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DERIVATIONS ON $\Gamma-$RINGS, PRIME $\Gamma-$RINGS AND SEMI-PRIME $\Gamma-$RINGS

By

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Dedication

I dedicate this work to my father, my mother,

to my wife Shireen, and to my sons, Ahmed and Fatima

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Lastly, I offer my regards and blessing to all of those who supported me in any way during the completion of this thesis.

اقـــــرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان :

الاشتقاقات المعرفة على الحلقات جاما و الحلقات جاما الأولية و الشبه أولية

DERIVATIONS ON $\Gamma-$RINGS, PRIME $\Gamma-$RINGS

AND SEMI-PRIME $\Gamma-$RINGS

أقر بأن ما اشتملت عليه هذه الرسالة انما هو نتاج جهدي الخاص، باستثناء ما تمت الاشارة اليه حيثما ورد، و أن هذه الرسالة ككل لم تقدم من قبل لنيل أية درجه علمية أو بحث علمي أو بحثـي لـــدى أية مؤســـســـــة تعـليمية او بحـثية أخــرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the result of the researcher's work, and has not been submitted elsewhere for any other degree or qualification.

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Contents

Abstract

The concept of derivations on $\Gamma-$rings, prime $\Gamma-$rings and semi-prime $\Gamma-$rings is being studied here. A generalization to these concepts was also introduced, the conditions that makes these rings commutative were also studied. Finally, the ΓM left R_Γ modules were introduced as well as the concept of derivations on left ΓM modules.

Introduction

An extensive generalized concept of classical ring is the notion of a gamma ring. As an emerging field of research, the research work of classical ring theory opposed to the gamma ring theory has been drawn interest of many algebraists and prominent mathematicians over the world to determine many basic properties of gamma ring and to enrich the world of algebra. The different researchers on this field have been doing significant contributions to this field. In recent years, a large number of researchers are engaged in increasing the efficiency of the results of gamma ring theory over the world.

The concept of a $\Gamma-$ring was introduced by Nobusawa [15] as a generalization of ring in 1964. Barnes [4] slightly weakened the conditions in the definition of $\Gamma-$ring in the sense of Nobusawa. After the study of $\Gamma-$rings by Nobusawa [15] and Barnes [4] many researchers have done a lot of work and have obtained some generalizations of the corresponding results in ring theory (see [5],[10],[13],[22] and references therein). They obtain large number of important basic properties of $\Gamma-$rings in various ways and determined some more remarkable results of $\Gamma-$rings. We start with the following necessary definitions.

In this thesis M denotes a $\Gamma-$ring in the sense of Barnes [4].

This thesis consists of five chapters: in chapter one we have reviewed some known definitions, some necessary lemmas and theorems which will be used in the next chapters, some basic definitions are presented which can be found in the indicated reference, we start our study with definition of a $\Gamma-$ring, several examples on $\Gamma-$rings, definition of subring and definition of center of $\Gamma-$ring. In section two we give some results on ideal of $\Gamma-$ring. In section three we give the definition of prime ideal and prime $\Gamma-$rings, some theorems and some lemmas about prime $\Gamma-$rings. In section four we give the definition of semi-prime ideal and semi-prime $\Gamma-$rings, some theorems and some lemmas about semi-prime $\Gamma-$rings. In section five we introduce and study the notion of modules over a fixed $\Gamma-$ring.

Chapter two consists of three sections: in section one we define a derivation on a $\Gamma-$ring, Jordan derivation on a $\Gamma-$ring, generalized derivation on $\Gamma-$ring, Jordan generalized derivation on a $\Gamma-$ring and some theorems and results on Jordan generalized left derivations in $\Gamma-$rings. In section two we define a $\Gamma-$semi-derivation, generalized inner derivation, $\Gamma-$homomorphism and projective product of $\Gamma-$rings, some theorems and results on $\Gamma-$derivations in the projective product of $\Gamma-$rings. In section three we define reverse derivation on $\Gamma-$ring, generalized reverse derivation on $\Gamma-$ring, Jordan generalized reverse derivation on $\Gamma-$ring and Jordan generalized triple reverse derivation on $\Gamma-$ring, several examples and theorems on Jordan generalized reverse derivations on $\Gamma-$rings.

Chapter three consists of three sections: in section one we prove that a prime $\Gamma-$ring M is commutative if f is a generalized derivation on M with an associated non-zero derivation D on M such that f is centralizing and commuting on a left ideal J of M. In section two we define a permuting tri-additive and the trace of a permuting tri-additive mapping, some theorems and results on permuting tri-derivation in prime $\Gamma-$rings. In section three we introduce the concept of triple higher derivation on a prime $\Gamma-$ring M and prove that every Jordan triple higher derivation on a prime $\Gamma-$ring M of characteristic different from two is a triple higher derivation on M and finally, it is shown that every Jordan triple higher derivation is a higher derivation on M.

Chapter four consists of four sections: in section one, the purpose of this section is to notions of generalized $I-$derivation and generalized reverse $I-$derivation on $\Gamma-$rings and to prove some remarkable results involving these mapping. In section two we presents the definition of orthogonal reverse derivations; some characterizations of semi-prime $\Gamma-$rings are obtained by using of orthogonal reverse derivations. We also investigate conditions for two reverse derivations to be orthogonal. In section three we extend the existing notions of derivations and generalized derivations in semi-prime $\Gamma-$ring. In section four we study and investigate some results concerning a permuting tri-derivation D on non-commutative 3-torsion free semi-prime $\Gamma-$rings M. Some characterizations of semi-prime $\Gamma-$rings are obtained by means of permuting tri-derivations.

Chapter five consists of two sections: in section one we present and study the concepts of a left $\Gamma M-$module, left derivation of a left $\Gamma M-$module. In section two we will define generalized left derivation and generalized Jordan left derivation.

Chapter One

Preliminaries

In this chapter we present some definitions and theorems on $\Gamma-$rings, ideal of $\Gamma-$rings, prime ideal, prime $\Gamma-$rings, semi-prime ideal, semi-prime $\Gamma-$rings and gamma modules which will be needed in the next chapters.

1.1 $\Gamma-$Ring

The concept of a $\Gamma-$ring (was first introduced by Nobusawa [15]) as a generalization of rings. Barnes [4] weakend slightly the conditions in the definition of $\Gamma-$ring in the sense of Nobusawa. Barnes obtain large number of important basic properties of $\Gamma-$rings in various ways and determined some more remarkable results of $\Gamma-$rings. We start with the following necessary definitions.

Definition 1.1.1: [4] Let M and Γ be additive abelian groups. If there exists a mapping $(x,\alpha,\mathrm{y}) \mapsto x\alpha y$ of $\mathrm{M}\times\Gamma\times\mathrm{M}\to\mathrm{M}$, satisfying the following conditions:

(i) $x\alpha y \in \mathrm{M}$;

(ii) $(x+\mathrm{y})\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)z = x\alpha z + x\beta z$; $x\alpha(y+z) = x\alpha y + x\alpha z$;

(iii) $(x\alpha \mathrm{y})\beta z = x\alpha(y\beta z)$ for all $x, y, z \in \mathrm{M}$ and $\alpha, \beta \in \Gamma$,

then M is called a $\Gamma-$*ring*.

Every ring M is a $\Gamma-$ring with $\mathrm{M}=\Gamma$. However a $\Gamma-$ring need not be a ring and many notions on the ring theory are generalized to the $\Gamma-$ring. i.e. gamma rings are more general than rings.

Examples 1.1.2:

1. Let R be an integral domain with the identity element 1. Take $\mathrm{M}=\mathrm{M}_{1\times 2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \text{ is an integer} \right\}$. Then M is a $\Gamma-$ring. If we assume that $\mathrm{N}=\{(a,a): a\in R\}\subset \mathrm{M}$, then it is easy to verify that N is also a $\Gamma-$ring(in fact, N is a subring of M).
2. (*Matrix Gamma Ring*): Let M be a $\Gamma-$ring. We denote the set of $m\times n$ matrices with entries from M and the set of $n\times m$ matrices with entries from Γ by $\mathrm{M}_{m\times n}$ and $\Gamma_{n\times m}$, respectively, then $\mathrm{M}_{m\times n}$ is a $\Gamma_{n\times m}-$ring with the multiplication defined by $(x_{ij})(\alpha_{ij})(y_{ij}) = (c_{ij})$, where $c_{ij} = \sum_p \sum_q x_{ip}\alpha_{pq}y_{qj}$.

For example, let R be any ring, and let $\mathrm{M} = \left\{ \begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix} : a,\ b,\ c,\ x,\ y,\ z \in R \right\}$, $\Gamma = \left\{ \begin{pmatrix} l & 0 & m \\ 0 & 0 & 0 \end{pmatrix} : l, m \in R \right\}$.

Then $\mathrm{M}_{3\times 2}$ is a $\Gamma_{2\times 3}-$ring.

Remark 1.1.4: If $m=n$, then M_n is a $\Gamma_n -$ ring .

Definition 1.1.5: [28] An additive subgroup S of a $\Gamma-$ring M is called $\Gamma-$*subring* of M if $S\Gamma S \subset S$.

Example 1.1.6:

Let R be any ring, $\mathrm{M}=\left\{\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : a,b,c\in R\right\}$ and $\Gamma=\left\{\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha\in R\right\}$ then M and Γ are both abelian groups under matrix addition.

Now it is easy to show that M is a $\Gamma-$ring under matrix multiplication, also we can prove that Γ is subring of M .

Definition 1.1.7: [4] A $\Gamma-$ring M is said to be *commutative* if $x\alpha y = y\alpha x$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.

Definition 1.1.8: [27] Let M be a $\Gamma-$ring. Then the set

$Z(\mathrm{M})=\{x\in\mathrm{M} : x\alpha y = y\alpha x \text{ for all } y\in\mathrm{M} \text{ and } \alpha\in\Gamma\}$ is called the *center* of the $\Gamma-$ring M .

Remark 1.1.9: If M is a $\Gamma-$ring, then $Z(M)$ is a $\Gamma-$subring of M .

Definition 1.1.10: [27] Let M be a $\Gamma-$ring. Then $[x,y]_\alpha = x\alpha y - y\alpha x$ is called the *commutator of* x and y with respect to α , where $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.

The following commutator identities follow easily from the above definition

(i) $[x\alpha y,z]_\beta = [x,z]_\beta\,\alpha y + x[\alpha,\beta]_z\, y + x\alpha[y,z]_\beta$ and

(ii) $[x,y\alpha z]_\beta = [x,y]_\beta\,\alpha z + y[\alpha,\beta]_x\, z + y\alpha[x,z]_\beta$, for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$.

Remarks 1.1.11:

1) Under the assumption:

   $(*)$ $\quad x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$.

   The above two identifies reduce to

   $[x\alpha y,z]_\beta = [x,z]_\beta\,\alpha y + x\alpha[y,z]_\beta$ and $[x,y\alpha z]_\beta = [x,y]_\beta\,\alpha z + y\alpha[x,z]_\beta$,

   which we shall use extensively.

2) M is called a $\Gamma-$ring with unit, if there exist element $1\in\mathrm{M}$ and $\alpha_0\in\Gamma$ such that for any $y\in\mathrm{M}$, $1\alpha_0 y = y = y\alpha_0 1$.
3) If A and B are subsets of the $\Gamma-$ring M and $\Lambda\subseteq\Gamma$, we denote by $A\Lambda B$ the subset of M consisting of all finite sums of the form $\sum_{i=1}^{n} a_i\gamma_i b_i$ where $(a_i,\gamma_i,b_i)\in A\times\Lambda\times B$.

For singleton subsets we abbreviate this notation, for example $\{a\}\Lambda B = a\Lambda B$.

1.2 Ideal of $\Gamma-$Ring

In this section, we introduce the notions of ideals and nilpotent $\Gamma-$rings.

Definition 1.2.1: [22] Let M be a $\Gamma-$ring. A subring I of M is an additive subgroup which is also a $\Gamma-$ring. An additive subgroup I of M is called a *left(right) ideal* of M if $\mathrm{M}\Gamma I \subseteq I \left(I\Gamma\mathrm{M} \subseteq I\right)$, where $\mathrm{M}\Gamma I = \{x\alpha a : x \in \mathrm{M}, \alpha \in \Gamma, a \in I\}$. If I is both a left and a right ideal, then I is called an *ideal* of M, or $\Gamma-$ideal I of M.

We denote an ideal I in M by $I \lhd \mathrm{M}$. An ideal $I \lhd \mathrm{M}$ is called *a proper ideal*, if $I \underset{\neq}{\subset} \mathrm{M}$. For each subset S of the $\Gamma-$ring M, the smallest ideal containing S is denoted by $\langle S \rangle$ and is called the *ideal generated* by S.

If S is finite, $\langle S \rangle$ is called *finitely generated*.

For each a of a $\Gamma-$ring M the smallest left ideal containing a is called the *principal left ideal generated* by a and is denoted by $\langle a \rangle_l$ or

$$\langle a| = \left\{ ma + \sum_{j=1}^{n} x_j \alpha_j a : m \in \mathbb{Z}^+ \bigcup \{0\}, n \in \mathbb{Z}^+, x_j \in S, \alpha_j \in \Gamma \right\}.$$

Similarly, we define the *principal right ideal generated* by a, by $\langle a \rangle_r$ or $|a\rangle = \left\{ ma + \sum_{i=1}^{n} a\beta_i y_i : m \in \mathbb{Z}^+ \bigcup \{0\}, n \in \mathbb{Z}^+, y_i \in S, \beta_i \in \Gamma \right\}$.

The *principal two-sided ideal generated* by a is denoted by $\langle a \rangle$, and is defined by $\langle a \rangle = \left\{ ma + \sum_{k=1}^{p} a\gamma_k \mathrm{z}_k + \sum_{t=1}^{s} \omega_t \delta_t a + \sum_{j=1}^{q} u_j \lambda_j a \mu_j v_j : m \in \mathbb{Z}^+ \bigcup \{0\}, p,s,q \in \mathbb{Z}^+, \mathrm{z}_k, \omega_t, u_j, v_j \in S \text{ and } \gamma_k, \delta_t, \lambda_j, \mu_j \in \Gamma \right\}$
where $\mathbb{Z}^+$ is the set of all positive integers.

Let I be an ideal of a $\Gamma-$ring M. If for each $a+I, b+I$ in the factor group M/I, and each $\alpha \in \Gamma$, we define $(a+I)\alpha(b+I) = a\alpha b + I$, then M/I is a $\Gamma-$ring which we shall call *the* $\Gamma-$*residue class ring* of M with respect to I.

Example 1.2.2: Let R be a ring and $(\mathbb{Z},+)$ be the group of integer numbers, we put $\mathrm{M} = \mathrm{M}_{2\times 2}(R)$ and $\Gamma = \mathrm{M}_{2\times 2}(\mathbb{Z})$, then M is a $\Gamma-$ring. We use the usual addition and multiplication on matrices of $\mathrm{M}\times\Gamma\times\mathrm{M}$.

Let $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a,b \in R \right\}$, clearly I is a right $\Gamma-$ideal of M but not a left $\Gamma-$ideal of M.

Definition 1.2.3: [25] Let M be a $\Gamma-$ring. An element x of M is called *nilpotent* if for some $\gamma \in \Gamma$, there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = \underbrace{(x\gamma x\gamma \ldots \gamma x\gamma)}_{n-times} x = 0$.

Definition 1.2.4: [25] An ideal A of a $\Gamma-$ring M is called *nilpotent* if $\exists\, n \in \mathbb{Z}^+$ s.t $(A\Gamma)^n A = \underbrace{(A\Gamma A\Gamma \ldots \Gamma A\Gamma)}_{n-times} A = 0$.

Theorem 1.2.5: [26] Let M be a $\Gamma-$ring and let N_1 and N_2 be two nilpotent left (right) ideals. Then $N_1 + N_2$ is a nilpotent left (right) ideal.

Proof: Let M be a $\Gamma-$ring. Let N_1 and N_2 be two nilpotent left ideals of M. Then there exist two least positive integers q and n such that

$$(N_1\Gamma)^q N_1 = \underbrace{(N_1\Gamma N_1\Gamma \ldots \Gamma N_1\Gamma)}_{q-\text{times}} N_1 = 0$$

and

$$(N_2\Gamma)^n N_2 = \underbrace{(N_2\Gamma N_2\Gamma \ldots \Gamma N_2\Gamma)}_{\text{n-times}} N_2 = 0$$

Then $N_1 + N_2$ is also a left ideal of M. Every element of $\{(N_1+N_2)\Gamma\}^{q+n+1}(N_1+N_2)$ is a sum of products $x_1\gamma x_2\gamma \ldots\ldots \gamma x_{q+n+2}$ in which either at least $(s+1)$ factors belong to N_1 or $(r+1)$ factors belong to N_2. In the former case, the above product may be written as $(x_1\gamma x_2\gamma \ldots \gamma x_{i_1})\gamma(x_{i_1+1}\gamma x_{i_1+2}\gamma \ldots \gamma x_{i_2})\gamma(x_{i_2+1}\gamma x_{i_2+2}\gamma \ldots \gamma x_{i_3})\gamma \ldots \gamma(x_{i_s+1}\gamma x_{i_s+2}\gamma \ldots \gamma x_{i_{s+1}})\gamma \ldots$, where $x_{i_1}, x_{i_2}, \ldots, x_{i_{s+1}} \in N_1$ and $s+1 \geq n+1$. Each group in parenthesis belongs to N_1, since N_1 is a left ideal of M. However, the product of any $s+1$ elements of N_1 is 0 and so the above product is 0. A similar argument holds when at least $(r+1)$ factors belong to N_2.

Thus $\{(N_1+N_2)\Gamma\}^{s+n+1}(N_1+N_2) = \underbrace{\{(N_1+N_2)\Gamma(N_1+N_2)\Gamma \ldots \Gamma(N_1+N_2)\Gamma\}}_{(s+n+1)-times}(N_1+N_2) = 0$ Hence (N_1+N_2) is nilpotent. Thus the theorem is proved. □

Corollary 1.2.6: Let M be a $\Gamma-$ring and let $N_1, N_2, \ldots, N_n$ be nilpotent left (right) ideals in M. Then $\sum N_\lambda$ is a nilpotent left (right) ideal in M.

Theorem 1.2.7 : [26] Let A be a nilpotent left (right) ideal in a $\Gamma-$ring M. Then $A\Gamma M$ $(M\Gamma A)$ is a nilpotent ideal in M.

Proof: Since A is a left ideal, so is $A\Gamma\mathrm{M}$, and since M is a right ideal then so is $A\Gamma\mathrm{M}$. Thus $A\Gamma\mathrm{M}$ is an ideal in M. If $(A\Gamma)^n A = 0$, then $\{(A\Gamma\mathrm{M})\Gamma\}^n (A\Gamma\mathrm{M})$

$$
\begin{aligned}
&= \underbrace{(A\Gamma\mathrm{M})\Gamma(A\Gamma\mathrm{M})\Gamma\ldots\Gamma(A\Gamma\mathrm{M})\Gamma}_{n-times}(A\Gamma\mathrm{M}) \\
&= A\Gamma\left[(\mathrm{M}\Gamma A)\Gamma(\mathrm{M}\Gamma A)\Gamma\ldots\Gamma(\mathrm{M}\Gamma A)\Gamma\right]\mathrm{M} \\
&= A\Gamma\{(\mathrm{M}\Gamma A)\Gamma\}^{n-1}(\mathrm{M}\Gamma A)\Gamma\mathrm{M} \\
&\subseteq A\Gamma\{(A\Gamma)\}^{n-1} A\Gamma\mathrm{M} \\
&= (A\Gamma)^n A\Gamma\mathrm{M} \\
&= 0\Gamma\mathrm{M} \\
&= 0
\end{aligned}
$$

Hence $A\Gamma\mathrm{M}$ is nilpotent . □

1.3 Prime Ideal And Prime $\Gamma-$Ring

In this section we present some definitions and theorems on prime ideal and prime $\Gamma-$rings.

Definition 1.3.1: [27] An ideal P of a $\Gamma-$ring M is said to be *prime* if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Definition 1.3.2: [14] A $\Gamma-$ring M is said to be *prime* if the zero ideal is prime .

Theorem 1.3.3: [14] If M is a $\Gamma-$ring, the following conditions are equivalent:

(i) M is a prime $\Gamma-$ring.

(ii) if $a,b \in \mathrm{M}$ and $a\Gamma\mathrm{M}\Gamma b = \{0\}$, then $a = 0$ or $b = 0$.

(iii) if $\langle a\rangle$ and $\langle b\rangle$ are principal ideals in M such that $\langle a\rangle\Gamma\langle b\rangle = \{0\}$ then $a = 0$ or $b = 0$.

(iv) If A and B are right ideals in M such that $A\Gamma B = \{0\}$, then $A = \{0\}$ or $B = \{0\}$.

(v) If A and B are left ideals in M such that $A\Gamma B = \{0\}$, then $A = \{0\}$ or $B = \{0\}$.

Proof: (i)$\rightarrow$ (ii)

Assume that a,b are non-zero elements in M therefore the ideals generated by a and b are non-zero ideals, thus $\langle a\rangle\Gamma\langle b\rangle \neq \{0\}$. But $a\Gamma\mathrm{M}\Gamma b \subset \langle a\rangle\Gamma\mathrm{M}\Gamma\langle b\rangle \subset \langle a\rangle\Gamma\langle b\rangle \neq \{0\}$.

Therefore, $a\Gamma\mathrm{M}\Gamma b \neq \{0\}$, but $a\Gamma\mathrm{M}\Gamma b = \{0\}, \forall a,b \in \mathrm{M}$. $\rightarrow\leftarrow$.

Thus, either $a = 0$ or $b = 0$.

(ii)$\rightarrow$ (iii)

Since $\langle a\rangle\Gamma\langle b\rangle=\{0\}$. But $a\Gamma M\Gamma b\subset\langle a\rangle\Gamma M\Gamma\langle b\rangle\subset\langle a\rangle\Gamma\langle b\rangle=\{0\}$.

Then $a\Gamma M\Gamma b=\{0\},\forall a,b\in M.$ by (ii) either $a=0$ or $b=0$.

(iii)$\rightarrow$ (iv)

Let $a\in A$ and $b\in B$ then $a\Gamma b\subset A\Gamma B=\{0\}$. Therefore $a\Gamma b=\{0\}$. Now, since A is right ideal, then $A\Gamma M\Gamma B\subset A\Gamma B=\{0\}$, then $a\Gamma M\Gamma b=\{0\},\forall a,b\in M.$ Now we claim that $\langle a\rangle\Gamma\langle b\rangle=\{0\}$. Assume not $\langle a\rangle\Gamma\langle b\rangle\neq\{0\}$. But $a\Gamma b\subset\langle a\rangle\Gamma\langle b\rangle\neq\{0\}$.

Therefore $a\Gamma b\neq\{0\}$, contradiction

With (i). Then $\langle a\rangle\Gamma\langle b\rangle=\{0\}$ and by assumption (iii) either $a=0,\ A=\{0\}$

or $b=0, B=\{0\}$.

(iv)$\rightarrow$ (v)

Let $a\in A$, $b\in B$ and $a\Gamma b=\{0\}$. Suppose that $\langle a\rangle\Gamma\langle b\rangle=\{0\}$. If not, then $\langle a\rangle\Gamma\langle b\rangle\neq\{0\}$, but $a\Gamma b\subset\langle a\rangle\Gamma\langle b\rangle\neq\{0\}$, we get $a\Gamma b\neq\{0\}$, which contradiction with (i), but $\langle a\rangle$, $\langle b\rangle$ are right ideal, then by (iv), we get either $\langle b\rangle=\{0\}\Rightarrow b=0, B=\{0\}$ or $\langle a\rangle=\{0\}\Rightarrow a=0, A=\{0\}$.

(v)$\rightarrow$ (i)

We want prove that $\{0\}$ is prime ideal.

Let A and B be ideals in M with $A\Gamma B=\{0\}$, but A and B are left ideal, then by (v) either $A=\{0\}$ or $B=\{0\}$. □

We say that an element a in a $\Gamma-$ring M centralizes a non-zero right(left) ideal I of M if $[a,x]_\alpha\in Z(M)$,$\forall x\in I,\forall\alpha\in\Gamma$.

Lemma 1.3.4: [18] Let M be a prime $\Gamma-$ring and suppose that $a\in M$ centralizes a non-zero right ideal of M. Then $a\in Z(M)$.

Proof: Suppose that a centralizes a non-zero right ideal A of M. If $x\in M, r\in A$, then $r\alpha x\in A$ for every $\alpha\in\Gamma$, hence $a\alpha(r\beta x)=r\beta x\alpha a$, for $\alpha,\beta\in\Gamma$. But $a\alpha r=r\alpha a$, for $\alpha\in\Gamma$, we thus get that $r\alpha(a\beta x-x\beta a)=0$ which is to say that $r\alpha[a,x]_\beta=0$, for all $x\in M$ and $\alpha,\beta\in\Gamma$.

Since M is prime and $A\neq 0$, we conclude that $[a,x]_\beta=0$ for all $x\in M,\beta\in\Gamma$, hence $a\in Z(M)$. □

Lemma 1.3.5: [18] Let M be any $\Gamma-$ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in \mathrm{M}$ and $\alpha,\beta \in \Gamma$, and let $u \in \mathrm{M}$. Then the set $V = \left\{ a \in \mathrm{M} : a\alpha[u,x]_\beta = 0, \text{ for all } x \in \mathrm{M} \text{ and } \alpha,\beta \in \Gamma \right\}$ is an ideal of M.

Proof: It is clear that V is a left ideal of M. Now, we show that V also is a right ideal. Let $a \in V$ and $x,\ r \in \mathrm{M}$. For all $\alpha,\beta,\delta \in \Gamma$, we have $a\alpha\left[u\beta(r\delta x)-(r\delta x)\beta u\right]=0$. The Jacobi identity for the commutators gives, $a\alpha[u\beta r\delta x - r\delta x\beta u] = (u\beta r - r\beta u)\delta x + r\beta(u\delta x - x\delta u)$, then using the condition, we get

$$0 = a\alpha\left[u\beta(r\delta x)-(r\delta x)\beta u\right] = a\alpha[u,r]_\delta \beta x + a\alpha r\beta[u,x]_\delta$$

That is, $a\alpha r\beta[u,x]_\delta = 0$, for any $\alpha,\beta,\delta \in \Gamma$. Hence $a\alpha r \in V$ and V is a right ideal of M. □

Definition 1.3.6: [28] Let M be a $\Gamma-$ring and I be a subset of M. The subset $Ann_l(I) = \{a \in \mathrm{M} : a\Gamma I = 0\}$ of M is called *a left annihilator* of I. A right annihilator $Ann_r(I)$ is defined similarly. If I is a non-zero ideal of M then $Ann_l(I) = Ann_r(I)$ and we denote it by $Ann(I)$.

Lemma 1.3.7: Let M be a prime $\Gamma-$ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in \mathrm{M}$ and $\alpha,\beta \in \Gamma$, and suppose that $0 \neq u \in \mathrm{M}$ satisfies $a\alpha[u,x]_\beta = 0,$ for all $x \in \mathrm{M}$ and $\alpha,\beta \in \Gamma$,Then $u \in Z(\mathrm{M})$.

Proof: By Lemma 1.3.5,

$V = \left\{ a \in \mathrm{M} : a\alpha[u,x]_\beta = 0, \text{ for all } x \in \mathrm{M} \text{ and } \alpha,\beta \in \Gamma \right\}$ is an ideal of M. Since, M is prime and $u\alpha x - x\alpha u \in Ann_r V$ we have $u\alpha x - x\alpha u = 0$, for all $x \in \mathrm{M}, \alpha \in \Gamma$, hence $u \in Z(\mathrm{M})$. □

Note: Let G be an additive group. We shall denote by $G_{m,n}$ the additive group of all $m \times n$ matrices over the group G. For $1 \le i \le m,\ 1 \le j \le n,$ and $a \in G,$ let aE_{ij} denote the matrix having a at the ith row and jth column, and 0 elsewhere.

Theorem 1.3.8: [14] If M is a $\Gamma-$ring, the matrix ring $\mathrm{M}_{m,n}$ is a prime $\Gamma_{n,m}-$ring if and only if M is a prime $\Gamma-$ring.

Proof: Let us prove that if M is not prime, then $\mathrm{M}_{m,n}$ is not prime. If M is not prime, there exist non-zero elements a and b of M such that $a\Gamma\mathrm{M}\Gamma b = 0$. Then, we have, for example, $aE_{11}\Gamma_{n,m}\mathrm{M}_{m,n}\Gamma_{n,m}bE_{11} = 0$ with aE_{11} and bE_{11} non-zero elements of $\mathrm{M}_{m,n}$. Hence, $\mathrm{M}_{m,n}$ is not prime. Conversely, suppose that $\mathrm{M}_{m,n}$ is not prime, and hence that there exist non-zero matrices $\sum_{i,j} a_{ij}E_{ij}$ and $\sum_{i,j} b_{ij}E_{ij}$ such that $\left(\sum_{i,j} a_{ij}E_{ij}\right)\Gamma_{n,m}\mathrm{M}_{m,n}\Gamma_{n,m}\left(\sum_{i,j} b_{ij}E_{ij}\right) = 0$. Let p,q,r and s be fixed positive integers such that $a_{pq} \neq 0$ and $b_{rs} \neq 0$. As a special case of the preceding equation, we find that for each $x \in \mathrm{M}$, each $\gamma,\ \eta \in \Gamma$,

$$\left(\sum_{i,j} a_{ij}E_{ij}\right)(\gamma E_{qp})(xE_{ps})(\eta E_{sr})\left(\sum_{i,j} b_{ij}E_{ij}\right)=\sum a_{iq}\gamma x\eta b_{rj}E_{ij}=0.$$

In particular, the (p,s) element must be zero, that is, $a_{pq}\gamma x\eta b_{rs}=0$.

Since this is true for every $x\in\mathrm{M}$ and every $\gamma,\eta\in\Gamma$, we have $a_{pq}\Gamma\mathrm{M}\Gamma b_{rs}=0$, and M is not prime. This completes the proof. □

Definition 1.3.9: [14] ($\Gamma-$*homomorphism*). Let M_i be a Γ_i-ring for $i=1,\ 2,$ an ordered pair (φ,ϕ) of mappings is called a *homomorphism* of M_1 onto M_2 if it satisfies the following properties :

1. φ is a group homomorphism from M_1 onto M_2.
2. ϕ is a group isomorphism from Γ_1 onto Γ_2.
3. For every $x,y\in\mathrm{M}_1,\ \gamma\in\Gamma_1,\ \varphi(x\gamma y)=\varphi(x)\phi(\gamma)\varphi(y)$.

Remarks 1.3.10: [14]

1. The kernel of the homomorphism (φ,ϕ) is defined to be $K=\{x\in\mathrm{M}:\varphi(x)=0\}$.
2. It is easy to show that K is an ideal of M.
3. If φ is a group isomorphism, that is, if $K=0$, then (φ,ϕ) is called *an isomorphism* from the Γ_1- ring M_1 onto the Γ_2-ring M_2.
4. In the special case where $\Gamma_1=\Gamma_2=\Gamma$, a $\Gamma-$homomorphism from M_1 to M_2 is a map φ from M_1 to M_2 such that $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\varphi(x\gamma y)=\varphi(x)\gamma\varphi(y)$ for all $x,y\in\mathrm{M}_1$ and all $\gamma\in\Gamma$, where the second map ϕ is taken to be the identity.

Example 1.3.11: Let φ be a homomorphism from a ring R into itself . Let $\mathrm{M}=\mathrm{M}_{1\times 2}(R)$ and $\Gamma=\left\{\begin{pmatrix} m \\ 0 \end{pmatrix}: m \text{ is an integer number}\right\}$. Then M is $\Gamma-$ring, where we use usual addition and multiplication on matrices of $\mathrm{M}\times\Gamma\times\mathrm{M}$. Let $\phi:\mathrm{M}\to\mathrm{M}$ be the additive map defined by

$\phi((a,b))=(\varphi(a),\varphi(b))$, for all $(a,b)\in\mathrm{M}$, then ϕ is a $\Gamma-$homomorphism on the $\Gamma-$ring M.

Example 1.3.12: [14] Let M be a $\Gamma-$ring, and I be an ideal in M. Then the ordered pair (Ψ,i) of mappings, where $\Psi:\mathrm{M}\to\mathrm{M}/I$ defined by $\Psi(x)=x+I$ for all $x\in\mathrm{M}$, and i is the identity mapping of Γ, is a $\Gamma-$homomorphism called *the natural homomorphism* from M onto M/I.

Theorem 1.3.13: If (φ,ϕ) is a homomorphism from a Γ_1-ring M_1 onto a Γ_2-ring M_2 with kernel K, then M_1/K and M_2 are isomorphic.

Proof: Define an ordered pair (f,ϕ) where $f:\mathrm{M}_1/K\to\mathrm{M}_2$, by $f(x+K)=\varphi(x)$ for all $x\in\mathrm{M}_1$. Then f is will defined and,

(1) f is group homomorphism, since

$f((x+K)+(y+K))=f(x+y+K)=\varphi(x+y)=\varphi(x)+\varphi(y)=f(x+K)+f(y+K).$

(2) f is onto since if we pick $z\in\mathrm{M}_2$ then as φ is onto, there exists $x\in\mathrm{M}_1$, such that $\varphi(x)=z$ then there exists $x+K\in\mathrm{M}_1/K$ such that $f(x+K)=\varphi(x)=z$.

(3) f is one-one since for $x+K\in Ker(f)$ where $x\in\mathrm{M}_1$ then $0=f(x+K)=\varphi(x)$, then $x\in Ker(\varphi)=K$, i.e. $x+K=K$, thus f is one-one.

(4) $f((x+K)\gamma(y+K))=f(x\gamma y+K)=\varphi(x\gamma y)=\varphi(x)\varphi(\gamma)\varphi(y)$

$$=f(x+K)\varphi(\gamma)f(y+K).\ \square$$

Lemma 1.3.14: [14] Let (Ψ,i) be a *homomorphism* of a $\Gamma-$ring M onto a $\Gamma-$ring N, with kernel K. Then each of the following is true:

(1) If I is an ideal (right ideal) in M, then $I\Psi$ is an ideal (right ideal) in N.
(2) If J is an ideal (right ideal) in N, then $J\Psi^{-1}$ is an ideal (right ideal) in M which contains K.
(3) If I is an ideal (right ideal) in M which contains K, then $I=(I\Psi)\Psi^{-1}$.
(4) The mapping $I\to I\Psi$ defines a one-one mapping of the set of ideals (right ideals) in M which contains K onto the set of all ideals (right ideals) in N.

Theorem 1.3.15: [14] If P is an ideal in the $\Gamma-$ring M, then the $\Gamma-$residue class ring M/P is a prime $\Gamma-$ring if and only if P is a prime ideal in M.

Proof: Let M/P be prime and A, B be ideals of M such that $A\Gamma B\subseteq P$. Let (ρ,i) be the natural homomorphism from M onto M/P. Then by Lemma 1.3.14 , $A\Psi$ and $B\Psi$ are ideals of M/P such that $A\Psi\Gamma B\Psi=\{0\}$. Since M/P is prime, it follows that $A\Psi=\{0\}$ or $B\Psi=\{0\}$, that is $A\subseteq P$ or $B\subseteq P$. Thus P is a prime ideal in M.

Conversely, let P be a prime ideal in M. Lemma 1.3.14 shows that each ideal in M/P is of the form A/P, where A is an ideal in M which contains P . Thus we may assume that A/P, B/P to be ideals of M/P such that $(A/P)\Gamma(B/P)=\{0\}$, which implies $A\Gamma B\subseteq P$. Then by the primeness of P we have $A\subseteq P$ or $B\subseteq P$. Hence $A=P$ or $B=P$ and so $A/P=\{0\}$ or $B/P=\{0\}$. This completes the proof. $\square$

Lemma 1.3.16: If I is an ideal in the $\Gamma-$ring M, then the matrix $\Gamma_{n,m}-$ring $(\mathrm{M}/I)_{m,n}$ is isomorphic to the $\Gamma_{n,m}-$ring $\mathrm{M}_{m,n}/I_{m,n}$.

Proof: Let φ be a mapping of the $\Gamma_{n,m}-$ring $(\mathrm{M}/I)_{m,n}$ to the $\Gamma_{n,m}-$ring $\mathrm{M}_{m,n}/I_{m,n}$ such that

$(x_{ij}+I)\varphi=(x_{ij})+I_{m,n}$. Clearly, φ is a group isomorphism from $(\mathrm{M}/I)_{m,n}$ onto $\mathrm{M}_{m,n}/I_{m,n}$. Let i be an identity mapping from $\Gamma_{n,m}$ onto $\Gamma_{n,m}$. By the definition of multiplication of the $\Gamma-$residue class ring, we have that

$$\left[(x_{ij}+I)(\gamma_{ij})(y_{ij}+I)\right]\varphi=(z_{ij}+I)\varphi, \text{ where } (z_{ij})=(x_{ij})(\gamma_{ij})(y_{ij})$$

$$=(x_{ij})(\gamma_{ij})(y_{ij})+I_{m,n}$$

$$=\left[(x_{ij})+I_{m,n}\right](\gamma_{ij})\left[(y_{ij})+I_{m,n}\right]$$

$$=(x_{ij}+I)\varphi(\gamma_{ij})i(y_{ij}+I)\varphi.$$

This shows that (φ,i) is an isomorphism of $(\mathrm{M}/I)_{m,n}$ onto $\mathrm{M}_{m,n}/I_{m,n}$. □

Definition 1.3.17: A $\Gamma-$ring M is said to be *right(left) strongly prime* if for each $0\neq a\in\mathrm{M}$, there exist finite subsets F and H of M and Γ respectively such that for any $b\in\mathrm{M}, a\alpha f\beta b=0\,(b\alpha f\beta a=0)$ for all $\alpha,\beta\in H, f\in F$ implies $b=0$.

A $\Gamma-$ring M is said to be *strongly prime* if it is both left and right strongly prime .

Example 1.3.18: Let $\mathrm{M}=\mathrm{M}_{1,2}(\mathbb{Z}),\ \Gamma=\mathrm{M}_{2,1}(\mathbb{Z})$ and $(a,b)\neq 0\in\mathrm{M}$.

Choose c,d,e,f,g and $l\in\mathbb{Z}$ such that $ac\neq -bd$ and $eg\neq -fl$. Consider $F=\{(e,f)\}$ and $H=\left\{\begin{pmatrix}c\\d\end{pmatrix},\begin{pmatrix}g\\l\end{pmatrix}\right\}$.

Then it can be easily checked that M is right and left strongly prime $\Gamma-$ring .

Therefore M is a strongly prime $\Gamma-$ring .

1.4 Semi-Prime Ideal And Semi-Prime $\Gamma-$Ring

In this section we present some notions and known results which will be used in the sequel.

Definition 1.4.1: [27] A $\Gamma-$ring M is said to be *2-torsion free* if $2x=0$ implies $x=0$ for all $x\in M$. A $\Gamma-$ring M is *n-torsion free*, where n is a positive integer, if $nx=0 \Rightarrow x=0, \forall x\in M.$

Definition 1.4.2: [27] An ideal I of a $\Gamma-$ring M is said to be *semi-prime* if for any ideal A of M, $A\Gamma A\subseteq I$ implies $A\subseteq I$.

Definition 1.4.2: [27] A $\Gamma-$ring M is said to be *semi-prime* if $a\Gamma M\Gamma a=\{0\}$, $a\in M$ implies $a=0$.

Lemma 1.4.3: [11] Suppose M is a semi-prime $\Gamma-$ring such that $x\alpha y\beta z = x\beta y\alpha z,$ for all $x,y,z\in M,$ and $\alpha,\beta\in\Gamma$. And suppose that the relation $a\alpha x\beta b+b\alpha x\beta c=0$ holds for all $x\in M$, some $a,b,c\in M$ and $\alpha,\beta\in\Gamma$. Then $(a+c)\alpha x\beta b=0$ is satisfied for all $x\in M$ and $\alpha,\beta\in\Gamma$.

Proof : Putting $x=x\beta b\alpha y$ in the relation $a\alpha x\beta b+b\alpha x\beta c=0$ (1.4.1)

We have $a\alpha x\beta b\alpha y\beta b+b\alpha x\beta b\alpha y\beta c=0$ (1.4.2)

On the other hand, a right multiplication by $\alpha y\beta b$ of (1.4.1) gives

$$a\alpha x\beta b\alpha y\beta b+b\alpha x\beta c\alpha y\beta b=0 \quad (1.4.3)$$

Subtraction (1.4.3) from (1.4.2), we have

$$b\alpha x\beta(b\alpha y\beta c-c\alpha y\beta b)=0 \quad (1.4.4)$$

Putting $x=y\beta c\alpha x$ in (1.4.4) gives

$$b\alpha y\beta c\alpha x\beta(b\alpha y\beta c-c\alpha y\beta b)=0 \quad (1.4.5)$$

Left multiplication by $c\alpha y\beta$ of (1.4.4) gives

$$c\alpha y\beta b\alpha x\beta(b\alpha y\beta c-c\alpha y\beta b)=0 \quad (1.4.6)$$

Subtracting (1.4.6) from (1.4.5), we obtain $(b\alpha y\beta c-c\alpha y\beta b)\alpha x\beta(b\alpha y\beta c-c\alpha y\beta b)=0$, which gives

$$b\alpha y\beta c=c\alpha y\beta b, y\in M \text{ and } \alpha,\beta\in\Gamma \quad (1.4.7)$$

Therefore, $b\alpha x\beta c$ can be replaced by $c\alpha x\beta b$ in (1.4.1), which gives $a\alpha x\beta b+c\alpha x\beta b=0$

i.e. $(a+c)\alpha x\beta b=0$. Hence, the proof is complete. □

The following lemmas will be used in proving theorem 4.3.5.

Lemma 1.4.4: [7] Let M be a 2-torsion free semi-prime $\Gamma-$ring. If $x\alpha x=0$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$, then $x\in Z(\mathrm{M})$.

Proof : We have $x\alpha x=0$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$. Replacing x by $x+y$, we get $x\alpha y+y\alpha x=0$ for all $x,\ y\in\mathrm{M},\ \alpha\in\Gamma$.

Right-multiplying by βx we obtain $x\alpha y\beta x=0$ for all $x,y\in\mathrm{M},\alpha,\beta\in\Gamma$. Replacing y by $y\gamma z$ and right-multiplying by αy we get $x\alpha y\gamma z\beta x\alpha y=0$ for all $x,y,z\in\mathrm{M},\alpha,\beta,\gamma\in\Gamma$. Since M is a semi-prime $\Gamma-$ring, we obtain $x\alpha y=0$ for all $x,y\in\mathrm{M},\alpha\in\Gamma$. By the same method, we get $y\alpha x=0$ for all $x,y\in\mathrm{M},\alpha\in\Gamma$, subtracting we obtain $[x,y]_\alpha=0$, for all $x,\ y\in\mathrm{M},\ \alpha\in\Gamma$. Then $x\in Z(\mathrm{M})$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$. □

Lemma 1.4.5: [6] Let M be a 2-torsion free semi-prime $\Gamma-$ring. If $a,b\in\mathrm{M}$ such that $a\Gamma m\Gamma b+b\Gamma m\Gamma a=0$ for all $m\in\mathrm{M}$, then $a\Gamma m\Gamma b=b\Gamma m\Gamma a=0$.

