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A continuous-time examination of the last buy problem

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A CONTINUOUS-TIME EXAMINATION OF THE LAST BUY PROBLEM

by

Nicholas William Leifker

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Business Administration
in the Graduate College of
The University of Iowa

July 2010

Thesis Supervisors: Professor Timothy J. Lowe
Professor Philip C. Jones

ABSTRACT

The last buy problem is a stochastic inventory management problem that occurs at the end of a product's life cycle. When production of a given product ceases, it may become necessary to shut down manufacture of all parts of the product. However, there will likely still be demand for spare parts of the product, due to part failure from the product still in use. To meet this demand, a one-time order of spare parts - a last buy - is made to satisfy the demand for all spare parts going forward. Thus, the last buy problem seeks to maximize a company's profits with respect to the number of spare parts manufactured.

Several different forms of the last buy problem exist, depending on the relationship between the manufacturer and the customer and the type of cost that occurs once the inventory has been depleted. In some cases, law or contract defines and limits the costs and revenues the manufacturer incurs due to the last buy order; in other cases, a manufacturer's own policies dictate the costs and revenues involved. As a result, we explore three main types of last buy problem, and the different methods used to solve for each.

In the last buy problem with incremental replenishment, individual parts demanded beyond the last buy are fabricated individually at significantly greater cost; as the total profit is concave with respect to the order amount, the optimal order amount can be found by analysis of the rate of change of the profit. The last buy problem with no replenishment occurs when there is no effective way to replenish part inventory beyond the last buy; as the total profit is not concave, an upper bound on the optimal order amount is determined, thus limiting the candidate solutions. Difficulties exist in calculating the optimal order amount in the last buy problem with batch replenishment, as the size of the replenishment batch is itself a last buy problem; we solve for a special case of the problem using renewal theory. We also examine the possibility of contract extensions in last buy problems, and their effect on the optimal order amount calculations.

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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I think the defining moment for this thesis occurred during the exploration phase of the project, when I was working with early formulations of the problem and coming to grips with the distributions involved. As I was giving my presentation on what would eventually become the Last Buy Problem with Incremental Replenishment, one of my advisors, Philip C. Jones, expressed the question as to why another batch wasn't made. Shortly after that, another of my advisors, Timothy Lowe, asked why replenishment was occurring at all - after all, wasn't this the "Last Buy" problem? As I thought about their questions, I realized that there were cases where each of the scenarios presented would occur - and that our examination of the Last Buy Problem would need to take all of these into account.

As such, I would like to start by thanking my current advisors, Philip C. Jones and Timothy Lowe, for their insight into the problem and for their guidance through the doctoral program. Having both of them as my advisors was a blessing, as both had beautiful and different methods of looking at a problem. That diversity of thought allowed for a far richer examination of the Last Buy Problem than what would have occurred otherwise.

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ABSTRACT

The last buy problem is a stochastic inventory management problem that occurs at the end of a product's life cycle. When production of a given product ceases, it may become necessary to shut down manufacture of all parts of the product. However, there will likely still be demand for spare parts of the product, due to part failure from the product still in use. To meet this demand, a one-time order of spare parts - a last buy - is made to satisfy the demand for all spare parts going forward. Thus, the last buy problem seeks to maximize a company's profits with respect to the number of spare parts manufactured.

Several different forms of the last buy problem exist, depending on the relationship between the manufacturer and the customer and the type of cost that occurs once the inventory has been depleted. In some cases, law or contract defines and limits the costs and revenues the manufacturer incurs due to the last buy order; in other cases, a manufacturer's own policies dictate the costs and revenues involved. As a result, we explore three main types of last buy problem, and the different methods used to solve for each.

In the last buy problem with incremental replenishment, individual parts demanded beyond the last buy are fabricated individually at significantly greater cost; as the total profit is concave with respect to the order amount, the optimal order amount can be found by analysis of the rate of change of the profit. The last buy problem with no replenishment occurs when there is no effective way to replenish part inventory beyond the last buy; as the total profit is not concave, an upper bound on the optimal order amount is determined, thus limiting the candidate solutions. Difficulties exist in calculating the optimal order amount in the last buy problem with batch replenishment, as the size of the replenishment batch is itself a last buy problem; we solve for a special case of the problem using renewal theory. We also examine the possibility of contract extensions in last buy problems, and their effect on the optimal order amount calculations.

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CHAPTER 1
INTRODUCTION

1.1 Background

Consider the case of a company that has decided to stop the manufacture of one of its products. Thus, the company will shut down the process used in the product's manufacture, and devote the labor and equipment used on that product to manufacture other products. However, due to government regulation, contractual obligations, and/or service considerations, the company will often keep spare parts available for sale so that customers are able to repair the product in the case of a part failure. As a result, the company will make one last order for parts before ceasing production (the last buy order), and use this to satisfy demand for spare parts in the future. Considering the various costs and revenues that come from manufacturing the spare parts, storing the parts, and selling the parts, the company must determine the optimal number of spare parts that should be manufactured.

At this point, the company faces questions that will determine the form of the last buy problem. How long are we required to serve the customer? Once the part manufacturing process has been shut down, can it be restarted at a reasonable cost? Can the spare part be fabricated individually without using the original manufacturing process? Is the cost resulting from not being able to satisfy demand greater than the cost of fabricating the part individually? All of these questions must be considered when determining the nature of the problem that must be solved.

Due to the multiple forms of part replenishment available and the possible time frames available, there is no single "last buy" formula; rather, it is necessary to choose the most appropriate last buy formulation based on the circumstances. As such, we will provide multiple formulations to match the different possible circumstances of the problem.

In the Last Buy Problem, parts are manufactured, and are then placed in inventory until they are sold. If the spare part is still in inventory when the decision is made to stop selling the part, the part will be salvaged. If the part on a working assembly fails, a spare part will be demanded by the customer. If the manufacturer is obligated to serve customer demand but no more spare parts are in inventory when a demand occurs, either a new batch will be made, a spare part will be fabricated individually at higher cost, or a penalty will be paid to the customer. These costs and revenues occur at different times; manufacturing costs occur at the time of the last buy order, while revenue, salvage, holding, and other costs occur over the time horizon of the problem. Thus, any revenues and costs will need to be discounted over time in order to maximize the net present value of the profit function.

The last buy problem can vary in two significant areas. These are:

- *Contract:* The contract defines the business relationship between the manufacturer and the customer. If a formal contract exists between the customer and the manufacturer, any penalties or replenishment costs will be limited to the length of the contract, which we denote as t_c. Also, it is assumed that information regarding the status of any assemblies will be available to the manufacturer. Thus, the manufacturer will have knowledge of the time at which the last assembly fails, and will be able to salvage any remaining spare parts at that time. On the other hand, if no formal contract exists between the customer and the manufacturer, the time frame under which penalties and replenishment will occur is assumed to be undefined. Moreover, because the manufacturer does not have data in regards to the status of assemblies still in operation, salvage will occur so far in the future as to be negligible.

- *Replenishment:* Many of the unique characteristics of a last buy problem come from the form of replenishment that occurs once the last-buy order has run out.
  - In incremental replenishment, items are fabricated individually as demanded. A replenishment cost is incurred for each part manufactured through this method, as the efficient processes used to manufacture the parts in the last-buy order are no longer available.
  - In batch replenishment, the same manufacturing process used for the last-buy order is set up, and another last-buy order is manufactured. In this case, the variable cost to manufacture the parts themselves is not materially higher; however, additional batch setup costs to restart the manufacturing process are incurred.
  - In some contract cases, it is not feasible to manufacture the parts individually or to set up the manufacturing process for another batch. In these cases, no replenishment is made. Existing customers are paid a penalty due to the manufacturer's inability to supply any more spare parts.

In the last buy problem with incremental replenishment, parts from the last buy order are used to replace failed parts until the last buy order is depleted. Once this occurs, demand will be met by manufactured parts built individually at a cost materially higher than the cost of last-buy manufacture. This is analogous to ordering a fabricated part from a machine shop to replace a failed part. Incremental replenishment will most often occur in cases where the total number of parts demanded is low, startup costs for the manufacturing process are high, and individual parts may be feasibly manufactured at a cost higher than the unit cost at the last-buy point.

The last buy problem with batch replenishment differs from the last buy problem with incremental replenishment in what occurs once the last buy order is depleted. Once this occurs, the manufacturing process by which the last buy was manufactured will be set up, incurring a setup cost, and another batch manufactured. In essence, another 'last buy' order is manufactured. Batch replenishment will most often occur in cases involving large production runs, where the setup costs are small in comparison to the value of the spare parts to be manufactured and sold.

The last buy problem with no replenishment also differs from the other forms of the last buy problem in what occurs once the last buy order is depleted. Once the order is depleted, no cost-efficient method exists to replenish the inventory; the manufacturer has no way of supplying the part to customers. Thus, when the part fails, the assembly fails as well, as the part cannot be replaced. The manufacturer pays a penalty to the customer for its inability to meet the customer's demand. This occurs in cases where the setup costs for the manufacturing process are prohibitively expensive, and where the part may not be fabricated individually at a reasonable cost.

The Last Buy Problem is not a trivial problem. Moncrief, Schroder, and Reynolds [7] have noted the cost of inventory of spare parts: they report that companies waste more than $1 million a year per industrial plant on spare parts - and roughly 30% of the spare parts in these plants are unnecessary. Most of the existing literature on the last buy problem has focused on the problem within a finite time frame and with time discretized into units. What we propose is a continuous-time optimization framework for the last buy problem in which the time frame of the problem may be undefined. A contract may define the length of time for which replenishment or penalty costs may be incurred, but it does not limit the length of time for which the customer is served. Also, our research method applies to any general distribution of the spare part demand over time.

1.2 Overview and Contribution

In this study, we look at the different forms of the last buy problem from a continuous-time perspective. In Chapter 2, we discuss the relevant literature for the problem, and indicate how our work fits into that literature. In Chapter 3, we define the notation we use in our models. Chapter 4 demonstrates the Last Buy Problem with Incremental Replenishment. We first separate the problem into parts according to the type of cost or revenue, and find the properties of each of these parts. As these properties demonstrate the concavity of the overall profit function with respect to the order amount, this allows us to quickly find a solution to the optimal order amount. Once we have presented our solution method, we present managerial insights based on our results, and give our conclusions.

In Chapter 5, we explore the properties of the Last Buy Problem with No Replenishment. As before, we separate the problem into parts according to the type of cost or revenue, and find properties for each. Unlike with the Last Buy Problem with Incremental Replenishment, the problem is nonconcave with respect to the order amount. Therefore, we find an upper bound on the rate of change of the profit, one we can prove decreases with increasing order amount. This in turn provides an upper bound on the optimal order amount. Once we have significantly reduced the number of potential solutions, we can find the optimal order amount from the remaining potential solutions. We then present managerial insights based on our results, and give our conclusions.

Due to the complexities inherent in the Last Buy Problem with Batch Replenishment, we are unable to give continuous-time solutions for the general case. The only case in which we might find solutions is in the exponential case, where the distributions of both part and assembly lifetime are exponential. In such cases, it is possible to use renewal theory [9] to find a solution. In Chapter 6, we demonstrate a renewal theory method for the batch replenishment case. The main problem with the batch replenishment case is that, if the last buy is ever depleted, the result is the manufacture of another batch - in effect, another last buy problem. As such, iteration is required to find the optimal order amount and value of each potential batch. By finding the value of each potential batch, the optimal solution for the initial last buy order may be found. We first present the parts of the revenue and costs in a renewal-theory format, and demonstrate the properties of the overall profit. Once the formula for the overall profit is presented, we then give three renewal-theory algorithms to demonstrate various possible forms of the last buy problem with batch replenishment.

Finally, we explore a problem discovered from our discussions with Rockwell Collins. Rockwell has encountered cases in which the customer had been served through the life of the contract - then wished to extend the terms of the contract once it was finished. As a result, in Chapter 7, we present the last buy problem with contract extension. We separate the penalty portion of the profit function into two parts: the penalty that occurs during the initial contract, and the potential penalties that could occur during the contract extension. The first task is to examine all cases and determine the cases in which the contract would be extended, and the cases where it would not. Once this is done, the optimal order amount is determined.

After these cases are presented, the thesis concludes with a discussion of future areas of research in Chapter 8.

CHAPTER 2
LITERATURE REVIEW

2.1 Early Research

One of the first treatments of the Last Buy problem was done by Fortuin [3]. In Fortuin, the spare part demand is assumed to have a nonstationary normal distribution whose mean decreases over time, the time frame under consideration for the post-manufacture period is finite and known, and the goal of the optimization method is to maintain a desired service level. The mean and variance of the probability of running out of parts is determined; based on this, the order amount is determined by the service level on the resulting normal distribution. These results are extended in a follow-up paper [4] to adjust for the observation that the results of [3] produced excessive safety stocks; in the follow-up paper, the minimum service level requirement is reduced to only the first few years of the post-manufacture period.

2.2 Discrete-Time Models

In a series of papers, Teunter, along with several colleagues, examined the Last Buy problem from different angles. Teunter and Klein Haneveld [13] first looked at the problem from a buyer's perspective, in which an owner with a single assembly, multiple parts per assembly, is seeking the optimal number of parts to stock up for the life of the assembly. This version of the problem seeks to balance purchase, discounted holding, and discounted 'out of order' costs over a finite time frame; if the supply of parts runs out before the time frame is completed, a cost is incurred for the remaining time. A decomposition method by which time is separated into discrete units is then used to determine approximate results for the optimal.

Teunter and Fortuin [12] extends these results to a case similar to the last buy problem, where a producer must supply parts without cost for customers as part of service contracts; in essence, the finite-time-frame final order problem with multiple assemblies instead of a single assembly, with no revenue involved. A follow-up paper[11] provides a case study using the methods in [12].

Teunter and Klein Haneveld expand on these results in a subsequent paper [14]. In this case, the model of service contracts is continued, in which a finite planning horizon is used, time is viewed continuously, demand follows a Poisson process, additional parts may be ordered with a lead time, and all stock is removed at the end of all service contracts. The goal is to minimize nondiscounted costs of replenishment, holding, backorder, and disposal. The model is not a strict final order model in that it is expected that multiple orders of parts will be given during the course of the service contracts, but at a cost greater than that of the initial order.

In other papers, Cattani and Souza[1] examine the trade-off involved in delaying the last buy order. In their model, they use a multivariate normal distribution of demand, and discrete time periods. They measure the total discounted cost of manufacturing, holding, and out-of-stock loss of sales, and compare these results to determine the optimal delay. Hong et al[5] calculate the Last Buy amount by discretizing the post-production phase of the product life cycle.

2.3 Renewal Theory Models

van Kooten and Tan [15] expand on the idea first presented by Ritchie and Wilcox [9] of using renewal theory to model the last buy problem under condemnation. In their model, an exponential distribution is used for the lifetime of each spare part and assembly, as well as the sales lifetime of the spare part before it is considered obsolete, and a transient Markov model used to find the optimal order amount.

2.4 Similar Problems

Feng et al[2] examines a problem similar to the Last Buy problem known as the Lifetime Buy Problem. In this case, manufacture of the assembly is still occurring; however, the manufacturer wishes to make an order of parts that will satisfy both manufacturing demand and repair demand. A Monte Carlo analysis is used to simulate both obsolescence date, forecasted demand, and associated costs over time. These results are then used and applied to the model by Teunter and Fortuin[12] to determine an approximate optimal-cost order amount.

Kim and Park[6] give another variation to the Last Buy problem by seeking to determine all end-of-life decisions, including parts inventory and warranty length. They set up a two-stage constrained nonlinear program involving both the manufacturing period and the post-manufacture period, and optimize for all variables, including life cycle, warranty length, and cost parameters to manufacture parts at given points in time.

Pourakbar et al [8] present a variation of the last buy problem for a specific application - consumer electronics. In their model, the value of producing a replacement spare part decreases over time, to the point that it would eventually be optimal to salvage any remaining inventory and manufacture replacement spare parts as needed. The purpose of Pourakbar's research is to find the optimal point at which the salvage should occur.

2.5 Our Research

Our work differs from the previous research in that it fills several gaps in the literature. First, our problem has no finite and defined time frame; even in contract cases in which penalties or replenishment costs are only incurred during the life of the contract, customer service will continue indefinitely. This makes sense, as the manufacturer is still likely to serve the customer even if not obligated by contract. Second, the solution methods presented in existing literature tends to fall into one of two categories: discrete-time models and renewal theory models. Our incremental-replenishment and no-replenishment models are designed to have characteristics of both discrete-time and renewal-theory models: our models allow for non-exponential lifetime distributions like a discrete-time model while still allowing for a continuous-time framework. Finally, we expand the existing knowledge base in the area by supplying a renewal-theory model for the last buy problem with batch replenishment and a model for the possibility of contract extension in the last buy problem.

CHAPTER 3
DEFINITIONS AND NOTATION

3.1 Definitions

In this thesis, the term ”assembly” or ”product” refers to the completed product that has been sold to the customer and for which spare parts are demanded. The term ”part” or ”spare part” refers to the part of the assembly that is under examination by the last buy problem.

3.2 Notation

This set involves the models to be found in the last buy problem with incremental replenishment and the last buy problem with batch replenishment.

- t: Time. For the problems we will be using, the time at which $t = 0$ represents the moment of the last-buy decision.
- t_f: The time at which service for the customer ceases. For the last buy problem with incremental replenishment, this occurs when the last assembly fails. For the last buy problem with no replenishment, this occurs when the last assembly fails or when the last buy order is depleted.
- t_c: In last buy problems involving a contract, t_c refers to the time at which the contract expires, after which no replenishment costs or penalties will be incurred.
- q: The number of parts manufactured in the last buy order.

- l, k: Variables used to represent the number of assemblies still in use at some arbitrary time.
- n: A variable used to represent the number of parts still in inventory at some arbitrary time.
- j: The number of assemblies functioning at the time of the last buy order.
- A_t: The number of assemblies currently functioning at time t.
- $P\{A_t = l\}$: The probability that the number of working assemblies in operation at time t is equal to l units.
- D_t: The number of spare parts demanded from time 0 up to a given time t.
- $P\{D_t = n\}$: The probability that demand for spare parts from time 0 up to time t is equal to n units.
- $P\{D_{t_1} = n|A_{t_2} = l\}$: The conditional probability that demand for spare parts from time 0 up to time t_1 is equal to n units, given that the number of assemblies still in operation at another time t_2 is equal to l. In the case most commonly used, $P\{D_t = n|A_{t_f} = 1\}$ represents the conditional probability that demand from time 0 up to time t is n, given that there is exactly 1 assembly remaining at time t_f.
- $R(q)$, $R(l, n)$: $R(q)$ represents the total expected discounted revenue (heretofore known simply as the "revenue"), given a last-buy order amount of q. In the last buy problem with batch replenishment, $R(l, n)$ represents the future expected discounted revenue given that l assemblies remain in operation and n parts

remain in inventory.

- $M(q)$: The total manufacturing cost for a last-buy order amount of q.
- $H(q)$, $H(l, n)$: $H(q)$ represents the total expected discounted holding cost (heretofore known as the "holding cost"), given a last-buy order amount of q. This includes the cost to maintain and operate any buildings and equipment used to store the items. The time value of money should not be included in the holding cost, as all costs are discounted over time; thus, any costs resulting from the time value of money are taken care of by the discount rate. In the last buy problem with batch replenishment, $H(l, n)$ represents the future expected discounted holding costs given that l assemblies remain in operation and n parts remain in inventory.
- $Z(q)$, $Z(l, n)$: $Z(q)$ represents the total expected discounted cost due to the last buy order running out, given a last-buy order amount of q. In the last buy problem with incremental replenishment, this is known as the "replenishment cost"; in the last buy problem with no replenishment, it is known as the "penalty cost". In the last buy problem with batch replenishment, $Z(l, n)$ represents the future expected discounted batch replenishment costs given that l assemblies remain in operation and n parts remain in inventory.
- $S(q)$, $S(l, n)$: $S(q)$ represents the total expected discounted salvage cost or value (heretofore known simply as the "salvage"), given a last-buy order amount of q. In the last buy problem with batch replenishment, $S(l, n)$ represents the future expected discounted salvage given that l assemblies remain in operation and n

parts remain in inventory.

- $\pi(q)$, $\pi(l, q)$: $\pi(q)$ is the total expected discounted profit (heretofore known simply as the "profit") of the last buy decision, given a last buy order amount of q. In the last buy problem with batch replenishment, $\pi(l, n)$ represents the future expected discounted profit given that q parts were manufactured with l assemblies remaining in operation.
- r: The amount of revenue generated per spare part sold.
- m: The individual cost to manufacture a single spare part in the last-buy order.
- h: The amount of cost per unit per unit time to hold a single spare part in inventory.
- p: The replenishment/penalty cost p has different definitions according to the type of last buy problem. In the last buy problem with incremental replenishment (Chapter 4), p represents the cost per part to fabricate a part to meet demand after the last buy order has been depleted. In the last buy problem with no replenishment (Chapter 5), p is the cost per assembly that the manufacturer must pay a penalty to for failing to meet demand.
- s: The salvage per unit; if the salvage is a revenue, then $s > 0$; if the salvage is a cost e.g. a disposal cost, then $s < 0$.
- β: If the lifetime of the assembly is represented as either an exponential or Weibull distribution, β represents the failure rate parameter of an assembly.
- ω The Weibull exponential parameter of the lifetime distribution of an assembly.

- λ: The exponential failure rate of the part.
- μ: If a Normal distribution is used, μ represents the mean of the distribution.
- σ: If a Normal distribution is used, σ represents the standard deviation of the distribution.
- α: The rate at which future costs and revenues are discounted over time.

3.3 Additional Notation: Chapter 5

- $\pi_p(q)$: In the last buy problem with no replenishment, $\pi_p(q)$ represents a portion of the total expected discounted profit - specifically, the total profit without the penalty cost subtracted. Thus, $\pi(q) + Z(q) = \pi_p(q)$.
- $A_{k,t}$: In cases where the assemblies cannot be evaluated as a group, $A_{k,t}$ represents the status of assembly k at time t. If $A_{k,t} = 1$, then assembly k is still functioning at time t; if $A_{k,t} = 0$, then assembly k has failed by time t.
- $D_{k,t}$: In cases where the assemblies cannot be evaluated as a group, $D_{k,t}$ represents the number of parts demanded by assembly k up to time t.
- $K(l, t_f)$: The expected standardized ($p = 1$) penalty, discounted to time t_f, given that l assemblies remain in operation at time t_f and the last buy order is depleted at time t_f. $K(l, t_f)$ can be thought of as the expected number of assemblies, discounted to time t_f, that the manufacturer will have to pay a penalty for given that the manufacturer is unable to meet customer demand starting at time t_f.
- $\Delta\pi_{ub}(q)$: An upper bound on the rate of change of $\pi(q)$.

3.4 Additional Notation: Chapter 6

- q_l^*: The optimal order amount for a batch at some arbitrary time, given that l assemblies are still in operation at that time.
- $C(l,n)$: The expected discounted value of a part of the total profit function for the remainder of the last buy period - specifically, the total profit without the manufacturing costs - given that l assemblies and n parts are still in operation. For the last buy decision or any batch decision with l assemblies remaining, $\pi(l,q) = C(l,q) - M(q)$
- $B(l)$: A Boolean variable that represents whether or not the optimal decision with l assemblies remaining and no parts remaining in inventory is to make another batch or to pay a penalty to the remaining customers. If $B(l) = TRUE$, then buying out the remaining customers is the optimal decision with l assemblies remaining. If $B(l) = FALSE$, manufacturing another batch is the best option with l assemblies remaining.
- p_{batch}: The fixed batch replenishment cost per batch; the fixed cost to set up and manufacture another batch. This does not include the variable cost per unit to manufacture the parts.
- p_{buyout}: The penalty cost per assembly to refuse to continue serving the customer.

3.5 Additional Notation: Chapter 7

This set involves the models to be found in the last buy problem with optional contract extension.

- t_{c1}: The time at which the initial contract expires, at which point the customer and manufacturer both have the choice of extending the contract or letting it expire. After the contract expires, no replenishment costs or penalties will be incurred, unless the initial contract is extended.
- t_{c2}: The time at which the extended contract expires; $t_{c2} > t_{c1}$.
- $T(l, n)$: The probability that both the customer and the manufacturer will agree to the contract extension, given the number of parts in inventory n and number of assemblies still in operation l at time t_{c1}.
- $V(l, n)$: The expected value of the penalty incurred during the contract extension period, given the number of parts in inventory n and number of assemblies still in operation l at time t_{c1}. Note that $V(l, n)$ is discounted to time 0.
- $X(q)$: The total expected discounted profit generated due to the contract extension (heretofore known simply as the "extension"), given a last-buy order amount of q. Note that $X(q)$ is discounted to time 0.
- X_{max}: The maximum possible value of the extension. Note that X_{max} is discounted to time 0.
- p_x: The revenue paid per assembly to the manufacturer to extend the supply contract from time t_{c1} to time t_{c2}.

- p_C: The maximum amount that the customer would be willing to pay per assembly to extend the supply contract from time t_{c1} to time t_{c2}.
- $p_M(q)$: The minimum amount that the manufacturer would be willing to accept that could possibly cause the manufacturer to extend the supply contract from time t_{c1} to time t_{c2}.
- $W(t)$: The customer's amortized value for a single assembly at time t.

CHAPTER 4
THE LAST BUY PROBLEM WITH INCREMENTAL REPLENISHMENT

4.1 Introduction

In the last buy problem with incremental replenishment, parts from the last buy order are used to replace failed parts until the last buy order is depleted. Once this occurs, demand will be met by manufactured parts built individually at a cost materially higher than the cost of last-buy manufacture. This is analogous to ordering a fabricated part from a machine shop to replace a failed part. Incremental replenishment will most often occur in cases where the total number of parts demanded is low, startup costs for the manufacturing process are high, and individual parts may be feasibly manufactured at a cost higher than the unit cost at the last-buy point.

4.2 Components of the Last Buy Problem

Using the notation presented in Chapter 3, the total expected profit, $\pi(q)$, is the value of the expected revenues minus the value of the expected costs:

$$\pi(q) = R(q) + S(q) - M(q) - H(q) - Z(q). \tag{4.1}$$

We now provide details on each of the terms in (4.1).

4.2.1 Revenue

The expected revenue, discounted over time, is generated by finding the expected value of the revenue by time t and determining the rate of change of this expected value over time. Given a probability distribution of demand over time $P\{D_t = n\}$, we have

$$R(q) = -r \sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt. \tag{4.2}$$

Lemma 4.1. *The revenue function has the following properties with respect to q:*

- *(a) The revenue $R(q)$ is nonnegative.*
- *(b) The revenue $R(q)$ is nondecreasing as q increases.*
- *(c)The revenue $R(q)$ is integer concave - that is, the rate of change of $R(q)$ is nonincreasing with increasing q.*

All of these results are fairly intuitive. A greater amount manufactured in the last buy order is not going to result in a decrease in sales revenue. At the same time, each additional part manufactured is less likely to be sold than the last - and, thus, will be more difficult to sell.

Proofs for all parts of Lemma 4.1 may be found in Appendix A.

4.2.2 Manufacturing Costs

The total manufacturing cost is simply the product of the last buy order amount q and the per-unit manufacturing cost m:

$$M(q) = mq. \tag{4.3}$$

4.2.3 Holding Costs

The holding costs are the associated labor, facility, and infrastructure costs incurred by holding a part in inventory. This cost is designated to have a cost of h per unit per unit time. Thus, if one spare part is held in inventory for one unit of time, a cost of h would be incurred.

The total discounted holding cost used depends on the specifics of the problem. In a non-contract problem where we would have no salvage, the expected value of the holding cost becomes

$$H(q) = h \sum_{n=0}^{q-1} (q-n) \int_{t=0}^{\infty} e^{-t\alpha} P\{D_t = n\} dt. \tag{4.4}$$

In a contract problem in which the salvage occurs when the last assembly fails, the holding cost is

$$H(q) = h \sum_{n=0}^{q-1} (q-n) \int_{t=0}^{\infty} \frac{dP\{A_{t_f} = 0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = n | A_{t_f} = 1\} dt dt_f. \tag{4.5}$$

Lemma 4.2. *The holding cost has the following properties, regardless of the form used, assuming* $h \geq 0$*:*

- *(a) The holding cost $H(q)$ is nondecreasing with increasing values of q.*
- *(b) The holding cost $H(q)$ is integer convex with respect to q - that is, the rate of increase of the holding cost is nondecreasing with increasing values of q.*

The addition of a part in the last buy order means that another part is going to be in holding - which means more holding costs will be generated. At the same time, that additional part will be less likely to sell than the last - which means it is likely to remain in inventory longer, and thus will generate more holding costs.

Proofs for both parts of Lemma 4.2 may be found in Appendix A.

4.2.4 Replenishment Costs

The replenishment cost is the cost to order the fabrication of a replacement part at significantly greater cost than the manufacturing cost. The formulation of this replenishment cost, regardless of whether the replenishment period is defined by contract (t_c finite) or undefined ($t_c = \infty$) is

$$Z(q) = -p \sum_{n=q}^{\infty} \int_{t=0}^{t_c} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt. \tag{4.6}$$

Lemma 4.3. *The properties of the incremental replenishment cost with respect to q are:*

- *(a) The incremental replenishment cost $Z(q)$ is nonnegative.*
- *(b) As q increases, the replenishment cost $Z(q)$ decreases.*
- *(c) The replenishment cost $Z(q)$ is integer convex with respect to q.*

With the last buy problem, a larger last buy order means a smaller likelihood that the order will run out - and, thus, a smaller likelihood that replenishment costs will be incurred. However, with each additional part ordered, it is less likely that the part will be demanded - and, thus, each additional part manufactured prevents a smaller expected replenishment cost than the one before.

Proofs for all parts of Lemma 4.3 may be found in Appendix A.

4.2.5 Salvage

The salvage differs from most other costs and revenues in that it does not occur gradually over the course of the post-production phase; rather, it occurs when the last assembly has failed at time t_f. Once this occurs, the remaining inventory is discarded at a revenue of s per unit. (If the financial event of scrapping each unit generates a cost instead of a revenue, then s is a negative number.)

For the last buy problem, the salvage is.

$$S(q) = s\sum_{n=0}^{q-1}(q-n)\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=n|A=1\}dt_f. \tag{4.7}$$

The reason for the $A = 1$ in the $P\{D_{t_f} = n|A = 1\}$ portion of the formula is that, in order for the last assembly to fail, the system needs to have only one assembly left - so that the next assembly failure is its last.

The properties of the salvage:

Lemma 4.4. *The properties of the salvage with respect to the order amount q are:*

- *(a) If the salvage per unit s is greater than 0, the salvage $S(q)$ increases with increasing q*
- *(b) If the salvage per unit s is greater than 0, the salvage $S(q)$ is integer convex with respect to q.*
- *(c) While the salvage $S(q)$ may be integer convex, the sum $(S(q) + R(q))$ is integer concave, provided that $s < r$.*
- *(d) If s is sufficiently large, then the total expected discounted profit $\pi(q)$ will continue to increase with increasing q; thus, the solution is unbounded.*

The goal of our model is to maximize the overall profit $\pi(q)$ in Equation (4.1). Because of this, it is desirable that $\pi(q)$ - and each part of $\pi(q)$ - be integer concave with respect to q. Requiring that $s < r$ to maintain the integer concavity of $\pi(q)$ is certainly reasonable since otherwise the manufacturer would have incentive to refuse sale of a part in order to retain it for salvage. Thus henceforth we assume $s < r$, as required by Lemma 4.4(c). Lemma 4.4(d) determines the conditions under which a finite optimal solution for the order amount q does not exist. Proofs for all parts of Lemma 4.4 can be found in Appendix A.

4.2.6 Profit Function

Now that the components of the Last Buy Problem have been formulated, we can now combine those components into full cost and revenue functions for the entire problem. The goal is to take the components found in the previous section, take the sum of these components as an overall profit function, then find the maximum of this function. The basic structure of the problem is given by (4.1):

$$\pi(q) = R(q) + S(q) - H(q) - Z(q) - M(q)..$$

The goal will be to find the value of q that maximizes $\pi(q)$.

Theorem 4.5. *The profit function is concave with respect to the positive integers.*

The theorem can be proven based on Lemmas 4.1 - 4.4. Each of the parts of the profit function - $(R(q) + S(q))$, $(-H(q))$, $(-Z(q))$, and $(-M(q))$ - are integer concave with respect to q; thus, the profit function is integer concave. As a result of this property, we can find the global maximum by examining the rate of change of the total cost and finding where it changes from positive to negative.

With $\Phi(\cdot)$ as a function defined in the nonnegative integers, letting $\Delta\Phi(q) = \Phi(q+1) - \Phi(q)$, in our problem we seek to find

$$q^* = \min\{q \geq 0 | \Delta R(q) + \Delta S(q) - \Delta H(q) - \Delta Z(q) - \Delta M(q) < 0\}. \quad (4.8)$$

Below are two examples of the last buy problem:

4.2.6.1 Incremental Replenishment with No Contract

In the case presented here, no contract exists between the manufacturer and the customer. However, the manufacturer still wants to supply spare parts to customers, even if the last buy order is depleted. The manufacturer knows the number of assemblies still in operation at the last buy decision point, but will have no knowledge of the number of assemblies in operation beyond that point. Because of this, no salvage will take place, and all spare parts are assumed to be held until sale; as a result, the salvage $S(q)$ is not included in (4.1). This makes the rate of change of the profit with respect to q

$$\begin{aligned}\Delta\pi(q) = p\int_{t=0}^{\infty}\sum_{n=q+1}^{\infty} e^{-t\alpha}\frac{dP\{D_t = n\}}{dt}dt + r\int_{t=0}^{\infty}\sum_{n=0}^{q} e^{-t\alpha}\frac{-dP\{D_t = n\}}{dt}dt \\ -m - h\int_{t=0}^{\infty}\sum_{n=0}^{q} e^{-t\alpha}P\{D_t = n\}dt.\end{aligned} \tag{4.9}$$

Since we seek the smallest value of q such that $\Delta\pi(q) < 0$, equation (4.9) can be rearranged to form the following inequality to find the optimal order amount, q^*:

$$q^* = \min_{q\in Z^+}\{\frac{p+r-m}{p+r+\frac{h}{\alpha}} \leq 1 + \int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t \leq q\}}{dt}dt\}. \tag{4.10}$$

This formula is the result of a newsvendor analysis with the following conditions and modifications:

- The overage cost C_o is equal to $m + \frac{h}{\alpha}$.
- The shortage cost C_u is equal to $p + r - m$.
- Any costs or revenues are discounted to time 0 according to the time at which they occur. The value of the holding cost per unit $\frac{h}{\alpha}$ is equivalent to the net present value of a perpetuity of size h - which is what the manufacturer commits to paying due to holding costs.

A discussion of the newsvendor model as it applies to the last buy problem is given in Appendix A.

