Cone Metric Spaces And Fixed Point Theorems

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Declaration

I declare that the master thesis entitled Cone Metric Spaces and Fixed Point Theorems, is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Hiba Qabaja

Signature:________________ Date:______________

Dedications

To my dearest people, who believed in me and led me to the road of success, to my parents, my child Ameer, my brothers, and sister, also I'll never forget my best, friends all dears, to the wonder of your hearts I send this dedication.

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Abstract

In this work, we study algebraic, geometrical, and topological properties of cone metric spaces. Also, we introduce fixed point theorems in cone metric spaces.
In fact, our work is a survey of the main results on cone metric spaces.

Introduction

Cone metric spaces was firstly introduced in 2007 by Huang and Zhang by means of partially ordering real Banach spaces, where they also proved some fixed point theorems for mappings satisfying different contractive conditions. The normality property of a cone was an important ingredient in their results, and they believe that their results generalized some fixed point theorems in metric spaces. Their work gave a base for more research. In 2008 authors of [26] analyze the existence of fixed points for a self-map defined on a complete, (sequentially compact) cone metric space (X, d) satisfying the T-contraction and T-contractive condition.
Later, in [30] authors improved some results in [13], by considering (X, d) complete and omitting the normality assumption of the cone P. After that the regularity condition of the cone P was omitted in sequentially compact cone metric space and considered the weaker condition of normality on the cone P. see [25]. Many studies appears during 2008 and 2009 about fixed point in cone metrics and coupled fixed point theorems.
In 2010 M.A.Khamsi [19] gives a memorable result, he claims that; most of the cone fixed point results are merely copies of the classical ones, and that any extension of known fixed point results to cone metric spaces is redundant and that underlying Banach spaces and the associated cone subsets are not necessary.
This approach included a small class of results and is very limited since it is requires only normal cones.

The classical contraction mapping principle of Banach is one of the most powerful theorems in fixed point theory because of its simplicity and usefulness.[18]
In recent times, fixed point theory has developed rapidly in partially ordered metric spaces, because of the importance of its applications in diverse disciplines of mathematics, statistics, chemistry, biology, computer science, engineering and economics, and also in dealing with problems arising in approximation theory, potential theory, game theory, mathematical economics, theory of differential equations,theory of integral equations, etc.(see [37]).

Weak contraction principle is a generalization of Banach's contraction principle which was first given by Albert et al. see [4].
Fixed point problems involving weak contractions and mappings satisfying weak

contraction type inequalities have been considered in the work of many researchers see the introduction of [9]. Authors of [9] show that certain functions will have fixed points if they satisfy certain weak contractive inequalities.
In 2013, there were many results and generalizations of some fixed point theorems, as in [10, 17]

Also, there were many attempts and studies a bout topological cone metric spaces, as in [38], where the authors proved that cone metric spaces are topological spaces. Moreover, compactness, boundedness, first countability were discussed there.

This thesis is mainly concern ed with providing some substantial results that came into view while mathematicians attempts to prove that cone metric spaces generalizes metric spaces. Even the results that have been established in 2017, which proved that cone metric spaces is a real generalization of metric spaces.
Chapter(I) consists of four sections, where we introduced, some definitions and basic theorems about cones, normal cones, regular cones, minihedral cones, strong minihedral cones, solid cones, some examples of cones, cone metric spaces, convergent sequences, T-contractive mappings and cone normed spaces. Also some properties of Metrizability of cone metric spaces.
Chapter(II) there are six sections, where we discussed TVS-cone metric spaces, and basic theorems on TVS-cone metric spaces, in addition to complete metric spaces, and TVS-cone normed spaces. Finally, we talk about Minkowski functional on solid vector spaces and TVS-cone metrics.
In the last chapter, chapter(III), we tried to trace some important results of fixed point theory, some examples, common fixed point theorems in cone metric spaces, and finally some coupled fixed point theorems in cone metric spaces.

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Chapter 1

Cone metric spaces

In this chapter, we will introduce some definitions and basic theorems about cones, normal cones, regular cones, minihedral cones, strongly minihedral cones, solid cones, cone metric spaces, convergent sequences, T-contractive mappings and cone normed spaces. Also this chapter contains some properties of metrizability of cone metric spaces.

1.1 Preliminaries

Definition 1.1. *A set A in X is said to be convex if, for all $x, y \in A$ and $t \in (0, 1)$, then the line segment $(1-t)x + ty \in A$.*

Definition 1.2. *A set A is called closed set if it contains its own boundary.*

Definition 1.3. *Banach space is a complete normed linear space.*

Definition 1.4. *l_∞ is the set of all bounded sequences of real numbers or complex numbers. That is, all sequences $x = \{x_i\}_1^\infty$ such that $\sup_i(x_i) < \infty$.*

Definition 1.5. *For $1 \leq p < \infty$, l_p denotes the set of all sequences of real numbers or complex numbers, such that $\sum_{i=1}^{\infty} \mid x_i \mid^p < \infty$.*

1.2 Cones

Definition 1.6. *:[13] Let E be a real Banach space. A nonempty subset $P \subset E$ is called a cone in E if it satisfies :*

i: P *is closed, convex, and* $P \neq 0$, *(where 0 is the zero vector of* P*) .*

ii: $0 \leq a, b \in I\!R$ *and* $x, y \in P$, *imply that* $ax + by \in P$.

iii: $x \in P$ *and* $-x \in P$ *imply that* $x = 0$

Given a cone $P \subset E$, we define a Partial ordering relation $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$.
We write $x \preceq y$ (x is away-behind y) if $y - x \in P^o$, where P^o denotes the interior of P. The relation away-behind is transitive and antisymmetric but not in general reflexive [21]. We write $x \prec y$ when $x \preceq y$ and $x \neq y$.

Examples of cones:

Example 1.7. *: Let* $E = I\!R^n$, *then* $P = \{(x_1, x_2, ..., x_n) \in E; x_i \geq 0; \forall i = 1, 2, ..., n\}$ *is a cone.*

Example 1.8. *: In* ℓ^p *spaces including* ℓ^∞, *the set* $P = \{x_n \in \ell_p : x_n \geq 0\}$
is a cone

Example 1.9. *:If* $E = C[0, 1]$ *with the supremum norm,* $(\parallel f \parallel = \sup_{x \in E} \mid f(x) \mid)$ *the set* $P = \{f \in E : f \geq 0\}$ *is a cone.*

Next, we introduce definitions of types of cones.

Definition 1.10. *:[40] A cone P in $(E, \|.\|)$ is called :*

i: ***Normal (N)****: If there exist a constant $k > 0$ such that if $0 \leq x \leq y$ then $\| x \| \leq k \| y \|$,*
the least positive number satisfying the above inequality is called the normal constant of P.

ii: ***Minihedral (M)****: If sup $\{x, y\}$ exists for all $x, y \in E$.*

iii: ***Strongly Minihedral (S)****: If every subset of E which is bounded above has a supremum .*

iv: ***Solid****: If $P^o \neq \emptyset$.*

v: ***Regular (R)****: If every increasing sequence which is bounded above in E is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq, ..., \leq y$ for some $y \in E$, then $\exists x \in E$ such that:* $\lim_{n\to\infty} \|x_n - x\| = 0$.

Equivalently; the cone P is regular if and only if every decreasing sequence which is bounded below is convergent. [14]

The next propositions give us an idea about the relation between Normal and Regular cones.

Proposition 1.11. *[30] Every regular cone is normal.*

Proof. : Let P a regular cone which is not normal. For each $n \geq 1$, choose $t_n, s_n \in P$ such that $t_n - s_n \in P$ and $n^2 \| t_n \| < \| s_n \|$.
For each $n \geq 1$, put $y_n = \frac{t_n}{\|t_n\|}$ and $x_n = \frac{s_n}{\|t_n\|}$. Then, x_n, y_n ,$y_n - x_n \in P$, $\| y_n \| = 1$ and $n^2 < \| x_n \|$, for all $n \geq 1$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \| y_n \|$ is convergent and P is closed, there is an element $y \in P$ such that $\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y$. Now, note that:

$$0 \leq x_1 \leq x_1 + \frac{1}{2^2} x_2 \leq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2 x_3} \leq ... \leq y.$$

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ is convergent because P is regular. Hence,

$$\lim_{n\to\infty} \frac{\| x_n \|}{n^2} = 0$$

which is a contradiction. $\square$

The following example shows that the converse of the proposition 1.11 is not true:

Example 1.12. *:[30] Consider the space $E = C[0,1]$ with the supremum norm, and let $P = \{f \in E : f \geq 0\}$. Then, P is a cone with normal constant $k = 1$.*
Now, consider the following sequence of elements of E $x \geq x^2 \geq x^3 \geq, ..., \geq 0$. which is decreasing and bounded from below but it is not convergent in E.

Proposition 1.13. *[30] There is no normal cone with normal constant $k < 1$.*

Proof. Let P a normal cone with normal constant $k < 1$, choose a nonzero element $x \in P$ and $0 < \varepsilon < 1$, such that $k < (1-\varepsilon)$ then, $(1-\varepsilon)x \leq x$ but
$(1-\varepsilon)\|x\| > k\|x\|$, this is contradiction. Hence there is no normal cones with normal constant $k < 1$. □

Proposition 1.14. *[30] For each $k > 1$, there is a normal cone with normal constant $M > k$.*

Proof. Let $k > 1$ be given. Consider:

$$E = \{ax + b : a, b \in I\!R; x \in [1 - (\frac{1}{k}), 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. First, we will show that P is regular (and hence normal by proposition 1.11). Let, $\{a_n x + b_n\}_{n\geq 1}$ be an increasing sequence which is bounded from above, that is, there is an element $cx + d \in E$ such that :

$$a_1 x + b_1 \leq a_2 x + b_2 \leq ... \leq a_n x + b_n \leq ... \leq cx + d$$

for all $x \in [1 - \frac{1}{k}, 1]$. Then, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two sequences in $I\!R$ such that:
$b_1 \leq b_2 \leq ... \leq d$, $a_1 \geq a_2 \geq ... \geq c$
Thus, $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are convergent.
Let $a_n \to a$ and $b_n \to b$. Then, $ax + b \in E$ and $a_n x + b_n \to ax + b$. Therefor, P is regular, and then by proposition 1.11, there is $M \geq 1$ such that $0 \leq g \leq f$ implies that:
$\| g \| \leq k \| f \|$, for all $g, f \in E$.
Now, we show that $M > k$.

First, note that $f(x) = kx + k \in P$, $g(x) = k \in P$ and $f - g \in P$. So, $0 \leq g \leq f$. Therefor,

$$k = \| g \| \leq M(\| f \|) = M.$$

On the other hand, if we conceder $f(x) = -(k + \frac{1}{k})x + k$ and $g(x) = k$, then $f \in P$, $g \in P$ and $f - g \in P$. Also, $\| g \| = k$ and $\| f \| = 1 - \frac{1}{k} + \frac{1}{k^2}$.
Thus, $k = \| g \| > k \| f \| = k + \frac{1}{k} - 1$.
This shows that $M > k$. $\square$

Proposition 1.15. *[31] Every strongly minihedral normal cone is regular.*

Here are some examples of normal and regular cones :

Example 1.16. *[6] $E = \mathbb{R}^2$ and $P = \{(x, 0) : x \geq 0\}$ is strongly minihedral but not minihedral, by definition 1.10.*

Example 1.17. *[30] Let $E = C^2_{\mathbb{R}}([0, 1])$ with the norm:*

$$\| f \| = \| f \|_\infty + \| f^{'} \|_\infty,$$

where $\| f \|_\infty = \max\{f(x) : x \in [0, 1]\}$ and,
$\| f^{'} \|_\infty = \max\{f^{'}(x) : x \in [0, 1]\}$. And consider the cone $P = \{f \in E : f \geq 0\}$.

For each $k \geq 1$, put $f(x) = x$ and $g(x) = x^{2k}$. Then $0 \leq g \leq f$, $\| f \| = 2$

and $\| g \| = 2k + 1$, as
$(\| g \| = \max\{x^{2k} : x \in [0, 1]\} + \max\{2kx^{2k-1} : x \in [0, 1]\} = 1 + 1 = 2)$.
Since $k \| f \| < \| g \|$, k is not normal constant of P. Therefor, P is a non-normal cone.

This example shows that there are non-normal cones.

1.3 Cone Metric spaces

In this section we will define cone metric spaces, and prove some theorems and properties about cone metric spaces, also we will discuss the relation between metric spaces and cone metric spaces.

Consider E to be a real Banach space, and P be a cone of E. We define *cone metric spaces* as follows :

Definition 1.18. *: [13] Let X be a nonempty set. Consider the mapping $d : X \times X \longrightarrow E$ which satisfies :*

i: $0 < d(x, y)$; $\forall x, y \in X$ *and* $d(x, y) = 0 \Leftrightarrow x = y$

ii: $d(x, y) = d(y, x)$; $\forall x, y \in X$

iii: $d(x, y) \leq d(x, z) + d(z, y)$; $\forall x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space. Note from the definition that cone metric spaces generalize metric spaces.

Definition 1.19. *[11] We say that the cone metric space (X, d) is :*

Cone rectangular metric space*: if we replace (iii) in definition 1.18 with:*

$d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$; $\forall x, y, \in X$.
And distinct points $w, z \in X - \{x, y\}$.[7]

Cone pentagonal metric space*: if we replace (iii) in definition 1.18 with:*

$d(x, y) \leq d(x, w) + d(w, u) + d(u, z) + d(z, y)$; $\forall x, y, \in X$. *And distinct points $w, z, u \in X - \{x, y\}$.[11]*

Cone hexagonal metric space*: if we replace (iii) in definition 1.18 with:*

$d(x, y) \leq d(x, w) + d(w, u) + d(u, v) + d(v, z) + d(z, y)$; $\forall x, y, \in X$.
And distinct points $w, z, u, v \in X - \{x, y\}$.[11]

Here are some familiar examples of cone metric spaces:

Example 1.20. *[5] Let $E = \mathbb{R}^n$ with $P = \{(x_1, x_2, ..., x_n) : x_i \geq 0,\ \forall i = 1, ..., n\}$ $X = \mathbb{R}^n$, and $D = X \times X \longrightarrow E$ such that:*

$$D(x, y) = \{|x - y|, \alpha_1|x - y|, ..., \alpha_{n-1}|x - y|\}$$

where $\alpha_i > 0$ for all $1 \leq i \leq n - 1$. Then (X, D) is a cone metric space.

Example 1.21. *[5] Let* $E = C_{\mathbb{R}}[0,1]$ *with the supremum norm and*

$$P = \{f \in E : f(t) \geq 0\}.$$

Then, P *is a cone with normal constant* $k = 1$*. Now, define* $D = X \times X \longrightarrow E$ *by* $D(x,y) = | x - y | \varphi$ *Where* $\varphi : [0,1] \to \mathbb{R}^+$ *such that* $\varphi(t) = e^t$*. Then* D *is a cone metric on* X*.*

Example 1.22. *[14] Let* $E = l^q$*, for* $q > 0$*,* $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0; \forall n\}$*.*
Let (X, ρ) *be a metric space and* $d : X \times X \longrightarrow E$ *defined by:*

$$d(x,y) = \{(\frac{\rho(x,y)}{2^n})^{\frac{1}{q}}\}_{n\geq 1}.$$

Then, (X,d) *is a cone metric space and the normal constant of* P *is equal to* 1*.*

Moreover, note the last example shows that the category of cone metric spaces is bigger than the category of metric spaces.
The next example is not a cone metric space.

