Modified Adomian Decomposition Method For Differential Equations

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Declaration

I declare that the master thesis entitled ”Modified Adomian Decomposition Method For Differential Equations” is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Dedications

To my parents, my wife Maha, my children, my brother Hazem and my sisters.

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Abstract

Nonlinear differential equations arise in all fields of applied mathematics, physical science and Engineering, hence being of fundamental importance the existence of methods to find their solutions. In the 1980's, George Adomian introduced a semi-analytical technique known as, Adomian decomposition method, for solving linear and nonlinear differential equations.

In this thesis, some modifications of the Adomian decomposition method are presented.

In chapter one, we explained the Adomian decomposition method and how to use it to solve linear and nonlinear differential equations and present few examples .

Modifications based on assumptions made by Adomian for solving differential equations are explained in chapter two as well as a comparison of the results found to those found by ADM were presented.

In chapter three, some modifications based on operators were presented and we compare the results found to those found by ADM.

Arabic Page

” łŁ Ł Ł łŁŁ ” łł Ł Ł łŁŁ Ł ł łŁ Ł Ł Ł, Ł ł łŁł Ł Ł łŁŁ . Ł Ł Ł łŁŁ ŁŁ ł ł łŁ ł łŁ Ł Ł Ł. Ł Ł łŁł Ł Ł łŁŁ, Ł ł łŁ łł łŁ Ł ł łŁ, ł Ł Ł Ł ł łŁ ł Ł ł Ł Ł łŁŁ Ł.

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Chapter 1

Introduction To The Modified Adomian Decomposition Method

1.1 Introduction

George Adomian established the Adomian decomposition method (ADM) in the 1980s, the ADM has been paid much attention in the recent years in applied mathematics, and in the field of series solution particularly. Moreover, it is a fact that this method is powerful, effective, as well easily solves many types of linear or nonlinear ordinary or partial differential equations, and integral equations [1, 2, 3, 4].

The ADM solves the problems in direct way and in an uncomplicated fashion without using linearization, perturbation or any other unpreferable assumptions that may change the physical behavior of the model, also the method is capable of greatly reducing the size of the computational work while still maintaining high accuracy of the numerical solution.

THE ADM has led to several modifications on the method made by various researchers in an attempt to improve the accuarcy or expand the application of the original method. To begin with, Adomian and Rach [5] introduced modified Adomian polynomials which converge slightly faster than the original or classical Adomian polynomials and are convenient for computer generation. The modified polynomials are defined using a differencing method. The first few terms of the modified Adomian polynomials generated are identical to the original Adomian polynomials, but higher order terms do exhibit differences. In addition to the classical and modified Adomian polynomials, Adomian also introduced accelerated Adomian polynomials [5, 24]. These Adomian polynomials provide faster convergence; however, they are less convenient computationally [5]. Despite the various types of the Adomian polynomials, the original Adomian polynomials are more generally used based on the advantage of a convenient algorithm which is easily remembered [24]. They are easily generated without a computer and converge rapidly enough for most problems [5] .

Proposed modifications to the standard ADM can be as simple as the following; Wazwaz [30] presented a reliable modification of the ADM. The modified ADM proposed by Wazwaz divides the original function f into two parts, one assigned to the initial term of the series and the other to the second term. All remaining terms of the recursive relationship are defined as previously, but the modification results in a different series being generated. This method has been shown to be computationally effecient; however, it does not always minimize the size of calculations needed and even requires much more calculations than the standard ADM. The success of the modified method depends mainly on the proper choice of the parts into which to divide the original function.

In 2001 Wazwaz and Al-sayed [29] presented a new modification of the ADM for linear and nonlinear operator, in the new modification Wazwaz replaced the process of dividing f into two components by a series of infinite components, the new modification introduce a promizing tool for linear and nonnlinear operator.

In 2005, Wazwaz [27] presented another type of modification to the ADM. The purpose of this new approach was to overcome the difficulties that arise when singular points are present. The modification arises in the initial definition of the operator when applying the ADM to the Emden-Fowler equation. According to Wazwaz [27], the ADM usually starts by defining the equation in an operator form by considering the lowest-ordered derivative in the problem. However, by defining the differential operator in terms of both derivatives in the equation, the singularity behavior was easily overcome. The most striking advantage of using this choice for the operator L is that it attacks the Emden-Fowler equation directly without any need for a transformation formula. This modification could prove useful for similar models with singularities.

In [13, 14, 17] Y. Q. Hasan, solved first and second-order ordinary differential equations by Modified ADM, the difficulty in using ADM directly to this type of equations, due to the existence of singular point at $x = 0$, is overcome here. He defined a new differential operator which can be used for singular and nonsingular ODEs.

Another modification was proposed by Luo [23]. This variation separates the ADM into two steps and therefore is termed the two-step ADM (TSAMD), the purpose behind the proposed scheme is to identify the exact solution more readily and eliminate some calculations. The two steps proposed by Luo [23] are as follows:

(1) Firstly, apply the inverse operator and the given conditions. Then, define a function, u_0, "where u_0 is the first term of the solution" based on the resulting terms. If this satisfies the original equation and the conditions as checked by substiution, it is considered the exact solution and the calcuations terminated. Otherwise, continue on to step two.

(2) Continue with the standard Adomian recursive relationship. As one can see, this modification involves verifying that the zeroth component of the series solution includes the exact solution [23]. As such, it offers the advantage of requiring less calculations than the standard ADM.

Another recent modification is termed the restarted Adomian method [9]. This method involves repeatedly updating the initial term of the series generated rather than calculating additional terms of the solution by determining Adomian polynomials for large indexes.

other researchers have developed modifications based on the operators to the (ADM).

1.2 Operator

An operator is a function that takes a function as an argument instead of numbers as we are used to dealing with in functions. We already know a couple of operators even if we did not know that they were operators. Here are some examples of operators

$$L = \frac{d}{dx}, \qquad L = \frac{\partial}{\partial x}, \qquad L = \int dx, \qquad L = \int_a^b dx.$$

If a function is plugged in, say in each of the above, then the following can be obtained

$$L(u) = \frac{du}{dx} \qquad L(u) = \frac{\partial u}{\partial x} \qquad L(u) = \int (u) dx \qquad L(u) = \int_a^b (u) dx.$$

These are simple examples of operators derivative and integral. A more complicated operator would be the heat operator. The heat operator can be found from a slight rewrite of the heat equation without sources. The heat operator is then

$$L = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}.$$

Also

$$L = \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right),$$

is another differential operator for a particular second order differential equation.

The operator L in second order differential equations is a twice differentiable function. The domain of L is the twice differentiable functions on an open interval I. The terminology L of the function u is used to describe $L(u)$, and the range of the functions on I (and hence $L(u)$ is itself a function on I). Generally, L is chosen

$$L(.) = \frac{d^p}{dx^p}(.),$$

for the p^{th} order differential equations and thus its inverse L^{-1} follows as the p-fold definite integration operator from x_0 to x. The operator L defined has the following basic property: If u_1 and u_2 are twice differentiable functions on I and c_1 and c_2 are constants, then

$$L\left[c_1 u_1 + c_2 u_2\right] = c_1 L\left[u_1\right] + c_2 L\left[u_2\right],$$

note: an operator L satisfying property above is called linear operator.

Example 1.2.1. *The differential equation*

$$u'' - \frac{1}{x}u' = 0,$$

can be rewritten in compact form as

$$\left(\frac{1}{x}u'\right)' = 0, \tag{1.1}$$

so from this, an operator can be generated to be:

$$L = \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right), \tag{1.2}$$

so that (1.2) can be written in an operator form as:

$$L[u] = \frac{d}{dx}\left(\frac{1}{x}\frac{du}{dx}\right),$$

after this, it can be verified that the operator is linear, bearing the basic properties of derivation of ordinary differential equations in mind and plugging in the expression $c_1u_1 + c_2u_2$ into the operator above it can be shown as:

$$\begin{aligned}
L[c_1u_1 + c_2u_2] &= \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}(c_1u_1 + c_2u_2)\right), \\
&= \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}(c_1u_1) + \frac{1}{x}\frac{d}{dx}(c_2u_2)\right), \\
&= \frac{d}{dx}\left(c_1\frac{1}{x}\frac{d}{dx}u_1 + c_2\frac{1}{x}\frac{d}{dx}u_2\right), \\
&= c_1\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}u_1\right) + c_2\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}u_2\right), \\
&= c_1L[u_1] + c_2L[u_2],
\end{aligned}$$

thus the operator at (1.2) is linear operator. An operator that is not linear is known as nonlinear operator.

In this thesis, nonlinear operators in which nonlinear functions are plugged are symbolized by some various representations like $N(u)$ and $F(u)$. These operators are used to determine the Adomian polynomials by the help of Adomian formula which is briefly discussed in the following chapter.

1.3 Adomian polynomials

The main part of ADM method is calculating Adomian polynomials for nonlinear polynomials.

In this section, we will obtaining the Adomian general formula for Adomian polynomials.

The decomposition method decomposes the solution $u(x)$ and the nonlinearity $N(u)$ into series

$$u(x) = \sum_{n=0}^{\infty} u_n \quad , N(u) = \sum_{n=0}^{\infty} A_n,$$

where A_n are the Adomian polynomials.
To compute A_n take $N(u) = f(u)$ to be a nonlinear function in u, where $u = u(x)$, and consider the Taylor series expansion of $f(u)$ around u_0

$$f(u) = f(u_0) + f'(u_0)(u - u_0) + \frac{1}{2!} f''(u_0)(u - u_0)^2 + \frac{1}{3!} f'''(u_0)(u - u_0)^3 + \cdots ,$$

but $u = u_0 + u_1 + u_2 + \cdots$,

then

$$f(u) = f(u_0) + f'(u_0)(u_1 + u_2 + u_3 + \cdots) + \frac{1}{2!} f''(u_0)(u_1 + u_2 + u_3 + \cdots)^2$$
$$+ \frac{1}{3!} f'''(u_0)(u_1 + u_2 + u_3 + \cdots)^3 + \cdots ,$$

by expanding all terms we get

$$f(u) = f(u_0) + f'(u_0)(u_1) + f'(u_0)(u_2) + f'(u_0)(u_3) + \cdots + \frac{1}{2!} f''(u_0)(u_1)^2$$
$$+ \frac{2}{2!} f''(u_0)(u_1 u_2) + \frac{1}{2!} f''(u_0)(u_1 u_3) + \cdots + \frac{1}{3!} f'''(u_0)(u_1)^3 + \frac{3}{3!} f'''(u_0) u_1^2 u_2$$

$$+\frac{1}{3!}f'''(u_0)u_1^2u_3+\cdots,$$

now, let li be the order of u_l^i and $l(i)+m(j)$ be the order of $u_l^iu_m^j$. Then A_n consists of all terms of order n, so we have

$$A_0 = f(u_0),$$

$$A_1 = u_1 f'(u_0),$$

$$A_2 = u_2 f'(u_0) + \frac{1}{2!}u_1^2 f''(u_0),$$

$$A_3 = u_3 f'(u_0) + \frac{2}{2!}u_1u_2 f''(u_0) + \frac{1}{3!}u_1^3 f''(u_0),$$

$$A_4 = u_4 f'(u_0) + \left[\frac{1}{2!}u_2^2 + u_1u_3\right] f''(u_0) + \frac{1}{2!}u_1^2u_2 f'''(u_0) + \frac{1}{4!}u_1^4 f''''(u_0).$$

$\vdots$

Hence,

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0}, n \geq 0. \tag{1.3}$$

To find the A_n's by Adomian general formula, these polynomials will be computed as follows:

$$A_0 = N(u_0),$$

$$A_1 = N(u_0)u_1 = \frac{d}{d\lambda}N(u_0+\lambda u_1)\bigg|_{\lambda=0},$$

$$A_2 = N'(u_0)u_2 + \frac{1}{2!}N''(u_0)u_1^2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}N(u_0+\lambda u_1+\lambda^2 u_2)\bigg|_{\lambda=0},$$

$$A_3 = N'(u_0)u_3 + \frac{2}{2!}N''(u_0)u_1u_2 + \frac{1}{3!}N'''(u_0)u_1^3 = \frac{1}{3!}\frac{d^2}{d\lambda^2}N(u_0+\lambda u_1+\lambda^2 u_2 + +\lambda^3 u_3)\bigg|_{\lambda=0},$$

$\vdots$

Example 1.3.1. *The Adomian polynomials of*

$$f(u) = u^5$$

are

$A_0 = u_0^5,$

$A_1 = 5u_0^4 u_1,$

$A_2 = 5u_0^4 u_2 + 10u_0^3 u_1^2,$

$A_3 = 5u_0^4 u_3 + 20u_0^3 u_1 u_2 + 10u_0^3 u_1^3,$

$\vdots$

for more example see [12, 34].

1.4 Analysis of the ADM

As well the ADM consist of decomposing the unknown function $u(x, y)$ of any equation into a sum of an infinite number of components defined by the decomposition series

$$u(x, y) = \sum_{i=1}^{\infty} u_n(x, y), \tag{1.4}$$

where the components $u_n(x, y), n \geq 0$ are to be determined in a recursive manner. The ADM concerns itself with finding the components
$u_0, u_1, u_2, \cdots$ individually.

The determinant of these component can be achieved in any easy way through a recursive relation that usually involve simple integrals. This technique is very simple in an abstract formulation but the difficulty arises in calculating the Adomian polynomials and proving the convergence of the series of the function.

The ADM consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian polynomials and finding the successive terms of the series solution by recurrent relation using Adomian polynomials. The solution is found as an infinite series in which each term can be easily determined and that converges quickly towards an accurate solution.

The ADM is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation and continuous with no need to discretization and consequent computer-intensive calculations.

ADM for ODEs

To give a clear overview of ADM, we consider a differential equation

$$F(u(t)) = g(t),$$

where F represents a general nonlinear ordinary or partial differential operator including both linear and nonlinear terms. Linear terms are decomposed into $L + R$, where L is invertible and is taken as the highest order derivative, and R is the remainder of the linear operator. Thus the equation may be written as

$$Lu + Nu + Ru = g, \tag{1.5}$$

where $N(u)$ represents the nonlinear terms. Solving for Lu, we obtain

$$Lu = g - Nu - Ru. \tag{1.6}$$

Operating on both sides of eq. (1.6) with L^{-1} we have,

$$L^{-1}Lu = L^{-1}g - L^{-1}Nu - L^{-1}Ru. \tag{1.7}$$

The decomposition method represents the solution $u(x,t)$ as a series of this form,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{1.8}$$

The nonlinear term Nu is decomposed as

$$N(u) = \sum_{n=0}^{\infty} A_n. \tag{1.9}$$

Substitute eq. (1.8) and eq. (1.9) into eq. (1.7) we get,

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi_0 + L^{-1}g(x) - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{1.10}$$

where,

$$\varphi_0 = \begin{cases} u(0), & \text{if } L = \dfrac{d}{dx}, \\ u(0) + xu'(0), & \text{if } L = \dfrac{d^2}{dx^2}, \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0), & \text{if } L = \dfrac{d^3}{dx^3}, \\ \qquad\vdots & \\ u(0) + xu'(0) + \dfrac{x^2}{2!}u''(0) + \cdots + \dfrac{x^n}{n!}u^{(n)}(0), & \text{if } L = \dfrac{d^{n+1}}{dx^{n+1}}. \end{cases} \tag{1.11}$$

Therefore

$$\begin{cases} u_0 = \varphi_0 + L^{-1}g(x), \\ u_1 = -L^{-1}Ru_0 - L^{-1}A_0, \\ u_2 = -L^{-1}Ru_1 - L^{-1}A_1, \\ \qquad\vdots \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n, n \geq 0, \end{cases} \tag{1.12}$$

where A_n are the Adomian polynomials generated for each nonlinearity so that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1, u_2 and etc.