Example: Ten assemblies ($j = 10$) are still in operation at the last-buy decision point; the distribution of the lifetime of each of these assemblies is exponential with a parameter $\beta = 2$. Each assembly has a single part for which we wish to stock parts; the lifetime of each part is exponential with a parameter $\lambda = 1$. Our applicable discount rate is $\alpha = 0.2$. The cost per unit manufactured $m = 5$; the revenue that results from the sale of the part is $r = 15$. The holding cost per unit per unit time is $h = 0.5$, and the replenishment cost incurred for each unit of demand not met is $p = 30$. What is the optimal order amount for spare parts?

Based on our last buy formula, our critical ratio is

$$\frac{p+r-m}{p+r+\frac{h}{\alpha}} = \frac{30+15-5}{30+15+\frac{0.5}{0.2}} = 0.842. \tag{4.11}$$

Therefore, we need the smallest value of q such that $1 + \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \le q\}}{dt} dt > 0.842$. We calculate the values of $\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \le q\}}{dt} dt$ for increasing values of q until our inequality is met.

Based on this, our optimal order amount should be 7 units, as it is the smallest value of q for which $1 + \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \le q\}}{dt} dt$ is greater than our critical ratio of 0.842. This bears out when we calculate results for each cost. Calculations of the last buy profit $\pi(q)$ for different values of q gives us an optimal order amount of 7 units, with a profit of 11.0.

4.2.6.2 Incremental Replenishment with Contract

In this example, the manufacturer is bound by contract to serve the customer up to a fixed time t_c. As such, the manufacturer is aware of the number of assemblies in operation at any given time. Should a part fail before time t_c, the manufacturer is obligated to supply a replacement part; however, the manufacturer is not obligated to supply a replacement part beyond time t_c. Also, when the last assembly has failed at time t_f, any parts remaining in inventory will be salvaged.

The formula for the rate of change of the last buy profit $\Delta\pi(q)$ is

$$\begin{aligned}\Delta\pi(q) = m + p\int_{t=0}^{t_c} e^{-t\alpha}\frac{dP\{D_t \le n\}}{dt}dt \\ +\sum_{n=0}^{q}\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}(se^{-t_f\alpha}P\{D_{t_f}=n|A=1\} \\ +h\int_{t=0}^{t_f} e^{-t\alpha}P\{D_t=n|A_{t_f}=1\}dt)dt_f \\ +r\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t \le n\}}{dt}dt.\end{aligned} \tag{4.12}$$

The solution method is to determine $\Delta\pi(q)$ for increasing values of q until $\Delta\pi(q) < 0$, which occurs at the optimal last buy order amount.

Example: Ten assemblies ($j = 10$) are in operation at the last buy order point; the lifetime of each assembly is normally distributed with a mean of $\mu = 4$ and a standard deviation of $\sigma = 2$. The lifetime of each part is exponentially distributed with a mean of $\frac{1}{\lambda} = 3.33$. The discount rate of any costs or revenues is $\alpha = 0.08$. Each part in the last buy order is manufactured at a cost of 12, and is sold for 25. Should the last buy order run out before time $t_c = 10$, a replacement part may be manufactured at a net cost of 25. Any parts remaining when the last assembly has failed will be sold for scrap for 4. The holding cost is 1 per part per unit time. What is the optimal order amount?

Again, we find the smallest value of q such that $\pi(q+1) - \pi(q) < 0$. Our results can be found in Table 4.1:

Table 4.1: Rate of Change of Last Buy Components

q	1	2	3	4	5	6	7	8	9	10
$\Delta\pi(q)$	18.1	17.6	17.0	16.2	15.2	13.8	12.1	10.1	7.9	5.6
q	11	12	13	14	15	16	17	18	19	20
$\Delta\pi(q)$	3.4	1.4	-0.4	-1.7	-2.8	-3.6	-4.1	-4.5	-4.7	-4.8

We can stop our search at 13 units, as the rate of change of the total cost begins to decrease at that point. We therefore find that 13 units is our optimal order amount.

When we look at the overall costs given values of q, shown in Figure 4.1, we find this to be the case. The optimal order amount occurs with a last buy order at 13 units; at this order amount, we incur a cost of 66.2 units, which is the best option available.

4.3 Numerical Experiments

As an example of the effectiveness of our method in comparison to other reasonable approaches, we present the following case: Ten assemblies are in operation

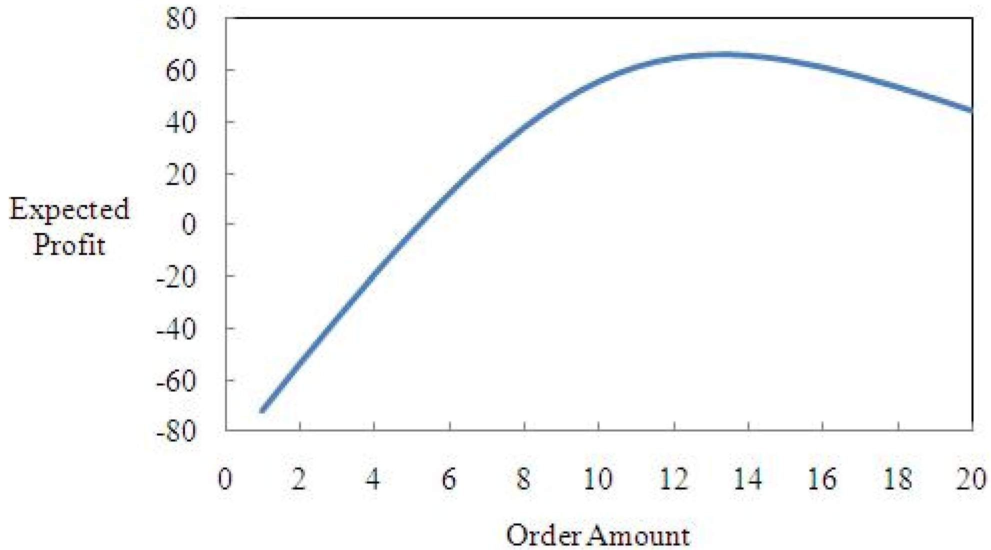

Figure 4.1: Total Profit, Given Values of q, for the Last Buy Problem with Incremental Replenishment

at the last-buy decision point. The lifetime of the assemblies is distributed exponentially with a mean of $\frac{1}{\beta} = 5$; the lifetime of the spare part in question is distributed exponentially with a mean of $\frac{1}{\lambda} = 4$. For our purposes, the discount rate is $\alpha = 0.08$. There is no fixed time limit on the overall service time; we consider $t_c = \infty$.

Each assembly is manufactured at a cost of $m = 4$ units, and sold for $r = 18$ units of revenue. While each part is held in inventory, it will generate a holding cost of $h = 0.5$ per unit per unit time. If the last buy order is depleted, any unit demanded beyond that generates a cost of $p = 5$ units. If all the assemblies fail before the last buy order is depleted, spare units will be salvaged, generating $s = 1$ units of revenue.

We calculate the optimal order amount using four methods - our optimal method, as well as three other methods that a typical business manager might use for the problem: (1) We calculate a solution using the overall expected cost with no discount. (2) We calculate a solution using a standard newsvendor formula (as opposed to formula (4.10) presented earlier). (3) We calculate a solution by using the average demand beyond the last buy order.

Table 4.2: Solution Results for Different Calculation Methods, Last Buy Problem with Incremental Replenishment

Method	Solution q	Profit $\pi(q)$	Difference in $\pi(q)$	% loss
Our method	10	24.52		
No discount	12	21.78	-2.74	11.2
Average demand	12	21.78	-2.74	11.2
Newsvendor	13	18.69	-5.83	23.8

Our results, including the difference in expected discounted profits between our method and the other methods, are shown in Table 4.2. As we see in this example, other solution methods result in expected profit losses of between 11 and 24 percent. It should be noted that the standard newsvendor method in the above table used a value of $\frac{h}{\alpha}$ for the holding cost per unit, which matches the value of a perpetuity of size h. A holding cost for a finite contract would have produced a smaller overage cost and a higher critical ratio - and, thus, a result further away from the result of our method.

In another experiment, we have found that unless the holding cost is unusually large, the larger the discount rate α, the smaller the optimal q^* The reason q^* decreases with increasing α stems from the effect of α on the net present value of the revenue, the salvage, and the incremental replenishment cost. An increase in α decreases the net present value of any cost/revenue event that occurs beyond time 0. Thus, the expected revenue, the expected salvage, and the expected replenishment cost are all reduced. A decrease in either the revenue or the salvage decreases the incentive to manufacture another spare part. A decrease in the replenishment cost, while increasing the overall profit, decreases the incentive to avoid the replenishment cost - and, thus, also decreases the incentive to manufacture another spare part. Thus, a decrease in the revenue from sales, a decrease in the salvage, and a decrease in the replenishment cost produce the same result with respect to q^* - a decreased incentive to manufacture more parts, and thus a lower q^*. While the holding cost produces an opposite effect - an increase in α produces a smaller overall holding cost, and thus a greater incentive to manufacture - this effect is usually dominated by the effect from the revenue, salvage, and replenishment. Thus, any action which would decrease the revenue, the salvage, and the replenishment cost - such as an increase in the discount rate α - would decrease the optimal q^*.

To demonstrate the above, consider the following example. The distribution of the lifetime of the assembly is a Weibull with rate parameter $\beta = 0.2$ and exponential parameter of $\omega = 1.25$; the distribution of the lifetime of the spare part is an exponential with mean $\frac{1}{\lambda} = 4$. The holding cost is \$0.5 per part per unit time, the salvage is \$1 per part, and the manufacturing cost is \$4 per part. In the example, an incremental replenishment cost of \$6 per part demanded and sales revenue of \$15 per part is used. The optimal order amount and optimal profit is calculated for both cases at different values of α, from $\alpha = 0.05$ to $\alpha = 0.2$. In this case, 25 assemblies

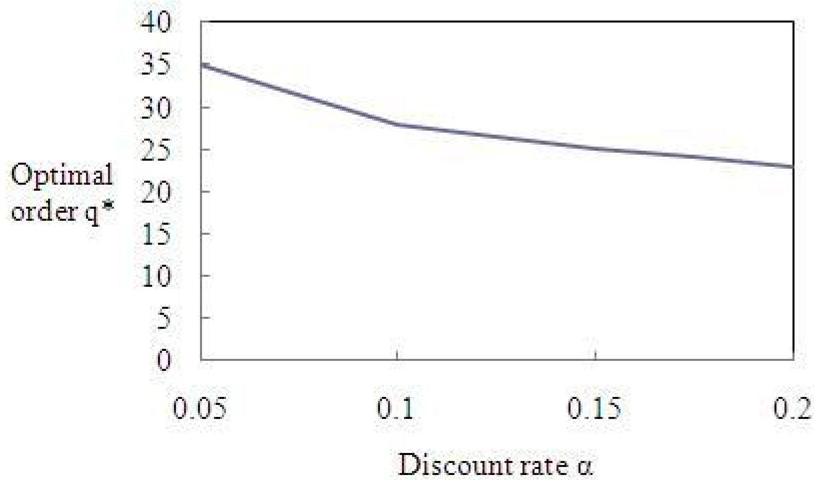

Figure 4.2: Change in Optimal Order Quantity with Discount Rate, Last Buy Problem with Incremental Replenishment

are in operation at time $t = 0$.

The results can be found in Figure 4.2. As we see, the optimal order amount q^* decreases steadily from 35 for $\alpha = 0.05$ to 23 for $\alpha = 0.2$.

Another numerical result: If both incremental and batch replenishment are possible, incremental replenishment would be preferred for cases involving smaller numbers of assemblies while batch replenishment would be preferred for cases involving large numbers of assemblies. This observation is the result of basic operations: smaller fixed costs and larger variable costs are preferred if small amounts are to be demanded, while larger fixed costs and smaller variable costs are preferred if large amounts are to be demanded. Thus, the incremental replenishment cost would be preferred for smaller numbers of assemblies remaining, while a batch setup cost would be preferred for larger numbers of assemblies.

Determining the distribution of demand for large numbers of assemblies may produce computation difficulties not found in smaller problems. To illustrate this, consider the use of factorials in the calculation of some members of the exponential family of probability distributions; exact calculation of probabilities in binomial and gamma distributions becomes difficult for large sample sizes. Because of this, it becomes important to see if the demand may be approximated by determining the solution for a smaller number of assemblies, then scaling the result for a larger number of assemblies. The results of approximation to scaling are mixed. While scaling produces reasonable approximations for the optimal solution, the problem is clearly not perfectly linearly scalable, because not all of the components of the profit function are related to the demand of the part over time.

We demonstrate this problem with an example designed to stretch the limits of approximation through scaling for the problem. In our demonstration, both part and assembly lifetime are exponentially distributed, with the mean assembly lifetime $\frac{1}{\beta} = 2.857$ and the mean part lifetime $\frac{1}{\lambda} = 2.5$. We use values of $r = 15$ and $p = 6$ for the revenue and the replenishment cost per unit demanded; manufacturing cost is $m = 4$ per unit. We use unusually large values for holding cost ($h = 3$) and salvage ($s = 3$) to emphasize costs that do not depend solely on spare part demand over time. Based on this, we find the optimal order amount for $j = 100$ assemblies. We then find the solution for 50, 25, 10, and 5 assemblies, then scale up the solutions of these smaller problems to approximate the solution with 100 assemblies.

Our results can be found in Table 4.3. Note that the scaling is not perfect; a

Table 4.3: Solution Results from Scaling

Scale size	Solution	Aggregated Soln	Profit	Difference in $\pi(q)$	% loss
100	83	-	299.6	0	0
50	40	80	297.7	1.9	0.7
25	19	76	290.0	9.6	3.3
10	7	70	268.7	30.9	11.5
5	4	80	297.7	1.9	0.7

solution of 83 spare parts for 100 assemblies does not translate to a solution of 8 or 9 spare parts for 10 assemblies. The solutions found through scaling produced profit losses of anywhere from 0.7% to 11.5%. Thus, while scaling can produce solutions for problems that might otherwise be intractable, it needs to be used with the understanding that while the solution found may be close to the optimal, some loss in the expected value of the profit may occur.

One last numerical result relates to the method of calculation used to solve the last buy problem. Infinite-timeframe integrations in the last buy problem may be approximated by integrating up to finite but sufficiently large values of t, and vice versa. One of the problems involved with the integration of $P\{D_t = n\}$ or $\frac{dP\{D_t=n\}}{dt}$ as a solution method is that the integration of either term may prove intractable, depending on the formulation of $P\{D_t = n\}$. For instance, a polynomial formulation of $P\{D_t = n\}$ may only provide a useable formula in finite-time integrations; conversely, exponential-based formulations of $P\{D_t = n\}$ tend to work best with infinite-time integrations. As such, it is useful to see the difference in the solution found between finite-time and infinite-time formulations, to test whether or not one may be used as an approximation for the other.

We therefore present the following case. We have 10 assemblies still in operations at the last buy order. The lifetime of the assembly is represented as a normal with mean $\mu = 4$, and standard deviation $\sigma = 1$. The lifetime of a spare part is represented as an exponential with mean lifetime $\frac{1}{\lambda} = 2$. The discount rate for all costs and revenues is $\alpha = 0.1$. Each part is manufactured at a cost of $m = 4$. The revenue from each part sold is $r = 15$, while the replenishment cost is $p = 6$. The holding cost per unit per unit time is $h = 1$; there is no salvage in this case. We designate the value of t that the formula will be integrated up to as t_{end}. Given the problem, we find the optimal order amount by integrating up to five values of t_{end}: 4, 5, 6, 7, and ∞.

Table 4.4: Solution Results from Integral Substitution

Time t_{end}	Solution $q*$	Difference
4	18	2
5	19	1
6	20	0
7	20	0
∞	20	0

The results are given in Table 4.4. The results are somewhat surprising, as it appears that what would seem to be a partial examination of the part demand would lead to results very close to the optimal order amount. However, it makes more sense when considered in the context of the lifetime of the assembly. By $t = 4$, half of the assemblies, on average, have already failed. The remaining assemblies are almost certain to fail within one or two more units of time. Thus, by $t = 4$, most of the demand for spare parts has already occurred.

As a result of this, it is possible to substitute infinite integration for finite integration if the time period used in the finite integration is sufficiently large. Based on the results in Table 4.4, it appears that we can safely replace finite integration with infinite integration if the finite integration covers at least to two standard deviations beyond the mean lifetime of the assembly.

4.4 Managerial Insights

The last buy problem with incremental replenishment has been shown to be a useful tool for managers, in particular for problems in which batch replenishment is too expensive to use effectively. It provides a more accurate representation of the optimal order amount in comparison to many of the standard tools that are currently used by managers, including average demand, no-discount, and newsvendor calculations.

Also, we have determined some useful properties for the last buy formulation that a manager should be aware of. The result determined through the continuous-time formulation is sensitive to the discount rate α that is used in the calculation; in general, a larger discount rate will mean a smaller optimal order amount. As such, the discount rate used in calculating the optimal order amount should be chosen with care.

Another issue that should be considered in setting up the calculations for the optimal order amount is if incremental replenishment is the best method available. In cases where it is possible to set up another batch run rather than using incremental replenishment, it may be more economical to use batch replenishment in cases involving large numbers of assemblies.

If the last buy problem involves a large number of assemblies, a reasonable approximation to the solution may be found by solving the last buy problem for a smaller number of assemblies, then scaling the result of the smaller problem to match the larger problem. We found in our numerical results that solutions within roughly 10% of the optimal solution had expected profits that were reasonably close to the optimal. As such, if scaling is required, scaling by a factor of 10 or less to find a solution should produce satisfactory results for the optimal order amount.

Finally, if the formula for $P\{D_t \leq n\}$ or $\frac{dP\{D_t \leq n\}}{dt}$ is proving difficult to integrate, one solution may be to substitute an infinite-timeframe integration for a sufficiently-large finite-timeframe integration, or vice versa. When substituting a finite-timeframe integration for an infinite-timeframe integration, we found that the finite time used should be greater than twice the mean lifetime of the assembly.

4.5 Conclusion

In this paper we have demonstrated a continuous-time method for calculating the optimal last-buy order amount in cases where demand is satisfied with incremental replenishment should the last-buy order run out. We find the discounted expected value of the total cost over time given a last buy order amount of q. We then measure the rate of change of this expected value with respect to q to find the minimum; as each of the components of the total cost is concave with respect to the integers, any local maximum found is a global maximum, and therefore the optimum. From there, we demonstrated types of last buy problems that involve incremental replenishment.

Overall, the model as defined in this chapter takes the rate of change of demand over time and uses it to determine the expected value of costs and revenues in the last buy period - and, from this, the optimal order amount. The model is variable in complexity - the model is as simple as a newsvendor equation for a manager seeking a quick calculation, but can be made more complex if a manager wishes to take into account additional details of the problem, such as salvage and time of service. As a result, inventory managers have a more effective tool at their disposal, combining accuracy and ease of use to make an effective last-buy decision.

CHAPTER 5
THE LAST BUY PROBLEM WITH NO REPLENISHMENT

5.1 Introduction

In the last buy problem with no replenishment, parts from the last buy order are used to replace failed parts until the last buy order is depleted. Once this occurs, no future demand will be met; any part failure that occurs will not be repaired, the corresponding assembly will be scrapped, and the manufacturer will pay a penalty to the customer as recompense for the manufacturer's inability to meet demand. These cases occur when any replenishment method available - whether through setting up the original manufacturing process or using some other form of fabrication - is either infeasible to implement or is too expensive.

5.2 Components of the Last Buy Problem

As with the last buy problem with incremental replenishment, the total expected profit, $\pi(q)$, is the value of the expected revenues minus the value of the expected costs:

$$\pi(q) = R(q) + S(q) - M(q) - H(q) - Z(q). \tag{5.1}$$

We now provide details on each of the terms in (5.1). As this is one variation of the last buy problem, some components of the last buy problem with no replenishment are identical to those used in the last buy problem with incremental replenishment. Thus, the properties for the revenue, the salvage, the holding costs, and the manufacturing costs can be found in Chapter 4 of this thesis.

Before discussing the parts of the last buy problem with no replenishment, a review of some notation specific to Chapter 5:

- $\pi_p(q)$: In the last buy problem with no replenishment, $\pi_p(q)$ represents a portion of the total expected discounted profit - specifically, the total profit without the penalty cost subtracted. Thus, $\pi(q) + Z(q) = \pi_p(q)$.
- $A_{k,t}$: In cases where the assemblies cannot be evaluated as a group, $A_{k,t}$ represents the status of assembly k at time t. If $A_{k,t} = 1$, then assembly k is still functioning at time t; if $A_{k,t} = 0$, then assembly k has failed by time t.
- $D_{k,t}$: In cases where the assemblies cannot be evaluated as a group, $D_{k,t}$ represents the number of parts demanded by assembly k up to time t.
- $K(l, t_f)$: The expected standardized ($p = 1$) penalty, discounted to time t_f, given that l assemblies remain in operation at time t_f and the last buy order is depleted at time t_f.$K(l, t_f)$ can be thought of as the expected number of assemblies, discounted to time t_f, that the manufacturer will have to pay a penalty for given that the manufacturer is unable to meet customer demand starting at time t_f.
- $\Delta\pi_{ub}(q)$: An upper bound on the rate of change of $\pi(q)$.

5.2.1 Revenue

The function for the expected revenue, discounted over time, is identical to the function used for revenue in the last buy problem with incremental replenishment, as presented in Subsection 4.2.1. Given a probability distribution of demand over time $P\{D_t = n\}$, our expected discounted revenue is

$$R(q) = -r \sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt. \tag{5.2}$$

The properties for the revenue can be found in Subsection 4.2.1.

5.2.2 Manufacturing Costs

The total manufacturing cost is simply the product of the last buy order amount q and the per-unit manufacturing cost m:

$$M(q) = mq. \tag{5.3}$$

5.2.3 Holding Costs

The holding costs are the associated labor, facility, and infrastructure costs incurred by holding a part in inventory. This cost is designated to have a cost of h per unit per unit time. Thus, if one spare part is held in inventory for one unit of time, a cost of h would be incurred.

The holding costs for the last buy problem with no replenishment are identical to the last buy problem with incremental replenishment. Also, the total discounted holding cost used depends on the specifics of the problem. In a non-contract problem where we would have no salvage, the expected value of the holding cost becomes

$$H(q) = h\sum_{n=0}^{q-1}(q-n)\int_{t=0}^{\infty} e^{-t\alpha}P\{D_t = n\}dt. \tag{5.4}$$

In a contract problem in which the salvage occurs when the last assembly fails, the holding cost is

$$H(q) = h\sum_{n=0}^{q-1}(q-n)\int_{t=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f} e^{-t\alpha}P\{D_t = n|A_{t_f}=1\}dtdt_f. \tag{5.5}$$

The properties of the holding costs can be found in Subsection 4.2.3.

5.2.4 Penalty Costs

The penalty cost is the amount paid to the customer that demands a spare part after the last buy order has been depleted. The formulation of this replenishment cost, regardless of whether the replenishment period is defined by contract (t_c finite) or undefined ($t_c = \infty$), has the following form:

$$Z(q) = -p\sum_{l=1}^{j}\int_{t_f=0}^{t_c} e^{-t_f\alpha}\frac{dP\{D_{t_f}\le q|A_{t_f}=l\}}{dt_f}P\{A_{t_f}=l\}K(l,t_f)dt_f. \tag{5.6}$$

The $K(l, t_f)$ function in the penalty cost represents the expected penalty cost, discounted to time t_f and standardized to $p = 1$, given that l assemblies remain in operation at time t_f, which is the time at which the $(q+1)$th part is demanded. This is multiplied by the penalty cost per assembly p to get the expected penalty cost, discounted to t_f, given l assemblies remain. The $\sum_{l=1}^{j} P\{A_{t_f} = l\} \frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f}$ terms represent the rate of increase in the probability of the $(q+1)$th part demand occurring.

The structure of the term $K(l, t_f)$ varies according to the type of problem. In the contract case, the structure of $K(l, t_f)$ is

$$K(l, t_f) = \{ \begin{matrix} l \; if \; t_f \leq t_c \\ 0 \, otherwise \end{matrix} . \tag{5.7}$$

The reason for this structure is that, once the manufacturer is unable to meet the requirements of a contract, the manufacturer must immediately pay a contract penalty proportional to the number of assemblies still under contract - that is, all assemblies still operational at time t_f

The structure of the non-contract case is more difficult. Once the last buy order is depleted, each assembly will fail with either a part failure event or assembly failure event. If the assembly fails because of a part failure event, the manufacturer must pay a penalty to the customer due to the failure of the manufacturer to meet part demand. If the assembly fails for any other reason, no penalty will be paid for that particular assembly. One assembly has already failed due to part failure - the assembly that demanded the $(q+1)$th part. Thus, for each assembly that remains in operation beyond time t_f, it becomes a question as to which fails first - the part or the assembly.

Thus, for the non-contract case, the structure of $K(l, t_f)$ is

$$K(l,t_f) = 1 - \sum_{k=1}^{l-1} \int_{t_f}^{\infty} e^{(t_f - t)\alpha} P\{A_{k,t} = 1 | A_{k,t_f} = 1\} \frac{dP\{D_{k,t} = n | D_{k,t_f} = n\}}{dt} dt. \quad (5.8)$$

This could be considered the long form of the $K(l, t_f)$ term. Because of the possibility that the individual assembly and part lifetimes are not IID at this point, each assembly is viewed individually. Fortunately, this is not too onerous a task, as each assembly is reduced to a binary question: Which fails first, the part or the assembly? If the assembly fails first, then no penalty will be incurred; if the part fails first, then a penalty of p units will be assessed at some future time. The 1 at the beginning of the term reflects the demand of the $(q+1)$th part; as it is this demand that begins the penalty portion of the last buy period, the penalty occurs immediately, and the assembly that demanded the part immediately fails.

If the lifetime of the parts follows a Poisson process with parameter λ, the structure of each part still in operation beyond time t_f can be considered IID. As a result, the structure of $K(l, t_f)$ for the non-contract case is

$$K(l,t_f) = 1 + (l-1) \int_{t_f}^{\infty} (e^{(t_f - t)\alpha} - e^{(t_f - t)(\alpha+\lambda)}) P_1\{A_t = 1 | A_{t_f} = 1\} dt. \quad (5.9)$$

The 1 in the formula is a result of being unable to satisfy the demand resulting from the $(q+1)$th part. The integral $\int_{t_f}^{\infty}(e^{(t_f-t)\alpha}-e^{(t_f-t)(\alpha+\lambda)})P\{A_{1,t}=1|A_{1,t_f}=1\}dt$ represents the expected penalty, discounted to time t_f and standardized to $p=1$, from a single assembly remaining in operation beyond time t_f. This integral is then multiplied by $l-1$ - the number of assemblies that remain in operation beyond time t_f.

The properties of the penalty cost with respect to q are:

Lemma 5.1. *The penalty cost has the following properties, assuming* $p \geq 0$*:*

- (a) As q increases, the penalty cost $Z(q)$ decreases.
- (b) The convexity of the penalty cost $Z(q)$ cannot be determined.
- (c) For any $q \geq 0$, the negative of the rate of change of the penalty $-(Z(q+1)-Z(q))$ is less than the penalty $Z(q)$.
- (d) As q becomes large ($q \to \infty$), the penalty $Z(q) \to 0$.
- (e) The penalty cost $Z(q)$ is non-negative.

The result for the penalty cost means that concavity properties alone are insufficient for determining the optimal order amount. However, because the penalty must remain positive and because it decreases with increasing q, an upper bound on the rate of change of the penalty cost exists. By extension, this upper bound on the rate of change of the penalty can be used to find an upper bound on the rate of change of the profit $\pi(q)$ - an upper bound that can be demonstrated to decrease with increasing q. The last part of the lemma is used to tie the upper bound of the penalty cost with the penalty itself; as q becomes large, both the penalty cost $Z(q)$ and the rate of change of the penalty $(Z(q+1) - Z(q))$ approach 0.

The proofs for Lemma 5.1 can be found in Appendix B.

5.2.5 Salvage

The salvage differs from most other costs and revenues in that it does not occur gradually over the course of the post-production phase; rather, it occurs when the last assembly has failed at time t_f. Once this occurs, the remaining inventory is discarded at a revenue of s per unit. (If the financial event of scrapping each unit generates a cost instead of a revenue, then s is a negative number.) As with the revenue, the manufacturing cost, and the holding cost, the formula for the salvage in the last buy problem with no replenishment is the same as in the last buy problem with incremental replenishment.

For the last buy problem with no replenishment, the salvage is.

$$S(q) = s\sum_{n=0}^{q-1}(q-n)\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=n|A_{t_f}=1\}dt_f. \tag{5.10}$$

As stated earlier, the properties for the salvage can be found in Subsection 4.2.5.

5.2.6 Profit Function

Now that the components of the Last Buy Problem have been formulated, we can now combine those components into full cost and revenue functions for the entire problem. The goal is to take the components found in the previous section, take the sum of these components as an overall profit function, then find the maximum of this function. The basic structure of the problem is given by (5.1):

$$\pi(q) = R(q) + S(q) - H(q) - M(q) - Z(q)..$$

The goal will be to find the value of q that maximizes $\pi(q)$.

Theorem 5.2. *The concavity of the profit function cannot be determined.*

The theorem comes as a result of Lemma 5.1. Because the penalty cost is neither concave nor convex, we cannot use concavity alone to find the optimal.

Theorem 5.3. *The portion of the profit function* $\pi_p(q) = R(q) + S(q) - H(q) - M(q)$ *is concave with respect to the positive integers - and therefore its rate of change of this portion of the profit function decreases with increasing* q*.*

The theorem can be proven based on Lemmas 4.1, 4.2, and 4.4. Each of the parts of this portion of the profit function are integer concave with respect to q - and, thus, the sum of these parts of the profit function is integer concave. As a result of this property and the properties of concave functions, the rate of change of this portion of the profit function decreases with increasing q.

At this point, it is necessary to establish the formula of the rate of change of the profit.

With $\Phi(\cdot)$ as a function defined in the nonnegative integers, letting $\Delta\Phi(q) = \Phi(q+1) - \Phi(q)$, the rate of change of our profit function is

$$\Delta\pi(q) = \Delta R(q) + \Delta S(q) - \Delta H(q) - \Delta Z(q) - \Delta M(q). \tag{5.11}$$

This leads to our last theorems:

Theorem 5.4. *The formula* $\Delta\pi_{ub}(q) = \Delta R(q) + \Delta S(q) - \Delta H(q) - \Delta M(q) + Z(q)$ *is an upper bound on* $\Delta\pi(q)$*, and decreases over time.*

Because $Z(q) \geq -(Z(q+1) - Z(q)) = -\Delta Z(q)$, the value for $\Delta\pi_{ub}(q)$ must be greater than or equal to $\Delta\pi(q)$, making it an upper bound. Moreover, because both $Z(q)$ and the rate of change of the portion of the profit function $\Delta\pi_p(q) = \Delta R(q) + \Delta S(q) - \Delta H(q) - \Delta M(q)$ are decreasing with increasing q, the sum $\Delta\pi_{ub}(q) = \Delta R(q) + \Delta S(q) - \Delta H(q) - \Delta M(q) + Z(q)$ must also be decreasing with increasing q.

In other words, we have established an upper bound on the rate of change of $\pi(q)$ - one that decreases as q increases.

Theorem 5.5. *If, for any order amount* $\hat{q}$*,* $\Delta\pi_{ub}(\hat{q}) < 0$*, then any order amount* $q^+ > \hat{q}$ *cannot be the optimal order amount.*

This comes as a result of Theorem 5.4. Because $\Delta\pi_{ub}(q)$ is decreasing with increasing q, once it becomes negative at any $\hat{q}$, $\Delta\pi_{ub}(q^+)$ cannot be positive for any $q^+ > \hat{q}$. Moreover, because $\Delta\pi_{ub}(q) > \Delta\pi(q)$, the rate of change of the profit function

$\Delta\pi(q)$ cannot be positive for any $q^+ > \hat{q}$. Thus, for any $q^+ > \hat{q}$, $\pi(\hat{q}) > \pi(q^+)$ - and any $q^+ > \hat{q}$ cannot be an optimal order amount.

The result of this is the establishment of an upper bound on the optimal order amount. Since any $q^+ > \hat{q}$ cannot be an optimal order amount if $\Delta\pi_{ub}(\hat{q}) < 0$, we only need to search $\{q : q \leq \hat{q}\}$ for an optimal solution.

Theorem 5.6. *If a finite optimal order amount exists, an upper bound for the optimal order amount q^* can be found by using $\Delta\pi_{ub}(q)$*

This can be found by looking at the limit of $\Delta\pi_{ub}(q)$ as $q \to \infty$. Because both the penalty cost $Z(q)$ and the rate of change of the penalty cost $(Z(q+1) - Z(q))$ approach 0 as q becomes large, $\Delta\pi_{ub}(q) \to \Delta\pi(q)$ as $q \to \infty$. Thus, the only case in which $\Delta\pi_{ub}(q)$ would not eventually become negative is if $\Delta\pi(q)$ does not become negative - which occurs only in cases where no finite optimal solution exists.

Thus, we have a solution method. We calculate $\Delta\pi(q)$ and $\Delta\pi_{ub}(q)$ for increasing values of q. Any potential solution $\hat{q}$ will have two properties. If q does not meet all of these properties, then it cannot be an optimal solution, because another q would have a greater value of $\pi(q)$.

1. $\Delta\pi(\hat{q}) \leq 0$: The value of any optimal solution $\pi(q^*)$ must be greater than or equal to $\pi(q^* + 1)$; thus, the rate of change $\Delta\pi(\hat{q})$ must be non-positive.
2. $\Delta\pi(\hat{q} - 1) \geq 0$: Similarly, the value of any optimal solution $\pi(q^*)$ must be greater than or equal to $\pi(q^* - 1)$. Thus, the rate of change $\Delta\pi(\hat{q} - 1)$ must be non-negative for any potential solution.

As a result, we can limit our search of solutions to only those values of $\hat{q}$ that have these properties. Moreover, once we find a $\hat{q}$ such that $\Delta\pi_{ub}(\hat{q}-1) < 0$, we can stop our search; whichever potential solution that fits the above criteria and provides the highest profit is our optimal.

Thus, our solution method is developed. We find $\Delta\pi(q)$ and $\Delta\pi_{ub}(q)$ for increasing values of q. Any $\hat{q}$ that meets our above criteria is given as a potential solution. Once a given value of $\bar{q}$ generates a value of $\Delta\pi_{ub}(\bar{q})$ such that $\Delta\pi_{ub}(\bar{q}) < 0$, then we can stop our search for potential solutions, as $q^* \leq \bar{q}$. We then evaluate any potential solutions already identified; the optimal order amount q^* is the potential solution such that $\pi(q^*) \geq \pi(\hat{q})$ for all other potential solutions $\hat{q}$.

We demonstrate this in the following examples:

5.2.6.1 No Replenishment with Contract

In this example, the manufacturer is bound by contract to serve the customer up to a fixed time t_c. As such, the manufacturer is aware of the number of assemblies in operation at any given time. Should a part fail before time t_c, the manufacturer is obligated to supply a replacement part; however, the manufacturer is not obligated to supply a replacement part beyond time t_c. Also, when the last assembly has failed at time t_f, any parts remaining in inventory will be salvaged.