Example 1.23. *[11] Let* $X = \mathbb{N}$*,* $E = \mathbb{R}^2$ *and* $P = \{(x,y) : x, y \geq 0\}$*.*
Define $d : X \times X \to E$ *as follows:* $d(x,y) = (0,0)$ *if* $x = y$*;* $d(x,y) = (9,15)$ *if* x *and* y *are in* $\{3,4\}$*,* $x \neq y$*;* $d(x,y) = (3,5)$ *if* x *and* y *cannot both at a time in* $\{3,4\}$*,* $x \neq y$*. Then* (X,d) *is a cone hexagonal (or pentagonal or rectangular) metric space, but not a cone metric because it lacks the triangular property:* $(9,15) = d(3,4) > d(3,5) + d(5,4) = (3,5) + (3,5) = (6,10)$ *as* $(9,15) - (6,10) = (3,6) \in P$*.*

Definition 1.24. *[39] Let* (X,d) *be a cone metric space and* $A \subseteq X$*. Then* A *is said to be bounded above if* $\exists e \in E$*;* $e \succeq 0$ *such that* $d(x,y) \preceq e$*;* $\forall x, y \in A$*. also,* A *is called bounded if,* $\delta(A) = sup\{d(x,y) : x, y \in A\}$ *exists in* E*.*

Thus if P is strongly minihedral, then being bounded is the same as being bounded above.

Example 1.25. *[28] Let* X *be a nonempty set and let* $(Y, \preceq)$ *be a solid Banach space. Suppose* $a \in Y$ *such that* $a \succ 0$*. Define the cone metric* $d = X \times X \to Y$ *by:*

$$d(x,y) = \begin{cases} a, & x \neq y; \\ 0, & x = y. \end{cases}$$

Then, (X,d) *is a cone metric space over* Y*. This space is called (discrete cone metric space).*

1.4 Convergence in Cone Metric Spaces

Below, we present the notion of convergence of sequences in cone metric spaces.

Definition 1.26. *:[13] Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then :*

i: $\{x_n\}_{n\geq 1}$ *converges to* x *whenever* $\forall c \in E$ *with* $0 \preceq c$*, there is a natural number* N *such that* $d(x_n, x) \preceq c, \forall n \geq N$*. We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x$.

ii: $\{x_n\}_{n\geq 1}$ *is Cauchy sequence whenever for every* $c \in E$ *with* $0 \preceq c$ *there is a natural number* N *such that* $d(x_n, x_m) \preceq c, \forall n, m \geq N$.

iii: (X, d) *is complete cone metric space if every Cauchy sequence is convergent in* X.

Lemma 1.27. *:[13] Let (X, d) be a cone metric space, P a normal cone with normal constant k and $\{x_n\}$ is a sequence in X, then*

i: $\{x_n\}$ *converge to* $x \Leftrightarrow \lim_{n\to\infty} d(x_n, x) = 0$.

ii: If $\{x_n\}$ *is convergent, then it is Cauchy sequence.*

iii: $\{x_n\}$ *is a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0$$

.

iv: If $\{x_n\} \to x$ *and* $\{x_n\} \to y, (n \to \infty)$ *then* $x = y$.

v: If $\{x_n\} \to x$ *and* $\{y_n\}$ *is another sequence in* X *such that* $\{y_n\} \to y$ *, then* $d(x_n, y_n) \to d(x, y)$.

Proof. i: Suppose that $\{x_n\}$ converges to x. For every real $\epsilon > 0$; choose $c \in E$ with $0 \ll c$ and $k \parallel c \parallel < \epsilon$. Then there is N, such that for all $n > N$, $d(x_n, x) \preceq c$, so when $n > N$,
$\parallel d(x_n, x) \parallel \leq k \parallel c \parallel < \epsilon$. This means that $d(x_n, x) \to 0$ as $(n \to \infty)$.

ii: For any $c \in E$ with $0 \preceq c$, there is N such that for all $n, m > N$, $d(x_n, x) \preceq \frac{c}{2}$ and $d(x_m, x) \preceq \frac{c}{2}$.
Hence $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) \preceq c$. Therefor $\{x_n\}$ is a Cauchy sequence.

iii: Suppose that $\{x_n\}$ is a Cauchy sequence. For every $\epsilon > 0$,
choose $c \in E$ with $0 \preceq c$ and $k \parallel c \parallel < \epsilon$. Then there is N, for all $n, m > N$, $d(x_n, x_m) \preceq c$. So that when $n, m > N$, $\parallel d(x_n, x_m) \parallel \leq k \parallel c \parallel < \epsilon$. This means that $d(x_n, x_m) \to 0$ as $(n, m \to \infty)$.
Conversely, suppose that $d(x_n, x_m) \to 0$ as $(n, m \to \infty)$. For $c \in E$ with $0 \preceq c$, there is $\delta > 0$, s.t $\parallel x \parallel < \delta \Rightarrow c - x \in P^0$. For this δ there is N, such that for all $n, m > N$, $\parallel d(x_n, x_m) \parallel < \delta$ So $c - d(x_n, x_m) \in P^0$. This means $d(x_n, x_m) \preceq c$. Hence $\{x_n\}$ is a Cauchy sequence.

iv: For any $c \in E$ with $0 \preceq c$, there is N such that for all $n > N$, $d(x_n, x) \ll c$ and $d(x_n, y) \preceq c$. We have

$$d(x, y) \leq d(x_n, x_) + d(x_n, y) \leq 2c.$$

Hence,

$$\parallel d(x, y) \parallel \leq 2k \parallel c \parallel .$$

Since c is arbitrary $d(x, y) = 0$ therefor $x = y$.

v: For every $\epsilon > 0$, choose $c \in E$ and $\parallel c \parallel < \frac{\epsilon}{4k+2}$. From $x_n \to x$ and $y_n \to y$, there is N such that for all $n > N$, $d(x_n, x) \ll c$, and $d(y_n, y) \preceq c$. We have

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y) \leq d(x, y) + 2c$$

Hence,

$$0 \leq d(x, y) + 2c - d(x_n, y_n) \leq 4c$$

and

$$\parallel d(x_n, y_n) - d(x, y) \parallel \leq \parallel d(x, y) + 2c - d(x_n, y_n) \parallel + \parallel 2c \parallel \leq (4k + 2) \parallel c \parallel < \epsilon.$$

Therefor, $d(x_n, y_n) \to d(x, y)$ as $(n \to \infty)$.

$\square$

Proposition 1.28. *[39] Every Cauchy sequence in a cone metric space over a strongly minihedral cone is bounded.*

Proof. Let $\{x_n\}$ be Cauchy sequence in cone P. Fix $c \ll 0$, choose $n_0 \in I\!N$ such that $m, n \geq n_0 \Rightarrow d(x_m, x_n) \ll c$.
Let $c^{'} = \sup\{c, d(x_m, x_n) : m, n < n_0\}$, note that $c^{'}$ exists since P is strongly minihedral. Hence, $d(x_m, x_n) \preceq c^{'}$; $\forall m, n$. So, $\{x_n\}$ is bounded. $\square$

Definition 1.29. *:[26] Let (X, d) be a cone metric space, P a normal cone with normal constant k and $T : X \to X$. Then T is said to be continuous if* $\lim_{n\to\infty} x_n = x$*, implies that* $\lim_{n\to\infty} Tx_n = Tx$, $\forall \{x_n\}$ *in X.*
T is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, when $\{T(y_n)\}$ is convergent , then (y_n) is also convergent.

Definition 1.30. *[13] Let (X, d) be a cone metric space, if for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that,*
$\{x_{n_i}\}$ is convergent in X. Then X is called a sequentially compact cone metric space.

Lemma 1.31. *:[13] If (X, d) be a sequentially compact cone metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent .*

Proposition 1.32. *[39] The cone metric d induced by a cone norm on a cone normed space satisfies* lemma of translation invariance*:*

$$d(x + a, y + a) = d(x, y)$$

$$d(ax, ay) = \mid a \mid d(x, y)$$

for all $x, y \in X$ and every scalar a.

Proof. We have $d(x + a, y + a) = \parallel (x + a) - (y + a) \parallel_c = \parallel x - y \parallel_c = d(x, y)$ and $d(ax, ay) = \parallel ax - ay \parallel_c = \mid a \mid\parallel x - y \parallel_c = \mid a \mid d(x, y)$. $\square$

1.5 Metrizability of cone metric space

The history of metrizability of cone metric spaces starts with the initial paper of Huang and Zhang in 2007 [13], after that there were many papers that deals with the theory of cone metric spaces, a basic question has been raised, that whether cone metric spaces is a real generalization of metric spaces? This question has been investigated in many papers as [6, 21].
Many authors showed that the cone metric spaces are metrizable and defined the equivalent metric using different approaches, and so, consequently due to those approaches every cone metric space is really a metric one, and every theorem in metric space is valid for cone metric spaces automatically.
Now, we will review some of those approaches. Firstly in [6] the authors showed that by renorming an ordered Banach space, every cone P can be converted to a normal cone with constant $k = 1$.

Theorem 1.33. *[6] Let $(E, \| \, . \, \|)$ be a real Banach space with a positive cone P. Then there exists a norm on E such that P is a normal cone with constant $k = 1$, with respect to this norm.*

Proof. Define $||| \, . \, ||| := E \to [0, \infty)$ by :

$$||| \, x \, ||| := \inf\{\| \, u \, \| : x \preceq u\} + \inf\{\| \, v \, \| : v \preceq x\} + \| \, x \, \|,$$

for all $x \in E$.
Let us show that $||| \, . \, |||$ is a norm on E. Firstly, by definition of $||| \, . \, |||$ it is clear that, $||| \, x \, ||| = 0$ if and only if $x = 0$ for all $x \in E$. Also

$$\begin{aligned} ||| -x ||| = & \inf\{\| \, u \, \| : -x \preceq u\} + \inf\{\| \, v \, \| : v \preceq -x\} + \| -x \, \| \\ = & \inf\{\| \, u \, \| : -u \preceq x\} + \inf\{\| \, v \, \| : x \preceq -v\} + \| \, x \, \| \\ = & \inf\{\| \, v' \, \| : v' \preceq x\} + \inf\{\| \, u' \, \| : x \preceq u'\} + \| \, x \, \| \\ = & ||| \, x \, ||| \, . \end{aligned}$$

For $\lambda > 0$,

$$\begin{aligned} ||| \, \lambda x \, ||| &= \inf\{\| \, u \, \| : \lambda x \preceq u\} + \inf\{\| \, v \, \| : v \preceq \lambda x\} + \| \, \lambda x \, \| \\ &= \inf\{\lambda \, \| \, \frac{1}{\lambda} u \, \| : x \preceq \frac{1}{\lambda} u\} + \inf\{\lambda \, \| \, \frac{1}{\lambda} v \, \| : \frac{1}{\lambda} v \preceq x\} + \lambda \, \| \, x \, \| \\ &= \lambda \, ||| \, x \, ||| \\ Therefor, ||| \, \lambda x \, ||| &= | \, \lambda \, ||| \, x \, \|, \forall x \in E; \lambda \in I\!R. \end{aligned}$$

To prove the triangle inequality of $\||\ .\ \||$, let $x, y \in E$,

$$\forall \epsilon > 0\ \exists u_1, v_1 : v_1 \preceq x \preceq u_1; \| u_1 \| + \| v_1 \| + \| x \| - \epsilon < \||\ x\ \||$$

$$\forall \epsilon > 0\ \exists u_2, v_2 : v_2 \preceq y \preceq u_2; \| u_2 \| + \| v_2 \| + \| y \| - \epsilon < \||\ y\ \||\ .$$

Therefor $v_1 + v_2 \preceq x + y \preceq u_1 + u_2$, hence

$$\||\ x + y\ \|| \leq \| v_1 + v_2 \| + \| u_1 + u_2 \| + \| x + y \| \leq \||\ x\ \|| + \||\ y\ \|| + 2\epsilon$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\||\ x + y\ \|| \leq \||\ x\ \|| + \||\ y\ \||\ .$$

So, $\||\ .\ \||$ is a norm on E.
Now, we shall show that P, with the norm $\||\ .\ \||$, is a normal cone with constant $k = 1$, that is for all $x, y \in E$

$$0 \preceq x \preceq y \Rightarrow \||\ x\ \|| \leq \||\ y\ \||\ .$$

Suppose that $0 \preceq x \preceq y$. Then,

$$0 \leq \||\ x\ \|| \leq \| 0 \| + \| y \| + \| x \| = \| y \| + \| x \|\ . \tag{1.1}$$

if we put $A := \{\| v \| : v \preceq y\}$ then, by 1.1, $\||\ x\ \||$ is a lower bound for $A + \| y \|$. So,

$$\||\ x\ \|| \leq \inf(A + \| y \|) = \inf A + \| y \| \leq \| y \|\ .$$

□

The next corollary is one of many results and notes that we can observe from the last theorem.

Corollary 1.34. *[6] Every cone metric space (X, D) is metrizable, with metric defined by $d(x, y) = ||| D(x, y) |||$.*

But we must mention that this result in [6] has been disproved in [17] and authors gave the next result as a counter example to show that the main theorem in [6] does not hold.

Example 1.35. *[30] Let $E = C_R^2([0,1])$ with the norm $\| f \| = \| f \|_\infty + \| f' \|_\infty$ and consider the cone $P = \{f \in E : f \geq 0\}$ then P is a non-normal cone. Let $f(x) = x$ and $g(x) = x^2$, $\forall x \in [0,1]$. Then,*
clearly, $0 \leq g \leq f$. Further $\| f \| = \| f \|_\infty + \| f' \|_\infty = 1 + 1 = 2$

and, $\| g \| = \| g \|_\infty + \| g' \|_\infty = 1 + 2 = 3$ Since $f \in P$,

$$\begin{aligned} ||| f ||| &= \inf\{\| u \| : f \leq u\} + \inf\{\| v \| : v \leq f\} + \| f \| \\ &= \inf\{\| u \| : f \leq u\} + 0 + \| f \| \\ &= \| f \|_\infty + 2 = 1 + 2 = 3 \end{aligned}$$

Also, $g \in P$

$$\begin{aligned} ||| g ||| &= \inf\{\| u \| : g \leq u\} + \inf\{\| v \| : v \leq g\} + \| g \| \\ &= \inf\{\| u \| : g \leq u\} + 0 + \| g \| \\ &= \| g \|_\infty + 3 = 1 + 3 = 4 \end{aligned}$$

Which contradicts results in [6], as P is normal cone with normal constant $K = 1$, but $0 \leq g \leq f$ does not implies that
$||| g ||| \leq ||| f |||$.

After that there were many attempts to find out whether cone metric spaces is a real generalization of metric spaces or not.
During my research, A.A.Hakawati and H.D.Sarris [14] continue trying to straighten the path of the renorming process in [6], by converting every strongly minihedral normal cone to a normal cone with constant $k = 1$ by giving a new norm to the Banach space, and using the following lemmas.

Lemma 1.36. *[14] Suppose P is strongly minihedral cone in real Banach space E. Then : for $x, y \in E$; we have:*

i: $\inf\{w : w \geq x + y\} = \inf\{u : u \geq x\} + \inf\{v : v \geq y\}$

ii: $\sup\{w : w \leq x + y\} = \sup\{u : u \leq x\} + \sup\{v : v \leq y\}$

for proof see[14].

Lemma 1.37. *[14] For $0 \leq x \leq y$ we have:*

i: $\| \sup\{u : u \leq x\} \| \leq \| \sup\{u\prime : u\prime \leq y\} \|$

ii: $\| \inf\{v : x \leq v\} \| \leq \| \inf\{v\prime : y \leq v\prime\} \|$

for proof we refer the reader to [14].

