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Nonlinear analysis of a two- and three-degree-of-freedom aeroelastic system with rotational stiffness free-play

by

David C. Asjes

A dissertation submitted to the graduate faculty in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

Major: Mechanical Engineering

Program of Study Committee: Atul Kelkar, Co-major Professor Umesh Vaidya, Co-major Professor Jerald Vogel Greg R. Luecke

C

 $Baskar\ Ganapathy subramanian$

Iowa State University

Ames, Iowa

2015

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DEDICATION

I would like to dedicate this thesis to Joseph, husband, father and saint, and a true help in time of need.

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ACKNOWLEDGEMENTS

I would like to take this opportunity to express my gratitude to all the people who have helped me through the doctoral program in general, and the research and writing of this theses in particular. First, my thanks to Dr. Atul Kelkar, whose generous support and patience has made this all possible in the first place. I would also like to thank Dr. Umesh Vaidya for his time and guidance. The insights of Dr. Kelkar and Dr. Vaidya have set the path for my investigation and research development. I would also like to thank Dr. Greg Luecke and Dr. Baskar Ganapathysubramanian for their guidance and interest in my research. I greatly appreciate the time they have been willing to give me on this endeavor. I cannot give enough thanks to Dr. Jerry Vogel, who in addition to his unfailing encouragement, insight into aerodynamic problems and ready willingness to brainstorm any obstacle, has been a true mentor to me. I can only hope someday to be one tenth the engineer that he is. My special thanks to Amit Diwadkar. I am sure there is nothing he cannot explain. And my deepest thanks to my wife Kate, for the loads she has carried while I studied, and for the unending support she has provided. She has been the one true essential for the whole journey.

ABSTRACT

Under the right parameters, flutter occurs in an airfoil when aerodynamic forces drive a dynamic structure to an oscillatory, possibly divergent condition. The presence of a rotational stiffness nonlinearity at the root of an all-moving airfoil has been shown to decrease the freestream velocity at which flutter occurs. Since this is a somewhat common configuration for flight structures and other aerodynamic machinery, a large amount of research has been devoted to understanding it over the several decades. Attempts to characterize it, however, have mostly resulted in methods that provide numerical simulation, validated by experimental results, rather than a nonlinear systems analysis approach. This research addresses the problem of characterizing the phenomenon of flutter in an all moving airfoil that has a rotational stiffness free-play nonlinearity. Application is made to both a rigid two-dimensional model and a flexible three-dimensional model. A system theory approach is used to model a typical airfoil system with rotational free-play nonlinearity so that analysis can be performed with necessarily conducting numerical time domain simulations of the model. The main contributions of this research are the introduction and validation of a nonlinear freeplay model that allows better exploitation of nonlinear systems analysis techniques, the design and validation of the subsequent two-dimensional model, the application of new system identification tools to provide an aerodynamic reduced order model that is reasonably accurate for three-dimensional modeling and computationally efficient, and the introduction of two new approaches to the three-dimensional modeling problem. This research introduces the use of a hyperbola function to model the free-play nonlinearity, allowing a system that is both continuous, responsive to changes in the free-play region width, and physically representative. For the two-dimensional case, the nonlinearity is modeled as a feedback interconnection of linear system and static nonlinearity. The feedback interconnection structure is exploited to analyze the system dynamics, consisting of unique stable fixed points, multiple steady states and limit cycle oscillations. A Hopf bifurcation is identified by analysis, and the results of the derivation are demonstrated to provide accurate predictions of flutter behavior. For the three-dimensional model, two candidate models are presented and analyzed. The first addresses the nonlinearity as a rigid body input, which is then superimposed upon the structural and aerodynamic systems, themselves connected in a loop. The second separates the nonlinearity out within a contained structural model, which is in turn looped with the aerodynamic model. In both cases, the aerodynamics are modeled by providing dynamic modal airfoil motion to a panel method flow code, and using the resulting aerodynamic force outputs in a system identification scheme. Finally, the results of the three-dimensional modeling are compared to experimental wind tunnel data for a flutter airfoil of similar physical properties.

CHAPTER 1. INTRODUCTION

The object of this paper is to explore the well known phenomenon of aeroelastic flutter for simplified two-dimensional and three-dimensional airfoil models with a specific and commonly encountered type of structural free-play nonlinearity in rotational stiffness by applying system analysis techniques novel approaches to modeling and design validation. Flutter has been observed in physical airfoils since the earliest days on aeronautical design and flight. General methods of predicting and characterizing flutter behavior have existed for decades, however the nonlinear nature of both the airfoil structure and the aerodynamic environment continue to make this phenomenon difficult to model precisely, especially during the design phase. Additionally, the presence of some fairly typical structural nonlinearities can have an appreciable effect on flutter behavior that further complicates prediction and characterization.

Aeroelasticity is a well-established field involving the dynamic interaction between inertia, structural and aerodynamic forces of a fluid-structural system —specifically, an airfoil in the presence of a fluid flow. Under certain conditions, the airfoil experiences oscillations known as flutter, an unstable self-feeding behavior in which aerodynamic forces couple with the elastic and inertial nature of the structure. The topic has been addressed in several classic texts including Bisplinghoff et al. (1996), Dowell et al. (1995), and Wright and Cooper (2007). Typical fundamental analysis techniques assume a linear structural model and a simplified aerodynamic model that nonetheless retains a nonlinear dependence on freestream velocity. These models can reliably predict the onset velocity and frequency of flutter from a simple eigenvalue solution of the general aeroelastic equations for a variety of fluid-structural systems, particularly in the region of inviscid, incompressible flow. Several useful surveys such as Mukhopadhyay (2003) and Marshall and Imregun (1996) provide an overview of the development of aeroelastic theory and research.

1.1 Definition of Flutter

Early studies of the flutter phenomenon were by necessity based on flight observation and wind tunnel experiment, and so the classic definitions of flutter were somewhat vague until the actual mechanisms of flutter were understood. In terms of a combined fluid-structural system, particularly in an aerodynamic setting, flutter is defined as an unstable self-feeding behavior in which aerodynamic forces couple with the elastic and inertial nature of the structure to produce oscillatory motion. Flutter therefore falls under the larger category of aeroelasticity, a multidisciplinary field pertaining to the interaction of inertia, structural and aerodynamic forces. Due to the nonlinear nature of the airfoil system, this oscillation can be bounded, divergent or slowly convergent. In the classic understanding, aerodynamic forces excite the natural frequencies related to the natural modes of the structure, producing an aeroelastic phenomenon. This motion can lead to structural failure by accelerating the fatigue of the constituent material of the load-bearing members, or by the outright divergence of the oscillations.

Bounded oscillations fall under the definition of limit cycle oscillation (LCO), a behavior that is only possible within a nonlinear system (see, e.g., Khalil (2002)). Typically, the nonlinearity responsible for the LCO lies either in some characteristic of the structure, or in some unsteady flow condition such as gust turbulence. The presence of nonlinearities in the system, especially the structure, can have observable effects on the dynamics of flutter for a given system, adding complexity to the model and making the flutter behavior difficult to predict. In practice, many airfoils are attached to a supporting object with the ability to rotate along an axis roughly perpendicular to the freestream airflow (e.g. aircraft control surfaces, wind turbine blades). A free-play deadband in this rotational axis adds a nonlinear characteristic to the structural stiffness in this degree of freedom, and is a common defect in such aerodynamic systems.

1.1.1 Historical Development and Literature Survey

In practice, an airfoil structure will involve nonlinearities, which can have observable effects on the dynamics of flutter for a given system. The nature of dynamic analysis for a physical system almost always involves some sort of linearization, whether in regards to the model for stiffness or energy dissipation, or any number of commonly accepted assumptions in the development of standard equations of motion. Many of these smaller nonlinear effects are removed by assuming linear behavior at some operating point of interest and restricting the model to a suitably small region about that point. Some structural characteristics, however, do not present such a near-linear behavior at the operating point, and to assume so removes a significant contributor to the dynamic behavior of the system. For an airfoil that is designed to rotate along a hinge line that runs perpendicular to the freestream velocity, such as a conventional control surface, an all-moving horizontal tail or a wind turbine blade, one of the most common structural nonlinearities is free-play in the rotational stiffness of the airfoil attachment. Rotational free-play nonlinearity lowers the flutter onset velocity, and typically introduces a LCO mode of flutter. While a flutter LCO is not divergent, it can significantly accelerate the fatigue of the structure and degrade controllability. Simple eigenvalue analysis of an aero-structural system with free-play dynamics becomes more difficult. Traditionally, design limits were determined from experimental wind tunnel results Hoffman and Speilberg (1954); Cooley (1958), and their conservative application constrained the final design.

The Wright Air Development Center (WADC) reports are significant regarding the motivation of this research, as discussed below. The reports Hoffman and Speilberg (1954) and Cooley (1958) were commissioned by the U.S. Air Force in the mid-1950s in anticipation of new designs for transonic aircraft that involved all-moving horizontal tail surfaces. These surfaces would combine the functions of a horizontal stabilizer with that of the elevator control surface, comprising the so-called stabilator. As the 1958 report stated, aircraft designers were anticipating "severe reductions in aerodynamic effectiveness of trailing edge control surfaces in the transonic speed region." Pitch control authority would be enhanced by the increased surface area of the stabilator, however the surface would need to be attached to the fuselage by a single pivot or hinge point. This hinge connection presented the probability that a free-play deadband would exist in the rotational connection, due initially to the limits of manufacturing and exacerbated by operational use. It had long been known that the presence of such a free-play nonlinearity would lower the velocity at which an airfoil with such a connection



Figure 1.1 Airfoil plan forms used in the WADC wind tunnel experiments: (a) The tapered airfoil studied in 1954; (b) The trapezoidal airfoil studied in 1958.

would enter into flutter–such issues were already a long-standing problem with many similarly attached control surfaces such as ailerons and elevators. The question was, how would the free-play affect these relatively larger airfoils, and what specifications needed to be placed on the manufacturing process?

Although at least five studies of this topic were taken to the WADC wind tunnels in the 1950s, this research relates in particular to the two cited as they most closely match the types of airfoils under consideration for certain current designs. The 1954 study combined eight structurally independent segments, each attached to a single perpendicular spar, making up a slightly tapered airfoil. The 1958 study involved a trapezoidal plan form, again with eight segments on a single spar, but placing the spar at a swept angle, forcing the edges of the segments to not be parallel to the freestream flow vector. Both are shown in Figure 1.1. As part of the research, the airfoils were mounted in the wind tunnel with a fixture that allowed a measurable amount of rotational stiffness free-play in the attachment, and the effects on flutter behavior were observed. The results quantified the reduction in flutter velocity remains constant.

Since the time of the WADC studies, all-moving airfoils have become common design attributes of many military and several commercial civilian aircraft, leading to ongoing research on the topic, both in wind tunnel experimentation and numerical analysis. In Carlton Schlomach (2009), the authors present the results of numerical simulation and wind tunnel experiment for an all-moveable control surface free-play model for the F-35 Air Vehicle Analysis program at Lockheed Martin Aeronautics Company. The study is motivated by what are termed restrictive free-play allowances in manufacturing. Their numerical program was for simulation only, rather than systems analysis, and relies on a switching algorithm to implement a piecewise linear free-play model. Their procedure begins with a validation of frequency domain behavior in the linear airfoil model before running numerical simulations with free-play present. The free-play nonlinearity is added to the aeroelastic system through a rigid body mode added on top of a finite element modal model of the structure system. Through this approach, the team was able to reasonably replicate flutter behavior observed in experimental data.

Recent studies have significantly advanced the characterization and analysis of the complex behavior of these nonlinear aeroelastic systems, particularly in light of expanding computational abilities. Experimental analysis of flutter and the effects of spring stiffness is presented in several sources. The authors of Trickey (2000) and Trickey et al. (2002) investigate a twodimensional (2D) airfoil with three degrees of freedom (a pivoting flap is placed at the trailing edge of the airfoil) through numerical simulation and physical experiment. One immediate observation is the experimental variability of the system parameters as observed in wind tunnel tests. These effects, along with the dynamics inserted by the turbulent airflow in the tunnel represented aspects that could not be modeled in the numerical simulation. Nonetheless, LCO behavior is observed in the wind tunnel and some limited numerical analysis is performed. The free-play nonlinearity is a piecewise linear scheme contained within the structural model, and requires a switching algorithm as the structure passes into and out of free-play. Of particular interest, this research applies the use of time-delay embedded coordinates to replicate the system dynamics, a technique which this research seeks to expand upon through spectral analysis. Work presented in Conner et al. (1997) adapts an already-proven 2D, three degree of freedom airfoil to a structural free-play nonlinearity in the trailing edge flap. Results are used to validate a numerical simulation based on 2D incompressible flow as modeled in a standard state space approximation of Theodorsen aerodynamics, and a piecewise linear function for the control surface rotational nonlinearity. Model validation is applied to time histories of the numerical simulation, and the authors recognize the inability to determine quantitatively the stability of the system when using a piecewise linear approximation. Tang et al. (2000) examines a typical 2D airfoil with control surface free-play, i.e., a three degree of freedom system, with emphasis on the behavior response to harmonic and continuous frequency sweep gust loads. Again, a piecewise linear free-play model is incorporated, and numerical simulations produce time histories that generally match wind tunnel results. The research presented in O'Neil and Strganac (1998) involves a 2D, two degree of freedom model but examines a cubic nonlinearity in addition to the piecewise linear model. Particular emphasis is placed on the requirement to model viscous and Coulomb damping forces within the structure in order to achieve more realistic simulation results.

In work related to this research, Whitmer et al. (2012) present results of a three-dimensional (3D) modeling methodology for the all-moving control surface with rotational free-play. In this study, a full 3D model is constructed using a segmented, lumped parameter structure, and the free-play nonlinearity is implemented as a piecewise linear switching scheme that transitions the system between an inner state and an outer state (i.e., inside the deadband and outside the deadband, respectively). The method requires an enforcement of boundary conditions at the switching, and the application of a loss factor to account for discontinuities in the energy content of the signal during switching. Several of the segmented model considerations informed the path of this research.

Bifurcation analysis of nonlinear aeroelastic systems from stable behavior to LCO and subsequent bifurcation into chaotic behavior is studied extensively in several sources. Lee et al. (1999b) provides a very comprehensive review of current practices in all the fundamental areas of aeroelastic study, describing the most typical system nonlinearities (cubic, free-play and hysteresis-note that cubic response is distinguished from free-play response), most common techniques for developing the structural and aerodynamic equations of motion, and some basic techniques for identifying Hopf bifurcation and eventual chaotic behavior. A piecewise linear free-play model is used, and conclusions regarding system behavior with free-play nonlinearity are based on numerical simulation and time histories. Bifurcation is demonstrated but not discussed within an analytical framework. A relationship between flutter onset velocities and free-play width is not explored. In Lee et al. (1999a), the authors examine a 2D airfoil in incompressible flow using a cubic function to model the rotational free-play nonlinearity. By adjusting the coefficients of the nonlinearity, both a softening spring and a hardening spring may be modeled. The hard spring model produces LCO behavior, but not a lowering of flutter velocity with free-play. The authors also perform an analytical determination of a Hopf bifurcation, although not a full application of the Poincare-Andronov-Hopf bifurcation theorem. A more detailed analysis of system bifurcation to limit cycle behavior based on center manifold theory is presented in Liu et al. (2000), where an aerodynamic force model based on Theodorsen's aeroelastic theory is used. The free-play nonlinearity is modeled again by a cubic function, however the results advance the usefulness of using the principle of normal form to validate the Hopf bifurcation and provide analytic insight into the characteristics of the system LCO behavior.

Tang and Dowell (2006) is a good example of analysis comparing a numerical model involving free-play with experimental results. The structure in question is a delta-wing and external store combination, with free-play in the load attachment, rather than airfoil rotational stiffness. A piecewise linear model is used for the free-play nonlinearity, and results are again based on numerical simulation and time histories, rather than a systems analysis approach. A modal Lagrangian development is used for the structure and an eigenmode reduction of the vortexlattice model is used for the aerodynamic model. The success of these techniques in mimicking the wind tunnel results indicates the validity of using these fundamental approaches in the structural and aerodynamic modeling.

It is worth noting that experimental results such those obtained by the Air Force in Hoffman and Speilberg (1954); Cooley (1958) are rarely matched by the numerical methods described above, and never in a manner based on a systems analysis approach. Wind tunnel experiments involving a 3D airfoil with a rotational rigid body mode demonstrate that the presence of rotational stiffness free-play exhibits a lower flutter onset velocity than the case of divergent flutter for a linear spring stiffness. Furthermore, these results show a clear dependence of flutter velocity on the angular width of the free-play region. The use of a cubic nonlinearity to model spring stiffness systems such as those analyzed in Liu et al. (2000); Lee et al. (1999a) are useful up to a point, but have so far failed to fully capture wind tunnel behavior. A cubic nonlinearity entails ever-increasing stiffness with increasing pitch as well as a spring force constant that saturates at infinity, which is not physically consistent. The free-play nonlinearity ought only to affect stiffness behavior about the origin, so a physically consistent free-play model requires the stiffness to saturate close to a linear spring stiffness outside the free-play region. The effect of practical free-play modeling on the flutter behavior has not been fully addressed.

1.2 Motivation

This research was initiated in response to a request by the Naval Air Systems Command (NAVAIR), who have an ongoing concern regarding the identification and avoidance of flutter modes on air vehicles acquired by the Department of the Navy. Many military aircraft – past, current and under design – operate with an all-moving airfoil surface as both a horizontal stabilizer and a pitch control surface. Free-play in the rotational hinge connection of that airfoil has long been known to advance the onset of flutter to lower freestream airspeeds. As a result, military specifications have been in place limiting the allowable amount of rotational free-play in the installation of such surfaces, both in design and while in operation. The specification for that free-play limit was established in the late 1950s, when all-moving stabilizers were first being designed for military aircraft. The WADC data mentioned above was collected by the Air Force in order to establish that specification, and due to engineering practices of the time were set at a fairly conservative value. NAVAIR engineers are interested in finding reliable computational methods for predicting with reasonable confidence the flutter speeds for existing platforms with all-moving airfoils that experience rotational free-play, and refining the design process for such airfoil surfaces, with the hope of relaxing the specification currently in place. Such a modification would reduce cost in both manufacturing tolerances and maintenance requirements, as well as open up design avenues, however small they may be.

This goal also matches up well with the current state of flutter research. Aeroelasticity is a well-established field, with fully developed classical techniques, which is to say linear, low fidelity models involving several reducing assumptions. Nonetheless, these techniques have been quite adequate when used with proper safety margins. Flutter research has also traditionally been advanced through extensive wind tunnel experimentation, and much of the classic literature is based on these data. Current state of the art, however, seeks to leverage ever-increasing numerical capabilities. Computational fluid dynamics and finite element analysis have been established for decades, but each have been computationally expensive when seeking detailed results. To combine the two fields in a meaningful way, as is required for aeroelastic research, has proven to be computationally prohibitive for all but the most powerful and dedicated numerical resources. There is great demand in the field of aeroelasticity, therefore, to find effective flutter models that can function with more modest computational resources.

