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Essays on Product Acquisition for Value Recovery

Dwayne D. Cole
Syracuse University

ESSAYS ON PRODUCT ACQUISTION FOR VALUE RECOVERY

ABSTRACT

This dissertation studies decision problems facing the manufacturer that offers cash incentive to encourage a fraction of its install base to return end-of-use devices. Marketing managers often use such tactics as a promotion tool to motivate sales of new products. Supply chain managers often use such tactics to obtain used products for profitable recovery operations.

The first essay, “Product Acquisition for Remanufacturing: A Dynamic Analysis,” analyzes the performance of buyback and trade-in policies for acquiring products to be remanufactured. A key distinguishing feature of this analysis is the consideration of time dynamics. In particular, both the quantity-condition profile of used products and the market interest in remanufactured products evolve over time, and the manner of evolution is influenced by new product sales. Essay 1 introduces and analyzes a series of models that reflect the dynamics of customer willingness-to-return and willingness-to-pay attitudes, the size and condition of the OEM product install base, the demand for remanufactured product, and the demand for new product. Conventional approaches set trade-in and buyback prices to maximize profits in a single period; however, our analysis show that companies can earn higher profits by adopting a proactive approach.

The second essay “Final Purchase and End-of-Use Acquisition Decisions in Response to a Component Phase-Out Announcement” is motivated by informal talks with supply chain executives from the computer industry. Essay 2 investigates a problem faced by a durable-goods manufacturer of a product that is no longer manufactured but still under

warranty. A supplier announces that a component of the product will be phased out and specifies a deadline for the final order. In addition to determining the final order quantity from the supplier, the manufacturer may introduce a trade-in program to generate an alternative supply of the component for the purpose of satisfying warranty claims. We analyze how industry and market characteristics influence the manufactures optimal decisions and profits. The analysis in the second essay lends insight into the determinants of the initial order quantity, the characteristics of a well-designed trade-in program to support component harvesting, and the cost of ignoring a trade-in program for component harvesting. We find that launching a trade-in program and harvesting spare-parts from the returned device is not only a viable response to a supplier's component phase out announcement, under certain conditions, launching a trade-in program is actually profitable.

ESSAYS ON PRODUCT ACQUISTION FOR VALUE RECOVERY

BY

DWAYNE DONELL COLE

B.S. Florida Agriculture & Mechanical University, Tallahassee, 1991
M.S. in MIS Florida State University, Tallahassee, 1999
MBA Florida Agriculture & Mechanical University, Tallahassee, 2000

Dissertation

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DEDICATION

Taj, Malcom, and Kalonji; do the best you can with what you have.

CHAPTER 1: INTRODUCTION

1. Introduction

Increasing concerns about the environment and the rise of extended producer responsibility (EPR) legislation has generated an interest in the recovery of used products from consumers. The recovery of used products supports corporate sustainability initiatives. For example, when Sharp Electronics announced its nation-wide TV and electronics take-back initiative, chairman and CEO Doug Koshina said "In all aspects of our business, we continuously seek ways to reduce impact on the environment" (Sharp Electronics, 2008). Recovering used products extends the product's useful life and keeps potentially hazardous materials from entering and polluting the waste stream. For the growing number of firms in industries affected by increasing EPR directives, proactively taking back used products is one way to address both international and domestic EPR mandates.

Aside from environmental concerns, a growing number of companies have introduced recovery programs because of economic benefits. Research in both the marketing (Guiltinan 2009, Jacoby et al. 1977, Nes and Cramer 2005) and operations (Linton and Jayaraman 2005) literature show that the economic value of most durable goods generally outlasts the owner's usage desires—consumers will stop using a product even though it still works. These end-of-use (EOU), yet functioning devices often have residual economic value. Gently used products can be resold through secondhand markets and working components can serve as replacement parts. This is particularly true for durable goods with rapidly changing features (e.g., mp3 players, flat-screen television, gaming consoles, PCs) and durable goods with infrequent, yet breakthrough technology upgrades (e.g., medical imaging equipment, wheel-balancing equipment, lawn mowers, kitchen appliances). For example, when Apple Inc. released the iPhone 4, third-party product take back and recovery intermediaries were willing to pay top dollar to acquire the iPhone 3 from their owners.[1] When Snap On launched its 2010 trade-in program, they used spare-parts recovered from returned wheel balancing systems to meet international service demand.[2] Cisco discovered internal use for its end-of-use returns and as result, reduced recycling cost by 40% (Nidumolu et. al 2009). In summary, gently used devices such as iPhones can be refurbished and sold through secondhand channels, well used devices such as wheel-balancing systems can be remanufactured and sold oversees in emerging markets, and non-working devices can be disassembled and harvested for working components to meet post-sales service requirements (e.g., spare-parts for warranty claims).

The environmental and economic benefits of product recovery programs have persuaded a growing number of large manufacturers and retailers to launch trade-in and/or buyback programs (e.g., Best Buy, Caterpillar, Dell, Herman Miller, General Electric, Pitney Bowes, Sony). For firms adopting such programs, a major managerial decision is determining the right time to buy and how much to pay for used product in a given condition. Determining the right time to buy is complicated by gaps between the available supply of used products and the demand for remanufactured product.

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[1] According to a 2010 press release, NextWorth (www.NextWorth.com), a firm that manages consumer electronic upgrade and trade-in programs, announced its new trade-in program for Apple iPhone owners. The program provided between $208 and $260 in exchange for the iPhone 3. The iPhone 4 sold for $199. Thus the buyback price fully covered the price of upgrading to the new iPhone 4.

[2] This statement comes from direct talks with senior executive from Snap On’s Repair Service and Information Division.

Determining how much to pay for product in a given condition is complicated by condition-dependent recovery cost and evolving consumers' willingness to accept return incentives.

1.2 Research Objectives

The primary purpose of this dissertation is to gain insight into the main economic trade-offs related to used product acquisition decisions for value recovery. Towards this end, this dissertation has two key objectives. The first objective is to develop decision frameworks that incorporate marketing and operational considerations into product acquisition policies in two different settings—one where products are acquired for the purposes of remanufacturing and another where products are acquired for the purposes of component harvesting to support warranty claims. For each setting, we seek to develop a rich framework suitable for investigating a range of specific questions related to product acquisition. The second objective is to provide guidance to managers on the design and implementation of alternative acquisition policies. We seek to identify types of acquisition policies that do, and do not, work well in different settings, and key factors under management control that drive performance.

In both essays, we focus on an integrated framework because, although trade-in and buyback programs have been studied in the marketing and operations literature independently, there is a need to develop decision frameworks that take both approaches into consideration.

1.3 Overview of Essay 1

The first essay addresses the dynamic pricing of trade-in and buyback offer menus. We consider a firm that engages in remanufacturing and new product manufacturing. In order to remanufacture a product, the firm must have a supply of used products. We consider the question of how to design an effective buyback program for product acquisition, and we consider the question of how to design an effective trade-in program for product acquisition. We also evaluate the relative performance of the two types of programs. There has been work that has examined trade-in policies and work that has examined buyback policies, but no work that has compared the two policies.

The sales rate of a product over its life-cycle follows a pattern of growth, maturity, and decline as a new generation of the product is introduced. A key distinguishing feature of the essay is the consideration of quality and volume dynamics. In particular, we account for the relationship between the life-cycles of new and remanufactured product. By defining and examining models that account for this relationship, we elucidate the time-dependent linkages between the new product sales, the supply of used products, the consumer's willingness to accept a buyback or trade-in offer, and the demand for remanufactured product. The life-cycle of a new product affects availability of used products and demand for recovered products, yet there have only been a few attempts to incorporate these effects into the trade-in or buyback pricing models.

Essay 1 introduces a framework that allows investigation of such questions as how the phase of the life-cycle influences issues such as the relative performance of buyback and trade-in programs, optimal acquisition prices, the value of coordinating marketing and operational considerations in used product acquisition, and the penalty for ignoring time-dynamics. To address these questions, Essay 1 introduces and analyzes a series of models that reflect the dynamics of customer willingness-to-return and willingness-to-pay attitudes, the size and condition of the OEM product install base, the demand for remanufactured product, and the demand for the new product. From a methodological perspective, this essay defines and analyzes a parsimonious model of the problem. Characterizations of optimal solutions are developed and solution methods for identifying optimal decisions and corresponding costs/profits are devised.

Our results contribute to the growing body of closed-loop supply chain research. First we introduce two algorithms that can be used to determine a menu of buyback prices and trade-in credits that maximize myopic profits. These two algorithms are significant to the literature because the resulting trade-in and buyback offer prices are based on consumer replacement purchase characteristics, which drive the availability and condition profile of used products. Second, we introduce two math programming formulations to calculate near optimal or dynamic trade-in and buyback prices. These math programming formulations account for the size and age-structure of the install base. This contribution is significant and as a result, we expose a "sweet-spot" age where used products are most amenable to profitable recovery operations. Furthermore, we show that even when remanufacturing is profitable, the firm may choose not to satisfy all demand for the remanufactured product. Finally, we contribute to the literature by comparing proactive and myopic product acquisition strategies. We find that when the time lag between new product sales and the demand for remanufactured product matches the sweet-spot age, myopic pricing policies perform nearly as well as the proactive policies.

1.4 Overview of Essay 2

The second essay addresses the issue of designing a trade-in program to acquire end-of-use components to support warranty claims. We consider the challenge that arises when a sole-source supplier phases out production of a key component used in a product with an install base under warranty. The supplier releases a component phase out announcement (CPOA), outlining final production plans and final order due dates. The importance and prevalence of this final order problem has increased over time due to two long-term trends—shrinking product life-cycles and growth in outsourcing. And the problem of sourcing spare-parts after the product life-cycle is an important issue in practice. Cattani (2005) describes how an end-of-life buy left the firm with millions of dollars of worthless inventory at a cost of 1% of sales.

The traditional response to a CPOA by the OEM is to place a final order that is large enough to cover all warranty requirements. However, our conversations with a large computer manufacturer indicate an interest in the possibility of using a trade-in program to supplement the final order quantity. Designing an appropriate trade-in policy is complicated because it involves consumer choice models and price-dependent return volumes. The design of a trade-in program to support warranty claims is also complicated, and unique, because the warranty claims on the obsolete product are influenced by the volume of product acquired through the trade-in program.

In Essay 2, we address these complications and consider the merits of using a trade-in program to obtain obsolete products under warranty for purposes of component harvesting. Specifically, we consider two types of trade-in policies—*full* and *matching*. The full trade-in policy offers a trade-in credit to the entire warranty population at a single point in time. The optimal trade-in quantity may or may not match warranty demand; units in excess of demand are salvaged and shortages incur a penalty cost (e.g., cost premium for a component obtained from a third party or for replacing the customer's product with a current generation model). Trade-in units are received and warranty claims are satisfied from inventory over the duration of the warranty horizon. The matching trade-in policy matches the component supply rate with the warranty claim demand rate over the duration of the warranty horizon.

