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Vanishing of Ext and Tor over complete intersections

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VANISHING OF EXT AND TOR OVER COMPLETE INTERSECTIONS

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Let $(R, \mathfrak{m})$ be a local complete intersection, that is, a local ring whose $\mathfrak{m}$-adic completion is the quotient of a complete regular local ring by a regular sequence. Let M and N be finitely generated R-modules. This dissertation concerns the vanishing of $\operatorname{Tor}_i^R(M, N)$ and $\operatorname{Ext}_R^i(M, N)$.

In this context, M satisfies Serre's condition (S_n) if and only if M is an nth syzygy. The complexity of M is the least nonnegative integer r such that the nth Betti number of M is bounded by a polynomial of degree $r-1$ for all sufficiently large n. We use this notion of Serre's condition and complexity to study the vanishing of $\operatorname{Tor}_i^R(M, N)$. In particular, building on results of C. Huneke, D. Jorgensen and R. Wiegand [32], and H. Dao [21], we obtain new results showing that good depth properties on the R-modules M, N and $M \otimes_R N$ force the vanishing of $\operatorname{Tor}_i^R(M, N)$ for all $i \geq 1$. We give examples showing that our results are sharp. We also show that if R is a one-dimensional domain and M and $M \otimes_R \operatorname{Hom}_R(M, R)$ are torsion-free, then M is free if and only if M has complexity at most one.

If R is a hypersurface and $\operatorname{Ext}_R^i(M, N)$ has finite length for all $i \gg 0$, then the Herbrand difference [18] is defined as $\operatorname{length}(\operatorname{Ext}_R^{2n}(M, N)) - \operatorname{length}(\operatorname{Ext}_R^{2n-1}(M, N))$ for some (equivalently, every) sufficiently large integer n. In joint work with Hailong Dao, we generalize and study the Herbrand difference. Using the Grothendieck group of finitely generated R-modules, we also examined the number of consecutive vanishing of $\operatorname{Ext}_R^i(M, N)$ needed to ensure that $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$. Our results

recover and improve on most of the known bounds in the literature, especially when R has dimension two.

DEDICATION

For Ela

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Contents

Chapter 1

Introduction

If R is a commutative ring, and M and N are R-modules, then $M \otimes_R N$ often has torsion even when M and N are torsion-free. Perhaps the first organized investigation of torsion in tensor products of modules was started by Auslander [3] in 1961. The main tool in his investigation was his famous rigidity theorem; if R is an unramified regular local ring, and M and N are finitely generated R-modules, then the vanishing of $\operatorname{Tor}_n^R(M,N)$ for some nonnegative integer n forces the vanishing of $\operatorname{Tor}_i^R(M,N)$ for all $i \geq n$. This result was extended to all regular local rings by Lichtenbaum in [42], where the ramified case was proved. An easy corollary of this rigidity theorem shows that, over a regular local ring R, if $M \otimes_R N$ is torsion-free, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$ and both M and N are torsion-free [3], [42]. As such a rigidity result does not hold in general over complete intersections of positive codimension, torsion properties in tensor products of modules over such rings are more mysterious than those over regular local rings.

Three decades later Huneke and R. Wiegand [33] extended Auslander's ideas and studied tensor product of finitely generated modules over a hypersurface (which is a complete intersection of codimension one). They proved, for instance, that if $M \otimes_R N$

is maximal Cohen-Macaulay over a hypersurface domain R, then both M and N are maximal Cohen-Macaulay, and either M or N is free. Some of these results of Huneke and Wiegand on the tensor products of modules were then generalized by Jorgensen in [37] and [36]. Jorgensen used the notion of *complexity* and determined several sufficient conditions for the vanishing of $\operatorname{Tor}_i^R(M,N)$ over complete intersections of arbitrary codimension. The complexity of M is the least nonnegative integer r such that the nth Betti number of M is bounded by a polynomial of degree $r-1$ for all sufficiently large n. It follows from [30] that if R is a complete intersection, then $\operatorname{cx}_R(M)$ cannot exceed the codimension of R for all finitely generated R-modules M.

A module M satisfies Serre's condition (S_n) if $\operatorname{depth}(M_q) \geq \min\{n, \dim(R_q)\}$ for all prime ideals q of R. It turns out that, if R is Gorenstein, M satsifies (S_1) if and only if it is torsion-free, and M satisfies (S_2) if and only if it is reflexive, that is, the natural map $M \to M^{**}$ is bijective, where $M^* = \operatorname{Hom}_R(M,R)$. More generally, M satisfies (S_n) if and only if M is an nth syzygy [26].

An important theorem proved by Huneke and Wiegand [33, 2.7] states that if $M \otimes_R N$ is reflexive over a hypersurface domain, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. Considering this and Auslander's result discussed above, Huneke, Wiegand and Jorgensen asked the following question in their paper [32]:

> Let R be a complete intersection of codimension c. Assume $M \otimes_R N$ satisfies (S_{c+1}) and some extra mild conditions. Then is $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$?

Several affirmative answers to this question were proved in the same paper [32] over complete intersections of codimension two and three. More recently Dao [21, 7.7] proved that if R has codimension c and is a quotient of an unramified regular local ring, M and N satisfy (S_c), $M \otimes_R N$ satisfies (S_{c+1}) and M_p is a free R_p-module for all prime ideals p of R with $\dim(R_p) \leq c$, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Chapter 3 of this thesis mainly serves to improve this result.

In the first part of Chapter 3 we collect some known results on the vanishing of Tor over complete intersections. Section 3.2 contains the definitions of *pushforwards* and *quasi-liftings*, and some preliminary results about them. The pushforward of a finitely generated torsion-free module is defined in [26]. Quasi-liftings were defined in [33] by Huneke and Wiegand, and were studied in the papers [32] and [34]. In section 3.3 we prove the following result that is stated as Theorem 3.3.2:

Theorem 1.0.1. *Let R be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$ (so that the codimension of R is c). Let M and N be finitely generated R-modules. Assume M and N satisfy (S_{c-1}), $M \otimes_R N$ satisfies (S_c), and M_p is a free R_p-module for all prime ideals p of R with $\dim(R_p) \leq c$. Then either $\operatorname{cx}_R(M) = \operatorname{cx}_R(N) = c$, or $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.*

This improves Dao's theorem discussed above for modules of small complexities. The non-triviality of our result follows from the fact that if R is a complete intersection, then there are many finitely generated modules of infinite projective dimension that have complexities strictly smaller than the codimension of R. We improve our result in Theorem 3.3.7 in case the module M considered is maximal Cohen-Macaulay. We also show in Examples 3.3.11 and 3.3.12 that our results are sharp.

Chapter 4 of this thesis is a joint work with Hailong Dao. It concerns the vanishing of $\operatorname{Ext}_R^i(M, N)$ for finitely generated modules M and N when M has finite *complete intersection dimension* (see Definition 4.3.1). This situation is slightly more general than assuming R is a complete intersection; modules over complete intersections have finite complete intersection dimension and modules of finite complete intersection dimension behave homologically like modules over complete intersections. For instance, a module M has finite complexity if it has finite complete intersection dimension, cf. Theorem 4.3.2. One of the main technical tools in this chapter is a generalization of the *Herbrand difference* (see Definition 4.2.1) that was first introduced by Buchweitz [18] over hypersurfaces that have isolated singularities. This approach yields sharper results than most of the bounds for the vanishing pattern of $\operatorname{Ext}_R^i(M,N)$ previously known in the literature.

In Section 4.1 we prove and record some preliminary results. In Section 4.2 we follow the arguments in [21] and define a pairing, denoted by $\mathrm{h}_e^R(M,N)$ for a positive integer e, that generalizes the Herbrand difference (see Definition 4.2.3). This function can be defined for a pair of finitely generated modules over a local ring R provided M has finite complete intersection dimension and $\operatorname{Ext}_R^i(M,N)$ has finite length for all $i \gg 0$. We also often use the *Grothendieck group* $G(R)$ of finitely generated R-modules. In some special cases, vanishing of $\mathrm{h}_e^R(M,N)$ yields nice consequences for the vanishing pattern of Ext modules. These are studied in Theorem 4.3.5 and Proposition 4.3.11.

Suppose R is a local ring and M and N are finitely generated R-modules. Assume M has finite complete intersection dimension, and let c be the complexity of M. Then $\operatorname{Ext}_R^n(M,N) = \cdots = \operatorname{Ext}_R^{n+c}(M,N) = 0$ for some $n > \operatorname{depth}(R) - \operatorname{depth}(M)$ if and only if $\operatorname{Ext}_R^i(M,N) = 0$ for all $i > \operatorname{depth}(R) - \operatorname{depth}(M)$. This result was proved by Avramov and Bucweitz [9], and Jorgensen [38] independently. Section 4.3 contains various vanishing results that generalize and improve upon this theorem. In Theorem 4.3.5 we replace c with the complexity of the pair (M,N), $\operatorname{cx}_R(M,N)$, a nonnegative integer that is bounded by the complexity of M (see Definition 2.2.2). We also improve Theorem 4.3.5 over one dimensional complete intersection domains, cf. Theorem 4.3.15. In particular, in Proposition 4.3.16, we prove the following:

Proposition 1.0.2. *Let R be an one-dimensional local hypersurface domain and let M and N be finitely generated R-modules. If $\operatorname{Ext}_R^n(M, N) = 0$ for some positive integer n, then either M or N has finite projective dimension.*

Chapter 2

Preliminaries

2.1 Notation and definitions

All rings are assumed to be local, that is, commutative and Noetherian with a unique maximal ideal, and all modules are assumed to be finitely generated. Our standard references for terminology are the books [17] and [43].

Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be a finitely generated R-module.

The *codimension* of R is defined to be the nonnegative integer $\operatorname{embdim}(R) - \dim(R)$ where $\operatorname{embdim}(R)$, the *embedding* dimension of R, is the k-vector space dimension of $\mathfrak{m}/\mathfrak{m}^2$. $\hat{R}$ will denote the m-adic completion of R. We will call R a *complete intersection* if $\hat{R}$ is of the form $S/(\underline{f})$ where $(S, \mathfrak{n})$ is a complete regular local ring and $\underline{f}$ is a regular sequence of S contained in $\mathfrak{n}$. Since $S/(g)$ is again a regular local ring if $g \in \mathfrak{n} - \mathfrak{n}^2$, we can always assume, by shortening the sequence if necessary, that $(\underline{f}) \subseteq \mathfrak{n}^2$, and in this case the codimension of R is equal to the length of the regular sequence $\underline{f}$ (cf. [17, Section 2.3]). A complete intersection of codimension one is called a *hypersurface.*

The depth of M, denoted by $\operatorname{depth}(M)$, is the length of a maximal M-regular sequence contained in $\mathfrak{m}$. (The depth of the zero module is defined to be ∞.) We say that M is *maximal Cohen-Macaulay* if M is nonzero and $\operatorname{depth}(M) = \dim(R)$.

Let S be the set of non-zerodivisors of R, and let $K = S^{-1}R$ be the total quotient ring of R. Then the torsion submodule of M, $t(M)$, is the kernel of the natural map $M \to M \otimes_R K$. M is called *torsion* provided $t(M) = M$ and *torsion-free* provided $t(M) = 0$.

Let $\mathbf{F} : \ldots \to F_2 \to F_1 \to F_0 \to 0$ be a minimal free resolution of M over R. Then the rank of F_n, denoted by $\beta_n^R(M)$, is the nth *Betti* number of M and is equal to $\dim_k(\operatorname{Tor}_n^R(M,k)) = \dim_k(\operatorname{Ext}_R^n(M,k))$. The nth *syzygy* of M, denoted by $\operatorname{syz}_n^R(M)$, is the image of the map $F_n \to F_{n-1}$ and is unique up to isomorphism. (We set $\operatorname{syz}_0^R(M) = M$.)

2.2 Complexity

The notion of *complexity*, which is a homological characteristic of modules, was first introduced by Alperin in [1] to study minimal projective resolutions of modules over group algebras. It was then brought into local algebra by Avramov [7].

Definition 2.2.1. *([7, 3.1]) The module M has* complexity r*, written as* $\operatorname{cx}_R(M) = r$*, provided r is the least nonnegative integer for which there exists a real number γ such that $\beta_n^R(M) \leq \gamma \cdot n^{r-1}$ for all $n \gg 0$. If there are no such r and γ, then one sets* $\operatorname{cx}_R(M) = \infty$.

The complexity of M measures how the Betti sequence $\beta_0^R(M), \beta_1^R(M), \ldots$ behaves with respect to polynomial growth. In general complexity may be infinite; for example, if $R = k[X,Y]/(X^2, XY, Y^2)$, then $\operatorname{cx}_R(M) \in \{0, \infty\}$ [8, 4.2.2]. It follows from the definition that M has finite projective dimension if and only if $\operatorname{cx}_R(M) = 0$, and M has bounded Betti numbers if and only if $\operatorname{cx}_R(M) \leq 1$.

Avramov and Buchweitz [9] defined the complexity for a pair of finitely generated R-modules (M, N) as follows:

Definition 2.2.2. *([9]) The pair (M, N) has* complexity r*, written as* $\mathrm{cx}_R(M, N) = r$*, provided r is the least nonnegative integer for which there exists a real number γ such that* $\dim_k(\mathrm{Ext}^n_R(M, N) \otimes_R k) \leq \gamma \cdot n^{r-1}$ *for all $n \gg 0$. If there are no such r and γ, then one sets* $\mathrm{cx}_R(M, N) = \infty$.

Thus the complexity $\mathrm{cx}_R(M, N)$ measures the size of $\mathrm{Ext}^i_R(M, N)$ for sufficiently large i. It follows from the definition of complexity that $\mathrm{cx}_R(M) = \mathrm{cx}_R(M, k)$, and $\mathrm{cx}_R(M, N) = 0$ if and only if $\mathrm{Ext}^i_R(M, N) = 0$ for all $i \gg 0$.

Chapter 3

Theorems on the vanishing of Tor

The contents of this chapter are contained in the author's paper:
Vanishing of Tor *over complete intersections*, to appear in J. Commutative Algebra.

3.1 Preliminary Results

We record some of the major theorems about the vanishing of Tor that will be used in this chapter.

Recall that a regular local ring $(S, \mathfrak{n})$ is called *unramified* [43, Chapter 29] if it is either equicharacteristic (that is, it contains a field) or is of mixed characteristic p and $p \notin \mathfrak{n}^2$. Moreover, a complete unramified regular local ring is a formal power series over either its residue field or a complete unramified discrete valuation ring [43, 29.7]

The rigidity of Tor starts with the following famous theorem of Auslander and Lichtenbaum:

Theorem 3.1.1. *([3, Corollary 2.2] and [42, Corollary 1]) Let $(R, \mathfrak{m})$ be a regular local ring, and let M and N be finitely generated R-modules. If $\operatorname{Tor}_n^R(M, N) = 0$ for some nonnegative integer n, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq n$.*

This result was first proved by Auslander [3] for unramified regular local rings and then extended to all regular local rings by Lichtenbaum in [42], where the ramified case was proved. Murthy [45] proved that a similar rigidity theorem holds over an arbitrary complete intersection of codimension c, provided one assumes the vanishing of $c+1$ consecutive Tor modules:

Theorem 3.1.2. *([45, 1.6]) Let $(R, \mathfrak{m})$ be a local complete intersection of codimension c, and let M and N be finitely generated R-modules. If*

$$\operatorname{Tor}_n^R(M,N) = \operatorname{Tor}_{n+1}^R(M,N) = \cdots = \operatorname{Tor}_{n+c}^R(M,N) = 0$$

for some $n \geq 1$, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq n$.