Proof : Let m and m' be two arbitrary elements of M. Then by using $a\Gamma m\Gamma b=-b\Gamma m\Gamma a$, we obtain $(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b)=-b\Gamma(m\Gamma a\Gamma m')\Gamma a\Gamma m\Gamma b$

$$=a\Gamma(m\Gamma a\Gamma m')\Gamma b\Gamma m\Gamma b$$
$$=-(a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b).$$

Therefore, we get $2((a\Gamma m\Gamma b)\Gamma m'\Gamma(a\Gamma m\Gamma b))=0$.

Since M is a *2-torsion free* semi-prime $\Gamma-$ring, then $a\Gamma m\Gamma b=0$ for all $m\in\mathrm{M}$. □

Lemma 1.4.6: [29] Let M be a semi-prime $\Gamma-$ring and I a non-zero ideal of M. Then $Ann_l I=Ann_r I$.

Proof : $Ann_r I=\{a\in\mathrm{M}: I\Gamma a=0\}$ is a right ideal of M, that is, $(Ann_r I)\Gamma\mathrm{M}\subset Ann_r I$. Similarly for $Ann_l I$ we can write $\mathrm{M}\Gamma(Ann_l I)\subset Ann_l I$. Since M is a semi-prime $\Gamma-$ring, $(Ann_r I)\Gamma I=\{0\}$, so $Ann_r I\subset Ann_l I$. In the same manner $I\Gamma(Ann_l I)\Gamma I\Gamma(Ann_l I)=\{0\}$ give us that $I\Gamma(Ann_l I)=\{0\}$ as M is a semi-prime $\Gamma-$ring. That is, $Ann_l I\subset Ann_r I$. So $Ann_l I=Ann_r I$. □

Lemma 1.4.7: [29] Let M be a semi-prime $\Gamma-$ring and I a non-zero ideal of M. Then

(i) $Ann I$ is an ideal of M.
(ii) $(Ann I)\cap I=\{0\}$.

Proof : (i) Let $a\in Ann I$. So by Lemma 1.4.6 $a\Gamma I=0=I\Gamma a$. If $a,b\in Ann I$, then $x\alpha(a-b)=x\alpha a-x\alpha b=0$ and $(a-b)\alpha x=a\alpha x-b\alpha x=0$ for all $x\in I$ and $\alpha\in\Gamma$. So we have $a-b\in Ann I$.

For all $a \in AnnI$, $x \in I$, $m \in M$ and $\alpha, \beta \in \Gamma$, $(a\alpha m)\beta x = a\alpha(m\beta x) = 0$ and $x\beta(a\alpha m) = (x\beta a)\alpha m = 0\alpha m = 0$, and so we get $(AnnI)\Gamma M \subset AnnI$. Similarly we get $M\Gamma(AnnI) \subset AnnI$

(ii) Since $(AnnI) \cap I$ is an ideal of M and $((AnnI) \cap I)\Gamma((AnnI) \cap I) \subset I\Gamma(AnnI) = \{0\}$, we have $((AnnI) \cap I)\Gamma((AnnI) \cap I) = \{0\}$ and since M is a semi-prime $\Gamma-$ring we get $(AnnI) \cap I = \{0\}$. □

Lemma 1.4.8: [29] Let M be a 2-torsion free semi-prime $\Gamma-$ring, I a non-zero ideal of M and $a, b \in M$. Then the following are equivalent ,

(i) $a\alpha x\beta b = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$.

(ii) $b\alpha x\beta a = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$.

(iii) $a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$.

If one of the conditions is fulfilled and $Ann_l I = \{0\}$, then $a\alpha b = 0 = b\alpha a$ for all $\alpha \in \Gamma$, moreover if M is a prime $\Gamma-$ring then $a = 0$ or $b = 0$.

Proof : (i) $\rightarrow$ (ii)

Suppose that $a\alpha x\beta b = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. Then $b\alpha x\beta a\gamma y\beta' b\alpha x\beta a = 0$ for all $x, y \in I$ and $\alpha, \beta, \gamma, \beta' \in \Gamma$. By writing $y\gamma' m$ for y, we get $b\alpha x\beta a\gamma y\gamma' m\beta' b\alpha x\beta a = 0$ where $m \in M$ and $\gamma' \in \Gamma$, hence $b\alpha x\beta a\gamma y\gamma' m\beta' b\alpha x\beta a\gamma y = 0$. Now since M is a semi-prime $\Gamma-$ring we have $b\alpha x\beta a\gamma y = 0$ for all $x, y \in I$ and $\alpha, \beta, \gamma \in \Gamma$. That is $b\alpha x\beta a \in Ann_l I$. Therefore $b\alpha x\beta a \in (Ann_l I) \cap I = \{0\}$ by Lemma 1.4.6 and Lemma 1.4.7 .

(ii) $\rightarrow$ (i) This can be done similarly .

(iii) $\rightarrow$ (i) Suppose that $a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x \in I$ and $\alpha, \beta \in \Gamma$. In the above equation, writing $x\beta b\alpha' m\beta' a\alpha x$ for x, then

$$
\begin{aligned}
a\alpha(x\beta b\alpha' m\beta' a\alpha x)\beta b &= -b\alpha(x\beta b\alpha' m\beta' a\alpha x)\beta a \\
&= -(b\alpha(x\beta b\alpha' m)\beta' a\alpha x\beta a) \\
&= a\alpha(x\beta b\alpha' m)\beta'(b\alpha x\beta a) \\
&= -(a\alpha x\beta b)\alpha' m\beta'(a\alpha x\beta b)
\end{aligned}
$$

Then we have $2a\alpha(x\beta b\alpha' m\beta' a\alpha x)\beta b = 0$. Since M is *2-torsion free* , we get $(a\alpha x\beta b)\alpha' m\beta'(a\alpha x\beta b) = 0$ for all $x \in I, \alpha, \beta, \alpha', \beta' \in \Gamma$ and $m \in M$. Next, since M is semi-prime $\Gamma-$ring, then $a\alpha x\beta b = 0$ for all $x \in I, \alpha, \beta \in \Gamma$.

If $a\Gamma I\Gamma b=\{0\}$, then we also have $(b\Gamma a)\Gamma I\Gamma(b\Gamma a)=\{0\}$ and $(a\Gamma b)\Gamma I\Gamma(a\Gamma b)=\{0\}$. Hence $b\alpha a\beta x=0$ and $a\alpha b\beta x=0$ for all $x\in I,\alpha,\beta\in\Gamma$, since M is a semi-prime $\Gamma-$ring and I is a non-zero ideal of M. This says that $a\alpha b, b\alpha a\in Ann_l I$. Since $Ann_l I=\{0\}$, we have $a\Gamma b=0=b\Gamma a$. Finally if $a\Gamma I\Gamma b=\{0\}$, then $a=0$ or $b=0$ as M is a prime $\Gamma-$ring . □

Theorem 1.4.9: [14] If I is an ideal in a $\Gamma-$ring M, all the following conditions are equivalent:

(i) I is a semi-prime ideal .
(ii) If $a\in\mathrm{M}$ such that $a\Gamma\mathrm{M}\Gamma a\subseteq I$, then $a\in I$.
(iii) If $\langle a\rangle$ is a principal ideal in M such that $\langle a\rangle\Gamma\langle a\rangle\subseteq I$, then $a\in I$.
(iv) If U is a right ideal in M such that $U\Gamma U\subseteq I$, then $U\subseteq I$.
(v) If V is a left ideal in M such that $V\Gamma V\subseteq I$, then $V\subseteq I$.

Proof : (i)$\rightarrow$ (ii)

Let $a\in\mathrm{M}$ and $\langle a\rangle$ is a principal ideal in M, suppose that $\langle a\rangle\Gamma\langle a\rangle\not\subseteq I$. But $a\Gamma\mathrm{M}\Gamma a\subseteq\langle a\rangle\Gamma\mathrm{M}\Gamma\langle a\rangle$. Now since $\langle a\rangle$ is an ideal of M, therefore by (i), we get $\langle a\rangle\Gamma\mathrm{M}\Gamma\langle a\rangle\subseteq\langle a\rangle\Gamma\langle a\rangle\not\subseteq I$, then $\langle a\rangle\Gamma\mathrm{M}\Gamma\langle a\rangle\not\subseteq I$, contradiction.

Thus $\langle a\rangle\Gamma\langle a\rangle\subseteq I$, but I is a semi-prime ideal. Therefore $\langle a\rangle\in I$ and $a\in I$.

(ii)$\rightarrow$ (iii)

Since $\langle a\rangle$ is a principle ideal in M such that $\langle a\rangle\Gamma\langle a\rangle\subseteq I$. Therefore $a\Gamma\mathrm{M}\Gamma a\subseteq\langle a\rangle\Gamma\mathrm{M}\Gamma\langle a\rangle\subseteq\langle a\rangle\Gamma\langle a\rangle\subseteq I$. Thus, $a\Gamma\mathrm{M}\Gamma a\subseteq I$, then by (ii) we get, $a\in I$.

(iii)$\rightarrow$ (iv)

Let $a\in U$ and $a\notin I$. Now let $\langle a\rangle$ be a principal ideal generated by a, then $\langle a\rangle\Gamma\langle a\rangle\not\subseteq I$. If $\langle a\rangle\Gamma\langle a\rangle\subseteq I$, then we get $a\in I\rightarrow\leftarrow$. Therefore $a\Gamma a\subseteq\langle a\rangle\Gamma\langle a\rangle\not\subseteq I$, then $a\Gamma a\not\subseteq I$. But $a\Gamma a\subseteq U\Gamma U\subseteq I$. We have $a\Gamma a\subseteq I\rightarrow\leftarrow$. Therefore $a\in I$.

(iv)$\rightarrow$ (v)

If $a\in V$, then $a\Gamma a\subseteq V$. Therefore $\langle a\rangle$ is a principal ideal in M generated by a. Suppose that $\langle a\rangle\Gamma\langle a\rangle\not\subseteq I$, then $a\Gamma a\not\subseteq I\rightarrow\leftarrow$. So $\langle a\rangle\Gamma\langle a\rangle\subseteq I$, but $\langle a\rangle$ is a right ideal, then by assumption of (iv) we have, $a\in I$.

(v)$\rightarrow$ (i)

Let A be an ideal of a $\Gamma-$ring M such that $A\Gamma A\subseteq I$. Since A is a left ideal, then by (v), we have $A\subseteq I$ thus I is semi-prime ideal. □

Theorem 1.4.10: [25] An ideal Q in a $\Gamma-$ring M is a semi-prime ideal in M if and only if M/Q contains no non-zero nilpotent ideals.

Proof : Let f be the natural $\Gamma-$homomorphism of M onto M/Q, with kernel Q. Suppose Q is a semi-prime ideal in M and U is a nilpotent ideal in M/Q, say $(U\Gamma)^n U=0$.

Then $f^{-1}\left((U\Gamma)^n U\right)=Q$ and it follows that $\left(f^{-1}(U)\Gamma\right)^n f^{-1}(U)\subseteq f^{-1}\left((U\Gamma)^n U\right)=Q$ and hence $U=\{0\}$.

Conversely, suppose that M/Q contains no non-zero nilpotent ideals and that A is an ideal in M such that $A\Gamma A\subseteq Q$. Then $f(A)\Gamma f(A)=f(A\Gamma A)=0$. Hence $f(A)=0$ and $A\subseteq Q$. □

Lemma 1.4.11: [7] Let M be a semi-prime $\Gamma-$ring. Then M contains no non-zero nilpotent ideal.

Proof: Let I be a nilpotent ideal of M. Then $(I\Gamma)^n I=0$ for some positive integer n. Let us assume that n is minimum. Now suppose that $n\geq 1$. Since $I\Gamma M\subset I$, we then have $(I\Gamma)^{n-1} I\Gamma M\Gamma(I\Gamma)^{n-1} I$ $\subset(I\Gamma)^{n-1} I(I\Gamma)^n I=(I\Gamma)^n I(I\Gamma)^{n-2} I=0$. Hence by the semi-primeness of M we get $(I\Gamma)^{n-1} I=0$, a contradiction to the minimality of n. Therefore $n=1$. Thus $I\Gamma I=0$.

Then $I\Gamma M\Gamma I\subset I\Gamma I=0$.

Since M is semi-prime, it gives $I=0$. This completes the proof. □

Remark 1.4.12: The above lemma gives that every prime $\Gamma-$ring has no nilpotent ideals.

Lemma 1.4.13: Let $G_1,\ldots,G_n$ be additive groups, and M a semi-prime $\Gamma-$ring. Suppose that the mappings $f:G_1\times\ldots\times G_n\to M$ and $g:G_1\times\ldots\times G_n\to M$ are additive in each argument. If $f(a_1,\ldots,a_n)\Gamma m\Gamma g(a_1,\ldots,a_n)=0$ for all $m\in M$ and $a_i\in G_i$, $i=1,\ldots,n$, then $f(a_1,\ldots,a_n)\Gamma m\Gamma g(b_1,\ldots,b_n)=0$ for all $m\in M$ and $a_i,b_i\in G_i$, $i=1,\ldots,n$.

Proof : It suffices to prove the case $n=1$. The mappings are then $f:G_1\to M$ and $g:G_1\to M$ such that $f(a)\Gamma m\Gamma g(a)=0$ and $f(b)\Gamma m\Gamma g(b)=0$ for all $a,b\in G_1$ and $m\in M$. Thus, we have

$$0=f(a+b)\Gamma m\Gamma g(a+b)$$

$$=f(a)\Gamma m\Gamma g(a)+f(a)\Gamma m\Gamma g(b)+f(b)\Gamma m\Gamma g(a)+f(a)\Gamma m\Gamma g(a)$$

$$=f(a)\Gamma m\Gamma g(b)+f(b)\Gamma m\Gamma g(a).$$

Let $m'\in M$. Then by the assumption, we get

$\left(f(a)\Gamma m\Gamma g(b)\right)\Gamma m'\Gamma\left(f(a)\Gamma m\Gamma g(b)\right)=-f(a)\Gamma\left(m\Gamma g(b)\Gamma m'\Gamma f(b)\Gamma m\right)\Gamma g(a)=0$. Hence, by the semi-primeness of M, we have $f(a)\Gamma m\Gamma g(b)=0$. This completes the proof of the lemma. □

Definition 1.4.14: [22] A subset N of a $\Gamma-$ring M is said to be an $n-system$ if $N=\phi$ or if $a\in N$ implies $\langle a\rangle\Gamma\langle a\rangle\cap N\neq\phi$.

Lemma 1.4.15: [22] Let M be a $\Gamma-$ring. Then an ideal Q in M is semi-prime if and only if Q^C is an $n-$ system, where Q^C is the complement of Q.

Proof : Suppose that Q is a semi-prime ideal and let $a\in Q^C$, then $a\notin Q$. Since Q is semi-prime, it follows from Theorem 1.4.9 that $\langle a\rangle\Gamma\langle a\rangle\not\subset Q$. This implies that $\langle a\rangle\Gamma\langle a\rangle\cap Q^C\neq\phi$, so that Q^C is an $n-system$.

Conversely, suppose Q^C is an $n-system$ and let $a\notin Q$. We shall prove that $\langle a\rangle\Gamma\langle a\rangle\not\subset Q$. Since Q^C is an $n-system$, $\langle a\rangle\Gamma\langle a\rangle\cap Q^C\neq\phi$. Take $z\in\langle a\rangle\Gamma\langle a\rangle\cap Q^C$ so that $z\in\langle a\rangle\Gamma\langle a\rangle$ and $z\notin Q$. Hence $\langle a\rangle\Gamma\langle a\rangle\not\subset Q$. Thus Q is a semi-prime ideal . □

Definition 1.4.16: [15] Let M be a $\Gamma-$ring. If for any non-zero element a of M there exists such an element γ (depending on a) in Γ such that $a\gamma a\neq 0$, we say that M is *semi-simple*. If for any non-zero elements a and b of M there exists γ (depending on a and b) in Γ such that $a\gamma b\neq 0$, we say that M is *simple.*

Theorem 1.4.17: [22] Let M be a $\Gamma-$ring. Then M is semi-simple if and only if M is semi-prime.

Proof: Suppose that $\langle a\rangle\Gamma\langle a\rangle=\{0\}$ for any $a\in M$. Since $a\Gamma a\subseteq\langle a\rangle\Gamma\langle a\rangle, a\Gamma a=\{0\}$. Since M is semi-simple, $a\Gamma a=\{0\}$ implies that $a=0$. Hence $\langle a\rangle=0$, so that M is semi-prime.

Conversely, suppose $a\Gamma a=\{0\}$ for any $a\in M$. Since $a\Gamma M\Gamma a\subseteq a\Gamma a, a\Gamma M\Gamma a=\{0\}$. Since M is semi-prime, it follows that $a=0$. Hence M is semi-simple. □

Corollary 1.4.18: [22] M is semi-prime if and only if for any ideals U,V in M, $U\Gamma V=\{0\}$ implies that $U\cap V=\{0\}$.

Proof: Suppose that M is semi-prime. Let U,V be ideals in M such that $U\Gamma V=\{0\}$ and let $x\in U\cap V$. Since $x\Gamma x\subseteq U\Gamma V$, $x\Gamma x=\{0\}$. Since M is semi-prime, M is semi-simple by Theorem 1.4.17. hence $x\Gamma x=\{0\}$ implies that $x=0$ and consequently $U\cap V=\{0\}$.

Conversely, suppose $U\Gamma U=\{0\}$ implies $U\cap U=\{0\}$ by hypothesis. Hence $U=\{0\}$, so that M is semi-prime. □

Definition 1.4.19: [17] An element a of a $\Gamma-$ring M is called *strongly nilpotent* if there exist a positive integer n such that $(a\Gamma)^n a = (a\Gamma a\Gamma a\Gamma ... a\Gamma)a = 0$. A subset S of M is *strongly nil* if each of its elements is strongly nilpotent. S is strongly nilpotent if there exist a positive integer n such that $(S\Gamma)^n S = 0$.

Clearly a strongly nilpotent set is also strongly nil.

Definition 1.4.20: The *strongly nilpotent radical,* denoted by S_{M} of a $\Gamma-$ring M is defined as the sum of all strongly nilpotent ideals of M.

Theorem 1.4.21: Every prime gamma ring is simple.

Proof: Let M be a prime $\Gamma-$ring. We show M is simple. If possible, let M be not simple. Then there exists two non-zero elements $x, y \in \mathrm{M}$ such that, $x\gamma y = 0$ for all $\gamma \in \Gamma$. Let $A = \langle x \rangle$ and $B = \langle y \rangle$. Then A and B are ideals of M. Let $a \in A\Gamma B$ be any element. Then $a \in \langle x \rangle \Gamma \langle y \rangle$ since $x\gamma y = 0$ for all $\gamma \in \Gamma$, so $a = 0$. Thus we get, $A\Gamma B = \{0\}$.

Since M is a prime $\Gamma-$ring, so $A\Gamma B = \{0\} \Rightarrow A = \{0\}$ or $B = \{0\}$.

Without loss of generality, let $A = \{0\}$. Then $\langle x \rangle = 0 \Rightarrow x = 0$, which contradicts that x is non-zero.

Thus M is simple. □

Remark 1.4.22: Every prime gamma ring is semi-simple.

Theorem 1.4.23: [17] Every simple gamma ring is a prime gamma ring.

Proof: Let M be a simple $\Gamma-$ring. Then for any two non-zero elements $x, y \in \mathrm{M}$, there exist $\gamma \in \Gamma$ such that $x\gamma y \neq 0$.

Let U, V be two ideals of M such that $\Gamma UV = \{0\}$. We show $U = \{0\}$ or $V = \{0\}$. If possible let $U \neq \{0\}$ and $V \neq \{0\}$. Then there exist $0 \neq x \in U$ and $0 \neq y \in V$. Since M is simple so there exist $\gamma \in \Gamma$ such that $x\gamma y \neq 0$.

Now $x\gamma y \in \Gamma UV = \{0\} \Rightarrow x\gamma y = 0$, which is a contradiction. So our supposition is wrong. So we must have $U = \{0\}$ or $V = \{0\}$. Hence M is a prime $\Gamma-$ring. □

Definition 1.4.24: [22] Let M be a $\Gamma-$ring. Then a left ideal I of M is said to be *essential* if $I \cap J \neq \{0\}$ for all non-zero left ideals J of M.

Definition 1.4.25: [12] Let M be a $\Gamma-$ring. Then the mapping $I:M\to M$ is called an *involution* if

(i) $II(a)=a$;

(ii) $I(a+b)=I(a)+I(b)$;

(iii) $I(a\alpha b)=I(b)\alpha I(a)$.

For all $a,b\in M$ and $\alpha\in\Gamma$.

Example 1.4.26: Let M be a $\Gamma-$ring. Define $M_1=\{(a,b):a,b\in M\}$ and $\Gamma_1=\{(\alpha,\alpha):\alpha\in\Gamma\}$. The addition and multiplication on M_1 are defined as follows:

$$(a,b)+(c,d)=(a+c,b+d) \text{ and } (a,b)(\alpha,\alpha)(c,d)=(a\alpha c,d\alpha b).$$

Under these addition and multiplication M_1 is a Γ_1-ring.

Define $I:M_1\to M_1$ by $I((a,b))=(b,a)$. Then

$$II((a,b))=I((b,a))=(a,b)$$

$$\begin{aligned} I((a,b)+(c,d)) &= I((a+c,b+d)) \\ &= (b+d,a+c) \\ &= (b,a)+(d,c) \\ &= I((a,b))+I((c,d)) \end{aligned}$$

$$\begin{aligned} I((a,b)(\alpha,\alpha)(c,d)) &= I((a\alpha c,d\alpha b)) \\ &= (d\alpha b,a\alpha c) \\ &= (d,c)(\alpha,\alpha)(b,a) \\ &= I((c,d))(\alpha,\alpha)I((a,b)) \end{aligned}$$

Therefore, I is an involution of the Γ_1-ring M_1. □

1.5 Gamma Modules

In this section we introduce and study the notion of modules over a fixed $\Gamma-$ ring.

Definition 1.5.1: [2] Let R be a $\Gamma-$ring. A (left) $R_\Gamma-$module is an additive abelian group M together with a mapping $.:R\times\Gamma\times\mathrm{M}\rightarrow\mathrm{M}$ (the image of (r,γ,m) being denoted by $r\gamma m$), such that for all $m,m_1,m_2\in\mathrm{M}$ and $\gamma,\gamma_1,\gamma_2\in\Gamma,r,r_1,r_2\in R$ the following hold :

(i) $r\gamma(m_1+m_2)=r\gamma m_1+r\gamma m_2;$

(ii) $(r_1+r_2)\gamma m=r_1\gamma m+r_2\gamma m;$

(iii) $r(\gamma_1+\gamma_2)m=r\gamma_1 m+r\gamma_2 m;$

(iv) $r_1\gamma_1(r_2\gamma_2 m)=(r_1\gamma_1 r_2)\gamma_2 m.$

A right $R_\Gamma-$module is defined in analogous manner.

Definition 1.5.2: [2] A (left) $R_\Gamma-$module M is *unitary* if there exist elements, say 1 in R and $\gamma_0\in\Gamma$, such that, $1\gamma_0 m=m$ for every $m\in\mathrm{M}$. We denote $1\gamma_0$ by 1_{γ_0}, so $1_{\gamma_0}m=m$ for all $m\in\mathrm{M}$.

Remark 1.5.3: [2] If M is a left $R_\Gamma-$module then it is easy to verify that $0\gamma m=r0m=r\gamma 0=0_{\mathrm{M}}$.

If R and S are $\Gamma-$rings then an $(R,S)_\Gamma-$bimodule M is both a left $R_\Gamma-$module and right $S_\Gamma-$module and simultaneously such that $(r\alpha m)\beta s=r\alpha(m\beta s)$, $\forall m\in\mathrm{M}$, $\forall r\in R$, $\forall s\in S$ and $\alpha,\beta\in\Gamma$.

Example 1.5.4: If R is a $\Gamma-$ring, then every abelian group M can be made into an $R_\Gamma-$module with trivial module structure by defining $r\gamma m=0$,$\forall r\in R,\forall\gamma\in\Gamma,\forall m\in\mathrm{M}$.

Example 1.5.5: Let R be an arbitrary commutative $\Gamma-$ring with identity. A polynomial in one indeterminate x with coefficients in R is an expression $P(x)=a_n x^n+a_{n-1}x^{n-1}+...+a_2x^2+a_1x+a_0$ with $a_i\in R$. The set $R[x]$ of all polynomials is then an abelian group. Now $R[x]$ becomes an $R_\Gamma-$module under the mapping $.:R\times\Gamma\times R[x]\rightarrow R[x]$, $(r,\gamma,f(x))\mapsto r\gamma f(x)=\sum_{i=1}^{n}(r\gamma a_i)x^i$.

Example 1.5.6: If R is a $\Gamma-$ring and M is an $R_\Gamma-$module. Set $\mathrm{M}[x]=\left\{\sum_{i=0}^{n}a_i x^i : a_i\in\mathrm{M}\right\}$. For $f(x)=\sum_{j=0}^{n}b_j x^j$ and $g(x)=\sum_{i=0}^{m}a_i x^i$, define the mapping $.:R[x]\times\Gamma\times\mathrm{M}[x]\rightarrow\mathrm{M}[x]$ $(g(x),\gamma,f(x))\mapsto g(x)\gamma f(x)=\sum_{k=1}^{m+n}(a_k\gamma b_k)x^k$. It is easy to verify that $\mathrm{M}[x]$ is an $R[x]_\Gamma-$module .

Example 1.5.7: Let I be an ideal of a $\Gamma-$ring R. Then R/I is an $R_\Gamma-\text{module}$, where the mapping $.: R\times\Gamma\times R/I \to R/I$ is defined by $(r,\gamma,r'+I)\mapsto(r\gamma r')+I$.

Example 1.5.8: Let M be an $R_\Gamma-\text{module}$, $m\in\mathrm{M}$. Letting $T(m)=\{t\in R: t\gamma m=0\ \forall\gamma\in\Gamma\}$. Then $T(m)$ is an $R_\Gamma-\text{module}$.

Proposition 1.5.9: [2] Let R be a $\Gamma-$ring and $(\mathrm{M},+,.)$ be an $R_\Gamma-\text{module}$. Set $Sub(\mathrm{M})=\{X:\mathrm{X}\subseteq\mathrm{M}\}$, then $Sub(\mathrm{M})$ is an $R_\Gamma-\text{module}$.

Proof : Define $\oplus:(A,B)\to A\oplus B$ by $A\oplus B=(A\setminus B)\cup(B\setminus A)$ for $A,B\in Sub(\mathrm{M})$. Then $(Sub(\mathrm{M}),\oplus)$ is an additive group with identity element ϕ and the inverse of each element A is itself. Consider the mapping $\circ: R\times\Gamma\times Sub(\mathrm{M})\to Sub(\mathrm{M})$

$(r,\gamma,X)\mapsto r\circ\gamma\circ X=r\gamma X$, where $r\gamma X=\{r\gamma x: x\in X\}$. Then we have

(i) $r\circ\gamma\circ(X_1\oplus X_2)=r.\gamma.(X_1\oplus X_2)=r.\gamma.((X_1\setminus X_2)\cup(X_2\setminus X_1))$

$$=r.\gamma.\{a:a\in(X_1\setminus X_2)\cup(X_2\setminus X_1)\}=\{r.\gamma.a:a\in(X_1\setminus X_2)\cup(X_2\setminus X_1)\}.$$

And

$$r\circ\gamma\circ X_1\oplus r\circ\gamma\circ X_2=r.\gamma.X_1\oplus r.\gamma.X_2=(r.\gamma.X_1\setminus r.\gamma.X_2)\cup(r.\gamma.X_2\setminus r.\gamma.X_1)$$

$$=\{r.\gamma.x:x\in(X_1\setminus X_2)\}\cup\{r.\gamma.x:x\in(X_2\setminus X_1)\}$$
$$=\{r.\gamma.x:x\in(X_1\setminus X_2)\cup(X_2\setminus X_1)\}$$

(ii) $(r_1+r_2)\circ\gamma\circ X=(r_1+r_2).\gamma.X=\{(r_1+r_2).\gamma.x:x\in X\}=\{r_1.\gamma.x+r_2.\gamma.x:x\in X\}$

$$=r_1.\gamma.X+r_2.\gamma.X=r_1\circ\gamma\circ X+r_2\circ\gamma\circ X$$

(iii) $r\circ(\gamma_1+\gamma_2)\circ X=r.(\gamma_1+\gamma_2).X=\{r.(\gamma_1+\gamma_2).x:x\in X\}$

$$=\{r.\gamma_1.x+r.\gamma_2.x:x\in X\}=r.\gamma_1.X+r.\gamma_2.X=r\circ\gamma_1\circ X+r\circ\gamma_2\circ X.$$

(iv) $r_1\circ\gamma_1\circ(r_2\circ\gamma_2\circ X)=r_1.\gamma_1.(r_2\circ\gamma_2\circ X)=\{r_1.\gamma_1.(r_2\circ\gamma_2\circ x):x\in X\}=\{r_1.\gamma_1.(r_2.\gamma_2.\mathrm{x}):x\in X\}$

$$=\{(r_1.\gamma_1.r_2).\gamma_2.x:x\in X\}=(r_1.\gamma_1.r_2).\gamma_2.X. \ \square$$

Example 1.5.10: Let $(R,\circ)$ be a $\Gamma-$ring. Then $R\oplus\mathbb{Z}=\{(r,s):r\in R,s\in\mathbb{Z}\}$ is a left $R_\Gamma-$module, where $\oplus'$ addition operation is defined by $(r,n)\oplus'(r',n')=(r+_R r',n+_\mathbb{Z} n')$ and the product $.:R\times\Gamma\times(R\oplus\mathbb{Z})\to R\oplus\mathbb{Z}$ is defined by $r'.\gamma.(r,n)\mapsto(r'\circ\gamma,r\circ n)$.

Example 1.5.11: Let $(R,.)$ and $(\mathrm{S},\circ)$ be $\Gamma-$rings. Then

i. The product $R\times S$ is a $\Gamma-$ring, under the mapping

$$((r_1,s_1),\gamma,(r_2,s_2))\mapsto(r_1.\gamma.r_2,s_1\circ\gamma\circ s_2).$$

ii. For $A=\left\{\begin{pmatrix} r & 0\\ 0 & s\end{pmatrix}:r\in R,s\in S\right\}$ there exist a mapping $R\times S\to A$, such that $(r,s)\mapsto\begin{pmatrix} r & 0\\ 0 & s\end{pmatrix}$ and A is a $\Gamma-$ring. Moreover, A is an $(R\times S)_\Gamma-$module under the mapping $.:(R\times S)\times\Gamma\times A\to A$.

$$\left((r_1,s_1),\gamma,\begin{pmatrix} r_2 & 0\\ 0 & s_2\end{pmatrix}\right)\mapsto\begin{pmatrix} r_1.\gamma.r_2 & 0\\ 0 & s_1\circ\gamma\circ s_2\end{pmatrix}.$$

Definition 1.5.12: $[2]$ Let $(\mathrm{M},+)$ be an $R_\Gamma-$module. A nonempty subset N of $(\mathrm{M},+)$ is said to be a (left) $R_\Gamma-$*submodule* of M if N is a subgroup of M and $R\Gamma N\subseteq N$, where $R\Gamma N=\{r\gamma n:\gamma\in\Gamma,r\in R,n\in N\}$, that is for all $n,n'\in N$ and for all $\gamma\in\Gamma,r\in R;n-n'\in N$ and $r\gamma n\in N$. In this case we write $N\leq\mathrm{M}$.

Remark 1.5.13: (i) Clearly $\{0\}$ and M are two trivial $R_\Gamma-$submodules of an $R_\Gamma-$module M called the trivial $R_\Gamma-$submodules.

(ii) Consider R as $R_\Gamma-$module. Clearly, every ideal of $\Gamma-$ring R is a submodule of R.

Theorem 1.5.14: $[2]$ Let N be an $R_\Gamma-$submodules of M. Then every $R_\Gamma-$submodule of M/N is of the form K/N, where K is an $R_\Gamma-$submodule of M containing N.

Proof : For all $x,y\in K,x+N,y+N\in K/N;(x+N)-(y+N)=(x-y)+N\in K/N,$ we have $x-y\in K$, and $\forall r\in R,\forall\gamma\in\Gamma,\forall x\in K$, we have $r\gamma(x+N)=r\gamma x+N\in K/N\Rightarrow r\gamma x\in K$. Then K is a $R_\Gamma-$ submodule of M. Conversely, it is easy to verify that $N\subseteq K\leq\mathrm{M}$ then K/N is $R_\Gamma-$submodule of M/N. This complete the proof. □

Proposition 1.5.15: Let M be an $R_\Gamma-$module and I be an ideal of R. Let X be a nonempty subset of M. Then $I\Gamma X=\left\{\sum_{i=1}^{n}a_i\gamma_i x_i:a_i\in I,\gamma_i\in\Gamma,x_i\in X,n\in\mathbb{N}\right\}$ is an $R_\Gamma-$submodule of M.

Proof : For elements $x=\sum_{i=1}^{n} a_i\alpha_i x_i$ and $y=\sum_{j=1}^{m} a'_j\beta_j y_j$ of $I\Gamma X$, we have $x-y=\sum_{k=1}^{m+n} b_k\gamma_k z_k \in I\Gamma X$. Now we consider the following cases:

Case (1): If $1\le k\le n$, then $b_k=a_k, \gamma_k=\alpha_k, z_k=x_k$.

Case (2): If $n+1\le k\le m+n$, then $b_k=-a'_{k-n}, \gamma_k=\beta_{k-n}, z_k=y_{k-n}$.

Now, $\forall r\in R, \forall\gamma\in\Gamma, \forall x=\sum_{i=1}^{n} a_i\gamma_i x_i \in I\Gamma X$, we have $r\gamma x=\sum_{i=1}^{n} r\gamma\left(a_i\gamma_i x_i\right)=\sum_{i=1}^{n}\left(r\gamma a_i\right)\gamma_i x_i$. Thus $I\Gamma X$ is an $R_\Gamma-$submodule of M. □

Definition 1.5.16: [2] Let M be an $R_\Gamma-$module and $\emptyset\ne X\subseteq\mathrm{M}$. Then the generated by X $R_\Gamma-$submodule of M, denoted by $\langle X\rangle$ is the smallest $R_\Gamma-$submodule of M containing X, i.e. $\langle X\rangle=\bigcap\{N : N\le\mathrm{M}\}$, X is called the *generator* of $\langle X\rangle$; and $\langle X\rangle$ is finitely generated if $|X|\prec\infty$. If $X=\{x_1,\ldots,x_n\}$ we write $\langle x_1,\ldots,x_n\rangle$ instead of $\langle\{x_1,\ldots,x_n\}\rangle$. In particular, if $X=\{x\}$ then $\langle x\rangle$ is called the *cyclic submodule* of M, generated by x.

Definition 1.5.17: [2] Let M and N be arbitrary $R_\Gamma-$modules. A mapping $f:\mathrm{M}\to N$ is a *homomorphism* of $R_\Gamma-$modules (or an $R_\Gamma-$homomorphism) if for all $x,y\in\mathrm{M}$ and $\forall r\in R, \forall\gamma\in\Gamma$ we have

(i) $f(x+y)=f(x)+f(y)$;

(ii) $f(r\gamma x)=r\gamma f(x)$.

A homomorphism f is monomorphism if f is one-to-one and f is *epimorphism* if f is onto. f is called *isomorphism* if f is both monomorphism and epimorphism . We denote the set of all $R_\Gamma-$homomorphisms from M into N by $Hom_{R_\Gamma}(\mathrm{M},N)$ or shortly by $Hom(\mathrm{M},N)$. In particular if $\mathrm{M}=N$ we denote $Hom(\mathrm{M},N)$ by $End(\mathrm{M})$.

Remark 1.5.18: If $f:\mathrm{M}\to N$ is an $R_\Gamma-$homomorphism, then $Kerf=\{x\in\mathrm{M} : f(x)=0\}$ is an $R_\Gamma-$homomorphism of M and $Imf=\{y\in N : \exists x\in\mathrm{M}; y=f(x)\}$ is an $R_\Gamma-$submodules of N.

Example 1.5.19: For all $R_\Gamma-$modules A,B, the zero map $0:A\to B$ is an $R_\Gamma-$homomorphism .

Example 1.5.20: Let R be a $\Gamma-$ring. Fix $r_0 \in \Gamma$ and consider the mapping

$\phi : R[x] \to R[x]$ by $f \mapsto f\gamma_0 x$. Then ϕ is an $R_\Gamma-$module homomorphism, because $\forall r \in R, \forall \gamma \in \Gamma$ and $\forall f, g \in R[x]$;

$\phi(f+g) = (f+g)\gamma_0 x = f\gamma_0 x + g\gamma_0 x = \phi(f) + \phi(g)$ and

$\phi(r\gamma f) = r\gamma f\gamma_0 x = r\gamma\phi(f)$.

Proposition 1.5.21: Let R be a $\Gamma-$ring. If $f : \mathrm{M} \to N$ is an $R_\Gamma-$homomorphism and $C \leq Kerf$, then there exists a unique $R_\Gamma-$homomorphism $\overline{f} : \mathrm{M}/C \to N$, such that for every $x \in \mathrm{M}$; $Ker\overline{f} = Kerf/C$ and $Im\overline{f} = Imf$, and $\overline{f}(x+C) = f(x)$, also $\overline{f}$ is

an $R_\Gamma-$isomorphism if and only if it is an $R_\Gamma-$epimorphism and $C = Kerf$. In particular $\mathrm{M}/Kerf \cong Imf$.

Proof : Let $b \in x+C$ then $b = x+c$ for some $c \in C$, also $f(b) = f(x+c)$.

We know f is $R_\Gamma-$homomorphism , therefore $f(b) = f(x+c) = f(x) + f(c) = f(x) + 0 = f(x)$ (since $C \leq Kerf$) then $\overline{f} : \mathrm{M}/C \to N$ is well defined function.

Also $\forall x+C, y+C \in \mathrm{M}/C$ and $\forall r \in R, \gamma \in \Gamma$ we have

(i) $\overline{f}((x+C)+(y+C)) = \overline{f}((x+y)+C) = f(x+y) = f(x) + f(y) = \overline{f}(x+C) + \overline{f}(y+C)$

(ii) $\overline{f}(r\gamma(x+C)) = \overline{f}(r\gamma x+C) = f(r\gamma x) = r\gamma f(x) = r\gamma\overline{f}(x+C)$.

Then $\overline{f}$ is a homomorphism of $R_\Gamma-$modules, also it is clear $Im\overline{f} = Imf$ and $\forall(x+C) \in Ker\overline{f}; x+C \in Ker\overline{f} \Leftrightarrow \overline{f}(x+C) = 0 \Leftrightarrow f(x) = 0 \Leftrightarrow x \in Kerf$, then $Ker\overline{f} = Kerf/C$.

The definition of $\overline{f}$ depends only on f , then $\overline{f}$ is unique.

$\overline{f}$ is an epimorphism if and only if f is an epimorphism. $\overline{f}$ is a monomorphism if and only if $Ker\overline{f}$ is a trivial $R_\Gamma-$submodule of M/C .

Actually if $Kerf = C$ then $\mathrm{M}/Kerf \cong Imf$. □

Chapter Two

Derivations On $\Gamma-$ Rings

2.1 Jordan Generalized Left Derivations On $\Gamma-$ Rings

Throughout the following, we assume that M is an arbitrary $\Gamma-$ring and F a generalized Jordan derivation on M. Clearly, every generalized derivation on M is a Jordan generalized derivation. The converse in general is not true. In the present section, it is shown that every Jordan generalized derivation on certain $\Gamma-$rings is a generalized derivation.

Definition 2.1.1: [27] An additive mapping $D:\mathrm{M}\to\mathrm{M}$ is called a *derivation* ($\Gamma-$ *derivation*) on a $\Gamma-$ ring M if $D(x\alpha y)=D(x)\alpha y+x\alpha D(y)$ holds for all $x,y\in\mathrm{M}$, and $\alpha\in\Gamma$.

Definition 2.1.2: [27] An additive mapping $D:\mathrm{M}\to\mathrm{M}$ is called a *Jordan derivation* on a $\Gamma-$ ring M if $D(x\alpha x)=D(x)\alpha x+x\alpha D(x)$ holds for all $x\in\mathrm{M}$, and $\alpha\in\Gamma$.

Definition 2.1.3: [5] An additive mapping $F:\mathrm{M}\to\mathrm{M}$ is called a *generalized derivation* (*generalized* $\Gamma-$ *derivation*) on a $\Gamma-$ring M if there exists a derivation $D:\mathrm{M}\to\mathrm{M}$ such that $F(x\alpha y)=F(x)\alpha y+x\alpha D(y)$ for all $x,y\in\mathrm{M}$, and $\alpha\in\Gamma$.

Definition 2.1.4: [5] An additive mapping $F:\mathrm{M}\to\mathrm{M}$ is called a *Jordan generalized derivation* on a $\Gamma-$ ring M if there exists a derivation $D:\mathrm{M}\to\mathrm{M}$ such that $F(x\alpha x)=F(x)\alpha x+x\alpha D(x)$ for all $x\in\mathrm{M}$, and $\alpha\in\Gamma$.

Example 2.1.5: Let $f:R\to R$ be a generalized derivation on a ring R. Then there exists a derivation $d:R\to R$ such that $f(xy)=f(x)y+xd(y)$ for all $x,y\in R$. Taking $\mathrm{M}=\mathrm{M}_{1\times 2}(R)$ and $\Gamma=\left\{\begin{pmatrix} n.1 \\ 0 \end{pmatrix}: n \text{ is an integer}\right\}$.

Then M is a $\Gamma-$ring. If we define the map $D:\mathrm{M}\to\mathrm{M}$ by $D((x,y))=(d(x),d(y))$ then D is a derivation on M. Let $F:\mathrm{M}\to\mathrm{M}$ be the additive map defined by $F((x,y))=(f(x),f(y))$.

Then F is a generalized derivation on M. Let N be the subset $\{(x,x):x\in R\}$ of M. Then N is a $\Gamma-$ ring, and the map $F:N\to N$ defined in terms of the generalized Jordan derivation $f:R\to R$ on R by $F((x,x))=(f(x),f(x))$ is a generalized Jordan derivation on N.

Definition 2.1.6: $[27]$ Let M be a $\Gamma-$ring and $D:\mathrm{M}\to\mathrm{M}$ be an additive map. D is called a *left derivation* if for all $x,y\in\mathrm{M},\alpha\in\Gamma$

$$D(x\alpha y)=x\alpha D(y)+y\alpha D(x).$$

A right derivation is defined similarly.