Now, the authors of [14] gave their belief in the following theorem

Theorem 1.38. *[14] Let $(E, \| \, . \, \|)$ be a real Banach space with strongly minihedral normal cone P. Then there exist a norm $[.]$ on E with respect to P which is a normal cone with normal constant $k = 1$.*

Proof. Define : $[.] : E \to [0, \infty)$ by:
$[x] = \| \inf\{u : x \leq u\} \| + \| \sup\{v : v \leq x\} \|, \forall x \in E$. It is clear that, if $x = 0$ then $[x] = 0, \forall x \in E$.
If $[x] = 0 \Rightarrow \exists u_n, v_n \in E$ such that : $v_n \leq x \leq u_n$ where $u_n \to 0, v_n \to 0$ as $n \to \infty$.
Since P is a normal cone then we get $x = 0$. Therefor $[x] = 0 \Leftrightarrow x = 0$.
Now,

$$\begin{aligned}
[-x] &= \| \inf\{u : -x \leq u\} \| + \| \sup\{v : v \leq -x\} \| \\
&= \| -\sup\{-u : -x \leq u\} \| + \| -\inf\{-v : v \leq -x\} \| \\
&= \| -\sup\{u : -u \leq x\} \| + \| \inf\{-v : x \leq -v\} \| \\
&= \| \sup\{u : u \leq x\} \| + \| \inf\{v : x \leq v\} \| = [x]
\end{aligned}$$

For $\lambda > 0$,

$$
\begin{aligned}
[\lambda x] &= \| \inf\{u : \lambda x \leq u\} \| + \| \sup\{v : v \leq \lambda x\} \| \\
&= \| \inf\{\lambda(\frac{1}{\lambda}u) : x \leq \frac{1}{\lambda}u\} \| + \| \sup\{\lambda(\frac{1}{\lambda})v : (\frac{1}{\lambda})v \leq x\} \| \\
&= \lambda \| \inf\{\frac{1}{\lambda}u : x \leq \frac{1}{\lambda}u\} \| + \lambda \| \sup\{\frac{1}{\lambda}v : (\frac{1}{\lambda})v \leq x\} \| = \lambda[x].
\end{aligned}
$$

There for $[\lambda x] = \mid \lambda \mid [x]$ $\forall x \in E$, and $\lambda \in I\!R$
Now, to prove the triangle inequality using lemma1.36 ,

$$
\begin{aligned}
[x+y] &= \| \inf\{u : x+y \leq u\} \| + \| \sup\{v : v \leq x+y\} \| \\
&= \| \inf\{u : x \leq u\} \| + \| \inf\{u : u \leq y\} \| + \| \sup\{v : v \leq x\} + \| + \| \sup\{v : v \leq y\} \| \\
&\leq \| \inf\{u : x \leq u\} \| + \| \inf\{u : u \leq y\} \| + \| \sup\{v : v \leq x\} + \| + \| \sup\{v : v \leq y \| \\
&= \| \inf\{u : x \leq u\} \| + \| \sup\{v : v \leq x\} + \| + \| \inf\{u : u \leq y\} \| + \| \sup\{v : v \leq y\} \| \\
&= [x] + [y].
\end{aligned}
$$

Therefor $[x+y] \leq [x] + [y]$ and hence, $[.]$ is a norm on E.
Finally, with respect to this norm $[.]$ and by lemma1.37, P is a normal cone with normal constant $k = 1$. □

Chapter 2

Topological Cone metric spaces

As one of the results of studding metric spaces was considering certain topological groups in place of Banach spaces in definition of cone metric spaces, many authors introduced some generalized topological concepts and definitions in cone metric spaces, as well as authors in [38].
In This chapter, we will introduce some of these topological concepts and definitions, and define the distance between two sets in cone metric spaces. Moreover, we will prove many theorems in topological cone metric space.

2.1 Topological Concepts

Definition 2.1. *[39] Let $A \neq \phi$ and $B \neq \phi$ be two subsets of cone metric space (X, d). The distance between A and B, denoted by $d(A, B)$, is defined by :*

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

If $A = \{a\}$, we write $d(a, B)$ for $d(A, B)$.

Definition 2.2. *[39] Let (X, d) be a cone metric space.A subset U of X is called bounded above if there exists $c \in E$, $c \succeq 0$ such that $d(x, y) \leq c$ for all $x, y \in U$, and is called bounded if $\delta(U) = \sup\{d(x, y) : x, y \in U\}$ exists in E. If the supremum does not exists, we say that U is unbounded.*

Definition 2.3. *[28] Let Y be vector space:*

i: A subset A of Y is said to be (sequentially) open if $x_n \to x$ and $x \in A$ imply $x_n \in A$ for all but finitely many n.

ii: A subset A of Y is said to be (sequentially) closed if $x_n \to x$ and $x_n \in A$ for all n imply $x \in A$.

Lemma 2.4. *[28] Let Y be a vector space. Suppose U and V are nonempty subsets of Y. Then:*

i: If U is open and $\lambda > 0$, then $(\lambda U = \{\lambda u : u \in U\})$ is open.

ii: If U or V is open, then $(U + V = \{x + y : x \in U, y \in V\})$ is open.

Definition 2.5. *[28] Let A be a subset of a vector space Y. The interior of A (A^0) is called the biggest open subset contained in A, that is, $A^0 = \cup U$ where $\cup$ ranges through the family of all open subsets of Y contained in A.*

Definition 2.6. *Given a vector space V over a field $I\!R$, and the partial order $(\leq)$ over V, the pair $(V, \leq)$ is called an ordered vector space if, for all $x, y, z \in V$ and $0 \leq \lambda$ in $I\!R$ the following axioms hold:*

i: $x \leq y \Rightarrow x + z \leq y + z$.

ii: $y \leq x \Rightarrow \lambda y \leq \lambda x$.

Definition 2.7. *[28] An ordered topological space E over a filed $I\!F$ is a topological vector space, and ordered vector space over the field $I\!F$, such that the positive cone E^+ is a closed subset of E, the last statement means that, if a sequence of non-negative elements x_i of E converges to an element x, then x is non-negative.*

Definition 2.8. *[28] Let(X, d) be a cone metric space over an ordered vector space $(Y, \preceq)$. For a point $x_0 \in X$ and a vector $r \in Y$ with $r \succeq 0$, the set*

$$\bar{U}(x_0, r) = \{x \in X : d(x, x_0) \preceq r\}$$

is called a closed ball *with center x_0 and radius r.*

Definition 2.9. *[28] Let(X, d) be a cone metric space over an ordered vector space $(Y, \preceq)$. For a point $x_0 \in X$ and a vector $r \in Y$ with $r \succeq 0$, the set*

$$U(x_0, r) = \{x \in X : d(x, x_0) \prec r\}$$

is called an open ball *with center x_0 and radius r.*

Definition 2.10. *[28] Let X be a cone metric space.*

a: A subset A of X is called bounded if it is contained in some closed ball.

b: A sequence $\{x_n\}$ is called bounded if the set of its terms is bounded.

Lemma 2.11. *[38] Let (X, d) be a cone metric space with cone P and a real Banach space E.*
Then for each $c \in E$ with $c \succeq 0$ there is a real number $\epsilon > 0$ such that for any $x \in E$ with $\| x \| < \epsilon$, we have $x \preceq c$.

Proof. Since $c \succeq 0$ and $c \in P^0$. Then, we can find $\epsilon > 0$ such that:

$$\{x \in E : \| x - c \| < \epsilon\} \subset P^0.$$

Now, if $\| x \| < \epsilon$. Then

$$\| (c - x) - c \| = \| -x \| = \| x \| < \epsilon,$$

and then $(c - x) \in P^0$. □

Lemma 2.12. *[38] Let (X, d) be a cone metric space. Then for each $c_1 \succeq 0$ and $c_2 \succeq 0$, there is $c \succeq 0$ such that $c \succeq c_1$ and $c \succeq c_2$.*

Proof. Since $c_2 \succeq 0$, then by lemma 2.11, we can find $\epsilon > 0$ such that $\| x \| < \epsilon$ this implies that $x \preceq c_2$. Choose n_0 such that $\frac{1}{n_0} < \frac{\epsilon}{\|c_1\|}$. Take $c = \frac{c_1}{n_0}$. Then

$$\| c \| = \| \frac{c_1}{n_0} \| = \frac{\| c_1 \|}{n_0} < \epsilon.$$

Therefor, $c \preceq c_2$. But also it is clear that $c \succeq 0$ and $c \preceq c_1$ □

Lemma 2.13. *[23] Let (E, P) be an ordered topological vector space. Then the following hold.*

1: If $\alpha \succeq \theta$, then $r\alpha \succeq \theta$ for each $r \in \mathbb{R}^+$.

2: If $\alpha \succeq \theta$, then $\alpha \succeq \frac{\alpha}{2} \succeq ... \succeq \frac{\alpha}{n} \succeq ... \succeq \theta$.

3: If $\alpha_1 \succeq \beta_1$, and $\alpha_2 \succeq \beta_2$, then $\alpha_1 + \alpha_2 \succeq \beta_1 + \beta_2$.

4: If $\alpha \succeq \beta \succeq \gamma$ or $\alpha \geq \beta \geq \gamma$, then $\alpha \succeq \gamma$.

5: *If* $\alpha \succeq \theta$ *and* $\beta \in E$*, then there is* $n \in \mathbb{N}$ *such that* $\frac{\beta}{n} \preceq \alpha$*.*

6: *If* $\alpha \succeq \theta$ *and* $\beta \succeq \theta$*, then there is* $\gamma \succeq \theta$ *such that* $\gamma \preceq \alpha$ *and* $\gamma \preceq \beta$*.*

7: *If* $\epsilon \succeq \theta$ *and* $\theta \preceq \alpha \preceq \frac{\epsilon}{n}$ *for each* $n \in \mathbb{N}$*, then* $\alpha = \theta$*.*

For proof see [23]

Definition 2.14. *A topological space* X *is said to be compact if for each open covering* $\{U_\alpha\}_{\alpha \in I}$ *of* X *there is a finite subcovering* $\{U_\beta\}_{\beta \in I}$*.*

Definition 2.15. *[13] Let* (X, d) *be a cone metric space. If for any sequence* $\{x_n \in X$*, there is a subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$ *such that* $\{x_n\}$ *is convergent in* X*. Then* X *is called a sequentially compact cone metric space.*

Definition 2.16. *A topological space* X *is Hausdorff (i.e* T_2*-space) if for any* $x, y \in X$*, with* $x \neq y$*, there exist open sets* U *containing* x *and* V *containing* y *such that* $U \cap V = \phi$*.*

Definition 2.17. *A first countable space is a topological space in which there exist a countable local base at each of its points.*

Definition 2.18. *A subset* E *of a topological space* X *is said to be second category in* X *if* E *cannot be written as the countable union of subsets which are nowhere dense (i.e* $Int(\overline{U}_i = \phi)$ *in* X*.*

Definition 2.19. *An isomorphism is a one-to-one correspondence between two sets, especially: a homomorphism that is one-to-one.*

2.2 TVS-Cone Metric Spaces

Definition 2.20. *[12] Let* E *be a topological vector space with its zero vector* θ*. A subset* P *of* E *is called a TVS-cone in* E *if the following are satisfied:*

i: P *is closed in* E *with a nonempty interior* P^0*.*

ii: α*,* $\beta \in P$ *and* $a, b \in \mathbb{R}^* \Rightarrow a\alpha + b\beta \in P$*.(where* $\mathbb{R}^*$ *is the nonnegative real numbers).*

iii: α*,* $-\alpha \in P \Rightarrow \alpha = \theta$*.*

Remark 2.21. *[12] It is clear from definition 2.20 that $\theta \in P$, but θ does not belong to P^0.*
In fact, pick $\alpha \in E - \{\theta\}$. Then, $\{\frac{1}{n}\alpha\} \to \theta$ and $\{-\frac{1}{n}\alpha\} \to \theta$ when $n \to \infty$. If $\theta \in P^0$, then there is $n \in \mathbb{N}$ such that $\{\frac{1}{n}\alpha, -(\frac{1}{n}\alpha)\} \subseteq P^0 \subseteq P$. By definition 2.20 (iii), we have $(\frac{1}{n})\alpha = \theta$. This contradicts that $\alpha \neq \theta$. So, θ does not belong to P^0.[12]

Now, we define some partial orderings as follows.

Definition 2.22. *[23] Let P be a TVS-cone in a topological vector space E. Some partial orderings $\leq$, $<$, and $\preceq$ on E with respect to P are defined as follows. For each $\alpha, \beta \in E$,*

- *i: $\alpha \leq \beta$ if $\beta - \alpha \in P$.*
- *ii: $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.*
- *iii: $\alpha \preceq \beta$ if $\beta - \alpha \in P^0$.*

Then, a pair (E, P) is called an ordered topological vector space.

For an ordered topological vector space (E, P), unless otherwise specified, we always suppose that E is a topological vector space with the zero vector θ and P is a TVS-cone in E with nonempty interior P^0.

Definition 2.23. *[27] Let X be a nonempty set. And $d : X \times X \to E$ be a mapping that satisfies :*

i: $d(x, y) \succeq \theta$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.

ii: $d(x, y) = d(y, x)$ for all $x, y \in X$.

iii: $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a topological vector space valued cone metric (TVS-cone metric for short)on X, and (X, d) is said to be a topological vector space valued cone metric space.

Note E in definition 2.23 considered as usual a real Banach space. Thus, a cone metric space in sense of Huang and Zhang [13] is a special case of TVS-cone metric space.[2]

Now, we discuss the relation between cone metric spaces and topological spaces.

Theorem 2.24. *[23] Let(X, d) be a TVS-cone metric space.*
Put $\beta = \{B(x, \epsilon) : x \in X; \epsilon \succeq 0\}$ then β is a base for some topology on X.

Proof. It is clear that $X = \cup\, \beta$. Let $B(x, \alpha)$, $B(y, \beta) \in \beta$ and $z \in B(x, \alpha) \cap B(y, \beta)$.
Since $z \in B(x, \alpha), d(x, z) \preceq \alpha$.
Put $\gamma_1 = \alpha - d(x, z)$; then $\gamma_1 \succeq 0$. We claim that $B(z, \gamma_1) \subseteq B(x, \alpha)$.
In fact, if $u \in B(z, \gamma_1)$, then $d(z, u) \preceq \gamma_1$, hence

$$d(x, u) \leq d(x, z) + d(z, u) \preceq d(x, z) + \gamma_1 = \alpha,$$

and so $u \in B(x, \alpha)$. Using the same way, we can obtain that there is $\gamma_2 \succeq 0$ such that $B(z, \gamma_2) \subseteq B(y, \beta)$. By lemma 2.13(6), there is $\gamma \succeq 0$ such that $\gamma \preceq \gamma_1$ and $\gamma \preceq \gamma_2$. Let $v \in B(z, \gamma)$; then $d(z, v) \preceq \gamma \preceq \gamma_1$ and $d(z, v) \preceq \gamma \preceq \gamma_2$, so

$$v \in B(z, \gamma_1) \cap B(z, \gamma_2) \subseteq B(x, \alpha) \cap B(y, \beta).$$

This proves that $B(z, \gamma) \subseteq B(x, \alpha) \cap B(y, \beta)$. Note that $x \in B(z, \gamma) \in\!\beta$.
Consequently,β is a base for topology on X. In fact, put $\tau = \{U \subseteq X : \exists \beta\prime\!\subseteq \beta\}$ $:\!U = \cup\beta\prime$; then τ is a topology on X and β is a base for τ.

