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The Cohomology of Modules over a Complete Intersection Ring

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COHOMOLOGY OF MODULES OVER COMPLETE INTERSECTION RINGS

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COHOMOLOGY OF MODULES OVER COMPLETE INTERSECTION RINGS

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We investigate the cohomology of modules over commutative complete intersection rings. The first main result is that if M is an arbitrary module over a complete intersection ring R, and if $\operatorname{Ext}_R^{2n}(M, M) = 0$ for some $n \geq 1$ then M has finite projective dimension. The second main result gives a new proof of the fact that the support variety of a Cohen-Macaulay module whose completion is indecomposable is projectively connected.

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Chapter 1

Introduction

Commutative algebra is a classical subject which originated in the study of topics such as algebraic curves, factorization in number fields, and the invariant theory of groups. Homological algebra is an abstraction of methods used in algebraic topology. In the mid 1950's Auslander, Buchsbaum and Serre used homological algebra to solve several long-open conjectures in commutative algebra. Since that groundbreaking work, homological techniques have been very important in the study of commutative algebra.

In this thesis we study the class of commutative rings known as *complete intersection rings.* We make now the assumption that all rings are noetherian and commutative. For ease of exposition we will also assume in this introduction that all rings are local. A ring R with a presentation as the quotient of a regular local ring Q by an ideal I,

$$R = Q/I,$$

is a complete intersection if I can be generated by a Q-regular sequence. This class of rings is particularly amenable to homological methods. To give an idea of what position these rings occupy among all commutative rings let us recall that there is a

"homological" hierachy of local rings:

$$\text{regular} \Rightarrow \text{complete intersection} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay}.$$

Regular local rings are the simplest rings homologically. In the work of Auslander, Buchsbaum and Serre mentioned above, regular local rings are characterized by the fact that every module has a finite free resolution. Thus a *singular* complete intersection, by which we mean one that's not regular, has at least one module whose minimal free resolution goes on forever. Understanding the infinite free resolutions that occur over a complete intersection is a classical problem.

One of the first steps in this program was taken by Tate [25] where he explicitly constructed a free resolution of the residue field over a complete intersection. Shamash [24] used the finite Q-free resolution of a finitely generated R-module to construct an R-free resolution of M. These results led to subsequent work in such papers as Gulliksen [19], Eisenbud [17], Avramov [3] and Avramov, Gasharov and Peeva [7]. These all study free resolutions over complete intersections. In particular this work has shown that the infinite resolutions that occur over a complete intersection have a different character than resolutions over non-complete intersection rings.

One of the themes that ties together the work after Shamash, listed above, is the use of *cohomology operators*. These operators have become central to the study of homological behavior of modules over complete intersection rings. In this thesis in particular we make use of them in a vital way. Because of this, and because they will lead us to a description of the results contained herein, we now describe them in detail.

For the rest of this introduction we assume that R is a complete intersection of the form $Q/(f_1, \ldots, f_c)$, where Q is a regular local ring and $\mathbf{f} = f_1, \ldots, f_c$ is an Q-regular

sequence. The *ring of cohomology operators* is the polynomial ring over R

$$\mathcal{S} = R[\chi_1, \ldots, \chi_c],$$

graded by giving each of the c polynomial indeterminates degree 2. Recall that $\text{Ext}_R^*(M, M)$ is a graded algebra for any R-module M. The multiplication is given by Yoneda products. The cohomology operators act via central homogeneous maps of *R-algebras*

$$\eta_M : \mathcal{S} \to \text{Ext}_R^*(M, M).$$

for each R-module M.

The usefulness of the operators is due to the non-triviality of the maps η_M. In general the Ext-algebra of a module over a commutative noetherian ring is highly non-commutative and non-noetherian. However, under the assumption that R is a complete intersection and M is finitely generated, the algebra $\text{Ext}_R^*(M, M)$ is a finitely generated $\mathcal{S}$-module via the map η_M. Thus this noncommutative algebra is module finite over a commutative noetherian ring and in particular is left and right noetherian. A direct consequence of this gives strong results about the Betti numbers of M. Recall that the *nth Betti number of M*, denoted $\beta_n(M)$ and calculated by $\dim_k \text{Ext}_R^n(M, k)$, is the rank of the nth free module in a minimal free resolution of M. Using basic commutative algebra and the finite generation of $\text{Ext}_R^*(M, M)$ over $\mathcal{S}$, one can show that the sequence $\{\beta_n(M)\}$, with $n \in \mathbb{N}$, satisfies a linear recurrence relation and is bounded by a polynomial in n of degree at most $c - 1$. For a module over a general local ring this sequence may grow exponentially and need not satisfy any linear recurrence relation.

Let us decribe another application of the action of the operators that will lead us immediately to a description of our first result. Avramov and Buchweitz showed that

if M is a finitely generated module with $\text{Ext}_R^{2n}(M, M) = 0$ for some $n \geq 1$ then M has finite projective dimension [4, 4.3]. This generalized greatly [2, 1.8] which was the case $n = 1$. While they do not appear in the statement of the result the action of the cohomology operators is essential to the proof.

The first result of this thesis generalizes the Avramov-Buchweitz theorem from finitely generated modules to arbitrary modules:

Theorem A. *Let $R = Q/(f_1, \ldots, f_c)$ where Q is a regular ring and $f_1, \ldots, f_c$ is an Q-regular sequence. Let M be an arbitrary R-module such that $\text{Ext}_R^{2n}(M, M) = 0$ for some $n \geq 1$. Then M has finite projective dimension.*

The proof of the Avramov-Buchweitz theorem relies on the finite generation of $\text{Ext}_R^*(M, M)$ over $\mathcal{S}$; see 4.1.1. However when M is not finitely generated over R then $\text{Ext}_R^*(M, M)$ may not be finitely generated over $\mathcal{S}$: already $\text{Ext}_R^0(M, M) = \text{Hom}_R(M, M)$ may not be finitely generated over $\mathcal{S}^0 = R$. To overcome this difficulty we work "globally." We proceed via the embedding of the category of R-modules into the *homotopy category of injective R-modules*, denoted $\mathsf{K}(\mathsf{Inj}\, R)$, whose objects are chain complexes of injective modules. This category is *triangulated* and *compactly generated* by the images of finitely generated R-modules; two properties that are essential to our argument. Indeed, the Avramov-Buchweitz result shows that the property we hope to prove holds for all "generators" of $\mathsf{K}(\mathsf{Inj}\, R)$. Using techniques from homotopy theory we are able to show that the property holds for a large class of objects of $\mathsf{K}(\mathsf{Inj}\, R)$. In particular it holds for all modules.

The proof of Theorem A was inspired by [11]. This methods in loc. cit. generalize the classical concept of support variety to compactly generated triangulated categories. Support varieties attach a geometric object to a representation, or equivalently module, that gives information about the free resolution of the module. Support

varieties were first defined by Carlson for representations of finite groups and have proliferated throughout algebra to include among others restricted Lie algebras, finite group schemes, and most relevant to us, finitely generated modules over complete intersections.

A general guideline is that the support set for an object X should closely reflect the structure of X. A famous instance of this is Carlson's connectedness theorem [16], which shows that the support variety of an indecomposable group representation is projectively connected. Bergh proved an analogue of this for support varieties over complete intersections. In the second half of this thesis we prove a generalization of Bergh's theorem to the setup of [11]:

Theorem B. *Let X be an object of $\mathsf{K}(\mathsf{Inj}\,R)$ such that the image of X is indecomposable in $\mathsf{K}(\mathsf{Inj}\,R)/\mathsf{loc}_{\mathsf{K}(\mathsf{Inj}\,R)}(\mathsf{i}R)$. Then the support set of X is a connected subset of $\mathrm{Proj}\,\mathcal{S}$.*

In the statement of the theorem $\mathsf{loc}_{\mathsf{K}(\mathsf{Inj}\,R)}(\mathsf{i}R)$ is the localizing subcategory of K generated by $\mathsf{i}R$, an injective resolution of R, and $\mathsf{K}(\mathsf{Inj}\,R)/\mathsf{loc}_{\mathsf{K}(\mathsf{Inj}\,R)}(\mathsf{i}R)$ is the Verdier quotient; the terminology is recalled in Chapter 2.

The support set of a module provides a measurement of the entire free resolution of the module. Thus for a connectedness result we need a stronger assumption than that the module is indecomposable: we need to ensure that its syzygies are indecomposable as well. When X is the injective resolution of a module, the assumption that X is indecomposable in $\mathsf{K}(\mathsf{Inj}\,R)/\mathsf{loc}_{\mathsf{K}(\mathsf{Inj}\,R)}(\mathsf{i}R)$ forces the syzygies of the module to be indecomposable. When the ring is zero dimensional, e.g. a group ring as in the case of Carlson's result, a module being indecomposable forces its syzygies to be indecomposable but this is no longer the case in higher dimensions. When X is the injective resolution of a maximal Cohen-Macaulay module M this condition boils

down to assuming that M is indecomposable after removing free summands.

As to the conclusion of the theorem, note that if the ring R is local then $\mathcal{S}$ has a unique maximal homogeneous ideal and hence every subset of $\operatorname{Spec} \mathcal{S}$ is connected. Thus being connected in $\operatorname{Proj} \mathcal{S}$ is a stronger statment.

A connection between Theorems A and B is that we work in the category $\mathsf{K}(\mathsf{Inj}\, R)$ to prove them. Compactly generated triangulated categories, such as $\mathsf{K}(\mathsf{Inj}\, R)$, behave very much like the homotopy category studied by algebraic topologists. In particular there is a rich theory of useful techniques imported from algebraic topology. In Chapter 2 we review the relevant defintions and techniques that we will use in the sequel. In Chapter 3 we sketch a construction of the action the ring of operators and discuss how this action is used to define the two notions of support mentioned above. Chapters 4 is devoted to proving Theorem A above. There is a result of independent interest in the first section. In Chapter 5 we prove Theorem B and show how it may be used to give a new proof of Bergh's result on support varieties over complete intersections.

Chapter 2

Background on triangulated categories

We make use of various triangulated categories throughout this thesis. In this chapter we review relevant properties of triangulated categories and discuss in detail the particular triangulated categories that we employ in the sequel.

2.1 Definitions

2.1.1. An *additive category* is a category in which all Hom-sets are abelian groups and the composition maps are homomorphisms of abelian groups. Moreover, finite products and coproducts exist and coincide. Let T be an additive category with a fixed autoequivalence Σ_{T}. A *triangle* in T is a diagram of the form:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma_{\mathsf{T}} X$$

where X, Y, Z are objects of T.

The category T is *triangulated with shift functor* Σ_{T} if there exists an autoequiv-

alence Σ_T and a class of triangles, the *distinguished triangles*, that satisfy the axioms (TR1)-(TR4) listed in e.g. [26].

When T is triangulated, a *triangulated subcategory* S of T is a full additive subcategory such that for all distringuished triangles $X \to Y \to Z \to \Sigma_T X$ with at least two of X, Y and Z in S then the third is as well. This implies in particular that S is closed under Σ_T.

Let T and D be triangulated categories with shift functors Σ_T and Σ_D respectively. A functor $F : T \to D$, equipped with an isomorphism of functors $\Sigma_D \circ F \cong F \circ \Sigma_T$, is a *triangulated functor* from T to D if F maps distinguished triangles in T to distinguished triangles in D.