We introduce a two-stage decision model where first-stage decision is the number of components in the final order. The second-stage decision, if necessary, is the design of a trade-in program. The second-stage decision is required if or when component supply approaches zero. The objective is to minimize total discounted product acquisition, warranty service, and component holding cost. Characterizations of optimal solutions are developed and solution methods for identifying optimal decisions and corresponding costs are devised.

Essay 2 contributes to operations literature related to managing spare parts inventory when the primary product is no longer manufactured. This period of the production life cycle is referred to as the final-phase or end-of-life phase. Although several authors have considered remanufacturing as an alternative supply source, this paper is the first to consider proactive marketing efforts to acquire used products from a segment of the warranty populations.

In our analysis of full and matching trade-in policies, we find that the use of a full trade-in policy is relatively robust—it will not perform much worse than the matching trade-in policy as long as the discount rate (net of inflation) and the holding cost rate are not too high, and will potentially lead to much greater savings. While a full trade-in policy is generally a safe choice, we find that three key indicators of when a matching trade-in policy is likely to be preferred: (1) positive trade-in potential, (2) low warranty service cost per unit, and (3) a high inventory holding cost rate. Finally, by incorporating market and consumer related elements into our model, this work exposes an interesting marketing result. We find that the single most important driver of trade-in policy value is the trade-in potential—a simple measure that is the difference between the increase in profit from locking in disloyal customers by means of a trade-in transaction and the minimum markdown required for a customer to accept a trade-in offer. The nature of this contribution is novel and raises additional questions about how market and consumer characteristics influence final-phase sourcing policies.

CHAPTER 2: PRODUCT ACQUISITION FOR REMANUFACTURING: A DYNAMIC ANALYSIS

1. Introduction

Sustainability concerns and the rise of extended producer responsibility (EPR) legislation have generated a growing interest in product take-back and recovery practices (e.g., product reuse, remanufacturing, recycling, spare parts harvesting, incineration for energy recovery). Such practices are attractive for two reasons. First, product take-back and recovery programs increase and extend the economic value of products in the market. Second, product take-back and recovery programs positively affect the environment (e.g., landfill avoidance) and support EPR legislation that requires collection of end-of-use product. This is especially true for durable goods such as automobiles, home appliances, and computers that are costly to dispose due to size and/or the presence of hazardous materials. Indeed, durable goods producers often offer product acquisition schemes such as trade-in discounts on a new product, cash buyback, and deposit fees to encourage returns.

The importance of product take-back and recovery programs is reflected in a growing literature on product acquisition policies for remanufacturing. For example, Guide et al. (2003) define an economic model of a remanufacturing firm and derive the optimal condition-dependent buyback prices and the optimal selling price for the remanufactured product. Bakal and Akcali (2006) study a similar problem but with a focus on the effects of uncertain yield of the return supply. Ray et al. (2005) consider a firm that produces both new and remanufactured products. The authors derive optimal selling prices for new and remanufactured product and the optimal condition-dependent trade-in discount function. The results in the product acquisition for remanufacturing literature are based on models of profit in a single period.

In this essay, we investigate the performance of buyback and trade-in policies for acquiring used product to be remanufactured. A key distinguishing feature of our analysis is the consideration of time dynamics. In particular, both the quantity-condition profile of used product and the market interest in remanufactured product evolve over time, and the manner of evolution is influenced by new product sales. In addition, the introduction of a new generation of the product can cause a shift in both return attitudes and remanufactured product purchase attitudes—customers become more willing to exchange old for new and the desirability of remanufactured product from the old generation decreases (Atasu et al. 2010). We introduce and analyze a series of models that reflect the dynamics of customer willingness-to-return and willingness-to-pay attitudes, the size and condition of the OEM product install base, the demand for remanufactured product, and the demand for new product.

The motivation for our study comes from our discussions with management at a computer manufacturer that sells both new and remanufactured product. However, we begin our analysis by studying the case where the remanufacturing division is run as a separate and independent entity. By contrasting decisions and profits of independent firms with that of a combined firm, we are able assess the value and impact of coordinating the operations of new and remanufactured product.

We find that although trade-in pricing schemes may lead to higher acquisition prices, the trade-in policies will generally result in higher profits from product take-back and recovery activities. In our analysis we also compare myopic product acquisition strategies with proactive strategies. We find that the relative performance of myopic and proactive pricing strategies is tied to the lag between new product sales and demand for remanufactured products and to a “sweet-spot” age. The “sweet-spot” age is the age where the combined take back and recovery cost are minimized.

The remainder of the essay is organized into four sections. Section 2 outlines the related literature and how our investigation differs from past research. Section 3 presents a model and analysis of a buyback program. Section 4 presents a model and analysis of a trade-in program. Section 5 investigates the relative performance of buyback and trade-in programs, and Section 6 provides a summary and suggestions for future research. A list of notation and assumptions, as well as derivations and proofs, can be found in the appendix.

2. Related Literature

Product trade-in and buyback strategies are broadly practiced in durable goods industries (i.e., automotive, consumer electronics, computers, and industrial equipment), and are well studied in the marketing literature. Earlier works in this area show that the benefits from offering trade-in programs are due largely to market segmentation and price discrimination, as the firm is able to price discriminate between owners and non-owners. Van Ackere and Reyniers (1995) develop a two-period pricing model. The first-period purchase decision segments the second-period market into owners and non-owners. The trade-in price can be set to take advantage of the fact that, by virtue of a past purchase decision, the owners have a relatively high valuation of the product Okada (2001, 2006). In addition to serving as a market segmentation and price discrimination mechanism, trade-in and buyback programs can reduce cannibalization of new product sales from secondhand markets. Levinthal and Purohit (1989) examine the competition between the new and old version of durable good and show how a monopolist who introduces a new product and continues to sell the old model can improve profits by buying back some of the old products. Fudenberg and Tirole (1998) show how used textbook and software purchases by the OEM helps to increase new product sales by reducing cannibalization. Bruce et al. (2006) consider how trade-in discounts can spur demand and increase profit when consumers are otherwise reluctant to purchase a new product due to the burden of paying off an outstanding loan (e.g., as in the auto industry). Rao et al. (2009) show how a trade-in program can be used to reduce the inefficiencies associated with the "lemon problem"—the trade-in opportunity motivates more owners to purchase new goods by reducing their proclivity to hold on to purchased goods because of the low price the latter would fetch in a lemon market. The above marketing papers are relevant because they provide guidance on modeling return volume as a function of trade-in and buyback prices. However, these papers only consider a two-period setting where the trade-in prices are set in period two. We consider a multi-period setting. Furthermore, none of these papers incorporate the economics of product take-back for remanufacturing, which is an element that is central to our analysis.

The marketing literature focuses on how trade-in and buyback programs affect the profitability of new product sales, or the forward flow of new products. The closed-loop supply chain (CLSC) literature, on the other hand, focuses on how trade-in and buyback incentives affect the reverse flow of used products, e.g., for the purposes of product recovery to support remanufacturing and/or respond to take-back legislation. Product acquisition management is an important function of CLSC systems with remanufacturing.

Remanufacturing depends on timely access to a reliable supply of used products that are still in relatively good condition. In the CLSC setting, trade-in and buyback prices incentivize product returns and thus act as product acquisition mechanisms. As such, they are critical because of their role in (i) shaping the quality of the returned products, and (ii) influencing the return volumes to align supply with demand (Thierry et al. 1995). Guide and Jayaraman (2000) and Flapper (2001) provide an overview of how product acquisition incentives influence the profitability of a remanufacturing firm. Klausner and Hendrickson (2000) study a German remanufacturer that once acquired used power-tools from the waste stream. They show how the firm improves profits by buying gently used tools from current holders instead of collecting used products from the waste stream.

Several recent studies model the volume of product returns as a function of acquisition price, either under a buyback program (e.g., Bakal and Akcali 2006, Guide et al. 2003, Karakayali et al. 2007) or a trade-in program (e.g., Ray et al. 2005). Guide et al. (2003) and Karakayali et al. (2007) consider models with discrete classes for the condition of used products whereas Bakal and Akcali (2006) and Ray et al. (2006) model the condition of used product as a continuous variable. All of these papers identify optimal condition-dependent acquisition prices. In addition, product selling prices—for remanufactured product in the case of Bakal and Akcali (2006), Guide et al. (2003), and Karakayali et al. (2007) and for new product in the case of Ray et al. (2006)—are endogenous. Acquisition and selling prices are set to maximize profit in a single period. The main distinguishing features between these papers and our work are (1) we compare and contrast both buyback and trade-in programs, (2) we consider the acquisition pricing problem over the life-cycle of a product rather than a single period, and (3) we consider settings where the firm is a price-taker rather than a price-setter in the new and remanufactured product markets.

Both forward and reverse operations are influenced by new product diffusion and product life-cycle dynamics. The product life-cycle explains how the sale of a new product grows, matures, and declines over time (Mahajan et.al. 1990). The product life-cycle also affects the availability of used products and the demand for remanufactured products over time. Tibben-Lembke (2002) and Östlin et al. (2009) describe the relationship between new product sales, used product returns, and recovered product demand, and how these relationships vary over different stages of product life-cycle (see Figure 2.1). We make use of this relationship in our model of remanufactured product demand.

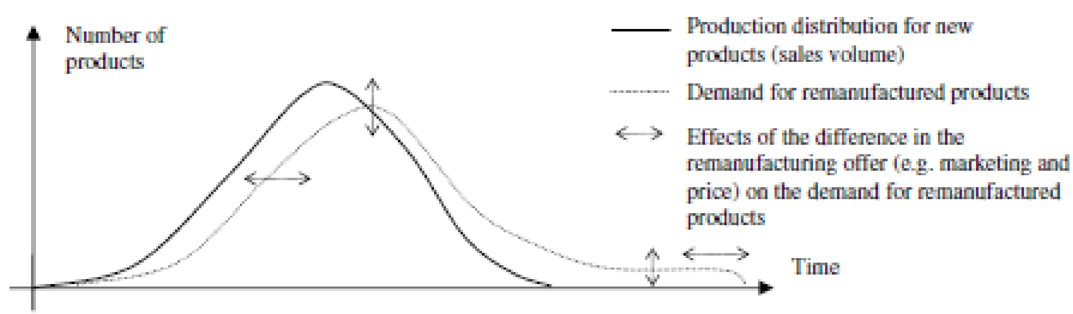

Figure 2.1. Illustration of the theoretical relationship between new product sales and the demand for a remanufactured version of the product (adapted from Östlin et al. 2009).

