $y \in (x_1, \ldots, x_n)$. In this case there is an isomorphism of complexes $K(x_1, \ldots, x_n, y) \cong K(x_1, \ldots, x_n, 0)$, see [3, 1.6.21]. As complexes,

$$K(x_1, \ldots, x_n, 0) \cong K(x_1, \ldots, x_n) \oplus \Sigma K(x_1, \ldots, x_n).$$

Thus we have

$$\mathrm{d}\ell\ell(K(x_1, \ldots, x_n)) \leq \mathrm{d}\ell\ell(K(x_1, \ldots, x_n, y))$$

as $K(x_1, \ldots, x_n)$ is a retract of $K(x_1, \ldots, x_n, 0)$.

If $K(x_1, \ldots, x_n)$ is a retract of V in $\mathcal{D}(R)$ then $K(x_1, \ldots, x_n, 0)$ is a retract of $V \oplus \Sigma V$, so

$$\mathrm{d}\ell\ell(K(x_1, \ldots, x_n, y)) \leq \mathrm{d}\ell\ell(K(x_1, \ldots, x_n)). \qquad \square$$

4.2 Bounds on Loewy lengths

Given the variety of notions for extending the definition of Loewy length, it is natural to compare them. An upper bound for $\mathrm{h}\ell\ell(K^R)$ is given by [2] in the form of a new invariant.

Definition 4.2.1. Since $\partial^{K^R}(\mathfrak{m}^i K_n) \subset \mathfrak{m}^{i+1} K_{n-1}$, for each i we get a complex

$$I^i_{K^R} = \ 0 \longrightarrow \mathfrak{m}^{i-d} K^R_d \longrightarrow \mathfrak{m}^{i-s+1} K^R_{d-1} \longrightarrow \cdots \longrightarrow \mathfrak{m}^{i-1} K^R_1 \longrightarrow \mathfrak{m}^i K^R_0 \longrightarrow 0 \,.$$

Define the *spread* of R to be

$$\mathrm{spread}(R) = \inf\{i \in \mathbb{Z} \mid H(I^j_{K^R}) = 0 \text{ for all } j \geq i\}.$$

Proposition 4.2.2 ([2, 6.2.2])**.** *We have*

$$\mathrm{h}\ell\ell_R(K^R) \leq \mathrm{spread}(R).$$

Proof. Let $s = \mathrm{spread}(R)$. Then

$$I^s_{K^R} = \ 0 \longrightarrow \mathfrak{m}^{s-d} K^R_d \longrightarrow \cdots \longrightarrow \mathfrak{m}^s K^R_0 \longrightarrow 0$$

is an exact subcomplex of K^R. So $K^R \simeq K^R / I^s_{K^R}$, but $\mathfrak{m}^s \cdot (K^R / I^s_{K^R}) = 0$. $\square$

By work of Serre, we know that $\mathrm{spread}\, R < \infty$. Part of the interest in focusing on the

Koszul complex K^R comes from the following theorem. The equivalence of the first three conditions is due to Avramov, Iyengar and Miller [2, 6.2.3].

Theorem 4.2.3. *The following are equivalent:*

1. *R is regular,*

2. $\operatorname{spread} R \leq 1$*,*

3. $\mathrm{h}\ell\ell(K^R) \leq 1$*,*

4. $\mathrm{d}\ell\ell(K^R) \leq 1$*.*

Part of the proof of this theorem relies on having an explicit collection of objects to test the derived Loewy length.