Definition 2.1.7: $[23]$ Let M be a $\Gamma-$ring and $D:\mathrm{M}\to\mathrm{M}$ be an additive map. D is called a *Jordan left derivation* if for all $x\in\mathrm{M},\alpha\in\Gamma$

$$D(x\alpha x)=2x\alpha D(x) .$$

Definition 2.1.8: $[23]$ Let M be a $\Gamma-$ring and $D:\mathrm{M}\to\mathrm{M}$ be an additive map. D is called a *Generalized left derivation* if there exist a left derivation $d:\mathrm{M}\to\mathrm{M}$ such that for all $x,y\in\mathrm{M},\alpha\in\Gamma$

$$D(x\alpha y)=x\alpha D(y)+y\alpha d(x) .$$

Definition 2.1.9: $[23]$ Let M be a $\Gamma-$ring and $D:\mathrm{M}\to\mathrm{M}$ be an additive map. D is called a *Generalized Jordan left derivation* if there exist a Jordan left derivation $d:\mathrm{M}\to\mathrm{M}$ such that for all $x\in\mathrm{M},\alpha\in\Gamma$

$$D(x\alpha x)=x\alpha D(x)+x\alpha d(x) .$$

Proposition 2.1.10: Let M be an arbitrary $\Gamma-$ring, $D:\mathrm{M}\to\mathrm{M}$ be a generalized Jordan left derivation and $d:\mathrm{M}\to\mathrm{M}$ be its associated Jordan left derivation. Then for all $x,y\in\mathrm{M},\alpha\in\Gamma$, we have

$$D(x\alpha y+y\alpha x)=x\alpha D(y)+x\alpha d(y)+y\alpha D(x)+y\alpha d(x).$$

Proof : By Definition 2.1.9, we have

$$\begin{aligned}D((x+y)\alpha(x+y))&=(x+y)\alpha D(x+y)+(x+y)\alpha d(x+y)\\&=(x+y)\alpha\big(D(x)+D(y)\big)+(x+y)\alpha\big(d(x)+d(y)\big)\\&=x\alpha D(x)+x\alpha D(y)+y\alpha D(x)+y\alpha D(y)\\&\quad+x\alpha d(x)+x\alpha d(y)+y\alpha d(x)+y\alpha d(y). \qquad (2.1.1)\end{aligned}$$

Also,

$$D\big((x+y)\alpha(x+y)\big)=D\big(x\alpha x+(x\alpha y+y\alpha x)+y\alpha y\big)$$

$$=D(x\alpha x)+D(x\alpha y+y\alpha x)+D(y\alpha y)\ .$$

By using Definition 2.1.9, we get

$$D\big((x+y)\alpha(x+y)\big)=x\alpha D(x)+x\alpha d(x)+D(x\alpha y+y\alpha x)+y\alpha D(y)+y\alpha d(y)\ . \quad (2.1.2)$$

In view of $(2.1.1)$ and $(2.1.2)$, we get

$$D(x\alpha y+y\alpha x)=x\alpha D(y)+x\alpha d(y)+y\alpha D(x)+y\alpha d(x)\ .$$

This completes the proof. □

Corollary 2.1.11: Let M be an arbitrary $\Gamma-$ ring and $d:\mathrm{M}\to\mathrm{M}$ be a Jordan left derivation. Then for all $x,y\in\mathrm{M},\alpha\in\Gamma$, we have

$$d(x\alpha y+y\alpha x)=2x\alpha d(y)+2y\alpha d(x)\ .$$

Proposition 2.1.12: $[23]$ Let M be a 2-torsion free $\Gamma-$ ring and $x\alpha y\beta z=x\beta y\alpha z$ holds for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$. Let $D:\mathrm{M}\to\mathrm{M}$ be a Generalized Jordan left derivation and $d:\mathrm{M}\to\mathrm{M}$ be an associated Jordan left derivation. Then the following statements hold for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$.

(i) $D(x\alpha y\beta x)=x\alpha y\beta D(x)+(2x\alpha y-y\alpha x)\beta d(x)+x\alpha x\beta d(y)\ .$

(ii) $D(x\alpha y\beta z+z\alpha y\beta x)=x\alpha y\beta D(z)+z\alpha y\beta D(x)+(2z\alpha y-y\alpha z)\beta d(x)$
$+(2x\alpha y-y\alpha x)\beta d(z)+(x\alpha z+z\alpha x)\beta d(y)\,.$

Proof : In view of Proposition 2.1.10, consider the following

$$D(x\beta y+y\beta x)=x\beta D(y)+x\beta d(y)+y\beta D(x)+y\beta d(x)\ .$$

The replacement of y by $x\alpha y+y\alpha x$ in the last relation yields

$$D\big(x\beta(x\alpha y+y\alpha x)+(x\alpha y+y\alpha x)\beta x\big)$$

$$=x\beta D(x\alpha y+y\alpha x)+x\beta d(x\alpha y+y\alpha x)+(x\alpha y+y\alpha x)\beta D(x)+(x\alpha y+y\alpha x)\beta d(x)\,.$$

By using Proposition 2.1.10 and Corollary 2.1.11 in the last relation, we get

$$D\left(x\beta\left(x\alpha y+y\alpha x\right)+\left(x\alpha y+y\alpha x\right)\beta x\right)$$

$$=x\beta\left\{x\alpha D\left(y\right)+x\alpha d\left(y\right)+y\alpha D\left(x\right)+y\alpha d\left(x\right)\right\}+x\beta\left\{2x\alpha d\left(y\right)+2y\alpha d\left(x\right)\right\}$$

$$+\left(x\alpha y+y\alpha x\right)\beta D\left(x\right)+\left(x\alpha y+y\alpha x\right)\beta d\left(x\right).$$

That is,

$$2D\left(x\alpha y\beta x\right)+D\left(x\beta x\alpha y+y\alpha x\beta x\right)$$

$$=x\beta x\alpha D\left(y\right)+x\beta x\alpha d\left(y\right)+x\beta y\alpha D\left(x\right)+x\beta y\alpha d\left(x\right)+2x\beta x\alpha d\left(y\right)$$

$$+2x\beta y\alpha d\left(x\right)+x\alpha y\beta D\left(x\right)+x\alpha y\beta d\left(x\right)+y\alpha x\beta D\left(x\right)+y\alpha x\beta d\left(x\right).$$

or,

$$2D\left(x\alpha y\beta x\right)+\left(x\beta x\right)\alpha D\left(y\right)+\left(x\beta x\right)\alpha d\left(y\right)+y\alpha D\left(x\beta x\right)+y\alpha d\left(x\beta x\right)$$

$$=x\beta x\alpha D\left(y\right)+x\beta x\alpha d\left(y\right)+x\beta y\alpha D\left(x\right)+x\beta y\alpha d\left(x\right)+2x\beta x\alpha d\left(y\right)$$

$$+2x\beta y\alpha d\left(x\right)+x\alpha y\beta D\left(x\right)+x\alpha y\beta d\left(x\right)+y\alpha x\beta D\left(x\right)+y\alpha x\beta d\left(x\right) .$$

In view of Definitions 2.1.7 and 2.1.9, the last expression becomes

$$2D\left(x\alpha y\beta x\right)+\left(x\beta x\right)\alpha D\left(y\right)+x\beta x\alpha d\left(y\right)+y\alpha\left\{x\beta D\left(x\right)+x\beta d\left(x\right)\right\}+2y\alpha x\beta d\left(x\right)$$

$$=x\beta x\alpha D\left(y\right)+x\beta x\alpha d\left(y\right)+x\beta y\alpha D\left(x\right)+x\beta y\alpha d\left(x\right)+2x\beta y\alpha d\left(x\right)$$

$$+2x\beta x\alpha d\left(y\right)+x\alpha y\beta D\left(x\right)+x\alpha y\beta d\left(x\right)+y\alpha x\beta D\left(x\right)+y\alpha x\beta d\left(x\right) .$$

By canceling identical terms and using the given condition $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$, we get

$$2D\left(x\alpha y\beta x\right)+x\beta x\alpha D\left(y\right)+x\beta x\alpha d\left(y\right)+y\alpha x\beta D\left(x\right)+y\alpha x\beta d\left(x\right)+2y\alpha x\beta d\left(x\right)$$

$$=x\beta x\alpha D\left(y\right)+x\beta x\alpha d\left(y\right)+x\beta y\alpha D\left(x\right)+x\beta y\alpha d\left(x\right)+2x\beta y\alpha d\left(x\right)$$

$$+2x\beta x\alpha d\left(y\right)+x\alpha y\beta D\left(x\right)+x\alpha y\beta d\left(x\right)+y\alpha x\beta D\left(x\right)+y\alpha x\beta d\left(x\right).$$

Consequently,

$$2D(x\alpha y\beta x)+2y\alpha x\beta d(x)=2x\alpha y\beta D(x)+4x\alpha y\beta d(x)+2x\alpha x\beta d(y)$$

As M is a 2-torsion free $\Gamma-$ ring so,

$$D(x\alpha y\beta x)=x\alpha y\beta D(x)+(2x\alpha y-y\alpha x)\beta d(x)+x\alpha x\beta d(y).$$

This completes the proof of (i).

(ii) The replacement of x by $x+z$ in (i), gives

$$D((x+z)\alpha y\beta(x+z))$$

$$=(x+z)\alpha y\beta D(x+z)+(2(x+z)\alpha y-y\alpha(x+z))\beta d(x+z)+(x+z)\alpha(x+z)\beta d(y),$$

$$D(x\alpha y\beta z+z\alpha y\beta x+x\alpha y\beta x+z\alpha y\beta z)$$

$$=(x+z)\alpha y\beta(D(x)+D(z))+\{2x\alpha y+2z\alpha y-y\alpha x-y\alpha z\}\beta(d(x)+d(z))$$

$$+x\alpha x\beta d(y)+x\alpha z\beta d(y)+z\alpha x\beta d(y)+z\alpha z\beta d(y),$$

or,

$$D(x\alpha y\beta z+z\alpha y\beta x)+D(x\alpha y\beta x)+D(z\alpha y\beta z)$$

$$=x\alpha y\beta D(x)+x\alpha y\beta D(z)+z\alpha y\beta D(x)+z\alpha y\beta D(z)+2x\alpha y\beta d(x)$$

$$+2x\alpha y\beta d(z)+2z\alpha y\beta d(x)+2z\alpha y\beta d(z)-y\alpha x\beta d(x)-y\alpha x\beta d(z)$$

$$-y\alpha z\beta d(x)-y\alpha z\beta d(z)+x\alpha x\beta d(y)+x\alpha z\beta d(y)+z\alpha x\beta yd(y)+z\alpha z\beta d(y).$$

The application of Proposition 2.1.12 (i) in the last relation, gives

$$D(x\alpha y\beta z+z\alpha y\beta x)+x\alpha y\beta D(x)+2x\alpha y\beta d(x)-y\alpha x\beta d(x)+x\alpha x\beta d(y)$$

$$+z\alpha y\beta D(x)+2z\alpha y\beta d(z)-y\alpha z\beta d(z)+z\alpha z\beta d(y)$$

$$=x\alpha y\beta D(x)+x\alpha y\beta D(z)+z\alpha y\beta D(x)+z\alpha y\beta D(z)+2x\alpha y\beta d(x)$$

$$+2x\alpha y\beta d(z)+2z\alpha y\beta d(x)+2z\alpha y\beta d(z)$$

$$-y\alpha x\beta d(x)-y\alpha x\beta d(z)-y\alpha z\beta d(x)-y\alpha z\beta d(z)+x\alpha x\beta d(y)$$

$$+x\alpha z\beta d(y)+z\alpha x\beta d(y)+z\alpha z\beta d(y).$$

Consequently,

$$D(x\alpha y\beta z+z\alpha y\beta x)=x\alpha y\beta D(z)+z\alpha y\beta D(x)+2z\alpha y\beta d(x)+2x\alpha y\beta d(z)$$

$$-y\alpha z\beta d(x)-y\alpha x\beta d(z)+x\alpha z\beta d(y)+z\alpha x\beta d(y),$$

or

$$D(x\alpha y\beta z+z\alpha y\beta x)=x\alpha y\beta D(z)+z\alpha y\beta D(x)+(2z\alpha y-y\alpha z)\beta d(x)$$

$$+(2x\alpha y-y\alpha x)\beta d(z)+(x\alpha z+z\alpha x)\beta d(y).$$

This completes the proof of (ii) . □

Corollary 2.1.13: [23] Let M be a 2-torsion free $\Gamma-$ring and $x\alpha y\beta z=x\beta y\alpha z$ hold for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$. Let $d:\mathrm{M}\to\mathrm{M}$ be Jordan left derivation.

Then the following statements hold for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$.

(i) $d(x\alpha y\beta x)=x\alpha x\beta d(y)+3x\alpha y\beta d(x)-y\alpha x\beta d(x)$.

(ii) $d(x\alpha y\beta z+z\alpha y\beta x)=(x\alpha z+z\alpha x)\beta d(y)+(3x\alpha y-y\alpha x)\beta d(z)$
$+(3z\alpha y-y\alpha z)\beta d(x)$.

Proposition 2.1.14: [23] Let M be a 2-torsion free $\Gamma-$ring and $x\alpha y\beta z=x\beta y\alpha z$ hold for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$. Let $D:\mathrm{M}\to\mathrm{M}$ be a generalized Jordan left derivation and $d:\mathrm{M}\to\mathrm{M}$ be an associated Jordan left derivation. Then the following statements hold for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$.

(i) $(x\alpha y-y\alpha x)\beta x\alpha d(x)=x\alpha(x\alpha y-y\alpha x)\beta d(x)$.

(ii) $(x\alpha y-y\alpha x)\beta\{d(x\alpha y)-x\alpha d(y)-y\alpha d(x)\}=0$.

2.2 Gamma-Derivations On The Projective Product Of $\Gamma-$ Rings

This section highlights many enlightening results on various gamma-derivations in the projective product of gamma-rings.

Definition 2.2.1: [16] Let M be a $\Gamma-$ring, then an additive mapping $d:\mathrm{M}\to\mathrm{M}$ is called a $\Gamma-$*semi-derivation* associated with a function $g:\mathrm{M}\to\mathrm{M}$ if for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$,

$$d(x\alpha y)=d(x)\alpha g(y)+x\alpha d(y)=d(x)\alpha y+g(x)\alpha d(y) \text{ and } d(g(x))=g(d(x)).$$

If $g=1$ i.e. the identity mapping on M, then all $\Gamma-$ semi-derivations associated with g are merely ordinary $\Gamma-$ derivations.

If g is an endomorphism of M, then other examples of semi-derivations are of the form $d(x)=x-g(x)$.

Definition 2.2.2: [16] A $\Gamma-$derivation D is said to be *inner* if $\exists\, a\in\mathrm{M}$ s.t $D(x\alpha x)=a\alpha x-x\alpha a$. A mapping $x\alpha x\mapsto a\alpha x+x\alpha b$, where a,b are fixed elements in M and for all $\alpha\in\Gamma$ is called a *generalized inner derivation*.

Definition 2.2.3: [16] Let S be a nonempty subset of M and let d be a $\Gamma-$derivation on M. If $d(x\alpha y)=d(x)\alpha d(y)$ [or $d(x\alpha y)=d(y)\alpha d(x)$], for all $x,y\in S,\alpha\in\Gamma$, then d is said to be a $\Gamma-$ *homomorphism* [or an *anti* $\Gamma-$ *homomorphism*] on S.

Definition 2.2.4: [16] Let M_1 a Γ_1-ring and M_2 a Γ_2-ring. Let $\mathrm{M}=\mathrm{M}_1\times\mathrm{M}_2$ and $\Gamma=\Gamma_1\times\Gamma_2$. Then we define addition and multiplication on M and Γ by, $(x_1,x_2)+(y_1,y_2)=(x_1+y_1,x_2+y_2)$,
$(\alpha_1,\alpha_2)+(\beta_1,\beta_2)=(\alpha_1+\beta_1,\alpha_2+\beta_2)$ and $(x_1,x_2)(\alpha_1,\alpha_2)(y_1,y_2)=(x_1\alpha_1y_1,x_2\alpha_2y_2)$ for every $(x_1,x_2),(y_1,y_2)\in\mathrm{M}$ and $(\alpha_1,\alpha_2),(\beta_1,\beta_2)\in\Gamma$.

With respect to this addition and multiplication M is a $\Gamma-$ring. We call this $\Gamma-$ring the *Projective product of* $\Gamma-$rings.

Since M_1, M_2 and Γ_1, Γ_2 are additive abelian groups, so obviously $\mathrm{M}=\mathrm{M}_1\times\mathrm{M}_2$ and $\Gamma=\Gamma_1\times\Gamma_2$ are additive abelian groups. To show M is a $\Gamma-$ring, we need to show the following properties:

Let $x=(x_1,x_2),y=(y_1,y_2),\mathrm{z}=(\mathrm{z}_1,\mathrm{z}_2)\in\mathrm{M}$ and $\alpha=(\alpha_1,\alpha_2),\beta=(\beta_1,\beta_2),\gamma=(\gamma_1,\gamma_2)\in\Gamma$ be any elements .

Property (i) :

$\forall x,y\in\mathrm{M},\alpha\in\Gamma$ we have $x\alpha y=(x_1,x_2)(\alpha_1,\alpha_2)(y_1,y_2)$

$$=(x_1\alpha_1y_1,x_2\alpha_2y_2)$$

as $x_1\alpha_1y_1\in\mathrm{M}_1$, $x_2\alpha_2y_2\in\mathrm{M}_2$ [since M_1 is a Γ_1-ring and M_2 is a Γ_2-ring] then $x\alpha y\in\mathrm{M}$.

Property (ii) :

$$(x+y)\alpha z=((x_1,x_2)+(y_1,y_2))(\alpha_1,\alpha_2)(\mathrm{z}_1,\mathrm{z}_2)$$

$$=((x_1+y_1),(x_2+y_2))(\alpha_1,\alpha_2)(\mathrm{z}_1,\mathrm{z}_2)$$

$$=\left(\left(x_1+y_1\right)\alpha_1 z_1,\left(x_2+y_2\right)\alpha_2 z_2\right)$$

$$=\left(x_1\alpha_1 z_1+y_1\alpha_1 z_1, x_2\alpha_2 z_2+y_2\alpha_2 z_2\right)$$

$$=\left(x_1\alpha_1 z_1, x_2\alpha_2 z_2\right)+\left(y_1\alpha_1 z_1, y_2\alpha_2 z_2\right)$$

$$=\left(x_1, x_2\right)\left(\alpha_1,\alpha_2\right)\left(z_1, z_2\right)+\left(y_1, y_2\right)\left(\alpha_1,\alpha_2\right)\left(z_1, z_2\right)=x\alpha z+y\alpha z$$

Thus we get, $(x+y)\alpha z=x\alpha z+y\alpha z$. Similarly, $x(\alpha+\beta)z=x\alpha z+x\beta z$ and $x\alpha(y+z)=x\alpha y+x\alpha z$.

Property (iii) :

$$(x\alpha y)\beta z=\left(\left(x_1, x_2\right)\left(\alpha_1,\alpha_2\right)\left(y_1, y_2\right)\right)\left(\beta_1,\beta_2\right)\left(z_1, z_2\right)$$

$$=\left(x_1\alpha_1 y_1, x_2\alpha_2 y_2\right)\left(\beta_1,\beta_2\right)\left(z_1, z_2\right)=\left(\left(x_1\alpha_1 y_1\right)\beta_1 z_1,\left(x_2\alpha_2 y_2\right)\beta_2 z_2\right)$$

$$=\left(x_1\alpha_1\left(y_1\beta_1 z_1\right), x_2\alpha_2\left(y_2\beta_2 z_2\right)\right) \text{[since } M_1 \text{ is a } \Gamma_1-\text{ring and } M_2 \text{ is a } \Gamma_2-\text{ring]}$$

$$=\left(x_1, x_2\right)\left(\alpha_1,\alpha_2\right)\left(y_1\beta_1 z_1, y_2\beta_2 z_2\right)$$

$$=\left(x_1, x_2\right)\left(\alpha_1,\alpha_2\right)\left(\left(y_1, y_2\right)\left(\beta_1,\beta_2\right)\left(z_1, z_2\right)\right)=x\alpha(y\beta z)$$

Thus we get, $(x\alpha y)\beta z=x\alpha(y\beta z)$. Similarly, $x\alpha(y\beta z)=x(\alpha y\beta)z$.

Let $x\alpha y=0\ \forall x, y\in M\Rightarrow\left(x_1, x_2\right)\left(\alpha_1,\alpha_2\right)\left(y_1, y_2\right)=0$

$$\Rightarrow\left(x_1\alpha_1 y_1, x_2\alpha_2 y_2\right)=0=(0,0)$$

$$\Rightarrow x_1\alpha_1 y_1=0\ ,\ x_2\alpha_2 y_2=0\ ,\ \forall x_1, y_1\in M_1 \text{ and } x_2, y_2\in M_2$$

$$\Rightarrow \alpha_1=0\ ,\ \alpha_2=0 \text{ [since } M_1 \text{ is a } \Gamma_1-\text{ring and } M_2 \text{ is a } \Gamma_2-\text{ring]}$$

$$\Rightarrow\left(\alpha_1,\alpha_2\right)=(0,0)\Rightarrow\alpha=0$$

Thus we get, $x\alpha y=0\ \forall x, y\in M\Rightarrow\alpha=0$.

Hence M is a gamma ring which is known as the projective product of gamma rings.

Theorem 2.2.5: [16] Let M_1 a Γ_1-ring and M_2 a Γ_2-ring and Γ be their projective product. Then we get the following results :

(i) Every pair of $\Gamma-$derivations D_1 and D_2 on M_1 and M_2 respectively give rise to a $\Gamma-$derivation D on M.

(ii) Two $\Gamma-$semi-derivations d_1 and d_2 on M_1 and M_2 respectively give rise to a $\Gamma-$semi-derivation d on M.

(iii) For every generalized $\Gamma-$derivations f_1 and f_2 on M_1 and M_2 respectively give rise to a generalized $\Gamma-$derivation f on M.

(iv) Two inner $\Gamma-$derivations d_1 and d_2 on M_1 and M_2 respectively give rise to an inner $\Gamma-$derivation d on M.

(v) Every two Jordan derivations j_1 and j_2 on M_1 and M_2 respectively give rise to a Jordan derivation j on M defined by j_1 and j_2.

(vi) Every two generalized Jordan derivations j_1 and j_2 on M_1 and M_2 respectively give rise to a generalized Jordan derivation j on M constructed with the help of j_1 and j_2.

(vii) Every two generalized inner derivations on M_1 and M_2 respectively give rise to a generalized inner derivation on M.

(viii) If ϕ_1 and ϕ_2 be two homomorphisms on M_1 and M_2 respectively, then there exist a homomorphism on M constructed with the help of ϕ_1 and ϕ_2.

Proof : (i) We define a mapping $D: M \to M$ by $D(x)=D((x_1,x_2))=(D_1(x_1),D_2(x_2))$. Clearly, D is well defined mapping. We show that D is a derivation on a $\Gamma-$ring M.

Let $x=(x_1,x_2)$, $y=(y_1,y_2)\in M$ and $\alpha=(\alpha_1,\alpha_2)\in\Gamma$ be any elements. Then $D(x+y)=D((x_1,x_2)+(y_1,y_2))=D((x_1+y_1),(x_2+y_2))=(D_1(x_1+y_1),D_2(x_2+y_2))$

$=(D_1(x_1)+D_1(y_1),D_2(x_2)+D_2(y_2))$[since D_1 and D_2 are additive mappings]

$=(D_1(x_1),D_2(x_2))+(D_1(y_1),D_2(y_2))=D((x_1,x_2))+D((y_1,y_2))=D(x)+D(y)$

Thus, $D(x+y)=D(x)+D(y)$ $\forall x,y\in M$ which implies that D is additive.

Again, $D(x\alpha y)=D((x_1,x_2)(\alpha_1,\alpha_2)(y_1,y_2))=D((x_1\alpha_1 y_1,x_2\alpha_2 y_2))=(D_1(x_1\alpha_1 y_1),D_2(x_2\alpha_2 y_2))$

$=(D_1(x_1)\alpha_1 y_1+x_1\alpha_1 D_1(y_1),D_2(x_2)\alpha_2 y_2+x_2\alpha_2 D_2(y_2))$ [since D_1 and D_2 are gamma-derivations on M_1 and M_2 respectively]

$$=\left(D_1(x_1)\alpha_1 y_1, D_2(x_2)\alpha_2 y_2\right)+\left(x_1\alpha_1 D_1(y_1), x_2\alpha_2 D_2(y_2)\right)$$

$$=\left(D_1(x_1), D_2(x_2)\right)(\alpha_1,\alpha_2)(y_1,y_2)+(x_1,x_2)(\alpha_1,\alpha_2)\left(D_1(y_1), D_2(y_2)\right)$$

$$=D(x)\alpha y+x\alpha D(y)$$

Thus, $D(x\alpha y)=D(x)\alpha y+x\alpha D(y)$ $\forall x,y\in \mathrm{M}$ and $\alpha\in\Gamma$. So D is a gamma-derivation on M .

(ii) Let d_1 be a $\Gamma-$semi-derivation on M_1 associated with the function $g_1:\mathrm{M}_1\to\mathrm{M}_1$ and d_2 be a $\Gamma-$semi-derivation on M_2 associated with the function $g_2:\mathrm{M}_2\to\mathrm{M}_2$.

We define the functions $d:\mathrm{M}\to\mathrm{M}$ and $\mathrm{g}:\mathrm{M}\to\mathrm{M}$ by

$d(x)=d\left((x_1,x_2)\right)=\left(d_1(x_1), d_2(x_2)\right)$ and

$g(x)=g\left((x_1,x_2)\right)=\left(g_1(x_1), \mathrm{g}_2(x_2)\right)$ for all $x=(x_1,x_2)\in\mathrm{M}$

Then clearly d and g are well defined as well as d is additive. Let $x=(x_1,x_2), y=(y_1,y_2)\in\mathrm{M}$ and $\alpha=(\alpha_1,\alpha_2)\in\Gamma$ be any elements. Then,
$d(x\alpha y)=d\left((x_1,x_2)(\alpha_1,\alpha_2)(y_1,y_2)\right)=d\left((x_1\alpha_1 y_1, x_2\alpha_2 y_2)\right)=\left(d_1(x_1\alpha_1 y_1), d_2(x_2\alpha_2 y_2)\right)$

$=\left(d_1(x_1)\alpha_1 g_1(y_1)+x_1\alpha_1 d_1(y_1), d_2(x_2)\alpha_2 g_2(y_2)+x_2\alpha_2 d_2(y_2)\right)$ [since d_1 and d_2 are $\Gamma-$semi-derivations on M_1 and M_2 respectively] .

$$=\left(d_1(x_1)\alpha_1 g_1(y_1), d_2(x_2)\alpha_2 g_2(y_2)\right)+\left(x_1\alpha_1 d_1(y_1), x_2\alpha_2 d_2(y_2)\right)$$

$$=\left(d_1(x_1), d_2(x_2)\right)(\alpha_1,\alpha_2)\left(g_1(y_1), g_2(y_2)\right)+(x_1,x_2)(\alpha_1,\alpha_2)\left(d_1(y_1), d_2(y_2)\right)$$

$$=d\left((x_1,x_2)\right)(\alpha_1,\alpha_2)\,g\left((y_1,y_2)\right)+(x_1,x_2)(\alpha_1,\alpha_2)\,d\left((y_1,y_2)\right)$$

$$=d(x)\alpha g(y)+x\alpha d(y)$$

Thus, $d(x\alpha y)=d(x)\alpha g(y)+x\alpha d(y)$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.

Similarly, we can show that, $d(x\alpha y)=d(x)\alpha y+g(x)\alpha d(y)$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.

Again, $d\left(g(x)\right)=d\left(g\left((x_1,x_2)\right)\right)=d\left(\left(g_1(x_1), g_2(x_2)\right)\right)=\left(d_1\left(g_1(x_1)\right), d_2\left(g_2(x_2)\right)\right)$

$=\left(g_1\left(d_1\left(x_1\right)\right),g_2\left(d_2\left(x_2\right)\right)\right)$ [Since d_1 and d_2 are semi-derivations on M_1 and M_2 respectively]

$$=g\left(\left(d_1\left(x_1\right),d_2\left(x_2\right)\right)\right)=g\left(d\left(\left(x_1,x_2\right)\right)\right)=g\left(d\left(x\right)\right)$$

Thus we get, $d\left(g\left(x\right)\right)=g\left(d\left(x\right)\right)$, $\forall x\in\mathrm{M}$

Hence d is a $\Gamma-$ semi-derivation on M associated with the function g and hence the required result.

(iii) using same method in part (ii).

(iv) Let d_1 be an inner $\Gamma-$ derivation on M_1 with respect to the element $a\in\mathrm{M}_1$ and d_2 be an inner $\Gamma-$ derivation on M_2 with respect to the element $b\in\mathrm{M}_2$. We defined a mapping $d:\mathrm{M}\to\mathrm{M}$ by $d\left(x\right)=d\left(\left(x_1,x_2\right)\right)=\left(d_1\left(x_1\right),d_2\left(x_2\right)\right)$, $\forall x=\left(x_1,x_2\right)\in\mathrm{M}$. Then, d is well defined as well as additive.

Let $x=\left(x_1,x_2\right)\in\mathrm{M}$ and $\alpha=\left(\alpha_1,\alpha_2\right)\in\Gamma$ be any two elements. Then $d\left(x\alpha x\right)=d\left(\left(x_1,x_2\right)\left(\alpha_1,\alpha_2\right)\left(x_1,x_2\right)\right)=d\left(\left(x_1\alpha_1x_1,x_2\alpha_2x_2\right)\right)=\left(d_1\left(x_1\alpha_1x_1\right),d_2\left(x_2\alpha_2x_2\right)\right)$

$=\left(a\alpha_1x_1-x_1\alpha_1a,b\alpha_2x_2-x_2\alpha_2b\right)$ [Since d_1 and d_2 are inner derivations on M_1 and M_2 w.r.t a and b respectively]

$$=\left(a\alpha_1x_1,b\alpha_2x_2\right)-\left(x_1\alpha_1a,x_2\alpha_2b\right)=\left(a_1,a_2\right)\left(\alpha_1,\alpha_2\right)\left(x_1,x_2\right)-\left(x_1,x_2\right)\left(\alpha_1,\alpha_2\right)\left(a_1,a_2\right)$$

$=m\alpha x-x\alpha m$ where $m=\left(a_1,a_2\right)\in\mathrm{M}$

Thus d is an inner derivation on M with respect to the element $m\in\mathrm{M}$. Similarly we can show $(v),(vi),(vii)$ and (vii).□

Theorem 2.2.6: [16] For every derivation D on M, there exist derivations D_1 and D_2 on M_1 and M_2 respectively, where M is the projective product of M_1 and M_2.

Furthermore, if D is semi-derivation/ generalized $\Gamma-$ derivation /inner $\Gamma-$ derivation/ Jordan derivation/ generalized Jordan derivation, then D_1 and D_2 are also so.

Proof : Let D be a derivation on M. Let x_1 be any element of M_1 and let $D\left(\left(x_1,0\right)\right)=\left(u_1,u_2\right)$

We define a map $D_1:\mathrm{M}_1\to\mathrm{M}_1$ by $D_1\left(x_1\right)=u_1$, i.e. by $D_1\left(x\right)=fD\left(\left(x,0\right)\right)$ [i.e. the first component of $D\left(\left(x,0\right)\right)$]. We shall show that D_1 is a derivation on M_1.

Let $x_1, x_2 \in \mathrm{M}_1$ be any two elements and $\alpha_1 \in \Gamma_1$, then $D_1(x_1+x_2) = fD((x_1+x_2,0)) = fD((x_1+x_2,0+0))$

$$= fD((x_1,0)+(x_2,0)) = f\left[D((x_1,0))+D((x_2,0))\right] \text{[since } D \text{ is additive]}$$

$$= fD((x_1,0)) + fD((x_2,0)) = D_1(x_1)+D_1(x_2)$$

Thus we get, $D_1(x_1+x_2) = D_1(x_1)+D_1(x_2)\ \forall x_1, x_2 \in \mathrm{M}_1$ i.e. D_1 is an additive.

Now, $D_1(x_1\alpha_1 x_2) = fD((x_1\alpha_1 x_2, 0)) = fD((x_1\alpha_1 x_2, 0\alpha_2 0)), \alpha_2 \in \Gamma_2$

$= fD((x_1,0)(\alpha_1,\alpha_2)(x_2,0)) = fD(x\alpha y)$, where $x=(x_1,0), y=(x_2,0)$ and $\alpha = (\alpha_1,\alpha_2)$,

$= f\left[D(x)\alpha y + x\alpha D(y)\right]$ [since D is a derivation on M]

$= f\left[D(x)\alpha y\right] + f\left[x\alpha D(y)\right] = f\left[D((x_1,0))(\alpha_1,\alpha_2)(x_2,0)\right] + f\left[(x_1,0)(\alpha_1,\alpha_2)D((x_2,0))\right]$

$= f\left[D(x)\alpha y\right] + f\left[x\alpha D(y)\right] = fD(x)\, f\alpha\, fy + fxf\alpha\, fD(y) = D_1(x_1)\alpha_1 x_2 + x_1\alpha_1 D_1(x_2)$

Thus we get, $D_1(x_1\alpha_1 x_2) = D_1(x_1)\alpha_1 x_2 + x_1\alpha_1 D_1(x_2)\ \forall x_1, x_2 \in \mathrm{M}_1$ and $\alpha_1 \in \Gamma_1$.

So D_1 is a derivation on M_1 defined by the derivation D on M.

Similarly defining a mapping, $D_2 : \mathrm{M}_2 \to \mathrm{M}_2$ by $D_2(x) = sD((0,x))$, where s represents the second component of $D((0,x))$, we can show that D_2 is a derivation on M_2.

Thus for every derivation D on M there exist derivations D_1 on M_1 and D_2 is on M_2 and hence the desired result. □

Remark 2.2.7: [16] The above results can be extended to the projective product of n number of Gamma-rings.

2.3 Jordan Generalized Reverse Derivations On $\Gamma-$Rings

In this section we introduce and study the concepts of reverse derivation, generalized reverse derivation, Jordan generalized reverse derivation, higher reverse derivation and generalized higher reverse derivation of $\Gamma-$ring.

Definition 2.3.1: [8] Let M be a $\Gamma-$ring and $d:\mathrm{M}\to\mathrm{M}$ be an additive mapping then d is called *reverse derivation* if $d(x\alpha y) = d(y)\alpha x + y\alpha d(x)$, for all $x, y \in \mathrm{M}, \alpha \in \Gamma$.

Let $D=(d_i)_{i\in\mathbb{N}}$ be additive mappings on a ring R then D is called *higher reverse derivation of* $\Gamma-$ ring M if $d_n(x\alpha y)=\sum_{i+j=n} d_i(y)\alpha d_j(x)$, $\forall x,y\in\mathrm{M},\alpha\in\Gamma$ and $n\in\mathbb{N}$.

D is called a *Jordan higher reverse derivation of* $\Gamma-$ring M if $d_n(x\alpha x)=\sum_{i+j=n} d_i(x)\alpha d_j(x)$, $\forall x\in\mathrm{M},\alpha\in\Gamma$ and $n\in\mathbb{N}$.

D is called a *Jordan triple higher reverse derivation of* $\Gamma-$ring M if $d_n(x\alpha y\beta x)=d_n(x)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} d_i(x)\beta d_j(y)\alpha d_r(x)$, $\forall x,y\in\mathrm{M},\alpha,\beta\in\Gamma$ and $n\in\mathbb{N}$.

Remark 2.3.2: [8] If M is commutative, then both a derivation and the reverse derivation are the same.

Example 2.3.3: Let R be an associative ring with 1, $d:R\to R$ be a reverse derivation. By Example 2.1.5 define $D:N\to N$ by $D((x,x))=(d(x),d(x))$. If $a=(x_1,x_1),b=(x_2,x_2)$ and $\alpha=\begin{pmatrix} n.1 \\ 0 \end{pmatrix}\in\Gamma$. Then we have

$$D(a\alpha b)=D\left((x_1,x_1)\begin{pmatrix} n.1 \\ 0 \end{pmatrix}(x_2,x_2)\right)$$

$$=D(x_1nx_2,x_1nx_2)=(d(x_1nx_2),d(x_1nx_2))$$

$$=(d(x_2)nx_1+x_2nd(x_1),d(x_2)nx_1+x_2nd(x_1))$$

$$=(d(x_2)nx_1,d(x_2)nx_1)+(x_2nd(x_1),x_2nd(x_1))$$

$$=(d(x_2),d(x_2))\begin{pmatrix} n.1 \\ 0 \end{pmatrix}(x_1,x_1)+(x_2,x_2)\begin{pmatrix} n.1 \\ 0 \end{pmatrix}(d(x_1),d(x_1))$$

$$=D((x_2,x_2))\alpha a+b\alpha D((x_1,x_1))=D(b)\alpha a+b\alpha D(a).$$

Hence D is a reverse derivation on the $\Gamma-$ ring N .

Example 2.3.4: Let R be a ring and $\mathrm{M}=\left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}: x,y\in R\right\}$, where $R^2\neq 0$, and $\Gamma=\left\{\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}: n \text{ is an integer}\right\}$.

Then M is a $\Gamma-$ring. Let $d:\mathrm{M}\to\mathrm{M}$ defined by $d\left(A\right)=d\left(\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}\right)=\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. It is easy to show that d is derivation but not reverse derivation.

Example 2.3.5: Let R be a ring and $\mathrm{M}=\left\{\begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}: x,y,z\in R\right\}$, and $\Gamma=\left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & n \end{pmatrix}: n \text{ is an integer}\right\}$. Then M is a $\Gamma-$ring. Let $d:\mathrm{M}\to\mathrm{M}$ defined by $d\left(A\right)=d\left(\begin{pmatrix} 0 & x & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)=\begin{pmatrix} 0 & 0 & 0 & -z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix}$. It is easy to show that d is a reverse derivation but not a derivation.

Definition 2.3.6: [21] Let M be a $\Gamma-$ ring and $f:\mathrm{M}\to\mathrm{M}$ be an additive mapping then f is called *generalized reverse derivation on* M if there exists a reverse derivation $d:\mathrm{M}\to\mathrm{M}$ such that $f\left(x\alpha y\right)=f\left(y\right)\alpha x+y\alpha d\left(x\right)$ for every $x,y\in\mathrm{M},\alpha\in\Gamma$.

f is said to be *a Jordan generalized reverse derivation of* M if there exists a Jordan reverse derivation such that $f\left(x\alpha x\right)=f\left(x\right)\alpha x+x\alpha d\left(x\right)$ for every $x\in\mathrm{M},\alpha\in\Gamma$.

f is said to be *a Jordan generalized triple reverse derivation of* M if there exists Jordan triple higher reverse derivation of M such that:

$$f\left(x\alpha y\beta x\right)=f\left(x\right)\beta x\alpha y+x\beta d\left(y\right)\alpha x+x\beta y\alpha d\left(x\right) \text{ for every } x,y\in\mathrm{M},\alpha,\beta\in\Gamma.$$

Remark 2.3.7: As shown in the examples above, the reverse derivation is not a derivation in general, but it is a Jordan derivation .

Remark 2.3.8: Every generalized reverse derivation of a $\Gamma-$ring M is Jordan generalized reverse derivation of M.

Lemma 2.3.9: [8] Let M be a $\Gamma-$ring and let f be a Jordan generalized reverse derivation of M then for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$, the following statements hold:

(i) $f(x\alpha y+y\alpha x)=f(y)\alpha x+y\alpha d(x)+f(x)\alpha y+x\alpha d(y)$

(ii) $f(x\alpha y\beta x+x\beta y\alpha x)=f(x)\beta x\alpha y+x\beta d(y)\alpha x+x\beta y\alpha d(x)+f(x)\alpha x\beta y$
$+x\alpha d(y)\beta x+x\alpha y\beta d(x)$

(iii) $f(x\alpha y\alpha x)=f(x)\alpha x\alpha y+x\alpha d(y)\alpha x+x\alpha y\alpha d(x)$

(iv) $f(x\alpha y\alpha z+z\alpha y\alpha x)=f(z)\alpha x\alpha y+z\alpha d(y)\alpha x+z\alpha y\alpha d(x)+f(x)\alpha z\alpha y$
$+x\alpha d(y)\alpha z+x\alpha y\alpha d(z)$

(v) $f(x\alpha y\beta z)=f(z)\beta x\alpha y+z\beta d(y)\alpha x+z\beta y\alpha d(x)$

(vi) $f(x\alpha y\beta z+z\alpha y\beta x)=f(z)\beta x\alpha y+z\beta d(y)\alpha x+z\beta y\alpha d(x)+f(x)\beta z\alpha y$
$+x\beta d(y)\alpha z+x\beta y\alpha d(z)$.

Definition 2.3.10: [21] Let f be a Jordan generalized reverse derivation of a $\Gamma-$ring M, then for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$ we define:

$$\delta(x,y)_\alpha=f(x\alpha y)-f(y)\alpha x-y\alpha d(x)$$

Lemma 2.3.11: If f is a Jordan generalized reverse derivation of $\Gamma-$ring M, then for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$ we get :

(i) $\delta(x,y)_\alpha=-\delta(y,x)_\alpha$

(ii) $\delta(x+y,z)_\alpha=\delta(x,z)_\alpha+\delta(y,z)_\alpha$

(iii) $\delta(x,y+z)_\alpha=\delta(x,y)_\alpha+\delta(x,z)_\alpha$

(iv) $\delta(x,y)_{\alpha+\beta}=\delta(x,y)_\alpha+\delta(x,y)_\beta$

Proof: (i) By Lemma 2.3.9 (i) and since f is additive mapping of M we get :

$$f(x\alpha y+y\alpha x)=f(y)\alpha x+y\alpha d(x)+f(x)\alpha y+x\alpha d(y)$$

$$f(x\alpha y)+f(y\alpha x)=\big(f(y)\alpha x+y\alpha d(x)\big)+\big(f(x)\alpha y+x\alpha d(y)\big)$$

$$f(x\alpha y)-f(y)\alpha x-y\alpha d(x)=-f(y\alpha x)+f(x)\alpha y+x\alpha d(y)$$

$$f(x\alpha y)-f(y)\alpha x-y\alpha d(x)=-\big(f(y\alpha x)-f(x)\alpha y-x\alpha d(y)\big)$$

Then we get $\delta(x,y)_\alpha=-\delta(y,x)_\alpha$.

(ii) $\delta(x+y,z)_\alpha = f((x+y)\alpha z)-(f(z)\alpha(x+y)+z\alpha d(x+y))$

$$= f(x\alpha z+y\alpha z)-(f(z)\alpha x+f(z)\alpha y+z\alpha d(x)+z\alpha d(y))$$

Since f is additive mapping of the $\Gamma-$ring

$$= f(x\alpha z)-f(z)\alpha x-z\alpha d(x)+f(y\alpha z)-f(z)\alpha y-z\alpha d(y)$$

$$=\delta(x,z)_\alpha+\delta(y,z)_\alpha$$

(iii) $\delta(x,y+z)_\alpha = f(x\alpha(y+z))-(f(y+z)\alpha x+(y+z)\alpha d(x))$

$$= f(x\alpha y)-f(y)\alpha x-y\alpha d(x)+f(x\alpha z)-f(z)\alpha x-z\alpha d(x)$$

$$=\delta(x,y)_\alpha+\delta(x,z)_\alpha$$

(iv) $\delta(x,y)_{\alpha+\beta} = f(x(\alpha+\beta)y)-(f(y)(\alpha+\beta)x+y(\alpha+\beta)d(x))$

$$= f(x\alpha y+x\beta y)-(f(y)\alpha x+f(y)\beta x+y\alpha d(x)+y\beta d(x))$$

Since f is additive mapping of a $\Gamma-$ring.

$$= f(x\alpha y)-f(y)\alpha x-y\alpha d(x)+f(x\beta y)-f(y)\beta x-y\beta d(x)$$

$$=\delta(x,y)_\alpha+\delta(x,y)_\beta . \ \square$$

Remark 2.3.12: Note that f is generalized reverse derivation of a $\Gamma-$ring M if and only if $\delta(x,y)_\alpha=0$ for all $x,y\in\mathrm{M},\alpha\in\Gamma$.