Several studies have addressed the implications of the product life-cycle on CLSC collection, recovery, and remarketing strategies. Debo et al. (2006) consider remanufacturing production strategies when the firm selects the degree of product remanufacturability in settings where the product is collected and remanufactured many times. Geyer et al. (2007) also consider a setting where the product is remanufactured many times. They focus on the economic feasibility of recovery operations and how life-cycle dynamics influence remanufacturability levels, collection rates, and optimal cost savings from remanufacturing. Umeda et al. (2006) consider the question of how many times a device can be used over the life-time of the product. Our work is similar to these papers in that we examine the evolution of the install base of new product over time and the associated evolution of demand for remanufactured product. Our work differs from these papers in that we concentrate on the impact of life-cycle dynamics on optimal acquisition prices and on the relative merits of buyback and trade-in programs. These elements are not considered within this stream of literature.

The cost to recover used products depends on the product's condition and on the targeted return volume. Higher trade-in prices increase the flow of reusable returns (Klausner and Hendrickson 2000). Galbreth and Blackburn (2006) analyze optimal acquisition and sorting policies. They show how acquiring more used products may reduce the average cost of remanufacturing. When the difference in the cost to recover high vs. low quality products is significant, the firm can potentially benefit from acquiring more used products in order to secure better quality products. Zikopoulos and Tagaras (2007) identify optimal procurement quantities from multiple alternative sources. In contrast to these papers, we consider condition-dependent acquisition pricing. In addition, our research is distinct in that we incorporate how the condition and quantity of the install base change over time.

Finally, a few recent studies have addressed the dynamic aspects of CLSC systems. Lee et al. (2010) examine a decentralized two-echelon distribution channel where the manufacturer provides a retailer with a fixed incentive to collect used products. The retailer determines a time-variant pricing rule that includes the price of the new product and the trade-in incentive. The manufacturer chooses a constant wholesale price and offers a fixed payment for the returned products. The authors show how the retailer's behavior over time affects collection rates. Although Lee et al. (2010) develop a time-variant model, their model does not incorporate how the recovery cost changes as a function of the age or condition of the trade-in returns. This is an important feature that underlies the motivation for our study because the OEM has information on the profile of the install base (e.g., quantity and age)—information that is highly relevant for setting acquisition prices.

3. Buyback Policies for a Producer of Remanufactured Product

The sales rate of a product over its life-cycle follows a pattern of growth, maturity, and decline as a new generation of the product is introduced. Both the supply and the demand of a remanufactured version of the product are influenced by historical sales of the new product. In particular, the sales history exposes the quantity-age profile of the install base (e.g., number of products in the market for less than one year, less than two years, etc.) and, as a measure of market response to the new product, is an indicator of market interest in the remanufactured product.

Section 3.1 addresses the relationship between new product sales over its life-cycle and the supply of used product for remanufacturing. Section 3.2 attends to the relationship between new product sales and the demand for remanufactured product. Section 3.3 identifies the optimal myopic age-dependent buyback pricing function and characterizes the optimal profits over the remanufactured product life-cycle. A myopic buyback price function sets prices to maximize profit in each period, with no consideration of future periods. Section 3.4 considers how the optimal buyback price function and profits are impacted when the firm is proactive in its pricing. Section 3.5 presents numerical analyses that compare and contrast the two pricing policies.

3.1. Remanufactured Product Supply

In this section we develop a model of return volume as a function of age-dependent buyback prices. We begin by describing the consumer-choice model that we use to define new product purchase probabilities and product return probabilities. We model the diffusion of new product sales in markets of two segments. Our model results in sales that are consistent with the diffusion model of Bass (1969), a parsimonious model that is empirically well-established for sales of consumer durables.

A generation of a product is introduced in period $t = 1$. The market is divided into two segments—innovators and imitators—that are distinguished by their valuation of the new product. The innovator segment (segment 1) valuation at the time of purchase is V_1, which is uniformly distributed. We normalize the support to [0, 1], i.e.,

$$V_1 \sim U[0, 1]. \tag{1.1}$$

Uniformly distributed valuation is common in the literature (e.g., Mussa and Rosen 1978, Purhoit and Staelin 1984). Note that the new product price is less than the maximum innovator valuation (i.e., $p_n < 1$); otherwise there would be no sales.

The imitator segment (segment 2) valuation at the time of purchase is V_2, which is also uniformly distributed. However, the upper support of V_2 in period t is the new product purchase price plus a term that is proportional to cumulative sales at the start of the period $D_n(t-1)$, i.e.,

$$V_2 \sim U[0, p_n + \iota D_n(t-1)]. \tag{1.2}$$

Note that the innovator segment valuation is unaffected by historical sales whereas the maximum valuation of a consumer in the imitator segment increases with the number of users. The maximum possible imitator valuation occurs at the end of the life-cycle when $D_n(t-1)$ is close to M, and its value depends upon model parameters. In the data underlying Figure 2.2, for example, $V_2 \leq p_n + \iota M = 0.92$.

The innovator segment makes up fraction θ of consumers who consider purchasing the product in each period, and the imitator segment makes up the balance $1 - \theta$. A consumer makes a purchase if the difference between valuation and purchase price is nonnegative. The total number of purchases over the life of the product is M. Accordingly, the sales in period t is

$$d_n(t) = \theta(1-p_n)\left(M - D_n(t-1)\right) + (1-\theta)\iota D_n(t-1)\left(M - D_n(t-1)\right), \tag{1.3}$$

which matches the form of the classic Bass model:

$$d_n(t) = a\left(M - D_n(t-1)\right) + b\left(\frac{D_n(t-1)}{M}\right)\left(M - D_n(t-1)\right)$$

where

$$a = \theta(1 - p_n)$$

is the coefficient of external influence (e.g., individual conversion ratio in the absence of adopter’s influence) and

$$b = (1 - \theta)\iota M$$

is the coefficient of internal influence (e.g., effect of each adopter on each non-adopter) (Jeuland, 1981). We approach the new product sales process as a two-step flow of information between two segments. This approach is consistent with marketing literature (see Van den Bulte and Joshi, 2007). Figure 2.2 illustrates sales over time according to (1.3).

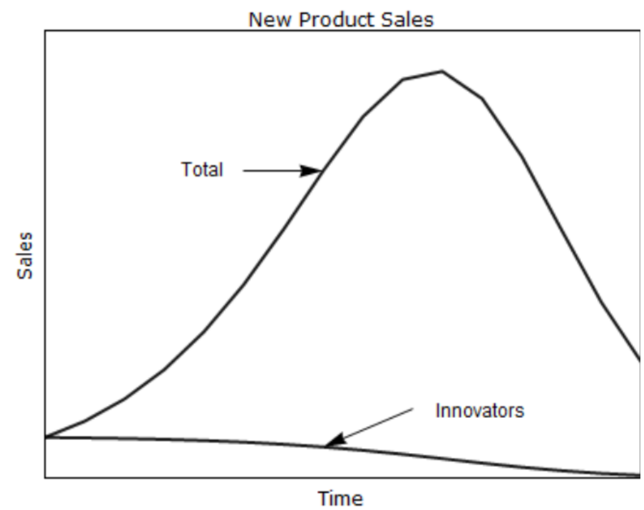

Figure 2.2 Illustration of data presented in Bass (1969). Sales per period with $\theta = 0.0218$, $\iota = 0.0000248$, $p_n = 0.5$, $M = 16{,}895$.This data yield the curve presented in Bass (1969) that closely matches the sales of room air conditioners between 1946 and 1961. The upper border of the shaded region shows sales to the innovator segment.

A product that has been used for i periods is said to be of age i. During ownership, the value of product in the eyes of the user declines with age. We let function $\upsilon(i)$ denote the valuation fraction of a product of age i, with $\upsilon(0) = 1$ and $\upsilon(i) \leq \upsilon(i-1)$. We assume that the function $\upsilon(i)$ applies to both segments. Thus, an individual with valuation V at the time of purchase has residual valuation $V\upsilon(i)$ when the product reaches age i. The function $\upsilon(i)$ characterizes the residence index which is the ratio of the time a product is used before reaching end-of-use and the length of the product lifecycle (Georgiadis et al., 2006). The value of $V\upsilon(i)$ can also be interpreted as the disutility associated with giving up a product through a buyback program, or buyback disutility.

A1. The valuation fraction of a product of age i, $\upsilon(i)$, is the same for the innovator and imitator segments.

At time t, a customer with product of age i will return the product if the difference between the buyback price and the buyback disutility is nonnegative. The implicit assumption is that customers are not strategic in their return decision. In our setting, the firm sets the age-dependent buyback prices dynamically, and consequently there is little basis for a customer to predict how the age-dependent buyback price function will change in the future. The setting is consistent with the firm in the computer industry that motivates this work—a customer can go to a website, enter the product and condition, and receive a real-time quote of the credit for a return.[3]

A2. Customers are not strategic in their return decision.

We let $c_b(t, i)$ denote the buyback price at time t for a product of age i. The probability distributions of the segment valuations in period t conditioned on a purchase transaction at time t are

$$V_1 \sim U[p_n, 1] \tag{1.4}$$

$$V_2 \sim U[p_n, p_n + \iota D_n(t-1)]. \tag{1.5}$$

Thus, the probability distributions of the residual value of a product purchased in period $t - i + 1$ that is now of age i at the end of period t are

[3] Based on discussions with executives from several third-party recovery firms, most firms determine these prices by using proprietary software to scan the Internet for the Lowest Internet Price (LIP) for a new product, estimate the resale price for the recovered product at 20 – 60% of LIP, then set trade-in and buyback prices to ensure profitability.

$$V_1\upsilon(i) \sim U[p_n\upsilon(i),\ \upsilon(i)] \tag{1.6}$$

$$V_2\upsilon(i) \sim U[p_n\upsilon(i),\ (p_n + \iota D_n(t-i))\upsilon(i)] \tag{1.7}$$

and the return probabilities in response to buyback price function $c_b(t, i)$ offered at the end of period t are

$$P\left[V_1\upsilon(i) \le c_b(t,i)\right] = \min\left\{\left(\frac{c_b(t,i) - p_n\upsilon(i)}{\upsilon(i)(1-p_n)}\right)^+, 1\right\} \tag{1.8}$$

$$P\left[V_2\upsilon(i) \le c_b(t,i)\right] = \min\left\{\left(\frac{c_b(t,i) - p_n\upsilon(i)}{\upsilon(i)\iota D_n(t-i)}\right)^+, 1\right\} \tag{1.9}$$

for $i \in [1, t]$.

