METRIC PROJECTIONS ONTO SUBSPACES OF FINITE CODIMENSION

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1. Introduction. Let E be a real normed vector space and let S be a subset of E. For x in E, we denote by $T(x)$ the set of nearest points in S to x, i.e. $T(x) = \{y \text{ in } S: ||x - y|| = \text{dist}\ (x, S)\}$. If $T(x)$ is nonvoid for all x in E, then S is an *E-subset*. If $T(x)$ contains at most one point for x in E, then S is a *U-subset*. If S is both an E-subset and a U-subset, then S is an EU-subset. (An EU-subset is often called a *Chebychev* set. Our terminology is that of [3].) The set-valued mapping T is called the *metric projection onto* S. In case S is an EU-subset, we regard T as a point-valued mapping.

The principal result of this paper is the following (which will follow from the slightly stronger Theorem 3):

Theorem. *If X is an infinite compact Hausdorff space and P is an EU-space of finite codimension greater than one in $C(X)$, then the metric projection onto P is discontinuous.*

This result is somewhat surprising since the first example of a discontinuous metric projection onto an EU-subset was given only recently [7; 87]. Other examples have since been given by Holmes and Kripke [6] and by Cheney and Wulbert [3].

Throughout this paper we are concerned with metric projections onto subspaces of finite codimension. In §2, we take E to be a space of the type $C(X)$, for X a compact Hausdorff space. Other results on this topic are found in [10] and [5]. Besides the theorem stated above, we obtain (Theorem 1) a characterization of those finite codimensional E-subspaces which have the property that $T(f)$ is finite dimensional for every f in $C(X)$.

In §3, we are concerned with spaces of the type $L_1(S, \Sigma, \mu)$ where (S, Σ, μ) is a σ-finite measure space. We characterize the U-subspaces of finite codimension and also characterize those EU-subspaces of finite codimension for which the metric projection is continuous and those for which the metric projection is linear.

In dealing with a space of the type $C(X)$, we identify the dual space $C(X)^*$ with the space $rca(X)$ of regular Borel measures on X. For μ in $C(X)^*$, μ^+, μ^- denote, respectively, the positive and negative parts of μ and supp (μ) denotes the support of μ [4].

For P a subspace of a normed vector space E with dual E^* and T the metric projection onto P, $P^\perp = \{x^* \text{ in } E^*: x^*(x) = 0 \text{ for all } x \text{ in } P\}$ and $P^0 = \{x \text{ in } E: 0 \text{ is in } T(x)\}$. A subset S of E is *boundedly compact* if $S \cap \{y: ||x - y|| \leq r\}$ is compact for all x in E and $r > 0$.

We state the following known results here for convenience.

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THEOREM A. ([11], also see [2]) *Let P be a subspace of E. Then* $P^0 = \bigcup\{W_f : f \text{ in } P^\perp, \|f\| = 1\}$, *where* $W_f = \{x \text{ in } E: f(x) = \|x\|\}$. *Also, for each* x *in* E, $T(x) = (x + P^0) \cap P$.

THEOREM B. [10] *Let P be an E-subspace of finite codimension in $C(X)$. Then each μ in $P^\perp$ attains its supremum on the unit sphere of $C(X)$, and therefore* $\operatorname{supp}(\mu^+) \cap \operatorname{supp}(\mu^-) = \phi$.

THEOREM C. [3] *Let P be an EU-subspace of finite codimension in E. Then the metric projection onto P is continuous if and only if P^0 is boundedly compact.*

We shall prove a result (Theorem 2) slightly stronger than Theorem C.

We denote the set-theoretic difference of the sets A and B by $A \setminus B$. Throughout, N denotes the set of positive integers. The cardinality of a set A is denoted by card (A). If A is contained in a topological space, its closure is denoted by $\mathrm{Cl}A$.

2. Spaces of the type $C(X)$.

Throughout this section, X denotes a compact Hausdorff space and P denotes a closed subspace of finite codimension in the real space $C(X)$.