As discussed in Chapter 2, the complexity of a module can be infinite. The following result of Gulliksen shows that this cannot happen over complete intersections:

Theorem 3.1.3. *([30]) Let $(R, \mathfrak{m})$ be a local complete intersection of codimension c, and let M be a finitely generated R-module. Then $\operatorname{cx}_R(M) \leq c$.*

It is worth noting that, over a complete intersection of codimension c, there exist modules of complexity r for any non-negative integer $r \leq c$; such modules were first constructed over local rings by Avramov [7, 6.6] (see also [10, 3.1-3.3]).

We will use several results of D. A. Jorgensen. The next one we record is a generalization of Murthy's theorem (Theorem 3.1.2).

Theorem 3.1.4. *([37, 2.3]) Let $(R, \mathfrak{m})$ be a local complete intersection of dimension d, and let M and N be finitely generated R-modules. Set $r = \min\{\operatorname{cx}_R(M), \operatorname{cx}_R(N)\}$ and $b = \max\{\operatorname{depth}_R(M), \operatorname{depth}_R(N)\}$. If*

$$\operatorname{Tor}_n^R(M,N) = \operatorname{Tor}_{n+1}^R(M,N) = \ldots = \operatorname{Tor}_{n+r}^R(M,N) = 0$$

for some $n \geq d-b+1$, then $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geq d-b+1$.

Theorem 3.1.5. *([37, 1.3]) Let $(R,\mathfrak{m})$ be a local complete intersection of codimension $c \geq 1$, and let F be a finite set of R-modules. Assume R is complete and has infinite residue field. Then there exists a complete intersection R_1 of codimension $c-1$, and a non-zerodivisor x of R_1 such that $R=R_1/(x)$ and, for all $M \in F$,*

$$\operatorname{cx}_{R_1}(M)=\begin{cases}\operatorname{cx}_R(M)-1, & \text{if } \operatorname{cx}_R(M)>0, \text{ and}\\ 0, & \text{if } \operatorname{cx}_R(M)=0.\end{cases}$$

As stated in [37], Theorem 3.1.5 also follows from a more general result of Avramov [7, 3.2.3 & 3.6].

Theorem 3.1.6. *([37, 2.7], cf. also [25]) Let $(R,\mathfrak{m})$ be a d-dimensional local complete intersection, and let M and N be finitely generated R-modules, at least one of which has complexity one. Set $b=\max\{\operatorname{depth}_R(M),\operatorname{depth}_R(N)\}$. Then $\operatorname{Tor}_i^R(M,N)\cong \operatorname{Tor}_{i+2}^R(M,N)$ for all $i \geq d-b+1$.*

Another important result that we will use is the depth formula. Auslander [3, 1.2] proved that if $(R,\mathfrak{m})$ is a local ring, M and N are finitely generated R-modules such that M has finite projective dimension and $q=\sup\{i : \operatorname{Tor}_i^R(M,N)\neq 0\}$, then the equality

$$\operatorname{depth}(M)+\operatorname{depth}(N)=\operatorname{depth}(R)+\operatorname{depth}(\operatorname{Tor}_q^R(M,N))-q$$

holds, provided either $q=0$ or $\operatorname{depth}(\operatorname{Tor}_q^R(M,N))\leq 1$. We refer the above equality as *depth formula*. This remarkable equality, for the case where $q=0$, was later obtained by Huneke and Wiegand for complete intersections without the finite projective dimension restriction on M. (See also [2] and [35] for some of the generalizations of the depth formula.)

Theorem 3.1.7. *[33, 2.5] Let $(R, \mathfrak{m})$ be a local complete intersection, and let M and N be finitely generated R-modules. If $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$, then the depth formula for M and N holds:*

$$\operatorname{depth}(M) + \operatorname{depth}(N) = \operatorname{depth}(R) + \operatorname{depth}(M \otimes_R N)$$

We will also make use of the following result of H. Dao.

Theorem 3.1.8. *([21, 7.7]) Let $(R, \mathfrak{m})$ be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$, for $c > 0$, is a regular sequence of S contained in $\mathfrak{n}^2$. Let M and N be finitely generated R-modules. Assume the following conditions hold:*

1. $\operatorname{Tor}_1^R(M,N) = \operatorname{Tor}_2^R(M,N) = \ldots = \operatorname{Tor}_c^R(M,N) = 0$.
2. $\operatorname{depth}(N) > 0$ *and* $\operatorname{depth}(M \otimes_R N) > 0$.
3. $\operatorname{Tor}_i^R(M,N)$ *has finite length for all* $i \gg 0$.

Then $\operatorname{Tor}_i^R(M,N) = 0$ *for all* $i \geq 1$.

3.2 Pushforward and Quasi-lifting

We recall the definitions of pushforward [26] and quasi-lifting [32]:

Let R be a Gorenstein ring, M a finitely generated torsion-free R-module, and $\{f_1, f_2, \ldots, f_m\}$ a minimal generating set for M^*. Let $\delta : R^{(m)} \twoheadrightarrow M^*$ be defined by

$\delta(e_i) = f_i$ for $i = 1, 2, \ldots, m$ where $\{e_1, e_2, \ldots, e_m\}$ is the standard basis for $R^{(m)}$. Then, composing the natural map $M \hookrightarrow M^{**}$ with δ^*, we define M_1 so that

$$\text{(PF)} \quad 0 \to M \xrightarrow{u} R^{(m)} \to M_1 \to 0$$

is a short exact sequence, where $u(x) = (f_1(x), f_2(x), \ldots, f_m(x))$ for all $x \in M$. Any module M_1 obtained in this way is referred to as a *pushforward* of M. We should note that such a construction is unique, up to a non-canonical isomorphism (cf. page 62 of [26]). Indeed, suppose $\{g_1, g_2, \ldots, g_m\}$ is another minimal generating set for M^*. Then, by the uniqueness of minimal resolutions, there exists an isomorphism φ so that the following diagram commutes,

$$\begin{tikzcd}
R^{(m)} \arrow[r, "\delta"] \arrow[d, dotted, "\varphi"', "\cong"] & M^* \arrow[r] \arrow[d, equal] & 0 \\
R^{(m)} \arrow[r, "\chi"'] & M^* \arrow[r] & 0
\end{tikzcd}$$

where $\chi(e_i) = g_i$ for $i = 1, 2, \ldots, m$. It follows that $\varphi^t v = u$ where φ^t is the transpose of φ and $v(x) = (g_1(x), g_2(x), \ldots, g_m(x))$. Hence we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "v"] \arrow[d, equal] & R^{(m)} \arrow[r] \arrow[d, "\varphi^t"', "\cong"] & M_1' \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & M \arrow[r, "u"] & R^{(m)} \arrow[r] & M_1 \arrow[r] & 0
\end{tikzcd}$$

Assume now $R = S/(f)$, where S is a Gorenstein ring and f is a non-zerodivisor of S. Let $S^{(m)} \twoheadrightarrow M_1$ be the composition of the canonical map $S^{(m)} \twoheadrightarrow R^{(m)}$ and the map $R^{(m)} \twoheadrightarrow M_1$ in (PF). Then a *quasi-lifting* of M with respect to the presentation

$R = S/(f)$ is the S-module E in the following short exact sequence:

$$\text{(QL)} \quad 0 \to E \to S^{(m)} \to M_1 \to 0$$

Therefore the quasi-lifting of M is unique, up to an isomorphism of S-modules.

In the next section we will study the vanishing of Tor for the modules satisfying the following condition (cf. [26]):

Definition 3.2.1. *For a non-negative integer n, we say M satisfies* Serre's condition (S_n) *if* $\operatorname{depth}_{R_q}(M_q) \geq min\{n, \dim(R_q)\}$ *for all* $q \in \operatorname{Spec}(R)$.

Note that this definition is different from the one given in [29, 5.7.2.I]. If R is Cohen-Macaulay, then Samuel proved in [47] that M satisfies (S_n) if and only if every R-regular sequence $x_1, x_2, \ldots, x_k$, with $k \leq n$, is also an M-regular sequence. This implies, in particular, if R is Cohen-Macaulay, then M satisfies (S_1) if and only if it is torsion-free. Moreover, if R is Gorenstein, then M satisfies (S_2) if and only if it is reflexive, that is, the natural map $M \to M^{**}$ is bijective, where $M^* = \operatorname{Hom}_R(M, R)$ (see [26, 3.6]).

We collect several properties of the pushforward and quasi-lifting from [32].

Proposition 3.2.2. *([32, 1.6 - 1.8]) Let $R = S/(f)$ where S is a Gorenstein ring and f is a non-zerodivisor of S. Assume M and N are finitely-generated torsion-free R-modules. Let M_1 and N_1 denote the pushforwards and E and F the quasi-liftings of M and N, respectively. Then one has the following properties:*

1. *Suppose $q \in \operatorname{Spec}(R)$ and M_q is maximal Cohen-Macaulay over R_q. If $(M_1)_q \neq 0$, then $(M_1)_q$ is maximal Cohen-Macaulay over R_q.*

2. *Suppose n is a positive integer. If M satisfies (S_n) as an R-module, then M_1 satisfies (S_{n-1}) as an R-module.*

3. *There is a short exact sequence of R-modules:* $0 \to M_1 \to E/fE \to M \to 0$.

4. *If* $p \in \operatorname{Spec}(S)$ *and* $f \notin p$, *then* E_p *is free over* S_p.

5. *Suppose* $p \in \operatorname{Spec}(S)$, $f \in p$ *and* $q = p/(f)$. *If* M_q *is free over* R_q, *then* E_p *is free over* S_p.

6. *Suppose* $p \in \operatorname{Spec}(S)$, $f \in p$ *and* $q = p/(f)$. *If* $(M_1)_q \neq 0$, *then* $\operatorname{depth}_{S_p}(E_p) = \operatorname{depth}_{R_q}((M_1)_q) + 1$.

7. *Suppose* S *is a complete intersection ring and* v *is a positive integer. Assume that both* M *and* N *satisfy* (S_v) *as R-modules and that* $M \otimes_R N$ *satisfies* (S_{v+1}) *as an R-module. If* $\operatorname{Tor}_i^R(M,N)_q = 0$ *for all* $i \geq 1$ *and all* $q \in X^v(R)$, *then* $E \otimes_S F$ *satisfies* (S_v).

The following proposition is embedded in the proofs of [32, 1.8] and [32, 2.4].

Proposition 3.2.3. *([32]) Let* $R = S/(f)$ *where* $(S, \mathfrak{n})$ *is a complete intersection and* f *is a non-zerodivisor of* S *contained in* $\mathfrak{n}$. *Assume* M *and* N *are finitely generated torsion-free R-modules. Let* M_1 *and* N_1 *denote the pushforwards and* E *and* F *the quasi-liftings of* M *and* N, *respectively.*

1. $\operatorname{Tor}_i^R(E/fE, N) \cong \operatorname{Tor}_i^S(E, F)$ *for all* $i \geq 1$.

2. *For each* $i \in \mathbb{Z}$ *there exists an exact sequence*
$$\operatorname{Tor}_{i+2}^R(E/fE, N) \to \operatorname{Tor}_{i+2}^R(M, N) \to \operatorname{Tor}_{i+1}^R(M_1, N) \to \operatorname{Tor}_{i+1}^R(E/fE, N)$$
$$\to \operatorname{Tor}_{i+1}^R(M, N).$$

3. *Assume* $\operatorname{Tor}_i^R(M,N)_q = 0$ *for all* $i \geq 1$ *and all* $q \in X^1(R)$.

    a) *If* $M \otimes_R N$ *is torsion-free, then* $\operatorname{Tor}_1^R(M_1, N) = 0$.

b) Assume $M \otimes_R N$ is reflexive. Then $M_1 \otimes_R N$ is torsion-free. Moreover, if $\operatorname{Tor}_i^S(E, F) = 0$ for all $i \geq 1$, then $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

4. *Let w be a positive integer. Assume $M \otimes_R N$ is torsion-free, and that $\operatorname{Tor}_i^R(M, N)_q = 0$ for all $i \geq 1$ and all $q \in X^w(R)$. Then $\operatorname{Tor}_i^S(E, F)_p = 0$ for all $i \geq 1$ and all $p \in X^{w+1}(S)$.*

Proof. Consider the pushforward and quasi-lifting of N:

$$(3.2.3.1)\ \ 0 \to N \to R^{(n)} \to N_1 \to 0$$

$$(3.2.3.2)\ \ 0 \to F \to S^{(n)} \to N_1 \to 0$$

Tensoring (3.2.3.1) with E/fE, we have that $\operatorname{Tor}_{i+1}^R(E/fE, N_1) \cong \operatorname{Tor}_i^R(E/fE, N)$ for all $i \geq 1$. Therefore [43, Chapter 18, Lemma 2] and (3.2.3.2) yield the isomorphism in (1).

Statement (2) follows at once from the exact sequence in Proposition 3.2.2(3).

For (3), consider the pushforward of M:

$$(3.2.3.3)\ \ 0 \to M \to R^{(m)} \to M_1 \to 0$$

Tensoring (3.2.3.3) with N, we get

$$(3.2.3.4)\ \ \operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+1}^R(M_1, N) \text{ for all } i \geq 1.$$

Let $q \in X^1(R)$. Then, since N is torsion-free, N_q is maximal Cohen-Macaulay over R_q. Moreover, (3.2.3.4) implies that $\operatorname{Tor}_i^R(M_1, N)_q = 0$ for all $i \geq 2$. Therefore, by

Theorem 3.1.4, we have

$$\text{(3.2.3.5)}\quad \operatorname{Tor}_i^R(M_1, N)_q = 0 \text{ for all } i \geq 1.$$

Note that (3.2.3.3) implies that there is an injection $\operatorname{Tor}_1^R(M_1, N) \hookrightarrow M \otimes_R N$. Thus part ($a$) follows from (3.2.3.5). Assume now $M \otimes_R N$ is reflexive. We will prove that $M_1 \otimes_R N$ is torsion-free. If $\dim(R) = 1$, then, by assumption, $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$. Therefore the claim follows from Proposition 3.2.2(1) and Theorem 3.1.7. Thus we may assume $\dim(R) \geq 2$. Let q be a prime ideal of R such that $(M_1 \otimes_R N)_q \neq 0$. Assume $\dim(R_q) \leq 1$. Then, by (3.2.3.5) and Theorem 3.1.7, $(M_1 \otimes_R N)_q$ is maximal Cohen-Macaulay. Assume now $\dim(R_q) \geq 2$. Note that (3.2.3.3) yields the following exact sequence:

$$\text{(3.2.3.6)}\quad 0 \to M \otimes_R N \to N^{(m)} \to M_1 \otimes_R N \to 0$$

Since $M \otimes_R N$ is reflexive, localizing (3.2.3.6) at q, we see that the depth lemma implies $\operatorname{depth}_{R_q}((M_1 \otimes_R N)_q) \geq 1$. This proves that $M_1 \otimes_R N$ is torsion-free. Suppose now that $\operatorname{Tor}_i^S(E, F) = 0$ for all $i \geq 1$. Then, by (1) and (2), we have

$$\text{(3.2.3.7)}\quad \operatorname{Tor}_{i+2}^R(M, N) \cong \operatorname{Tor}_{i+1}^R(M_1, N) \text{ for all } i \geq 0.$$

In particular, $\operatorname{Tor}_2^R(M, N) = 0$. Note that, by (3.2.3.4) and (3.2.3.7), we have that $\operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_{i+2}^R(M, N)$ for all $i \geq 1$. Since $\operatorname{Tor}_1^R(E/fE, N) = 0$ by (1), letting $i = -1$ in (2), we see that $\operatorname{Tor}_1^R(M, N) = 0$. Therefore $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. This proves (3).