The Adomian polynomials are obtained from the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0}, n = 0, 1, 2, \cdots. \tag{1.13}$$

We write the first five Adomian polynomials

$$\begin{cases} A_0 = N(u_0), \\ A_1 = u_1 N'(u_0), \\ A_2 = u_2 N'(u_0) + \dfrac{1}{2!} u_1^2 N''(u_0), \\ A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \dfrac{1}{3!} u_1^3 N'''(u_0), \\ A_4 = u_4 N'(u_0) + \left[\dfrac{1}{2!} u_2^2 + u_1 u_3 \right] N''(u_0) + \dfrac{1}{2!} u_1^2 u_2 N'''(u_0) + \dfrac{1}{4!} u_1^4 N''''(u_0), \\ \vdots \end{cases} \tag{1.14}$$

So, the practical solution for the n terms approximation is

$$\phi_n = \sum_{i=0}^{n-1} u_i, \tag{1.15}$$

where

$$u(x,t) = \lim_{n \to \infty} \phi_n(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{1.16}$$

We now demonstrate the ADM on the following illustrative examples.

Example 1.4.1. *Consider the second order linear ordinary differential equation*

$$u^{''} - u = 1, \tag{1.17}$$

subject to the initial conditions

$$u(0) = 0, \quad u^{'}(0) = 1.$$

In operator form, eq. (1.17) can be written as

$$Lu = 1 + u, \tag{1.18}$$

where L is the second order differential operator $Lu = u^{''}$, so $L^{-1}$is given by

$$L^{-1}(.) = \int_0^x \int_0^x (.) dx dx.$$

Applying L^{-1} to both sides of (1.18) and using the initial conditions into (1.19) gives

$$u = u(0) + xu'(0) + L^{-1}(1) + L^{-1}(u) = x + \frac{x^2}{2} + L^{-1}(u),$$

applying eq. (1.8) to the last eq. we have

$$\sum_{n=0}^{\infty} u_n = x + \frac{x^2}{2} + L^{-1}(\sum_{n=0}^{\infty} u_n),$$

this leads to the recursive relation

$$\begin{cases} u_0 = x + \frac{x^2}{2}, \\ u_{n+1} = L^{-1}(u_n), \quad n \geq 0. \end{cases}$$

The first few components are thus determined as follows:

$$u_0 = x + \frac{x^2}{2},$$

$$u_1 = \frac{x^3}{6} + \frac{x^4}{24},$$

$$u_2 = \frac{x^5}{5!} + \frac{x^6}{6!}.$$

Consequently, the solution in a series form is given by

$$u(x) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

and clearly in a closed form is given by

$$u(x) = e^x - 1.$$

ADM for PDEs

Previously, we applied the (ADM) to ordinary differential equations. Now, we will show how the method can be implemented to partial differential equations as well. Consider the general partial differential equation written in operator form:

$$L_x u + L_t u + Ru + Fu = g, \tag{1.19}$$

where $L_x$is the highest order differential in x, L_t is the highest order differential in t, R is the remainder of differential operator consisting of lower order derivatives, $F(u)$ is an analytic nonlinear term, and g is the specified inhomogeneous term. Applying the inverse operator L_x^{-1}, the equation (1.19) becomes

$$u = \varphi_0 - L_x^{-1}L_t u - L_x^{-1}Ru - L_x^{-1}F(u) + L_x^{-1}g, \tag{1.20}$$

where

$$\varphi_0 = \begin{cases} u(0,t), & \text{if } L = \dfrac{\partial}{\partial x}, \\ u(0,t) + xu_x(0,t), & \text{if } L = \dfrac{\partial^2}{\partial x^2}, \\ u(0,t) + xu_x(0,t) + \dfrac{x^2}{2!}u_{xx}(0,t), & \text{if } L = \dfrac{\partial^3}{\partial x^3}, \\ \qquad\vdots & \\ u(0,t) + xu_x(0,t) + \dfrac{x^2}{2!}u_{xx}(0,t) + \cdots + \dfrac{x^n}{n!}u_{xx\ldots(ntimes)}(0,t), & \text{if } L = \dfrac{\partial^{n+1}}{\partial x^{n+1}}. \end{cases}$$

The method admits the decomposition of $u(x,t)$ into an infinite series of terms expressed as:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{1.21}$$

and the nonlinear term $F(u)$ is to be equated to an infinite series of polynomials

$$F(u(x,t)) = \sum_{n=0}^{\infty} A_n, \tag{1.22}$$

where A_n are the Adomian polynomials that represent the nonlinear term $F(u(x,t))$, insertinging (1.21) and (1.22) into (1.4) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi_0 + L_x^{-1}L_t \sum_{n=0}^{\infty} u_n(x,t) - L_x^{-1}R \sum_{n=0}^{\infty} u_n(x,t) - L_x^{-1} \sum_{n=0}^{\infty} A_n(x,t) + L_x^{-1}g.$$

The various terms $u_n(x,t)$ of the solution $u(x,t)$ can be easily determined by using the recursive relation

$$\begin{cases} u_0(x,t) = \varphi_0 + L_x^{-1}g, \\ u_{n+1}(x,t) = -L_x^{-1}L_t u_n(x,t) - L_x^{-1}Ru_n(x,t) - L_x^{-1}A_n, \quad n \geq 0. \end{cases} \tag{1.23}$$

Consequently, the first few terms of the solution are given by

$u_0 = \varphi_0 + L_x^{-1} g,$

$u_1 = -L_x^{-1} L_t u_0(x,t) - L_x^{-1} R u_0(x,t) - L_x^{-1} A_0,$

$u_2 = -L_x^{-1} L_t u_1(x,t) - L_x^{-1} R u_1(x,t) - L_x^{-1} A_1,$

$u_3 = -L_x^{-1} L_t u_2(x,t) - L_x^{-1} R u_2(x,t) - L_x^{-1} A_2,$

$\vdots$

Example 1.4.2. *Consider the following homogeneous partial differential equation*

$$u_x - u_y = 0, \quad u(0,y) = y, \quad u(x,0) = x. \tag{1.24}$$

In an operator form, eq. (1.24) becomes

$$L_x u(x,y) = L_y u(x,y), \tag{1.25}$$

where the operator L_x and L_y are defined by

$$L_x = \frac{d}{dx} \ \ and \ \ L_y = \frac{d}{dy}.$$

Applying the inverse operator L^{-1} to both side of (1.31) and using the given condition $u(0,y) = y$ yields

$$u(x,y) = y + L_x^{-1}(L_y u), \tag{1.26}$$

define the unknown function $u(x,y)$ by the decomposition series

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y), \tag{1.27}$$

inserting (1.27) into both sides of (1.26) gives

$$\sum_{n=0}^{\infty} u_n(x,y) = y + L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right), \tag{1.28}$$

by considering few terms of the decomposition of $u(x,y)$, eq. (1.34) becomes

$$u_0 + u_1 + u_2 + \cdots = y + L_x^{-1}(L_y(u_0 + u_1 + u_2 + \cdots)),$$

proceeding as before, we identify the zeroth component u_0 by

$$u_0(x, y) = y,$$

after identifying the zeroth component $u_0(x, y)$, we obtain the recursive scheme

$$\begin{cases} u_0(x, y) = y, \\ u_{n+1}(x, y) = L_x^{-1} L_y(u_n), n \geq 0. \end{cases} \tag{1.29}$$

The components $u_0, u_1, u_2, \cdots$ are thus determined as follows :

$u_0(x, y) = y,$

$u_1(x, y) = L_x^{-1} L_y u_0 = L_x^{-1} L_y(y) = x,$

$u_2(x, y) = L_x^{-1} L_y u_1 = L_x^{-1} L_y(x) = 0,$

it is obvious that the component $u_n(x, y) = 0, n \geq 2$. Consequently, the solution is given by

$$u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + \cdots = u_0(x, y) + u_1(x, y) = y + x,$$

hence the exact solution of the homogeneous partial deferential equation in eq.(1.24) is given by

$$u(x, y) = x + y.$$

Example 1.4.3. *Consider the initial value problem of nonlinear partial differential equation*

$$u_{xx} + \frac{1}{4} u_t^2 = u(x, t), \quad u(0, t) = 1 + t^2, \quad u_x(0, t) = 1. \tag{1.30}$$

We first rewrite eq. (1.30) in an operator form as

$$L_x u = u - \frac{1}{4} u_t^2$$

where L_x is a second order partial differential operator. Operating with L_x^{-1} both sides of the last PDE and using the initial conditions gives

$$u = 1 + t^2 + x + L_x^{-1} u - \frac{1}{4} L_x^{-1} u_t^2.$$

Applying eq. (1.8) and (1.9) we have

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + t^2 + x + L_x^{-1}(\sum_{n=0}^{\infty} u_n(x,t)) - \frac{1}{4}L_x^{-1}(\sum_{n=0}^{\infty} A_n(x,t))$$

Recursively we determine $u_0, u_1, u_2,$,to obtain

$$u_0(x,t) = 1 + t^2 + x,$$

$$u_{n+1}(x,t) = L_x^{-1}u_n(x,t) - \frac{1}{4}L_x^{-1}A_n, \quad n \geq 0,$$

where A_n are the Adomian polynomials. The first few polynomials for the nonlinear quadratic term u_t^2 are given by

$A_0 = u_{0t}^2,$

$A_1 = 2u_{0t}u_{1t},$

$A_2 = 2u_{0t}u_{2t} + u_{1t}^2.$

$\vdots$

Consequently, the first three terms of the solution $u(x,t)$ are given by

$u_0(x,t) = 1 + t^2 + x,$

$u_1(x,t) = L_x^{-1}u_0 - \frac{1}{4}L_x^{-1}A_0 = L_x^{-1}(1+x) = \frac{x^2}{2!} + \frac{x^3}{3!},$

$u_2(x,t) = L_x^{-1}u_1 - \frac{1}{4}L_x^{-1}A_1 = L_x^{-1}(\frac{x^2}{2!} + \frac{x^3}{3!}) = \frac{x^4}{4!} + \frac{x^5}{5!},$

$\vdots$

thus, the infinite solution in a series form is given by

$$u(x,t) = t^2 + (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots).$$

Note that infinite series is the McLaurin series expansion of e^x. Indeed, the latter equation leads to the exact solution of our initial value problem which is given by

$$u(x,t) = t^2 + e^x,$$

for more example see [31, 32].

1.5 Modified Adomian polynomials

In this section a new class of the Adomian Polynomials is defined, denoted by$\overline{A}_n$. In the ADM for solving nonlinear differential or partial differential equations [1, 2, 3, 4, 32], Several studies such as Rach [5], zhu [34], Wazwaz [28], Duan [10], [11] have been proposed to modifiy the regular Adomian polynomials A_n, a rapidly converging approximant to the solution u denoted by $\varphi_m[u] = \sum_{n=0}^{m-1} u_n$. Then $u(x,t) = \lim_{m\to\infty} \varphi_m[u] = \lim_{m\to\infty} \sum_{n=0}^{m-1} u_n = u$,
where u_n are components to be determined such that we have convergence to u.
Now we make an analogous definition that just as $\varphi_m[u]$ or simply φ_m approximates u, i.e.,

$$\varphi_m[u] = \sum_{n=0}^{m-1} u_n,$$

$\Phi_m[f(u)]$ similarly approximates $f(u)$, or

$$\phi_m[f(u)] = \sum_{n=0}^{m-1} \overline{A}_n.$$

The $\overline{A}_n$, represent a new class of the Adomian polynomials and the $\lim_{m\to\infty}[\Phi_m = \sum_{n=0}^{m-1} \overline{A}_n] = f(u)$. Thus we view $\Phi_m = f(u)$ and $\varphi_m = u$ as truncated representations of $f(u)$ and u. The $\overline{A}_n$, can now be defined by:

$$\overline{A}_m = \Phi_{m+1} - \Phi_m, \tag{1.31}$$

just as

$$u_m = \varphi_{m+1} - \varphi_m.$$

From $\Phi_m = \sum_{n=0}^{m-1} \overline{A}_n$, we see that $\Phi_1 = \overline{A}_0$. for $m \geq 1$,

$$\overline{A}_m = \varphi_{m+1}[f(u)] - \varphi_m[f(u)],$$

thus

$$\overline{A}_1 = \varphi_2[f(u)] - \varphi_1[f(u)],$$

$$\overline{A}_2 = \varphi_3[f(u)] - \varphi_2[f(u)],$$

$$\overline{A}_3 = \varphi_4[f(u)] - \varphi_3[f(u)],$$
$$\vdots$$

we can also write from (1.31)

$$\overline{A}_1 = \Phi_2 - \Phi_1,$$

$$\overline{A}_2 = \Phi_3 - \Phi_2,$$

$$\overline{A}_3 = \Phi_4 - \Phi_3,$$
$$\vdots$$

the Φ_m are conveniently defined as:

$$\Phi_m = \sum_{n=0}^{m-1} \frac{(\varphi_{m-n+1} - u_0)^n}{n!} f^n(u_0),$$

hence,

$$\Phi_1 = f(u_0),$$

$$\Phi_2 = f(u_0) + u_1 f^1(u_0),$$

$$\Phi_3 = f(u_0) + (\varphi_3 - u_0) f^1(u_0) + \frac{(\varphi_2 - u_0)^2}{2!} f^2(u_0),$$
$$\vdots$$

from which

$$\overline{A}_0 = f(u_0),$$

$$\overline{A}_1 = \Phi_2 - \Phi_1 = u_1 f^1(u_0),$$

$$\overline{A}_2 = \Phi_3 - \Phi_2 = u_2 f^1(u_0) + (\frac{u_1^2}{2!}) f^2(u_0),$$

which so far, are identical to the classical or original A_0, A_1, A_2, respectively. For $m \geq 3$, $\overline{A}_m = A_m$. To see this, we calculate Φ_4 and A_3.

$$\Phi_4 = f(u_0) + (\varphi_4 - u_0) f^1(u_0) + \frac{(\varphi_3 - u_0)^2}{2!} f^2(u_0) + \frac{(\varphi_2 - u_0)^3}{3!} f^3(u_0),$$

since

$$\overline{A}_3 = \Phi_4 - \Phi_3 = u_3 f^1(u_0) + (\frac{u_2^2}{2!} + u_1 u_2) f^2 u_0 + (\frac{u_1^3}{3!}) f^3 u_0,$$

but

$$A_3 = u_3 f^1(u_0) + u_1 u_2 f^2 u_0 + (\frac{u_1^3}{3!}) f^3 u_0,$$

clearly, then the decomposition components u_n of the solution u of a differential equation using the A_n for nonlinearities are equal to the components using the $\overline{A}_n$ for u_0, u_1, u_2, u_3 but not for $u_4, u_5, \cdots$.