1.3 Research Objectives

With the above motivation in mind, the objectives of this research are as follows:

- Characterize the equations of motion for an aeroelastic system, linearized in aerodynamics but nonlinear in structural dynamics, such that the structural nonlinearity may be separated out for the purposes of providing a suitable framework for modeling and systems analysis.
- Develop a new nonlinear stiffness model for both 2D and 3D modeling that is
 - 1. physically representative,
 - 2. mathematically accurate and tractable, and
 - 3. suitable for supporting traditional nonlinear analysis techniques.
- Develop 2D and 3D models that are capable of demonstrating
 - 1. the existence of fixed points in the airfoil state,
 - 2. a δ dependence of flutter onset velocity,
 - 3. the existence of LCO in the nonlinear model, and
 - 4. reasonable fidelity to experimental results.
- Validate the Hopf bifurcation of the nonlinear system with LCO

1.4 Approach

Through this research, an attempt is made to address some of the more difficult aspects of analysis and characterization of flutter within a nonlinear system, and in the process come up with a new approach to flutter modeling and analysis. A new nonlinear stiffness model is introduced involving a hard spring with a limiting stiffness coefficient that effectively captures the δ -dependence of flutter velocity, previously observed in experiment Hoffman and Speilberg (1954). The Hopf bifurcation exhibited by the resulting system is also characterized through both derivation and simulation results, and show it to be a function of velocity. Resonance between pitch and plunge frequencies is also demonstrated for this new nonlinear stiffness model and shown to be consistent with classical flutter literature.

In the analysis, the airfoil system is modeled with the rotational free-play nonlinearity as a feedback interconnection between a coupled linearized fluid-structural system and a static structural nonlinearity, similar to Brockett (1982). This allows us to use systems theory tools for stability analysis. A simple two-dimensional, two-degree-of-freedom airfoil is employed that is allowed to move in pitch and plunge, has a rotational free-play band of angular width $\pm \delta$, and which is exposed to a steady freestream airflow. The resulting aerodynamic loads (lift and moment) act as the forcing function to the system dynamics of the airfoil. Lift and moment are calculated using general lift and moment coefficients as developed from two-dimensional panel method code. The resulting state space model maintains the intrinsic coupling of the fluid flow with the airfoil structure.

The time-domain behavior of the airfoil system is consistent with the behavior of the eigenvalues of the system Jacobian taken about the equilibrium fixed points of the model. As freestream velocity is varied, the system behavior shows a transition from a region of stability to a stable limit cycle at a critical airspeed dependent on the size of the free-play region. To the best of the author's knowledge, this is the first result demonstrating free-play dependence of critical flutter airspeed. Results of this analysis indicate an airfoil system susceptible to flutter that exhibits a pitchfork bifurcation at a relatively slow airspeed and a Hopf bifurcation at a moderate airspeed. The results of the two-dimensional modeling are then used to inform the development and assessment of a three-dimensional model. This model necessarily requires the generation of a structural model, an aerodynamic model, and a modeling approach to incorporate the freeplay nonlinearity. The structural model is developed through a lumped mass method. An aerodynamic reduced order model is generated by first using oscillating mode shapes as inputs to a suitable fluid dynamics flow code, then using the resulting input-output data sets to accomplish a reasonably accurate system identification technique. Two methods are evaluated for the incorporation of the free-play nonlinearity – applying the deadband model to the rigid body as a whole and superimposing it onto the structural/aero system, and by separating the free-play mode out from the structure and implementing the nonlinearity without removing it from the overall structural model. Both approaches are evaluated for their fidelity to expected results, as well as recently acquired wind tunnel data. The experimental data has been collected in support of the NAVAIR research, and uses an airfoil specifically designed to closely match the airfoil used in Hoffman and Speilberg (1954) in all aspects.

1.5 Contribution to the thesis

The results of this work provide the basis for an efficient and reliable method to model an aeroelastic system involving a free-play nonlinearity in rotational stiffness. The introduction of a continuous differentiable model for the free-play nonlinearity allows a straightforward identification of the system fixed points, both at and away from the state space origin, and thereby facilitating analysis of the system linearized about those fixed points. This approach is reliable and cannot be achieved by traditional piecewise linear modeling of the nonlinearity. Previous models, most notably the cubic nonlinearity, are not physically representative.

This approach is first validated in the 2D aeroelastic model, where reasonable results for nonlinear behavior are obtained. Flutter onset in the presence of the free-play nonlinearity occurs at velocities below the flutter velocity of the linear structure, with a clear dependence of onset velocity with free-play width δ . The approach is then extended to two approaches for a 3D model, with both approaches able to replicate aspects of 3D flutter behavior observed in experimental results. These 2D and 3D modeling and analysis results therefore advance the research in the area of nonlinear aeroelastic modeling. Specifically, this research has produced the following key results:

- The coupled 2D model of an airfoil with the two-degree-of-freedom (plunge, h and pitch, θ) is developed such that the rotational free-play nonlinearity is separated out from the otherwise linearized model, and modeled as a feedback component, per the Luré form.
- 2. A hyperbola function works well to model the rotational free-play nonlinearity within the Luré form.
- 3. The resulting analytical framework for the 2D aeroelastic system successfully forecasts nonlinear flutter behavior, especially the δ dependence for both the non-origin fixed point locations and the flutter onset velocity.
- 4. The flutter behavior of the 2D nonlinear system is confirmed to be an LCO associated with a supercritical Poincare-Andronov-Hopf bifurcation by analytic derivation.
- 5. Flutter behavior results are further confirmed by spectral analysis.
- 6. The development of new 3D models introduces reasonable methods of representing the system in Luré form, allowing the same analytical framework methods developed in the 2D model
- 7. The successful incorporation of recently developed, computationally low-cost system identification tools has led to the efficient development of a reasonably accurate aerodynamic reduced order model for use in the 3D models.
- 8. System analysis of the 3D models provide qualified results regarding the stability and LCO behavior required to characterize flutter behavior. It is noted, however, that efforts to further improve the 3D model are continuing.

9. The numerical models developed compare favorably to the WADC wind tunnel results described as one of the motivations for this research.

Nomenclature

A	=	state matrix, coupled system
В	=	input matrix, coupled system
C	=	output matrix, coupled system
C_L	=	aerodynamic lift coefficient
C_M	=	aerodynamic moment coefficient
\bar{c}	=	chord length
D_L	=	mass-normalized damping matrix
$E_{L,NL}$	=	mass-normalized stiffness matrix, linear and nonlinear, respectively
e	=	dislocation of center of mass from elastic axis
fps	=	feet per second
e_{ac}	=	dislocation of aerodynamic center from elastic axis
h	=	vertical displacement (plunge), positive down
J	=	system Jacobian matrix
K_1	=	translational stiffness
K_2	=	rotational stiffness
K_L	=	linear stiffness matrix
K_{NL}	=	nonlinear stiffness vector
L_A	=	aerodynamic lift
M	=	mass matrix
M_A	=	aerodynamic moment
m	=	mass
q	=	pitch rate
\bar{q}	=	dynamic pressure
R	=	damping (rates) matrix
V	=	freestream velocity
α	=	angle of attack
$lpha_\gamma$	=	hyperbola asymptote slope parameter

 $\alpha_{\lambda}, \beta_{\lambda} =$ real and imaginary components of the hyperbolic fixed point eigenvalue $\delta =$ half-width of rotational free-play band, radians (unless otherwise noted) $\gamma =$ slope of a hyperbola asymptote $\kappa =$ equilibrium point parameter for the hyperbola model $\phi =$ a nonlinear function

 θ = rotational displacement (pitch), positive nose-up

CHAPTER 2. TWO-DIMENSIONAL MODEL DEVELOPMENT

The fundamental basis for the application of systems analysis techniques on the analysis of the flutter phenomenon in a nonlinear system is the derivation of a reasonable fluid-structural model and the development of a feedback interconnection model. The analysis also requires the development of an analytical nonlinearity function that governs the modeling of rotational freeplay. These modeling requirements apply to both the two-dimensional and three-dimensional systems.

2.1 Basic structural model

The analysis of this study is based on a simple two-dimensional airfoil constrained to two degrees of freedom, namely motion in pitch θ and plunge h, Figure 2.1. The structural forces are developed using Lagrange's method per, e.g. Inman (1996), and take into account force and moment contributions due to the dislocation of the center of gravity from the elastic axis (e). For the 2D airfoil, it is assumed that the elastic axis coincides with the axis of rotation. This distinction becomes significant when the 2D model is expanded into a segmented model of a three-dimensional airfoil, particularly in the case of an all-moving airfoil. The structural restoring forces are applied at the elastic axis of the model, therefore stiffness in plunge (K_h) and pitch (K_{θ}) are decoupled.

With the frame of reference located at the elastic axis, the dislocation of the center of mass introduces a coupling of pitch and plunge inertia sometimes referred to as static unbalance. The energy based approach determines the system equations of motion, resulting in a standard set of equations where the structural forces are balanced against the existing aerodynamic forces.



Figure 2.1 Schematic of the Two Degree of Freedom Airfoil Model. (1) Aerodynamic Center, (2) Elastic Axis, (3) Center of Gravity

$$M_{s} \begin{bmatrix} \ddot{h} \\ \ddot{\theta} \end{bmatrix} + K_{s} \begin{bmatrix} h \\ \theta \end{bmatrix} = \mathbf{F}_{aero}$$
(2.1)

$$M_s = \begin{bmatrix} m & -me \\ -me & me^2 + J \end{bmatrix}, \quad K_s = \begin{bmatrix} K_h & 0 \\ 0 & K_\theta \end{bmatrix}, \quad \mathbf{F}_{aero} = \begin{bmatrix} -L_a \\ M_a \end{bmatrix}$$

The aerodynamic forces and moments on the right-hand side of 2.1 act as the forcing functions to the elastic structure, and are functions of angle of attack α , angle of attack rate $\dot{\alpha}$ and pitch rate q. Far a reasonably faithful dynamic model, these forces and moments must capture the unsteady terms of the fluid flow response. A simple but effective way to define these forces and moments is through the use of classical non-dimensional force and moment coefficients, which are valid for both steady and unsteady aerodynamics. Total lift and moment contributions are then determined for the aerodynamic states of α , $\dot{\alpha}$ and q by using these coefficients.

$$L_{a} = \bar{q}S\left(C_{L\alpha}\alpha + C_{L\dot{\alpha}}\dot{\alpha} + C_{Lq}q\right)$$

$$M_{a} = \bar{q}S\bar{c}\left(C_{M\alpha}\alpha + C_{M\dot{\alpha}}\dot{\alpha} + C_{Mq}q\right) + \bar{q}Se_{ac}\left(C_{L\alpha}\alpha + C_{L\dot{\alpha}}\dot{\alpha} + C_{Lq}q\right)$$

$$(2.2)$$

where \bar{q} is dynamic pressure, \bar{c} is chord length. The aerodynamic moment equation includes terms that describe moments due to the dislocation of the aerodynamic center from the elastic axis e_{ac} . The use of aerodynamic coefficients indicates that the model is a linearization of the nonlinear aerodynamics (linearized about the origin in the case of a symmetric airfoil). Analysis is therefore restricted to a region about the steady trim condition. This will not prove to be a constraint to the simulation and analysis of this model.

To employ this method, accurate aerodynamic derivatives must be obtained. Modern computational fluid dynamics packages are able to provide these values for any given airfoil shape and set of atmospheric conditions. This study relies on PMARC 12, a panel code method developed by NASA Ames Research Center (Ashby et al. (1992)). PMARC provides results for incompressible, inviscid flow, so results are valid up to 0.3 M. These terms are nonlinear, but can be safely assumed to be linear for the flight conditions around the stable equilibrium points of the aerodynamic system. They are also Reynolds number dependent.

Most methods for determining the aerodynamic forces on an airfoil system place the lift and moment at the aerodynamic center, the chord-wise location where the aerodynamic moment has no dependence on angle of attack. For most typical airfoils, this point lies in the vicinity of the quarter-chord point $(\frac{c}{4})$. In the above model, the aerodynamic forces are resolved back to the elastic axis including the induced moment from the lift applied off of the elastic axis. The e_{ac} term is defined as the distance from the aerodynamic center to the elastic axis.

The first order terms in α and θ (since $q = \dot{\theta}$) capture the transient response characteristics involved with the flow response to a change in airfoil state, that is, the unsteady aerodynamics. To provide the full aeroelastic model, the dependent variables of the aerodynamic equations are converted to the state variables of the structure by making the following substitution:

$$\alpha = \theta + \frac{\dot{h}}{V}, \quad \dot{\alpha} = \dot{\theta} + \frac{\ddot{h}}{V}, \quad q = \dot{\theta}$$
(2.3)

This allows the coupled aeroelastic model to be resolved to a common set of state variables, so that

$$L_a = \bar{q}S\left[C_{L\alpha}\left(\theta + \frac{\dot{h}}{V}\right) + C_{L\dot{\alpha}}\left(\dot{\theta} + \frac{\ddot{h}}{V}\right) + C_{Lq}\dot{\theta}\right]$$
(2.4)

$$M_{a} = \bar{q}S\bar{c}\left[C_{M\alpha}\left(\theta + \frac{\dot{h}}{V}\right) + C_{M\dot{\alpha}}\left(\dot{\theta} + \frac{\ddot{h}}{V}\right) + C_{Mq}\dot{\theta}\right] \\ + \bar{q}Se_{ac}\left[C_{L\alpha}\left(\theta + \frac{\dot{h}}{V}\right) + C_{L\dot{\alpha}}\left(\dot{\theta} + \frac{\ddot{h}}{V}\right) + C_{Lq}\dot{\theta}\right]$$

The system remains first order in plunge velocity, \dot{h} and pitch angle, θ , but the aerodynamic force equations represented by structural state variables produce aerodynamic contributions to the apparent mass, stiffness and damping of the coupled system.

$$F_{aero} = M_{app} \begin{bmatrix} \ddot{h} \\ \ddot{\theta} \end{bmatrix} + B_a \begin{bmatrix} \dot{h} \\ \dot{\theta} \end{bmatrix} + K_a \begin{bmatrix} h \\ \theta \end{bmatrix}$$
(2.5)

$$M_{app} = \bar{q}S \begin{bmatrix} \frac{-C_{L\dot{\alpha}}}{V} & 0\\ \frac{\bar{c}C_{M\dot{\alpha}} + e_{ac}C_{L\dot{\alpha}}}{V} & 0 \end{bmatrix}, \ K_a = \bar{q}S \begin{bmatrix} 0 & C_{L\alpha}\\ 0 & \bar{c}C_{M\alpha} + e_{ac}C_{L\alpha} \end{bmatrix}$$

$$B_{a} = \bar{q}S \begin{bmatrix} \frac{-C_{L\alpha}}{V} & -(C_{L\dot{\alpha}} + C_{Lq}) \\ \frac{\bar{c}C_{M\alpha} + e_{ac}C_{l\alpha}}{V} & \bar{c}\left(C_{M\dot{\alpha}} + C_{Mq}\right) + e_{ac}\left(C_{L\dot{\alpha}} + C_{Lq}\right) \end{bmatrix}$$

When 2.1 and 2.5 are combined, a coupled system in h and θ results.

$$(M_s - M_{app}) \begin{bmatrix} \ddot{h} \\ \ddot{\theta} \end{bmatrix} = B_a \begin{bmatrix} \dot{h} \\ \dot{\theta} \end{bmatrix} + (K_a - K_s) \begin{bmatrix} h \\ \theta \end{bmatrix}$$
(2.6)

$$\begin{bmatrix} \ddot{h} \\ \ddot{\theta} \end{bmatrix} = (M_s - M_{app})^{-1} B_a \begin{bmatrix} \dot{h} \\ \dot{\theta} \end{bmatrix} + (M_s - M_{app})^{-1} (K_a - K_s) \begin{bmatrix} h \\ \theta \end{bmatrix} = D \begin{bmatrix} \dot{h} \\ \dot{\theta} \end{bmatrix} + E \begin{bmatrix} h \\ \theta \end{bmatrix}$$

A state space representation is obtained by assigning the state vector $\mathbf{x} = \begin{bmatrix} h \theta \dot{h} \dot{\theta} \end{bmatrix}^T = [x_1 x_2 x_3 x_4]$ so that

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & I \\ E & D \end{bmatrix} \mathbf{x}$$
(2.7)

The nonlinear structural stiffness behavior is contained in the original K_{θ} term, which in turn is now embedded in the *E* matrix derived above. Specifically,

$$E_{11} = \frac{-K_h V(me^2 + J)}{|M_s - M_{app}|}$$

$$E_{12} = \frac{-(K_\theta Vem) - \left[VS\bar{q}\left[C_{L\alpha}(J + e^2, -mee_{ac}) - C_{m\alpha}\bar{c}em\right]\right]}{|M_s - M_{app}|}$$

$$E_{21} = \frac{-K_h (Vem + C_{L\dot{\alpha}}S\bar{q}e_{ac} + C_{M\dot{\alpha}}Sqc)}{|M_s - M_{app}|}$$
(2.8)

$$E_{22} = \frac{-K_{\theta}(Vm + C_{L\dot{\alpha}}S\bar{q}) - \left[S^2\bar{q}^2\bar{c}(C_{L\alpha}C_{M\dot{\alpha}} - C_{L\dot{\alpha}}C_{M\alpha})S\bar{q}Vm\left[C_{L\alpha}(e - e_{ac}) - C_{M\alpha}\bar{c}\right]\right]}{|M_s - M_{app}|}$$

$$|M_s - M_{app}| = JVm + \bar{q}S\left[C_{L\dot{\alpha}}(J + e^2m - ee_{ac}m) - C_{m\dot{\alpha}}\bar{c}em\right]$$
(2.9)

Rotational stiffness only affects the elements E_{12} and E_{22} , and can be separated out into the elements

$$E_{12L} = \frac{-\left[VS\bar{q}\left[C_{L\alpha}(J+e^{2},-mee_{ac})-C_{m\alpha}\bar{c}em\right]\right]}{|M_{s}-M_{app}|}$$
(2.10)
$$E_{22L} = \frac{-\left[S^{2}\bar{q}^{2}\bar{c}(C_{L\alpha}C_{M\dot{\alpha}}-C_{L\dot{\alpha}}C_{M\alpha})S\bar{q}Vm\left[C_{L\alpha}(e-e_{ac})-C_{M\alpha}\bar{c}\right]\right]}{|M_{s}-M_{app}|}$$
(2.11)

and

$$E_{12NL} = \frac{-Vem}{|M_s - M_{app}|} K_{\theta}$$

$$E_{22NL} = \frac{-(Vm + C_{L\dot{\alpha}}S\bar{q})}{|M_s - M_{app}|} K_{\theta}$$
(2.12)

With this expanded construct, the state space representation developed above in equations 2.7 become

$$\dot{x}_{1} = x_{3}$$

$$\dot{x}_{2} = x_{4}$$

$$\dot{x}_{3} = E_{11}x_{1} + (E_{12L}x_{2} + E_{12NL}\hat{x}_{2}) + D_{11}x_{3} + D_{12}x_{4}$$

$$\dot{x}_{4} = E_{21}x_{1} + (E_{22L}x_{2} + E_{22NL}\hat{x}_{2}) + D_{21}x_{3} + D_{22}x_{4}$$
(2.13)

or equivalently,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & I \\ E_L & D \end{bmatrix} \mathbf{x} + E_{NL} \hat{\mathbf{x}}_2$$
(2.14)

The terms in 2.14 can be simplified to the nominal state equation, and $\hat{\mathbf{x}}_2$ is now defined as $\phi(\theta)$. Note also that for the state variable as defined above, $\theta = x_2$, or more specifically, $\theta = [0 \ 1 \ 0 \ 0] \mathbf{x}^T$. This produces the baseline state space formulation for the nonlinear system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\phi(\theta) \tag{2.15}$$
$$\theta = \mathbf{y} = C\mathbf{x}$$

In this manner, the stiffness constant K_{θ} is now replaced with a general nonlinear function $\phi(\theta)$.