For the rest of the section T will be a triangulated category with shift functor $\Sigma = \Sigma_T$. By a triangle in T we always mean a distinguished triangle.

2.1.2. Let S be a triangulated subcategory of T. We say S is a *thick subcategory* if $X \oplus X' \in S$ implies that $X \in S$. If furthermore S is closed under all set-valued coproducts then we say S is *localizing*. We will often use the fact that if T is itself closed under coproducts then a triangulated subcategory S is localizing if it is closed under coproducts [22, 1.5], i.e. S is already thick in this case.

For a class of objects $\mathcal{C}$ in T the *thick subcategory generated by $\mathcal{C}$*, denoted $\mathsf{thick}_T(\mathcal{C})$, is the smallest thick subcategory containing $\mathcal{C}$. Analagously one defines the *localizing subcategory generated by $\mathcal{C}$*, denoted $\mathsf{loc}_T(\mathcal{C})$.

2.1.3. Let X, Y be objects in T. By $\mathrm{Hom}_T^*(X, Y)$ we denote the $\mathbb{Z}$-graded abelian group which in degree n is $\mathrm{Hom}_T(X, \Sigma^n Y)$.

The group $\mathrm{Hom}_T^*(X, X)$ is a graded ring with multiplication given by composition. Also $\mathrm{Hom}_T^*(X, Y)$ is a bimodule with left action by $\mathrm{Hom}_T^*(Y, Y)$ and right action by $\mathrm{Hom}_T^*(X, X)$.

2.1.4. An object X of T is *compact* if the natural map

$$\bigoplus_{i \in I} \mathrm{Hom}_{\mathsf{T}}(X, Y_i) \to \mathrm{Hom}_{\mathsf{T}}\Big(X, \bigoplus_{i \in I} Y_i\Big)$$

is an isomorphism for all sets of objects $\{Y_i\}_{i \in I}$ of T. We denote the full subcategory of compact objects of T by T^c.

2.1.5. The category T is *compactly generated* if there is a set of compact objects $\mathcal{C}$ such that

$$\mathrm{loc}_{\mathsf{T}}(\mathcal{C}) = \mathsf{T}.$$

In this case we say $\mathcal{C}$ is a *set of compact generators for* T.

When $\mathcal{C}$ is a set of compact objects a result of Neeman [22, 2.2] shows that

$$\mathrm{loc}_{\mathsf{T}}(\mathcal{C})^c = \mathrm{thick}_{\mathsf{T}}(\mathcal{C}).$$

In particular if $\mathcal{C}$ is a set of compact generators of T then the full subcategory of compact objects of T is completely determined by $\mathcal{C}$:

$$\mathsf{T}^c = \mathrm{thick}_{\mathsf{T}}(\mathcal{C}).$$

The following is well known. We give a proof for lack of a reference, and to give a flavor of the type of argument used in this context.

Lemma 2.1.6. *Let* T *be compactly generated with a set of generators* $\mathcal{C}$ *and let* X *be an object of* T. *Then* $X \neq 0$ *if and only if there exists a* $C \in \mathcal{C}$ *such that* $\mathrm{Hom}_{\mathsf{T}}^*(C, X) \neq 0$.

Proof. Clearly if $\mathrm{Hom}_{\mathsf{T}}^*(C, X)$ is nonzero for any object C then X is nonzero. To see

the other direction we consider the full subcategory

$$\mathcal{B} = \{\, Z \mid \operatorname{Hom}_{\mathsf{T}}^*(Z, X) = 0 \,\}.$$

We claim it is localizing. It is clearly closed under Σ as

$$\operatorname{Hom}_{\mathsf{T}}^*(Z, X) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(Z, \Sigma^n X) \cong \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(\Sigma^{-n} Z, X).$$

Given an exact triangle $Z' \to Z \to Z'' \to$ there is a long exact sequence

$$\ldots \to \operatorname{Hom}_{\mathsf{T}}^n(Z'', X) \to \operatorname{Hom}_{\mathsf{T}}^n(Z, X) \to \operatorname{Hom}_{\mathsf{T}}^n(Z', X) \to \operatorname{Hom}_{\mathsf{T}}^{n+1}(Z'', X) \to \ldots$$

from which one sees that if $\operatorname{Hom}_{\mathsf{T}}^*(-, X)$ is zero for two of the three objects in the triangle it must be zero on the third. Finally $\mathcal{B}$ is closed under coproducts as there is an isomorphism

$$\prod_{i \in I} \operatorname{Hom}_{\mathsf{T}}^*(Z_i, X) \to \operatorname{Hom}_{\mathsf{T}}^*(\bigoplus_{i \in I} Z_i, X)$$

for any set of objects Z_i in K.

If $\operatorname{Hom}_{\mathsf{T}}^*(C, X) = 0$ for all C in $\mathcal{C}$ then all objects of $\mathcal{C}$ are in $\mathcal{B}$. But since $\mathcal{B}$ is localizing we have $\mathsf{T} = \mathsf{loc}_{\mathsf{T}}(\mathcal{C}) \subseteq \mathcal{B}$. Thus $\mathcal{B} = \mathsf{T}$ and in particular $X \in \mathcal{B}$. This shows that $\operatorname{Hom}_{\mathsf{K}}^*(X, X) = 0$ and hence X is zero itself. $\square$

2.2   Homotopy categories

In this section R denotes a left noetherian associative ring and $\operatorname{Mod} R$ denotes the category of left R-modules.

2.2.1. Let $\mathcal{A}$ be an additive subcategory of $\operatorname{Mod} R$. By an $\mathcal{A}$-*complex* we mean a

diagram

$$\ldots \to X^n \xrightarrow{\partial_X^n} X^{n+1} \xrightarrow{\partial_X^{n+1}} X^{n+2} \to \ldots$$

of R-modules with $X^n \in \mathcal{A}$ and $\partial_X^{n+1}\partial_X^n = 0$ for all $n \in \mathbb{Z}$. When $\mathcal{A}$ is $\operatorname{Mod} R$ we simply say R-complex.

2.2.2. Let X be an R-complex. We write $\operatorname{H}^n(X)$ for the nth cohomology group of X and $\operatorname{H}(X)$ for the graded R-module which in degree n is $\operatorname{H}^n(X)$. We say X has *finitely generated total cohomology* if $\operatorname{H}(X)$ is a finitely generated R-module.

The *Hom-complex* between R-complexes X, Y, denoted $\operatorname{Hom}_R(X, Y)$, has components and differential given by

$$\operatorname{Hom}_R(X, Y)^n = \prod_{i \in \mathcal{Z}} \operatorname{Hom}_R(X^i, Y^{i+n}) \qquad \partial(f) = \partial_Y \circ f - (-1)^{|f|} f \circ \partial_X.$$

A *morphism* is a degree zero cycle $f : X \to Y$ of $\operatorname{Hom}_R(X, Y)$; thus a morphism is a degree preserving map from X to Y that commutes with the differentials. We denote the induced map on cohomology by $\operatorname{H}(f)$. It is a *quasi-isomorphism* when $\operatorname{H}(f) : \operatorname{H}(X) \to \operatorname{H}(Y)$ is an isomorphism.

2.2.3. The category of $\mathcal{A}$-complexes and morphisms is denoted $\mathsf{C}(\mathcal{A})$. The *homotopy category of $\mathcal{A}$*, denoted $\mathsf{K}(\mathcal{A})$, has the same objects as $\mathsf{C}(\mathcal{A})$ with morphisms given by

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y) := \operatorname{H}^0(\operatorname{Hom}_R(X, Y)).$$

Thus morphisms in this category are morphisms of complexes modulo homotopy equivalence.

2.2.4. The shift functor Σ on $\mathsf{K}(\mathcal{A})$ is defined by $(\Sigma X)^n = X^{n+1}$ and $\partial_{\Sigma X} = -\partial_X$.

Given a morphism

$$f : X \to Y$$

in $\mathsf{K}(\mathcal{A})$ the *mapping cone* fits into a triangle $X \xrightarrow{f} Y \to \mathrm{cone}(f) \to \Sigma X$; see section 1.5 of [26] for a reference. A triangle $X' \to Y' \to Z' \to \Sigma X'$ in $\mathsf{K}(\mathcal{A})$ is *distinguished* if there is a commutative diagram

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'
\end{array}
$$

in which the vertical maps are isomorphisms. These structures make $\mathsf{K}(\mathcal{A})$ into a triangulated category with shift and distinguished triangles as above; see section 10.2 of [26] for a proof.

The following is a subcategory of $\mathsf{K}(\mathcal{A})$ that we will use frequently in the sequel.

2.2.5. A complex X is *acyclic* if $\mathrm{H}(X) = 0$. The category $\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$ is the full subcategory of $\mathsf{K}(\mathcal{A})$ whose objects are the acyclic $\mathcal{A}$-complexes. Since homology commutes with coproducts one easily checks that $\mathsf{K}_{\mathsf{ac}}(\mathcal{A})$ is a localizing subcategory of $\mathsf{K}(\mathcal{A})$.

2.3   Localization in triangulated categories

In this section we review the two types of localization for triangulated categories. Throughout T is a triangulated category and S is a thick subcategory of T.

2.3.1. The *kernel* of a triangulated functor $F : \mathsf{T} \to \mathsf{D}$, denoted $\ker F$, is the full subcategory of T with objects all X such that $F(X) = 0$; this is a thick subcategory.

A *Verdier localization* of S is a triangulated functor

$$Q : \mathsf{T} \to \mathsf{T}/\mathsf{S}$$

from T to a triangulated category T/S such that $\ker Q = \mathsf{S}$ and every triangulated functor $F : \mathsf{T} \to \mathsf{D}$ with $\mathsf{S} \subseteq \ker F$ factors uniquely through Q. By the universal property it is easy to see that the category S is unique up to equivalence.

To construct a Verdier localization one takes T/S to be the category whose objects are the same as those of T. The morphisms from X to Y is given by equivalence classes of diagrams of the form

$$
\begin{array}{ccc}
 & X' & \\
{}^{\alpha}\swarrow & & \searrow{}^{\beta} \\
X & & Y
\end{array}
$$

such that the cone of α is in S. The equivalence relation, namely that two diagrams are equivalent if there exists a third that both factor through, ensures that composition is well-defined in the category T/S. A reference for this construction is section 2 of [23]. The localizations we encounter in this thesis have no set-theoretic issues as they may be realized as subcategories of existing categories.

The next construction allows us to "separate" an object X into a piece that lies in a subcategory S and a piece that lies away from S.

2.3.2. Let S be a localizing subcategory of T and let

$$\mathsf{S}^{\perp} := \{\, X \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}^{*}(S, X) = 0 \text{ for all } S \in \mathsf{S}\}.$$

When one can, for any object X in T, find a triangle

$$X' \to X \to X'' \to \Sigma X'$$

with X' in S and X'' in $\mathsf{S}^{\perp}$ one says that *Bousfield localization* at S exists. By [23, 8.4.5] Bousfield localization exists if S is the localizing subcategory of a set of compact objects of T; see 2.1.4 for the definition.

When a triangle as above exists, the map $X' \to X$ is universal with respect to maps from objects in S to X and the map $X \to X''$ is universal with respect to maps from X to objects in $\mathsf{S}^{\perp}$.