Proposition 4.2.4. *If F is a complex of free R-modules then*

$$\mathrm{d}\ell\ell(F) = \inf\{n \geq 0 \mid F \textit{ is a retract of } F/\mathfrak{m}^n F \textit{ in } \mathcal{D}(R)\}.$$

Proof. Clearly, $\mathrm{d}\ell\ell(F)$ is at most the right-hand side. So it suffices to check the other inequality. Suppose that F is a retract of V in $\mathcal{D}(R)$ with $\ell\ell(V) = n$. So we have morphisms $f : F \to V$ and $g : V \to F$ in $\mathcal{D}(R)$ such that gf is the identity on F. Since F is a complex of free modules we can assume that the morphism $f : F \to V$ is an ordinary morphism of complexes. As $\mathfrak{m}^n V = 0$, we have $\mathfrak{m}^n F$ in the kernel of f, so we get an induced map $f' : F/\mathfrak{m}^n F \to V$.

We claim that F is a retract of $F/\mathfrak{m}^n F$ via the canonical map $\pi : F \to F/\mathfrak{m}^n F$. We have

$$(gf')\pi = g(f'\pi) = gf,$$

and gf is the identity on F. □

Corollary 4.2.5. *For F a complex of free R-modules we have*

$$\mathrm{d}\ell\ell(F) > \sup\{n \mid z \in Z(F),\ [z] \neq 0 \in H(F) \text{ and } z \in \mathfrak{m}^n F\}.$$

Proof. Let $m = \mathrm{d}\ell\ell(F)$. Then F is a retract of $F/\mathfrak{m}^m F$. In particular, $H(F)$ embeds in $H(F/\mathfrak{m}^m F)$. Thus every cycle in F representing a non-zero homology element must map to a non-zero element of $F/\mathfrak{m}^m F$. So if z is a cycle of F that is not a boundary and $z \in \mathfrak{m}^n$ then we must have $m > n$. □

With this in hand, the proof of Theorem 4.2.3 is straight-forward.

Proof of 4.2.3.

$(1 \Rightarrow 2)$. Since R is regular, K^R is a free resolution of the residue field k. Thus $\partial(K_1^R) = \mathfrak{m}K_0^R$, so $I^1_{K^R}$ is exact. Thus $\operatorname{spread} R \leq 1$.

$(2 \Rightarrow 3)$. Clear as $\mathrm{h}\ell\ell(K^R) \leq \operatorname{spread} R$.

$(3 \Rightarrow 4)$. Clear as $\mathrm{d}\ell\ell(K^R) \leq \mathrm{h}\ell\ell(K^R)$.

$(4 \Rightarrow 1)$. By Corollary 4.2.5, there cannot be any cycles in $\mathfrak{m}K^R$ that represent non-zero homology elements. However, for $i > 0$ a non-zero element of $H_i(K^R)$ must be represented by some cycle in $\mathfrak{m}K^R$. Therefore $H_i(K^R) = 0$ for $i > 0$, and so K^R is a finite free resolution of the residue field k. □

The lower bound in Proposition 4.2.4 can be tight and, in fact it is strong enough to allow us to compute $\mathrm{d}\ell\ell(K^R)$ and $\mathrm{h}\ell\ell(K^R)$ for any artinian ring.

Proposition 4.2.6. *Let $(R, \mathfrak{m})$ be an artinian ring. Then $\mathrm{d}\ell\ell(K^R) = \mathrm{h}\ell\ell(K^R) = \ell\ell(R)$.*

Proof. Let $n = \ell\ell(R)$. Then $\mathfrak{m}^n = 0$ and $\mathfrak{m}^{n-1} \neq 0$. For e the embedding dimension, We have $H_e(K^R) = (0 : \mathfrak{m})$, but $\mathfrak{m}^{n-1} \subseteq (0 : \mathfrak{m})$ and $\mathfrak{m}^{n-1}$ is non-zero. So there is a non-zero element in $H_e(K^R)$ represented by a cycle in $\mathfrak{m}^{n-1}$. Thus $\mathrm{d}\ell\ell(K^R) > n-1$ and

$\mathrm{d}\ell\ell(K^R) \leq \ell\ell(R) = n$. So $\mathrm{d}\ell\ell(K^R) = \ell\ell(R)$. Since $\mathrm{d}\ell\ell(K^R) \leq \mathrm{h}\ell\ell(K^R) \leq \ell\ell(R)$ we have $\mathrm{h}\ell\ell(K^R) = \ell\ell(R)$ as well. □

While the spread is a convenient upper bound, it is often too large. Another upper bound can be found by extending Loewy length to give a finite number for non-artinian rings. Ding [8] defines the generalized Loewy length of a ring as follows.