DEFINITION. Let T be the metric projection onto a subspace Q in the space E. Then Q is an *EF-subspace* if $T(x)$ is a nonvoid finite dimensional set for every x in E.

EF-subspaces played an important role in [1].

We have the following

THEOREM 1. *Let P be an E-subspace. Then P is an EF-subspace if and only if* $X \setminus \operatorname{supp}(\mu)$ *is finite for every μ in* $P^\perp \setminus \{0\}$.

Proof. Let T be the metric projection onto P. ($\Leftarrow$) Assume $X \setminus \operatorname{supp}(\mu)$ is finite for every μ in $P^\perp \setminus \{0\}$. For any f in $C(X)$ and p in $T(f)$, $f - p$ is in P^0 and $T(f - p) = T(f) - p$. Hence it suffices to show that $T(f)$ is finite-dimensional for f in P^0. We may assume that $\|f\| = 1$. There exists (Theorem A) μ in $P^\perp$ such that $\|\mu\| = 1$ and $\int_X f\, d\mu = 1$. Then $f = 1$ on $\operatorname{supp}(\mu^+)$ and $f = -1$ on $\operatorname{supp}(\mu^-)$. Let p be in $T(f)$. Since $\|f - p\| = 1$, $p \geq 0$ on $\operatorname{supp}(\mu^+)$ and $p \leq 0$ on $\operatorname{supp}(\mu^-)$. However, $\int_X p\, d\mu = 0$, so that p vanishes identically on $\operatorname{supp}(\mu)$. Thus $T(f) \subseteq \{h \text{ in } C(X): h \text{ vanishes on } \operatorname{supp}(\mu)\}$, and this set is finite-dimensional because $X \setminus \operatorname{supp}(\mu)$ is finite.
($\Rightarrow$) Let n be the codimension of P. Suppose $X \setminus \operatorname{supp}(\mu)$ is infinite for some nonzero μ in $P^\perp$. We can assume that $\|\mu\| = 1$. Let U_1 be a non-void open subset of X such that $\mathrm{Cl}U_1 \subseteq X \setminus \operatorname{supp}(\mu)$ and such that $X \setminus (\operatorname{supp}(\mu) \cup \mathrm{Cl}U_1)$ is infinite. Then let U_2 be a non-void open subset such that $\mathrm{Cl}U_2 \subseteq X \setminus (\operatorname{supp}(\mu) \cup \mathrm{Cl}U_1)$ and such that $X \setminus (\operatorname{supp}(\mu) \cup \mathrm{Cl}U_1 \cup \mathrm{Cl}U_2)$ is infinite. Continuing in this fashion, we obtain a sequence $\{U_1, U_2, \cdots\}$ of non-void open sets such that $\mathrm{Cl}U_i \subseteq X \setminus \operatorname{supp}(\mu)$ for each i in N, and such that $\mathrm{Cl}U_i \cap$

$\mathrm{Cl}U_j = \phi$, for i, j in N, $i \neq j$. For each i in N let h_i be in $C(X)$ such that $||h_i|| = 1$, h_i is 1 on supp (μ^+), h_i is -1 on supp (μ^-), and such that h_i vanishes on $\mathrm{Cl}U_i$. The existence of h_i is ensured by Theorem B. Let $h = \sum_1^\infty (\frac{1}{2})^i h_i$ in $C(X)$. Then h is 1 on supp (μ^+) and h is -1 on supp (μ^-). Since, also, $||h|| = 1$, h is in P^0. Also, for each i in N, $M_i = \max \{|h(x)|: x \text{ in } \mathrm{Cl}U_i\} < 1$. For each i in N let p_i be a non-zero function in $C(X)$ which vanishes outside U_i. For each k in N, choose a non-zero linear combination q_k of functions $p_{kn+1}, p_{kn+2}, \cdots, p_{kn+n}$ such that $||q_k|| \leq 1 - M_i$, $i = kn + 1, kn + 2, \cdots, kn + n$, and such that q_k is in P. This is possible since $P^\perp$ is n-dimensional and since p_i vanishes on supp (μ), for each i in N. Now $||h - q_k|| = 1$ for all k in N. Therefore the infinite linearly independent set $\{q_1, q_2, \cdots\}$ is contained in $T(h)$ so that $T(h)$ is infinite dimensional.