The formula for the rate of change of the last buy profit $\Delta\pi(q)$ is

$$\Delta\pi(q) = m + p\sum_{l=1}^{j}\int_{t_f=0}^{t_c} e^{-t_f\alpha}\frac{dP\{D_{t_f} = q+1|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l,t_f)dt_f$$
$$+\sum_{n=0}^{q}\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f} = 0\}}{dt_f}(se^{-t_f\alpha}P\{D_{t_f} = n|A_{t_f} = 1\}$$
$$+h\int_{t=0}^{t_f} e^{-t\alpha}P\{D_t = n|A_{t_f} = 1\}dt)dt_f$$
$$+r\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t \le n\}}{dt}dt. \tag{5.12}$$

The upper bound on the rate of change, $\Delta\pi_{ub}(q)$, is

$$\Delta\pi_{ub}(q) = m - p\sum_{l=1}^{j}\int_{t_f=0}^{t_c} e^{-t_f\alpha}\frac{dP\{D_{t_f} \le q|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l,t_f)dt_f$$
$$+\sum_{n=0}^{q}\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f} = 0\}}{dt_f}(se^{-t_f\alpha}P\{D_{t_f} = n|A_{t_f} = 1\} \tag{5.13}$$
$$+h\int_{t=0}^{t_f} e^{-t\alpha}P\{D_t = n|A_{t_f} = 1\}dt)dt_f$$
$$+r\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t \le n\}}{dt}dt.$$

In a contract case, the penalty formula $K(l, t_f)$ is simply l.

Example: An assembly for a part from time $t = 0$ has an expected lifetime represented as a Weibull distribution with rate parameter $\beta = \frac{1}{3}$ and exponential parameter $\gamma = 1.5$. The failure rate for each part is an exponential with rate parameter $\lambda = 0.5$.

Ten assemblies are in operation at time $t = 0$, with one part used per assembly. What is the optimal order amount for spare parts in the last buy period?

Solution: We find the values of $\Delta\pi(q)$ and $\Delta\pi_{ub}(q)$ for increasing values of q. The results are given in Table 5.1.

Table 5.1: Rate of Change and Upper Bound on the Rate of Change of Last Buy Components, Contract Example

Last Buy q	1	2	3	4	5	6	7	8	9	10
$\Delta\pi(q)$	25.3	25.7	25.4	24.5	23.2	21.6	19.6	17.3	14.7	11.9
$\Delta\pi_{ub}(q)$	183.5	167.0	149.8	132.5	115.5	99.1	83.7	69.3	56.2	44.3
Last Buy q	11	12	13	14	15	16	17	18	19	20
$\Delta\pi(q)$	9.0	6.2	3.5	1.1	-1.0	-2.7	-4.0	-5.0	-5.6	-6.1
$\Delta\pi_{ub}(q)$	33.8	24.7	17.1	10.8	5.8	1.9	-0.9	-3.0	-4.3	-5.3

As we see, the rate of change of the profit $\Delta\pi(q)$ goes from positive to negative only at $q = 15$; our upper bound becomes negative at $q = 18$. Thus, our optimal order amount should be found at $q = 15$. This is found to be the case, as our optimal solution is at $q = 15$, with a total profit π^* of 64.5. It is interesting to note the behavior of $\Delta\pi(q)$ in this case. As our proofs of the penalty cost indicate, the profit function is not concave; this is demonstrated by the increase in $\Delta\pi(q)$ from $q = 1$ to $q = 2$.

5.2.6.2 No Replenishment with No Contract

In the case presented here, no contract exists between the manufacturer and the customer. However, the manufacturer still wants to supply spare parts to customers, even if the last buy order is depleted. We assume that the manufacturer knows the number of assemblies still in operation at the last buy decision point, but will have no knowledge of the number of assemblies in operation beyond that point. Because of this, no salvage will take place, and all spare parts are assumed to be held until sale; as a result, the salvage $S(q)$ is not included in (5.1).

Our formula for the rate of change of profit, $\Delta\pi(q)$, is

$$\Delta\pi(q) = m + h\sum_{n=0}^{q-1}\int_{t=0}^{\infty}(q-n)e^{-t\alpha}P\{D_t = n\}dt$$
$$+p\sum_{l=1}^{j}\int_{t_f=0}^{\infty}e^{-t_f\alpha}\frac{dP\{D_{t_f} = q+1|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l,t_f)dt_f \tag{5.14}$$
$$+r\int_{t=0}^{\infty}e^{-t\alpha}\frac{dP\{D_t \leq n\}}{dt}]dt.$$

Our upper bound on the rate of change, $\Delta\pi_{ub}(q)$, is

$$\Delta\pi_{ub}(q) = m + h\sum_{n=0}^{q-1}\int_{t=0}^{\infty}(q-n)e^{-t\alpha}P\{D_t = n\}dt$$
$$-p\sum_{l=1}^{j}\int_{t_f=0}^{\infty}e^{-t_f\alpha}\frac{dP\{D_{t_f} \leq q|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l,t_f)dt_f \tag{5.15}$$
$$+r\int_{t=0}^{\infty}e^{-t\alpha}\frac{dP\{D_t \leq n\}}{dt}]dt.$$

In this case, our multiplier $K(l, t_f)$, from formula (5.9), is

$$K(l,t_f) = 1 + (l-1)\int_{t_f}^{\infty}(e^{(t_f-t)\alpha} - e^{(t_f-t)(\alpha+\lambda)})P_1\{A_t = 1|A_{t_f} = 1\}dt.$$

Example: An assembly for a part from time $t = 0$ has an expected lifetime represented as a Normal distribution with mean $\mu = 6$ and standard deviation $\sigma = 1$. The failure rate for each part is an exponential with rate parameter $\lambda = 0.2$.

Fifteen assemblies are in operation at time $t = 0$, with one part used per assembly. The part is manufactured at a cost of 3 units, and is sold for a cost 10 units. The cost of holding the part in inventory is 0.75 per part per unit time. If a part is demanded by an assembly after the last buy order has been depleted, then the assembly fails and a cost of 10 units is incurred. What is the optimal order amount for spare parts in the last buy period?

Table 5.2: Rate of Change and Upper Bound on the Rate of Change of Last Buy Components, Non-Contract Example

Last Buy q	1	2	3	4	5	6	7	8	9	10
$\Delta\pi(q)$	11.7	11.4	11.1	11.0	10.8	10.7	10.5	10.1	9.7	9.0
$\Delta\pi_{ub}(q)$	113.0	107.1	101.1	95.0	88.7	82.1	75.3	68.3	61.0	53.6
Last Buy q	11	12	13	14	15	16	17	18	19	20
$\Delta\pi(q)$	8.0	6.6	4.9	2.9	0.5	-2.0	-4.6	-7.0	-9.4	-11.4
$\Delta\pi_{ub}(q)$	46.1	38.6	31.1	23.9	17.1	10.7	5.0	-0.1	-4.4	-8.0

The results of our calculations are found in Table 5.2. As $\Delta\pi(q)$ goes from positive to negative only at $q = 16$, it is the only possible solution of the ones we have searched; as $\Delta\pi_{ub}(q)$ becomes negative at $q = 18$, we know that we do not need to search any values of q above that. Thus, our optimal solution can be found at $q = 16$. This proves to be the case, with a total profit at $q^* = 16$ of $\Delta\pi(q^*) = 28.5$.

5.3 Numerical Experiments

For our first numerical experiment to test the effectiveness of our solution method, we first compare it to other methods that a manager could potentially use to solve for the last buy problem with no replenishment. These methods include:

- Average demand.
- Newsvendor formula, with the penalty cost per assembly used in the shortage cost.
- Newsvendor formula, in which the penalty used in the shortage cost is the penalty cost divided by the expected number of parts demanded per assembly. This is to attempt to represent a shortage cost per unit penalty closer to the actual penalty shortage cost per spare part.

To that end, we consider the following example. Fifteen assemblies, each with an exponential distribution (mean $\frac{1}{\beta} = 4$). Each part lifetime also has an exponential distribution (mean $\frac{1}{\lambda} = 2$). The holding cost is 1 per spare part per unit time, salvage is 4 units per part, manufacturing cost is 12 units per part, and revenue is 25 units per part. If inventory runs out, a penalty cost of 40 per assembly is incurred. All revenues and costs occur with a discount rate of $\alpha = 0.08$.

The results are found in Table 5.3. As we see, other estimation methods, such as newsvendor and average demand, result in profit losses ranging from 3% to 28%. Thus, our solution method can provide significant savings over other forms of estimation. We note that it appears that a scaled version of the newsvendor, one

Table 5.3: Solution Results for Different Calculation Methods, Last Buy Problem with No Replenishment

Method	Solution q	Profit $\pi(q)$	Difference in $\pi(q)$	% loss
Our method	28	63.6		
Average demand	30	61.6	-2.0	3.1
Newsvendor (unscaled)	33	45.5	-18.1	28.5
Newsvendor (scaled)	31	57.8	-5.8	9.1

where the penalty cost is closer to a representation of the actual cost of shortage, seems to produce better results than an unscaled newsvendor.

In another experiment, we found that, unless the holding cost is unusually large, an increase in the discount rate α used will result in a decrease in the optimal order amount q^*. The reason for this is that, with the exception of the holding cost, a decrease in the present value of any costs and revenues that are incurred beyond time $t = 0$ will result in a decrease in the optimal order amount. A decrease in revenue or salvage will decrease the optimal order amount; a decrease in the penalty, while increasing the overall profit, reduces the incentive on the part of the manufacturer to supply more parts - and, thus, also reduces the optimal order amount. Thus, a decrease in the revenue from sales, a decrease in the salvage, and a decrease in the penalty costs will all result in a lower optimal order amount q^*. As an increase in the discount rate results in a decrease in the total revenue from sales, total salvage, and total penalty cost, such an increase will cause a decrease in the optimal order amount q^*.

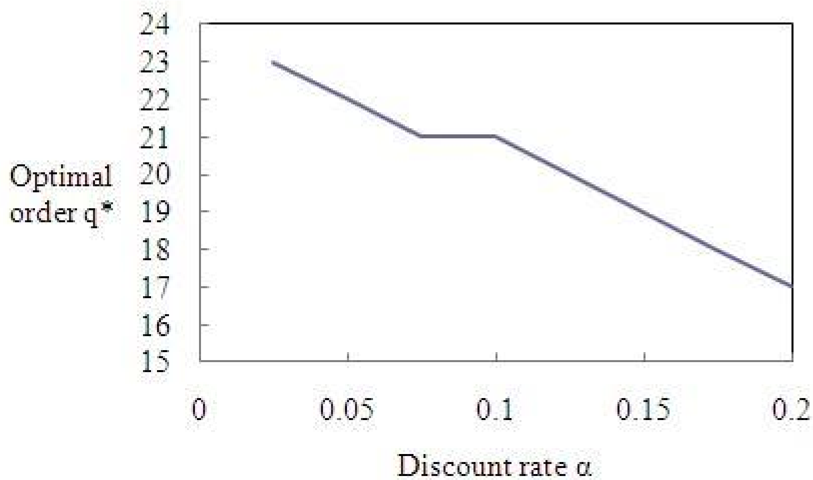

Figure 5.1: Change in Optimal Order Quantity with Discount Rate, Last Buy Problem with No Replenishment

We demonstrate this with the following example. The distribution of the lifetime of an assembly from time $t = 0$ is represented as a Weibull distribution with rate parameter $\beta = 0.3$ and exponent parameter $\gamma = 1.5$. The distribution of the lifetime of the spare part is represented as an exponential distribution with mean $\frac{1}{\lambda} = 3.333$. The penalty per assembly is $p = 20$, the revenue per unit sold is $r = 20$, the salvage cost per unit is $s = 3$, the holding cost is $h = 1$ per part per unit time, and the manufacturing cost per unit is $m = 5$.

We show the optimal order amounts for the example with varying values of α ranging from $\alpha = 0.025$ to $\alpha = 0.2$. The results for these are found in Figure 5.1. As we see, the result decreases with increasing values of α, from $q^* = 23$ at $\alpha = 0.025$ to $q^* = 17$ at $\alpha = 0.2$.

Determining the distribution of demand for large numbers of assemblies may produce computation difficulties not found in smaller problems. To illustrate this, consider the use of factorials in the calculation of some members of the exponential family of probability distributions; exact calculation of probabilities in binomial and gamma distributions becomes difficult for large sample sizes. Because of this, it becomes important to see if the demand might be approximated by determining the solution for a smaller number of assemblies, then scaling the result for a larger number of assemblies.

What we found differed from our results for the last buy problem with incremental replenishment. Unlike with incremental replenishment, scaled results for no replenishment cases will tend to overestimate the optimal order amount for larger numbers of assemblies. We use three cases to illustrate this, each with different types of distributions for the lifetime of the assembly, and find the optimal order amount for cases with 5, 10, 20, and 40 assemblies at time $t = 0$:

- *Case 1*: Exponential distribution for the assembly lifetime with mean time $\frac{1}{\beta} = 3.333$, exponential part lifetime with mean $\frac{1}{\lambda} = 3.333$, discount rate $\alpha = 0.05$. Unit holding cost $h = 1$, penalty cost $p = 50$, salvage cost $s = 4$, manufacturing cost $m = 10$, revenue $r = 10$.

- *Case 2*: Weibull distribution for the assembly lifetime with rate parameter $\beta = 0.125$ and exponent parameter $\gamma = 1.25$, exponential part lifetime with mean $\frac{1}{\lambda} = 6$, discount rate $\alpha = 0.1$. Unit holding cost $h = 0.8$, penalty cost $p = 50$, salvage cost $s = 2$, manufacturing cost $m = 5$, revenue $r = 20$.

- *Case 3*: Normal distribution for the assembly lifetime with mean $\beta = 0.125$ and exponent parameter $\gamma = 1.25$, exponential part lifetime with mean $\frac{1}{\lambda} = 6$, discount rate $\alpha = 0.1$. Unit holding cost $h = 0.8$, penalty cost $p = 50$, salvage cost $s = 2$, manufacturing cost $m = 5$, revenue $r = 20$.

The results of scaling all results to 40 assemblies is represented in Table 5.4. Note the results for smaller numbers of assemblies as they are scaled to larger results. In every case, linear scaling of the optimal results for small numbers of assemblies produces an estimate greater than the optimal order amount for larger numbers of assemblies.

Table 5.4: Solution Results from Scaling for Different Distributions

Distribution type	# of Assys	Solution	Soln Scaled to 40 Assys
Exponential	40	43	43
Exponential	20	23	46
Exponential	10	12	48
Exponential	5	7	56
Weibull	40	39	39
Weibull	20	21	42
Weibull	10	11	44
Weibull	5	6	48
Normal	40	34	34
Normal	20	19	38
Normal	10	10	40
Normal	5	6	48

To further demonstrate the effect of the penalty cost on scaling, we produce an additional experiment. We perform the scaling experiment for three cases, identical in every way except the value of the unit penalty cost p, by taking Case 3 (Normal distribution of assembly lifetime) for different values of p.

The results of this can be found in Table 5.5. As we see, for cases with small values of p (in this case, $p = 10$), linear scaling of results for smaller numbers of assemblies produces results close to the optimal for larger numbers of assemblies. As p increases, linear scaling of smaller numbers of assemblies tends to produce over-estimation of the optimal order amount for larger numbers of assemblies. As such, any solution method to the last buy problem with no replenishment that involves the scaling of answers should be done with caution, especially if the penalty cost is high.

Table 5.5: Solution Results from Scaling for Different Values of p

Distribution type	# of Assys	Solution	Soln Scaled to 40 Assys
Normal, $p = 10$	40	30	30
Normal, $p = 10$	20	16	32
Normal, $p = 10$	10	8	32
Normal, $p = 10$	5	4	32
Normal, $p = 25$	40	32	32
Normal, $p = 25$	20	17	34
Normal, $p = 25$	10	9	36
Normal, $p = 25$	5	5	40
Normal, $p = 50$	40	34	34
Normal, $p = 50$	20	19	38
Normal, $p = 50$	10	10	40
Normal, $p = 50$	5	6	48

This result is due to the effect of scaling on the standard deviation of summed results, combined with the penalty cost pushing the optimal order amount to the extreme right end of the demand probability curve. In essence, if it is unlikely for the demand of one assembly to reach q^*, it is even more unlikely for the demand of two identical assemblies to reach $2q^*$. As a result, a linear scaling of the result for one assembly will tend to be overly conservative for two assemblies in avoiding the penalty cost.

One last numerical result relates to the method of calculation used to solve the last buy problem. Infinite-timeframe integrations in the last buy problem may be approximated by integrating up to finite but sufficiently large values of t, and vice versa. One of the problems involved with the integration of $P\{D_t = n\}$ or $\frac{dP\{D_t=n\}}{dt}$ as a solution method is that the integration of either term may prove intractable, depending on the formulation of $P\{D_t = n\}$. For instance, a polynomial formulation of $P\{D_t = n\}$ may only provide a useable formula in finite-time integrations; conversely, exponential-based formulations of $P\{D_t = n\}$ tend to work best with infinite-time integrations. As such, it is useful to see the difference in the solution found between finite-time and infinite-time formulations, to test whether or not one may be used as an approximation for the other.

As a result of this, we present the following case. The manufacturer is contracted to supply spare parts for the customer for five years i.e. $t_c = 5$. 25 assemblies are still in operation at time $t = 0$, with the distribution of the lifetime of the assembly represented as a Weibull with rate parameter $\beta = 0.2$ and exponential parameter $\gamma = 1.25$. The spare part is represented as an exponential with mean lifetime $\frac{1}{\lambda} = 3.333$. The discount rate is $\alpha = 0.05$. The revenue generated per part supplied is $r = 20$, the penalty per assembly for being unable to supply the spare part during the contract period is $p = 40$, the parts are manufactured at a cost of $m = 10$ per part, and any parts remaining after all the assemblies have failed will be salvaged at a revenue of $s = 3$ per part. Parts in inventory generate a holding cost of $h = 1$ per part per year. Instead of integrating up to $t = \infty$ for the revenue, holding, and salvage, we integrate only up to a finite time t_{end}, and compare the results. The results are found in Table 5.6.

Table 5.6: Solution Results from Finite-Time Integrations

Time t_{end}	Solution $q*$	Difference
7.5	30	2
10	29	1
20	28	0
40	28	0
∞	28	0

Given the results, it is apparent that the primary purpose of the integration is to cover a feasible region during which assemblies could, within a reasonable probability, be operating. In this case, by time $t_{end} = 10$, each assembly has only a 9.3% chance of still operating; by $t_{end} = 20$, each assembly only has a 0.4% chance of still operating. Thus, most part demands will have occurred before time $t_{end} = 10$ - and virtually all part demands will have occurred by time $t_{end} = 20$. To compound the matter, all costs and revenues are discounted over time; as a result, the value of each unit of revenue or cost at time $t_{end} = 20$ will be worth only only 36.7% of the value of any cost or revenue at time 0. Thus, we are able to find the optimal solution to the problem while integrating over a finite horizon, provided that the finite horizon is largely beyond the reach of the lifetimes of the assemblies.

5.4 Managerial Insights

Our solution for the last buy problem with no replenishment is a useful tool for managers to use in cases where forms of replenishment are either infeasible or too costly to be effective. It provides a more accurate representation of the optimal order amount than other available calculation methods.

When calculating the optimal order amount, a manager should be aware of the effect of the discount rate α on the optimal solution. The manufacture of spare parts in the last buy is like any other investment; money is used to manufacture the spare parts and to pay for its storage so that, later, the part can be sold to a customer to avoid any significant penalties from running out. The effect of a larger α on the optimal solution is that the benefits from the sale and from avoiding the penalty costs are reduced; therefore, the optimal order amount is also reduced with larger α.

One noted difference between the calculation methods for the last buy problem with incremental replenishment and the last buy problem with no replenishment is that solutions for the last buy problem with no replenishment do not scale linearly, i.e. if the optimal order amount for j assemblies is q^*, then the optimal order amount for $2j$ assemblies is likely to be less than $2q^*$. This is due to the nature of the penalty cost in the no-replenishment case, as it tends to increase at a slower rate than the other costs with increasing numbers of assemblies. Thus, the last buy problem with no replenishment does not lend itself to scaling, which would normally ease the calculation of difficult problems with large numbers of assemblies.

Finally, if the integration of $P\{D_t \leq n\}$ or $\frac{dP\{D_t \leq n\}}{dt}$ proves to be intractable, it may be eased by possibly substituting a finite-timeframe integration for an infinite-timeframe integration, if the timeframe is sufficiently large. When substituting a finite-timeframe integration for an infinite-timeframe integration or vice versa, it appears from our limited numerical experiments that the finite time used should be at least greater than twice the mean lifetime of the assembly.

5.5 Conclusion

In this chapter we have demonstrated a continuous-time method for calculating the optimal last-buy order amount in cases where demand cannot be effectively satisfied should the last-buy order run out. We find the discounted expected value of the total profit over time given a last buy order amount of q. Because the total profit is neither concave nor convex with respect to the order amount q, we must limit the possible solutions available, as a local maximum is not necessarily a global maximum. We therefore find an upper bound on the rate of change of the total revenue for increasing values of q; this gives us an upper bound on our optimal order amount q^*. We then examine the rate of change of the total revenue for values of $q \leq q^*$, and check any potential solutions (defined as values of q where the rate of change goes from positive to negative). The maximum solution among any remaining candidates is our optimal order amount q^*.

Overall, the model for the last buy problem with no replenishment is somewhat more difficult than the solution methods for the last buy problem with incremental replenishment. For instance, linear scaling of solutions is not recommended for the last buy problem with no replenishment. However, there are tools available to mitigate some of these difficulties. For instance, as it is only necessary to integrate over the feasible life of the assembly, in most cases solutions can be reasonably approximated by integrating only over a limited lifetime. This allows for some tailoring of the integrations, to ease overall calculation.

CHAPTER 6
A RENEWAL-THEORY SOLUTION TO THE LAST BUY PROBLEM WITH BATCH REPLENISHMENT

In the last buy problem with batch replenishment, parts from the last buy order are used to replace failed parts until the last buy order is depleted. Once this occurs, the manufacturing process by which the original last buy was manufactured will be set up again, and a new batch manufactured. Setting up the new batch requires an additional setup cost, as well as the cost to manufacture the spare parts. This method of replenishment is most common in cases involving large numbers of assemblies in operation at the last buy point, where the setup costs are small in comparison to the variable cost of manufacture.

Unfortunately, determining the optimal order amount in such cases is difficult because each batch is itself a last buy problem. As a result, it becomes impossible to evaluate all possible outcomes, as the value and probability of each potential batch references the value and probability of each subsequent potential batch. Because of this, we must simplify the problem by making the following assumptions:

- The lifetime of each part and each assembly can be represented as an exponential distribution.
- The lifetime of the contract (the time during which the manufacturer is expected to serve the customer) is considered long enough that all assemblies are likely to fail before the contract expires.

Both of these assumptions allow for the memoryless properties of exponential distributions to come into play, effectively turning the problem into a Markov decision process. Thus, time is no longer required to be a variable in the problem. This simplifies calculations considerably, as the only variables remaining to determine the state of the problem are the number of parts remaining in inventory and the number of assemblies still in operation.

6.1 Renewal Theory

The mathematical basis behind renewal theory lies in the structure and properties of the exponential distribution. The exponential distribution is the only distribution that has the following property:

$$F\{t + t_1 | t_1\} = F\{t\} \, \forall \, t, t_1 \geq 0. \tag{6.1}$$

This property, known as the memoryless property [10], means that the rate of an event occurring does not change as the time t advances. As a result, time no longer becomes a determining factor in determining the expected future states of the system - only the current state. Thus, the expected value of a current state can be determined by representing the problem as a continuous-time Markov chain. We find the total expected value by finding the expected value generated during the current state plus the probability-weighted expected values of all of the future states that the current state can transition into.

$$G(i) = g(i) + \sum_{h} P(h|i)G(h). \tag{6.2}$$

In our model, we will represent each of the revenues and costs of the current state using a renewal theory setup. Thus, each revenue and cost for a given state will consist of the expected value generated during that state plus the probability-weighted expected values of all future states. However, additional calculations will be needed for our problem, as some of the transition probabilities $P(h|i)$ are unknown.

6.2 States in the Last Buy Problem

Given the memoryless property of the system, the state of the last buy problem can be categorized by two variables:

- The number of assemblies still in operation.
- The number of spare parts remaining in inventory.

Thus, we will characterize the state of the system in the last buy problem with variables (l, n), with l indicating the number of assemblies still in operation and n indicating the number of spare parts remaining in inventory. The variable j is used to indicate the number of assemblies in operation at the beginning of the last buy period, while some form of the variable q is used to indicate the size of either the last buy order or a subsequent batch.

A review of notation specific to Chapter 6:

- q_l^*: The optimal order amount for a batch at some arbitrary time, given that l

assemblies are still in operation at that time.

- $C(l, n)$: The expected discounted value of a part of the total profit function for the remainder of the last buy period - specifically, the total profit without the manufacturing costs - given that l assemblies and n parts are still in operation. For the last buy decision or any batch decision with l assemblies remaining, $\pi(l, q) = C(l, q) - M(q)$

- $B(l)$: A Boolean variable that represents whether or not the optimal decision with l assemblies remaining and no parts remaining in inventory is to make another batch or to pay a penalty to the remaining customers. If $B(l) = TRUE$, then buying out the remaining customers is the optimal decision with l assemblies remaining. If $B(l) = FALSE$, manufacturing another batch is the best option with l assemblies remaining.

- p_{batch}: The fixed batch replenishment cost per batch; the fixed cost to set up and manufacture another batch. This does not include the variable cost per unit to manufacture the parts.

- p_{buyout}: The penalty cost per assembly to refuse to continue serving the customer.

6.3 Components of the Last Buy Problem

Given that j assemblies are in operation at the beginning of the last buy period and q spare parts are manufactured in the last buy order, we represent the total expected profit, $\pi(j, q)$, as the value of the expected revenues minus the value of the expected costs:

$$\pi(j,q) = R(j,q) + S(j,q) - M(q) - H(j,q) - Z(j,q). \tag{6.3}$$

For ease of calculation, we also define $C(j,q)$ as the total expected profit without the manufacturing cost factored in.

$$C(j,q) = R(j,q) + S(j,q) - H(j,q) - Z(j,q). \tag{6.4}$$

The rate transition of the process can be represented by Figure 6.1. The process starts at state (j,q), once the last buy order is manufactured. From there, either a part will fail, at which time the state will decay to $(j,q-1)$, or an assembly will fail, at which time the state will decay to $(j-1,q)$. These states, in turn, will decay to smaller numbers of assemblies and parts. Should the last part in inventory be demanded, leaving the process in a state $(l,0)$, a new batch would be manufactured (as represented by the dashed arrows in Figure 6.1), bringing the state of the process back to some state (l,q_l^*). Thus, the value of $C(j,q)$ (and, by extension, $\pi(j,q)$) can be found by examining the value of the states (j,q) decays into.

We now provide details on each of the terms in (6.3) and (6.4). The proofs of any stated results are available in Appendix C.

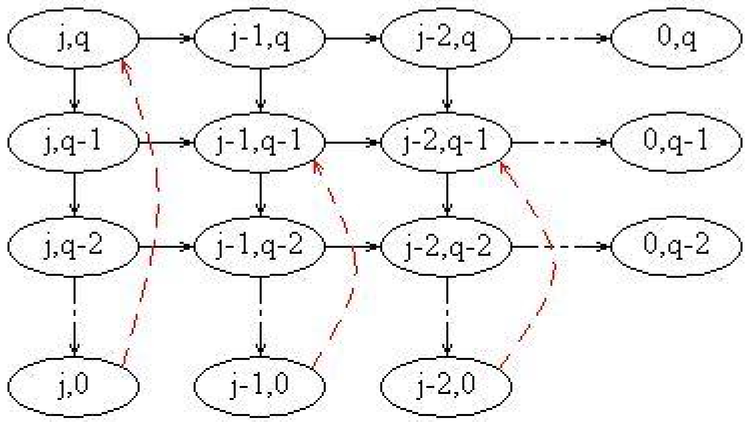

Figure 6.1: Rate Transition Diagram Between States in the Last Buy Problem with Batch Replenishment

6.3.1 Revenue

The expected discounted revenue is generated through renewal theory to be the following for any number of assemblies l and any number of parts in inventory n:

$$R(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}\left(\frac{\lambda}{\beta+\lambda}(r + R(l,n-1)) + \frac{\beta}{\beta+\lambda}R(l-1,n)\right). \tag{6.5}$$

The parts of this revenue formulation can be viewed as follows:

- $\frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}$: The effect of the discount rate on the expected future revenue from any transition states.
- $\frac{\lambda}{\beta+\lambda}$: The probability of transitioning from state (l,n) to state $(l,n-1)$.
- $(r + R(l,n-1))$: The expected revenue as a result of transitioning to $(l,n-1)$: the immediate revenue from selling a part, and the expected future revenue from state $(l,n-1)$ to any future states.

- $\frac{\beta}{\beta+\lambda}$: The probability of transitioning from state (l, n) to state $(l-1, n)$.
- $R(l-1, n)$: The expected revenue as a result of transitioning to $(l-1, n)$, which consist only of the expected future revenue from state $(l-1, n)$ to any future states.

If the last assembly fails, leaving us at any state $(0, n)$, no further revenue will take place. Thus,

$$R(0, n) = 0. \tag{6.6}$$

If we run out of spare parts, thus leaving us at a state $(l, 0)$, we set up another batch order, and manufacture a batch size that maximizes our profit going forward. Thus,

$$R(l, 0) = R(l, q_l^*). \tag{6.7}$$

where q_l^* is the optimal order amount for a batch in which l assemblies are in operation at the time the batch is manufactured.

It is this transition from state $(l, 0)$ to (l, q_l^*) that causes the Markov process to be insufficient by itself for solving the problem. At the start of the problem, q_l^* is unknown for all numbers of assemblies l. It is necessary to find q_l^* for all values of l from 1 to j - the number of assemblies still in operation at the time of the last buy - before a solution can be found. Except for the manufacturing cost, this transition is found in all of the costs and revenues involved in the last buy problem with batch replenishment.

6.3.2 Manufacturing Costs

The total manufacturing cost is simply the product of a batch order amount q and the per-unit manufacturing cost m:

$$M(q) = mq. \tag{6.8}$$

The manufacturing cost is the only part of the profit function not determined by renewal theory. Thus, it is necessary to find a numerical result for part of the profit using renewal theory, then add the manufacturing cost in afterwards.

6.3.3 Holding Costs

The holding costs are the associated labor, facility, and infrastructure costs incurred by holding a part in inventory. This cost is designated to have a cost of h per unit per unit time. Thus, if one spare part is held in inventory for one unit of time, a cost of h would be incurred.

The renewal-theory form of the holding cost is

$$H(l,n) = \frac{nh}{l(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}\left(\frac{\lambda}{\beta+\lambda}(H(l,n-1)) + \frac{\beta}{\beta+\lambda}H(l-1,n)\right). \tag{6.9}$$

Unlike any of the other portions of the total cost, the holding cost actually generates costs while in a given state. Thus, $\frac{nh}{l(\beta+\lambda)+\alpha}$ represents the expected discounted holding cost generated while in state (l, n). The explanation for all other parts of the holding cost match the explanations for the revenue function (6.5).

For the end state of $H(0, n)$, the value of the holding cost is 0:

$$H(0, n) = 0. \tag{6.10}$$

If the last buy is depleted and a state of $(l, 0)$ reached, another batch is manufactured. Thus, for the state of $(l, 0)$, the holding cost $H(l, 0)$ is

$$H(l, 0) = H(l, q_l^*). \tag{6.11}$$

6.3.4 Batch Replenishment Costs

The batch replenishment costs represent amount paid by the manufacturer to replenish the inventory once it has depleted. Once the inventory is reduced to 0, a new batch is set up, generating a fixed cost of p_{batch}, and q_l^* assemblies are manufactured.

The expected batch replenishment cost, discounted over time, is generated through renewal theory to be the following for any number of assemblies l and any number of parts in inventory n:

$$Z(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\lambda}{\beta+\lambda}(Z(l,n-1)) + \frac{\beta}{\beta+\lambda}Z(l-1,n)). \tag{6.12}$$

If the last assembly has failed, the value of the batch replenishment cost is 0, as no further replenishment is needed:

$$Z(0,n) = 0. \tag{6.13}$$

If we run out of parts in inventory, however, we need to make a new batch. Thus, we incur a batch setup cost of p_{batch} and manufacture a new batch of size q_l^*, thus replenishing our inventory.

$$Z(l,0) = p_{batch} + M(q_l^*) + Z(l,q_l^*). \tag{6.14}$$

6.3.5 Salvage

The salvage differs from most other costs and revenues in that it does not occur gradually over the course of the post-production phase; rather, it occurs only once when the last assembly has failed. Once this occurs, the remaining inventory is discarded at a revenue of s per unit. (If the financial event of scrapping each unit generates a cost instead of a revenue, then s is a negative number.)

For the last buy problem at a state with l assemblies still in operation and n

parts in inventory, the renewal-theory form of the salvage is

$$S(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}\left(\frac{\lambda}{\beta+\lambda}(S(l,n-1)) + \frac{\beta}{\beta+\lambda}S(l-1,n)\right). \qquad (6.15)$$

If the last assembly has failed, then the value of the salvage $S(0,n)$ at that state is equal to the salvage per part s multiplied by the number of parts still in inventory n:

$$S(0,n) = sn. \qquad (6.16)$$

If the inventory is depleted with l assemblies remaining in operation, another batch will be manufactured, thus replenishing the inventory:

$$S(l,0) = S(l,q_l^*). \qquad (6.17)$$

6.4 Calculation of the Optimal Order Amount

Now that the components of the Last Buy Problem have been formulated, we can now combine those components into full cost and revenue functions for the entire problem. The goal is to take the components found in the previous section, take the sum of these components as an overall profit function, then find the maximum of this function. The basic structure of the problem is given by (6.3):

$$\pi(j,q) = R(j,q) + S(j,q) - H(j,q) - M(q) - Z(j,q).$$

To simplify this, we combine most of the terms of this formula into $C(j,q)$ so that

$$\pi(j,q) = C(j,q) - M(q). \tag{6.18}$$

Since $C(j,q) = R(j,q) + S(j,q) - H(j,q) - Z(j,q)$, we take the sum of these parts from (6.5), (6.9), (6.12), and (6.15) to get $C(l,n)$ for any $l > 0$ and $n > 0$:

$$\begin{aligned} C(l,n) = \frac{-nh}{l(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\lambda}{\beta+\lambda} \\ (r + R(l,n-1) + S(l,n-1) - H(l,n-1) - Z(l,n-1)) \\ + \frac{\beta}{\beta+\lambda} R(l-1,n) + S(l-1,n) - H(l-1,n) - Z(l-1,n)). \end{aligned} \tag{6.19}$$

We can combine the individual costs and revenues in formula (6.19) to find

$$C(l,n) = \frac{-nh}{j(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\lambda}{\beta+\lambda}(r + C(l,n-1)) + \frac{\beta}{\beta+\lambda} C(l-1,n)). \tag{6.20}$$

Should the last assembly fail, leaving the process in a state $(0,n)$, the parts remaining in inventory would be salvaged:

$$C(0,n) = ns. \tag{6.21}$$

If the last part in inventory fails, a new batch would be set up at a cost of p_{batch}, a new batch of size q_l^* would be manufactured, and go to state (l, q_l^*):

$$C(l,0) = -p_{batch} - M(q_l^*) + (R(l,q_l^*) + S(l,q_l^*) - H(l,q_l^*) - Z(l,q_l^*)). \tag{6.22}$$

Equation (6.22) can be restructured as:

$$C(l,0) = -p_{batch} - M(q_l^*) + C(l,q_l^*). \tag{6.23}$$

Another representation of equation (6.23) can be found by combining terms:

$$C(l,0) = -p_{batch} + \pi(l,q_l^*). \tag{6.24}$$

Given the structure of the problem as shown in Figure 6.1, the value of $C(l,n)$ for any value of l assemblies and n parts can be calculated from the value of the states it decays into, $C(l,n-1)$ and $C(l-1,n)$. As time passes, a process in any state (l,n) will decay into either state $(l-1,n)$ or $(l,n-1)$. These states, in turn, will continue to decay to smaller numbers of assemblies and parts until either the last part in inventory has been sold, in which case another batch of parts will be manufactured and the system will return to a state (l, q_l^*), or the last assembly has failed, at which point the part will no longer be sold and the remaining inventory will be salvaged.