For (4), let $p \in X^{w+1}(S)$. By Proposition 3.2.2(4), we may assume $f \in p$. Let $q = p/(f)$. Then $q \in X^w(R)$. Recall that, by ($3a$), $\operatorname{Tor}_1^R(M_1, N) = 0$. Moreover, (3.2.3.4) implies that $\mathrm{Tor}_i^R(M_1,N)_q = 0$ for all $i \geq 2$. Therefore $\mathrm{Tor}_i^R(M_1,N)_q = 0$ for all $i \geq 1$. Now the short exact sequence in Proposition 3.2.2(3) yields that $\mathrm{Tor}_i^R(E/fE,N)_q = 0$ for all $i \geq 1$. Thus (4) follows from the isomorphism in (1). $\square$

3.3 Depth of tensor products of modules and the vanishing of Tor

We start examining certain conditions on the modules M, N and $M \otimes_R N$ that imply the vanishing of homology modules $\mathrm{Tor}_i^R(M,N)$ for all $i \geq 1$. Our main instruments will be pushforwards and quasi-liftings.

Following [32] we denote by $X^n(R)$ the set $\{q \in \mathrm{Spec}(R) : \mathrm{depth}(R_q) \leq n\}$ and say M is free on $X^n(R)$ if M_q is a free R_q-module for all $q \in X^n(R)$. We will usually assume that R is a complete intersection, in which case $\mathrm{depth}(R_q) = \dim(R_q)$ for all $q \in \mathrm{Spec}(R)$.

Our results are motivated by the following theorem due to H. Dao. Recall from Definition 3.2.1 that M satisfies (S_n) if $\mathrm{depth}_{R_q}(M_q) \geq \min\{n, \dim(R_q)\}$ for all $q \in \mathrm{Spec}(R)$.

Theorem 3.3.1. *([21, 7.6]) Let $(R,\mathfrak{m})$ be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S,\mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$. Let M and N be finitely generated R-modules. Assume M is free on $X^c(R)$, M and N satisfy (S_c) and $M \otimes_R N$ satisfies (S_{c+1}). Then* $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i \geq 1$.

Although Theorem 3.3.1 is a powerful tool, it has no content when $c \geq \dim(R)$. (The assumption that M is free on $X^c(R)$ forces M to be free). We will prove variations of this result that give useful information even when $c \geq \dim(R)$.

Note that, if one assumes M is free on $X^{c-1}(R)$ instead of $X^c(R)$ in Theorem 3.3.1, then it is not necessarily true that $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geq 1$: Let $R = k[[X,Y]]/(XY)$ and $M = R/(x)$. Then M is torsion-free and is free on $X^0(R)$; moreover, $\operatorname{Tor}_i^R(M,M) \neq 0$ if and only if i is a positive odd integer or zero. Assuming M is free on $X^{c-1}(R)$, we show in Theorem 3.3.2 that non-vanishing homology can occur only if the modules considered have maximal complexities (see Definition 2.2.1). This improves Theorem 3.3.1 for modules of small complexities (see also Corollary 3.3.14).

Theorem 3.3.2. *Let $(R,\mathfrak{m})$ be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S,\mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$. Let M and N be finitely generated R-modules. Assume the following conditions hold:*

1. *M and N satisfy (S_{c-1}).*
2. *$M \otimes_R N$ satisfies (S_c).*
3. *If $c \geq 2$, then assume further that $\operatorname{Tor}_i^R(M,N)_q = 0$ for all $i \geq 1$ and all $q \in X^{c-1}(R)$ (e.g., M is free on $X^{c-1}(R)$).*

Then either $\operatorname{cx}_R(M) = \operatorname{cx}_R(N) = c$, or $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.

Proof. Note that if $R \to A$ is a flat local homomorphism of Gorenstein rings, b is a positive integer, and X is a finitely generated R-module satisfying (S_b) as an R-module, then $X \otimes_R A$ satisfies (S_b) as an A-module. (This follows from Proposition 3.2.2(2); see [40, 1.3] or the proof of [26, 3.8] for a stronger result.) Moreover, an unramified regular local ring (S,n) remains unramified when we extend its residue field by using the faithfully flat extension $S \hookrightarrow S[z]_{\mathfrak{n}S[z]}$ where z is an indeterminate over S. Therefore, without loss of generality, we may assume R is complete and has infinite residue field.

We will use the same notations for the pushforwards and quasi-liftings of M and N as in the proof of Proposition 3.2.3.

If $c = 0$, then $\mathrm{cx}_R(M) = \mathrm{cx}_R(N) = 0$ (cf. Theorem 3.1.3), and so we may assume $c \geq 1$. Without loss of generality, we will assume $\mathrm{cx}_R(M) < c$ and prove that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. We proceed by induction on c. Suppose $c = 1$. Then, by assumption, M has finite projective dimension. Since $M \otimes_R N$ is torsion-free, [34, 2.3] implies that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. Assume now $c \geq 2$. By the proof of [37, 1.3] (Theorem 3.1.5), there exists a regular sequence $\underline{x} = x_1, x_2, \ldots, x_c$ generating $(\underline{f})$ such that $R = R_1/(x)$ and $\mathrm{cx}_{R_1}(M_1) < \mathrm{codim}(R_1) = c-1$, where $R_1 = S/(x_2, x_3, \ldots, x_c)$ and $x = x_1$. It follows that $\mathrm{cx}_{R_1}(E) < \mathrm{codim}(R_1)$. Note that (2) and (6) of Proposition 3.2.2 imply that E and F satisfy (S_{c-1}). Moreover, letting $w = c-1$ in Proposition 3.2.3(4), we have $\mathrm{Tor}_i^{R_1}(E,F)_p = 0$ for all $i \geq 1$ and all $p \in X^c(R_1)$. Finally, setting $v = c-1$ in Proposition 3.2.2(7), we conclude that $E \otimes_{R_1} F$ satisfies (S_{c-1}). Hence, if we replace M and N by E and F and c by $c-1$, the induction hypothesis implies that $\mathrm{Tor}_i^{R_1}(E,F) = 0$ for all $i \geq 1$. Therefore, by Proposition 3.2.3(3b), $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. $\square$

Corollary 3.3.3. *Let $(R, \mathfrak{m})$ be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$. Let M be a finitely generated R-module. Assume the following conditions hold:*

1. *M satisfies (S_{c-1}).*
2. *M is free on $X^{c-1}(R)$.*
3. *$M \otimes_R M$ satisfies (S_c).*

Then either $\mathrm{cx}_R(M) = c$, or M has finite projective dimension.

Proof. The case where $c \leq 1$ is trivial. Suppose $c \geq 2$ and $\mathrm{cx}_R(M) < c$. Then, by Theorem 3.3.2, $\mathrm{Tor}_i^R(M, M) = 0$ for all $i \geq 1$. Therefore, by [36, 1.2], M has finite projective dimension. $\square$

Remark 3.3.4. *Note that, in the previous corollary, if $c \geq 1$ and $\mathrm{cx}_R(M) < c$, then $\mathrm{Tor}_i^R(M, M) = 0$ for all $i \geq 1$ and hence the localization of the depth formula of Theorem 3.1.7 shows that M satisfies (S_c). It is not known (at least to the author) whether one can conclude the same thing for the module M in Theorem 3.3.2. More specifically, if $(R, \mathfrak{m})$ is a local complete intersection, and M and N are non-zero finitely generated R-modules such that $M \otimes_R N$ satisfies (S_n) for some positive integer n and $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$, then does M satisfy (S_n)? [2, 2.8] asserts a positive answer to this question, but the proof is flawed. The localization of the depth formula at a prime ideal which is not in the support of N does not reveal anything about the depth of M.*

Next we examine Theorem 3.3.2 when one of the modules considered is maximal Cohen-Macaulay. We will use the following variant of Theorem 3.1.4.

Proposition 3.3.5. [1] *Let (R, m) be a d-dimensional local complete intersection ring, and let M and N be finitely generated R-modules. Set $r = \min\{\mathrm{cx}_R(M), \mathrm{cx}_R(N)\}$ and $b = \max\{\mathrm{depth}_R(M), \mathrm{depth}_R(N)\}$. Assume $r \geq 1$ and*

$$\mathrm{Tor}_n^R(M, N) = \mathrm{Tor}_{n+1}^R(M, N) = \ldots = \mathrm{Tor}_{n+r-1}^R(M, N) = 0$$

for some $n \geq d - b + 1$.

1. *If r is odd, then $\mathrm{Tor}_{n+2i}^R(M, N) = 0$ for all $i \geq 0$.*

2. *If r is even, then $\mathrm{Tor}_{n+2i+1}^R(M, N) = 0$ for all $i \geq 0$.*

[1] After this result was posted, a more general version of it appeared in [15].

Proof. Without loss of generality we may assume $r = \mathrm{cx}_R(M)$. Moreover, by passing to $R[z]_{\mathfrak{m}R[z]}$ and then completing, we may assume that R is complete and has infinite residue field. We proceed by induction on r. Assume $r = 1$. Then, by Theorem 3.1.6, $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i+2}^R(M,N)$ for all $i \geq d-b+1$. Since $\mathrm{Tor}_n^R(M,N) = 0$ by assumption, we conclude that $\mathrm{Tor}_{n+2i}^R(M,N) = 0$ for all $i \geq 0$. Assume now that $r \geq 2$. Since R is complete, [14, 2.1.i] (cf. also the proofs of [10, 7.8 & 8.6(2)]) provides a short exact sequence

$$(3.3.5.1) \quad 0 \to M \to K \to \mathrm{syz}_1^R(M) \to 0$$

where K is a finitely generated R-module such that $\mathrm{cx}_R(K) = r-1$ and $\mathrm{depth}_R(K) = \mathrm{depth}_R(M)$. We now have the following exact sequence induced by (3.3.5.1):

$$(3.3.5.2) \quad \mathrm{Tor}_{j+1}^R(K,N) \to \mathrm{Tor}_{j+2}^R(M,N) \to \mathrm{Tor}_j^R(M,N) \to \mathrm{Tor}_j^R(K,N) \to \mathrm{Tor}_{j+1}^R(M,N) \quad \text{for all } j \geq 1.$$

This shows that $\mathrm{Tor}_n^R(K,N) = \mathrm{Tor}_{n+1}^R(K,N) = \ldots = \mathrm{Tor}_{n+r-2}^R(K,N) = 0$. If r is even, then the induction hypothesis implies $\mathrm{Tor}_{n+2i}^R(K,N) = 0$ for all $i \geq 0$. Therefore, using (3.3.5.2), we have an injection $\mathrm{Tor}_{n+2i+1}^R(M,N) \hookrightarrow \mathrm{Tor}_{n+2i-1}^R(M,N)$ for all $i \geq 1$, and (2) follows. Similarly, if r is odd, then the induction hypothesis implies that $\mathrm{Tor}_{n+2i+1}^R(K,N) = 0$ for all $i \geq 0$. Hence, by (3.3.5.2), $\mathrm{Tor}_{n+2i+2}^R(M,N) \hookrightarrow \mathrm{Tor}_{n+2i}^R(M,N)$ is an injection for all $i \geq 0$, and we have (1). $\square$

In the proof of Theorem 3.3.7, we will use the following result: If $(R,\mathfrak{m})$ is a local complete intersection, M a maximal Cohen-Macaulay R-module and N is a finitely generated R-module that has finite projective dimension, then $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$. Note that this follows from Theorem 3.1.4, or the fact that, over a Gorenstein ring R, a maximal Cohen-Macaulay R-module is a dth syzygy where $d = \dim(R)$. It is worth noting that this result also holds over any local ring [49, 2.2]. Here we include an elementary proof for the general case and refer the interested reader to [5, 4.9] for a more general result.

Theorem 3.3.6. *([49, 2.2]) Let (R, m) be a local ring, and let M and N be non-zero finitely generated R-modules. If M is maximal Cohen-Macaulay and N has finite projective dimension, then $\operatorname{Tor}_i^R(M, N) = 0$ for $i \geq 1$.*

Proof. We will first show that $\operatorname{Tor}_1^R(M, N) = 0$ by induction on $\dim(R)$. Note that, by the Auslander-Buchsbaum equality, the result holds if $\operatorname{depth}(R) = 0$. In particular the case where $\dim(R) = 0$ holds. Suppose now $\operatorname{depth}(R) > 0$. Then, by the induction hypothesis, $\operatorname{Tor}_1^R(M, N)$ has finite length. Let $N' = \operatorname{syz}_1^R(N)$ and choose a non-zerodivisor on R and N'. Since M/xM is maximal Cohen-Macaulay and N'/xN' has finite projective dimension over R/xR, the induction hypothesis implies that $\operatorname{Tor}_1^{R/xR}(M/xM, N'/xN') = 0$. Consider the short exact sequence

$$(3.3.6.1) \quad 0 \to M \xrightarrow{x} M \to M/xM \to 0$$

Tensor (3.3.6.1) with N' to get the exact sequence

$$(3.3.6.2) \quad \operatorname{Tor}_1^R(M/xM, N') \to M \otimes_R N' \xrightarrow{x} M \otimes_R N'$$

Note that $\operatorname{Tor}_1^{R/xR}(M/xM, N'/xN') \cong \operatorname{Tor}_1^R(M/xM, N')$ by [43, Chapter 18, Lemma 2]. Thus $\operatorname{Tor}_1^R(M/xM, N') = 0$. Hence (3.3.6.2) shows that $\operatorname{depth}(M \otimes_R N') > 0$. Now consider the short exact sequence

$$(3.3.6.3) \quad 0 \to N' \to R^{(t)} \to N \to 0$$

Since $\operatorname{depth}(M \otimes_R N') > 0$ and $\operatorname{Tor}_1^R(M,N)$ has finite length over R, tensoring (3.3.6.3) with M, we conclude that $\operatorname{Tor}_1^R(M,N) = 0$. Now induction on the projective dimension of N shows that $\operatorname{Tor}_i^R(M,N) = 0$ for $i \geq 1$. $\square$

Theorem 3.3.7. *Let $(R, \mathfrak{m})$ be a local complete intersection, and let M and N be finitely generated R-modules. Assume M is maximal Cohen-Macaulay. Set $r = \min\{\operatorname{cx}_R(M), \operatorname{cx}_R(N)\}$.*

1. *Assume M is free on $X^r(R)$, N satisfies (S_r) and $M \otimes_R N$ satisfies (S_{r+1}). Then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.*
2. *Assume M is free on $X^{r-1}(R)$, N satisfies (S_{r-1}) and $M \otimes_R N$ satisfies (S_r). Then $\operatorname{Tor}_i^R(M,N) = 0$ for all even $i \geq 2$. Furthermore, if $\operatorname{Tor}_j^R(M,N) = 0$ for some odd $j \geq 1$, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.*

Proof. As in the proof of Theorem 3.3.2, we may assume R is complete and has infinite residue field. If M has finite projective dimension, then M is free by the Auslander-Buchsbaum formula so there is nothing to prove. If N has finite projective dimension, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$ by Theorem 3.3.6. Thus we may assume $\operatorname{cx}_R(M) > 0$ and $\operatorname{cx}_R(N) > 0$. We will use the same notations for the pushforwards and quasi-liftings of M and N as in the proof of Proposition 3.2.3.