2.2 Free-play nonlinearity

Candidate models are now considered for $\phi(\theta)$, the stiffness free-play nonlinearity in pitch. For the purposes of numerical simulation, this nonlinearity has been typically modeled by either a piecewise linear function or a cubic function. Theoretical results developed by Tang and Dowell (1993) using both of these methods match reasonably well with experimental results, predicting ranges of LCO and possibly chaotic behavior, and indicating a strong dependence on initial conditions of the structure.



Figure 2.2 free-play model with piecewise linear stiffness.

2.2.1 Piecewise linear

The piecewise linear model allows for an adjustable free-play region $\pm \delta$, with essentially no stiffness inside θ and constant linear stiffness outside, as shown in Figure 2.2.

$$\phi_{pwl}(\theta) = \begin{cases} \theta + \delta & \text{if } \theta < -\delta \\ 0 & \text{if } |\theta| \le \delta \\ \theta - \delta & \text{if } \theta > \delta \end{cases}$$
(2.16)

A piecewise linear function, though easy to implement in a model, does not provide much insight into the flutter characteristics from a systems analysis perspective. Studies such as Hoffman and Speilberg (1954); Cooley (1958) show that flutter onset velocity V_f is dependent upon the size of the free-play region. As the free-play region increases, flutter onset occurs at a lower velocity, saturating at some minimum value above which any increase in free-play region does not affect V_f . However, the piecewise linear free-play model does not show any δ -dependence in V_f when flutter onset is determined by eigenvalue analysis of the local linearization at the equilibrium points. Therefore the piecewise linear model is not useful for a systems approach to flutter prediction and analysis.

2.2.2 Cubic function

To facilitate nonlinear systems analysis techniques, a model function is required that is continuous, differentiable and Lipschitz, which is to say for a function f(x),

$$||f(x_1) - f(x_2)|| \le L||x_1 - x_2|$$

for some $L \ge 0$, where $\| * \|$ indicates any suitable p-norm. Additionally, the analysis will require a monotonically increasing function with a distinct, unique crossing of the domain axis at $\theta = 0$, ensuring passivity and a distinct fixed point at the origin. It is also desired that the function can be adjusted so that various widths of the free-play region can be implemented, and that the function be relatively tractable through its first derivative, for ease of implementation.

A cubic function has often been used to model structural nonlinearity in torsion, using both hard spring and soft spring models. Numerical simulation by Lee, et al. Lee et al. (1999a) examined both forms for their ability to model flutter behavior with a nonlinear structural model. The general formulation for this nonlinear rotational stiffness is

$$\phi_{cub}(\theta) = \beta_1 \theta + \beta_3 \theta^3 \tag{2.17}$$

To provide very light stiffness in the free-play region, $\beta_1 \ll 1$, and if desired, a piecewise continuous approach can be used to insert an almost-flat function for the free-play region. A hardening spring is established for $\beta_3 > 1$, while $\beta_3 < 1$ establishes a soft spring.

Results by Lee, et al. for the soft spring model showed a clear dependence on the initial conditions of the structure, as well as a sub-critical Hopf bifurcation, however it was not an accurate representation of free-play. The hard spring model showed no initial condition dependence, but a clear presence of LCO behavior. LCO onset, however, occurred at the linear flutter onset velocity, which again is not consistent with experimental results such as Hoffman and Speilberg (1954); Cooley (1958), which show a clear reduction in V_f in the presence of free-play. While the soft spring cubic function is still relatively easy to implement in numerical simulation and provides a δ -dependence for V_f , the model is not readily tailored to a true free-play condition.

2.2.3 Sigmoidal

This difference in observed behavior and theoretical analysis indicates that the model for the nonlinear spring can be improved, particularly one that admits to a systems analysis approach. A model based on a continuous function involving a gradual variation in stiffness is proposed, to capture the phenomenon of δ -dependence for the flutter velocity. Another candidate for modeling the rotational free-play nonlinearity involves use of a sigmoidal function, specifically one that is based on the logistic function

$$f(y) = \frac{K}{1 + e^{-a(y-y_0)}}$$
(2.18)

where K defines the maximum value of the logistic curve, a defines the steepness of the transition from 0 (for $y \ll y_0$) to K (for $y \gg y_0$), and y_0 defines the transition point.

A rotational stiffness free-play model is constructed by pairing two logistic functions, which act as on-off switches for the edges of the free-play region, and offsetting their transition points by $\pm \delta$ as appropriate.

$$\phi_{sig}(\theta) = (\theta - \delta) \left(\frac{1}{1 + e^{-a(\theta - \delta - 0.1s)}}\right) + (\theta + \delta) \left(\frac{1}{1 + e^{-a(\theta - \delta - 0.1s)}}\right) + s\theta \tag{2.19}$$

This function allows the structural rotational stiffness to be tailored for variation in angular width of the free-play region (via δ), and provides the ability to control some non-negative slope within the free-play region (via $s\theta$, $s \ge 0$), thus allowing the existence of finitely many fixed points.

The parameter *a* controls the sharpness of the transition at the knee of the function where θ is in the vicinity of δ . The amount of slope *s* introduced to the function is an order of magnitude less than the half-width of the free-play region itself (δ). Without this slope, the free-play model (similar to the piecewise linear model) will be nearly flat in the free-play region and as with the piecewise linear model, would present a continuum of of fixed points for $|\theta| \leq \delta$. Using 2.19 ensures the existence of a finite number of unique equilibrium fixed points (not an unrealistic property for the model). This feature allows the correlation of the time-domain behavior to the movement of the system eigenvalues as freestream velocity *V* is varied.
2.2.4 Hyperbola

While the sigmoidal model provides a physically realizable nonlinearity function, is continuous and involves a gradual variation in stiffness at the crossover region between free-play and normal stiffness, it is a difficult function to work with in nonlinear analysis, particularly due to the complexity of first and second derivatives, which are necessary for both linearization and confirmation of a Hopf bifurcation. Adaptation of a hyperbola function addresses these concerns. When stiffness is modeled by a hyperbola function, the output $\phi(\theta)$ asymptotically saturates to a finite linear stiffness. The full nonlinear behavior is modeled by joining at the origin two antisymmetric hyperbolas from the first and third quadrants. The hyperbola nonlinearity can be thought of as a practical approximation of a free-play nonlinearity where the free-play region transitions continuously to a linear stiffness region as shown in Figure 2.3. The equation is derived for the first quadrant; the function in the third quadrant is an inverted mirror image.



Figure 2.3 Comparison of free-play nonlinearity models. Hyperbola models are shown in blue for various values of δ , along with a piece-wise linear model and the linear case.

To design a hyperbola function that models a free-play nonlinearity, with asymptote slopes γ_1 and γ_2 , the following criteria are applied:

- The vertex of the hyperbola is chosen at the edge of the free-play region, $(\delta, 0)$;
- The asymptotic slope of the hyperbola as $\theta \to \infty$ is γ_2 (this sets the slope in linear stiffness regime);
- The hyperbola must pass through the origin i.e. $\phi_h(0) = 0$;
- The asymptotic slope in the free-play region is γ_1 .

The free-play nonlinearity is then modeled as

$$\phi_h(y) = \operatorname{sgn}(y) \left(\left(\frac{\gamma_1 + \gamma_2}{2} \right) (|y| - \delta) + \left[\frac{(\gamma_2 - \gamma_1)^2 (|y| - \delta)^2 + 4\gamma_1 \gamma_2 \delta^2}{4} \right]^{\frac{1}{2}} \right)$$
(2.20)

The following conditions on the nonlinearity $\phi_h(y)$ are applied-

- 1. The limiting stiffness as $y \to \infty$ is the linear stiffness of the spring $\implies \gamma_2 = 1$.
- 2. As free-play is increased $(\delta \to \infty)$, nonlinear stiffness converges to a piecewise linear stiffness $\implies \lim_{\delta \to \infty} \gamma_1 = 0.$
- 3. As free-play is decreased $(\delta \to 0)$ nonlinear stiffness converges to linear stiffness $\implies \lim_{\delta \to 0} \gamma_1 = \gamma_2 = 1.$

These conditions then allow the definition of $\gamma_1 = \frac{1}{1+\alpha_\gamma \delta^r}$. The values of α_γ and r are then used to tailor the model to reflect the free-play condition. By choosing $\alpha_\gamma = 100$, r = 0.1, a representative free-play model is maintained as δ is varied.

2.3 Aerodynamic model

Aerodynamic forces and moments are generated by the pressure distribution along the surfaces of an airfoil, which is in turn dependent upon the direction and magnitude of the flow field. As a result, the aerodynamic component of an aeroelastic system is nonlinear. In addition, realistic aerodynamic flow is typically unsteady. An extensive body of literature exists in the field. Classic treatments such as Bisplinghoff et al. (1996) addressed the complexity of the problem by assuming small disturbances from steady flow, i.e. perturbations which are easily linearized. Also, classic aeroelastic studies at least began in the incompressible region, using potential flow theory when possible. The results were generally useful. Lee et al. (1999b) provides a useful summary of modern methods of modeling the compressible flow regime for aeroelastic studies, specifically as they address the more common contributors to flutter such as shock wave propagation and flow separation. With the advent of modern computational methods, and the ever increasing capabilities of present-day processors, numerical methods are becoming increasingly relevant to the modeling of nonlinear flow to the level that aeroelastic response can be approximated, however comprehensive flow models for aeroelastic systems are still too computationally expensive for most analysis, which has therefore driven much recent research into economical computational methods (Dowell et al. (2003)).

2.3.1 Use of CFD-derived aerodynamic coefficients

As this research is intended to validate new tools for a systems analysis approach to the study of flutter, it begins with a relatively simple flow model, using many of the assumptions of the classic aeroelastic studies along with low-order computational fluid dynamics processes. The derivation of the aerodynamic contribution to the model is provided above. A full description of the use of dimensional analysis to develop the equations relating the aerodynamic forces and moments on the airfoil to the parameters of the fluid flow can be found in Appendix D of Blakelock (1991). The applicable standard lift and moment coefficients are determined by applying a standard aerodynamic potential flow code (PMARC 12, Ashby et al. (1992)) to simulate dynamic flow over a NACA 0010 airfoil. The model used for this research is therefore limited to inviscid, incompressible flow. It is worth noting that the inclusion of the terms for lift and aerodynamic moment due to $\dot{\alpha}$ and q allow for the capture of effects due to unsteady flow and force lag, even though they are replaced through the quasi-steady flow assumptions of $\dot{\alpha} = \dot{\theta} + (\ddot{h}/V)$ and $q = \dot{\theta}$.

2.3.2 Classic analytical modeling techniques

Several approaches are available for development of an aerodynamic model that will effectively represent the system that outputs aerodynamic forces in response to a structural shape input. Many classical developments of unsteady aerodynamic response for a 2D airfoil in incompressible flow exist, including Wagner's function for the time-domain response to a unit step variation in angle of attack (Wagner (1925)), and Theodorsen's function for the frequency domain response to a sinusoidal pitching motion (Theodorsen (1935)). These provide good results for the prediction of the flutter velocity and frequency for airfoils, however their results are only valid at zero velocity, flutter velocity or in the presence of a sinusoidal excitation. Eigenvalue behavior therefore cannot be adequately evaluated against the parametrization of the model.

To model system behavior for a more general set of inputs (e.g. step and impulse), a modified method is required. The p-k method (Wright and Cooper (2007)) and Roger's approximation (Roger (1977)) provide frequency domain and time domain methods, respectively, for adapting the Theodorsen model into a frequency response function for the aerodynamic model, and are commonly used in industry. The ability of these methods to characterize system damping and damped natural frequency with respect to subcritical freestream velocities is improved, but still not accurate.

These methods are all easily adapted to system identification techniques. In general, these analytical models are manipulated to a rational function approximation of the desired order in the frequency domain, and the system coefficients are determined from the input/output relationship provided by a suitable aerodynamic flow code (examples include Brunton and Rowley (2013) for approximations of the Theodorsen model, and Tiffany and Adams, Jr. (1988) for approximations of models in the form of Roger's approximation).

CHAPTER 3. BEHAVIOR AND ANALYSIS OF THE TWO-DIMENSIONAL MODEL

3.1 Introduction

Classic study and research into aeroelastic effects and the phenomenon of flutter is based on the simple 2D model. The model developed in the previous chapter is based on several simplifying assumptions. For instance, as a 2D system, it contains no span-wise flow, and due to the flow code used to determine the aerodynamic force and moment coefficients, it is limited to the incompressible flow region. Nonetheless, this model is still able to produce valid, verifiable results to help establish the utility of this systems approach. Moreover, several practical applications exist such as the operation of wind turbine blades where such a 2D model can be quite realistic.

3.2 Stability Analysis

3.2.1 Equilibrium Points

The equilibrium points of the system are determined from 2.15. Clearly, the origin is always an equilibrium state for the system. This is referred to as the origin equilibrium point. By manipulating 2.15 and defining $\kappa = \frac{-1}{CA^{-1}B}$, any non-zero equilibrium points, i.e. equilibrium point(s) not located at the state space origin, are found by solving $\kappa y = \phi(y)$. The system will exhibit two additional equilibrium states with non-zero θ^*) if

$$\kappa > \frac{2\gamma_1\gamma_2}{\gamma_1+\gamma_2} = \frac{2}{2+\alpha_\gamma\delta^r}$$

From 2.20 developed previously and the definition of κ , the non-zero equilibrium points can be determined analytically. The three equilibria are defined at

$$\theta_o^* = 0, \qquad \theta_{nz}^* = \pm \delta \left(\frac{\gamma_1}{\gamma_1 - \kappa} + \frac{\gamma_2}{\gamma_2 - \kappa} \right)$$
(3.1)

The non-zero equilibria are symmetric and have a V_{∞} dependence via κ . By extension, when (2.15) is linearized about the system equilibrium points, system eigenvalues will also have a dependence on δ and V_{∞} . These relationships are depicted below in the discussion of simulation results.

Behavior of the system about the origin equilibrium point is determined by an eigenvalue analysis of the corresponding system Jacobian, J = A as defined in (2.15). The dominant eigenvalues for this system (a velocity-dependent complex conjugate pair) become unstable at a relatively low velocity V_{pf} , which is also the velocity at which the non-zero equilibrium points appear. That is to say, the sign of each θ_{nz}^* switches at V_{pf} , and $\theta_{nz}^*(V_{pf} = 0)$. A pitchfork bifurcation occurs at this velocity. For $\delta = 0.01$, $V_{pf} = 36.6 fps$.

Regarding the characterization of the non-zero equilibrium points, the Hartman-Grobman theorem allows us to linearize along a range of freestream velocities for any fixed δ . As long as the equilibrium points remain hyperbolic, the eigenvalues of the resulting Jacobian matrices will indicate system behavior for that free-play value. For the range of velocities under consideration, the linearized system presents two sets of complex conjugate eigenvalues ($\lambda_{1,2}, \lambda_{3,4}$). The results shown in Figure 3.1 demonstrate this relationship.

3.2.2 Establishing a Hopf Bifurcation

For those conditions where a set of complex eigenvalues of the linearized system approach the imaginary axis, however, the Center Manifold theorem must be applied to properly characterize any bifurcation that is exhibited by the system with small variations in velocity about that point. The development follows the general procedure outlined in Wiggins (2003) and demonstrated in Liu et al. (2000).

With a specified free-play value, system bifurcation occurs at a specific velocity, and is associated with a particular non-zero equilibrium point. Without loss of generality, the system is translated so that this equilibrium point is moved to the origin of the phase space. The system is then linearized and transformed into Jordan canonical form so that the four-dimensional state



Figure 3.1 Velocity-dependent behavior of linearized system eigenvalues for various values of δ (radians). (a) All four eigenvalues. (b) Detail of dominant roots $\lambda_{1,2}$.

system can be identified as two separable two-dimensional systems – one possessing stable eigenvalues and one with purely imaginary eigenvalues, i.e. a non-hyperbolic system. From this form, the so-called center manifold is determined. We know from the Center Manifold theorem that a center manifold exists for this system, and for sufficiently small variations in velocity, the dynamics of (A.1) will be given by the dynamics of this non-hyperbolic system restricted to the center manifold. If we designate the new variables as $u \in \mathbb{R}^2$, we can write the model as

$$\dot{u} = J_{cm}u + \begin{bmatrix} h_1(u_1, u_2) \\ h_2(u_1, u_2) \end{bmatrix}$$
 (3.2)

where h_1 and h_2 include second- and third-order terms in u.

The transformation to the center manifold simplifies the linear portion of (2.15) as much as possible. By application of the Normal Form theorem (Wiggins (2003)), the system is transformed into complex space, and two near-identity transformations are performed that map the system to spaces, each with a new basis selected to simplify the complex system as much as possible. Through these mappings, all second-order and most third-order terms of the nonlinear portion of (3.2) are eliminated, greatly simplifying the system to the form

$$\dot{y} = \lambda y + c(V)y^2\bar{y} + \mathcal{O}(4)$$

where $\lambda = \alpha_{\gamma}(V) + i\beta_{\gamma}(V)$ is the dominant eigenvalue of the original non-hyperbolic system, and c(V) = a(V) + ib(V) is a parameter-dependent complex constant that results from the above simplification. Finally, the system is converted into polar coordinates on the complex plain, resulting in a system of the form

$$\dot{r} = \alpha_{\gamma}r + ar^3 + \mathcal{O}(r^5),$$

$$\dot{\theta} = \beta_{\gamma} + br^2 + \mathcal{O}(r^4),$$
(3.3)

which allows the identification of the Poincaré–Andronov–Hopf (PAH) Bifurcation and its subor super-criticality. The full derivation of the center manifold of the system and the transformation to normal form is presented in detail in Appendix A.

Once the normal form is obtained, one more Taylor expansion is carried out so that 3.3 becomes

$$\dot{r} = \alpha'(0)_{\gamma}r + a(0)r^3 + \mathcal{O}(\gamma^2 r, \gamma r^3, r^5), \dot{\theta} = \beta(0) + \beta'(0)\gamma + b(0)r^2 + \mathcal{O}(\gamma^2, \gamma r^2, r^4)$$
(3.4)

where " \prime " indicates differentiation with respect to perturbations in the parameter of interest, V.