Because of this, we can find the revenue value for each $C(l, n)$ up to $C(j, q)$ by working backwards. The value of $C(1, 1)$ can be solved using equation (6.19) and the values of $C(1, 0)$ and $C(0, 1)$ given in (6.21) and (6.23). Once $C(1, 1)$ is solved, the value for $C(1, 2)$ may be solved by using $C(1, 1)$ and $C(0, 2)$. Similarly, the revenue value for $C(2, 1)$ may be found by using $C(1, 1)$ and $C(2, 0)$. This will continue for larger values of l and n until a solution for $C(j, q)$ is determined.

One problem exists for this method, however. At the beginning of the process, we do not know the values for $C(l, 0)$ for any value of l from 1 to j because the batch order amount q_l^* is unknown. Because of this, we must add an additional iteration step in the process.

For one assembly, we find the values of $C(1, n)$ for values of n from 1 to some large number - for our case we use $\frac{2j\beta}{\lambda}$, which is twice the expected demand for all j assemblies. We then use these values of $C(1, n)$ to find expected values of the profit $\pi(1, n)$ for all n from 1 to $\frac{2j\beta}{\lambda}$. These values of $\pi(1, n)$ are then used to find a value for $C(1, 0)$:

$$C(1, 0) = -p_{batch} + \max_n \{\pi(1, n)\}. \tag{6.25}$$

We then use this new value of $C(1, 0)$ to calculate new values of $\pi(1, n)$. We repeat the process until $C(1, 0)$ converges to a value. Once $C(1, 0)$ has converged, we acknowledge that the value of n that produces the maximum value for $\pi(1, n)$ is the optimal order amount for one assembly. Once we have converged to an answer for one assembly, the process is repeated for two assemblies, three assemblies, etc., until an optimal order amount for all j assemblies is found.

Based on this, we have derived two algorithms to find the optimal order amount for the last buy problem in cases where batch replenishment is an option. The first of these is the case where batch replenishment is the only option - where, once the inventory is depleted, a new batch will be manufactured and a setup cost incurred. We will also explore a case in which the manufacturer may also buy the customer out instead of manufacturing another batch.

6.4.1 The Optimal Order Amount for the Last Buy Problem with Batch Replenishment

In the case where batch replenishment is the only option, demand will occur until either the last assembly has failed or the inventory is depleted. If the inventory has depleted, a new batch will be set up (incurring a cost of p_{batch}) and a new 'last buy' will be manufactured.

The method to solve this problem can be found in Algorithm 6.1. This algorithm provides a decision strategy for all numbers of assemblies from 1 to j in the form of $q^*(l)$ - the optimal order amount with l assemblies remaining. It also provides a value of $\pi_{max}(j)$ as the expected discounted value of the optimal order amount with j assemblies.

Because the optimal order amount and profit for $l+1$ assemblies is not likely to

Algorithm 6.1 Calculation of Optimal Order Amount q^* with Batch Replenishment

Require: j, α, β, λ, s, p_{batch}, m, r, h, ϵ_{min}

$n_{max} \leftarrow \frac{2j\beta}{\lambda}$ {Establishes an upper limit on the number of parts}

$n \leftarrow 1$

while $n < n_{max}$ **do**

$C(0, n) \leftarrow sn$ {Sets values for $C(0, n)$}

$n \leftarrow n + 1$

end while

$C(1, 0) \leftarrow -p_{batch}$

$\pi_{max}(0) \leftarrow -jp_{batch}$ {Initializes value of $\pi_{max}(0)$}

for $l = 1$ to j **do**

$\epsilon \leftarrow 1$ {Initializes error value ϵ}

$\pi_{max}(l) \leftarrow \pi_{max}(l-1)$ {Initial value of $\pi_{max}(l)$ pegged by $\pi_{max}(l-1)$}

$C(l, 0) \leftarrow \pi_{max}(l) - p_{batch}$

while $\epsilon > \epsilon_{min}$ **do**

$\pi_{old}(l) \leftarrow \pi_{max}(l)$ {Copies current value of $\pi_{max}(l)$ for later comparison with changed $\pi_{max}(l)$}

$\pi_{max}(l) \leftarrow -jp_{batch}$ {Resets value of $\pi_{max}(l)$ to low value so that largest value can be found}

$q^*(l) = 0$ {Initializes value of order amount}

$n \leftarrow 1$

while $n < n_{max}$ **do**

$C(l, n) = \frac{nh}{l(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\lambda}{\beta+\lambda}(r + C(l, n-1)) + \frac{\beta}{\beta+\lambda}C(l-1, n))$ {Calculates profit without manufacturing costs for l assemblies and n parts}

if $C(l, n) - mn > \pi_{max}(l)$ **then**

$\pi_{max}(l) \leftarrow C(l, n) - mn$ {If profit from n parts is larger than for any previous order amount, will make new profit $\pi_{max}(l)$ from order amount of n}

$q^*(l) = n$

end if

$n \leftarrow n + 1$

end while

$\epsilon \leftarrow |\pi_{max}(l) - \pi_{old}(l)|$ {Change in $\pi_{max}(l)$ from iteration}

end while

end for

be very different than the result for l assemblies, we reduce the number of iterations required by taking the profit for l assemblies and using it as a starting point in the iteration for $l+1$ assemblies. Thus, the starting point for the values of $C(l,0)$ for all $l>1$ start close to its optimal result.

That said, while the algorithm produces the optimal order amount in the batch replenishment case, batch replenishment may not be the best option. Because of this, we find the following observation:

Observation 6.1. *The value of the optimal solution for the last buy problem with batch replenishment and a buyout option is greater than or equal to the value of the optimal solution without a buyout option.*

Consider what happens in the batch replenishment case when inventory is depleted and very few assemblies remain. A large fixed batch setup cost is incurred, while a small number of parts are produced. If other options exist to satisfy the customer, it may be more economical to use those methods instead. For instance, a manufacturer may simply pay off the customer (as in the no replenishment case) or fabricate the parts individually at higher cost, without incurring the setup costs that would come from the batch method.

The result of this is the addition of a max function for all states $(l,0)$. In the last buy problem with batch replenishment and no buyout option, the value of $C(l,0)$ is $-p_{batch} - mq_l^* + C(l, q_l^*)$ - the cost of manufacturing another batch, as well as the expected discounted costs of any future states. If a buyout option of cost p_{buyout} per assembly were instituted, the value of $C(l,0)$ would become

$$C(l,0) = \max[-lp_{buyout}, -p_{batch} - mq_l^* + C(l, q_l^*)]. \tag{6.26}$$

Thus, because of the addition of the max functions in (6.26), the optimal solution for a last buy problem with batch replenishment and buyout option must be greater than or equal to the optimal solution in a problem without the buyout option.

Because of this, we have developed one other algorithm to allow for options on what to do when the last buy order is depleted.

6.4.2 The Optimal Order Amount for the Last Buy Problem with Batch Replenishment and Buyout Option

The buyout option allows for a much more realistic representation of what would happen in a real-life situation. It is unlikely that a manufacturer would resort to a costly batch setup if there are few assemblies remaining to demand parts. Therefore, the manufacturer is likely to take an alternate method of satisfying the customer - either by buying the customer off or by meeting the demand through other channels.

If a buyout option is allowed, then one of two events may take place when inventory is depleted. Under normal conditions, a new batch will be set up and manufactured. However, if it is cheaper to simply buy out the remaining customer than to manufacture a new batch, then the manufacturer will buy out the customer instead.

Fortunately, calculation of the value of the buyout option is more straightforward than the value of an optimal batch. For l surviving assemblies, the total buyout cost is lp_{buyout}, where p_{buyout} is the buyout cost per assembly.

As a result, we present Algorithm 6.2. It is a modification to Algorithm 6.1 in that it provides for the buyout option.

Like Algorithm 6.1, the solutions for Algorithm 6.2 provide a decision scheme for 1 to j assemblies. If the Boolean variable $B(l) = TRUE$, then the optimal decision if the inventory is depleted with l assemblies remaining is to buy off the customer. If $B(l) = FALSE$, then the optimal decision with l assemblies remaining is to manufacture another batch. In this second case, the algorithm provides a decision strategy for all numbers of assemblies from 1 to j in the form of $q^*(l)$ - the optimal batch order amount with l assemblies remaining. It also provides a value of $\pi_{max}(j)$ as the expected discounted value of the optimal order amount with j assemblies.

The buyout option algorithm can also be used to find the optimal solution if incremental replenishment is viewed as a possible alternative to batch replenishment. In that case, the value of the buyout cost per assembly p_{buyout} is $\frac{p_i\lambda}{\beta+\alpha}$, where p_i represents the incremental replenishment cost per unit. A proof demonstrating $\frac{p_i\lambda}{\beta+\alpha}$ as the expected discounted value of the incremental replenishment cost per assembly can be found in Appendix C. Thus, the buyout option algorithm may be used to provide either the no-replenishment option or the incremental-replenishment option as a complement to the batch replenishment case.

Algorithm 6.2 Calculation of Optimal Order Amount q^* with Batch Replenishment and Buyout Option

Require: j, α, β, λ, s, p_{batch}, p_{buyout}, m, r, h, ϵ_{min}
  $n_{max} \leftarrow \frac{2j\beta}{\lambda}$ {Establishes an upper limit on the number of parts}
  $n \leftarrow 1$
  while $n < n_{max}$ **do**
    $C(0, n) \leftarrow sn$ {Sets values for $C(0, n)$}
    $n \leftarrow n + 1$
  end while
  $C(1, 0) \leftarrow -p_{buyout}$
  $\pi_{max}(0) \leftarrow -jp_{batch}$ {Initializes value of $\pi_{max}(0)$}
  for $l = 1$ to j **do**
    $B(l) \leftarrow TRUE$ {Initializes buyout decision variable}
    $\epsilon \leftarrow 1$ {Initializes error value ϵ}
    $\pi_{max}(l) \leftarrow \pi_{max}(l-1)$ {Initial value of $\pi_{max}(l)$ pegged by $\pi_{max}(l-1)$}
    $C(l, 0) \leftarrow \pi_{max}(l) - p_{batch}$
    while $\epsilon > \epsilon_{min}$ **do**
      $\pi_{old}(l) \leftarrow \pi_{max}(l)$ {Copies current value of $\pi_{max}(l)$ for later comparison with changed $\pi_{max}(l)$}
      $\pi_{max}(l) \leftarrow -lp_{buyout}$ {Resets value of $\pi_{max}(l)$ to buyout value}
      $q^*(l) = 0$ {Initializes value of order amount}
      $n \leftarrow 1$
      while $n < n_{max}$ **do**
        $C(l, n) = \frac{nh}{l(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}\left(\frac{\lambda}{\beta+\lambda}(r + C(l, n-1)) + \frac{\beta}{\beta+\lambda}C(l-1, n)\right)$ {Calculates profit without manufacturing costs for l assemblies and n parts}
        if $C(l, n) - mn > \pi_{max}(l)$ **then**
          $B(l) \leftarrow FALSE$
          $\pi_{max}(l) \leftarrow C(l, n) - mn$ {If profit from n parts is larger than for any previous order amount, will make new profit $\pi_{max}(l)$ from order amount of n}
          $q^*(l) = n$
        end if
        $n \leftarrow n + 1$
      end while
      $\epsilon \leftarrow |\pi_{max}(l) - \pi_{old}(l)|$ {Change in $\pi_{max}(l)$ from iteration}
    end while
  end for

6.5 Numerical Experiments

We start our numerical experiments to see if the sensitivity results found for other forms of the last buy problem also apply to the last buy problem with batch replenishment. The first is to see the effect of the discount rate α on the optimal order amount.

Suppose we have 75 assemblies at our last buy order point; the lifetime of each assembly is distributed exponentially with a mean of $\frac{1}{\beta} = 4$. Each assembly contains a spare part, the lifetime of which is distributed exponentially with a mean of $\frac{1}{\lambda} = 2$. The setup cost for each additional batch is $p_{batch} = 100$; each part is manufactured at a cost of $m = 4$. If a part fails, a new one will be ordered from the manufacturer at a cost of $r = 15$. The part will generate a holding cost per unit time of $h = 0.5$; if the part fails to sell before all assemblies have failed, then it will be salvaged, generating $s = 1$ unit of revenue. We find the optimal order amount for different discount rates, ranging from $\alpha = 0.05$ to $\alpha = 0.20$.

Table 6.1: Effect of Discount Rate α on Optimal Order Amount

Discount Rate α	0.05	0.075	0.01	0.125	0.15	0.175	0.20
Order Amount q^*	94	88	82	78	74	71	68
Value $\pi(q^*)$	1035.23	948.54	874.12	809.66	753.35	703.78	659.82

The results can be found in Table 6.1. As found in numerical experiments for other types of last buy problems, increasing the discount rate α produced a noticeable decrease in the optimal order amount q^*. As with the other types of last buy problem, this is largely due to the fact that most revenues and costs that encourage higher order amounts, such as revenues from sales and batch setup costs, occur beyond time $t = 0$, while the manufacturing costs, one of the main costs that discourage higher order amounts, occurs at time $t = 0$.

The first property to note from our numerical experiments is that the value of the optimal solution in which a buyout option is available will always be larger than the optimal solution for the case in which no buyout option exists. Thus, as long as the buyout option exists, no matter the size, the manufacturer's expected value will always be at least as high as without. This makes sense as the no-buyout option is a restriction of the buyout option problem.

We demonstrate this property with the following example. The lifetime of the assembly is represented as an exponential with a mean lifetime of $\frac{1}{\beta} = 4$; the lifetime of the part is represented as an exponential with a mean lifetime of $\frac{1}{\lambda} = 2$. The revenue generated by the sale of a spare part is $r = 20$; if the part doesn't sell, the salvage is $s = 3$; the setup cost of manufacturing another batch is $p_{batch} = 200$; if a buyout option is available, the buyout cost is $p_{buyout} = 40$. The holding cost for each unit per unit time is $h = 1$, while the cost to manufacture each part is $m = 5$. All costs and revenues are discounted at a rate of $\alpha = 0.05$. We find the optimal order amount for different numbers of assemblies for both the buyout and no-buyout cases.

As we see in Table 6.2, the results for the last buy problem with batch replenishment and a buyout option invariably produces higher expected profits than an

Table 6.2: Increase in Expected Profit as a Result of Buyout Option

Number of Assemblies	5	10	25	50
No-buyout Order Amount q^*	14	23	46	79
No-buyout Value $\pi(q^*)$	11.03	73.35	310.85	762.41
Buyout Order Amount q^*	12	21	45	80
Buyout Value $\pi(q^*)$	35.25	98.29	334.19	801.69

identical last buy problem without a buyout option. The reason for this lies in the structure of the problem under renewal theory, and is explained in Appendix C.

The second property relates to the situations under which the buyout would be taken, rather than a new batch manufactured. A buyout would be taken for smaller numbers of assemblies remaining, while the buyout would be refused for larger numbers of assemblies. If a buyout is the best option for l assemblies, then it is the best option for any number of assemblies less than l. If a buyout is not the best option for l assemblies, then it will not be the best option for any number of assemblies greater than l.

Because of this, it is of interest to find some form of mathematical relationship for the crossover point - the number of assemblies required to make assembling another batch the optimal decision, rather than buying the customer out. While we are unlikely to find an easy formula that will give us an exact answer, an approximate solution that brings us close to a solution would be beneficial.

As a result of this, we developed a tool which a manager may use to find the crossover point, and thus be able to make decisions at a glance as to whether or not to make another batch. The batch-or-buyout decision may be represented graphically in much the same way that complex chemical engineering concepts such as non-ideal gas properties and the turbulence of fluid flow may be represented.

To demonstrate this, we give four graphs, showing the crossover point from batch to buyout under different states. All of the lines in each graph are developed for a given value of $\frac{r}{m}$, the markup ratio between the revenue from sales and the manufacturing cost per part, and $\frac{\lambda}{\beta}$, the ratio between the failure rates of the assembly and part. Each of the different lines in the graph represent a different ratio of p_{batch}/r, the ratio between the batch setup cost and the revenue per part demanded.

Each last buy problem for a manager will contain the following information: r, the revenue per unit demanded; m, the manufacturing cost per unit of the part; p_{batch}, the batch setup cost per batch; p_{buyout}, the buyout cost per assembly; β, the assembly failure rate; and λ, the part failure rate. Other parts of the formula, such as the salvage, the holding cost, and the salvage, were not included in the formula as they had little effect on the crossover point, but if a manager wishes to customize their own chart with these, they may do so.

We make these charts by finding the crossover point for the following examples. We remove the discount rate, holding cost, and salvage from the formula - thus, all that remains is the manufacturing cost, the revenue from sales, the batch setup cost, and the buyout cost. We set the revenue per unit sold at 1, and change the other costs to match. As our discount rate is set at 0, only the ratio of part assembly failure rate to assembly failure rate is important, rather than the absolute values for these failure rates. We then measure the crossover point for different values of p_{batch} and p_{buyout}. These graphs are presented as Figures 6.2 through 6.5. The buyout area is represented by the area underneath the curve; the batch replenishment area is represented by the area above the curve.

What we find in our graphs is that, because the buyout cost and the setup cost tend to be much larger than any other costs and revenues that can be generated, the ratio of the setup cost to the buyout cost $\frac{p_{batch}}{p_{buyout}}$ tends to provide a base value for the crossover point. This is understandable given that the total buyout cost for l assemblies remaining is lp_{buyout} - the number of assemblies remaining multiplied by the buyout cost per assembly. Thus, once the total buyout cost is greater than the batch setup cost, it is likely that the preference will be to make another batch. If the other costs and revenues, such as the manufacturing costs and revenue from the sale of spare parts, are significantly smaller than the buyout and batch costs, then these revenues and costs are not likely to provide much effect to the crossover point.

The charts provided also show some important properties in regards to the crossover point. The first of these is that the larger the buyout and batch costs are relative to the revenue, the more likely the crossover point will remain near the $\frac{p_{batch}}{p_{buyout}}$ ratio. This makes sense, as the larger the buyout and batch costs are in comparison to the manufacturing cost and revenue, the more these will dominate the calculation of the crossover point. Also, the larger the markup ratio $\frac{r}{m}$ and the larger the ratio of the part failure rate to the assembly failure rate $\frac{\lambda}{\beta}$, the smaller the crossover point, and the more the crossover point deviated from the $\frac{p_{batch}}{p_{buyout}}$ ratio. This is understandable,

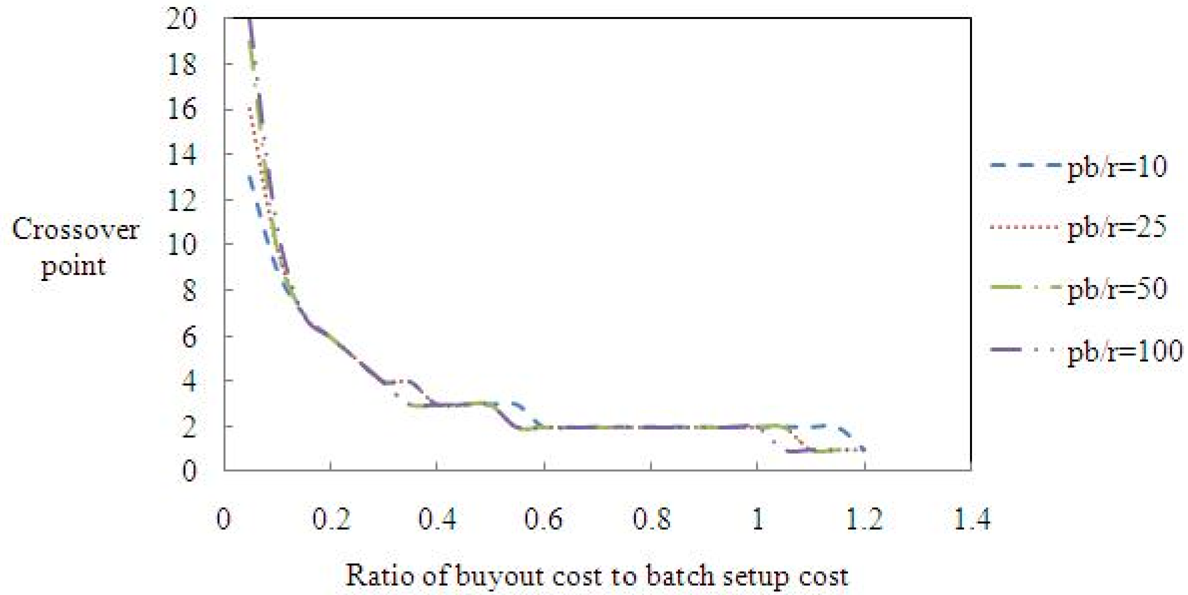

Figure 6.2: Crossover Point from Buyout to Batch, Markup Ratio $\frac{r}{m} = 2$, Expected Demand per Assembly $\frac{\lambda}{\beta}$=1

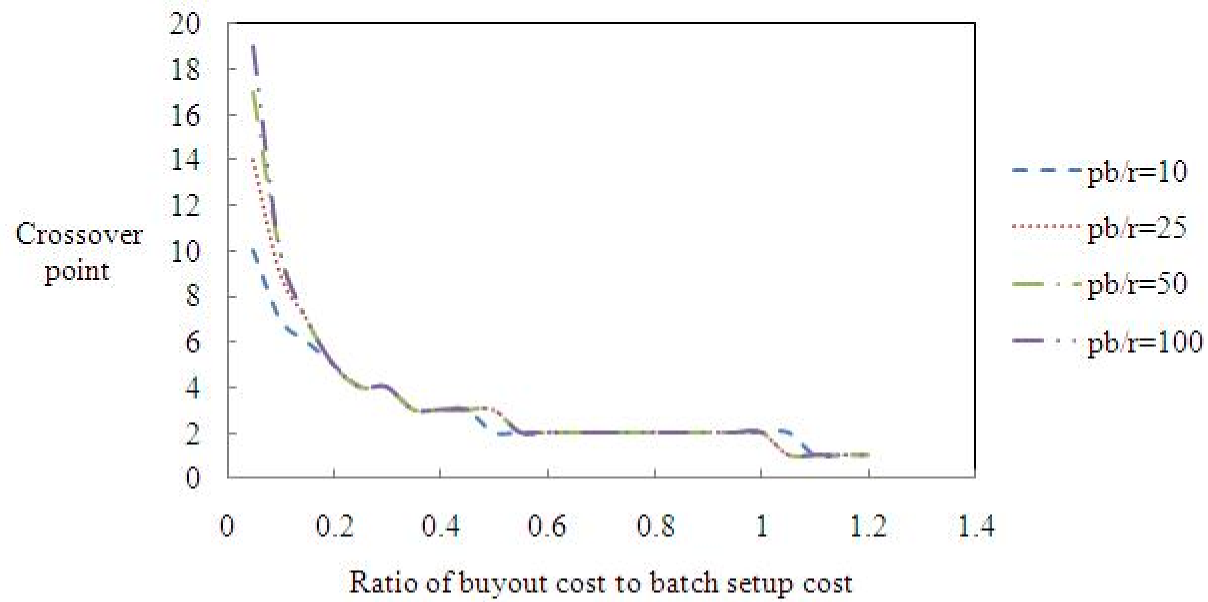

Figure 6.3: Crossover Point from Buyout to Batch, Markup Ratio $\frac{r}{m} = 4$, Expected Demand per Assembly $\frac{\lambda}{\beta}$=1

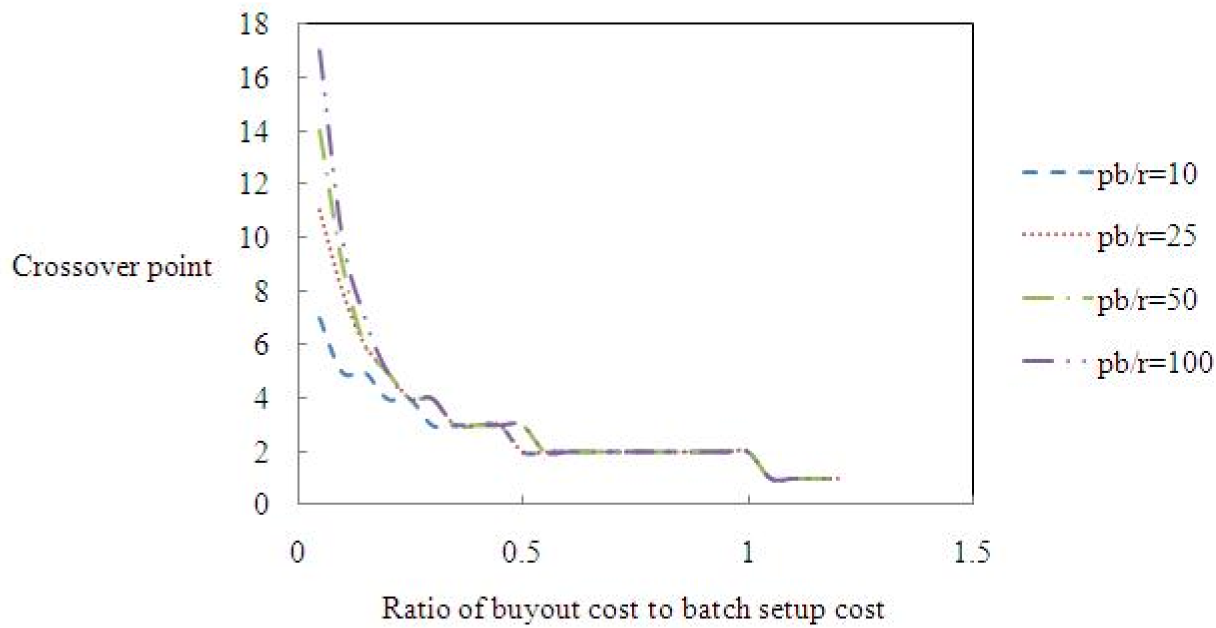

Figure 6.4: Crossover Point from Buyout to Batch, Markup Ratio $\frac{r}{m} = 2$, Expected Demand per Assembly $\frac{\lambda}{\beta}$=2

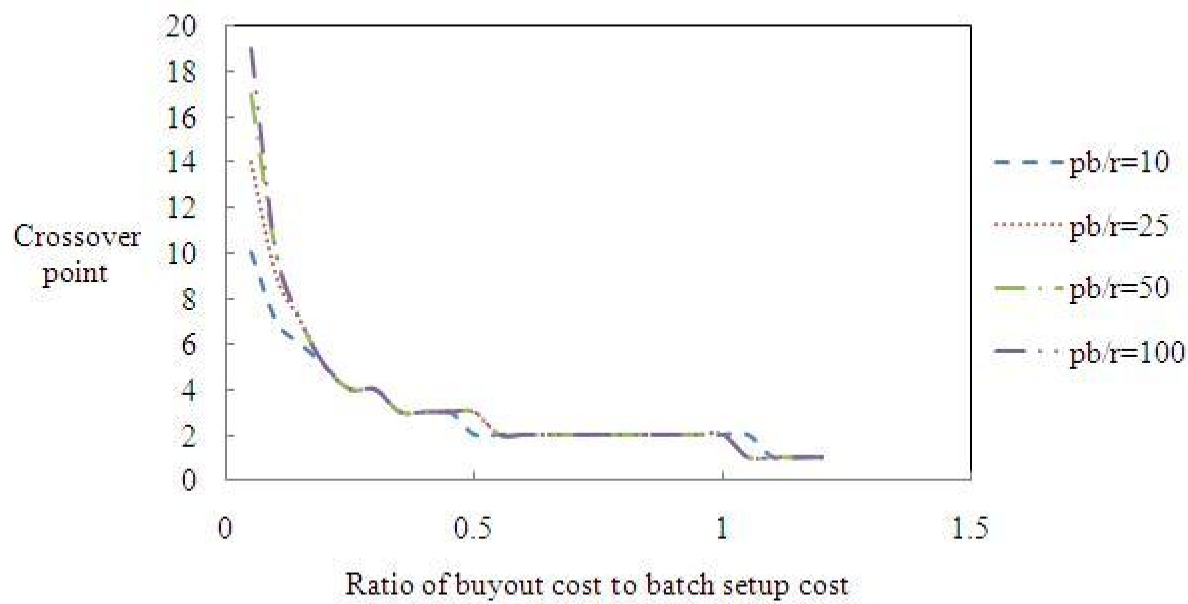

Figure 6.5: Crossover Point from Buyout to Batch, Markup Ratio $\frac{r}{m} = 4$, Expected Demand per Assembly $\frac{\lambda}{\beta}$=2

as both the markup ratio and the part-assembly failure rate ratio favor the profit to be made from manufacturing more spare parts - the greater markup ratio indicates a greater profit from each part demanded, while a larger failure ratio indicates a greater expected demand for parts per assembly.

The chart may be used by a manager as a quick reference to help determine a batch-or-buyout decision by finding their location on the graph. Over time, a particular spare part's position on the graph will descend as more assemblies fail. If the manager is on or above the crossover point line in terms of the number of assemblies remaining once the inventory for that part has been depleted, then the optimal decision is to make a new batch, and the manager can immediately begin preparations to make a new batch. If the manager is below the line, then the optimal decision is to buy out all remaining customers.

Because of the results regarding $\frac{p_{batch}}{p_{buyout}}$, it becomes important to find out what happens when the buyout cost $p_{buyout} = 0$ - that is, no penalty occurs for "buying" the customer off - and the setup cost p_{batch} is large. Despite such a small buyout, there will always be a finite number of assemblies l for which the batch option is the best choice, provided that there is any possibility of profit to be made from its manufacture (that is, $r > m$).

The reason for this can be found by exploring what happens as the number of assemblies l becomes very large. The cumulative distribution before a single assembly is demanded, given l assemblies, is $e^{-lt\lambda}$. As l becomes very large, $e^{-lt\lambda}$ approaches 0 even for small values of t. In other words, the more assemblies there are that have a part to fail, the more likely that one of those parts will fail, and demand for that part will occur. Thus, a large number of assemblies are likely to be demanded quickly. Eventually, if we make l large enough, we will find a case such that the revenue generated from part sales would make it optimal for a batch to be made.

6.6 Managerial Insights

Our solution for the last buy problem with batch replenishment is a useful tool for managers to use in cases where the expected method of replenishment is to manufacture another batch, if the distributions of the lifetimes of the parts and assemblies can be reasonably approximated by an exponential distribution, and the length of any potential contract period is long enough that the contract's termination is not likely to affect the outcome.

While these assumptions may seem restrictive, the solution method is also an easily-scalable solution that not only provides the optimal order amount for the initial last buy decision, but also supplies the decision scheme for all potential states in which the inventory has been depleted. For a strictly batch-replenishment case, this includes the number of parts to make in the new batch; for a case in which a buyout option is available, it also includes the situations under which the buyout should occur. This solution method is such that it can effectively be used for large-scale problems, even with thousands of assemblies in operation.

We also provide the manager with a fast method by which they can determine the best course of action should they find themselves with a batch-or-buyout decision. By using one of the graphs supplied here (or by developing one for their own purposes), a manager has the solution at a glance whether or not a new batch should be manufactured, given the conditions of the problem.

6.7 Conclusions

The last buy problem with batch replenishment is the most complex of the standard forms of the last buy problem. Should inventory ever be depleted, the response is to manufacture another batch - the size of which is itself another last buy problem. As a result, it is necessary to look at a simplified form of the problem to find a solution. We limit the problem to exponential distributions, and treat the problem as a Markov decision process.

Unfortunately, because of the iterative nature of the problem, a Markov decision process by itself is insufficient to solve the problem, because the response once the inventory is depleted is to manufacture another batch. Therefore, adding an iteration step is necessary to determine both the optimal order amount and the value of the new batch.

While the iteration step adds a level of complexity to the problem, the problem still solves reasonably quickly, even for large numbers of assemblies. Moreover, our solution is more than just an order amount at the initial stage - our solution set is a set of decisions to be made should the inventory be depleted.

We also looked at the possibility of a variant of the last buy problem, one where the remaining customers could be bought out if the cost of buyout was less than the cost of making another batch. Our solutions in these cases involved not only the optimal order amount, but whether or not it was optimal to even make another batch as opposed to buying the customers out.

CHAPTER 7
THE EFFECT OF OPTIONAL CONTRACT EXTENSIONS ON THE LAST BUY PROBLEM

7.1 Problem Definition and Variants

In this chapter, we consider the possibility of contract extension - increasing the amount of time of spare part coverage. For a fee, the manufacturer agrees to extend its guarantee to supply parts to the customer, agreeing to pay a penalty if it is unable to satisfy the customer's demand for spare parts in the extension period. This adds a level of complexity to the last buy problem. The possibility of contract extension adds the following questions to the problem:

- Under what conditions will the contract be extended? For the cases we will explore in this thesis, we assume that the contract will be extended if the customer still has assemblies in operation at time t_{c1}, if the manufacturer's expected value of extending the contract is positive, and if the cost per assembly of the extension p_x is less than the maximum amount that the customer is willing to pay.
- What is the expected value of the possible contract extension, given a last-buy order amount of q? In order to determine the expected value of the possible extension, we need to find the probabilities of the number of assemblies still in operation and the number of parts in inventory at time t_{c1}, given that j assemblies are in operation when a last buy order of q units is manufactured.

