NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS, THEIR SOLUTIONS, AND PROPERTIES

by

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DEDICATION

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ABSTRACT

Although valuable understanding of real-world phenomena can be gained experimentally, it is often the case that experimental investigations can be found to be limited by financial, ethical, or other constraints making such an approach impractical or, in some cases, even impossible. To nevertheless understand and make predictions of the natural world around us, countless processes encountered in the physical and biological sciences, engineering, economics, and medicine can be efficiently described by means of mathematical models written in terms of ordinary or/and partial differential equations or their systems. Fundamental questions that arise in the modeling process need care that relies on the use of mathematical analysis. It is also the case that more realistic models directly relevant to the specific area of application are often nonlinear, calling for a robust treatment of general classes of differential equations. This thesis is devoted to developing a range of proof techniques for the mathematical analysis of general classes of both linear and nonlinear and both ordinary and partial differential equations that help in gaining an understanding of the fundamental properties of their solutions.

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CHAPTER 1

INTRODUCTION

A broad range of phenomena occurring throughout nature can be modeled by making the assumption that the particles forming the system exist as a continuum. In this macroscopic approach, the resulting theoretical description of the physical system involves functions of one or more continuous independent variables. Even in simple cases, such functions are unknown and have to be determined by considering a mathematical model consisting of ordinary or partial differential equations. Such equations involve derivatives of the unknown functions and have for many years been broadly applied in a variety of disciplines including the physical sciences and biological life sciences, engineering, economics, and medicine. Various studies that can give an outlook of the depth and breadth of the use of both ordinary and partial differential equations used in a broad range of fields can be found in monographs, such as [9], [3], [4], [5], [11].

To form a complete mathematical formulation, differential equations are supplemented by initial and/or boundary conditions, which can be decisive in determining the qualitative structure of the corresponding solutions. When constructing problems described by differential equations, one should make sure that the problems, consisting of differential equations together with the corresponding initial and/or boundary conditions, as appropriate, are well-posed, see e.g., [5]. Well-posedness, or more precisely a lack of it, has for years baffled physicists, engineers, and others using mathematical models to formulate real-world problems, as well as computational mathematicians trying to solve them. It is, therefore, important to understand what well-posedness is and to mathematically quantify models that possess this property and those that do not. To do so, it is necessary to engage in *mathematical analysis*. There are three natural questions concerning differential equations

- existence: does the problem have a solution?
- uniqueness: does the problem have exactly one solution?
- stability: is the problem stable, that is, do all solutions that are close to each other initially remain close for all time?

Answers to these questions are often non-trivial and, depending on the form of the differential equations, need special investigation that relies on the use of mathematical analysis and a range of related proof techniques. Although numerical computations are helpful and can be informative when faced with a well-posed problem, they may lead nowhere and often may be a waste of time because of ill-posedness. It is also the case that many results on numerical methods for differential equations would not be possible to obtain without the use of mathematical analysis. Rigorous, analytical approaches, therefore, help in gaining understanding that is otherwise not possible to gain through computations alone.

To proceed, we define general classes of differential equations that will aid in formulating the subject of this thesis. A general ordinary differential equation can be written concisely in the form

$$F\Big(t, u, \frac{du}{dt}, \text{model parameters}\Big) = 0,$$

involving the continuous independent and dependent variables and its ordinary derivative. Importantly, the differential equation may also depend on model parameters, which can decisively change the qualitative form of the solution and may influence well-posedness properties. A general partial differential equation can be written in the form

$$F\Big(t, x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial^k u}{\partial x_i^{k_1} \ldots \partial x_j^{k_m}}, \text{model parameters}\Big) = 0,$$

where t represents time, $x = (x_1, x_2, \ldots, x_n)$ usually represents space, the function

$$u = u(t, x_1, x_2, \ldots, x_n)$$

is the unknown solution to be determined and $k = k_1 + \cdots + k_m$. As in the case of ordinary differential equations, the given function F often depends on unknown model parameters that have to be estimated according to experimental, field, or clinical data, which can greatly influence the behavior of the resulting problem. In this thesis, we investigate general classes of both ordinary and partial differential equations written in the forms

$$\frac{du}{dt} = f(t, u(t))$$

and

$$\frac{\partial u}{\partial t} = f\Big(t, x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}\Big)$$

and proceed with developing proof techniques and applying methods of mathematical analysis to obtain results critical to the understanding of a range of important properties of both classes of differential equations.

The techniques employed in this thesis, and more generally the techniques of mathematical analysis, present an important and necessary tool not only for non-mathematicians and mathematical modelers as an aid in formulating well-posed models for real-world phenomena, but also for numerical analysts as it is often necessary to understand the fundamental structure of the differential problem in question and the qualitative behavior of its solutions before attempting quantitative computations. These techniques can also be used in studies of yet more generalized equations beyond the scope of the above general classes of differential questions. The more generalized equations arise as a result of a curiosity observed in nature involving past phases. In many cases, ordinary and partial differential equations depend not only on the behavior of their solutions at a present stage but also at some past stage or stages. Many of these natural phenomena are described by such more general differential equations as investigated, for example, in [1], [2], [6], [7], [8], [10], [12], [13].

This thesis builds on results presented in the monograph [11]. Not all of the results have been proved in [11] and those that have been proved have in many cases been proved under different assumptions. Moreover, the proofs presented in this thesis include significantly more intermediate steps and helpful explanations needed for a full understanding and to increase accessibility to a broader range of readers.

CHAPTER 2

THEOREMS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS AND THEIR SYSTEMS

In this chapter, we present and prove a selection of theorems on nonlinear ordinary differential equations and their systems. The results of the theorems and central ideas behind some of the proofs will be applied in the remaining chapters.

The first of these theorems involves the relationship between different differentiable functions and will be helpful in classifying the difference in behavior between different solutions of differential equations.

Theorem 2.1. *Suppose $u, v : (0, T] \to \mathbb{R}$ are differentiable functions that satisfy the following condition for all $t_0 \in (0, T]$:*

$$u(t_0) = v(t_0) \quad \textit{implies} \quad u'(t_0) < v'(t_0). \tag{2.1}$$

Then, u and v satisfy exactly one of the cases:

(I) $u(t) < v(t)$, for all $t \in (0, T]$,

(II) there exists $\bar{t} \in (0, T]$ such that $u(t) \geq v(t)$, for all $0 < t < \bar{t}$.

Proof. Suppose case (I) is not true. Then, there exists $\bar{t} \in (0, T]$ such that $u(\bar{t}) \geq v(\bar{t})$. We now want to show that, in this case, there is no $t_* < \bar{t}$ that would satisfy the inequality $u(t_*) < v(t_*)$. By contradiction, suppose that such t_* exists. Then, there would exist $t_0 > t_*$ such that

$$u(t_0) = v(t_0) \quad \text{and} \quad u(t) < v(t) \quad \text{for all} \quad t_* \leq t < t_0.$$

Then,

$$\frac{u(t) - u(t_0)}{t - t_0} > \frac{v(t) - v(t_0)}{t - t_0}, \quad t_* \leq t < t_0$$

and taking the limit with $t \to t_0$, we get

$$u'(t_0) \geq v'(t_0). \tag{2.2}$$

Since $u(t_0) = v(t_0)$, inequality (2.2) contradicts the assumption (2.1). Therefore, $u(t) \geq v(t)$, for all $0 < t \leq \bar{t}$, which proves (II). Since there are only two cases: either (I) is satisfied or (I) is not satisfied, the proof of the Theorem 2.1 is finished. □

In the next step, we apply Theorem 2.1 and show strict inequality between two functions.

Theorem 2.2. *Suppose* $f : \mathbb{R}^2 \to \mathbb{R}$ *is continuous and* $u, v : [0, T] \to \mathbb{R}$ *are continuous on* $[0, T]$ *and differentiable on* $(0, T]$*. Moreover, the following conditions are satisfied:*

(1) there exists $\epsilon > 0$ *such that* $u(t) < v(t)$ *for all* $t \in (0, \epsilon)$*,*

(2) $u'(t) - f(t, u(t)) < v'(t) - f(t, v(t))$ *for all* $t \in (0, T]$*.*

Then, $u(t) < v(t)$ *for all* $t \in (0, T]$*.*

Proof. We apply Theorem 2.1. Suppose $t_0 \in (0,T]$ is such that $u(t_0) = v(t_0)$. Then, $f(t_0, u(t_0)) = f(t_0, v(t_0))$ and from *(2)* we get

$$u'(t_0) - f(t_0, u(t_0)) < v'(t_0) - f(t_0, v(t_0)).$$

Therefore, $u'(t_0) < v'(t_0)$ and the assumptions of Theorem 2.1 are satisfied. By Theorem 2.1, either case (I) or case (II) is true. From *(1)*, it is seen that case (II) does not hold. Therefore, case (I) is valid and $u(t) < v(t)$ for all $t \in (0,T]$, which finishes the proof. $\square$

We now apply Theorem 2.2 to prove the following result on lower and upper bounds for solutions of initial value problems, useful in evaluating stability properties of the corresponding class of ordinary differential equations.

Theorem 2.3. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\eta \in \mathbb{R}$, and $w : [0,T] \to \mathbb{R}$ solves the problem*

$$\begin{cases} w'(t) &=& f(t, w(t)), \quad t \in (0,T], \\ w(0) &=& \eta. \end{cases}$$

Moreover, suppose that $u, v : [0,T] \to \mathbb{R}$ are continuous on $[0,T]$ and differentiable on $(0,T]$, and satisfy the conditions:

(1) $u(0) < \eta < v(0)$,

(2) $u'(t) < f(t, u(t)), \quad v'(t) > f(t, v(t)), \quad$ *for all* $\ t \in (0,T]$.

Then, the strict inequalities

$$u(t) < w(t) < v(t) \tag{2.3}$$

hold for all $t \in (0,T]$.

Proof. We apply Theorem 2.2 twice with u and w for the first time and with w and v for the second time. Since $u(0) < w(0)$ and the functions u and w are continuous, there exists $\epsilon > 0$ such that $u(t) < w(t)$ for all $t \in [0, \epsilon)$ and condition *(1)* of Theorem 2.2 is satisfied. Moreover, from *(2)*, we get

$$u'(t) - f(t, u(t)) < 0 = w'(t) - f(t, w(t)),$$

for all $t \in (0, T]$, and condition *(2)* of Theorem 2.2 is satisfied. Therefore, by Theorem 2.2, we conclude that $u(t) < w(t)$, for $t \in (0, T]$, and the first inequality in (2.3) is proved. In a similar way, it can be shown that $w(t) < v(t)$, for $t \in (0, T]$, and the proof of (2.3) is finished. □

The central idea behind the proof relies on the use of the previous theorem, which in turn can be traced back to rely on Theorem 2.1. Although not immediately relevant, we have seen through the sequence of proofs how the results of Theorem 2.1 contribute to the development of a useful result for evaluating stability properties of a class of initial value problems.

The next theorem presents relations between functions that satisfy generalized Lipschitz conditions.