This completes the proof.

We can easily give an example of an EF-subspace P which is not an EU-subspace. We work with the sequence space $l_\infty = C(X)$, where X is the Stone-Čech compactification of N. Let $m_1 = (0, 0, \frac{1}{2}, \frac{1}{4}, \cdots)$ and let $m_2 = (0, 0, \frac{1}{3}, \frac{1}{9}, \cdots)$. Then m_1, m_2 are in l_1. Since $l_1^* = l_\infty$, we will consider elements in l_∞ as acting as functionals on l_1. Define $P_i = \{x \text{ in } l_\infty : x(m_i) = 0\}$, $i = 1, 2$, and let $P = P_1 \cap P_2$. Then P is a weak*-closed subspace of codimension 2 in l_∞. Hence it is an E-subspace. By Theorem 1, P is an EF-subspace. However it is not an EU-subspace by Proposition 6 of [10]. It is clear that, in a similar manner, subspaces of all finite codimensions may be constructed which are EF-subspaces but not EU-subspaces.

Definition. Suppose F is a mapping from a topological space Y into the set of subsets of Y. Then F is *upper semi-continuous* [8] if $\{y \text{ in } Y : F(y) \subseteq U\}$ is open whenever U is open.

For results concerning upper semi-continuity and other continuity conditions on set-valued metric projections, see [1] and the references cited there.

Theorem 2. *Let E be a normed vector space and let Q be an E-subspace of finite codimension in E. Let T be the metric projection onto Q. Then Q^0 is boundedly compact if and only if T is upper semi-continuous, and $T(x)$ is compact for each x in E.*

Proof. ($\Rightarrow$) Assume Q^0 is boundedly compact. Then $T(x) = (x + Q^0) \cap Q$ (Theorem A) is clearly compact for each x in E. Suppose T is not upper semi-continuous. Then there exists an open set U, x_0 in E, and a sequence $\{x_n\}$ converging to x_0 such that $T(x_0) \subseteq U$ but $T(x_n) \not\subseteq U$ for all n in N. For n in N, define $S_n = x_n - T(x_n)$. Then $S_n \subseteq Q^0$. Since $T(x_n) \not\subseteq U$, $S_n \not\subseteq x_n - U$. For each n in N chose w_n in $S_n \setminus (x_n - U)$. Since $\{w_n\}$ is a bounded sequence in Q^0, it has a subsequence $\{w_{n_i}\}$ which converges to, say, w in Q^0. We show that w is not in $x_0 - U$. Suppose, on the contrary, that w is in $x_0 - U$ and let $\epsilon > 0$ be such that $\{y: ||w - y|| < 2\epsilon\} \subseteq x_0 - U$. Now choose n in N such that $||x_0 - x_n|| < \epsilon$ and such that $||w_n - w|| < \epsilon$. Then

$$||w - w_n + x_n - x_0|| \leq ||w_n - w|| + ||x_n - x_0|| < 2\epsilon.$$

Hence $w_n - x_n + x_0$ is in $x_0 - U$ so that $w_n - x_n$ is in $-U$. Thus w_n is in $x_n - U$. But this contradicts the choice of w_n. Therefore w is not in $x_0 - U$. Now we show that $x_0 - w$ is in $T(x_0)$. Now, for each i in N, $x_{n_i} - w_{n_i}$ is in $T(x_{n_i})$ so that