□

Theorem 2.25. *[38] Every cone metric space (X, d) is a topological space.*

Proof. For $c \in E$, $c \succeq 0$, let $B(x, c) = \{y \in X : d(x, y) \preceq c\}$ and $\beta = \{B(x, c) : x \in X, c \succeq 0\}$ so,$T_c = \{U \subset X : \forall x \in U, \exists B \in \beta, x \in B \subset U\}$ is a topology on X. In fact, we have:

i: ϕ, $X \in T_c$.

ii: Let $U, V \in T_c$ and let $x \in U \cap V$. Then, $x \in U$ and $x \in V$, find $c_1 \succeq 0, c_2 \succeq 0$ such that $x \in B(x, c_1) \subset U$ and $x \in B(x, c_2) \subset V$, by lemma 2.12 find $c \succeq 0$ such that $c \preceq c_1$ and $c \preceq c_2$. Then, clearly

$$x \in B(x, c) \subset B(x, c_1) \cap B(x, c_2) \subset U \cap V.$$

Hence, $U \cap V \in T_c$.

iii: Let $U_\alpha \in T_c$ for each $\alpha \in \Gamma$ and let $x \in \cup_{\alpha \in \Gamma} U_\alpha$. Then, $\exists \alpha_0 \in \Gamma$ such that $X \in U_{\alpha_0}$. Hence, find $c \succeq 0$ such that $x \in B(x, c) \subset U_\alpha \subset \cup_{\alpha \in \Gamma} U_\alpha$.
That is, $\cup_{\alpha \in \Gamma} U_\alpha \in T_c$.

$\square$

2.3 Some Theorems on TVS-Cone Metric Spaces

Here some theorems that have been proved ,and some definitions about TVS-cone spaces are presented.

Theorem 2.26. *[3] Every cone metric space (X, d) is a Hausdorff space. (i.e T_2 space).*

Proof. Let $x, y \in X$ such that $x \neq y$. Then, $d(x, y) = c$ where $c \succeq 0$, so that $[B(x, \frac{c}{2}) \cap B(y, \frac{c}{2})] \in T_c$ and $B(x, \frac{c}{2}) \cap B(y, \frac{c}{2}) = \phi$. □

Theorem 2.27. *[6] Every cone metric space (X, d) is a first countable.*

Proof. Let $q \in X$ and fix $c \succeq 0$ where $c \in E$. We see that

$$\beta_q = \{B(q, \frac{c}{n}) : n \in I\!N\}$$

is local base at q. Let U be open with $q \in U$.
Find $c_1 \succeq 0$ such that $q \in B(q, c_1) \subset U$, also by 2.13(2) , find $n_0 \in I\!N$ such that $\frac{c}{n_0} \preceq c_1$. Therefor,

$$B(q, \frac{c}{n_0}) \subset B(q, c_1) \subset U.$$

□

Corollary 2.28. *[2] A mapping from TVS-cone metric space to an arbitrary topological space is continuous if and only if it is sequentially continuous.*

Corollary 2.29. *[2] Every TVS-cone metric space (X, P) is topologically isomorphic to its correspondent metric space (X, d_p)*

Note that corollary 2.29 explains many topological properties of TVS-cone metrics, such as:

Corollary 2.30. *[2] A TVS-cone metric space is compact if and only if it is sequentially compact.*

As compactness and sequentially compactness are topological properties .

Corollary 2.31. *[2] Every TVS-cone metric is second category.*

Also, second category is topological property.

Theorem 2.32. *[39] Let (X, d) be a cone metric space and $A(\neq \phi)$ a subset of X. Then, $x \in \bar{A}$ if and only if $d(x, A) = 0$.*

Proof. Suppose $x \in \bar{A}$. Then, for fixed $c \succeq 0$ and each $n \in I\!N$, we have $B(x, \frac{c}{n}) \cap A \neq \phi$. Therefor, for each $n \in I\!N$ there exist $a_n \in A$ such that

$$0 \leq d(x, A) \leq d(x, a_n) < \frac{c}{n}.$$

Hence, $\| d(x, A) \| \leq k(\frac{\|c\|}{n})$, for all $n \in I\!N$. Our conclusion then is that $d(x, A) = 0$. Conversely, let $U \in \tau_c$ be open in (X, d) such that $x \in U$, then find $c \succeq 0$ such that $B(x, c) \subset U$. But, since $0 = d(x, A) < c$, find $a \in A$ such that $d(x, a) < c$. That is $a \in A \cap B(x, c) \subset A \cap U$. $\square$

Lemma 2.33. *[39] Let $c_1, c_2 \in P$ such that $c_1 < c_2 + s$ for all $s \succeq 0$. Then, $c_1 \leq c_2$.*

Theorem 2.34. *[39] Every cone metric space is a T_4-space.*

Proof. We first show that (X, d) is a Hausdorff space and hence is a T_1-space. Let $x \neq y$ be two points in X. Then $d(x, y) = c > 0$ so that $B(x, \frac{c}{2}) \cap B(y, \frac{c}{2}) = \phi$, implying (X, d) is Hausdorff space. To show that X is normal, let A and B be two closed disjoint subsets of X and define:
$U = \{x \in X : d(x, A) < d(x, B)\}$ and $V = \{x \in X : d(x, A) > d(x, B)\}$.
From the definition of U and V, $U \cap V = \phi$. Furthermore, if $a \in A$, then $d(a, A) = 0$, $a \in B$ and since B is closed, $d(a, B) > 0$. According to 2.32, $0 = d(a, A) < d(a, B)$, so that $a \in U$, It follows that $A \subset U$. Similarly, $B \subset V$.
Now, if we can show that U and V are open we will done. To show that U is open, let $x_0 \in U$ then $c_1 = d(x_0, A) < d(x_0, B) = c_2$.
Since, $(c_2 - c_1) > 0$ $(i.e(c_2 - c_1) \in P, c_2 \neq c_1)$,
we may define $c = \frac{1}{2}(c_2 - c_1)$ and consider the basic open $B(x_0, \frac{c}{2})$.
Let $x \in B(x_0, \frac{c}{2})$. Then, for each $s \succeq 0$, by the definition of $d(x_0, A)$, there exists $a \in A$ such that $d(x_0, a) < c_1 + s$. Therefor,

$$d(x, A) \leq d(x, a) \leq d(x, x_0) + d(x_0, a) < \frac{c}{2} + c_1 + s = (\frac{c}{2} + c_1) + s.$$

Then, by 2.33, it follows that $d(x, A) \leq \frac{c}{2} + c_1 = \frac{1}{4}(c_2 + 3c_1)$. Also, for every $b \in B$, we have $d(b, x_0) \leq d(b, x) + d(x, x_0)$ and, since $d(b, x_0) \geq d(x_0, B)$ and $d(x, x_0) < \frac{c}{2}$, we may write $d(b, x) + \frac{c}{2} > d(x_0, B) = c_2$. Thus, $d(b, x) > c_2 - \frac{c}{2} = \frac{1}{4}(3c_2 + c_1)$. Then, by noting that $c_2 + 3c_1 < 3c_2 + c_1$ we conclude that $d(x, A) < d(x, B)$. That is, $x \in U$ and hence U is open. The same reasoning shows V is also an open subset of X. $\square$

2.4 TVS-Cone Normed Spaces

It is clear that each cone normed space is a cone metric space. With the cone metric is given by $d(x,y) = \| x-y \|_c$.

Definition 2.35. *[27] Let X be a linear space over a field $I\!F$, a norm on X is a function:*
$\| \,.\, \| : X \to I\!R$, *s.t* $\forall x,y \in X$ *and* $a \in I\!F$, *we have:*

i: $\| x \| \geq 0 and \| x \| = 0 \Leftrightarrow x = 0$.

ii: $\| ax \| = | a | \| x \|$.

iii: $\| x+y \| \leq \| x \| + \| y \|$.

$(X, \| \,.\, \|)$ *is called normed linear space.*

Definition 2.36. *A cone normed space $(X, \| \,.\, \|_c)$ is called cone Banach space if every Cauchy sequence in X convergent in X.*

Proposition 2.37. *[22]*

i: A norm is a real valued continuous function .

ii: Every normed space is a metric space with respect to the metric $d(x,y) = \| x-y \|$, and is called metric induced by the norm.

iii: For any two elements x and y of a normed space we have,

$$|\| x \| - \| y \|| \leq \| x-y \|$$

For the proof we curious the readers to [22].

Definition 2.38. *[27] Let X be a vector space, where $\| \,.\, \|_{c_1} : X \to E$ and $\| \,.\, \|_{c_2} : X \to E$ be two TVS- cone norms on X.*
then $\| \,.\, \|_{c_1}$ is said to be equivalent to $\| \,.\, \|_{c_2}$ if there exist $\alpha, \beta > 0$ such that :

$$\alpha \| x \|_{c_1} \preceq \| x \|_{c_2} \preceq \beta \| x \|_{c_1}$$

for each $x \in X$.

Theorem 2.39. *[27] Let X be a vector space. if $\| \, . \, \|_{c_1}$ and $\| \, . \, \|_{c_2}$ are two equivalent TVS-cone norms on X. Then $\tau_{c_1} = \tau_{c_2}$. Moreover, the converse is valid if all elements of P are comparable. (i.e for all $c_1, c_2 \in P, c_1 \preceq c_2$ or $c_2 \preceq c_1$).*

Proof. Fixe $\in P^0$ and suppose that $\| \, . \, \|_i = \xi_e(\| \, . \, \|_{c_i}), i = 1, 2$. We know there exist $\alpha, \beta > 0$ (where ξ_e is nonlinear scalarization function defined as
$\xi_e(y) = \inf\{e \in I\!R : y \in re - P\}, \forall y \in E$ such that $\alpha \| x \|_{c_1} \leq \| x \|_{c_2} \leq \beta \| x \|_{c_1}$, for each $x \in X$, also as (If $x \in y + P$, then $\xi_y \leq \xi_x$) (see lemma 1.2 [27]), ξ_e is an increasing function on E, thus:

$$\alpha \| x \|_1 \preceq \| x \|_2 \preceq \beta \| x \|_1$$

for each $x \in X$. Hence $\| \, . \, \|_1$ and $\| \, . \, \|_2$ are equivalent norms on X, so they induce same topology on X. On the other hand, $\| \, . \, \|_i$ induces $\tau_{c_i}, i = 1, 2$. Therefor, $\tau_{c_1} = \tau_{c_2}$.
Conversely, let $\tau_{c_1} = \tau_{c_2}$, then $\| \, . \, \|_1$ and $\|, \|_2$ are equivalent norms on X. Therefor, there exist scalers $\alpha, \beta > 0$ such that :

$$\alpha \| x \|_1 \leq \| x \|_2 \leq \beta \| x \|_1,$$

for each $x \in X$. So we have:

$$\alpha \xi_e(\| x \|_1) \leq \xi_e(\| x \|_2) \leq \beta \xi_e(\| x \|_1)$$

for each $x \in X$. On the other hand, the elements of P are comparable with each other and ξ_e is increasing on E, hence:

$$\alpha \| x \|_{c_1} \preceq \| x \|_{c_2} \preceq \beta \| x \|_{c_1}$$

for each $x \in X$. $\square$

Definition 2.40. *[27] Let $(X, \| \, . \, \|_c)$ and $(Y, \| \, . \, \|_c)$ be two TVS-cone normed spaces and T be a linear map from X into Y. T is called a cone bounded linear map, if there exists $M > 0$ such that
$\| Tx \|_c \preceq M \| x \|_c$ for all $x \in X$. We denote by $||| T |||$ the infimum of such M, i.e, $||| T ||| = \inf\{M > 0 : \| Tx \|_c \preceq M \| x \|_c\}$.*

Example 2.41. *[27] Let $E = \ell^1$ and $P = \{\{x_n\} \in \ell^1 : x_n \geq 0, \forall n\}$. Then P is a cone in E. Put $X = C^1[0,1]$ and $Y = C[0,1]$. Moreover, define $\| \,.\, \|_{c_1}: X \to E$ and $\| \,.\, \|_{c_2}: Y \to E$ as follows:*
$\| f \|_{c_1} = \{\frac{\|f\|_1}{2^n}\}_{n=1}^{\infty}$ and $\| g \|_{c_2} = \{\frac{\|g\|_2}{2^n}\}_{n=1}^{\infty}$
where $\| f \|_1 = \| f \|_\infty + \| f' \|_\infty$ and $\| g \|_2 = \| g \|_\infty$ for all $f \in X$ and $g \in Y$. Obviously, $\| \,.\, \|_{c_1}$ and $\| \,.\, \|_{c_2}$ are two cone norms on X and Y respectively. Now define $T : (X, \| \,.\, \|_{c_1}) \to (Y, \| \,.\, \|_{c_2})$ by $Tf = f'$. Therefor, $\| Tf \|_{c_2} \preceq \| f \|_{c_1}$ implies that T is a cone bounded linear map.

Theorem 2.42. *[27] (Open mapping theorem) Let $(X, \| \,.\, \|_{c_1})$ and $(Y, \| \,.\, \|_{c_2})$ be two complete TVS-cone normed spaces and $Y : X \to Y$ be a surjective cone-bounded linear map, then T is open mapping (i.e $T(G)$ is an open set in (Y, τ_c) whenever G is an open set in (X, τ_c))*

For proof we refer the reader to [20]

Theorem 2.43. *[27] (The inverse mapping theorem) If X and Y are two cone Banach spaces and $T : X \to Y$ is a bijective cone-bounded linear map, then $T^{-1} : Y \to X$ is continuous.*

For proof see [20]

Theorem 2.44. *[27] (The closed graph theorem) If X and Y are two complete TVS-cone normed spaces and $T : X \to Y$ is a linear map such that the graph of T is*

$$Gr(T) = \{(x, Tx) \in X \times Y : x \in X\}$$

is closed, then T is continuous.

For proof [20]

Theorem 2.45. *[27] Suppose that $(X, \| \,.\, \|_c)$ is a cone normed space and τ_c is the cone topology on X. Define $f : X \to E$ by $f(x) = \| x \|_c$, then f is $(\tau_c, \| \,.\, \|)$ continuous.*

Proof. Let $\{x_n\} \subseteq X, x \in X$ and $\| x_n - x \|_c \to \theta$ as $n \to \infty$. Then by the triangle inequality we have:

$$- \| x_n - x \|_c \preceq \| x_n \|_c - \| x \|_c \preceq \| x_n - x_n \|_c .$$

It follows from the sandwich theorem that $\lim_{n\to\infty} \| x_n \|_c = \| x \|_c$ in E. $\square$

2.5 Minkowski functional on solid vector space and TVS-cone metrics

We will continue with properties of TVS-cone metric spaces, with functional view.

Definition 2.46. *[28] Let X be a real vector space and $A \subset X$ such that for all $x \in X$: there exist $\lambda > 0$ such that $x \in \lambda A$ (called absorbing). Then the functional $\| \, . \, \|: X \to I\!R$ defined by:*
$\| x \| = \inf\{\lambda \geq 0 : x \in \lambda A\}$, is called Minkowski functional of A.

Lemma 2.47. *[28] Let X be a solid vector space ($X^0 \neq \phi$). Let $\| \, . \, \|: X \to I\!R$ be the Minkowski functional of $[-b, b]$ for some vector $b \in X$ with $b \succ 0$. Then:*

i: $\| \, . \, \|$ is a monotone norm on X which can be defined as:

$$\| x \| = \min\{\lambda \geq 0 : -\lambda b \leq x \leq \lambda b\}.$$

ii: For $x \in X$ and $\epsilon > 0$,

$$\| x \| < \epsilon \Leftrightarrow -\epsilon b < x < \epsilon b.$$

For proof we refer the reader to [28].

Theorem 2.48. *[28] Let (X, d) be a cone metric space over a solid vector space $(Y, \preceq)$. Suppose $\| \, . \, \|: Y \to I\!R$ is the Minkowski functional of $[-b, b]$ for some $b \in Y$ with $b > 0$. Then :*

i: The metric $\rho : X \times X \to I\!R$ defined by $\rho(x, y) = \| d(x, y) \|$ generates the cone metric topology on X.

ii The cone metric space (X, d) is complete if and only if the metric space (X, ρ) is complete.

iii For $x_i, y_i \in X$ and $\lambda_i \in I\!R$ $(i = 0, 1, ..., n)$,
$d(x_0, y_0) \preceq \lambda_0 + (\sum_{i=1})^n \lambda_i d(x_i, y_i)$ implies
$\rho(x_0, y_0) \leq \| \lambda_0 \| + (\sum_{i=1})^n \lambda_i \rho(x_i, y_i)$.