Theorem 2.3.13: [21] Let f be a Jordan generalized reverse derivation of M then $\delta(x,y)_\alpha=0$ for all $x,y\in\mathrm{M},\alpha\in\Gamma$.

Proof: By Lemma 2.3.9 (i) we get:

$$f(x\alpha y+y\alpha x)= f(y)\alpha x+y\alpha d(x)+f(x)\alpha y+x\alpha d(y) \tag{2.3.1}$$

on the other hand, since f is additive mapping of the $\Gamma-$ring M we have:

$$f(x\alpha y+y\alpha x)= f(x\alpha y)+f(y\alpha x)= f(x\alpha y)+f(x)\alpha y+x\alpha d(y) \tag{2.3.2}$$

Comparing $(2.3.1)$ and $(2.3.2)$ we get

$f(x\alpha y)=f(y)\alpha x+y\alpha d(x) \Rightarrow f(x\alpha y)-f(y)\alpha x-y\alpha d(x)=0$ by Definition 2.3.10 we get , $\delta(x,y)_\alpha=0$. □

Corollary 2.3.14: [21] Every Jordan generalized reverse derivation of $\Gamma-$ ring M is generalized reverse derivation of M .

Proof: By Theorem 2.3.13 we get $\delta(x,y)_\alpha=0$ and Remark 2.3.12 the proof done. □

Proposition 2.3.15: [21] Every Jordan generalized reverse derivation of a 2-torision free $\Gamma-$ ring M where $x\alpha y\beta z=x\beta y\alpha z$ is Jordan generalized triple reverse derivation of M .

Proof: Let f be a Jordan generalized reverse derivation of M , replace y by $(x\beta y+y\beta x)$ in Lemma 2.3.9 (i) we get $f(x\alpha(x\beta y+y\beta x)+(x\beta y+y\beta x)\alpha x)=f(x\alpha(x\beta y)+x\alpha(y\beta x)+(x\beta y)\alpha x+(y\beta x)\alpha x)$

$$=f((x\alpha x)\beta y+(x\alpha y)\beta x+(x\beta y)\alpha x+(y\beta x)\alpha x)$$

$$=f(y)\beta x\alpha x+y\beta d(x\alpha x)+f(x)\beta(x\alpha y)+x\beta d(x\alpha y)+f(x)\alpha(x\beta y)+x\alpha d(x\beta y)$$

$$+f(x)\alpha(y\beta x)+x\alpha d(y\beta x)$$

$$=f(y)\beta x\alpha x+y\beta d(x)\alpha x+y\beta x\alpha d(x)+f(x)\beta x\alpha y+x\beta d(y)\alpha x+x\beta y\alpha d(x)+f(x)\alpha x\beta y$$
$$+x\alpha d(y)\beta x+x\alpha y\beta d(x)+f(x)\alpha y\beta x+x\alpha d(x)\beta y+x\alpha x\beta d(y) \qquad (2.3.3)$$

On the other hand:

$$f(x\alpha(x\beta y+y\beta x)+(x\beta y+y\beta x)\alpha x)=f(x\alpha x\beta y+x\alpha y\beta x+x\beta y\alpha x+y\beta x\alpha x)$$

$$=f((x\alpha x\beta y+y\beta x\alpha x)+(x\alpha y\beta x+x\beta y\alpha x)). \quad (2.3.4)$$

Comparing $(2.3.3)$ and $(2.3.4)$, and since $x\alpha y\beta z=x\beta y\alpha z$ we get

$$f(x\alpha y\beta x+x\alpha y\beta x)=2f(x\alpha y\beta x)=2(f(x)\beta x\alpha y+x\beta d(y)\alpha x+x\beta y\alpha d(x))$$

Since M is a 2-torision free then we have

$$f(x\alpha y\beta x)=f(x)\beta x\alpha y+x\beta d(y)\alpha x+x\beta y\alpha d(x).$$

Definition 2.3.16: [20] Let M be a $\Gamma-$ ring and $F=(f_i)_{i\in\mathbb{N}}$ be a family of additive mappings of M such that $f_0=id_{\mathrm{M}}$ then F is called *generalized higher reverse derivation of* M if there exists a higher reverse derivation $D=(d_i)_{i\in\mathbb{N}}$ of M such that for all $n\in\mathbb{N}$ we have :

$$f_n(x\alpha y)=\sum_{i+j=n} f_i(y)\alpha d_j(x) \qquad (2.3.5)$$

For every $x,y\in M,\alpha\in\Gamma$.

F is called *a Jordan generalized higher reverse derivation of* M if there exists a Jordan higher reverse derivation $D=(d_i)_{i\in\mathbb{N}}$ of M such that for all $n\in\mathbb{N}$ we have :

$$f_n(x\alpha x)=\sum_{i+j=n} f_i(x)\alpha d_j(x) \qquad (2.3.6)$$

For every $x\in M,\alpha\in\Gamma$.

F is said to be *a Jordan generalized triple higher reverse derivation of* M if there exists a Jordan triple higher reverse derivation $D=(d_i)_{i\in\mathbb{N}}$ of M such that for all $n\in\mathbb{N}$ we have :

$$f_n(x\alpha y\beta x)=f_n(x)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x) \qquad (2.3.7)$$

For every $x,y\in M,\alpha,\beta\in\Gamma$.

Example 2.3.17: Let $F=(f_i)_{i\in\mathbb{N}}$ be a generalized higher reverse derivation on a ring R then there exists a higher reverse derivation $d=(f_i)_{i\in\mathbb{N}}$ of R such that

$$f_n(xy)=\sum_{i+j=n} f_i(y)d_j(x)$$

Then by Example 2.3.3 we define $D=(D_i)_{i\in\mathbb{N}}$ be a family of additive mappings of M such that $D_n(a,b)=(d_n(a),d_n(b))$ then D is higher reverse derivation of M.

Let $F=(f_i)_{i\in\mathbb{N}}$ be a family of additive mappings of M defined by $F_n(a,b)=(f_n(a),f_n(b))$ then F is a generalized higher reverse derivation of M.

It is clear that every generalized higher reverse derivation of a $\Gamma-$ring M is Jordan generalized higher reverse derivation of M, but the converse is not true in general.

Lemma 2.3.18: [20] Let M be a $\Gamma-$ring and let $F=(f_i)_{i\in\mathbb{N}}$ be a Jordan generalized higher reverse derivation of M then for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$, the following statements hold:

(i) $$f_n(x\alpha y+y\alpha x)=\sum_{i+j=n}^{i\prec n} f_i(y)\alpha d_j(x)+f_i(x)\alpha d_j(y)$$

In particular if $y\in Z(M)$.

(ii) $$f_n(x\alpha y\beta x+x\beta y\alpha x)=f_n(x)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x)+f_n(x)\alpha x\beta y$$

$$+\sum_{i+j+r=n}^{i\prec n} f_i(x)\alpha d_j(y)\beta d_r(x)$$

(iii) $$f_n(x\alpha y\alpha x)=f_n(x)\alpha x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(x)$$

(iv) $$f_n(x\alpha y\alpha z+z\alpha y\alpha x)=f_n(z)\alpha x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\alpha d_j(y)\alpha d_r(x)+f_n(x)\alpha z\alpha y$$

$$+\sum_{i+j+r=n}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(z)$$

(v) $$f_n(x\alpha y\beta z)=f_n(z)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(x)$$

(vi) $$f_n(x\alpha y\beta z+z\alpha y\beta x)=f_n(z)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(x)+f_n(x)\beta z\alpha y$$

$$+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(z)$$

Proof: (i) Replace $(x+y)$ for x and y in Definition 2.3.16 (2.3.5) we get:

$$f_n((x+y)\alpha(x+y))=\sum_{i+j=n} f_i(x+y)\alpha d_j(x+y)$$

$$=\sum_{i+j=n}\left(f_i(x)\alpha d_j(x)+f_i(y)\alpha d_j(x)+f_i(x)\alpha d_j(y)+f_i(y)\alpha d_j(y)\right) \qquad (2.3.8)$$

On the other hand:

$$f_n((x+y)\alpha(x+y))=f_n(x\alpha x+x\alpha y+y\alpha x+y\alpha y)=f_n(x\alpha x+y\alpha y)+f_n(x\alpha y+y\alpha x)$$

$$=\sum_{i+j=n} f_i(x)\alpha d_j(x)+f_i(y)\alpha d_j(y)+f_n(x\alpha y+y\alpha x) \qquad (2.3.9)$$

Comparing $(2.3.8)$ and $(2.3.9)$ we get:

$$f_n(x\alpha y+y\alpha x)=\sum_{i+j=n} f_i(y)\alpha d_j(x)+f_i(x)\alpha d_j(y).$$

(ii) Replacing $x\beta y+y\beta x$ for y in 2.3.18(i) we get:

$$f_n(x\alpha(x\beta y+y\beta x)+(x\beta y+y\beta x)\alpha x)=f_n(x\alpha(x\beta y)+x\alpha(y\beta x)+(x\beta y)\alpha x+(y\beta x)\alpha x)$$

$$=f_n((x\alpha x)\beta y+(x\alpha y)\beta x+(x\beta y)\alpha x+(y\beta x)\alpha x)$$

$$=\sum_{i+j=n} f_i(y)\beta d_j(x\alpha x)+f_i(x)\beta d_j(x\alpha y)+f_i(x)\alpha d_j(x\beta y)+f_i(x)\alpha d_j(y\beta x)$$

$$= \sum_{i+j+r=n} f_i(y)\beta d_j(x)\alpha d_r(x) + f_i(x)\beta d_j(y)\alpha d_r(x) + f_i(x)\alpha d_j(y)\beta d_r(x) + f_i(x)\alpha d_j(x)\beta d_r(y)$$

$$= f_n(y)\beta x\alpha x + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(y)\beta d_j(x)\alpha d_r(x) + f_n(x)\beta x\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x) + f_n(x)\alpha x\beta y$$

$$+ \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(y)\beta d_r(x) + f_n(x)\alpha y\beta x + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(x)\beta d_r(y) \qquad (2.3.9)$$

On the other hand:

$$f_n\left(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x\right) = f_n\left(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x\right) = f_n(y)\beta x\alpha x$$

$$+ \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(y)\beta d_j(x)\alpha d_r(x) + f_n(x)\alpha y\beta x + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(x)\beta d_r(y) + f_n(x\alpha y\beta x + x\beta y\alpha x) \quad (2.3.10)$$

Comparing $(2.3.9)$ and $(2.3.10)$ we get the require result.

(iii) Replacing α for β in 2.3.18(ii) we have:

$$f_n(x\alpha y\alpha x + x\alpha y\alpha x) = 2 f_n(x\alpha y\alpha x) = 2\left(f_n(x)\alpha x\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(x)\right)$$

Since M is 2-torsion free then we get:

$$f_n(x\alpha y\alpha x) = f_n(x)\alpha x\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(x).$$

(iv) Replacing $x+z$ for x in 2.3.18(iii) we have:

$$f_n\left((x+z)\alpha y\alpha(x+z)\right) = f_n(x+z)\alpha(x+z)\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x+z)\alpha d_j(y)\alpha d_r(x+z) = f_n(x)\alpha x\alpha y$$

$$+ \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(x) + f_n(z)\alpha x\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(z)\alpha d_j(y)\alpha d_r(x) + f_n(x)\alpha z\alpha y$$

$$+ \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(z) + f_n(z)\alpha z\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(z)\alpha d_j(y)\alpha d_r(z) \qquad (2.3.11)$$

On the other hand:

$$f_n\left((x+z)\alpha y\alpha(x+z)\right) = f_n(x\alpha y\alpha x + x\alpha y\alpha z + z\alpha y\alpha x + z\alpha y\alpha z) = f_n(x)\alpha x\alpha y$$

$$+ \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(x)\alpha d_j(y)\alpha d_r(x) + f_n(z)\alpha z\alpha y + \sum_{\substack{i+j+r=n}}^{i\prec n} f_i(z)\alpha d_j(y)\alpha d_r(z) + f_n(x\alpha y\alpha z + z\alpha y\alpha x) \quad (2.3.12)$$

Comparing $(2.3.11)$ and $(2.3.12)$ we get the require result.

(v) Replace $(x+z)$ for x in Definition 2.3.16(2.3.7) we have:

$$f_n\left((x+z)\alpha y\beta(x+z)\right)=f_n(x+z)\beta(x+z)\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x+z)\beta d_j(y)\alpha d_r(x+z)=f_n(x)\beta x\alpha y$$

$$+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x)+f_n(z)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(x)+f_n(x)\beta z\alpha y$$

$$+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(z)+f_n(z)\beta z\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(z) \qquad (2.3.13)$$

On the other hand:

$$f_n\left((x+z)\alpha y\beta(x+z)\right)=f_n(x\alpha y\beta x+x\alpha y\beta z+z\alpha y\beta x+z\alpha y\beta z)=f_n(x\alpha y\beta x+z\alpha y\beta x+z\alpha y\beta z)$$

$$+f_n(x\alpha y\beta z)=f_n(x)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x)+f_n(x)\beta z\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(z)$$

$$+f_n(z)\beta z\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(z)+f_n(x\alpha y\beta z) \qquad (2.3.14)$$

comparing $(2.3.13)$ and $(2.3.14)$ we get:

$$f_n(x\alpha y\beta z)=f_n(z)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(x).$$

(vi) Replace $(x+z)$ for x in Definition 2.3.16(2.3.7) we have:

$$f_n\left((x+z)\alpha y\beta(x+z)\right)=f_n(x+z)\beta(x+z)\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x+z)\beta d_j(y)\alpha d_r(x+z)$$

$$=\left(f_n(x)+f_n(z)\right)\beta(x+z)\alpha y+\sum_{i+j+r=n}^{i\prec n}\left(f_i(x)+f_i(z)\right)\beta d_j(y)\alpha\left(d_r(x)+d_r(z)\right)$$

$$=f_n(x)\beta x\alpha y+f_n(z)\beta x\alpha y+f_n(x)\beta z\alpha y+f_n(z)\beta z\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x)$$

$$+f_i(z)\beta d_j(y)\alpha d_r(x)+f_i(x)\beta d_j(y)\alpha d_r(z)+f_i(z)\beta d_j(y)\alpha d_r(z) \qquad (2.3.15)$$

On the other hand:

$$f_n\left((x+z)\alpha y\beta(x+z)\right)=f_n(x\alpha y\beta x+x\alpha y\beta z+z\alpha y\beta x+z\alpha y\beta z)=f_n(x\alpha y\beta x+z\alpha y\beta z)$$

$$+f_n(x\alpha y\beta z+z\alpha y\beta x)=f_n(x)\beta x\alpha y+\sum_{i+j+r=n}^{i\prec n} f_i(x)\beta d_j(y)\alpha d_r(x)+f_n(z)\beta z\alpha y$$

$$+\sum_{i+j+r=n}^{i\prec n} f_i(z)\beta d_j(y)\alpha d_r(z)+f_n(x\alpha y\beta z+z\alpha y\beta x) \qquad (2.3.16)$$

comparing $(2.3.15)$ and $(2.3.16)$ we get the require result. □

Definition 2.3.19: [20] Let $F=(f_i)_{i\in\mathbb{N}}$ be a Jordan generalized higher reverse derivation of a $\Gamma-$ring M, then for all $x,y\in M$ and $\alpha\in\Gamma$ we define:

$$\delta_n(x,y)_\alpha = f_n(x\alpha y)-\sum_{i+j=n} f_i(y)\alpha d_j(x)$$

Lemma 2.3.20: [20] If $F=(f_i)_{i\in\mathbb{N}}$ is a Jordan generalized higher reverse derivation of $\Gamma-$ring M, then for all $x,y,z\in M$, $\alpha,\beta\in\Gamma$ and $n\in\mathbb{N}$ we get:

(i) $\delta_n(x,y)_\alpha=-\delta_n(y,x)_\alpha$

(ii) $\delta_n(x+y,z)_\alpha=\delta_n(x,z)_\alpha+\delta_n(y,z)_\alpha$

(iii) $\delta_n(x,y+z)_\alpha=\delta_n(x,y)_\alpha+\delta_n(x,z)_\alpha$

(iv) $\delta_n(x,y)_{\alpha+\beta}=\delta_n(x,y)_\alpha+\delta_n(x,y)_\beta$

Remark 2.3.21: Many notions on the Jordan generalized reverse derivations on $\Gamma-$rings are generalized to the Jordan generalized higher reverse derivations on $\Gamma-$rings.

Remark 2.3.22: [20] Note that $F=(f_i)_{i\in\mathbb{N}}$ is a generalized higher reverse derivation of a $\Gamma-$ring M if and only if $\delta_n(x,y)_\alpha=0$ for all $x,y\in M$, $\alpha\in\Gamma$ and $n\in\mathbb{N}$.

Theorem 2.3.33: [20] Let $F=(f_i)_{i\in\mathbb{N}}$ be a Jordan generalized higher reverse derivation of a $\Gamma-$ring M then $\delta_n(x,y)_\alpha=0$ for all $x,y\in M$, $\alpha\in\Gamma$ and $n\in\mathbb{N}$.

Proof: By Lemma 2.3.18(i) we get:

$$f_n(x\alpha y+y\alpha x)=\sum_{i+j=n}^{i\prec n} f_i(y)\alpha d_j(x)+f_i(x)\alpha d_j(y) \qquad (2.3.17)$$

On the other hand:

Since f_n is additive mapping of a $\Gamma-$ring M we have:

$$f_n(x\alpha y+y\alpha x)=f_n(x\alpha y)+f_n(y\alpha x)=f_n(x\alpha y)+\sum_{i+j=n}^{i\prec n} f_i(x)\alpha d_j(y) \qquad (2.3.18)$$

Compare $(2.3.17)$ and $(2.3.18)$ we get:

$$f_n(x\alpha y)=\sum_{i+j=n}^{i\prec n} f_i(y)\alpha d_j(x)\Rightarrow f_n(x\alpha y)-\sum_{i+j=n}^{i\prec n} f_i(y)\alpha d_j(x)=0$$

By Definition 2.3.19 we get:

$\delta_n(x,y)_\alpha=0$. □

Chapter Three

Derivations On Prime $\Gamma-$Rings

3.1 Generalized Derivations On Prime $\Gamma-$Rings

In this section, we prove that a prime $\Gamma-$ring M is commutative if f is a generalized derivation on M with an associated non-zero derivation D on M such that f is centralizing and commuting on a left ideal J of M.

A mapping f is said to be commuting on a left ideal J of M if $\left[f(x),x\right]_{\alpha}=0$ for all $x\in J,\alpha\in\Gamma$ and f is said to be centralizing if $\left[f(x),x\right]_{\alpha}\in Z(\mathrm{M})$ for all $x\in J,\alpha\in\Gamma$.

Remark 3.1.1: Let M be a prime $\Gamma-$ring and J a nonzero left ideal of M. If D is a nonzero derivation on M, then D is also a nonzero derivation on J.

Remark 3.1.2: Let M be a prime $\Gamma-$ring and J a nonzero left ideal of M. If J is commutative, then M is also commutative.

Lemma 3.1.3: [13] Suppose M is a prime $\Gamma-$ring such that $x\alpha y\beta z=x\beta y\alpha z$, for all $x,y,z\in\mathrm{M}$ $\alpha,\beta\in\Gamma$, and $D:\mathrm{M}\to\mathrm{M}$ be a derivation. For an element $a\in\mathrm{M}$, if $a\alpha D(x)=0$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$, then either $a=0$ or $D=0$.

Proof: By our assumption, $a\alpha D(x)=0$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$. We replace x by $x\beta y$, then $a\alpha D(x\beta y)=0\Rightarrow a\alpha D(x)\beta y+a\alpha x\beta D(y)=0\Rightarrow a\alpha x\beta D(y)=0$ for all $x,y\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$. If D is not zero, that is, if $D(y)\neq 0$ for some $y\in\mathrm{M}$, then by definition of prime $\Gamma-$ ring, $a=0$. □

Lemma 3.1.4: [13] Suppose M is a prime $\Gamma-$ ring such that $x\alpha y\beta z=x\beta y\alpha z$, for all $x,y,z\in\mathrm{M}$ $\alpha,\beta\in\Gamma$, and J a nonzero left ideal of M. If M has a derivation D which is zero on J, then D is zero on M.

Proof: By the hypothesis, $D(J)=0$. Replacing J by $\mathrm{M}\Gamma J$, we have $0=D(\mathrm{M}\Gamma J)$ $=D(\mathrm{M})\Gamma J+\mathrm{M}\Gamma D(J)=D(\mathrm{M})\Gamma J$. Hence by Lemma 3.1.3, D must be zero, since J is nonzero.

Lemma 3.1.5: [13] Suppose M is a prime $\Gamma-$ring such that $x\alpha y\beta z=x\beta y\alpha z$, for all $x,y,z\in\mathrm{M}$ $\alpha,\beta\in\Gamma$, and J a nonzero left ideal of M. If J is commutative, then M is commutative.

Proof: Suppose that x is a fixed element in J. Since J is commutative, so for all $y\in J$ and $\alpha\in\Gamma$, $[x,y]_{\alpha}=0$ and consequently, $[x,J]_{\Gamma}=0$. Hence by Lemma 3.1.4, $[x,J]_{\Gamma}=0$ on M and $x\in Z(\mathrm{M})$. Thus $[x,\mathrm{M}]_{\Gamma}=0$ for every $x\in J$ and hence $[J,y]_{\Gamma}=0$ for all $y\in\mathrm{M}$. Again Lemma 3.1.4, $[J,y]_{\Gamma}=0$ and $y\in Z(\mathrm{M})$ for all $y\in\mathrm{M}$. Therefore M is commutative.

Lemma 3.1.6: [13] Let M be a prime $\Gamma-$ring and $f:\mathrm{M}\to\mathrm{M}$ be an additive mapping. If f is centralizing on a left ideal J of M, then $f(a)\in Z(\mathrm{M})$ for all $a\in J\cup Z(\mathrm{M})$.

Proof: By our assumption, f is a centralizing on a left ideal J of M. Thus we have, $\left[f(a),a\right]_\alpha\in Z(\mathrm{M})$ for all $a\in J$ and $\alpha\in\Gamma$. By linearization, for all $a,b\in J$ and $\alpha\in\Gamma$, we have

$$\left[f(a),b\right]_\alpha+\left[f(b),a\right]_\alpha\in Z(\mathrm{M}) \tag{3.1.1}$$

If $a\in Z(\mathrm{M})$, then equation (3.1.1) implies $\left[f(a),b\right]_\alpha\in Z(\mathrm{M})$. Now replacing b by $f(a)\beta b$, we have $\left[f(a),f(a)\beta b\right]_\alpha\in Z(\mathrm{M})$, this implies $f(a)\beta\left[f(a),b\right]_\alpha\in Z(\mathrm{M})$. If $\left[f(a),b\right]_\alpha=0$, then $f(a)\in C_{\Gamma\mathrm{M}}(J)$, the centralizer of J in M and hence $f(a)\in Z(\mathrm{M})$. Otherwise, if $\left[f(a),b\right]_\alpha\neq 0$, then $f(a)\in Z(\mathrm{M})$. □

Theorem 3.1.7: [13] Let M be a prime $\Gamma-$ring such that $x\alpha y\beta z=x\beta y\alpha z$, for all $x,y,z\in\mathrm{M}$, $\alpha,\beta\in\Gamma$ and D a nonzero derivation on M. If f is a generalized derivation on a left ideal J of M such that f is commuting on J, then M is commutative.

Proof: By our hypothesis, f is commuting on J. Thus we have $\left[f(a),a\right]_\alpha=0$ for all $a\in J$ and $\alpha\in\Gamma$. By linearizing this relation, we get $\left[f(a),b\right]_\alpha+\left[f(b),a\right]_\alpha=0$. Putting $b=b\beta a$ and simplifying, we obtain $\left[b\beta D(a),a\right]_\alpha=0$. Replacing b by $r\gamma b$, we have $[r,a]_\alpha\,\beta a\gamma D(a)=0$ for all $a\in J, r\in\mathrm{M}$ and $\alpha,\beta,\gamma\in\Gamma$. Since M is prime $\Gamma-$ring, thus $[r,a]_\alpha=0$ or $D(a)=0$. Therefore for any $a\in J$, either $a\in Z(\mathrm{M})$ or $D(a)=0$. Since D is nonzero derivation on M, then by Lemma 3.1.4, D is nonzero on J. Suppose $D(a)\neq 0$ for some $a\in J$, then $a\in Z(\mathrm{M})$. Let $c\in J$ with $c\neq Z(\mathrm{M})$. Then $D(c)=0$ and $a+c\notin Z(\mathrm{M})$, that is, $D(a+c)=0$ and so $D(a)=0$, which is a contradiction. Thus $c\in Z(\mathrm{M})$ for all $c\in J$. Hence J is commutative and hence by Lemma 3.1.5, M is commutative. □

Theorem 3.1.8: [13] Let M be a prime $\Gamma-$ring such that $x\alpha y\beta z=x\beta y\alpha z$, for all $x,y,z\in\mathrm{M}$, $\alpha,\beta\in\Gamma$ and J a left ideal of M with $J\cap Z(\mathrm{M})\neq 0$. If f is a generalized derivation on M with associated nonzero derivation D such that f is commuting on J, then M is commutative.

Proof: We claim that $Z(\mathrm{M})\neq 0$ because of f is commuting on J and the proof is complete. Now by linearization, for all $a,b\in J$ and $\alpha\in\Gamma$, we have

$$\left[f(a),b\right]_\alpha+\left[f(b),a\right]_\alpha\in Z(\mathrm{M}),$$

If we replace x by $c\beta b$ with $0\neq c\in Z(\mathrm{M})$, then we have
$\left[f(c),b\right]_\alpha\beta b+c\beta\left[D(b),b\right]_\alpha+c\beta\left[f(b),b\right]_\alpha\in Z(\mathrm{M})$. From lemma 3.1.3, $f(c)\in Z(\mathrm{M})$ and hence $c\beta\left[D(b),b\right]_\alpha+c\beta\left[f(b),b\right]_\alpha\in Z(\mathrm{M})$.

Since f is a centralizing on J, we have $c\beta\left[f(b),b\right]_{\alpha}\in Z(\mathrm{M})$ and consequently $c\beta\left[D(b),b\right]_{\alpha}\in Z(\mathrm{M})$ As c is nonzero, Lemma 1.3.4 follows that $\left[D(b),b\right]_{\alpha}\in Z(\mathrm{M})$. This implies D is centralizing on J and hence we conclude that M is commutative. □

Remark 3.1.9: Let M be a prime $\Gamma-$ring such that $\mathrm{M}\Gamma\mathrm{M}\neq\mathrm{M}$ and let A be the set of all ideals of M which have zero annihilator in M, in this case, the set A is closed under multiplication. Indeed, let U and V be in A . The equality $U\Gamma V\beta x=0$ for $x\in\mathrm{M}$ and all $\beta\in\Gamma$ yields $V\beta x\subseteq Ann_rU=\langle 0\rangle$,i.e, $V\beta x=0$ and so $x\in Ann_rV=\langle 0\rangle$ which implies $x=0$ Then we get that $U\Gamma V\in\mathrm{A}$.

Denote $\Delta=\{(U,f):U(\neq 0)$ is an ideal of M and $f:U\to\mathrm{M}$ is a right M-module homomorphism for all $U\in\mathrm{A}\}$
Define a relation $\sim$ on Δ by $(U,f)\sim(V,g)\Leftrightarrow\exists W(\neq 0)\subset U\bigcap V$ such that $f=g$ on W.

Since the set A is closed under multiplication, it is possible to find such an ideal $W\in\mathrm{A}$ and so "$\sim$" is an equivalence relation. This gives a chance for us to get a partition of Δ. We then denote the equivalence class by $Cl(U,f)=\hat{f}$, where $\hat{f}=\{g:V\to\mathrm{M}\mid(U,f)\sim(V,g)\}$, and denote by Q the set of all equivalence classes. Then Q is a $\Gamma-$ring, which is called the *quotient* $\Gamma-$*ring* of M. The set $C_{\Gamma}=\{g\in Q\mid g\gamma f=f\gamma g$ for all $f\in Q$ and $\gamma\in\Gamma\}$ is called *extended centroid* of M.

Lemma 3.1.10: Suppose that the elements a_i,b_i in the central closure of a prime $\Gamma-$ring M satisfy $\sum a_i\alpha_i x\beta_i b_i=0$ for all $x\in\mathrm{M}$ and $\alpha_i,\beta_i\in\Gamma$. If $b_i\neq 0$ for some i, then a_i's are $C-$dependent, where C is the extended centroid of M.

Proof: Let M be a prime $\Gamma-$ring and let $C_{\Gamma}=C$ be the extended centroid of M. If a_i and b_i are non-zero elements of M such that $\sum a_i\alpha_i x\beta_i b_i=0$ for all $x\in\mathrm{M}$ and $\alpha_i,\beta_i\in\Gamma$, then a_i's (also b_i's) are linearly dependent over C. Moreover, if $a\alpha x\beta b=b\alpha x\beta a$ for all $x\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$, where a,b are fixed and $a\neq 0$, then there exists $\lambda\in C$ such that $a=\lambda\alpha b$ for $\alpha\in\Gamma$. Clearly, the lemma is proved. □

Lemma 3.1.11: [30] Let M be a prime $\Gamma-$ring of characteristic 2. Let d_1 and d_2 two non-zero derivations of M and right $\mathrm{M}-$module homomorphisms. If $d_1d_2(x)=0$ $\quad(3.1.2)$ for all $x\in\mathrm{M}$, then there exists $\lambda\in C_{\Gamma}$ such that $d_2(x)=\lambda\alpha d_1(x)$ for all $\alpha\in\Gamma$ and $x\in\mathrm{M}$.

Proof: Let $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$. Replacing x by $x\gamma y$ in $(3.1.2)$, it follows from $char\mathrm{M}=2$ that for all $x,y\in\mathrm{M}$ and $\gamma\in\Gamma$

$$d_1(x)\gamma d_2(y)=d_2(x)\gamma d_1(y)\qquad(3.1.3)$$

Replacing x by $x\beta z$ in $(3.1.3)$, we get

$$d_1(x)\beta z\gamma d_2(y) = d_2(x)\beta z\gamma d_1(y) \qquad (3.1.4)$$

for all $x \in M$ and $\gamma \in \Gamma$. Now, if we replace y by x in $(3.1.4)$, then we obtain

$$d_1(x)\beta z\gamma d_2(x) = d_2(x)\beta z\gamma d_1(x) \qquad (3.1.5)$$

for all $x \in M$ and $\gamma, \beta \in \Gamma$. If $d_1(x) \neq 0$, then there exists $\lambda(x) \in C_\Gamma$ such that $d_2(x) = \lambda(x)\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$ by Lemma 3.1.10. Thus, if $d_1(x) \neq 0 \neq d_1(y)$, then $(3.1.4)$ implies that

$$(\lambda(y) - \lambda(x))\alpha d_1(x)\beta z\gamma d_2(x) = 0 \qquad (3.1.5)$$

Since M is a prime $\Gamma-$ring, we conclude by using Lemma 3.1.3 that $\lambda(y) = \lambda(x)$ for all $x, y \in M$. Hence we proved that there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$ with $d_1(x) \neq 0$. On the other hand, if $d_1(x) = 0$, then $d_2(x) = 0$ as well. Therefore, $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. This complete the proof. □

Proposition 3.1.12: [30] Let M be a prime $\Gamma-$ring of characteristic 2 and d a non-zero derivation of M. If $d(x) \in Z(M)$ for all $x \in M$, then there exists $\lambda(m) \in C_\Gamma$ such that $d(m) = \lambda(m)\alpha d(z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or M is commutative.

Proof: Since $d(x) \in Z(M)$ for all $x \in M$, we have

$$[d(x), y]_\beta = 0 \text{ for all } x, y \in M \text{ and } \beta \in \Gamma \qquad (3.1.6)$$

Replacing x by $x\gamma z$ in $(3.1.6)$, we have

$$d(x)\gamma[z, y]_\beta + d(z)\gamma[x, y]_\beta = 0 \qquad (3.1.7)$$

for all $x, y, z \in M$ and $\gamma, \beta \in \Gamma$. Replace z by $d(z)$ in $(3.1.7)$, we obtain

$$d^2(z)\gamma[x, y]_\beta = 0, \ \forall x, y, z \in M, \ \gamma, \beta \in \Gamma \qquad (3.1.8)$$

Now, substituting $z\alpha m$ for z in $(3.1.8)$, it follows $char M = 2$ that

$$d^2(z)\alpha m\gamma[x, y]_\beta = 0, \ \forall x, y, z, m \in M, \ \gamma, \beta, \alpha \in \Gamma \qquad (3.1.9)$$

Since M is a prime $\Gamma-$ring, we obtain

$$d^2(z) = 0, \ \forall z \in M \text{ or } [x, y]_\beta = 0, \ \forall x, y \in M, \beta \in \Gamma \qquad (3.1.10)$$

from $(3.1.10)$, if $d^2(z) = 0$ for all $z \in M$, then replacing z by $z\gamma m$ in this last relation, it follows from $d(x) \in Z(M)$ that

$$d(z)\gamma d(m)=d(m)\gamma d(z),\ \forall z,m\in M,\gamma\in\Gamma \qquad (3.1.11)$$

Replacing z by $z\alpha n$ in $(3.1.11)$, it follows from $d(x)\in Z(M)$ that

$$d(z)\alpha n\gamma d(m)=d(m)\alpha n\gamma d(z),\ \forall z,m,n\in M,\gamma,\alpha\in\Gamma \qquad (3.1.12)$$

If $d(z)\neq 0$, then there exists $\lambda(m)\in C_\Gamma$ such that $d(m)=\lambda(m)\alpha d(z)$ for all $z,m\in M$ and $\alpha\in\Gamma$ by Lemma 3.1.10. on the other hand, it follows from $(3.1.10)$ that if $[x,y]_\beta=0$ for all $x,y\in M$ and $\beta\in\Gamma$, then M is commutative. □

Theorem 3.1.13: [30] Let M be a prime $\Gamma-$ring of characteristic 2, d_1 and d_2 two non-zero derivations of M and U a non-zero ideal of M. If

$$d_1d_2(u)=0 \text{ for all } u\in U \qquad (3.1.13)$$

then there exists $\lambda\in C_\Gamma$ such that $d_2(x)=\lambda\alpha d_1(x)$ for all $\alpha\in\Gamma$ and $x\in M$.

Proof: Let $u,v\in U$ and $\gamma\in\Gamma$. Replacing u by $d_2(u)\gamma v$ in $(3.1.13)$, we get

$$d_2^2(u)\gamma d_1(v)=0 \text{ for all } u,v\in U \text{ and } \gamma\in\Gamma \qquad (3.1.14)$$

Since $d_1\neq 0$, it follows from Lemma 3.1.3 that $d_2^2(u)=0$ for all $u\in U$, so from $char M=2$ that $d_2^2=0$. Now, substituting $u\gamma d_2(x)$ for u in (1), we get

$$d_2(u)\gamma d_1(d_2(x))=0 \text{ for all } u\in U,x\in M \text{ and } \gamma\in\Gamma\ (3.1.15)$$

Since $d_2\neq 0$, we get $d_1(d_2(x))=0$ for all $x\in M$ by Lemma 3.1.3. Hence there exists $\lambda\in C_\Gamma$ such that $d_2(x)=\lambda\alpha d_1(x)$ for all $\alpha\in\Gamma$ and $x\in M$ by Lemma 3.1.11. □

Theorem 3.1.14: [30] Let M be a prime $\Gamma-$ring, U a non-zero right ideal of M and d a non-zero derivation of M. If

$$d(u)\gamma a=0 \text{ for all } u\in U \text{ and } \gamma\in\Gamma \qquad (3.1.16)$$

Where a is a fixed element of M, then there exists an element q of Q such that $q\gamma a=0$ and $q\gamma u=0$ for all $u\in U$ and $\gamma\in\Gamma$.

Proof: Let $u\in U$, $x\in M$ and $\beta\in\Gamma$. Since U is a right ideal of M, we have $u\beta x\in U$. Replacing u by $u\beta x$ in $(3.1.16)$, we get

$$d(u)\beta x\gamma a+u\beta d(x)\gamma a=0 \tag{3.1.17}$$

for all $u\in U$, $x\in \mathrm{M}$ and $\gamma,\beta\in\Gamma$. Hence $d(u)\beta x\gamma a\alpha m+u\beta d(x)\gamma a\alpha m=0$ for any $m\in \mathrm{M}$ and $\alpha\in\Gamma$, and so $d(u)\beta\left(\sum x\gamma a\alpha m\right)=-\left(u\beta\left(\sum d(x)\gamma a\alpha m\right)\right)$. Therefore, for any $v\in V=\mathrm{M}\Gamma a\Gamma\mathrm{M}$ which is a non-zero ideal of M, we have

$$d(u)\beta v=u\beta f(v) \tag{3.1.18}$$

for all $u\in U$. $f(v)$ is independent of u but it is dependent on v. Since M is a prime $\Gamma-$ring, $f(v)$ is well-defined and unique for all $v\in V$. Note that $v\alpha y\in V$ for any $y\in \mathrm{M}$, $v\in V$ and $\alpha\in\Gamma$. Replacing v by $v\alpha y$ in $(3.1.18)$ we get

$$d(u)\beta(v\alpha y)=u\beta f(v\alpha y) \tag{3.1.19}$$

for all $y\in \mathrm{M}$ and so by using $(3.1.18)$ and $(3.1.19)$, we have

$$\begin{aligned}(d(u)\beta v)\alpha y=u\beta f(v\alpha y)&\Rightarrow(u\beta f(v))\alpha y=u\beta f(v\alpha y)\\ &\Rightarrow u\beta f(v)\alpha y=u\beta f(v\alpha y)\\ &\Rightarrow u\beta\left(f(v)\alpha y-f(v\alpha y)\right)=0\end{aligned}$$

which implies from Theorem 1.3.3 that

$$f(v\alpha y)=f(v)\alpha y \tag{3.1.20}$$

for all $y\in \mathrm{M}$, $v\in V$ and $\alpha\in\Gamma$. It follows from $(3.1.20)$ that $f:V\to \mathrm{M}$ is a right $\mathrm{M}-$module homomorphism. In this case, $q=Cl(V,f)\in Q$. Moreover, $f(v)=q\beta v$ for all $v\in V$ and $\alpha\in\Gamma$. Let $x\in \mathrm{M}$, $v\in V$, $u\in U$ and $\gamma,\beta\in\Gamma$. Replacing v by $x\gamma v$ in $(3.1.18)$, we get

$$d(u)\beta(x\gamma v)=u\beta f(x\gamma v)=u\beta(q\beta x\gamma v) \tag{3.1.21}$$

Also, replacing u by $u\gamma x$ in $(3.1.18)$, we get

$$d(u)\gamma x\beta v=u\gamma x\beta q\beta v-u\gamma d(x)\beta v \tag{3.1.22}$$

Now, replacing β by γ and replacing γ by β in $(3.1.22)$, we get

$$d(u)\beta x\gamma v=u\beta x\gamma q\gamma v-u\beta d(x)\gamma v \tag{3.1.23}$$

Thus, from $(3.1.21)$ and $(3.1.22)$ we obtain

$$u\beta\left(q\beta x-x\gamma q+d(x)\right)\gamma v=0 \qquad (3.1.24)$$

for all $x\in M, v\in V, u\in U$ and $\gamma,\beta\in\Gamma$. Hence $d(x)=x\gamma q-q\beta x$ for all $x\in M$ and $\gamma,\beta\in\Gamma$ by Theorem 1.3.3. now, we shall prove that q can be chosen in Q such that $q\gamma a=0$ and $q\gamma u=0$ for all $u\in U$ and $\gamma\in\Gamma$. Let $u\in U$ and $x\in M, d(u)=q\alpha u-u\beta q$ and $d(x)=q\beta x-x\alpha q$. Then we have $0=d(u\beta x)\gamma a=\left(q\alpha(u\beta x)-(u\beta x)\alpha q\right)\gamma a$. Thus, $q\alpha u\beta x\gamma a=u\beta x\alpha q\gamma a$. If $q\gamma a=0$, then $q\alpha u\beta x\gamma a=0$, and so since M is prime $\Gamma-$ring, we get $q\Gamma U=\{0\}$. On the other hand, if $q\gamma a\neq 0$, then $q\gamma u\neq 0$. In fact, if $q\gamma u=0$, then $q\gamma a=0$ since $q\alpha u\beta x\gamma a=u\beta x\alpha q\gamma a$. Thus, we may suppose that $q\gamma a\neq 0$ and $q\gamma u\neq 0$ for all $u\in U$ and $\gamma\in\Gamma$. In this case, we get $q\alpha u\beta x\gamma a=u\beta x\alpha q\gamma a$ for all $x\in M, u\in U$ and $\gamma,\beta,\alpha\in\Gamma$. It follows from Lemma 3.1.10 that there exists $\lambda\in C_\Gamma$ such that $q\gamma a=\lambda\delta a$ and $q\gamma u=\lambda\delta u$ for all $u\in U$ and $\gamma,\delta,\alpha\in\Gamma$. Hence, if $q'=q-\lambda$, then $q'\Gamma a=0$ and $q'\Gamma U=\{0\}$. This completes the proof. □

Lemma 3.1.15: [30] Let M be a prime $\Gamma-$ring, U a non-zero right (resp. left) ideal of M and $a\in M$. If $U\Gamma a=\{0\}$ (resp. $a\Gamma U=\{0\}$), then $a=0$.

Theorem 3.1.16: [30] Let M be a prime $\Gamma-$ring with $char M\neq 2$, U a non-zero right ideal of M and d a non-zero derivation of M. Then the subring of M generated by $d(U)$ contains no non-zero right ideals of M if and only if $d(U)\Gamma U=\{0\}$.