To apply these tools, we identify the deviation of the parameter V from the velocity at which bifurcation occurs, $\mu = V - V_f$, and the slope of the real part of the eigenvalue trajectory as a function of V in the vicinity of V_f , $\alpha'_{\lambda}(V) = \frac{\partial \alpha_{\gamma}(V)}{\partial V}$. With the center manifold determined for the system, the reduced normal form (3.3) can be numerically evaluated over a range of V within a sufficiently small neighborhood of V_f . For the 2D model developed for a free-play of $\delta = 0.01 \ rad$, plots of α vs. V and a vs. V are shown in Figure 3.2

With these data, we can establish the existence of a periodic orbit in the model for the range of velocities that satisfy $-\infty < \frac{\mu d}{a} < 0$, that is, when $V > V_f$ since Figure 3.2 demonstrates



Figure 3.2 Velocity-dependent behavior of normal form coefficients (a) Real part of dominant eigenvalue, $\alpha_{\gamma}(V)$; (b) Real part of surviving third order term, a(V).

that $\alpha'_{\lambda}(V_f) > 0$ and a < 0 at the bifurcation point. Furthermore, the negative value of $a(V_f)$ establishes that the periodic orbit is asymptotically stable. While the role of a in this derivation is apparent from the derivation shown in Appendix A, its actual calculation through the numerous transforms is impractical to the point of impossibility. Fortunately, it has been shown by Guckenheimer and Holmes (1983) that following explicit calculation may be used. At the bifurcation point, where $\alpha(V^*)to0$, A.20 reduces to

$$\dot{y}_1 = -\omega y_2 + f(y_1, y_2, 0),$$

$$\dot{y}_2 = \omega y_1 + g(y_1, y_2, 0)$$
(3.5)

and the coefficient $a(0) \equiv a$ is given by

$$a = \frac{1}{16} \left[f_{y_1y_1y_1} + f_{y_1y_2y_2} + g_{y_1y_1y_2} + g_{y_2y_2y_2} \right] + \frac{1}{16\omega} \left[f_{y_1y_2}(f_{y_1y_1} + f_{y_2y_2}) - g_{y_1y_2}(g_{y_1y_1} + g_{y_2y_2}) - f_{y_1y_1}g_{y_1y_1} + f_{y_2y_2}g_{y_2y_2} \right]$$

where the subscripts denote partial derivatives evaluated at the bifurcation point (0, 0, 0).

Finally, LCO amplitude and frequency can be predicted from the normal form results through the relationships

$$r = \sqrt{\frac{-\mu \alpha'_{\lambda}(V_f)}{a}}, \quad \dot{\theta} = \beta_{\lambda} + (\beta'_{\lambda}(V_f) - \frac{b(V_f)\alpha'_{\lambda}(V_f)}{a})\mu$$

These predictions are shown in Figure 3.3, along with amplitude and frequency data obtained from time histories of the model and FFT results of the same. Frequency predictions are very close. Amplitude results agree qualitatively, however time history amplitudes must be scaled by a factor of 38.85 to reach quantitative agreement with the predicted values. As Strogatz (1994) points out, however, the development of the normal form, due to the simplifications involved in its derivation, assumes an idealized behavior of the LCO radius that is seldom matched by the behavior of the LCO in practice.



Figure 3.3 LCO characterization over a range of velocities in the vicinity of the flutter velocity (i.e. bifurcation): (a) Pitch angle amplitude; (b) Frequency.

These results lead to a schematic diagram of the Hopf bifurcation behavior as shown in Figure 3.4. The simulation results are deferred to the succeeding simulation section as we will now discuss another unique approach and data analysis tools used to characterize the flutter behavior of the system.

3.2.3 Spectral Analysis

In this section we discuss spectral linear transfer operator-based methods for the analysis of time series data obtained from simulation. The proposed methods can also be used for



Figure 3.4 Schematic of Hopf bifurcation in the system dynamics as a function of the airspeed.

the analysis of experimental data. The spectral methods are useful for the analysis of steady state dynamics consisting of LCO behavior. In particular, information about the location and frequency of LCOs can be obtained using the spectral method presented in Mehta and Vaidya (2005). The basic idea behind the proposed approach is to embed the time series data in appropriate high dimensional space using the technique of time delayed embedding. Let $\theta(\Delta t)$ for $t = 1, 2, \dots, M$ be the time series data. If N is the embedding dimension, then any point in the embedded space will be of the form $(\theta(\Delta m), \theta(\Delta(m-1)), \dots, \theta(\Delta(m-N+1)))$. N is typically determined by the minimum number of dimensions required to capture the anticipated limit cycle behavior in the state or physical space. If the embedded space is then partitioned into L boxes $\{D_1, \ldots, D_L\}$, the transition probability of any point from box D_i to box D_j , denoted by p_{ij} , is obtained by determining the average number of points that make the transition from box D_i to D_j . The matrix obtained using this procedure, $[P]_{ij} = p_{ij}$, is a Markov matrix, and is row stochastic (Mehta and Vaidya (2005)). Spectral analysis of this Markov matrix provides useful information about the steady state dynamics of the system. In particular, the eigenfunction of the Markov matrix associated with eigenvalue $\lambda = 1 \pm 0j$ is the one supported in the steady state dynamics, the so-called invariant measure of the nonlinear system. Similarly, the eigenvalue of the Markov matrix on the complex plane unit circle (actually a complex conjugate pair), with the second largest real part, carries information about the dominant frequency in the system.

3.3 Simulation

3.3.1 Flutter Onset Behavior

The simulation results for the 2D model with hyperbola nonlinearity validate the results of the system analysis performed above. Simulations are first performed for a system with linear rotational stiffness; the free-play region is set to zero. This system exhibits stable behavior at lower velocities with oscillatory transient responses that converge to the state space origin. As velocity is increased, settling time grows, until V = 128.85 fps, where the system becomes divergent. We establish this as the nominal linear flutter velocity, V_f . For simulations of the nonlinear system, velocities will be normalized with respect to V_f .

With the free-play model included, system behavior exhibits the presence of a single equilibrium point at zero pitch for the lowest velocities. As velocity is increased, a velocity is reached where transient responses converge to one of two distinct equilibrium points, both at nonzero pitch angles, with equal amplitudes but opposite signs $(\pm \theta^*)$. This pitchfork bifurcation is considered an artifact of the system model, and the velocity at which it occurs (V_{pf}) decreases as the width of the free-play region decreases. Above V_{pf} , the origin becomes an unstable equilibrium point. As velocity is further increased, a critical velocity is reached where steady state model behavior no longer converges to a single value, but converges to a LCO orbit. Although not divergent, this sustained oscillation is considered to be flutter, and the associated velocity at which LCO is manifest is considered to be the flutter onset value, or critical velocity V_c , for the nonlinear structure. V_c decreases as the amount of free-play increases. Figure 3.5(a) shows this relationship for the model used. The results for free-play variation on the flutter onset velocity show a qualitative match with results obtained experimentally in Hoffman and Speilberg (1954), as shown in Figure 3.5(b). The 2D model results are therefore in general agreement with the WADC data by indicating a relationship between normalized flutter velocity and the size of the free-play region. This is one of the key results of this research. Quantitative agree-



Figure 3.5 Flutter onset as a function of free-play, with the velocity normalized to the linear model onset velocity (a) in the 2D model and (b) in the WADC results (Hoffman and Speilberg (1954)).

ment was not expected, since the physical parameters for the 2D model could not be expected to replicate those for a 3D physical model.

Results from FFT analysis shown in Figure 3.6 indicate the dominant frequency of the pitch and plunge data (dashed green and dotted blue lines, respectively) versus velocity. These frequencies coalesce as soon as the non-origin equilibrium points are created (i.e. at the pitchfork bifurcation), then increase together until rapidly converging to the frequency of the emerging unstable eigenvalue at the critical velocity ($V_c/V_f = 0.69$ for $\delta = 0.01$ radians). This behavior matches the resonance observed in pitch and plunge dynamics in classical flutter literature, and may also be used to determine a flutter boundary based upon eigenvalue convergence.

Simulation results are also consistent with the analytical identification of the Hopf bifurcation. Figure 3.7 shows the bifurcation plot of the non-zero equilibrium points for $\delta = 0.01$ rad. as the normalized airspeed V/V_f is varied. For $V/V_f \leq 0.69$, the black line denotes a stable equilibrium point. Above this velocity, the equilibrium point undergoes a super-critical Hopf bifurcation to a stable limit cycle. The red dashed line denotes that at bifurcation, the equilibrium point becomes unstable.



Figure 3.6 The imaginary part of the non-hyperbolic eigenvalue (red) and the dominant frequencies from FFT of pitch (dashed green) and plunge (dotted blue) data for $\delta = 0.01$ radians.



Figure 3.7 The underlying Poincare-Andronov-Hopf bifurcation in the system dynamics at $\delta=0.01$ rad.

3.3.2 Spectral Analysis

A spectral analysis of the system data was conducted based on the theory presented in Mehta and Vaidya (2005) and using the time series simulation results. The embedding dimension was chosen as $d_e = 2$ since, for the pitch angle θ , we expect to observe a 2-dimensional limit cycle in the state space of the original system. The limit cycle oscillations in the pitch and plunge motion were introduced by the nonlinearity in the θ dynamics. Hence we use delay co-ordinate embedding of the θ dynamics to reconstruct the structure of $\theta - \dot{\theta}$ space as described in Section 3.2.3. The time step of embedded data was $T_e = \delta t = 0.01$, where $\delta t = 0.01$ was the time step of simulation for the data. The partition of the space was constructed by a grid of $N_x \times N_y = 4900$ cells with $N_x = N_y = 70$ partitions on the θ and $\dot{\theta}$ dimensions. As noted in Section 3.2.3, the dominant frequency of the dynamics is obtained from the unit magnitude complex eigenvalue (complex conjugate pair) with the second largest real part. In practice, due to a finite time series data and a finite partition, there is degradation of information, which may lead to the complex eigenvalue with the frequency information not having unit magnitude. Hence, in the finite approximation of the PF-operator as a Markov matrix, the dominant frequency of the dynamics is obtained from the largest complex eigenvalue $\lambda_2 = \lambda_{2_{Re}} + \lambda_{2_{Im}} j = 0.9679 + 0.2492 j$, as shown in Figure 3.8 for V = 90 m/s. We have $|\lambda_2| = 0.9995$, and the dominant frequency is then given by $\omega_d \approx \frac{\lambda_{2_{Im}}}{T_w} = 24.92 \frac{rad}{sec}$. The dominant frequency obtained from FFT analysis of the data is $\omega_d = 25.1956 \frac{rad}{sec}$. As we do a finer partition with $N_x, N_y > 70$, spectral analysis results will asymptotically approach the actual dominant frequency of oscillation.

In Figure 3.9(a) we plot the dynamics of the pitch data as embedded in two dimensions at V = 90 and $\delta = 0.01$. We see that this data demonstrates the existence of a limit cycle as expected. We plot the steady state behavior at V = 90 just above the flutter velocity for $\delta = 0.01$ in Figure 3.9b. This invariant measure plot encompasses the partitioned space and indicates the limit cycle as a stable attractor, confirming the behavior of the embedded data and time domain dynamics. In this plot, the color values on the limit cycle indicate time spent in that region of space.



Figure 3.8 (a) Spectral plot of the discrete PF operator (Markov matrix) showing eigenvalues,(b) Zoomed in version indicating the complex conjugate pair that captures the dominant oscillation frequency.



Figure 3.9 (a) Embedding at $\delta = 0.01$ and V = 90 indicates stable behavior of the limit cycle, (b) Invariant measure indicates the stable limit cycle as the attractor.



Figure 3.10 (a) Embedding at $\delta = 0.01$ and V = 130 implies stable behavior on a period-2 orbit compared to V = 90. (b) Invariant measure at V = 130 clearly indicates a period-2 orbit borne out of a period-doubling bifurcation at some V > 90.

In Figure 3.10(a) we plot the embedded data for V = 130 and $\delta = 0.01$, which is approximately V_f for the linear system (i.e. no free-play), and at which we observe pure divergence for the linear system. With free-play present, the system does not diverge. In Figure 3.10(b) we plot the invariant measure for V = 130. Here we observe the limit cycle split into a periodic orbit that can be characterized as a period two limit cycle. Each loop of this period two limit cycle is almost periodic and indicates a bifurcation in the limit cycle.

The results of the spectral analysis therefore confirm the results obtained from both the Poincare-Andronov-Hopf bifurcation analysis, particularly the behavior of the dynamics on the center manifold when converted to the normal form, but also of the observed time domain behavior presented by numerical simulation. The benefit of the spectral analysis is that system behavior can be depicted at a much lower computational cost, and provide valuable detail into the nature of LCO dynamics.

CHAPTER 4. THREE-DIMENSIONAL MODEL DEVELOPMENT

The significant results from the 2D modeling and analysis in this research include the development of a novel characterization of the free-play nonlinearity, which allows relatively easy application of eigenvalue analysis techniques, the validation of the Hopf bifurcation when using this model, which provides a tool for predicting flutter onset, frequency and amplitude, and the application of spectrum analysis techniques. These results are now carried over to the development and analysis of a 3D model.

2D flutter dynamics can be modeled fairly easily with a coupled fluid-structural system. Structural nonlinearities can be separated from the coupled model and the system can be converted to Luré form to more easily facilitate system analysis. The 3D model does not lend itself as easily to modeling with a coupled system. Studies of flutter involving a full 3D airfoil have therefore traditionally been done in the wind tunnel. The motivation for this research, in fact, is the desire to computationally match the results of experimental data collected by the US Air Force in the mid-1950s (Hoffman and Speilberg (1954), Cooley (1958)). As theoretical results for low Mach numbers have come to closely match experimental data, more recent wind tunnel testing has begun to focus on higher Mach numbers, the influence of turbulence, and other highly nonlinear phenomenon that make theoretical analysis difficult and computationally prohibitive. Studies such as Schairer and Hand (1999) are typical. Theoretical modeling of flutter on a 3D airfoil has therefore been most commonly done through the linkage of separate, decoupled structural and aerodynamic models. This approach can be found in fundamental texts such as Bisplinghoff et al. (1996), as well as more recent studies such as Silva and Bartels (2004) and Preidikman and Mook (2000). This research explores two methods of modeling the total system. One is to include a separate rigid body system to incorporate the mechanism for the rotational stiffness nonlinearity. The key questions in this process involve the determination of which methods to use for modeling of the structure and the flow field, and how those models might be linked along with the free-play nonlinearity model to form the comprehensive model. Constraints involve the level of fidelity required from the resulting simulation, and the level of computational resources available to carry out the numerical requirements.

Several tools are readily available for the development of both a structural model of a 3D airfoil and a model of the surrounding aerodynamics. Closed-form solutions of airfoil structures are not available, so approximate models are used that discretize the structural domain. In the simplest case, the airfoil can be reduced to a single point mass located at the airfoil center of mass, and connected to the system boundary by a single stiffness connection (and possibly a single damping connection), but modern numerical methods allow a 3D airfoil to be modeled with standard finite element techniques.

4.1 Structural Model

For this research, a two degree of freedom (2DOF) lumped parameter model is used to generate the structural system. The main motivation for this strategy is to allow close matching to the WADC model of 1954 (Hoffman and Speilberg (1954)), where an eight segment model was constructed and used in the wind tunnel, and the available data is provided in a persegment fashion. This research is closely tied to the objective of building a replica of that WADC model and using it to validate the theoretical results of this research. The replica was therefore designed to a set of segment masses, moments of inertia, bending and rotational stiffness. This naturally led to the development of a lumped mass model. The use of lumped mass model is common in basic aeroelastic studies, often in conjunction with the Rayleigh–Ritz method of approximating structural response, and are known to produce reasonably accurate results at a relatively low computational cost.

The theoretical model represents a segmented airfoil as illustrated in Figure 4.1, which is comprised of eight sections. Each section is largely similar to the 2D model represented in Chapter 3, with an aerodynamic center, x_{ac} at the quarter-chord point, an elastic axis, x_{ea} at $\bar{c}/3$ (or 33% of the chord), and a center of mass, x_{cg} at some location behind (towards the trailing edge from) the elastic axis, all per Figure 2.1. Each segment is designed to be four inches in span, so these locations are considered to lie along a span-wise line parallel to the elastic axis. The segments in the airfoil are designed and assembled so that the elastic axes are aligned from segment to segment and run perpendicular to the direction of the freestream velocity. Each segment has a slightly shorter chord length than the segment immediately inboard, producing a taper in the wing and a slight sweep angle to the collective leading edge. Chord length and the chord-wise locations of x_{ac} , x_{ea} , x_{cg} are established on the chord line running down the center-line of each segment. The coordinate system is established so that plunge, h is positive in the down direction (making normal lift forces negative), and pitch, θ is positive in the leading-edge-up direction.



Figure 4.1 Segmented airfoil with mass and stiffness elements depicted.

Each segment has two degrees of freedom with a mass and moment of inertia, M_i and J_i respectively, and each is connected to its adjoining segment by a dual spring component, providing stiffness in both plunge and pitch that are designated K_{hi} and $K_{\theta i}$. Due to the dislocation of the center of mass from the axis of rotation, e, a static unbalance term appears in the mass matrix, resulting in inertial coupling between plunge and pitch motion within each segment, however there is no inertial coupling between segments. The axis of rotation is

assumed to be along the elastic axis, so no stiffness coupling exists between pitch and plunge modes, however stiffness effects between segments propagate though the model so that each stiffness interface influences the dynamics at each segment. In fact, the stiffness terms for the model are derived from the influence matrices (design or measured) for the model. The influence matrix in bend I_h indicates the vertical displacement of each segment due to a vertical force applied at one segment. The torsional influence matrix I_{θ} indicates the angle of rotation at each segment due to a moment input at one. The stiffness matrices are then taken as the inverse of the influence matrices, so that $K_h = I_h^{-1}$ and $K_{\theta} = I_{\theta}^{-1}$. As a result, the equations of motion for the *i*th segment of an n-segment lumped parameter airfoil are

$$m_{i}\ddot{h}_{i} - m_{i}e_{i}\ddot{\theta}_{i} + \sum_{j=1}^{n} K_{h_{ij}}h_{j} = -L_{i}$$

$$J_{i}\ddot{\theta}_{i} - m_{i}e_{i}\ddot{h}_{i} + \sum_{j=1}^{n} K_{\theta_{ij}}\theta_{j} = M_{a,i} \quad for \ i = 1, 2, \dots n$$
(4.1)

or in matrix format for an eight segment model,

$$\begin{bmatrix} M & S \\ S & J \end{bmatrix}_{16\times 16} \begin{bmatrix} \ddot{\mathbf{h}} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} K_h & 0_8 \\ 0_8 & K_\theta \end{bmatrix}_{16\times 16} \begin{bmatrix} \mathbf{h} \\ \theta \end{bmatrix} = \begin{bmatrix} -L \\ M_a \end{bmatrix}_{16\times 1}$$
(4.2)

where M, J and S are the diagonal 8×8 mass, moment of inertia and static unbalance matrices, K_h and K_{θ} are the full 8×8 stiffness matrices, 0_8 is a square 8×8 matrix of zeros, and -Land M_a are the 8×1 aerodynamic lift and moment matrices that provide the input to the structural system.