Theorem 2.4. *Suppose* $u, v, \rho, \bar{\rho} : [0, T] \to \mathbb{R}$ *are continuous on* $[0, T]$ *and differentiable on* $(0, T]$*. Moreover, suppose that the following conditions are satisfied:*

(1) there exists $\epsilon > 0$ *such that the inequalities*

$$-\bar{\rho(t)} < v(t) - u(t) < \rho(t) \tag{2.4}$$

hold for all $t \in (0, \epsilon)$,

(2) the function u solves the differential equation

$$u'(t) = f(t, u(t)) \tag{2.5}$$

and the function v solves it with a defect no less than $-\bar{\delta}(t)$ and no greater than $\delta(t)$; that is,

$$-\bar{\delta}(t) \leq v'(t) - f(t, v(t)) \leq \delta(t), \tag{2.6}$$

for all $t \in (0, T]$, where $\delta, \bar{\delta} : (0, T] \to \mathbb{R}$ are continuous functions,

(3) suppose that $\omega, \bar{\omega} : [0, T] \times \mathbb{R} \to \mathbb{R}$ and ρ, $\bar{\rho}$ satisfy

$$\begin{cases} \rho'(t) > \omega(t, \rho(t)) + \delta(t), \\ \bar{\rho}'(t) > \bar{\omega}(t, \bar{\rho}(t)) + \bar{\delta}(t), \end{cases} \tag{2.7}$$

for all $t \in (0, T]$,

(4) the function f satisfies the inequalities

$$\begin{aligned} f(t, v(t)) - f(t, v(t) - \rho(t)) &\leq \omega(t, \rho(t)) \\ f(t, v(t) + \bar{\rho}(t)) - f(t, v(t)) &\leq \bar{\omega}(t, \bar{\rho}(t)), \end{aligned} \tag{2.8}$$

for $t \in (0, T]$.

Then, the inequalities

$$-\bar{\rho}(t) < v(t) - u(t) < \rho(t) \tag{2.9}$$

hold for all $t \in (0, T]$.

Proof. We apply Theorem 2.1 with u replaced by $v - u$ and v replaced by ρ for the right-hand side inequality in (2.9). Suppose $t_0 \in (0, T]$ is such that $v(t_0) - u(t_0) = \rho(t_0)$. Then, from (2.4)-(2.8), we get

$$\begin{aligned} v'(t_0) - u'(t_0) &\leq \delta(t_0) + f(t_0, v(t_0)) - u'(t_0) \\ &= \delta(t_0) + f(t_0, v(t_0)) - f(t_0, u(t_0)) \\ &= \delta(t_0) + f(t_0, v(t_0)) - f(t_0, v(t_0) - \rho(t_0)) \\ &\leq \delta(t_0) + \omega(t_0, \rho(t_0)) < \rho'(t_0) \end{aligned}$$

and the assumptions of Theorem 2.1 are satisfied. Therefore, from the assertion of Theorem 2.1, either case (I) or case (II) is true. If case (II) holds, then for all $t \in (0, \bar{t}]$ the inequality $v(t) - u(t) \geq \rho(t)$ holds, where $\bar{t} > 0$ is a certain point in $(0, T]$, which contradicts condition (2.4). Therefore, case (I) holds and $v(t) - u(t) < \rho(t)$, for all $t \in (0, T]$, which finishes the proof of the right-hand side inequality in (2.9). The proof of the left-hand side inequality in (2.9) is similar. □

Although the last theorem involves more assumptions and therefore narrows down the class of functions that the theorem applies to, the result of the theorem gives useful information about the 'closeness' of a class of functions - a result useful in evaluating one of the properties of well-posedness.

We now present a generalization of Theorem 2.1 to higher dimensional functions. Rather than being relevant for problems composed of a single ordinary differential equation, the results of the following theorem will be useful in determining the properties of systems of two or more ordinary differential equations. We will follow and expand on the proof techniques of Theorem 2.1 to generalize to higher dimensions and prove the following:

Theorem 2.5. *Suppose* $u = (u_1, u_2, \ldots, u_n) : (0,T] \to \mathbb{R}^n$*,* $v = (v_1, v_2, \ldots, v_n) : (0,T] \to \mathbb{R}^n$*, and* u_j, v_j *are differentiable functions that satisfy the following condition. Let* $t_0 \in (0,T]$ *and* $i \in \{1,2,\ldots,n\}$ *be arbitrary and the following implication be satisfied:*

$$\textit{if } u_i(t_0) = v_i(t_0) \textit{ and } u_j(t_0) \le v_j(t_0) \textit{ for all } j \in \{1,2,\ldots,n\}, \textit{ then } u_i'(t_0) < v_i'(t_0).$$

Then, either

(I) $u_j(t) < v_j(t)$ *for all* $t \in (0,T]$ *and all* $j \in \{1,2,\ldots,n\}$ *or*

(II) there exists $\bar{t} > 0$ *such that for all* $t \in (0,\bar{t}]$ *there exists* $i \in \{1,2,\ldots,n\}$ *such that* $u_i(t) \ge v_i(t)$

and both cases do not hold together.

Proof. By contradiction, suppose neither (I) nor (II) is true. Then, by a continuity argument, there exists $t_0 \in (0,T]$ such that the following three conditions are satisfied:

- $u_i(t_0) = v_i(t_0)$ for a certain $i \in \{1,2,\ldots,n\}$,
- $u_j(t_0) \le v_j(t_0)$ for all $j \in \{1,2,\ldots,n\}$,
- $u_j(t) < v_j(t)$ for all $t \in (0,t_0)$ and all $j \in \{1,2,\ldots,n\}$.

This is because the negation of (I) or (II) gives that there is at least one value of i for which there exist two corresponding values of t such that $u_i - v_i$ is positive for the larger value of t and strictly negative for the smaller value of t. Hence, by continuity, there must exist a value of t for which $u_i = v_i$. To prove the remaining two bullet points, we take the smallest such t across all such i.

From this we get,

$$\frac{u_i(t) - u_i(t_0)}{t - t_0} > \frac{v_i(t) - v_i(t_0)}{t - t_0},$$

for $t \in (0, t_0)$. We now take $t \to t_0$ in the above inequality and get

$$u_i'(t_0) \geq v_i'(t_0),$$

which contradicts the assumption of the theorem. Therefore, we conclude that either (I) or (II) holds, which finishes the proof. □

We now apply Theorem 2.5 to prove results on the relationship between two functions of time in $\mathbb{R}$, which will be useful in determining the relationship between two different solutions of a system of ordinary differential equations.

Theorem 2.6. *Suppose* $u = (u_1, u_2, \ldots, u_n) : [0, T] \to \mathbb{R}^n$, $\quad v = (v_1, v_2, \ldots, v_n) : [0, T] \to \mathbb{R}^n$ *and* u_j, v_j *are continuous on* $[0, T]$ *and differentiable on* $(0, T]$ *for all* $j \in \{1, 2, \ldots, n\}$. *Moreover, the following conditions are satisfied:*

(1) there exists $\epsilon > 0$ *such that the inequality*

$$u_j(t) < v_j(t) \tag{2.10}$$

holds for all $t \in (0, \epsilon)$ *and all* $j \in \{1, 2, \ldots, n\}$,

(2) for any $t \in [0, T]$, *if* $z = (z_1, z_2, \ldots, z_n)$ *is such that* $u_j(t) \leq z_j \leq v_j(t)$, *for all* $j \in \{1, 2, \ldots, n\}$, *and* $z_i = u_i(t)$ *for a certain* $i \in \{1, 2, \ldots, n\}$, *then*

$$u_i'(t) - f_i(t, z) < v_i'(t) - f_i(t, v(t)). \tag{2.11}$$

Then, the inequality $u_j(t) < v_j(t)$ *holds for all* $t \in (0, T]$ *and all* $j \in \{1, 2, \ldots, n\}$.

Proof. Either the conclusion of the theorem is true or (since u and v are continuous) there exists $t_0 \in (0, T]$ such that the following two conditions are satisfied:

- $u_j(t_0) \leq v_j(t_0)$ for all $j \in \{1, 2, \ldots, n\}$,
- $u_i(t_0) = v_i(t_0)$ for a certain $i \in \{1, 2, \ldots, n\}$.

Then, from (2.11) with $z = v(t_0)$, we get

$$u_i'(t_0) - f_i(t_0, v(t_0)) < v_i'(t_0) - f_i(t_0, v(t_0))$$

thus

$$u_i'(t_0) < v_i'(t_0) \tag{2.12}$$

and the assumptions of Theorem 2.5 are satisfied. Therefore, by Theorem 2.5, either case (I) or case (II) is true. If case (I) is true, then we get the assertion of Theorem 2.6. On the other hand, condition (2.10) contradicts and eliminates case (II), which finishes the proof of Theorem 2.6. $\square$

Remark 2.1. Theorem 2.6 is valid also if the condition *(2)* is replaced by the following condition:

(3) for any $t \in [0, T]$, if $z = (z_1, z_2, \ldots, z_n)$ is such that $u_j(t) \leq z_j \leq v_j(t)$, for all $j \in \{1, 2, \ldots, n\}$, and $z_i = v_i(t)$ for a certain $i \in \{1, 2, \ldots, n\}$, then

$$u_i'(t) - f_i(t, u(t)) < v_i'(t) - f_i(t, z). \tag{2.13}$$

generalizing the scope of the theorem.

The difference between the two conditions *(2)* and *(3)* resides in the arguments of the functions f_i for $i \in \{1, 2, \ldots, n\}$ on either the left-hand side or the right-hand side of the strict inequality.

To see how the proof of Theorem 2.6 changes if these two conditions are interchanged, we first consider such a replacement and then we apply the inequality (2.13) with the value $z = u(t_0)$. We then get the following

$$u_i'(t_0) - f_i(t_0, u(t_0)) < v_i'(t_0) - f_i(t_0, u(t_0)),$$

which implies the inequality (2.12), that is, we arrive at the result of the theorem. After this modification to the proof of Theorem 2.6, the rest of the proof remains as before.

Finally, we apply Theorem 2.6 to prove the following theorem giving upper and lower bounds to the solution of a higher dimensional initial value problem consisting of a system of two or more ordinary differential equations. This result on the upper and lower bounds is useful in determining stability properties of such systems of differential equations.

Theorem 2.7. *Suppose* $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$*,* $\eta \in \mathbb{R}^n$*, and* $w : [0, T] \to \mathbb{R}^n$ *is a solution to the initial value problem*

$$\begin{cases} w_i'(t) & = & f_i(t, w_1(t), \ldots, w_n(t)), \quad t \in (0, T] \\ w_i(0) & = & \eta_i, \quad i = 1, 2, ..., n. \end{cases}$$

Moreover, suppose $u = (u_1, u_2, \ldots, u_n) : [0, T] \to \mathbb{R}^n$*,* $v = (v_1, v_2, ..., v_n) : [0, T] \to \mathbb{R}^n$*,* u_j, v_j *are continuous on* $[0, T]$ *and differentiable on* $(0, T]$ *for all* $j = 1, 2, \ldots, n$

and satisfy the following conditions:

(1) there exists $\epsilon > 0$ *such that the inequalities*

$$u_j(t) < w_j(t) < v_j(t) \tag{2.14}$$

hold for all $t \in (0, \epsilon)$ *and all* $j \in \{1, 2, \ldots, n\}$*,*

(2) for any $t \in [0, T]$ *and* $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ *the two conditions are satisfied:*

- *if* $z_i = u_i(t)$ *and* $z_j \geq u_j(t)$*, for all* $j \in \{1, 2, \ldots, n\}$*, then* $u_i'(t) < f_i(t, z)$*,*
- *if* $z_i = v_i(t)$ *and if* $z_j \leq v_j(t)$*, for all* $j \in \{1, 2, \ldots, n\}$*, then* $v_i'(t) > f_i(t, z)$*.*

Then,

$$u_j(t) < w_j(t) < v_j(t) \tag{2.15}$$

for all $t \in (0, T]$ *and all* $j \in \{1, 2, \ldots, n\}$*.*

Proof. From (2.14), condition (*1*) of Theorem 2.6 is satisfied. We now verify condition (*2*) for u and w. Let $z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$ be such that

- $u_j(t) \leq z_j \leq w_j(t)$, for all $j \in \{1, 2, \ldots, n\}$, and
- $z_i = u_i(t)$, for a certain $i \in \{1, 2, \ldots, n\}$.