Proof: Let A be the subring generated by $d(U)$. Let $S=A\cap U$, $u\in U$, $s\in S$ and $\gamma\in\Gamma$. Then $d(s\gamma u)=d(s)\gamma u+s\gamma d(u)\in A$, and so we have $d(s)\gamma u\in S$. Thus $d(S)\Gamma U$ is a right ideal of M. In this case, $d(S)\Gamma U=\{0\}$ by hypothesis. $d(u\gamma a)=d(u)\gamma a+u\gamma d(a)\in S$ and $d(u)\gamma a\in S$ where $u\in U, a\in A$. Thus, we have $u\gamma d(a)\in S$. Therefore, $0=d\left(u\gamma d(a)\right)\beta u=\left(u\gamma d^2(a)+d(u)\gamma d(a)\right)\beta u$. Since M is a prime $\Gamma-$ring, it follows from Lemma 3.1.15 that

$$u\gamma d^2(a)+d(u)\gamma d(a)=0 \qquad (3.1.25)$$

for all $u\in U, a\in A$ and $\gamma\in\Gamma$. Replacing u by $u\beta v$ where $v\in U,\beta\in\Gamma$ in $(3.1.25)$ we get,

$$d(u)\beta v\gamma d(a)=0 \qquad (3.1.26)$$

Since M is a prime $\Gamma-$ring, we get $d(U)\Gamma U=\{0\}$ or $d(A)\Gamma U=\{0\}$. If $d(A)\Gamma U=\{0\}$, then $d^2(U)\Gamma U=\{0\}$.

Let $u,v\in U$ and $\beta\in\Gamma$, then $0=d\left(d\left(u\beta v\right)\right)=u\beta d^{2}\left(v\right)+d\left(u\right)\beta d\left(v\right)+d\left(v\right)\beta d\left(u\right)+d^{2}\left(u\right)\beta v$, and so we have $d\left(u\right)\beta d\left(v\right)=0$ for all $u,v\in U$ and $\beta\in\Gamma$ by $char\mathrm{M}\neq 2$. Replacing u by $u\gamma w$ where $w\in U,\gamma\in\Gamma$ in the last relation, we have $d\left(u\right)\gamma w\beta d\left(v\right)=0$ which yields $d\left(u\right)\gamma v=0$ for all $u,v\in U$ and $\gamma\in\Gamma$.

Conversely assume that $d\left(U\right)\Gamma U=\{0\}$. Then $A\Gamma U=\{0\}$. Since M is a prime $\Gamma-$ring, A contains no non-zero right ideals. □

Theorem 3.1.17: [30] Let M be a prime $\Gamma-$ring with $char\mathrm{M}\neq 2$, U a nonzero right ideal of M and d_1,d_2 are two non-zero derivations of M. If $d_1d_2\left(U\right)=\{0\}$, then there exists two elements p,q of Q such that $q\Gamma U=\{0\}$ and $p\Gamma U=\{0\}$.

Proof: If $d_1d_2\left(U\right)=\{0\}$, then $d_1\left(A\right)=\{0\}$ where A is a subring generated by $d_2\left(U\right)$. Since $d\neq 0$, A contains no non-zero right ideals of M. Thus, from Theorem 3.1.16, we have $d_2\left(u\right)\gamma v=0$ for all $u,v\in U$ and $\gamma\in\Gamma$. Also, there exists $q\in Q$ such that $q\Gamma U=\{0\}$ by Theorem 3.1.14. Therefore $d_2\left(u\gamma v\right)=u\gamma d_2\left(v\right)$ for all $u,v\in U$ and $\gamma\in\Gamma$. In this case, $0=d_1d_2\left(u\gamma v\right)=d_1\left(u\gamma d_2\left(v\right)\right)=d_1\left(u\right)\gamma d_2\left(v\right)$, and since M is a prime $\Gamma-$ring, we get $d_2\left(u\right)\gamma v=0$ for all $u,v\in U$ and $\gamma\in\Gamma$. Again, by Theorem 3.1.14, there exists $p\in Q$ such that $p\Gamma U=\{0\}$. This completes the proof. □

Remark 3.1.18: (a) Consider the following example. Let R be a ring. A derivation $d:R\to R$ is called *an inner derivation* if there exists $a\in R$ such that $d\left(x\right)=\left[a,x\right]=ax-xa$ for all $x\in R$. Let S be the 2×2 matrix ring over Galois field $\{0,1,w,w^2\}$, with inner derivations d_1 and d_2 defined by

$$d_1\left(x\right)=\left[\begin{bmatrix}0&1\\0&0\end{bmatrix},x\right],d_2\left(x\right)=\left[\begin{bmatrix}0&w\\0&0\end{bmatrix},x\right]$$

for all $x\in S$. Then the characteristic of S is 2 and we have $d_1\neq 0,d_2\neq 0,d_1d_2=0$ and $d_2^2=0$. Also, if we take $\mathrm{M}=\mathrm{M}_{1\times 2}\left(S\right)=\{\left(a,b\right)\mid \mathrm{a,b}\in\mathrm{S}\}$ and $\Gamma=\left\{\begin{bmatrix}n\\0\end{bmatrix}\mid n \text{ is an integer}\right\}$,

then M is a prime $\Gamma-$ring of characteristic 2. Define an additive map $D_1:\mathrm{M}\to\mathrm{M}$ by $D_1\left(x,y\right)=\left(d_1\left(x\right),d_1\left(y\right)\right)$. Since $\left(x,y\right)\begin{bmatrix}n\\0\end{bmatrix}\left(a,b\right)=\left(nxa,nxb\right)$, therefore D_1 is a derivation on M. Similarly $D_2:\mathrm{M}\to\mathrm{M}$ given by $D_2\left(x,y\right)=\left(d_2\left(x\right),d_2\left(y\right)\right)$ is a derivation. In this case, we have $D_1\neq 0,D_2\neq 0,D_1D_2=0$ and $D_2^2=0$. Thus we know that there exist two derivations D_1,D_2 of M such that $D_1D_2\left(\mathrm{M}\right)=\{0\}$ but $D_1\left(\mathrm{M}\right)\Gamma\mathrm{M}\neq\{0\}$ and $D_2\left(\mathrm{M}\right)\Gamma\mathrm{M}\neq\{0\}$. Therefore the condition of $char\mathrm{M}\neq 2$ in Theorems 3.1.15 and 3.1.16 is necessary.

(b) In Theorems 3.1.14 and 3.1.17, if $a\gamma(c\beta b)=a\beta(c\gamma b)$ for all $a,b,c\in M$ and $\gamma,\beta\in\Gamma$, then $d(x)=[q,x]_\gamma=q\gamma x-x\gamma q$ for all $x\in M,\gamma\in\Gamma$ and for some $q\in Q$ is inner derivation and also $d_1(x)=[q,x]_\gamma$ and $d_2(x)=[q,x]_\beta$ for all $x\in M,\gamma,\beta\in\Gamma$ and for some elements $q,p\in Q$ are inner derivations .

3.2 Permuting Tri-Derivation On Prime $\Gamma-$ Rings

Let M be a $\Gamma-$ring. A mapping $D:M\times M\times M\to M$ is said to be *tri-additive* if it satisfies:

1. $D(x+w,y,z)=D(x,y,z)+D(w,y,z)$,
2. $D(x,y+w,z)=D(x,y,z)+D(x,w,z)$,
3. $D(x,y,z+w)=D(x,y,z)+D(x,y,w)$.

for all $x,y,z,w\in M$. A tri-additive mapping D is said to be *permuting tri-additive* if $D(x,y,z)=D(x,z,y)=D(y,x,z)=D(y,z,x)=D(z,x,y)=D(z,y,x)$ for all $x,y,z\in M$. A mapping $d:M\to M$ defined by $d(x)=D(x,x,x)$ is called the *trace* of D, where D is a permuting tri-additive mapping. It is obvious that if D is a permuting tri-additive mapping, then the trace of D satisfies the relation

$$d(x+y)=d(x)+d(y)+3D(x,x,y)+3D(x,y,y) \qquad (3.2.1)$$

for all $x,y\in M$. A permuting tri-additive mapping D is called a *permuting tri-derivation* if $D(x\alpha w,y,z)=D(x,y,z)\alpha w+x\alpha D(w,y,z)$ for all $x,y,z,w\in M$ and $\alpha\in\Gamma$. Then the relations

$$D(x,y\alpha w,z)=D(x,y,z)\alpha w+y\alpha D(w,y,z)$$

and

$$D(x,y,z\alpha w)=D(x,y,z)\alpha w+z\alpha D(w,y,z)$$

are fulfilled for all $x,y,z,w\in M$ and $\alpha\in\Gamma$. Let D be a permuting tri-additive mapping of M, where M is a $\Gamma-$ring. Since

$$D(0,x,y)=D(0+0,x,y)=D(0,x,y)+D(0,x,y),$$

We have $D(0,x,y)=0$ for all $x,y\in M$. Thus

$$0=D(0,y,z)=D(-x+x,y,z)=D(-x,y,z)+D(x,y,z),$$

and so $D(-x,y,z)=-D(x,y,z)$ for all $x,y,z\in M$. Therefore the mapping $d:M\to M$ defined by $d(x)=D(x,x,x)$ is an odd function.

Example 3.2.1: For a commutative ring R, let

$$\mathrm{M}=\left\{\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a,b,c\in R\right\} \text{ and } \Gamma=\left\{\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha\in R\right\}.$$

It is obvious that M and Γ are both abelian groups under matrix addition. Now it is easy to show that M is a $\Gamma-$ring under matrix multiplication. A map $D:\mathrm{M}\times\mathrm{M}\times\mathrm{M}\to\mathrm{M}$ defined by $\left(\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \mapsto \begin{pmatrix} 0 & 0 & a_1\alpha a_2\beta a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a permuting tri-derivation.

Lemma 3.2.2: [28] Let M be a 2,3-torsion free $\Gamma-$ring and I a non-zero one-side ideal of M. Let D be a permuting tri-derivation with the trace d. Consider the following conditions:

i. $d(x)=0$ for all $x\in I$
ii. $D(x,y,z)=0$ for all $x,y,z\in I$
iii. $D(m,x,y)=0$ for all $x,y\in I$ and $m\in\mathrm{M}$
iv. $D(m,n,x)=0$ for all $x\in I$ and $m,n\in\mathrm{M}$
v. $D(m,n,r)=0$ for all $m,n,r\in\mathrm{M}$.

Then (i) and (ii) are equivalent. Moreover if M is a prime $\Gamma-$ring or $Ann_r I=0$ (or $Ann_l I=0$), the above conditions are all equivalent.

Proof: Let I be a right ideal of M and let $m,n,r\in\mathrm{M}, x,y,z\in I$ and $\alpha,\beta,\gamma\in\Gamma$. Since M is 3-torsion free, it follows from $(3.2.1)$ that

$$D(x,x,y)+D(x,y,y)=0 \tag{3.2.2}$$

Writing $y+z$ for y in $(3.2.2)$ and using the fact that M is 2-torsion free, we know that (i) and (ii) are equivalent. Replacing z by $z\alpha m$ in (ii) implies that

$$0=D(x,y,z\alpha m)=D(x,y,z)\alpha m+z\alpha D(m,x,y)=z\alpha D(m,x,y) \tag{3.2.3}$$

If M is a prime $\Gamma-$ring then by Lemma 3.1.3, we get (ii) and (iii) are equivalent. If $Ann_r I=0$, then from $(3.2.3)$ we get (ii) and (iii) are equivalent. Replacing y by $y\beta n$ in (iii), we have

$$0=D(m,x,y\beta n)=D(m,x,y)\beta n+y\beta D(m,n,x)=y\beta D(m,n,x) \tag{3.2.4}$$

If M is a prime $\Gamma-$ring then by Lemma 3.1.3, we get (iii) and (iv) are equivalent. If $Ann_r I=0$, then from $(3.2.4)$ we get (iii) and (iv) are equivalent. Replacing x by $x\gamma r$ in (iv), we have

$$0 = D(m,n,x\gamma r) = D(m,n,x)\gamma n + x\gamma D(m,n,r) = x\gamma D(m,n,r) \qquad (3.2.5)$$

If M is a prime $\Gamma-$ring then by Lemma 3.1.3, we obtain (iv) and (v) are equivalent. If $Ann_r I = 0$, then from $(3.2.5)$ we have (iv) and (v) are equivalent. Similarly we can prove the result for a left ideal I. □

Theorem 3.2.3: [28] Let M be a 2,3-torsion free prime $\Gamma-$ring, I a nonzero ideal of M. Let D_1 and D_2 be permuting tri-derivations of M with traces d_1 and d_2 respectively. If $D_1(d_2(x),x,x) = 0$ for all $x \in I$, then $D_1 = 0$ or $D_2 = 0$.

Proof: Assume that $D_1(d_2(x),x,x) = 0$ for all $x \in I$. For any $x, y \in I$ we have

$$D_1(d_2(x+y),x+y,x+y) + D_1(d_2(-x+y),x+y,x+y) = 0.$$

Since M is 2-torsion free, it follows that

$$2D_1(d_2(x),x,y) + D_1(d_2(y),x,x) + 3D_1(D_2(x,x,y),x,x) + 3D_1(D_2(x,x,y),y,y)$$

$$+6D_1(D_2(x,y,y),x,y) = 0 \qquad (3.2.6)$$

for all $x, y \in I$. Writing $x+y$ for y in $(3.2.6)$ and using the fact that M is 3-torsion free, we get

$$D_1(d_2(x),y,y) + 4D_1(d_2(x),x,y) + 6D_1(D_2(x,x,y),x,x) + 6D_1(D_2(x,x,y),x,y)$$

$$+3D_1(D_2(x,y,y),x,x) = 0 \qquad (3.2.7)$$

for all $x, y \in I$. Writing $-x$ for x in $(3.2.7)$ and using the fact that M is 2-torsion free, we get

$$4D_1(d_2(x),x,y) + 6D_1(D_2(x,x,y),x,x) = 0 \qquad (3.2.8)$$

for all $x, y \in I$. Replacing y for $x\alpha y$ in $(3.2.8)$ and using the hypothesis and the fact that M is 2,3-torsion free, we get

$$d_2(x)\alpha D_1(x,x,y) + d_1(x)\alpha D_2(x,x,y) = 0 \qquad (3.2.9)$$

for all $x, y \in I$ and $\alpha \in \Gamma$. Writing $y\beta z$ for y in $(3.2.9)$ implies that

$$d_2(x)\alpha y\beta D_1(x,x,z) + d_1(x)\alpha y\beta D_2(x,x,z) = 0 \qquad (3.2.10)$$

for all $x, y, z \in I$ and $\alpha, \beta \in \Gamma$. Writing x for z in $(3.2.10)$ and using Lemma 1.4.8, we have

$$d_1(x)\alpha y\beta d_2(x) = 0 \qquad (3.2.11)$$

for all $x,y,z\in I$ and $\alpha,\beta\in\Gamma$. In this case, suppose that d_1 and d_2 are both different from zero. Then there exist $x_1,x_2\in I$ such that $d_1(x_1)\neq 0$ and $d_2(x_2)\neq 0$. In particular, $d_1(x_1)\alpha y\beta d_2(x_1)=0$ for all $y\in I$ and $\alpha,\beta\in\Gamma$. Since $d_1(x_1)\neq 0$ and M is prime $\Gamma-$ring we have $d_2(x_1)=0$. Similarly, we get $d_1(x_2)=0$. Then the relation $(3.2.10)$ reduces to the equation $d_1(x_1)\alpha y\beta D_2(x_1,x_1,z)=0$ for all $y,z\in I$ and $\alpha,\beta\in\Gamma$. Using this relation and Lemma 3.1.3 we obtain that $D_2(x_1,x_1,z)=0$ for all $z\in I$ because of $d_1(x_1)\neq 0$ (the mapping $z\to D_2(x_1,x_1,z)$ is a derivation). Thus, we have $D_2(x_1,x_1,z)=0$. In the same way, we get $D_1(x_1,x_1,z)=0$. Substituting x_1+x_2 for z, we obtain

$$d_1(z)=d_1(x_1+x_2)=d_1(x_1)+d_1(x_2)+3D_1(x_1,x_1,x_2)+3D_1(x_1,x_2,x_2)=d_1(x_1)\neq 0$$

and

$$d_2(z)=d_2(x_1+x_2)=d_2(x_1)+d_2(x_2)+3D_2(x_1,x_1,x_2)+3D_2(x_1,x_2,x_2)=d_2(x_2)\neq 0$$

Therefore we have $d_1(z)\neq 0$ and $d_2(z)\neq 0$, a contradiction. Hence, we get $d_1(x)=0$ for all $x\in I$ or $d_2(x)=0$ for all $x\in I$. Thus $D_1=0$ or $D_2=0$. □

Remark 3.2.4: Let M be a 2,3-torsion free prime $\Gamma-$ring. Let D be permuting tri-derivation of M with trace d. If $a\alpha d(x)=0$ for all $x\in\mathrm{M},\alpha\in\Gamma$, where a is a fixed element of M, then either $a=0$ or $D=0$.

Theorem 3.2.5: [28] Let M be a prime $\Gamma-$ring of characteristic not 2 and 3, 5-torsion free, I a non-zero ideal of M. Let D_1 and D_2 be permuting tri-derivations of M and let d_1 and d_2 be traces of D_1 and D_2, respectively, such that $d_2(I)\subset I$. If $Ann_l I=0$ and $D_1(d_2(x),d_2(x),x)=0$ for all $x\in I$, then $D_1=0$ or $D_2=0$.

Proof: For any $x,y\in I$, we have

$$D_1(d_2(x+y),d_2(x+y),x+y)+D_1(d_2(-x+y),d_2(-x+y),-x+y)=0$$

Since $Char\mathrm{M}\neq 2$, it follows that

$$\begin{aligned}
&2D_1(d_2(y),d_2(x),x)+6D_1(D_2(x,x,y),d_2(x),x)\\
&+6D_1(D_2(x,y,y),d_2(y),x)+18D_1(D_2(x,x,y),D_2(x,y,y),x)\\
&+D_1(d_2(x),d_2(x),y)+6D_1(D_2(x,y,y),d_2(x),y)\\
&+6D_1(D_2(x,x,y),d_2(y),y)+9D_1(D_2(x,y,y),D_2(x,y,y),y)\\
&+9D_1(D_2(x,x,y),D_2(x,x,y),y)=0 \text{ for all } x,y\in I. \qquad (3.2.12)
\end{aligned}$$

Writing $2x$ for x in $(3.2.12)$ and using the fact that $Char\mathrm{M}\neq 2$ and M is 3-torsion free, we get

$$\begin{aligned}&2D_1\left(d_2\left(y\right),d_2\left(x\right),x\right)+30D_1\left(D_2\left(x,x,y\right),d_2\left(x\right),x\right)\\&+5D_1\left(d_2\left(x\right),d_2\left(x\right),y\right)+18D_1\left(D_2\left(x,x,y\right),D_2\left(x,y,y\right),x\right)\\&+6D_1\left(D_2\left(x,y,y\right),d_2\left(x\right),y\right)+9D_1\left(D_2\left(x,x,y\right),D_2\left(x,x,y\right),y\right)=0 \text{ for all } x,y\in I.\end{aligned} \quad (3.2.13)$$

Writing $2x$ for x in $(3.2.13)$ and using the fact that $Char\mathrm{M}\neq 2$ and M is 3,5-torsion free, we get

$$6D_1\left(D_2\left(x,x,y\right),d_2\left(x\right),x\right)+D_1\left(d_2\left(x\right),d_2\left(x\right),y\right)=0 \text{ for all } x,y\in I. \quad (3.2.14)$$

Replacing y for $y\beta x$ in $(3.2.14)$ implies that

$$D_2\left(x,x,y\right)\beta D_1\left(d_2\left(x\right),x,x\right)+D_1\left(d_2\left(x\right),x,y\right)\beta d_2\left(x\right)=0 \text{ for all } x,y\in I,\beta\in\Gamma. \quad (3.2.15)$$

Replacing y for $x\alpha y$ in $(3.2.15)$ induces

$$d_2\left(x\right)\alpha y\beta D_1\left(d_2\left(x\right),x,x\right)+D_1\left(d_2\left(x\right),x,x\right)\alpha y\beta d_2\left(x\right)=0 \text{ for all } x,y\in I,\alpha,\beta\in\Gamma. \quad (3.2.16)$$

We now show that $D_1\left(d_2\left(x\right),x,x\right)=0$ for all $x\in I$. Assume that there exists $x_1\in I$ such that $D_1\left(d_2\left(x_1\right),x_1,x_1\right)\neq 0$.

Replacing x by x_1 in $(3.2.16)$, then $d_2\left(x_1\right)=0$ by Lemma 1.4.8. Therefore $D_1\left(d_2\left(x_1\right),x_1,x_1\right)=D_1\left(0,x_1,x_1\right)=0$, a contradiction. It follows from Theorem 3.2.3 that $D_1=0$ or $D_2=0$. □

Corollary 3.2.6: Let M be a prime $\Gamma-$ring of characteristic not 2, 3 and 5, 7-torsion free, I a nonzero ideal of M. Let D_1 and D_2 be permuting tri-derivations of M and let d_1 and d_2 be traces of D_1 and D_2, respectively, such that $d_2\left(I\right)\subset I$. If $d_1\left(d_2\left(x\right)\right)=f\left(x\right)$ for all $x\in I$ then $D_1=0$ or $D_2=0$, where a permuting tri-additive mapping $F:\mathrm{M}\times\mathrm{M}\times\mathrm{M}\rightarrow\mathrm{M}$ and f is the trace of F.

3.3 Jordan Triple Higher Derivation On Prime $\Gamma-$ Rings

In this section we introduce the concept of triple higher derivation on a prime $\Gamma-$ring M and prove that every Jordan triple higher derivation on a prime $\Gamma-$ring M of characteristic different from two is a triple higher derivation on M and then, it is shown that every Jordan triple higher derivation is a higher derivation on M.

Definition 3.3.1: [3] Let M be a $\Gamma-$ring and $D=\{d_n\}_{n\in\mathbb{N}}$ be a family of additive mappings $d_n : M \to M$ such that $d_0 = I_M$. Then D is said to be

a) a *higher derivation* on M if for each $n\in\mathbb{N}$,
$d_n(a\alpha b)=\sum_{p+q=n} d_p(a)\alpha d_q(b)$ for all $a,b\in M$, and $\alpha\in\Gamma$;

b) a *Jordan higher derivation* on M if for each $n\in\mathbb{N}$,
$d_n(a\alpha a)=\sum_{p+q=n} d_p(a)\alpha d_q(a)$ for all $a\in M$, and $\alpha\in\Gamma$;

c) a *triple higher derivation* on M if for each $n\in\mathbb{N}$,
$d_n(a\alpha b\beta c)=\sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c)$ for all $a,b,c\in M$, and $\alpha,\beta\in\Gamma$;

d) a *Jordan triple higher derivation* on M if for each $n\in\mathbb{N}$,
$d_n(a\alpha b\beta a)=\sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(a)$ for all $a,b\in M$, and $\alpha,\beta\in\Gamma$.

Example 3.3.2: [3] By Example 1.1.2 (2), let $f : R\to R$ be a triple derivation on R. Now define $F : M\to M$ such that $F((x,y))=(f(x),f(y))$. Then F is a triple derivation on M. In fact, if

$a=(x_1,y_1), b=(x_2,y_2), c=(x_3,y_3)\in M, \alpha=\begin{pmatrix} n_1.1 \\ 0\end{pmatrix}$ and $\beta=\begin{pmatrix} n_2.1 \\ 0\end{pmatrix}\in\Gamma$, then

$a\alpha b\beta c=(x_1n_1x_2n_2x_3, x_1n_1x_2n_2y_3)$, and then we get $F(a\alpha b\beta c)=F(a)\alpha b\beta c+a\alpha F(b)\beta c+a\alpha b\beta F(c)$, for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$.

Define $d_n=\dfrac{F^n}{n!}$, for all $n\in\mathbb{N}$, where F is a triple derivation on M.

Claim: $D=\{d_n\}_{n\in\mathbb{N}}$ is a triple higher derivation on M.

We shall use induction on n to prove the claim:

For $n=0, d_0(a\alpha b\beta c)=\dfrac{F^0(a\alpha b\beta c)}{0!}=a\alpha b\beta c$.

For $n=1, d_1(a\alpha b\beta c)=\dfrac{F^1(a\alpha b\beta c)}{1!}=F(a\alpha b\beta c)=F(a)\alpha b\beta c+a\alpha F(b)\beta c+a\alpha b\beta F(c)$. Suppose that,

$d_m = \dfrac{F^m}{m!}$ defines a triple higher derivation on M for each $m \prec n$. Consider,

$d_n(a\alpha b\beta c) = \dfrac{F^n(a\alpha b\beta c)}{n!} = \dfrac{1}{n}\left(F\left(\dfrac{F^{n-1}(a\alpha b\beta c)}{(n-1)!}\right)\right) = \dfrac{1}{n}F\left(d_{n-1}(a\alpha b\beta c)\right)$. Applying the hypothesis of induction on d_{n-1} , we have

$$d_n(a\alpha b\beta c) = \frac{F}{n}\sum_{p+q+r=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)$$

$$= \frac{F}{n}\sum_{p+q+r=n-1} \frac{F^p(a)}{p!}\alpha\frac{F^q(b)}{q!}\beta\frac{F^r(c)}{r!}$$
$$= \frac{1}{n}\sum_{p+q+r=n-1}\left(\frac{F^{p+1}(a)}{p!}\alpha\frac{F^q(b)}{q!}\beta\frac{F^r(c)}{r!} + \frac{F^p(a)}{p!}\alpha\frac{F^{q+1}(b)}{q!}\beta\frac{F^r(c)}{r!} + \frac{F^p(a)}{p!}\alpha\frac{F^q(b)}{q!}\beta\frac{F^{r+1}(c)}{r!}\right)$$
$$= \frac{1}{n}\sum_{p+q+r=n-1}\left(d_{p+1}(a)\alpha d_q(b)\beta d_r(c)(p+1) + d_p(a)\alpha d_{q+1}(b)\beta d_r(c)(q+1) + d_p(a)\alpha d_q(b)\beta d_{r+1}(c)(r+1)\right)$$
$$= \frac{1}{n}\left\{\sum_{j=0}^{n-1}\left(\sum_{i=0,i\prec j}^{j} d_{i+1}(a)\alpha d_{j-i}(b)\right)\beta d_{n-1-j}(c)(i+1) + \sum_{j=0}^{n-1}\left(\sum_{i=0,i\prec j}^{j} d_i(a)\alpha d_{j-i+1}(b)\right)\beta d_{n-1-j}(c)(j-i+1)\right.$$
$$\left. + \sum_{j=0}^{n-1}\left(\sum_{i=0,i\prec j}^{j} d_i(a)\alpha d_{j-i}(b)\right)\beta d_{n-j}(c)(n-j)\right\}$$
$$= \frac{1}{n}\sum_{j=2}^{n-2}\sum_{i=2}^{j-1} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)i + \frac{1}{n}\sum_{i=2}^{n-1} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)i - \frac{1}{n}\sum_{j=2}^{n-2}\sum_{i=2}^{j-1} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)$$
$$- \frac{1}{n}\sum_{i=2}^{n-2} d_i(a)\alpha d_{n-i-1}(b)\beta d_1(c) + \frac{1}{n}\sum_{j=2}^{n-1} d_j(a)\alpha b\beta d_{n-j}(c)j - \frac{1}{n}\sum_{j=2}^{n-1} d_j(a)\alpha b\beta d_{n-j}(c)$$
$$+ \frac{1}{n}\sum_{j=2}^{n-2} d_j(a)\alpha d_{n-j}(b)\beta cj + \frac{1}{n}d_n(a)\alpha b\beta cn + \frac{1}{n}d_{n-1}(a)\alpha d_1(b)\beta c(n-1) - \frac{1}{n}d_n(a)\alpha b\beta c$$
$$- \frac{1}{n}d_{n-1}(a)\alpha d_1(b)\beta c - \frac{1}{n}\sum_{j=2}^{n-2} d_j(a)\alpha d_{n-j}(b)\beta c + \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{j} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)$$

$$- \frac{1}{n}\sum_{j=2}^{n-2}\sum_{i=1}^{j-1} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c)i + \frac{1}{n}\sum_{i=0}^{n-2} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)n - \frac{1}{n}\sum_{i=0}^{n-2} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)$$
$$- \frac{1}{n}\sum_{i=2}^{n-2} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)i - \frac{1}{n}d_1(a)\alpha d_{n-2}(b)\beta d_1(c) + \sum_{i=0}^{n-1} d_i(a)\alpha d_{n-j}(b)\beta c - \frac{1}{n}\sum_{i=2}^{n-1} d_i(a)\alpha d_{n-i}(b)\beta ci$$
$$- \frac{1}{n}d_1(a)\alpha d_{n-1}(b)\beta c + \sum_{j=0}^{n-1}\sum_{i=0}^{j} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c) - \frac{1}{n}d_1(a)\alpha b\beta d_{n-1}(c) - \sum_{i=0}^{n-1} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c)$$
$$+ \frac{1}{n}\sum_{i=0}^{n-1} d_i(a)\alpha d_{n-1-i}(b)\beta d_1(c) = \sum_{j=0}^{n-1}\sum_{i=0}^{j} d_i(a)\alpha d_{j-i}(b)\beta d_{n-j}(c) + d_n(a)\alpha b\beta c + \sum_{i=0}^{n-1} d_i(a)\alpha d_{n-i}(b)\beta c$$
$$= \sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c).$$

Thus, the family $D=\{d_n\}_{n\in\mathbb{N}}$ where, $d_n=\dfrac{F^n}{n!}$ defines a triple higher derivation on M.

Similarly, if $f:R\to R$ is considered to be a Jordan triple derivation on R then using similar procedure one can find an example of Jordan triple higher derivation on M.

Remark 3.3.3: In the above example if we consider $f:R\to R$ as derivation (resp. Jordan derivation), then using similar arguments as given in the above example with necessary variations, one can construct an example of higher derivation (resp. Jordan higher derivation) on M.

It can be easily seen that every triple higher derivation is a Jordan triple higher derivation. But the converse is not true in general. In the present section we establish the converse of the above statement under certain conditions.

Definition 3.3.4: [3] Let M be a $\Gamma-$ring. Then for all $a,b,c\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$ we define

$$[a,b,c]_{\alpha,\beta}=a\alpha b\beta c-c\alpha b\beta a.$$

Lemma 3.3.5: [3] If M is a $\Gamma-$ring, then for all $a,b,c,d\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$

i. $[a,b,c]_{\alpha,\beta}+[c,b,a]_{\alpha,\beta}=0$

ii. $[a+c,b,d]_{\alpha,\beta}=[a,b,d]_{\alpha,\beta}+[c,b,d]_{\alpha,\beta}$

iii. $[a,b,c+d]_{\alpha,\beta}=[a,b,c]_{\alpha,\beta}+[a,b,d]_{\alpha,\beta}$

iv. $[a,b+d,c]_{\alpha,\beta}=[a,b,c]_{\alpha,\beta}+[a,d,c]_{\alpha,\beta}$

v. $[a,b,c]_{\alpha+\beta,\gamma}=[a,b,c]_{\alpha,\gamma}+[a,b,c]_{\beta,\gamma}$

vi. $[a,b,c]_{\alpha,\beta+\gamma}=[a,b,c]_{\alpha,\beta}+[a,b,c]_{\alpha,\gamma}$.

Proof: direct application of Definition 3.3.4. □

Lemma 3.3.6: [3] Let M be a $\Gamma-$ring and $D=\{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple higher derivation on M. Then for all $a,b,c\in\mathrm{M}$, and for all $\alpha,\beta\in\Gamma$, we have

$$d_n(a\alpha b\beta c+c\alpha b\beta a)=\sum_{p+q+r=n}d_p(a)\alpha d_q(b)\beta d_r(c)+\sum_{p+q+r=n}d_p(c)\alpha d_q(b)\beta d_r(a).$$

Proof: Since $d_n(a\alpha b\beta c)=\sum_{p+q+r=n}d_p(a)\alpha d_q(b)\beta d_r(c)$. Linearizing on a we get, $d_n((a+c)\alpha b\beta(a+c))=\sum_{p+q+r=n}d_p(a+c)\alpha d_q(b)\beta d_r(a+c)$. Computing and canceling the like terms from both sides, the proof will be complete. □

Let $D=\{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple higher derivation of a $\Gamma-$ring M. Then for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ we define

$$G^n_{\alpha,\beta}(a,b,c)=d_n(a\alpha b\beta c)-\sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c) \text{ for all } n\in\mathbb{N}.$$

Lemma 3.3.7: Let $D=\{d_n\}_{n\in\mathbb{N}}$ be a Jordan triple higher derivation of a $\Gamma-$ring M. Then for all $a,b,c\in M$, $\alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$, we have

i. $G^n_{\alpha,\beta}(a,b,c)+G^n_{\alpha,\beta}(c,b,a)=0$,

ii. $G^n_{\alpha,\beta}(a+c,b,e)=G^n_{\alpha,\beta}(a,b,e)+G^n_{\alpha,\beta}(c,b,e)$,

iii. $G^n_{\alpha,\beta}(a,b,c+e)=G^n_{\alpha,\beta}(a,b,c)+G^n_{\alpha,\beta}(a,b,e)$,

iv. $G^n_{\alpha,\beta}(a,b+c,e)=G^n_{\alpha,\beta}(a,b,e)+G^n_{\alpha,\beta}(a,c,e)$,

v. $G^n_{\alpha+\gamma,\beta}(a,b,c)=G^n_{\alpha,\beta}(a,b,c)+G^n_{\gamma,\beta}(a,b,c)$,

vi. $G^n_{\alpha,\beta+\gamma}(a,b,c)=G^n_{\alpha,\beta}(a,b,c)+G^n_{\alpha,\gamma}(a,b,c)$.

Proof: Proof of part (i) is obvious by Lemma 3.3.6, while the proofs of parts (ii)-(vi) can be obtained easily by using additivity of d_n.

Lemma 3.3.8: Let M be a 2-torsion free semi-prime $\Gamma-$ring. If $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=0$, then $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,v,w]_{\alpha,\beta}=0$, for all $a,b,c,u,v,w,x\in M,\alpha,\beta,\gamma,\delta\in\Gamma$ and for all $n\in\mathbb{N}$.

Proof: Replacing a by $a+u$ in the hypothesis we get $G^n_{\alpha,\beta}(a+u,b,c)\gamma x\delta[a+u,b,c]_{\alpha,\beta}=0$. Hence using Lemma 3.3.5. we find that $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}+G^n_{\alpha,\beta}(u,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=0$.

Now consider; $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}\gamma x\delta G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}$

$=-G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}\gamma x\delta G^n_{\alpha,\beta}(u,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=0$ using hypothesis. Since M is semi-prime $\Gamma-$ring, then $G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[u,b,c]_{\alpha,\beta}=0$. Similarly, replacing b by $b+v$ and c by $c+w$ and using semiprimeness of M, we get the required result. □

Now we are well equipped to prove the main result, which is:

Theorem 3.3.9: [3] Let M be a prime $\Gamma-$ ring of characteristic different from two, then every Jordan triple higher derivation on M is a triple higher derivation on M.

Proof: we are given that the family $D=\{d_n\}_{n\in\mathbb{N}}$ of additive mappings on M satisfies $d_n(a\alpha b\beta a)=\sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(a)$ for all $a,b\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$, and for all $n\in\mathbb{N}$. Now we compute $\Delta=d_n\big(a\alpha(b\beta c\gamma x\delta c\alpha b)\beta a+c\alpha(b\beta a\gamma x\delta ac\alpha b)\beta c\big)$ where

$$\Delta=\sum_{p+i+v=n} d_p(a)\alpha d_i(b\beta c\gamma x\delta c\alpha b)\beta d_v(a)+\sum_{p+i+v=n} d_p(c)\alpha d_i(b\beta a\gamma x\delta a\alpha b)\beta d_v(c)$$

$$=\sum_{p+q+j+u+v=n} d_p(a)\alpha d_q(b)\beta d_j(c\gamma x\delta c)\alpha d_u(b)\beta d_v(a)$$

$$+\sum_{p+q+j+u+v=n} d_p(c)\alpha d_q(b)\beta d_j(a\gamma x\delta a)\alpha d_u(b)\beta d_v(c)$$

$$=\sum_{p+q+r+s+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_s(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+\sum_{p+q+r+s+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_s(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c).$$

On the other hand,

$\Delta=d_n\big((a\alpha b\beta c)\gamma x\delta(c\alpha b\beta a)+(c\alpha b\beta a)\gamma x\delta(a\alpha b\beta c)\big)$ and using Lemma 3.3.6 we get

$$\Delta=\sum_{i+s+j=n} d_i(a\alpha b\beta c)\gamma d_s(x)\delta d_j(c\alpha b\beta a)+\sum_{i+s+j=n} d_i(c\alpha b\beta a)\gamma d_s(x)\delta d_j(a\alpha b\beta c).$$

On comparing the above two equalities we get,

$$\sum_{i+s+j=n} d_i(a\alpha b\beta c)\gamma d_s(x)\delta d_j(c\alpha b\beta a)+\sum_{i+s+j=n} d_i(c\alpha b\beta a)\gamma d_s(x)\delta d_j(a\alpha b\beta c)$$

$$=\sum_{p+q+r+s+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_s(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a) \quad (3.3.1)$$

$$+\sum_{p+q+r+s+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_s(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c).$$

In $(3.3.1)$, put $n=1$ and cancel the like terms from both sides of this equality and then arrange them, to obtain

$$G^1_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}+[a,b,c]_{\alpha,\beta}\gamma x\delta G^1_{\alpha,\beta}(a,b,c)=0 \tag{3.3.2}$$

In view of Lemma 1.4.5 and the above equation, we obtain

$$G^1_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}=[a,b,c]_{\alpha,\beta}\gamma x\delta G^1_{\alpha,\beta}(a,b,c)=0$$

Also using Lemma 3.3.8 we find that

$$G^1_{\alpha,\beta}(a,b,c)\gamma x\delta[u,v,w]_{\alpha,\beta}=0\text{, for all } a,b,c,u,v,w\in \mathrm{M},\alpha,\beta,\gamma,\delta\in\Gamma.$$

Therefore using primeness of M, we get either $G^1_{\alpha,\beta}(a,b,c)=0$ or $\delta[u,v,w]_{\alpha,\beta}=0$, for all $a,b,c,u,v,w\in \mathrm{M},\alpha,\beta,\gamma,\delta\in\Gamma$. If we suppose that $\delta[u,v,w]_{\alpha,\beta}=0$, for all $u,v,w\in \mathrm{M},\alpha,\beta\in\Gamma$, then we have $u\alpha v\beta w=w\alpha v\beta u$. Hence by Lemma 3.3.6 we find that,

$$d_1(a\alpha b\beta c+a\alpha b\beta c)=\sum_{p+q+r=1}d_p(a)\alpha d_q(b)\beta d_r(c)+\sum_{p+q+r=1}d_p(c)\alpha d_q(b)\beta d_r(a).$$

Since M is 2-torsion free, we get $d_1(a\alpha b\beta c)=\sum_{p+q+r=n}d_p(a)\alpha d_q(b)\beta d_r(c)$ or we may say that for $n=1, D=\{d_n\}_{n\in\mathbb{N}}$ is a triple higher derivation. On the other hand if $G^1_{\alpha,\beta}(a,b,c)=0$, then $d_1(a\alpha b\beta c)=\sum_{p+q+r=1}d_p(a)\alpha d_q(b)\beta d_r(c)$, which again implies that $D=\{d_n\}_{n\in\mathbb{N}}$ is triple higher derivation.

Now let the result holds for $n-1$, i.e., $G^1_{\alpha,\beta}(a\alpha b\beta c)=0$ for all $a,b,c\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$.

Also (3.3.1) can be rewritten as

$$\sum_{i+j=n}d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a)+\sum_{i+j=n-1}d_i(a\alpha b\beta c)\gamma d_1(x)\delta d_j(c\alpha b\beta a)$$

$$+\ldots+\sum_{i+j=1}d_i(a\alpha b\beta c)\gamma d_{n-1}(x)\delta d_j(c\alpha b\beta a)+\sum_{i+j=0}d_i(a\alpha b\beta c)\gamma d_n(x)\delta d_j(c\alpha b\beta a)$$

$$+\sum_{i+j=n}d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c)+\sum_{i+j=n-1}d_i(c\alpha b\beta a)\gamma d_1(x)\delta d_j(a\alpha b\beta c)$$

$$+\ldots+\sum_{i+j=1}d_i(c\alpha b\beta a)\gamma d_{n-1}(x)\delta d_j(a\alpha b\beta c)+\sum_{i+j=0}d_i(c\alpha b\beta a)\gamma d_n(x)\delta d_j(a\alpha b\beta c)$$

$$= \sum_{p+q+r+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+ \sum_{p+q+r+t+u+v=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_1(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+\ldots+ \sum_{p+q+r+t+u+v=1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_{n-1}(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+ \sum_{p+q+r+t+u+v=0} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_n(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+ \sum_{p+q+r+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c)$$

$$+ \sum_{p+q+r+t+u+v=n-1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_1(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c)$$

$$+\ldots+ \sum_{p+q+r+t+u+v=1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_{n-1}(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c)$$

$$+ \sum_{p+q+r+t+u+v=0} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_n(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c).$$

Also, we have

$$\sum_{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a) + \sum_{i+j=n-1} d_i(a\alpha b\beta c)\gamma d_1(x)\delta d_j(c\alpha b\beta a)$$

$$+\ldots+ d_1(a\alpha b\beta c)\gamma d_{n-1}(x)\delta c\alpha b\beta a + a\alpha b\beta c d_{n-1}(x)\delta d_1(c\alpha b\beta a)$$

$$+ a\alpha b\beta c\gamma d_n(x)\delta c\alpha b\beta a + \sum_{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c)$$

$$+ \sum_{i+j=n-1} d_i(c\alpha b\beta a)\gamma d_1(x)\delta d_j(a\alpha b\beta c) + \ldots + d_1(c\alpha b\beta a)\gamma d_{n-1}(x)\delta a\alpha b\beta c$$

$$+ c\alpha b\beta a\gamma d_{n-1}(x)\delta d_1(a\alpha b\beta c) + c\alpha b\beta a\gamma d_n(x)\delta a\alpha b\beta c$$

$$= \sum_{p+q+r+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+ \sum_{p+q+r+t+u+v=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma d_1(x)\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+\ldots+d_1(a)\alpha b\beta c\gamma d_{n-1}(x)\delta c\alpha b\beta a+a\alpha d_1(b)\beta c\gamma d_{n-1}(x)\delta c\alpha b\beta a$$

$$+a\alpha b\beta d_1(c)\gamma d_{n-1}(x)\delta c\alpha b\beta a+a\alpha b\beta c\gamma d_{n-1}(x)\delta d_1(c)\alpha b\beta a$$

$$+a\alpha b\beta c\gamma d_{n-1}(x)\delta c\alpha d_1(b)\beta a+a\alpha b\beta c\gamma d_{n-1}(x)\delta c\alpha b\beta d_1(a)$$

$$+a\alpha b\beta c\gamma d_n(x)\delta c\alpha b\beta a+\sum_{p+q+r+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c)$$

$$+\sum_{p+q+r+t+u+v=n-1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma d_1(x)\delta d_t(a)\alpha d_u(b)\beta d_v(c)$$

$$+\ldots+d_1(c)\alpha b\beta a\gamma d_{n-1}(x)\delta a\alpha b\beta c+c\alpha d_1(b)\beta a\gamma d_{n-1}(x)\delta a\alpha b\beta c$$

$$+c\alpha b\beta d_1(a)\gamma d_{n-1}(x)\delta a\alpha b\beta c+c\alpha b\beta a\gamma d_{n-1}(x)\delta d_1(a)\alpha b\beta c$$

$$+c\alpha b\beta a\gamma d_{n-1}(x)\delta a\alpha d_1(b)\beta c+c\alpha b\beta a\gamma d_{n-1}(x)\delta a\alpha b\beta d_1(c)$$

$$+c\alpha b\beta a\gamma d_n(x)\delta a\alpha b\beta c\,.$$

Now since $d_{n-1}(a\alpha b\beta c)=\sum_{p+q+r=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)$ for all $a,b,c\in \mathrm{M},\alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$, the above expression reduces to

$$\sum_{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a)+\sum_{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c)$$

$$=\sum_{p+q+r+t+u+v=n} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+\sum_{p+q+r+t+u+v=n} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c).$$

It can also be written as

$$d_n(a\alpha b\beta c)\gamma x\delta c\alpha b\beta a+a\alpha b\beta c\gamma x\delta d_n(c\alpha b\beta a)+\sum_{0<i,j\le n-1}^{i+j=n} d_i(a\alpha b\beta c)\gamma x\delta d_j(c\alpha b\beta a)$$

$$+d_n(c\alpha b\beta a)\gamma x\delta a\alpha b\beta c+c\alpha b\beta a\gamma x\delta d_n(a\alpha b\beta c)+\sum_{0<i,j\le n-1}^{i+j=n} d_i(c\alpha b\beta a)\gamma x\delta d_j(a\alpha b\beta c)$$

$$=\sum_{p+q+r=0,t+u+v=n} a\alpha b\beta c\gamma x\delta d_t(c)\alpha d_u(b)\beta d_v(a)$$

$$+\sum_{p+q+r=0,t+u+v=0} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta c\alpha b\beta a$$

$$+\sum_{0<p+q+r,t+u+v\leq n-1} d_p(a)\alpha d_q(b)\beta d_r(c)\gamma x\delta d_t(c)\alpha d_u(b)\beta d_r(a)$$

$$+\sum_{p+q+r=0,t+u+v=n} c\alpha b\beta a\gamma x\delta d_t(a)\alpha d_u(b)\beta d_v(c)$$

$$+\sum_{p+q+r=n,t+u+v=0} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta a\alpha b\beta c$$

$$+\sum_{0<p+q+r,t+u+v\leq n-1} d_p(c)\alpha d_q(b)\beta d_r(a)\gamma x\delta d_t(a)\alpha d_u(b)\beta d_r(c).$$

On using $d_{n-1}(a\alpha b\beta c)=\sum_{p+q+r=n-1} d_p(a)\alpha d_q(b)\beta d_r(c)$ for all $a,b,c\in\mathrm{M},\alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$, we get,

$G^n_{\alpha,\beta}(a,b,c)\gamma x\delta[a,b,c]_{\alpha,\beta}+[a,b,c]_{\alpha,\beta}\gamma x\delta G^n_{\alpha,\beta}(a,b,c)=0$ for all $a,b,c\in\mathrm{M},\alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$.