Example 2.3.3. Let R be a commutative noetherian ring. Consider the homotopy category $\mathsf{K}(\operatorname{Mod} R)$ and the thick subcategory $\mathsf{K}_{\mathsf{ac}}(\operatorname{Mod} R)$ of acyclic R-complexes. The Verdier localization of $\mathsf{K}_{\mathsf{ac}}(\operatorname{Mod} R)$ in $\mathsf{K}(\operatorname{Mod} R)$ is denoted $\mathsf{D}(R)$. This is the classical *derived category* of R. Since a morphism α in $\mathsf{K}(\operatorname{Mod} R)$ is a quasi-isomorphism if and only the cone of α is acyclic, we see from the construction above that every quasi-isomorphism is invertible in $\mathsf{D}(R)$.

The module R, viewed as a complex concentrated in degree 0, is compact in $\mathsf{D}(R)$. To see this, note that $\operatorname{Hom}^{n}_{\mathsf{D}(R)}(R, X) \cong \mathrm{H}^{n}(X)$ and use the fact that cohomology commutes with direct sums. Moreover there is a nonzero map from $\Sigma^{i} R \to X$ for every non-zero X in $\mathsf{D}(R)$ and some $i \in \mathbb{Z}$. This shows that $(\mathsf{loc}_{\mathsf{D}(R)}(R))^{\perp} = 0$. The Bousfield localization at $\mathsf{loc}_{\mathsf{D}(R)}(R)$ exists by the above; however since $(\mathsf{loc}_{\mathsf{D}(R)}(R))^{\perp} = 0$ this shows that every object $X \in \mathsf{D}(R)$ is in $\mathsf{loc}_{\mathsf{D}(R)}(R)$. Thus R is a compact generator of $\mathsf{D}(R)$. From Neeman's theorem, recalled in 2.1.5 above, we see that the compact objects of $\mathsf{D}(R)$ are exactly those in the thick subcategory generated by R:

$$\mathsf{D}(R)^{\mathsf{c}} = \mathsf{thick}_{\mathsf{D}(R)}(R).$$

Also $\mathsf{thick}_{\mathsf{D}(R)}(R)$ consists of all complexes which are quasi-isomorphic to finite complexes of finitely generated projective modules; such a complex is *perfect*. In particular the image of a finitely generated module in $\mathsf{D}(R)$ is compact if and only if the module has finite projective dimension.

The connection between Verdier and Bousfield localization is given by the following

Theorem 2.3.4. *Let* S *be a triangulated subcategory of a compactly generated triangulated category* T. *The following are equivalent:*

(i) Bousfield localization at S *exists;*

(ii) the Verdier localization map $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ *has a right adjoint;*

(iii) the inclusion $\mathsf{S} \to \mathsf{T}$ *has a right adjoint.*

In case one of these conditions hold, the right adjoint to $\mathsf{T} \to \mathsf{T}/\mathsf{S}$ *is a fully faithful functor whose image is the subcategory* $\mathsf{S}^\perp$. *Moreover the triangle* (2.3.2.1) *for an object* X *is given by the canonical maps of the adjunction.*

Proof. The equivalence of (i) and (ii) is [23, 9.1.13] while [21, 3.2] shows that (ii) and (iii) are equivalent. By 9.1.16 and 9.1.8 of [23] the last two statements hold in case Bousfield localization exists. $\qquad\square$

2.4 Homotopy category of injectives

In this section we discuss the category in which we work throughout the sequel. We assume that R is a left noetherian associative ring and $\operatorname{Mod} R$ again denotes the category of left R-modules.

2.4.1. Let $\operatorname{Inj} R$ be the full additive subcategory of $\operatorname{Mod} R$ whose objects are the injective R-modules. As in 2.2.1 we consider the homotopy category of this additive subcategory, denoted $\mathsf{K}(\operatorname{Inj} R)$, which is called the *homotopy category of injective R-modules*. When the ring is clear from the context we abbreviate $\mathsf{K}(\operatorname{Inj} R)$ to K.

We say an R-complex I is *semi-injective* if I^n is an injective R-module for all $n \in \mathbb{N}$ and if the Hom-complex $\operatorname{Hom}_R(-, I)$ preserves quasi-isomorphisms. Let K_{ac} be the category of acyclic complexes of injectives, see 2.2.5. From the definition it follows that a complex I in K is semi-injective if and only if it is in the subcategory $\mathsf{K}_{\mathsf{ac}}^{\perp} := \{\, X \in \mathsf{K} \mid \operatorname{Hom}_{\mathsf{K}}^*(A, X) = 0 \text{ for all } A \in \mathsf{K}_{\mathsf{ac}} \,\}$.

Let X be an arbitrary R-complex. A *semi-injective resolution* of X is a semi-injective complex $\mathsf{i}X$ and a quasi-isomorphism $X \to \mathsf{i}X$. Such a resolution always exists; see [5] or [6].

The definition of semi-injective resolutions is formulated to give these complexes the lifting properties required to do homological algebra. In particular they are used in two ways: to define injective dimension and derived functors.

2.4.2. A complex X has *finite injective dimension* if there exists a semi-injective resolution of X, say $\mathsf{i}X$, such that $(\mathsf{i}X)^n = 0$ for all $n \gg 0$.

For complexes X and Y define the *derived functor of the* Hom-*complex* to be

$$\operatorname{Ext}_R^*(X, Y) := \operatorname{Hom}_{\mathsf{K}}^*(\mathsf{i}X, \mathsf{i}Y)$$

where $\mathsf{i}X$ and $\mathsf{i}Y$ are semi-injective resolutions of X and Y respectively. Using the definition of semi-injective complex one can show that the above definition is independent of the resolutions chosen and that $\operatorname{Ext}_R^*(X, Y) \cong \operatorname{Hom}_{\mathsf{K}(\operatorname{Mod} R)}^*(X, \mathsf{i}Y)$.

When X and Y are complexes concentrated in degree 0, i.e. modules, these notions

are the usual ones.

2.4.3. The diagram

$$\mathsf{K}_{\mathsf{ac}} \to \mathsf{K} \to \mathsf{D}(R)$$

is a Bousfield localization at K_{ac}; see for instance [21, 3.6]. In particular the natural map $\mathsf{K} \to \mathsf{D}(R)$ has a fully-faithful right adjoint Q_ρ whose image is $\mathsf{K}_{\mathsf{ac}}^\perp$, i.e. the subcategory of semi-injective complexes. One can show that for a complex X, the complex $Q_\rho X$ is a semi-injective resolution of X. Thus Q_ρ is an embedding of $\mathsf{D}(R)$ into K that sends every complex to a semi-injective resolution.

Krause went on to show that the map $\mathsf{K} \to \mathsf{D}(R)$ restricts to give an equivalence:

(2.4.3.1) $$\mathsf{K}^c \to \mathsf{D}^f(R)$$

where $\mathsf{D}^f(R)$ is the full subcategory of $\mathsf{D}(R)$ whose objects are those complexes with finitely generated total cohomology. Thus the compact objects of K are exactly the semi-injective resolutions of R-complexes with finitely generated total cohomology.

Remark. The equivalence 2.4.3.1 is the central reason we work in K. In the derived category the image of a finitely generated module is compact if and only if it has finite projective dimension. However when constructing Bousfield localizations we need the image of every finitely generated module to be compact. As the above shows K is a category, which has a subcategory equivalent to $\mathsf{D}(R)$, satisfying this property.

Chapter 3

Support over complete

intersections

In this chapter we first discuss the cohomology operators for a complete intersection. These operators provide a theory of support and local cohomology via [11]. We discuss the parts of loc. cit. that we will need in the sequel. Finally we present details on Koszul objects. We assume throughout this chapter that $R = Q/(f_1, \ldots, f_c)$ where Q is a commutative noetherian ring and $f_1, \ldots, f_c$ is a Q-regular sequence. We define $\mathcal{S}$ to be the polynomial ring, $R[\chi_1, \ldots, \chi_c]$, graded by declaring that χ_i has cohomological degree 2 for $i = 1, \ldots, c$. As in Chapter 2 $\mathsf{K} := \mathsf{K}(\mathsf{Inj}\, R)$ is the homotopy category of injective R-modules.

3.1 Cohomology operators

Avramov has shown in [3, section 1] that for a complex of injective R-modules X there are elements $\zeta_1, \ldots, \zeta_c$ in $\mathrm{H}^2(\mathrm{Hom}_R(X, X)) = \mathrm{Hom}_{\mathsf{K}}(X, \Sigma^2 X)$.

For proofs of the following properties see [3] and [9].

3.1.1. Let X and Y be complexes of injective R modules. Recall that $\operatorname{Hom}^*_{\mathsf{K}}(X, Y)$ is a left $\operatorname{Hom}^*_{\mathsf{K}}(Y, Y)$-module and a right $\operatorname{Hom}^*_{\mathsf{K}}(X, X)$-module.

The ζ_i, from above, lie in the center of the algebra $\operatorname{Hom}^*_{\mathsf{K}}(X, X)$ and hence determine a map of R-algebras

$$\eta_X : \mathcal{S} \cong \operatorname{Sym}_R(R^c) \to \operatorname{Hom}^*_{\mathsf{K}}(X, X).$$

By restriction $\operatorname{Hom}^*_{\mathsf{K}}(X, Y)$ is an $\mathcal{S}$-module via the maps η_Y and η_X. These actions coincide up to a sign:

$$\eta_Y(s)\xi = (-1)^{|s|}\xi\eta_X(s)$$

for $s \in \mathcal{S}$ and $\xi \in \operatorname{Hom}^*_{\mathsf{K}}(X, Y)$.

Assume that X and Y are compact in K. So $X \cong iM$ and $Y \cong iN$ for finitely generated R-modules M and N respectively. The action of $\mathcal{S}$ on $\operatorname{Hom}^*_{\mathsf{K}}(X, Y) \cong \operatorname{Ext}^*_R(M, N)$ coincides with the action of the Eisenbud operators of [17]. If either $\operatorname{proj dim}_Q M$ or $\operatorname{inj dim}_Q N$ is finite then $\operatorname{Hom}^*_{\mathsf{K}}(X, Y) \cong \operatorname{Ext}^*_R(M, N)$ is a finitely generated $\mathcal{S}$-module. This was originally proven by Gulliksen in [19]; a different action of $\mathcal{S}$ was used but [9] shows that this action coincides with the one defined above up to sign.

3.2   Support and local cohomology

In this section we sketch the definition of support and local cohomology from [11]. We also discuss the relation of support to the support varieties of [4].

3.2.1. Set $\mathcal{S}^+ = \oplus_{i \geq 1} \mathcal{S}^i$. The set $\operatorname{Spec}^+ \mathcal{S}$ denotes the set of homogeneous prime ideals of $\mathcal{S}$. By $\operatorname{Proj} \mathcal{S}$ we denote the subset of $\operatorname{Spec}^+ \mathcal{S}$ given by those $\mathfrak{p}$ such that $\mathfrak{p} \not\supseteq \mathcal{S}^+$.

Recall that $\mathrm{Spec}^+ \mathcal{S}$ is equipped with a topology, the Zariski topology. The closed sets are $\mathcal{V}(I) := \{\, \mathfrak{p} \in \mathrm{Proj}\, \mathcal{S} \mid I \subseteq \mathfrak{p} \,\}$ for a homogeneous ideal I of $\mathcal{S}$. We give $\mathrm{Proj}\, \mathcal{S}$ the subset topology.

A subset $\mathcal{V}$ of $\mathrm{Spec}^+ \mathcal{S}$ is *specialization closed* if $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{p} \in \mathcal{V}$ then $\mathfrak{q} \in \mathcal{V}$.