Example 1.5.1. $\dfrac{du}{dx} = u^2, \quad u(0) = 1.$

In an operator form write

$$Lu = u^2.$$

Applying L^{-1} to both sides yeild

$$u = u(0) + L^{-1} u^2,$$

$$u = 1 - L^{-1} \sum_{n=0}^{\infty} A_n.$$

Using the original A_n,

$A_0 = u_0^2,$

$A_1 = 2u_0 u_1,$

$A_2 = u_1^2 + 2u_0 u_2,$

$A_3 = 2u_1 u_2 + 2u_0 u_3,$

$A_4 = u_2^2 + 2u_1 u_3 + 2u_0 u_4,$

$A_5 = 2u_2 u_3 + 2u_1 u_4 + 2u_0 u_3,$

$\vdots$

if we use the $\overline{A}_n$, we have

$\overline{A}_0 = u_0^2,$

$\overline{A}_1 = 2u_0 u_1,$

$\overline{A}_2 = u_1^2 + 2u_0 u_2,$

$$\overline{A}_3 = u_2^2 + 2u_1u_2 + 2u_0u_3,$$
$$\overline{A}_4 = u_3^3 + 2u_0u_4 + 2u_1u_3 + 2u_2u_3,$$
$$\overline{A}_5 = u_2^4 + 2u_0u_5 + 2u_1u_4 + 2u_2u_4 + 2u_3y_4,$$
$$\vdots$$

we note a difference from the original A_n, beginning with $\overline{A}_3$ which appears in the fourth term of the decomposition. The regular polynomials A_n have generally been used because they are simply generated, The convergence of the $\overline{A}_n$ is slightly faster than for the A_n since the two are identical until $\overline{A}_3$.

Chapter 2

Some Modifications of the ADM Based on the Assumptions

Several researchers have developed modifications to the ADM [8, 13, 23, 27]. The modifications arise from evaluating difficulties specific for the type of problem under consideration. Usually the modification involves only a slight change and is aimed at improving the convergence or accuracy of the series solution. This further demonstrates the wide applicability that the ADM has, as well as its simplicity since it can be easily modified for the situation at hand. In this chapter we present some modifications of the ADM where the assumptions made by Adomian were modified.

Note that, the modified ADM will be applied wherever it is appropriate, to all partial differential equations of any order. The modified ADM may give the exact solution after just two iterations only and without using the Adomian polynomials.

2.1 The modified decomposition method by Wazwaz

The assumptions made by Adomian were modified in (1999) by Wazwaz [30]. In (2001) Wazwaz and Al-sayed considered a new modification [29]. In this section we present these two modifications.

2.1.1 The first modified (MADM1)

Wazwaz presented a reliable modification of the Adomian decomposition method. As we know the ADM suggest that the zeroth component u_0 usually defined by function $f = \varphi + L^{-1}g$. But the modified decomposition method proposed by Wazwaz was established based on the assumption that the function f can be divided into two parts one assigned to the initial term of the series and the other to the second term. All remaining terms of the recursive relationship are defined as previously, but the modification results in a different series being generated. This method has been shown to be computationally efficient; however, it does not always minimize the size of calculations needed. The success of the modified method depends mainly on the proper choice of the parts into which to divide the original function. Under this assumption we set

$$f = f_0 + f_1.$$

Based on this, we formulate the modified recursive relation as follows:

$$\begin{cases} u_0(x) = f_0, \\ u_1(x) = f_1 - L^{-1}(Ru_0) - L^{-1}(A_0), \\ u_{n+1}(x) = -L^{-1}(Ru_n) - L^{-1}(A_n), \quad n \geq 0. \end{cases} \tag{2.1}$$

Having calculated the component $u_n(x, y)$, the solution in a series form follows immediately.

Although this variation in the formation of u_0 and u_1 is slight, however it plays a major role in accelerating the convergence of the solution and in minimizing the size of calculations.

Furthermore, there is no need sometimes to evaluate the so-called Adomian polynomials required for nonlinear operators. Two important remarks related to the modified method were made in this section. First, by proper selection of the function f_0 and f_1 , the exact solution u may be obtained by using very few iterations, and sometimes by evaluating only two components. The success of this modification depends only on the choice of f_0 and f_1 , and this can be made through trials, that are the only criteria which can be applied so far. Second, if g consists of one term only, the standard decomposition method should be employed in this case, see [29].

2.1.2 The second modified (MADM2)

As indicated earlier, although the modified decomposition method may provide the exact solution by using two iterations only, and sometimes without any need for Adomian polynomials, but its effectiveness depends on the proper choice of f_0 and f_1 . In the new modification, Wazwaz and Al-sayed [33] replaces the process of dividing f into two components by a series of infinite components. He suggests that f be expressed in Taylor series

$$f = \sum_{n=0}^{\infty} f_n. \tag{2.2}$$

Moreover, he suggest a new recursive relationship expressed in the form

$$\begin{cases} u_0 = f_0, \\ u_{n+1} = f_{n+1} - L^{-1}\left(Ru_n\right) - L^{-1}\left(A_n\right), \; n \geq 0. \end{cases} \tag{2.3}$$

having calculated the component $u_n(x, y)$, the solution in a series form follows immediately.

We can observe that algorithm (2.3) reduces the number of terms involved in each standard ADM only. Moreover this reduction of terms in each component facilitates the construction of Adomian polynomials for nonlinear operators. The new modification overcomes the difficulty of decomposing $f(x)$, and introduces an efficient algorithm that improves the performance of the standard ADM.

Note: If f consists of one term only, then scheme (2.3) reduces to ADM relation . Moreover, if f consists of two terms, then relation (2.3) reduces to the modified relation (2.1), see [30].

2.2 The two-step ADM (TSADM)

Although the modified decomposition method may provide the exact solution by using two iteration only, the criterion of dividing the function f into two practical parts, and the case where f consists only of one term remain unsolved so far.In fact, as will be seen from the examples below, the effort to divide f into two parts is useless and may even decelerates the convergence sometimes.

Another modification of ADM was proposed by Luo [23]. This variation separates the ADM into two steps and therefore is termed the two-step ADM. The purpose behind the proposed scheme is to identify the exact solution more readily and eliminate some calculations as such. The two steps proposed by Luo are as follows: Firstly, apply the inverse operator and the given conditions. Then, define a function, u_0, based on the resulting terms. If this satisfies the original equation and the conditions as checked by substitution, it is considered the exact solution and the calculations terminated. Otherwise, continue on to step two. In step tow we are continue with the standard Adomian recursive relationship. As one can see, this modification involves "verifying that the zeroth component of the series solution includes the exact solution". As such, it offers the advantage of requiring less calculations than the standard ADM.

The main ideas of the TSADM method are:

(1) Applying the inverse operator L^{-1} and using the given condition we obtain

$$\Phi = \varphi + L^{-1}g, \tag{2.4}$$

where the function φ represents the terms arising from using the given conditions, all are assumed to be prescribed.To achieve the objectives of this study, we set

$$\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_m, \tag{2.5}$$

where $\Phi_0,\ \Phi_1,\ \cdots\ \Phi_m$ are the terms arising from integrating the source term and from using the given conditions. Based on this, we define

$$u_0 = \Phi_n + \cdots + \Phi_{n+s}, \tag{2.6}$$

where $n = 0, 1, 2, \cdots, m$, and $s = 0, 1, 2, \cdots, m - n$, then we verify that u_0 satisfies the original equation and the given condition by substitution, once the exact solution is obtained we finish. otherwise, we go to following step two.

(2) We set $u_0 = \Phi$ and continue with the standard Adomian recursive relation

$$u_{n+1} = -L^{-1}(Ru_n) - L^{-1}(A_n), \; n \geq 0. \tag{2.7}$$

Compared to the standard Adomian method and the modified method, we can see that the TSADM may provide the solution by using one iteration only. It is important to note that the procedure of verification in the first step can be significantly effective in many practical cases. This can be seen from the examples below by taking full advantage of the property of the original equation and the given conditions. Further, the TSADM method avoids the difficulties arising in the modified method.

2.3 Restarted ADM (RADM)

Basically the RADM has the same structure as that of the ADM but the ADM is used more than once. In practice, after applying the ADM and calculating m terms of the series solution, for arbitrary m, the summation of these terms is taken as the first term of the solution of the ADM and then the method starts again for arbitrary m' times. In other words, to apply the RADM, firstly we apply the ADM and set $\phi_m = u_0 + u_1 + \ldots + u_{m-1}$. Then the RADM begins when we choose $\phi_m(t)$ as the first term of the solution in the ADM; hence, in essence, we reset the initial term. The RADM can be summarized in the following algorithm, see [6, 9, 26].

The algorithm

Consider the differential equation

$$Lu + Ru + Nu = g. \tag{2.8}$$

Step 1. Choose positive natural numbers m, n, m'.

Step 2. Use the ADM to solve the differential equation and obtain $\Phi_m(t)$, then let $G(t) = \Phi_m(t)$.

Step 3. Add and substract $G(t)$ to right side of eq. (2.8)

For $k = 1$ to n, do Step 4. Let $u^{res}_{k,0}(t) = G(t)$.

Step 5. $u^{res}_{k,1}(t) = \varphi_0 + L^{-1}g(x) - L^{-1}Ru_0 - L^{-1}A_0 - G(t)$.

Step 6. $u^{res}_{k,n+1}(t) = L^{-1}g(x) - L^{-1}Ru_n - L^{-1}A_n$.

Step 7. Let

$$x^{res}(t) = \sum_{n=0}^{m'} u^{res}_{k,n}(t),$$
$$G(t) = u^{res}(t),$$

End for Step 8. Consider the approximate solution of the problem as $\Phi(t) = G(t)$.

See [26].

2.4 Examples

In this section, some initial value problems are considered to show the efficiency of each modified.

Example 2.4.1.

$$u' - u = x\cos x - x\sin x + \sin x, \quad u(0) = 0.$$

Applying $L^{-1}$to both sides yields

$$u(x) = x\sin x - x\cos x - sinx + L^{-1}u(x),$$

where $L(.) = \dfrac{d}{dx}(.)$, and $L^{-1}(.) = \int_0^t (.)dt.$

Then we have recursive relationship

$$\begin{cases} u_0 = xsinx - xcosx - sinx, \\ u_{n+1} = L^{-1}u_n, n \geq 0. \end{cases}$$

By using MADM1:

$u_0 = x \sin x,$

$u_1 = x \cos x - \sin x + L^{-1}u_0 = 0,$

$\vdots$

$u_{n+1} = L^{-1}u_n = 0, n \geq 0.$

Then the exact solution is $u(x) = x \sin x$

By using MADM2: the Taylor expansion for

$f(x) = x \sin x + x \cos x - \sin x$ is given by

$$f(x) = x^2 - \frac{2x^3}{3!} - \frac{x^4}{4!} + \frac{4x^5}{5!} + \frac{x^6}{6!} + \cdots,$$

then the recursive relationship

$u_0 = x^2,$

$u_1 = \dfrac{-2x^3}{3!} + L^{-1}u_0 = 0,$

$u_2 = \dfrac{-x^4}{3!} + L^{-1}u_1 = \dfrac{-x^4}{3!},$

$u_3 = 0,$

$u_4 = \dfrac{-x^6}{5!}.$

$\vdots$

The solution in a series form is given by $u(x) = x^2 - \dfrac{x^4}{3!} + \dfrac{x^6}{5!} + \cdots = x\left[x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots\right] =$

$x \sin x$.

By using TSADM: Let

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2,$$

$$\Phi_0 = x \sin x, \;\; \Phi_1 = x \cos x, \; \Phi_2 = -\sin x,$$

clearly Φ_0, Φ_1 and Φ_2 satisfy, by selecting $u_0 = \Phi_0$ and by verifying that u_0 justified the differential equation.

Then, the exact solution is obtained immediately $u = x \sin x$.

By using RADM: Let

$$\begin{cases} u_0 = x \sin x, \\ u_1 = x \cos x - \sin x + L^{-1} A_0 = 0. \end{cases}$$

Then

$$\Phi^1 = x \sin x,$$

then the exact solution is $u(x) = x \sin x$.

The following table display a comparison of absolute errors between the exact solution and approximate solutions by ADM and MADM2.

x	U Exact	ADM	MADM2	ADM Abs. error	MADM2 Abs. error
0.0	0.0	$3.10862 * 10^{-15}$	0.0	$3.10862 * 10^{-15}$	$1.54811 * 10^{-11}$
0.1	-0.00500836	-0.00500836	-0.00500836	$1.48548 * 10^{-11}$	$1.9028 * 10^{-9}$
0.2	-0.0201348	-0.0201348	-0.0201348	$2.19203 * 10^{-8}$	$7.46429 * 10^{-7}$
0.3	-0.0456917	-0.0456934	-0.0456917	$1.75982 * 10^{-6}$	$2.90542 * 10^{-5}$
0.4	-0.082229	-0.0822708	-0.082229	$4.18148 * 10^{-5}$	$4.39705 * 10^{-4}$
0.5	-0.130584	-0.131082	-0.130584	$4.98158 * 10^{-4}$	$3.96416 * 10^{-3}$

Table 2.1: Comparison of absolute errors between the exact solution and approximate solutions by ADM and MADM2.

Note: MADM2 does not always have higher accuracy than the ADM but we shown here that MADM2 are successfully applied to solve differential equation and minimize the size of calculations.

Example 2.4.2. *Consider the linear partial differential equation [8]*

$$u_{tt} + u_{xx} + u = 0, \tag{2.9}$$

with initial conditions $u(x, 0) = 1 + \sin x$*,* $u_t(x, 0) = 0$.

In an operator form the eq. (2.9) becomes

$$L_{tt}u(x, y) = -(u_{xx} + u), \tag{2.10}$$

where $L_{tt} = \dfrac{d^2}{dt^2}$, and $L_{tt}^{-1} = \int_0^1 \int_0^1 (.)dtdt$.