We next introduce expressions for the size of the install base of age i product in each segment at time t. From (1.3), it follows that the fraction of sales to the innovator segment in period t is

$$\theta(t) = \frac{\theta(1-p_n)}{\theta(1-p_n) + (1-\theta)\iota D_n(t-1)} \tag{1.10}$$

and the fraction of sales to the imitator segment in period t is $1 - \theta(t)$. The number of products of age i in the segment j install base at the end of period t is $N_{bj}(t, i)$. The number of products of age i returned in period t from segment j is $s_{bj}(t, i)$. For the purposes of defining age in functions $N_{bj}(\cdot)$ and $s_{bj}(\cdot)$, we assume that all sales occur at the beginning of the period and all returns occur at the end of the period.[4] In particular, the timing of events in each period is as follows:

[4] Some firms allow returns for a full refund within a specified time period after purchase for product that is in "like new" condition. These products are reintroduced to the market with minimal investment and do not enter the remanufacturing process. The volume of sales in period t, $d_n(t)$, is net of these returns.

Start of period *t*

1. Demand for new product occurs

End of period *t*

1. Buyback price schedule is posted and returns occur
2. Returns are remanufactured
3. Demand for remanufactured product occurs
4. Incur cost h per unit on inventory of remanufactured product

We assume that the populations of consumers who consider purchasing new products and remanufactured products are distinct. This is consistent with observations by Guide and Li (2010) who study bidding behavior of participants in eBay auctions for a consumer good (i.e., Skil jigsaw). They found that customers who bid on the new product never bid on the remanufactured version of the product, and customers who bid on the remanufactured product never bid on the new product. Feedback from each group showed awareness of both new and remanufactured versions of the product, and that the products were not perceived as substitutes.

A3. The new product and the remanufactured product serve distinct markets—the products are not substitutes.

A certain fraction of customers who return their product of age i in response to the buyback offer at the end of the period replace their product by purchasing a new version of the product from the firm at the beginning of the next period (i.e., customers who accept the buyback offer do not consider purchasing a remanufactured product). The repurchase fraction is a measure of customer brand loyalty that we assume is not affected by the number of periods the customer has used the product. We let γ denote this fraction. As an example, the number of products of age 1 in the install base at the end of period t is the new customer sales plus the buyback customers sales at the beginning of period t less the buybacks at the end of period t. The buyback customer sales to segment j at the beginning of period t is the number of buybacks at the end of the previous period adjusted by the repurchase fraction, i.e.,

$$d_{bj}(t) = \gamma \sum_{k=1}^{t-1} s_{bj}(t-1,k).^{5} \tag{1.11}$$

for $t \geq 1$. For the computation of $d_{bj}(0, 0)$, we define $s_{bj}(0, 0) = 0$.

A4. The repurchase rate, γ, is independent of the age of the product when returned.

Recall that $d_n(t)$ denotes sales to new customers. Formulas for $N_{b1}(t, i)$ are illustrated below for the first few periods

$$N_{b1}(1,1) = \theta(1)d_n(1) - s_{b1}(1,1)$$

$$N_{b1}(2,1) = \theta(2)d_n(2) + \gamma s_{b1}(1,1) - s_{b1}(2,1)$$

$$N_{b1}(2,2) = N_{b1}(1,1) - s_{b1}(2,2)$$

$$N_{b1}(3,1) = \theta(3)d_n(3) + \gamma[s_{b1}(2,1) + s_{b1}(2,2)] - s_{b1}(3,1)$$

$$N_{b1}(3,2) = N_{b1}(2,1) - s_{b1}(3,2)$$

$$N_{b1}(3,3) = N_{b1}(2,2) - s_{b1}(3,3),$$

(see Figure 2.3). The only differences between the $N_{b1}(t, i)$ formulas and the $N_{b2}(t, i)$ formulas are that $j = 1$ is replaced with $j = 2$ and $\theta(t)$ is replaced with $1 - \theta(t)$.

---

[5] The number of distinct owners of the product does not change when a buyback credit is used for a replacement purchase. Consequently, these repeat purchases do not impact sales to the imitator segment (any segment?).

To simplify the presentation of a general expression for $N_{bj}(t, i)$, we define $N_{bj}(t, 0)$ as units that enter the segment j install base at the beginning of period $t + 1$ (equivalently, at the end of period t), which are of age 0 at the moment of entry, i.e.,

$$N_{b1}(t,0) = \theta(t+1)d_n(t+1) + d_{b1}(t+1) = \theta(t+1)d_n(t+1) + \gamma\sum_{k=1}^{t} s_{b1}(t,k)$$

$$N_{b2}(t,0) = \left[1-\theta(t+1)\right]d_n(t+1) + d_{b2}(t+1) = \left[1-\theta(t+1)\right]d_n(t+1) + \gamma\sum_{k=1}^{t} s_{b2}(t,k)$$

for $t \geq 0$. A general expression for the number of age i units in the segment j install base at time t is simply the segment j install base of age $i - 1$ product at the end of period $t - 1$ (and beginning of period t) less the number of age i units returned by segment j customers at the end of period t, i.e.,

$$N_{bj}(t, i) = N_{bj}(t-1, i-1) - s_{bj}(t, i) \text{ for } j \in \{1, 2\},\ i \in [1, t],\ t \geq 1. \tag{1.12}$$

Based on the consumer choice model, the volume of age i product that is returned by each segment at the end of period t is given by

$$s_{bj}(t, i) = P\left[V_j \upsilon(i) \leq c_b(t,i)\right] N_{bj}(t-1, i-1) \text{ for } j \in \{1, 2\},\ i \in [1, t],\ t \geq 1.^{6} \tag{1.13}$$

[6] In practice, there may be some fraction of products in the install base that cannot be returned (e.g., lost or discarded products, customers who would never consider returning product due to lack of awareness, interest, etc.). The right-hand side of (1.13) could be proportionally reduced to account for this effect and the following results carry through. In the interest of parsimony, we do not include this parameter in the expressions.

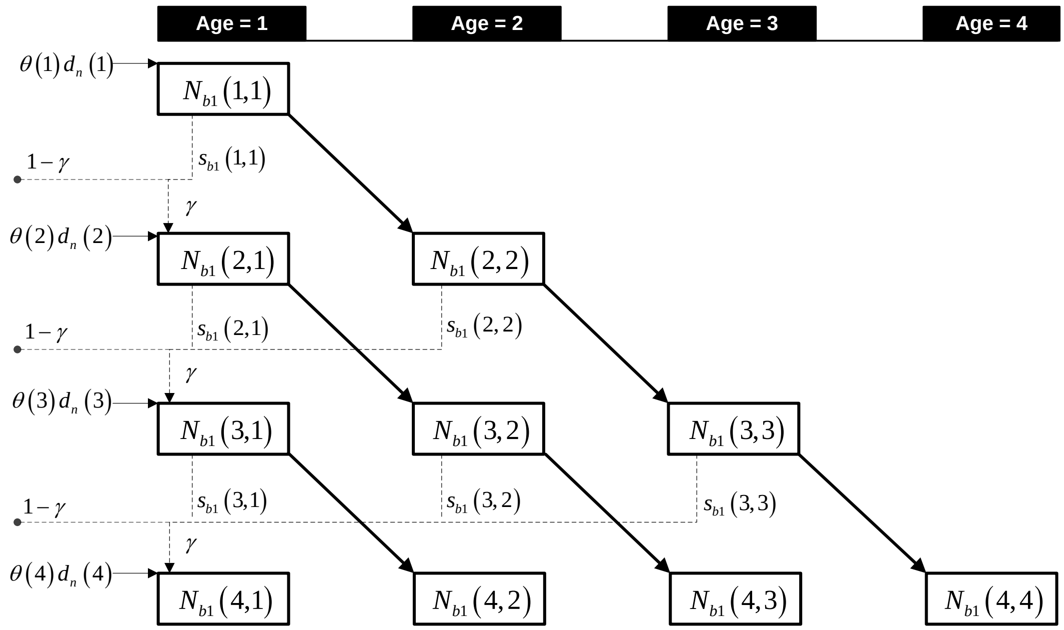

Figure 2.3. Illustration of the evolution of the innovator segment install base.

3.2. Remanufactured Product Demand

Section 3.1 describes a model for the relationship between new product sales over its life-cycle and the supply of used products for remanufacturing. In this section we turn our attention to the relationship between new product sales and the demand for remanufactured products. Figure 2.4 shows monthly new product sales and remanufactured product demand (actual and projected) of a model of a photocopy machine between 2005 and 2012. The sales patterns in the figure are consistent with the linkage between new product sales and remanufactured product demand illustrated in Figure 2.1.

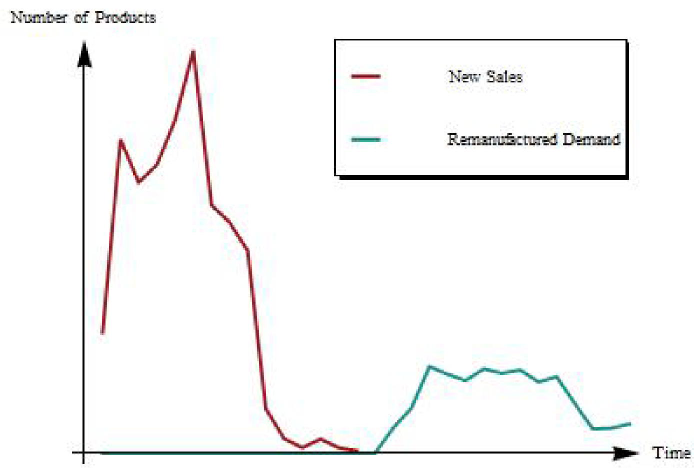

Figure 2.4. Relationship between new product sales of a photocopy machine and the demand for a remanufactured version of the photocopier. The time frame is 2005 –2012.

We use two parameters to specify the remanufactured product demand in terms of new product sales—a relative-size parameter α and a time-lag parameter τ. The value of α defines the size of the remanufactured product market relative to the new product market. The value of τ is the number of periods that the remanufactured product demand is behind the new product demand. In Figure 2.4, for example, the value of τ is about four years and the value of α is approximately 25%.