We set $M_0 = M$ and consider the pushforwards for $i = 1, 2, \ldots, r+1$:

$$(3.3.7.1)\quad 0 \to M_{i-1} \to G_i \to M_i \to 0$$

Note that, since we assume M is maximal Cohen-Macaulay, Proposition 3.2.2(1) implies that M_i is maximal Cohen-Macaulay for all $i = 1, 2, \ldots, r+1$.

(1) The assumptions and [32, 2.1] imply that $\operatorname{Tor}_i^R(M_{r+1}, N) = 0$ for all $i = 1, 2, \ldots, r+$

1. Since $\min\{\mathrm{cx}_R(M_{r+1}), \mathrm{cx}_R(N)\} = r$, $\mathrm{Tor}_i^R(M_{r+1}, N) = 0$ for all $i \geq 1$ by Theorem 3.1.4. This implies that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

(2) The assumptions and [32, 2.1] imply that $\mathrm{Tor}_i^R(M_r, N) = 0$ for all $i = 1, 2, \ldots, r$. Therefore, by Proposition 3.3.5, $\mathrm{Tor}_i^R(M_r, N) = 0$ for all even $i \geq 2$ if r is even, and $\mathrm{Tor}_i^R(M_r, N) = 0$ for all odd $i \geq 1$ if r is odd. Hence, by shifting along the sequences (3.9.1), we conclude that $\mathrm{Tor}_i^R(M, N) = 0$ for all even $i \geq 2$.

Suppose now $\mathrm{Tor}_j^R(M, N) = 0$ for some odd $j \geq 1$. To prove the second claim in (2), we proceed by induction on r. Assume $r = 1$. Then, by Theorem 3.1.6, $\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_{i+2}^R(M, N)$ for all $i \geq 1$, and hence the result follows. Assume now $r \geq 2$. Recall that M_1 and N_1 denote the pushforwards of M and N, respectively. (Note that, since $r \geq 2$, we can construct the pushforward of N.) As in the proof of Theorem 3.3.2, we choose, using Theorem 3.1.5, a complete intersection S, and a non-zerodivisor f of S such that $R = S/(f)$ and $\min\{\mathrm{cx}_S(M_1), \mathrm{cx}_S(N_1)\} = r - 1$. Now, with respect to the presentation $R = S/(f)$, we construct the quasi-liftings E and F of M and N, respectively:

$$(3.3.7.2)\ \ 0 \to E \to S^{(m)} \to M_1 \to 0$$

$$(3.3.7.3)\ \ 0 \to F \to S^{(n)} \to N_1 \to 0$$

Thus $\min\{\mathrm{cx}_S(E), \mathrm{cx}_S(F)\} = r - 1$. Note that $\mathrm{Tor}_i^R(M_1, N) = 0$ for all odd $i \geq 1$. Therefore, by (1) and (2) of Proposition 3.2.3, we see that $\mathrm{Tor}_j^S(E, F) = 0$. Now, replacing M and N by E and F, and using the induction hypothesis with Proposition 3.2.3($3b$), we conclude that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. $\square$

Remark 3.3.8. *It is shown in [33, 4.1] that the assumptions, in* (1) *and* (2) *of Theorem* 3.3.7*, that $M \otimes_R N$ satisfies (S_r), respectively (S_{r+1}), cannot be removed.*

It should be pointed out that if M and N are two finitely generated modules over a complete intersection R such that M is maximal Cohen-Macaulay and $\operatorname{Tor}_i^R(M,N)=0$ for all even $i \geq 2$, then the vanishing of $\operatorname{Tor}_j^R(M,N)$ for some odd $j \geq 1$ does not in general imply $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geq 1$. The following example, verified by *Macaulay 2* [28], is a special case of [37, 4.1].

Example 3.3.9. ([37, 4.1]) Let k be a field and put $R = k[[X,Y,Z,U]]/(XY,ZU)$. Then R is a complete intersection of dimension two and codimension two. Let $M = R/(y,u)$, and let N be the cokernel of the following map:

$$R^{(2)} \xrightarrow{\begin{bmatrix} 0 & u \\ -z & x \\ y & 0 \end{bmatrix}} R^{(3)}$$

Then M and N are maximal Cohen-Macaulay, $\operatorname{Tor}_1^R(M,N)=\operatorname{Tor}_2^R(M,N)=0$ and $\operatorname{cx}_R(M)=\operatorname{cx}_R(N)=2$. Moreover, by Proposition 3.3.5, $\operatorname{Tor}_i^R(M,N)=0$ for all even $i \geq 2$. Therefore, if there is an odd $j \geq 3$ such that $\operatorname{Tor}_j^R(M,N)=0$, then Theorem 3.1.2 implies $\operatorname{Tor}_i^R(M,N)=0$ for all $i \gg 0$. This shows, by [44, 2.1], that $\operatorname{cx}_R(M)+\operatorname{cx}_R(N) \leq 2$, which is false. Thus $\operatorname{Tor}_i^R(M,N) \neq 0$ for all odd $i \geq 3$.

As an immediate corollary of Theorem 3.3.7, we have:

Corollary 3.3.10. *Let $(R,\mathfrak{m})$ be local complete intersection, and let M and N be finitely generated R-modules. Assume M, N and $M \otimes_R N$ are maximal Cohen-Macaulay. Set $r = \min\{\operatorname{cx}_R(M), \operatorname{cx}_R(N)\}$.*

1. *If M is free on $X^r(R)$, then $\operatorname{Tor}_i^R(M,N)=0$ for all $i \geq 1$.*

2. *Assume M is free on $X^{r-1}(R)$. Then $\operatorname{Tor}_i^R(M,N) = 0$ for all even $i \geq 2$. Moreover, if $\operatorname{Tor}_j^R(M,N) = 0$ for some odd $j \geq 1$, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \geq 1$.*

The assumptions in (1) and (2) of Corollary 3.3.10 that M is free on $X^{r-1}(R)$, respectively on $X^r(R)$, cannot be removed.

Example 3.3.11. Let R and M be as in Example 3.3.9, and let $q = (y,u,x)$. Then $\dim(R_q) = 1$ and $M_q = R_q/(y)$ is not a free R_q-module. Thus M is not free on $X^1(R)$. It can be checked that a minimal resolution of M is:

$$\cdots \longrightarrow R^{(4)} \xrightarrow{\begin{bmatrix} u & -y & 0 & 0 \\ 0 & z & x & 0 \\ 0 & 0 & u & y \end{bmatrix}} R^{(3)} \xrightarrow{\begin{bmatrix} 0 & -u & x \\ z & y & 0 \end{bmatrix}} R^{(2)} \xrightarrow{\begin{bmatrix} y & u \end{bmatrix}} R \longrightarrow 0$$

Using the resolution above, we see that $\operatorname{Tor}_2^R(M,M) \neq 0$.

Example 3.3.12. Let R be as in Example 3.3.9, and let $M = R/(x)$ and $N = R/(xz)$. Then M, N and $M \otimes_R N$ are maximal Cohen-Macaulay. A minimal resolution of M is:

$$\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow 0$$

It is easy to see that $\operatorname{Tor}_1^R(M,N) \neq 0$, $\operatorname{Tor}_2^R(M,N) = 0$ and $\operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_{i+2}^R(M,N)$ for all $i \geq 1$. One can also see that $\operatorname{Tor}_1^R(M,N) \cong R/(x,y,u) \cong k[[Z]]$. In particular, $\operatorname{depth}(\operatorname{Tor}_i^R(M,N)) = 1$ if i is a positive odd integer. Hence M and N are not free on $X^1(R)$.

Our next theorem can be established by modifying the proof of [21, 7.6] (stated as Theorem 3.3.1). Here we give a different proof using the quasi-liftings as in Theorem

3.3.2 and Theorem 3.3.7. We will use it to make a further observation in Corollary 3.3.14 which improves our main result, Theorem 3.3.2.

Theorem 3.3.13. *(H. Dao) Let $(R, \mathfrak{m})$ be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$ is a regular sequence of S contained in $\mathfrak{n}^2$. Let M and N be finitely generated R-modules, and n be an integer such that $n \neq c$ if n is positive. Assume the following conditions hold:*

1. *M and N satisfy (S_{c-n}).*

2. *M is free on $X^{c-n}(R)$.*

3. *$M \otimes_R N$ satisfies (S_{c-n+1}).*

4. *$\mathrm{Tor}_1^R(M, N) = \mathrm{Tor}_2^R(M, N) = \ldots = \mathrm{Tor}_n^R(M, N) = 0$.*

Then $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

Proof. Without loss of generality we may assume R is complete. We will use the same notations for the pushforwards and quasi-liftings of M and N as in the proof of Proposition 3.2.3. Note that, if $n \leq 0$, then the result follows from Theorem 3.3.1. Moreover, if $c < n$, then (4) and Theorem 3.1.2 imply that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Therefore we may assume $c > n \geq 1$.

Assume $c = n + 1$. Then M and N are torsion-free, M is free on $X^1(R)$, $M \otimes_R N$ is reflexive and $\mathrm{Tor}_1^R(M, N) = \mathrm{Tor}_2^R(M, N) = \ldots = \mathrm{Tor}_{c-1}^R(M, N) = 0$. Consider the pushforward of M:

$$(3.3.13.1) \quad 0 \to M \to R^{(m)} \to M_1 \to 0$$

Note that, by Proposition 3.2.3(3), $\operatorname{Tor}_1^R(M_1, N) = 0$ and $M_1 \otimes_R N$ is torsion-free. Moreover, since $c \geq 2$, we have

$$(3.3.13.2) \quad \operatorname{Tor}_1^R(M_1, N) = \operatorname{Tor}_2^R(M_1, N) = \ldots = \operatorname{Tor}_c^R(M_1, N) = 0.$$

We proceed by induction on $d = \dim(R)$. The case where $d \leq 1$ follows from the fact that M is free on $X^1(R)$. Assume $d \geq 2$. Then the induction hypothesis and (3.3.13.1) imply that $\operatorname{Tor}_i^R(M_1, N)$ has finite length for all $i \geq 1$. (If R_p has codimension less than c, we use Theorem 3.1.2 and (3.3.13.2)) Now applying Theorem 3.1.8 to M_1 and N, we conclude that $\operatorname{Tor}_i^R(M_1, N) = 0$ for all $i \geq 1$. This proves the case where $c = n + 1$.

Assume now $c \geq n + 2$. Let $R = S/(f)$ where S is an unramified complete intersection of codimension $c - 1$, and f is a non-zerodivisor of S. Then E and F satisfy (S_{c-n}) and E is free on $X^{c-n+1}(S)$ (cf. Proposition 3.2.2). Moreover, by Proposition 3.2.2(7), $E \otimes_S F$ satisfies (S_{c-n}). Note also that (1) and (2) of Proposition 3.2.3 show that $\operatorname{Tor}_1^S(E, F) = \operatorname{Tor}_2^S(E, F) = \ldots = \operatorname{Tor}_n^S(E, F) = 0$. Therefore, replacing M and N by E and F and c by $c - 1$, the induction hypothesis on c implies that $\operatorname{Tor}_i^S(E, F) = 0$ for all $i \geq 1$. Thus, by Proposition 3.2.3(3b), $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. $\square$

As a corollary of Theorem 3.3.2 and Theorem 3.3.13 we have:

Corollary 3.3.14. *Let $(R, \mathfrak{m})$ be a local ring such that $\hat{R} = S/(\underline{f})$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $\underline{f} = f_1, f_2, \ldots, f_c$, for $c \neq 1$, is a regular sequence of S contained in $\mathfrak{n}^2$. Let M and N be finitely generated R-modules. Assume the following conditions hold:*

1. *M and N satisfy (S_{c-1}).*

2. M is free on $X^{c-1}(R)$.

3. $M \otimes_R N$ satisfies (S_c).

Then either (a) $\mathrm{cx}_R(M) = \mathrm{cx}_R(N) = c$ *and* $\mathrm{Tor}_1^R(M,N) \neq 0$*, or (b)* $\mathrm{Tor}_i^R(M,N) = 0$ *for all* $i \geq 1$*.*

We do not know whether Theorem 3.3.13 holds if $c = n \geq 1$. In particular, it seems reasonable to ask the following question (see also [22, 4.1].):

Question 3.3.15. Let $(R, \mathfrak{m})$ be a local ring such that $\hat{R} = S/(f)$ where $(S, \mathfrak{n})$ is a complete unramified regular local ring and $0 \neq f \in \mathfrak{n}^2$. Let M and N be finitely generated R-modules such that M is free on $X^0(R)$ and $M \otimes_R N$ is torsion-free. If $\mathrm{Tor}_1^R(M,N) = 0$, then is $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq 1$?

Next we consider a question of Huneke and R. Wiegand [33, page 473] that is similar to Question 3.3.15. Recall that $M^* = \mathrm{Hom}_R(M,R)$.

Question 3.3.16. (Huneke and R. Wiegand) If $(R, \mathfrak{m})$ is a one-dimensional Gorenstein domain, and M is a torsion-free R-module such that $M \otimes_R M^*$ is torsion-free, then is M free?

In their remarkable paper [33, 3.1], Huneke and Wiegand proved that over a local hypersurface, if the tensor product of two modules, at least one of which has constant rank, is maximal Cohen-Macaulay, then one of them must be free. Here a finitely generated R-module M has constant rank if there exists an r such that $M_p \cong R_p^{(r)}$ for all $p \in X^0(R)$. Using this result, they proved in [33, 5.2] that Question 3.3.16 has a positive answer over any domain R satisfying (S_2) (not necessarily Gorenstein and one-dimensional) provided $M \otimes_R M^*$ is reflexive and R_p is a hypersurface for all $p \in X^1(R)$. However, if the ring is not assumed to be a hypersurface (in codimension one), it is not known (at least to the author) whether Question 3.3.16 has an affirmative answer, even over a complete intersection domain of codimension two. Following the same induction argument as in [33, 5.2], we will now establish a consequence of a theorem of Avramov–Buchweitz [9, 4.2] and Huneke–Jorgensen [31, 5.9]. Since a finitely generated module over a hypersurface has complexity at most one (cf. Theorem 3.1.3), this will generalize the result discussed above [33, 5.2] over complete intersection rings.

Recall that M is reflexive if and only the natural map $M \to M^{**}$ is bijective.

Proposition 3.3.17. *Let $(R, \mathfrak{m})$ be a local complete intersection, and let M be a finitely generated torsion-free R-module such that $M \otimes_R M^*$ is reflexive. Assume one of the following holds:*

1. *For every $q \in X^1(R)$, $\operatorname{Tor}_i^R(M, M^*)_q = 0$ for some even $i \geq 2$ and $\operatorname{Tor}_j^R(M, M^*)_q = 0$ for some odd $j \geq 1$.*

2. *M has constant rank and $cx_{R_q}(M_q) \leq 1$ (that is, M_q has bounded Betti numbers) for every $q \in X^1(R)$.*

Then M is free.