Applying L_{tt}^{-1} to both sides of (2.10) and using the initial condition we obtain

$$u(x, t) = 1 + \sin x - L_{tt}^{-1}(u + u_{xx}).$$

By using (MADM1)

we divide $f(x)$ into two parts,

$f_0 = 1$, and $f_1 = \sin x$,then we have from the recursive relation

$$\begin{cases} u_0 = 1, \\ u_1 = \sin x - L_{tt}^{-1}\left(u_0 + u_{0|_{xx}}\right), n \geq 0 \\ u_n + 1 = L_{tt}^{-1}\left(u_n + u_{n|_{xx}}\right), n \geq 0. \end{cases}$$

The first few component from the last recursive relation are

$$u_0 = 1,$$
$$u_1 = \sin x - L_{tt}^{-1}\left(u_0 + u_{xx}\right) = \sin x - \frac{1}{2!}t^2,$$
$$u_2 = -L_{tt}^{-1}\left(u_1 + u_{1|_{xx}}\right) = \frac{1}{4!}t^4,$$
$$u_3 = -L_{tt}^{-1}\left(u_2 + u_{2|_{xx}}\right) = -\frac{1}{6!}t^6,$$
$$\vdots$$
$$u(x,t) = u_0 + u_1 + u_2 + \cdots$$
$$= \sin x + 1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots$$
$$= \sin x + \cos t.$$

By using (MADM2) :

the Taylor expansion for $f(x) = 1 + \sin x$ is :

$$f(x) = 1 + x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots,$$

then we have from the recursive relation

$$u_0 = 1,$$
$$u_1 = x - L_{tt}^{-1}\left(u_0 + u_{xx}\right) = x - \frac{1}{2!}t^2,$$
$$u_2 = -\frac{1}{3!}x^3 - L_{tt}^{-1}\left(u_1 + u_{1|_{xx}}\right) = -\frac{1}{3!}x^3 - \frac{1}{2}xt^2 + \frac{1}{4!}t^4,$$
$$u_3 = \frac{1}{5!}x^5 - L_{tt}^{-1}\left(u_2 + u_{2|_{xx}}\right) = \frac{1}{5!}x^5 - \frac{1}{6!}t^6 + \frac{1}{24}xt^4 + \frac{1}{2}xt^2 + \frac{1}{12}x^3t^2,$$
$$\vdots$$

$$
\begin{aligned}
u(x,t) &= u_0 + u_1 + u_2 + \cdots, \\
&= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) + \left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4 - \frac{1}{6!}t^6 + \cdots\right), \\
&= \sin x + \cos t.
\end{aligned}
$$

An important conclusion that can be made here is that the exact solution was accelerate by using the modification more than the standard Adomian method .

Example 2.4.3. ***By using (TSADM) :***

Consider the partial differential equation

$$u_{xx} + (1-2x)u_{xy} + (x^2 - x - 2)u_{yy} = 0, \tag{2.11}$$

with the initial condition $u(x,0) = x, u_y(x,0) = 1.$

In an operator form the equation (2.11) becomes

$$L_{yy}u(x,y) = -\frac{(1-2x)}{(x^2-x-2)}u_{xy} - \frac{1}{(x^2-x-2)}u_{xx}, \tag{2.12}$$

where $L_{yy}(.) = \dfrac{d^2}{dy^2}(.)$, and $L_{yy}^{-1}(.) = \int_0^y \int_0^y (.)dydy.$

Applying L_{yy}^{-1} to both side of (2.12) and using the initial condition we obtain

$$u(x,y) = x + y - L_{yy}^{-1}\left[\frac{(1-2x)}{(x^2-x-2)}u_{xy} - \frac{1}{(x^2-x-2)}u_{xx}\right], \tag{2.13}$$

using the eq. (2.13) gives: $\Phi = \Phi_0 + \Phi_1 = x + y$

$\Phi_0 = x$, $\Phi_1 = y$,

by select $u_0 = x + y$ and verify that u_0 satisfies the eq. (2.11) and the given conditions.

Then the exact solution is

$$u(x,y) = x + y.$$

Example 2.4.4. ***By using (RADM) [7]:***

Consider the boundary value problem

$$u'' + \lambda\left(1 + u + u^2 + u^3\right) = 0, \quad 0 < x < 1, \tag{2.14}$$

$$u(0) = u(1) = 0.$$

Applying the standard ADM in eq.(2.14), we get

$$\sum_{n=0}^{\infty} u_n = -\lambda\left(L^{-1}(1) + L^{-1}\sum_{n=0}^{\infty} u_n + \frac{1}{2!}L^{-1}\sum_{n=0}^{\infty} A_n + \frac{1}{3!}L^{-1}\sum_{n=0}^{\infty} B_n\right), \quad n \geq 0. \tag{2.15}$$

This gives

$$\begin{aligned} u_0 &= -\frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda x, \\ u_{n+1} &= -\lambda\left(L^{-1}u_n + \frac{1}{2!}L^{-1}A_n + \frac{1}{3!}L^{-1}B_n\right), \quad n \geq 0. \end{aligned} \tag{2.16}$$

Adding and subtracting $g(x)$ to right side of eq. (2.15) to obtain

$$\sum_{n=0}^{\infty} u_n = G(x) - \lambda\left(L^{-1}(1) + L^{-1}\sum_{n=0}^{\infty} u_n + \frac{1}{2!}L^{-1}\sum_{n=0}^{\infty} A_n + \frac{1}{3!}L^{-1}\sum_{n=0}^{\infty} B_n\right) - G(x), \quad n \geq 0, \tag{2.17}$$

by equating the terms we can get

$$\begin{aligned} u_0 &= G(x), \\ u_1 &= -\lambda\left(L^{-1}(1) + L^{-1}u_0 + \frac{1}{2!}L^{-1}A_0 + \frac{1}{3!}L^{-1}B_0\right) - G(X), \\ u_{n+2} &= -\lambda\left(L^{-1}u_{n+1} + \frac{1}{2!}L^{-1}A_{n+1} + \frac{1}{3!}L^{-1}B_{n+1}\right), \quad n \geq 0, \end{aligned} \tag{2.18}$$

Step 1: In this step, $g(x)$ is calculated from eq. (2.16) as follows:

$$\begin{aligned} u_0 &= -\frac{1}{2}\lambda x^2 + \frac{1}{2}\lambda x, \\ u_1 &= -\frac{1}{2688}\lambda^4 x^8 + \frac{1}{672}\lambda^4 x^7. \end{aligned}$$

So

$$G(x) = \phi^1(x) = u_0 + u_1,$$

Step 2: Now, components of the RADM is computed from eq. (2.18) as follows:

$$
\begin{aligned}
u_0 &= G(x), \\
u_1 &= -\frac{1}{757447262208}\lambda^{13}x^{26} + \frac{1}{58265174016}\lambda^{13}x^{25} - \cdots, \\
u_2 &= \frac{1}{20709139646480822304768}\lambda^{22}x^{44} - \frac{1}{941324529385491922944}\lambda^{22}x^{34} + \cdots, \\
u_3 &= -\frac{1921}{110351113887194570160875411425 19808}\lambda^{31}x^{62} + \cdots, \\
u_4 &= -\frac{29478690975630536257743382096194416238070333 44}{\lambda}^{40} x^{80} + \cdots,
\end{aligned}
$$

so,

$$\phi^2 = u_0 + u_1 + u_2 + u_3 + u_4,$$

The approximate solution $u(x)$ is obtained in a series form

$$u(x) = \phi^2 = u_0 + u_1 + u_2 + u_3 + u_4.$$

x	Exact	ADM Abs. error	RADM Abs. error
0	0	0	0
0.1	0.0498467900	$2.9 * 10^{-5}$	$4.8 * 10^{-7}$
0.2	0.0891899350	$5.6 * 10^{-5}$	$9.4 * 10^{-7}$
0.3	0.1176090956	$7.9 * 10^{-5}$	$1.3 * 10^{-7}$
0.4	0.1347902526	$9.4 * 10^{-5}$	$1.6 * 10^{-7}$
0.5	0.1405392142	$9.9 * 10^{-5}$	$1.7 * 10^{-7}$
0.6	0.1347902526	$9.4 * 10^{-5}$	$1.6 * 10^{-7}$
0.7	0.1176090956	$7.9 * 10^{-5}$	$1.3 * 10^{-7}$
0.8	0.0891899350	$5.6 * 10^{-5}$	$9.4 * 10^{-7}$
0.9	0.0498467900	$2.9 * 10^{-5}$	$4.8 * 10^{-7}$
1.0	0	$2.5 * 10^{-11}$	$1 * 10^{-11}$

Table 2.2: Comparison of absolute errors between the exact solution and approximate solutions by ADM and RADM.

The obtained results indicate that the new techniques give more suitable and accurate solutions compared with the ADM.

Chapter 3

Some Modifications of ADM Based On The Operators

3.1 MADM3

In this section, we present a reliable modification of the ADM to solve singular and nonsingular initial value problems of the first, second and high order ordinary differential equations. Theoretical considerations have been discussed and the solutions are constructed in the form of a convergent series. Some examples are presented to show the ability of the method for linear and nonlinear problems.

We will show that, with the proper use of MADM3, it is possible to obtain an analytic solution to first order differential equation, singular or nonsingular. The difficulty in using ADM directly to this type of equations, due to the existence of singular point at $x = 0$, is overcome . Here we use the MADM3 for solving singular and nonsingular initial value problem of order one and two. It is demonstrated that this method has the ability of both linear and nonlinear ordinary differential equation.

3.1.1 First order ODEs

The first order ordinary differential equation can be consider as:

$$u' + p(x)u + F(x, u) = g(x), \tag{3.1}$$

with boundary condition $u(0) = A$,

where A is constant, $p(x)$ and $g(x)$ are given functions and $F(x, u)$ is a real function.

The ADM can not find the solution of (3.1) directly at $x = 0$. For example, we cannot find the solution of $u' + \dfrac{\sec^2 x}{\tan x}u = 2\sec^2 x$ at $x = 0$ by ADM.

For this reason, Hasan in [14] introduced a new modification of ADM (MADM3), he proposed a new differential operator which can be used for singular and nonsingular ODEs.

Method of solution

Define a new differential operator L in terms of the one derivative contained in the problem. Rewrite (3.1) in the form

$$Lu = g(x) - F(x, u), \tag{3.2}$$

where the differential operator is defined by

$$L(u) = e^{-\int p(x)dx}\frac{d}{dx}\left(e^{\int p(x)dx}u\right). \tag{3.3}$$

The inverse operator L^{-1} is therefore consider a one-fold integral operator, as below,

$$L^{-1}(.) = e^{-\int p(x)dx}\int_0^x e^{\int p(x)dx}(.)dx. \tag{3.4}$$

Applying L^{-1} of (3.4) to the first tow terms $u' + p(x)u$ of eq. (3.1). We find

$$L^{-1}(u' + p(x)u) = e^{-\int p(x)dx}\int_0^x e^{\int p(x)dx}(u' + p(x)u)dx = u - u(0)\phi(0)e^{-\int p(x)dx}$$

where $\phi(x) = e^{\int p(x)dx}$.

By operating L^{-1} on (3.3), we have

$$u(x) = u(0)\phi(0)e^{-\int p(x)dx}dx + L^{-1}g(x) - L^{-1}F(x,u). \qquad (3.5)$$

The ADM introduces the solution $u(x)$ by an infinite series of components

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \qquad (3.6)$$

and the nonlinear function $F(x,u)$ by an infinite series of polynomials

$$F(x,u) = \sum_{n=0}^{\infty} A_n, \qquad (3.7)$$

where the components $u_n(x)$ of the solution $u(x)$ will be determined recurrently and A_n are Adomian polynomial that can be constructed for various classes of nonlinearity according to specific algorithms set by Wazwaz [28]. For a nonlinear $F(u)$, the first few polynomials are given by

$$\begin{cases} A_0 = F(y_0), \\ A_1 = u_1 F'(u_0) \\ A_2 = u_2 F'(u_0) + \frac{1}{2!}u_1^2 F''(u_0), \\ A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F''(u_0), \\ \qquad \vdots \end{cases} \qquad (3.8)$$

Substituting (3.6) and (3.7) into (3.5) gives

$$\sum_{n=0}^{\infty} u_n(x) = u(0)\phi(0)e^{-\int p(x)dx}dx + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n.$$

To determine the components $u_n(x)$, we use ADM that suggests the use of the recursive relation

$$u_0(x) = u(0)\phi(0)e^{-\int p(x)dx}dx + L^{-1}g(x),$$

$$u_{n+1}(x) = -L^{-1}(A_n), n \geq 0,$$

which gives

$$\begin{cases} u_0(x) = u(0)\phi(0)e^{-\int p(x)dx}dx + L^{-1}g(x), \\ u_1(x) = -L^{-1}(A_0), \\ y_2(x) = -L^{-1}(A_1), \\ u_3(x) = -L^{-1}(A_2), \\ \qquad \vdots \end{cases} \tag{3.9}$$

from (3.8) and (3.9), we can determine the components $u_n(x)$ and hence the series solution of $u(x)$ in (3.6) can be immediately obtained. For numerical purposes, the n^{th}-term approximant

$$\Phi_n = \sum_{n=0}^{n-1} u_n(x),$$

can be used to approximate the exact solution.

Example 3.1.1. *We consider the linear singular initial value problem*

$$u' + \frac{\sec^2 x}{\tan x} u = 2\sec^2 x, \tag{3.10}$$

$$u(0) = 0.$$

we put

$$L(.) = \frac{1}{\tan x}\frac{d}{dx}\tan x(.),$$

so

$$L^{-1}(.) = \frac{1}{\tan x}\int_0^x \tan x(.)dx,$$

in an operator form eq. (3.10) becomes

$$Lu = 2\sec^2 x, \tag{3.11}$$

applying L^{-1} to both sides of (3.11) we have

$$LL^{-1}u = \frac{1}{\tan x}\int_0^x 2\tan x(\sec^2 x)dx,$$

$$\Rightarrow u(x) = \tan x.$$

Example 3.1.2. *Consider the nonlinear initial value problem*

$$u' + 2xu = 1 + x^2 + u^2, \tag{3.12}$$

$$u(0) = 1.$$

We put

$$L(.) = e^{-x^2} \frac{d}{dx} e^{x^2}(.),$$

so

$$L^{-1}(.) = e^{-x^2} \int_0^x e^{x^2}(.).$$

In an operator form, eq. (3.12) becomes

$$Lu = 1 + x^2 + u^2. \tag{3.13}$$

Applying the inverse operator L^{-1} to be the sides of eq.(3.13) we get:

$$u(x) = e^{-x^2} + L^{-1}(1 + x^2) + L^{-1}(u^2),$$

$$u_0 = e^{-x^2} + e^{-x^2} \int_0^x e^{x^2}(1 + x^2)dx,$$

by using Taylor series of e^{-x^2} and e^{x^2} with order 6 and Adomain polynomials mentioned we obtain

$$u_0 = 1 + x - x^2 - \frac{x^3}{3} + \frac{x^4}{2} + \frac{2x^5}{15} - \frac{x^6}{6} - \frac{4x^7}{105} - \frac{143x^9}{3780} + \cdots,$$

$$u_1 = x + x^2 - x^3 - \frac{7x^4}{6} + \frac{2x^5}{3} + \frac{32x^6}{45} - \frac{103x^7}{315} - \frac{383x^8}{1260} + \cdots,$$

$$u_2 = x^2 + \frac{4x^3}{3} - x^4 - \frac{29x^5}{15} + \frac{5x^6}{9} + \frac{14x^7}{9} - \frac{619x^8}{2520} + \cdots,$$

$$u_3 = x^3 + \frac{5x^4}{3} - \frac{13x^5}{15} - \frac{253x^6}{90} + \frac{7x^7}{45} - \frac{79x^8}{30} + \cdots,$$