Now upon using the same methods as used after $(3.3.2)$, we find that, either $G^n_{\alpha,\beta}(a,b,c)=0$ or $[a,b,c]_{\alpha,\beta}=0$, for all $a,b,c\in\mathrm{M},\alpha,\beta\in\Gamma$ and for all $n\in\mathbb{N}$. If $G^n_{\alpha,\beta}(a,b,c)=0$, then by definition of $G^n_{\alpha,\beta}(a,b,c)$, $D=\{d_n\}_{n\in\mathbb{N}}$ becomes a triple higher derivation. Whereas if $[a,b,c]_{\alpha,\beta}=0$, in view of Lemma 3.3.5 and using torsion restriction on M again $D=\{d_n\}_{n\in\mathbb{N}}$ becomes a triple higher derivation. Hence the required result is proved. □

Let $D=\{d_n\}_{n\in\mathbb{N}}$ be a Jordan higher derivation of a $\Gamma-$ring M. Then for all $a,b\in\mathrm{M}$ and $\alpha\in\Gamma$ we define

$$F^n_{\alpha,\beta}(a,b)=d_n(a\alpha b)-\sum_{p+q=n} d_p(a)\alpha d_q(b) \text{ for all } n\in\mathbb{N}.$$

It can be easily seen that every higher derivation on a $\Gamma-$ring M is a triple higher derivation on M. But the converse is not true in general. The theorem given below provides the necessary condition such converse holds for a prime $\Gamma-$ring M.

Theorem 3.3.10: [3] Any triple higher derivation of a prime $\Gamma-$ring M of characteristic different from two is a higher derivation on M.

Proof: Given that $D=\{d_n\}_{n\in\mathbb{N}}$ is a triple higher derivation on M, i.e.,

$$d_n(a\alpha b\beta c)=\sum_{p+q+r=n} d_p(a)\alpha d_q(b)\beta d_r(c) \text{ for each } a,b,c\in M, \alpha,\beta\in\Gamma \text{ and for all } n\in\mathbb{N}.$$

Now consider

$$\Delta=d_n\big(a\alpha(b\gamma x\delta a)\alpha b\big)=\sum_{p+i+t=n} d_p(a)\alpha d_i(b\gamma x\delta a)\alpha d_t(b)$$

$$=\sum_{p+q+r+s+t=n} d_p(a)\alpha d_q(b)\gamma d_r(x)\delta d_s(a)\alpha d_t(b).$$

Again,

$$\Delta=d_n\big((a\alpha b)\gamma x\delta(a\alpha b)\big)=\sum_{i+r+j=n} d_i(a\alpha b)\gamma d_r(x)\delta d_j(a\alpha b)$$

Comparing the above two expressions so obtained for Δ, we obtain

$$\sum_{i+r+j=n} d_i(a\alpha b)\gamma d_r(x)\delta d_j(a\alpha b)=\sum_{p+q+r+s+t=n} d_p(a)\alpha d_q(b)\gamma d_r(x)\delta d_s(a)\alpha d_t(b) \quad (3.3.3)$$

Hence, for $n=1$ the above equation becomes,

$$d_1(a\alpha b)\gamma x\delta a\alpha b+a\alpha b\gamma d_1(x)\delta a\alpha b+a\alpha b\gamma x\delta d_1(a\alpha b)$$

$$=d_1(a)\alpha b\gamma x\delta a\alpha b+a\alpha d_1(b)\gamma x\delta a\alpha b+a\alpha b\gamma d_1(x)\delta a\alpha b$$

$$+a\alpha b\gamma x\delta d_1(a)\alpha b+a\alpha b\gamma x\delta a\alpha d_1(b).$$

This yields that, $F^1_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b+a\alpha b\gamma x\delta F^1_{\alpha,\beta}(a,b)=0$.

On using Lemma 1.4.5 we have $F^1_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b=0$ for all $a,b,x\in M$ and $\alpha,\beta\in\Gamma$, which further on linearizing becomes $F^1_{\alpha,\beta}(a,b)\gamma x\delta c\alpha d$ for all $a,b,x,c,d\in M$ and $\alpha,\beta\in\Gamma$. Again since M is prime, we get $F^1_{\alpha,\beta}(a,b)=0$ for all $a,b,x\in M$ and $\alpha,\beta\in\Gamma$, or we can say that $d_1(a\alpha b)=\sum_{i+j=1} d_i(a)\alpha d_j(b)$ for all $a,b\in M$ and $\alpha,\beta\in\Gamma$. Let the result hold for $n-1$, i.e.,

$$d_{n-1}(a\alpha b)=\sum_{i+j=n-1} d_i(a)\alpha d_j(b) \text{ for all } a,b\in M \text{ and } \alpha,\beta\in\Gamma. \quad (3.3.4)$$

$(3.3.3)$ can be rewritten as

$$\sum_{i+j=n} d_i(a\alpha b)\gamma x\delta d_j(a\alpha b)+\sum_{i+j=n-1} d_i(a\alpha b)\gamma d_1(x)\delta d_j(a\alpha b)$$

$$+\ldots+\sum_{i+j=1} d_i(a\alpha b)\gamma d_{n-1}(x)\delta d_j(a\alpha b)+\sum_{i+j=0} d_i(a\alpha b)\gamma d_n(x)\delta d_j(a\alpha b)$$

$$=\sum_{p+q+s+t=n} d_p(a)\alpha d_q(b)\gamma x\delta d_s(a)\alpha d_t(b)$$

$$+\sum_{p+q+s+t=n-1} d_p(a)\alpha d_q(b)\gamma d_1(x)\delta d_s(a)\alpha d_t(b)$$

$$+\sum_{p+q+s+t=1} d_p(a)\alpha d_q(b)\gamma d_{n-1}(x)\delta d_s(a)\alpha d_t(b)$$

$$+\sum_{p+q+s+t=0} d_p(a)\alpha d_q(b)\gamma d_n(x)\delta d_s(a)\alpha d_t(b).$$

On using $(3.3.4)$ we get,

$$\sum_{i+j=n} d_i(a\alpha b)\gamma x\delta d_j(a\alpha b)=\sum_{p+q+s+t=n} d_p(a)\alpha d_q(b)\gamma x\delta d_s(a)\alpha d_t(b).$$

Also

$$d_n(a\alpha b)\gamma x\delta(a\alpha b)+a\alpha b\gamma x\delta d_n(a\alpha b)+\sum_{0<i,j\le n-1} d_i(a\alpha b)\gamma x\delta d_j(a\alpha b)$$

$$=\sum_{s+t=n} a\alpha b\gamma x\delta d_s(a)\alpha d_t(b)+\sum_{p+q=n} d_p(a)\alpha d_q(b)\gamma x\delta a\alpha b$$

$$+\sum_{0<p+q,s+t\le n-1} d_p(a)\alpha d_q(b)\gamma x\delta d_s(a)\alpha d_t(b),$$

And again using $(3.3.4)$ we have,

$$d_n(a\alpha b)\gamma x\delta(a\alpha b)+a\alpha b\gamma x\delta d_n(a\alpha b)=\sum_{s+t=n} a\alpha b\gamma x\delta d_s(a)\alpha d_t(b)$$

$$+\sum_{p+q=n} d_p(a)\alpha d_q(b)\gamma x\delta a\alpha b,$$

or, $F^n_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b+a\alpha b\gamma x\delta F^n_{\alpha,\beta}(a,b)=0$. On using Lemma 1.4.5 we have $F^n_{\alpha,\beta}(a,b)\gamma x\delta a\alpha b=0$ for all $a,b,x\in \mathrm{M}$ and $\alpha,\beta\in\Gamma$, which further becomes $F^n_{\alpha,\beta}(a,b)\gamma x\delta c\alpha d=0$ for all $a,b,x,c,d\in \mathrm{M}$ and $\alpha,\beta\in\Gamma$.

Again since M is prime, we get $F_{\alpha,\beta}^{n}(a,b)=0$ for all $a,b\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$, or we can say that $d_n(a\alpha b)=\sum_{i+j=n} d_i(a)\alpha d_j(b)$ for all $a,b\in\mathrm{M}$ and $\alpha\in\Gamma$, and for each $n\in\mathbb{N}$. Therefore $D=\{d_n\}_{n\in\mathbb{N}}$ becomes a higher derivation on M. □

In view of Theorems 3.3.9 and 3.3.10 one can easily conclude the following:

Corollary 3.3.11: Any Jordan triple higher derivation of a prime $\Gamma-$ring M of characteristic different from two is a higher derivation on M.

Chapter Four

Derivations On Semi-Prime $\Gamma-$ Rings

4.1 Generalized Derivations On Semi-Prime $\Gamma-$ Rings With Involution

The purpose of this section is the notions of generalized $I-$derivation and generalized reverse $I-$derivation on $\Gamma-$ rings and to prove some remarkable results involving these mappings.

Let M be a $\Gamma-$ ring with involution I. An additive mapping $D:M\to M$ is called an $I-$*derivation* if $D(a\alpha b)=D(a)\alpha I(b)+a\alpha D(b)$ for all $a,b\in M$, $\alpha\in\Gamma$ and D is called a *reverse* $I-$*derivation* if $D(a\alpha b)=D(b)\alpha I(a)+b\alpha D(a)$ for all $a,b\in M$, $\alpha\in\Gamma$. An additive mapping $T:M\to M$ is called a *left (right)* $I-$*centralizer* if $T(a\alpha b)=T(a)\alpha I(b)$ (resp. $T(a\alpha b)=I(a)\alpha T(b)$) for all $a,b\in M$, $\alpha\in\Gamma$. An additive mapping $F:M\to M$ is called a *generalized* $I-$*derivation* if $F(a\alpha b)=F(a)\alpha I(b)+a\alpha D(b)$ for all $a,b\in M$ and $\alpha\in\Gamma$, D an $I-$derivation on M. An additive mapping $F:M\to M$ is called a *generalized reverse* $I-$ *derivation* if $F(a\alpha b)=F(b)\alpha I(a)+b\alpha D(a)$ for all $a,b\in M$ and $\alpha\in\Gamma$, D a reverse $I-$derivation on M.

Theorem 4.1.1: [12] Suppose that M is a semi-prime $\Gamma-$ ring with involution I and $D:M\to M$ is an $I-$derivation. If F is a generalized $I-$derivation on M, then F maps M into $Z(M)$.

Proof : By definition of F, we have

$$F(a\alpha b)=F(a)\alpha I(b)+a\alpha D(b) \qquad (4.1.1)$$

For all $a,b\in M,\alpha\in\Gamma$. Putting $b=b\beta c$ in $(4.1.1)$, we have

$$F(a\alpha b\beta c)=F(a)\alpha I(c)\beta I(b)+a\alpha D(b\beta c)$$

$$=F(a)\alpha I(c)\beta I(b)+a\alpha D(b)\beta I(c)+a\alpha b\beta D(c) \qquad (4.1.2)$$

Also, we can write

$$F(a\alpha b\beta c)=F((a\alpha b)\beta c)=F(a)\alpha I(b)\beta I(c)+a\alpha D(b)\beta I(c)+a\alpha b\beta D(c) \qquad (4.1.3)$$

Hence, from $(4.1.2)$ and $(4.1.3)$, we obtain $F(a)\alpha[I(c),I(b)]_\beta=0$ $\qquad (4.1.4)$

For $I(b)=b$ and $I(c)=c$, $(4.1.4)$ becomes $F(a)\alpha[c,b]_\beta=0$ $\qquad (4.1.5)$

Putting $c = c\gamma F(a)$ in $(4.1.5)$, we have

$$F(a)\alpha c\gamma\left[F(a),b\right]_\beta + F(a)\alpha[c,d]_\beta \gamma F(a) = 0 \Rightarrow F(a)\alpha c\gamma\left[F(a),b\right]_\beta = 0 \qquad (4.1.6)$$

Left multiplication of $(4.1.6)$ by $b\beta$, we get $b\beta F(a)\alpha c\gamma\left[F(a),b\right]_\beta = 0 \qquad (4.1.7)$

Putting $c = b\beta c$ in $(4.1.6)$, we have $F(a)\alpha b\beta c\gamma\left[F(a),b\right]_\beta = 0 \qquad (4.1.8)$

Subtracting $(4.1.7)$ from $(4.1.8)$ and let $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in \mathrm{M}, \alpha,\beta \in \Gamma$, we obtain $\left[F(a),b\right]_\beta \alpha c\gamma\left[F(a),b\right]_\beta = 0 \qquad (4.1.9)$

For all $a,b,c \in \mathrm{M}, \alpha,\beta,\gamma \in \Gamma$. Hence, by semiprimeness of M, we have $\left[F(a),b\right]_\beta = 0$ for all $a,b \in \mathrm{M}$ and $\beta \in \Gamma$. Therefore F maps M into $Z(\mathrm{M})$. Hence the theorem is complete. □

Theorem 4.1.2: [12] Suppose that M is a semi-prime $\Gamma-$ring with involution I. If the additive mapping $T:\mathrm{M} \to \mathrm{M}$ is defined by $T(a\alpha b) = T(a)\alpha I(b)$ for all $a,b \in \mathrm{M}$ and $\alpha \in \Gamma$ then T maps M into $Z(\mathrm{M})$.

Proof: By the hypothesis, we get $T(a\alpha b) = T(a)\alpha I(b) \qquad (4.1.10)$

Putting $b = c\beta b$ in $(4.1.10)$, we have $T(a\alpha c\beta b) = T(a)\alpha I(b)\beta I(c) \qquad (4.1.11)$

Also, we can write

$$T(a\alpha c\beta b) = T((a\alpha c)\beta b) = T(a\alpha c)\beta I(b) = T(a)\alpha I(c)\beta I(b) \qquad (4.1.12)$$

Hence from $(4.1.11),(4.1.12)$ and let $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in \mathrm{M}, \alpha,\beta \in \Gamma$, we obtain $T(a)\beta\left[I(c),I(b)\right]_\alpha = 0 \qquad (4.1.13)$

The equation $(4.1.13)$ is similar to the equation $(4.1.4)$ with the exception that the left $I-$centralizer T instead of generalized $I-$derivation F. Thus the same approach, we have used after the equation $(4.1.4)$ in Theorem 4.1.1, we obtain the required result $\left[T(a),b\right]_\alpha = 0$ for all $a,b \in \mathrm{M}$ and $\alpha \in \Gamma$. Hence the theorem is proved. □

Corollary 4.1.3: Suppose that M is a prime $\Gamma-$ring with involution I and D an $I-$derivation on M. If F is a generalized $I-$derivation on M, then either $F = 0$ or M is commutative.

Proof: According to Theorem 4.1.1, we have $F(a)\beta[b,c]_\alpha=0$ for all $a,b,c\in M$ and $\alpha,\beta\in\Gamma$. Putting $b=b\gamma e$, we obtain $F(a)\beta b\gamma[e,c]_\alpha+F(a)\beta[b,c]_\alpha\gamma e=0$ for all $a,b,c,e\in M$ and $\alpha,\beta,\gamma\in\Gamma$, which implies $F(a)\beta b\gamma[e,c]_\alpha=0$. Hence by the primeness of M, we have $F(a)=0$ or $[e,c]_\alpha=0$, that is, $F=0$ or M is commutative. □

Corollary 4.1.4: Suppose that M is a semi-simple $\Gamma-$ring with involution I and D an $I-$derivation on M. If F is a generalized $I-$derivation on M, then F maps M into $Z(M)$.

Proof: Since every semi-simple $\Gamma-$ring with involution is semi-prime $\Gamma-$ring with the involution, so according to the theorem 4.1.1, the corollary is nothing to prove. □

Corollary 4.1.5: Suppose that M is a $\Gamma-$ring with involution I. If D is a nonzero $I-$derivation on M, then D maps M into $Z(M)$.

Proof: The corollary is nothing to prove if we consider $F=D$ in the proof of theorem 4.1.1. □

Theorem 4.1.6: [12] Suppose that M is a semi-prime $\Gamma-$ring with involution I and D a reverse $I-$ derivation on M. If F is a general reverse $I-$derivation on M, then $[D(a),c]_\alpha=0$ for all $a,c\in M$ and $\alpha\in\Gamma$.

Proof: By the definition of generalized $I-$derivation F on M, we have

$$F(a\alpha b)=F(b)\alpha I(a)+b\alpha D(a) \tag{4.1.14}$$

For all $a,b\in M$ and $\alpha\in\Gamma$. Replacing a by $a\beta c$ in $(4.1.14)$, we have

$$F(a\beta c\alpha b)=F(b)\alpha I(c)\beta I(a)+b\alpha D(c)\beta I(a)+b\alpha c\beta D(a) \tag{4.1.15}$$

Also, we can write

$$F(a\beta c\alpha b)=F(a\beta(c\alpha b))=F(b)\alpha I(c)\beta I(a)+b\alpha D(c)\beta I(a)+c\alpha b\beta D(a) \tag{4.1.16}$$

Comparing $(4.1.15)$, $(4.1.16)$ and let $a\alpha b\beta c=a\beta b\alpha c$ for all $a,b,c\in M,\alpha,\beta\in\Gamma$, we have $[b,c]_\alpha\beta D(a)=0$ $(4.1.17)$. putting $b=D(a)\gamma b$ in $(4.1.17)$, we have

$$D(a)\gamma[b,c]_\alpha\beta D(a)+[D(a),c]_\alpha\gamma b\beta D(a)=0 \tag{4.1.18}$$

Using $(4.1.17)$, we obtain $[D(a),c]_\alpha\gamma b\beta D(a)=0$ $\quad(4.1.19)$

Putting $b=b\alpha c$ in $(4.1.19)$, we have $[D(a),c]_\alpha\gamma b\alpha c\beta D(a)=0$ $\quad(4.1.20)$

Right multiplication of $(4.1.19)$ by αc, we have

$$[D(a),c]_\alpha \gamma b\beta D(a)\alpha c = 0 \qquad (4.1.21)$$

Subtracting $(4.1.20)$ from $(4.1.21)$ and let $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M, \alpha,\beta \in \Gamma$, we obtain

$$[D(a),c]_\alpha \beta b\gamma [D(a),c]_\alpha = 0.$$

Hence by semiprimeness of M, we have $[D(a),c]_\alpha = 0$ for all $a,c \in M$ and $\alpha \in \Gamma$, and the theorem is complete. □

Corollary 4.1.7: [12] Suppose that M is a non-commutative prime $\Gamma-$ring with involution I and D a reverse $I-$derivation on M. If F is a generalized reverse $I-$derivation on M, then F is a reverse left $I-$centralizer on M.

Proof: If we replace b by $a\gamma b$, the relation $(4.1.17)$ gives $a\gamma[b,c]_\alpha \beta D(a) + [a,c]_\alpha \gamma b\beta D(a) = 0$ and using $(4.1.17)$, the relation implies $[a,c]_\alpha \gamma b\beta D(a) = 0$ for all $a,b,c \in M$ and $\alpha,\beta,\gamma \in \Gamma$. Hence by primeness of M, either $[a,c]_\alpha = 0$ or $D(a) = 0$. If we consider, $U = \{a \in M : [a,c]_\alpha = 0 \text{ for all } c \in M, \alpha \in \Gamma\}$ and $V = \{a \in M : D(a) = 0\}$. Then clearly U and V are additive subgroups of M and $U \cup V = M$. Therefore by Brauer's trick, either $U = M$ or $V = M$. If $U = M$, then $[a,c]_\alpha = 0$ for all $a,c \in M$ and $\alpha \in \Gamma$. That is, M is commutative which gives a contradiction. On the other hand, if $V = M$, then $D(a) = 0$ for all $a \in M$. Therefore by definition of F gives $F(a\alpha b) = F(b)\alpha I(a)$ for all $a,b \in M$ and $\alpha \in \Gamma$. Hence the proof is complete. □

Corollary 4.1.8: Suppose that M is a semi-prime $\Gamma-$ring with involution I. If D is a reverse $I-$derivation on M, then D maps M into $Z(M)$.

Proof: If we consider $F = D$, Theorem 4.1.6 gives the result. □

4.2 Semi-Prime $\Gamma-$ Rings With Orthogonal Reverse Derivations

This section presents the definition of orthogonal reverse derivations; some characterizations of semi-prime $\Gamma-$ rings are obtained by using orthogonal reverse derivations. We also investigate conditions for two reverse derivations to be orthogonal.

Definition 4.2.1: [8] Let d and g be two reverse derivations on M. If

$$d(x)\Gamma M\Gamma g(y) = 0 = g(y)\Gamma M\Gamma d(x) \text{ for all } x,y \in M. \qquad (4.2.1)$$

Then d and g are said to be orthogonal.

Remark 4.2.2: [8] A non-zero reverse derivation can not be orthogonal on itself.

Example 4.2.3: Let M_1 be a Γ_1-ring and let M_2 be a Γ_2-ring. Consider $M=M_1\times M_2$ and $\Gamma=\Gamma_1\times\Gamma_2$. The addition and multiplication on M and Γ are defined as follows:

$(a,b)+(c,d)=(a+c,b+d)$, $(a,b)(\alpha,\beta)(c,d)=(a\alpha c,b\beta d)$ for every $a,b\in M_1,c,d\in M_2$ $\alpha\in\Gamma_1$ and $\beta\in\Gamma_2$.

Under these operations M is a $\Gamma-$ ring. Let d_1 be a reverse derivation on M_1. Define a derivation d on M by $d((a,b))=(d_1(a),0)$. Then d is a reverse derivation on M. Let d_2 be a reverse derivation on M_2. Define a derivation g on M by $g((a,b))=(0,d_2(b))$. Then g is a reverse derivation on M. It is clear that d and g are orthogonal reverse derivation on M.

Lemma 4.2.4: [8] Let M be a semi-prime $\Gamma-$ ring and suppose that additive mappings d and g of M into itself satisfy $d(x)\Gamma M\Gamma g(x)=0$, for all $x\in M$. Then $d(x)\Gamma M\Gamma g(y)=0$, for all $x,y\in M$.

Proof: Suppose that $d(x)\alpha m\beta g(x)=0$, for all $x,m\in M,\alpha,\beta\in\Gamma$. Replace x by $x+y$ in the above relation, we get

$$0=d(x+y)\alpha m\beta g(x+y)=(d(x)+d(y))\alpha m\beta(g(x)+g(y))$$

$$=d(x)\alpha m\beta g(x)+d(x)\alpha m\beta g(y)+d(y)\alpha m\beta g(x)+d(y)\alpha m\beta g(y)$$

$=d(x)\alpha m\beta g(y)+d(y)\alpha m\beta g(x)$. Thus $d(x)\alpha m\beta g(y)=-d(y)\alpha m\beta g(x)$.

Now

$$(d(x)\alpha m\beta g(y))\gamma n\delta(d(x)\alpha m\beta g(y))=(d(x)\alpha m\beta g(y))\gamma n\delta(-d(y)\alpha m\beta g(x))$$

$$=-(d(x)\alpha m\beta g(y)\gamma n\delta d(y)\alpha m\beta g(x))=0$$

For all $x,y,m,n\in M$ and $\alpha,\beta,\delta\in\Gamma$.

Thus $d(x)\Gamma M\Gamma g(y)=0$, for all $x,y\in M$. □

Lemma 4.2.5: [8] Let M be a 2-torsion free semi-prime $\Gamma-$ ring. Let d and g be reverse derivations of M. Then

$$d(x)\Gamma g(y)+g(x)\Gamma d(y)=0 \text{ , for all } x,y\in M . \qquad (4.2.2)$$

if and only if d and g are orthogonal.

Proof: Suppose that $d(x)\alpha g(y)+g(x)\alpha d(y)=0$, for all $x,y\in M$ and $\alpha\in\Gamma$. Consider the substituting $y=x\beta y$ in $(4.2.2)$. Then we obtain

$$0=d(x)\alpha g(x\beta y)+g(x)\alpha d(x\beta y),$$

$$0=d(x)\alpha\big(g(y)\beta x+y\beta g(x)\big)+g(x)\alpha\big(d(y)\beta x+y\beta d(x)\big),$$

$$0=\big(d(x)\alpha g(y)+g(x)\alpha d(y)\big)\beta x+d(x)\alpha y\beta g(x)+g(x)\alpha y\beta d(x).$$

Using $(4.2.2)$, we have $d(x)\alpha y\beta g(x)+g(x)\alpha y\beta d(x)=0$. Then due to Lemma $(4.2.2)$, we get $d(x)\alpha y\beta g(x)=0$, which gives the orthogonality of d and g.

Conversely, if d and g are orthogonal, we get $d(x)\alpha m\beta g(y)=g(x)\alpha m\beta d(y)=0$ for all $m\in M,\alpha,\beta\in\Gamma$. Then by using Lemma 1.4.5, we obtain $d(x)\alpha g(y)=g(x)\alpha d(y)=0$, for all $x,y\in M,\alpha\in\Gamma$. Thus $d(x)\alpha g(y)+g(x)\alpha d(y)=0$, for all $x,y\in M,\alpha\in\Gamma$ which completes the proof. □

Remark 4.2.6: Suppose that d and g are reverse derivations of a $\Gamma-$ring M. The following identities are immediate from the definition of reverse derivation.

$$(dg)(x\alpha y)=d\big(g(x\alpha y)\big)=d\big(g(y)\alpha x+y\alpha g(x)\big)=(dg)(x)\alpha y+d(x)\alpha g(y)$$

$$+g(x)\alpha d(y)+x\alpha(dg)(y) \text{ for all } x,y\in M,\alpha\in\Gamma . \qquad (4.2.3)$$

Similarly,

$$(gd)(x\alpha y)=g\big(d(x\alpha y)\big)=g\big(d(y)\alpha x+y\alpha d(x)\big)=(gd)(x)\alpha y+g(x)\alpha d(y)$$

$$+d(x)\alpha g(y)+x\alpha(gd)(y) \text{ for all } x,y\in M,\alpha\in\Gamma . \qquad (4.2.4)$$

Theorem 4.2.7: [8] Let M be a 2-torsion free semi-prime $\Gamma-$ ring. Let d and g be reverse derivations on M. Then the following conditions are equivalent:

i. d and g are orthogonal.
ii. $dg=0$.
iii. $gd=0$.
iv. $dg+gd=0$.
v. dg is a derivation.
vi. gd is a derivation.

Proof: $(ii)\Rightarrow(i)$**.** Suppose $dg=0$. Then by using the identity $(4.2.3)$, we obtain

$$d(x)\alpha g(y)+g(x)\alpha d(y)=0, \text{ for all } x,y\in M,\alpha\in\Gamma.$$

Therefore by Lemma $(4.2.5)$, d and g are orthogonal.

$(i)\Rightarrow(ii)$. Consider $d(x)\alpha y\beta g(z)=0$, for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Then

$$\begin{aligned}0=d\big(d(x)\alpha y\beta g(z)\big)&=d\big(y\beta g(z)\big)\alpha d(x)+y\beta g(z)\alpha d^{2}(x)\\&=(dg)(z)\beta y\alpha d(x)+g(z)\beta d(y)\alpha d(x)+y\beta g(z)\alpha d\big(d(x)\big)\end{aligned}$$

Owing to (i), the second and third summands are zero. Therefore we obtain $(dg)(z)\beta y\alpha d(x)=0$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Now take $x=g(z)$ and we obtain

$$(dg)(z)\beta y\alpha(dg)(z)=0, \text{ for all } z\in M \text{ and } \alpha,\beta\in\Gamma.$$

Since M is semi-prime, we get $(dg)(z)=0$, for all $z\in M$, that is $dg=0$.

The proof of the parts $(iii)\Rightarrow(i)$ and $(i)\Rightarrow(iii)$ are similar.

$(iv)\Rightarrow(i)$. If d and g are any reverse derivations, then by (ii) and (iii), $dg=0$ and $gd=0$.

Now using the equation $(4.2.3)$, we obtain,

$$\begin{aligned}(dg+gd)(x\alpha y)=(dg)(x\alpha y)+(gd)(x\alpha y)&\\=(dg)(x)\alpha y+d(x)\alpha g(y)+g(x)\alpha d(y)+x\alpha(dg)(y)&\\+(gd)(x)\alpha y+g(x)\alpha d(x)+d(x)\alpha g(y)+x\alpha(gd)(y)&\end{aligned}$$

$$= (dg+gd)(x)\alpha y + 2d(x)\alpha g(y) + 2g(x)\alpha d(y)$$

$$+x\alpha\big((dg)(y)+(gd)(y)\big) \text{ for all } x,y\in \mathrm{M}, \alpha\in\Gamma.$$

Thus, if $dg+gd=0$, then the above relation reduces to $2\big(d(x)\alpha g(y)+g(x)\alpha d(y)\big)=0$, for all $x,y\in \mathrm{M},\alpha\in\Gamma$. Since M is 2-torsion free, we get

$d(x)\alpha g(y)+g(y)\alpha d(y)=0$, for all $x,y\in \mathrm{M},\alpha\in\Gamma$. By Lemma 4.2.5, we get that d and g are orthogonal.

$(i)\Rightarrow(iv)$. From the parts (ii) and (iii), we get $dg+gd=0$.

$(v)\Rightarrow(i)$. Since dg is a derivation, we have $(dg)(x\alpha y)=(dg)(x)\alpha y+x\alpha(dg)(y)$. Comparing this expression with $(4.2.3)$, we obtain $d(x)\alpha g(y)+g(x)\alpha d(y)=0$.

The proof of $(vi)\Rightarrow(i)$ is the similar to that of $(v)\Rightarrow(i)$.

$(iii)\Rightarrow(vi)$. Obvious. This completes the proof. □

Corollary 4.2.8: Let M be a prime 2-torsion free $\Gamma-$ring. Suppose that d and g are orthogonal reverse derivations of M. Then either $d=0$ or $g=0$.

The proof is immediate from Theorem 4.2.7.

Theorem 4.2.9: [8] Let M be a 2-torsion free semi-prime $\Gamma-$ring satisfying the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z\in \mathrm{M}$ and $\alpha,\beta\in\Gamma$. Let d and g be reverse derivations on M. Then the following conditions are equivalent:

i. d and g are orthogonal.
ii. $d(x)\Gamma g(x)=0$, for all $x\in \mathrm{M}$.
iii. $g(x)\Gamma d(x)=0$, for all $x\in \mathrm{M}$.
iv. $d(x)\Gamma g(x)+g(x)\Gamma d(x)=0$, for all $x\in \mathrm{M}$.

Proof: $(ii)\Rightarrow(i)$. The linearization of $d(x+y)\alpha g(x+y)=0$ gives

$$d(x)\alpha g(y)+d(y)\alpha g(x)=0 \text{ , for all } x,y\in \mathrm{M},\alpha\in\Gamma \qquad (4.2.5)$$

take $y\beta z$ as y in $(4.2.5)$, we obtained $d(x)\alpha g(y\beta z)+d(y\beta z)\alpha g(x)=0$ for all $x,y,z\in \mathrm{M},\alpha,\beta\in\Gamma$.

$$d(x)\alpha g(z)\beta y+d(x)\alpha z\beta g(y)+d(z)\beta g(x)+z\beta d(y)\alpha g(x)=0 \qquad (4.2.6)$$

for all $x,y,z\in M,\alpha,\beta\in\Gamma$.

Since, $d(x)\alpha g(z)=-d(z)\alpha g(x)$ and $d(y)\alpha g(x)=-d(x)\alpha g(y)$ and so $(4.2.6)$ becomes $-d(z)\alpha g(x)\beta y+d(x)\alpha z\beta g(y)+d(z)\beta y\alpha g(x)-z\beta d(x)\alpha g(y)=0$ for all $x,y,z\in M,\alpha,\beta\in\Gamma$.

Now, since $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$,we get

$$d(z)\beta[y,g(x)]\alpha+[d(x),z]\alpha\beta g(y)=0 \qquad (4.2.7)$$

Replacing z by $d(x)$ in $(4.2.7)$ we obtained $d^2(x)\beta[y,g(x)]\alpha=0$ $(4.2.8)$ for all $x,y\in M$, $\alpha,\beta\in\Gamma$.

Letting $y=y\delta w$ in $(4.2.8)$, we get

$$0=d^2(x)\beta[y\delta w,g(x)]\alpha$$

$$=d^2(x)\beta y\delta[w,g(x)]\alpha+d^2(x)\beta y\delta[w,g(x)]\alpha$$

$$=d^2(x)\beta y\delta[w,g(x)]\alpha \text{ for all } x,y,w\in M,\alpha,\beta,\delta\in\Gamma.$$

Then by Lemma 4.2.4, we obtain

$$d^2(x)\beta y\delta[w,g(y)]\alpha=0 \text{ for all } x,y,w\in M,\alpha,\beta,\delta\in\Gamma \qquad (4.2.9)$$

Replacing x by $x\lambda u$ in $(4.2.9)$ we get,

$$0=d^2(x\lambda u)\beta y\delta[w,g(y)]\alpha$$

$$=\left(d^2(x)\lambda u+2d(x)\lambda d(u)+x\lambda d^2(u)\right)\beta y\delta[w,g(y)]\alpha \qquad (4.2.10)$$

For all $x,u\in M,\alpha,\beta,\delta,\lambda\in\Gamma$.

By $(4.2.9)$ the relation $(4.2.10)$ reduced to $2d(x)\lambda d(u)\beta y\delta[w,g(y)]\alpha=0$. Since M is 2-torsion free, we have

$$d(x)\lambda d(u)\beta y\delta[w,g(y)]\alpha=0 \text{ for all } x,y\in M,\alpha,\beta,\delta,\lambda\in\Gamma \qquad (4.2.11)$$

Taking $x\gamma z$ for x in $(4.2.11)$, we get

$$0 = d(x\gamma z)\lambda d(u)\beta y\delta[w, g(y)]\alpha$$

$$= d(z)\gamma x\lambda d(u)\beta y\delta[w, g(y)]\alpha + z\gamma d(x)\lambda d(u)\beta z\delta[w, g(y)]\alpha$$

and $d(z)\gamma x\lambda d(u)\beta y\delta[w, g(y)]\alpha = 0$. (by using $(4.2.11)$)

In particular, $d(z)\gamma x\lambda d(x)\beta y\delta[w, g(y)]\alpha = 0$.

The replacement $d(z) = d(x)\beta y\delta[w, g(y)]\alpha$, gives

$d(x)\beta y\delta[w, g(y)]\alpha\gamma x\lambda d(x)\beta y\delta[w, g(y)]\alpha = 0$.

Since M is semi-prime, we get $d(x)\beta y\delta[w, g(y)]\alpha = 0$. Using $(4.2.9)$ and $(4.2.11)$ we obtain by replacing $d(x)$ for w, $[d(x), g(y)]\alpha\gamma y\delta[d(x), g(y)]\alpha = 0$, for all $x, y \in \mathrm{M}, \alpha, \beta, \delta, \gamma \in \Gamma$.

Hence, $d(x)\alpha g(y) = g(y)\alpha d(x)$, for all $x, y \in \mathrm{M}, \alpha \in \Gamma$. Thus $(4.2.5)$ can be written in the form $g(y)\alpha d(x) + d(y)\alpha g(x) = 0$, for all $x, y \in \mathrm{M}, \alpha \in \Gamma$. Now use Lemma 4.2.5 to get the required relation.

$(i) \Rightarrow (iii)$. If d and g are orthogonal then we have $d(x)\Gamma\mathrm{M}\Gamma g(x) = 0$, for all $x \in \mathrm{M}$. Then we get $d(x)\alpha g(x) = 0$, for all $x \in \mathrm{M}, \alpha \in \Gamma$.

$(iii) \Rightarrow (ii)$. Take $y = x$ in $(4.2.3)$. Then we see that $(dg)(x\alpha x) = (dg)(x)\alpha x + d(x)\alpha g(x)$ $g(x)\alpha d(x) + x\alpha(dg)(x)$. Thus we obtain

$$(dg)(x\alpha x) = (dg)(x)\alpha x + x\alpha(dg)(x) \text{ for all } x \in \mathrm{M}, \alpha \in \Gamma. \qquad (4.2.12)$$

Equation $(4.2.12)$ implies that dg is a Jordan derivation. We know that if M is semi-prime $\Gamma-$ring, then every Jordan derivation is a derivation.

$(iii) \Rightarrow (ii)$. This follows from Lemma 4.2.5. □

Corollary 4.2.10: [8] Let M be a 2-torsion free semi-prime $\Gamma-$ring and let d be a reverse derivation on M. If d^2 is also a derivation, then $d = 0$.

The proof follows from part (ii) of Theorem 4.2.9.

Theorem 4.2.11: [8] Let M be a 2-torsion free semi-prime $\Gamma-$ring. Let d and g be a reverse derivation on M. Then the following conditions are equivalent:

i. d and g are orthogonal.

ii. There exist ideals K_1 and K_2 of M such that:

a) $K_1 \cap K_2 = 0$ and $K = K_1 \oplus K_2$ is a non-zero ideal of M.

b) d maps M into K_1 and g maps M into K_2.

c) The restriction of d to $K = K_1 \oplus K_2$ is a direct sum $d_1 \oplus 0_2$, where $d_1 : K_1 \to K_1$ is a reverse derivation of K_1 and $0_2 : K_2 \to K_2$ is zero. If $d_1 = 0$ then $d = 0$.

d) The restriction of g to $K = K_1 \oplus K_2$ is a direct sum $0_1 \oplus g_2$, where $g_2 : K_2 \to K_2$ is a reverse derivation of K_2 and $0_1 : K_1 \to K_1$ is zero. If $g_2 = 0$ then $g = 0$.

Proof: $(ii) \Rightarrow (i)$**.** Obvious.

$(i) \Rightarrow (ii)$. Let K_1 be an ideal of M generated by all $d(x), x \in M$, and let K_2 be $Ann(K_1)$, the annihilator of K_1. From equation $(4.2.1)$ we see that $g(x) \in K_2$, for all $x \in M$. Whenever K_1 is an ideal in a semi-prime $\Gamma-$ring, we have $K_1 \cap K_2 = 0$ and $K = K_1 \oplus K_2$ is a non-zero ideal. Thus a) and b) are proved.

Our next goal is to show that d is zero on K_2. Take $k_2 \in K_2$. Then $k_1 \alpha k_2 = 0$, for all $k_1 \in K_1, \alpha \in \Gamma$. Hence $0 = d(k_1 \alpha k_2) = d(k_2)\alpha k_1 + k_2 \alpha d(k_1)$. It is obvious from the definition of K that d leaves K_1 invariant and hence $k_2 \alpha d(k_1) = 0$. Then the above relation reduces to $d(k_2)\alpha k_1 = 0$. Since in a semi-prime $\Gamma-$ring the left, right and two-sided annihilators of an ideal coincide, we then have $d(k_2) \in Ann(K_1) = K_2$. But on the other hand $d(K_2)$ belongs to the set of generating elements of K_1. Thus $d(k_1) \in K_1 \cap K_2 = 0$, which means that d is zero on K_2. As we have mentioned above d leaves K_1 invariant. Therefore we may define a mapping $d_1 : K_1 \to K_1$ as a restriction of d to K_1.

Suppose that $d_1 = 0$. Then d is zero on $K = K_1 \oplus K_2$. Take $k \in K$ and $y \in M$, we have $d(y\alpha k) = d(k)\alpha y + k\alpha d(y)$. But $d(y\alpha k) = d(k) = 0$ since $k\alpha y, k \in K, \alpha \in \Gamma$. Consequently $k\alpha d(y) = 0$, for all $y \in M, \alpha \in \Gamma$. Thus $d(y) \in Ann(K)$. But ideal K is a non-zero and therefore $Ann(K) = 0$. Hence $d(y) = 0$, for all $y \in M$. Then c) is thereby proved.

It remains to prove d). First we show that g is zero on K_1. Take $x, y, z \in M, \alpha, \beta \in \Gamma$ and set $k_1 = z\alpha d(y)\beta x$. Then

$$g(k_1) = g(x)\beta(z\alpha d(y)) + x\beta g(z\alpha d(y))$$

$$= g(x)\beta z\alpha d(y) + x\beta(gd)(y)\alpha z + x\beta d(y)\alpha g(z).$$

Since d and g are orthogonal we have $g(x)\alpha z\beta d(y)=0, d(y)\alpha g(z)=0$ and $gd=0$. Hence $g(k_1)=0$. In a similar fashion we see that $g(z\alpha d(y))=0, g(d(y)\alpha x)=0$ and $g(d(y))=0$. Then h is zero on K_1. Recall that g maps M into K_2. In particular, it leaves K_2 invariant. Thus we may define $g_2: K_2 \to K_2$ as a restriction of g to K_2. The proof that $g_2=0$ implies $g=0$ is the same as the proof that $d_1=0$ implies $d=0$. This completes the proof. □

Corollary 4.2.12: Let M be a 2-torsion free semi-prime $\Gamma-$ring and let d be a reverse derivation of M. If $d(x)\alpha d(x)=0$ for all $x\in \mathrm{M}, \alpha\in\Gamma$, then $d=0$.