Proof. We will use the fact that M has finite projective dimension if and only if $\operatorname{Ext}_R^{2n}(M, M) = 0$ for some nonnegative n [9, 4.2]. Suppose first that $d := \dim(R) \leq 1$. First assume (1). If $d = 0$, then $\operatorname{Ext}_R^i(M, M)$ is the Matlis dual of $\operatorname{Tor}_i^R(M, M^*)$ (cf. [17]). Thus, by assumption, $\operatorname{Ext}_R^i(M, M) = 0$ for some even $i \geq 2$. So, by [9, 4.2], M has finite projective dimension, that is, M is free by the Auslander-Buchsbaum formula. Suppose now $d = 1$. Consider the exact sequence

$$(3.3.17.1) \quad 0 \to \operatorname{syz}_j^R(M) \to F \to \operatorname{syz}_{j-1}^R(M) \to 0$$

where F is a free R-module. Tensoring (3.3.17.1) by M^* we get the exact sequence

$$(3.3.17.2)\ \ 0 \to \operatorname{Tor}_1^R(\operatorname{syz}_{j-1}^R(M), M^*) \to \operatorname{syz}_j^R(M) \otimes_R M^* \to F \otimes M^*$$

Since $\operatorname{Tor}_1^R(\operatorname{syz}_{j-1}^R(M), M^*) \cong \operatorname{Tor}_j^R(M, M^*) = 0$ and M^* is torsion-free, the depth lemma implies that $\operatorname{depth}(\operatorname{syz}_j^R(M) \otimes_R M^*) > 0$. As R has dimension one, it follows that $\operatorname{syz}_j^R(M) \otimes_R M^*$ is maximal Cohen-Macaulay. Since $\operatorname{Ext}_R^1(\operatorname{syz}_j^R(M), M)$ has finite length, [31, 5.9] implies that $\operatorname{Ext}_R^1(\operatorname{syz}_j^R(M), M) = 0$, that is, $\operatorname{Ext}_R^{j+1}(M, M) = 0$. As j is odd, using [9, 4.2] one more time, we conclude that M is free.

Next assume (2). If $d = 0$, then the result follows by the assumption of constant rank. Suppose now $d = 1$. If $\operatorname{cx}_R(M) = 0$, that is, if M has finite projective dimension, then M is free by the Auslander-Buchsbaum formula. Suppose now that $\operatorname{cx}_R(M) = 1$. It follows from [14, Lemma 3.3] that if N is a finite length module, then the vanishing of $\operatorname{Tor}_n^R(M, N)$ for some non-negative integer n forces the vanishing of $\operatorname{Tor}_i^R(M, N)$ for all $i \geq 1$. Since M has constant rank, so does M^* and hence there is a short exact sequence [33, 1.3]

$$(3.3.17.3)\ \ 0 \to M^* \to G \to N \to 0$$

where G is a free module and N is torsion. Note that, since $d = 1$, N has finite length. Furthermore $\operatorname{Tor}_1^R(M, N) = 0$; this follows from (3.3.17.3) and the fact that $M \otimes_R M^*$ is torsion-free. Hence, by [14, Lemma 3.3] (stated above), $\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Thus $\operatorname{Tor}_i^R(M^*, M) = 0$ for all $i \geq 1$. However, Jorgensen proved in [36, 1.3] that $\operatorname{Tor}_i^R(M, M^*) \neq 0$ for infinitely many i. This contradiction shows that $\operatorname{cx}_R(M) = 0$, and hence M is free. This proves the proposition for the case where $d \leq 1$.

Suppose now $d \geq 2$, and proceed by induction on d. The induction hypothesis implies that M is free on $X^1(R)$. We now follow [33, 5.2] and [3, 3.3]. It is known that

[6, A.1] the map $\alpha_M : M \otimes_R M^* \to \operatorname{Hom}_R(M, M)$, given by $\alpha_M(a \otimes f)(x) = f(x)a$ for all a and x in M and f in M^*, is surjective if and only if M is free. Consider the exact sequence

$$(3.3.17.4)\quad 0 \to B \to M \otimes_R M^* \xrightarrow{\alpha_M} \operatorname{Hom}_R(M, M) \to C \to 0$$

Note that, since M is free on $X^1(R)$, $(\alpha_M)_q$ is an isomorphism for all $q \in X^1(R)$. Therefore, since $M \otimes_R M^*$ is torsion-free, $B = 0$. So, if $C \neq 0$, localizing (3.3.17.4) at an associated prime ideal of C, we see that the depth lemma gives a contradiction. Thus $C = 0$ and hence M is free. $\square$

Corollary 3.3.18. *Let $(R, \mathfrak{m})$ be a local complete intersection domain, and let M be a finitely generated torsion-free R-module. Assume that $M \otimes_R M^*$ is reflexive. If M has bounded Betti numbers, then M is free.*

If R is a complete intersection and M is a finitely generated R-module of complexity at least two, then it is known that the Betti sequence of M is eventually strictly increasing [10, 8.1]. Therefore, by Corollary 3.3.18, our contribution for the question of Huneke and Wiegand (Question 3.3.16) can also be stated as follows:

Corollary 3.3.19. *Let $(R, \mathfrak{m})$ be an one-dimensional local complete intersection domain. If there exits a non-free finitely generated torsion-free R-module M such that $M \otimes_R M^*$ is torsion-free, then the Betti sequence of M must be eventually strictly increasing.*

Considering the results in Proposition 3.3.17, it seems reasonable to ask the following weaker form of Question 3.3.16 for complete intersection domains:

Question 3.3.20. Let $(R, \mathfrak{m})$ be a one-dimensional local complete intersection domain, and let M be a finitely generated R-module such that M and $M \otimes_R M^*$ are torsion-free. If $\operatorname{Tor}_i^R(M, M^*) = 0$ for some $i \geq 1$, then is M free?

Chapter 4

Theorems on the vanishing of Ext

The contents of this chapter are contained in the author's paper:

Asymptotic Behavior of Ext *functors for modules of finite complete intersection dimension*, joint work with Hailong Dao[1], preprint.

4.1 Preliminary Results

We give a brief account of the *cohomology operators* originally introduced by Eisenbud [25]. There are several definitions for these operators, but it was proved in [11] that they all agree up to sign (cf. also [7]). The one explained here is from [25].

Assume $R = Q/(\underline{f})$ where Q is a local ring and $\underline{f} = f_1, f_2, \ldots, f_r$ is a regular sequence of Q. Let M and N be finitely generated R-modules.

Let $\mathbf{F} : \ldots \longrightarrow F_{i+1} \xrightarrow{\partial} F_i \xrightarrow{\partial} F_{i-1} \longrightarrow \ldots$ be a complex of free R-modules, and let $(\widetilde{\mathbf{F}}, \widetilde{\partial})$ be a *lifting* of $(\mathbf{F}, \partial)$, that is, $\widetilde{\mathbf{F}} : \ldots \longrightarrow \widetilde{F}_{i+1} \xrightarrow{\widetilde{\partial}} \widetilde{F}_i \xrightarrow{\widetilde{\partial}} \widetilde{F}_{i-1} \longrightarrow \ldots$ is a sequence of free Q-modules with maps $\widetilde{\partial}$ between them so that $(\widetilde{\mathbf{F}} \otimes_Q R, \widetilde{\partial} \otimes_Q R) \cong (\mathbf{F}, \partial)$. Such a lifting always exists; $\widetilde{\partial}$ can be thought as matrices with entries in Q of preimages of the entries of the matrices representing ∂. Since $\widetilde{\partial}^2 \equiv 0 \pmod{(\underline{f})}$, one can write

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$\widetilde{\partial}^2 = \sum_{j=1}^{r} f_j \widetilde{t}_j$, where $\widetilde{t}_j$ are degree -2 endomorphisms of the graded Q module $\widetilde{\mathbf{F}}$ for all $j = 1, 2, \ldots, r$. Now set $t_j = \widetilde{t}_j \otimes_Q R$. The maps t_j are independent of the choice of $\widetilde{t}_j$, and are chain maps (that is, they commute with the differentials ∂) on $\mathbf{F}$ of degree -2 [25, 1.2].

It was proved in [25, 1.3-1.5] that t_j are natural and commute, up to homotopy. More specifically, if $f : \mathbf{F} \longrightarrow \mathbf{G}$ is a chain map of complexes of free R-modules, then ft_j is homotopic to $s_j f$ where s_j, for $j = 1, 2, \ldots r$, are the Eisenbud operators defined on the complex $\mathbf{G}$. In particular, $t_j t_k$ and $t_k t_j$ are homotopic.

Now let $\mathbf{F} \xrightarrow{\simeq} M$ be a free resolution of M over R, and t_j be the Eisenbud operators defined on $\mathbf{F}$. Then the map $\chi_j : \operatorname{Ext}^i_R(M, N) \longrightarrow \operatorname{Ext}^{i+2}_R(M, N)$, for $j = 1, 2, \ldots r$, on cohomology induced by t_j is defined as $\mathrm{H}^i(\operatorname{Hom}(t_j, N))$. The properties of t_j discussed above show that the action of χ_j on $\operatorname{Ext}^*_R(M, N) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_R(M, N)$ is defined and independent of the choice of the free resolution of M. Since $\chi_j \chi_k = \chi_k \chi_j$ on cohomology, the action of χ_j shows that $\operatorname{Ext}^*_R(M, N)$ has a graded module structure over the ring of cohomology operators $\mathcal{S} = R[\underline{\chi}]$ where $\underline{\chi} = \chi_1, \chi_2, \ldots, \chi_r$ and each χ_j has degree two.

The non-triviality of the action of χ_j is expressed by the following theorem:

Theorem 4.1.1. *([10, 4.2] and [30, 3.1]) Let R be a ring such that $R = Q/(\underline{f})$ where Q is a local ring and $\underline{f}$ is a regular sequence of Q, and let M and N be finitely generated R-modules. Let $\mathcal{S} = R[\underline{\chi}]$ be the ring of cohomology operators defined by the regular sequence $\underline{f}$. Then $\operatorname{Ext}^*_R(M, N)$ is a finitely generated graded module over $\mathcal{S}$ if and only if $\operatorname{Ext}^i_Q(M, N) = 0$ for all $i \gg 0$.*

We should mention that if Q is regular, then it follows from Theorem 4.1.1 that $\operatorname{Ext}^*_R(M, N)$ is a finitely generated graded module over $\mathcal{S}$. In the next proposition, $\operatorname{Ann}_R(X)$ denotes the annihilator of an R-module X.

Proposition 4.1.2. *([21, 2.4], cf. also [39,* Theorem 1*]) Let $T = \bigoplus_{i\geq 0} T_i$ be a graded Noetherian module over the polynomial ring $R[X_1, X_2, \ldots, X_r]$. Assume that each X_i has positive degree n. Then the sequence of ideals $\{\mathrm{Ann}_R(T_i)\}$ eventually becomes periodic of period n.*

If the length of the modules $\mathrm{Ext}^i_R(M,N)$ is finite for all $i \gg 0$, then we let $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) = \inf\{s : \lambda_R(\mathrm{Ext}^i_R(M,N) < \infty \text{ for all } i \geq s\}$, where $\lambda_R(X)$ denotes the length of an R-module X.

Recall from Definition 2.2.2 that $\mathrm{cx}_R(M,N) = s$, if s is the least nonnegative integer for which there exists a real number γ such that $\dim_k(\mathrm{Ext}^n_R(M,N) \otimes_R k) \leq \gamma \cdot n^{s-1}$ for all $n \gg 0$.

Proposition 4.1.3. *Let R be a ring such that $R = Q/(\underline{f})$ where Q is a local ring and $\underline{f}$ is a regular sequence of Q. Let M and N be finitely generated R-modules. Assume $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$ and $\mathrm{Ext}^i_Q(M,N) = 0$ for all $i \gg 0$. Then $\mathrm{cx}_R(M,N)$ is the least nonnegative integer s for which there exists a real number γ such that $\lambda_R(\mathrm{Ext}^i_R(M,N)) \leq \gamma \cdot n^{s-1}$ for all $n \gg 0$.*

Proof. Theorem 4.1.1 shows that the graded module $\mathrm{Ext}^*_R(M,N)$ is Noetherian over the ring $\mathcal{S} = R[\underline{\chi}]$ of cohomology operators defined by the regular sequence $\underline{f}$. Recall that each χ_j has degree two. Thus the sequence of ideals $\{\mathrm{Ann}_R(\mathrm{Ext}^i_R(M,N))\}$ eventually becomes periodic of period two by Proposition 4.1.2. Hence, since $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$, one can find a positive integer h such that $\mathfrak{m}^h \mathrm{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, where $\mathfrak{m}$ is the unique maximal ideal of R. Now the claim is proved in [24, 2.2 & 2.5]; since $\frac{1}{\lambda_R(R/\mathfrak{m}^h)} \cdot \lambda_R(\mathrm{Ext}^i_R(M,N)) \leq \dim_k(\mathrm{Ext}^n_R(M,N) \otimes_R k) \leq \lambda_R(\mathrm{Ext}^i_R(M,N))$ for all $i \gg 0$, the result follows from the definition of complexity. □

Proposition 4.1.4. *([46, 11.65]) Let $R = Q/(x)$ where Q is a commutative ring and*

x is a non-zerodivisor of Q. If M and N are R-modules, then one has the change of rings long exact sequence of Ext*:*

$$\begin{aligned} 0 \to \operatorname{Ext}^1_R(M,N) \to \operatorname{Ext}^1_Q(M,N) \to \operatorname{Ext}^0_R(M,N) \to \cdots \to \\ \operatorname{Ext}^n_R(M,N) \to \operatorname{Ext}^n_Q(M,N) \to \operatorname{Ext}^{n-1}_R(M,N) \to \\ \operatorname{Ext}^{n+1}_R(M,N) \to \operatorname{Ext}^{n+1}_Q(M,N) \to \operatorname{Ext}^n_R(M,N) \to \dots \end{aligned}$$

Recall that if R is a normal domain, that is, if R is a domain and is integrally closed in its field of fractions K, then, for $x \in R$, we can write

$$(x)R = \bigcap_{\dim(R_p)=1} p^{(v_p(x))}$$

Here p is a prime ideal of R and $v_p : K \to \mathbb{Z}$ is the associated valuation of the discrete valuation ring $R_p = \{\alpha \in K : v_p(\alpha) \geq 0\}$. Let $X^1(R)$ be the set of all prime ideals p of R such that $\dim(R_p) = 1$, and let $X(R)$ be the free abelian group on $X^1(R)$. Set $P(R)$ to be the set of elements of the form $\sum_{p\in X^1(R)} v_p(x)p$. Then $P(R)$ is a subgroup and the class group $\mathrm{Cl}(R)$ of R is defined to be $X(R)/P(R)$ (cf. [16, Chapter 7]).

We denote by $G(R)$ the *Grothendieck group* of finitely generated R-modules, that is, the quotient of the free abelian group of all isomorphism classes of finitely generated R-modules by the subgroup generated by the relations coming from short exact sequences of finitely generated R-modules [12]. We write $[M]$ for the class of M in $G(R)$ and denote by $\overline{G}(R)$ the group $G(R)/\mathbb{Z}\cdot[R]$, the reduced Grothendieck group of R. We set $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$ for an abelian group G.

Some facts about the group $\overline{G}(R)_{\mathbb{Q}}$ are collected in the next proposition.

Proposition 4.1.5. *Let R be a local ring, and let N be a finitely generated R-module. Then $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$ for each one of the following cases:*

1. *N has finite length, or N is a syzygy of some finite length R-module.*

2. *R is Artinian.*

3. *R is a one-dimensional domain.*

4. *R is a two-dimensional normal domain with torsion class group.*

Proof. Let $\mathfrak{m}$ denote the unique maximal ideal of R. Set $k = R/\mathfrak{m}$. Assume N has finite length. We claim $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. Note that $[N] = l \cdot [k]$ where $l = \lambda_R(N)$. Therefore it suffices to prove $[X] = 0$ in $\overline{G}(R)$ for some finite length R-module X. If $\dim R = 0$, then there is nothing to prove as we kill the class $[R]$ of R. Suppose now $\dim R > 0$. Choose a prime ideal p and an element x in m such that $x \notin p$ and $\dim(R/p) = 1$. Then the short exact sequence $0 \to R/p \xrightarrow{x} R/p \to R/(p + x) \to 0$ implies that $[R/(p + (x))] = 0$. This proves the claim. Therefore (1) and (2) follow.