The system in 4.2 is an ideal one, with both inertial and stiffness coupling, although the stiffness matrix development segregates between bending and torsional stiffness effects. This structure is a classic set of equations leading to modal analysis, which will be address in the following section. Practically speaking, however, some small amount of damping is known to exist in the system. In fact, useful numerical analysis and simulation will not be possible with the addition of some structural damping, and so for full modeling, 4.2 becomes

$$\begin{bmatrix} M & S \\ S & J \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{h}} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} C_h & 0_8 \\ 0_8 & C_\theta \end{bmatrix} \begin{bmatrix} \dot{\mathbf{h}} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} K_h & 0_8 \\ 0_8 & K_\theta \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \theta \end{bmatrix} = \begin{bmatrix} -L \\ M_a \end{bmatrix}$$
(4.3)

To develop the terms for the damping matrix, the natural frequencies are determined from the structural mass and stiffness matrices as

$$\omega_{n,i}^2 = \sqrt{\frac{k_{ii}}{m_{ii}}} \quad for \ i = 1, \dots n \tag{4.4}$$

With the natural frequencies known, a light damping ration of $\zeta_i = 0.1 \rightarrow 1$ is selected, and the individual terms are calculated to be $2\zeta_i \omega_{n,i}$.

Let $z = [h \ \theta]^T$, so that the system may be more compactly defined as

$$M\ddot{\mathbf{z}} + C\dot{\mathbf{z}} + K\mathbf{z} = \begin{bmatrix} -L\\ M_a \end{bmatrix}$$
(4.5)

For the theoretical model under study, all of these physical parameters were designed to closely approximate the WADC model. Appendix B provides data on the physical parameters of the design.

To complete the structural model, the equations of motion in 4.5 are converted to a state space representation. With the definition of $x_s = [h_1 \cdots h_8 \ \theta_1 \cdots \theta_8 \ \dot{h}_1 \cdots \dot{h}_8 \ \dot{\theta}_1 \cdots \dot{\theta}_8]^T$, the system becomes

$$\dot{x}_s = A_s x_s + B_s u_s(t)$$

$$y_s(t) = C_s x_s$$
(4.6)

4.2 Aerodynamic Model

As with the 2D model, development of the aerodynamics begins in the low speed regime, and is derived from the results of PMARC, a panel-method flow code (Ashby et al. (1992)). Constant values for C_L and C_{M_a} are sufficient, when combined with dynamic pressure, to model the aerodynamic forces and moments. The 3D model, however, entails three-dimensional flow, in particular a span-wise component that is not present in the 2D case. Dynamic modeling of the aerodynamics is also required to capture the lag in pressure filed response to airfoil motion in the flow. This latter concern is traditionally addressed through the use of Wagner's function or the Theodorsen function when dealing with analytic aerodynamic solutions. By using numerical methods, these same considerations are addressed, and a 3D aerodynamic model can be developed through some sort of system realization technique. Lee et al. (1999b) provides an overview of computational flow codes commonly used to model both subsonic and transonic flow, and makes the general point that flow code solutions are typically either limited in their fidelity or range of validity, or computationally expensive. Current advanced work in the field, such as Silva and Bartels (2004) use full Navier Stokes codes such as CFL3D.

Typically a black box approach is employed whereby a suitable flow code is used to pair known structural inputs to aerodynamic force outputs, in either a SISO or MIMO scheme. For this research, PMARC is used. The structural inputs are based on the dominant mode shapes of the structure. Any dynamics of the structure will necessarily be comprised of the superposition of simple harmonic motion of mode shapes, and acceptably reasonable dynamic response can be captured by combining a small set of the most dominant shapes. The relevant mode shapes are those associated with natural frequencies that fall below a reasonable threshold. Classic linear aeroelastic analysis shows that flutter onset occurs when the velocity-dependent natural frequencies of the first two structural modes converge to resonance. Natural frequencies and mode shapes are obtained by solving the generalized eigenvalue problem for the zero damping structural model, derived from 4.5 (Inman (1996)). Figures 4.2 and 4.3 illustrate the mode shapes in reference to the nominal airfoil shape. While none of the modes is pure bending or pure torsion, these actions are clearly dominant in the first two modes.



Figure 4.2 First two mode shapes for the wind tunnel design (nominal airfoil shape in blue, mode shape in red): (a) Mode 1, 1^{st} bending, 3.62 Hz; (b) Mode 2, 1^{st} torsion, 10.91 Hz.

For each mode shape, an aerodynamic response was generated for harmonic motion of that mode using PMARC. The amplitude of the mode shape was varied sinusoidally, and with a



Figure 4.3 Secondary mode shapes for the wind tunnel design (nominal airfoil shape in blue, mode shape in red): (a) Mode 3, 2^{nd} bending, 16.40 Hz; (b) Mode 4, 2^{nd} torsion, 23.10 Hz.

time-varying frequency. Frequencies were swept from 1 Hz to 30 Hz over a seven second frame at a sampling rate of 300 s^{-1} . Simulations were conducted with constant freestream velocities of 60, 80, 100, 120, 140, 160, 180 and 200 fps. Using the two dominant mode shapes, at total of sixteen simulations were conducted. While these simulations produced modest resolution, the results are adequate and were achieved with current workstation computational capacities. These flow code results provide input-output time history data sets that match \dot{h} , θ and $\dot{\theta}$ dynamics to responses in C_L and C_{M_a} . Input and output data were parsed by segment so that $u(t) = [\dot{h}_1 \ \theta_1 \ \dot{\theta}_1 \ \cdots \ \dot{h}_8 \ \theta_8 \ \dot{\theta}_8]^T$ and $y(t) = [C_{L1} \ C_{M_a1} \ \cdots \ C_{L8} \ C_{M_a8}]^T$.

The development of the aerodynamic reduced-order model (aero ROM) from these data proceeds along the lines of the techniques presented in Ljung (1999). System identification is carried out by prediction error minimization (PEM), and begins by first assuming a linear uniformly stable solution model \mathcal{M} . A set of parameters are then determined that minimize the error of the system estimate with respect to the output data. For this research, a two dimension linear time-invariant state space model is assumed for the solution, such that

$$\dot{x}_a = A_a x_a + B_a u_a(t)$$

$$y_a(t) = C_a x_a + D_a u_a(t)$$
(4.7)

with $A_a \in \mathbb{R}^{32 \times 32}$, $B_a \in \mathbb{R}^{32 \times 24}$, $C_a \in \mathbb{R}^{16 \times 32}$, and $D_a \in \mathbb{R}^{16 \times 24}$.

This is a reasonable estimate considering the general first-order behavior of standard fluid flow system. We assume the existence of LTI coefficient matrices A, B, C, and D that form a true model of the aerodynamics, and which are comprised of elements that form the parameter vector θ_N . To find these parameters, we seek the estimated parameter vector

$$\hat{\theta}_N = \underset{\theta \in \mathcal{D}_\mathcal{M}}{\arg\min} V_N \tag{4.8}$$

where $\mathcal{D}_{\mathcal{M}}$ is the space of all possible solutions of the designated form (e.g., second order state space) and V_N is a cost function of the parameter estimation process based on the error function e(t), defined as

$$V_N(G, H) = \sum_{t=1}^{N} e^2(t)$$
$$e(t) = H^{-1}(q) \left[y(t) - G(q)u(t) \right]$$
(4.9)

The functions y(t) and u(t) are the time histories of the output and the input, respectively, G(q) is the discrete domain input transfer function, and H(q) is the discrete domain noise input transfer function. The variable q is the forward shift of a set time unit, and acts as the differentiation operator for the sampled data systems. The transfer functions can be obtained by the relationships

$$G(q) = G(q, \theta) = C(\theta) [qI - A(\theta)]^{-1} B(\theta) + D(\theta)$$
$$H(q) = H(q, \theta) = C(\theta)$$

The solution for $\hat{\theta}_N$ given a cost function V_N exists under the following assumptions:

- 1. V_N converges to a limit function $\bar{V}(\theta)$. This also implies that the minimizing argument $\hat{\theta}_N$ of V_N also converges to the minimizing argument θ^* of \bar{V} .
- 2. The signals y(t) and u(t) are quasi-stationary.
- 3. \mathcal{M} is a linear, uniformly stable solution model.

For PEM to proceed, an initial set of estimated parameters is required. The technique used for this research uses a non-iterative subspace approach to construct this estimate. The subspace approach assumes a state space model of a certain order for the solution, and applies that model to the known data sets for y(t) and u(t), and an estimated initial state vector x_0 . This initial state is estimated to be one of the following values:

- Let x_0 be uniformly zero; or
- Let x_0 be treated as an independent estimation parameter; or
- Use a least squares fit such that, given $u(t), y_{est} \rightarrow y(t)$.

With the parameter estimates initialized through this subspace approach, parameter estimate values are refined through PEM. It is important to note that nothing in the derivation of PEM guarantees that $\bar{V}(\theta)$ has a unique global minimum. This consideration must therefore be accounted for when applying this technique.

This system identification was performed in MATLAB using the function **ssest** on each mode for all eight airspeeds, on a segment-by-segment basis. Results for the segments were then concatenated into one ABCD quadruple, generating sixteen sets of results. For the aero ROM to be useful across the range of velocities, it was necessary to ensure that the minima for the limit function \overline{V}_N (itself a function of freestream velocity V) would vary continuously from velocity to velocity. To enforce this criteria, system solutions for both modes were derived for the velocity V = 80 fps. These systems (one for each mode) were used to provide initial parameter estimations for the derivations at the other seven airspeeds. In this manner, the state space parameters were made to vary more or less monotonically from airspeed to airspeed. To further optimize the algorithm, focus was placed on the 3.5 to 8 Hz range. This captured the natural frequencies of the first two modes shapes. Feedthrough was also selected for all three input signals, thus ensuring a non-zero D matrix. This was done to provide for the influence of second order terms $(\ddot{h} \text{ and } \ddot{\theta})$ directly into the formulation of the aerodynamic coefficients. This can be considered a necessary modification to the form of the assumed solution, and reflects classical aerodynamic theory on the development aeroelastic equations. See, for instance, Wright and Cooper (2007). Finally, since the output data were derived from flow code, it was assumed that there was no disturbance input. Removing this aspect allowed the algorithm to proceed more quickly.



Results of the system identification process were generally acceptable. An example for both C_L and C_{M_a} are shown in Figures 4.4 and 4.5.

Figure 4.4 System ID results for C_L response to a frequency sweep of mode shape 2, segment 4. Results for all eight velocities are shown (60 fps upper right to 200 fps lower left), with output data represented in blue and system ID response in dashed red. Functions in gray are input signals.

Finally, aero ROM solutions are required at freestream velocities other than the eight selected for the system identification process. An interpolation algorithm was therefore developed to provide aerodynamic model estimates for velocities between 60 fps and 200 fps. Again, it was desired that the model parameters would vary monotonically between airspeeds, in particular the eigenvalues and eigenvectors of the A_a matrices, and the actual values of the B_a , C_a and D_a matrices. Results at the eight airspeeds confirmed that this was achieved. To achieve parameter continuity at the intermediate airspeeds, A_a matrices were parsed into eigenvalues and modal matrices. These eigenvalues and eigenvectors were then interpolated using a piecewise cubic hermite interpolating polynomial, with elements calculated at steps of 1 fps, allowing for the reconstruction of a consistent A_a matrix at any intermediate airspeed. In a similar fashion,



Figure 4.5 System ID results for C_{M_a} response to a frequency sweep of mode shape 2, segment 4. Results for all eight velocities are shown (60 fps upper right to 200 fps lower left), with output data represented in blue and system ID response in dashed green. Functions in gray are input signals.

coefficients of the C_a and D_a matrices were interpolated. Due to the very small variance in B_a matrix coefficients, a simple linear interpolation was used.

4.3 Full System Model

One of the main goals of this research is the development of an efficient model for the prediction of flutter in the presence of a free-play nonlinearity. The modeling and results of the 2D system were specifically intended to validate on a fundamental level the utility of the hyperbola free-play model, at least insofar as these results compare to those of other candidate systems such as a cubic function, Lee et al. (1999a), or a careful switching within a piecewise linear model, Conner et al. (1997). This requires that the linkage of the structural model with the aerodynamic model will necessarily include the modeling of rotational free-play nonlinearity. For the purposes of this research, the free-play is assumed to be at the point of rotation at the

root of the airfoil. The effects of the free-play nonlinearity in rotational stiffness can be seen to affect the airfoil in one of two manners. The first modeling method considers the free-play nonlinearity to affect the rigid body as a whole through the stiffness parameter at the root. The second method considers only a single structural system, but with the free-play nonlinear portion of the model separated out from the linear portion. Numerical results for each method indicate strengths and weaknesses for both, and therefore show the worth of pursuing these models.

An important note—the axis of rotation is co-located with the elastic axis. Analysis of a 2DOF nonlinear aeroleastic system where the airfoil elastic axis does not coincide with the axis of rotation is left for further study.

4.3.1 Rigid Body Model

The rigid body model assumes a nonlinear subsystem encompassing the entire rigid body dynamics, which are then overlaid onto the structural model, as shown in Figure 4.6.



Figure 4.6 The decoupled aeroelastic system using the rigid body approach.

The state space representation of the rigid body model is

$$\dot{x}_r = A_r x_r + B_{re} u_{re} + B_{ri} u_{ri}$$

$$y_{re} = C_{re} x_r$$

$$y_{ri} = C_{ri} x_r$$
(4.10)

The subscript e indicates inputs and outputs with the aeroelastic system external to the rigid body, and the subscript i indicates those internal to the rigid body model. The rigid body state vector $x_r = [\theta_r \ \dot{\theta}_r]^T$, so that $A_r \in \mathbb{R}^{2\times 2}$. The free-play nonlinearity resides in the internal feedback loop such that $u_{ri} = \phi(y_{ri}) = \phi(C_{ri}x_r)$. Accordingly, $B_{ri} \in \mathbb{R}^{2\times 1}$ and $C_{ri} \in \mathbb{R}^{1\times 2}$. The external input to the rigid body model is the same as that for the structural model, so that $B_{re} \in \mathbb{R}^{2\times 16}$. The external output is added to the output of the structural model, so that $C_{re} \in \mathbb{R}^{24\times 2}$.

The parameters of the rigid body model are taken from the equation of motion

$$J_{\Sigma}\ddot{\theta}_r + C_r\dot{\theta}_r + K_r\phi(\theta) = F_{aero}$$
(4.11)

where C_r is an assumed damping term at the root of the airfoil, K_r is the stiffness term, J_{Σ} is the airfoil total moment of inertia, F_{aero} is the overall lift and aerodynamic moment acting on the airfoil, and $\phi(\theta)$ represents the hyperbola nonlinearity model, equation 2.20 presented in Chapter 3. As with the structure, the damping term is developed by assuming a ζ representative of very light damping (0.1 to 1%), a natural frequency $\omega_n = \sqrt{K/J}$, and a damping term $2\zeta\omega_n$. This leads to the state equation

$$\dot{x}_{r} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-C_{r}}{J_{\Sigma}} \end{bmatrix} x_{r} + \frac{1}{J_{\Sigma}} \begin{bmatrix} 0 \cdots \cdots 0 \\ e_{1 \times 8} & \bar{c}_{1 \times 8} \end{bmatrix} u_{re} + \begin{bmatrix} 0 \\ \frac{-K_{r}}{J_{\Sigma}} \end{bmatrix} \phi(C_{ri}x_{r})$$
$$y_{ri} = \begin{bmatrix} 1 & 0 \end{bmatrix} x_{r}$$
$$y_{re} = C_{re}x_{r}$$
(4.12)

 C_{re} is a 24 × 2 matrix that extends x_r so that the rigid body dynamics may be conveyed to all segments of the airfoil.

The full system matrix is now derived with straightforward algebra. First, recognize that

$$u_s = u_{re} = K_s y_a = K_s (C_a x_a + D_a u_a)$$

and

 $u_a = y_s + y_{re}$

This leads to a set of combined state equations where, with the exception of the nonlinear term, the inputs and outputs have been replaced by the state vectors.

$$\dot{x}_{a} = A_{a}x_{a} + B_{a}u_{a} = A_{z}x_{a} + B_{a}(y_{s} + y_{re})$$

= $A_{a}x_{a} + B_{a}(C_{s}x_{s} = C_{re}x_{r})$ (4.13)

$$\dot{x}_{s} = A_{s}x_{s} + B_{s}u_{s}$$

$$= A_{s}x_{s} + B_{s}[K_{s}(C_{a}x_{a} + D_{a}u_{a})]$$

$$= A_{s}x_{s} + B_{s}[K_{s}(C_{a}x_{a} + D_{a}(C_{s}x_{s} + C_{re}x_{r}))]$$

$$= (B_{s}K_{s}C_{a})x_{a} + (A_{s} + B_{s}K_{s}D_{a}C_{s})x_{s} + (B_{s}K_{s}D_{a}C_{re})x_{r}$$
(4.14)

$$\dot{x}_{r} = A_{r}x_{r} + B_{re}u_{re} + B_{ri}u_{ri}$$

$$= A_{r}x_{r} + B_{re}[K_{s}(C_{a}x_{a} + D_{a}(C_{s}x_{s} + C_{re}x_{r}))] + B_{ri}u_{ri}$$

$$= (B_{re}K_{s}C_{a})x_{a} + (B_{re}K_{s}D_{a}C_{s})x_{s} + (A_{r} + B_{re}K_{s}D_{a}C_{re})x_{r} + B_{ri}u_{ri}$$
(4.15)

These restated systems 4.13, 4.14 and 4.15, along with the relationship $u_{ri} = \phi(C_{ri}x_r)$, combine to form the overall 66 × 66 system

$$\begin{pmatrix} \dot{x}_{a} \\ \dot{x}_{s} \\ \dot{x}_{r} \end{pmatrix} = \begin{bmatrix} A_{a} & B_{a}C_{s} & B_{a}C_{re} \\ (B_{s}K_{s}C_{a}) & (A_{s} + B_{s}K_{s}D_{a}C_{s}) & (B_{s}K_{s}D_{a}C_{re}) \\ (B_{re}K_{s}C_{a}) & (B_{re}K_{s}D_{a}C_{s}) & (A_{r} + B_{re}K_{s}D_{a}C_{re}) \end{bmatrix} \begin{pmatrix} x_{a} \\ x_{s} \\ x_{r} \end{pmatrix} + \begin{pmatrix} 0_{32\times1} \\ 0_{32\times1} \\ B_{ri} \end{pmatrix} \phi([0_{1\times32}\ 0_{1\times32}\ C_{ri}]x_{r})$$

$$(4.16)$$

While the structural and rigid body parameters are constants, all of the aerodymic matrices and the dynamic pressure matrix K_s are velocity dependent; δ dependence enters into the overall system through the nonlinear function $\phi(C_{ri}x_r)$.