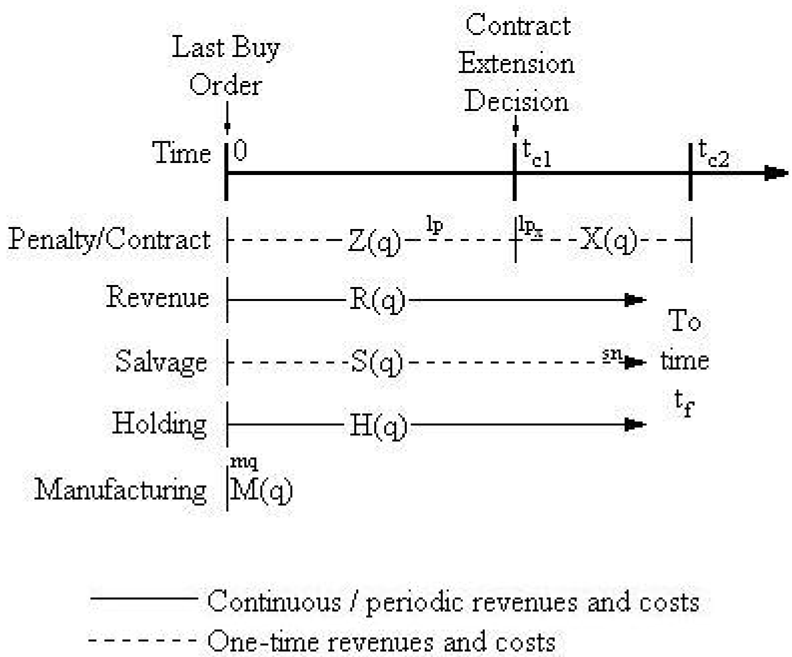

Figure 7.1: Timeline of Decisions, Costs, and Revenues in the Last Buy Problem with Optional Contract Extension

Thus, our problem can be viewed as a two-stage stochastic problem. First, we must find the expected value of the contract extension for each possible number of assemblies in operation and parts in inventory at time t_{c1}. Once these values are determined, they are then used in the overall profit function to find the optimal last-buy order amount.

The revenues, costs, and decisions in the last buy problem with optional contract extension can be viewed as a timeline in Figure 7.1. At time 0, the manufacturer has to decide how many spare parts to manufacture in the last buy. This order amount is designated as q. The last buy order is manufactured, incurring a cost of $M(q)$, after which no more parts will be made. This begins the post-production phase of the last buy problem.

Starting at time 0, all parts in inventory generate a holding cost of h per part per unit time, thus generating the overall holding cost $H(q)$. When a part fails, a replacement part will be taken out of inventory and sold to the customer for a price of r, thus generating the overall revenue of $R(q)$. Regardless of whether or not the contract is extended, the manufacturer will continue to stock parts in inventory and sell them to the customer until such time as either the inventory is depleted or the last assembly has failed. Should the last assembly fail before the last spare part has been sold, then the remaining inventory will be salvaged, generating revenue of s per spare part remaining, thus generating the salvage $S(q)$. As all of the revenue of $S(q)$ occurs once the last assembly fails, $S(q)$ is viewed as a one-time revenue.

From time 0 to time t_{c1}, the length of the initial contract, if the manufacturer is ever unable to satisfy demand for the spare part before time t_{c1}, a penalty cost of p will be paid to the customer for each assembly still in operation, thus generating the total expected discounted penalty cost of $Z(q)$. The penalty $Z(q)$ all occurs at one time once the manufacturer is unable to meet the customer's demand.

At time t_{c1}, both the customer and the manufacturer have a second decision: whether or not to extend the length of the contract from time t_{c1} to time t_{c2}. The manufacturer, with information on the maximum value that the customer is willing to pay, will use the expected value of the penalty in the contract extension period to determine whether or not to make a contract extension offer to the customer. If the manufacturer makes an offer and the customer accepts, the customer will pay p_x

units to the manufacturer for each assembly still in operation in order to extend the contract. This revenue to the manufacturer is part of $X(q)$, the value of the extension to the manufacturer.

If both the customer and the manufacturer have agreed to extend the contract, the manufacturer will continue to pay a penalty cost of p per assembly if the manufacturer is ever unable to satisfy demand for the spare part before time t_{c2}. The expected value of this penalty, represented by $V(l, n)$, is a part of the expected value of the extension $X(q)$. All of the penalty will occur at one time once the manufacturer is unable to meet the customer's demand, assuming that the contract has been extended.

However, if either the customer or the manufacturer decide not to extend the contract, then the manufacturer will still continue to sell spare parts to the customer beyond time t_{c1}. Should the manufacturer be unable to supply any more spare parts to the customer beyond time t_{c1}, no penalty will be paid to the customer. Also, the customer will not pay the manufacturer p_x for each of l assemblies still in operation at time t_{c1} to extend the contract.

In summary, two decisions must be made. The first decision, made solely by the manufacturer at time 0, is the last buy order amount; how many parts should be manufactured in the last buy? The second, made jointly by the manufacturer and the customer at time t_{c1}, involves the contract extension. Will the contract be extended from time t_{c1} to time t_{c2}? In order to find the optimal order amount for the last buy, it is necessary to first determine the conditions at t_{c1} in which both the customer and manufacturer will agree to extend the contract.

7.2 Parts of the Last Buy Problem with Optional Contract Extension

As this is a variant of the last buy problem, most of the parts used in this chapter are identical to parts used in previous chapters of this thesis. The properties for the revenue, the salvage, the holding costs, and the manufacturing costs can be found in Chapter 4 of this thesis; the properties for the penalty cost can be found in Chapter 5.

7.2.1 Revenue

The expected revenue, discounted over time, is generated by finding the expected value of the revenue by time t and determining the rate of change of this expected value over time. Given a probability distribution of demand over time $P\{D_t = n\}$, we have

$$R(q) = -r \sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt \tag{7.1}$$

The properties for the revenue can be found in Subsection 4.2.1.

7.2.2 Manufacturing Costs

The total manufacturing cost is simply the product of the last buy order amount q and the per-unit manufacturing cost m:

$$M(q) = mq \tag{7.2}$$

7.2.3 Holding Costs

The holding costs are the associated labor, facility, and infrastructure costs incurred by holding a part in inventory. This cost is designated to have a cost of h per unit per unit time. Thus, if one spare part is held in inventory for one unit of time, a cost of h would be incurred.

The total discounted holding cost used depends on the specifics of the problem. In a non-contract problem where we would have no salvage, the expected value of the holding cost becomes

$$H(q) = h \sum_{n=0}^{q-1} (q-n) \int_{t=0}^{\infty} e^{-t\alpha} P\{D_t = n\} dt \tag{7.3}$$

In a contract problem in which the salvage occurs when the last assembly fails, the holding cost is

$$H(q) = h \sum_{n=0}^{q-1} (q-n) \int_{t=0}^{\infty} \frac{dP\{A_{t_f} = 0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = n | A_{t_f} = 1\} dt dt_f \tag{7.4}$$

The properties of the holding costs can be found in Subsection 4.2.3.

7.2.4 Initial Penalty Costs

The penalty cost is the amount paid to the customer that demands a spare part after the last buy order has been depleted, during the time of the initial contract. The formulation of this penalty cost has the following form:

$$Z(q) = -p \sum_{l=1}^{j} l \int_{t_f=0}^{t_c} e^{-t_f \alpha} \frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\} dt_f \tag{7.5}$$

The properties for the penalty costs during the initial contract period can be found in Subsection 5.2.4.

7.2.5 Salvage

The salvage differs from most other costs and revenues in that it does not occur gradually over the course of the post-production phase; rather, it occurs when the last assembly has failed at time t_f. Once this occurs, the remaining inventory is discarded at a revenue of s per unit. (If the financial event of scrapping each unit generates a cost instead of a revenue, then s is a negative number.)

For the last buy problem, the salvage is

$$S(q) = s \sum_{n=0}^{q-1} (q-n) \int_{t_f=0}^{\infty} e^{-t_f \alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = n | A = 1\} dt_f \tag{7.6}$$

As stated earlier, the properties for the salvage can be found in Subsection 4.2.5.

7.2.6 Extension

The decision to extend the contract is partially based on the expected value of any potential revenues and penalties that occur due to the decision to extend the contract. If a contract has been extended, then the manufacturer will pay a penalty if it is unable to satisfy customer demand during the extension period. However, the renewal decision is dependent on the state of the system at the time of the decision. If the customer has no assemblies remaining in operation or if the cost to extend the contract is too high, the customer will not pay to extend the contract. At the same time, if the manufacturer is likely to lose money from the contract extension, the manufacturer is likely to refuse the extension. Thus, it becomes necessary to define the conditions under which both the customer and the manufacturer both agree to the extension.

To refresh, some notation specific to Chapter 7:

- t_{c1}: The time at which the initial contract expires, at which point the customer and manufacturer both have the choice of extending the contract or letting it expire. After the contract expires, no replenishment costs or penalties will be incurred, unless the initial contract is extended.
- t_{c2}: The time at which the extended contract expires; $t_{c2} > t_{c1}$.
- $T(l, n)$: The probability that both the customer and the manufacturer will agree to the contract extension, given the number of parts in inventory n and number of assemblies still in operation l at time t_{c1}.
- $V(l, n)$: The expected value of the penalty incurred during the contract exten-

sion period, given the number of parts in inventory n and number of assemblies still in operation l at time t_{c1}. Note that $V(l,n)$ is discounted to time 0.

- $X(q)$: The total expected discounted profit generated due to the contract extension (heretofore known simply as the "extension"), given a last-buy order amount of q. Note that $X(q)$ is discounted to time 0.
- X_{max}: The maximum possible value of the extension. Note that X_{max} is discounted to time 0.
- p_x: The revenue paid per assembly to the manufacturer to extend the supply contract from time t_{c1} to time t_{c2}. This revenue is paid at time t_{c1}.
- p_C: The maximum amount per assembly that the customer would be willing to pay per assembly to extend the supply contract from time t_{c1} to time t_{c2}.
- $p_M(q)$: The minimum amount that the manufacturer would be willing to accept that could possibly cause the manufacturer to extend the supply contract from time t_{c1} to time t_{c2}.
- $W(t)$: The customer's amortized value for a single assembly at time t.

The value of the extension, in general form, is

$$X(q) = p\sum_{l=1}^{j}\sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\}P\{D_{t_{c1}} = n|A_{t_{c1}} = l\}T(l,n)[e^{-t_{c1}\alpha}lp_x - V(l,n)] \quad (7.7)$$

where $T(l,n)$ represents the probability of both manufacturer and customer agreeing to the extension, and $V(l,n)$ represents the expected value of the penalty incurred in the extension period given the number of parts and assemblies remaining at the extension point.

Unfortunately, this formula is too general to be able to develop any useful properties. Therefore, we will explore a special case of the problem and define $T(l,n)$ as follows:

- $T(l,n) = 1$ if $V(l,n) \leq e^{-t_{c1}\alpha} l p_x$ (Expected profit from the contract extension is nonnegative), $l \geq 1$ (the customer has an assembly they want to protect), $D_{t_{c1}} < q$ (the manufacturer still has parts remaining to supply), and $p_x \leq p_C$ (the value of the contract extension per assembly is less than the customer's maximum allowed value).
- $T(l,n) = 0$ if $V(l,n) > e^{-t_{c1}\alpha} l p_x$, or $l = 0$, or $D_{t_{c1}} \geq q$, or $p_x > p_C$

If we define $T(l,n)$ in this manner, as long as $p_x \leq p_C$, the value of the extension becomes:

$$X(q) = \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = n | A_{t_{c1}} = l\} \max[e^{-t_{c1}\alpha} l p_x - V(l,n), 0] \quad (7.8)$$

The parts of $X(q)$ can be explained as follows:

- $P\{A_{t_{c1}} = l\}$: The probability that l assemblies have survived to time t_{c1}.
- $P\{D_{t_{c1}} = n | A_{t_{c1}} = l\}$: The probability that n spare parts have been demanded up to time t_{c1}, given that l assemblies have survived to time t_{c1}. Along with $P\{A_{t_{c1}} = l\}$, these two probabilities together provide the probability that l

assemblies are still functioning and $q-n$ parts are in inventory at time t_{c1}.

- $e^{-t_{c1}\alpha} l p_x$: The revenue, discounted to time 0, to extend the contract out to time t_{c2}, given that l assemblies remain at time t_{c1}.
- $V(l,n)$: The expected value, discounted to time 0, of any penalty or replenishment costs incurred during the contract extension, under the conditions of l assemblies remaining in operation and n parts remaining in inventory at time t_{c1}.

Thus, $X(q)$ represents the sum of all probabilities of the number of parts in inventory and the number of assemblies still in operation at time t_{c1}, multiplied by the net expected discounted value of all revenues and costs, given the number of parts in inventory and the number of assemblies in operation at time t_{c1}.

In the no-replenishment case, $V(l,n)$ is

$$V(l,n) = -p \sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt. \tag{7.9}$$

The $V(l,n)$ equation is similar to the contract-case penalty cost $Z(q)$ represented in (5.6). Once the $(q+1)$th part is demanded, the contract is broken, and the manufacturer must pay a penalty for all assemblies remaining in operation. It should be noted that $V(l,n)$ is non-negative.

The structure of the contract extension revenue per assembly p_x is a compromise between the customer and the manufacturer. If p_x is low, then the manufacturer is unlikely to agree to any extension; if p_x is high, then the customer is unlikely to agree. From the customer's perspective, if p_x is too high, then the optimal decision for the customer may be to purchase a new assembly to replace the old should the part fail, rather than agree to the contract extension. For instance, an upper bound on p_x may be found by finding the amortized value of a single assembly through the contract extension period:

$$p_C \leq -\sum_{n=0}^{\infty} \int_{t_{c1}}^{t_{c2}} e^{\alpha(t_{c1}-t)} \frac{dP\{D_{1,t} = n | D_{1,t_{c1}} = n\}}{dt} W(t)dt. \tag{7.10}$$

If the amortized value of a single assembly over time, weighted by the likelihood of part failure, is less than the cost to the customer to extend the contract for that assembly, then the customer is not going to extend the contract. Thus, if $p_x > p_C$, no contract extension will take place. In our model, we assume that the manufacturer as of time 0 has perfect information in regards to the customer's upper bound, and, if feasible, selects a maximal p_x for which the customer will be certain to agree to the extension at time t_{c1}

The decision from the manufacturer in regards to p_x is more complicated. To determine whether or not to extend the contract, the manufacturer uses the net expected discounted value of any penalties and revenues given the number of assemblies still in operation and the number of parts still in inventory at time t_{c1}. As long as a case exists for which this expected value is positive, then the manufacturer has a case in which the contract may be extended. As such, for an order amount of q, the

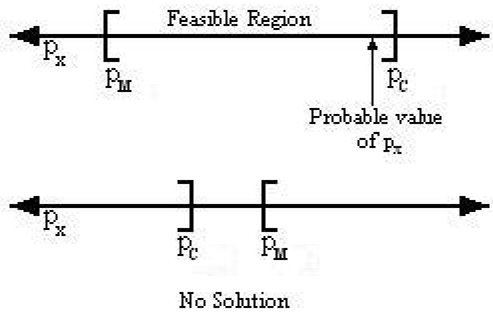

Figure 7.2: Determination of the Contract Extension Revenue Per Assembly

minimal value of p_x that will allow for any possibility of extension is

$$p_M(q) = -p \int_{t_{c1}}^{t_{c2}} e^{-\alpha(t-t_{c1})} \frac{dP\{D_t \leq q | A_t = 1, A_{t_{c1}} = 1, D_{t_{c1}} = 0\}}{dt} P\{A_t = 1 | A_{t_{c1}} = 1, D_{t_{c1}} = 0\} dt. \tag{7.11}$$

This is the best-case scenario for the manufacturer in regards to contract extension: no parts demanded up to time t_{c1}, and only one assembly remaining. As such, the expected value for the penalty is at its lowest with 1 assembly remaining and q parts in inventory. If the contract extension revenue per assembly p_x is unable to meet expected costs even in this best case scenario, then no combination of parts and assemblies remaining will result in a contract extension. Thus, if $p_x < p_M(q)$, no contract extension will take place. Note that while Equation (7.11) provides an absolute lower bound on p_x, a larger value of p_x will mean that the manufacturer is more likely to accept the contract extension.

The value of p_x in our model can be demonstrated by Figure 7.2. In this model, the manufacturer is assumed to have some knowledge of the customer's decision process in regards to contract extension. Thus, in negotiations, the resulting value of p_x will be close to the maximum value of p_x allowed by the customer. This is depicted in the first example: the likely value of p_x is close to the upper bound p_C. The second example depicted in Figure 7.2 shows the possibility of no solution for p_x. It is possible that the lower bound $p_M(q)$ may be greater than the upper bound p_C; in such a case, no contract extension will take place.

The extension has the following properties with respect to the order amount q:

Lemma 7.1. *The extension has the following properties:*

- *(a) As q increases, the extension $X(q)$ increases.*
- *(b)* $0 \leq X(q) \leq e^{-t_{c1}\alpha} \sum_{l=1}^{j} P\{A_{t_{c1}} = l\} l p_x$
- *(c) The extension $X(q)$ is neither concave nor convex.*

These properties are important for solving the problem, because they both eliminate the possibility of using concavity to find the solution and allow for the ability to find an upper bound on the profit function. The proofs for these properties can be found in Appendix D.

7.3 Solution Method

The total profit can be represented as the sum of each of the costs and revenues:

$$\pi(q) = R(q) + S(q) + X(q) - M(q) - H(q) - Z(q) \tag{7.12}$$

With the exception of the extension, all of the other parts of the total profit are identical to those used in (5.1), the total profit of the last buy problem with no replenishment. In both cases, the penalty cost $Z(q)$ only examines the possibility of running out to to time t_{c1} (or t_c as represented in Chapter 5). The manufacturing cost occurs at the last buy point in either formula. Revenue and holding occur until either the last part has sold or the last assembly has failed; the salvage occurs if the last assembly has failed. Thus, other parts are not affected by the extension.

Because of the nature of the extension, it is necessary to solve for $X(q)$ before we solve for the other portions of the profit. More specifically, it is necessary to solve for the parts of $X(q)$, starting with equation (7.9) for $V(l, n)$ for different values of l and n.

$$V(l,n) = -p\sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$$

$$P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt.$$

Because of the properties of the extension are similar to the properties of the penalty cost, it can be solved in a manner similar to the last buy problem with no replenishment, with a few adjustments. The extension, given a value of l assemblies at time t_c, increases to lp_x as $q \to \infty$. Thus, by separating the extension into parts according to the number of assemblies remaining at time t_c, we can find an upper bound on the rate of change of $X(q)$.

Thus, we can establish an upper bound on the rate of change of the extension

- an upper bound that decreases with increasing values of q.

$$\Delta X(q) \leq \sum_{l=1}^{j} P\{A_{t_{c1}} = l\} e^{-t_{c1}\alpha} l p_x - X(q). \tag{7.13}$$

The reason this inequality exists can be found in Lemma 7.1 and in Appendix D. As q becomes large, the probability $P\{D_t \leq q\}$ not only starts off close to 1, but stays close to 1 for the entire extension time period. Thus, the derivative $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ for any reasonably possible value of n approaches 0 for all t - and, thus, $V(l,n)$ approaches 0. This leaves only the profit to be gained by accepting the contract - which is the expected value of the number of assemblies still functioning at time t_{c1} multiplied by the extension payoff p_x. Thus, as a result, as q gets large, $X(q)$ increases, but cannot increase above $\sum_{l=1}^{j} P\{A_{t_{c1}} = l\} e^{-t_{c1}\alpha} l p_x$. Thus, we have an upper bound on both the extension $X(q)$ and the rate of change of the extension $\Delta X(q)$:

$$X_{max} = \sum_{l=1}^{j} P\{A_{t_{c1}} = l\} e^{-t_{c1}\alpha} l p_x. \tag{7.14}$$

Once values for $V(l,n)$ are found from equation (7.9), calculation of the expected value of the extension is simplified. In this case, the calculation is simplified by the fact that, no matter the initial last buy order amount q, if there are l assemblies and $q - n$ parts remaining at time t_{c1}, the above equation (7.9) is unchanged. The only thing that changes is the probability of $q - n$ parts and l assemblies remaining.

Thus, it becomes a relatively simple exercise to calculate $X(q)$ or the rate of change $\Delta X(q)$ for increasing values of q.

From (7.10), the rate of change of the profit $\Delta\pi(q)$ is

$$\Delta\pi(q) = \Delta R(q) + \Delta S(q) + \Delta X(q) - \Delta H(q) - \Delta Z(q) - \Delta M(q). \tag{7.15}$$

However, because the formula is not concave, we cannot guarantee that any local maximum found on the profit curve is a global maximum. Therefore, we find an upper bound on the optimal solution to the problem in much the same way as the last buy problem with no replenishment was solved: we find both the rates of change $\Delta\pi(q)$, and an upper bound on the rate of change $\Delta\pi_{ub}(q)$:

$$\Delta\pi_{ub}(q) = \Delta R(q) + \Delta S(q) + [\sum_{l=1}^{j} P\{A_{t_{c1}} = l\}e^{-t_{c1}\alpha} l p_x - X(q)] - \Delta H(q) + Z(q) - \Delta M(q). \tag{7.16}$$

Because all of the parts of $\Delta\pi_{ub}(q)$ decrease with increasing q and because $\Delta\pi_{ub}(q) > \Delta\pi(q)$, we know that the smallest value of $\Delta\pi_{ub}(q)$ such that $\Delta\pi_{ub}(q) < 0$ establishes an upper bound on the optimal order amount q^*, because we know that $\Delta\pi(q) < 0$ for all higher values of q. Once we have this upper bound, we search the values of q less than this upper bound for the optimal, in the same way that we found the solution to the standard last buy problem with no replenishment.

Example: We have a set of 10 assemblies still in operation at the last buy point. The lifetime of each assembly is represented as a Weibull with rate parameter $\beta = 0.2$ and exponential parameter $\omega = 1.05$. The lifetime of each part is represented as an exponential with a mean lifetime of $\frac{1}{\lambda} = 3.33$. The initial contract period is six years so that $t_{c1} = 6$; the extension would cause the contract to go out to $t_{c2} = 10$. To extend the contract, the customer will have to pay $x = 3$ units for each assembly still in operation.

Beyond that, the cost to manufacture each spare part is $m = 10$; each part sells for $r = 20$, and each part that doesn't sell is salvaged for $s = 3$. The penalty, whether it occurs in the initial contract or the extension, is $p = 40$ per assembly in operation once the last buy runs out. All costs and revenues, including the extension payments, are discounted at a rate of $\alpha = 0.05$.

Our first step is to calculate the expected maximum value of the extension. From (7.14):

$$X_{max} = \sum_{l=0}^{j} e^{-t_{c1}\alpha} l p_x P\{A_{t_{c1}} = l\} = 8.94. \tag{7.17}$$

Once this is finished, we can find both our rate of change and the upper bound on our rate of change. This is presented in Table 7.1.

From the results presented in Table 7.1, the upper bound on the rate of change becomes negative at $q = 15$ units; therefore, our optimal order amount must be less than or equal to 15. As the only point where our rate of change goes from positive to negative occurs at $q = 13$, this is our optimal order amount.

Table 7.1: Calculation of the Optimal Order Amount

Order amount, q	1	2	3	4	5	6	7	8
Rate of change $\Delta\pi(q)$	51.1	46.9	43.3	40.2	36.9	32.9	28.0	22.2
Upper Bound $\Delta\pi_{ub}(q)$	321.2	277.9	237.9	200.6	165.4	132.4	102.2	75.6
Order amount, q	9	10	11	12	13	14	15	16
Rate of change $\Delta\pi(q)$	16.1	10.3	5.1	0.9	-2.4	-4.8	-6.6	-7.9
Upper Bound $\Delta\pi_{ub}(q)$	53.3	35.5	22.1	12.4	5.5	0.7	-2.7	-5.1

This matches our optimal solution for the problem, as demonstrated in Figure 7.3. The optimal order amount for the problem occurs at $q^* = 13$ units, with a value of 39.5.

7.4 Numerical Experiments

The first thing to note in regards to the calculation of the extension is that its calculation can be greatly simplified with very little loss of accuracy. While an exhaustive examination of all potential combinations of parts and assemblies remaining at time t_{c1} would produce an exact solution, it is possible to significantly reduce the number of combinations used without significantly compromising the overall result.

To understand why, we present a simple scenario: Suppose the lifetime of each assembly is identically and normally distributed, and the contract time t_{c1} extends to a point that is one standard deviation beyond the mean lifetime of an assembly. Thus, each assembly has only a 15.9% chance of surviving to the end of the initial contract. If there are ten such assemblies, the probability of the number of assemblies surviving to the end of the initial contract are presented in Table 7.2.

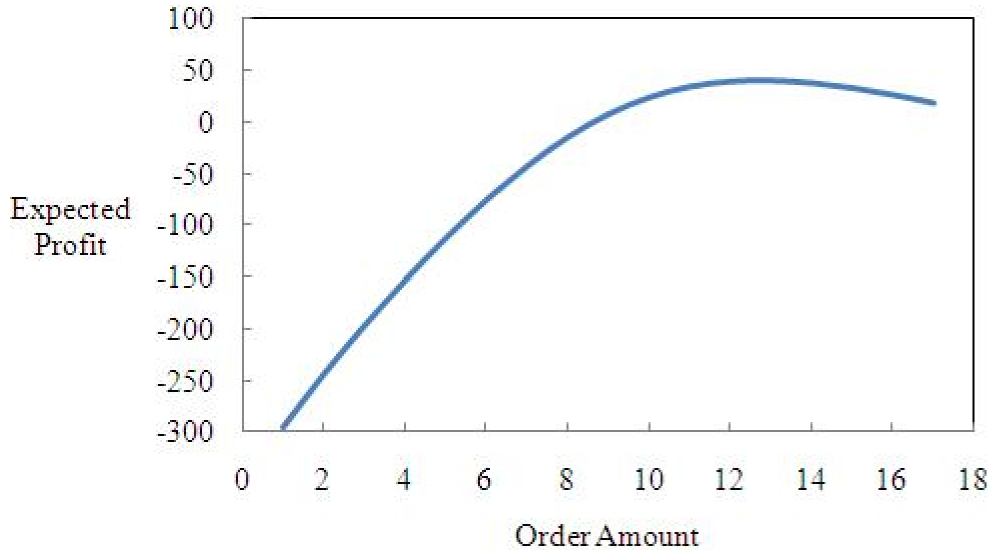

Figure 7.3: The Total Profit, Given Values of q, for the Last Buy Problem with Optional Contract Extension

Clearly, we can exclude the extension calculations in cases where 6 or greater assemblies survive to the end of the initial contract. As these comprise less than 0.3% of our possibilities, we can ignore these cases without incurring significant errors in our model.

Table 7.2: Probabilities of the Number of Assemblies Outlasting the Initial Contract

Number of Assys, l	0	1	2	3	4	5	6	7-10
$P\{A_{t_{c1}} = l\}$	0.177	0.335	0.285	0.144	0.047	0.011	0.002	0.000

This exclusion of calculations also applies to the number of parts n demanded during the initial contract, given that l assemblies survived to the end of the contract. If the number of parts demanded given the number of assemblies that survived, $P\{D_{t_{c1}} = n | A_{t_{c1}} = l\}$ is small, these events can be similarly removed from our calculations. The size of the probability $P\{D_{t_{c1}} = n | A_{t_{c1}} = l\}$ that allows for its exclusion varies from problem to problem; a manager might seek to include at least 98% of the possible outcomes for each number of assemblies remaining in order to ensure an accurate solution.

The second experiment to note in regards to the extension is the effect of the discount rate α on the extension. If the manufacturer uses a larger discount rate, it is more likely that the manufacturer will accept the contract extension.

To understand why, we need to understand the order in which events occur in the last buy extension. The incentive for the manufacturer to accept the contract extension occurs at time t_{c1}, when the manufacturer accepts the customer's payment for accepting the contract. The penalties for accepting the contract occur during the contract extension between times t_{c1} and t_{c2}, at which point the manufacturer may be forced to pay a penalty to the customer for failing to meet demand if the extension is accepted. Because the penalties associated with the contract extension occur later than the payoff associated with the extension, a larger discount rate will decrease the value of the penalty in relation to the payoff. Thus, the manufacturer will be more likely to accept the contract extension if a larger discount rate is used.

We demonstrate this by finding the values of the extension to the manufacturer for three cases; the results are given in Tables 7.3 through 7.5. In all three cases, the lifetime of the assembly is represented as a Weibull with rate parameter $\beta = 0.1$ and exponential parameter $\omega = 1.1$. The lifetime of the spare part is represented as an exponential with mean $\frac{1}{\lambda} = 4$. The payout to the manufacturer for extending the contract is $x = 3$ per assembly still in operation at time t_{c1}, the end of the initial contract. If the manufacturer is unable to meet demand for spare parts while under contract, either during the initial contract or during the contract extension, the manufacturer must pay a cost of $p = 50$ for each assembly still in operation at that point. Eight assemblies are in operation at time $t = 0$. Given these conditions, we find the value of the contract extension decision, $\max[e^{-t_{c1}\alpha} l p_x - V(l,n), 0]$, for different values of l, the number of assemblies still in operation at time t_{c1}, and n, the number of parts remaining in inventory at time t_{c1}. These form the values presented in Tables 7.3 through 7.5.

In the examples, we note the number of parts in inventory that would allow for the manufacturer to accept the contract extension while still making a profit, given the number of assemblies still in operation. For all numbers of assemblies, the number of parts in inventory that would allow the manufacturer to accept the extension decreases as the discount rate α increases. If 8 assemblies are in operation at time t_{c1} and a discount rate of only $\alpha = 0.1$ is used, an inventory of 10 spare parts is required for the manufacturer to accept the extension; with a discount rate of $\alpha = 0.3$, only 7 parts are required in inventory. Thus, a manufacturer needs to be conscious of the discount rate used in calculations, and its effect on the contract-extension decision.

Table 7.3: Extension Values to Manufacturer, Given $\alpha = 0.1$

Number of Spare Parts at t_{c1}	1	2	3	4	5	6	7	8	9	10
1 assembly	0	0	0.485	1.121	1.335	1.397	1.413	1.416	1.417	1.417
2 assemblies	0	0	0	0.809	2.057	2.568	2.752	2.811	2.828	2.833
3 assemblies	0	0	0	0	1.131	2.902	3.722	4.062	4.189	4.233
4 assemblies	0	0	0	0	0	1.502	3.710	4.824	5.334	5.546
5 assemblies	0	0	0	0	0	0	1.935	4.511	5.894	6.574
6 assemblies	0	0	0	0	0	0	0	2.434	5.319	6.946
7 assemblies	0	0	0	0	0	0	0	0	2.997	6.946
8 assemblies	0	0	0	0	0	0	0	0	0	3.622

Table 7.4: Extension Values to Manufacturer, Given $\alpha = 0.2$

Number of Spare Parts at t_{c1}	1	2	3	4	5	6	7	8	9	10
1 assembly	0	0	0.402	0.588	0.648	0.664	0.668	0.669	0.669	0.669
2 assemblies	0	0	0	0.769	1.129	1.269	1.317	1.333	1.337	1.338
3 assemblies	0	0	0	0.122	1.145	1.648	1.871	1.960	1.993	2.004
4 assemblies	0	0	0	0	0.382	1.543	2.160	2.460	2.593	2.647
5 assemblies	0	0	0	0	0	0.702	1.962	2.672	3.041	3.218
6 assemblies	0	0	0	0	0	0	1.068	2.404	3.189	3.619
7 assemblies	0	0	0	0	0	0	0	1.471	2.862	3.712
8 assemblies	0	0	0	0	0	0	0	0	1.905	3.340

Table 7.5: Extension Values to Manufacturer, Given $\alpha = 0.3$

Number of Spare Parts at t_{c1}	1	2	3	4	5	6	7	8	9	10
1 assembly	0	0.066	0.237	0.294	0.310	0.315	0.316	0.316	0.316	0.316
2 assemblies	0	0	0.206	0.468	0.575	0.614	0.627	0.631	0.632	0.632
3 assemblies	0	0	0	0.390	0.705	0.851	0.912	0.936	0.945	0.947
4 assemblies	0	0	0	0	0.601	0.950	1.126	1.208	1.243	1.257
5 assemblies	0	0	0	0	0.185	0.832	1.203	1.402	1.502	1.548
6 assemblies	0	0	0	0	0	0.440	1.076	1.462	1.680	1.795
7 assemblies	0	0	0	0	0	0	0.695	1.331	1.726	1.959
8 assemblies	0	0	0	0	0	0	0.035	0.966	1.594	1.996

Our final numerical experiment notes the profitability of the contract extension in the last buy. Given that the agreement with the customer has expired, is it in the manufacturer's best interest to continue serving the customer at all?

We present the following example. The lifetime of the assembly is represented as a Weibull with a rate parameter of $\beta = \frac{1}{7}$ and an exponential parameter of $\omega = 1.08$. The lifetime of each spare part is represented as an exponential with mean $\frac{1}{\lambda} = 3.33$. The discount rate used for all revenues and costs is $\alpha = 0.1$. Parts are manufactured at a cost of $m = 10$ per part and sold at a value of $r = 20$ per part; any parts not sold are salvaged at a value of $s = 5$ per part. If the inventory is depleted while the manufacturer is under contract, the manufacturer must pay $p = 50$ for each assembly still in operation. The holding cost per part per unit time is $h = 1.5$. The length of the initial contract is $t_{c1} = 7.5$; if the customer and the manufacturer agree to extend the contract to time $t_{c2} = 15$, the customer will pay $x = 4$ for each assembly still in operation.

Given these conditions, we run two different scenarios. In the first, the contract is extended if the manufacturer is able to make a profit by extending the contract. In the second, the manufacturer not only refuses to extend the contract, but salvages all parts still in inventory and ceases any further service to the customer.

The results are in Table 7.6. As we see, if only optimal conditions are considered, the best option for the manufacturer is to cease selling the part to the customer and to salvage what parts remain in inventory. Thus, provided that no loss of sales or goodwill would occur from the salvage, the best move for a manufacturer to make

Table 7.6: Comparison of Contract Extension and Salvage options

	Optimal order amount q^*	Value of optimal solution $\pi(q^*)$
Extension option	16	1.4
Salvage option	17	14.3

is to stop serving the customer once the contract expires. A manufacturer should consider this salvage option, as well as any potential consequences, when weighing a contract extension.

7.5 Managerial Insights

When a manager begins to calculate for the optimal last buy order in a case where contract extensions are a possibility, the first quality to note about the problem is that not every potential end-of-contract scenario needs to be calculated in order to determine an optimal order amount. If the probability of a specific scenario occurring is so small that the expected value of that scenario is reduced to near zero, then those calculations can be effectively ignored. This is an effective method to reduce the number of calculations required without seriously compromising the accuracy of the solution. This is particularly useful for reducing calculations in cases where large numbers of possible inventory-assembly combinations at time t_{c1} are presented.

When a manager calculates for an optimal order amount, some consideration should be given as to the effect of the discount rate on the contract extension decisions. A higher discount rate will always result in a greater probability that the extension will be accepted, as the higher discount rate will reduce the value of the expected penalty from extending the contract in comparison to the payoff to the manufacturer for extending the contract.

Finally, a manager should keep in mind that extending the contract is not necessarily a requirement - indeed, that continuing to serve the customer beyond the contract period is not a requirement. In fact, the optimal choice for the manufacturer may be to simply sell any remaining inventory off once the initial contract has expired. A manager should weigh all costs, revenues, and options, both at the initial last buy and the end of the initial contract, to determine the best course of action for the manufacturing firm to take.

7.6 Conclusion

One of the issues that complicates the last buy problem is the possibility of contract extensions. At the end of a manufacturer's contract with a customer, it is possible that a customer could request a contract extension to ensure that the manufacturer will continue to serve the customer. In such a case, the manufacturer, for a fee, ensures the customer that any demand for a particular spare part will be met.