As noted above, the populations of consumers who consider purchasing new products and remanufactured products are distinct (Guide and Li 2010). The remanufactured product demand in period t is

$$d_r(t) = \alpha d_n(t - \tau) \text{ for } \tau < t \tag{1.14}$$

and the total sales in periods 1 through t is

$$D_r(t) = \alpha D_n(t - \tau) = \sum_{j=1}^{t} d_r(j) \text{ for } \tau < t. \tag{1.15}$$

Recall that in our model, demand for new product occurs at the beginning of the period, whereas returns occur at the end of the period (i.e., so that the install base at the time that buybacks take place is net of any “like new” returns to the retailer that take place shortly after purchase). We assume that demand for remanufactured product occurs at the end of the period, and thus can be satisfied using the buyback returns in the period. If the duration of a period is long, then it is possible that $\tau = 0$. However, there is typically a time lag for products to be returned (Umeda et al. 2005), which can lead to a multi-period time lag between the patterns of new and remanufactured product demand. In practice, the magnitude of the time lag may evolve over time, though in the interest of parsimony, we assume a fixed value over the duration of the life-cycle.

3.3 Remanufacturing Cost and the Optimal Myopic Buyback Price Schedule

The net out-of-pocket costs of transforming a returned unit into a remanufactured product of age i is $c_m(i)$, is which is nondecreasing in age (i.e., age is a proxy for the condition of a returned product). This cost is comprised of the pure remanufacturing cost, less the profit benefit of advancing the sale of unit loyal consumers who would have purchased the product at some point in the future if the product acquisition program did not exist. Guide et al. (2003), for example, report that ReCellular’s remanufacturing cost is convex increasing in i, where the larger the value of i, the worse the condition of the returned product. The value includes the discounted benefit associated with advancing replacement purchase. According to our notation, the total cost to acquire and remanufacture a product of age i in period t is

$$c_b(t, i) + c_m(i). \qquad (1.16)$$

We assume that each unit acquired through the buyback program is remanufactured (e.g., some portion of each returned product is reused). Of course, once product is beyond a certain age, the value of $c_m(i)$ may reach a point where it is not profitable to remanufacture the product and the firm will not offer to buyback these products.

The total return volume in period t of product of age i is

$$s_b(t, i) = s_{b1}(t, i) + s_{b2}(t, i) \tag{1.17}$$

or in expanded form (see (1.13)),

$$s_b(t, i) = \begin{array}{l} \min\left\{\left(\dfrac{c_b(t,i) - p_n\upsilon(i)}{\upsilon(i)(1-p_n)}\right)^+, 1\right\} N_{b1}(t-1, i-1) + \\ \min\left\{\left(\dfrac{c_b(t,i) - p_n\upsilon(i)}{\upsilon(i)\iota D_n(t-i)}\right)^+, 1\right\} N_{b2}(t-1, i-1) \end{array}. \tag{1.18}$$

We denote the total number of units of age i in the install base just prior to returns at the end of period t as $N_b(t, i)$, i.e.,

$$N_b(t, i) = N_{b1}(t, i) + N_{b2}(t, i). \tag{1.19}$$

Note that if $N_b(t-1, i-1) = 0$, then there can be no returns of age i products in period t (i.e., because there are no such products in the install base). We let $\Omega(t)$ denote the set of product ages for which returns are possible in our optimization problem, i.e., $\Omega(t) = \{i : N_b(t-1, i-1) > 0\}$.

The return volume function, $s_b(t, i)$, is constant for any $c_b(t, i) \geq \max\{\upsilon(i), \upsilon(i)[p_n + \iota D(t-i)]\}$, e.g., 100% of the age i product owned by the innovator segment is returned if $c_b(t, i) \geq \upsilon(i)$, and 100% of the age i product owned by the imitator segment is returned if

$c_b(t, i) \geq \upsilon(i)[p_n + \iota D(t-i)]$. For any age $i \in \Omega(t)$, we see from (1.18) that $s_b(t, i)$ is strictly increasing over the range of viable buyback prices

$$c_b(t, i) \in \Omega_{c_b}(t,i) \equiv \left[p_n \upsilon(i), \upsilon(i) \max\left\{1, p_n + \iota D_n(t-i)\right\}\right], \; i \in \Omega(t). \qquad (1.20)$$

Thus, we can invert (1.18) to express buyback price as a function of volume over the corresponding range of viable volumes, denoted $\Omega_{s_b}(t,i)$ for product of age i, i.e.,

$$\Omega_{s_b}(t,i) \equiv \left[0, N_b(t-1, i-1)\right] \qquad (1.21)$$

If $1 \geq p_n + \iota D(t-i)$, we say that the age i cost structure conforms to Regime 1; otherwise the age i cost structure conforms to Regime 2. If Regime 1 applies, then at $c_b(t, i) = \upsilon(i)[p_n + \iota D(t-i)]$, 100% of age i product owned by the imitator segment is returned and the total return volume is

$$A_{b1}(t, i) = \left(\frac{\iota D_n(t-i)}{1-p_n}\right) N_{b1}(t-1, i-1) + N_{b2}(t-1, i-1) \qquad (1.22)$$

(obtained by substituting $c_b(t, i) = \upsilon(i)[p_n + \iota D(t-i)]$ into (1.18)). At $c_b(t, i) = \upsilon(i)$, the entire age i install base is returned.

If Regime 2 applies, then at $c_b(t, i) = \upsilon(i)$, 100% of age i product owned by the innovator segment is returned and the total return volume is

$$A_{b2}(t, i) = N_{b1}(t-1, i-1) + \left(\frac{1-p_n}{\iota D_n(t-i)}\right) N_{b2}(t-1, i-1) \qquad (1.23)$$

(obtained by substituting $c_b(t, i) = \upsilon(i)$ into (1.18)). At $c_b(t, i) = \upsilon(i)[p_n + \iota D(t-i)]$, the entire age i install base is returned. Inverting (1.18), the buyback price under Regime $k \in \{1, 2\}$ is

$$c_b(t,i) = \begin{cases} a_b(t,i)s_b(t,i)+b_b(t,i), \ s_b(t,i)\in\left[0, A_{bk}(t,i)\right] \\ a_{bk}(t,i)s_b(t,i)+b_{bk}(t,i), \ s_b(t,i)\in\left[A_{bk}(t,i), N_b(t-1,i-1)\right] \end{cases} \tag{1.24}$$

where

$$a_b(t,i) = \frac{\upsilon(i)(1-p_n)\iota D_n(t-i)}{(1-p_n)N_{b2}(t-1,i-1)+\iota D_n(t-i)N_{b1}(t-1,i-1)}$$

$$b_b(t,i) = p_n\upsilon(i)$$

$$a_{b1}(t,i) = \frac{\upsilon(i)(1-p_n)}{N_{b1}(t-1,i-1)} \qquad b_{b1}(t,i) = \upsilon(i)\left(p_n - \frac{(1-p_n)N_{b2}(t-1,i-1)}{N_{b1}(t-1,i-1)}\right)$$

$$a_{b2}(t,i) = \frac{\upsilon(i)\iota D_n(t-i)}{N_{b2}(t-1,i-1)} \qquad b_{b2}(t,i) = \upsilon(i)\left(p_n - \frac{\iota D_n(t-i)N_{b1}(t-1,i-1)}{N_{b2}(t-1,i-1)}\right).$$

The value of $b_b(t, i)$ is the upper limit of the buyback price $c_b(t, i)$ for which there is no return volume (i.e., $c_b(t, i)$ must be above $b_b(t, i)$ in order for some customers to accept the buyback offer). The value of $a_b(t, i)$ is the increase in the buyback price required to generate an additional unit in return volume once the buyback price passes the threshold value $b_b(t, i)$. Depending on the regime, the slope of the buyback price function shifts to either $a_{b1}(t, i)$ or $a_{b2}(t, i)$ once all customers in a segment with age i product have accepted the buyback offer.

The objective of the firm is to maximize its total profit $\Pi_b(t)$, which is the sum of the profit from remanufactured products and the revenue earned from replacement purchases. The remanufactured product's selling price is p_r and revenue earned from the replacement purchase is m with a probability of γ (e.g., a fraction γ of loyal consumers will replace their existing product with a new product once they are finished using it). The profit due to buyback and remanufacturing in period t is

$$\Pi_b(t) = \sum_{i\in\Omega(t)} \left(p_r - c_b(t,i) - c_m(i) \right) s_b(t,i) . \tag{1.25}$$

The value of $p_r - c_b(t, i) - c_m(i)$ is the profit on each unit that is returned and sold as a remanufactured product. An alternative representation of (1.25) could include the term $m\gamma$, which represents the marginal contribution from the fraction of consumers who will replace their existing product with a new product once they are finished using it. We omit this term from (1.25) because our comparative analysis focus on the relative performance of buyback and trade-in transaction. In section 4.2, we present the profit due to trade-in and remanufacturing.

The myopic pricing problem treats each period independently. We assume that unmet demand in a period is not backordered. The optimal remanufactured product profit in period t is

$$\Pi_b^m(t) = \max_{s_b(t,i)\in\Omega_{s_b}(t,i), i\in\Omega(t)} \left\{ \Pi_b(t) \mid \sum_{i\in\Omega(t)} s_b(t,i) \le d_r(t) \right\} . \tag{1.26}$$

By substituting (1.24) into (1.25), we see that $\Pi_b(t)$ is a piecewise concave function that is separable in the decision vector $\mathbf{s_b}(t) = (s_b(t, 1), \ldots, s_b(t, t))$. The marginal profit associated with age i product under Regime $k \in \{1, 2\}$ is

$$\frac{\partial \Pi_b(t)}{\partial s_b(t,i)} = \begin{cases} p_r - c_m(i) - b_b(t,i) - 2a_b(t,i)s_b(t,i), \; s_b(t,i) \in \left[0, A_{bk}(t,i)\right) \\ p_r - c_m(i) - b_{bk}(t,i) - 2a_{bk}(t,i)s_b(t,i), \; s_b(t,i) \in \left(A_{bk}(t,i), N_b(t-1,i-1)\right] \end{cases} . \tag{1.27}$$

A5. Unsatisfied demand in a period results in a lost sale (i.e., no backorders).

The profit function is non-differentiable at $s_b(t, i) = A_{bk}(t, i)$. However, as we will show below, the marginal profit at $s_b(t, i) = A_{bk}(t, i)^-$ is not less than the marginal profit at

$s_b(t, i) = A_{bk}(t, i)^+$. We can use this fact to specify a simple greedy algorithm that solves (1.26).