4.3.2 Separated System Model

Although the rigid body model successful demonstrates LCO behavior, it fails to reflect a δ dependence of the flutter onset velocity and does not show convergence to a stable nonorigin fixed point in numerical time simulation. As a result, a second modeling approach was developed. This second approach is more straightforward than the previous scheme. The method follows the approach of the 2D model by identifying and separating out the nonlinear components of the physical model and developing them in the Luré form during the structural model development. The general block diagram is depicted in Figure 4.7.



Figure 4.7 The decoupled aeroelastic system using the rigid body approach.

The development begins with the equations of motion shown above in 4.5 for the general structural model, however the linear and nonlinear stiffness terms are separated.

$$M\ddot{\mathbf{z}} + C\dot{\mathbf{z}} + K_L \mathbf{z}_L + K_{NL} \mathbf{z}_N L = \mathbf{F}_{\mathbf{a}}$$

$$\tag{4.17}$$

so that

$$\ddot{\mathbf{z}} = -M^{-1}C\dot{\mathbf{z}} - M^{-1}K_L\mathbf{z}_L + M^{-1}\mathbf{F}_{\mathbf{a}} - M^{-1}K_{NL}\mathbf{z}_NL$$
(4.18)

Converting to state space produces a model similar to the one developed for the 2D case, which was one of the motivations for developing this 3D approach.

$$\dot{x}_{s} = \begin{bmatrix} 0_{16} & I_{16} \\ -M^{-1}K_{L} & -M^{-1}C \end{bmatrix} x_{s} + \begin{bmatrix} 0_{16} \\ M^{-1} \end{bmatrix} + \begin{bmatrix} 0_{16} \\ -M^{-1}K_{NL} \end{bmatrix}$$
$$= A_{s}x_{s} + B_{s}u_{s} + B_{si}\phi(y_{si})$$
(4.19)

$$y_s = C_s x_s \; ; \quad y_{si} = C_{si} x_s \tag{4.20}$$

The same aerodynamic model is used in this process, as shown in 4.7, so that

$$u_s = K_s y_a = K_s (C_a x_a + D_a u_a)$$
$$u_a = K_a y_s = K_a (C_s x_s)$$
$$y_{si} = C_{si} x_s$$

The state equations become

$$\begin{split} \dot{x}_a &= A_a x_a + B_a k_a C_s x_s \\ \dot{x}_s &= (A_s + B_s K_s D_a K_a C_s) x_s + (B_s K_s C_a) x_a + B_{si} \phi(C_{si} x_s) \end{split}$$

and the combined 64×64 system becomes

$$\begin{pmatrix} \dot{x}_a \\ \dot{x}_s \end{pmatrix} = \begin{bmatrix} A_a & (B_a K_a C_s) \\ (B_s K_s C_a) & (A_s + B_s K_s D_a K_a C_s) \end{bmatrix} \begin{pmatrix} x_a \\ x_s \end{pmatrix} + \begin{bmatrix} 0_{32 \times 16} \\ B_{si} \end{bmatrix} \phi([0_{16 \times 16} C_{si}] \begin{pmatrix} x_a \\ x_s \end{pmatrix})$$

$$(4.21)$$

As with the rigid body model above, the aerodynamic terms and the dynamic pressure matrix K_s are all velocity dependent, and the δ dependency is contained in $\phi = \phi(\delta)$.

CHAPTER 5. ANALYSIS OF THE THREE-DIMENSIONAL MODEL

The systems analysis approach to the flutter problem in a 3D airfoil seeks to use standard and advanced linear and nonlinear analysis techniques to evaluate a numerical model such as the one presented in Chapter 5. The parameters used for the structure and the airflow is presented in Appendix B. The complete system equations for both the rigid body model (4.16) and the separated system model (4.21) have been modeled in MATLAB. Each model relies on a common aero ROM derivation function that uses the system identification results from the flow code data, as detailed in Chapter 5. The airspeed range of the aero ROM is from 60 to 200 fps.

5.1 Rigid Body Model

5.1.1 Eigenvalue Analysis

The first step in evaluating the rigid body model is to validate the results for a zero free-play condition, or in other words, the linear condition. The main parameters for performing this calibration are the damping ratio of the structure and the loop gain of the system. Classical analysis such as Bisplinghoff et al. (1996) and Wright and Cooper (2007) have shown that the existence of flutter, whether LCO or divergent, is strongly related to the level of damping in the system, both due to the aerodynamic flow and to the structure itself. The damping ratio of the structural model for this study, with respect to both the WADC model and the airfoil constructed for wind tunnel testing, is due mainly to the internal damping of the material used in the spar, typically very low for aluminum. Conversely, consider also that traditional analysis has produced reliable models with the no-damping assumption, indicating that any assumed level of damping should be very small. For these reasons, the damping ratio for the structural model is kept as low as possible, preferably around 0.1 to 3%.

The implementation of a loop gain parameter addresses an important consideration for this process. The numerical model in this study relies on an aero ROM derived from flow code data, which in turn was derived by using harmonic motion of the mode shapes. The magnitude of oscillation used in the flow code runs was limited to 10% of the full eigenvector value, to keep full motion of the airfoil in the flow code environment from becoming too excessive and causing the aerodynamic response to exceed the roughly linear response region. In other words, airfoil motion and deflection needed to not lead to separation of airflow, wing stall (global or local), or other nonlinear behavior. Even so, the amplitude of pitch and plunge oscillations used in the aero ROM development were much greater than the motion observed in the WADC report, various numerical research results, and in the results of this research. These considerations reflect the concern that the aerodynamic response of the flow code represented an overexcited input. The large input to the aero ROM process contributed to resolution of the input-output data, which was beneficial for the system identification process, however it was expected that when combined into the overall system, some amount of attenuation in loop gain would be required to achieve stable performance and reasonable flutter velocities.

Calibrating the model therefore consists of adjusting the structural damping and loop gain in order to achieve reasonable behavior of the real value of the system eigenvalues with respect to velocity for the linear model. Eigenvalues for the system are taken from the rigid body overall system 4.16. As $\delta \to 0$, the hyperbola function ϕ collapses to a linear relationship, and the product $B_{ri}C_{ri}$ may simply be added to the bottom left 2 × 2 element of the system Amatrix. It is the behavior of the eigenvalues of this matrix with which we are concerned.

A linear aeroelastic model will exhibit stability at lower airspeeds and typically have one eigenvalue move past one or more other, less stable eigenvalues to cross into the right hand plane. This velocity can be predicted from linear analysis of the quasi-steady model (see e.g., Wright and Cooper (2007)), or from experimental data if available. We note that with the configuration of the first WADC model (Hoffman and Speilberg (1954)), which this research seeks to replicate, demonstrated a zero-free-play flutter onset velocity of 142 fps, while the wind tunnel model employed in relation to this research exhibited a zero free-play flutter velocity of 80 fps. The linear free-play models were therefore adjusted to achieve a zero free-play flutter velocity between these two benchmarks. Figure 5.1 shows the velocity-dependent stability of the rigid body model with a flutter onset speed $V_f = 94$ fps. This result was achieved with a loop gain of $K_a = 0.002$ and a damping ratio of $\zeta = 0.002$.



Figure 5.1 The real parts of the five dominant eigenvalues (one pair are complex conjugates) for the rigid body model. The linear system goes unstable at 94 fps, and diverges.

Qualitatively, this eigenvalue behavior closely resembles the behavior of the linear horizontal tail model of the F-35 design stage as presented in Carlton Schlomach (2009).

With the introduction of free-play, the eigenvalue plots show a different behavior. In Figure 5.2, the system demonstrates stable behavior for $V/V_f < 0.86$. Below this velocity, the disturbed aeroelastic system settles back to the origin of the physical space at zero pitch and zero plunge. From $V/V_f = 0.86 \rightarrow 1.0$, the origin eigenvalue of the nonlinear system is unstable. In this velocity region, the model exhibits LCO behavior. An example is shown in Figures 5.3 and 5.4.

The behavior of the system in LCO presents some interesting detail. From the mass and stiffness matrices used in building the structural model, the first (lowest) four natural frequen-


Figure 5.2 The real parts of the rigid body system dominant eigenvalues. Blue dashed indicates the eigenvalue of the linear system. Gold dashed indicates the eigenvalue of the nonlinear system at the origin fixed point. Gold solid indicates the eigenvalue at the non-origin fixed points. Red and green lines indicate the velocity at which the origin (green) and non-origin (red) system fixed points become unstable. The green line at the lower velocity is associated with the origin fixed point of the nonlinear system, while the green line at the higher velocity is associated with the unique fixed point of the linear system.

cies are 3.28 Hz, 8.89 Hz, 14.8 Hz and 18.7 Hz. The highest frequency for the sixteenth mode is 1.3 kHz. Several segments in Figures 5.3 and 5.4 show a variety of frequency components present. FFT results show that the oscillations in plunge for the root segment (segment 1) have a primary frequency component at 404 Hz and a secondary component at 1330 Hz. Segments 5 and 6 show flutter components at 6.1 Hz and 55.7 Hz. The variations in flutter amplitude between segments indicates that a combination of mode shapes are present in the flutter motion. It should also be noted that time history results of LCO parameters appears to be dependent on initial conditions.

Above $V/V_f = 1.0$. the system becomes divergent, as with the linear system. Figure 5.2 also indicates that while the rigid body model shows a clear reduction in flutter velocity with the introduction of the free-play nonlinearity, it does not exhibit any dependence on the width of the free-play region δ . Analytically, the system exhibits non-trivial fixed points. Figure 5.5



Figure 5.3 Time history of vertical displacement in inches for the nonlinear system with $\delta = 0.0025 rad$. and $V/V_f = 0.97$.

shows their magnitude for θ_1 of the rigid body model. In simulation time histories, however, model behavior does not bear this out.

5.2 Separated Structural System Model

5.2.1 Eigenvalue Analysis

As with the rigid body model, a small amount of internal structural damping is assumed for the separated system model, and the level of loop gain is available as a tuning mechanism for tailoring the behavior of the combined structural/aerodynamic model. For the separated system 4.21, as with the rigid body model, the hyperbola function collapses to $\phi(\theta_1) \rightarrow \theta_1$ as $\delta \rightarrow 0$ and $B_{si}C_{si}$ is combined appropriately with the overall system A matrix as above.



Figure 5.4 Time history of pitch rotation in radians for the nonlinear system with $\delta = 0.0025 rad$. and $V/V_f = 0.97$.

This model shows the linear behavior of the system. Eigenvalue behavior for the zero free-play instance of the separated system is shown in Figure 5.6. Flutter speed is found to be $V_f = 101$ fps. This was achieved with an internal damping ratio of $\zeta = 0.0014$ and a loop gain of K = 1. Even more than the rigid body approach, this eigenvalue behavior closely matches the results in Carlton Schlomach (2009).

Again, free-play is introduced to the system, however the eigenvalue behavior is somewhat different from the rigid body model, as shown in Figure 5.7. At all velocities below $V/V_f = 1.0$, the non-origin fixed points remain stable, with all eigenvalues maintaining negative real parts. The origin fixed point, however, is unstable. In the velocity region below $V/V_f = 1.0$, time histories indicate that the segments converge via unique trajectories to the non-origin fixed point state. Like the previous model, the system exhibits non-trivial fixed points. The results



Figure 5.5 Rigid body model fixed points for θ_1 at various free-play widths, $\delta = 0 \rightarrow 0.01$. presented in Figure 5.8 indicates the magnitude of θ_1 of the separated system model at its convergent fixed point value.

Figure 5.9 illustrates this condition for $V/V_f = 0.89$ fps and $\delta = 0.004$ rad. Plunge values (Figure 5.9(a)) converge to 0, but the pitch values (Figure 5.9(b)) converge to non-zero values. The pitch angle of the root segment in particular converges to a value just outside the free-play window, as indicated by the analytic solution. It is also noted that the inboard-most segments exhibit a low amplitude high frequency behavior in pitch. FFT results of the time history data of $\theta_1(t)$ are centered on 127.4 Hz, however the shape of the frequency spectrum resembles Gaussian noise.

Both the rigid body model and the separated system model exhibit some aspects of the 2D flutter model. The rigid body model presents LCO behavior when a rotational stiffness freeplay exists, with the LCO occurring at velocities well below the flutter velocity of the model with no free-play. The LCO is centered on the origin, however, and at no velocity indicates the existence of a non-origin fixed point. The separated system model clearly shows the system converging to a non-origin fixed point at all velocities below the linear flutter velocity, but at



Figure 5.6 Similar to the rigid body model, the separated system model shows standard eigenvalue behavior for the linear condition. The linear system goes unstable at 101 fps, and diverges.

no point exhibits LCO behavior. Neither model indicated a δ dependence for the flutter onset velocity. This is a deficiency in the model that requires further research.

5.3 Experimental Results

This research was conducted in conjunction with separate but related research that involved the fabrication and wind tunnel testing of an airfoil specifically designed to exhibit flutter under controlled parameters. This model was built in the Fall of 2013, and wind tunnel experiments were conducted in November and December of that year using the Bill James Open-Return Wind Tunnel, located in the Wind Simulation and Testing Laboratory (WiST Lab) in the Department of Aerospace Engineering at Iowa State University. The wind tunnel test section is 3.0 ft (0.915 m) wide and 2.5 ft (0.762 m) high, and is capable of a maximum operational freestream velocity of 200 fps (61 m/s). This section provides a brief description of the model, the results of the wind tunnel testing, and how the behavior of the 3D models developed through this research compare to those of the wind tunnel.



Figure 5.7 The real parts of the separated system dominant eigenvalues. Blue dashed indicates the eigenvalue of the linear system. Gold dashed indicates the eigenvalue of the nonlinear system at the origin fixed point. Gold solid (inderlying the blue dashed line) indicates the eigenvalue at the non-origin fixed points. Red and green lines indicate the velocity at which the origin (green) and non-origin (red) system fixed points become unstable.

5.3.1 Model Description

The wind tunnel model was comprised of eight segments of varying chord lengths attached independently to a single aluminum spar, rectangular in cross-section and tapered in a stairstep fashion from the root attachment point to the tip attachment point. Each segment was built around an aluminum center rib, with milled wood skins providing the outer mold line of the airfoil and two additional aluminum ribs acting as end caps for each side of the segment, as shown in Figure 5.10(a). The segments were attached at the center rib only, as shown in Figure 5.10(b), effectively concentrating the force and moment inputs onto discrete locations along the spar. Each segment was mounted to the spar at the one-third chord location, thus providing a straight elastic and rotational axis that extended through the airfoil on a line perpendicular to the freestream flow vector.

Each segment was a 4 in. (0.102 m) wide NACA0010 airfoil section, giving the airfoil a full span of 32 in. (0.813 m). Mass and moment of inertia were distributed within the segments to



Figure 5.8 Separate system model fixed points for θ_1 at various free-play widths, $\delta = 0 \rightarrow 0.01$.

approximate the inertia properties of the original WADC model. Total mass and moment of inertia for all segments was 0.2258 slugs (3.30 kg) and 0.1421 sl ft² (0.1927 kg m²), respectively. Full mass and stiffness details of the model design may be found in Appendix B. The model was mounted vertically in the wind tunnel using the free-play control mechanism adapted from results presented in Fichera et al. (2012). The mechanism as adapted for these wind tunnel experiments is depicted in Figure 5.11. The free-play in shown was connected by a lever arm to the airfoil spar. By adjusting the top plate fore and aft, the pin was allowed a specified amount of free-play before engaging the sides of the top plate slot. Once against the edge of the slot, the pin then encountered the rotational root stiffness provided by the compliance springs attached to the base plate. Zero free-play was achieved by fixing the top plate fully aft, thus engaging the free-play pin with the narrow end of the slot. In this manner, free-play widths of 0, 0.072, 0.105, 0.154, 0.236, 0.393, 0.675 and 1.23 degrees were tested. The airfoil was instrumented with a potentiometer at the root to measure angular position and eight accelerometers positioned one per segment, on the center rib and offset from the spar. Force transducers were placed at the root compliance springs. Impulse excitations were applied manually to the airfoil to provoke a dynamic response. Figure 5.12 shows the model as installed in the wind tunnel.



Figure 5.9 Time histories of the separated system model at V = 90 fps $(V/V_f = 0.89)$ with $\delta = 0.004$ rad. The fixed point pitch angle for θ_1 is clearly visible, however all pitch and plunge states show some non-zero fixed point location. (a) Plunge response (b) Pitch response.



Figure 5.10 Illustration of the components comprising the wind tunnel model. (a) A representative segment. (b) A top view of the eight segments attached to the single spar, absent the top and bottom skins.



Figure 5.11 Top view of mechanism used to control the width of the rotational free-play region for the wind tunnel model. The free-play region for the model was adjusted by moving the top plate fore and aft.

5.3.2 Wind Tunnel Results

When tested with 0 degrees free-play with the large (i.e., most stiff) root compliance springs in place, the wind tunnel model exhibited divergent flutter at a freestream velocity of 80.2 fps (24.5 m/s), with no flutter behavior at lower velocities. Figure 5.13 shows that as freestream velocity increases, the frequency components of the two dominant mode shapes converge until the flutter velocity is reached, at which point they become equal. This is the classic behavior for an airfoil with zero free-play.

As free-play is introduced, LCO behavior is demonstrated. LCO onset occurs at velocities below the linear flutter velocity, with the onset speeds decreasing as free-play width increases. For each free-play width, a velocity is reached where the LCO becomes no longer stable, and airfoil oscillations become divergent. This divergence velocity also decreases with free-play width. Data for the model with the large root compliance spring are shown in Figure 5.14. These results match well with those documented in Hoffman and Speilberg (1954).



Figure 5.12 The 3D model as installed in the wind tunnel.

Several key observations were made regarding the behavior of the wind tunnel model and the effects of rotational free-play at the root.

- The presence of free-play in the root rotational stiffness caused LCO onset, which then progressed to divergence as velocity increased.
- LCO onset and eventual divergence occurred as lower speeds as free-play increased.
- The velocity separation between LCO onset velocity and divergence velocity appeared to increase with root stiffness outside the free-play region.
- The effect of free-play on onset velocities for LCO and divergence increased with root stiffness.

One other important observation relates to the manifestation of a non-origin fixed point in the model behavior. For those velocities where the system was stable and the root pitch angle converged, the steady-state angle was not zero, clearly demonstrating a non-origin fixed point. When the airfoil entered a LCO for a given free-play width, however, oscillations appeared to be centered on the origin (i.e., 0 degrees pitch), rather than a non-zero value. A representative example is shown in Figure 5.15 for a free-play region of 0.236 deg., however this behavior was observed at all free-play widths.



Figure 5.13 FFT results for the first two mode shapes of the no-free-play model in the wind tunnel. Frequencies migrate together as freestream velocity increases.

5.3.3 Comparison of Numerical results to Experimental Results

The wind tunnel results of the model described above are in accord with other previous wind tunnel experiments of flutter models with an option to inject rotational free-play nonlinearity, including the WADC model. These systems share the following common behaviors.