$$u_4 = x^4 + 2x^5 - \frac{28x^6}{45} - \frac{236x^7}{63} - \frac{28x^8}{45} + \cdots,$$

$$u_5 = x^5 + \frac{7x^6}{3} - \frac{4x^7}{15} - \frac{2963x^8}{630} + \cdots,$$

$$\vdots$$

this means that the solution in a series form is given by

$$u(x) = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

$$= 1 + 2x + x^2 + x^3 + x^4 + x^5$$

and in the closed form

$$u(x) = x + \frac{1}{1-x}.$$

Example 3.1.3. *Consider the nonlinear initial value problem*

$$u' + 3x^2 u = e^x + 3u(\ln u)^2, \tag{3.14}$$

$$u(0) = 1.$$

We put

$$L(.) = e^{-x^3} \frac{d}{dx} e^{x^3}(.)$$

$$L^{-1}(.) = e^{-x^3} \int_0^x e^{x^3}(.)dx.$$

In an operator form, eq. (3.14) becomes

$$Lu = e^x + 3u(\ln u)^2 \tag{3.15}$$

Applying the inverse operator L^{-1} to both sides of eq. (3.15), we have

$$u(x) = e^{-x^3} + L^{-1}(e^x) + 3L^{-1}u(\ln u)^2,$$

$$u_0 = e^{-x^3} + L^{-1}(e^x),$$

$$= e^{-x^3} + e^{-x^3} \int_0^x e^{x^3+x} dx.$$

By using Taylor series of e^{-x^3} and e^{x^3+x} with order 8 and Adomain polynomials mentioned we obtain

$$u_0 = 1 + x + \frac{x^2}{2} - \frac{5x^3}{6} - \frac{17x^4}{24} - \frac{7x^5}{24} + \frac{301x^6}{6!} + \frac{1531x^7}{7!} + \frac{4411x^8}{8!} + \cdots,$$

$$
\begin{aligned}
u_1 &= x^3 + \frac{3x^4}{4} - \frac{9x^5}{10} - \frac{5x^6}{3} - \frac{127x^7}{280} + \frac{353x^8}{320} + \cdots ,\\
u_2 &= \frac{6x^5}{5} + \frac{5x^6}{4} - \frac{183x^7}{140} + \frac{235x^8}{80} + \cdots ,\\
u_3 &= \frac{51x^7}{35} + \frac{39x^8}{20} - \frac{1027x^9}{560} - \frac{15531x^10}{2800} + \cdots ,\\
&\vdots
\end{aligned}
$$

this means that the solution in a series form is given by

$$
\begin{aligned}
u(x) &= u_0 + u_1 + u_2 + u_3 + \cdots\\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}
\end{aligned}
$$

and in the closed form

$$u(x) = e^x.$$

3.1.2 Second order ODEs [19].

Consider the initial value problem in the second order ordinary differential equation in the form

$$
\begin{cases}
u'' + p(x)u' + F(x,u) = g(x),\\
u(0) = A, u'(0) = B,
\end{cases}
\tag{3.16}
$$

where $F(x,u)$ is a real function, $p(x)$ and $g(x)$ are given functions and A and B are constants.

Method of solution

Here, we propose the new differential operator, as below

$$L(.) = e^{-\int p(x)dx}\frac{d}{dx}\left(e^{\int p(x)dx}\frac{d}{dx}\right)(.),$$

so, the problem (3.16) can be written as,

$$Lu = g(x) - F(x,u). \tag{3.17}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below,

$$L^{-1}(.) = \int_0^x e^{-\int p(x)dx}\int_0^x e^{\int p(x)dx}(.)dx\,dx.$$

By applying L^{-1} on (3.17), we have

$$u(x) = \varphi(x) + L^{-1}g(x) - L^{-1}F(x,u), \tag{3.18}$$

such that

$$L(\varphi(x) = 0).$$

Recall that the ADM introduce the solution $y(x)$ and the nonlinear function $F(x,y)$ by infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{3.19}$$

and

$$F(x,y) = \sum_{n=0}^{\infty} A_n, \tag{3.20}$$

where the components $u_n(x)$ of the solution $u(x)$ will be determined recurrently as seen in the previous section.

$$\begin{cases} A_0 = F(u_0), \\ A_1 = u_1 F'(u_0), \\ A_2 = u_2 F'(u_0) + \dfrac{1}{2!}u_1^2 F''(u_0), \\ A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \dfrac{1}{3!}u_1^3 F'''(u_0), \\ \qquad \vdots \end{cases}$$

which can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function. By substituting (3.19) and (3.20) into (3.18), we get

$$\sum_{n=0}^{\infty} u_n(x) = \varphi(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n. \tag{3.21}$$

Through using ADM, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0(x) = \varphi(x) + L^{-1}g(x), \\ u_{n+1}(x) = -L^{-1}\sum_{n=0}^{\infty} A_n, \quad n \geq 0, \end{cases} \tag{3.22}$$

which gives

$$\begin{cases} u_0(x) = \varphi(x) + L^{-1}g(x), \\ u_1(x) = -L^{-1}(A_0), \\ u_2(x) = -L^{-1}(A_1), \\ u_3(x) = -L^{-1}(A_2), \\ \qquad \vdots \end{cases} \tag{3.23}$$

From (3.22) and (3.23), we can determine the components $u_n(x)$ and hence the series solution of $u(x)$ in (3.21) can be immediately obtained.

Example 3.1.4. *Consider the Lane-Emden equation formulated as, [27] ,*

$$\begin{cases} u'' + \frac{2}{x}u' + F(x,u) = g(x), 0 \leq x \leq 1 \\ u(0) = A, u'(0) = B. \end{cases} \tag{3.24}$$

where A and B are constants, $F(x,u)$ is a real function and $g(x) \in [0.1]$ is given.

Usually, the standard ADM is divergent to solve singular LaneEmden equations. To overcome the singularity behavior, Wazwaz [27] defined the differential operator L in terms of two derivatives contained in the problem. He rewrote (3.24) in the form

$$Lu = -F(x,u) + g(x),$$

where the differential operator L is defined by

$$L = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right).$$

Note that the above operator is a special kind of the proposed operator (3.1.2), since for LaneEmden problem (3.24), $p(x)$ is equal to $\frac{2}{x}$, so,

$$e^{-\int p(x)dx} = x^{-2},$$

and

$$e^{\int p(x)dx} = x^2,$$

therefore we have

$$L = e^{-\int p(x)dx} \frac{d}{dx} \left(e^{\int p(x)dx} \frac{d}{dx} \right) = x^{-2} \frac{d}{dx} \left(x^2 \frac{d}{dx} \right),$$

and

$$L^{-1}(.) = \int_0^x x^{-2} \int_0^x x^2(.)dx\, dx.$$

Example 3.1.5. *Consider the linear singular initial value problem*

$$\begin{cases} u'' + \dfrac{\cos x}{\sin x} u' = -2\cos x, \\ u(0) = 1,\ u'(0) = 0. \end{cases} \tag{3.25}$$

We put

$$L(.) = \frac{1}{\sin x} \frac{d}{dx} \sin x \frac{d}{dx}(.),$$

so

$$L^{-1}(.) = \int_0^x \frac{1}{\sin x} \int_0^x \sin x(.)dx\, dx.$$

In an operator form, eq. (3.25) becomes

$$Lu = -2\cos x, \tag{3.26}$$

Now, by applying L^{-1} to both sides of (3.26) we have

$$L^{-1}Lu = -2 \int_0^x \frac{1}{\sin x} \int_0^x \sin x(\cos(x))dx\, dx,$$

and it implies,

$$u(x) = u(0) + xu'(0) + \cos(x) - 1 \Rightarrow u(x) = \cos(x).$$

So, the exact solution is easily obtained by modified Adomian decomposition method.

Example 3.1.6. *Consider the linear nonsingular initial value problem*

$$\begin{cases} u'' + u' = 2x + 2, \\ u(0) = 0, u'(0) = 0. \end{cases} \tag{3.27}$$

According to (3.1.2), we put

$$L = e^{-x}\frac{d}{dx}e^{x}\frac{d}{dx},$$

so

$$L^{-1}(.) = \int_0^x e^{-x}\int_0^x e^{x}(.)dx\,dx.$$

In an operator form, eq. (3.27) becomes

$$Lu = 2x + 2. \tag{3.28}$$

Now, by applying L^{-1} to both sides of (3.28), we have

$$L^{-1}Lu = \int_0^x e^{-x}\int_0^x e^{x}(2x+2)dx\,dx,$$

and it implies that

$$u(x) = u(0) + u'(0) + x^2 = x^2.$$

So, the exact solution is easily obtained by proposed Adomian method.

Example 3.1.7. *Consider the nonlinear initial value problem*

$$\begin{cases} u'' + xu' + x^2u^3 = (2+6x^2)e^{x^2} + x^2e^{3x^2}, \\ u(0) = 1,\ u'(0) = 0. \end{cases} \tag{3.29}$$

with the exact solution $u(x) = e^{x^2}$.

According to (3.1.2), we put

$$L = e^{\frac{-x^2}{2}}\frac{d}{dx}e^{\frac{x^2}{2}}\frac{d}{dx},$$

so

$$L^{-1}(.) = \int_0^x e^{\frac{-x^2}{2}}\int_0^x e^{\frac{x^2}{2}}(.)dx\,dx.$$

In an operator from, eq. (3.29) becomes

$$Lu = \left(2+6x^2\right)e^{x^2} + x^2 e^{3x^2} - x^2 u^3$$

Now, by applying L^{-1} to both sides , we have

$$L^{-1}Lu = u(0) + xu'(0) + L^{-1}g - L^{-1}\left(x^2u^3\right).$$

And we have,

$$\begin{cases} u_0 = u(0) + xu'(0) + L^{-1}\left(g(x)\right), \\ u_{n+1} = -L^{-1}\left(A_n\right),\ n \geq 0. \end{cases}$$

We compute A_n's Adomian polynomials of nonlinear term x^2y^3, as below

$$\begin{cases} A_0 = x^2u^3, \\ A_1 = x^2(3u_0^2u_1), \\ A_2 = x^2(3u_0^2u_2 + 3u_0u_1^2), \\ A_3 = x^2(3u_0^2u_3 + 3u_0u_1^2 + 6u_0y_1u_2 + u_1^3), \\ \qquad \vdots \end{cases} \tag{3.30}$$

by using Taylor series of $g(x)$, $e^{\frac{-x^2}{2}}$ and $e^{\frac{x^2}{2}}$ with order 10 and Adomian polynomials mentioned in (3.30), we obtain,

$$\begin{cases} u_0 = 1 + x^2 + \frac{7x^4}{12} - \frac{7x^5}{24} + \frac{23x^6}{96} + \cdots, \\ u_1 + u_0 = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots, \\ u_0 + u_1 + u_2 = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + \frac{731x^{12}}{44354} + \cdots, \end{cases}$$

note that the Taylor series of the exact solution $y(x) = e^{x^2}$ with order 10 is as below

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120} + O(x^{11}).$$

Note that: the Adomian decomposition method is divergent to solve these type of second order ordinary differential equation.

3.1.3 High-order and system of nonlinear differential equations [18]

This section extends MADM3 for specific second order ordinary differential equations to high order and system of differential equations.

Consider the initial value problem in the n-order differential equation in the form:

$$\begin{cases} u^{(n)} + p(x)u^{(n-1)} + Nu = g(x), \\ u(0) = \alpha_0,\ u'(0) = \alpha_1, \ldots,\ u^{(n-1)}(0) = \alpha_{n-1}, \end{cases} \tag{3.31}$$

where N is a nonlinear differential operator of order less than $n-1$, $p(x)$ and $g(x)$ are given functions and $\alpha_0,\ \alpha_1, \ldots, \alpha_{n-1}$ are given constants.

Here, consider the differential operator, as below:

$$L = e^{-\int p(x)dx} \frac{d}{dx} \left(e^{\int p(x)dx} \frac{d^{n-1}}{d^{n-1}x} \right), \tag{3.32}$$

so, the problem (3.31) can be written as

$$Lu = g(x) - Nu. \tag{3.33}$$

The inverse operator L^{-1} is therefore considered an n-fold integral operator, as below:

$$L^{-1}(.) = \int_0^x \int_0^x \cdots \int_0^x e^{-\int p(x)dx} \int_0^x e^{\int p(x)dx} (.) dx \cdots dx. \tag{3.34}$$

By operating L^{-1} on (3.33), we have

$$u(x) = \varphi(x) + L^{-1}g(x) - L^{-1}Nu,$$

such that

$$L\varphi(x = 0),$$

so, we have

$$\sum_{n=0}^{\infty} u(x) = \varphi(x) + L^{-1}g(x) - L^{-1} \sum_{n=0}^{\infty} A_n.$$

Through using ADM, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0 = \varphi(x) + L^{-1}g(x), \\ u_{n+1} = -L^{-1}A_n, \quad n \geq 0. \end{cases}$$

The mentioned method can be used for solving system of differential equation in the following form

$$\begin{cases} u_1^{(n)} + p(x)u_1^{(n-1)} + F_1(x, y_1, \cdots, y_1^{(n-2)}, y_2, \cdots, y_2^{n-2}, y_n, \cdots, y_n^{(n-2)}) = g_1(x), \\ u_2^{(n)} + p(x)u_2^{(n-1)} + F_2(x, y_1, \cdots, y_1^{(n-2)}, y_2, \cdots, y_2^{n-2}, y_n, \cdots, y_n^{(n-2)}) = g_2(x), \\ \vdots \\ u_n^{(n)} + p(x)u_n^{(n-1)} + F_n(x, y_1, \cdots, y_1^{(n-2)}, y_2, \cdots, y_2^{n-2}, y_n, \cdots, y_n^{(n-2)}) = g_n(x), \end{cases}$$

Example 3.1.8. *Consider linear singular initial value problem in third order ordinary differential equation,*

$$\begin{cases} u^{'''} + \dfrac{cos(x)}{sin(x)}u^{'} = sin(x)cos(x), \\ u(0) = 0, \quad u^{'}(0) = -2, \ \ u^{''}(0) = 0. \end{cases} \tag{3.35}$$

According to (3.32) and (3.34) we put

$$L(.) = \frac{1}{sin(x)}\frac{d}{dx}(sin(x)\frac{d^2}{dx^2})(.),$$

so

$$L^{-1}(.) = \int_0^x \int_0^x \frac{1}{sin(x)}[\int_0^x sin(x)(.)dx]dxdx,$$

In an operator form, eq. (3.35) becomes

$$Lu = sin(x)cos(x). \tag{3.36}$$

Now by applying L^{-1} on both side of (3.36), one gets

$$L^{-1}Lu = \int_0^x \int_0^x \frac{1}{sin(x)}[\int_0^x sin(x)(sin(x)cos(x))dx]dxdx,$$

and this implies

$$u(x) = u(0) + xu^{'}(0) + \frac{3x^2}{4} - \frac{3}{8}cos(2x) = 1 - 2x + \frac{3x^2}{4} - \frac{3}{8}cos(2x).$$

Example 3.1.9. *Consider the nonlinear system of differential equation,*

$$\begin{cases} u'' + \tan u' + z^2 = g(x), \quad u(0) = 0, \ u'(0) = 0, \\ z'' + 100z' + y^2 = h(x), \quad z(0) = 0, \ z'(0) = 0, \end{cases} \tag{3.37}$$

where $g(x)$ and $h(x)$ are compatible to exact solutions

$$u(x) = x\sin x \; and \; z(x) = x\tan x.$$

Here, we use Taylor series of $g(x)$, $h(x)$ and $\tan x$ with order 9.