$$||w_{n_i}|| \leq ||x_{n_i} - q||, \quad \text{for all } q \text{ in } Q.$$

Taking the limit as $i \to \infty$, we get

$$||w|| \leq ||x_0 - q||, \quad \text{for all } q \text{ in } Q.$$

Therefore, $x_0 - w$ is in $T(x_0)$. But w is not in $x_0 - U$ so that $x_0 - w$ is not in U. This is impossible since $T(x_0) \subseteq U$. Thus T is upper semi-continuous. ($\Leftarrow$) Suppose $T(x)$ is compact for each x in E and T is upper semi-continuous, but Q^0 is not boundedly compact. Let $\{x_n\}$ be a bounded sequence in Q^0 without a convergent subsequence. Since the space E/Q is finite dimensional and since $\{x_n + Q\}$ is bounded in E/Q, $\{x_n + Q\}$ has a convergent subsequence and we assume, without loss of generality, that $x_n + Q \to x + Q$. For each n in N, choose q_n in $T(x_n - x)$. Then

$$||x_n - q_n - x|| = \text{dist}\,(x_n - x, Q) \to 0.$$

Therefore, $x_n - q_n \to x$. Now $T(x_n - q_n) = T(x_n) - q_n$ so that $T(x_n - q_n)$ contains $-q_n = (x_n - x - q_n) + (x - x_n)$. By the upper semi-continuity of T, we may, by passing to a subsequence if necessary, find, for each n in N, z_n in $T(x)$ such that

$$||(x_n - x - q_n) + (x - x_n) - z_n|| < 1/n.$$

Since $T(x)$ is compact, we may, without loss of generality, assume that $\{z_n\}$ converges to, say, z in $T(x)$. We have

$$||(x - z) - x_n|| \leq ||(x_n - x - q_n) + (x - x_n) - z_n|| + ||z - z_n|| + ||x_n - x - q_n|| \to 0.$$

Hence $x_n \to x - z$, which contradicts the assumption that $\{x_n\}$ has no convergent subsequence. Thus Q^0 is boundedly compact.

This completes the proof.

We remark that the proof given above is a modification of the proof of Theorem C given in [3].

THEOREM 3. *Suppose P is an EF-subspace of finite codimension greater than* 1 *and let T be the metric projection onto P. If X is infinite, then T is not upper semi-continuous.*

Proof. Suppose P^0 is boundedly compact. Define, for x, y in X, $x \sim y$ if and only if $h(x) = h(y)$ for all h in P^0. Then $\sim$ is an equivalence relation and the quotient space $Y = X/\sim$ is a compact Hausdorff space. Under the natural identification, $C(X) \supseteq C(Y) = \{f \text{ in } C(X): x \sim y \text{ implies } f(x) = f(y)\}$. Now P^0 separates the points of Y. We want to show that P^0 actually contains enough extreme points of the unit sphere of $C(X)$ (i.e. functions e in $C(X)$ such that $|e| = 1$) to separate the points of Y. Let x, y be points of X belonging to different $\sim$-equivalence classes. By definition of the relation $\sim$, there exists a function h in P^0 such that $h(x) \neq h(y)$. We may assume that $||h|| = 1$. By Theorem A, there exists μ in $P^\perp$ such that $\int_X h\, d\mu = 1 = ||\mu||$. Thus $h = 1$ on supp (μ^+) and $h = -1$ on supp (μ^-). By Theorem 1, $X \setminus (\text{supp } (\mu^+) \cup \text{supp } (\mu^-))$ is a finite set. It is also open and therefore each of its points is isolated. It is now clear that we may choose a function e in $C(X)$ such that $|e| = 1$, e agrees with h on supp $(\mu^+) \cup$ supp (μ^-), and such that $e(x) \neq e(y)$. Since $\int_X e\, d\mu = \int_X h\, d\mu = 1$, e is in P^0, and we have proved that P^0 contains enough extreme points of the unit sphere of $C(X)$ to separate the points of Y. However the distance between two distinct extreme points is two. Therefore, since P^0 is boundedly compact, it contains only finitely many extreme points. From these facts, we see that Y is finite. Since X is infinite, there exists an infinite $\sim$-equivalence class $E \subseteq X$. Clearly E is both open and closed. We now show that if μ is in $P^\perp \setminus \{0\}$, then $\mu(E) \neq 0$. To do this, suppose $\mu(E) = 0$. Since, by Theorem 1, $X \setminus$supp (μ) is finite, $E \cap$ supp (μ^+) and $E \cap$ supp (μ^-) must both be non-void. By Theorem B, supp $(\mu^+) \cap$ supp $(\mu^-) = \phi$, and therefore there exists a function h in $C(X)$ such that $||h|| = 1$, $h = 1$ on supp (μ^+), and $h = -1$ on supp (μ^-). Hence $\int_X h\, d\mu = ||\mu||$ and therefore h is in P^0. However h is not constant on E, which is impossible since E is a $\sim$-equivalence class. Thus we have proved that $\mu(E) \neq 0$ for μ in $P^\perp \setminus \{0\}$. Now let μ_1, μ_2 be linearly independent measures in $P^\perp$ and let $\mu = \mu_1(E)\mu_2 - \mu_2(E)\mu_1$. Then μ is in $P^\perp \setminus \{0\}$ but $\mu(E) = 0$, and so we reach a contradiction. Thus P^0 is not boundedly compact, and therefore T is not upper semi-continuous by Theorem 2.