If $d^2=g^2$ or if $d(x)\alpha d(x)=g(x)\alpha g(x)$, for every $x\in \mathrm{M}, \alpha\in\Gamma$, then we obtain the relation between the reverse derivations d and g of a $\Gamma-$ring.

Theorem 4.2.13: [19] Let M be a 2-torsion free semi-prime $\Gamma-$ring. Let d and g be reverse derivations on M. Suppose that $d^2=g^2$, then $d+g$ and $d-g$ are orthogonal. Thus, there exist ideals K_1 and K_2 of M such that $K=K_1\oplus K_2$ is a non-zero ideal which is direct sum in M, $d=g$ on K_1 and $d=-g$ on K_2.

Proof: From $d^2=g^2$ it follows immediately that $(d+g)(d-g)+(d-g)(d+g)=0$. Hence $d+g$ and $d-g$ are orthogonal by the part (iv) of Theorem 4.2.7. Another part of Theorem 4.2.13, follows from (ii) of Theorem 4.2.11. □

From Theorem 4.2.13 we get the following

Corollary 4.2.14: Let M be a prime 2-torsion free $\Gamma-$ring. Let d and g be derivations of M. If $d^2=g^2$, then either $d=-g$ or $d=g$.

Theorem 4.2.15: [8] Let M be a 2-torsion free semi-prime $\Gamma-$ring. Let d and g be reverse derivations of M. If $d(x)\alpha d(x)=g(x)\alpha g(x)$, for all $x\in \mathrm{M}, \alpha\in\Gamma$, then $d+g$ and $d-g$ are orthogonal. Thus, there exist ideals K_1 and K_2 of M such that $K=K_1\oplus K_2$ is an essential direct sum in M, $d=g$ on K_1 and $d=-g$ on K_2.

Proof: Note that $(d+g)(x)(d-g)(x)+(d-g)(x)(d+g)(x)=0$, for all $x\in \mathrm{M}, \alpha\in\Gamma$. Now applying parts (ii) and (iii) of Theorem 4.2.9, we obtain the required result. □

Corollary 4.2.16: Let M be a prime 2-torsion free $\Gamma-$ring. Let d and g be reverse derivations of M. If $d(x)\alpha d(x)=g(x)\alpha g(x)$, for all $x\in \mathrm{M}, \alpha\in\Gamma$, then either $d=g$ or $d=-g$.

The proof is immediate from Theorem 4.2.15. □

4.3 Orthogonal $(\sigma,\tau)-$ Derivations On Semi-Prime $\Gamma-$ Rings

The objective of this section is to extend the existing notions of derivations and generalized derivations in semi-prime $\Gamma-$ring.

Definition 4.3.1: [1] Let σ and τ be endomorphisms of M. Motivated by the concepts of $(\sigma,\tau)-$ derivation and generalized $(\sigma,\tau)-$ derivation in rings, the notions of $(\sigma,\tau)-$ derivation and generalized $(\sigma,\tau)-$ derivation in $\Gamma-$ rings are defined as follows:

An additive mapping $d:M\to M$ is called a $(\sigma,\tau)-$ *derivation* if $d(x\alpha y)=d(x)\alpha\sigma(y)+\tau(x)\alpha d(y)$ holds for all $x,y\in M$ and $\alpha\in\Gamma$. An additive map F of M is a *generalized* $(\sigma,\tau)-$ *derivation* if there exists a $(\sigma,\tau)-$ derivation d of M such that $F(x\alpha y)=F(x)\alpha\sigma(y)+\tau(x)\alpha d(y)$ holds for all $x,y\in M$ and $\alpha\in\Gamma$.

Remarks 4.3.2:

1. The notion of generalized $(\sigma,\tau)-$ derivation includes those of $(\sigma,\tau)-$ derivation when $F=d$, of derivation when $F=d$, and $\sigma=\tau=I_M$, the identity map on M, and of generalized derivation, which is the case when $\sigma=\tau=I_M$. Note that, a generalized $(I_M,I_M)-$derivation is just a generalized derivation.
2. Every generalized derivation is a generalized $(\sigma,\tau)-$ derivation with $\sigma=\tau=I_M$, the identity map on M, but the converse need not be true in general. The following example shows that the notion of a generalized $(\sigma,\tau)-$ derivation in fact generalizes that of a generalized derivation.

Example 4.3.3: Let R be any ring, and let $M=\left\{\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}: a,b,c,x,y,z\in R\right\}$, $\Gamma=\left\{\begin{pmatrix} l & 0 & m \\ 0 & 0 & 0 \end{pmatrix}: l,m\in R\right\}$.

Then M is a $\Gamma-$ ring. Further, the mappings $\sigma,\tau:M\to M$ defined by

$\sigma\left(\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\right)=\begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix}$, $\tau\left(\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\right)=\begin{pmatrix} a & 0 \\ 0 & 0 \\ c & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\in M$ are endomorphisms of M. Next, define the map $d:M\to M$ such that $d\left(\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\right)=\begin{pmatrix} 0 & 0 \\ b & 0 \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\in M$. Clearly, d is a $(\sigma,\tau)-$derivation but not a derivation on M. Moreover, consider the map $F:M\to M$ defined as $F\left(\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\right)=\begin{pmatrix} a & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & x \\ b & y \\ c & z \end{pmatrix}\in M$.

Then F is a generalized $(\sigma,\tau)-$derivation on M induced by d. However, F is not a generalized derivation on M.

Note: From now and until the end of this section, M is always a 2-torsion free semi-prime $\Gamma-$ring while σ and τ are automorphisms of M.

Lemma 4.3.4: [1] Let M be a 2-torsion free semi-prime $\Gamma-$ring, and d,g be $(\sigma,\tau)-$derivation of M. Then d and g are orthogonal if and only if $d(x)\alpha g(y)+g(x)\alpha d(y)=0$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.

The proof is immediate from Lemma 4.2.5. □

Theorem 4.3.5: [1] Let M be a 2-torsion free semi-prime $\Gamma-$ring, such that $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$. Further, suppose d and g are $(\sigma,\tau)-$derivation of M such that $d\sigma=\sigma d,\ d\tau=\tau d$. Then d and g are orthogonal if and only if $d(x)\alpha g(x)=0$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$.

Proof: Suppose that $d(x)\alpha g(x)=0$ for all $x\in\mathrm{M}$ and $\alpha\in\Gamma$. Linearizing this relation, we get

$$d(x)\alpha g(y)+d(y)\alpha g(x)=0 \text{ for all } x,y\in\mathrm{M} \text{ and } \alpha\in\Gamma. \qquad (4.3.1)$$

Replacing y by $y\beta z$ in $(4.3.1)$, we get

$$0=d(x)\alpha g(y\beta z)+d(y\beta z)\alpha g(x)$$

$$=d(x)\alpha g(y)\beta\sigma(z)+d(x)\alpha\tau(y)\beta g(z)+d(y)\beta\sigma(z)\alpha g(x)+\tau(y)\beta d(z)\alpha g(x)$$

In view of $(4.3.1)$, we have $d(x)\alpha g(y)=-d(y)\alpha g(x)$ and $d(z)\alpha g(x)=-d(x)\alpha g(z)$, and hence the above expression reduces to

$$d(y)\beta[\sigma(z),g(x)]_\alpha=[\tau(y),d(x)]_\alpha\beta g(z) \text{ for all } x,y,z\in\mathrm{M} \text{ and } \alpha,\beta\in\Gamma. \quad (4.3.2)$$

Replacing y by $\tau^{-1}(d(x))$ in $(4.3.2)$, we obtain

$$d(\tau^{-1}(d(x)))\beta[\sigma(z),g(x)]_\alpha=0 \text{ for all } x,z\in\mathrm{M} \text{ and } \alpha,\beta\in\Gamma.$$

This implies that

$$\tau^{-1}(d^2(x))\beta[z_1,g(x)]_\alpha=0 \text{ for all } x,z_1\in\mathrm{M} \text{ and } \alpha,\beta\in\Gamma. \qquad (4.3.3)$$

Replacing z_1 by $z\gamma s$ in $(4.3.3)$ and using Lemma 1.4.5 and relation $(4.3.3)$, we obtain

$$\tau^{-1}\left(d^2(x)\right)\beta z\gamma\left[s,g(y)\right]_\alpha = 0 \text{ for all } s,x,y,z \in \mathrm{M} \text{ and } \alpha,\beta,\gamma \in \Gamma. \qquad (4.3.4)$$

Replacing x by $x\delta u$ in $(4.3.4)$ and using it, we get

$$2\left(d(x)\delta d\left(\tau^{-1}\left(\sigma(u)\right)\right)\beta z\gamma\left[s,g(y)\right]_\alpha\right) = 0 \ \forall s,u,x,y,z \in \mathrm{M},\ \alpha,\beta,\gamma,\delta \in \Gamma.$$

Putting $u = \sigma^{-1}\left(\tau(u)\right)$ in above and using the fact that M is 2-torsion free, we find that

$$d(x)\delta d(u)\beta z\gamma\left[s,g(y)\right]_\alpha = 0,\ \forall s,u,x,y,z \in \mathrm{M},\ \alpha,\beta,\gamma,\delta \in \Gamma. \qquad (4.3.5)$$

Substituting $x\alpha_1 t$ for x in $(4.3.5)$ and using it, we find that

$$d(x)\alpha_1\sigma(t)\delta d(u)\beta z\gamma\left[s,g(y)\right]_\alpha = 0,\ \forall s,t,u,x,y,z \in \mathrm{M},\ \alpha_1,\alpha,\beta,\gamma,\delta \in \Gamma.$$

The above expression yields that

$$d(x)\beta z\gamma\left[s,g(y)\right]_\alpha \alpha_1 \mathrm{M}\delta d(x)\beta z\gamma\left[s,g(y)\right]_\alpha = 0,\ \forall s,x,y,z \in \mathrm{M},\ \alpha_1,\alpha,\beta,\gamma,\delta \in \Gamma.$$

Semiprimeness of M implies that

$$d(x)\beta z\gamma\left[s,g(y)\right]_\alpha = 0,\ \forall s,x,y,z \in \mathrm{M},\ \alpha,\beta,\gamma \in \Gamma, \text{ and hence}$$

$$d(x)\alpha z\gamma\left[d(x),g(y)\right]_\alpha = 0,\ \forall x,y,z \in \mathrm{M},\ \alpha,\beta \in \Gamma. \qquad (4.3.6)$$

Replacing z by $g(y)\beta z$, we get

$$d(x)\alpha g(y)\beta z\gamma\left[d(x),g(y)\right]_\alpha = 0,\ \forall x,y,z \in \mathrm{M},\ \alpha,\beta,\gamma \in \Gamma. \qquad (4.3.7)$$

Also, from $(4.3.6)$, we have

$$g(y)\alpha d(x)\beta z\gamma\left[d(x),g(y)\right]_\alpha = 0,\ \forall x,y,z \in \mathrm{M},\ \alpha,\beta,\gamma \in \Gamma. \qquad (4.3.8)$$

Subtracting $(4.3.8)$ from $(4.3.7)$, we get

$$\left[d(x),g(y)\right]_\alpha \beta \,\mathrm{M}\, \gamma\left[d(x),g(y)\right]_\alpha = 0,\ \forall x,y,z \in \mathrm{M},\ \alpha,\beta,\gamma \in \Gamma.$$

Semiprimeness of M yields that $\left[d(x),g(y)\right]_\alpha = 0,\ \forall x,y \in \mathrm{M},\ \alpha \in \Gamma$. That is, $d(x)\alpha g(y) = g(y)\alpha d(x)$ for all $x,y \in \mathrm{M}$ and $\alpha \in \Gamma$. Thus, $(4.3.1)$ can be written as $d(x)\alpha g(y) + g(x)\alpha d(y) = 0$ for all $x,y \in \mathrm{M}$ and $\alpha \in \Gamma$. By Lemma 4.3.4, d and g are orthogonal.

Conversely, suppose that d and g are orthogonal. Then $d(x)\beta \mathrm{M} \gamma g(x)=0$ for all $x \in \mathrm{M}$ and $\beta,\gamma \in \Gamma$. Therefore, $d(x)\alpha g(x)=0$ for all $x \in \mathrm{M}$ and $\alpha \in \Gamma$ by Lemma 1.4.8 .□

Theorem 4.3.6: [1] Let M be a 2-torsion free semi-prime $\Gamma-$ring. Suppose d and g are $(\sigma,\tau)-$ derivations of M such that $d\sigma=\sigma d,\ g\sigma=\sigma g,\ d\tau=\tau d,\ g\tau=\tau g$. Then the following conditions are equivalent:

i. d and g are orthogonal.
ii. $dg=0$.
iii. $gd=0$.
iv. $dg+gd=0$.
v. dg is a $(\sigma^2,\tau^2)-$derivation of M.

Note: The following example shows that the hypothesis of semiprimeness in Theorem 4.3.6 is essential.

Example 4.3.7: Let R be any 2-torsion free ring and let $\mathrm{M}=\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a,b,c \in R\right\}$, $\Gamma=\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,y \in R\right\}$. Then M is a 2-torsion free $\Gamma-$ring. It can be easily seen that M is not semi-prime. Take $\sigma=\tau=I_{\mathrm{M}}$, where I_{M} is the identity map on M. Define the maps $d,g:\mathrm{M}\to\mathrm{M}$ such that

$$d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)=\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},\ g\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)=\begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in \mathrm{M}.$$

Then it is straightforward to check that d and g are $(\sigma,\tau)-$derivations on M. Also, d and g are orthogonal, and dg is a $(\sigma^2,\tau^2)-$derivation on M. However, $dg\neq 0,\ gd\neq 0$ and $dg+gd\neq 0$.

Remark 4.3.8: Two generalized derivations (F,d) and (G,g) of M are called orthogonal If $F(x)\Gamma\mathrm{M}\Gamma G(y)=\{0\}=G(y)\Gamma\mathrm{M}\Gamma F(x)$ holds for all $x,y\in\mathrm{M}$.

Lemma 4.3.9: [1] Suppose that two generalized $(\sigma,\tau)-$derivations (F,d) and (G,g) of M are orthogonal. Then following relations hold:

i. $F(x)\alpha G(y)=G(x)\alpha F(y)=0$, and hence $F(x)\alpha G(y)+G(x)\alpha F(y)=0$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.
ii. d and G are orthogonal and $d(x)\alpha G(y)=G(y)\alpha d(x)=0$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.
iii. g and F are orthogonal and $g(x)\alpha F(y)=F(y)\alpha g(x)=0$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.
iv. d and g are orthogonal.
v. If $F\sigma=\sigma F,\ F\tau=\tau F,\ G\sigma=\sigma G,\ G\tau=\tau G$ and $d\sigma=\sigma d,\ d\tau=\tau d,\ g\sigma=\sigma g,\ g\tau=\tau g,$ then $dG=Gd=0,\ gF=Fg=0$ and $FG=GF=0$.

Proof: (i). By the hypothesis, we have $F(x)\alpha z\beta G(y)=0$ for all $x,y,z\in \mathrm{M}$ and $\alpha,\beta\in\Gamma$. Application of Lemma 1.4.8 yields that $F(x)\gamma G(y)=0=G(y)\gamma F(x)$. Therefore, $F(x)\gamma G(y)+G(y)\gamma F(x)=0$ for all $x,y\in\mathrm{M}$ and $\gamma\in\Gamma$.

(ii). By (i), we have $F(x)\alpha G(y)=0$ and $F(x)\beta z\gamma G(y)=0$ for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta,\gamma\in\Gamma$. Hence

$0=F(z\beta x)\alpha G(y)=F(z)\beta\sigma(x)\alpha G(y)+\tau(z)\beta d(x)\alpha G(y)=\tau(z)\beta d(x)\alpha G(y)$. Since τ is an automorphism of M, the last expression yields that

$$d(x)\alpha G(y)\gamma\mathrm{M}\beta d(x)\alpha G(y)=\{0\} \text{ for all } x,y\in\mathrm{M} \text{ and } \alpha,\beta,\gamma\in\Gamma.$$

Thus, the semiprimeness of M forces that

$$d(x)\alpha G(y)=0 \text{ for all } x,y\in\mathrm{M} \text{ and } \alpha\in\Gamma. \tag{4.3.9}$$

Replacing x by $x\beta s$ in $(4.3.9)$, we get

$$0=d(x\beta s)\alpha G(y)=d(x)\beta\sigma(s)\alpha G(y)+\tau(x)\beta d(s)\alpha G(y).$$

Using $(4.3.9)$ and fact that σ is an automorphism of M, we obtain

$$d(x)\Gamma\mathrm{M}\Gamma G(y)=\{0\} \text{ for all } x,y\in\mathrm{M}.$$

Application of Lemma 1.4.8 yields that d and G are orthogonal, and hence $d(x)\alpha G(y)=G(y)\alpha d(x)=0$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$.

(iii). Using similar approach as we have used in (ii).

(iv). By the assumption, we have $F(x)\alpha G(y)=0$ for all $x,y\in\mathrm{M}$ and $\alpha\in\Gamma$. This implies that $0=F(x\beta z)\alpha G(y\gamma w)=\left(F(x)\beta\sigma(z)+\tau(x)\beta d(z)\right)\alpha\left(G(y)\gamma\sigma(w)+\tau(y)\gamma g(w)\right)$

$=F(x)\beta\sigma(z)\alpha G(y)\gamma\sigma(w)+F(x)\beta\sigma(z)\alpha\tau(y)\gamma g(w)+\tau(x)\beta d(z)\alpha G(y)\gamma\sigma(w)$

$+\tau(x)\beta d(z)\alpha\tau(y)\gamma g(w)$.

Using (ii) and (iii), we find that

$$\tau(x)\beta d(z)\alpha\tau(y)\gamma g(w)=0 \text{ for all } w,x,y,z\in\mathrm{M} \text{ and } \alpha,\beta,\gamma\in\Gamma.$$

Since τ is an automorphism of M, so the last expression yields that

$$d(z)\alpha\mathrm{M}\gamma g(w)\delta\mathrm{M}\beta d(z)\alpha\mathrm{M}\gamma g(w)=\{0\} \text{ for all } w,z\in\mathrm{M} \text{ and } \alpha,\beta,\gamma,\delta\in\Gamma.$$

The semiprimeness of M forces that

$$d(z)\alpha\mathrm{M}\gamma g(w)=\{0\} \text{ for all } w,z\in\mathrm{M} \text{ and } \alpha,\gamma\in\Gamma.$$

Hence by Lemma 1.4.8, d and g are orthogonal.

(v). In view of (ii) d and g are orthogonal. Hence, $0=G\big(d(x)\alpha z\beta G(y)\big)$

$$=Gd(x)\alpha\sigma(z)\beta\sigma\big(G(y)\big)+\tau\big(d(x)\big)\alpha g(z)\beta\sigma\big(G(y)\big)+\tau\big(d(x)\big)\alpha\tau(z)\beta g\big(G(y)\big).$$

Since $d\tau=\tau d$, $G\sigma=\sigma G$ and d,g are orthogonal, so we obtain

$$Gd(x)\alpha z_1\beta G(y_1)=0 \text{ for all } x,y_1,z_1\in\mathrm{M} \text{ and } \alpha,\beta\in\Gamma. \qquad (4.3.10)$$

Replacing y_1 by $d(x)$ in $(4.3.10)$ and using the semiprimeness of M, we get $Gd=0$. Similarly, since each of the equalities $d\big(G(x)\alpha z\beta d(y)\big)=0$, $F\big(g(x)\alpha z\beta F(y)\big)=0$, $g\big(F(x)\alpha z\beta g(y)\big)=0$, $F\big(G(x)\alpha z\beta F(y)\big)=0$, and $G\big(F(x)\alpha z\beta G(y)\big)=0$ hold for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$, we concloud that $dG=Fg=gF=FG=GF=0$, respectively. □

In view of Theorem 4.3.6(ii) and Lemma 4.3.9, we have the following corollary:

Corollary 4.3.10: [1] Let (F,d) and (G,g) be orthogonal generalized $(\sigma,\tau)-$ derivations of M such that $F\sigma=\sigma F,\ F\tau=\tau F,\ G\sigma=\sigma G,\ G\tau=\tau G$ and $d\sigma=\sigma d,\ d\tau=\tau d,\ g\sigma=\sigma g,\ g\tau=\tau g$, then dg is a $(\sigma^2,\tau^2)-$ derivations of M and $(FG,dg)=(0,0)$ is a generalized $(\sigma^2,\tau^2)-$ derivations of M.

Theorem 4.3.11: [1] Suppose (F,d) and (G,g) are generalized $(\sigma,\tau)-$ derivations of M such that $F\sigma=\sigma F,\ F\tau=\tau F,\ G\sigma=\sigma G,\ G\tau=\tau G$ and $d\sigma=\sigma d,\ d\tau=\tau d,\ g\sigma=\sigma g,\ g\tau=\tau g$. Then (F,d) and (G,g) are orthogonal if and only if one of the following holds:

i. (a) $F(x)\gamma G(y)+G(x)\gamma F(x)=0,\ \forall x,y\in\mathrm{M},\gamma\in\Gamma$.
   (b) $d(x)\gamma G(y)+g(x)\gamma F(y)=0,\ \forall x,y\in\mathrm{M},\gamma\in\Gamma$.

ii. $F(x)\gamma G(y)=d(x)\gamma G(y)=0,\ \forall x,y\in\mathrm{M},\gamma\in\Gamma$.

iii. $F(x)\gamma G(y)=0,\ \forall x,y\in\mathrm{M},\gamma\in\Gamma$ and $dG=dg=0$.

iv. (FG,dg) is a generalized $(\sigma^2,\tau^2)-$ derivation and $F(x)\gamma G(y)=0,\forall x,y\in\mathrm{M},\gamma\in\Gamma$.

Proof: In view of Lemma 4.3.9, Corollary 4.3.10 and the orthogonality of (F,d) and $(G,g) \Rightarrow$ $(i),(ii),(iii)$ and (iv). Now, we establish

$(i) \Rightarrow$ "(F,d) and (G,g) are orthogonal." By the hypothesis, we have

$$F(x)\gamma G(y)+G(x)\gamma F(y)=0 \text{ for all } x,y\in M \text{ and } \gamma\in\Gamma. \qquad (4.3.11)$$

Replacing x by $x\alpha z$ in $(4.3.11)$, we find that

$$0=F(x\alpha z)\gamma G(y)+G(x\alpha z)\gamma F(y)$$
$$=F(x)\alpha\sigma(z)\gamma G(y)+\tau(x)\alpha d(z)\gamma G(y)+G(x)\alpha\sigma(z)\gamma F(y)+\tau(x)\alpha g(z)\gamma F(y)$$

Using (b) in last expression, we get

$$F(x)\alpha\sigma(z)\gamma G(y)+G(x)\alpha\sigma(z)\gamma F(y)=0 \text{ for all } x,y,z\in M \text{ and } \alpha,\gamma\in\Gamma. \qquad (4.3.12)$$

Since σ is an automorphism of M, the relation $(4.3.12)$ can be rewritten as

$$F(x)\alpha z_1\gamma G(x)+G(x)\alpha z_1\gamma F(x)=0 \text{ for all } x,z_1\in M \text{ and } \alpha,\gamma\in\Gamma.$$

By Lemma 1.4.8, we conclude that $F(x)\alpha z_1\gamma G(x)=0$ and $G(x)\alpha z_1\gamma F(x)=0$ for all $x,z_1\in M$ and $\alpha,\gamma\in\Gamma$. Using Lemma 1.4.5, we have $F(x)\alpha z_1\gamma G(y)=0$ for all $x,y,z_1\in M$ and $\alpha,\gamma\in\Gamma$. Therefore, F and G are orthogonal.

$(ii) \Rightarrow$ "(F,d) and (G,g) are orthogonal." Given that $F(x)\gamma G(y)=0$. Putting $x\alpha z$ for x, we get

$$0=F(x\alpha z)\gamma G(y)$$
$$=F(x)\alpha\sigma(z)\gamma G(y)+\tau(x)\alpha d(z)\gamma G(y)$$
$$=F(x)\alpha\sigma(z)\gamma G(y)$$

Using Lemma 1.4.8 and the fact that σ is an automorphism of M, we conclude that (F,d) and (G,g) are orthogonal.

$(iii) \Rightarrow$ "(F,d) and (G,g) are orthogonal." By the assumption, we have

$$0=dG(x\alpha y)$$

$$= d\left(G(x)\alpha\sigma(y) + \tau(x)\alpha g(y)\right)$$

$$= dG(x)\alpha\sigma^2(y) + \tau\left(G(x)\right)\alpha d\left(\sigma(y)\right) + d\left(\tau(x)\right)\alpha\sigma\left(g(y)\right) + \tau^2(x)\alpha dg(y)$$

$$= \tau\left(G(x)\right)\alpha d\left(\sigma(y)\right) + d\left(\tau(x)\right)\alpha\sigma\left(g(y)\right).$$

Since $G\tau = \tau G,\ g\sigma = \sigma g$ and σ, τ are automorphisms of M, we have

$$G(x_1)\alpha d(y_1) + d(x_1)\alpha g(y_1) = 0 \text{ for all } x_1, y_1 \in \mathrm{M}, \alpha \in \Gamma.$$

Application of Theorem 4.3.6(iv) and Lemma 1.4.8 yields that

$$G(x_1)\alpha d(y_1) = 0 \text{ for all } x_1, y_1 \in \mathrm{M}, \alpha \in \Gamma.$$

Replacing x_1 by $x\beta z$ and using Theorem 4.3.6(iv) and Lemma 1.4.8, we obtain

$$G(x)\beta\sigma(z)\alpha d(y_1) = 0 \text{ for all } x, y_1, z \in \mathrm{M} \text{ and } \alpha, \beta \in \Gamma.$$

By Lemma 1.4.8, we have $d(y_1)\gamma G(x) = 0$ for all $x, y_1 \in \mathrm{M}$ and $\gamma \in \Gamma$, which satisfies (ii). Therefore, (iii) implies that (F, d) and (G, g) are orthogonal.

$(iv) \Rightarrow$ "(F, d) and (G, g) are orthogonal." Since (FG, dg) is a generalized $(\sigma^2, \tau^2)-$derivation and dg is a $(\sigma^2, \tau^2)-$derivation, we have

$$FG(x\gamma y) = FG(x)\gamma\sigma^2(y) + \tau^2(x)\gamma dg(y) \text{ for all } x, y \in \mathrm{M} \text{ and } \gamma \in \Gamma. \quad (4.3.13)$$

Also

$$FG(x\gamma y) = FG(x)\gamma\sigma^2(y) + \tau\left(G(x)\right)\gamma d\left(\sigma(y)\right) + F\left(\tau(x)\right)\gamma\sigma\left(g(y)\right) + \tau^2(x)\gamma dg(y) \quad (4.3.14)$$

Comparing $(4.3.13)$ and $(4.3.14)$, we get

$$\tau\left(G(x)\right)\gamma d\left(\sigma(y)\right) + F\left(\tau(x)\right)\gamma\sigma\left(g(y)\right) = 0 \text{ for all } x, y \in \mathrm{M} \text{ and } \gamma \in \Gamma.$$

Since σ, τ are automorphisms of M and noting that $G\tau = \tau G, g\sigma = \sigma g$, we have

$$G(x_1)\gamma d(y_1) + F(x_1)\gamma g(y_1) = 0 \text{ for all } x_1, y_1 \in \mathrm{M} \text{ and } \gamma \in \Gamma. \quad (4.3.15)$$

Since, $F(x_1)\gamma G(y_1) = 0$, we get

$$0 = F(x_1)\gamma G(y_1\alpha z_1)$$

$$=F(x_1)\gamma G(y_1)\alpha\sigma(z_1)+F(x_1)\gamma\tau(y_1)\alpha g(z_1)=F(x_1)\gamma\tau(y_1)\alpha g(z_1)$$

By Lemma 1.4.8, we have $g(z_1)\gamma F(x_1)=0$ for all $x_1,z_1\in M$ and $\gamma\in\Gamma$. Replace z_1 by $y_1\beta z_1$ to get

$$0=g(y_1\beta z_1)\gamma F(x_1)$$

$$=g(y_1)\beta\sigma(z_1)\gamma F(x_1)+\tau(y_1)\beta g(z_1)\gamma F(x_1)$$

$$=g(y_1)\beta\sigma(z_1)\gamma F(x_1)$$

Since σ is an automorphism of M and using Lemma 1.4.8, we find that $F(x_1)\gamma g(y_1)=0$ for all $x_1,y_1\in M$ and $\gamma\in\Gamma$. Now from $(4.3.15)$, we get $G(x_1)\gamma d(y_1)=0$ for all $x_1,y_1\in M$ and $\gamma\in\Gamma$. Putting $z_1\alpha y_1$ for y_1 in the last relation, we get

$$0=G(x_1)\gamma d(z_1\alpha y_1)$$

$$=G(x_1)\gamma d(z_1)\alpha\sigma(y_1)+G(x_1)\gamma\tau(z_1)\alpha d(y_1)$$

$$=G(x_1)\gamma\tau(z_1)\alpha d(y_1).$$

Since τ is an automorphism of M, the above expression forces that $G(x_1)\gamma z_2\alpha d(y_1)=0$ for all $x_1,y_1,z_2\in M$ and $\alpha,\gamma\in\Gamma$. Again using Lemma 1.4.8, we obtain $d(y_1)\gamma G(x_1)=0$ for all $x_1,y_1\in M$ and $\gamma\in\Gamma$. By (ii), (F,d) and (G,g) are orthogonal. □

Theorem 4.3.12: [1] Let (F,d) and (G,g) be generalized $(\sigma,\tau)-$derivations of M such that $d\sigma=\sigma d,\ d\tau=\tau d,\ g\sigma=\sigma g,\ g\tau=\tau g$. Then the following conditions are equivalent:

i. (FG,dg) is a generalized $(\sigma^2,\tau^2)-$derivation.
ii. (GF,gd) is a generalized $(\sigma^2,\tau^2)-$derivation.
iii. F and g are orthogonal, also G and d are orthogonal.

Proof: $(i)\Rightarrow(iii)$. Suppose (FG,dg) is a generalized $(\sigma^2,\tau^2)-$derivation. We have $G(x)\gamma d(y)+F(x)\gamma g(y)=0$ for all $x,y\in M$ and $\gamma\in\Gamma$. Replacing y by $y\beta z$, we obtain

$$0=G(x)\gamma d(y\beta z)+F(x)\gamma g(y\beta z)$$

$$=G(x)\gamma d(y)\beta\sigma(z)+G(x)\gamma\tau(y)\beta d(z)+F(x)\gamma g(y)\beta\sigma(z)+F(x)\gamma\tau(y)\beta g(z)$$

$$=G(x)\gamma\tau(y)\beta d(z)+F(x)\gamma\tau(y)\beta g(z).$$

Since τ is an automorphism of M, the above relation yields that

$G(x)\gamma y_1\beta d(z)+F(x)\gamma y_1\beta g(z)=0$ for all $x, y_1, z\in\mathrm{M}$ and $\beta,\gamma\in\Gamma$. $(4.3.16)$

Since dg is a $(\sigma^2,\tau^2)-$derivation, so d and g are orthogonal by Theorem 4.3.6. replacing y_1 by $g(z)\alpha y$ and using the orthogonality of d and g. We get

$$0=G(x)\gamma g(z)\alpha y\beta d(z)+F(x)\gamma g(z)\alpha y\beta g(z)$$

$$=F(x)\gamma g(z)\alpha y\beta g(z).$$

Again replacing y by $y\delta F(x)$ and β by γ and using the semiprimeness of M, we obtain

$F(x)\gamma g(z)=0$ for all $x,z\in\mathrm{M}$ and $\gamma\in\Gamma$. $(4.3.17)$

Substituting $y\alpha z$ for z in $(4.3.17)$, we find that

$F(x)\gamma g(y)\alpha\sigma(z)+F(x)\gamma\tau(y)\alpha g(z)=0$ for all $x,y,z\in\mathrm{M}$ and $\alpha,\gamma\in\Gamma$.

Using $(4.3.17)$ and the fact that τ is an automorphism of M, we get

$F(x)\gamma y_1\alpha g(z)=0$ for all $x,y_1,z\in\mathrm{M}$ and $\alpha,\gamma\in\Gamma$.

Therefore by Lemma 1.4.8, F and g are orthogonal. Hence $(4.3.16)$ becomes $G(x)\gamma y_1\beta d(z)=0$ for all $x,y_1,z\in\mathrm{M}$ and $\beta,\gamma\in\Gamma$. Thus, G and d are orthogonal.

$(iii)\Rightarrow(i)$. By the orthogonality of F and g, we have

$F(x)\alpha y\beta g(z)=0$ for all $x,y,z\in\mathrm{M}$ and $\alpha,\beta\in\Gamma$. $(4.3.18)$

Replacing x by $s\gamma x$, we get

$$0=F(s\gamma x)\alpha y\beta g(z)$$

$$=F(s)\gamma\sigma(x)\alpha y\beta g(z)+\tau(s)\gamma d(x)\alpha y\beta g(z)$$

$$=\tau(s)\gamma d(x)\alpha y\beta g(y).$$

Since τ is an automorphism of M and using the semiprimeness of M, we get $d(x)\alpha y\beta g(z)=0$ for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. By Lemma 1.4.8, d and g are orthogonal. Thus, by Theorem 4.3.6, dg is a $(\sigma^2,\tau^2)-$ derivation. Now, replacing y by $g(z)\gamma y\delta F(x)$ and β by α in $(4.3.18)$, we get

$$F(x)\alpha g(z)\gamma y\delta F(x)\alpha g(z)=0 \text{ for all } x,y,z\in M \text{ and } \alpha,\beta,\delta\in\Gamma.$$

By the semiprimeness of M, we have $F(x)\alpha g(z)=0$ for all $x,z\in M$ and $\alpha\in\Gamma$. Similarly, by the orthogonality of G and d, we have $G(x)\alpha d(z)=0$ for all $x,z\in M$ and $\alpha\in\Gamma$. Thus,

$$FG(x\alpha y)=FG(x)\alpha\sigma^2(y)+\tau^2(x)\alpha dg(y) \text{ for all } x,y\in M \text{ and } \alpha\in\Gamma.$$

Hence (FG,dg) is a generalized $(\sigma^2,\tau^2)-$derivation.

$(ii)\Leftrightarrow(iii)$. Using similar approach as we have used to prove $(i)\Leftrightarrow(iii)$. □

Corollary 4.3.13: Let (F,d) and (G,g) be generalized derivations of M. Then the following conditions are equivalent:

i. (FG,dg) is a generalized derivation.
ii. (GF,gd) is a generalized derivation.
iii. F and g are orthogonal, also G and d are orthogonal.

Note: The following example shows that Theorem 4.3.12 does not hold for arbitrary $\Gamma-$ rings.

Example 4.3.14: Let R be any 2-torsion free ring and let $M=\left\{\begin{pmatrix} a\\ b\\ c\\ f\\ h\end{pmatrix}: a,b,c,f,h\in R\right\}$, $\Gamma=\{(l\quad 0\quad 0\quad 0\quad m): l,m\in R\}$. Then M is a 2-torsion free $\Gamma-$ ring which is not semi-prime. Define the map $\sigma: M\to M$ such that $\sigma\left(\begin{pmatrix} a\\ b\\ c\\ f\\ h\end{pmatrix}\right)=\begin{pmatrix} a\\ c\\ b\\ f\\ h\end{pmatrix}$. Clearly, σ is an automorphism of M and take $\tau=I_M$, where I_M is the identity map of M. Next, define the maps $d,g: M\to M$ such that,

$d\left(\begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f \\ 0 \end{pmatrix}$, $g\left(\begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix}\right) = \begin{pmatrix} 0 \\ c \\ b \\ 0 \\ 0 \end{pmatrix}$ for all $\begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} \in \mathrm{M}$. It can be easily verified that d and g are $(\sigma,\tau)-$ derivations of M such that $d\sigma = \sigma d,\ d\tau = \tau d,\ g\sigma = \sigma g,\ g\tau = \tau g$. Now, consider the maps $F, G : \mathrm{M} \to \mathrm{M}$ such that $F\left(\begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix}\right) = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $G\left(\begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ h \end{pmatrix}$ for all $\begin{pmatrix} a \\ b \\ c \\ f \\ h \end{pmatrix} \in \mathrm{M}$.

It can be easily checked that (F,d) and (G,g) are generalized $(\sigma,\tau)-$ derivations of M. Also, (FG,dg) and (GF,gd) are generalized $(\sigma^2,\tau^2)-$ derivations of M but neither F and g are orthogonal nor G and d are orthogonal.

Corollary 4.3.15: Let (F,d) be generalized $(\sigma,\tau)-$ derivations of M. If $F(x)\gamma F(y) = 0$ for all $x, y \in \mathrm{M}$ and $\gamma \in \Gamma$, then $F = d = 0$.

Proof: Notice that $F(x)\gamma F(y) = 0$ for all $x, y \in \mathrm{M}$ and $\gamma \in \Gamma$. Replacing y by $y\beta z$, we get

$$0 = F(x)\gamma F(y\beta z) = F(x)\gamma F(y)\beta\sigma(z) + F(x)\gamma\tau(y)\beta d(z) = F(x)\gamma\tau(y)\beta d(z).$$

Since τ is an automorphism of M and using Lemma1.4.8, we have $d(z)\gamma F(x) = 0$ for all $x, z \in \mathrm{M}$ and $\gamma \in \Gamma$. Now, replacing x by $x\alpha z$, we get

$$0 = d(z)\gamma F(x\alpha z) = d(z)\gamma F(x)\alpha\sigma(z) + d(z)\gamma\tau(x)\alpha d(z) = d(z)\gamma\tau(x)\alpha d(z).$$

By the semiprimeness of M, we get $d(z) = 0$ for all $z \in \mathrm{M}$. Therefore, $d = 0$. Again

$$0 = F(x\gamma z)\alpha F(y) = F(x)\gamma\sigma(z)\alpha F(y) + \tau(x)\gamma d(z)\alpha F(y) = F(x)\gamma\sigma(z)\alpha F(y).$$

In particular, we have

$$F(x)\gamma z_1\alpha F(x) = 0 \text{ for all } x, z_1 \in \mathrm{M} \text{ and } \alpha, \gamma \in \Gamma.$$

Using the semiprimeness of M, we get $F(x) = 0$ for all $x \in \mathrm{M}$ and hence $F = 0$. □

Example 4.3.16: Let R be any 2-torsion free ring and let

$\mathrm{M}=\left\{\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix} : a,b,c,f \in R\right\}$, $\Gamma=\{(0 \quad x \quad 0 \quad 0) : x \in R\}$. Then M is a 2-torsion free $\Gamma-$ring which is not semi-prime. Define the mappings $\sigma,\tau:\mathrm{M}\to\mathrm{M}$ such that $\sigma\left(\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\right)=\begin{pmatrix} c \\ b \\ a \\ f \end{pmatrix}$, $\tau\left(\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\right)=\begin{pmatrix} f \\ b \\ c \\ a \end{pmatrix}$ for all $\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\in\mathrm{M}$. Clearly, σ and τ are automorphism of M. Next, define the map $d:\mathrm{M}\to\mathrm{M}$ such that, $d\left(\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\right)=\begin{pmatrix} 0 \\ 0 \\ c \\ f \end{pmatrix}$ for all $\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\in\mathrm{M}$. It can be easily verified that d is a $(\sigma,\tau)-$derivation of M.Further, consider the map $F:\mathrm{M}\to\mathrm{M}$ such that $F\left(\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\right)=\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$ for all $\begin{pmatrix} a \\ b \\ c \\ f \end{pmatrix}\in\mathrm{M}$.

Then it is straightforward to check that F is a generalized $(\sigma,\tau)-$derivation of M. Moreover, F satisfies the relation $F(x)\gamma F(y)=0$ for all $x,y\in\mathrm{M}$ and $\gamma\in\Gamma$, but neither $F=0$ nor $d=0$.

4.4 Permuting Tri-Derivations On Semi-Prime $\Gamma-$Rings

In this section, we investigate some results concerning a permuting tri-derivation D on non-commutative 3-torsion free semi-prime $\Gamma-$rings M. Some characterizations of semi-prime $\Gamma-$rings are obtained by means of permuting tri-derivations.

Let I be a nonempty subset of M. Then a map $d:\mathrm{M}\to\mathrm{M}$ is said to be *commuting* (resp. centralizing) on I if $[d(x),x]_\alpha=0$ for all $x\in I,\alpha\in\Gamma$ (resp. $[d(x),x]_\alpha\in Z(\mathrm{M})$ for all $x\in I,\alpha\in\Gamma$), and is called *central* if $d(x)\in Z(\mathrm{M})$ for all $x\in\mathrm{M},\alpha\in\Gamma$.

Every central mapping is obviously commuting but not conversely in general, and d is called skew-centralizing on a subset I of M (resp. skew-commuting on a subset I of M) if $d(x)\alpha x+x\alpha d(x)\in Z(\mathrm{M})$ holds for all $x\in I,\alpha\in I$ (resp. $d(x)\alpha x+x\alpha d(x)=0$ holds for all $x\in I,\alpha\in\Gamma$).

Note: In this section we shall assume $(*)$ $x\alpha y\beta z=x\beta y\alpha z$ for all $x,y,z\in\mathrm{M},\alpha,\beta\in\Gamma$.

Theorem 4.4.1: Let M be a 3-torsion free semi-prime $\Gamma-$ ring satisfying the condition $(*)$ and let I be a non-zero ideal of M. If there exists a permuting tri-derivation $D: M\times M\times M \to M$ such that d is an automorphism commuting on I, where d is the trace of D, then I is a non-zero commutative ideal.