By using standard ADM. Here, we have

$$\begin{aligned} u_0 &= L^{-1}g(x) = x^2 + \frac{1}{24}x^6 + \frac{3}{224}x^8 + \cdots, \\ z_0 &= L^{-1}h(x) = x^2 + \frac{100}{3}x^3 +, \frac{1}{3}x^4 + \frac{20}{3}x^5 + \frac{1}{3}x^4 + \frac{1}{6}x^6 - \frac{40}{21}x^7 + \frac{121}{2520}x^8 + \cdots, \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} u_{n+1} &= -L^{-1}(\tan u_n') - L^{-1}A_n, \quad n \geq 0 \\ z_{n+1} &= -L^{-1}(100z_n') - L^{-1}B_n, \quad n \geq 0 \end{aligned} \tag{3.39}$$

where A_n and B_n are the Adomian polynomials of nonlinear terms y^2 and z^2. Also, $f(x)$ denoted the taylor series of $\tan x$ with order 9.

In this case, through considering (3.38) and (3.39), we have

$$\begin{cases} u_0 = x^2 + \frac{1}{24}x^6 + \frac{3}{224}x^8 + \cdots, \\ u_0 + u_1 = x^2 - \frac{1}{6}x^4 - \frac{1}{72}x^6 - \frac{100}{63}x^7 + \cdots, \\ \vdots \\ u_0 + u_1 + \cdots + u_6 = x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \frac{1}{5040}x^8 + \frac{200000}{81}x^9 + \cdots. \end{cases}$$

and

$$\begin{cases} z_0 = x^2 + \frac{100}{3}x^3 + \frac{1}{3}x^4 + \frac{20}{3}x^5 + \cdots, \\ z_0 + z_1 = x^2 - 833x^4 - \frac{4994}{45}x^6 - \frac{10}{21}x^7 + \cdots, \\ \vdots \\ z_0 + z_1 + \cdots + z_6 = x^2 + \frac{1}{3}x^4 + \frac{2}{15}x^6 + \frac{17}{315}x^8 + \frac{312500000000}{567}x^9 + \cdots. \end{cases}$$

So, the standard ADM converges to Taylor expansion of exact solution.

By using MADM3. By applying MADM to problem (3.37), we obtain:

$$\begin{cases} u_0 = x^2 - \frac{1}{6}x^4 + \frac{1}{24}x^6 + \frac{41}{5040}x^8 + \cdots, \\ \vdots \\ u_0 + u_1 = x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \frac{1}{5040}x^8 + \frac{1339}{1814400}x^{10} + \cdots. \end{cases}$$

and

$$\begin{cases} z_0 = x^2 + \frac{1}{3}x^4 + \frac{1}{6}x^6 + \cdots, \\ \vdots \\ z_0 + z_1 = x^2 + \frac{1}{3}x^4 + \frac{2}{15}x^6 + \frac{17}{315}x^8 + \frac{28350}{28350}x^{10} + \cdots. \end{cases}$$

which is quite close to Taylor expansion of exact solution. For more example see [22]. The obtained result show that the rate of convergence of MADM4 is higher than standard ADM for this problem.

x	U Exact	ADM Abs error	MADM4 absolute error
0.0	0	0	0
0.2	0.039733866159012	0.001264197530582	$7.528800499700949 * 10^{-11}$
0.4	0.155767336923460	$1.058797809982970 * 10^{-7}$	$7.709478999640140 * 10^{-8}$
0.6	0.338785484037021	$4.445704692990216 * 10^{-6}$	$4.064858429964069 * 10^{-6}$
0.8	0.573884872719618	$7.894635701799491 * 10^{-5}$	$5.406399990504074 * 10^{-5}$
1.0	0.841470984807897	$7.352541691750814 * 10^{-4}$	$4.022691603570161 * 10^{-4}$

Table 3.1: Comparison of absolute errors between the exact solution and approximate solutions by ADM and MADM4.

3.2 MADM for singular ordinary differential equations (MADM4) [25]

In this section, an efficient modification of ADM with another inverse differential operator is introduced for solving second order singular initial value problems of ordinary differential equations. The proposed method is tested on several linear and non-linear boundary value problems. All the numerical results obtained by using modified Adomian decomposition (MADM4) show very good agreement with the exact solutions for only a few terms. In addition, we use this method to overcome the singularity difficulty for higher-order boundary value problems. The proposed method is tested for some examples and the obtained results show the advantage of using this method.

3.2.1 (MADM4) for second ODEs

Consider the singular initial value problem in the second order ordinary differential equation in the form

$$\begin{cases} u'' + \dfrac{2}{x}u' + F(x,u) = g(x), \\ u(0) = A, u'(0) = B, \end{cases} \tag{3.40}$$

where $F(x,u)$ is a real function, $g(x)$ is given function and A and B are constants. Here, we present another differential operator, as below

$$L = x^{-1}\frac{d^2}{dx^2}xu, \tag{3.41}$$

so, the problem (3.40) can be written as,

$$Lu = g(x) - F(x,u).$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as below

$$L^{-1}(.) = x^{-1}\int_0^x\int_0^x x(.)dxdx. \tag{3.42}$$

Applying L^{-1} of (3.42) to the first two terms $u'' + \frac{2}{x}u'$ of equation (3.40) we find

$$\begin{aligned} L^{-1}\left(u'' + \frac{2}{x}u'\right) &= x^{-1}\int_0^x\int_0^x x\left(u'' + \frac{2}{x}u'\right)dxdx. \\ &= x^{-1}\int_0^x (xu' + u - u(0))dx = u - u(0). \end{aligned}$$

By operating L^{-1} on (3.41), we have

$$u(x) = A + L^{-1}g(x) - L^{-1}F(x,u).$$

Recall that the ADM introduce the solution $u(x)$ and the nonlinear function $F(x,u)$ by infinity series

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

and

$$F(x,u) = \sum_{n=0}^{\infty} A_n,$$

where,

$$\begin{cases} A_0 = F(u_0), \\ A_1 = u_1 F'(u_0), \\ A_2 = u_2 F'(u_0) + \frac{1}{2!} u_1^2 F''(u_0), \\ A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!} u_1^3 F'''(u_0), \\ \qquad\qquad \vdots \end{cases}$$

which can be used to construct Adomian polynomials, when $F(u)$ is a nonlinear function.

so, we get

$$\sum_{n=0}^{\infty} u(x) = A + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n.$$

Through using ADM, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0(x) = A + L^{-1}g(x), \\ u_{n+1}(x) = -L^{-1}(A_n), n \geq 0, \end{cases}$$

which gives

$$\begin{cases} u_1(x) = -L^{-1}(A_0), \\ u_2(x) = -L^{-1}(A_1), \\ u_3(x) = -L^{-1}(A_2), \\ \qquad \vdots \end{cases}$$

Example 3.2.1. *Consider the linear singular initial value problem [16]*

$$u'' + \frac{2}{x}u' + u = 6 + 12x + x^2 + x^3 \tag{3.43}$$

$$u(0) = u'(0) = 0.$$

In an operator form, equation (3.43) becomes

$$Lu = 6 + 12x + x^2 + x^3 - u, \tag{3.44}$$

Applying L^{-1} on both sides of (3.44) we find

$$u(x) = L^{-1}(6 + 12x + x^2 + x^3) - L^{-1}(u),$$

where

$$L^{-1}(.) = x^{-1} \int_0^t \int_0^t x(.)\, dt \;\; dt.$$

Therefore,

$$u_(x) = x^2 + x^3 + \frac{x^4}{20} + \frac{x^5}{30},$$

we divided $x^2 + x^3 + \frac{x^4}{20} + \frac{x^5}{30}$ in two parts

$$\begin{aligned} u_0 &= x^2 + x^3, \\ u_1 &= \frac{x^4}{20} + \frac{x^5}{30} - L^{-1}u_0, \\ u_{n+1} &= -L^{-1}(u_n). \end{aligned} \tag{3.45}$$

This in turn gives

$$u_0 = x^2 + x^3,$$

and

$$\begin{aligned} u_1 &= \frac{x^4}{20} + \frac{x^5}{30} - L^{-1}(y_0), \\ u_{n+1} &= 0, \; n \geq 0. \end{aligned}$$

In view of (3.45), the exact solution is given by

$$u = x^2 + x^3.$$

A generalization of equation (3.40) has been studied by Wazwaz [27]. In a parallel manner, we replace the standard coefficients of u' and u by $\dfrac{2n}{x}$ and $\dfrac{n(n-1)}{x^2}$ respectively, for real n, $n \geq 0$.

In other words, a general equation

$$u'' + \frac{2n}{x}u' + \frac{n(n-1)}{x^2}u + F(x,u) = g(x), n \geq 0, \tag{3.46}$$
$$u(0) = A, \;\; u'(0) = B.$$

he propose the new differential operator, as below

$$L(.) = x^{-n}\frac{d^2}{dx^n}x^n(.),$$

so, the problem (3.46) can be written as,

$$Lu = g(x) - F(x,u). \tag{3.47}$$

The inverse operator L^{-1} is therefore considered a two-fold integral operator, as

$$L^{-1}(.) = x^{-n}\int_0^x\int_0^x x^n(.)dxdx. \tag{3.48}$$

Applying L^{-1} of (3.48) to the first three terms of equation (3.46) we find

$$\begin{aligned} L^{-1}\left(u'' + \frac{2n}{x}u' + \frac{n(n-1)}{x^2}u\right) &= x^{-n}\int_0^x\int_0^x x^n\left(u'' + \frac{2n}{x}u' + \frac{n(n-1)}{x^2}u\right)dxdx \\ &= x^{-n}\int_0^x (x^n u^{'} + nx^{n-1}u)dx = u. \end{aligned}$$

By operating L^{-1} on (3.47), we have

$$u(x) = A + L^{-1}g(x) - L^{-1}F(x,u),$$

proceeding as before we obtain through using ADM, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0(x) = A + L_n^{-1}g(x), \\ u_{n+1}(x) = -L_n^{-1}(A_n), n \geq 0, \end{cases}$$

where A_n are Adomian polynomials that represent the nonlinear term $F(x,u)$:

Example 3.2.2. *Consider the linear singular initial value problem*

$$u'' + \frac{4}{x}u' + \frac{2}{x^2}u = 12, \tag{3.49}$$

$$u(0) = 0, \;\; u'(0) = 0.$$

According to (3.2.1) we put

$$L(.) = x^{-2}\frac{d^2}{dx^2}x^2(.),$$

so

$$L^{-1}(.) = x^{-2}\int_0^x\int_0^x x^2(.).$$

In an operator form, equation (3.49) becomes

$$Ly = 12.$$

Now, by applying L^{-1} to both sides we have

$$L^{-1}Lu = x^{-2}\int_0^x\int_0^x 12x^2\; dxdx,$$

and it implies,

$$u(x) = x^2.$$

Example 3.2.3. *Consider the nonlinear singular initial value problems*

$$u'' + \frac{6}{x}u' + \frac{6}{x^2}u + u^2 = 20 + x^4, \tag{3.50}$$

$$u(0) = 0, \;\; u'(0) = 0.$$

According to (3.2.1). We put

$$L(.) = x^{-3}\frac{d^2}{dx^2}x^3(.),$$

so

$$L^{-1}(.) = x^{-3}\int_0^x\int_0^x x^3(.).$$

In an operator form, equation (3.50) becomes

$$Lu = 20 + x^4 - u^2. \tag{3.51}$$

Now, by applying L^{-1} to both sides of (3.51) we have

$$u = L^{-1}(20 + x^4) - L^{-1}(u^2).$$

Therefore

$$u = x^2 + \frac{x^6}{72} - L^{-1}y^2,$$

by divided $x^2 + \dfrac{x^6}{72}$ into two parts and we obtain the recursive relationship

$$\begin{cases} u_0 = x^2, \\ u_1 = \dfrac{x^6}{72} - L^{-1}A_0, \\ \qquad \vdots \\ u_{n+1} = -L^{-1}(A_n). \end{cases} \tag{3.52}$$

Which implies

$$u_{n+1} = 0, n \geq 0.$$

In view of (3.52) the exact solution is given by

$$u(x) = x^2.$$

And so, the exact solution is easing obtained by proposed Adomian method.

A generalization of second order ODEs [13].

$$\begin{cases} u'' + \dfrac{n}{x}u' + \dfrac{m}{x^2}u = g(x) + F(x,u), n \geq 1, m \geq 0, \\ u(0) = A, u'(0) = B. \end{cases} \tag{3.53}$$

Where $F(x,u)$ and $g(x)$ are given functions $A,\ B,\ m$ and n are constants.

We define the differential operator L in the terms of the three part $u'' + \dfrac{n}{x}u' + \dfrac{m}{x^2}u$ contained in the problem.

Under the transformations $2h+k=n$ and $(h-1)(h+k)=m$, the equation (3.53) is transformed to

$$u''+\frac{2h+k}{x}u'+\frac{(h-1)(h+k)}{x^2}u=g(x)+F(x,u),\ n\geq 1,\ m\geq 0,$$

where h and k are constants. Rewrite (3.53) in the form

$$Lu=g(x)+F(x,u),$$

where the differential operator L is defined by

$$L(.)=x^{-h}\frac{d}{dx}\left(x^{-k}\frac{d}{dx}x^{h+k}\right)(.). \tag{3.54}$$

The inverse L^{-1} is therefore considered a twofold integral operator defined by

$$L^{-1}(.)=x^{-(h+k)}\int_0^x x^k\int_0^x x^h(.)dx\,dx. \tag{3.55}$$

Applying L^{-1} defined in (3.55) to both side of eq. (3.53) we get

$$u=\varphi(x)+L^{-1}\left(g(x)\right)+L^{-1}\left(F(x,u)\right). \tag{3.56}$$

such that $L\left(\varphi(x)\right)=0$.