Then, by assumptions of Theorem 2.7, we get $u_i'(t) < f_i(t, z)$ and

$$u_i'(t) - f_i(t, z) < 0 = w_i'(t) - f_i(t, w(t)),$$

which shows that condition (*2*) of Theorem 2.6 is satisfied. By Theorem 2.6, $u_j(t) < w_j(t)$, for all $t \in (0, T]$ and all $j \in \{1, 2, , n\}$. Therefore, the first inequality in (2.15) is satisfied. We now verify condition (*2*) for w and v. Let $z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n$ be such that

- $w_j(t) \le z_j \le v_j(t)$, for all $j \in \{1, 2, \ldots, n\}$, and
- $z_i = v_i(t)$ for a certain $i \in \{1, 2, \ldots, n\}$.

Then, from condition (*2*) of Theorem 2.7, we get $v_i'(t) > f_i(t, z)$. Therefore,

$$v_i'(t) - f_i(t, z) > 0 = w_i' - f_i(t, w(t))$$

and condition (*2*) is satisfied. By Theorem 2.6, we find that $w_j(t) < v_j(t)$, for all $j \in \{1, 2, ..., n\}$ and $t \in (0, T]$, which proves the assertion of Theorem 2.7 and finishes the proof. $\square$

CHAPTER 3

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: STRICT INEQUALITIES

The following notation, definitions, and assumptions will be used in the next chapters. Suppose $u : \mathbb{R}^{\mathrm{n}} \times \mathbb{R} \to \mathbb{R}$ is twice differentiable with respect to x and once differentiable with respect to t. Then,

$$\frac{\partial u}{\partial x} = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\Big), \qquad \frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1}^{n}.$$

Suppose $M, \tilde{M} \in \mathbb{R}^{n,n}$ are symmetric matrices. Then,

$$M \leq \tilde{M} \quad \text{if and only if} \quad \sum_{i,j=1}^{n} (\tilde{M}_{ij} - M_{ij})\alpha_i \alpha_j \geq 0, \quad \text{for all } \alpha \in \mathbb{R}^n.$$

The nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = f\Big(x_1, x_2, ..., x_n, t, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}\Big)$$

will be shortly written in the form

$$\frac{\partial u}{\partial t} = f\Big(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\Big),$$

where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$.

Let $b = (b_1, \ldots, b_n) \in \mathbb{R}^n_+$ and $T \in \mathbb{R}_+$ be fixed, where $\mathbb{R}_+ = (0, \infty)$. Then, we define the set

$$S = \Big\{(x,t) : \quad 0 < t < T, \quad -b_i < x_i < b_i, \quad i = 1, \ldots, n\Big\}$$

and its boundaries

$$\begin{aligned} \delta_0 S &= [-b, b] \times \{0\}, \\ \delta_1 S &= \{(x,t) : \quad 0 < t < T, \quad x_i = \pm b_i, \quad i = 1, \ldots, n\}, \\ \delta_2 S &= [-b, b] \times \{T\}, \\ B &= \delta_0 S \cup \delta_1 S, \end{aligned}$$

and closure $\bar{S} = S \cup B \cup \delta_2 S$, where $[-b, b] = [-b_1, b_1] \times \cdots \times [-b_n, b_n]$.

Throughout this chapter and the following chapters, we will make the assumption that the function f (a function of several variables) is continuous and satisfies the condition

$$f(x,t,p,q,M) \le f(x,t,p,q,\tilde{M}), \tag{3.1}$$

for all $M, \tilde{M} \in \mathbb{R}^{n,n}$ such that $M \le \tilde{M}$, and all $(x,t) \in \bar{S}$, $p \in \mathbb{R}$, $q \in \mathbb{R}^n$.

The following results will be applied to prove subsequent theorems. The first of these specifies the relationship between two higher dimensional functions under a set of constraints on their partial derivatives.

Theorem 3.1. *Suppose that*

(1) $u, v : \bar{S} \to \mathbb{R}$ are continuous functions and have continuous first order partial derivatives with respect to t and first and second order partial derivatives with

respect to x in $S \cup \delta_2 S$

(2) *if $(x,t) \in S \cup \delta_2 S$, $u(x,t) = v(x,t)$, $\dfrac{\partial u}{\partial x}(x,t) = \dfrac{\partial v}{\partial x}(x,t)$, and $\dfrac{\partial^2 u}{\partial x^2}(x,t) \leq \dfrac{\partial^2 v}{\partial x^2}(x,t)$ then $\dfrac{\partial u}{\partial t}(x,t) < \dfrac{\partial v}{\partial t}(x,t)$.*

Then, u and v satisfy exactly one of the following cases:

(I) *$u(x,t) < v(x,t)$, for all $(x,t) \in S \cup \delta_2 S$,*

(II) *there is a maximal $\bar{t} \in [0, T)$ such that*

$$u(x,t) < v(x,t),$$

for all $(x,t) \in S \cup \delta_2 S$ with $t \leq \bar{t}$; $\bar{t}$ is maximal in the sense that there is a sequence $\Big\{(x_k, t_k)\Big\}_{k=1}^{\infty} \subseteq S \cup \delta_2 S$ with $t_k > \bar{t}$ and $u(x_k, t_k) \geq v(x_k, t_k)$, $k = 1, 2, \ldots$, such that

$$\lim_{k \to \infty} (x_k, t_k) = (\bar{x}, \bar{t}) \in \delta_0 S \cup \delta_1 S$$

(the points (x_k, t_k) tend to a boundary point).

Proof. Suppose $\bar{t}$ is the largest number such that

$$u(x,t) < v(x,t), \quad \text{for all } (x,t) \in S \cup \delta_2 S \text{ with } t < \bar{t}.$$

Since u and v are continuous,

$$u(x,\bar{t}) \leq v(x,\bar{t}), \quad \text{for all } x \text{ such that } \ (x,\bar{t}) \in S \cup \delta_2 S.$$

Define $S_{\bar{t}} = \{(x,\bar{t}) \in S \cup \delta_2 S \ : \ u(x,\bar{t}) \leq v(x,\bar{t})\}$. Since it may happen that $\bar{t} = 0$, it is possible that $S_{\bar{t}} = \emptyset$. We now want to prove that

$$u(x,\bar{t}) < v(x,\bar{t}), \quad \text{for all } (x,\bar{t}) \in S_{\bar{t}}. \tag{3.2}$$

Suppose that there exists $(\bar{x},\bar{t}) \in S_{\bar{t}}$ such that $u(\bar{x},\bar{t}) = v(\bar{x},\bar{t})$. Then, $\varphi(\bar{x},\bar{t}) = v(\bar{x},\bar{t}) - u(\bar{x},\bar{t}) = 0$ and $\varphi(x,\bar{t}) \geq 0$, for $(x,\bar{t}) \in S_{\bar{t}}$. Therefore, $\varphi(\cdot,\bar{t})$ has a minimum at $\bar{x}$, which implies that

$$\varphi(\bar{x},\bar{t}) = 0, \quad \frac{\partial \varphi}{\partial x}(\bar{x},\bar{t}) = 0, \quad \frac{\partial^2 \varphi}{\partial x^2}(\bar{x},\bar{t}) \geq 0;$$

that is,

$$u(\bar{x},\bar{t}) = v(\bar{x},\bar{t}), \quad \frac{\partial u}{\partial x}(\bar{x},\bar{t}) = \frac{\partial v}{\partial x}(\bar{x},\bar{t}), \quad \frac{\partial^2 u}{\partial x^2}(\bar{x},\bar{t}) \leq \frac{\partial^2 v}{\partial x^2}(\bar{x},\bar{t})$$

and by assumption *(2)*, we get

$$\frac{\partial u}{\partial t}(\bar{x},\bar{t}) < \frac{\partial v}{\partial t}(\bar{x},\bar{t}). \tag{3.3}$$

From the definition of $\bar{t}$, we get

$$u(\bar{x},t) < v(\bar{x},t), \text{ for all } t < \bar{t}, \text{ and } u(\bar{x},\bar{t}) = v(\bar{x},\bar{t}),$$

which, as in the proof of Theorem 2.1, imply $\frac{\partial u}{\partial t}(\bar{x},\bar{t}) \geq \frac{\partial v}{\partial t}(\bar{x},\bar{t})$ and contradict inequality (3.3). From this contradiction, we conclude that inequality (3.2) is true. If $\bar{t} = 0$, that is $S_{\bar{t}} = \emptyset$, then inequality (3.2) is satisfied in a trivial way. If $\bar{t} = T$, then from the inequality between the functions u and v given by (3.2) and the definition of $\bar{t}$, we get

$$u(x,t) < v(x,t), \text{ for all } (x,t) \in S \cup \delta_2 S,$$

which finishes the proof of (I). If $\bar{t} < T$, then from the definition of $\bar{t}$ and by compactness, there exists a sequence

$$\Big\{(x_k, t_k)\Big\}_{k=1}^{\infty} \subseteq S \cup \delta_2 S$$

such that $t_k > \bar{t}$, $u(x_k, t_k) \geq v(x_k, t_k)$, for all k, $\big\{t_k\big\}_{k=1}^{\infty}$ is strictly decreasing and $\lim\limits_{k\to\infty} t_k = \bar{t}$, $\lim\limits_{k\to\infty} x_k = \bar{x}$. Therefore, $u(\bar{x}, \bar{t}) \geq v(\bar{x}, \bar{t})$ and $(\bar{x}, \bar{t}) \in \delta_0 S \cup \delta_1 S$, which finishes the proof of (II). $\square$

The following result determines the relationship between two continuous functions that satisfy suitable differentiability conditions and a class of partial differential inequalities.

Theorem 3.2. *Suppose $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies assumption (3.1) and $u, v : \bar{S} \to \mathbb{R}$ are continuous, have continuous partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$, and satisfy the conditions*

(1) $u(x,t) - v(x,t) \leq 0$, for all $(x,t) \in B$,

(2) the strict inequality

$$\frac{\partial u}{\partial t}(x,t) - f\Big(x, t, u(x,t), \frac{\partial u}{\partial x}(x,t), \frac{\partial^2 u}{\partial x^2}(x,t)\Big) < \frac{\partial v}{\partial t}(x,t) - f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big)$$

is satisfied for all $(x,t) \in S \cup \delta_2 S$.

Then,

$$u(x,t) < v(x,t), \quad \textit{for all } (x,t) \in S \cup \delta_2 S.$$

Proof. We apply Theorem 3.1. Note that assumption *(1)* of Theorem 3.1 is satisfied. We now verify whether assumption *(2)* of Theorem 3.1 is satisfied. Let $(x,t) \in S \cup \delta_2 S$ be such that

$$u(x,t) = v(x,t), \quad \frac{\partial u}{\partial x}(x,t) = \frac{\partial v}{\partial x}(x,t), \quad \frac{\partial^2 u}{\partial x^2}(x,t) \leq \frac{\partial^2 v}{\partial x^2}(x,t).$$

Then, from (*2*) and assumption (3.1), we get

$$\begin{aligned} 0 &< \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\ &\quad -f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ &= \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\ &\quad -f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ &\leq \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t); \end{aligned}$$

that is, $\frac{\partial u}{\partial t}(x,t) < \frac{\partial v}{\partial t}(x,t)$, which shows that condition (*2*) of Theorem 3.1 is satisfied. By Theorem 3.1, only one of the two cases (I), (II) is true. If (II) is true, $\limsup_{k\to\infty} \Big[u(x_k,t_k) - v(x_k,t_k)\Big] \geq 0$, which contradicts assumption (*1*) of Theorem 3.2. Therefore, only (I) is true; that is, $u(x,t) < v(x,t)$, for all $(x,t) \in S \cup \delta_2 S$, which finishes the proof of Theorem 3.2. □

Theorems and proof techniques involving partial differential inequalities underly fundamental results addressing partial differential equations and the behavior of their solutions. Obtaining a grasp of the analysis of partial differential inequalities, therefore, is central to gaining an understanding of the many partial differential equations that model nature and help in addressing fundamental questions raised by the modeling process. A variety of the main concepts and questions are discussed in the following chapter by using this technique.