3.2.2. For a graded $\mathcal{S}$-module $\mathcal{N}$, we set

$$\mathrm{Supp}_{\mathcal{S}} \mathcal{N} := \{\, \mathfrak{p} \in \mathrm{Spec}^+ \mathcal{S} \mid \mathcal{N}_{\mathfrak{p}} \neq 0 \,\}.$$

Recall that for objects X and Y of K, the set $\mathrm{Hom}^*_{\mathsf{K}}(X, Y)$ is an $\mathcal{S}$-module **??**. For $\mathcal{V} \subseteq \mathrm{Spec}^+ \mathcal{S}$ specialization closed we define the following full subcategory of K:

(3.2.2.1) $$\mathsf{K}_{\mathcal{V}} := \{\, X \in \mathsf{K} \mid \mathrm{Supp}_{\mathcal{S}} \mathrm{Hom}^*_{\mathsf{K}}(C, X) \subseteq \mathcal{V} \text{ for all } C \in \mathsf{K}^{\mathsf{c}} \,\}.$$

Lemma 3.2.3. *Let $\mathcal{V} \subseteq \mathrm{Spec}^+ \mathcal{S}$ be specialization closed and $C \in \mathsf{K}^c$.*

1. *The category $\mathsf{K}_{\mathcal{V}}$ is localizing.*

2. *The object C is in $\mathsf{K}_{\mathcal{V}}$ if and only if $\mathrm{Supp}_{\mathcal{S}} \mathrm{Hom}^*_{\mathsf{K}}(C, C) \subseteq \mathcal{V}$.*

These are proved in [11]; we give a quick proof for the sake of exposition.

Proof. (1) An exact triangle $X' \to X \to X'' \to$ induces a long exact sequence of $\mathcal{S}$-modules

$$\ldots \to \mathrm{Hom}^*_{\mathsf{K}}(C, X') \to \mathrm{Hom}^*_{\mathsf{K}}(C, X) \to \mathrm{Hom}^*_{\mathsf{K}}(C, X'') \to \mathrm{Hom}^*_{\mathsf{K}}(C, X')[1] \to \ldots$$

where $M[i]_j = M_{i+j}$ for an $\mathcal{S}$-module M. Assume that X', X'' are in $\mathsf{K}_{\mathcal{V}}$. Since $\mathrm{Supp}_{\mathcal{S}} M \subseteq \mathrm{Supp}_{\mathcal{S}} M' \cup \mathrm{Supp}_{\mathcal{S}} M''$ for an exact sequence of $\mathcal{S}$-modules $M' \to M \to M''$ we see that $X \in \mathsf{K}_{\mathcal{V}}$.

Given a family $\{X_i\}$ of objects in K there is an isomorphism $\bigoplus_i \operatorname{Hom}^*_\mathsf{K}(C, X_i) \to \operatorname{Hom}^*_\mathsf{K}(C, \bigoplus X_i)$ since C is compact. For a family of $\mathcal{S}$-modules $\{M_i\}$ we have $\operatorname{Supp}_\mathcal{S}(\bigoplus M_i) = \bigcup \operatorname{Supp}_\mathcal{S} M_i$. Thus if each X_i is in $\mathsf{K}_\mathcal{V}$ from the isomorphism we see that so will $\bigoplus X_i$ be.

(2) Assume $C \in \mathsf{K}_\mathcal{V}$. Then by the definition of $\mathsf{K}_\mathcal{V}$ we have that $\operatorname{Supp}_\mathcal{S} \operatorname{Hom}^*_\mathsf{K}(C, C) \subseteq \mathcal{V}$. For the other inclusion, assume that $\operatorname{Supp}_\mathcal{S} \operatorname{Hom}^*_\mathsf{K}(C, C) \subseteq \mathcal{V}$. The action of $\mathcal{S}$ on $\operatorname{Hom}^*_\mathsf{K}(D, C)$, for any object $D \in \mathsf{K}$, factors through $\operatorname{Hom}^*_\mathsf{K}(C, C)$. In particular $\operatorname{Supp}_\mathcal{S} \operatorname{Hom}^*_\mathsf{K}(D, C) \subseteq \operatorname{Supp}_\mathcal{S} \operatorname{Hom}^*_\mathsf{K}(C, C)$.

<div style="text-align: right">□</div>

More generally, the objects of $\mathsf{K}_\mathcal{V}$ are exactly those with "support" contained inside $\mathcal{V}$. This is made precise in the sequel.

3.2.4. By [11, 4.5], Bousfield localization exists at $\mathsf{K}_\mathcal{V}$. Thus we have a diagram

$$\mathsf{K}_\mathcal{V} \overset{\Gamma_\mathcal{V}}{\underset{\longrightarrow}{\longleftarrow}} \mathsf{K} \overset{L_\mathcal{V}}{\underset{\longrightarrow}{\longleftarrow}} \mathsf{K}/\mathsf{K}_\mathcal{V}$$

where the unlabeled arrows are the natural functors. In particular for every X in K there is a triangle

$$\Gamma_\mathcal{V} X \to X \to L_\mathcal{V} X \to .$$

The functors $\Gamma_\mathcal{V}$ are the *local cohomology* functors. Note that $\Gamma_\mathcal{V} X$ is in $\mathsf{K}_\mathcal{V}$ by definition.

3.2.5. For $\mathfrak{p} \in \operatorname{Spec}^+ \mathcal{S}$

$$\mathcal{Z}(\mathfrak{p}) := \{ \mathfrak{q} \in \operatorname{Spec} \mathcal{S} \mid \mathfrak{q} \nsubseteq \mathfrak{p} \}.$$

This is specialization closed. For $X \in \mathsf{K}$, following [11], the *support of X* is

$$\mathcal{V}_{\mathcal{S}}(X) := \{\, \mathfrak{p} \in \operatorname{Spec} \mathcal{S} \mid L_{\mathcal{Z}(\mathfrak{p})} \Gamma_{\mathcal{V}(\mathfrak{p})} X \neq 0 \}.$$

By [11, 5.7] an object X is in $\mathsf{K}_{\mathcal{V}}$ if and only if $\mathcal{V}_{\mathcal{S}}(X) \subseteq \mathcal{V}$; moreover X is nonzero if and only if $\mathcal{V}_{\mathcal{S}}(X)$ is nonempty.

3.2.6. The functors $\Gamma_{\mathcal{V}}, L_{\mathcal{V}}$ are important in the proof of the connectedness theorem. We use a special case of the Mayer-Vietoris triangle [11, 7.5]. Let $X \in \mathsf{K}$ and $\mathcal{V}_1, \mathcal{V}_2 \subseteq \operatorname{Spec} \mathcal{S}$ specialization closed with $\mathcal{V}_{\mathcal{S}}(X) \subseteq \mathcal{V}_1 \cup \mathcal{V}_2$. There is an exact triangle

$$\Gamma_{\mathcal{V}_1 \cap \mathcal{V}_2} X \to \Gamma_{\mathcal{V}_1} X \oplus \Gamma_{\mathcal{V}_2} X \to X \to .$$

3.2.7. Let M be a finitely generated R-module such that $\operatorname{proj dim}_Q M < \infty$. Let $X = \mathrm{i}M$, an injective resolution of M. Since $X \in \mathsf{K}$, the support $\mathcal{V}_{\mathcal{S}}(X)$ is defined, as above.

The set $\mathcal{V}_{\mathcal{S}} X$ decomposes in the following way [11, 11.3]:

$$\mathcal{V}_{\mathcal{S}}(X) = \bigcup_{\mathfrak{p} \in \operatorname{Spec} R} \operatorname{Supp}_{\mathcal{S} \otimes_R k(\mathfrak{p})} \operatorname{Ext}^*_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, k(\mathfrak{p}))$$

where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p} R_{\mathfrak{p}}$. This is actually a "fibering" over R in the following sense [11, 11.3]:

$$\operatorname{Supp}_{\mathcal{S} \otimes_R k(\mathfrak{p})} \operatorname{Ext}^*_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, k(\mathfrak{p})) = \mathcal{V}_{\mathcal{S}}(X) \cap \pi^{-1}(\mathfrak{p})$$

, where $\pi : \operatorname{Proj} \mathcal{S} \to \operatorname{Spec} R$ is the map induced by the inclusion $R \to R[\chi] = \mathcal{S}$.

Assume $(R, \mathfrak{m}, k)$ is local with maximal ideal $\mathfrak{m}$ and residue field k. We see that

$$\mathcal{V}_{\mathcal{S}}(X) \cap \pi^{-1}(\mathfrak{m}) = \operatorname{supp}_{k[\chi]} \operatorname{Ext}^*_R(M, k).$$

We define $\mathcal{V}_{\mathcal{R}}(M) := \operatorname{supp}_{k[\boldsymbol{\chi}]} \operatorname{Ext}_R^*(M, k)$, where $\mathcal{R} := \mathcal{S} \otimes_R k \cong k[\chi_1, \ldots, \chi_c]$. The set $\mathcal{V}_{\mathcal{R}}(M)$ recovers the *support variety* of M introduced by Avramov and Buchweitz; see 5.5.

3.3 Koszul objects

The following simple and useful construction has been used in e.g. [8], [11], [13].

3.3.1. Let $s \in \mathcal{S}^n$ and $X \in \mathsf{K}$. Recall that there is a map of graded R-algebras $\eta_X : \mathcal{S} \to \operatorname{Hom}_{\mathsf{K}}^*(X, X)$. Since s the map is homogenous $\eta_X(s)$ is an element of $\operatorname{Hom}_{\mathsf{K}}(X, \Sigma^n X)$. The *Koszul object* of s on X, denoted $X/\!/s$, is the mapping cone of $\eta_X(s)$; by definition there is an exact triangle

$$(3.3.1.1) \qquad\qquad X \xrightarrow{\eta_X(s)} \Sigma^n X \to X/\!/s \to .$$

For a sequence of homogeneous elements $\mathbf{s} = s_1, \ldots, s_r$ in $\mathcal{S}$ the Koszul object of $\mathbf{s}$ on X, denoted $X/\!/\mathbf{s}$, is defined inductively as the Koszul object of s_r on $X/\!/s_1, \ldots, s_{r-1}$.

If I is a homogenous ideal of $\mathcal{S}$, we define $X/\!/I$ to be $X/\!/s_1, \ldots, s_r$ for some generating set $s_1, \ldots, s_r$ of I. This may depend on the generators chosen, but by [10, 2.6.1] all such objects generate the same localizing subcategory.

Note that if X is compact then the triangle (3.3.1.1) shows that so is $X/\!/I$.

Let $\mathbf{s} = s_1, \ldots, s_r$ be a sequence of homogeneous elements of $\mathcal{S}$ and X and Y objects of K. We will use the following properties of Koszul objects [11, 5.11]:

3.3.1.1. For all X, Y in K there exists an integer $n \geq 0$ such that

$$(\mathbf{s})^n \operatorname{Hom}_{\mathsf{K}}^*(X/\!/(\mathbf{s}), Y) = 0 = (\mathbf{s})^n \operatorname{Hom}_{\mathsf{K}}^*(Y, X/\!/(\mathbf{s})).$$

3.3.1.2. If $\mathrm{Hom}^*_{\mathsf{K}}(Y, X/\!/(\mathbf{s})) = 0$ and the $\mathcal{S}$-module $\mathrm{Hom}^*_{\mathsf{K}}(Y, X)$ is $(\mathbf{s})$-torsion then $\mathrm{Hom}^*_{\mathsf{K}}(Y, X) = 0$. Recall for an ideal $I \subseteq \mathcal{S}$, an $\mathcal{S}$-module $\mathcal{M}$ is *I-torsion* if for each $m \in \mathcal{M}$ there exists an integer n such that $I^n m = 0$.