Suppose now R is a domain. Then there is an exact sequence

$$0 \to K \to N \to R^{(t)} \to C \to 0$$

where K and C are torsion R-modules [33, 1.3]. If $\dim(R) = 1$, then K and C have finite length, and hence $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. This proves (3).

Next assume that R is a two-dimensional normal domain. We will show that $\overline{G}(R)_{\mathbb{Q}} \cong \mathrm{Cl}(R)_{\mathbb{Q}}$. As $\mathrm{Cl}(R)$ is torsion, this implies $\overline{G}(R)_{\mathbb{Q}} = 0$ and hence proves (4). Since $\dim(R) = 2$, there is a well-defined map $\alpha : \mathrm{Cl}(R) \to \overline{G}(R)_{\mathbb{Q}}$ given by $\alpha([p]) = [R/p]$ for height one prime ideals p of R. The maps γ and δ in the localization exact sequence [12, 6.2]

$$F^{\times} \xrightarrow{\gamma} G(\mathrm{tor}(R)) \longrightarrow G(R) \xrightarrow{\delta} \mathbb{Z} \longrightarrow 0$$

are defined as $\gamma\,(a/b) = [R/a] - [R/b]$ and $\delta\,([M]) = \dim_F(M \otimes_R F)$. Here F is the field of fractions of R and $G(\mathrm{tor}(R))$ is the Grothendieck group of finitely generated torsion R-modules. This shows that $\overline{G}(R)$ is isomorphic to the free abelian group on finitely generated torsion R-modules modulo the classes of the form $[R/x]$ where $x \in R - \{0\}$, and the relations coming from short exact sequences of torsion R-modules. Hence one has a well-defined map $\beta : \overline{G}(R) \to \mathrm{Cl}(R)$, where $\beta\,([M]) = \sum \lambda_{R_p}(M_p)[p]$ with the sum is taken over all height-one prime ideals p of R. Therefore the isomorphism $\overline{G}(R)_{\mathbb{Q}} \cong \mathrm{Cl}(R)_{\mathbb{Q}}$ follows from the fact that $\alpha \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\beta \otimes_{\mathbb{Z}} \mathbb{Q}$ are inverses to each other. $\square$

An important class of rings satisfing the hypotheses of Proposition 4.1.5(4) is recorded next:

Remark 4.1.6. *Let $(R, \mathfrak{m}, k)$ be a two-dimensional local complete normal domain. Assume k is either finite or is the algebraic closure of a finite field. Then it follows from [20, Theorem 4] and [27, 4.5] that R has torsion class group.*

4.2 Asymptotic Behavior of Ext and the generalized Herbrand function

In this section we will adapt the arguments of [21] and define an asymptotic function associated to $\mathrm{Ext}^i_R(M, N)$ for a pair of finitely generated R-modules (M, N). This function can be viewed as a natural generalization of the notion of *Herbrand* difference, defined by Buchweitz.

Recall that if the length of the modules $\mathrm{Ext}^i_R(M, N)$ is finite for all $i \gg 0$, then we denote by $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M, N)$ the set $\inf\{s : \lambda_R(\mathrm{Ext}^i_R(M, N) < \infty \text{ for all } i \geq s\}$, where $\lambda_R(X)$ denotes the length of an R-module X.

We write $\beta_i^R(M,N) = \lambda_R(\mathrm{Ext}_R^i(M,N))$ for all $i \geq \mathrm{f}_{\mathrm{ext}}^{\mathrm{R}}(M,N)$ for notational convenience.

We now state a slightly modified version of Buchweitz's definition of the Herbrand difference for modules that may not be maximal Cohen-Macaulay (cf. Section 10.2 of [18]).

Definition 4.2.1. *([18]) Let R be a local hypersurface with an isolated singularity, that is, R_p is regular for all non-maximal prime ideals p of R. For a pair of finitely generated R-modules (M,N), the sequence of modules $\{\mathrm{Ext}_R^i(M,N)\}$ is eventually periodic of period at most two, and $\mathrm{Ext}_R^i(M,N)$ has finite length for all $i > \operatorname{depth} R - \operatorname{depth} M$. The Herbrand difference $\mathrm{h}^R(M,N)$ of (M,N) is defined as:*

$$\mathrm{h}^R(M,N) = \beta_n^R(M,N) - \beta_{n-1}^R(M,N)$$

where n is any even number such that $n > \operatorname{depth} R - \operatorname{depth} M + 1$.

The Herbrand difference is relevant when proving results for the vanishing pattern of Ext modules because of the following simple observation: Suppose that $\mathrm{h}^R(M,N) = 0$. If $\mathrm{Ext}_R^n(M,N) = 0$ for some $n > \operatorname{depth} R - \operatorname{depth} M$, then $\mathrm{Ext}_R^i(M,N) = 0$ for all $i \geq n$. To generalize this function, we first prove that the numbers $\beta_i^R(M,N)$ share properties similar to those of the Betti numbers of the module M (cf. also [7, 9.2.1]).

Proposition 4.2.2. *Let R be a ring such that $R = Q/(\underline{f})$ where Q is a local ring and $\underline{f} = f_1, ..., f_r$ is a regular sequence in Q. Let M and N be finitely generated R-modules. Assume $\mathrm{f}_{\mathrm{ext}}^{\mathrm{R}}(M,N) < \infty$ and $\mathrm{Ext}_Q^i(M,N) = 0$ for all $i \gg 0$ (which is automatic if $\mathrm{pd}_Q(M) < \infty$). Set*

$$P_{M,N}^R(t) = \sum_{i=\mathrm{f}_{\mathrm{ext}}^{\mathrm{R}}(M,N)}^{\infty} \beta_i^R(M,N)t^i$$

Then the following hold:

1. *There is a polynomial* $p(t) \in \mathbb{Z}[t]$, *with* $p(\pm 1) \neq 0$, *such that:*

$$P^R_{M,N}(t) = \frac{p(t)}{(1-t)^c(1+t)^d}.$$

2. $\beta_i^R(M,N) = \dfrac{m_0}{(c-1)!} i^{c-1} + (-1)^i \dfrac{n_0}{(d-1)!} i^{d-1} + q_{(-1)^i}(i)$ *for all* $i \gg 0$, *where* m_0 *is a non-negative rational number and* $g_{\pm}(t) \in \mathbb{Q}[t]$ *are polynomials that have degrees strictly less than* $\max\{c, d\} - 1$.

3. $d \leq c = \mathrm{cx}_R(M,N) \leq r$.

Proof.

1. Set $\xi = \bigoplus_{i=\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N)}^{\infty} \mathrm{Ext}^i_R(M,N)$. Let $\mathfrak{m}$ be the unique maximal ideal of R. Then Theorem 4.1.1 and Proposition 4.1.2 show that there exists a positive integer h such that ξ is a finitely generated graded module over the ring $\mathcal{T} = (R/\mathfrak{m}^h)[\chi_1, \chi_2, \ldots, \chi_r]$, where each χ_i has degree 2. Therefore the Hilbert-Serre Theorem [43, 13.2] applies to the module ξ over $\mathcal{T}$. This shows that $P^R_{M,N}(t) = \dfrac{h(t)}{(1-t^2)^r}$ for some polynomial $h(t) \in \mathbb{Z}[t]$. Now, cancelling the powers of $1-t$ and $1+t$, one can find a polynomial $p(t) \in \mathbb{Z}[t]$ such that $P^R_{M,N}(t) = \dfrac{p(t)}{(1-t)^c(1+t)^d}$ where $p(\pm 1) \neq 0$.

2. From part (1), we can decompose $P^R_{M,N}(t)$ into partial fractions and write :

$$\sum_{i=\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N)}^{\infty} \beta_i^R(M,N) t^i = \frac{p(t)}{(1-t)^c(1+t)^d} = \sum_{l=0}^{c-1} \frac{m_l}{(1-t)^{c-l}} + \sum_{l=0}^{d-1} \frac{n_l}{(1+t)^{d-l}} + q(t)$$

Here $q(t) \in \mathbb{Z}[t]$. Then, by comparing coefficients from both sides, we get the desired formula for $\beta_i^R(M,N)$. Since $\beta_i^R(M,N) \geq 0$, m_0 must be a non-negative rational number.

3. That $c \leq r$ is obvious by the proof of (1). Since the sign of $\beta_i^R(M,N)$ for odd i is positive only if $c \geq d$, the first inequality is also clear. The size of $\beta_i^R(M,N)$ behaves like a polynomial of degree $\max\{c,d\}-1 = c-1$. Therefore, by Proposition 4.1.3, we see that $\mathrm{cx}_R(M,N) = c$.

□

Definition 4.2.3. *Let R be a local ring, and let M and N be finitely generated R-modules. Assume that $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$. Then, for a non-negative integer e, $\mathrm{h}_e^R(M,N)$ is defined as follows:*

$$\mathrm{h}_e^R(M,N) = \lim_{n\to\infty} \frac{\sum_{i=\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N)}^{n} (-1)^i \beta_i^R(M,N)}{n^e}$$

The behavior of $\beta_i^R(M,N)$, proved in Proposition 4.2.2, shows that the function $\mathrm{h}_\bullet^R(M,N)$ behaves quite well:

Theorem 4.2.4. *Let R be a ring such that $R = Q/(\underline{f})$ where Q is a local ring and $\underline{f} = f_1, ..., f_r$ is a regular sequence of Q. Let M and N be finitely generated R-modules. Assume $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$ and $\mathrm{Ext}_Q^i(M,N) = 0$ for all $i \gg 0$. Set $c = \mathrm{cx}_R(M,N)$.*

1. *If e is an integer such that $e \geq c$, then $\mathrm{h}_e^R(M,N)$ is finite. Moreover, if $e > c$, then $\mathrm{h}_e^R(M,N) = 0$.*

2. *(Biadditivity)*

(i) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequences of finitely generated* R*-modules. Assume* $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M_j, N) < \infty$ *and* $\mathrm{Ext}^i_Q(M_j, N) = 0$ *for all* $i \gg 0$ *and for all* j*. Assume further that* e *is an integer such that* $e \geq \mathrm{cx}_R(M_j, N)$ *for all* j*. If* $e \geq 1$*, then*

$$\mathrm{h}^R_e(M_2, N) = \mathrm{h}^R_e(M_1, N) + \mathrm{h}^R_e(M_3, N).$$

Moreover, if $e = 0$ *and* $\lambda_R(M \otimes_R N) < \infty$*, then*

$$\mathrm{h}^R_0(M_2, N) = \mathrm{h}^R_0(M_1, N) + \mathrm{h}^R_0(M_3, N).$$

(ii) Let $0 \to N_1 \to N_2 \to N_3 \to 0$ *be an exact sequence of finitely generated* R*-modules. Assume* $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M, N_j) < \infty$ *and* $\mathrm{Ext}^i_Q(M, N_j) = 0$ *for all* $i \gg 0$ *and for all* j*. Assume further that* e *is an integer such that* $e \geq \mathrm{cx}_R(M, N_j)$ *for all* j*. If* $e \geq 1$*, then*

$$\mathrm{h}^R_e(M, N_2) = \mathrm{h}^R_e(M, N_1) + \mathrm{h}^R_e(M, N_3).$$

Moreover, if $e = 0$ *and* $\lambda_R(M \otimes_R N) < \infty$*, then*

$$\mathrm{h}^R_0(M, N_2) = \mathrm{h}^R_0(M, N_1) + \mathrm{h}^R_0(M, N_3).$$

3. (Change of rings)

Suppose that $r \geq 1$ *and set* $R' = Q/(f_1, ..., f_{r-1})$*. Let* e *be a positive integer such that* $e \geq c$*. Assume* $\mathrm{cx}_{R'}(M, N) \leq e - 1$*. If* $e \geq 2$*, or* $e = 1$ *and* $\lambda_R(M \otimes_R N) < \infty$*, then* $2 \cdot \mathrm{h}^R_e(M, N) = \mathrm{h}^{R'}_{e-1}(M, N)$*.*

Proof. Let n and h be integers such that $n > h$. Set $g^R_{M,N}(h,n) = \sum_{i=h}^{n}(-1)^i\beta_i^R(M,N)$. Assume $e \geq 1$. Then, for a fixed h, it is clear that:

$$\mathrm{h}_e^R(M,N) = \lim_{n\to\infty}\frac{g^R_{M,N}(h,n)}{n^e}$$

(1) If $e = 0$, then $c = 0$ and hence $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \gg 0$. Thus $\mathrm{h}_e^R(M,N)$ is finite. Assume now $e \geq 1$. We choose a sufficiently large integer h so that the formula for $\beta_i^R(M,N)$ in Proposition 4.2.2(2) is true for all $i \geq h$. Then,

$$\begin{aligned} g^R_{M,N}(h,n) &= \sum_{i=h}^{n}(-1)^i\beta_i^R(M,N) \\ &= \frac{m_0}{(c-1)!}\sum_{i=h}^{n}(-1)^i i^{c-1} + \frac{n_0}{(d-1)!}\sum_{i=h}^{n} i^{d-1} + \sum_{i=h}^{n}(-1)^i g_{(-1)^i}(i) \end{aligned} \tag{4.2.1}$$

Note that $\sum_{i=h}^{n}(-1)^i i^{c-1}$ and $\sum_{i=h}^{n}(-1)^i g_{(-1)^i}(i)$ are polynomials in n of degree $c-1$ and at most $c-2$, respectively. Since $e \geq c$, it follows from (4.2.1) that:

$$\mathrm{h}_e^R(M,N) = \lim_{n\to\infty}\frac{g^R_{M,N}(h,n)}{n^e} = \lim_{n\to\infty}\frac{n_0}{(d-1)!}\cdot\frac{\sum_{i=h}^{n} i^{d-1}}{n^e} \tag{4.2.2}$$

Using (4.2.2) and the equality $\sum_{i=h}^{n} i^{d-1} = \frac{n^d}{d} +$ lower order terms, we have:

$$\mathrm{h}_e^R(M,N) = \lim_{n\to\infty}\frac{n_0}{d!}n^{d-e} \tag{4.2.3}$$

The claim now follows from the fact that $e-d$ is a non-negative integer (see also Proposition 4.2.2(3)).