CHAPTER 4

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: WEAK INEQUALITIES

We now apply Theorem 3.1 and variations of it to prove a sequence of results involving partial differential inequalities under suitable continuity and differentiability conditions. The first result classifies the difference in the behavior between two different functions of $n+1$ independent variables satisfying specified partial differential inequalities. The result holds for the full domain of interest.

Theorem 4.1. *Suppose $u, v : \bar{S} \to \mathbb{R}$ are continuous and have continuous first order partial derivatives with respect to t and first and second order partial derivatives with respect to x in $S \cup \delta_2 S$. Suppose also that $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies assumption (3.1) and the condition*

$$f\Big(x, t, v(x,t) + z, \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) \le \omega(t, z), \tag{4.1}$$

for all $(x,t) \in S \cup \delta_2 S$ and all $z > 0$, with $\omega : (0,T] \times [0,\infty) \to \mathbb{R}$ (being a function of only t and z), such that for all $\varepsilon > 0$, there exist $\delta > 0$ and a continuous function $\rho : [0,T] \to [0,\infty)$, which is differentiable in $(0,T]$, and for all $t \in (0,T]$ the inequalities $\delta \le \rho(t) \le \varepsilon$, $\rho'(t) > \omega(t, \rho(t))$ are satisfied. Moreover, suppose that

(1) $u(x,t) \le v(x,t)$, for all $(x,t) \in B$,

(2) and that the inequality

$$\begin{aligned}&\frac{\partial u}{\partial t}(x,t) - f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \leq \\ &\frac{\partial v}{\partial t}(x,t) - f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)\end{aligned} \tag{4.2}$$

holds for all $(x,t) \in S \cup \delta_2 S$.

Then, $u(x,t) \leq v(x,t)$ *for all* $(x,t) \in \bar{S}$.

Proof. Let $\varepsilon > 0$ be arbitrary and $\rho : [0,T] \to [0,\infty)$ be chosen for this ε according to condition (4.1). We now want to prove that $u(x,t) < v(x,t) + \rho(t)$ for all $(x,t) \in S \cup \delta_2 S$. For this, we apply for Theorem 3.1 with u and $v+\rho$. We note that it is possible to verify that condition (*1*) of Theorem 3.1 is satisfied with u and $v+\rho$. To verify condition (*2*) of Theorem 3.1, suppose that $(x,t) \in S \cup \delta_2 S$ is such that

$$u(x,t) = v(x,t) + \rho(t), \quad \frac{\partial u}{\partial x}(x,t) = \frac{\partial v}{\partial x}(x,t), \quad \frac{\partial^2 u}{\partial x^2}(x,t) \leq \frac{\partial^2 v}{\partial x^2}(x,t).$$

Then, from condition (*2*) of Theorem 4.1 and assumptions (3.1) and (4.1), we get

$$\begin{aligned} 0 \;\leq\;& \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\ & -f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ =\;& \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\ & -f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ & +f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ \leq\;& \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ & -f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \end{aligned}$$

and

$$\begin{aligned} 0 &\leq \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + \omega(t,\rho(t)) \\ &< \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + \rho'(t) = \frac{\partial}{\partial t}\Big(v(x,t)+\rho(t)\Big) - \frac{\partial}{\partial t}u(x,t). \end{aligned}$$

Therefore, $\frac{\partial}{\partial t}u(x,t) < \frac{\partial}{\partial t}\big(v(x,t)+\rho(t)\big)$, and Theorem 3.1 applies. If case (*II*) from Theorem 3.1 is true, then taking $k \to \infty$, we get $u(\bar{x},\bar{t}) \geq v(\bar{x},\bar{t}) + \rho(\bar{t})$, for $(\bar{x},\bar{t}) \in B$. Since $\rho(\bar{t}) > 0$, $u(\bar{x},\bar{t}) > v(\bar{x},\bar{t})$, for $(\bar{x},\bar{t}) \in B$, which contradicts assumption (*1*) of Theorem 4.1. Therefore, only case (*I*) is true; that is,

$$u(x,t) < v(x,t) + \rho(t), \quad \text{for all } (x,t) \in S \cup \delta_2 S.$$

Since $\lim_{\varepsilon \to 0} \rho(t) = 0$, when taking $\varepsilon \to 0$ in the above inequality, we get

$$u(x,t) \leq v(x,t), \quad \text{for all } (x,t) \in S \cup \delta_2 S,$$

which finishes the proof of Theorem 4.1 as from (*1*), the above inequality is satisfied on B. $\square$

Remark 4.1. The inequality in condition (4.1) of Theorem 4.1 can be replaced by the inequality

$$f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x,t,v(x,t)-z,\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \leq \omega(t,z),$$

for all $(x,t) \in S \cup \delta_2 S$ and all $z > 0$.

Assumption 4.1. *Suppose* $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ *satisfies condition (3.1) and* $\eta : B \to \mathbb{R}$ *is continuous. We consider the initial-boundary value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) &= f\Big(x,t,u(x,t),\dfrac{\partial u}{\partial x}(x,t),\dfrac{\partial^2 u}{\partial x^2}(x,t)\Big), \quad (x,t)\in S\cup\delta_2 S, \\ u(x,t) &= \eta(x,t), \quad (x,t)\in B, \end{cases} \tag{4.3}$$

where the solution $u:\bar{S}\to\mathbb{R}$ *is continuous on* $\bar{S}$ *and has continuous partial derivatives* $\dfrac{\partial}{\partial t},\dfrac{\partial}{\partial x},\dfrac{\partial^2}{\partial x^2}$ *in* $S\cup\delta_2 S$.

The following theorem derives upper and lower bounds for problem (4.3) given in terms of a system of partial differential equations supplemented by appropriate boundary conditions. The bounds are derived by considering a relevant system of partial differential inequalities.

Theorem 4.2. *Suppose*

(1) $f:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^{n,n}\to\mathbb{R}$ *satisfies assumption (3.1) and*

$$f(x,t,z,p,q)-f(x,t,\tilde{z},p,q)\le\omega(t,z-\tilde{z}),\quad \textit{for all } z\ge\tilde{z},$$

where $\omega:(0,T]\times[0,\infty)\to\mathbb{R}$ *is as in condition (4.1) of Theorem 4.1,*

(2) $\eta:B\to\mathbb{R}$ *is as in Assumption 4.1*

(3) $\varphi,\psi:\bar{S}\to\mathbb{R}$ *are continuous and have continuous derivatives* $\dfrac{\partial\varphi}{\partial t}$, $\dfrac{\partial\psi}{\partial t}$, $\dfrac{\partial\varphi}{\partial x}$, $\dfrac{\partial\psi}{\partial x}$, $\dfrac{\partial^2\varphi}{\partial x^2}$, $\dfrac{\partial^2\psi}{\partial x^2}$ *in* $S\cup\delta_2 S$,

(4) $\varphi(x,t)\le\eta(x,t)\le\psi(x,t)$, *for all* $(x,t)\in B$,

(5) the inequalities

$$\begin{aligned} \frac{\partial\varphi}{\partial t}(x,t) &\le f\Big(x,t,\varphi(x,t),\frac{\partial\varphi}{\partial x}(x,t),\frac{\partial^2\varphi}{\partial x^2}(x,t)\Big) \\ \frac{\partial\psi}{\partial t}(x,t) &\ge f\Big(x,t,\psi(x,t),\frac{\partial\psi}{\partial x}(x,t),\frac{\partial^2\psi}{\partial x^2}(x,t)\Big), \end{aligned}$$

hold for all $(x,t) \in S \cup \delta_2 S$.

Then, the solution u *of problem (4.3) satisfies the inequalities*

$$\varphi(x,t) \leq u(x,t) \leq \psi(x,t),$$

for all $(x,t) \in \bar{S}$.

Proof. To prove the inequality $\varphi(x,t) \leq u(x,t)$, we apply Theorem 4.1 with u replaced by φ and v replaced by u. To verify condition (*1*) of Theorem 4.1, from condition (*4*) and (4.3), we get $\varphi(x,t) \leq \eta(x,t) = u(x,t)$, for $(x,t) \in B$. We now verify condition (*2*) of Theorem 4.1:

$$\begin{aligned}
&\frac{\partial \varphi}{\partial t}(x,t) - f\Big(x,t,\varphi(x,t),\frac{\partial \varphi}{\partial x}(x,t),\frac{\partial^2 \varphi}{\partial x^2}(x,t)\Big) \leq 0 \\
&= \frac{\partial u}{\partial t}(x,t) - f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big),
\end{aligned}$$

where the equality comes from the fact that u solves (4.3). Therefore, by Theorem 4.1, $\varphi(x,t) \leq u(x,t)$, for all $(x,t) \in S \cup \delta_2 S$. To prove the inequality $u(x,t) \leq \psi(x,t)$, we apply Theorem 4.1 with u and v replaced by ψ. As before, the condition (*4*) implies that condition (*1*) of Theorem 4.1 is satisfied. To verify condition (*2*) of Theorem 4.1, we get

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) - f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) = 0 \\
&\leq \frac{\partial \psi}{\partial t}(x,t) - f\Big(x,t,\psi(x,t),\frac{\partial \psi}{\partial x}(x,t),\frac{\partial^2 \psi}{\partial x^2}(x,t)\Big)
\end{aligned}$$

and, by Theorem (4.2), $u(x,t) \leq \psi(x,t)$ for all $(x,t) \in S \cup \delta_2 S$, which finishes the proof as (*1*) and (*3*) imply the rest of the assumptions of Theorem 4.2. $\square$

We now prove a result on the 'closeness' of a set of exact and approximate solutions to a system of partial differential equations.

Theorem 4.3. *Suppose $u, v : \bar{S} \to \mathbb{R}$ are continuous, have continuous partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$, $\rho, \bar{\rho} : [0,T] \to [0,T]$ are continuous on $[0,T]$ and differentiable on $(0,T]$, and the functions $\delta, \bar{\delta} : [0,T] \to \mathbb{R}$, $\omega, \bar{\omega} : (0,T] \times [0,T] \to \mathbb{R}$ satisfy the conditions:*

(1) $-\bar{\rho}(t) < v(x,t) - u(x,t) < \rho(t)$, for all $(x,t) \in B$,

(2) u and v are exact and approximate solutions, respectively; that is,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = f\Big(x, t, u(x,t), \dfrac{\partial u}{\partial x}(x,t), \dfrac{\partial^2 u}{\partial x^2}(x,t)\Big), \\ -\bar{\delta}(t) \le \dfrac{\partial v}{\partial t}(x,t) - f\Big(x, t, v(x,t), \dfrac{\partial v}{\partial x}(x,t), \dfrac{\partial^2 v}{\partial x^2}(x,t)\Big) \le \delta(t), \end{cases}$$

for all $(x,t) \in S \cup \delta_2 S$,

(3) the strict inequalities

$$\begin{cases} \rho'(t) & > & \omega\big(t, \rho(t)\big) + \delta(t), \\ \bar{\rho}'(t) & > & \bar{\omega}\big(t, \bar{\rho}(t)\big) + \bar{\delta}(t), \end{cases}$$

hold for all $t \in (0,T]$,

(4) $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies assumption (3.1) and the conditions

$$f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)-f\Big(x,t,v(x,t)-\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)$$
$$\leq \omega(t,\rho(t)),$$
$$f\Big(x,t,v(x,t)+\bar{\rho}(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)-f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)$$
$$\leq \bar{\omega}(t,\bar{\rho}(t)),$$

for all $(x,t)\in S\cup\delta_2 S$.

Then,

$$-\bar{\rho}(t)<v(x,t)-u(x,t)<\rho(t), \tag{4.4}$$

for all $(x,t)\in\bar{S}$.