This completes the proof.

In case P is an EU-subspace, upper semi-continuity of T is equivalent to continuity. Thus we have the result mentioned earlier:

THEOREM 4. *If X is an infinite compact Hausdorff space and P is an EU-subspace of finite codimension greater than one in $C(X)$, then the metric projection onto P is discontinuous.*

We remark that EU-subspaces of finite codimension greater than one in spaces of the type $C(X)$ are not at all rare. For example, A. L. Garkavi [5] has shown that if X is a compact metric space with the property that its isolated points are dense, then $C(X)$ contains EU-subspaces of all finite codimensions.

3. Spaces of the type $L_1(S, \Sigma, \mu)$. In this section, (S, Σ, μ) denotes a σ-finite measure space, i.e. S is a non-void set, Σ is a σ-algebra of subsets of S, and μ is a σ-finite measure defined on Σ. If a in Σ is such that $0 < \mu(a) < \infty$ and such that B in Σ, $B \subseteq a$ implies $\mu(B) = 0$ or $\mu(B) = \mu(a)$, then a is an *atom*

[4]. It is well known that a measurable function is constant a.e. on an atom. For B in Σ, at B denotes the set $\{a$ in Σ: a is an atom and a is contained in $B\}$. Since (S, Σ, μ) is σ-finite, at S is countable. We call the set B in Σ *purely atomic* if $\mu(B \setminus \bigcup \{a: a \text{ in } atB\}) = 0$.

Throughout this section, P denotes a closed subspace of finite codimension in $L_1(S, \Sigma, \mu)$. We identify the dual $L_1(S, \Sigma, \mu)^*$ with $L_\infty(S, \Sigma, \mu)$. For f in $L_\infty(S, \Sigma, \mu)$, the set crit (f) is defined, up to a set of measure zero, by: crit $(f) = \{s \text{ in } S: |f(s)| = ||f||\}$. Finally, if h is in $L_1(S, \Sigma, \mu)$, supp (h) is defined, up to a set of measure zero, by supp $(h) = \{s \text{ in } S: h(s) \neq 0\}$.

THEOREM 5. *The subspace P is a U-space if and only if, for each f in $P^\perp \setminus \{0\}$, the following conditions hold:* (i) crit (f) *is purely atomic;* (ii) *if $A \subseteq$* crit (f) *and* $0 <$ card $(atA) = k < \infty$, *then the set of restrictions $P^\perp \mid A$ is k-dimensional.*

Note. Condition (ii) implies that card (at crit (f)) does not exceed the codimension of P.