Koszul objects give a description of the subcategory $\mathsf{K}_{\mathcal{V}}$, defined in 3.2.2, which we need in the sequel.

3.3.2. Let I be an ideal of $\mathcal{S}$ and $\mathcal{V}(I)$ the closed subset of $\mathrm{Spec}^+ \mathcal{S}$ determined by I. By [10, 2.7] there is an equality

$$\mathsf{K}_{\mathcal{V}(I)} = \mathsf{loc}_{\mathsf{K}}(\, C/\!/I \,|\, C \in \mathsf{K}^c\,).$$

Chapter 4

Finite injective dimension via vanishing of self-extensions

In this chapter we assume that $R = Q/(f_1, \ldots, f_c)$, where Q is a commutative noetherian ring regular of finite Krull dimension and $f_1, \ldots, f_c$ is a Q-regular sequence. While we do not assume that Q is local, note that it does have finite global dimension; i.e. there is an integer N such that every Q-module has projective dimension at most N. As before $\mathcal{S} = R[\chi_1, \ldots, \chi_c]$ is the ring of operators and $\mathsf{K} = \mathsf{K}(\mathsf{Inj}\,R)$ is the homotopy category of injective R-modules. We set $\mathcal{S}^+ = \oplus_{i \geq 1} \mathcal{S}^i$.

The goal of this chapter is to proof Theorem A from the introduction. In the first section we prove a preliminary result of independent interest.

4.1 Systems of parameters and Koszul objects

We say that a complex M is *perfect* when $M \in \mathsf{thick}_{\mathsf{D}(R)}(R)$; see 2.3.3. This implies in particular that M has finitely generated cohomology.

The following is well-known. We reprove it here because we could not find the

formulation that we need. It was originally proved in the case M is a module and $n = 1$ by Auslander, Ding and Solberg [2]; Avramov and Buchweitz proved it for any module of finite complete-intersection dimension and any $n \geq 1$ in [4].

Proposition 4.1.1. *Let M be an R-complex with finitely generated total cohomology. If $\operatorname{Ext}_R^{2n}(M, M) = 0$ for some $n \geq 1$, then M is perfect.*

Recall that for every complex M there is a homogeneous map of R-algebras $\eta_M : \mathcal{S} \to \operatorname{Ext}_R^*(M, M)$.

Proof. We will prove the statement in case R is a local ring with residue field k. The reduction to this case is standard commutative algebra.

We will use the fact that over a local ring, M is perfect if $\operatorname{Ext}_R^n(M, k)$ for $n \gg 0$ [5].

Now for all $m \geq n$ and $1 \leq i \leq c$ we see that

$$\eta_M(\chi_i^m) \operatorname{Ext}_R^*(M, M) = \eta_M(\chi_i^{m-n}) \eta_M(\chi_i^n) \operatorname{Ext}_R^*(M, M) = 0.$$

This first equality follows from the fact that η_M is a ring map; the second since $\eta_M(\chi_i^n) \in \operatorname{Ext}_R^{2n}(M, M) = 0$. This shows that the finitely generated $\mathcal{S}$-module $\operatorname{Ext}_R^*(M, M)$ is $(\boldsymbol{\chi}) = (\chi_1, \dots, \chi_c)$-torsion. Since the action of $\mathcal{S}$ on $\operatorname{Ext}_R^*(M, k)$ factors through $\operatorname{Ext}_R^*(M, M)$ we see that $\operatorname{Ext}_R^*(M, k)$ is also $(\boldsymbol{\chi})$-torsion. In particular since each member of a finite set of generators is killed by a power of $(\boldsymbol{\chi})$, we see that $\operatorname{Ext}_R^n(M, k) = 0$ for $n \gg 0$. Since R is local this shows that M is perfect. $\square$

Let $\nu_1, \dots, \nu_l$ be homogeneous elements of positive degree in the ring of operators $\mathcal{S}$. Consider the subring $R[\nu_1, \dots, \nu_l]$ of $\mathcal{S}$ generated by the elements $\nu_1, \dots, \nu_l$. The $\mathcal{S}$-module $\operatorname{Ext}_R^*(M, M)$ is an $R[\nu_1, \dots, \nu_l]$-module via the inclusion map of rings

$R[\nu_1, \ldots, \nu_l] \subseteq \mathcal{S}$. The proof of the following statement uses an idea from the argument of [3, 2.1]. For an R-complex M, recall that $M /\!\!/ (\nu_1, \ldots, \nu_l)$ denotes the Koszul object of $\nu_1, \ldots, \nu_l$ on M; see 3.3.1.

Theorem 4.1.2. *Let $\boldsymbol{\nu} = \nu_1, \ldots, \nu_l$ be a sequence of homogeneous elements of positive degree in the ring of operators $\mathcal{S}$ and let M be an R-complex with finitely generated total cohomology.*

The $R[\boldsymbol{\nu}]$-module $\mathrm{Ext}_R^*(M, M)$ is finitely generated if and only if $M /\!\!/ (\boldsymbol{\nu})$ is perfect.

Proof. We assume first that $\mathrm{Ext}_R^*(M, M)$ is a finitely generated $R[\boldsymbol{\nu}]$ module. Since $\mathrm{Ext}_R^*(M, M /\!\!/ \boldsymbol{\nu})$ is a finitely generated $\mathrm{Ext}_R^*(M, M)$ module [9, 5.1] it is also a finitely generated $R[\boldsymbol{\nu}]$ module. But by 3.3.1.1 there is an $n \geq 1$ such that

$$(\boldsymbol{\nu})^n \, \mathrm{Ext}_R^*(M, M /\!\!/ \boldsymbol{\nu}) = 0.$$

We claim that $\mathrm{Ext}_R^*(M, M /\!\!/ \boldsymbol{\nu})$ is a finitely generated R-module via the embedding $R \subseteq R[\boldsymbol{\nu}]$.

To see this assume $\mathcal{M}$ is a finitely generated $R[\boldsymbol{\nu}]$-module such that $(\boldsymbol{\nu})^n \mathcal{M} = 0$ for some $n \geq 1$. We induce on n. When $n = 1$ then $\mathcal{M}$ is a finitely generated $R[\boldsymbol{\nu}]/(\boldsymbol{\nu}) \cong R$ module. Since the composition

$$R \subseteq R[\boldsymbol{\nu}] \to R[\boldsymbol{\nu}]/(\boldsymbol{\nu}) \cong R$$

is the identity, we see that $\mathcal{M}$ is a finitely generated R-module under the action via the inclusion. Now assume the statement holds for all finitely generated $R[\boldsymbol{\nu}]$-modules and all integers less than n and let $\mathcal{M}$ be a finitely generated $R[\boldsymbol{\nu}]$-module with $(\boldsymbol{\nu})^n \mathcal{M} = 0$. By induction the quotient $\mathcal{M}/(\boldsymbol{\nu})^{(n-1)} \mathcal{M}$ is finitely generated since it is annihilated by $(\boldsymbol{\nu})^{(n-1)}$. Also the submodule $(\boldsymbol{\nu})^{n-1} \mathcal{M}$ is finitely generated over

R since it is annihilated by $(\boldsymbol{\nu})$. Thus from the exact sequence

$$0 \to (\boldsymbol{\nu})^{n-1}\mathcal{M} \to \mathcal{M} \to \mathcal{M}/(\boldsymbol{\nu})^{(n-1)}\mathcal{M} \to 0$$

we see that $\mathcal{M}$ is finitely generated over R.

Since $\operatorname{Ext}_R^*(M, M/\!\!/\boldsymbol{\nu})$ is finitely generated over R we must have $\operatorname{Ext}_R^n(M, M/\!\!/\boldsymbol{\nu}) = 0$ for $n \gg 0$. We claim the full subcategory

$$\mathcal{A} = \{\, X \in \mathsf{D}^f(R) \mid \operatorname{Ext}_R^n(X, M/\!\!/\boldsymbol{\nu}) = 0 \text{ for } n \gg 0\}$$

is thick. Recall that $\mathsf{D}^f(R)$ is the full subcategory of $\mathsf{D}(R)$ with objects complexes with finitely generated total cohomology. An exact triangle

$$X' \to X \to X'' \to$$

induces a long exact sequence

$$\ldots \to \operatorname{Ext}_R^n(X'', M) \to \operatorname{Ext}_R^n(X, M) \to \operatorname{Ext}_R^n(X', M) \to \operatorname{Ext}_R^{n+1}(X'', M) \to \ldots$$

From this one sees that $\mathcal{A}$ is closed under exact triangles. It also clearly closed under direct summands.

Since M is in the thick subcategory $\mathcal{A}$ so is every object in $\mathsf{thick}_{\mathsf{D}(R)}(M)$. In particular $M/\!\!/\boldsymbol{\nu}$ is in this subcategory and hence

$$\operatorname{Ext}_R^n(M/\!\!/\boldsymbol{\nu}, M/\!\!/\boldsymbol{\nu}) = 0 \text{ for } n \gg 0.$$

This shows that $M/\!\!/\boldsymbol{\nu})$ is perfect by 4.1.1.

To prove the converse we induce on l, the length of the sequence $\boldsymbol{\nu}$. The base case $l = 0$ implies that M is perfect; it's easy to see that this implies that $\operatorname{Ext}_R^n(M, M) = 0$ for $n \gg 0$ and hence $\operatorname{Ext}_R^*(M, M)$ is finitely generated over R. We now assume that $M/\!\!/\nu_1, \dots, \nu_l$ is perfect and $l \geq 1$. Since the Koszul object of the sequence $\nu_2, \dots, \nu_l$ on $M/\!\!/\nu_1$ is perfect, we see that $\operatorname{Ext}_R^*(M/\!\!/\nu_1, M/\!\!/\nu_1)$ is a finitely generated $R[\nu_2, \dots, \nu_l]$ module by induction. Since $\operatorname{Ext}_R^*(M/\!\!/\nu_1, M)$ is a finitely generated $\operatorname{Ext}_R^*(M/\!\!/\nu_1, M/\!\!/\nu_1)$-module, it is also finitely generated over $R[\nu_2, \dots, \nu_l]$. From the exact triangle $M \xrightarrow{\nu_1} \Sigma^{|\nu_1|} M \to M/\!\!/\nu_1 \to$ in $\mathsf{D}^f(R)$ we arrive at the diagram of graded R-modules below, in which the image of every arrow is the kernel of the proceeding arrow:

$$\operatorname{Ext}_R^*(M, M) \xrightarrow{\hspace{3em} \nu_1 \hspace{3em}} \operatorname{Ext}_R^{*+|\nu_1|}(M, M) \, .$$

$$\operatorname{Ext}_R^*(M/\!\!/\nu_1, M) \quad \phi$$

The image of ϕ is a submodule of the noetherian $R[\nu_2, \dots, \nu_l]$-module $\operatorname{Ext}_R^*(M/\!\!/\nu_1, M)$. Pick a generating set

$$\phi(e_1), \dots, \phi(e_m)$$

for this image. Let H be the $R[\nu_2, \dots, \nu_l]$-submodule of $\operatorname{Ext}_R^*(M, M)$ generated by $e_1, \dots, e_m$. By exactness $\operatorname{Ext}_R^*(M, M) = H + \nu_1 \operatorname{Ext}_R^*(M, M)$. Iterating, we have that

$$\operatorname{Ext}_R^*(M, M) = \Sigma_{i=0}^j \nu_1^i H + \nu_1^j \operatorname{Ext}_R^*(M, M)$$

for all $j \geq 1$. But $\cap_{i=1}^\infty \nu_1^i \operatorname{Ext}_R^*(M, M) = 0$ since $|\nu_1| > 0$. Thus $\operatorname{Ext}_R^*(M, M) = \Sigma_{i=0}^\infty \nu_1^i H$, and hence $\operatorname{Ext}_R^*(M, M)$ is a finitely generated $R[\nu_1, \dots, \nu_l]$-module. $\qquad \square$