Proof. To prove the right-hand side inequality in (4.4), we apply Theorem 3.1 with u replaced by $v-\rho$ and v replaced by u, which satisfy condition (*1*) of Theorem (3.1). To verify whether they satisfy condition (*2*) of Theorem 3.1, we take $(x,t)\in S\cup\delta_2 S$ (arbitrary) and assume that

$$v(x,t)-\rho(t)=u(x,t),\quad \frac{\partial v}{\partial x}(x,t)=\frac{\partial u}{\partial x}(x,t),\quad \frac{\partial^2 v}{\partial x^2}(x,t)\leq\frac{\partial^2 u}{\partial x^2}(x,t).$$

Then, from conditions (*2*), (3.1), (*4*), and (*3*), respectively, we get

$$\begin{aligned}\frac{\partial v}{\partial t}(x,t)-\frac{\partial u}{\partial t}(x,t) &\leq f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)+\delta(t)\\ &- f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big)\\ &= f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)+\delta(t)\\ &- f\Big(x,t,v(x,t)-\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big)\end{aligned}$$

$$\begin{aligned}
&\le \; f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)+\delta(t)\\
&- \; f\Big(x,t,v(x,t)-\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)\\
&\le \; \omega(t,\rho(t))+\delta(t)<\rho'(t).
\end{aligned}$$

Thus, $\frac{\partial v}{\partial t}(x,t)-\rho'(t)<\frac{\partial u}{\partial t}(x,t)$ and Theorem 3.1 applies. Case (II) from the assertion of Theorem 3.1 means that there exists $(\bar{x},\bar{t})\in B$ such that $v(\bar{x},\bar{t})-\rho(\bar{t})\ge u(\bar{x},\bar{t})$, which contradicts (*1*) and shows that case (II) is not possible. Therefore, case (I) holds, which means that

$$v(x,t)-\rho(t)<u(x,t),\quad \text{for all } (x,t)\in S\cup\delta_2 S.$$

To prove the inequality $-\bar{\rho}(t)<v(x,t)-u(x,t)$, we apply Theorem 3.1 with u and v replaced by $v+\bar{\rho}$, which satisfy condition (*1*) of Theorem 3.1. To verify whether they satisfy condition (*2*) of Theorem 3.1, we take $(x,t)\in S\cup\delta_2 S$ (arbitrary) and assume that

$$u(x,t)=v(x,t)+\bar{\rho}(t),\quad \frac{\partial u}{\partial x}(x,t)=\frac{\partial v}{\partial x}(x,t),\quad \frac{\partial^2 u}{\partial x^2}(x,t)\le\frac{\partial^2 v}{\partial x^2}(x,t).$$

Then,

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) - \frac{\partial v}{\partial t}(x,t) &\leq f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) + \bar{\delta}(t) \\
&- f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&= f\Big(x,t,v(x,t)+\bar{\rho}(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) + \bar{\delta}(t) \\
&- f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)
\end{aligned}$$

$$\begin{aligned}
&\leq f\Big(x,t,v(x,t)+\bar{\rho}(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) + \bar{\delta}(t) \\
&- f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&\leq \bar{\omega}(t,\bar{\rho}(t)) + \bar{\delta}(t) < \bar{\rho}'(t).
\end{aligned}$$

Therefore, $\frac{\partial u}{\partial t}(x,t) < \frac{\partial v}{\partial t}(x,t) + \bar{\rho}'(t)$ and Theorem 3.1 applies. Case (II) from the assertion of Theorem 3.1 means that there exists $(\bar{x},\bar{t}) \in B$ such that $u(\bar{x},\bar{t}) \geq v(\bar{x},\bar{t}) + \bar{\rho}(\bar{t})$; that is, $-\bar{\rho}(\bar{t}) \geq v(\bar{x},\bar{t}) - u(\bar{x},\bar{t})$, which contradicts (*1*) and shows that (II) is impossible. Therefore, case (I) is true, which shows that $u(x,t) < v(x,t)+\bar{\rho}(t)$, for all $(x,t) \in S \cup \delta_2 S$. In summary, $v(x,t) - \rho(t) < u(x,t) < v(x,t) + \bar{\rho}(t)$, for all $(x,t) \in S \cup \delta_2 S$, and the proof is finished. □

Example. Note that if, for $n = 1$, we take $f(x,t,r,p,q) = q + p + r - x - t$, $u(x,t) = x + t$, $v(x,t) = x$, $\rho(t) = \exp(2t)$, $\bar{\rho}(t) = \exp(3t)$, $\delta(t) = t + 1$, $\bar{\delta}(t) = t + 2$, and $\omega(t,y) = \bar{\omega}(t,y) = y$, then all assumptions of Theorem 4.3 are satisfied. To verify condition (*1*), note that

$$-\bar{\rho}(t) = -\exp(3t) < v(x,t) - u(x,t) = x - x - t = -t < \exp(2t) = \rho(t),$$

for all $(x,t) \in B$. To verify condition (*2*) of Theorem 4.3, note that

$$\frac{\partial u}{\partial t}(x,t) = 1 = \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial u}{\partial x}(x,t) + u(x,t) - x - t$$

and u satisfies the partial differential equation. Moreover,

$$-\bar{\delta}(t) = -t - 2 \le -1 - x + x + t =$$
$$\frac{\partial v}{\partial t}(x,t) - f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) \le t + 1 = \delta(t).$$

To verify condition (*3*) of Theorem 4.3, note that

$$\rho'(t) = 2\exp(2t) > \exp(2t) + t + 1 = \rho(t) + \delta(t) = \omega\big(t, \rho(t)\big) + \delta(t)$$

and

$$\bar{\rho}'(t) = 3\exp(3t) > \exp(3t) + t + 2 = \bar{\rho}(t) + \bar{\delta}(t) = \omega\big(t, \rho(t)\big) + \delta(t).$$

To verify condition (*4*) of Theorem 4.3, note that f satisfies assumption (3.1). Moreover,

$$f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x, t, v(x,t) - \rho(t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) =$$
$$v(x,t) + \rho(t) - v(x,t) = \rho(t) = \omega\big(t, \rho(t)\big)$$

and

$$f\Big(x, t, v(x,t) + \bar{\rho}(t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) =$$
$$v(x,t) + \bar{\rho}(t) - v(x,t) = \bar{\rho}(t) = \omega\big(t, \bar{\rho}(t)\big)$$

confirming the result of the theorem.

Theorem 4.4. *Suppose* $u, v : \bar{S} \to \mathbb{R}$ *are continuous, have continuous partial derivatives* $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ *in* $S \cup \delta_2 S$*,* $\rho : [0,T] \to [0,\infty)$ *is continuous on* $[0,T]$ *and differentiable on* $(0,T]$*, and the functions* $\delta : [0,T] \to \mathbb{R}$*,* $\omega : (0,T] \times [0,\infty) \to \mathbb{R}$ *satisfy the* *conditions:*

(1) $|u(x,t) - v(x,t)| < \rho(t)$*, for all* $(x,t) \in B$*,*

(2)

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = f\Big(x, t, u(x,t), \dfrac{\partial u}{\partial x}(x,t), \dfrac{\partial^2 u}{\partial x^2}(x,t)\Big), \\ \left| \dfrac{\partial v}{\partial t}(x,t) - f\Big(x, t, v(x,t), \dfrac{\partial v}{\partial x}(x,t), \dfrac{\partial^2 v}{\partial x^2}(x,t)\Big) \right| \le \delta(t), \end{cases}$$

for all $(x,t) \in S \cup \delta_2 S$*,*

(3) $\rho'(t) > \omega(t, \rho(t)) + \delta(t)$*, for all* $t \in (0,T]$*,*

(4) $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ *satisfies assumption (3.1) and*

$$f\Big(x, t, z + \rho(t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x, t, z, \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) \le \omega(t, \rho(t)),$$

for all $z \in \mathbb{R}$*.*

Then,

$$|u(x,t) - v(x,t)| < \rho(t),$$

for all $(x,t) \in S \cup \delta_2 S$*.*

Proof. We apply Theorem 4.3 with $\bar{\rho} = \rho$, $\bar{\delta} = \delta$, and $\bar{\omega} = \omega$. Then, assumptions (*1*), (*2*), (*3*) of Theorem 4.3 are satisfied. To verify assumption (*4*) of Theorem 4.3, we take $z = v(x,t) - \rho(t)$ in (*4*) and get the first inequality in (*4*) of Theorem 4.3. Taking $z = v(x,t)$ in (*4*), we get the second inequality in (*4*) of Theorem 4.3. Therefore, Theorem 4.3 applies and its assertion finishes the proof of Theorem 4.4. □

Example. Note that if, for $n = 1$, $f(x,t,r,p,q) = q + p + r - x - t$, $u(x,t) = x + t$, $v(x,t) = x$, $\rho(t) = \exp(2t)$, $\delta(t) = t + 1$, and $\omega(t,y) = y$, then all assumptions of Theorem 4.4 are satisfied. To verify condition (*1*), note that

$$|u(x,t) - v(x,t)| = |t| = t < \exp(2t) = \rho(t),$$

for all $(x,t) \in B$. To verify condition (*2*), note that

$$\frac{\partial u}{\partial t}(x,t) = 1 = \frac{\partial^2 u}{\partial x^2}(x,t) + \frac{\partial u}{\partial x}(x,t) + u(x,t) - x - t$$

and u satisfies the partial differential equation. Moreover,

$$\left| \frac{\partial v}{\partial t}(x,t) - f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) \right| = |t-1| \leq t + 1 = \delta(t).$$

To verify condition (*3*) of Theorem 4.4, note that

$$\rho'(t) = 2\exp(2t) > \exp(2t) + t + 1 = \rho(t) + \delta(t) = \omega\big(t, \rho(t)\big) + \delta(t).$$

To verify condition (*4*) of Theorem 4.4, note that f satisfies assumption (3.1) and that

$$\begin{aligned} f\Big(x, t, z + \rho(t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x, t, z, \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) &= \\ z + \rho(t) - z = \rho(t) = \omega\big(t, \rho(t)\big)&, \end{aligned}$$

confirming the validity of the theorem in this case.

In what follows, we prove a theorem that addresses both the question of uniqueness of solutions and the question of continuous dependence of the solution on initial and boundary values. The proof of this important theorem cumulates previous concepts and uses the results developed thus far.

Theorem 4.5. *Suppose $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies assumption (3.1) and the condition*

$$f(t, x, z, p, r) - f(t, x, \tilde{z}, p, r) \leq \ \omega(t, z - \tilde{z}), \tag{4.5}$$

for all $z \geq \tilde{z}$, where $\omega : (0, T] \times [0, \infty) \to \mathbb{R}$ is such that for all $\epsilon > 0$ there exist $\delta > 0$ and a function $\rho : [0, T] \to [0, \infty)$ continuous on $[0, T]$ and differentiable on $(0, T]$ such that for all $t \in (0, T]$ the inequalities

$$\begin{cases} \delta \leq \rho(t) & \leq \ \epsilon \\ \rho'(t) > \omega\big(t, \rho(t)\big) \end{cases} \tag{4.6}$$

are satisfied. Moreover, suppose that $\eta : B \to \mathbb{R}$ is continuous.

Then, the initial-boundary value problem (4.3) has at most one solution $u : \bar{S} \to \mathbb{R}$ such that it is continuous on $\bar{S}$ and has continuous partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$. Moreover u depends continuously on the initial and boundary values.

Proof. Let u and v be two solutions to (4.3). Then, u and v satisfy the assumptions of Theorem 4.4 with $\delta \equiv 0$ and ρ defined in (4.6) with $\epsilon \to 0$. Since $\lim_{\epsilon \to 0} \rho(t) = 0$, for each $t \in (0, T]$, from the assertion of Theorem (4.4), we get $v \equiv u$ and uniqueness of the solution is proved. Continuous dependence of u on the initial and boundary values is defined in the following way: for all $\epsilon > 0$ there exists $\delta > 0$ such that for all functions $v : \bar{S} \to \mathbb{R}$ continuous on $\bar{S}$ and with continuous partial derivatives $\frac{\partial}{\partial t}$,

$\dfrac{\partial}{\partial x}, \dfrac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$ if

$$\frac{\partial v}{\partial t}(x,t) = f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big),$$

for all $(x,t) \in S \cup \delta_2 S$, and if

$$|u(x,t) - v(x,t)| < \delta,$$

for all $(x,t) \in B$, then

$$|u(x,t) - v(x,t)| < \epsilon,$$

for all $(x,t) \in \bar{S}$.