- 1. Without free-play in the system, the models demonstrate a divergent flutter onset velocity V_f . Below this velocity, the models are stable for all initial conditions within the stable region of the aerodynamics. That is, initial conditions do not put the airfoil in stall or flow separation.
- 2. With rotational free-play, the models demonstrate a stable LCO with an onset velocity V_{LCO} that is well below V_f . The LCO amplitude increases with freestream velocity. At some velocity, the LCO becomes unstable and the structural dynamics become unstable. This velocity is at or slightly less than V_f .



Figure 5.14 Wind tunnel results for the large root compliance stiffness, showing a decrease in both LCO onset velocity and divergence velocity as free-play width increases.

- 3. For a given system, V_{LCO} decreases as the width of the free-play region increases, showing a clear δ dependence (using the parlance of this research).
- 4. Below V_{LCO} , structural dynamics will converge to a stable equilibrium point not located at the state space origin. This equilibrium, or fixed, point will include a pitch angle at the span-wise location of the free-play nonlinearity (typically the root) that is greater than the angle of the free-play limit. Once past V_{LCO} , the limit cycle oscillations will not be centered on the state space origin. This last point is ambiguous in older wind tunnel experiments such as the WADC work, where precise angular position data was not available.

The two models developed in this research each exhibit several of these characteristics, however neither provides a full replication of typical wind tunnel model behavior. The rigid body model exhibits LCO onset when rotational free-play is present in the model. The onset velocity for the LCO is below the onset velocity for divergent flutter in the model with no free-play, and oscillations are centered on the state space origin. for example, the wind tunnel model demonstrated a zero free-play divergent flutter velocity of V = 80.2 fps (24.4 m/s). With



Figure 5.15 Presence of fixed point behavior in the wind tunnel model for $\delta = \pm 0.236 rad$. (a) System converges to a non-zero fixed point in the stable velocity region. (b) System appears to oscillate about the origin in the LCO velocity region.

a free-play deadband of $\pm \delta = 0.154$ deg, or 0.0027 rad, LCO flutter onset occurred at 64.5 fps (19.5 m/s), which is 80.4% of V_f . The rigid body model exhibited zero free-play flutter at 94 fps (28.7 m/s), and for all free-play widths, LCO flutter onset of 81 fps (24.7 m/s), or 86.2% of V_f .

Unlike the wind tunnel model, however, LCO onset velocities for the rigid body model do not vary with the width of the free-play region. At velocities below free-play onset, the eigenvalues of the rigid body model are all stable, and the system converges to the state space origin, never manifesting convergence to a non-origin fixed point. Additionally, the amplitude of oscillation for the rigid body model only increases modestly with velocity as compared to the amplitude variations in the free-play model. Finally, divergence in the rigid body model always occurs at the linear structure flutter velocity V_f , never at a lower velocity.

The separated system model never exhibits LCO behavior, and only shows divergence at a fixed velocity V_f , regardless of the presence or absence of rotational free-play. Unlike the rigid body model, however, the separated system model clearly shows convergence to a non-origin fixed point at velocities below V_f , as can be observed in the wind tunnel model behavior.

CHAPTER 6. CONCLUSIONS AND FUTURE RESEARCH

A 2D model for flutter behavior under incompressible flow in an airfoil with a rotational stiffness free-play nonlinearity in pitch has been developed that builds on models developed in early work. This new model more faithfully represents the behavior of a physical system with these characteristics while still allowing the application of nonlinear analysis tools. Specifically, the feedback interconnection model developed per Brockett (1982) allows us to apply systems theory tools to the problem, and to pursue a clear method for observing the presence of a Hopf bifurcation related to the phenomenon of flutter in an airfoil with two degrees of freedom. These results inform the development of two accurate, computationally economical 3D flutter models that incorporates a rotational free-play nonlinearity. Although not completely accurate in their flutter behavior of the flutter problem in the presence of a rotational free-play nonlinearity. The results of this research measurably advance ongoing efforts to provide numerical analysis tools capable of providing reliable design insight to airfoil systems without the need to rely on experimental data or overly conservative specifications.

6.1 Key Contributions

The objectives accomplished in this research are to characterize an aeroelastic system so that a structural nonlinearity can be separated out for systems analysis, develop a workable nonlinear stiffness model that is an improvement of those used in the current literature, develop full 2D and 3D models that are capable of demonstrating representative nonlinear behavior, and validating the apparent LCO behavior of the nonlinear aeroelastic model as a Hopf bifurcation. The key contributions of this research are summarized in the following key results: The coupled 2D model of an airfoil with the two-degree-of-freedom (plunge, h and pitch, θ) is developed such that the rotational free-play nonlinearity is separated out from the otherwise linearized model, and modeled as a feedback component, per the Luré form.

With the 2D model in the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\phi(\theta)$$
$$\theta = \mathbf{y} = C\mathbf{x} ,$$

the nonlinear system can be used to solve for the system fixed points, either through graphical techniques or analytically. The system can then be linearized by a truncated Taylor series expansion about those fixed points. Both the nonlinear system and the linearized approximation have freestream velocity as a parameter. As a result, the fixed point locations will be velocity dependent and the resulting eigenvalue analysis can be conducted to determine the velocity-dependent stability of the system. The use of the traditional piecewise linear model does not support this type of nonlinear analysis.

2. A hyperbola function works well to model the rotational free-play nonlinearity within the Luré form.

For an input y, the free-play nonlinearity may be modeled by the hyperbola function

$$\phi_h(y) = \operatorname{sgn}(y) \left(\left(\frac{\gamma_1 + \gamma_2}{2} \right) (|y| - \delta) + \left[\frac{(\gamma_2 - \gamma_1)^2 (|y| - \delta)^2 + 4\gamma_1 \gamma_2 \delta^2}{4} \right]^{\frac{1}{2}} \right)$$

where γ_1 represents the hyperbola asymptote within the free-play region and γ_2 represents the asymptote outside the free-play. In addition to being continuous, differentiable and Lipschitz, this model is easy to adjust for different values of free-play width δ without compromising monotonicity. The slopes of both the interior asymptote and exterior asymptote are also easily adjusted without compromising the desired δ of the system, allowing control of system passivity and the 'hardening' or 'softening' of the response (ala the cubic nonlinearity). As a result, the hyperbola model demonstrates a δ dependence for the flutter velocity, something that the piecewise linear model is unable to reproduce. These qualities also give the hyperbola function important advantages over the cubic and sigmoidal nonlinear models. Namely, the free-play width is easily adjusted in the hyperbola model without compromising monotonicity as with the sigmoidal model, and it models the free-play region as gradually and asymptotically transitioning into linear spring behavior outside the free-play region, unlike the cubic nonlinearity.

3. The resulting analytical framework for the 2D aeroelastic system successfully forecasts nonlinear flutter behavior, especially the δ dependence for both the non-origin fixed point locations and the flutter onset velocity.

Stability analysis of the 2D system clearly indicates that the flutter onset velocity for a system with a rotational free-play nonlinearity can be reliably predicted by system analysis. Furthermore, that onset velocity

- will be lower that the divergent flutter velocity of the linear structural case,
- will decrease as free-play width δ increases, and
- will manifest as a LCO for a region of velocities, with an upper limit velocity that results in divergence.

The analytic framework also provides an accurate prediction of flutter frequency based on the eigenvalue behavior of the system linearized about the fixed point.

4. The flutter behavior of the 2D nonlinear system is confirmed to be an LCO associated with a supercritical Poincare-Andronov-Hopf bifurcation by analytic derivation.

The existence of a stable oscillation in the 2D model with a hyperbola function nonlinearity is validated through the analytical framework by determining the center manifold for the system about its non-hyperbolic fixed points at bifurcation, simplifying the linear and nonlinear portions of the resulting center manifold dynamics and applying the Poincare-Andronov-Hopf bifurcation theorem. The analysis is performed on a system that is parametrized by freestream velocity. These results rigorously establish system behavior above the bifurcation velocity as an LCO. While wind tunnel results and numerical simulations can qualitatively indicate stable oscillatory behavior, this conclusion is now validated.

5. Flutter behavior results are further confirmed by spectral analysis.

Spectral analysis methods are successfully employed on numerical simulation results. Time history data has been embedded to produce a valid two-dimensional $(\theta, \dot{\theta})$ and associated Markov transition matrix. This matrix in turn has produced valid indicators of the flutter behavior of the system as well as spatial probability density information that indicates limit cycle behavior. In addition to the trajectory profile, the steady state density within the gridded phase space indicates the time spent in various regions of the reduced state space, helping to characterize the nonlinear limit cycle behavior. With this information further insight into system trajectories can be obtained. From spectral theory we know that the eigenvector of the Markov matrix with unit eigenvalue characterizes steady state behavior and indicates the presence of limit cycle oscillations beyond the flutter velocity.

6. The development of new 3D models introduces reasonable methods of representing the system in Luré form, allowing the same analytical framework methods developed in the 2D model

Two different modeling approaches are presented to carry forward the results of the 2D model into a 3D airfoil system. In developing the models, a relatively simple but representative lumped mass structural model is combined with an aerodynamic reduced-order model (aero ROM) developed from computationally low cost methods (discussed below). The rigid body approach recognizes the relationship between a rotational free-play non-linearity at the root of the 3D airfoil, and assumes a nonlinear subsystem encompassing the entire rigid body dynamics, which are then overlaid onto the structural model. The associated nonlinearity is easily separable from the system model, producing a framework similar to that used in the 2D model. The separated system approach is developed more closely to the derivation of the 2D model, whereby the nonlinear rotational stiffness com-

ponent, specifically the rotational stiffness at the root segment, is identified and separated out during the development of the structural model and kept distinct when the structure is combined with the aero ROM. In this manner, the nonlinearity interacts with the system as a feedback component to the structure, which again is easily separable from the system model.

These approaches are both physically realistic and analytically useful.

7. The successful incorporation of recently developed, computationally low-cost system identification tools has led to the efficient development of a reasonably accurate aero ROM for use in the 3D models.

The literature describes many techniques for performing system identification for the generation of an aero ROM, however they typically rely on flow code results from packages that require a great deal of time or computational budget, such as in Silva and Bartels (2004). The method developed for this research, on the other hand, relies on a fairly simple and fast panel method CFD code, and leverages the capabilities of prediction error minimization (PEM) techniques. Although not recently reported, the techniques can still be considered fairly new, and to the best of the author's knowledge, have never been used for the purposes of developing a reliable aero ROM. In addition the system identification techniques, an algorithm was established to provide reasonable aero ROM interpolation of the state space coefficient matrices for the spectrum of the velocity parameter that preserved continuity of both the input and output matrices as well as the system eigenvalues and eigenvectors. As a result, analytical results for both of the full 3D models indicated smooth behavior of the system with variations in velocity.

8. System analysis of the 3D models provide qualified results regarding the stability and LCO behavior required to characterize flutter behavior. The difficulty in fully replicating a δ dependence for the onset velocity is a motivation for continued research in 3D modeling refinements.

Analysis of the rigid body model demonstrates a distinct velocity V_f at which the zero freeplay system becomes unstable, resulting in divergent flutter. When free-play is introduced to the model, eigenvalue analysis of the system as linearized about the origin and nonorigin fixed points indicate a range of velocities below V_f where the origin is unstable, but the the non-origin fixed point remains stable. Numerical simulation confirms that apparent LCO behavior occurs at those velocities, demonstrating the lower LCO flutter onset velocity expected. System analysis also predicts the location of the non-origin fixed point at the root segment of the lumped mass model, although numerical simulation fails to converge to any location other than the origin when velocities are below the LOC onset velocity. The separated system model, on the other hand, both predicts and simulates convergence to the non-origin fixed points when operating at velocities within the stable region of the parametrized system. The separated system model, however, does not predict and does not exhibit LCO behavior at any velocity below V_f for the linear system. Although a fully successful 3D model was not demonstrated, the techniques presented produced limited results that nonetheless advance the pursuit of a faithful 3D aeroelastic model that is accurate and computationally economical.

9. The numerical models developed compare favorably to the WADC wind tunnel results described as one of the motivations for this research.

One of the primary motivations for this research was the interest in the Naval Air Systems Command in finding reliable computational methods for refining the design process for airfoils with respect to flutter. The benchmark for any newly developed numerical method was the ability to replicate the experimental data generated by Hoffman and Speilberg (1954). Results of the 2D model qualitatively match the data from the WADC tests in demonstrating a stable LCO when rotational free-play is present, a δ dependence for flutter onset velocity, reasonable frequencies for both the primary modes and the LCO oscillations, and divergence at higher velocities. These results validate the use of the hyperbola function for the nonlinearity and the analytical framework used to evaluate the model.

As indicated above, results of the 3D model, while not completely matching the WADC data, still demonstrate the existence of LCO behavior at velocities below the nominal

linear system divergent flutter velocity, and the tendency of the system to converge to a non-origin fixed point at lower velocities where eigenvalue analysis indicates stability for all identified fixed points.

6.2 Future Research

The novel approaches developed in this research have successfully modeled aeroelastic flutter for a 2D, two-degree-of-freedom with a rotational stiffness free-play in incompressible flow. Extension of the approach to two different 3D models has produced qualified but important success in accurately matching wind tunnel results. The results of this research provide a clear path towards realizing an accurate, computationally efficient method for producing numerical analysis capable of reliably informing airfoil system design.

The first step is the refinement of the current 3D models such that LCO behavior is retained while a dependence of LCO flutter onset velocity on free-play width δ is produced. The first avenue for improvement lies in an enhanced structural model. The lumped mass model has the benefit of being relatively low order. The 3D structural models developed in this research used a 32 degree of freedom state space representation. Using a higher order finite element model for the 3D problem would be the first avenue to pursue for improving the fidelity of the analysis. One consideration is that a finite element model of the structural system allows a greater multiplicity of paths for energy transfer within the structure, increasing the physical opportunities for constructive and destructive dynamics throughout the model physical space. Another is that finite element modeling leads naturally to analysis in the modal domain of the structure, which aligns better with the scheme of the aero ROM development. Transmission of modal coordinates from the structural model to the aerodynamic model better reflects advances in the flutter analysis literature, such as Silva and Bartels (2004). The challenge of the research is in keeping the overall 3D system tractable in light of the (much) higher order system likely to result from finite element analysis.

While the aero ROM development has proven to be adequate for the modeling performed in this research, it is limited to the incompressible flow regime. While many real-world aeroelastic problems operate in this regime, such as wind turbine blades, the most demanding design problems remain in the transonic and supersonic regimes, where the flutter mechanism tends to be shock wave related. This calls for research into the incorporation of more advanced flow codes in the system identification process. Transonic small disturbance methods hold promise in the compressible flow regime if airfoil motion can be contained within reasonable limits. Of course, higher fidelity results can be attained through the use of full Euler-Navier Stokes flow codes, however their computational expense is prohibitive for the purposes of developing an efficient aeroelastic analysis tool for airfoil design.

Although experimental validation was not a primary goal of this research, the availability of recent wind tunnel experiments greatly aided the evaluation of the 3D models. This leads to another research opportunity—the generation of more advanced wind tunnel results in cooperation with the advanced modeling techniques described above. The experimental data used in conjunction with this research was hindered by a lack of sufficient measurement, which prevented the recovery of structural mode shapes from the airfoil dynamics. In addition, static inertia and stiffness measurements were insufficient to for building a structural model for the 3D systems used. Beyond the improvement in test technique, however, is the opportunity to model more advanced airfoils, especially ones of trapezoidal shape along the lines of Cooley (1958). This shape persists in many current aircraft designs. Wind tunnel experiments should also be conducted in the compressible flow regime if results of a 3D numerical model are to be validated for those freestream velocities and Mach numbers.

And finally, flutter does not have to be an undesirable phenomenon. Research opportunities exist for developing a system that enters into a stable LCO over a range of flow velocities. The structure under oscillation would be designed in such a manner so that the mechanical motion and kinetic energy can be converted to electrical energy, thus harvesting energy from fluid flow. The design tools presented in this research would contribute to the process of enhancing the nonlinear flutter behavior of an aeroelastic system, maximizing flutter amplitude and the range of velocities over which LCO occurs. In particular, a control scheme can be designed that adjusts the width of the rotational freeplay region, providing optimum oscillatory behavior for the given ambient flow. This research would combine especially well with the employment of magnetostrictive materials.

APPENDIX A. VALIDATING THE POINCARE-ANDRONOV-HOPF BIFURCATION

The basic model for the dynamics of the physical 2D system is given by

$$\dot{x} = Ax + B\phi(y), \quad y = Cx \tag{A.1}$$

where $x = [h \ \theta \ \dot{h} \ \dot{\theta}]^T$, the *A*, *B* and *C* matrices proceed from the coupled aeroelastic development, and $\phi(\theta)$ captures the rotational free-play nonlinearity, specifically via the hyperbola function. Both of these are presented in Chapter 3.

$$\phi(y) = \operatorname{sgn}(y) \left(\left(\frac{\gamma_1 + \gamma_2}{2}\right) \left(|y| - \delta \right) + \left[\frac{(\gamma_2 - \gamma_1)^2 (|y| - \delta)^2 + 4\gamma_1 \gamma_2 \delta^2}{4} \right]^{\frac{1}{2}} \right)$$
(A.2)

This system exhibits three equilibrium points in the $h-\theta$ space, one at the origin and two spaced symmetrically about the origin at nonzero values of h and θ . The nature of the system behavior with respect to these equilibrium, or fixed, points changes as the parameter of freestream velocity, V varies. Time domain simulations indicate LCO behavior within a certain velocity range, and so a motivation exists to verify by the application of existing theorems that the bifurcation that leads to LCO behavior is in fact a Poincare-Andronov-Hopf (PAH) bifurcation. This development follows the general procedure outlined in Wiggins (2003) and demonstrated in Liu et al. (2000).

A.1 Center Manifold

The procedure begins by assuming a nonlinear system

$$\dot{x} = f(x) \quad x \in \mathbb{R}$$

with equilibrium points at $x = x^*$. These equilibrium points may be located anywhere in the domain. f(x) is a vector field in the state space, and with the assumption that trajectories are

smooth enough in a neighborhood about a given equilibrium point $(f \in C^r)$, a smooth local flow on a mapping Φ_t exists such that

$$x(t) = \Phi_t(x_0, t)$$

Equilibrium points are determined through the solution of $f(x^*) = 0$. If the system is expanded about the equilibrium point(s):

$$x = x^* + \delta x$$

$$\delta \dot{x} = \frac{\partial f}{\partial x}\Big|_{x = x^*} \cdot \delta x \quad \Rightarrow J(x^*) = \frac{\partial f}{\partial x}\Big|_{x = x^*}$$

 $J(x^*)$ comprises the system Jacobian evaluated at the equilibrium point(s), and is subject to the two following theorems:

Theorem A. 1. : Lyapunov's first theorem. If $J(x^*)$ has no eigenvalues with zero real parts, the $\Phi_t \to e^{Jt}$ smoothly in some region about x^* .

Theorem A. 2. If $J(x^*)$ has no eigenvalues with zero real parts, then there exist stable and unstable manifolds of the nonlinear system, W^S and W^U , which are tangent to the stable and unstable trajectories E^S , E^U of the original system f(x) at x^* .