Proof. $(\Rightarrow)$ Suppose P is a U-space and let f be in $P^\perp \setminus \{0\}$. We may assume that $||f|| = 1$. First suppose crit (f) is not purely atomic. Then there exists a set B in Σ such that $0 < \mu(B) < \infty$, $atB = \phi$, and $|f| = 1$ a.e. on B. Let $B_1, B_2, \cdots, B_{n+1}$ be pairwise disjoint subsets of B such that $\mu(B_i) > 0$ for $i = 1, 2, \cdots n + 1$. Let $f_1 = f$ and let $\{f_1, f_2, \cdots, f_n\}$ be a basis for $P^\perp$. There exist numbers $r_1, r_2, \cdots, r_{n+1}$, not all zero, such that

$$\sum_{i=1}^{n+1} r_i \int_{B_i} f_j \, d\mu = 0, \qquad j = 1, 2, \cdots, n.$$

We may assume that $|r_i| < 1/\sum_{j=1}^{n+1} \mu(B_j)$, for $i = 1, 2, \cdots, n + 1$. Define p by:

$$p(x) = \begin{cases} r_i, x \text{ in } B_i, & i = 1, 2, \cdots, n+1 \\ 0, \text{ otherwise.} \end{cases}$$

Then p is in P. Define h by:

$$h(x) = \begin{cases} f(x) \Big/ \sum_{j=1}^{n+1} \mu(B_j), & x \text{ in } B_i, \quad i = 1, 2, \cdots, n+1 \\ 0, \text{ otherwise.} \end{cases}$$

Then $||h|| = 1$ and $\int_S hf \, d\mu = 1$, so that h is in P^0. However,

$$||h - p|| = \int_S |h - p| \, d\mu = \sum_{i=1}^{n+1} \int_{B_i} |h - p| \, d\mu = \sum_{i=1}^{n+1} \int_{B_i} (fh - fp) \, d\mu$$

$$= 1 - \sum_{i=1}^{n+1} \int_{B_i} r_i f \, d\mu = 1.$$

Thus p is in $T(h)$ and $p \neq 0$. Therefore the assumption that P is a U-space is contradicted. Hence crit (f) is purely atomic.

Now let $A \subseteq$ crit (f) with $0 < \text{card}\,(atA) = k < \infty$. Suppose the dimension of $P^\perp \mid A < k$. Write $atA = \{a_1, a_2, \cdots, a_k\}$. There exist $r_1, r_2, \cdots, r_k$, not all zero, such that

$$\sum_{i=1}^{k} r_i g(a_i) = 0, \qquad \text{for all} \quad g \quad \text{in} \quad P^\perp.$$

We can assume that $|r_i| < \mu(a_i)/\sum_{j=1}^{k} \mu(a_j)$, for all $i = 1, 2, \cdots, k$. Define p in $L_1(S, \Sigma, \mu)$ by:

$$p(x) = \begin{cases} r_i/\mu(a_i), & \text{if} \quad x \text{ is in } a_i, \qquad i = 1, 2, \cdots, k. \\ 0, & x \text{ not in } A. \end{cases}$$

Then p is in P. Define h by:

$$h(x) = \begin{cases} f(a_i) \Big/ \sum_{j=1}^{k} \mu(a_j), & x \text{ in } a_i, \qquad i = 1, 2, \cdots, k \\ 0, & x \text{ not in } A. \end{cases}$$

Then h is in P^0 and $||h|| = 1$. However,

$$\begin{aligned} ||h - p|| = \sum_{i=1}^{k} |h(a_i) - p(a_i)|\, \mu(a_i) &= \sum_{i=1}^{k} f(a_i)(h(a_i) - p(a_i))\mu(a_i) \\ &= \sum_{i=1}^{k} \left[\mu(a_i) \Big/ \sum_{j=1}^{k} \mu(a_j) \right] - \sum_{i=1}^{k} r_i f(a_i) = 1. \end{aligned}$$

Since $p \neq 0$, this contradicts the assumption that P is a U-subspace.

Therefore (i) and (ii) are satisfied.