Let $\epsilon > 0$ be arbitrary and δ chosen from (4.6). We now take $v : \bar{S} \to \mathbb{R}$ continuous on $\bar{S}$ and with partial derivatives $\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}, \dfrac{\partial^2}{\partial x^2}$ continuous on $S \cup \delta_2 S$ and such that the sentence before the above implication is satisfied. Below, we verify point by point that the assumptions of Theorem 4.4 are satisfied with ρ defined in (4.6):

(1) $|u(x,t) - v(x,t)| < \delta \leq \rho(t)$, for all $(x,t) \in B$,

(2) is satisfied as both u and v satisfy the partial differential equation on $S \cup \delta_2 S$, here $\delta(t) \equiv 0$,

(3) is satisfied as, from (4.6), $\rho'(t) > \omega(t, \rho(t))$. Since $\delta(t) \equiv 0$, we get $\rho'(t) > \omega(t, \rho(t)) + \delta(t)$,

(4) from (4.5), we get

$$\begin{aligned}&f\Big(x, t, z + \rho(t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x, t, z, \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ &\leq \omega(t, z + \rho(t) - z) = \omega(t, \rho(t)),\end{aligned}$$

for all $z \in \mathbb{R}$.

Therefore, Theorem 4.4 applies, and from its assertion and from (4.6), we get

$$|u(x,t) - v(x,t)| < \rho(t) \leq \epsilon, \quad \text{for all } (x,t) \in S \cup \delta_2 S,$$

which proves the continuous dependence of u on the initial and boundary values (as in the inequality with δ it may be defined as $0 < \delta < \epsilon$ and $|u(x,t) - v(x,t)| < \epsilon$ is satisfied for $(x,t) \in \bar{S}$). $\square$

CHAPTER 5

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS: FURTHER ANALYSIS

In this chapter, we build on the results and proof techniques of the previous chapters and formulate and prove theorems on the well-posedness of classes of partial differential equations under a range of natural conditions.

The first result characterizes when one can expect to have uniqueness and continuous dependence on initial data given certain regularity conditions and assumptions on the right-hand side function f.

Theorem 5.1. *Suppose $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies assumption (3.1) and the condition*

$$f(x,t,z,p,r) - f(x,t,\tilde{z},p,r) \leq \omega(t, z - \tilde{z}), \tag{5.1}$$

for all $z \geq \tilde{z}$, where $\omega : (0,T] \times [0,\infty) \to \mathbb{R}$ is such that for all $\epsilon > 0$ there exist $\delta > 0$ and a function $\rho : [0,T] \to [0,\infty)$ continuous on $[0,T]$ and differentiable on $(0,T]$ such that for all $t \in (0,T]$ the inequalities

$$\begin{cases} \delta \leq \rho(t) \leq \epsilon \\ \rho'(t) > \omega\big(t, \rho(t)\big) + \delta \end{cases} \tag{5.2}$$

are satisfied. Moreover, suppose that $\eta : B \to \mathbb{R}$ *is continuous.*

Then, the initial-boundary value problem (4.3) has at most one solution $u : \bar{S} \to \mathbb{R}$ *such that it is continuous on* $\bar{S}$ *and has partial derivatives* $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ *that are continuous in* $S \cup \delta_2 S$*. Moreover,* u *depends continuously on the initial and boundary values and the right-hand side function* f*; that is, for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all functions* $v : \bar{S} \to \mathbb{R}$ *continuous on* $\bar{S}$ *and with partial derivatives* $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}$, $\frac{\partial^2}{\partial x^2}$ *continuous in* $S \cup \delta_2 S$ *if*

$$\left| \frac{\partial v}{\partial t}(x,t) - f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \right| < \delta,$$

for all $(x,t) \in S \cup \delta_2 S$*, and if*

$$|u(x,t) - v(x,t)| < \delta,$$

for all $(x,t) \in B$*, then*

$$|u(x,t) - v(x,t)| < \epsilon,$$

for all $(x,t) \in \bar{S}$*.*

Proof. Uniqueness comes from the same arguments as in the proof of Theorem 4.5. To prove that u depends continuously on the initial and boundary values and the right-hand side function f, we apply Theorem 4.4. Condition (*1*) of Theorem 4.4 is satisfied as it is shown in the proof of Theorem 4.5. Condition (*2*) of Theorem 4.4 is satisfied with $\delta(t) \equiv \delta$. Condition (*3*) is satisfied because of (5.2). To verify condition (*4*), we use (5.1) and, for $z \in \mathbb{R}$, we get

$$f\Big(x,t,z+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x,t,z,\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \leq \omega(t,\rho(t)).$$

Therefore, by the assertion of Theorem 4.4, we get $|v(x,t) - u(x,t)| < \rho(t)$, for $(x,t) \in S \cup \delta_2 S$, and from (5.2), we get $|v(x,t) - u(x,t)| < \epsilon$, which is also satisfied for $(x,t) \in B$ and δ may be defined as $0 < \delta \leq \epsilon$. □

We now apply Theorem 3.1 to prove a series of results on the solutions of a class of partial differential equations and their relation to approximate solutions defined in a weak sense.

Theorem 5.2. *Suppose the functions $f, \omega, \bar{\omega} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfy assumption (3.1), the functions $u, v, \rho, \bar{\rho} : \bar{S} \to \mathbb{R}$ are continuous and have continuous partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$, $\delta, \bar{\delta} : S \cup \delta_2 S \to \mathbb{R}$. Suppose also that the following four conditions are satisfied:*

(1) the strict inequalities

$$-\bar{\rho}(x,t) < v(x,t) - u(x,t) < \rho(x,t),$$

hold for all $(x,t) \in B$,

(2) u is a solution of the following partial differential equation and v satisfies the weak inequalities

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = f\Big(x, t, u(x,t), \dfrac{\partial u}{\partial x}(x,t), \dfrac{\partial^2 u}{\partial x^2}(x,t)\Big), \\ -\bar{\delta}(x,t) \leq \dfrac{\partial v}{\partial t}(x,t) - f\Big(x, t, v(x,t), \dfrac{\partial v}{\partial x}(x,t), \dfrac{\partial^2 v}{\partial x^2}(x,t)\Big) \leq \delta(x,t), \end{cases}$$

for all $(x,t) \in S \cup \delta_2 S$, *(3) ρ and $\bar\rho$ satisfy the strict inequalities*

$$\begin{cases} \dfrac{\partial\rho}{\partial t}(x,t) > \delta(x,t) + \omega\Big(x,t,\rho(x,t),\dfrac{\partial\rho}{\partial x}(x,t),\dfrac{\partial^2\rho}{\partial x^2}(x,t)\Big) \\ \dfrac{\partial\bar\rho}{\partial t}(x,t) > \bar\delta(x,t) + \bar\omega\Big(x,t,\bar\rho(x,t),\dfrac{\partial\bar\rho}{\partial x}(x,t),\dfrac{\partial^2\bar\rho}{\partial x^2}(x,t)\Big), \end{cases}$$

for all $(x,t) \in S \cup \delta_2 S$,

(4) v satisfies the two weak inequalities

$$\begin{cases} f\Big(x,t,v(x,t),\dfrac{\partial v}{\partial x}(x,t),\dfrac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ -f\Big(x,t,v(x,t)-\rho(x,t),\dfrac{\partial v}{\partial x}(x,t)-\dfrac{\partial\rho}{\partial x}(x,t),\dfrac{\partial^2 v}{\partial x^2}(x,t)-\dfrac{\partial^2\rho}{\partial x^2}(x,t)\Big) \\ \leq \omega\Big(x,t,\rho(x,t),\dfrac{\partial\rho}{\partial x}(x,t),\dfrac{\partial^2\rho}{\partial x^2}(x,t)\Big), \\ f\Big(x,t,v(x,t)+\bar\rho(x,t),\dfrac{\partial v}{\partial x}(x,t)+\dfrac{\partial\bar\rho}{\partial x}(x,t),\dfrac{\partial^2 v}{\partial x^2}(x,t)+\dfrac{\partial^2\bar\rho}{\partial x^2}(x,t)\Big) \\ -f\Big(x,t,v(x,t),\dfrac{\partial v}{\partial x}(x,t),\dfrac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ \leq \bar\omega\Big(x,t,\bar\rho(x,t),\dfrac{\partial\bar\rho}{\partial x}(x,t),\dfrac{\partial^2\bar\rho}{\partial x^2}(x,t)\Big), \end{cases}$$

for all $(x,t) \in S \cup \delta_2 S$.

Then,

$$-\bar\rho(x,t) < v(x,t) - u(x,t) < \rho(x,t), \tag{5.3}$$

for all $(x,t) \in S \cup \delta_2 S$.

Proof. We first prove the right-hand side inequality in (5.3). For this, we apply Theorem 3.1 with u replaced by $v - \rho$ and v replaced by u. Both functions satisfy condition (*1*) of Theorem 3.1. We now verify whether they satisfy condition (*2*) of Theorem 3.1. Let $(x,t) \in S \cup \delta_2 S$ be arbitrary such that

$$v(x,t) - \rho(x,t) = u(x,t), \quad \frac{\partial v}{\partial x}(x,t) - \frac{\partial \rho}{\partial x}(x,t) = \frac{\partial u}{\partial x}(x,t),$$
$$\frac{\partial^2 v}{\partial x^2}(x,t) - \frac{\partial^2 \rho}{\partial x^2}(x,t) \leq \frac{\partial^2 u}{\partial x^2}(x,t).$$

We need to show that

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial \rho}{\partial t}(x,t) < \frac{\partial u}{\partial t}(x,t).$$

From (*2*), (3.1), (*4*), and (*3*), we get

$$\begin{aligned}
\frac{\partial v}{\partial t}(x,t) &\leq \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) + \delta(x,t) \\
&- f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\
&= f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) + \delta(x,t) \\
&- f\Big(x,t,v(x,t)-\rho(x,t),\frac{\partial v}{\partial x}(x,t)-\frac{\partial \rho}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\
&\leq f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) + \delta(x,t) \\
&- f\Big(x,t,v(x,t)-\rho(x,t),\frac{\partial v}{\partial x}(x,t)-\frac{\partial \rho}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)-\frac{\partial^2 \rho}{\partial x^2}(x,t)\Big) \\
&\leq \omega\Big(x,t,\rho(x,t),\frac{\partial \rho}{\partial x}(x,t),\frac{\partial^2 \rho}{\partial x^2}(x,t)\Big) + \delta(x,t) < \frac{\partial \rho}{\partial t}(x,t).
\end{aligned}$$

Therefore, condition (*2*) of Theorem 3.1 is satisfied and from its assertion $v - \rho$ and u satisfy only one of the cases $(I), (II)$. Case (II) means that there is $(\bar{x},\bar{t}) \in B$ such that $v(\bar{x},\bar{t}) - \rho(\bar{x},\bar{t}) \geq u(\bar{x},\bar{t})$ but it contradicts condition (*1*) of Theorem 5.2. Therefore, only case (I) is valid, which implies that $v(x,t) - \rho(x,t) < u(x,t)$, for all $(x,t) \in S \cup \delta_2 S$, and finishes the proof of the right-hand side inequality in (5.3). We now prove the inequality $-\bar{\rho}(x,t) < v(x,t) - u(x,t)$ and apply Theorem 3.1 with u and v replaced by $v + \bar{\rho}$, which satisfy condition *(1)* of Theorem 3.1. To verify whether they also satisfy condition *(2)*, we take an arbitrary $(x,t) \in S \cup \delta_2 S$ and assume that

$$\begin{aligned}
&u(x,t) = v(x,t) + \bar{\rho}(x,t), \quad \frac{\partial u}{\partial x}(x,t) = \frac{\partial v}{\partial x}(x,t) + \frac{\partial \bar{\rho}}{\partial x}(x,t),\\
&\frac{\partial^2 u}{\partial x^2}(x,t) \leq \frac{\partial^2 v}{\partial x^2}(x,t) + \frac{\partial^2 \bar{\rho}}{\partial x^2}(x,t).
\end{aligned}$$