If a manufacturer believes that a customer is likely to ask for such a contract extension, the manufacturer may use the expected value of the contract extension as part of their calculations for the optimal order amount. In order to determine the optimal order amount with the possibility of contract extensions, it is necessary to treat the value of the extension in the same manner as the penalty cost. Neither the extension nor the penalty cost are concave with respect to the order amount q. However, the value of both in the total profit function increases to a specific value with increasing q - in the case of the extension, up to $\sum_{l=0}^{j} e^{-t_{c1}\alpha} lp(x)P\{A_{t_{c1}} = l\}$. Thus, like with the penalty cost, we can use the value of the extension to establish an upper bound on the rate of change of the total cost - an upper bound that decreases with increasing q. The point at which this upper bound becomes negative establishes an upper bound on the optimal order amount; we can then search the remaining possible solutions to find the optimal order amount.

When a manager calculates the optimal solution to the last buy problem with optional contract extension, the calculations can be greatly simplified by examining the probability of each scenario at t_{c1}, the end of the initial contract period. If a scenario is so unlikely that its effect on the expected value is negligible, then it can be ignored in calculations, thus simplifying the overall calculations. The manager should also be aware not only of the costs and values of an optional contract extension, but should evaluate other customer service levels and options as well, as our experiments showed that, in some cases, the optimal move for a manufacturer may be to salvage all remaining inventory and cease serving the customer altogether.

CHAPTER 8
FUTURE WORK

The research contained herein allows for several areas of future exploration. In many cases, the last buy occurs with several mitigating circumstances that affect how the problem should be addressed. As such, there are problems beyond the last buy that should be addressed.

8.1 Contract Pooling

In many cases, a manufacturer will have multiple contracts with different customers. Each of these contracts will have different requirements: different contract lengths, different sales values for the spare part, and different penalties for failing to meet customer demand.

All of the other models demonstrated here operate under a basic assumption: If there is demand from a customer and a part sits in inventory, then that part will go to the customer. In a case with pooled contracts, this is not necessarily the case. A difference in contract length could cause meeting demand from a customer no longer under contract to be suboptimal, as meeting that demand could result in paying a penalty to another customer still under contract. If one customer pays less for a spare part than another, it is possible that satisfying demand from the lesser-paying customer would result in less revenue overall. Finally, if one customer negotiated a higher penalty for failing to meet demand, it may be optimal to refuse demand from other customers to ensure demand from that customer is met.

8.2 Valuation and Principal-Agent Issues

In some cases, a manufacturer will sometimes wish to avoid dealing with the problem of supplying spare parts by contracting another company to manufacture and supply spare parts. This provides two problems that need to be addressed in the business relationship between the outsourcing company and the part-supplying company.

At a bare minimum, a solution to a last buy problem is not one number, but two: the optimal order amount, and the expected discounted value of that solution. That expected discounted value provides a value to a company to sell off the responsibilities of supplying spare parts. Thus, the last buy problem can potentially be used as a form of valuation.

The other problem occurs as a result of last buy problems with no form of replenishment. The ”penalty cost” incurred in a last buy problem with no replenishment may not simply be a dollar value. While some of it may be a material penalty that can be transferred to the agent company, some of it could be represented as a loss of goodwill. In such a case, a problem exists: What incentives and penalties need to be put in the contract so that the contracted company orders what is, for the manufacturer, an optimal order amount? These are problems that must be addressed as the manufacturer and the contracted company work out an agreement.

8.3 Contract Extensions and Consumer Behavior

In the model we used for our contract extension research, we simplified our calculations by providing a specific rule for acceptance of the contract extension. This leaves unsaid what would occur if a different model regarding contract acceptance were used. While some argument would be needed as to why a profit-based decision model would not be used from the manufacturer's perspective, the method used by the customer - that the customer would agree as long as the extension cost was reasonable and assemblies remained to be served - leaves room for potential improvement. One possible change to the contract extension case involves the use of probabilities for contract extension based on the number of assemblies remaining at the conclusion of the initial contract. In other words, instead of being a simple yes/no decision regarding whether or not the customer would desire a contract extension, an estimate of the probability of the customer accepting a contract extension is used. Based on past events, the manufacturer would estimate the probability that the customer would seek a contract extension given the number of assemblies the customer still has in operation. Such a method would be necessitated in any case where the manufacturer has only imperfect information regarding the extension price that the customer would be willing to accept. Another possibility in regards to contract extension lies in the nature of the type of payment made from the customer to the manufacturer as incentive to agree to the contract extension. In our model, we varied the cost from customer to manufacturer based on the number of assemblies the contract extension would serve. One possible alternative to this would be to dictate a flat rate for the price of a contract extension, regardless of the number of assemblies remaining. Overall, we intend to vary the behavior model used to represent the customer to determine its overall effect on the use of contract extensions.

8.4 Integrated Post-production Spare Part Management

The research in this thesis focuses on one spare part of one product. Products often contain multiple different types of spare parts, and in some cases certain spare parts can be found on multiple different products. Therefore, what is ultimately needed is an integrated spare part management system, one that is able to combine the information from multiple potential last buy problems into a single system.

To understand the mechanism of how such a spare part management system would operate, first consider that each customer faces choices in regards to the product. The customer has a probability each day of deciding to replace the product with a new model, thus taking the old product out of service. The customer also has a choice with each mechanical failure that occurs to the product: replace the broken part, thus generating demand for the spare part, or replacing the product with a new model. Thus, the probability distribution of assembly failure for a given product over time is the sum of these individual probabilities. Based on this probability distribution and the distribution of the lifetime of each spare part, a company may generate a demand distribution for the spare part over time. This can be done for all of the spare parts in the mechanism, provided that the product can be repaired by replacing the part.

APPENDIX A
PROOFS AND ANALYSES FOR THE LAST BUY PROBLEM WITH INCREMENTAL REPLENISHMENT

Revenue Proofs

The proofs for the properties of the sales revenue:

Proof: The revenue $R(q)$ is non-negative.

The equation for the revenue can be found from (4.2):

$$R(q) = -r \sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt.$$

By looking at the parts of (4.2) and determining if each part is non-negative or non-positive, we can determine if (4.2) in its entirety is non-positive or non-negative.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $\frac{dP\{D_t \leq n\}}{dt}$: The probability $P\{D_t \leq n\}$ decreases as t increases, as we do not allow for negative demand. Thus, once the demand up to a specific time $D_{\hat{t}} > n$, the demand can never be less than or equal to n at any point in time beyond $\hat{t}$. Thus, because the probability $P\{D_t \leq n\}$ can only decrease as t increases, the rate of change $\frac{dP\{D_t \leq n\}}{dt}$ is non-positive for all values of t.
- r: The revenue per part sold r is non-negative, according to the requirements of the problem.
- $\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt$: All parts of this integral are non-negative for all values of t except for $\frac{dP\{D_t \leq n\}}{dt}$, which is non-positive for all values of t. As a result, the

entire integral must be non-positive.

Because $\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt$ is non-positive and r is non-negative, the negative of the product of these three parts must be non-negative. Thus,

$$R(q) = -r \sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt \geq 0. \tag{A-1}$$

Thus, $R(q)$ is non-negative. ■

Proof: The sales revenue $R(q)$ is nondecreasing as q increases.

The strategy used for this proof is to find the rate of change of the revenue with respect to q, $R(q+1) - R(q)$, and see if this rate of change is non-negative by examining each of its parts. For the revenue model with a discount rate, the difference in the revenue $R(q)$ with a change in the order amount from q to $q+1$ is

$$\begin{aligned} R(q+1) - R(q) = r[&\sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt \\ &- \sum_{n=0}^{q} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt]. \end{aligned} \tag{A-2}$$

This can be restated as

$$R(q+1) - R(q) = -r \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-3}$$

From this, we need to look at each of the parts of the formula to determine whether it is positive or negative. As r is always non-negative, $(-r)$ is always non-positive. The discount $e^{-t\alpha}$ is always positive, as $0 < e^{-t\alpha} \leq 1$ for all $t \geq 0$. Because we assume that we will never have a negative net demand, the final part, $\frac{dP\{D_t \leq q\}}{dt}$, is always non-positive, as $P\{D_t \leq q\} = 1$ at $t = 0$, and gradually decreases as t increases. Thus, as we have two negative terms in our formula, the overall formula will not be negative. As a result, $R(q+1) - R(q) \geq 0$ for all feasible solutions of q. ■

Proof: The sales revenue function $R(q)$ is concave with respect to q - that is, the rate of increase of $R(q)$ is nonincreasing as q increases.

In this case, our goal is to examine the convexity function $R(q+2) - 2R(q+1) + R(q)$. If this convexity function can be demonstrated to be non-positive by examining its parts, then the revenue is concave.

The determination of concavity with respect to $R(q+2) - 2R(q+1) + R(q)$ is complicated by the fact that $\frac{d(P\{D_t = q+1\})}{dt}$ may be either positive or negative at different values of t. Thus, it becomes necessary to determine whether or not the integral $\int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt$ is positive or negative. We determine that $\int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt$ is non-negative through an induction proof using the points at which $\frac{d(P\{D_t = q+1\})}{dt}$ goes from negative to positive. Once it is determined that $\int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > 0$, the concavity of $R(q)$ can be proven.

The proof starts by stating $R(q+2) - 2R(q+1) + R(q)$:

$$R(q+2) - 2R(q+1) + R(q) = \\ r\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t \leq q\}}{dt}dt - r\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t \leq (q+1)\}}{dt}dt. \tag{A-4}$$

By combining the two terms, we simplify our result.

$$R(q+2) - 2R(q+1) + R(q) = -r\int_{t=0}^{\infty} e^{-t\alpha}\frac{d(P\{D_t = q+1\})}{dt}dt. \tag{A-5}$$

Thus, the convexity or concavity of the revenue is dependent on whether or not $\int_{t=0}^{\infty} e^{-t\alpha}\frac{d(P\{D_t=q+1\})}{dt}dt$ is positive or negative. Unfortunately, $\frac{d(P\{D_t=q+1\})}{dt}$ can be either positive or negative at any given value of $t > 0$. Fortunately, while there may be possible values of $t > 0$ for which $\frac{d(P\{D_t=q+1\})}{dt}$ is negative, for any $\hat{t} > 0$ and any integer $n > 0$, $\int_{t=0}^{\hat{t}} \frac{d(P\{D_t=n\})}{dt}dt \geq 0$, as any negative result would result in a negative value for $P\{D_x = n\}$, which is an impossibility.

We assume, for some $\frac{d(P\{D_t=n\})}{dt}dt$, $n > 0$, that it goes from positive to negative at points $t_{x1}, t_{x2}, t_{x3}, ...$ and from negative to positive at points $t_{y1}, t_{y2}, t_{y3}....$ Note that the last time in the set $(t_{y1}, t_{y2}, t_{y3}...)$ may approach infinity.

Thus, $P\{D_t = n\}$ goes from positive to negative at time t_{x1}. Then, from the above, $\int_{t=0}^{t_{x1}} \frac{d(P\{D_t=n\})}{dt}dt + \int_{t=t_{x1}}^{t_{y1}} \frac{d(P\{D_t=n\})}{dt}dt \geq 0$, or

$$\int_{t=0}^{t_{x1}} \frac{d(P\{D_t = n\})}{dt}dt \geq -\int_{t=t_{x1}}^{t_{y1}} \frac{d(P\{D_t = n\})}{dt}dt. \tag{A-6}$$

Since $e^{-t_{x1}\alpha} > 0$ for all α,

$$\int_{t=0}^{t_{x1}} e^{-t_{x1}\alpha} \frac{d(P\{D_t = n\})}{dt} dt \geq - \int_{t=t_{x1}}^{t_{y1}} e^{-t_{x1}\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-7}$$

For $0 < t < t_{x1}$ and for $\alpha > 0$, $e^{-t\alpha} > e^{-t_{x1}\alpha}$. Also, for $t_{x1} < t < t_{y1}$, $e^{-t\alpha} < e^{-t_{x1}\alpha}$. Therefore,

$$\int_{t=0}^{t_{x1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > \int_{t=0}^{t_{x1}} e^{-t_{x1}\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-8}$$

and

$$- \int_{t=t_{x1}}^{t_{y1}} e^{-t_{x1}\alpha} \frac{d(P\{D_t = n\})}{dt} dt > - \int_{t=t_{x1}}^{t_{y1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-9}$$

Equations (A-8) and (A-9) can be combined as

$$\begin{aligned} &\int_{t=0}^{t_{x1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{x1}}^{t_{y1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt \\ &> \int_{t=0}^{t_{x1}} e^{-t_{x1}\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{x1}}^{t_{y1}} e^{-t_{x1}\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \end{aligned} \tag{A-10}$$

This simplifies to

$$\int_{t=0}^{t_{y1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > e^{-t_{x1}\alpha} \int_{t=0}^{t_{y1}} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-11}$$

Since $e^{-t_{x1}\alpha} > 0$ and $\int_{t=0}^{t_{y1}} \frac{d(P\{D_t=n\})}{dt} dt = P_{t_{y1}}[D = n\} \geq 0$, then

$$\int_{t=0}^{t_{y1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt \geq 0. \tag{A-12}$$

The next step is to establish that $\int_{t=0}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t=n\})}{dt} dt \geq 0$ given that $\int_{t=0}^{t_{y(l)}} e^{-t\alpha} \frac{d(P\{D_t=n\})}{dt} dt \geq 0$.

As before, since $\int_{t=0}^{t_{x(l+1)}} \frac{d(P\{D_t=n\})}{dt} dt + \int_{t=t_{x(l+1)}}^{t_{y(l+1)}} \frac{d(P\{D_t=n\})}{dt} dt \geq 0$, we start with

$$\int_{t=0}^{t_{x(l+1)}} \frac{d(P\{D_t = n\})}{dt} dt \geq -\int_{t=t_{x(l+1)}}^{t_{y(l+1)}} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-13}$$

We split up the left hand side of equation (A-13) to get

$$\int_{t=0}^{t_{y(l)}} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{y(l)}}^{t_{x(l+1)}} \frac{d(P\{D_t = n\})}{dt} dt \geq -\int_{t=t_{x(l+1)}}^{t_{y(l+1)}} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-14}$$

Since $e^{-t_{x(l+1)}\alpha} > 0$ for all α, we can distribute this through (A-14) to get

$$\int_{t=0}^{t_{y(l)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{y(l)}}^{t_{x(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt \\ \geq - \int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \quad \text{(A-15)}$$

Now that we have this, we need to look at each part of the formula and compare $\int e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t=n\})}{dt} dt$ with $\int e^{-t\alpha} \frac{d(P\{D_t=n\})}{dt} dt$

First, because $t_{x(l+1)} > t_{x(l)}$, the exponential

$$e^{-t_{x(l+1)}\alpha} < e^{-t_{x(l)}\alpha}. \quad \text{(A-16)}$$

Thus, because $\int_{t=0}^{t_{y(l)}} \frac{d(P\{D_t=n\})}{dt} dt \geq 0$, we multiply both exponents by this integral:

$$\int_{t=0}^{t_{y(l)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=0}^{t_{y(l)}} e^{-t_{x(l)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \quad \text{(A-17)}$$

Moreover, we assume the following as a result of the induction proof:

$$\int_{t=0}^{t_{y(l)}} e^{-t_{x(l)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=0}^{t_{y(l)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \quad \text{(A-18)}$$

We combine these two statements (A-17) and (A-18) to find

$$\int_{t=0}^{t_{y(l)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=0}^{t_{y(l)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-19}$$

For $t_{y(l)} < t < t_{x(l+1)}$ and for $\alpha > 0$, $e^{-t\alpha} > e^{-t_{x(l+1)}\alpha}$. Also, for $t_{x(l+1)} < t < t_{y(l+1)}$, $e^{-t\alpha} < e^{-t_{x1}\alpha}$. Therefore,

$$\int_{t=t_{y(l)}}^{t_{x(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > \int_{t=t_{y(l)}}^{t_{x(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-20}$$

Also, for $t_{x(l+1)} < t < t_{y(l+1)}$,

$$-\int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt > -\int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-21}$$

Or, to rephrase,

$$\int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-22}$$

As has been shown before,

$$e^{-t_{x(l+1)}\alpha} \int_{t=0}^{t_{y(l+1)}} \frac{d(P\{D_t = n\})}{dt} dt \geq 0. \tag{A-23}$$

This integral can be divided into parts:

$$\int_{t=0}^{t_{y(l)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{y(l)}}^{t_{x(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt \geq 0. \tag{A-24}$$

However, based on the previous inequalities (A-18), (A-20), and (A-22),

$$\int_{t=0}^{t_{y(l)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{y(l)}}^{t_{x(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=0}^{t_{y(l)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{y(l)}}^{t_{x(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt + \int_{t=t_{x(l+1)}}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-25}$$

Or, to simplify,

$$\int_{t=0}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=0}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt. \tag{A-26}$$

Thus,

$$\int_{t=0}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > 0 \tag{A-27}$$

Therefore, as we have shown, to restate the results of (A-11) and (A-12),

$$\int_{t=0}^{t_{y1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > e^{-t_{x1}\alpha} \int_{t=0}^{t_{y1}} \frac{d(P\{D_t = n\})}{dt} dt.$$

and

$$\int_{t=0}^{t_{y1}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt \geq 0.$$

Also, if we take the results of (A-11) and (A-12), then, to restate (A-26) and (A-27),

$$\int_{t=0}^{t_{y(l+1)}} e^{-t_{x(l+1)}\alpha} \frac{d(P\{D_t = n\})}{dt} dt < \int_{t=0}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt.$$

and

$$\int_{t=0}^{t_{y(l+1)}} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > 0.$$

As such, we prove by induction that for all t_y,

$$\int_{t=0}^{t_y} e^{-t\alpha} \frac{d(P\{D_t = n\})}{dt} dt > 0. \tag{A-28}$$

Since any $\int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t=q+1\})}{dt} dt > 0$, the convexity of the revenue function is negative:

$$R(q+2) - 2R(q+1) + R(q) = -r \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = q+1\})}{dt} dt \leq 0. \qquad \text{(A-29)}$$

Thus, the rate of change of $R(q)$ does not increase as q increases, thus proving concavity of $R(q)$ with respect to q. ■

Holding Cost Proofs

The properties of the holding cost as q increases are as follows:

Proof: The holding cost $H(q)$ is nondecreasing with increasing values of q

To prove that the holding cost is nondecreasing with increasing values of q, we must demonstrate that the rate of change of the holding cost $H(q+1) - H(q)$ is non-negative for any possible value of q. If $H(q+1) - H(q)$ is always greater than or equal to 0, then the holding cost cannot decrease with increasing q, thus completing our proof. As there are two forms of the holding cost formula (one for cases where salvage occurs, and one for cases where salvage does not occur), we must prove the formula for both cases. Thus, we separate our proof into two parts - one in which we prove the holding cost $H(q)$ is nondecreasing for the case without salvage, and one in which we prove the holding cost $H(q)$ is nondecreasing for the case with salvage.

Holding cost without salvage

For the holding cost in the case where no salvage occurs, our strategy is to find the formula for the rate of change $H(q+1) - H(q)$, separate the rate of change into parts, and determine if each part is positive or negative. If each part is non-negative, then the product of those parts must in turn be non-negative. The general statement of the holding cost $H(q)$ for a given value of q is

$$H(q) = h \sum_{n=0}^{q-1} (q-n) \int_{t=0}^{\infty} e^{-t\alpha} P\{D_t = n\} dt. \tag{A-30}$$

The change in $H(q)$ from q to $q+1$ is

$$H(q+1) - H(q) = h \int_{t=0}^{\infty} e^{-t\alpha} \sum_{n=0}^{q} P\{D_t = n\} dt. \tag{A-31}$$

If we can demonstrate that (A-31) is never negative, then we have our proof for the no-salvage case. We combine the summation statement in (A-31) to get

$$H(q+1) - H(q) = h \int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\} dt. \tag{A-32}$$

We separate the formula (A-32) into its component parts and determine whether each part is positive or negative. If each of the parts is determined to be non-negative, then the product of those parts - the rate of change - is also non-negative.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $P\{D_t \leq q\}$: This is a probability. By definition, a probability must be between 0 and 1. Therefore, $P\{D_t \leq q\}$ is non-negative for all values of t.
- h: The holding cost per unit per unit time is non-negative, according to the requirements of the problem.
- $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\} dt$: Because $e^{-t\alpha} > 0$ and $P\{D_t \leq q\} \geq 0$ for all values of t, then the integral $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\} dt$ must also be non-negative.

Because each of the parts of the formula are non-negative, the product of those parts must also be non-negative. Thus,

$$H(q+1) - H(q) = h \int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\} dt \geq 0. \tag{A-33}$$

Since the rate of change of $H(q)$ with respect to q is non-negative, then $H(q)$ is non-decreasing with increasing values of q. Thus, we have proven that the holding cost $H(q)$ is nondecreasing with increasing values of q for the non-salvage case.

Holding cost with salvage

The proof for the holding cost for the case where the last buy problem could have a salvage is nearly identical, and follows the same pattern as the proof for the holding cost without salvage. We take the rate of change $H(q+1) - H(q)$, separate it into parts, and determine if the parts are non-positive.

The holding cost for the last buy problem in a case in which salvage occurs is

$$H(q) = h\sum_{n=0}^{q-1}(q-n)\int_{t=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=n|A_{t_f}=1\}dtdt_f. \quad \text{(A-34)}$$

Thus, the rate of change of the holding cost $H(q+1) - H(q)$ is

$$H(q+1)-H(q) = h\sum_{n=0}^{q}\int_{t=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=n|A_{t_f}=1\}dtdt_f. \quad \text{(A-35)}$$

If we can demonstrate that (A-35) is never negative, then we have our proof for the no-salvage case. We combine the summation statement in (A-35) to get

$$H(q+1)-H(q) = h\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t\leq q|A_{t_f}=1\}dtdt_f. \quad \text{(A-36)}$$

As with the proof for the holding cost where no salvage occurs, we separate the rate of change as presented in (A-36) into its component parts and determine if each part is positive or negative. If each of the parts is determined to be non-negative, then the product of those parts - the rate of change - is also non-negative.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $P\{D_t \leq q|A_{t_f}=1\}$: This is a probability. By definition, a probability must be between 0 and 1. Therefore, $P\{D_t \leq q|A_{t_f}=1\}$ is non-negative for all values of t.

- h: The holding cost per unit per unit time is non-negative, according to the requirements of the problem.
- $\frac{dP\{A_{t_f}=0\}}{dt_f}$: The state $A_{t_f} = 0$ can be considered a trapping state. Once the last assembly fails, the number of assemblies remaining in operation will be 0 and will remain at 0. Thus, the probability $P\{A_{t_f} = 0\}$ is nondecreasing with increasing t_f. Thus, its derivative $\frac{dP\{A_{t_f}=0\}}{dt_f}$ is non-negative for all possible values of t_f.
- $\int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt$: Because $e^{-t\alpha} > 0$ and $P\{D_t \leq q | A_{t_f} = 1\} \geq 0$ for all values of t, then the integral $\int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt$ must also be non-negative for all possible values of t_f.
- $\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt dt_f$: Also, because $\int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt$ and $\frac{dP\{A_{t_f}=0\}}{dt_f}$ are non-negative for all possible values of t_f, the rate of change integral $\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt dt_f$ must also be non-negative.

Because both $\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt dt_f$ and h are non-negative, then the product of those parts must also be non-negative. Thus,

$$H(q+1) - H(q) = h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f} = 0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t \leq q | A_{t_f} = 1\} dt dt_f \geq 0. \tag{A-37}$$

Thus, we have proven that the rate of change of the holding cost is non-negative for the salvage case. As we have proven that the rate of change of the holding cost is non-negative for both the salvage and non-salvage cases, we have proven that the holding cost $H(q)$ is nondecreasing for increasing values of q. ■

Proof: The holding cost $H(q)$ is convex with respect to q - that is, the rate of increase of the holding cost is non-decreasing with increasing values of q.

Here, we consider the convexity of the holding cost with respect to q by looking at the convexity formula for the holding cost $H(q+2) - 2H(q+1) + H(q)$. If the formula is positive, the holding cost is convex with respect to q; if the formula is negative, the holding cost is concave with respect to q.

To prove that the holding cost is convex with respect to q, we must demonstrate that the convexity of the holding cost $H(q+2)-2H(q+1)+H(q)$ is non-negative for any possible value of q. If $H(q+2)-2H(q+1)+H(q)$ is always greater than or equal to 0, then the rate of change of the holding cost cannot decrease with increasing q, thus completing our proof. As there are two forms of the holding cost formula (one for cases where salvage occurs, and one for cases where salvage does not occur), we must prove the formula for both cases. Thus, we separate our proof into two parts - one in which we prove the holding cost $H(q)$ is convex for the case without salvage, and one in which we prove the holding cost $H(q)$ is convex for the case with salvage.

Holding cost without salvage

For the holding cost in the case where no salvage occurs, our strategy is to find the formula for the convexity $H(q+2) - 2H(q+1) + H(q)$, separate the convexity into parts, and determine if each part is positive or negative. If each part is non-negative, then the product of those parts must in turn be non-negative. By using the formulation of the rate of change of the holding cost $H(q+1) - H(q)$ for a given value of q given as (A-31), the convexity function $H(q+2) - 2H(q+1) + H(q)$ for a case without salvage is

$$\begin{aligned} H(q+2) - 2H(q+1) + H(q) = h \int_{t=0}^{t_f} e^{-t\alpha} \sum_{n=0}^{q+1} P\{D_t = n\} dt \\ -h \int_{t=0}^{t_f} e^{-t\alpha} \sum_{n=0}^{q} P\{D_t = n\} dt. \end{aligned} \tag{A-38}$$

We subtract the two parts of (A-38) to get

$$H(q+2) - 2H(q+1) + H(q) = h \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\} dt. \tag{A-39}$$

We then separate the convexity function (A-39) into parts. If all of its parts are non-negative, then the product of those parts - the convexity function - is also non-negative. Thus, we examine each part of the convexity function, then put the parts back together once its sign is determined.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $P\{D_t = q+1\}$: This is a probability. By definition, a probability must be between 0 and 1. Therefore, $P\{D_t = q+1\}$ is non-negative for all values of t.
- h: The holding cost per unit per unit time is non-negative, according to the requirements of the problem.

- $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t = q+1\}dt$: Because $e^{-t\alpha} > 0$ and $P\{D_t = q+1\} \geq 0$ for all values of t, then the integral $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t = q+1\}dt$ must also be non-negative.

Since $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t = q+1\}dt$ and h are both non-negative for all possible values of t, we find that the product of these two parts must also be non-negative:

$$H(q+2) - 2H(q+1) + H(q) = h \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dt \geq 0. \tag{A-40}$$

Thus, since $H(q+2) - 2H(q+1) + H(q) \geq 0$ for all possible values of q, the convexity of the holding cost is non-negative with increasing values of q - and, thus, for the case in which no salvage occurs, our holding cost $H(q)$ is convex with respect to q.

Holding cost with salvage

For the holding cost in the case where salvage can occur, our strategy is to find the formula for the convexity $H(q+2) - 2H(q+1) + H(q)$, separate the convexity into parts, and determine if each part is positive or negative. If each part is non-negative, then the product of those parts must in turn be non-negative. By using the formulation of the rate of change of the holding cost $H(q+1) - H(q)$ for a given value of q given as (A-35), the convexity function $H(q+2) - 2H(q+1) + H(q)$ for a case in which salvage can occur is

$$H(q+2) - 2H(q+1) + H(q) = h\sum_{n=0}^{q+1}\int_{t=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=n\}dtdt_f$$
$$-h\sum_{n=0}^{q}\int_{t=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=n\}dtdt_f. \tag{A-41}$$

We subtract the two portions of equation (A-41) to get

$$H(q+2) - 2H(q+1) + H(q) = h\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=q+1\}dtdt_f. \tag{A-42}$$

Again, we separate the convexity function (A-42) into parts and determine if each part is positive or negative. If all parts are non-negative, then the product of those parts - our convexity formula - must also be non-negative, thus proving convexity of the holding cost in the case in which salvage occurs. Thus, we examine each part of the convexity function, then put the parts back together once its sign is determined.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $P\{D_t = q+1\}$: This is a probability. By definition, a probability must be between 0 and 1. Therefore, $P\{D_t = q+1\}$ is non-negative for all values of t.
- h: The holding cost per unit per unit time is non-negative, according to the requirements of the problem.
- $\frac{dP\{A_{t_f}=0\}}{dt_f}$: The state $A_{t_f} = 0$ can be considered a trapping state. Once the

last assembly fails, the number of assemblies remaining in operation will be 0 and will remain at 0. Thus, the probability $P\{A_{t_f} = 0\}$ is nondecreasing with increasing t_f. As a result, its derivative $\frac{dP\{A_{t_f}=0\}}{dt_f}$ is non-negative for all possible values of t_f.

- $\int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dt$: Because $e^{-t\alpha} > 0$ and $P\{D_t = q+1\} \geq 0$ for all values of t, then the integral $\int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dt$ must also be non-negative for all possible values of t_f.
- $\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dtdt_f$: Because $\int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dt$ and $\frac{dP\{A_{t_f}=0\}}{dt_f}$ are non-negative for all possible values of t_f, the integral $\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dtdt_f$ must also be non-negative.

Since $\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dtdt_f$ and h are both non-negative, we find that the product of these two parts must also be non-negative:

$$\begin{aligned} H(q+2) - 2H(q+1) + H(q) = \\ h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} P\{D_t = q+1\}dtdt_f \geq 0. \end{aligned} \tag{A-43}$$

Thus, since $H(q+2) - 2H(q+1) + H(q) \geq 0$ for all possible values of q, the convexity of the holding cost is non-negative with increasing values of q - and, thus, for the case in which salvage can occur, our holding cost $H(q)$ is convex with respect to q.

Because the holding cost $H(q)$ has been proven to be convex for both the case in which salvage can occur and the case where salvage cannot occur, the holding cost

$H(q)$ is proven to be convex with respect to q. ■

Incremental Replenishment Proofs

Proof: The incremental replenishment cost $Z(l, n)$ is non-negative.

The equation for the revenue can be found from (4.6):

$$Z(q) = -p \sum_{n=q}^{\infty} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt.$$

By looking at the parts of (4.6) and determining if each part is non-negative or non-positive, we can determine if (4.6) in its entirety is non-positive or non-negative.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $\frac{dP\{D_t \leq n\}}{dt}$: The probability $P\{D_t \leq n\}$ decreases as t increases, as we do not allow for negative demand. Thus, once the demand up to a specific time $D_{\hat{t}} > n$, the demand can never be less than or equal to n at any point in time beyond $\hat{t}$. Thus, because the probability $P\{D_t \leq n\}$ can only decrease as t increases, the rate of change $\frac{dP\{D_t \leq n\}}{dt}$ is non-positive for all values of t.
- r: The revenue per part sold r is non-negative, according to the requirements of the problem.
- $\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt$: All parts of this integral are non-negative for all values of t except for $\frac{dP\{D_t \leq n\}}{dt}$, which is non-positive for all values of t. As a result, the entire integral must be non-positive.

Because $\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt$ is non-positive and rp is non-negative, the negative of the product of these three parts must be non-negative. Thus,

$$Z(q) = -p \sum_{n=q}^{\infty} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt \geq 0. \tag{A-44}$$

Thus, $Z(q)$ is non-negative. ■

Proof: The incremental replenishment cost $Z(q)$ is non-increasing with increasing q.

To determine whether or not the incremental replenishment cost is nonincreasing with increasing q, it is necessary to take the rate of change of the incremental replenishment $Z(q+1) - Z(q)$, separate the rate of change into parts, and determine if each part is positive or negative. If the product of the parts is non-positive, then the incremental replenishment cost is nonincreasing with increasing q. Thus, in order to determine whether or not $Z(q)$ increases or decreases with respect to q, we need to look at the rate of change of $Z(q)$ with respect to q.

$$Z(q+1) - Z(q) = p \sum_{n=q}^{\infty} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt - p \sum_{n=q+1}^{\infty} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt. \tag{A-45}$$

We subtract these two parts to get

$$Z(q+1) - Z(q) = p \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-46}$$

From here, we look at each part to determine whether it is positive or negative. The incremental replenishment cost per unit p is always non-negative. The discount factor $e^{-t\alpha}$ is always positive. The rate of change of demand $\frac{dP\{D_t \leq q\}}{dt}$ is always negative; the probability $P\{D_t \leq q\}$ starts at 1 at $t = 0$, and slowly reduces as time increases. Thus, because one part of the statement is non-positive and the other parts of the statement are non-negative, the statement in its entirety is non-positive. Thus, the rate of change of $Z(q)$ is

$$Z(q+1) - Z(q) = p \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt \leq 0. \tag{A-47}$$

Because $Z(q+1) - Z(q) \leq 0$, the incremental replenishment cost $Z(q)$ is non-increasing as q increases. ■

Proof: The incremental replenishment cost $Z(q)$ is convex with respect to q.

Similarly, in order to determine the convexity or concavity of $Z(q)$, we look at the convexity function $Z(q+2) - 2Z(q+1) + Z(q)$ with respect to q. If the concavity function $Z(q+2) - 2Z(q+1) + Z(q)$ is non-negative for any value of q, then the incremental replenishment cost $Z(q)$ is convex; if the concavity function can be proven to be non-positive for all possible values of q, then the incremental cost is concave.

Thus, we take the concavity function $Z(q+2) - 2Z(q+1) + Z(q)$ and examine each of its parts. If the parts are all non-negative, then $Z(q+2) - 2Z(q+1) + Z(q)$ must also be non-negative - and, thus, the incremental replenishment cost must be convex.

$$Z(q+2) - 2Z(q+1) + Z(q) = p(\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt - \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt). \tag{A-48}$$

We subtract the two parts of the formula out to get

$$Z(q+2) - 2Z(q+1) + Z(q) = p \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t = q+1\}}{dt} dt. \tag{A-49}$$

As usual, we determine whether each part of the equation is positive or negative. The incremental replenishment cost p is positive. It was proven during the proof for the revenue that $\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t = q+1\}}{dt} dt \geq 0$. Therefore, as all parts of the formula are non-negative,

$$Z(q+2) - 2Z(q+1) + Z(q) = p \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t = q+1\}}{dt} dt \geq 0. \tag{A-50}$$

Since $Z(q+2) - 2Z(q+1) + Z(q) \geq 0$, the incremental replenishment cost is convex with respect to q. ■

Salvage Proofs

Proof: The salvage $S(q)$ increases with increasing q

In order to determine whether or not $S(q)$ increases or decreases as q increases, we need to look at the rate of change of $S(q)$ with respect to q. By examining the parts of $S(q+1)-S(q)$ to determine whether the parts are positive or negative, we may determine whether $S(q+1)-S(q)$ itself is positive or negative - and, thus, determine whether the salvage $S(q)$ increases or decreases.