Proof. i: It follows from (lemma 2.47 (i), and the definition of cone metric spaces) that ρ is a metric on X. Denoting by $B(x, \epsilon)$ an open ball in (X, ρ) and by

$U(x,c)$ an open ball in (X,d), we shall prove that each $B(x,\epsilon)$ contains some $U(x,c)$ and vice versa. First, we shall show that

$$B(x,\epsilon) = U(x,\epsilon b), \forall x \in X, \epsilon > 0 \tag{2.1}$$

$B(x,\epsilon) = U(x,\epsilon b)$, for all $x \in X$ and $\epsilon > 0$.
According to 2.47(ii), for all $x, y \in X$ and $\epsilon > 0$,

$$\| d(x,y) \| < \epsilon \Leftrightarrow d(x,y) \prec \epsilon b,$$

that is,

$$\rho(x,y) < \epsilon \Leftrightarrow d(x,y) \prec \epsilon b \tag{2.2}$$

which proves 2.1. Note that identity 2.1 means that every open ball in the metric space (X,ρ) is an open ball in the cone metric space (X,d).
Now, let $U(x,c)$ be an arbitrary open ball in (X,d). Choosing $\epsilon > 0$ such that $\epsilon b \prec c$, we conclude by 2.1 that $B(x,\epsilon) \subset U(x,c)$.

ii: Let $\{x_n\}$ be a sequence in X. We have to prove that $\{x_n\}$ is a d-Cauchy if and only if it is a ρ-Cauchy.
First note that 2.2 implies that for each $\epsilon > 0$ and all $m,n > N$.

$$\rho(x_n, x_m) < \epsilon \Leftrightarrow d(x_n, x_m) \prec \epsilon b.$$

Let $\{x_n\}$ be a d-Cauchy and $\epsilon > 0$ be fixed.Then there is an integer N such that $d(x_n, x_m) \prec \epsilon b$ for all $m, n > N$. Hence, $\rho(x_n, x_m) < \epsilon$ for all $m, n > N$ which means that $\{x_n\}$ is a ρ-Cauchy.
Now, let $\{x_n\}$ be a ρ-Cauchy and $c \succ 0$ be fixed, Choose $\epsilon > o$ such that $\epsilon b \prec c$. Then there is an integer N such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$. Therefor, for these n and m we get $d(x_n, x_m) \prec \epsilon b \prec c$ which means that $\{x_n\}$ is a d-Cauchy.

iii: Follows from the monotony of the norm $\| \,.\, \|$ and the definition of the metric ρ.

□

Note that 2.1 holds also for closed ball in the spaces (X,ρ) and (X,d). The main idea of theorem 2.48 formulated in the next theorem.

Theorem 2.49. *[28] Let (X, d) be a cone metric space over a solid vector space $(Y, \preceq)$. Then there exist a metric ρ on X such that the following statements hold true:*

i: The metric ρ generates the cone metric topology on X.

ii The cone metric space (X, d) is complete if and only if the metric space (X, ρ) is complete.

iii For $x_i, y_i \in X$ and $\lambda_i \in I\!R$ $(i = 0, 1, ..., n)$,
$d(x_0, y_0) \preceq (\sum_{i=1})^n \lambda_i d(x_i, y_i)$ implies $\rho(x_0, y_0) \leq (\sum_{i=1})^n \lambda_i \rho(x_i, y_i)$.

We proved that every cone metric space is T_2 space and first countable space, and as every first countable space is sequential space(see [35]), that leads to, every cone metric space is sequential space. The following corollary is a consequence of 2.49 :

Corollary 2.50. *[28] Let (X, d) be a cone metric space over a solid vector space Y. Then the following statements hold true.*

i: A subset of X is open if and only if it is sequentially open.

ii: A subset of X closed if and only if it is sequentially closed.

iii: A function $f : D \subset X \times X$ is continuous if and only if it is sequentially continuous.

Chapter 3

Fixed Point Theorems in Cone Metric Spaces

Fixed point theory occupies major place in cone metric space studies. Huang and Zhang in [13] obtained various fixed point theorems for contractive single-valued maps in cone metric spaces, also they studied the existence and uniqueness of the fixed point. Subsequently, many mathematicians investigate more results about fixed point theory.
In this chapter we will present some important results of fixed point theory, some examples, common fixed point theorems in cone metric spaces, and finally some coupled fixed point theorems in cone metric spaces.

3.1 Some Concepts of Fixed Point Theory

Definition 3.1. *[10] Let (X,d) be a cone metric space and $T,S : X \to X$ two functions. The mapping S is said to be T-contractive if for each $x,y \in X$ such that $Tx \neq Ty$ then: $d(TSx,TSy) < d(x,y)$.*

It is clear from the last definition that every T-contraction function is T-contractive, and its clear from the next example that the converse is not true.

Example 3.2. *: [10]*

1: *Let $E = (C_{[0,1]}, I\!R)$, $P = \{\gamma \in E : \gamma \geq 0\} \subset E$, $X = [1,+\infty)$, and $d : X \times X \to E$ defined by $d(x,y) = \mid x-y \mid e^t$, where $e^t \in E$. Then (X,d) is a cone metric space. Let $T,S : X \to X$ be two functions defined by $Sx = \sqrt{x}$ and $Tx = x$. Then:*

i: S is a T-contractive function.

ii: S is not a T-contraction mapping.

2: Let $E = (C_{[0,1]}, I\!R)$, $P = \{\gamma \in E : \gamma \geq 0\} \subset E$, $X = [0, \frac{1}{2}]$, and $d : X \times X \to E$ defined by $d(x, y) =\mid x - y \mid e^t$, where $e^t \in E$. Obviously (X, d) is a cone metric space and the function $S : X \to X$ is defined by $Sx = \frac{x^2}{\sqrt{2}}$ is not contractive. If $T : X \to X$ is defined by $Tx = x^2$, then S is a T-contractive, because

$$d(TSx, TSy) =\mid TSx - TSy \mid e^t =\mid \frac{x^4}{2} - \frac{y^4}{2} \mid e^t$$

$$\frac{1}{2} \mid x^2 + y^2 \mid\mid Tx - Ty \mid e^t$$

$$<\mid Tx - Ty \mid e^t = d(Tx, Ty).$$

Definition 3.3. *A vector valued function is a function where the domain is the set of real numbers and the range is a set of vectors.*

Definition 3.4. *f is said to be an alternating function if it changes the sign whenever two arguments are changed.*

3.2 Fixed Point Theorems of Contractive Mappings in Cone Metric Spaces

In this section we will study some fixed point theorems of contractive mappings in cone metric spaces.
First, here is some definitions that we need in this section.

Definition 3.5. *[10] Let (X, d) be a cone metric space and $T, S : X \to X$ two functions. A mapping S is said to be a T-contraction if there is a constant $\alpha \in [0, 1)$ such that:*
$d(TSx, TSy) \leq \alpha d(Tx, Ty)$, for all $x, y \in X$.

One of the very popular tools of a fixed point theory is the *Banach contraction principle* which first appeared in 1922 see [37]. It states that: **if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, then T has a unique fixed point.**

Now, we will start with the first theorems that generalized some fixed point theorems in metric spaces.

Theorem 3.6. *[13] Let (X, d) be a complete cone metric space, P be a normal cone with normal constant k. Suppose the mapping $T : X \to X$ satisfies the contractive condition:*
$d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point in X. And for any $x \in X$ iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, ...$
We have,

$$d(x_{n+1}, x_n) = d(Tx_n, T_{n-1}) \leq kd(x_n, x_{n-1}) \leq k^2 d(x_{n-1}, x_{n-2}) \leq ... \leq k^n d(x_1, x_0).$$

So, for $n > m$

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m)$$
$$\leq (k^{n-1} + k^{n-2} + ... + k^m)d(x_1, x_0) \leq \frac{k^m}{1-k} d(x_1, x_0).$$

We get $\| d(x_n, x_m) \| \leq \frac{k^m}{1-k} K \| d(x_1, x_0) \|$.
This implies $d(x_n, x_m) \to 0$ as $(n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the compactness of X, there is $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since

$$d(Tx^*, x^*) \leq d(Tx_n, Tx^*) + d(Tx_n, x^*) \leq kd(x_n, x^*) + d(x_{n+1}, x^*).$$

$$\| d(Tx^*, x^*) \| \leq K(k \| d(x_n, x^*) \| + \| d(x_{n+1}, x^*) \|) \to \infty.$$

Hence, $d(Tx^*, x^*) = 0$. This implies $Tx^* = x^*$. So x^* is a fixed point of T.
Now, for uniqueness, if y^* is another fixed point of T, then
$d(x^*, y^*) = d(Tx^*, Ty^*) \leq kd(x^*, y^*)$. Hence $\| d(x^*, y^*) \| = 0$ and $x^* = y^*$. Therefor the fixed point of T is unique. $\square$

Corollary 3.7. *[13] Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K. For $c \in E$ with $0 \ll c$ and $x_0 \in X$,*
set $B(x_0, c) = \{x \in X : d(x_0, x) \leq c\}$.
Suppose the mapping $T : X \to X$ satisfies the contractive condition

$d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in B(x_0, c)$ where $k \in [0, 1)$
and $d(Tx_0, x_0) \leq (1 - k)c$. Then T has a unique fixed point.

Proof. For proof we only need to prove that $B(x_0, c)$ is complete and $Tx \in B(x_0, c)$ for all $x \in B(x_0, c)$. Suppose $\{x_n\}$ is a Cauchy sequence in $B(x_0, c)$. Then $\{x_n\}$ is also a Cauchy sequence in X. By the completeness of X, there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. We have

$$d(x_0, x) \leq d(x_n, x_0) + d(x_n, x) \leq d(x_n, x) + c.$$

Since $x_n \to x, d(x_n, x) \to 0$. Hence $d(x_0, x) \leq c$, and $x \in B(x_0, c)$. Therefor $B(x_0, c)$ is complete.
For every $x \in B(x_0, c)$

$$d(x_0, Tx) \leq d(Tx_0, x_0) + d(Tx_0, Tx) \leq (1-k)c + kd(x_0, x) \leq (1-k)c + kc = c.$$

Hence $Tx \in B(x_0, c)$. $\square$

The next two theorems also generalized another fixed point theorem of metric spaces.

Theorem 3.8. *[13] Let (X, d) be a sequentially compact cone metric space, P be a regular cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$ and $x \neq y$.*
Then T has a unique fixed point in X.

For proof see [13]

Theorem 3.9. *[13] Let (X, d) be a complete cone metric space , P a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq k(d(Tx, x) + d(Ty, y))$, for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is constant. Then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the fixed point.*

For proof see [13].

Theorem 3.10. *[13] Let (X, d) be a complete cone metric space , P a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition $d(Tx, Ty) \leq k(d(Tx, y) + d(x, Ty))$, for all $x, y \in X$, where $k \in [0, \frac{1}{2})$ is constant. Then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

For proof see [13].

Example 3.11. *: [13] Let $E = I\!R^2$, the Euclidean plane, and*
$P = \{(x, y) \in I\!R^2 : x, y \geq 0\}$ a normal cone.
Let $X\{(x, 0) \in I\!R^2 : 0 \leq x \leq 1\} \cup \{(0, x) \in I\!R^2 : 0 \leq x \leq 1\}$.
The mapping $d : X \times X \to E$ is defined by :

$$d((x, 0), (y, 0)) = (\frac{4}{3} \mid x - y \mid, \mid x - y \mid),$$

$$d((0, x), (0, y)) = (\mid x - y \mid, \frac{2}{3} \mid x - y \mid),$$

$$d((x, 0), (0, y)) = d((0, y), (x, 0)) = (\frac{4}{3}x + y, x + \frac{2}{3}y).$$

Then (X, d) is a complete cone metric space.
Let mapping $T : X \to X$ with $T((x, 0)) = (0, x)$ and $T((0, x)) = (\frac{1}{2}x, 0)$. Then T

satisfies the contractive condition:
$d(T((x_1, x_2)), T((y_1, y_2))) \leq kd((x_1, x_2), (y_1, y_2))$, *for all* $(x_1, x_2), (y_1, y_2) \in X$, *with the constant* $k = \frac{3}{4} \in [0, 1)$. *It is obvious that* T *has a unique fixed point* $(0, 0) \in X$. *On the other hand, we see that* T *is not a contractive mapping in the Euclidean metric on* X.

Theorem 3.12. *[30] Let* (X, d) *be a complete cone metric space and the mapping* $T : X \to X$ *satisfy the contractive condition*
$d(Tx, Ty) \leq kd(x, y) + ld(y, Tx)$, *for all* $x, y \in X$, *where* $k, l \in [0, 1)$ *are constants. Then* T *has a fixed point in* X. *Also, the fixed point of* T *is unique whenever* $k+l < 1$.

Proof. For each $x_0 \in X$ and $n \geq 1$, set $x_1 = Tx_0$ and $x_{n+1} = Tx_n = T^{n+1}x_0$. Then, $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq k(d(x_n, x_{n-1}) + d(Tx_{n-1}, x_n)) = kd(x_n, x_{n-1}) \leq k^n d(x_1, x_0)$.
Thus for $n > m$, we have :
$d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \leq (k^{n-1} + k^{n-2} + ... + k^m)d(x_1, x_0) \leq \frac{k^m}{1-k}d(x_1, x_0)$. Let $0 \ll c$ be given. Choose a natural number N_1 such that $\frac{k^m}{1-k}d(x_1, x_0) \ll c$ for all $m \geq N_1$.
Thus ,$d(x_n, x_m) \ll c$ for $n > m$. Therefor, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, d). Since (X, d) is complete cone metric space, there exists $x^* \in X$ such that $x_n \to x^*$. Choose a natural number N_2 such that $d(x_n, x^*) \ll \frac{c}{3}$, for all $n \geq N_2$ we have :
$d(Tx^*, x^*) \leq d(x_n, Tx^*) + d(x_n, x^*) = d(Tx_{n-1}, Tx^*) + d(x_n, x^*) \leq kd(x_{n-1}, x^*) + ld(Tx_{n-1}, x^*) + d(x_n, x^*)$
$\leq d(x_{n-1}, x^*) + d(x_n, x^*) + d(x_n, x^*)$.
so,
$d(Tx^*, x^*) \ll \frac{c}{3} + \frac{c}{3} + \frac{c}{3} = c$.
Thus, $d(Tx^*, x^*) \ll \frac{c}{m}$, for all $m \geq 1$. Hence, $\frac{c}{m} - d(Tx^*, x^*) \in P$, for all $m \geq 1$. Since $\frac{c}{m} \to 0$ as $m \to \infty$ and P is closed, $-d(Tx^*, x^*) \in P$. But, $d(Tx^*, x^*) \in P$. Therefor, $d(Tx^*, x^*) = 0$, and so $Tx^* = x^*$.
Now, if y^* is another fixed point of T and $k + l < 1$, then:
$d(x^*, y^*) = d(Tx^*, y^*) \leq kd(x^*, y^*) + ld(Tx^*, y^*) = (k + l)d(x^*, y^*)$.
Hence, $d(x^*, y^*) = 0$ and so $x^* = y^*$. Therefor, the fixed point of T is unique whenever $k + l < 1$. □

In 2010 M.A.Khamsi [19] discuss the fixed point existence results of contractive mappings, and shows that most of the new results that he got are merely copies of the classical ones in metric spaces.