The ADM introduce the solution $u(x)$ and the nonlinear function $F(x,u)$ by infinite series

$$u(x)=\sum_{n=0}^{\infty}u_n(x), \tag{3.57}$$

and

$$F(x,u)=\sum_{n=0}^{\infty}A_n, \tag{3.58}$$

where the components $u_n(x)$ of the solution $u(x)$ will be determined recurrently. By substituting (3.57) and (3.58) into (3.56) gives,

$$\sum_{n=0}^{\infty}u_n(x)=\phi(x)+L^{-1}\left(g(x)\right)+L^{-1}\left(\sum_{n=0}^{\infty}A_n\right).$$

Through using ADM, the components $u_n(x)$ can be determined as

$$\begin{cases} u_0(x)=\phi(x)+L^{-1}g(x), \\ u_{n+1}(x)=L^{-1}A_n, n\geq 0, \end{cases}$$

which gives

$$\begin{cases} u_0(x) = \phi(x) + L^{-1}g(x), \\ u_1(x) = -L^{-1}(A_0), \\ u_2(x) = -L^{-1}(A_1), \\ u_3(x) = -L^{-1}(A_2), \\ \qquad \vdots \end{cases}$$

Example 3.2.4. *Consider the singular initial value problem*

$$\begin{cases} u'' + \dfrac{5}{x}u' + \dfrac{3}{x^2}u = 15, \\ \quad u(0) = 0, u'(0) = 0. \end{cases} \tag{3.59}$$

We put $2h + k = 5$ and $(h-1)(h+k) = 3$.

it follows that $k = 1, h = 2$, substitution of k and h in eq. (3.54) yields the operator

$$L(.) = x^{-2}\frac{d}{dx}\left(x^{-1}\frac{d}{dx}x^3(.)\right),$$

so

$$L^{-1}(.) = x^{-2}\int_0^x x\int_0^x x^2(.)dx\,dx.$$

In an operator form, eq. (3.59) becomes

$$Lu = 15, \tag{3.60}$$

applying L^{-1} on both sides of (3.60) to obtain

$$u = L^{-1}(15),$$

and it implies,

$$u = x^2.$$

So, the exact solution is very easily obtained by this method.

Example 3.2.5. *Consider the nonlinear initial value problem*

$$\begin{cases} u'' + \dfrac{3}{x}u' + \dfrac{1}{x^2}u = u^2 + e^x, \\ u(0) = 1, u'(0) = 1. \end{cases} \tag{3.61}$$

Here, we use Taylor series of $g(x)$ with order 9, we put

$2h + k = 3, \ \ (h-1)(h+k) = 1,$

it follows that $k = -1, h = 2$ substitution of h and k in eq. (3.54) yields the operator

$$L(.) = x^{-2}\frac{d}{dx}\left(x\frac{d}{dx}x(.)\right),$$

so

$$L^{-1}(.) = x^{-1}\int_0^x x^{-1}\int_0^x x^2(.)dx\, dx.$$

In an operator form, eq. (3.61) becomes

$$Lu = u^2 + g(x). \tag{3.62}$$

Applying L^{-1} on both sides of (3.62) to obtain

$$\begin{cases} u_0(x) = L^{-1}g(x), \\ u_{n+1}(x) = L^{-1}A_n, \quad n \geq 0, \end{cases} \tag{3.63}$$

A_n's are Adomian polynomials of nonlinear term y^2, as below:

$$\begin{cases} A_0 = u_0^2, \\ A_1 = 2u_0u_1, \\ A_2 = u_1^2u + 2u_0u_1, \\ A_3 = 2u_1u_2 + 2u_0u_3, \\ \qquad \vdots \end{cases} \tag{3.64}$$

So, by substituting (3.64) into (3.63), we have

$$\begin{cases} u_0 = 1 + x + \dfrac{7x^2}{18} - \dfrac{23x^4}{600} - \dfrac{431x^5}{1080} - \cdots, \\ u_1 = \dfrac{x^2}{9} + \dfrac{x^3}{8} + \dfrac{16x^4}{225} + \dfrac{31x^5}{1296} + \dfrac{1279x^6}{396900} - \cdots, \\ u_2 = \dfrac{2x^4}{225} + \dfrac{17x^5}{1296} + \dfrac{3877x^6}{396900} + \dfrac{1201x^7}{259200} + \cdots, \\ u_0 + u_1 + u_2 = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \dfrac{x^5}{120} + \cdots, \end{cases}$$

which $u_0 + u_1 + u_2$ is quite close to Taylor expansion of exact solution $u(x) = e^x$.

3.2.2 Higher-order boundary value problems [15]

Consider the singular boundary value problem of $(n+1)$-order ordinary differential equation in the form

$$u^{(n+1)} + \frac{m}{x}u^{(n)} + Nu = g(x), \tag{3.65}$$

with initial conditions

$$u(0) \;=\; a_0,\ u'(0) = a_1, \cdots, u^{(n-1)}(0) = a_{n-1},\ u'(b) = c,$$

where N is a nonlinear differential operator of order less than n, $g(x)$ is a given function, $a_0,\ a_1, ...,\ a_{n1},, c,\ b$ are given constants, where $m \leq n-1$, $n \geq 1$. We propose the new differential operator, as below

$$L(.) = x^{-1}\frac{d^{n-1}}{dx^{n-1}}\ x^{n-m}\frac{d}{dx}\ x^{m-n-1}\frac{d}{dx}(.), \tag{3.66}$$

so, the problem (3.65) can be written as,

$$Lu = g(x) - Nu. \tag{3.67}$$

the inverse operator L_1^{-1} is therefore considered a $n+1$-fold integral operator, as below,

$$L_1^{-1}(.) = \int_b^x x^{n-m-1}\int_0^x x^{m-n}(.)\int_0^x \cdots \int_0^x x(.)dx\cdots dx,$$

by applying L_1^{-1} on (3.67), we have

$$u(x) = \varphi + L^{-1}g(x) - L^{-1}Nu$$

such that $L(\varphi(x) = 0)$.

Hence,

$$\sum_{n=0}^{\infty} u(x) = \varphi_(x) + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} A_n,$$

so, we get

$$\begin{aligned} u_0 &= \varphi(x) + L^{-1}g(x) \\ u_{n+1} &= -L^{-1}\sum_{n=0}^{\infty} A_n, \quad n \geq 0, \end{aligned}$$

which gives

$$\begin{aligned} u_0 &= \varphi(x) + L^{-1}g(x) \\ u_1 &= -L^{-1}A_0, \\ u_2 &= -L^{-1}A_1, \\ u_3 &= -L^{-1}A_2, \\ &\vdots \end{aligned}$$

Example 3.2.6. *Consider the nonlinear boundary value problem*

$$u^{'''} - \frac{2}{x}u^{''} - u^3 = g(x), \;\; u(0) = 0, \quad u^{'}(0) = 0, \quad u(1) = 10.8731, \qquad (3.68)$$

where $g(x) = 7x^2e^x + 6xe^x - 6e^x - x^9e^{3x} + x^3e^x$.

We use the Taylor series of $g(x)$ with order 10, $g(x) = g_T = -6 + 10x^2 + 10x^3 + \frac{21}{4}x^4 + \frac{28}{15}x^5 + \frac{1}{2}x^6 + \frac{3}{28}x^7 + \frac{11}{576}x^8 - \frac{3769}{3780}x^9 - \frac{100787}{33600}x^10.$

We put

$$L. = x^{-1}\frac{d}{dx}x^4\frac{d}{dx}x^{-3}\frac{d}{dx}(.).$$

so that

$$L^{-1}(.) = \int_0^x x^3 \int_0^x x^{-4} \int_0^x x(.) \;\, dx \;\, dx \;\, dx,$$

in an operator form, (3.68) becomes

$$Lu = g(x) + u^3, \qquad (3.69)$$

applying L^{-1}, on both sides of (3.69) and then incorporating the given boundary conditions, we find

$u(x) = 2.71828x^4 + L^{-1}g(x) + L^{-1}y^3$

Proceeding as before we obtained the recursive relationship

$$u_0(x) = 2.71828x^4 + L^{-1}g(x) \tag{3.70}$$

$$u_{n+1}(x) = L^{-1}A_n, \quad n \geq 0 \tag{3.71}$$

computing the Adomian polynomials for the nonlinear term u^3 and Substituting into (3.71) gives the components of the solution which is in good agreement with the Taylor series of the exact solution $u(x) = x^3e^x$, see [18].

3.3 MADM for singular partial differential equations (MADM5)

3.3.1 (MADM5) for first order PDEs

Consider the following general first-order (in t) singular nonlinear PDE:

$$u_t + \frac{p}{t}u = F(x, u, u_x), \tag{3.72}$$

where t and x are independent variables, u is the dependent variable, F is a nonlinear function of x, u and u_x and p is a real constant: $p \geq 0$. The initial condition is as follows:

$$u(x, 0) = h(x). \tag{3.73}$$

In order to solve the PDE (3.72) with initial condition (3.73) by the modified decomposition method (MADM5), at first, the linear differential operator $L_t(.) = \dfrac{d(.)}{dt} + \left(\dfrac{p}{t}\right)(.)$ is defined, and the left-hand side of (3.72) is rewritten as

$$L_t u = \frac{du}{dt} + \frac{p}{t}u. \tag{3.74}$$

The inverse differential operator of L_t, that is L_t^{-1}, is defined such that $L_t^{-1}(L_t u) = u$.

$$L_t^{-1} = \frac{1}{t^p}\int_0^t t^p(.)dt. \tag{3.75}$$

Applying the inverse differential operator, defined in (3.74), to the left-hand side of (3.72) we get

$$\begin{aligned} L_t^{-1}\left(\frac{du}{dt} + \frac{p}{t}u\right) &= \frac{1}{t^p}\int_0^t t^p\left(\frac{du}{dt} + \frac{p}{t}u\right)dt, \\ &= \frac{1}{t^p}\int_0^t \left(t^p\frac{du}{dt} + pt^{p-1}u\right)dt, \\ &= \frac{1}{t^p}\int_0^t t^p\frac{d(t^p u)}{dt}dt, \\ &= \frac{1}{t^p}(t^p u)_0^t = \frac{1}{t^p}(t^p u) = u. \end{aligned}$$

The inverse differential operator of (3.75), defined in the present work, can be used to solve the general first-order singular nonlinear pdes. Applying (**??**) to (3.72) gives

$$L_t u = F(x, u, u_x). \tag{3.76}$$

Applying L_t^{-1} to both sides of (3.76) we obtain

$$u(x,t) = \varphi(x) + L_t^{-1}(F(x,u,u_x)), \tag{3.77}$$

where φ is obtained as the result of initial condition,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

and

$$F(x,u,u_x) = \sum_{n=0}^{\infty} A_n(x,t).$$

So eq. (3.77) can be written as

$$\sum_{n=0}^{\infty} u_n(x,t) = \varphi + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right),$$

according to the ADM, all terms of $u(x,t)$ except $u_0(x,t)$ are determined by recursive relation; that is,

$$u_0(x,t) = \varphi(x),$$

$$u_1(x,t) = L_t^{-1}\left(A_0(x,t)\right),$$

$$u_{n+1}(x,t) = L_t^{-1}\left(A_n(x,t)\right), n \geq 1.$$

By using the modified decomposition method (MADM4), $\varphi(x)$ splits into two parts:$\varphi(x) = \varphi_1(x) + \varphi_2(x)$ the first part $_1(x)$, is written with $u_0(x,t)$ and the second part $\varphi_2(x)$, is written with $u_1(x,t)$ as follows:

$$\begin{cases} u_0(x,t) = \varphi_1(x), \\ u_1(x,t) = \varphi_2(x) + L_t^{-1}\left(A_0(x,t)\right), \\ u_{n+1}(x,t) = L_t^{-1}\left(A_n(x,t)\right), n \geq 1. \end{cases}$$

Example 3.3.1. *Consider the following first-order (in t) non homogeneous singular nonlinear PDE with a homogeneous initial condition:*

$$\begin{cases} u_t + \dfrac{u}{2t} = 6uu_x - u_{xxx} - \dfrac{7}{2}t^2 + 9t^3. \\ u(x,0) = 0. \end{cases} \tag{3.78}$$

According to (**??**) in an operator form eq.(3.78) becomes

$$L_t u = 6uu_x - u_{xxx} - 7\frac{7}{2}t^2 + 9t^3. \tag{3.79}$$

Applying the inverse differential operator $L_t^{-1}(.) = \dfrac{1}{t}\int_0^{t^{1/2}} t^{1/2}(.)dt.$ defined in (3.75) with $p = \dfrac{1}{2}$ on (3.79) gives

$$u(x,t) = -t^3 + 2t^4 + 6L_t^{-1}\left(uu_x\right) - L_t^{-1}\left(u_{xxx}\right). \tag{3.80}$$

Now, according to the (ADM), the dependent variable $u(x,t)$ and the nonlinear term uu_x are substituted with the infinite series as follows:

$$\begin{cases} u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \\ uu_x = L_t^{-1}\left(A_n(x,t)\right). \end{cases} \tag{3.81}$$

Substituting the infinite series of (3.81) in (3.80) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = -t^3 + 2t^4 + 6L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right) - L_t^{-1}\left(u_{xxx}\right).$$

Hence,

$$\begin{cases} u_0(x,t) = -t^3, \\ u_1(x,t) = 2t^4 + 6L_t^{-1}\left(A_0(x,t)\right) - L_t^{-1}\left(u_{0|_{xxx}}\right), \\ u_{m+1}(x,t) = 6L_t^{-1}\left(A_n(x,t)\right) + L_t^{-1}\left(u_{n|_{xxx}}\right),\ n \geq 1. \end{cases} \tag{3.82}$$

The Adomian polynomials $A_n's$ are obtained as

$$\begin{cases} A_0(x,t) = u_0(x,t)u_{0|_x}(x,t) = 0, \\ A_1(x,t) = u_0(x,t)u_1(x,t) + u_1(x,t)u_{0|_x}(x,t), \\ A_2(x,t) = u_0(x,t)u_{2|_x}(x,t) + u_1(x,t)u_{1|_x}(x,t) + u_2(x,t)u_{1|_x}(x,t), \\ A_m(x,t) = 0, m \geq 3. \end{cases}$$

the first few component from recursive relation (3.82) are

$$\begin{cases} u_0(x,t) = -t^3, \\ u_1(x,t) = 2t^4, \\ u_2(x,t) = 6L_t^{-1}\left(A_1(x,t)\right) + L_t^{-1}\left(u_{1|_{xxx}}\right) = 0, \\ u_n(x,t) = 0,\ n \geq 3. \end{cases}$$

The solution of the first-order singular nonlinear initial-value problem of (3.78) by the use of (MADM5) is the sum of u_n, that is, $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$ such that

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots = -t^3 + 2t^4,$$

3.3.2 (MADM5) for second order PDEs

Consider the following general second-order (in t) singular nonlinear pdes:

$$\frac{d^2u}{dt^2} + \frac{p}{t}\frac{du}{dt} = F(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}), \tag{3.83}$$

with initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x),$$

where t and x are independent variables, u is the dependent variable, F is a nonlinear function of x, u, u_x and u_{xx} and p is a real constant: $p \geq 0$.