We start with the formula for the rate of change of the salvage, $S(q+1)-S(q)$:

$$
\begin{aligned}
S(q+1)-S(q) = {} & s\sum_{n=0}^{q}(q+1-n)\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=n|A=1\}dt_f \\
& -s\sum_{n=0}^{q-1}(q-n)\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=n|A=1\}dt_f.
\end{aligned} \tag{A-51}
$$

We combine the two sides of the formula to find

$$
S(q+1)-S(q) = s\sum_{n=0}^{q}\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=n|A=1\}dt_f. \tag{A-52}
$$

As usual, we look at each part of the formula and determine if it is positive or negative. The salvage cost per unit s is positive; the discount factor $e^{-t_f\alpha}$ is also positive for all values of t_f. The rate of increase of the probability of all assemblies failing $\frac{dP\{A_{t_f}=0\}}{dt_f}$ is non-negative, as $P\{A_{t_f}=0\}$ starts at 0 when $t_f=0$, and increases to 1 as t_f increases. Finally, the probability $P\{D_{t_f}=n|A=1\}$ is also non-negative, as it is a probability. As none of the parts of the rate of change of $S(q)$ are negative, the formula is non-negative - and, thus,

$$S(q+1) - S(q) = s \sum_{n=0}^{q} \int_{t_f=0}^{\infty} e^{-t_f \alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = n | A = 1\} dt_f \geq 0. \quad \text{(A-53)}$$

Since $S(q+1) - S(q) \geq 0$, $S(q)$ increases as q increases. ■

Proof: The salvage $S(q)$ is convex with respect to q.

In order to determine whether or not the salvage cost is convex with respect to q, we look at the convexity function $S(q+2) - 2S(q+1) + S(q)$ with respect to q. If $S(q+2) - 2S(q+1) + S(q)$ can be proven to be non-negative for all possible values of q, then the salvage $S(q)$ is convex with respect to q. By examining the parts of $S(q+2) - 2S(q+1) + S(q)$ to determine whether they are positive or negative, the convexity function $S(q+2) - 2S(q+1) + S(q)$ may be determined to be positive or negative - thus demonstrating whether or not the salvage is convex or concave.

Thus, we start with the convexity formula, $S(q+2) - 2S(q+1) + S(q)$:

$$\begin{aligned} S(q+2) - 2S(q+1) + S(q) = s &\sum_{n=0}^{q+1} \int_{t_f=0}^{\infty} e^{-t_f \alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = n | A = 1\} dt_f \\ -s &\sum_{n=0}^{q} \int_{t_f=0}^{\infty} e^{-t_f \alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = n | A = 1\} dt_f. \end{aligned} \quad \text{(A-54)}$$

By combining the two sides of the formula, we get

$$S(q+2) - 2S(q+1) + S(q) = s \int_{t_f=0}^{\infty} e^{-t_f \alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = q+1 | A = 1\} dt_f. \quad \text{(A-55)}$$

Again, we look at each part of the formula and determine if it is positive or negative. The salvage cost per unit s is positive; the discount factor $e^{-t_f \alpha}$ is also positive. The rate of increase of the probability of all assemblies failing $\frac{dP\{A_{t_f}=0\}}{dt_f}$ is non-negative. Finally, the probability $P\{D_{t_f} = q + 1|A = 1\}$ is also non-negative. As none of the parts of the rate of change of $S(q)$ are negative, the formula is non-negative. Therefore,

$$S(q+2) - 2S(q+1) + S(q) = s \int_{t_f=0}^{\infty} e^{-t_f \alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = q+1|A = 1\} dt_f \geq 0. \tag{A-56}$$

As $S(q+2) - 2S(q+1) + S(q) \geq 0$, the rate of change of $S(q)$ is non-decreasing over time - and, thus, the salvage cost $S(q)$ is convex. ■

Proof: If the unit salvage s is less than or equal to the unit revenue r generated from the demand of a part, then $(R(q) + S(q))$ is concave.

In order to understand why, it is necessary to look at the properties of both the salvage and the revenue functions. By summing the revenue and the salvage together, we can then examine each part of the resulting equation, as well as its convexity, by examining each part of the combined convexity function. Should the convexity of $(R(q) + S(q))$ be nonpositive, then the combined function would be concave with respect to q.

The salvage function $S(q)$ for the infinite-timeframe case is

$$S(q) = s\sum_{n=0}^{q-1}(q-n)\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=n|A=1\}dt_f. \qquad \text{(A-57)}$$

The convexity function for the salvage $S(q+2) - 2S(q+1) + S(q)$ is

$$S(q+2) - 2S(q+1) + S(q) = s\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=q+1|A=1\}dt_f. \qquad \text{(A-58)}$$

From (A-5) and (A-58), the convexity of $(R(q)+S(q))$ is

$$\begin{aligned} S(q+2) - 2S(q+1) + S(q) + R(q+2) - 2R(q+1) + R(q) = \\ -r\int_{t=0}^{\infty} e^{-t\alpha}\frac{d(P\{D_t=q+1\})}{dt}dt \\ +s\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f}P\{D_{t_f}=q+1|A=1\}dt_f. \end{aligned} \qquad \text{(A-59)}$$

Our first step is to split the revenue from demand into two parts. Since $r \geq s > 0$ according to the conditions of the proof, we can split the unit revenue r into two parts: the salvage revenue $s > 0$ and a 'profit' revenue $r' \geq 0$. In other words, $r = r' + s$. The combination of both the total revenue and the salvage is therefore

$$S(q+2) - 2S(q+1) + S(q) + R(q+2) - 2R(q+1) + R(q) =$$
$$-r' \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = q+1\})}{dt} dt - s \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = q+1\})}{dt} dt \quad \text{(A-60)}$$
$$+s \int_{t_f=0}^{\infty} e^{-t_f\alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = q+1 | A = 1\} dt_f.$$

The concavity demonstrated by $-r' \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t=q+1\})}{dt} dt$ - and, by extension, the convexity demonstrated its negative, $r' \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t=q+1\})}{dt} dt$ - is identical to a standard revenue function, and has already been determined for all $r' \geq 0$. For all $r' > 0$, $-r' \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t=q+1\})}{dt} dt < 0$, and thus the function $R(q)$ is concave. Therefore, that portion of the formula may be treated as a standard revenue function. We define the remainder of the convexity function - a modified salvage we designate as $CX(S(q))$ - to be

$$CX(S(q)) = -s \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = q+1\})}{dt} dt$$
$$+s \int_{t_f=0}^{\infty} e^{-t_f\alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = q+1 | A = 1\} dt_f. \quad \text{(A-61)}$$

We rearrange this to form

$$CX(S(q)) = s(- \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = q+1\})}{dt} dt$$
$$+ \int_{t_f=0}^{\infty} e^{-t_f\alpha} \frac{dP\{A_{t_f} = 0\}}{dt_f} P\{D_{t_f} = q+1 | A = 1\} dt_f). \quad \text{(A-62)}$$

The next step is to convert the revenue and the salvage portions of the formula into a format by which they can be combined.

In order to do this, it is important to understand what is going on within $\frac{d(P\{D_t=q+1\})}{dt}$. Any $P\{D_t = q+1\}$ may be considered dependent on the time of failure of the last assembly, t_f. For instance, once the last assembly fails, no additional demand will take place; thus, any integration according to time only needs to go from 0 to t_f, rather than from 0 to ∞. Also, if it is determined that the last assembly failed at time t_f, then the distribution of demand that occurs from time 0 to time t_f is dependent on all but one assembly failing before t_f, and the last assembly failing at time t_f.

In other words, if we know at what time t_f the last assembly failure occurs, we must change the probability distribution used from $P\{D_t = q+1\}$ to $P\{D_t = q+1|A_{t_f} = 1\}$. Thus, by making the revenue portion of our convexity function dependent on t_f and integrating over all possible values of t_f, we change the revenue portion of the convexity function to

$$\begin{aligned} &-\int_{t=0}^{\infty} e^{-t\alpha}\frac{d(P\{D_t = q+1\})}{dt}dt \\ &= -\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f} e^{-t\alpha}\frac{d(P\{D_t = q+1|A_{t_f}=1\})}{dt}dt dt_f. \end{aligned} \tag{A-63}$$

We can then substitute this into the convexity function to get

$$\begin{aligned} CX(S(q)) = s(&-\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f} e^{-t\alpha}\frac{d(P\{D_t = q+1|A_{t_f}=1\})}{dt}dt dt_f \\ &+\int_{t_f=0}^{\infty} e^{-t_f\alpha}\frac{dP\{A_{t_f}=0\}}{dt_f})P\{D_{t_f} = q+1|A=1\}dt_f. \end{aligned} \tag{A-64}$$

We then make one more substitution. As the conditional demand probability $P\{D_{t_f} = q+1|A = 1\} = \int_{t=0}^{t_f} \frac{d(P\{D_t=q+1|A_{t_f}=1\})}{dt}dt$, we can substitute this in our formula to get

$$\begin{aligned} CX(S(q)) = s(-\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} \frac{d(P\{D_t = q+1|A_{t_f}=1\})}{dt} dt dt_f \\ + \int_{t=0}^{\infty} e^{-t_f\alpha} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} \frac{d(P\{D_t = q+1|A_{t_f}=1\})}{dt} dt dt_f). \end{aligned} \tag{A-65}$$

This can be combined to form

$$\begin{aligned} S'(q+2) - 2S'(q+1) + S'(q) = s(\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \\ \int_{t=0}^{t_f} (e^{-t_f\alpha} - e^{-t\alpha}) \frac{d(P\{D_t = q+1|A_{t_f}=1\})}{dt} dt dt_f). \end{aligned} \tag{A-66}$$

The proof from this point is similar to the proof used to determine convexity or concavity for the standard revenue function. According to the assumptions of the problem, the unit salvage $s \geq 0$. The probability of all assemblies failing $\frac{dP\{A_{t_f}=0\}}{dt_f}$ increases as t_f increases; therefore, $\frac{dP\{A_{t_f}=0\}}{dt_f} \geq 0$. Therefore, what remains is to determine if $\int_{t=0}^{t_f}(e^{-t_f\alpha} - e^{-t\alpha})\frac{d(P\{D_t=q+1|A_{t_f}=1\})}{dt}dt$ is positive or negative to determine of the modified salvage is convex or concave.

The problem in this case is that $\frac{d(P\{D_t=q+1|A_{t_f}=1\})}{dt}$ may be negative for some values of $t > 0$. Fortunately, while there may be possible values of $t_x > 0$ for which $\frac{d(P_{t_x}[D=q+1|A_{t_f}=1\})}{dt}$ is negative, for any $t_x > 0$ and any integer $n > 0$, then the integral

$\int_{t=0}^{t_x} \frac{d(P\{D_t=n|A_{t_f}=1\})}{dt}dt \geq 0$, as any negative result would result in a negative value for $P_{t_x}[D=n\}$, which is an impossibility.

The proof follows in an almost identical manner to the proof for the standard revenue function. The proof of $\int_{t=0}^{t_f} e^{-t\alpha}\frac{d(P\{D_t=q+1|A_{t_f}=1\})}{dt}dt \geq 0$ was proven in the convexity proof for the standard revenue function; because $e^{-t_f\alpha} \leq e^{-t\alpha}$ for all t such that $0 \leq t \leq t_f$, the integral $\int_{t=0}^{t_f}(e^{-t_f\alpha} - e^{-t\alpha})\frac{d(P\{D_t=q+1|A_{t_f}=1\})}{dt}dt \leq 0$. Thus, it is proven by induction that

$$\int_{t=0}^{t_y}(e^{-t_f\alpha} - e^{-t\alpha})\frac{d(P\{D_t = n|A_{t_f} = 1\})}{dt}dt \leq 0. \tag{A-67}$$

Since any $\int_{t=0}^{t_f}(e^{-t_f\alpha} - e^{-t\alpha})\frac{d(P\{D_t=n|A_{t_f}=1\})}{dt}dt < 0$, the convexity function $CX(S(q))$ is non-positive:

$$\begin{aligned} CX(S(q)) = s(\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \\ \int_{t=0}^{t_f}(e^{-t_f\alpha} - e^{-t\alpha})\frac{d(P\{D_t = q+1|A_{t_f} = 1\})}{dt}dt dt_f) \leq 0. \end{aligned} \tag{A-68}$$

Since we already determined during the proof for the revenue that the integral $-r'\int_{t=0}^{\infty} e^{-t\alpha}\frac{d(P\{D_t=q+1\})}{dt}dt < 0$, the convexity for the sum of both the salvage and the revenue from demand must also be negative:

$$S(q+2) - 2S(q+1) + S(q) + R(q+2) - 2R(q+1) + R(q) =$$

$$-r' \int_{t=0}^{\infty} e^{-t\alpha} \frac{d(P\{D_t = q+1\})}{dt} dt + s(\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f} = 0\}}{dt_f}$$

$$\int_{t=0}^{t_f} (e^{-t_f\alpha} - e^{-t\alpha}) \frac{d(P\{D_t = q+1 | A_{t_f} = 1\})}{dt} dt dt_f) P\{D_{t_f} = q+1 | A = 1\} dt_f < 0. \tag{A-69}$$

In other words,

$$S(q+2) - 2S(q+1) + S(q) + R(q+2) - 2R(q+1) + R(q) < 0. \tag{A-70}$$

Thus, if $s \leq r$, the rate of change of $(R(q) + S(q))$ does not increase as q increases, thus proving concavity of $(R(q) + S(q))$ with respect to q. ■

Proof: If the salvage is sufficiently high, then the total profit will continue to increase as q increases.

The last buy problem with a non-terminating incremental penalty cost will, under normal circumstances, increase for a time, reach an optimal point, then decrease. However, when s is high enough that

$$s \geq \frac{m + h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} dt dt_f}{\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{A_t=0\}}{dt} dt}. \tag{A-71}$$

the profit will continue to increase with increasing q.

To prove the result, we need to look at the rate of change of the profit $\pi(q)$ with respect to q as q gets large.

At our optimal value of q, the rate of change $\pi(q+1)-\pi(q)$ goes from positive to negative. As our total profit function $\pi(q)$ is concave, once the point at which $\pi(q+1)-\pi(q)$ becomes negative is found, we have found the value of q that will maximize our overall profit. Thus, we want to find the smallest value of q such that $\pi(q+1)-\pi(q)<0$.

Which leaves us with a question. Is there a case where the rate of change $\pi(q+1)-\pi(q)$ never reaches 0 - where the profit keeps increasing as q increases? In order to find such a case, we need to see what happens to the rate of change of each of the costs and revenues as q gets large.

Holding Costs

The holding cost with salvage is

$$H(q) = h\sum_{n=0}^{q-1}(q-n)\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=n|A_{t_f}=1\}dtdt_f. \quad \text{(A-72)}$$

This leaves the rate of the change of the holding cost with respect to q as

$$H(q+1)-H(q) = h\sum_{n=0}^{q}\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f}=0\}}{dt_f}\int_{t=0}^{t_f}e^{-t\alpha}P\{D_t=n|A_{t_f}=1\}dtdt_f. \quad \text{(A-73)}$$

As $q \to \infty$, the value of term $\sum_{n=0}^{q} P\{D_t = n | A_{t_f} = 1\}$ approaches 1, as all possible values of n from 0 to ∞ are represented. Thus, the rate of change of our holding cost, at high values of q, becomes

$$\lim_{q\to\infty}(H(q+1) - H(q)) = h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f} = 0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} dt dt_f. \tag{A-74}$$

Penalty

The penalty cost is

$$Z(q) = p \sum_{n=q}^{\infty} \int_{t=0}^{t_c} e^{-t\alpha} \frac{dP\{D_t \le n\}}{dt} dt. \tag{A-75}$$

If no contract exists, then $t_c = \infty$; otherwise, t_c is positive and finite. The value of t_c has no bearing on the proof.

The rate of change of the penalty cost $Z(q+1) - Z(q)$ is

$$Z(q+1) - Z(q) = -p \int_{t=0}^{t_c} e^{-t\alpha} \frac{dP\{D_t \le q\}}{dt} dt. \tag{A-76}$$

As q grows large, $\frac{dP\{D_t \le q\}}{dt}$ goes to 0. The probability $P\{D_0 \le q\} = 1$, and as q grows large, it stays at 1 as t increases because it becomes impossible for the demand D_t to ever exceed q. Thus, since $\frac{dP\{D_t \le q\}}{dt} = 0$ for all t and large values of q,

$$\lim_{q \to \infty} (Z(q+1) - Z(q)) = 0. \tag{A-77}$$

Revenue

The revenue from demand is

$$R(q) = r \sum_{n=0}^{q-1} \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq n\}}{dt} dt. \tag{A-78}$$

The resulting rate of change of the penalty cost $R(q+1) - R(q)$ is

$$R(q+1) - R(q) = r \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-79}$$

As $\frac{dP\{D_t \leq q\}}{dt}$ goes to 0 as q grows large,

$$\lim_{q \to \infty} (R(q+1) - R(q)) = 0. \tag{A-80}$$

Salvage

The salvage as represented in the finite-time case is

$$S(q) = s \sum_{n=0}^{q-1} (q-n) \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{A_t = 0\}}{dt} P\{D_t = n | A_t = 1\} dt. \tag{A-81}$$

The rate of change of the salvage with respect to q is

$$S(q+1) - S(q) = s\sum_{n=0}^{q}\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{A_t = 0\}}{dt}.P\{D_t = n|A_t = 1\}dt. \tag{A-82}$$

As $q \to \infty$, the value of term $\sum_{n=0}^{q} P\{D_t = n|A_t = 1\}$ approaches 1, as all possible values of n from 0 to ∞ are represented. Similarly, the sum $\sum_{n=0}^{q} P\{D_{t_{end}} = n \cap A_{t_{end}} = l\}$ approaches $P\{A_{t_{end}} = l\}$, because all possible values of n from 0 to ∞ are represented. Thus, the rate of change of our holding cost, at high values of q, becomes

$$\lim_{q\to\infty}(S(q+1) - S(q)) = s\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{A_t = 0\}}{dt}dt. \tag{A-83}$$

Manufacturing Cost

The manufacturing cost $M(q)$ is

$$M(q) = mq. \tag{A-84}$$

Thus, the rate of change of $M(q)$ with respect to q is

$$M(q+1) - M(q) = m. \tag{A-85}$$

Total Profit

The total profit $\pi(q)$ is

$$\pi(q) = S(q) + R(q) - Z(q) - M(q) - H(q). \tag{A-86}$$

The rate of change of the total profit $\pi(q)$ with respect to q is

$$\begin{aligned}\pi(q+1) - \pi(q) = (S(q+1) - S(q)) + (R(q+1) - R(q)) - (Z(q+1) - Z(q))\\ -(M(q+1) - M(q)) - (H(q+1) - H(q)).\end{aligned} \tag{A-87}$$

Thus, as $q \to \infty$, this rate of change becomes

$$\begin{aligned}\lim_{q\to\infty}(\pi(q+1) - \pi(q)) = s\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{A_t = 0\}}{dt}dt\\ +0 - 0 - m - h\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f} = 0\}}{dt_f}\int_{t=0}^{t_f} e^{-t\alpha}dtdt_f.\end{aligned} \tag{A-88}$$

We take out any redundant features to get

$$\begin{aligned}\pi(q+1) - \pi(q) = s\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{A_t = 0\}}{dt}dt\\ -m - h\int_{t_f=0}^{\infty}\frac{dP\{A_{t_f} = 0\}}{dt_f}\int_{t=0}^{t_f} e^{-t\alpha}dtdt_f.\end{aligned} \tag{A-89}$$

This rate of change $\pi(q+1)-\pi(q)$ will normally be positive under small values of q, then will start to decrease with higher values of q. However, is there a case where $\pi(q+1)-\pi(q) \geq 0$ as $q \to \infty$ - where the total profit continues to increase?

As we see, this is potentially the case. This will occur if $\lim_{q\to\infty} \pi(q+1) - \pi(q) \geq 0$, or

$$0 \leq s \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{A_t = 0\}}{dt} dt - m - h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f} = 0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} dt dt_f. \qquad \text{(A-90)}$$

We move the holding and manufacturing terms to the left-hand side of the formula to get

$$m + h \int_{t_f=0}^{t_{end}} \frac{dP\{A_{t_f} = 0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} dt dt_f \leq s \int_{t=0}^{t_{end}} e^{-t\alpha} \frac{dP\{A_t = 0\}}{dt} dt. \qquad \text{(A-91)}$$

Since $e^{-t\alpha}$ and $\frac{dP\{A_t=0\}}{dt}$ are non-negative for all values of $t \geq 0$, we can isolate s to find:

$$\frac{m + h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} \int_{t=0}^{t_f} e^{-t\alpha} dt dt_f}{\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{A_t=0\}}{dt} dt} \leq s. \qquad \text{(A-92)}$$

This simplifies to

$$\frac{m + h \int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f} (1 - e^{-t_f\alpha}) dt_f}{\int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{A_t=0\}}{dt} dt} \leq s. \qquad \text{(A-93)}$$

Thus, if we have a case where s is so large that

$$\frac{m + h\int_{t_f=0}^{\infty} \frac{dP\{A_{t_f}=0\}}{dt_f}(1-e^{-t_f\alpha})dt_f}{\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{A_t=0\}}{dt}dt} \leq s.$$

the rate of change of the profit $\pi(q+1) - \pi(q)$ will always be positive, and the profit will always increase. Thus, our value of s should be less than the above formula if we want a finite solution. ■

Discounted Newsvendor Model

The structure of the solution is similar in nature to a newsvendor formula, with the addition of a discount rate to take into account the time between manufacture and demand. The goal of the classic newsvendor model is to find the lowest q such that

$$C_o P\{D_\infty \leq q\} \geq C_u P\{D_\infty > q\}. \tag{A-94}$$

where

- C_o - Overage cost: the cost of ordering one more unit than demanded.
- C_u - Shortage cost: the cost of ordering one unit fewer than demanded.
- $P\{D_\infty > q\}$ - Probability of shortage: the probability that the amount ordered will be less than demanded. The probability of shortage will be high with low

q, then decrease as q increases.

- $P\{D_\infty \leq q\}$ - Probability of overage: the probability that the amount ordered will not be less than demanded. The probability of overage will be low with low q, then increase as q increases.

This formula can be rearranged as

$$C_o(-P\{D_\infty \leq q\}) \leq C_u(P\{D_\infty \leq q\} - 1). \tag{A-95}$$

The shortage-cost side of the formula can be converted to

$$C_o(-P\{D_\infty \leq q\}) \leq C_u \int_{t=0}^{\infty} \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-96}$$

At this point, it's important to take a couple of things into account.

- $\frac{dP\{D_t \leq q\}}{dt} = -\frac{dP\{D_t > q\}}{dt}$. This is normal because, for any $t > 0$ and positive integer q, $P\{D_t \leq q\} + P\{D_t > q\} = 1$. As a result, any derivative $\frac{dP\{D_t \leq q\}}{dt} + \frac{dP\{D_t > q\}}{dt} = 0$.

- $\frac{dP\{D_t \leq q\}}{dt} \leq 0$. Because there cannot be "negative demand" in our model, it is impossible to go from $D \geq q$ to any $D < q$ as time goes forward. As a result, $P\{D_t \leq q\}$ can never increase, and the rate of change $\frac{dP\{D_t \leq q\}}{dt}$ can never be positive.

The result of this is that $\frac{dP\{D_t \leq q\}}{dt}$ represents the negative of the incremental rate of increase in the probability that the demand has exceeded the newsvendor amount ordered.

Suppose that, instead of C_u being a constant, at least part of C_u varied with time - for instance, suppose that some of the components of C_u were discounted by a factor of α. Then, instead of having C_u, we have $C_u(t)$. In this case, we would have

$$C_o(-P\{D_\infty \leq q\}) \leq \int_{t=0}^{\infty} C_u(t) \frac{dP\{D_t \leq q\}}{dt} dt. \qquad \text{(A-97)}$$

At this point, some rearrangement of the overage cost portion of the formula (A-97) is needed to match the shortage cost portion.

$$C_o(-1 - P\{D_\infty \leq q\} + 1) \leq \int_{t=0}^{\infty} C_u(t) \frac{dP\{D_t \leq q\}}{dt} dt. \qquad \text{(A-98)}$$

$$C_o(-1 - \int_{t=0}^{\infty} \frac{dP\{D_t \leq q\}}{dt} dt) \leq \int_{t=0}^{\infty} C_u(t) \frac{dP\{D_t \leq q\}}{dt} dt. \qquad \text{(A-99)}$$

By moving everything to one side, we get

$$0 \leq C_o + C_o \int_{t=0}^{\infty} \frac{dP\{D_t \leq q\}}{dt} dt + \int_{t=0}^{\infty} C_u(t) \frac{dP\{D_t \leq q\}}{dt} dt. \qquad \text{(A-100)}$$

By combining the integrals, we get

$$0 \leq C_o + \int_{t=0}^{\infty} (C_o + C_u(t)) \frac{dP\{D_t \leq q\}}{dt} dt. \qquad \text{(A-101)}$$

Here, we must make one more assumption: that the term $(C_o + C_u(t))$ can be separated into $(C_o + C_u)f(t)$ - that is, a constant $(C_o + C_u)$ multiplied by a function $f(t)$. In a discount-rate problem, this is not as restrictive as it may appear; it simply means that all surviving terms in $(C_o + C_u)$ should be discounted. The resulting formula becomes

$$0 \leq C_o + (C_o + C_u) \int_{t=0}^{\infty} f(t) \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-102}$$

By moving the overage and shortage costs to one side, we get

$$\frac{-C_o}{C_o + C_u} \leq \int_{t=0}^{\infty} f(t) \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-103}$$

By adding 1 to both sides, we come up with a discounted version of the newsvendor problem:

$$\frac{C_u}{C_o + C_u} \leq 1 + \int_{t=0}^{\infty} f(t) \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-104}$$

Thus, our goal is to find the smallest value of q for which the inequality is true. For the discounted newsvendor, $f(t) = e^{-t\alpha}$, where α is the discount rate. Thus, the discounted newsvendor formula is

$$\frac{C_u}{C_o + C_u} \leq 1 + \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-105}$$

For our model, the overage and shortage costs are as follows:

- *Overage cost*: The cost of manufacturing one unit more than demanded includes manufacturing one more part than needed (at a cost of m) and holding that part in inventory long-term (in this case, incurring a perpetuity at cost h per unit time, thus incurring a total discounted cost of $\frac{h}{\alpha}$). Thus, the total overage cost is $m + \frac{h}{\alpha}$.
- *Shortage cost*: The cost of manufacturing one unit less than demanded includes the penalty cost p incurred and the loss of revenue r, but is reduced by the initial cost of manufacture m. Thus, the total shortage cost is $p + r - m$

From this, our formulation becomes

$$\frac{C_u}{C_o + C_u} = \frac{p + r - m}{(p + r - m) + (m + \frac{h}{\alpha})} = \frac{p + r - m}{p + r + \frac{h}{\alpha}}. \tag{A-106}$$

Thus, the general formulation of the last buy formula is

$$\frac{p + r - m}{p + r + \frac{h}{\alpha}} \leq 1 + \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt. \tag{A-107}$$

This is demonstrated when we consider the last buy problem without contract. The rate of change of the total profit $\pi(q)$ with respect to the order amount q is

$$\pi(q+1)-\pi(q) = p\int_{t=0}^{\infty}\sum_{n=q+1}^{\infty} e^{-t\alpha}\frac{dP\{D_t=n\}}{dt}dt + r\int_{t=0}^{\infty}\sum_{n=0}^{q} e^{-t\alpha}\frac{-dP\{D_t=n\}}{dt}dt$$
$$-m-h\int_{t=0}^{\infty}\sum_{n=0}^{q} e^{-t\alpha}P\{D_t=n\}dt. \tag{A-108}$$

Because $\sum_{n=0}^{q}\frac{dP\{D_t=n\}}{dt}+\sum_{n=q+1}^{\infty}\frac{dP\{D_t=n\}}{dt}=0$, equation (A-108) simplifies to

$$\pi(q+1)-\pi(q) = -m-h\int_{t=0}^{\infty} e^{-t\alpha}P\{D_t\le q\}dt-(p+r)\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t\le q\}}{dt}dt. \tag{A-109}$$

Since $\int_{t=0}^{\infty}e^{-t\alpha}\frac{dP\{D_t\le q\}}{dt}dt=\alpha\int_{t=0}^{\infty}e^{-t\alpha}P\{D_t\le q\}dt-1$, we can simplify the formula further to

$$\pi(q+1)-\pi(q) = -m-\frac{h}{\alpha}-(p+r+\frac{h}{\alpha})\int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t\le q\}}{dt}dt. \tag{A-110}$$

What we seek is the lowest integer value of q at which $\pi(q+1)-\pi(q)<0$. At this point, the formula will have ceased increasing with increasing q, and will only decrease with increasing q. Thus, we move the terms to one side to find

$$\frac{-m-\frac{h}{\alpha}}{p+r+\frac{h}{\alpha}} \le \int_{t=0}^{\infty} e^{-t\alpha}\frac{dP\{D_t\le q\}}{dt}dt. \tag{A-111}$$

We add 1 to both sides to get the newsvendor formulation for the optimal order amount, as shown in equation (A-107):

$$\frac{p+r-m}{p+r+\frac{h}{\alpha}} \leq 1 + \int_{t=0}^{\infty} e^{-t\alpha} \frac{dP\{D_t \leq q\}}{dt} dt.$$

APPENDIX B
PROOFS FOR THE LAST BUY PROBLEM WITH NO REPLENISHMENT

Penalty Proofs

Proof: The expected value of the penalty cost $Z(q)$ is non-negative.

The equation for the expected value can be found from (5.6):

$$Z(q) = -p \sum_{l=1}^{j} \int_{t_f=0}^{t_c} e^{-t_f \alpha} \frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\} K(l, t_f) dt_f.$$

By looking at the parts of (5.6) and determining if each part is non-negative or non-positive, we can determine if (5.6) in its entirety is non-positive or non-negative.

- $e^{-t_f \alpha}$: This is an exponential; thus, it is positive for all values of t.
- $\frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f}$: The probability $P\{D_{t_f} \leq q | A_{t_f} = l\}$ decreases as t increases, as we do not allow for negative demand. Thus, once the demand up to a specific time $D_{\hat{t_f}} > q$, the demand can never be less than or equal to q at any point in time beyond $\hat{t_f}$. Thus, because the probability $P\{D_{t_f} \leq q | A_{t_f} = l\}$ can only decrease as t increases, the rate of change $\frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f}$ is non-positive for all values of t.
- p: The penalty per assembly p is non-negative, according to the requirements of the problem.
- $K(l, t_f)$: The expected discounted number of assemblies for which the manufac-

turer must pay a penalty should the manufacturer be unable to meet demand must be non-negative.

- $P\{A_{t_f} = l\}$: By definition, all probabilities must be between 0 and 1. Thus, $P\{A_{t_f} = l\}$ is non-negative.
- $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\}dt$: Because $e^{-t\alpha} > 0$ and $P\{D_t \leq q\} \geq 0$ for all values of t, then the integral $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\}dt$ must also be non-negative for all values of t.
- $\int_{t_f=0}^{t_c} e^{-t_f\alpha} \frac{dP\{D_{t_f} \leq q|A_{t_f}=l\}}{dt_f} P\{A_{t_f} = l\}K(l,t_f)dt_f$: All parts of this integral are non-negative for all values of t except for $\frac{dP\{D_{t_f} \leq q|A_{t_f}=l\}}{dt_f}$, which is non-positive for all values of t. As a result, the entire integral must be non-positive.

Because $\int_{t_f=0}^{t_c} e^{-t_f\alpha} \frac{dP\{D_{t_f} \leq q|A_{t_f}=l\}}{dt_f} P\{A_{t_f} = l\}K(l,t_f)dt_f$ is non-positive and p is non-negative, the negative of the product of these two parts must be non-negative. Thus,

$$Z(q) = -p \sum_{l=1}^{j} \int_{t_f=0}^{t_c} e^{-t_f\alpha} \frac{dP\{D_{t_f} \leq q|A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\}K(l,t_f)dt_f \geq 0. \quad \text{(B-1)}$$

Thus, $Z(q)$ is non-negative. ■

Proof: The penalty cost $Z(q)$ is non-increasing with increasing q.

Due to the nature of the penalty cost - where the assembly failure rate changes once penalties begin to occur - a slightly different tactic is used to determine if the penalty cost is non-increasing. We show the effect of increasing the order amount from

q units to $q+1$ units on the penalty cost. Because the contract and non-contract cases have different penalties, each of these cases must be investigated separately. Thus, we need to demonstrate the effect of increasing the order amount from q to $q+1$ for a given scenario.

Suppose that, for a given order amount q, the $q+1$th unit is demanded at time t_f. In such a case, the penalty becomes

$$Z_{t_f}(q) = pe^{-t_f\alpha} \sum_{l=1}^{j} P\{A_{t_f} = l\} K(l, t_f). \tag{B-2}$$

The formula $K(l, t_f)$ represents the expected standardized (i.e. p=1) penalty, discounted to time t_f, given that the $(q+1)$th unit failed at time t_f. As the structure of $K(l, t_f)$ is different for the contract case and the non-contract case, we will need to determine whether or not $Z_{t_f}(q)$ increases for both cases.

Non-Contract Case

With the value of $K(l, t_f)$ for the non-contract case, equation (B-2) becomes:

$$\begin{aligned} Z_{t_f}(q) = pe^{-t_f\alpha} \sum_{l=1}^{j} P\{A_{t_f} = l\} \\ (1 - \sum_{k=1}^{l-1} \int_{t_f}^{\infty} e^{(t_f - t)\alpha} P_k\{A_t = 1 | A_{t_f} = 1\} \frac{dP_k\{D_t = n | D_{t_f} = n\}}{dt} dt). \end{aligned} \tag{B-3}$$

Consider what would happen if the manufacturer had ordered $q+1$ units instead of q units. The demand of the $(q+2)$nd, $(q+3)$rd, etc. part all occur at the same time and from the same assembly; for these parts of the formula, nothing changes. What does change is that the $(q+1)$th demand is satisfied by the last buy order. Thus, the assembly that demanded the $(q+1)$th part does not generate a penalty cost at t_f, but could possibly generate a penalty cost in the future, in the same way that any other assemblies remaining may generate a penalty cost. As a result, the value for $Z_{t_f}(q+1)$ is

$$Z_{t_f}(q+1) = pe^{-t_f\alpha}\sum_{l=1}^{j} P\{A_{t_f} = l\}$$
$$(-\sum_{k=1}^{l}\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_k\{A_t = 1|A_{t_f} = 1\}\frac{dP_k\{D_t = n|D_{t_f} = n\}}{dt}dt). \tag{B-4}$$

Thus, by taking the formulas of (B-3) and (B-4), the rate of change $Z_{t_f}(q+1) - Z_{t_f}(q)$ becomes

$$Z_{t_f}(q+1) - Z_{t_f}(q) = pe^{-t_f\alpha}\sum_{l=1}^{j} P\{A_{t_f} = l\}$$
$$(-\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_l\{A_t = 1|A_{t_f} = 1\}\frac{dP_l\{D_t = n|D_{t_f} = n\}}{dt}dt - 1). \tag{B-5}$$

If the rate of change $Z_{t_f}(q+1) - Z_{t_f}(q)$ is non-positive for all values of q, then the penalty cost is non-increasing with increasing q. In order to find this, we look at the parts of $Z_{t_f}(q+1) - Z_{t_f}(q)$.