Observe that

$$a_b(t, i) \le a_{bk}(t, i) \text{ for } k \in \{1, 2\} \text{ and } i \in \{1, \dots, t\}. \tag{1.28}$$

The profit contribution of age i product under Regime $k \in \{1, 2\}$, denoted $\Pi_b\left(t, i, s_b\left(t, i\right)\right)$, is piecewise continuous, and at $s_b(t, i) = A_{bk}(t, i)$ we have

$$\begin{aligned}\Pi_b\left(t, i, A_k\left(t, i\right)\right) &= A_{bk}\left(t, i\right)\left[p_r - c_m\left(i\right) - b_b\left(t, i\right) - a_b\left(t, i\right) A_{bk}\left(t, i\right)\right] \\ &= A_{bk}\left(t, i\right)\left[p_r - c_m\left(i\right) - b_{bk}\left(t, i\right) - a_{bk}\left(t, i\right) A_{bk}\left(t, i\right)\right]\end{aligned}$$

which implies

$$p_r - c_m\left(i\right) - b_b\left(t, i\right) - a_b\left(t, i\right) A_{bk}\left(t, i\right) = p_r - c_m\left(i\right) - b_{bk}\left(t, i\right) - a_{bk}\left(t, i\right) A_{bk}\left(t, i\right). \tag{1.29}$$

Expression (1.28) says that the slope of the buyback price function is lower when there are age i products owned by both segments than when one segment has returned all age i product (e.g., each additional unit of return volume requires a lower increase in buyback price when the install base includes both segments). As noted above, the profit expression is a piecewise continuous function comprised of two curves that intersect at return volume $s_b(t, i) = A_k(t, i)$, yielding (1.29). Thus, from (1.28) and (1.29), it follows that

$$p_r - c_m\left(i\right) - b_b\left(t, i\right) - 2a_b\left(t, i\right) A_{bk}\left(t, i\right) \ge p_r - c_m\left(i\right) - b_{bk}\left(t, i\right) - 2a_{bk}\left(t, i\right) A_{bk}\left(t, i\right) \tag{1.30}$$

An optimization algorithm for solving (1.26) begins by ranking the values of $\left.\frac{\partial \Pi_b\left(t\right)}{\partial s_b\left(t, i\right)}\right|_{s_b(t,i)=0}$ from largest-to-smallest for $i \in \Omega(t)$. Of course, the value of $\frac{\partial \Pi_b\left(t\right)}{\partial s_b\left(t, i\right)}$ changes as return volume is allocated to $s_b(t, i)$. The algorithm tracks marginal profit and allocates volume in a manner that maximizes the increase in profit per unit increase in return volume. Volume is allocated as long as marginal profit is positive and the constraint $\sum_{i\in\Omega(t)} s_b(t,i) \le d_r(t)$ is satisfied. The algorithm steps are described below (see the appendix for details on the implementation of Step 3). We suppress the parameter t in our description.

Optimal Algorithm for the Myopic Buyback Problem

1. Initialize the decision vector, $\mathbf{s_b} = (0, \ldots, 0)$
2. From among the viable ages (contained in Ω), identify the age(s) with the maximum marginal profit, say age J.
3. Add volume to $s_b(J)$ in an amount that is the minimum of 5 values: (1) quantity that results in marginal profit of age J to equal the second-highest marginal profit, (2) quantity at which $s_b(J) = A_{bk}(J)$, (3) quantity at which $s_b(J) = N_b(J-1)$, (4) quantity at which the marginal profit of age J is zero, (5) quantity at which total return volume is equal to demand d_r.
4. If the quantity added to $s_b(J)$ is equal to the value given in either (4) or (5), then exit.
5. If the quantity added to $s_b(J)$ is equal to the value given in (3), then remove J from the viable age set Ω.
6. Go to step 2.

3.4 Optimal Proactive Buyback Price Schedule

The previous section solves a myopic optimization problem. In this section we consider the problem of setting a buyback price schedule in the current period so as to maximize profit over the duration of the product life-cycle. In effect, this requires determining the buyback price schedule in each future period, though in practice, such future period price schedules would be finalized once the period is reached based on the sales history and demand projections available at that time.

Myopic buyback pricing has the advantage of being relatively simple. One objective of our analysis of myopic and proactive pricing problems is to identify conditions under which myopic buyback pricing is nearly optimal and far from optimal.

We let T denote the last period in the remanufactured product life-cycle. The particular value of T is a management decision that we assume to be exogenous to our problem, though period T would typically be in the decline stage of the life-cycle, i.e., $T > t^*$ where t^* denotes the period of peak sales for the remanufactured product. For example, in the continuous-time analog of our Bass diffusion model of demand, remanufactured product sales reaches its peak in period

$$t^* = \left(\frac{1}{\theta\left(1-p_n\right)+\left(1-\theta\right)\iota M}\right)\ln\left(\frac{\left(1-\theta\right)\iota M}{\theta\left(1-p_n\right)}\right)+\tau \ . \tag{1.31}$$

The first term in (1.31) is the period of peak sales for new product, which is advanced by τ periods to yield the period of peak demand for the remanufactured version of the product.

The problem is to maximize total discounted profit over the planning horizon of T periods. We present two math programming formulations for this problem. An advantage of the first formulation is that it conveys the problem in a relatively simple manner. A disadvantage is that the objective function is not smooth in the decision variables, which creates difficulties for some nonlinear optimization algorithms. The second formulation is more complex but yields a smooth objective function.

Math Programming Formulation 1

We require the following additional notation:

h = inventory holding cost per unit-period for returned product (incurred at the end of the period after returns are received)

r = net discount rate, e.g., cost of capital less inflation

$x(t)$ = sales of remanufactured product in period t

$I(t)$ = inventory of remanufactured product at the end of period t, with $I(0) = 0$

The problem is

$$\Pi_b^p = \max \sum_{t=1}^{T} (1-r)^t \, \Pi_b(t)$$

where

$$\Pi_b(t) = p_r x(t) - h\,I(t) - \sum_{i=1}^{t} \left(c_b(t,i) + c_m(i) \right) s_b(t,i) \tag{1.32}$$

subject to

$$N_{bj}(t,i) + s_{bj}(t,i) - N_{bj}(t-1,i-1) = 0,\ i \in \{1,\dots,t\},\ j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.33}$$

$$I(t) + x(t) - \sum_{i=1}^{t} s_b(t,i) - I(t-1) = 0,\ t \in \{1,\dots,T\} \tag{1.34}$$

$$x(t) \le d_r(t),\ t \in \{1,\dots,T\} \tag{1.35}$$

$$s_{b1}(t,i) - \min\left\{ \left(\frac{c_b(t,i) - p_n \upsilon(i)}{\upsilon(i)(1-p_n)} \right)^+ ,1 \right\} N_{b1}(t-1,i-1) = 0,\ i \in \{1,\dots,t\}, t \in \{1,\dots,T\} \tag{1.36}$$

$$s_{b2}(t,i) - \min\left\{ \left(\frac{c_b(t,i) - p_n \upsilon(i)}{\upsilon(i)\iota D_n(t-i)} \right)^+ ,1 \right\} N_{b2}(t-1,i-1) = 0,\ i \in \{1,\dots,t\}, t \in \{1,\dots,T\} \tag{1.37}$$

$$s_b(t,i), N_{bj}(t,i), x(t), I(t) \ge 0,\ i \in \{1,\dots,t\},\ j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.38}$$

Constraints defined in (1.33) are the flow-balance constraints for the install base of each age and segment over time. Constraints defined in (1.34) are flow-balance constraints for remanufactured product inventory. Constraints defined in (1.35) state that sales can be no more than demand. Constraints defined in (1.36) and (1.37) specify returns by segment given the buyback price $c_b(t, i)$, as determined by total return volume $s_b(t, i)$. Constraints defined in (1.38) enforce nonnegativity of the decision variables. The expressions for $c_b(t, i)$ depend on the regime, and are given below (see (1.24)):

If $p_n + \iota D_n(t-i) \leq 1$, then

$$c_b(t,i) = \begin{cases} \upsilon(i)\left(\dfrac{(1-p_n)\iota D_n(t-i)s_b(t,i)}{(1-p_n)N_{b2}(t-1,i-1)+\iota D_n(t-i)N_{b1}(t-1,i-1)}+p_n\right), \\ \qquad \text{if } s_b(t,i)\in\left[0,\left(\dfrac{\iota D_n(t-i)}{1-p_n}\right)N_{b1}(t-1,i-1)+N_{b2}(t-1,i-1)\right] \\ \upsilon(i)\left(\dfrac{(1-p_n)s_b(t,i)}{N_{b1}(t-1,i-1)}+p_n-\dfrac{(1-p_n)N_{b2}(t-1,i-1)}{N_{b1}(t-1,i-1)}\right), \\ \qquad \text{if } s_b(t,i)\in\left[\begin{array}{l}\left(\dfrac{\iota D_n(t-i)}{1-p_n}\right)N_{b1}(t-1,i-1)+N_{b2}(t-1,i-1), \\ N_{b1}(t-1,i-1)+N_{b2}(t-1,i-1)\end{array}\right] \end{cases}$$

if $p_n + \iota D_n(t-i) \geq 1$, then

$$c_b(t,i) = \begin{cases} \upsilon(i)\left(\dfrac{(1-p_n)\iota D_n(t-i)s_b(t,i)}{(1-p_n)N_{b2}(t-1,i-1)+\iota D_n(t-i)N_{b1}(t-1,i-1)}+p_n\right), \\ \qquad \text{if } s_b(t,i)\in\left[0,N_{b1}(t-1,i-1)+\left(\dfrac{1-p_n}{\iota D_n(t-1)}\right)N_{b2}(t-1,i-1)\right] \\ \upsilon(i)\left(\dfrac{\iota D_n(t-i)s_b(t,i)}{N_{b2}(t-1,i-1)}+p_n-\dfrac{\iota D_n(t-i)N_{b1}(t-1,i-1)}{N_{b2}(t-1,i-1)}\right), \\ \qquad \text{if } s_b(t,i)\in\left[\begin{array}{l}N_{b1}(t-1,i-1)+\left(\dfrac{1-p_n}{\iota D_n(t-i)}\right)N_{b2}(t-1,i-1), \\ N_{b1}(t-1,i-1)+N_{b2}(t-1,i-1)\end{array}\right] \end{cases}$$

The positioning of $c_m(i)$ in $\Pi_b(t)$, is based on the assumption that the firm remanufactures the unit in the period in which it is returned. We make this assumption in order to simplify the problem while still capturing the essence of the anticipatory buyback pricing problem. Without this assumption, we would need to track inventory by condition and we would need to assure that units in inventory are remanufactured in order of best-to-worst condition (i.e., from smallest-to-largest $c_m(i)$). This simplifying assumption has no impact when $r = 0$ (i.e., the cost to remanufacture x units in a period is the same as the cost to remanufacture the x units n periods later).

A6. The firm remanufactures the unit in the period in which it is returned.