Then, from *(2)*, (3.1), *(4)*, and *(3)*, we get

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &\leq \frac{\partial v}{\partial t}(x,t) + f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) + \bar{\delta}(x,t)\\
&- f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)\\
&= f\Big(x,t,v(x,t)+\bar{\rho}(x,t),\frac{\partial v}{\partial x}(x,t)+\frac{\partial \bar{\rho}}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big)\\
&- f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) + \bar{\delta}(x,t)\\
&\leq f\Big(x,t,v(x,t)+\bar{\rho}(x,t),\frac{\partial v}{\partial x}(x,t)+\frac{\partial \bar{\rho}}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)+\frac{\partial^2 \bar{\rho}}{\partial x^2}(x,t)\Big)\\
&- f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) + \bar{\delta}(x,t)\\
&\leq \bar{\omega}\Big(x,t,\bar{\rho}(x,t),\frac{\partial \bar{\rho}}{\partial x}(x,t),\frac{\partial^2 \bar{\rho}}{\partial x^2}(x,t)\Big) + \bar{\delta}(x,t) < \frac{\partial \bar{\rho}}{\partial t}(x,t).
\end{aligned}$$

Therefore, $-\dfrac{\partial \bar{\rho}}{\partial t}(x,t) < \dfrac{\partial v}{\partial t}(x,t) - \dfrac{\partial u}{\partial t}(x,t)$ and Theorem 3.1 applies. From the assertion of Theorem 3.1, case *(II)* means that $-\bar{\rho}(\bar{x},\bar{t}) \geq v(\bar{x},\bar{t}) - u(\bar{x},\bar{t})$, for a certain $(\bar{x},\bar{t}) \in B$, but this contradicts condition *(1)* of Theorem 5.2. Therefore, case *(I)* is true; that is, $-\bar{\rho}(x,t) < v(x,t) - u(x,t)$, for all $(x,t) \in S \cup \delta_2 S$, and the left-hand side inequality in (5.3) is proved. $\square$

In what follows we prove a result on upper and lower bounds to solutions of partial differential inequalities, useful in the context of proving stability properties or continuous dependence to initial data for solutions to classes of partial differential equations.

Theorem 5.3. *Suppose* $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ *satisfies assumption (3.1),* $v : \bar{S} \to \mathbb{R}$ *are continuous and has continuous partial derivatives* $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$ *in* $S \cup \delta_2 S$*. Let*

$$\bar{A}(v) = \inf\{A : A > v(x,t), \quad \text{for all } (x,t) \in B\},$$
$$\underline{A}(v) = \sup\{A : A < v(x,t), \quad \text{for all } (x,t) \in B\}.$$

Then, the following two properties hold:

(i) if

$$\frac{\partial v}{\partial t}(x,t) \le f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big),$$

for all $(x,t) \in S \cup \delta_2 S$*, and there exists* $\epsilon > 0$ *such that*

$$\forall z \quad \bar{A}(v) < z < \bar{A}(v) + \epsilon \implies f(x,t,z,0,0) \le 0$$

then $v \le \bar{A}(v)$*,*

(ii) if

$$\frac{\partial v}{\partial t}(x,t) \ge f\Big(x, t, v(x,t), \frac{\partial v}{\partial x}(x,t), \frac{\partial^2 v}{\partial x^2}(x,t)\Big),$$

for all $(x,t) \in S \cup \delta_2 S$*, and there exists* $\epsilon > 0$ *such that*

$$\forall z \quad \underline{A}(v) - \epsilon < z < \underline{A}(v) \implies f(x,t,z,0,0) \geq 0$$

then $v \geq \underline{A}(v)$.

Proof. (i) Suppose

$$v(x,t) < A, \quad \text{for all } (x,t) \in B. \tag{5.4}$$

We want to show that

$$v(x,t) \leq A, \quad \text{for all } (x,t) \in S \cup \delta_2 S. \tag{5.5}$$

We apply Theorem 3.1 with u replaced by v and v replaced by $A + \delta t$, where $\delta > 0$ and A and δ are chosen in such a way that

$$A + \delta T < \bar{A}(v) + \epsilon,$$

where $\epsilon > 0$ is as in (i). Note that $A - \bar{A}(v) + \delta T$ can be arbitrarily small. The functions $v(x,t)$ and $A + \delta t$ satisfy condition *(1)* of Theorem 3.1. To verify condition *(2)*, suppose that $(x,t) \in S \cup \delta_2 S$ and

$$v(x,t) = A + \delta t, \quad \frac{\partial v}{\partial x}(x,t) = 0, \quad \frac{\partial^2 v}{\partial x^2}(x,t) \leq 0.$$

Then,

$$\begin{aligned}
\frac{\partial v}{\partial t}(x,t) &\leq f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&= f\Big(x,t,v(x,t),0,\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&\leq f(x,t,v(x,t),0,0) = f(x,t,A+\delta t,0,0) \leq 0 < \delta = \frac{d}{dt}\big(A + \delta t\big).
\end{aligned}$$

Therefore, by Theorem 3.1, there are two cases (I) and (II) and only one of them is true. If (II) is true, then there is $(\bar{x}, \bar{t}) \in B$ such that $v(\bar{x}, \bar{t}) \geq A + \delta\bar{t}$, which contradicts assumption (5.4). Therefore, (II) is not satisfied and (I) is true, which means that

$$v(x,t) < A + \delta t, \quad \text{for all } (x,t) \in S \cup \delta_2 S.$$

Taking $\delta \to 0$, gives $v(x,t) \leq A$ and (5.5) is proved. Since A can be arbitrarily close to $\bar{A}(v)$, we get

$$v(x,t) \leq \bar{A}(v), \quad \text{for all } (x,t) \in S \cup \delta_2 S,$$

and (i) is proved. The proof of (ii) is similar. □

In the subsequent parts of this chapter, we will be making use of the following definition:

Definition 5.1. Let $(\bar{x}, \bar{t}) \in \delta_1 S$. Then,

$$\frac{\partial u}{\partial n_a}(\bar{x}, \bar{t}) = -\lim_{k \to \infty} \frac{u(x_k, \bar{t}) - u(\bar{x}, \bar{t})}{\|x_k - \bar{x}\|}$$

is an outer normal derivative, where $(x_k, \bar{t}) \in S \cup \delta_2 S$, $\lim\limits_{k\to\infty} x_k = \bar{x}$, and $\|\cdot\|$ is the Euclidean norm. The minus sign reflects the convention that we work with outer normal derivatives.

In the next part of the chapter, we will use the following notation:

$$\begin{aligned} \delta_1^+ S &= \{(x,t) \in \delta_1 S : \text{ there exists } i \in \{1, 2, ..., n\} \text{ such that } x_i = b_i\}, \\ \delta_1^- S &= \{(x,t) \in \delta_1 S : \text{ there exists } i \in \{1, 2, ..., n\} \text{ such that } x_i = -b_i\}. \end{aligned}$$

Suppose $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies the assumption (3.1) and that $\eta : B \setminus \delta_1^+ S \to \mathbb{R}$ and $\xi : \delta_1^+ S \times \mathbb{R} \to \mathbb{R}$ are both continuous functions. If $u : \bar{S} \to \mathbb{R}$ is continuous, has continuous partial derivatives $\dfrac{\partial u}{\partial t}, \dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}$ in $S \cup \delta_2 S$, and satisfies the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = f\Big(x, t, u(x,t), \dfrac{\partial u}{\partial x}(x,t), \dfrac{\partial^2 u}{\partial x^2}(x,t)\Big), \quad (x,t) \in S \cup \delta_2 S, \\ u(x,t) = \eta(x,t), \quad (x,t) \in B \setminus \delta_1^+ S, \\ \dfrac{\partial u}{\partial n_a}(x,t) + \xi\big(x,t,u(x,t)\big) = 0, \quad (x,t) \in \delta_1^+ S, \end{cases} \tag{5.6}$$

then u is called a solution of the initial-boundary value problem (5.6).

Using the above formulations and definitions, we are able to prove a result characterizing the relationship between two solutions that satisfy natural partial differential inequalities and boundary inequalities at their modified boundaries. The assertion of the theorem holds for the entire domain of interest.

Theorem 5.4. *Suppose that the function $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$ satisfies assumption (3.1) and that the functions $u, v : \bar{S} \to \mathbb{R}$ are continuous, have continuous partial derivatives $\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}, \dfrac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$, and that their outer normal derivatives at points in the boundary set $\delta_1^+ S$ exist. Suppose also that the function $\xi : \delta_1^+ S \times \mathbb{R} \to \mathbb{R}$ is continuous, and that the following conditions in the form of strict inequalities on the functions u and v are satisfied:*

(1) u and v satisfy the following strict inequalities at the boundaries

$$\begin{cases} u(x,t) < v(x,t), \quad \textit{for all } (x,t) \in B \setminus \delta_1^+ S \\ \dfrac{\partial u}{\partial n_a}(x,t) + \xi\big(x,t,u(x,t)\big) < \dfrac{\partial v}{\partial n_a}(x,t) + \xi\big(x,t,v(x,t)\big), \quad \textit{for all } (x,t) \in \delta_1^+ S, \end{cases}$$

(2) u and v satisfy the following strict differential inequality

$$\begin{aligned}&\frac{\partial u}{\partial t}(x,t) - f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big)\\ &< \frac{\partial v}{\partial t}(x,t) - f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big),\end{aligned}$$

for all $(x,t) \in S \cup \delta_2 S$.

Then, $u(x,t) < v(x,t)$ for all $(x,t) \in S \cup \delta_2 S \cup \delta_1^+ S$.