Proof: Suppose that $\left[d(x),x\right]_\beta = 0$ for all $x\in I, \beta\in\Gamma$. $\quad (4.4.1)$

Substituting x by $x+y$ leads to

$$\left[d(x),y\right]_\beta + \left[d(y),x\right]_\beta + 3\left[D(x,x,y),x\right]_\beta + 3\left[D(x,y,y),x\right]_\beta + 3\left[D(x,x,y),y\right]_\beta$$

$$+3\left[D(x,y,y),y\right]_\beta = 0 \text{ for all } x,y\in I, \beta\in\Gamma. \qquad (4.4.2)$$

Putting $-x$ instead of x in $(4.4.2)$ we get

$$\left[D(x,y,y),x\right]_\beta + \left[D(x,x,y),y\right]_\beta = 0 \text{ for all } x,y\in I, \beta\in\Gamma. \qquad (4.4.3)$$

Since d is odd, we set $x = x+y$ in $(4.4.3)$ and then use $(4.4.1)$ and $(4.4.2)$ to obtain

$$\left[d(y),x\right]_\beta + 3\left[D(x,y,y),y\right]_\beta = 0 \text{ for all } x,y\in I, \beta\in\Gamma. \qquad (4.4.4)$$

Let us write $y\alpha x$ instead of x in $(4.4.4)$, we obtain

$$\left[d(y),y\alpha x\right]_\beta + 3\left[D(y\alpha x,y,y),y\right]_\beta = y\alpha\left[d(y),x\right]_\beta + 3d(y)\alpha\left[x,y\right]_\beta$$

$$+3y\alpha\left[D(x,y,y),y\right]_\beta = y\alpha\left(\left[d(y),x\right]_\beta + 3\left[D(x,y,y),y\right]_\beta\right)$$

$$+3d(y)\alpha\left[x,y\right]_\beta = 0, \ \forall x,y\in I, \alpha,\beta\in\Gamma.$$

Then $d(y)\alpha\left[x,y\right]_\beta = 0$ $\forall x,y\in I,\alpha,\beta\in\Gamma$, since d is an automorphism, we obtain $y\alpha\left[x,y\right]_\beta = 0$ $\forall x,y\in I,\alpha,\beta\in\Gamma$.

Replacing x by $y\alpha x$, we get

$$y\alpha x\gamma\left[x,y\right]_\beta = 0 \ \forall x,y\in I, \ \alpha,\beta,\gamma\in\Gamma. \qquad (4.4.5)$$

Again left-multiplying by x implies that

$$x\alpha y\gamma\left[x,y\right]_\beta = 0 \ \forall x,y\in I, \ \alpha,\beta,\gamma\in\Gamma. \qquad (4.4.6)$$

Subtracting $(4.4.5)$ and $(4.4.6)$ with using M is semi-prime $\Gamma-$ ring, we completes our proof. □

Corollary 4.4.2: [7] Let M be a 3-torsion free semi-prime $\Gamma-$ ring satisfying the condition $(*)$ and I be an ideal of M. If there exists a permuting tri-derivation $D: M \times M \times M \rightarrow M$ such that d is commuting on I, where d is the trace of D, then I is a central ideal.

Theorem 4.4.3: [7] Let M be a 3-torsion free semi-prime $\Gamma-$ ring satisfying the condition $(*)$. If there exists a permuting tri-derivation $D: M \times M \times M \rightarrow M$ such that d is an automorphism commuting on M, where d is the trace of D, then M is commutative.

Proof: For all $x \in M$, we have $d(x) \in Z(M)$, then $[d(x), x]_\beta = 0, \forall \beta \in \Gamma$ $\qquad (4.4.7)$

Substituting x by $x+y$, we obtained

$$[d(x), y]_\beta + [d(y), x]_\beta + 3[D(x,x,y), x]_\beta + 3[D(x,y,y), x]_\beta + 3[D(x,x,y), y]_\beta$$

$$+3[D(x,y,y), y]_\beta = 0 \text{ for all } x, y \in M, \beta \in \Gamma. \qquad (4.4.8)$$

Putting $-x$ instead of x in $(4.4.8)$ and comparing $(4.4.8)$ with the result, we arrive at

$$[D(x,y,y), x]_\beta + [D(x,x,y), y]_\beta = 0 \qquad (4.4.9)$$

Since d is odd, we set $x = x+y$ in $(4.4.9)$ and then use $(4.4.7)$ and $(4.4.9)$ to get

$$[d(y), x]_\beta + 3[D(x,y,y), y]_\beta = 0 \qquad (4.4.10)$$

Let us write $y\alpha x$ instead of x in $(4.4.10)$, we obtain

$$[d(y), y\alpha x]_\beta + 3[D(y\alpha x, y, y), y]_\beta = y\alpha[d(y), x]_\beta + 3d(y)\alpha[x,y]_\beta$$

$$+3y\alpha[D(x,y,y), y]_\beta = y\alpha\left([d(y), x]_\beta + 3[D(x,y,y), y]_\beta\right) + 3d(y)\alpha[x,y]_\beta = 0$$

Then $d(y)\alpha[x,y]_\beta = 0$. Since d is an automorphism, we obtain $y\alpha[x,y]_\beta = 0$. Replacing x by $y\alpha x$, we get

$$y\alpha x\gamma[x,y]_\beta = 0 \qquad (4.4.11)$$

Again left-multiplying by x implies that

$$x\alpha y\gamma[x,y]_\beta = 0 \qquad (4.4.12)$$

Subtracting $(4.4.11)$ and $(4.4.12)$ with using M is a semi-prime $\Gamma-$ring, we get the required result. □

Theorem 4.4.4: [7] Let M be a 6-torsion free semi-prime $\Gamma-$ring satisfying the condition $(*)$. If there exists a permuting tri-derivation $D:\mathrm{M}\times\mathrm{M}\times\mathrm{M}\to\mathrm{M}$ such that d is an automorphism centralizing on M, where d is the trace of D, then M is commutative.

Proof: Assume that

$$\left[d(x),x\right]_\beta \in Z(\mathrm{M}) \text{ for all } x\in\mathrm{M} \text{ and } \beta\in\Gamma \qquad (4.4.13)$$

Replacing x by $x+y$ and again using $(4.4.13)$, we obtain

$$\left[d(x),y\right]_\beta+\left[d(y),x\right]_\beta+3\left[D(x,x,y),x\right]_\beta+3\left[D(x,y,y),x\right]_\beta+3\left[D(x,x,y),y\right]_\beta$$
$$+3\left[D(x,y,y),y\right]_\beta \in Z(\mathrm{M}) ,\forall x,y\in\mathrm{M},\beta\in\Gamma \qquad (4.4.14)$$

Replacing x by $-x$ in $(4.4.14)$ we get

$$\left[D(x,y,y),x\right]_\beta+\left[D(x,x,y),y\right]_\beta \in Z(\mathrm{M}) ,\forall x,y\in\mathrm{M},\beta\in\Gamma \qquad (4.4.15)$$

Replacing x by $x+y$ in $(4.4.15)$, we obtain

$$\left[d(y),x\right]_\beta+3\left[D(x,y,y),y\right]_\beta \in Z(\mathrm{M}) ,\forall x,y\in\mathrm{M},\beta\in\Gamma \qquad (4.4.16)$$

Taking $x=y\alpha y$ in $(4.4.16)$ and invoking $(4.4.13)$, we get

$$\left[d(y),y\alpha y\right]_\beta+3\left[D(y\alpha y,y,y),y\right]_\beta=8\left[d(y),y\right]_\beta\alpha y\in Z(\mathrm{M}),\ \forall y\in\mathrm{M},\alpha,\beta\in\Gamma \ (4.4.17)$$

Now commuting $(4.4.17)$ with $d(y)$ yields

$$8\left[d(y),y\right]_\beta\alpha\left[d(y),y\right]_\beta = 0 ,\ \forall y\in\mathrm{M},\alpha,\beta\in\Gamma.$$

Again substituting x by $y\alpha x$ in $(4.4.16)$ gives

$$\left[d(y), y\alpha x\right]_\beta + 3\left[D(y\alpha x, y, y), y\right]_\beta = y\alpha\left(\left[d(y), x\right]_\beta + 3\left[D(x,y,y),y\right]_\beta\right) + 3d(y)\alpha[x,y]_\beta$$
$$+4\left[d(y), y\right]_\beta \alpha x \in Z(\mathrm{M}) \text{ for all } x, y \in \mathrm{M}, \alpha, \beta \in \Gamma .$$

Then $\left[y\alpha\left(\left[d(y),x\right]_\beta + 3\left[D(x,y,y),y\right]_\beta\right), y\right]_\beta + \left[3d(y)\alpha[x,y]_\beta + 4\left[d(y),x\right]_\beta \alpha x, y\right]_\beta = 0$ for all $x, y \in \mathrm{M}, \alpha, \beta \in \Gamma$. And so we get

$$3d(y)\alpha\left[[x,y]_\beta, y\right]_\beta + 7\left[d(y),y\right]_\beta \alpha[x,y]_\beta = 0 \text{ for all } x, y \in \mathrm{M}, \alpha, \beta \in \Gamma . \qquad (4.4.18)$$

Since d acts as an automorphism with M is 6-torsion free the relation $(4.4.18)$ reduces to $y\alpha\left[[x,y]_\beta, y\right]_\beta = 0$ for all $x, y \in \mathrm{M}, \alpha, \beta \in \Gamma$. Replacing x by $r\delta x$ we get

$$y\alpha x\delta\left[[x,y]_\beta, y\right]_\beta + 2y\alpha[x,y]_\beta = 0 \text{ for all } x, y \in \mathrm{M}, \alpha, \beta, \delta \in \Gamma . \qquad (4.4.19)$$

Replacing y by $-y$ in $(4.4.19)$ and subtracting with $(4.4.19)$, gives

$$4y\delta[x,y]_\beta = 0 \text{ for all } x, y \in \mathrm{M}, \beta, \delta \in \Gamma . \qquad (4.4.20)$$

Replacing x by $x\gamma r$ and left-multiplying by s, we obtain

$$4y\delta x\alpha[r,y]_\beta = 0 \text{ for all } x, y, r, s \in \mathrm{M}, \alpha, \beta, \delta \in \Gamma . \qquad (4.4.21)$$

Again in $(4.4.20)$ replacing x by $x\lambda m$ and x by $s\delta x$, we get

$$4y\gamma s\delta x\alpha[m,y]_\beta = 0 \text{ for all } x, y, m, s \in \mathrm{M}, \alpha, \beta, \delta, \gamma \in \Gamma . \qquad (4.4.22)$$

Subtracting $(4.4.21)$ and $(4.4.22)$ with using M is 6-torision free semi-prime, we obtain $[s,y]_\beta = 0$ for all $s, y \in \mathrm{M}$. Thus, we get M is commutative. □

Theorem 4.4.5: [7] Let M be a 3-torsion free semi-prime $\Gamma-$ring satisfying the condition $(*)$. If there exists a permuting tri-derivation $D: \mathrm{M}\times\mathrm{M}\times\mathrm{M} \to \mathrm{M}$ such that d is commuting on M, where d is the trace of D, then M is a central mapping.

Proof: we have $\left[d(x),x\right]_{\beta}=0$ for all $x\in\mathrm{M},\beta\in\Gamma$. $\qquad(4.4.23)$

Substituting x by $x+y$ leads to

$$\left[d(x),y\right]_{\beta}+\left[d(y),x\right]_{\beta}+3\left[D(x,x,y),x\right]_{\beta}+3\left[D(x,y,y),x\right]_{\beta}+3\left[D(x,x,y),y\right]_{\beta}$$

$$+3\left[D(x,y,y),y\right]_{\beta}=0 \text{ for all } x,y\in\mathrm{M},\beta\in\Gamma. \qquad (4.4.24)$$

Putting $-x$ instead of x in $(4.4.24)$ we get

$$\left[D(x,y,y),x\right]_{\beta}+\left[D(x,x,y),y\right]_{\beta}=0 \text{ for all } x,y\in\mathrm{M},\beta\in\Gamma. \qquad (4.4.25)$$

Since d is odd, we set $x=x+y$ in $(4.4.25)$ and then use $(4.4.23)$ and $(4.4.24)$ to obtain

$$\left[d(y),x\right]_{\beta}+3\left[D(x,y,y),y\right]_{\beta}=0 \text{ for all } x,y\in\mathrm{M},\beta\in\Gamma. \qquad (4.4.26)$$

Let us substitute $y\alpha x$ instead of x in $(4.4.26)$, since M is 3-torsion semi-prime, then $d(y)\alpha\left[x,y\right]_{\beta}=0$ $\forall x,y\in\mathrm{M},\alpha,\beta\in\Gamma$.

Applying Lemma 1.4.4, the above relation gives $d(y)\in Z(\mathrm{M})$ for all $y\in\mathrm{M}$, and this completes the proof of the theorem. □

Theorem 4.4.6: $[7]$ Let M be a 3-torsion free semi-prime $\Gamma-$ring. If there exists a permuting tri-derivation $D:\mathrm{M}\times\mathrm{M}\times\mathrm{M}\rightarrow\mathrm{M}$ such that d is commuting on M, where d is the trace of D, then D is commuting (resp. centralizing).

Proof: we can restrict our attention to relation, $\left[d(x),x\right]_{\beta}=0$ for all $x\in\mathrm{M},\beta\in\Gamma$. The substitution of $x+y$ for x in above relation gives

$$\left[d(x),y\right]_{\beta}+\left[d(y),x\right]_{\beta}+3\left[D(x,x,y),x\right]_{\beta}+3\left[D(x,y,y),x\right]_{\beta}+3\left[D(x,x,y),y\right]_{\beta}+3\left[D(x,y,y),y\right]_{\beta}=0$$

$$\forall x,y\in\mathrm{M},\beta\in\Gamma. \qquad (4.4.27)$$

Now, by the same method in Theorem 4.4.5, we arrive at

$$y\delta\left[d(y),x\right]_{\beta}+3d(y)\delta\left[x,y\right]_{\beta}+3y\delta\left[D(x,y,y),y\right]_{\beta}=0\ \forall x,y\in\mathrm{M},\beta,\delta\in\Gamma. \qquad (4.4.28)$$

Which implies that

$$d(y)\delta\left[x,y\right]_{\beta}=0 \text{ for all } x,y\in\mathrm{M},\beta,\delta\in\Gamma. \qquad (4.4.29)$$

The above relation gives $d(y) \in Z(\mathrm{M})$ for all $x \in \mathrm{M}$. By substitution the relation $d(y) \in Z(\mathrm{M})$ in $(4.4.29)$ with using replacing x by y and M is 3-torsion free semi-prime, we obtain

$[D(y,y,y),y]_\beta = 0$ for all $x, y \in \mathrm{M}, \beta \in \Gamma$. $\quad (4.4.30)$

Then D is commuting(resp. centralizing) of M. □

Theorem 4.4.7: [7] Let M be a non-commutative 3-torsion free semi-prime $\Gamma-$ring satisfying the condition $(*)$. If there exists a permuting tri-derivation $D:\mathrm{M}\times\mathrm{M}\times\mathrm{M}\to\mathrm{M}$ such that d is skew-commuting on M, where d is the trace of D, then d is commuting.

Proof: We have $d(x)\alpha x + x\alpha d(x) = 0$ for all $x \in \mathrm{M}$ and $\alpha \in \Gamma$. Replacing x by $x+y$, we obtain

$$d(y)\alpha x + 3D(x,x,y)\alpha x + 3D(x,y,y)\alpha x + d(x)\alpha y + 3D(x,x,y)\alpha y$$

$$+3D(x,y,y)\alpha y + x\alpha d(y) + 3x\alpha D(x,x,y) + 3x\alpha D(x,y,y) + y\alpha d(y)$$

$$+3y\alpha D(x,x,y) + 3y\alpha D(x,y,y) = 0 \text{ for all } x,y \in \mathrm{M}, \alpha \in \Gamma. \quad (4.4.31)$$

We substitute $-x$ for x in $(4.4.31)$ we get

$$3D(x,y,y)\alpha x + 3D(x,x,y)\alpha y + 3x\alpha D(x,y,y) + 3y\alpha D(x,x,y) = 0$$ for all $x, y \in \mathrm{M}, \alpha \in \Gamma$.

Since M is 3-torsion free, we obtain

$$D(x,y,y)\alpha x + D(x,x,y)\alpha y + x\alpha D(x,y,y) + y\alpha D(x,x,y) = 0 \quad (4.4.32)$$

for all $x, y \in \mathrm{M}, \alpha \in \Gamma$.

Again we substituting $x\beta y$ for x in $(4.4.32)$ then we get

$$x\alpha D(y,y,y)\beta y + D(x,y,y)\alpha x\beta y + x\alpha y\beta D(y,y,y) + D(x,y,y)\alpha y = 0 \quad (4.4.33)$$

for all $x, y \in \mathrm{M}, \alpha, \beta \in \Gamma$.

We substitute $-x$ for x in $(4.4.33)$ and compare $(4.4.33)$ with the result to get $D(x,y,y)\alpha x\beta y = 0$ for all $x, y \in \mathrm{M}, \alpha, \beta \in \Gamma$. Replacing x by y and since d is the trace of D, we obtain $d(y)\alpha y\beta y = 0$ for all $y \in \mathrm{M}$. Left-multiplying by y and right-multiplying by $d(y)\delta y$ with using Lemma 1.4.4, we obtain

$$y\delta d(y)\beta y = 0 \text{ for all } y \in M, \beta, \delta \in \Gamma. \tag{4.4.34}$$

Left-multiplying $(4.4.34)$ by $d(y)$ with using (Lemma 1.4.11 and Remark 3.2.4) gives

$$d(y)\beta y = 0 \text{ for all } y \in M, \beta \in \Gamma. \tag{4.4.35}$$

Right-multiplying $(4.4.34)$ by $d(y)$ with using (Lemma 1.4.11 and Remark 3.2.4) and subtracting the result with $(4.4.35)$, we obtain $\left[d(y), y\right]_{\beta} = 0$ for all $y \in M, \beta \in \Gamma$.

By Theorem 4.4.3, we complete our proof. □

Theorem 4.4.8: [7] Let M be a non-commutative 3-torsion free semi-prime $\Gamma-$ring satisfying the condition $(*)$ and I be a non-zero ideal of M. If there exists a permuting tri-derivation $D: M \times M \times M \to M$ such that d is skew-commuting on I, where d is the trace of D, then d is commuting on I.

Proof: Using same method in Theorem 4.4.7, we complete the proof. □

Theorem 4.4.9: [7] Let M be a non-commutative 3-torsion free semi-prime $\Gamma-$ring satisfying the condition $(*)$ and I be a non-zero ideal of M. If there exists a permuting tri-derivation $D: M \times M \times M \to M$ such that d is skew-centralizing on I, where d is the trace of D, then d is commuting on I.

Proof: Using same method in Theorem 4.4.7, we complete the proof. □

Chapter Five

Derivations On $\Gamma M-$ Modules

5.1 Jordan Left Derivations On Left $\Gamma M-$ Modules

In this section, we present the concepts of a left $\Gamma M-$ module, left derivation on left $\Gamma M-$ module and we will prove that; for M a $\Gamma-$ring such that $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M, \alpha,\beta \in \Gamma$ $(*)$, and X a left $\Gamma M-$ module, if $a\alpha x = 0$ with $a \in M, x \in X$ and $\alpha \in \Gamma$ then either $a = 0$ or $x = 0$. It is also shown that there exists a non-zero left derivation $d: M \to X$, such that

1. If $d: M \to X$ is a non-zero left derivation and X a left $\Gamma M-$ module. Then M is commutative.
2. If $d: M \to X$ is a non-zero Jordan left derivation and X is 2-torsion free. Then M is commutative.

Now, we start by the following definition.

Definition 5.1.1: [10] Let M be a $\Gamma-$ring, $(X,+)$ be an abelian group and X a left $\Gamma M-$module. An additive mapping $d: M \to X$ is a left derivation if $d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$ and a Jordan left derivation if $d(a\alpha a) = 2a\alpha d(a)$ for all $a,b \in M$ and $\alpha \in \Gamma$.

Lemma 5.1.2: [10] Let M be a $\Gamma-$ring satisfying $(*)$ and let X be a 2-torsion free left $\Gamma M-$module. Let $d: M \to X$ be a Jordan left derivation. Then, for all $a,b \in M$ and $\beta,\alpha \in \Gamma$.

i. $d(a\alpha b + b\alpha a) = 2a\alpha d(b) + 2b\alpha d(a)$.

ii. $d(a\alpha b\beta a) = a\beta a\alpha d(b) + 3a\alpha b\beta d(a) - b\alpha a\beta d(a)$.

iii. $$d(a\alpha b\beta c + c\alpha b\beta a) = (a\beta c + c\beta a)\alpha d(b) + 3a\alpha b\beta d(c) + 3c\alpha b\beta d(a) - b\alpha c\beta d(a) - b\alpha a\beta d(c).$$

iv. $(a\alpha b - b\alpha a)\beta a\alpha d(a) = a\alpha(a\alpha b - b\alpha a)\beta d(a)$.

v. $(a\alpha b - b\alpha a)\beta\big(d(a\alpha b) - a\alpha d(b) - b\alpha d(a)\big) = 0$.

Proof: By Corollary 2.1.11, Preposition 2.1.12, Corollary 2.1.13 and Proposition 2.1.14 the proof is complete. □

Lemma 5.1.3: [10] Let M be a $\Gamma-$ring satisfying $(*)$ and let X be a 2-torsion free $\Gamma M-$module. Then there exists a Jordan left derivation $d: M \to X$ such that

i. $d(a\alpha a\beta b) = a\alpha a\beta d(b) + (a\beta b + b\beta a)\alpha d(a) + a\alpha d(a\beta b - b\beta a)$.

ii. $d(b\alpha a\beta a) = a\alpha a\beta d(b) + (3b\beta a - a\beta b)\alpha d(a) - a\alpha d(a\beta b - b\beta a)$.

iii. $(a\alpha b - b\alpha a)\beta d(a\alpha b - b\alpha a) = 0$.

iv. $(a\alpha a\beta b - 2a\alpha b\beta a + b\alpha a\beta a)\alpha d(b) = 0$.

for all $a,b,c \in M$ and $\beta,\alpha \in \Gamma$.

Proof: Substituting $b\beta a$ and $a\beta b$ for b in Lemma 5.1.2 (i), we get

$$d(a\alpha b\beta a+b\beta a\alpha a)=2a\alpha d(b\beta a)+2b\beta a\alpha d(a) \tag{5.1.1}$$

and $$d(a\alpha a\beta b+a\beta b\alpha a)=2a\alpha d(a\beta b)+2a\beta b\alpha d(a) \tag{5.1.2}$$

Taking $(5.1.2)$ minus $(5.1.1)$ and then using $(*)$, we get

$$d(a\alpha a\beta b-b\alpha a\beta a)=2a\alpha d(a\beta b-b\beta a)+2(a\beta b-b\beta a)\alpha d(a) \tag{5.1.3}$$

Replacing a by $a\beta a$ in Lemma 5.1.2 (i) and then by $(*)$, we get

$$d(a\alpha a\beta b+b\alpha a\beta a)=2a\alpha a\beta d(b)+4b\beta a\alpha d(a) \tag{5.1.4}$$

By $(5.1.3)$ and $(5.1.4)$ with the condition that X is 2-torsion free, we have (i) .

Subtracting $(5.1.3)$ from $(5.1.4)$ and then applying the same condition, we obtain (ii).

By Lemma 5.1.2 (v), we have

$$(a\alpha b-b\alpha a)\beta(d(a\alpha b)-b\alpha d(a)-a\alpha d(b))=0 \tag{5.1.5}$$

Using Lemma 5.1.2 (i) in $(5.1.5)$, we get

$$(a\alpha b-b\alpha a)\beta(d(b\alpha a)-a\alpha d(b)-b\alpha d(a))=0 \tag{5.1.6}$$

Taking $(5.1.5)$ minus $(5.1.6)$, we obtain (iii).

By Lemma 5.1.2 (i), Lemma 5.1.2 (ii) and $(*)$, we have

$$d((a\alpha b-b\alpha a)\beta(a\alpha b-b\alpha a))=-3(a\alpha a\beta b-2a\alpha b\beta a+b\alpha a\beta a)\alpha d(b)$$
$$-(b\alpha b\beta a-2b\alpha a\beta b+a\alpha b\beta b)\alpha d(a).$$

On the other hand, using (iii), we have $d((a\alpha b-b\alpha a)\beta(a\alpha b-b\alpha a))=0$.

Thus we have

$$3(a\alpha a\beta b-2a\alpha b\beta a+b\alpha a\beta a)\alpha d(b)+(b\alpha b\beta a-2b\alpha a\beta b+a\alpha b\beta b)\alpha d(a)=0 \tag{5.1.7}$$

From Lemma 5.1.2 (iv),

$$\left(a\alpha a\beta b-2a\alpha b\beta a+b\alpha a\beta a\right)\alpha d\left(a\right)=0 \qquad (5.1.8)$$

Replacing a by $a+b$ in $(5.1.8)$, we obtain

$$\left(a\alpha a\beta b-2a\alpha b\beta a+b\alpha a\beta a\right)\alpha d\left(b\right)-\left(b\alpha b\beta a-2b\alpha a\beta b+a\alpha b\beta b\right)\alpha d\left(a\right)=0 \qquad (5.1.9)$$

Adding $(5.1.7)$ and $(5.1.9)$, and then using the condition that X is 2-torsion free, we get

$$\left(a\alpha a\beta b-2a\alpha b\beta a+b\alpha a\beta a\right)\alpha d\left(b\right)=0 \qquad (5.1.10)$$

Hence from $(5.1.9)$ and $(5.1.10)$, we obtain (iv). □

Theorem 5.1.4: [10] Let M be a $\Gamma-$ring satisfying $(*)$ and let X be a left $\Gamma\mathrm{M}-$module. Suppose that $a\alpha x=0$ with $a\in\mathrm{M}, x\in X$ and $\alpha\in\Gamma$ implies that either $a=0$ or $x=0$. If there exists a non-zero left derivation $d:\mathrm{M}\to X$. Then M is commutative.

Proof: Since $d:\mathrm{M}\to X$ is a non-zero left derivation, we have

$$d\left(a\alpha b\right)=a\alpha d\left(b\right)+b\alpha d\left(a\right), \text{ for all } a,b\in\mathrm{M} \text{ and } \alpha\in\Gamma. \qquad (5.1.11)$$

Replacing b by $b\beta a$ in $(5.1.11)$ for all $\beta\in\Gamma$, we have

$$d\left(a\alpha\left(b\beta a\right)\right)=a\alpha d\left(b\beta a\right)+b\beta a\alpha d\left(a\right)=a\alpha b\beta d\left(a\right)+a\alpha a\beta d\left(b\right)+b\beta a\alpha d\left(a\right) \qquad (5.1.12)$$

On the other hand

$$d\left(\left(a\alpha b\right)\beta a\right)=\left(a\alpha b\right)\beta d\left(a\right)+a\beta a\alpha d\left(b\right)+a\beta b\alpha d\left(a\right). \qquad (5.1.13)$$

Now from $(5.1.12)$ and $(5.1.13)$, we get $\left(a\alpha b-b\alpha a\right)\beta d\left(a\right)=0$. By assumption for each $a\in\mathrm{M}$ either $a\in Z\left(\mathrm{M}\right)$ or $d\left(a\right)=0$. But then $Z\left(\mathrm{M}\right)$ and $Kerd=\left\{m\in\mathrm{M}:d\left(m\right)=0\right\}$ are additive subgroups of M. Since $Z\left(\mathrm{M}\right)$ and $Kerd$ are proper subgroups of M, either $\mathrm{M}=Z\left(\mathrm{M}\right)$ or $\mathrm{M}=Kerd$. But $d\neq 0$, then $\mathrm{M}=Z\left(\mathrm{M}\right)$. This completes the proof. □

Theorem 5.1.5: [10] Let M be a $\Gamma-$ring satisfying $(*)$ and let X be a 2-torsion free left $\Gamma\mathrm{M}-$module. Suppose that $a\alpha x=0$ with $a\in\mathrm{M}$, $x\in X$ and $\alpha\in\Gamma$ implies that either $a=0$ or $x=0$. If there exists a non-zero Jordan left derivation $d:\mathrm{M}\to X$. Then M is commutative.

Proof: By Lemma 5.1.3 (iii), we have

$$\left(a\alpha b-b\alpha a\right)\beta d\left(a\alpha b-b\alpha a\right)=0, \text{ for all } a,b\in\mathrm{M} \text{ and } \beta,\alpha\in\Gamma.$$

Then by assumption either $a\alpha b-b\alpha a=0$ or $d(a\alpha b-b\alpha a)=0$. If $a\alpha b-b\alpha a=0$, then M is commutative.

If $d(a\alpha b-b\alpha a)=0$, then $2d(a\alpha b)=d(a\alpha b)+d(b\alpha a)$ $\qquad (5.1.14)$

In $(5.1.14)$ replace $a\beta b$ for b, for all $\beta\in\Gamma$, we obtain

$$2d(a\alpha a\beta b)=d(a\alpha a\beta b)+d(a\beta b\alpha a).$$

Now by Lemma 5.1.3 (i), Lemma 5.1.2 (ii) and above relation, we get

$$2d(a\alpha a\beta b)=a\alpha a\beta d(b)+(a\beta b+b\beta a)\alpha d(a)+a\alpha d(a\beta b-b\beta a)+a\beta a\alpha d(b)$$

$$+3a\alpha b\beta d(a)-b\alpha a\beta d(a)$$

$$2d(a\alpha a\beta b)=2a\alpha a\beta d(b)+4a\alpha b\beta d(a). \qquad (5.1.15)$$

But by Lemma 5.1.3 (i), we have

$$2d(a\alpha a\beta b)=2a\alpha a\beta d(b)+2(a\beta b+b\beta a)\alpha d(a)+2a\alpha d(a\beta b-b\beta a)$$

$$2d(a\alpha a\beta b)=2a\alpha a\beta d(b)+2(a\beta b+b\beta a)\alpha d(a) \qquad (5.1.16)$$

Replace $(5.1.16)$ in $(5.1.15)$, we get

$$2(a\beta b-b\beta a)\alpha d(a)=0. \qquad (5.1.17)$$

Since X be a 2-torsion free $\Gamma\mathrm{M}-$module then from $(5.1.17)$, we have $(a\beta b-b\beta a)\alpha d(a)\in\mathrm{M}\Gamma X\subseteq X$ and $(a\beta b-b\beta a)\alpha d(a)=0$, for all $a,b\in\mathrm{M},\beta,\alpha\in\Gamma$, therefore by assumption either $a\beta b-b\beta a=0$, then M is commutative or $d(a)=0$, a contradiction. □

5.2 Generalized Left Derivations On Left $\Gamma\mathrm{M}-$ Modules

In this section, we will define generalized left derivation, generalized Jordan left derivation and we will prove that; if M is a $\Gamma-$ring satisfying $(*)$ and X is a 2-torsion free left $\Gamma\mathrm{M}-$module. Suppose that, $a\alpha x=0$ with $a\in\mathrm{M},x\in X$ and $\alpha\in\Gamma$ implies that either $a=0$ or $x=0$. If $D:\mathrm{M}\to X$ is generalized left derivation with associated non-zero Jordan left derivation $d:\mathrm{M}\to X$. Then M is commutative.

Definition 5.2.1: [9] Let M be a $\Gamma-$ring and X be a left $\Gamma\mathrm{M}-$module, an additive mapping $D:\mathrm{M}\to X$ is called *generalized left derivation* if there exists a left derivation $d:\mathrm{M}\to X$, such that $D(a\alpha b)=a\alpha D(b)+b\alpha d(a)$ for all $a,b\in\mathrm{M}$ and $\alpha\in\Gamma$.

And D is called generalized Jordan left derivation if there exists a Jordan left derivation $d : \mathrm{M} \to X$ such that $D(a\alpha a) = a\alpha D(a) + a\alpha d(a)$ for all $a \in \mathrm{M}, \alpha \in \Gamma$.

Note: X is faithful if $X\Gamma a = \{0\}$ forces $a = 0$ for all $a \in \mathrm{M}$. X is prime if $m\Gamma\mathrm{M}\Gamma x = 0$, for $m \in \mathrm{M}$ and $x \in X$ implies that either $x = 0$ or $m\Gamma X = 0$.

Lemma 5.2.2: [9] Suppose that X is a faithful prime $\Gamma\mathrm{M}-$module. Let $a, b \in \mathrm{M}$ and $x \in X$. If (the prime $\Gamma-$ring) M is 2-torsion free satisfying $(*)$ and $a\alpha m\beta b\gamma m\delta x = 0$, for all $m \in \mathrm{M}$ and $\alpha, \beta, \gamma, \delta \in \Gamma$, then $a = 0$ or $b = 0$, or $x = 0$.

Proof: We use the hypothesis $a\alpha m\beta b\gamma m\delta x = 0$, for all $a, b, m \in \mathrm{M}, x \in X$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Replacing m by $u + v$ in the above equation and then putting $v = m\beta a\alpha m\beta b\gamma m$, we get $a\alpha u\beta b\gamma m\beta a\alpha m\beta b\gamma m\delta x + a\alpha m\beta a\alpha m\beta b\gamma m\beta b\gamma u\delta x = 0$. This gives $a\alpha m\beta a\alpha m\beta b\gamma m\beta b\gamma u\delta x = 0$, for all $a, b, m, u \in \mathrm{M}, x \in X$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. If $x = 0$, we are done.

Suppose that $x \neq 0$. Since X is faithful and prime, then $(a\alpha m\beta a)\alpha m\beta(b\gamma m\beta b) = 0$, for all $a, b, m \in \mathrm{M}$ and $\alpha, \beta, \gamma \in \Gamma$. Primeness of M gives $a\alpha m\beta a = 0$ or $b\gamma m\beta b = 0$, and consequently, $a = 0$ or $b = 0$. □

Defining $D_\alpha(x) = [a, x]_\alpha$, for all $a, x \in \mathrm{M}$ and $\alpha \in \Gamma$, we have

Lemma 5.2.3: [9] Let M be a $\Gamma-$ring which satisfies $(*)$ and let $a \in \mathrm{M}$ be a fixed element. Then :

i. $D_\alpha(x)$ is a derivation.
ii. $D_\alpha D_\beta(x) = a\alpha D_\beta(x) - D_\beta(x)\alpha a$.
iii. $D_\alpha D_\beta(x) = D_\beta D_\alpha(x)$.
iv. $D_\alpha D_\beta(x\gamma y) = D_\alpha D_\beta(x)\gamma y + 2D_\alpha(x)\beta D_\gamma(y) + x\gamma D_\alpha D_\beta(y)$.

for all $x, y \in \mathrm{M}$ and $\alpha, \beta, \gamma \in \Gamma$.

Proof: (i) For all $x, y \in \mathrm{M}$, $\alpha, \beta \in \Gamma$ and using $(*)$, we have

$$D_\alpha(x\beta y) = [a, x\beta y]_\alpha = [a, x]_\alpha \beta y + x\alpha[a, y]_\beta = D_\alpha(x)\beta y + x\alpha D_\beta(y).$$

(ii) By definition, we have

$D_\alpha D_\beta(x) = D_\alpha\left([a, x]_\beta\right) = \left[a, [a, x]_\beta\right]_\alpha = a\alpha[a, x]_\beta - [a, x]_\beta \alpha a = a\alpha D_\beta(x) - D_\beta(x)\alpha a$ for all $a, x \in \mathrm{M}$ and $\alpha, \beta \in \Gamma$.

(iii) Using $(*)$, we get

$$D_\alpha D_\beta(x)=D_\alpha\left([a,x]_\beta\right)=\left[a,[a,x]_\beta\right]_\alpha=a\alpha(a\beta x-x\beta a)-(a\beta x-x\beta a)\alpha a$$

$$=a\beta(a\alpha x-x\alpha a)-(a\alpha x-x\alpha a)\beta a=\left[a,[a,x]_\alpha\right]_\beta=D_\beta\left([a,x]_\alpha\right)=D_\beta D_\alpha(x)$$

for all $a,x\in M$ and $\alpha,\beta\in\Gamma$.

(iv) By (ii) and $(*)$, we have

$$D_\alpha D_\beta(x\gamma y)=a\alpha a\beta x\gamma y-a\alpha x\gamma y\beta a-a\beta x\gamma y\alpha a+x\gamma y\beta a\alpha a$$

$$=(a\alpha a\beta x-a\alpha x\beta a-a\beta x\alpha a+x\beta a\alpha a)\gamma y+2a\alpha x\beta(a\gamma y-y\gamma a)$$

$$-2x\alpha a\beta(a\gamma y-y\gamma a)+x\gamma(a\alpha a\beta y-a\alpha y\beta a-a\beta y\alpha a+y\beta a\alpha a)$$

$$=D_\alpha D_\beta(x)\gamma y+2(a\alpha x-x\alpha a)\beta(a\gamma y-y\gamma a)+x\gamma D_\alpha D_\beta(y)$$

$$=D_\alpha D_\beta(x)\gamma y+2D_\alpha(x)\beta D_\gamma(y)+x\gamma D_\alpha D_\beta(y).$$

For all $x,y\in M$ and $\alpha,\beta,\gamma\in\Gamma$. □

Lemma 5.2.4: Let M be a $\Gamma-$ring satisfying $(*)$ and of characteristic not 3, and $d:M\to X$ a Jordan left derivation, where X is faithful and prime $\Gamma M-$module. If $d(a)\neq 0$, for some $a\in M$, then $\left[a,[a,b]_\beta\right]_\alpha\gamma\left[a,[a,b]_\beta\right]_\alpha=0$, for all $b\in M$ and $\alpha,\beta,\gamma\in\Gamma$.

Proof: Let $a\in M$ be a fixed element. By Lemma 5.2.3, we have

$$D_\alpha D_\beta(x)=a\alpha(a\beta x-x\beta a)-(a\beta x-x\beta a)\alpha a \tag{5.2.1}$$

for all $x\in M$ and $\alpha,\beta\in\Gamma$.

Using $(*)$ in $(a\alpha b-b\alpha a)\beta a\alpha D(a)=a\alpha(a\alpha b-b\alpha a)\beta D(a)$, for all $a,b\in M$ and $\alpha,\beta\in\Gamma$, we obtain

$$\left(a\alpha(a\beta x-x\beta a)-(a\beta x-x\beta a)\right)\alpha d(a)=0 \tag{5.2.2}$$

for all $x\in M$ and $\alpha,\beta\in\Gamma$.

From $(5.2.1)$ and $(5.2.2)$, we get

$$D_\alpha D_\beta(x)\alpha d(a)=0, \tag{5.2.3}$$

By Lemma 5.2.3 (iv) and $(5.2.3)$, we have

$$\left(D_\alpha D_\beta(x)\gamma y+2D_\alpha(x)\beta D_\gamma(y)\right)\alpha d(a)=0 \tag{5.2.4}$$

For all $x,y\in M$ and $\alpha,\beta,\gamma\in\Gamma$.

Replacing y by $D_\alpha(y\beta z)$ in $(5.2.4)$ and by Lemma 5.2.3 (i), we obtain

$$\left(D_\alpha D_\beta(x)\gamma\left(D_\alpha(y)\beta z+y\alpha D_\beta(z)\right)+2D_\alpha(x)\beta D_\gamma\left(D_\alpha(y\beta z)\right)\right)\alpha d(a)=0 \tag{5.2.5}$$

Using Lemma 5.2.3 (iii) in $(5.2.5)$, and then using $(5.2.3)$, we get

$$\left(D_\alpha D_\beta(x)\gamma\left(D_\alpha(y)\beta z+y\alpha D_\beta(z)+D_\alpha D_\beta(x)\gamma y\alpha D_\beta(z)\right)\right)\alpha d(a)=0 \tag{5.2.6}$$

Replacing $D_\alpha(z)$ for z in $(5.2.6)$, and then by Lemma 5.2.3 (iii) and $(5.2.3)$, we obtain

$$\left(D_\alpha D_\beta(x)\gamma D_\alpha(y)\alpha D_\beta(z)\right)\alpha d(a)=0 \tag{5.2.7}$$

Replacing $D_\alpha(y)$ for y in $(5.2.6)$, and then by Lemma 5.2.3 (iii) in $(5.2.7)$, we obtain

$$\left(\left(D_\alpha D_\beta(x)\right)\gamma\left(D_\alpha D_\beta(y)\right)\right)\alpha z\alpha d(a)=0 \tag{5.2.8}$$

Since $(5.2.8)$ holds for all $z\in M$, we are forced to conclude that $d\neq 0$ implies

$$\left(D_\alpha D_\beta(x)\right)\gamma\left(D_\alpha D_\beta(y)\right)=0$$

for all $x,y\in M$ and $\alpha,\beta,\gamma\in\Gamma$.

In particular, $\left(D_\alpha D_\beta(b)\right)\gamma\left(D_\alpha D_\beta(b)\right)=0$, For all $b\in M$ and $\alpha,\beta,\gamma\in\Gamma$.

This gives $\left[a,[a,b]_\beta\right]_\alpha\gamma\left[a,[a,b]_\beta\right]_\alpha=0$, for all $b\in M$ and $\alpha,\beta,\gamma\in\Gamma$. □

Theorem 5.2.5: [9] Let M be a $\Gamma-$ring satisfying $(*)$ and X be a 2-torsion free left $\Gamma M-$module. Suppose that, $a\alpha x=0$ with $a\in M$, $x\in X$ and $\alpha\in\Gamma$ implies that either $a=0$ or $x=0$. If $D:M\to X$ is generalized left derivation with associated non-zero Jordan left derivation $d:M\to X$. Then M is commutative.

Proof: For all $a,b\in M$ and $\alpha,\beta\in\Gamma$, then we have

$$D\left((a\alpha a)\beta b\right)=a\alpha a\beta D(b)+b\beta d(a\alpha a)=a\alpha a\beta D(b)+2b\beta a\alpha d(a)$$

$$D\left((a\alpha a)\beta b\right)=a\alpha a\beta D(b)+2b\beta a\alpha d(a) \tag{5.2.9}$$

On the other hand

$$D\big((a\alpha a)\beta b\big)=D\big(a\alpha(a\beta b)\big)=a\alpha D(a\beta b)+a\beta b\alpha d(a)$$

$$=a\alpha\big(a\beta D(b)+b\beta d(a)\big)+a\beta b\alpha d(a)$$

$$=a\alpha a\beta D(b)+a\alpha b\beta d(a)+a\beta b\alpha d(a)$$

But M satisfying $(*)$, then

$$D\big((a\alpha a)\beta b\big)=a\alpha a\beta D(b)+2a\beta b\alpha d(a) \tag{5.2.10}$$

Now from $(5.2.9)$ and $(5.2.10)$, we get

$$2[a,b]_\beta\,\alpha d(a)=0 \tag{5.2.11}$$

But $2[x,y]_\beta\,\alpha d(a)\in \mathrm{M}\Gamma X\subset X$ (since X be a left $\Gamma\mathrm{M}-$module) and X be a 2-torsion free therefore from $(5.2.11)$, we have $[a,b]_\beta\,\alpha d(a)=0$, by assumption then either $[a,b]_\beta=0$ or $d(a)=0$. For each $a\in\mathrm{M}$ either $a\in Z(\mathrm{M})$ or $d(a)=0$. But then $Z(\mathrm{M})$ and $Kerd=\{m\in\mathrm{M}:d(m)=0\}$ are additive subgroups of M. Since $Z(\mathrm{M})$ and $Kerd$ are proper subgroups of M, either $\mathrm{M}=Z(\mathrm{M})$ or $\mathrm{M}=Kerd$. But $d\neq 0$, then $\mathrm{M}=Z(\mathrm{M})$. This completes the proof. □

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ملخص البحث

تعرضنا في هذا البحث الى مفهوم الاشتقاقات المعرفة على حلقات Γ ، المعرفة على حلقات Γ الأولية و حلقات Γ شبه الأولية. كما تم تعميم هذه المفاهيم و دراسة الخصائص الاساسية لكل منها اضافة الى الشروط التي تجعل هذه الحلقات تبديلية .

و في الختام تم التعرف الى مقاييس ΓM اليسرى على حلقات Γ و كذلك مفهوم الاشتقاقات على مقاييس ΓM اليسرى .