Math Programming Formulation 2

For the second formulation, we use $y_{-}(t)$ to denote unsatisfied demand in period t and we use $y(t)$ to denote the difference between supply and demand in period t. Thus, the inventory at the end of period t is $y(t) + y_{-}(t)$, which is initialized at 0, i.e., $y(0) + y_{-}(0) = 0$. A number of other intermediate variables are computed in the constraints. The problem is

$$\Pi_b^p = \max \sum_{t=1}^{T} (1-r)^t \, \Pi_b(t) \tag{1.39}$$

where

$$\Pi_b(t) = p_r\left[d_r(t) - y_{-}(t)\right] - h\left[y(t) + y_{-}(t)\right] - \sum_{i=1}^{t}\left(c_b(t,i) + c_m(i)\right)\left[s_{b1}(t,i) + s_{b2}(t,i)\right]$$

subject to

$$z1(t,i) \le \frac{c_b(t,i) - p_n \upsilon(i)}{\upsilon(i)(1-p_n)}, \; i \in \{1,\ldots,t\}, t \in \{1,\ldots,T\} \tag{1.40}$$

$$z2\left(t,i\right)\leq\frac{c_b\left(t,i\right)-p_n\upsilon\left(i\right)}{\upsilon\left(i\right)\iota D_n\left(t-i\right)},\ i\in\left\{1,...,t\right\},t\in\left\{1,...,T\right\} \tag{1.41}$$

$$zj\left(t,i\right)\geq 0,\ i\in\left\{1,...,t\right\},\ j\in\left\{1,2\right\},t\in\left\{1,...,T\right\} \tag{1.42}$$

$$zj\left(t,i\right)\leq 1,\ i\in\left\{1,...,t\right\},\ j\in\left\{1,2\right\},t\in\left\{1,...,T\right\} \tag{1.43}$$

$$s_{bj}\left(t,i\right)=zj\left(t,i\right)N_{bj}\left(t-1,i-1\right),\ i\in\left\{1,...,t\right\},\ j\in\left\{1,2\right\},t\in\left\{1,...,T\right\} \tag{1.44}$$

$$N_{bj}\left(t,i\right)=N_{bj}\left(t-1,i-1\right)-s_{bj}\left(t,i\right),\ i\in\left\{2,...,t\right\},\ j\in\left\{1,2\right\},t\in\left\{1,...,T\right\} \tag{1.45}$$

$$y\left(1\right)=s_{b1}\left(1,1\right)+s_{b2}\left(1,1\right)-d_r\left(1\right) \tag{1.46}$$

$$y\left(t\right)=y\left(t-1\right)+y_-\left(t-1\right)+\sum_{i=1}^{t}\left(s_{b1}\left(t,i\right)+s_{b2}\left(t,i\right)\right)-d_r\left(t\right)\ ,t\in\left\{2,...,T\right\} \tag{1.47}$$

$$y_-\left(1\right)\geq d_r\left(1\right)-\left(s_{b1}\left(1,1\right)+s_{b2}\left(1,1\right)\right) \tag{1.48}$$

$$y_-\left(t\right)\geq d_r\left(t\right)-\sum_{i=1}^{t}\left(s_{b1}\left(t,i\right)+s_{b2}\left(t,i\right)\right)-y\left(t-1\right)-y_-\left(t-1\right)\ ,t\in\left\{2,...,T\right\} \tag{1.49}$$

$$y_-\left(t\right)\geq 0,\ t\in\left\{1,...,T\right\} \tag{1.50}$$

Constraints (1.40) – (1.43) ensure that the min{(·)+, 1} terms in (1.18) take on the proper values. Constraints (1.44) – (1.45) are implementations of (1.18) and (1.12). Constraints (1.46) and (1.47) define the difference between supply and demand at the end of periods 1 though *T*. Constraints (1.48) and (1.49) define unsatisfied demand in periods 1 through *T*; the inequality is tight when demand is more than supply because of the cost.

3.5 Numerical Illustrations

This section compares the performance of myopic and proactive solutions through a few numerical illustrations. Note that the cost to acquire and remanufacture a unit of age *i* from a segment 1 customer is in the following range:

$$\upsilon(i)p_n + c_m(i) < c_b(t, i) + c_m(i) \leq \upsilon(i) + c_m(i). \tag{1.51}$$

The range follows from the fact that the range of valuations at the time of purchase is in the interval $[p_n, 1]$ (see (1.4)). Thus, the buyback price for age i must be at least $\upsilon(i)p_n$ before a segment 1 customer will accept the offer, and all segment 1 customers will accept the offer at buyback price $c_b(t, i) = \upsilon(i)$. Similarly, the cost to acquire and remanufacture a unit of age i from a segment 2 customer is in the following range:

$$\upsilon(i)p_n + c_m(i) < c_b(t, i) + c_m(i) \leq \upsilon(i)[p_n + \iota D(t-1)] + c_m(i). \tag{1.52}$$

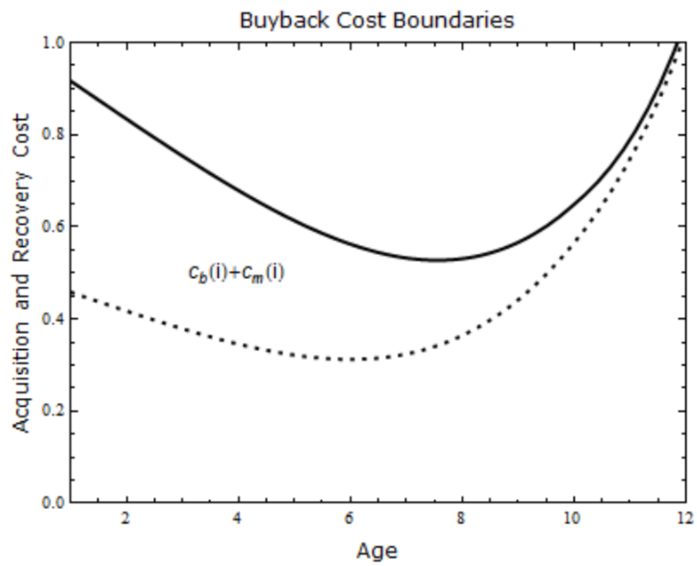

Figure 2.5. Upper and lower bounds on buyback acquisition and recovery cost

For given functions $\upsilon(i)$ and $c_m(i)$, we can examine how the lower and upper limits of the ranges vary as a function of i (see Figure 2.5). And for certain functional forms, there may exist a relatively narrow range of ages associated with lowest acquisition and remanufacturing cost, e.g., a sweet spot defining a band of ages that are most cost effective. Guide et al. (2003), for example, report ReCellular's cost and acquisition percentage data by condition (see Table 3 in Guide et al.). The data suggest a sweet spot around condition 3, and that the acquisition and remanufacture of very new and very old product is not cost effective. In settings with a pronounced sweet spot, say i^*, we find that the difference between i^* and the time lag τ is an indicator of the relative performance between a myopic solution and a proactive solution. In particular, when i^* is close to τ, then the profit of the myopic solution is close to the profit of the profit of the proactive solution. As these parameter values get farther apart, the use of a proactive solution method in place of the myopic algorithm is more likely to add value. This effect is illustrated in Table 2.1 below.

i^*	τ	Π_M/Π_P	$\%_M$	$\%_P$
6	0	0.877	41%	34%
6	6	0.973	83%	83%
6	10	0.820	33%	47%

Table 2.1. The ratio of optimal myopic to proactive buyback profits and the percent of remanufactured product demand satisfied by myopic and proactive buyback policies

The results in Table 2.1 correspond to a 12 year new product diffusion process. The diffusion parameters are based on sales data from Xerox DocuTech copiers sold from 1991 to 2000 where the market size, coefficient of innovation, and coefficient of imitation are M = 38,833 units, p =.015 and q =0.346 respectively (Van de Capelle, 2004). Given a value p_n =0.5, the values of θ =0.3 and ι =0.00001273 correspond with the values of p and q; however, we adjust these values and let θ = .15 and ι =0.00001329. These adjustments preserve the fundamental structure of the diffusion process, and yet allow us to generate a product lifecycle of approximately 12 years, ensuring that the maximum valuation for imitator type is not more than 1. We assume $\upsilon(i)$ is linear where $\upsilon(0)$ = 1, and $\upsilon(12)$ approaches 0.This assumption suggests a *residence index* = 1, which implies the typical consumer's usage cycle matches the product lifecycle. We assume $c_m(i)$ is convex and increasing such that at $c_m(12) = 1$ and we set its shape parameter equal to 4.[7] Finally we let $\gamma = 0.4$, $p_r = 0.4$ and $\alpha = 0.8$. Note, our assumptions regarding the functional form of $\upsilon(i)$ and $c_m(i)$, suggest a sweet-spot at age $i = 6$. Given these assumptions, we used the algorithm outlined in Section 3.3 to generate the optimal myopic profits. We used *Large Scale SQP* solver to search for optimal (near optimal) solutions to the math programming formulations developed in Section 3.4. *Large Scale SQP* uses genetic and evolutionary algorithms to find solutions to non-smooth optimization problems.

The results in Table 2.1 show that when the time lag matches the sweet spot age (i.e., $\tau = i^*$), there is little difference between optimal profits under the myopic and the proactive buyback pricing policies (i.e., 97.3%). This is because when $\tau = i^*$, the availability of used products that are both affordable and suitable for cost effective remanufacturing is high. Columns 4 and 5 of table 2.1 contain the percent of remanufactured demand satisfied by the myopic and proactive solutions respectively. The return volumes under the optimal myopic and proactive policies show that when the time lag matches the sweet spot age, both polices satisfy roughly 83% of the demand for remanufactured product. However, the proactive solution has a slight advantage when $\tau < i^*$ and when $\tau > i^*$, but for different reasons. When $\tau < i^*$, the proactive decision maker acquires fewer products in the early stages only to acquire them later at a lower price. The myopic decision maker will acquire end-of-use returns as long as marginal cost is less than marginal profits. When $\tau = 0$, the myopic decision maker satisfies 41%

[7]We assume the functional form for our condition dependent remanufacturing cost is $c_m(i) = (i/12)^4$, for $0 < i \leq 12$ $c_m(i) = 1$, otherwise.

of the remanufactured product demand while the proactive active decision maker only satisfies 34% of the remanufactured product demand, yet earns higher profits. When $\tau = 10 > i^*$, the opposite is true. The myopic decision maker is less able to accommodate the demand for the remanufactured product in later periods due to insufficient supply of products that are affordable to recover. On the other hand, the proactive decision maker accumulates an inventory of used products in earlier periods (e.g., where acquisition and recovery cost are low), in order to satisfy demand in later periods (e.g, where recovery cost are high). For instance, when $\tau = 10$, we see that in table 2.1 the optimal myopic policy satisfies 33% of the remanufactured product demand while the proactive policy satisfies 47% of total remanufactured product demand and earns larger profits.