($\Leftarrow$) Suppose P is not a U-space. Then there exists h in P^0, $||h|| = 1$, and p in P, $p \neq 0$, such that $||h \pm p|| = 1$. There exists f in $P^\perp$ such that $||f|| = 1$ and $\int_S fh\, d\mu = 1$. Suppose (i) and (ii) hold for f. Clearly supp $(h) \subseteq$ crit (f). We first show that supp $(h) \supseteq$ supp (p). Without loss of generality, we may assume that $|p(a)| < |h(a)|$, for every atom $a \subseteq$ supp (h). If supp $(p) \nsubseteq$ supp (h), then, letting Σ denote summation over the elements a in at (supp (h)), we have, for each r in $[-1, 1]$,

$$\sum |h(a) - rp(a)|\, \mu(a) < 1.$$

But sgn $(h(a) - rp(a))$ = sgn $h(a)$ = sgn $f(a)$, for a in at (supp (h)) and hence

$$\sum (f(a)h(a) - rf(a)p(a))\mu(a) < 1, r \quad \text{in} \quad [-1, 1].$$

Thus,

$$1 - r \sum f(a)p(a)\mu(a) < 1, r \quad \text{in} \quad [-1, 1].$$

Therefore,

$$r \sum f(a)p(a)\mu(a) > 0, r \quad \text{in} \quad [-1,1],$$

which is absurd. Thus supp $(p) \subseteq$ supp (h).

Let k = card (*at* supp (h)). Then $P^\perp \mid$ supp (h) is k-dimensional. But this leads to a contradiction because $p(a) \neq 0$ for some a in *at* (supp (h)) and the following linear equations hold, for $\{f_1^*, f_2^*, \cdots, f_k^*\}$ a basis for $P^\perp \mid$ supp (h) and Σ retaining its former meaning,

$$\sum f_i^*(a)p(a)\mu(a) = 0, \qquad i = 1, 2, \cdots, k.$$

This completes the proof.

We define $A_p = \cup\{at \text{ crit } (f): f \text{ in } P^\perp\setminus\{0\}\}$. We have:

THEOREM 6. *Let P be an EU-subspace. Then the metric projection T onto P is continuous if and only if A_P is finite.*

Proof. ($\Rightarrow$) Suppose T is continuous and A_P is infinite. Let $\{a_1, a_2, \cdots\}$ be distinct atoms in A_P and let, for i in N, f_i be in $P^\perp\setminus\{0\}$ such that $\|f_i\| = 1$ and $f_i = 1$ on a_i. For i in N, define h_i by:

$$h_i(x) = \begin{cases} 1/\mu(a_i), & x \text{ in } a_i \\ 0, & x \text{ not in } a_i . \end{cases}$$

Then, for each i in N, $\|h_i\| = 1$, and $\int_S h_i f_i \, d\mu = 1$, so that h_i is in P^0. If $i \neq j$, $\|h_i - h_j\| = 2$ so that the bounded sequence $\{h_1, h_2, \cdots,\}$ has no convergent subsequence. Thus P^0 is not boundedly compact. But this is a contradiction because of Theorem C.

($\Leftarrow$) Suppose A_P is finite. Then

$$P^0 \subseteq \{h \text{ in } L_1(S, \Sigma, \mu): h \text{ vanishes outside } A_P\}$$

Since the latter set is obviously finite-dimensional, P^0 is boundedly compact and T is continuous.

This completes the proof.

COROLLARY 7. *If atS is finite, then every metric projection onto a finite codimensional EU-subspace of $L_1(S, \Sigma, \mu)$ is continuous.*

R. R. Phelps [9] pointed out that if $atS = \phi$, then there are no EU-subspaces of finite codimension in $L_1(S, \Sigma, \mu)$. On the other hand, it is not difficult to show that if $atS \neq \phi$, then there are EU-subspaces of every finite codimension not exceeding card (atS). These can be chosen so as to have continuous metric projections. An example of a discontinuous metric projection onto an EU-subspace of codimension 2 in l_1 was given in [3]. Their example can easily be modified to show the following.

COROLLARY 8. *If atS is infinite, then there is an EU-subspace of codimension 2 in $L_1(S, \Sigma, \mu)$ with discontinuous metric projection.*

A metric projection onto an EU-subspace is always bounded and therefore continuous if linear. The converse is false as is shown by simple finite dimensional examples. We now prove the following.