Corollary 4.1.3. *Let M be a complex with finitely generated cohomology. Then $M/\!\!/\chi_1, \dots, \chi_c$ is perfect.*

Proof. By 3.1.1 we know that $\operatorname{Ext}_R^*(M, M)$ is a finitely generated $R[\chi_1, \ldots, \chi_c]$-module. Thus by Theorem 4.1.2 we see that $M/\!\!/(\chi_1, \ldots, \chi_c)$ is perfect. $\qquad\square$

4.2 Torsion in the module of self-extensions

In this section we prove the following:

Theorem 4.2.1. *Let M be an R-complex with nonzero cohomology such that $\mathrm{H}^n(M) = 0$ for all $n \gg 0$. If the $\mathcal{S}$-module $\operatorname{Ext}_R^*(M, M)$ is $\mathcal{S}^+$-torsion then M has finite injective dimension.*

The definition of Ext and injective dimension for complexes is recalled in 2.4.2. The action of $\mathcal{S}$ on $\operatorname{Ext}_R^*(M, M)$ is sketched in 3.1.1. Recall that a module over a commutative ring is *I-torsion*, for I an ideal, if every element of the module is annihilated by a power of I; see 3.3.1.2.

We need several preliminary results in the proof of the theorem. Recall that K^c is the subcategory of compact objects of K and $C/\!\!/(\chi_1, \ldots, \chi_c)$ is a Koszul object of $(\chi_1, \ldots, \chi_c)$ on C; see 3.3.1.

Proposition 4.2.2. *Let $\mathrm{i}R$ an injective resolution of R. There is an inclusion of subcategories:*

$$\operatorname{loc}_\mathsf{K}(C/\!\!/(\chi_1, \ldots, \chi_c) \mid C \in \mathsf{K}^\mathsf{c}) \subseteq \operatorname{loc}_\mathsf{K}(\mathrm{i}R).$$

Proof. Set $\boldsymbol{\chi} = \chi_1, \ldots, \chi_c$. It suffices to show that $C/\!\!/\boldsymbol{\chi}$ is in $\operatorname{loc}_\mathsf{K}(\mathrm{i}R)$ for all $C \in \mathsf{K}^c$.

Fix a compact object C of K. By 2.4.3.1 there exists a complex M with finitely generated total cohomology such that $C \cong \mathrm{i}M$, where $\mathrm{i}M$ is an injective resolution of M. By 4.1.3 $\operatorname{proj dim}_R M/\!\!/(\chi_1, \ldots, \chi_c)$ is perfect, i.e.

$$M/\!\!/(\chi_1, \ldots, \chi_c) \in \operatorname{thick}_{\mathsf{D}(R)}(R).$$

Applying the injective resolution functor $i(-)$ to the the triangle $M \xrightarrow{\chi_1} \Sigma^2 M \rightarrow M /\!/ \chi_1 \rightarrow$ gives a triangle

$$iM \xrightarrow{\chi_1} \Sigma^2 iM \rightarrow i(M /\!/ \chi_1) \rightarrow .$$

This shows that $i(M /\!/ \chi_1)$ is a Koszul object of χ_1 on $iM \cong C$. Repeating we see that $i(M /\!/ (\chi_1, \dots, \chi_c))$ is a Koszul object of $(\chi_1, \dots, \chi_c)$ on C. Thus $C /\!/ (\chi_1, \dots, \chi_c) \in \mathsf{thick}_\mathsf{K}(i(M /\!/ (\chi_1, \dots, \chi_c)))$ by [10, 2.6.1]. Taking injective resolutions is a triangulated functor and triangulated functors preserve thick subcategories. Thus we have that

$$i(M /\!/ (\chi_1, \dots, \chi_c)) \in \mathsf{thick}_\mathsf{K}(iR).$$

Taken together this shows $C /\!/ (\chi_1, \dots, \chi_c) \in \mathsf{thick}_\mathsf{K}(iR)$. Finally the inclusion $\mathsf{thick}_\mathsf{K}(iR) \subseteq \mathsf{loc}_\mathsf{K}(iR)$ is clear.

$\square$

The proposition below holds for any commutative Gorenstein ring of finite Krull dimension.

Proposition 4.2.3. *Let M be an R-complex with $\mathrm{H}^n(M) = 0$ for all $n \gg 0$. Let iR and iM be semi-injective resolutions of R and M respectively. If iM is in $\mathsf{loc}_\mathsf{K}(iR)$, then M has finite injective dimension.*

Proof. Under the hypothesis that R has finite injective dimension and that M has no cohomology in high degrees, there exists a Gorenstein injective resolution of M [1, 3.2]. This is a map $v : iM \rightarrow T$, where T is an acyclic complex of injective modules, such that $v^n : (iM)^n \rightarrow T^n$ is bijective for all $n \geq k$, for some $k \in \mathbb{Z}$. We have isomorphisms

$$\mathrm{Hom}_\mathsf{K}^*(iR, T) \cong \mathrm{Hom}_{\mathsf{K}(R)}^*(R, T) \cong \mathrm{H}^*(T) = 0.$$

The first is [21, 2.1]; the second is clear and the third is the fact that T is acyclic.

The class of complexes X such that $\mathrm{Hom}_{\mathsf{K}}^*(X, T) = 0$ is localizing. Since iR is in this class so will be the entire subcategory $\mathsf{loc}_{\mathsf{K}}(iR)$. In particular since $iM \in \mathsf{loc}_{\mathsf{K}}(iR)$ we see that

$$\mathrm{Hom}_{\mathsf{K}}^*(iM, T) = 0.$$

This shows that the map v above is nulhomotopic. Now we have a nullhomotopic map $v : iM \to T$ which is bijective in all high degress and has target an acyclic complex of injective modules. We'll show that this forces iM to have an injective cokernel in a high degree.

Since v is nullhomotopic there exists a map $s : iM \to T$ such that $\partial_T s + s \partial_{iM} = v$. Denote the component from $(iM)^n \to T^{n-1}$ by s^n. Since v^n is bijective for all $n \geq k$ we have that $(v^n)^{-1} \partial_T s^n + (v^n)^{-1} s^{n+1} \partial_{iM} = 1_{iM^n}$. Where $(v^n)^{-1}$ is the inverse of the bijective map v^n. One checks that v^{-1} commutes with the differentials in the degrees for which it is defined; this gives

$$\partial_{iM}(v^n)^{-1} s^n + (v^n)^{-1} s^{n+1} \partial_{iM} = 1_{iM^n}.$$

A simple diagram chase now shows that $\mathrm{Im}(\partial_{iM}^k)$ splits as a submodule of $(iM)^{k+1}$ and hence is injective.

The fact that v is a bijection for $n \geq k$ implies that $\mathrm{H}^n(iM) = 0$ for $n \geq k$. Thus iM has an injective cokernel in a degree higher than its last nonzero cohomology; by [5, 2.4.I] this implies that M has finite injective dimension. □

We now prove the main theorem of the chapter. Part of the proof reproves a special case of [11, 6.4].

Proof of Theorem 4.2.1. Set $X = iM$ and $\boldsymbol{\chi} = \chi_1, \ldots, \chi_c$. It suffices to show that

$X \in \mathsf{loc}_\mathsf{K}(C /\!\!/ \chi \mid C \in \mathsf{K}^c)$; for then by 4.2.2 and 4.2.3 this will show that M has finite injective dimension.

Set $\mathcal{C} = \mathsf{loc}_\mathsf{K}(C /\!\!/ \chi \mid C \in \mathsf{K}^c)$. Note that $\mathcal{C}$ is compactly generated, so Bousfield localization at $\mathcal{C}$ exists; see 2.3.2. It yields a triangle

$$(4.2.3.1) \qquad\qquad X' \to X \to X'' \to$$

with $X' \in \mathcal{C}$ and $X'' \in \mathcal{C}^\perp$.

Fix a compact object D. The action of $\mathcal{S}$ on $\mathrm{Hom}^*_\mathsf{K}(D, X)$ is via the map $\eta_X : \mathcal{S} \to \mathrm{Hom}^*_\mathsf{K}(X, X)$. By hypothesis $\mathrm{Hom}^*_\mathsf{K}(X, X)$ is $\mathcal{S}^+$-torsion, i.e. $\eta_X(s)^n = 0$ for all $s \in \mathcal{S}$ and some $n \geq 0$. Thus $\mathrm{Hom}^*_\mathsf{K}(D, X)$ is also $\mathcal{S}^+$-torsion.

Now consider the full subcategory

$$\mathcal{B} = \{\, Z \in \mathsf{K} \mid \mathrm{Hom}^*_\mathsf{K}(D, Z) \text{ is } \mathcal{S}^+\text{-torsion}\}.$$

We claim that it is localizing.

To see this, first note that it is clearly closed under Σ; given a triangle $Y \to Z \to W \to \Sigma Y$ in K there is an exact sequence of $\mathcal{S}$-modules:

$$\mathrm{Hom}^*_\mathsf{K}(D, Y) \to \mathrm{Hom}^*_\mathsf{K}(D, Z) \to \mathrm{Hom}^*_\mathsf{K}(D, W).$$

From this we see that if $\mathrm{Hom}^*_\mathsf{K}(D, Y)$ and $\mathrm{Hom}^*_\mathsf{K}(D, W)$ are $\mathcal{S}^+$-torsion then so is $\mathrm{Hom}^*_\mathsf{K}(D, Z)$. This shows that if Y and W are in $\mathcal{B}$ then so is Z; thus $\mathcal{B}$ is triangulated. Using that D is compact one sees that $\mathcal{B}$ is localizing. By 3.3.1.2, for every object C the module $\mathrm{Hom}^*_\mathsf{K}(D, C /\!\!/ \chi)$ is S^+-torsion. Thus $\mathcal{C} = \mathsf{loc}_\mathsf{K}(C /\!\!/ \chi \mid C \in \mathsf{K}^c) \subseteq \mathcal{B}$ since $\mathcal{B}$ is localizing and every object $C /\!\!/ \chi$ is in $\mathcal{B}$.

Now we have that $\mathrm{Hom}^*_\mathsf{K}(D, X')$ is $\mathcal{S}^+$-torsion since $X' \in \mathcal{C} \subseteq \mathcal{B}$. Also from

(4.2.3.1) we have that X'' is in $\mathcal{B}$ since the other two objects in the triangle are; to rephrase we have that $\mathrm{Hom}^*_{\mathsf{K}}(D, X'')$ is also $\mathcal{S}^+$-torsion. By (3.3.1.2) this implies that $\mathrm{Hom}^*_{\mathsf{K}}(D, X'') = 0$. Since D was an arbitrary compact object and K is compactly generated this shows that $X'' = 0$, see 2.1.6. From the triangle (4.2.3.1) we see that $X' \cong X$ is an element of $\mathcal{C} = \mathsf{loc}_{\mathsf{K}}(C /\!\!/ \boldsymbol{\chi} \mid C \in \mathsf{K}^c)$.