Proof. The functions u and v satisfy assumptions (*1*) and (*2*) of Theorem 3.1, which implies that there are two cases (I) and (II) and only one of them is true. If (II) is true, then $u(\bar{x},\bar{t}) = v(\bar{x},\bar{t})$, for some $(\bar{x},\bar{t}) \in B$. If $(\bar{x},\bar{t}) \in B \setminus \delta_1^+ S$, then we get a contradiction as, from (*1*), $u(\bar{x},\bar{t}) < v(\bar{x},\bar{t})$. Therefore, $(\bar{x},\bar{t}) \in \delta_1^+ S$ and $u(\bar{x},\bar{t}) = v(\bar{x},\bar{t})$. Also, $u(x,\bar{t}) < v(x,\bar{t})$, for $(x,\bar{t}) \in S \cup \delta_2 S$. Therefore,

$$\begin{aligned}\frac{\partial}{\partial n_a}\big(v(\bar{x},\bar{t}) - u(\bar{x},\bar{t})\big) &= -\lim_{k\to\infty} \frac{v(x_k,\bar{t}) - v(\bar{x},\bar{t}) - u(x_k,\bar{t}) + u(\bar{x},\bar{t})}{\|x_k - \bar{x}\|}\\ &= -\lim_{k\to\infty} \frac{v(x_k,\bar{t}) - u(x_k,\bar{t})}{\|x_k - \bar{x}\|} \le 0,\end{aligned}$$

where $(x_k,\bar{t}) \in S \cup \delta_2 S$ and $\lim\limits_{k\to\infty} x_k = \bar{x}$. Thus, we find the following relation between the outward normal derivatives of u and v

$$\frac{\partial v}{\partial n_a}(\bar{x},\bar{t}) \le \frac{\partial u}{\partial n_a}(\bar{x},\bar{t}). \tag{5.7}$$

Since $\xi\big(\bar{x},\bar{t},u(\bar{x},\bar{t})\big) = \xi\big(\bar{x},\bar{t},v(\bar{x},\bar{t})\big)$, from assumption (*1*), $\dfrac{\partial u}{\partial n_a}(\bar{x},\bar{t}) < \dfrac{\partial v}{\partial n_a}(\bar{x},\bar{t})$, which contradicts (5.7). Therefore, case (I) is true and, by Theorem (3.1), we get

$$u(x,t) < v(x,t), \quad \text{for all } (x,t) \in S \cup \delta_2 S. \tag{5.8}$$

To finish the proof, all that remains to be shown is that the inequality holds over a larger domain; that is,

$$u(x,t) < v(x,t), \quad \text{for all } (x,t) \in \delta_1^+ S.$$

Suppose, for the sake of contradiction, that there in fact exists some $(\bar{x}, \bar{t}) \in \delta_1^+ S$ such that

$$u(\bar{x}, \bar{t}) \geq v(\bar{x}, \bar{t}). \tag{5.9}$$

If $u(\bar{x}, \bar{t}) > v(\bar{x}, \bar{t})$, then, since u and v are continuous, we would have a contradiction with (5.8). Therefore, the inequality $u(\bar{x}, \bar{t}) > v(\bar{x}, \bar{t})$ cannot hold, and from (5.9) we get that $u(\bar{x}, \bar{t}) = v(\bar{x}, \bar{t})$, which is shown to be impossible. Therefore, we conclude with the following inequality

$$u(x,t) < v(x,t), \quad \text{for all } (x,t) \in S \cup \delta_2 S \cup \delta_1^+ S,$$

and the proof is finished. $\square$

Having developed the necessary proof techniques and collection of results proved so far in the previous parts of this chapter and in earlier chapters, we conclude with a theorem that classifies the inequality between two different solutions of partial differential inequalities and boundary inequalities, providing the underlying core to understanding the stability and uniqueness properties relevant to a general class of partial differential equations.

Theorem 5.5. *Suppose $u, v : \bar{S} \to \mathbb{R}$ are continuous, have continuous partial derivatives $\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial x}, \dfrac{\partial^2}{\partial x^2}$ in $S \cup \delta_2 S$, and that their outer normal derivatives at points in $\delta_1^+ S$*

exist. Suppose also that the function $\xi : \delta_1^+ S \times \mathbb{R} \to \mathbb{R}$ *is continuous and that it satisfies the condition*

$$\xi(x,t,z) < \xi(x,t,\bar{z}),$$

for $z < \bar{z}$ *and* $(x,t) \in \delta_1^+ S$. *Suppose also that the function*

$$f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n,n} \to \mathbb{R}$$

satisfies assumption (3.1) and the inequality

$$f\Big(x,t,v(x,t)+z,\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) - \quad f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\ \leq \omega(t,z), \tag{5.10}$$

for all $z > 0$ *and* $(x,t) \in S \cup \delta_2 S$. *Here, the function* $\omega : (0,T] \times [0,\infty) \to \mathbb{R}$ *is defined such a way that for all* $\epsilon > 0$ *there exists* $\delta > 0$ *and a function* $\rho : [0,T] \to [0,\infty)$, *which is both continuous in* $[0,T]$ *and differentiable in* $(0,T]$, *and satisfies the both of the inequalities*

$$\begin{cases} \delta \leq \rho(t) \leq \epsilon, \\ \rho'(t) > \omega(t,\rho(t)), \end{cases} \tag{5.11}$$

for all $t \in (0,T]$. *Moreover, suppose that the following conditions are satisfied*

(1) u *and* v *satisfy the following inequalities at the boundaries*

$$\begin{cases} u(x,t) \leq v(x,t), \quad \textit{for all } (x,t) \in B \setminus \delta_1^+ S, \\ \dfrac{\partial u}{\partial n_a}(x,t) + \xi\big(x,t,u(x,t)\big) \leq \dfrac{\partial v}{\partial n_a}(x,t) + \xi\big(x,t,v(x,t)\big), \quad \textit{for all } (x,t) \in \delta_1^+ S, \end{cases}$$

(2) u and v satisfy the following differential inequalities

$$\frac{\partial u}{\partial t}(x,t) - f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big)$$
$$\leq \frac{\partial v}{\partial t}(x,t) - f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big),$$

for all $(x,t) \in S \cup \delta_2 S$.

Then, $u(x,t) \leq v(x,t)$, *for all* $(x,t) \in S \cup \delta_2 S \cup \delta_1^+ S$.

Proof. For an arbitrary $\epsilon > 0$, we take $\delta > 0$ and $\rho : [0,T] \to [0,\infty)$ as in (5.11). The goal is to show that:

$$u(x,t) < v(x,t) + \rho(t),$$

for all $(x,t) \in S \cup \delta_2 S \cup \delta_1^+ S$. We apply Theorem 3.1 with u and $v+\rho$, which satisfy condition (*1*) of Theorem 3.1. We now verify condition (*2*) of Theorem 3.1. Let $(x,t) \in S \cup \delta_2 S$ be such that

$$u(x,t) = v(x,t) + \rho(t), \quad \frac{\partial u}{\partial x}(x,t) = \frac{\partial v}{\partial x}(x,t), \quad \frac{\partial^2 u}{\partial x^2}(x,t) \leq \frac{\partial^2 v}{\partial x^2}(x,t).$$

Then, from condition (*2*) of Theorem 5.5, assumption (3.1), and (5.10), we get

$$0 \leq \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,u(x,t),\frac{\partial u}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big)$$
$$-f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big)$$

$$
\begin{aligned}
&= \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 u}{\partial x^2}(x,t)\Big) \\
&-f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&+f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) - f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&\le \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + f\Big(x,t,v(x,t)+\rho(t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&-f\Big(x,t,v(x,t),\frac{\partial v}{\partial x}(x,t),\frac{\partial^2 v}{\partial x^2}(x,t)\Big) \\
&\le \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + \omega\big(t,\rho(t)\big) < \frac{\partial v}{\partial t}(x,t) - \frac{\partial u}{\partial t}(x,t) + \rho'(t) \\
&= \frac{\partial}{\partial t}\big(v(x,t)+\rho(t)\big) - \frac{\partial u}{\partial t}(x,t)
\end{aligned}
$$

and condition (*2*) of Theorem 3.1 is proved. From the assertion of Theorem 3.1, there are two cases, (I) and (II), and only one is true. If (II) is true, then

$$u(\bar{x},\bar{t}) = v(\bar{x},\bar{t}) + \rho(\bar{t}),$$

for a certain $(\bar{x},\bar{t}) \in B$. Since $\rho(\bar{t}) > 0$, $u(\bar{x},\bar{t}) > v(\bar{x},\bar{t})$. From condition (*1*) (first inequality), we conclude that $(\bar{x},\bar{t}) \in \delta_1^+ S$. Then, from the second inequality in condition (*1*) and the assumption for ξ, we get

$$
\begin{aligned}
\frac{\partial u}{\partial n_a}(\bar{x},\bar{t}) + \xi\big(\bar{x},\bar{t},u(\bar{x},\bar{t})\big) &\le \frac{\partial v}{\partial n_a}(\bar{x},\bar{t}) + \xi\big(\bar{x},\bar{t},v(\bar{x},\bar{t})\big) \\
&< \frac{\partial}{\partial n_a}\big(v(\bar{x},\bar{t})+\rho(\bar{t})\big) + \xi\big(\bar{x},\bar{t},v(\bar{x},\bar{t})+\rho(\bar{t})\big),
\end{aligned}
\tag{5.12}
$$

where the latter inequality comes from the fact that

$$\begin{aligned}\frac{\partial}{\partial n_a}\big(v(\bar{x},\bar{t})+\rho(\bar{t})\big) &= -\lim_{k\to\infty}\frac{v(x_k,\bar{t})+\rho(\bar{t})-v(\bar{x},\bar{t})-\rho(\bar{t})}{\|x_k-\bar{x}\|}\\ &= -\lim_{k\to\infty}\frac{v(x_k,\bar{t})-v(\bar{x},\bar{t})}{\|x_k-\bar{x}\|}=\frac{\partial v}{\partial n_a}(\bar{x},\bar{t}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\frac{\partial}{\partial n_a}\Big(v(\bar{x},\bar{t})+\rho(\bar{t})-u(\bar{x},\bar{t})\Big)\\ &=-\lim_{k\to\infty}\frac{v(x_k,\bar{t})+\rho(\bar{t})-u(x_k,\bar{t})-v(\bar{x},\bar{t})-\rho(\bar{t})+u(\bar{x},\bar{t})}{\|x_k-\bar{x}\|}\le 0,\end{aligned}$$

where $(x_k,\bar{t})\in S\cup\delta_2 S$ are such that $\lim_{k\to\infty} x_k=\bar{x}$ and $u(x_k,\bar{t})<v(x_k,\bar{t})+\rho(\bar{t})$. Therefore,

$$\frac{\partial}{\partial n_a}\big(v(\bar{x},\bar{t})+\rho(\bar{t})\big)\le\frac{\partial}{\partial n_a}u(\bar{x},\bar{t})$$

and since $u(\bar{x},\bar{t})=v(\bar{x},\bar{t})+\rho(\bar{t})$, we get

$$\begin{aligned}\frac{\partial}{\partial n_a}\big(v(\bar{x},\bar{t})+\rho(\bar{t})\big)+\xi\big(\bar{x},\bar{t},u(\bar{x},\bar{t})\big) &\le \frac{\partial}{\partial n_a}u(\bar{x},\bar{t})+\xi\big(\bar{x},\bar{t},u(\bar{x},\bar{t})\big),\\ \frac{\partial}{\partial n_a}\big(v(\bar{x},\bar{t})+\rho(\bar{t})\big)+\xi\big(\bar{x},\bar{t},v(\bar{x},\bar{t})+\rho(\bar{t})\big) &\le \frac{\partial}{\partial n_a}u(\bar{x},\bar{t})+\xi\big(\bar{x},\bar{t},u(\bar{x},\bar{t})\big),\end{aligned}$$

which contradicts (5.12). Therefore, case (I) is true and we get

$$u(x,t)<v(x,t)+\rho(t), \tag{5.13}$$

for all $(x,t)\in S\cup\delta_2 S$. We want to show this inequality also for all $(x,t)\in\delta_1^+ S$. By contradiction, suppose that there exists $(\bar{x},\bar{t})\in\delta_1^+ S$ such that

$$u(\bar{x},\bar{t})\ge v(\bar{x},\bar{t})+\rho(\bar{t}).$$

The strong inequality $u(\bar{x}, \bar{t}) > v(\bar{x}, \bar{t}) + \rho(\bar{t})$ contradicts (5.13) so we get $u(\bar{x}, \bar{t}) = v(\bar{x}, \bar{t}) + \rho(\bar{t})$, which is the case considered above and as seen above this case implies the above contradiction. Therefore, $u(x, t) < v(x, t) + \rho(t)$, for all $(x, t) \in S \cup \delta_2 S \cup \delta_1^+ S$. Taking $\epsilon \to 0$ in the above inequality, we get $u(x, t) \leq v(x, t)$, for all $(x, t) \in S \cup \delta_2 S \cup \delta_1^+ S$, as $\lim_{\epsilon \to 0} \rho(t) = 0$, for all $t \in [0, T]$, and the proof is finished. $\square$

The above theorem as well as the collection of results proved in this thesis and the proof techniques developed and elaborated in order to do so have come to the core of understanding important theoretical questions raised as mathematical models get built and analyzed. The results often rely on and get constructed by considering auxiliary systems of partial differential inequalities, which can, as we have seen in this thesis, provide useful insight into the behavior of ordinary and partial differential equations.

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