In order to use the modified decomposition method (MADM5), the left-hand side of PDE (3.83) is considered as the linear invertible operator L_t :

$$L_t u = \frac{d^2u}{dt^2} + \frac{p}{t}\frac{du}{dt} \Rightarrow L_t(.) = \frac{d^2(.)}{dt^2} + \frac{p}{t}\frac{d(.)}{dt}. \tag{3.84}$$

The inverse of the linear differential operator (3.84) is defined as

$$L_t^{-1} = \int_0^t \frac{1}{t^p} \int_0^t t^p dt dt. \tag{3.85}$$

Applying the inverse differential operator, defined in (3.85), to the left-hand side of (3.84) we get

$$\begin{aligned}
L_t^{-1}\left(\frac{d^2u}{dt^2} + \frac{p}{t}\frac{du}{dt}\right) &= \int_0^t \frac{1}{t^p} \int_0^t t^p \left(\frac{d^2u}{dt^2} + \frac{p}{t}\frac{du}{dt}\right) dt, \\
&= \int_0^t \frac{1}{t^p} \int_0^t \left(t^p \frac{d^2u}{dt^2} + pt^{p-1}\frac{du}{dt}\right) dt, \\
&= \int_0^t \frac{1}{t^p} \left(t^p \frac{du}{dt}\right) dt, \\
&= \int_0^t \left(\frac{du}{dt} dt\right) = (u)_0^t = u(x,t) - u(x,0).
\end{aligned}$$

The inverse differential operator of (3.85), defined in the present work, can be used to solve the general second-order singular nonlinear PDEs. Applying (3.84) to (3.83) gives

$$L_t u = F(x, u, u_x, u_{xx}, u_t). \tag{3.86}$$

Applying L_t^{-1} to both sides of (3.86) we obtain

$$u(x,t) = f(x) + tg(x) + L_t^{-1} F(x, u, u_x, u_{xx}, u_t) \tag{3.87}$$

The (ADM) states that the dependent variable $u(x,t)$ and F the nonlinear term should be written as the following infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

$$F(x, u, u_x) = \sum_{n=0}^{\infty} A_n(x,t)$$

Substituting the infinite series in (3.87) gives

$$\sum_{n=0}^{\infty} u_m(x,t) = f(x) + tg(x) + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right).$$

By using the modified decomposition method (MADM1) $f(x) + tg(x)$ splits into two parts; $f(x)$ is written with $u_0(x,t)$ and $tg(x)$ is written with $u_1(x,t)$ as follows:

$$\begin{cases} u_0(x,t) = f(x), \\ u_1(x,t) = tg(x) + L_t^{-1}\left(A_0(x,t)\right), \\ u_{n+1}(x,t) = L_t^{-1}\left(A_n(x,t)\right), n \geq 1. \end{cases}$$

Example 3.3.2. *Consider the following general second-order nonhomogeneous initial-value problem with the homogeneous initial conditions:*

$$\frac{d^2u}{dt^2} - \frac{1}{3t}\frac{du}{dt} + \left(\frac{du}{dx}\right)^2 = 1 + \frac{5}{3}xt - t^2, \tag{3.88}$$

$$u(x,0) = 0, u_t(x,0) = 0.$$

According to (3.84) in an operator form eq. (3.88) becomes

$$L_t u = 1 + \frac{5}{3}xt - t^2 - (\frac{du}{dx})^2.$$

Applying the inverse differential operator $L_t^{-1}(.) = \int_0^t t^{1/3} \int_0^t t^{-1/3}(.)dtdt.$ defined in (3.85), with $p = \dfrac{-1}{3}$ on the PDE (3.88) gives

$$u(x,t) = \frac{3t^2}{4} + \frac{t^3}{3}x - \frac{3t^4}{32}L^{-1}\left(u_x\right)^2.$$

Using of the modified method (MADM5) results

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{3t^2}{4} + \frac{t^3}{3}x - \frac{3t^4}{32} + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right),$$

where,

$$\begin{cases} u_0(x,t) = \dfrac{3t^2}{4}, \\ u_1(x,t) = \dfrac{t^3}{3}x - \dfrac{3t^4}{32} + L_t^{-1}\left(A_0(x,t)\right), \\ u_{n+1}(x,t) = L_t^{-1}\left(A_n(x,t)\right), \quad n \geq 1. \end{cases} \tag{3.89}$$

$A_n's$ are Adomian polynomial of nonlinear term $\left(\frac{du}{dx}\right)^2$ can be expressed as follows:

$A_0(x,t) = u_{0|_x}(x,t) = 0,$

$A_1(x,t) = 2u_{0|_x}(x,t)u_{1|_x}(x,t) = 0,$

$A_2(x,t) = 2u_{0|_x}(x,t)u_{2|_x}(x,t) + \left(u_{1|_x}(x,t)\right)^2 = \frac{t^6}{9},$

$A_3(x,t) = 2u_{0|_x}u_{3|_x}(x,t) + 2u_{1|_x}(x,t)u_{2|_x}(x,t) = 0,$

$A_4(x,t) = 2u_{0|_x}u_{4|_x}(x,t) + 2u_{1|_x}(x,t)u_{3|_x}(x,t)\left(u_{2|_x}(x,t)\right)^2 = 0,$

$A_n(x,t) = 0, \ n \geq 5.$

So, by substituting the last A_m's on (3.89),we have

$$\begin{cases} u_0(x,t) = \frac{3t^2}{4}, \\ u_1(x,t) = x\frac{t^3}{3} - \frac{3t^4}{32}, \\ u_2(x,t) = L_t^{-1}\left(A_1(x,t)\right) = 0, \\ u_3(x,t) = L_t^{-1}\left(A_2(x,t)\right) = \frac{1}{460}t^{\frac{23}{3}}, \\ u_4(x,t) = L_t^{-1}\left(A_3(x,t)\right) = 0, \\ u_n(x,t) = L_t^{-1}\left(A_{n-1}(x,t)\right) = 0, \ n \geq 5 \end{cases}$$

Therefore, solution of second-order initial-value problem of (3.83) is as follows:

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\ &= \frac{3}{4}t^2 + \frac{x}{3}t^3 - \frac{3t^4}{32} + \frac{1}{460}t^{\frac{23}{3}} \end{aligned}$$

which is the exact solution of the initial-value problem of (3.83).

3.3.3 (MADM5) for higher-order singular PDEs

$$\frac{d^{n+1}u}{dt^{n+1}} + \frac{p}{t}\frac{d^n u}{dt^n} = F(x,u,\frac{d^n u}{dx^n},\cdots,\frac{du}{dx},\frac{du}{dt}), \tag{3.90}$$

with initial conditions

$$u(x,0) = f(x), u_t(x,0) = g(x) \cdots u_{t \cdots (n+1 \ time) \ t} = h(x),$$

where t and x are independent variables, u is the dependent variable, F is a nonlinear function of x, u, u_x and u_{xx} and $u_{x \cdots (n+1 \ \ time) \ x}$ and p is a real constant: $p \geq 0$. In order to use the modified decomposition method (MADM5), the left-hand side of PDE (3.90) is considered as the linear invertible operator L_t :

$$L_t u = \frac{d^{n+1}u}{dt^{n+1}} + \frac{p}{t}\frac{d^n u}{dt^n} \Rightarrow L_t(.) = \frac{d^{n+1}(.)}{dt^{n+1}} + \frac{p}{t}\frac{d^n(.)}{dt^n}. \tag{3.91}$$

The inverse of the linear differential operator (3.91) is defined as

$$L_t^{-1} = \int_0^t \int_0^t \cdots (n) \ times \ \int_0^t \frac{1}{t^p} \int_0^t t^p dt \cdots (n+1) \ times \ dt. \tag{3.92}$$

The inverse differential operator of (3.92), defined in the present work, can be used to solve the general n+1-order singular nonlinear PDEs. Applying (3.91) to (3.90) gives

$$L_t u = F(x, u, \frac{du}{dx}, \cdots \frac{d^{n+1}u}{dx^{n+1}}, \frac{du}{dt}). \tag{3.93}$$

Applying L_t^{-1} to both sides of (3.93) we obtain

$$u(x,t) = f(x) + tg(x) + \cdots + \frac{t^n}{n!}h(x) + L_t^{-1}(x, u, \frac{du}{dx}, \cdots, \frac{d^{n+1}u}{dx^{n+1}}, \frac{du}{dt}). \tag{3.94}$$

where $f(x) + tg(x) + \cdots + \frac{t^n}{n!}h$)(xappears as the result of initial conditions. Using Adomian decomposition method, (3.94) can be rewritten as

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + \cdots + \frac{t^n}{n!}h(x) + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right).$$

the modified decomposition method (MADM1) splites $f(x) + tg(x) + \cdots + \frac{t^n}{n}h(x)$ into two parts:$f(x)$ is writen with $u_0(x,t)$ and $tg(x) + \cdots + \frac{t^n}{n}h(x)$ is writen with $u_1(x,t)$ as follows:

$$\begin{cases} u_0(x,t) = f(x), \\ u_1(x,t) = tg(x) + \cdots + \frac{t^n}{n!}h(x) + L_t^{-1}\left(A_0(x,t)\right), \\ u_{n+1}(x,t) = L_t^{-1}\left(A_n(x,t)\right), \ n \geq 1. \end{cases}$$

General complete second-order singular nonlinear PDEs.

Consider the general second-order (in t) singular nonlinear PDE in following form:

$$\frac{d^2u}{dt^2}+\frac{2p}{t}\frac{du}{dt}+\frac{p(p-1)}{t^2}u=F(x,u,\frac{du}{dx},\frac{d^2u}{dx^2},\frac{du}{dt}), \tag{3.95}$$

with initial conditions

$$u(x,0)=f(x),\ u_t(x,0)=g(x),$$

where t and x are independent variables, u is the dependent variable, F is a nonlinear function of x, u, u_x, u_{xx}, u_t and p is a real constant: $p\geq 1$.

Defining the linear differential operator $L_t(.)=\frac{d^2(.)}{dt^2}+\frac{2p}{t}\frac{d(.)}{dt}+\frac{p(p-1)}{t^2}(.)$ the left-hand side of (3.95) is rewritten as

$$Lu=\frac{d^2u}{dt^2}+\frac{2p}{t}\frac{du}{dt}+\frac{p(p-1)}{t^2}u \tag{3.96}$$

The inverse differential operator of L_t , that is, L^{-1} is defined such that:

$$L_t^{-1}(.)=\frac{1}{t^p}\int_0^t\int_0^t t^p(.)dtdt. \tag{3.97}$$

Applying the inverse differential operator, defined in (3.97), to the left-hand side of (3.95) we get

$$\begin{aligned}L_t^{-1}\left(\frac{d^2u}{dt^2}+\frac{2p}{t}\frac{du}{dt}+\frac{p(p-1)}{t^2}u\right)&=\frac{1}{t^p}\int_0^t\int_0^t t^p\left(\frac{d^2u}{dt^2}+\frac{2p}{t}\frac{du}{dt}+\frac{p(p-1)}{t^2}u\right)dtdt,\\&=\frac{1}{t^p}\int_0^t\int_0^t\left(t^p\frac{d^2u}{dt^2}+2pt^{p-1}\frac{du}{dt}+p(p-1_t^{(p-2)}u\right)dtdt,\\&=\frac{1}{t^p}\int_0^t\int_0^t\frac{d}{dt}\left(t^p\frac{d^2u}{dt^2}+pt^{(p-1)}u\right)dtdt,\\&=\frac{1}{t^p}\int_0^t\frac{d}{dt}\left(t^pu\right)dt.=\frac{1}{t^p}\left(t^pu\right)=u.\end{aligned}$$

The inverse differential operator of (3.96) can be used to solve the general complete second-order singular nonlinear PDEs.

Applying L_t^{-1} to both sides of (3.95) we obtain

$$u(x,t)=f(x)+tg(x)+L_t^{-1}\left(F(x,u,u_x,u_{xx},u_t)\right), \tag{3.98}$$

where $f(x) + tg(x)$ appears as the result of initial conditions. Using Adomian decomposition method, (3.98) can be rewritten as

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + tg(x) + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right).$$

The modified decomposition method (MADM1) splits $f(x) + tg(x)$ into two parts, $f(x)$ is written with $u_0(x,t)$ and $tg(x)$ is written with $u_1(x,t)$ as follows:

$$\begin{cases} u_0(x,t) = f(x), \\ u_1(x,t) = tg(x) + L_t^{-1}(A_0(x,t)), \\ u_{n+1}(x,t) = L_t^{-1}(A_n(x,t)),\ n \geq 1. \end{cases}$$

Example 3.3.3. *Consider the following second-order initial value problem*

$$\frac{d^2u}{dt^2} + \frac{2}{t}\frac{du}{dt} + \frac{du}{dx}\frac{d^2u}{dx^2} = 1 + x, \tag{3.99}$$

$$u(x,0) = 0, \quad u_t(x,0) = 0.$$

According to (3.96) in an operator form eq. (3.99)

$$L_t u = 1 + x - \frac{du}{dx}\frac{d^2u}{dx^2}. \tag{3.100}$$

Applying the inverse differential operator $L_t^{-1}(.) = \frac{1}{t}\int_0^t\int_0^t t(.)dtdt.$,define in (3.97) on the PDE (3.100) gives

$$u(x,t) = \frac{t^2}{6} + \frac{t^2}{6}x - L_t^{-1}(u_x u_{xx}) \tag{3.101}$$

Using the (ADM), (3.101) becomes

$$\sum_{n=0}^{\infty} u_n(x,t) = \frac{3t^2}{4} + \frac{t^3}{3}x - \frac{3t^4}{32} + L_t^{-1}\left(\sum_{n=0}^{\infty} A_n(x,t)\right),$$

$$\begin{cases} u_0(x,t) = \dfrac{t^2}{6}, \\ u_1(x,t) = \dfrac{t^2}{6}x + L_t^{-1}(A_0(x,t)), \\ u_{n+1}(x,t) = L_t^{-1}(A_n(x,t)), \quad n \geq 1. \end{cases} \tag{3.102}$$

A_n's are Adomian polynomial of nonlinear term can be expressed as follows

$A_0(x,t) = u_{0|_x}(x,t)u_{0|_{xx}}(x,t) = 0,$

$A_1(x,t) = u_{0|_x}(x,t)u_{1|_{xx}}(x,t) + u_{1|_x}(x,t)u_{0|_{xx}}(x,t) = 0,$

$A_2(x,t) = u_{0|_x}(x,t)u_{2|_{xx}}(x,t) + u_{1|_x}(x,t)u_{1|_{xx}}(x,t) + u_{2|_x}(x,t)u_{0|_{xx}}(x,t) = 0,$

$A_n(x,t) = 0,\ n \geq 3.$

The first few component from recursive relation (3.102) are

$$\begin{cases} u_0(x,t) = \dfrac{t^2}{6},, \\ u_1(x,t) = x\dfrac{t^2}{6}, \\ u_2(x,t) = -L_t^{-1}\left(A_1(x,t)\right) = 0, \\ u_{n+1}(x,t) = L_t^{-1}\left(A_n(x,t)\right) = 0, n \geq 3 \end{cases}$$

Therefore, solution of second-order initial-value problem of (3.101) by MDM is as follows:

$$u(x,t) = u_0(x,t) + u_1(x,t) + \cdots = \frac{t^2}{6} + \frac{t^2}{6}x,$$

which is the exact solution of the initial-value problem of (3.101), for more example see [21, 20].

In this thesis, the ADM and some modifications of ADM are successfully applied to solve many differential equations, we shown that the modified methods are simple, reliable, efficient and require fewer computations. We proposed an efficient modification of the standard ADM for solving singular and non singular partial differential equations. Furthermore, we made a comparison between some of these modifications and ADM showed that the accuracy and the rate of convergence of MADM is higher than standard ADM for many problems.

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