THEOREM 9. *Let P be an EU-subspace with metric projection T. Then T is linear if and only if there exists f in $P^{\perp}$ such that at* (crit (f)) $= A_P$.

Proof. It is known [6] that T is linear if and only if P^0 is a subspace.

$(\Rightarrow)$ Suppose *at* (crit (f)) $\neq A_P$ for all f in $P^{\perp}$. Then there exists a finite set $B \subseteq A_P$ such that *at* (crit (f)) $\neq B$ for all f in $P^{\perp}$. To see this, suppose A_P is finite. Then we may take $B = A_P$. Suppose A_P is infinite. Then let B be any finite subset of A_P with cardinality greater than the codimension of P. Such a set satisfies the stated condition because of condition (ii) of Theorem 5. For each a in B, define h_a to be $1/\mu(a)$ on a and 0 otherwise. Then each h_a is in P^0. Let $h = \sum_{a \text{ in } B} h_a$. Clearly $\int_S hf\, d\mu < ||h||$ for all f in $P^{\perp}$ such that $||f|| = 1$. Thus h is not in P^0 and therefore P^0 is not a subspace.
$(\Leftarrow)$ Suppose there exists f in $P^{\perp}$ such that *at* (crit (f)) $= A_P$. Let h_1, h_2 be in P^0. Then h_1 and h_2 are supported on $\bigcup\{a: a \text{ in } A_P\}$. To show that $h_1 + h_2$ is in P^0, it suffices to find g in $P^{\perp}$ such that $||g|| = 1$ and $g = \operatorname{sgn}(h_1 + h_2)$ on $\bigcup\{a: a \text{ in } A_P\}$. Now the dimension of $P^{\perp} \mid A_P$ is card (A_P) by Theorem 5. Thus there exists g in $P^{\perp}$ such that $g \mid A_P = \operatorname{sgn}(h_1 + h_2)$. Clearly $||g|| \geq 1$. Suppose $||g|| > 1$. By Proposition 2 of [10], g (as a functional on $L_1(S, \Sigma, \mu)$) attains its supremum on the unit sphere of $L_1(S, \Sigma, \mu)$. Also, the set $\{f$ in $L_1(S, \Sigma, \mu): ||f|| = 1$ and $\int_S gf\, d\mu = ||g||\}$ is a finite-dimensional [9; Theorem 1.4] extremal subset and therefore, using the Krein–Milman Theorem, contains an extreme point of the unit sphere of $L_1(S, \Sigma, \mu)$. Since each such extreme point vanishes outside an atom, we conclude that *at* (crit (g)) $\not\subseteq A_P$, which is absurd. Therefore $||g|| = 1$ and hence $h_1 + h_2$ is in P^0. Since P^0 is always closed under scalar multiplication, P^0 is a subspace.

This completes the proof.

We conclude by giving an example of an EU-subspace of codimension 2 in l_1 such that the metric projection T onto P is continuous but not linear. Let h_1 be the sequence $(1, 1, 0, 0, \cdots)$ And h_2 be $(1, 0, 1, 0, 0, \cdots)$. Then h_1 and h_2 are in c_0. Since $c_0^* = l_1$, the subspace

$$P = \{a = (a_1, a_2, \cdots) \text{ in } l_1 : a_1 + a_2 = a_1 + a_3 = 0\}$$

is weak*-closed and therefore an E-subspace. Moreover $P^{\perp}$ is the subspace of l_∞ spanned by h_1 and h_2. Now $l_1 = L_1(N, \Sigma, \mu)$ where N is the set of positive integers, Σ is the set of all subsets of N, and μ is the counting measure. If $\alpha h_1 + \beta h_2$ is in $P^{\perp} \setminus \{0\}$, then crit $(\alpha h_1 + \beta h_2)$ is one of the sets: $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1\}$, $\{2\}$, $\{3\}$. By Theorem 5, P is a U-subspace. Since $A_P = \{1, 2, 3,\}$, T is continuous but not linear.

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