One of the extensions for 3.6 was in [26] in 2008, after that in 2013 authors of [10] made another extension and improvement for the same result.

Theorem 3.13. *[10] Let (X, d) be a complete cone metric space, P be a normal cone with normal constant K, in addition let $T : X \to X$ be an one to one and continuous function, and $R, S : X \to X$ be pair of T-contraction continuous functions. Then:*

- *i: For every $x_0 \in X$, $\lim_{n\to\infty} d(TR^{2n+1}x_0, TR^{2n+2}x_0) = 0$ and $\lim_{n\to\infty} d(TS^{2n+2}x_0, TS^{2n+3}x_0) = 0$.*
- *ii: There is $\alpha \in X$ such that $\lim_{n\to\infty} TR^{2n+1}x_0 = \alpha = \lim_{n\to\infty} TS^{2n+2}x_0$.*
- *iii: If T is subsequentially convergent, then $\{R^{2n+1}x_0\}$ and $\{S^{2n+2}x_0\}$ have a convergent subsequences.*
- *iv: There is a unique common fixed point $u \in X$ such that $Ru = uSu$.(we will talk about common fixed points in the next section)*
- *v: If T is sequentially convergent, then for each $x_0 \in X$ the iterate sequences $\{R^{2n+1}x_0\}$ and $\{S^{2n+2}x_0\}$ converges to u.*

Proof. For every $x_1, x_2 \in X$,we have

$$d(Tx_1, Tx_2) \leq d(Tx_1, TRx_1) + d(TRx_1, TEx_2) + d(TRx_2, Tx_2)$$

$$\leq d(Tx_1, TRx_1) + ad(Tx_1, Tx_2) + d(TRx_2, Tx_2).$$

so,

$$d(Tx_1, Tx_2) \leq \frac{1}{1-a}[d(Tx_1, TRx_1) + d(TRx_2, Tx_2)]. \tag{3.1}$$

Now, choose $x_0 \in X$ and define the picard iteration associated to R, $\{x_{2n+1}\}$ given by :
$x_{2n+2} = Rx_{2n+1} = R^{2n+1}x_0$, $n = 0, 1, 2, \dots$.
Similarly, associated to S, $\{x_{2n+2}\}$ is given by:
$x_{2n+3} = Sx_{2n+2} = S^{2n+2}x_0$, $n = 0, 1, 2, \dots$.
Now, $d(Tx_{2n+1}, Tx_{2n+2}) = d(TR^{2n+1}x_0, TR^{2n+2}x_0) \leq ad(TR^{2n}x_0, TR^{2n+1}x_0)$,
hence,

$$d(TR^{n+1}x_0, TR^{2n+2}x_0) \leq a^{2n+1}d(Tx_0, TRx_0). \tag{3.2}$$

Similarly,

$$d(TS^{2n+2}x_0) \leq b^{2n+2}d(Tx_0, TSx_0).$$

Since P is a normal constant K, from 3.2 we get

$$\| d(TR^{2n+1}x_0, TR^{2n+2}x_0) \| \leq a^{2n+1}K \| d(Tx_0, TRx_0) \|$$

which implies: $\lim_{n\to\infty} d(TR^{2n+1}x_0, TR^{2n+2}x_0) = 0$.
Therefor, for $m, n \in I\!N$ with $m > n$, by 3.1 and 3.2 we have

$$d(Tx_{2n+1}, Tx_{2m+1}) = d(TR^{2n+1}x_0, TR^{2m+1}x_0)$$

$$\leq \frac{1}{1-a}[d(TR^{2n+1}x_0, TR^{2n+2}x_0) + d(TR^{2m+2}x_0, TR^{2m+1}x_0)]$$

$$\leq \frac{1}{1-a}[a^{2n+1}d(Tx_0, TRx_0) + a^{2m+1}d(Tx_0, TRx_0)].$$

Hence, $d(TR^{2n+1}x_0, TR^{2m+1}) \leq \frac{a^{2n+1}+a^{2m+1}}{1-a}d(Tx_0, TR_0)$.
Taking norm to inequality above, we obtain that

$$\| d(TR^{2n+1}x_0, TR^{2m+1}) \| \leq \frac{a^{2n+1}+a^{2m+1}}{1-a}K \| d(Tx_0, TR_0) \|$$

Consequently

$$\lim_{n,m\to\infty} d(TR^{2n+1}x_0, TR^{2m+1}x_0) = 0. \tag{3.3}$$

which prove(i).On the other hand, 3.3 implies that $\{TR^{2n+1}x_0\}$ is a Cauchy sequence in X. By the completeness of X, there is $\alpha \in X$ such that

$$\lim_{n\to\infty} TR^{2n+1}x_0 = \alpha \tag{3.4}$$

Proving in this way assertion (ii). Now, if T is subsequentially convergent, then $\{R^{2n+1}x_0\}$ has a convergent subsequence. So, there exist $u \in X$ and $\{(2n+1)_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} R^{(2n+1)_i}x_0 = u \tag{3.5}$$

Since T is continuous we have $\lim_{i\to\infty} TR^{(2n+1)_i}x_0 = Tu$ From 3.4 we conclude that $Tu = \alpha$. Since R is continuous, and also by 3.5 then, $\lim_{i\to\infty} R^{(2n+1)_{i+1}}x_0 = Ru$ as well as $\lim_{i\to\infty} TR^{(2n+1)_{i+1}}x_0 = TRu$. Again by 3.4, the following equality holds $\lim_{i\to\infty} TR^{(2n+1)_{i+1}}x_0 = \alpha$. Hence, $TRu = \alpha = Tu$. Since T is injective, then $Ru = u$. So, R has fixed point. Therefor assertion (iii), is proved.
On the other hand, since T is one to one and R is a T-contraction, R has a unique fixed point, i.e conclusion (iv).
Finally, if T is sequentially convergent, $\{R^{2n+1}x_0\}$ is convergent to u, that is $\lim_{n\to\infty} R^{2n+1}x_0 = u$. Proving in this way conclusion (v). Similarly, it can be proved that all five assertion for T-contraction function S.
Hence, u is a unique fixed common point of R and S. □

Theorem 3.14. *[26] Let (X, d) be a compact cone metric space, P be a normal cone with normal constant K, and $T, S : X \to X$ functions such that T is injective, continuous and S is a T-contractive mapping. Then,*

i: S has a unique fixed point.

ii: For any $x_0 \in X$ the sequence iterates $\{S^n x_0\}$ converges to the fixed point of S.

Proof. We first are going to show that S is continuous function. Let $\lim_{n\to\infty} x_n = x$, we want to prove that $\lim_{n\to\infty} Sx_n = Sx$. Since S is T-contractive, we get $d(TSx_n, TSx) \leq d(x_n, x)$, so, $\| d(TSx_n, TSx) \| \leq K \| d(Tx_n, Tx) \|$.
Now, since T is continuous, we have $\lim_{n\to\infty} \| d(TSx_n, TSx) \| = 0$,
also $\lim_{n\to\infty} d(TSx_n, TSx) = 0$, therefor,

$$\lim_{n\to\infty} TSx_n = TSx. \tag{3.6}$$

Let $\{Sx_n\}$ be an arbitrary convergent subsequence of $\{x_n\}$. There is a $y \in X$ such that $\lim_{n\to\infty} Sx_{n_i} = y$.
By continuity of T we infer,

$$\lim_{n\to\infty} TSx_{n_i} = Ty. \tag{3.7}$$

By 3.6 and 3.7 we conclude that $TSx = Ty$. Since T is one to one then, $Sx = y$. Hence, every convergent subsequence of $\{Sx_n\}$ converges to Sx. From the fact that X is a compact cone metric space, we arrive to the conclusion that S is a continuous function.
Now, because of T and S are continuous functions, then the function $\varphi : X \to P$ defined by $\varphi(y) = d(TSy, Ty)$, for all $y \in X$, is continuous on X and from compactness of X, the function φ attains its minimum, say at $x \in X$.
If $Sx \neq x$, then $\varphi(Sx) = d(TS^2x, TSx) < d(TSx, Tx) = \varphi(x)$, which is a contradiction, So $Sx = x$ proving in this part (i).
Choose $x_0 \in X$ and set $a_n = d(TS^n x_0, Tx)$. Since

$$a_{n+1} = d(TS^{n+1}x_0, Tx) = d(TS^{n+1}x_0, TSx) \leq d(TS^n x_0, Tx) = a_n.$$

Then $\{a_n\}$ is non increasing sequence of non negative real numbers, and so it has a limit, say a, that is
$a = \lim_{n\to\infty} a_n$ or $\lim_{n\to\infty} d(TS^{n+1}x_0, Tx) = a$.
By compactness, $\{TS^n x_0\}$ has a convergent subsequence $\{TS^{n_i} x_0\}$ i.e

$$\lim_{n\to\infty} TS^{n_i} x_0 = z \tag{3.8}$$

From the sequentially convergent of T, there exists $w \in X$ such that $\lim_{n\to\infty} S^{n_i}x_0 = w$ so,

$$\lim_{n\to\infty} TS^{n_i}x_0 = Tw \tag{3.9}$$

By 3.8 and 3.9, $tw = z$. Then $d(Tw, Tx) = a$. Now we will show that $Sw = x$. If $Sw \neq x$, then

$$a \lim_{n\to\infty} d(TS^n x_0, Tx) = \lim_{n\to\infty} d(TS^{n_i}x_0, Tx) = d(TSw, Tx)$$

$$= d(TSw, TSx) < d(Tw, Tx) = a$$

which is a contradiction. In this way, we get that $Sw = x$ and hence, $a = \lim_{n\to\infty} d(TS^{n_i+1}x_0, Tx) = d(TSw, Tx) = 0$ Therefor, $\lim_{n\to\infty} TS^{n_i}x_0 = Tx$. Finally condition T sequentially convergent implies $\lim_{n\to\infty} S^n x_0 = x$, which implies the proof. □

3.2.1 Examples

In this short section we will introduce some examples in cone metric spaces of functions have fixed point, and functions that have not fixed points.

Example 3.15. *[26] Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$,$X = [1, +\infty)$ and $d : X \times X \to E$ defined by $d(x, y) =\mid x - y \mid e^t$, where $\varphi(t) = e^t \in E$.*
Then (X, d) is a complete cone metric space. Now we will consider the following functions,
$T, S : X \to X$ defined by $Tx = 1 + \ln x$ and $Sx = \sqrt{x}$. It is clear that S is not a contraction mapping, but it is a T-contraction because,

$$d(TSx, TSy) =\mid TSx - TSy \mid e^t = \frac{1}{2} \mid \ln x - \ln y \mid e^t$$

$$=\mid Tx - Ty \mid e^t \leq \frac{1}{2} d(Tx, Ty).$$

Also, T is one to one, continuous and subsequentially convergent. Therefor, by theorem 3.13 T has a unique fixed point,y.

Example 3.16. *[26] Let $E = (C_{[0,1]}, \mathbb{R})$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $X = \mathbb{R}$ and $d : X \times X \to E$ defined by $d(x, y) =\mid x - y \mid e^t$, where $e^t \in E$. Then (X, d) is a complete cone metric space.*
Let $T, S : X \to X$ be two functions defined by $Tx = e^{-x}$ and $Sx = 2x + 1$.

It is clear that S is a T-contraction, but T is not subsequentially convergent, because $T_n \to 0$ as $n \to \infty$ but the sequence $\{n\}$ has not any convergent subsequence and S has not a fixed point.

This example shows that we can not omit the sequentially convergence of the function T in the theorem 3.13 (v).

Example 3.17. *[26] Let $E = (C_{[0,1]}, I\!R)$, $P = \{\varphi \in E : \varphi \geq 0\} \subset E$, $X = [0,1]$ and $d : X \times X \to E$ defined by $d(x,y) =\mid x - y \mid e^t$, where $e^t \in E$. It clear that X is a compact cone metric space.*
Let $T, S : X \to X$ be two functions defined by $Tx = x^2$ and $Sx = \frac{x^2}{\sqrt{2}}$.
Satisfy that T is injective and continuous, whereas S is a T-contractive. So by theorem 3.14 we have that S has a unique fixed point, $x = 0$.

3.3 Common Fixed Point Theorem in Cone Metric spaces

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. For more about first appearance of common fixed point theorems untill 2008 we refer the reader to introduction of [1], the authors of [1] presented and obtained several coincidence and common fixed point theorems for mappings defined on a cone metric space. After that authors of [15] made a generalization of preceding fixed point theorems on complete cone metric spaces. After that many mathematicians introduced different functions and prove fixed and common fixed point theorems, as MS-altering function in [24, 9], weakly compati/ble see [1, 16].
In this section we will give definitions of coincidence point, weakly compatible, and altering function. And we will introduce some important common fixed point theorems.

Definition 3.18. *[16] Let f and g be self mappings of a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*

Definition 3.19. *[24] A function $f : P \to P$ is called* subadditive *if for all $x, y \in P$, $f(x+y) \leq f(x) + f(y)$.*

Definition 3.20. *[24] If Y be any partially ordered set with relation $\leq$ and $\psi : Y \to Y$, we say that ψ is non decreasing if $x, y \in Y$, $x \leq y \Rightarrow \psi(x) \leq \psi(y)$.*

Definition 3.21. *[16] Two self mappings f and g of a set X are said to be weakly compatible if they commute at their coincidence points; that is, if $fu = gu$ for some $u \in X$, then $fgu = gfu$.*

Proposition 3.22. *[1] Let f and g be weakly compatible self mappings of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique fixed common fixed point of f and g.*

Proof. Since $w = fx = gx$ and f and g are weakly compatible, we have $fw = fgx = gfx = gw$: i.e,$fw = gw$ is a point of coincidence of f and g. But w is the only point of coincidence of f and g, so $w = fw = gw$. Moreover if $z = fz = gz$, then z is a point of coincidence of f and g, and therefor $z = w$ by uniqueness. Thus w is a unique common fixed point of f and g. $\square$

Theorem 3.23. *[16] Let (X, d) be a cone metric space, and P be a normal cone with normal constant K. Suppose that the mappings $f, g : X \to X$ satisfy the contractive condition:*

$$d(fx, fy) \leq r[d(fx, gy) + d(fy, gx) + d(fx, gx) + d(fy, gy)]$$

where $r \in [0, \frac{1}{4})$ is a constant. If the range of g contains the range of f, and $g(X)$ is complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. Then, since $fX \subset gX$, we choose a point $x_1 \in X$ such that $f(x_0) = g(x_1)$. Continuing in this process , having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $f(x_n) = g(x_{n+1})$. Then, $d(g_{n+1}, gx_n) = d(fx_n, fx_{n-1})$

$$\leq r[d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n) + d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1})]$$

$$\leq 2r[d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})].$$

So, we have $d(gx_{n+1}, gx_n) \leq hd(gx_n, gx_{n-1})$, with $h = \frac{2r}{1-2r}$.
Now, for $n > m$, we get

$$d(gx_n, gx_m) \leq d(gx_n, gx_{n-1}) + d(gx_{n-1}, gx_{n-2}) + ... + d(gx_{m+1}, gx_m)$$

$$\leq (h^{n-1} + h^{n-2} + ... + h^m) d(gx_1, gx_0)$$
$$\leq \frac{h^m}{1-h} d(gx_1, gx_0),$$

using the normality of cone P, implies that

$$\| d(gx_n, gx_m) \| \leq \frac{h^m}{1-h} K \| d(gx_1, gx_0) \| .$$

Then, $d(gx_n, gx_m) \to 0$ as $n, m \to \infty$, and so, $\{gx_n\}$ is a Cauchy sequence in X. Since $g(X)$ is a complete subspace of X, so there exists $q \in g(X)$ such that $gx_n \to q$, as $n \to \infty$. Consequently, we can find $p \in X$ such that $g(p) = q$. Thus,
$d(gx_n, fp) = d(fx_{n-1}, fp) \leq r[d(fx_{n-1}, gp) + d(fp, gx_{n-1}) + d(fx_{n-1}, gx_{n-1}) + d(fp, gp)]$ Using the normality of cone P, implies that
$\| d(gx_n, fp) \| \leq Kr \| d(gx_{n-1}, gp) \| = 0$, as $n \to \infty$.
Hence, $d(gx_n, fp) \to 0$ as $n \to \infty$. Also, we have $d(gx_n, gp) \to 0$ as $n \to \infty$. The uniqueness of limit in a cone metric space implies that $f(p) = g(p)$. Again, we show that f and g have a unique point of coincidence.
For this possible, assume that there exists an another point $t \in X$ such that
$f(t) = g(t)$. Then, we have

$$d(gt, gp) = d(ft, fp) \leq r[d(ft, fp) + d(fp.gt) + d(ft, gt) + d(fp, gp).$$

Now, using the normality of cone P, implies that $\| d(gt, gp) \| = 0$, and so, we have $gt = gp$. Finally, using 3.22, we conclude that f and g have a unique common fixed point. □

Example 3.24. *[16] Let $E = I^2$, for $I = [0, 1]$, $P = \{(x, y) \in E : x, y \geq 0\} \subset I^2$, $d : I \times I \to E$
such that $d(x, y) = (| x - y |, \alpha | x - y |)$, where $\alpha > 0$ is a constant. Define $fx = \frac{\alpha x}{1+\alpha x}$, for all $x \in I$ and $gx = \alpha x$ for all $x \in I$. Then, for $\alpha = 1$, both the mappings f and g are weakly compatible and satisfy all conditions of the above theorem with $x = 0$ as a unique common fixed pint.*

Definition 3.25. *[24] Let $\psi : P \to P$ be a vector valued function then ψ is called MS-Altering function if*

i: ψ is non decreasing, subadditive, continuous and sequentially convergent.

ii: $\psi(a) = 0$ if and only if $a = 0$.