$\square$

The following result, Theorem A of the introduction, is a corollary of Theorem 4.2.1.

Corollary 4.2.4. *Let M be an arbitrary R-module such that $\mathrm{Ext}^{2n}_R(M, M) = 0$ for some $n \geq 1$. Then M has finite projective dimension.*

Proof. It suffices to prove that $\mathrm{Ext}^*_R(M, M)$ is $\mathcal{S}^+$-torsion. To see this let $m \geq n$; we have that

$$\eta_M(\chi_i^m) \, \mathrm{Ext}^*_R(M, M) = \eta_M(\chi_i^{m-n}) \eta_M(\chi_i^n) \, \mathrm{Ext}^*_R(M, M) = 0 \text{ for all } i = 1, \ldots, c.$$

Thus by Theorem 4.2.1 the module M has finite injective dimension. Since R is a complete intersection it is Gorenstein; since it has finite Krull dimension it has finite injective dimension over itself. Using this one can show that when a module has finite injective dimension it must also have finite projective dimension. Thus M has finite projective dimension. $\square$

Remark.   1. For Theorem 4.2.1, it is not vital that R is commutative. The proof goes through for any left noetherian ring that has a ring of operators $\mathcal{S}$ over which the Ext-algebra of any finitely generated module is noetherian. This point of view is taken in [15].

2. The boundedness assumption on the cohomology of M in Theorem 4.2.1 is necessary: take for instance the complex $M = \oplus_{n \in \mathbb{Z}} E[n]$ which has an injective module E in each degree and zero differential. This does not have finite injective dimension by the definition of [5] since it has cohomology in arbitrarily high degrees.

Chapter 5

Connectedness of support varieties

In this chapter we prove Theorem B from the introduction and show that it specializes to give a different proof of a result of Bergh [12].

We assume that $R = Q/(f_1, \ldots, f_c)$ where Q is a commutative noetherian regular ring and $f_1, \ldots, f_c$ is an Q-regular sequence. As before $\mathcal{S} = R[\boldsymbol{\chi}]$ is the graded ring of operators.

5.1 Connectedness of BIK support

Definition. Let $\mathcal{V}$ be a subset of $\operatorname{Spec}^+ \mathcal{S}$. We say $\mathcal{V}$ is *projectively connected* if $\mathcal{V} \cap \operatorname{Proj} \mathcal{S}$ is a connected topological space of $\operatorname{Proj} \mathcal{S}$; in other words if there exist closed subsets $\mathcal{V}_1, \mathcal{V}_2$ of $\operatorname{Spec} \mathcal{S}$ such that

$$\mathcal{V} = (\mathcal{V}_1 \cup \mathcal{V}_2) \cap \mathcal{V} \quad \text{and} \quad \mathcal{V}_1 \cap \mathcal{V}_2 \subseteq V(\mathcal{S}_+)$$

then $\mathcal{V} \cap \mathcal{V}_i \subseteq V(S_+)$ for $i = 1$ or 2.

Remark. This definition applies to any positively graded homogeneous ring.

The main result of this section is the following:

Theorem 5.1. *Let iR be an injective resolution of R, $\mathsf{loc}_\mathsf{K}(iR)$ the localizing subcategory it generates, and $\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR)$ the Verdier quotient. If $X \in \mathsf{K}$ is such that the image of X in $\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR)$ is indecomposable then $\mathcal{V}_\mathcal{S}(X)$ is projectively connected.*

To prove Theorem 5.1 we need the following.

Theorem 5.2. *Let $\mathcal{V}(\boldsymbol{\chi}) = V(\chi_1, \ldots, \chi_c)$ be the closed subset of $\mathrm{Spec}^+\, \mathcal{S}$ defined by the homogeneous ideal $(\chi_1, \ldots, \chi_c)$ and let iR be an injective resolution of R. Then $\mathsf{K}_{\mathcal{V}(\boldsymbol{\chi})} = \mathsf{loc}_\mathsf{K}(iR)$.*

Proof. Let M be an R-complex with finitely generated total cohomology. Since R is Gorenstein M we have $\mathrm{Ext}_R^n(M, R) = 0$ for all $n \gg 0$ [18]. Since $|\chi_i| = 2 > 0$ this forces $\mathrm{Ext}_R^*(M, R)$ to be $(\boldsymbol{\chi})$-torsion. Thus by [11, 2.5]

$$\mathrm{Supp}_\mathcal{S} \mathrm{Hom}_\mathsf{K}^*(iM, iR) = \mathrm{Supp}_\mathcal{S} \mathrm{Ext}_R^*(M, R) \subseteq \mathcal{V}(\boldsymbol{\chi}).$$

Thus we have $iR \in \mathsf{K}_{\mathcal{V}(\boldsymbol{\chi})}$ by 3.2.2, noting that all compact objects of K are of the form iM for M a complex with finitely generated total cohomology; see 2.4.3.1. Since $\mathsf{K}_{\mathcal{V}(\boldsymbol{\chi})}$ is a localizing subcategory we see that $\mathsf{loc}_\mathsf{K}(iR) \subseteq \mathsf{K}_{\mathcal{V}(\boldsymbol{\chi})}$.

To show the other inclusion, by 3.3.2 we need only show that $C/\!\!/(\boldsymbol{\chi})$ is in $\mathsf{loc}_\mathsf{K}(iR)$ for every compact object C of K. But this is Theorem 4.2.2. $\qquad\square$

Proof of Theorem 5.1. Assume $\mathcal{V}_\mathcal{S}(X)$ is disconnected; we show that X is decomposable in $\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR)$. By assumption there exist nonempty closed sets $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathrm{Spec}^+\, \mathcal{S}$ with $\mathcal{V}_i \cap \mathcal{V}_\mathcal{S}(X) \not\subseteq V(\boldsymbol{\chi})$ for $i = 1, 2$ such that

$$\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_\mathcal{S}(X) \text{ and } \mathcal{V}_1 \cap \mathcal{V}_2 \subseteq V(\boldsymbol{\chi}).$$

The Mayer-Vietoris triangle, see (3.2.6), associated to $\mathcal{V}_1$ and $\mathcal{V}_2$ yields a triangle:

$$(*) \qquad \Gamma_{\mathcal{V}_1 \cap \mathcal{V}_2} X \to \Gamma_{\mathcal{V}_1} X \oplus \Gamma_{\mathcal{V}_2} X \to X \to .$$

By definition $\Gamma_{\mathcal{V}_1 \cap \mathcal{V}_2} X \in \mathsf{K}_{\mathcal{V}_1 \cap \mathcal{V}_2}$ and $\mathsf{K}_{\mathcal{V}_1 \cap \mathcal{V}_2} \subseteq \mathsf{K}_{\mathcal{V}(\chi)}$; see (3.2.6). Since $\mathsf{K}_{\mathcal{V}(\chi)} = \mathsf{loc}_{\mathsf{K}}(iR)$ by 5.2, $\Gamma_{\mathcal{V}_1 \cap \mathcal{V}_2} X$ is in the kernel of the quotient functor $\rho : \mathsf{K} \to \mathsf{K}/\mathsf{loc}_{\mathsf{K}}(iR)$. Thus one has an isomorphism

$$\Gamma_{\mathcal{V}_1} X \oplus \Gamma_{\mathcal{V}_2} X \xrightarrow{\cong} X$$

in $\mathsf{K}/\mathsf{loc}_{\mathsf{K}}(iR)$. Because $\mathcal{V}_i \cap \mathcal{V}_{\mathcal{S}}(X) \nsubseteq V(\chi)$, the object $\Gamma_{\mathcal{V}_i} X$ is nonzero in $\mathsf{K}/\mathsf{loc}_{\mathsf{K}}(iR)$ [11, 5.7.2]. This shows X is decomposable in $\mathsf{K}/\mathsf{loc}_{\mathsf{K}}(iR)$. $\square$

5.2 MCM modules

In this section we assume that Q, and hence R, is local. Let $\mathfrak{m}$ denote the unique maximal ideal and k the residue field.

Definition. The *stable category of maximal Cohen-Macaulay modules* has objects the maximal Cohen-Macaulay R-modules and the set of morphisms between two objects M and N is given by

$$\mathrm{Hom}_R(M, N)/P\,\mathrm{Hom}_R(M, N)$$

where $P\,\mathrm{Hom}_R(M, N)$ is the submodule of all R-linear maps that factor through a projective module. We denote the stable category by $\underline{\mathrm{MCM}}(R)$. See e.g. [14] for further details on the stable category.

There is a functor, $\mathrm{MCM}(R) \to \underline{\mathrm{MCM}}(R)$, which is the identity on objects which satisfies the universal property that any additive functor F from $\mathrm{MCM}(R)$ to an

additive category $\mathcal{A}$, such that $F(P) = 0$ for every projective module, factors through $\mathrm{MCM}(R) \to \underline{\mathrm{MCM}}(R)$. Moreover, a map $f : M \to N$ in $\underline{\mathrm{MCM}}(R)$ is an isomorphism if and only if there exist free modules P and Q and an isomorphism $g : M \oplus P \to N \oplus Q$ in $\mathrm{Mod}(R)$, where the component of g from M to N is a lifting of f; see [20].

The following result is in [21]; we restate and reprove it to fit our purposes more closely.

Proposition 5.3. *Let M be a maximal Cohen-Macaulay module that is indecomposable in $\underline{\mathrm{MCM}}(R)$ and let $\mathrm{i}M$ be an injective resolution of M. The image of $\mathrm{i}M$ under the quotient map $\mathsf{K} \to \mathsf{K}/\mathsf{loc}_{\mathsf{K}}(\mathrm{i}R)$ is indecomposable.*

Proof. Consider the natural map $F : \mathrm{MCM}(R) \to \mathsf{D}^f(R)/\mathsf{thick}_{\mathsf{D}(R)}(R)$ which sends a module to its image in the quotient $\mathsf{D}^f(R)/\mathsf{thick}_{\mathsf{D}(R)}(R)$. It sends projective modules to zero, hence factors through a map

$$(5.2.0.1) \qquad \underline{\mathrm{MCM}}(R) \to D^f(R)/\mathsf{thick}_{\mathsf{D}(R)}(R).$$

When R is Gorenstein this is an equivalence [14, 4.4.1] .

Consider also the equivalence $\mathsf{D}^f(R) \to \mathsf{K}^c$ from 2.4.3.1. Under the equivalence R gets mapped to an injective resolution $\mathrm{i}R$. Thus the equivalence restricts to an equivalence $\mathsf{thick}_{\mathsf{D}(R)}(R) \to \mathsf{thick}_{\mathsf{K}}(\mathrm{i}R)$. Taking quotients we have an equivalence

$$(5.2.0.2) \qquad D^f(R)/\mathsf{thick}_{\mathsf{D}(R)}(R) \to \mathsf{K}^c/\mathsf{thick}_{\mathsf{K}}(\mathrm{i}R).$$

Neeman's theorem recalled in 2.1.4 shows that $\mathsf{thick}_{\mathsf{K}}(\mathrm{i}R) = \mathsf{loc}_{\mathsf{K}}(\mathrm{i}R)^c$. Composing (5.2.0.1) and (5.2.0.2) gives an equivalence:

$$(5.2.0.3) \qquad \underline{\mathrm{MCM}}(R) \to \mathsf{K}^c/\mathsf{loc}_{\mathsf{K}}(\mathrm{i}R)^c$$

which sends a maximal Cohen-Macaulay module to the image of an injective resolution in $\mathsf{K}^c/\mathsf{loc}_\mathsf{K}(iR)^c$.