Definition 4.2.7. If $\dim R > 0$ define the *generalized Loewy length* to be

$$\mathrm{g}\ell\ell(R) = \inf\{\ell\ell(R/(x_1, \dots, x_n)) \mid x_1, \dots, x_n \text{ a system of parameters}\}.$$

If $\dim R = 0$ set $\mathrm{g}\ell\ell\, R = \ell\ell_R\, R$.

For Cohen-Macaulay rings we can use the generalized Loewy length as an upper bound on the derived Loewy length.

Proposition 4.2.8. *If R is Cohen-Macaulay then $\mathrm{d}\ell\ell(K^R) \leq \mathrm{g}\ell\ell(R)$.*

Proof. Take a system of parameters $x_1, \dots, x_d$. Since R is Cohen-Macaulay this is a regular sequence. Extend this to a sequence $x_1, \dots, x_d, x_{d+1}, \dots, x_e$ that generates $\mathfrak{m}$. By Prop. 4.1.5 we have

$$\mathrm{d}\ell\ell(K^R) = \mathrm{d}\ell\ell(K(x_1, \dots, x_e)).$$

Set $S = R/(x_1, \dots, x_d)$. The Koszul complex on $x_1, \dots, x_d$ is quasi-isomorphic to S, so $K(x_1, \dots, x_e)$ is quasi-isomorphic to $S \otimes K(x_{d+1}, \dots, x_e) = K^S$. Since $\mathcal{D}(S)$ embeds in $\mathcal{D}(R)$, if K^S is a retract of V in $\mathcal{D}(S)$ then K^S is also a retract of V in $\mathcal{D}(R)$. So $\mathrm{d}\ell\ell_R(K^S) \leq \mathrm{d}\ell\ell_S(K^S) = \ell\ell_S(S)$.

Thus for all systems of parameters $x_1, \dots, x_d$ we have

$$\mathrm{d}\ell\ell(K^R) \leq \ell\ell_{R/(x_1,\dots,x_d)}(R/(x_1, \dots, x_d)).$$

So $\mathrm{g}\ell\ell(R) \geq \mathrm{d}\ell\ell(K^R)$. $\square$

This inequality can be strict, as the following example, borrowed from [11], shows.

Example 4.2.9. Let $R = \mathbb{F}_2[[x,y]]/(xy^2 + x^2y)$. In this case, straight-forward computation shows that $\mathrm{spread}(R) = 4$, and that $(y^2 + xy, 0) \in R^2 = K_1^R$ is a cycle. So we must have $3 \leq \mathrm{d}\ell\ell(K^R) \leq 4$. The map $\mathbb{F}_2 \to \mathbb{F}_4$ is split, so $K^R = \mathbb{F}_2 \otimes_{\mathbb{F}_2} K^R$ is a retract of $\mathbb{F}_4 \otimes_{\mathbb{F}_2} K^R = K^{\mathbb{F}_4 \otimes_{\mathbb{F}_2} R}$. So

$$\mathrm{d}\ell\ell_R(K^R) \leq \mathrm{d}\ell\ell_{\mathbb{F}_4 \otimes R}(K^{\mathbb{F}_4 \otimes R}).$$

By computations of Hashimoto and Shida we have $\mathrm{g}\ell\ell(R) = 4$ and $\mathrm{g}\ell\ell(\mathbb{F}_4 \otimes_{\mathbb{F}_2} R) = 3$. By Prop. 4.2.8 we conclude that $\mathrm{d}\ell\ell_R(K^R) = 3$.

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