By replacing conditions (i) and (ii) by weaker conditions, define cone altering function as :

Definition 3.26. *[24] Let P be a normal cone and $\psi : P \to P$ be a vector valued function then ψ is called cone altering function if*

i: ψ is non decreasing and subadditive.

ii: $\psi(a_n) \to 0$ if and only if $a_n \to 0$, for any sequence $\{a_n\} \in P$.

Note that as for such ψ on normal cone P, $\psi(a) = 0 \Leftrightarrow a = 0$. For the main theorem in cone metric space by altering distances we need the following lemmas and proposition, for there proves see [24].

Lemma 3.27. *[24] Let (X, d) be a cone metric space and P be a normal cone in a real Banach space E, ψ is a cone altering function, and $k_1, k_2, k_3 > 0$. If $x_n \to x$, $y_n \to y$ in X and $ka \leq k_1\psi[d(x_n, x)] + k_2\psi[d(y_n, y)]$, then $a = 0$.*

Lemma 3.28. *[24] Let (X, d) be a cone metric space with normal cone P, and $f, g : X \to X$ be mappings such that, for all $x, y \in X$*

$$\psi[d(fx, fy)] \leq a_1\psi[d(gx, gy)]+a_2\psi[d(fx, gx)]+a_3\psi[d(fy, gy)]+a_4\psi[d(fx, gy)]+a_5\psi[d(fy, gx)] \tag{3.10}$$

where $a_i, i = 1, 2, 3, 4, 5$ are nonnegative constants such that $a_1+a_2+a_3+a_4+a_5 < 1$ and ψ is a cone altering function. If f and g have a point of coincidence then it is unique.

Theorem 3.29. *[24] Let (X, d) be a cone metric space with normal cone P, and $f, g : X \to X$ be mappings, $\psi : P \to P$ is a cone altering function such that, $f(X) \subset g(X)$, for all $x, y \in X$, 3.10 is satisfied and $f(X)$ or $g(X)$ is complete, then f and g have a unique point of coincidence. Furthermore, if (f, g) is weakly compatible pair then f, g have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrary, we define sequence $\{y_n\}$ such that $y_n = fx_n = gx_{n+1}$ for all $n \geq 0$.
If $y_n = y_{n+1}$ for any n, then $y_n = y_m$ for all $m > n$, hence $\{y_n\}$ is Cauchy sequence.
If $y_n \neq y_{n+1}$ for all n, then from 3.10 we have

$$\psi[d(fx_{n+1}, fx_n)] \leq a_1\psi[d(gx_{n+1}, gx_n)]+a_2\psi[d(fx_{n+1}, gx_{n+1})]+a_3\psi[d(fx_n, gx_n)]+a_4\psi[d(fx_{n+1}, gx_n)]+a_5$$

writing $d_n = d(y_n, y_{n+1})$ we have

$$\psi[d_n] \leq a_1\psi[d_{n-1}] + a_2\psi[d_n] + a_3\psi[d_{n-1}] + a_4\psi[d_n] + a_4\psi[d_{n-1}] \tag{3.11}$$

$$(1 - a_2 - a_4)\psi[d_n] \leq (a_1 + a_3 + a_4)\psi[d_{n-1}] \tag{3.12}$$

using symmetry of 3.10 in x, y we have

$$(1 - a_3 - a_5)\psi[d_n] \leq (a_1 + a_2 + a_5)\psi[d_{n-1}] \tag{3.13}$$

Combining (3.12) and (3.13)

$$\psi[d_n] \leq \frac{2a_1 + a_2 + a_3 + a_4 + a_5}{2 - (a_2 + a_3 + a_4 + a_5}\psi[d_{n-1}] = \lambda\psi[d_{n-1}]$$

and so, $\psi[d_n] \leq \lambda^n\psi[d_0]$, where $\lambda = \frac{2a_1+a_2+a_3+a_4+a_5}{2-(a_2+a_3+a_4+a_5} < 1$.
If $m > n$, we have

$$\psi[d(y_n, y_m)] \leq \psi[d(y_n, y_{n+1})] + \psi[d(y_{n+1}, y_{n+2})] + ... + \psi[d(y_{m-1}, y_m)]$$

$$\leq \psi[d_n] + \psi[d_{n+1}] + ... + \psi[d_{m-1}]$$

$$\leq \lambda^n\psi[d_0] + \lambda^{n+1}\psi[d_0] + ... + \lambda^{m-1}\psi[d_0]$$

$$\leq \frac{\lambda^n}{1-\lambda}\psi[d_0]$$

Since $0 \leq \lambda < 1$ hence by normality of the cone P,
$\| \psi[d(y_n, y_m)] \| \leq \frac{K\lambda^n}{1-\lambda} \| \psi[d_0] \| \to 0$, therefor, $\psi[d(y_n, y_m)] \to 0$ and so, $d(y_n, y_m) \to 0$ hence $\{y_n\}$ is a Cauchy sequence. Let $f(X)$ is complete then since $y_n = fx_n = gx_{n+1}$ and $\{y_n\}$ is Cauchy in $f(X)$, so it must be convergent in $f(X)$. Let $y_n \to u \in f(X)$ (note that it is also true if $g(X)$ is complete with $u \in g(X)$). Since $u \in f(X) \subset g(X)$, let $u \in g(v)$ for some $v \in X$.
We show that $gv = fv$. Now by 3.10

$$\psi[d(fv, u)] \leq \psi[d(fv, fx_n)] + \psi[d(fx_n, u)]$$

$$\leq a_1\psi[d(gv, gx_n)] + a_2\psi[d(fv, gv)] + a_3\psi[d(fx_n, gx_n)]a_4\psi[d(fv, gx_n)] + a_5\psi[d(fx_n, gv)] + \psi[d(fx_n, u)]$$

$$= a_1\psi[d(u, y_{n-1})] + a_2\psi[d(fv, u)] + a_3\psi[d(y_n, y_{n-1})] + a_4\psi[d(fv, y_{n-1})] + a_5\psi[d(y_n, u)] + \psi[d(y_n, u)]$$

$$(1 - a_2 - a_4)\psi[d(fv, u)] \leq (a_1 + a_3 + a_4)\psi[d(u, y_{n-1})] + (a_3 + a_5 + 1)\psi[d(y_n, u)]$$

Hence by lemma 2.1, $\psi[d(fv,u)] = 0$ and so $d(fv,u) = 0$ i.e. $fv = u = gv$. Thus u is point of coincidence of f and g, hence by lemma 2.2it is unique. Furthermore, if pair $f,g)$ is weakly compatible then by 3.22, u is unique common fixed point of f and g. □

There were an extension in [10] for theorem 3.14 in section 3.2, the following theorem shows that

Theorem 3.30. *[10] Let (X,d) be a compact cone metric space, P be a normal cone with normal constant K, in addition let $T : X \to X$ is injective and continuous function and $R,S : X \to X$ be a pair of T-contractive mappings, then:*

i: R and S have a unique common fixed point.

ii: For any $x_0 \in X$ the iterate sequences $\{R^{2n+1}x_0\}$ and $\{S^{2n+2}x_0\}$ converges to the common fixed point of R and S.

Proof. First, we are going to show that R and S are continuous functions. Let $\lim_{n\to\infty} x_{2n+1} = x$. We want to prove that $\lim_{n\to\infty} Rx_{2n+1} = Rx$. Since R is T-contractive, we get $d(TRx_{2n+1}, TRx) \le d(Tx_{2n+1}, Tx)$

so, $\| d(TRx_{2n+1}, TRx) \| \le K \| d(Tx_{2n+1}, Tx) \|$.
Now, since T is continuous, we have $\lim_{n\to\infty} \| d(TRx_{2n+1}, TRx) \| = 0$
also $\lim_{n\to\infty} d(TRx_{2n+1}, TRx) = 0$. Therefor,

$$\lim_{n\to\infty} TRx_{2n+1} = TRx.$$

Let $\{Rx_{(2n+1)_i}\}$ be an arbitrary convergent subsequence of $\{x_{2n+1}\}$. There is a $y \in X$ such that $\lim_{n\to\infty} RX_{(2n+1)_i} = y$.
By the continuity of T we have;

$$\lim_{n\to\infty} TRX_{(2n+1)_i} = Ty \tag{3.14}$$

By 3.3 and 3.14 we conclude that $TRx = Ty$. Since T is one to one then, $Rx = y$. Hence, every convergent subsequence of $\{Rx_{2n+1}\}$ converge to Rx. From the fact that X is compact cone metric space, we get the conclusion that R is a continuous function. Similarly we can show that S is also a continuous function.
Now, because of T and R are continuous functions, then the function $\gamma : X \to P$,

defined by $\gamma(y) = d(TRy, Ty)$, for all $y \in X$, is continuous on X and from the compactness of X, the function γ attains its minimum, say at $x \in X$.
If $Rx \neq x$, then $\gamma(Rx) = d(TR^3x, TRx) < d(TRx, Tx) = \gamma(x)$ which is a contradiction. So $Rx = x$. Similarly we have $Sx = x$, proving part (i).
Choose $x_0 \in X$ and set $a_{2n+1} = d(TR^{2n+1}x_0, Tx)$.
Since $a_{2n+2} = d(TR^{2n+2}x_0, Tx) = d(TR^{2n+2}x_0, TRx) \leq d(TR^{2n+1}x_0, Tx) = a_{2n+1}$, then $\{a_{2n+1}\}$ is a non increasing sequence of non negative real numbers and so it has a limit, say a, that is
$a = \lim_{n\to\infty} a_{2n+1}$ or $\lim_{n\to\infty} d(TR^{2n+1}x_0, Tx) = a$.
By compactness, $\{TR^{2n+1}x_0\}$ has a convergent subsequence $\{TR^{(2n+1)_i}x_0\}$ i.e.

$$\lim_{n\to\infty} TR^{(2n+1)_i}x_0 = Z \tag{3.15}$$

from the sequentially convergent of T, there exists $w \in X$ such that $\lim_{n\to\infty} R^{(2n+1)_i}x_0 = w$. So,

$$\lim_{n\to\infty} TR^{(2n+1)_i}x_0 = Tw. \tag{3.16}$$

By 3.15 and 3.16, $Tw = Z$. Then $d(Tw, Tx) = a$.
Now, we are going to show that $Rw = x$. If $Rw \neq x$, then

$$\begin{aligned} a &= \lim_{n\to\infty} d(TR^{2n+1}x_0, Tx) \\ &= \lim_{n\to\infty} d(TR^{(2n+1)_i}x_0, Tx) \\ &= d(TRw, TRx) \\ &< d(Tw, Tx) = a \end{aligned}$$

which is a contradiction. In this way, we get that $Rw = x$ and hence
$a = \lim_{n\to\infty} d(TR^{(2n+1)_{i+1}}x_0, Tx) = d(TRw, Tx) = 0$.
Therefor, $\lim_{n\to\infty} TR^{2n+1}x_0 = Tx$. Finally condition T sequentially convergent implies $\lim_{n\to\infty} R^{2n+1}x_0 = x$. Similarly it can be established that
$\lim_{n\to\infty} S^{2n+2}x_0 = x$, which means that the iterate sequences $\{R^{2n+1}x_0\}$ and $\{S^{2n+2}x_0\}$ converges to the common fixed point of R and S. $\square$

3.4 Some Coupled Fixed Point Theorems in Cone Metric Spaces

In 1987 Guo and Lakshmikantham introduced the concept of coupled fixed point for partially ordered set. By using the concept of mixed monotone property (Gnana Bhaskar and Lakshmikantham 2006 studied the existence and uniqueness of a coupled fixed point result in partially ordered metric space, and introduced the concept of a mixed monotone property for the first time and gave their classical coupled fixed point theorems for mappings which satisfy the mixed monotone property, see[18]. After that many researchers studied the coupled fixed point and discussed its application. See introduction of [33].
In this section we will introduce some coupled fixed point theorems in ordered metric spaces (which is play a major role to prove the existence and uniqueness of solutions for some differential and integral equations) see introduction of [29], some coupled fixed point theorems for mappings satisfy some contractive conditions on complete cone metric spaces, and coupled fixed point theorems for weak contractions under F-Invariant.

Definition 3.31. *[36] Let $(X, \sqsubseteq)$ be a partial ordered set. A mapping $F : X \times X \to X$ is said to have* a mixed monotone property *if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$*

$$x_1, x_2 \in X, x_1 \sqsubseteq x_2 \Rightarrow F(x_1, y) \sqsubseteq F(x_2, y),$$

$$y_1, y_2 \in X, y_1 \sqsubseteq y_2 \Rightarrow F(x, y_1) \sqsupseteq F(x, y_2).$$

Definition 3.32. *[36] Let (X, d) be a cone metric space, then a function $q : X\times X \to E$ is called a c-distance on X if:*

i: $q(x, y) \succeq \theta . \forall x, y \in X.$

ii: $q(x, y) \preceq q(x, z) + q(z, y).$

iii: $\forall x \in X$, $\exists u = u_x \in P$ *such that* $q(x, y_n) \preceq u$ *for each* $n \in I\!N$ *then,* $q(x, y) \preceq u$ *whenever* $\{y_n\}$ *is a sequence in* X *converging to* $y \in X$.

iv: $\forall c \in E$ *with* $\theta \preceq c$ $\exists e \in E$ *with* $\theta \preceq e$ *such that* $q(z, x) \preceq e$ *and* $q(z, y) \preceq c$ $\Rightarrow d(x, y) \preceq c.$

Definition 3.33. *[32]Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Example 3.34. *[37] Let $X = [0, \infty)$ and $F : X \times X \to X$ be defined by $F(x, y) = x + y$ for all $x, y \in X$. Then, F has a unique coupled fixed point $(0, 0)$.*

Definition 3.35. *[29] Let $F : X \times X \to X$ and $g : X \to X$. An element $(x, y) \in X^2$ is called* a coupled coincidence point *of F and g*
if $F(x, y) = gx$ and $F(y, x) = gy$.
While $(gx, gy) \in X^2$ is said a coupled point of coincidence of mappings F and g. Moreover, (x, y) is called a coupled common fixed point *of F and g if*
$F(x, y) = gx = x$ and $F(y, x) = gy = y$.