(2) It is enough to prove (i) since (ii) follows in an identical manner. Assume $e \geq \mathrm{cx}_R(M_j,N)$ for each j. The short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ gives

rise to the following long exact sequence

$$\cdots \to \mathrm{Ext}^i(M_3, N) \to \mathrm{Ext}^i(M_2, N) \to \mathrm{Ext}^i(M_1, N) \to \mathrm{Ext}^{i+1}(M_3, N) \to \dots \quad (4.2.4)$$

We truncate (4.2.4) and obtain the exact sequence

$$\begin{aligned} 0 \to B_h \to \mathrm{Ext}^h(M_3, N) \to \mathrm{Ext}^h(M_2, N) \to \mathrm{Ext}^h(M_1, N) \to \\ \cdots \to \mathrm{Ext}^n(M_3, N) \to \mathrm{Ext}^n(M_2, N) \to \mathrm{Ext}^n(M_1, N) \to C_n \to 0, \end{aligned} \quad (4.2.5)$$

where n and h are integers such that $n > h > \mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M_j, N)$ for each j. Taking the alternating sum of the lengths of the modules in (4.2.5), we obtain

$$g^R_{M_1,N}(h,n) - g^R_{M_2,N}(h,n) + g^R_{M_3,N}(h,n) = \pm\, \lambda_R(B_h) \pm \lambda_R(C_n). \quad (4.2.6)$$

Since $e \geq \mathrm{cx}_R(M_1, N)$, it follows from (4.2.5) that $\lambda_R(C_n) \leq \beta^R_n(M_1, N) \leq A \cdot n^{e-1}$ for some real number A and all $n \gg 0$. As h is fixed, (4.2.3) and (4.2.6) give the equality we seek:

$$\begin{aligned} \mathrm{h}^R_e(M_1, N) - \mathrm{h}^R_e(M_2, N) + \mathrm{h}^R_e(M_3, N) &= \\ \lim_{n\to\infty} \frac{g^R_{M_1,N}(h,n)}{n^e} - \lim_{n\to\infty} \frac{g^R_{M_2,N}(h,n)}{n^e} + \lim_{n\to\infty} \frac{g^R_{M_3,N}(h,n)}{n^e} &= \\ \lim_{n\to\infty} \frac{g^R_{M_1,N}(h,n) - g^R_{M_2,N}(h,n) + g^R_{M_3,N}(h,n)}{n^e} &= \\ \lim_{n\to\infty} \frac{\pm\, \lambda_R(B_h) \pm \lambda_R(C_n)}{n^e} &= 0. \end{aligned}$$

Suppose now $e = 0$ and $\lambda_R(M \otimes_R N) < \infty$. Then $\mathrm{cx}_R(M_j, N) = 0$, that is, $\mathrm{Ext}^i_R(M_j, N) = 0$ for all $i \gg 0$ and all j. Moreover $\lambda_R(\mathrm{Ext}^i_R(M, N)) < \infty$ for all i and j. Therefore, taking the alternating sum of the lengths of $\mathrm{Ext}^i_R(M, N)$ in (4.2.4), we conclude that $\mathrm{h}^R_0(M_2, N) = \mathrm{h}^R_0(M_1, N) + \mathrm{h}^R_0(M_3, N)$.

(3) Write $R = R'/(x)$, where $R' = Q/(f_1, ..., f_{r-1})$ and $x = f_r$. Then Proposition 4.1.4 gives the following long exact sequence:

$$... \to \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_{R'}(M,N) \to \operatorname{Ext}^{i-1}_R(M,N) \to \operatorname{Ext}^{i+1}_R(M,N) \to ... \quad (4.2.7)$$

Suppose now $e = 1$ and $\lambda_R(M \otimes_R N) < \infty$. Since $\operatorname{Ext}^i_{R'}(M,N) = 0$ for all $i \gg 0$, it follows from the exact sequence (4.2.7) that $2 \cdot \mathrm{h}^R_1(M,N) = \mathrm{h}^{R'}_0(M,N)$. Assume now $e \geq 2$. We truncate (4.2.7) and obtain the exact sequence

$$\begin{aligned} 0 \to B_h \to \operatorname{Ext}^h_R(M,N) \to \operatorname{Ext}^h_{R'}(M,N) \to \operatorname{Ext}^{h-1}_R(M,N) \to \\ \cdots \to \operatorname{Ext}^{n-1}_R(M,N) \to \operatorname{Ext}^{n+1}_R(M,N) \to C_n \to 0, \end{aligned} \quad (4.2.8)$$

where n and h are integers such that $n > h > \mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N)$. Taking the alternating sum of the lengths of the modules in (4.2.8) we get:

$$g^{R'}_{M,N}(h,n) = (-1)^n \beta^R_n(M,N) - (-1)^n \beta^R_{n+1}(M,N) \pm \beta^R_{h-1}(M,N) \pm \lambda_R(B_h) \pm \lambda_R(C_n).$$

Since C_n is a submodule of $\operatorname{Ext}^{R'}_{n+1}(M,N)$ and $\operatorname{cx}_{R'}(M,N) \leq e-1$, we have that $\lambda_R(C_n) \leq \beta^{R'}_{n+1}(M,N) \leq B \cdot n^{e-2}$ for some real number B and for all $n \gg 0$. Therefore the equality in Proposition 4.2.2(2) implies

$$g^{R'}_{M,N}(h,n) = \frac{2 \cdot n^{d-1} \cdot n_0}{(d-1)!} + f(n) \text{ for all } n \gg 0, \quad (4.2.9)$$

where $f(t) \in \mathbb{Z}[t]$ is a polynomial of order at most $e-2$. Now it follows from (4.2.9) that

$$\mathrm{h}^{R'}_{e-1}(M,N) = \lim_{n\to\infty} \frac{g^{R'}_{M,N}(h,n)}{n^{e-1}} = 2d \cdot \lim_{n\to\infty} \frac{n_0}{d!} n^{d-e} \quad (4.2.10)$$

Notice, if $d \neq e$, $\lim_{n\to\infty} \frac{n_0}{d!} n^{d-e} = 0$. Thus (4.2.10) shows that $\mathrm{h}^{R'}_{e-1}(M,N) = 2e \cdot \frac{n_0}{d!} n^{d-e}$. Therefore (4.2.3) gives the equality we seek: $\mathrm{h}^{R'}_{e-1}(M,N) = 2e \cdot \mathrm{h}^{R}_{e}(M,N)$. □

4.3 Complete intersection dimension and the vanishing of Ext

We will now prove various vanishing results for $\mathrm{Ext}^i_R(M,N)$. Our main tool will be the function $\mathrm{h}^R_\bullet(M,N)$. Throughout this section we assume that M has finite complete intersection dimension, a situation which is slightly more general than assuming R is a complete intersection.

Definition 4.3.1. *([10]) Let R be a local ring, and let M be finitely generated R-module. Then M is said to have finite complete intersection dimension, denoted by* $\text{CI-dim}_R(M) < \infty$*, if there exists a diagram of local homomorphisms $R \to S \twoheadleftarrow P$, where $R \to S$ is flat, $S \twoheadleftarrow P$ is surjective with kernel generated by a regular sequence of P contained in the maximal ideal of P, and* $\mathrm{pd}_P(M \otimes_R S) < \infty$.

It follows from the definition that modules of finite projective dimension and modules over complete intersection rings have finite complete intersection dimension. There are also local rings R that are not complete intersections, and finitely generated R-modules that do not have finite projective dimension but have finite complete intersection dimension [10, Chapter 4].

The following theorem of Avramov, Gasharov and Peeva shows that finite complete intersection dimension implies finite complexity (cf. Definitions 2.2.1 & 2.2.2; see also Theorem 3.1.3):

Theorem 4.3.2. *([9, 4.1.2] and [10, 5.6]) Let $(R, \mathfrak{m}, k)$ be a local ring, and let M be finitely generated R-module. If* $\text{CI-dim}_R(M) < \infty$*, then* $\mathrm{cx}_R(M,N) \leq \mathrm{cx}_R(M,k) =$

$\mathrm{cx}_R(M) \leq \mathrm{embdim}(R) - \mathrm{depth}(R)$.

Next we point out that the function $\mathrm{h}^R_\bullet(M,N)$ is finite (cf. Definition 4.2.3) when M has finite complete intersection dimension:

Remark 4.3.3. *Let $(R,\mathfrak{m})$ be a local ring, and let M and N be finitely generated R-modules such that $\text{CI-dim}_R(M) < \infty$. Assume that $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$ and e is an integer such that $e \geq \mathrm{cx}_R(M,N)$. Then $\mathrm{h}^R_e(M,N)$ is finite. One can see this as follows: Since $\text{CI-dim}_R(M) < \infty$, there exists a diagram $R \to S \overset{\alpha}{\twoheadleftarrow} P$ as in Definition 4.3.1 such that $\mathrm{pd}_P(M \otimes_R S) < \infty$. Let p be a minimal prime of $S/\mathfrak{m}S$ and set $q = \alpha^{-1}(p)$. Then the localized diagram $R \to S_p \twoheadleftarrow P_q$ has zero-dimensional closed fiber such that $\mathrm{pd}_{P_q}(M \otimes_R S_p) < \infty$ (cf. for example the proof of [48, 2.11]). Thus we may replace the original diagram with the localized one and assume that the closed fiber $S/\mathfrak{m}S$ is Artinian. Therefore, since $R \to S$ is flat, $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) = \mathrm{f}^{S}_{\mathrm{ext}}(M \otimes_R S, N \otimes_R S)$ and $\mathrm{cx}_R(M,N) = \mathrm{cx}_S(M \otimes_R S, N \otimes_R S)$. Write $S = P/(\underline{f})$ for some regular sequence $\underline{f}$ of P. Then it follows from Theorem 4.2.4(1) that $\mathrm{h}^S_e(M \otimes_R S, N \otimes_R S)$ is finite. One can now define $\mathrm{h}^R_e(M,N)$ as in Definition 4.2.3; it is a multiple of $\mathrm{h}^S_e(M \otimes_R S, N \otimes_R S)$ and hence is finite.*

The following result was proved by Jorgensen [38, 2.6(1)] and Avramov-Buchweitz [9, 4.7] independently. We will generalize it in Corollary 4.3.7.

Theorem 4.3.4. *(Jorgensen, Avramov-Buchweitz) Let M and N be finitely generated modules over a local ring R such that $\text{CI-dim}_R(M) < \infty$. Set $c = \mathrm{cx}_R(M)$. If $\mathrm{Ext}^n_R(M,N) = \cdots = \mathrm{Ext}^{n+c}_R(M,N) = 0$ for some $n > \mathrm{depth}(R) - \mathrm{depth}(M)$, then $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > \mathrm{depth}(R) - \mathrm{depth}(M)$.*

Theorem 4.3.5. *Let R be a local ring, and let M and N be finitely generated R-modules. Assume $\text{CI-dim}_R(M) < \infty$ and $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$. Let e be an integer such* *that* $e \geq \mathrm{cx}_R(M,N)$*. Assume further that* $\mathrm{h}_e^R(M,N) = 0$*. If* $\mathrm{Ext}_R^n(M,N) = \cdots = \mathrm{Ext}_R^{n+e-1}(M,N) = 0$ *for some* $n > \mathrm{depth}(R) - \mathrm{depth}(M)$*, then* $\mathrm{Ext}_R^i(M,N) = 0$ *for all* $i > \mathrm{depth}(R) - \mathrm{depth}(M)$*.*

The proof of Theorem 4.3.5 will depend on the following lemma:

Lemma 4.3.6. *Let* $R \twoheadleftarrow P$ *be a surjection of local rings, and let* M *and* N *be finitely generated* R*-modules. Assume* $\mathrm{pd}_P(M) < \infty$ *and the kernel of* $R \twoheadleftarrow P$ *is generated by a regular sequence of* P*. Assume further that the residue field* k *of* P *is infinite. Set* $c = \mathrm{cx}_R(M,N)$*. If* $c \geq 1$*, then the surjection* $R \twoheadleftarrow P$ *can be factored as* $R \twoheadleftarrow Q \twoheadleftarrow P$ *such that* $\mathrm{cx}_Q(M,N) = 0$ *and the kernel of* $R \twoheadleftarrow Q$ *is generated by a regular sequence of* Q *of length* c*. Furthermore, if* $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$ *(i.e.,* $\lambda_R(\mathrm{Ext}_R^i(M,N)) < \infty$ *for all* $i \gg 0$*), then there exits a local ring* R' *and a non-zerodivisor* x *of* R' *such that* $R = R'/(x)$*,* $\mathrm{CI\text{-}dim}_{R'}(M) < \infty$ *and* $\mathrm{cx}_{R'}(M,N) = \mathrm{cx}_R(M,N) - 1$*.*

Proof. The proof follows from that of [7, 9.3.1] and [10, 5.9]. Note that Theorem 4.1.1 shows the graded module $\mathcal{E} = \mathrm{Ext}_R^*(M,N) \otimes_R k$ is finitely generated over $\mathcal{R} = \mathcal{S} \otimes_R k$, where $\mathcal{S} = R[\underline{\chi}]$ is the ring of cohomology operators defined by the map $R \twoheadleftarrow P$. Then, as $\mathcal{E} \neq 0$, $\dim_{\mathcal{R}} \mathcal{E} = c \geq 1$ [9, 1.3]. The kernel of $R \twoheadleftarrow P$ can be generated by a regular sequence $f_1, f_2, \ldots, f_r$ that defines $\underline{\chi} = \chi_1, \chi_2, \ldots, \chi_r$ such that; (i) the ring of cohomology operators defined by the presentation $R = Q/(f_1, f_2, \ldots, f_c)$, where $Q = P/(f_{c+1}, f_2, \ldots, f_r)$, is identified with $\mathcal{R}' = k[\chi_1, \chi_2, \ldots, \chi_c] \subseteq \mathcal{R}$ and (ii) $\chi_1, \chi_2, \ldots, \chi_c$ form a system of parameters on $\mathcal{E}$. Since $\mathcal{E}$ is finitely generated over $\mathcal{R}'$, Nakayama's lemma and Theorem 4.1.1 imply that $\mathrm{cx}_Q(M,N) = 0$.

Now assume $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$ and write $R = R'/(x)$ where $R' = Q/(f_1, \ldots, f_{c-1})$ and $x = f_c$. Note that, since $\mathrm{pd}_P(M) < \infty$, the map $Q \twoheadleftarrow P$ implies that $\mathrm{CI\text{-}dim}_Q(M) < \infty$. Furthermore $\mathrm{CI\text{-}dim}_{R'}(M) + (c-1) \leq \mathrm{CI\text{-}dim}_Q(M)$ [10, 1.2.3]. Thus $\mathrm{CI\text{-}dim}_{R'}(M) < \infty$. Write $R = R'/(x)$ with $R' = Q/(f_1, \ldots, f_{c-1})$. Consider the long exact sequence that follows from Proposition 4.1.4:

$$(4.3.6.1) \quad \ldots \to \operatorname{Ext}^n_R(M,N) \to \operatorname{Ext}^n_{R'}(M,N) \to \operatorname{Ext}^{n-1}_R(M,N) \to \ldots$$

Proposition 4.1.3 shows that we can use the length function, which is additive on short exact sequences, in the definition of the complexity of (M,N). Therefore, taking the alternating sum of the lengths of the Ext modules in (4.3.6.1), we conclude that (cf. also the argument in [10, 1.5]):

$$\operatorname{cx}_R(M,N) - 1 \leq \operatorname{cx}_{R'}(M,N), \quad \operatorname{cx}_{R'}(M,N) \leq \operatorname{cx}_Q(M,N) + c - 1 = \operatorname{cx}_R(M,N) - 1.$$

Thus $\operatorname{cx}_{R'}(M,N) = \operatorname{cx}_R(M,N) - 1$. □

Proof of Theorem 4.3.5. Set $c = \operatorname{cx}_R(M,N)$. If $c = 0$, then $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$, and hence the result follows from Theorem 4.3.4. So we assume that $c \geq 1$. Since $\operatorname{CI-dim}_R(M) < \infty$, by Remark 4.3.3 and [10, 1.14], one can choose a diagram of local homomorphisms $R \to S \twoheadleftarrow P$ as in Definition 4.3.1 such that $\operatorname{pd}_P(M \otimes_R S) < \infty$, $S/\mathfrak{m}S$ is Artinian and P has infinite residue field. Note that $\mathrm{h}^R_e(M,N) = 0$ if and only if $\mathrm{h}^S_e(M \otimes_R S, N \otimes_R S) = 0$. Therefore we may assume $R = S$. One can now apply Lemma 4.3.6 to construct the rings Q and R'.