This development leads to

Theorem A. 3. : Center Manifold Theorem. If the assumptions of the previous two theorems apply, namely:

- $\dot{x} = f(x)$ $x \in \mathbb{R}^n$
- x^* is defined by $f(x^*) = 0$
- $J(x^*) = \left. \frac{\partial f}{\partial x} \right|_{x=x^*}$
- J is divided into three subspaces, E^S , E^U and E^C

Then the results of the previous two theorems apply, and in addition, there exists a center manifold W^C tangent to E^C at x^* . W^S and W^U are unique, but W^C need not be.

The manifolds, W are tangent to the state trajectories E at the equilibrium point(s) x^* . These results are applicable to the 2D aeroelastic system presented in this research. Since the main interest is in the behavior of the system at the equilibrium points not located at the origin, the process begins by translating system (A.1) so that one of the symmetric non-zero fixed points is transformed to the origin. By defining $q = x - x^*$ so that $q(x_*) = q_* = 0$,

$$\dot{q} = \dot{x} = Ax + B\phi(Cx)$$

$$= Aq + Ax^* + B\phi(Cq + Cx^*)$$

$$= Aq - B\phi(Cx^*) + B\phi(Cq + Cx^*)$$
(A.3)

and note that at the equilibrium point, $Ax^* + B\phi(Cx^*) = 0$.

The Taylor series expansion of A.3 at $q = q^*$ with the truncation of higher order terms leads to

$$\dot{q} = Aq + B\left(\frac{d\phi(Cx^*)}{d(Cq)}Cq + \frac{1}{2}\frac{d^2\phi(Cx^*)}{d(Cq)^2}(Cq)^2 + \frac{1}{6}\frac{d^3\phi(Cx^*)}{d(Cq)^3}(Cq)^3 + \dots\right)$$

$$= (A + \frac{d\phi(Cx^*)}{d(Cq)}BC)q + B\left(\frac{1}{2}\frac{d^2\phi(Cx^*)}{d(Cq)^2}(Cq)^2 + \frac{1}{6}\frac{d^3\phi(Cx^*)}{d(Cq)^3}(Cq)^3\right)$$

$$= Jq + f(Cq) \tag{A.4}$$

The next step is to define the transform matrix T such that $J = T\Lambda T^{-1}$ and $\Lambda = T^{-1}JT$, and the system can be transformed into the (real) Jordan canonical form under the relationship q = Tz. A.4 becomes

$$\dot{z} = \Lambda z + T^{-1} f(CTz) \tag{A.5}$$

where

$$\Lambda = \begin{bmatrix} \Lambda_c & 0_2 \\ 0_2 & \Lambda_s \end{bmatrix} = \begin{bmatrix} 0 & \beta_c & 0 & 0 \\ -\beta_c & 0 & 0 \\ 0 & 0 & -\alpha_s & \beta_s \\ 0 & 0 & -\beta_s & -\alpha_s \end{bmatrix}$$

with β_c , α_s , $\beta_s > 0$. It is now helpful to define the transformed system

$$T^{-1}f(CTz) = \begin{bmatrix} g_1(z_1, z_2) \\ g_2(z_1, z_2) \end{bmatrix}$$
(A.6)

After the transformation, this system is separable into distinct two-dimensional center and stable subsystems

$$\dot{z_1} = \Lambda_c z_1 + g_1(z_1, z_2), \quad \dot{z_2} = \Lambda_s z_2 + g_2(z_1, z_2)$$
(A.7)

where g_1 and g_2 represent the nonlinear portions of the two systems. Their development proceeds as follows. Prior to the transform to Jordan canonical form, the nonlinear portion of the system is described from the higher order terms of the Taylor ser4ies expansion as

$$f(Cq) = B[\gamma_2(\theta^*)(Cq)^2 + \gamma_3(\theta^*)(Cq)^3 + ...]$$
(A.8)

where $\gamma_i(\theta^*) = \frac{1}{i!} \frac{d^i \phi(\theta^*)}{d(Cq)^i}$ (*i* = 2, 3 after truncation). After the transform to Jordan canonical form, and noting from the original system that $C = [0 \ 1 \ 0 \ 0]$,

$$Cq = CTz = T_{21}z_{11} + T_{22}z_{12} + T_{23}z_{21} + T_{24}z_{22}$$
$$f(Cq) = f(CTz) = B[\gamma_2(\theta^*)(CTz)^2 + \gamma_3(\theta^*)(CTz)^3] = B\eta(\theta^*, z)$$

Furthermore, it is noted from the original system that $B = [0 \ 0 \ 0 \ K_{nl}]^T$, where K_{nl} represents rotational stiffness outside the deadband region, so that the vector $S = T^{-1}B$ can now be defined so as to represent the nonlinear portion of the canonical form as

$$T^{-1}f(CTz) = S\eta(\theta^*, z) = \begin{bmatrix} S_1\eta(\theta^*, z) \\ S_2\eta(\theta^*, z) \\ S_3\eta(\theta^*, z) \\ S_4\eta(\theta^*, z) \end{bmatrix} = \begin{bmatrix} g_{11}(z_1, z_2) \\ g_{12}(z_1, z_2) \\ g_{21}(z_1, z_2) \\ g_{22}(z_1, z_2) \end{bmatrix}$$
(A.9)

From the Center Manifold theorem, it is known that a center manifold exists for (A.7), which will allows the system dynamics to be analyzed in the vicinity of the original non-hyperbolic fixed point. The manifold is postulated as $z_2 = h(z_1)$, which can be approximated by a sum of polynomials in $z_1 (= [z_{11} \ z_{12}]^T)$.

$$h_i(z_{11}, z_{12}) = f_i^{(2)}(z_1^2) + f_i^{(3)}(z_1^3) + \mathcal{O}(4), \qquad i = 1, 2$$
 (A.10)

When the derivative of the center manifold relationship is taken,

$$\dot{z}_2 = \frac{\partial h(z_1)}{\partial z_1} \dot{z}_1$$

which, when combined with the separated system results shown in (A.7), produces the relationship

$$\Lambda_2 z_2 + g_2(z_1, z_2) = \frac{\partial h(z_1)}{\partial z_1} (\Lambda_1 z_1 + g_1(z_1, z_2))$$

where

$$\frac{\partial h(z_1)}{\partial z_1} = \begin{bmatrix} \partial h_1 / \partial z_{11} & \partial h_1 / \partial z_{12} \\ \partial h_2 / \partial z_{11} & \partial h_2 / \partial z_{12} \end{bmatrix}$$

When the truncated polynomial approximation of (A.10) is substituted into this relationship, the following relationship is obtained

$$\frac{dh(z_1)}{dz_1} \left[\Lambda_c z_1 + g_1 \left(z_1, h(z_1) \right) \right] = \Lambda_s h(z_1) + g_2 \left(z_1, h(z_1) \right)$$

from which the fourteen coefficients of the second- and third-order terms of $h(z_1)$ can be obtained. When this center manifold approximation is substituted back into the original canonical system (A.7), a reduced system representing the dynamics on the center manifold is obtained. Since the assumptions in this development necessarily preclude the resulting system from being identical to that of z_1 , a new space is denoted, $u(=[u_1 \ u_2]^T) \in \mathbb{R}^2$, so that the two-dimensional reduced system becomes

$$\dot{u} = \hat{B}u + K(u; V) \tag{A.11}$$

where

$$\hat{B} = \begin{bmatrix} \hat{b}_{11} & \hat{b}_{12} \\ \hat{b}_{21} & \hat{b}_{22} \end{bmatrix}, \quad K(u;V) = \begin{bmatrix} k_1(u_1, u_2; V) \\ k_2(u_1, u_2; V) \end{bmatrix}$$

The coefficients of the linear term \hat{B} are also functions of V, and as such, as $V \to V^*$

$$\hat{b}_{11}, \ \hat{b}_{22} \to 0$$

 $\hat{b}_{12}, \ \hat{b}_{21} \to \omega_0, \ -\omega_0$
and $\ \hat{B} \to \Lambda_c$

The system dependence on the freestream velocity parameter is made explicit in the notation used for (A.11).

A.2 Transformation to Normal Form

The next objectives of the system transformation are to simplify the linear portion of the system to the greatest extent, then to simplify the nonlinear remainder. Through the development of the reduced system on the center manifold, a system results whereby the behavior of the original system can be analyzed through the behavior of the reduces system on the center manifold. However, the coefficients of the linear portion of the reduced system, \hat{B} , are not guaranteed to behave in a symmetric fashion in the vicinity of $V = V^*$. A transformation is therefore applied to (A.11) such that $u = T_{cm}y$, in order to transform the center manifold system into the standard form. The transform matrix is defined as

$$T_{cm} = \begin{bmatrix} 0 & \hat{b}_{12} \\ \beta & \alpha - \hat{b}_{11} \end{bmatrix}, \quad T_{cm}^{-1} = \frac{1}{\beta \hat{b}_{12}} \begin{bmatrix} -\alpha + \hat{b}_{11} & \hat{b}_{12} \\ \beta & 0 \end{bmatrix}$$

where $\alpha = \frac{1}{2} \left(\hat{b}_{11} + \hat{b}_{12} \right)$ and $\beta = \sqrt{\hat{b}_{11}\hat{b}_{22} - \hat{b}_{12}\hat{b}_{21} - \alpha^2}$. This transformation returns the canonical form

$$\dot{y} = \Gamma y + g(y_1, y_2; V) \tag{A.12}$$

$$\Gamma = \begin{bmatrix} \alpha(V) & \beta(V) \\ -\beta(V) & \alpha(V) \end{bmatrix}, \ g(y) = T_{cm}^{-1}k_1(T_{cm}y)$$

At this point, the linear portion of (A.5) has been simplified as much as possible.

The final process in transforming the dynamical system to its final normal form requires the application of the normal form theorem, which is stated as follows:

Theorem A. 4. By a sequence of analytic coordinate changes, A.12 can be transformed to

$$\dot{y} = \Gamma y + F_2^r(y) + \dots + F_{r-1}^r(y) + \mathcal{O}(|y|^r),$$
 (A.13)

where $F_k^r(y) \in G_k$, $2 \le k \le r-1$, and G_k is a space complementary to $L_J^{(k)}(H_k)$ (i.e. a linear map on H_k via Lie algebra, where H_k is a space of vector-valued homogeneous polynomials of degree k). Equation A.13 is said to be in normal form through order r-1.

The scheme is to transform the system A.12 into A.13 beginning with k = 2 through a near-identity transform comprising a second order space and choosing (or rather assuming) a basis for that space that allows the transform to eliminate as many of the second order terms as possible. The process is repeated for successively higher integer values of k until the nonlinear portion is sufficiently simplified.

The process is facilitated by first changing the system to complex coordinates using the linear transformation

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix}; \quad \begin{pmatrix} x \\ \bar{x} \end{pmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

to obtain

$$\begin{pmatrix} \dot{x} \\ \dot{\bar{x}} \end{pmatrix} = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix} \begin{pmatrix} x \\ \bar{x} \end{pmatrix} + \begin{pmatrix} G_1(x,\bar{x}) \\ G_2(x,\bar{x}) \end{pmatrix}$$

where $G_1 = g_1(y_1(x, \bar{x}), y_2(x, \bar{x}); V) + ig_2(y_1(x, \bar{x}), y_2(x, \bar{x}); V)$, and G_2 is the complex conjugate. Due to the complex symmetry, analysis of the normal form needs only to be performed on one of the two equations, namely

$$\dot{x} = \lambda x + G_1(x, \bar{x} : V), \quad \text{where } \lambda = \alpha + \mathrm{i}\beta$$
 (A.14)

With a Taylor series expansion about the point $V = V^*$, (A.14) becomes

$$\dot{x} = \lambda x + G^{(2)} + G^{(3)} + \mathcal{O}(4)$$
 (A.15)

 $G^{(i)}$ are homogeneous polynomials in x, \bar{x} of order *i* representing the lower order terms of the nonlinear portion of the system, and whose coefficients depend on V.

To simplify the second order terms of (A.15), the near-identity transformation is now applied

$$x \mapsto x + f^{(2)}(x, \bar{x}) \tag{A.16}$$

so that

$$\dot{x}\left(1+\frac{\partial f^{(2)}}{\partial x}\right) + \frac{\partial f^{(2)}}{\partial \bar{x}}\dot{\bar{x}} = \lambda x + \lambda f^{(2)} + G^2(x,\bar{x}) + \mathcal{O}(3)$$
(A.17)

or

$$\dot{x} = \left(1 + \frac{\partial f^{(2)}}{\partial x}\right)^{-1} \left[\lambda x + \lambda f^{(2)} - \frac{\partial f^{(2)}}{\partial \bar{x}} \dot{\bar{x}} + G^2(x, \bar{x}) + \mathcal{O}(3)\right]$$
(A.18)

It can be shown that a linear map of $f^{(2)}$ can be found from the space of homogeneous polynomials in x and \bar{x} of degree 2, denoted as $F_2 = \text{span} \{x^2, x\bar{x}, \bar{x}^2\}$, onto that same space. Furthermore, it can be shown that for the results of this near-identity transformation, the map of $f^{(2)}$ forms a basis of F_2 , thereby allowing that, for V sufficiently close to V^* , all second-order terms in (A.15) can be eliminated.

One byproduct of this second order near-identity transform, however, is the generation of new third-order terms, such that $G^{(3)} \mapsto \tilde{G}^{(3)}$ with the second-order near-identity transform, and (A.15) becomes

$$\dot{x} = \lambda x + \tilde{G}^{(3)} + \mathcal{O}(4) \tag{A.19}$$

In a similar fashion to the second-order near-identity transform, let $x \mapsto x + f^{(3)}(x, \bar{x})$. After computing the linear map of $f^{(3)}$ onto the space $F_3 = \text{span} \{x^3, x^2 \bar{x}, x \bar{x}^2, \bar{x}^3\}$, it can be shown that the resulting map does not span F_3 . It can be shown that all terms of $\tilde{G}^{(3)}$ involving $x^3, x^2 \bar{x}$ and \bar{x}^3 can be eliminated, leaving only those terms involving $x^2 \bar{x}$.

Further derivation proves that all fourth-order terms may be similarly removed through the appropriate near-identity transform. The system is finally reduced to

$$\dot{x} = \lambda x + P_{21}x^2\bar{x} + \mathcal{O}(5)$$

in some neighborhood around $V = V^*$. With $\lambda(V) = \alpha(V) + i\beta(V)$ and $P_{21}(V) = a(V) + ib(V)$, the system can be reverted to the reduced canonical system in Cartesian coordinates by identifying $x = y_1 + iy_2$.

$$\dot{y}_1 = \alpha y_1 - \omega y_2 + (ay_1 - by_2)(y_1^2 + y_2^2) + \mathcal{O}(5),$$

$$\dot{y}_2 = \omega y_1 + \alpha y_2 + (by_1 + ay_2)(y_1^2 + y_2^2) + \mathcal{O}(5).$$
 (A.20)

Finally, to place the system in final normal form for the application of the Poincaré-Andronov-Hopf bifurcation theorem, it is converted to polar coordinates so that it can be expressed as

$$\dot{r} = \alpha r + ar^3 + \mathcal{O}(r^5),$$

$$\dot{\theta} = \beta + br^2 + \mathcal{O}(r^4)$$
(A.21)

APPENDIX B. MODEL PARAMETERS

Aerodynamic parameters

General

Standard gravity, g	=	$32.17 \; ft/s^2$
Standard day air density, ρ	=	$0.002378 \ slug/ft^3$

2D Model

$C_{L\alpha}$	=	3.5860
$C_{L\dot{lpha}}$	=	0.0230
C_{Lq}	=	0.0386
$C_{M\alpha}$	=	2.934
$C_{M\dot{lpha}}$	=	-0.0103
C_{Mq}	=	-0.025

Structural parameters

2D Model

Mass, m	=	$4.04\ slugs$
Moment of intertia, J	=	$16 \; slug \cdot ft$
Chord length, \bar{c}	=	10 in
Bending stiffness, K_h	=	$2,500 \; lb_f/in$
Torsional stiffness, K_{θ}	=	$15,000 \ ft \cdot lb_f/rad$
Displacement of aerodynamic center from elastic axis, e_{ac}	=	-1.5 in

	Mass	Moment of Inertia	Chord	Mass Moment, e_i	
Segment	$(\times 10^{-3} \text{ slugs})$	$(imes 10^{-3} ext{ slug ft}^2)$	(feet)	(feet)	
1	34.2	40.3	1.75	0.274	
2	35.9	33.5	1.65	0.258	
3	34.2	24.9	1.55	0.243	
4	30.8	17.2	1.45	0.227	
5	26.5	11.1	1.35	0.212	
6	23.1	7.1	1.25	0.196	
7	24.0	5.3	1.15	0.181	
8	17.1	2.7	1.05	0.165	
Total	225.8	142.1	-	-	
	Di	istances positive from the	leading edg	ge back.	

 Table B.1
 Design Inertia Properties of the Eight Segment Airfoil

All segments are attached to a straight aluminum spar at the $\bar{c}_i/3$ point.

Segment	1	2	3	4	5	6	7	8
1	0.0083	0.03915	0.07483	0.09166	0.1250	0.1543	0.1707	0.2000
2	0.03915	0.3330	0.6024	0.9212	1.2311	1.4659	1.8458	2.1156
3	0.07483	0.6024	1.3800	2.0840	3.0280	3.8420	4.5810	5.3800
4	0.09166	0.9212	2.0840	3.7800	5.5700	7.3800	9.1450	10.9000
5	0.1250	1.2311	3.02800	5.5700	8.7500	11.9600	14.9950	18.4300
6	0.1543	1.4659	3.8420	7.3800	11.9600	17.5000	22.7000	28.0500
7	0.1707	1.8458	4.5810	9.1450	14.9950	22.7000	31.3000	39.5500
8	0.2000	2.1156	5.3800	10.9000	18.4300	28.0500	39.5500	51.7000

Table B.2 Design Bending Influence Matrix $(ft/lb_f \times 10^{-3})$

Table B.3 Design Torsion Influence Matrix $(rad/ft \cdot lb_f)$

Segment	1	2	3	4	5	6	7	8
1	0.00047	0.00047	0.00047	0.00047	0.00047	0.00047	0.00047	0.00047
2	0.00047	0.00160	0.00160	0.00160	0.00160	0.00160	0.00160	0.00160
3	0.00047	0.00160	0.00308	0.00308	0.00308	0.00308	0.00308	0.00308
4	0.00047	0.00160	0.00308	0.00499	0.00499	0.00499	0.00499	0.00499
5	0.00047	0.00160	0.00308	0.00499	0.00745	0.00745	0.00745	0.00745
6	0.00047	0.00160	0.00308	0.00499	0.00745	0.01090	0.01090	0.01090
7	0.00047	0.00160	0.00308	0.00499	0.00745	0.01090	0.01550	0.01550
8	0.00047	0.00160	0.00308	0.00499	0.00745	0.01090	0.01550	0.02200

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