3.5 Coupled Fixed Point Under F-Invariant Set

First we will start with some important definitions and examples.

Definition 3.36. *[36] Let (X, d) be a cone metric space and $F : X \times X \to X$ a given mapping. Let M be a nonempty subset of X^4. One says that M is F-invariant subset of X^4 if and only if,*
for all $x, y, z, w \in X$, one has

i: $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$;

ii: $(x, y, z, w) \in M \Leftrightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

From the last definition we obtain that the set $M = X^4$ is trivially F-invariant.

Example 3.37. *[18] Let (X, d) be a cone metric space endowed with a partially order $\sqsubseteq$. Let $F : X \times X \to X$ be a mapping satisfying the mixed monotone property; that is, for all $x, y \in X$, we have*
$x_1, x_2 \in X,\ x_1 \sqsubseteq x_2 \Rightarrow F(x_1, y) \sqsubseteq F(x_2, y)$
and
$y_1, y_2 \in X,\ y_1 \sqsubseteq y_2 \Rightarrow F(x, y_1) \sqsupseteq F(x, y_2)$.
Define $M = \{(a, b, c, d) : c \sqsubseteq a, b \sqsubseteq d\} \subseteq X^4$. Then, M is F-invariant of X^4.

Theorem 3.38. *[36] Let (X,d) be a complete cone metric space. Let q be a c-distance on X, M a nonempty subset of X^4, and $F : X \times X \to X$ a continuous function such that*

$$q(F(x,y), F(x^*,y^*)) \preceq \frac{k}{2}(q(x,x^*) + q(y,y^*))$$

for some $k \in [0,1)$ and all $x, y, x^*, y^* \in X$ with
$(x,y,x^*,y^*) \in M$ or $(x^*,y^*,x,y) \in M$.
If M is F-invariant and there exist $x_0, y_0 \in X$ such that
$(F(x_0,y_0), F(y_0,x_0), x_0, y_0) \in M$
Then F has a coupled fixed point (u,v). Moreover, if $(u,v,u,v) \in M$, then $q(v,v) = \theta$ and $q(u,u) = \theta$.

For proof see [36].

Authors of [18] show the weakness of theorem 3.38 with an example, and give an extension of 3.38. We will start with the example.

Example 3.39. *[18] Let $E = C^1_{\mathbb{R}}[0,1]$ with $\| x \|_E = \| x \|_\infty + \| x' \|_\infty$ and*
$P = \{x \in E, x(t) \geq 0, t \in [0,1]\}$. Let $X = [0,\infty)$ (with the usual order $\sqsubseteq$), and let $d = X \times X \to E$ be defined by:
$d(x,y)(t) = \mid x - y \mid 2^t$. Then (X,d) is a complete cone metric space.
Let, further, $q : X \times X \to E$ be defined by $q(x,y)(t) = y2^t$. It is easy to check that q is a c-distance. Consider the mapping $F : X \times X \to X$ by:

$$F(x,y) = \begin{cases} \frac{3}{5}(x+y), & if x \geq y \\ 0, & if x < y \end{cases}$$

Let $M = X^4$ and so M is an F-variant subset of X. Now, we show that there is no $k \in [0,1)$ for which theorem 3.38 holds.
To prove this, suppose the contrary; that is, there is $k \in [0,1)$ such that

$$q(F(x,y), F(x^*,y^*)) \preceq \frac{k}{2}(q(x,x^*) + q(y,y^*))$$

for all $x, y, x^*, y^* \in X$ with $(x,y,x^*,y^*) \in M$ or $(x^*,y^*,x,y) \in M$.

Take $x = 0, y = 1, x^* = 1$ and $y^* = 0$. Then

$$q(F(0,1), F(1,0))(t) \preceq \frac{k}{2}(q(0,1) + q(1,0))(t)$$

This implies $F(1,0)2^t = \frac{3}{5}2^t \preceq \frac{k}{2}2^t$.
Hence, $k \geq \frac{6}{5}$ *is a contradiction. therefor, there is no* k *for which theorem 3.38 holds.Moreover, for* $y_1 = \frac{1}{3}$ *and* $y_2 = \frac{1}{2}$,
we have for $x = 1$, *we get* $y_1 = \frac{1}{3} \sqsubseteq \frac{1}{2} = y_2$ *but* $F(x, y_1) = \frac{4}{5} \sqsubseteq \frac{9}{10} = F(x, y_2)$. *So, the mapping* F *does not satisfy the mixed monotone property.*

Lemma 3.40. *[18] Let* (X, d) *be a cone metric space and* q *be a c-distance on* X. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in* X *and* $x, y, z \in X$. *Suppose that* $\{u_n\}$ *is a sequence in* P *converging to* θ. *Then the following holds:*

i: If $q(x_n, y)$ *and* $q(x_n, z) \preceq u_n$, *then* $y = z$.

ii: If $q(x_n, y_n)$ *and* $q(x_n, z) \preceq u_n$, *then* $\{y_n\}$ *converges to a point* $z \in X$.

iii: If $q(x_n, x_m) \preceq u_n$ *for each* $m > n$, *then* $\{x_n\}$*is a Cauchy sequence in* X.

iv: If $q(y, x_n) \preceq u_n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Now, we will provide the extension of 3.38.

Theorem 3.41. *[18] Let* (X, d) *be a complete cone metric space. Let* q *be a c-distance on* X, M *be a nonempty subset of* X^4 *and* $F : X \times X \to X$ *be a continuous function such that*

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \preceq k(q(x, x^*) + q(y, y^*))$$

for some $k \in [0, 1)$ *and all* $x, y, x^*, y^* \in X$ *with* $(x, y, x^*, y^*) \in M$ *or* $(x^*, y^*, x, y) \in M$. *If* M *is* F*-invariant and there exist* $x_0, y_0 \in X$ *such that* $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$,
then, F *has a coupled fixed point* (u, v). *Moreover, if* $(u, v, u, v) \in M$, *then* $q(u, u) = \theta$ *and* $q(v, v) = \theta$.

Proof. Since $F(X \times X) \subseteq X$, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_n = F(x_{n-1}, y_{n-1}) and y_n = F(y_{n-1}, x_{n-1}) \tag{3.17}$$

for all $n \in I\!N$. Since $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) = (x_1, y_1, x_0, y_0) \in M$ and M is F-invariant set, we get

$$(F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) = (x_2, y_2, x_1, y_1) \in M.$$

Again, using the fact that M is F-invariant set, we have

$$(F(x_2, y_2), F(y_2, x_2), F(x_1, y_1), F(y_1, x_1)) = (x_3, y_3, x_2, y_2) \in M.$$

By repeating the argument similar to the above, we get

$$(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}), x_{n-1}, y_{n-1}) = (x_n, y_n, x_{n-1}, y_{n-1}) \in M$$

For all $n \in I\!N$. From theorem 3.41, we have

$$\begin{aligned} q(x_n, x_{n+1}) + q(y_n, y_{n+1}) &\preceq q(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + q(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\preceq k(q(x_{n-1}, x_n), q(y_{n-1}, y_n)). \end{aligned}$$

We repeat the above process for n-times, we get

$$q(x_n, x_{n+1}) + q(y_n, y_{n+1}) \preceq k^n(q(x_0, x_1) + q(y_0, y_1)) \tag{3.18}$$

From 3.18, we can conclude that
$q(x_n, x_{n+1}) \preceq k^n(q(x_0, x_1) + q(y_0, y_1))$ and, $q(y_n, y_{n+1}) \preceq k^n(q(x_0, x_1) + q(y_0, y_1))$ Let $m, n \in I\!N$ with $m > n$.
Since $q(x_n, x_m) \preceq \sum_{i=n}^{m-1} q(x_i, x_{i+1})$ $q(y_n, y_m) \preceq \sum_{i=n}^{m-1} q(y_i, y_{i+1})$ and $0 \leq k < 1$, we have
$q(x_n, x_m) \preceq \frac{k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$ and
$q(y_n, y_m) \preceq \frac{k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$ Using lemma 3.40, we have $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d).
By completeness of X, we get $x_n \to u$ and $y_n \to v$ as $n \to \infty$ for some $u, v \in X$.
Since F is continuous, taking $n \to \infty$ in 3.17, we get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) = F(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n = F(u, v)$$

and

$$\lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) = F(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n = F(v, u)$$

By the uniqueness of the limits, we get $u = F(u, v)$ and $v = Fv, u)$. Therefor, (u, v) is a coupled fixed point of F.
Finally, we assume that $(u, v, u, v) \in M$. By theorem 3.41, we have

$$q(u, u) + q(v, v) = q(F, (u, v), F(v, u)) + q(F(v, u), F(u, v)) \preceq k(q(u, u) + q(v, v)).$$

Since $0 \leq k < 1$, we conclude that $q(u, u) + q(v, v) = \theta$ and hence $q(u, u) = \theta$ and $q(v, v) = \theta$. $\square$

Remark 3.42. *[18] Refering to example 3.39, we obtain that the mapping F has a coupled fixed point.*
Indeed, for all $x, y, x^*, y^* \in X$ with
$(x, y, x^*, y^*) \in M$ or $(x^*, y^*, x, y) \in M$
we have

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \preceq \frac{3}{5}(q(x, x^*) + q(y, y^*)).$$

Also, we note that there exists points $0, 1 \in X$ such that $(F(0, 1), F(1, 0), 0, 1) \in M$. Thus, by theorem 3.41, we have F has a coupled fixed point that is a point $(0, 0)$.

Theorem 3.43. *[18] In addition to the hypotheses of theorem 3.41, suppose that for any two elements x and y in X, we have*
$(y, x, x, y) \in M$ or $(x, y, y, x) \in M$.
Then, the coupled fixed point has the form (u, u), where $u \in X$.

Proof. As in the proof of theorem 3.41, there exists a coupled fixed point $(u, v) \in X \times X$. Hence, $u = F(u, v)$ and $v = F(v, u)$.
Moreover, $q(u, u) = \theta$ and $q(v, v) = \theta$ if $(u, v, v, u) \in M$.
From the addition hypothesis, we have $(u, v, v, u) \in M$ or $(v, u, u, v) \in M$. By theorem 3.41 we get:

$$q(u, v) + q(v, u) = q(F(u, v), F(v, u) + q(F(v, u), F(u, v)) \preceq k(q(u, v) + q(v, u)).$$

Since $0 \leq k < 1$, we get $q(u, v) + q(v, u) = \theta$. Therefor, $q(u, v) = \theta$ and $q(v, u) = \theta$.
Let $u_n = \theta$ and $x_n = u$. Then $q(x_n, u) \preceq u_n$ and $q(x_n, v) \preceq u_n$.
From example 3.40, we have $u = v$. Therefor, the coupled fixed point of F has the form (u, u). This completes the proof. □

Theorem 3.44. *[18] Let (X, d) be a complete cone metric space. Let q be a c-distance on X, M be a subset of X^4 and*
$F : X \times X \to X$ be a function such that

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \preceq k(q(x, x^*) + q(y, y^*)) \tag{3.19}$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with $(x, y, x^*, y^*) \in M$ or $(x^*, y^*, x, y) \in M$. Also, suppose that

i: there exist $x_0, y_0 \in X$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$.

ii: Two sequences $\{x_n\}$, $\{y_n\}$with $(x_{n+1}, y_{n+1}, x_n, y_n) \in M$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x$, $\{y_n\} \to y$, then $(x, y, x_n, y_n) \in M$ for all $n \in \mathbb{N}$.

If M is an F-invariant set, then F has a coupled fixed point. Moreover, if $(u, v, u, v) \in M$, then $q(u, u) = \theta$ and $q(v, v) = \theta$.

Proof. As in the proof of theorem 3.41, we have that $\{x_n\}$ and $\{y_n\}$ in X such that $(x_n, y_n, x_{n-1}, y_{n-1}) \in M$ for all $n \in \mathbb{N}$. Moreover, we have that $\{x_n\}$ converges to a point $u \in X$ and $\{y_n\}$ converges to $v \in X$,

$$q(x_n, x_m) \preceq \frac{k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

and

$$q(y_n, y_m) \preceq \frac{k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

for each $m > n \geq 1$. Since q is a c-distance, we have

$$q(x_n, u) \preceq \frac{k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

and

$$q(y_n, v) \preceq \frac{k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

and so

$$q(x_n, u) + q(y_n, v) \preceq \frac{2k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1)) \tag{3.20}$$

By assumption (ii), we have $(u, v, x_n, y_n) \in M$. From 3.19 and 3.20, we have

$$\begin{aligned} q(x_n, F(u, v)) + q(y_n, F(v, u)) &= q(F(x_{n-1}, y_{n-1}), F(u, v)) + q(F(y_{n-1}, x_{n-1}), F(v, u)) \\ &\preceq k(q(x_{n-1}, u) + q(y_{n-1}, v)) \\ &\preceq k.\frac{2k^{n-1}}{1-k}(q(x_0, x_1) + q(y_0, y_1)) \\ &= \frac{2k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1)). \end{aligned}$$

Therefor, we have

$$q(x_n, F(u, v)) \preceq \frac{2k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

and

$$q(y_n, F(v, u)) \preceq \frac{2k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

From 3.20, we get

$$q(x_n, u) \preceq \frac{2k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

and

$$q(y_n, v) \preceq \frac{2k^n}{1-k}(q(x_0, x_1) + q(y_0, y_1))$$

Since 3.5, 3.5, 3.5 and 3.5 hold, by using lemma 3.40, we get $u = F(u, v)$ and $v = F(v, u)$.
Therefor, (u, v) is a coupled fixed point of F. The proof of $q(u, u) = \theta$ and $q(v, v) = \theta$ is the same as the proof in 3.41. This completes the proof. □

Theorem 3.45. *[18] In addition to the hypothesis of theorem 3.44, suppose that for any two elements x and y in X, we have*
$(y, x, x, y) \in M$ or $(x, y, y, x) \in M$.
Then the coupled fixed point has the form (u, u), where $u \in X$.

The proof the same as proof of theorem 3.43. The next corollary considered as an direct result from theorem 3.41.

Corollary 3.46. *[18] Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a c-distance on X and $F : X \times X \to X$ be a continuous function having the mixed monotone property such that*

$$q(F(x, y), F(x^*, y^*)) + q(F(x, y), F(x^*, y^*)) \preceq k(q(x, x^*) + q(y, y^*))$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with
$(x \sqsubseteq x^*) \wedge (y \sqsupseteq y^*)$ or $(x \sqsupseteq x^*) \wedge (y \sqsubseteq y^*)$. If there exist $x_0, y_0 \in X$ such that $x_0 \sqsubseteq F(x_0, y_0)$ and $F(y_0, x_0) \sqsubseteq y_0$,
then F has a coupled fixed point (u, v). Moreover, we have $q(v, v) = \theta$ and $q(u, u) = \theta$.

Proof. Let $M = \{(a, b, c, d) : c \sqsubseteq a, b \sqsubseteq d\} \subseteq X^4$. We obtain that M is an F-invariant set. By3.46, we have

$$q(F(x, y), F(x^*, y^*)) + q(F(y, x), F(y^*, x^*)) \preceq k(q(x, x^*) + q(y, y^*))$$

for some $k \in [0, 1)$ and all $x, y, x^*, y^* \in X$ with $(x, y, x^*, y^*) \in M$ or $(x^*, y^*, x, y) \in M$. Now, all the hypotheses of theorem 3.41 holds. Thus, F has a coupled fixed point. □

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