- p: The unit penalty cost per assembly p is, by the conditions of the problem, non-negative.

- $e^{-t_f\alpha}$, $e^{(t_f-t)\alpha}$: These are exponentials; as such, these are positive for all values of t. Moreover, $e^{(t_f-t)\alpha} \leq 1$ for all values of $t \geq t_f$.
- $P\{A_{t_f} = l\}$, $P_l\{A_t = 1|A_{t_f} = 1\}$: These are probabilities. By definition, all probabilities are between 0 and 1. As such, these are non-negative.
- $\frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}$: This is non-positive for all values of t. The reason is that the probability $P_l\{D_t = n|D_{t_f} = n\}$ is equal to 1 at time t_f, then steadily decreases as t increases beyond t_f.
- $-\int_{t_f}^{\infty} \frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt$: This is positive, but less than 1. The reason for this is that $-\int_{t_f}^{\infty} \frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt \leq 1$, because $P_l\{D_t = n|D_{t_f} = n\}$ is a probability that decreases from 1 and, therefore, any integral of its rate of change over time must be between 0 and -1. Thus, its integral must be between 0 and 1.
- $-\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_l\{A_t = 1|A_{t_f} = 1\}\frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt$: This is also non-negative and between 0 and 1. The reason for this is that, for all $t \geq t_f$, $0 \leq P_l\{A_t = 1|A_{t_f} = 1\} \leq 1$ and $0 \leq e^{(t_f-t)\alpha} \leq 1$. Thus, because $P_l\{A_t = 1|A_{t_f} = 1\}$ and $e^{(t_f-t)\alpha}$ are both positive and less than 1 and because $\frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}$ is negative for all $t \geq t_f$, the integral $-\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_l\{A_t = 1|A_{t_f} = 1\}\frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt$ must be of less absolute value than the original, as well as be of the same sign. Thus, as $-\int_{t_f}^{\infty} \frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt \leq 1$, so must $-\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_l\{A_t = 1|A_{t_f} = 1\}\frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt \leq 1$

As a result, the $(-\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_l\{A_t = 1|A_{t_f} = 1\}\frac{dP_l\{D_t=n|D_{t_f}=n\}}{dt}dt - 1)$ portion of the rate of change must be non-positive. Thus, because p, $e^{-t_f\alpha}$, and $P\{A_{t_f} = l\}$ are all non-negative, the product of all of these parts - the rate of change - must be non-positive:

$$Z_{t_f}(q+1) - Z_{t_f}(q) = pe^{-t_f\alpha} \sum_{l=1}^{j} P\{A_{t_f} = l\}$$
$$(-\int_{t_f}^{\infty} e^{(t_f-t)\alpha} P_l\{A_t = 1|A_{t_f} = 1\} \frac{dP_l\{D_t = n|D_{t_f} = n\}}{dt} dt - 1) \leq 0. \tag{B-6}$$

As a result, for the non-contract case, the penalty cost $Z(q)$ is non-increasing with increasing q.

Contract Case

In the contract case, the result is a bit different. We install the value of $K(l, t_f)$ to get

$$Z_{t_f}(q) = pe^{-t_f\alpha} \sum_{l=1}^{j} P\{A_{t_f} = l\}l. \tag{B-7}$$

In this case, we need to find the result if our order amount goes from q to $q+1$. What we get is

$$Z_{t_f}(q+1) = pe^{-t_f\alpha} \sum_{l=1}^{j} P\{A_{t_f} = l\}(-\int_{t_f}^{\infty} e^{t_f-t} \frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt}$$
$$\sum_{i=1}^{l} iP\{A_t = i|A_{t_f} = l\}dt). \tag{B-8}$$

We can separate this out to

$$Z_{t_f}(q+1) = pe^{-t_f\alpha}\sum_{l=1}^{j} P\{A_{t_f} = l\}(\sum_{i=1}^{l} i \\ (-\int_{t_f}^{\infty} e^{t_f - t}\frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt} P\{A_t = i|A_{t_f} = l\}dt)). \tag{B-9}$$

As demonstrated earlier,

$$0 \leq -\int_{t_f}^{\infty} e^{t_f - t}\frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt} \leq 1. \tag{B-10}$$

Also, we establish that, for any time $t > t_f$,

$$\sum_{i=1}^{l} iP\{A_t = i|A_{t_f} = l\} \leq l. \tag{B-11}$$

as l is the maximum possible value of the number of assemblies after time t_f, and $\sum_{i=1}^{l} iP\{A_t = i|A_{t_f} = l\}$ represents the expected value of the number of assemblies for a time $t > t_f$. Thus, by taking equations (B-10) and (B-11), we find that

$$0 \leq (-\int_{t_f}^{\infty} e^{t_f - t}\frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt}\sum_{i=1}^{l} iP\{A_t = i|A_{t_f} = l\}dt)) \\ \leq -l\int_{t_f}^{\infty} e^{t_f - t}\frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt} \leq l. \tag{B-12}$$

Thus, we establish that, from the results of (B-12),

$$l \geq (\sum_{i=1}^{l} i(-\int_{t_f}^{\infty} e^{t_f - t}\frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt}P\{A_t = i|A_{t_f} = l\}dt)). \quad \text{(B-13)}$$

We put this into our penalty formulas of (B-7) and (B-9) and find that

$$pe^{-t_f\alpha}\sum_{l=1}^{j} P\{A_{t_f} = l\}l \geq pe^{-t_f\alpha}\sum_{l=1}^{j} P\{A_{t_f} = l\}(\sum_{i=1}^{l} i$$
$$(-\int_{t_f}^{\infty} e^{t_f - t}\frac{dP\{D_t = q+1|D_{t_f} = q+1\}}{dt}P\{A_t = i|A_{t_f} = l\}dt)). \quad \text{(B-14)}$$

Or, to summarize,

$$Z_{t_f}(q) \geq Z_{t_f}(q+1). \quad \text{(B-15)}$$

Thus, since we have proven the lemma for both the non-contract and contract case, the penalty cost is non-increasing with increasing q. ■

Proof: The convexity of the penalty cost $Z(q)$ is with respect to q cannot be determined.

We demonstrate this by giving an example: a simplified version of the contract-case penalty in which there is no discount rate ($\alpha = 0$).

$$Z(q) = -p\sum_{l=1}^{j} l \int_{t_f=0}^{t_c} \frac{dP\{D_{t_f} \leq q|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}dt_f. \quad \text{(B-16)}$$

This can be edited to

$$Z(q) = -p \sum_{l=1}^{j} l \int_{t_f=0}^{t_c} \frac{dP\{D_{t_f} \leq q \cap A_{t_f} = l\}}{dt_f} dt_f. \tag{B-17}$$

We solve for the integral to get

$$Z(q) = -p \sum_{l=1}^{j} l(P\{D_{t_c} \leq q \cap A_{t_c} = l\} - P\{D_0 \leq q \cap A_0 = l\}). \tag{B-18}$$

Since $P\{D_0 \leq q \cap A_0 = j\} = 1$ and $P\{D_0 \leq q \cap A_0 = l\} = 0$ for all other values of l, equation (B-18) simplifies to

$$Z(q) = pj - \sum_{l=1}^{j} lP\{D_{t_c} \leq q \cap A_{t_c} = l\}. \tag{B-19}$$

The rate of change of this statement with respect to q is

$$Z(q+1) - Z(q) = -\sum_{l=1}^{j} lP\{D_{t_c} = q+1 \cap A_{t_c} = l\}. \tag{B-20}$$

When we seek the convexity, we find

$$Z(q+2) - 2Z(q+1) + Z(q) = \sum_{l=1}^{j} l(P\{D_{t_c} = q+1 \cap A_{t_c} = l\} - P\{D_{t_c} = q+2 \cap A_{t_c} = l\}). \tag{B-21}$$

This illustrates the problem with convexity involved with this formula. In some cases, ($P\{D_{t_c} = q+1 \cap A_{t_c} = l\}$ is going to be greater than $P\{D_{t_c} = q+2 \cap A_{t_c} = l\}$); in other cases, less. Thus, we cannot effectively determine convexity for the penalty cost. ■

Proof: For any $q \geq 0$, the negative of the rate of change of the penalty $-(Z(q+1) - Z(q))$ is less than the penalty $Z(q)$.

This can be determined from two properties.

- The penalty cost is non-negative.
- The penalty cost is decreasing.

The decreasing nature of the penalty cost was demonstrated earlier in Appendix B. That the penalty cost is non-negative can be demonstrated as follows:

The expected discounted penalty cost is

$$Z(q) = -p \sum_{l=1}^{j} \int_{t_f=0}^{t_c} e^{-t_f \alpha} \frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\} K(l, t_f) dt_f. \qquad \text{(B-22)}$$

By definition, p and l are non-negative. Also by definition, $K(l, t_f) \geq 0$ for all possible values of l and t_f. As $P\{A_{t_f} = l\}$ is a probability, it is also non-negative. Also, the exponential $e^{-t_f \alpha} > 0$.

The structure of $\frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f}$ is such that it will be negative for all values of $t_f > 0$. Since we cannot have negative demand, the probability that demand is less than or equal to a given amount steadily decreases over time. Thus, $\frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f}$

is non-positive.

The result of these properties is that

$$0 \leq -\int_{t_f=0}^{t_c} pe^{-t_f\alpha}\frac{dP\{D_{t_f} \leq q|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l, t_f)dt_f. \tag{B-23}$$

Thus,

$$Z(q) = -p\sum_{l=1}^{j}\int_{t_f=0}^{t_c} e^{-t_f\alpha}\frac{dP\{D_{t_f} \leq q|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l, t_f)dt_f \geq 0. \tag{B-24}$$

and the penalty cost $Z(q)$ is non-negative for all values of $q \geq 0$.

Because $Z(q+1)$ is non-negative, $Z(q) - Z(q+1) \leq Z(q)$. Or,

$$-(Z(q+1) - Z(q)) \leq Z(q). \tag{B-25}$$

Thus, the negative of the rate of change of $Z(q)$ is always less than $Z(q)$. ∎

Proof: As q becomes large ($q \to \infty$), the penalty $Z(q) \to 0$

The formula for the penalty cost $Z(q)$ is

$$Z(q) = -p\sum_{l=1}^{j}\int_{t_f=0}^{t_c} e^{-t_f\alpha}\frac{dP\{D_{t_f} \leq q|A_{t_f} = l\}}{dt_f}P\{A_{t_f} = l\}K(l, t_f)dt_f. \tag{B-26}$$

At very large values of q, the probability $P\{D_{t_f} \leq q | A_{t_f} = l\}$ does not change except for values of $A_{t_f} > 0$ and for very large values of t_f. In other words, to get large values of q such that $D_{t_f} > q$ (and thus cause a change in $P\{D_{t_f} \leq q | A_{t_f} = l\}$), assemblies have to be operating for a very long time. Thus, if the assemblies are operating at large values of t_f, it is possible to have a nonzero value for $\frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f}$. However, as t_f increases, $P\{A_{t_f} = l\} \to 0$ for all $l > 0$. Eventually, all assemblies will fail. q can be made sufficiently large that the value of t_f at which $P\{D_{t_f} \leq q | A_{t_f} = l\}$ changes is so high that, for all $l > 0$, $P\{A_{t_f} = l\} \sim 0$. Thus, for all values of l and t_f,

$$\frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\} \sim 0. \tag{B-27}$$

Because of (B-27), the value of the integral in the penalty cost function $e^{-t_f \alpha} \frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\} K(l, t_f) dt_f$ approaches 0 for all values of t_f. Thus,

$$\lim_{q \to \infty} -p \sum_{l=1}^{j} \int_{t_f=0}^{t_c} e^{-t_f \alpha} \frac{dP\{D_{t_f} \leq q | A_{t_f} = l\}}{dt_f} P\{A_{t_f} = l\} K(l, t_f) dt_f = 0. \tag{B-28}$$

■

APPENDIX C
PROOFS AND ANALYSES FOR THE LAST BUY PROBLEM WITH BATCH REPLENISHMENT

Renewal Theory Formulation

A value for a state i in a Renewal Theory formulation, discounted over time, can be viewed as follows:

$$C(i) = \int_{t=0}^{\infty} e^{-t\alpha}(\iota_i(t)P\{S_t = i\} + \sum_{\forall j|i\to j} \frac{dP\{i \to j\}}{dt}(\nu(i,j) + C(j)))dt. \qquad \text{(C-1)}$$

In this formulation, $\iota_i(t)$ indicates the rate at which costs or revenues are generated while in state i, $P\{S_t = i\}$ represents the probability that the system is still in state i given that it started in state i at time 0, $\frac{dP\{i\to j\}}{dt}$ represents the rate of increase in the probability that the system has transitioned from state i to state j by time t, $\nu(i,j)$ represents the cost or revenue generated from the transition from state i to state j, and $C(j)$ represents the value of state j.

Based on this, we need to come up with values for the various states and transitions for each part of the Last Buy problem. We view the possible states for the Last Buy problem as (l, n), where l represents the number of assemblies still in operation and n represents the number of parts in inventory.

Formulation of the Revenue

For the revenue, we need to consider what happens at each possible value of (l, n). For starters, no revenue is generated while the system stays in a given state; thus, $\iota_{(l,n)}(t) = 0$ for all (l, n). At any given (l, n) where both l and n are positive, the system can transition from (l, n) to one of two possible states. If an assembly fails in state (l, n), then the system transitions to state $(l - 1, n)$. If a part fails in state (l, n), then a part is taken out of inventory to replace it, generating a revenue of r units and transitioning to state $(l, n - 1)$. Thus, if both l and n are positive, our revenue formulation under renewal theory becomes

$$\begin{aligned} R(l, n) = \int_{t=0}^{\infty} e^{-t\alpha}(&P\{D_t = 0\}P\{A_t < l\}R(l - 1, n) \\ &+P\{D_t > 0\}P\{A_t = l\}(r + R(l, n - 1))dt. \end{aligned} \tag{C-2}$$

The case where $l = 0$ is a trapping state, as this implies that all assemblies have failed; thus, for all inventory levels n, $R(0, n) = 0$. If, however, the inventory level is depleted and $n = 0$, then another batch will immediately be made. Thus, the value of any $R(l, 0) = R(l, q_l^*)$, where q_l^* is the optimal order amount for a batch if l assemblies remain.

When there are l assemblies remaining, this implies that there are l potential parts and l potential assemblies that could fail. Thus, the values of each of the probabilities in $R(l, n)$ are:

$$P\{D_t = 0\} = e^{-tl\lambda}. \tag{C-3}$$

$$P\{A_t = l\} = e^{-tl\beta}. \tag{C-4}$$

$$P\{D_t > 0\} = 1 - P\{D_t = 0\} = 1 - e^{-tl\lambda}. \tag{C-5}$$

$$P\{A_t < l\} = 1 - P\{A_t = l\} = 1 - e^{-tl\beta}. \tag{C-6}$$

By applying equations (C-3) through (C-6) into equation (C-2), we get

$$R(l,n) = \int_{t=0}^{\infty} e^{-t\alpha}(e^{-tl\lambda}(1-e^{-tl\beta})R(l-1,n)+(1-e^{-tl\lambda})e^{-tl\beta}(r+R(l,n-1))dt. \tag{C-7}$$

By solving for the integral in (C-7), we get:

$$R(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\beta}{\lambda+\beta}R(l-1,n) + \frac{\lambda}{\lambda+\beta}(r+R(l,n-1)). \tag{C-8}$$

Thus, we summarize the revenue formulation as follows:

If $l > 0$ and $n > 0$, then

$$R(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\beta}{\lambda+\beta}R(l-1,n) + \frac{\lambda}{\lambda+\beta}(r+R(l,n-1)). \tag{C-9}$$

If $l = 0$, then

$$R(0, n) = 0. \tag{C-10}$$

If $n = 0$, then

$$R(l, 0) = R(l, q_l^*). \tag{C-11}$$

Formulation of the Holding Cost

The holding cost is unique in that it is the only one of the costs that generates a revenue or cost for remaining in a given state (l, n). While there are n parts remaining in inventory, each part generates a holding cost of h units for each unit of time it remains in inventory. Thus, by adapting (C-1) to the holding cost, we must make $\iota_{(l,n)}(t) = hn$. Other than this, there is no cost or revenue for any change of state. Other than this, the rules and probabilities of the formulation are the same as with the revenue:

$$\begin{aligned} H(l, n) = \int_{t=0}^{\infty} e^{-t\alpha}(&P\{D_t = 0\}P\{A_t = l\}hn \\ +P\{D_t = 0\}P\{A_t < l\}H(l-1, n) + P\{D_t > 0\}&P\{A_t = l\}H(l, n-1))dt. \end{aligned} \tag{C-12}$$

By applying the values for the probabilities as presented in (C-3) through (C-6) into the holding cost formula of (C-12) and integrating, we get

$$H(l,n) = \frac{nh}{l(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\beta}{\lambda+\beta}H(l-1,n) + \frac{\lambda}{\lambda+\beta}(H(l,n-1)). \quad \text{(C-13)}$$

The cases at the extremes - when $l = 0$ or $n = 0$ - are the same as with the revenue. Therefore, a summary of the values of the holding cost at each state are:

If $l > 0$ and $n > 0$, then

$$H(l,n) = \frac{nh}{l(\beta+\lambda)+\alpha} + \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\beta}{\lambda+\beta}H(l-1,n) + \frac{\lambda}{\lambda+\beta}(H(l,n-1))dt. \quad \text{(C-14)}$$

If $l = 0$, then

$$H(0,n) = 0. \quad \text{(C-15)}$$

If $n = 0$, then

$$H(l,0) = H(l,q_l^*). \quad \text{(C-16)}$$

Formulation of the Batch Replenishment Cost

For the batch replenishment costs, no costs occur until the inventory is depleted. Therefore, the cost generated while in any state (l, n) where $n > 0$ is 0, as is any cost of transition from (l, n) to any other state. Thus, for any state where both l and n are positive,

$$\begin{aligned} Z(l,n) = \int_{t=0}^{\infty} e^{-t\alpha}(&P\{D_t = 0\}P\{A_t < l\}Z(l-1,n) \\ &+P\{D_t > 0\}P\{A_t = l\}Z(l,n-1))dt. \end{aligned} \tag{C-17}$$

By applying the values for the probabilities as presented in (C-3) through (C-6) into the batch replenishment formula of (C-17) and integrating, we get

$$Z(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}\left(\frac{\beta}{\lambda+\beta}Z(l-1,n) + \frac{\lambda}{\lambda+\beta}Z(l,n-1)\right). \tag{C-18}$$

The states where $l = 0$ are trapping states; at this point, no assemblies remain, so no further costs will be generated. On the other hand, when the inventory $n = 0$, a new batch will be immediately manufactured - and a batch setup cost of p units generated. Thus, the batch replenishment cost, as represented within renewal theory, becomes

Thus, if $l = 0$, then

$$Z(0,n) = 0. \tag{C-19}$$

If $n = 0$, then

$$Z(l,0) = p_{batch} + M(q_l^*) + Z(l, q_l^*). \tag{C-20}$$

Formulation of the Salvage

The renewal theory formulation for the salvage is very similar to the formulation for the penalty. In both cases, no costs are generated in the intermediate states where both l and n are positive - neither while remaining at state (l, n) nor in the transition to another state. Thus, for positive l and n,

$$\begin{aligned} S(l,n) = \int_{t=0}^{\infty} e^{-t\alpha}(&P\{D_t = 0\}P\{A_t < l\}S(l-1,n) \\ &+P\{D_t > 0\}P\{A_t = l\}S(l,n-1))dt. \end{aligned} \tag{C-21}$$

By applying the values for the probabilities as presented in (C-3) through (C-6) into the salvage formula of (C-21) and integrating, we get

$$S(l,n) = \frac{l(\beta+\lambda)}{l(\beta+\lambda)+\alpha}(\frac{\beta}{\lambda+\beta}S(l-1,n) + \frac{\lambda}{\lambda+\beta}S(l,n-1)). \tag{C-22}$$

Where the salvage differs from the penalty is that, instead of generating cost or revenue when the inventory is depleted, the salvage occurs when the last assembly has failed. At that point, the remaining n parts are sold out of inventory, generating a revenue of s per part. Thus, the value of the salvage at state $(0, n)$ is

$$S(0, n) = sn. \tag{C-23}$$

If the system runs out of parts, a new batch of parts is generated. Thus, once the system reaches the state $(l, 0)$, another batch is made, and the state of the system goes to (l, q_l^*), where q_l^* is the optimal order amount for a batch if l assemblies remain.

In summary, the value of the salvage at a given state (l, n) is:

If $l = 0$, then

$$S(0, n) = sn. \tag{C-24}$$

If $n = 0$, then

$$S(l, 0) = S(l, q_l^*). \tag{C-25}$$

Proof: If using incremental replenishment as a form of buyout in the last buy problem with batch replenishment and buyout option, the value of the total discounted incremental replenishment per assembly is $\frac{p_i \lambda}{\beta + \alpha}$, where p_i represents the cost per part to fabricate the parts individually at higher cost.

To demonstrate this, we use a simplified form of $C(l, n)$ formula as presented in (6.20). By using equation (6.20) to show what would happen to an incremental replenishment case for a single assembly, we can show why the value of the incremental replenishment per assembly is $\frac{p_i\lambda}{\beta+\alpha}$. The $\frac{p_i\lambda}{\beta+\alpha}$ is the result of a power series generated from $\frac{\lambda}{\lambda+\beta+\alpha}$. We start with (6.20) by making several alterations necessary to apply this formula to our example:

- $l = 1$. We are looking at each assembly individually. As there is no longer any inventory, as each part will be made to demand from that point, there is no need to look at all of the assemblies as a group.
- $h = 0$. No items will be in inventory from this point; thus, the holding cost can be removed from (6.20) for our example.
- $r = -p_i$. Whenever a part is sold, instead of generating revenue, a cost of $-p_i$ units is generated.
- $C(0, n) = 0$. Because no items are in inventory, no salvage will occur.
- $n \to \infty$. We can make parts on demand; thus, any demand will be met. We represent this by making n very large.

These changes cause our modified version of (6.20) to become

$$C(1, n) = \frac{(\beta + \lambda)}{(\beta + \lambda) + \alpha}(\frac{\lambda}{\beta + \lambda}(-p_i + C(1, n - 1))). \tag{C-26}$$

Equation (C-26) can be simplified to

$$C(1,n) = \frac{\lambda}{\beta+\lambda+\alpha}(-p_i + C(1,n-1)). \tag{C-27}$$

Or, to distribute the $\frac{\lambda}{\beta+\lambda+\alpha}$ term throughout (C-27):

$$C(1,n) = \frac{-p_i\lambda}{\beta+\lambda+\alpha} + \frac{\lambda C(1,n-1)}{\beta+\lambda+\alpha}. \tag{C-28}$$

From (C-28), we get the distribution for $C(1,n-1)$:

$$C(1,n-1) = \frac{-p_i\lambda}{\beta+\lambda+\alpha} + \frac{\lambda C(1,n-2)}{\beta+\lambda+\alpha}. \tag{C-29}$$

We insert (C-29) back into (C-28):

$$C(1,n) = \frac{-p_i\lambda}{\beta+\lambda+\alpha} - p_i(\frac{\lambda}{\beta+\lambda+\alpha})^2 + (\frac{\lambda}{\beta+\lambda+\alpha})^2 C(1,n-2). \tag{C-30}$$

Thus we demonstrate the power series in development. By extending $C(1,n)$ through all possible values of n, (C-30) becomes

$$C(1,n) = -p_i \sum_{m=1}^{\infty} (\frac{\lambda}{\beta+\lambda+\alpha})^m. \tag{C-31}$$

The power series in (C-31) solves to

$$C(1,n) = -p_i(\frac{1}{1-(\frac{\lambda)}{\beta+\lambda+\alpha})} - 1). \tag{C-32}$$

We use some algebraic manipulation of (C-32) to find

$$C(1,n) = -p_i(\frac{\beta+\lambda+\alpha}{\beta+\lambda} - 1). \tag{C-33}$$

By subtracting the 1 from (C-33), we get

$$C(1,n) = -p_i(\frac{\lambda}{\beta+\lambda}). \tag{C-34}$$

In short, the value of any future expected incremental replenishment from a single assembly is $-p_i(\frac{\lambda}{\beta+\lambda})$. Thus, if using incremental replenishment as a possible alternative to batch replenishment, a value of $p_{buyout} = p_i(\frac{\lambda}{\beta+\lambda})$ should be used as the value of the buyout cost. ∎

APPENDIX D
PROOFS FOR THE LAST BUY PROBLEM WITH OPTIONAL CONTRACT EXTENSION

Proofs

Proof: The value of the extension $X(q)$ satisfies $0 \leq X(q) \leq \sum_{l=1}^{j} lp_x P\{A_{t_{c1}} = l\}$.

There are two inequalities in the proof, both of which need to be evaluated separately. We start by examining the first half of this set of inequalities - that $0 \leq X(q)$.

The value of the extension is

$$X(q) = \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = n | A_{t_{c1}} = l\} \max[lp_x - V(l,n), 0]. \tag{D-1}$$

where

$$V(l,n) = -p \sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt. \tag{D-2}$$

$V(l,n)$ is the expected value, discounted to time t_{c1}, of the penalty costs that are incurred during the contract extension period.

As all probabilities are such that $0 \leq P\{\cdot\} \leq 1$ and $\max[lp_x - V(l,n), 0] \geq 0$, then the product of the max function and any probabilities must be non-negative. Therefore,

$$X(q) = \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = n | A_{t_{c1}} = l\} \max[lp_x - V(l,n), 0] \geq 0. \quad \text{(D-3)}$$

Thus, we have demonstrated that the extension $X(q)$ is non-negative.

Once that's done, we need to prove the other half of the inequality, that $X(q) \leq \sum_{l=1}^{j} lp_x P\{A_{t_{c1}} = l\}$. Before we start, we first need to prove that $V(l,n) \geq 0$. That way, we have an upper limit on the $\max[lp_x - V(l,n), 0]$ term - which, in turn, should provide an upper bound on the entirety of the $X(q)$ equation.

To prove that $V(l,n) \geq 0$, we need to examine all of the parts of $V(l,n)$, and prove that they are non-negative for all values of t.

- $e^{-\alpha t}$: This is an exponential, and thus is positive for all values of t.
- $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$: The state $D_t \leq q$ can be viewed as the opposite of a trapping state; once $D_t > q$, the total demand can never go below q again, because we cannot have negative demand. Therefore, the probability $P\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}$ can never increase - and the rate of change of that probability $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ can never be positive for any value of t. Thus, $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ is non-positive.
- $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$: This is a probability, and as such is between 0 and 1 for all values of t. Thus, $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$ is non-negative for all values of t.
- $\int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt$: Of the parts

of this integral, $e^{-\alpha t}$ is non-negative for all values of t, $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ is non-positive for all values of t, and $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$ is non-negative for all values of t. Thus, the integral of the product of these parts must be non-positive.

- $-p$: The penalty cost per assembly for failing to meet customer demand is non-negative according to the requirements of the problem; thus, its negative must be non-positive.
- k: The number of assemblies still in operation must always be non-negative.

Since $\int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt \leq 0$, $-p \leq 0$, and $k \geq 0$, their product must also be greater than 0:

$$V(l,n) = -p \sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt \geq 0. \tag{D-4}$$

Thus, we have demonstrated that $V(l,n) \geq 0$. As a result, the maximum value of $\max[lp_x - V(l,n), 0]$

$$\max_{l,n}[\max[lp_x - V(l,n), 0]] \leq lp_x. \tag{D-5}$$

Now that we have our maximum value for $\max[lp_x - V(l,n), 0]$, we can start to find out the maximum possible value for $X(q)$. Once we find this, we will have the upper bound required to finish our proof. We insert this maximum back into formula

(D-1) for $X(q)$:

$$X_{max} = \max[X(q)] \leq \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = n | A_{t_{c1}} = l\}(lp_x). \tag{D-6}$$

Equation (D-6) can be simplified to

$$X_{max} = \max[X(q)] \leq \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{D_{t_{c1}} = n \cap A_{t_{c1}} = l\}(lp_x). \tag{D-7}$$

By combining the summation of the demand into the probabilities in equation (D-7), we get

$$X_{max} = \max[X(q)] \leq \sum_{l=1}^{j} P\{D_{t_{c1}} < q \cap A_{t_{c1}} = l\}(lp_x). \tag{D-8}$$

As q becomes large, $P\{D_{t_{c1}} < q\} \to 1$ for all $A_{t_{c1}} = l$. Thus, from (D-8),

$$X_{max} = max[X(q)] \leq \sum_{l=1}^{j} P\{A_{t_{c1}} = l\}(lp_x). \tag{D-9}$$

By combining our results from (D-3) and (D-9), we get

$$0 \leq X(q) \leq \sum_{l=1}^{j} lp_x P\{A_{t_{c1}} = l\}. \tag{D-10}$$

Thus, both inequalities are proven. ∎

As an extension of this, it has become important to prove not only that $X(q) \leq \sum_{l=1}^{j} lp_x P\{A_{t_{c1}} = l\}$, but that, as q becomes large, $X(q) \to \sum_{l=1}^{j} lp_x P\{A_{t_{c1}} = l\}$. In order to demonstrate this, we take the limit of parts of $X(q)$ as $q \to \infty$.

Proof: The limit $\lim_{q\to\infty} X(q) = \sum_{l=1}^{j} lp_x P\{A_{t_{c1}} = l\}$

In order to complete this proof, we must examine what happens to each part of $X(q)$ as q becomes large. As it is the most complicated portion of $X(q)$, we start by examining the parts of $V(l,n)$ shown in (D-2), as $q \to \infty$:

- $-p$, k: As these are both stated by the problem, these are essentially constants.
- $e^{-\alpha t}$. As all times are positive, $0 < e^{-\alpha t} < 1$ for all possible values of t.
- $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$: The state $D_t \leq q$ can be viewed as the opposite of a trapping state; once $D_t > q$, the total demand can never go below q again, because we cannot have negative demand. However, as $q \to \infty$, the probability that the demand somehow exceeds q approaches 0, even at large values of t. Thus, as $q \to \infty$, $P\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}$ stays at 1 and does not change. As a result, $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ approaches 0 for all values of t.
- $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$: This is a probability, and thus is between 0 and 1 for all values of t.
- $\int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt$: Because $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ approaches 0 for all values of t and $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$ and $e^{-\alpha t}$ are finite, integral $\int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt$ approaches 0 as q becomes large.

Thus, as $q \to \infty$, because the value of the integral approaches 0, $V(l,n)$ also approaches 0:

$$\lim_{q\to\infty} V(l,n) = -p \sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} \\ P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt = 0. \tag{D-11}$$

Thus, because $V(l,n)$ approaches 0 as q becomes large, the value of the formula in (D-1) reduces to

$$\lim_{q\to\infty} X(q) = \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = n | A_{t_{c1}} = l\} l p_x. \tag{D-12}$$

From here, we see what happens to the other parts of $X(q)$ as $q \to \infty$. We start by combining the two probabilities in (D-12):

$$\lim_{q\to\infty} X(q) = \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{D_{t_{c1}} = n \cap A_{t_{c1}} = l\} l p_x. \tag{D-13}$$

As $q \to \infty$, the sum of probabilities $\sum_{n=0}^{q-1} P\{D_{t_{c1}} = n \cap A_{t_{c1}} = l\}$ approaches $P\{A_{t_{c1}} = l\}$. The reason for this is that, as $q \to \infty$, $\sum_{n=0}^{q-1} P\{D_{t_{c1}} = n\}$ approaches 1, as the summation takes up all possible values of $D_{t_{c1}}$. As a result of this, equation (D-13) becomes

$$\lim_{q\to\infty} X(q) = \sum_{l=1}^{j} P\{A_{t_{c1}} = l\} l p_x. \tag{D-14}$$

Thus, we have established the value of $X(q)$ as q becomes large. ■

Proof: The value of the extension $X(q)$ is non-decreasing with increasing q.

The value of the extension is

$$X(q) = \sum_{l=1}^{j} \sum_{n=0}^{q-1} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = n | A_{t_{c1}} = l\} \max[lp_x - V(l,n), 0]. \tag{D-15}$$

Thus, the rate of change of the extension with respect to q is

$$\Delta X(q) = X(q+1) - X(q) = \sum_{l=1}^{j} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = q | A_{t_{c1}} = l\} \max[lp_x - V(l,q), 0]. \tag{D-16}$$

As stated earlier, all probabilities $0 \leq P\{\cdot\} \leq 1$. Also, because of the definition of a max function, $0 \leq \max[lp_x - V(l,q), 0]$. Thus, all parts of $\Delta X(q)$ are non-negative. As a result,

$$\begin{aligned} \Delta X(q) &= X(q+1) - X(q) \\ &= \sum_{l=1}^{j} P\{A_{t_{c1}} = l\} P\{D_{t_{c1}} = q | A_{t_{c1}} = l\} \max[lp_x - V(l,q), 0] \geq 0. \end{aligned} \tag{D-17}$$

Thus, since the rate of change of the extension is non-negative with increasing q, the value of the extension $X(q)$ is non-decreasing with increasing q. ■

Proof: The expected value of the penalty in the contract extension period $V(l,n)$ is non-negative.

The equation for the expected value can be found from (7.9):

$$V(l,n) = -p\sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$$

$$P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\} dt.$$

By looking at the parts of (7.9) and determining if each part is non-negative or non-positive, we can determine if (7.9) in its entirety is non-positive or non-negative.

- $e^{-t\alpha}$: This is an exponential; thus, it is positive for all values of t.
- $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$: The probability $P\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}$ decreases as t increases, as we do not allow for negative demand. Thus, once the demand up to a specific time $D_{\hat{t}} > q$, the demand can never be less than or equal to q at any point in time beyond $\hat{t}$. Thus, because the probability $P\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}$ can only decrease as t increases, the rate of change $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$ is non-positive for all values of t.
- k: According to the summation that determines the values of k used, k is positive.
- p: The penalty per assembly p is non-negative, according to the requirements of the problem.
- $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$: By definition, all probabilities must be between 0 and 1. Thus, $P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}$ is non-negative.

- $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\}dt$: Because $e^{-t\alpha} > 0$ and $P\{D_t \leq q\} \geq 0$ for all values of t, then the integral $\int_{t=0}^{\infty} e^{-t\alpha} P\{D_t \leq q\}dt$ must also be non-negative for all values of t.

- $\int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}dt$: All parts of this integral are non-negative for all values of t except for $\frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt}$, which is non-positive for all values of t. As a result, the entire integral must be non-positive.

Because $\int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}dt$ is non-positive, p is non-negative, and k is non-negative, the negative of the product of these three parts must be non-negative. Thus,

$$V(l,n) = -p \sum_{k=1}^{l} k \int_{t_{c1}}^{t_{c2}} e^{-\alpha t} \frac{dP\{D_t \leq q | A_t = k, A_{t_{c1}} = l, D_{t_{c1}} = n\}}{dt} P\{A_t = k | A_{t_{c1}} = l, D_{t_{c1}} = n\}dt \geq 0. \tag{D-18}$$

Thus, $V(l,n)$ is non-negative. ■

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