We shall proceed by induction on e. We have already settled the case $c = 0$ or $e = 0$. So suppose $c = e = 1$. Then $R' = Q$ and $R = Q/(x)$. Since $\operatorname{Ext}^i_Q(M,N) = 0$ for all $i \gg 0$, Theorem 4.3.4 shows that $\operatorname{Ext}^i_Q(M,N) = 0$ for all $i > \operatorname{depth}(Q) - \operatorname{depth}(M)$. Thus Proposition 4.1.4 implies that $\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^{i+2}_R(M,N)$ for all $i > \operatorname{depth}(R) - \operatorname{depth}(M)$. Set $w = \operatorname{depth}(R) - \operatorname{depth}(M) + 1$. Since we assume $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M,N) < \infty$ there exist integers a and b such that $\beta^R_{w+2i}(M,N) = a$ and $\beta^R_{w+2i+1}(M,N) = b$ for all

$i \geq 0$. Now, since $\mathrm{h}_1^R(M,N)=0$, we have:

$$\lim_{n\to\infty} \frac{(-1)^w \cdot a + (-1)^{w+1}\cdot b + (-1)^{w+2}\cdot a + (-1)^{w+3}\cdot b + \cdots + (-1)^n \beta_n^R(M,N)}{n} = 0$$

The limit on the left is $(-1)^w(a-b)/2$, so $\beta_i^R(M,N) = \beta_{i+1}^R(M,N)$ for all $i \geq w$. Since $\mathrm{Ext}_R^j(M,N)=0$ for some integer $j \geq w$, we conclude that $\mathrm{Ext}_R^i(M,N)=0$ for all $i \geq w$, which is what we want. Assume now $e \geq 2$. Then Theorem 4.2.4(3) shows that $\mathrm{h}_{e-1}^{R'}(M,N)=0$. Moreover, by Proposition 4.1.4, we have that $\mathrm{Ext}_{R'}^{n+1}(M,N) = \cdots = \mathrm{Ext}_{R'}^{n+e-1}(M,N)=0$. Since $\mathrm{cx}_{R'}(M,N) = \mathrm{cx}_R(M,N)-1 \leq e-1$, the induction hypothesis implies that $\mathrm{Ext}_{R'}^i(M,N)=0$ for all $i > \mathrm{depth}(R') - \mathrm{depth}(M)$. Hence, using Proposition 4.1.4 and the fact that $\mathrm{Ext}_R^n(M,N) = \mathrm{Ext}_R^{n+1}(M,N) = 0$, we conclude $\mathrm{Ext}_R^i(M,N)=0$ for all $i \gg 0$. Thus $c=0$ and hence the result follows from Theorem 4.3.4. □

Since $\mathrm{cx}_R(M,N) \leq \mathrm{cx}_R(M)$ (cf. Theorem 4.3.2), our next result generalizes Theorem 4.3.4.

Corollary 4.3.7. *Let R be a local ring, and let M and N be finitely generated R-modules. Assume $\text{CI-dim}_R(M) < \infty$. Let e be an integer such that $e > \mathrm{cx}_R(M,N)$. If $\mathrm{Ext}_R^n(M,N) = \cdots = \mathrm{Ext}_R^{n+e-1}(M,N) = 0$ for some $n > \mathrm{depth}(R) - \mathrm{depth}(M)$, then $\mathrm{Ext}_R^i(M,N)=0$ for all $i > \mathrm{depth}(R) - \mathrm{depth}(M)$.*

Proof. If $\mathrm{f}_{\mathrm{ext}}^{\mathrm{R}}(M,N) < \infty$, then Theorem 4.2.4(1) shows that $\mathrm{h}_e^R(M,N)=0$ and hence the result follows from Theorem 4.3.5. Therefore it suffices to prove $\mathrm{f}_{\mathrm{ext}}^{\mathrm{R}}(M,N) < \infty$. We shall proceed by induction on $\dim(R)$. There is nothing to prove if $\dim(R)=0$, since Theorem 4.3.5 applies directly. Thus assume $\dim(R) \geq 1$ and let p be a non-maximal prime ideal of R. Since $\text{CI-dim}_{R_p}(M_p) \leq \text{CI-dim}_R(M)$ [10, 1.6] and

$\mathrm{cx}_{R_p}(M_p, N_p) \leq \mathrm{cx}_R(M, N)$, the induction hypothesis implies that $\mathrm{Ext}^i_{R_p}(M_p, N_p) = 0$ for all $i > \mathrm{depth}(R_p) - \mathrm{depth}(M_p)$. Therefore $\mathrm{f}^{\mathrm{R}}_{\mathrm{ext}}(M, N) < \infty$. □

Remark 4.3.8. *Avramov showed in [8, 9.3.7] that the conclusion of Corollary 4.3.7 is not necessarily true in case $e = \mathrm{cx}_R(M, N)$; there are finitely generated modules M and N over a complete intersection ring R such that $\mathrm{Ext}^n_R(M, N) = \cdots = \mathrm{Ext}^{n+e-1}_R(M, N) = 0$ for some $n > \mathrm{depth}(R) - \mathrm{depth}(M)$, where $e = \mathrm{cx}_R(M) = \mathrm{cx}_R(N) = \mathrm{cx}_R(M, N) > 1$.*

Proposition 4.3.9. *Let R be a local ring, and let M and N be finitely generated R-modules. Assume $\text{CI-dim}_R(M) < \infty$ and $\lambda_R(N) < \infty$. Let e be an integer such that $e \geq \max\{1, \mathrm{cx}(M)\}$. Then $\mathrm{h}^R_e(M, N) = 0$.*

Proof. Note that, if $\mathrm{cx}_R(M) = 0$, then the statement is obvious. Therefore we can assume $\mathrm{cx}_R(M) \geq 1$. We need to check the assertion only for the case $N = k$, the residue field of R, as N has a finite filtration by copies of k. In view of Theorem 4.2.4 and Lemma 4.3.6 we can assume $e = 1$. Then $\mathrm{cx}_R(M) = 1$. Moreover, by Lemma 4.3.6, we can write $R = R'/(x)$ where x is a non-zerodivisor of R' and $\mathrm{pd}_{R'} M < \infty$. Now Theorem 4.2.4(3) shows that $2 \cdot \mathrm{h}^R_1(M, k) = \mathrm{h}^{R'}_0(M, k) = \chi_{R'}(M)$, where $\chi_{R'}(M)$ is the Euler characteristic of M over R'. Since $x \in \mathrm{Ann}_{R'}(M)$, $\chi_{R'}(M) = 0$ (cf. [43, 19.8]). □

The following corollary now follows immediately from Theorem 4.3.5 and Proposition 4.3.9.

Corollary 4.3.10. *([13, 3.5]) Let R be a local ring, and let M and N be finitely generated R-modules. Assume $\text{CI-dim}_R(M) < \infty$ and $\lambda_R(N) < \infty$. If $\mathrm{Ext}^n_R(M, N) = \cdots = \mathrm{Ext}^{n+c-1}_R(M, N) = 0$ for some $n > \mathrm{depth}(R) - \mathrm{depth}(M)$, where $c = \mathrm{cx}_R(M)$, then $\mathrm{Ext}^i_R(M, N) = 0$ for all $i > \mathrm{depth}(R) - \mathrm{depth}(M)$.*

Recall that $\overline{G}(R)_{\mathbb{Q}} = (G(R)/\mathbb{Z} \cdot [R]) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $G(R)$ is the Grothendieck group of finitely generated R-modules, and $[M]$ denotes the class of M in $G(R)$.

Proposition 4.3.11. *Let $(R, \mathfrak{m})$ be a local ring, and let M and N be finitely generated R-modules. Assume the following conditions hold:*

1. $\text{CI-dim}_R(M) < \infty$.

2. $\text{pd}_{R_p}(M_p) < \infty$ *for all* $p \in \text{Spec}(R) - \{\mathfrak{m}\}$.

3. $[N] = 0$ *in* $\overline{G}(R)_{\mathbb{Q}}$.

Set $c = \text{cx}_R(M)$. *If* $\text{Ext}^n_R(M,N) = \cdots = \text{Ext}^{n+c-1}_R(M,N) = 0$ *for some* $n > \text{depth}(R) - \text{depth}(M)$*, then* $\text{Ext}^i_R(M,N) = 0$ *for all* $i > \text{depth}(R) - \text{depth}(M)$.

Proof. There is nothing to prove if $c = 0$. So we may assume $c \geq 1$. Let X be a finitely generated R-module. As $\text{pd}_{R_p}(M_p) < \infty$ for all $p \in \text{Spec}(R) - \{\mathfrak{m}\}$, $\text{f}^{\text{R}}_{\text{ext}}(M,X) < \infty$. Hence Theorem 4.2.4(1) shows that $\text{h}^R_c(M,X)$ is finite. Therefore $\text{h}^R_c(M,-) : G(R) \to \mathbb{Q}$ defines a linear map by Theorem 4.2.4(2). Note that, since $\text{CI-dim}_R(M) < \infty$, $\text{Ext}^i_R(M,R) = 0$ for all $i \gg 0$ (cf. [10, 1.4] and [4, ch.3]; see also [19, 1.2.7]). Thus one obtains an induced map $\text{h}^R_c(M,-) : \overline{G}(R)_{\mathbb{Q}} \to \mathbb{Q}$. This implies, since $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$, that $\text{h}^R_c(M,N) = 0$. The result now follows from Theorem 4.3.5. $\square$

Our next two results follow from Proposition 4.1.5 and Proposition 4.3.11. They improve Theorem 4.3.4 for finitely generated modules over certain local rings (see also Remark 4.1.6 concerning Corollary 4.3.12).

Corollary 4.3.12. *Let R be a two-dimensional local normal domain such that the class group of R is torsion. Let M and N be finitely generated R-modules. Assume that*

$\text{CI-dim}_R(M) < \infty$. *Set* $c = \text{cx}_R(M)$. *If* $\text{Ext}_R^n(M,N) = \cdots = \text{Ext}_R^{n+c-1}(M,N) = 0$ *for some* $n > 2 - \text{depth}(M)$, *then* $\text{Ext}_R^i(M,N) = 0$ *for all* $i > 2 - \text{depth}(M)$.

Corollary 4.3.13. *Let R be a one-dimensional local domain, and let M and N be finitely generated R-modules. Assume* $\text{CI-dim}_R(M) < \infty$. *Set* $c = \text{cx}_R(M)$. *If* $\text{Ext}_R^n(M,N) = \cdots = \text{Ext}_R^{n+c-1}(M,N) = 0$ *for some* $n > 1 - \text{depth}(M)$, *then* $\text{Ext}_R^i(M,N) = 0$ *for all* $i > 1 - \text{depth}(M)$.

We will make use of the following result, due to Araya and Yoshino, and improve Corollary 4.3.13 over complete intersections.

Theorem 4.3.14. *([2, 4.2], cf. also [9, 4.8]) Let R be a local ring, and let M and N be finitely generated R-modules. Assume* $\text{CI-dim}_R(M) < \infty$. *If* $\text{Ext}_R^i(M,N) = 0$ *for all* $i \gg 0$, *then* $\sup\{n : \text{Ext}_R^n(M,N) \neq 0\} = \text{depth}(R) - \text{depth}(M)$.

Recall that each finitely generated module over a complete intersection ring has finite complete intersection dimension and finite complexity (cf. Theorem 4.3.2).

Proposition 4.3.15. *Let R be a one-dimensional local complete intersection domain, and let M and N be finitely generated R-modules. Assume* $\text{Ext}_R^1(M,N) = \cdots = \text{Ext}_R^c(M,N) = 0$, *where* $c = \text{cx}_R(M) \geq 1$. *Then M is torsion-free and* $\text{Ext}_R^i(M,N) = 0$ *for all* $i \geq 1$.

Proof. If M is torsion-free, then the result follows from Corollary 4.3.13. Therefore suppose M has torsion, that is, $\text{depth}(M) = 0$ (since $\dim(R) = 1$). Then there exists an exact sequence (a maximal Cohen-Macaulay approximation)

$$(4.3.15.1)\quad 0 \to T \to X \to M \to 0$$

where X is torsion-free and T has finite injective dimension [5]. Since R is Gorenstein, T has also finite projective dimension [41, 2.2]. As R is one-dimensional and $\operatorname{depth}(M) = 0$, the depth lemma and (4.3.15.1) imply that T is free. Therefore, by (4.3.15.1), $\operatorname{Ext}^1_R(X,N) = \cdots = \operatorname{Ext}^c_R(X,N) = 0$ and $\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_R(X,N)$ for all $i \geq 2$. Since $\operatorname{cx}_R(M) = \operatorname{cx}_R(X)$, it now follows from Corollary 4.3.13 that $\operatorname{Ext}^i_R(X,N) = 0$ for all $i \geq 1$. Thus $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \geq 1$. However, since $\operatorname{depth}(M) = 0$, $\operatorname{Ext}^1_R(M,N) \neq 0$ by Theorem 4.3.14. Therefore M is torsion-free and $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \geq 1$. $\square$

As the vanishing of $\operatorname{Ext}^i_R(M,N)$ for all $i \gg 0$ over a hypersurface R forces M or N to have finite projective dimension [9, 5.12], Corollary 4.3.13 and Proposition 4.3.15 yield the following result:

Corollary 4.3.16. *Let R be a one-dimensional local hypersurface domain, and let M and N be finitely generated R-modules. If $\operatorname{Ext}^n_R(M,N) = 0$ for some positive integer n, then either M or N has finite projective dimension.*

Remark 4.3.17. *We note that the conclusion of Corollary 4.3.16 is not true over an arbitrary hypersurface; it is easy to see that $\operatorname{Ext}^1_R(M,M) = 0$ if $M = R/(x)$ and $R = k[[X,Y]]/(XY)$. Recall also from Question 3.3.16 that if M is a finitely generated torsion-free module over a one-dimensional complete intersection domain R, it is not known in general whether $\operatorname{Ext}^1_R(M,M) = 0$ (equivalently $M \otimes_R M^*$ is torsion-free [33, 4.6]) forces M to be free. For modules with bounded Betti numbers, we have the following result (see also [23, 5.5]).*

Proposition 4.3.18. *Let R be a local ring, and let M be finitely generated R-module. Assume $\operatorname{CI-dim}_R(M) < \infty$ and M has bounded Betti numbers. Assume further that $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. If $\operatorname{Ext}^n_R(M,M) = 0$ for some $n > \operatorname{depth}(R) - \operatorname{depth}(M)$, then M has finite projective dimension.*

Proof. Notice that, by definition, $\mathrm{cx}_R(M) \leq 1$ (see Definition 2.2.1). We will use the fact that M has finite projective dimension if and only if $\mathrm{Ext}_R^{2n}(M, M) = 0$ for some nonnegative integer n [9, 4.2]. We proceed by induction on $\dim(R)$. Assume first $\dim(R) = 0$. Then Corollary 4.3.10 implies that $\mathrm{pd}_R(M) < \infty$. Suppose now that $\dim(R) \geq 1$. Then the induction hypothesis implies that $\mathrm{pd}_{R_p}(M_p) < \infty$ for all non-maximal prime ideals p of R. Hence Proposition 4.3.11 and [9, 4.2] give the desired result. $\square$

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