We need to be careful as $\mathsf{K}^c/\mathsf{loc}_\mathsf{K}(iR)^c$ is not in general equivalent to $(\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR))^c$. However, the Neeman-Thomason-Trobaugh-Ravenal localization theorem [22, 2.1] shows that there is a fully faithful functor $G : \mathsf{K}^c/\mathsf{loc}_\mathsf{K}(iR)^c \to (\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR))^c$ making the following diagram commute:

(5.2.0.4)
$$
\begin{array}{ccc}
\mathsf{K}^c & \longrightarrow & (\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR))^c \\
\downarrow & \nearrow^{G} & \\
\mathsf{K}^c/\mathsf{loc}_\mathsf{K}(iR)^c & &
\end{array}
$$

The vertical arrow is the natural quotient map. By [22, 2.4] the quotient map $\mathsf{K} \to \mathsf{K}/\mathsf{loc}_\mathsf{K}(iR)$ preserves compactness; the horizontal functor is the restriction of this map to compact objects.

Now consider the object iM of K and assume that the image of iM under the map $\mathsf{K} \to \mathsf{K}/\mathsf{loc}_\mathsf{K}(iR)$ is decomposable. Since M is finitely generated iM is compact; moreover by the commutativity of (5.2.0.4) iM is in the image of the functor G. By the fully faithfulness of the functor G this implies that the image of iM in $\mathsf{K}^c/\mathsf{loc}_\mathsf{K}(iR)^c$ is decomposable. And hence M is decomposable in $\underline{\mathrm{MCM}}(R)$ by (5.2.0.3). $\qquad\square$

The set $\mathcal{V}_\mathcal{R}(M)$ is defined in 3.2.7.

Theorem 5.4. *Let M be a maximal Cohen-Macaulay R-module whose image in $\underline{\mathrm{MCM}}(R)$ is indecomposable. The set $\mathcal{V}_\mathcal{R}(M)$ of M is a projectively connected subset of $\mathbb{P}_k^{c-1}$.*

Proof. Let iM be an injective resolution of M. By Propsition 5.3 the image of iM in $\mathsf{K}/\mathsf{loc}_\mathsf{K}(iR)$ is indecomposable; by Theorem 5.1 the support set $\mathcal{V}_\mathcal{S}(iM)$ is a connected subset of $\mathrm{Proj}\,\mathcal{S}$. Recall that $\mathcal{V}_\mathcal{R}(M) = \pi^{-1}(\mathfrak{m}) \cap \mathcal{V}_\mathcal{S}(M)$; see 3.2.7.

We will again prove the contrapositive. Assume that $\mathcal{V}_\mathcal{R}(M) = \mathcal{W}_1 \cup \mathcal{W}_2$ with $\mathcal{W}_1 \cap \mathcal{W}_2 = (\boldsymbol{\chi})$ with $\mathcal{W}_i \nsubseteq \mathcal{V}(\boldsymbol{\chi})$. We will show this implies the disconnectedness of $\mathcal{V}_\mathcal{S}(M)$. By Proposition 5.3 and Theorem 5.1 this will show that M is decomposable.

Let $\mathcal{V}_i = \{\, \mathfrak{p} = \mathfrak{p}_0 + \mathfrak{p}_1 + \cdots \in \mathcal{V}_\mathcal{S}(M) \,|\, \mathfrak{p}_0 \in \operatorname{Spec} R \text{ and } \mathfrak{m} + \mathfrak{p}_1 + \cdots \in \mathcal{W}_i \}$. By construction $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_\mathcal{S}(M)$ and it is also clear that $\mathcal{V}_1 \cap \mathcal{V}_2 = \mathcal{V}(\boldsymbol{\chi})$. Finally $\mathcal{V}_i \nsubseteq \mathcal{V}(\chi)$ since $\mathcal{W}_i \nsubseteq (\boldsymbol{\chi})$. Thus $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}_\mathcal{S}(M)$ is a nontrivial disconnection. $\qquad\square$

5.3 Bergh's Result

Throughout the thesis so far we have assumed that R is the quotient of a regular ring. However there are examples of local rings that are not quotients of regular local rings. Recall that by Cohen's structure theorem every complete local ring is the quotient of a regular local ring.

Definition. A local ring $(R, \mathfrak{m}, k)$ is a *local complete intersection* ring if the $\mathfrak{m}$-adic completion of R, denoted $\widehat{R}$, is presented as

$$\widehat{R} = Q/(f_1, \ldots, f_c)$$

for some regular local ring Q and a Q-regular sequence $f_1, \ldots, f_c$. The codimension of R is equal to the smallest such c ranging over all regular local rings and regular sequences as above.

Remark. It is an open question whether every complete intersection is in fact already the quotient of a regular ring by a regular sequence.

In this section we show how 5.4 recovers a recent result of Bergh about support varieties over local complete intersections and discuss the necessity of one of the assumptions.

For the rest of the section we fix a local complete intersection R of codimension c and a presentation $\widehat{R} = Q/(f_1, \ldots, f_c)$, where Q is a regular local ring and $f_1, \ldots, f_c$ is a Q-regular sequence. There is a ring of operators $\widehat{R}[\chi_1, \ldots, \chi_c]$ which acts on $\operatorname{Ext}^*_{\widehat{R}}(\widehat{M}, \widehat{N})$ for every pair of finite R-modules M and N. Set $\mathcal{R} := k \otimes_{\widehat{R}} \widehat{R}[\chi_1, \ldots, \chi_c] = k[\chi_1, \ldots, \chi_c]$. Given an R-module M the *support variety of M* is

$$\mathcal{V}_{\mathcal{R}}(\widehat{M}) = \operatorname{supp}_{k[\chi]} \operatorname{Ext}^*_{\widehat{R}}(\widehat{M}, k) \subseteq \mathbb{P}^{c-1}_R$$

When R is the quotient of a regular local ring we recover nothing new:

Lemma 5.5. *Let $(R, \mathfrak{m}, k)$ be a local ring with a presentation*

$$R = Q/(f_1, \ldots, f_c)$$

where Q is a regular local ring and $(f_1, \ldots, f_c)$ is a Q-regular sequence.

Let M be a finitely generated R-module. Then the following subsets of $\operatorname{Spec}^+ k[\chi_1, \ldots, \chi_c]$ are equal:

$$\mathcal{V}_{\mathcal{R}}(M) = \mathcal{V}_{\mathcal{R}}(\widehat{M}).$$

Proof. First note that the presentation of R gives a presenation of the $\mathfrak{m}$-adic completion of R as $\widehat{R} = \widehat{Q}/(f_1, \ldots, f_c)$. By [4, 5.3] the support variety of M does not depend on the presentation of $\widehat{R}$ as a quotient of a regular local ring by a regular sequence, so we compute the support variety using the presentation $\widehat{R} = \widehat{Q}/(f_1, \ldots, f_c)$.

The maps $R \to \widehat{R}$ and $M \to \widehat{M}$ induce a map

$$\zeta : \operatorname{Ext}^*_{\widehat{R}}(\widehat{M}, k) \to \operatorname{Ext}^*_R(M, k).$$

Since k is $\mathfrak{m}$-adically complete the map ζ is an isormorphism in each degree. By [9,

3.1] this is a morphism of $\mathcal{R}$-modules and hence an isomorphism of $\mathcal{R}$-modules. In particular

$$\mathcal{V}_{\mathcal{R}}(M) := \operatorname{supp}_{\mathcal{R}} \operatorname{Ext}_R^*(M, k) = \operatorname{supp}_{\mathcal{R}} \operatorname{Ext}_{\widehat{R}}^*(\widehat{M}, k) =: \mathcal{V}_{\mathcal{R}}(\widehat{M})$$

which shows the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following was originallly proved in [12, 3.2].

Theorem 5.6. *Let $(R, \mathfrak{m}, k)$ be a local complete intersection of codimension c. Let M be a Cohen-Macaulay module whose completion in the $\mathfrak{m}$-adic topology is indecomposable. Then the support variety of M is a connected subset of $\mathbb{P}_k^{c-1}$.*

Proof. First, we may assume that M is maximal Cohen-Macaulay. Indeed, by [27, 8.17], every syzygy of M has an indecomposable syzygy. Also, the supporty variety of M and a syzygy of M coincide.

By assumption $\widehat{M}$ is an indecomposable $\widehat{R}$-module. Moreover, we may assume $\widehat{M}$ is indecomposable in $\underline{\operatorname{MCM}}(\widehat{R})$. For assume there is a decomposition

$$\widehat{M} \cong M_1 \oplus M_2 \in \underline{\operatorname{MCM}}(\widehat{R}).$$

Thus there exist projective $\widehat{R}$-modules P, Q such that as $\widehat{R}$-modules

$$\widehat{M} \oplus P \cong M_1 \oplus M_2 \oplus Q.$$

Since $\widehat{R}$ is complete the Krull-Remak-Schmidt theorem holds. Thus we have

$$\widehat{M} \,|\, M_1 \,, \ \widehat{M} \,|\, M_2 \ \text{or} \ \widehat{M} \,|\, Q$$

since $\widehat{M}$ is indecomposable. If $\widehat{M} \,|\, Q$ then $\widehat{M}$ is projective and $\mathcal{V}_{\mathcal{R}}(\widehat{M}) = 0$ and in particular is projectively connected. So we may assume $\widehat{M} \,|\, M_1$. Cancelling $\widehat{M}$ there exists N such that

$$P \cong N \oplus M_2 \oplus Q$$

which shows that M_2 is projective and hence zero in $\underline{\mathrm{MCM}}(S)$.

Now since we're assuming the image of $\widehat{M}$ is indecomposable in $\underline{\mathrm{MCM}}(\widehat{R})$, Theorem 5.4 applies to show that $\mathcal{V}_{\mathcal{R}}(\widehat{M})$, which is the support variety of M, is a connected subset of $\mathbb{P}_k^{c-1}$. $\qquad\square$

Remark.   1. Bergh has raised the question as to whether the assumption of completeness is necessary. The above two theorems show that if there exists a local complete intersection $(R, \mathfrak{m}, k)$ and a maximal Cohen-Macaulay R-module M, with M indecomposable in $\underline{\mathrm{MCM}}(R)$ and $\mathcal{V}_{\mathcal{R}}(\widehat{M})$ projectively disconnected, then R is not the quotient of a regular local ring. For if $R = Q/I$ is the quotient of a regular ring, then I must be generated by a regular sequence. And then by 5.4 the set $\mathcal{V}_{\mathcal{R}}(M)$ is projectively connected. But by 5.5, there is an equality $\mathcal{V}_R(M) = \mathcal{V}_{\mathcal{R}}(\widehat{M})$ which shows in particular that $\mathcal{V}_{\mathcal{R}}(\widehat{M})$ must be connected.

2. As the previous chapter, the methods used here apply more generally. Specifically they show that for a ring with a support theory, an indecomposable module which is the analogue of a maximal Cohen-Macaulay module, will have a connected support set.

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