4. Trade-in Policies for a Producer of New and Remanufactured Products

In this section, we consider the relative performance of myopic and proactive trade-in acquisition policies. A firm that sells both new and remanufactured product has the option to offer a trade-in price for a return.[8] A customer who accepts the trade-in offer receives a price discount on the purchase of the new product. In the next section, we will compare the profitability of buyback and trade-in policies for an OEM/remanufacturer.

The organization of this section parallels Section 3. In Section 4.1 we describe how return volumes are related to the trade-in credit. Section 4.2 examines the optimal myopic trade-in price schedule, Section 4.3 examines the optimal proactive trade-in price schedule, and Section 4.4 contains numerical analyses that compares and contrasts the two pricing policies.

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[8]The more common terms are trade-in *discount* or trade-in *credit*. We use *price* to be consistent with buyback terminology, which simplifies our wording later on when we discuss these policies together.

4.1 Remanufactured Product Supply

Recall from Section 3.1 that the buyback disutility for a customer with age i product and valuation V at the time of purchase is $V\upsilon(i)$. This means that a customer who is offered a cash amount of $c_b(t, i)$ satisfying $c_b(t, i) \geq V\upsilon(i)$ will return his or her product. A trade-in transaction, on the other hand, has strings attached. A customer who is offered a trade-in price of $c_t(t, i)$ must purchase a new product from the firm to receive the credit, e.g., the new product purchase price on a trade-in transaction is $p_n - c_t(t, i)$. We let $\phi \in [1, \infty)$ be a measure of the consumer's perceived cost of reduced flexibility associated with a trade-in transaction relative to a buyback transaction. More precisely, ϕ is the ratio of trade-in- to-buyback disutility, which we assume to be independent of product's age. Thus, at time t, a customer with product of age i will accept the trade-in offer and return the product if $c_t(t, i) \geq \phi V\upsilon(i)$.

A7. The ratio of trade-in- to-buyback disutility, ϕ, is independent of product age.

Recall from Section 3.1 that γ is the fraction of buyback customers with product of age i who prefer to replace their returned product with a new product from the firm. The parameters ϕ and γ are related. For example, if $\gamma = 1$, then a trade-in transaction offers no disadvantage relative to a buyback transaction, and $\phi = 1$; customers prefer to use the cash from a buyback transaction to purchase a new product from the firm, i.e.,

$$\gamma = 1 \Rightarrow \phi = 1 \tag{1.53}$$

In general, the values of ϕ and γ are inversely related; a small value of γ means that few customers prefer to replace their old product with a new product from the firm, which implies a high value ϕ, i.e., a high cost of the reduced flexibility associated with a trade-in transaction relative to a buyback transaction.

As one may expect (and as will be shown below), the buyback and trade-in profit expressions are equivalent when customers are very loyal to the firm (i.e., $\gamma = \phi = 1$). In essence, when $\gamma = 1$, customers perceive no difference between a buyback credit and a trade-in credit. Differences in the relative performance of buyback and trade-in programs arise in settings where $\gamma < 1$.

From the valuation probability distributions associated with customers who purchased the product (see (1.4) and (1.5)), it follows that

$$\phi V_1 \upsilon(i) \sim U[\phi p_n \upsilon(i),\ \phi\upsilon(i)] \tag{1.54}$$

$$\phi V_2 \upsilon(i) \sim U[\phi p_n \upsilon(i),\ \phi(p_n + \iota D_n(t-i))\upsilon(i)] \tag{1.55}$$

and the return probabilities in response to trade-in price function $c_t(t, i)$ offered at the end of period t are

$$P\left[\phi V_1 \upsilon(i) \le c_t(t,i)\right] = \min\left\{\left(\frac{c_t(t,i) - \phi p_n \upsilon(i)}{\phi\upsilon(i)(1-p_n)}\right)^+, 1\right\} \tag{1.56}$$

$$P\left[\phi V_2 \upsilon(i) \le c_t(t,i)\right] = \min\left\{\left(\frac{c_t(t,i) - \phi p_n \upsilon(i)}{\phi\upsilon(i)\iota D_n(t-i)}\right)^+, 1\right\} \tag{1.57}$$

for $i \in [1, t]$.

The number of products of age i in the segment j install base at the end of period t is $N_{tj}(t, i)$. The number of products of age i returned in period t from segment j is $s_{tj}(t, i)$. The expressions for $N_{tj}(t, i)$ are similar to the expressions for $N_{bj}(t, i)$. The only difference stems from the fact that each trade-in transaction is associated with the purchase of a new product from the firm, whereas only fraction γ of buybacks result in the purchase of a new product from the firm. Replacing γ with 1 in (1.11) yields the trade-in customer sales to segment j at the beginning of period t, i.e.,

$$d_{tj}(t) = \sum_{k=1}^{t-1} s_{tj}\left(t-1,k\right). \tag{1.58}$$

for $t \geq 1$. For the computation of $d_{tj}(0, 0)$, we define $s_{tj}(0, 0) = 0$.

Recall that $d_n(t)$ denotes sales to new customers in period t. Following Section 3.1, we define

$$N_{t1}(t, 0) = \theta\left(t+1\right)d_n\left(t+1\right) + d_{t1}\left(t+1\right) = \theta\left(t+1\right)d_n\left(t+1\right) + \sum_{k=1}^{t} s_{t1}\left(t,k\right)$$

$$N_{t2}(t, 0) = \left[1-\theta\left(t+1\right)\right]d_n\left(t+1\right) + d_{t2}\left(t+1\right) = \left[1-\theta\left(t+1\right)\right]d_n\left(t+1\right) + \sum_{k=1}^{t} s_{t2}\left(t,k\right)$$

for $t \geq 0$, and the general expression for the install base is

$$N_{tj}(t, i) = N_{tj}(t-1, i-1) - s_{tj}(t, i) \text{ for } j \in \{1, 2\} \text{ and } i \in [1, t],\ t \geq 1. \tag{1.59}$$

The volume of age i product that is returned by each segment at the end of period t is given by

$$s_{tj}(t, i) = P\left[\phi V_j \upsilon\left(i\right) \leq c_b\left(t,i\right)\right] N_{tj}\left(t-1, i-1\right) \text{ for } j \in \{1, 2\},\ i \in [1, t],\ t \geq 1. \tag{1.60}$$

4.2 The Optimal Myopic Trade-in Price Schedule

The total return volume in period t of product of age i is

$$s_t(t, i) = s_{t1}(t, i) + s_{t2}(t, i) \tag{1.61}$$

or in expanded form (see (1.60)),

$$s_t(t, i) = \begin{array}{l} \min\left\{\left(\dfrac{c_t(t,i)-p_n\phi\upsilon(i)}{\phi\upsilon(i)(1-p_n)}\right)^+, 1\right\} N_{t1}(t-1,i-1) + \\ \min\left\{\left(\dfrac{c_t(t,i)-p_n\phi\upsilon(i)}{\phi\upsilon(i)\iota D_n(t-i)}\right)^+, 1\right\} N_{t2}(t-1,i-1) \end{array}. \tag{1.62}$$

We denote the total number of units of age i in the install base just prior to returns at the end of period t as $N_t(t, i)$, i.e.,

$$N_t(t, i) = N_{t1}(t, i) + N_{t2}(t, i). \tag{1.63}$$

Recall from Section 3.3 that $\Omega(t)$ is the set of ages at the end of period t for which returns are viable, i.e., $\Omega(t) = \{i : N_t(t, i) > 0\}$. The return volume function, $s_t(t, i)$, is constant for any $c_t(t, i) \geq \max\{\phi\upsilon(i), \phi\upsilon(i)[p_n + \iota D(t-i)]\}$, e.g., 100% of the age i product owned by the innovator segment is returned if $c_t(t, i) \geq \phi\upsilon(i)$, and 100% of the age i product owned by the imitator segment is returned if $c_t(t, i) \geq \phi\upsilon(i)[p_n + \iota D(t-i)]$. For any age $i \in \Omega(t)$, we see from (1.62) that $s_t(t, i)$ is strictly increasing over the range of viable trade-in prices

$$c_t(t, i) \in \Omega_{c_t}(t,i) \equiv \left[p_n\phi\upsilon(i), \phi\upsilon(i)\max\{1, p_n + \iota D_n(t-i)\}\right],\ i \in \Omega(t). \tag{1.64}$$

Thus, we can invert (1.62) to express trade-in price as a function of volume over the corresponding range of viable volumes, denoted $\Omega_{s_t}(t,i)$ for product of age i, i.e.,

$$\Omega_{s_t}(t,i) \equiv \left[0, N_t(t-1,i-1)\right]. \tag{1.65}$$

If $1 \geq p_n + \iota D(t-i)$, we say that the age i cost structure conforms to Regime 1; otherwise the age i cost structure conforms to Regime 2. If Regime 1 applies, then at $c_t(t, i) = \phi\upsilon(i)[p_n + \iota D(t-i)]$, 100% of age i product owned by the imitator segment is returned and the total return volume is

$$A_{t1}(t,\,i) = \left(\frac{\iota D_n\left(t-i\right)}{1-p_n}\right) N_{t1}\left(t-1, i-1\right) + N_{t2}\left(t-1, i-1\right) \tag{1.66}$$

(obtained by substituting $c_t(t, i) = \phi\upsilon(i)[p_n + \iota D(t-i)]$ into (1.62); see (1.22)). At $c_t(t, i) = \phi\upsilon(i)$, the entire age i install base is returned.

If Regime 2 applies, then at $c_t(t, i) = \phi\upsilon(i)$, 100% of age i product owned by the innovator segment is returned and the total return volume is

$$A_{t2}(t,\,i) = N_{t1}\left(t-1, i-1\right) + \left(\frac{1-p_n}{\iota D_n\left(t-i\right)}\right) N_{t2}\left(t-1, i-1\right) \tag{1.67}$$

(obtained by substituting $c_t(t, i) = \phi\upsilon(i)$ into (1.62); see (1.23)). At $c_t(t, i) = \phi\upsilon(i)[p_n + \iota D(t-i)]$, the entire age i install base is returned. Inverting (1.62), the trade-in price under Regime $k \in \{1, 2\}$ is

$$c_t(t,\,i) = \begin{cases} a_t\left(t,i\right)s_t\left(t,i\right) + b_t\left(t,i\right),\ s_t\left(t,i\right) \in \left[0, A_{tk}\left(t,i\right)\right] \\ a_{tk}\left(t,i\right)s_t\left(t,i\right) + b_{tk}\left(t,i\right),\ s_t\left(t,i\right) \in \left[A_{tk}\left(t,i\right), N_t\left(t-1,i-1\right)\right] \end{cases} \tag{1.68}$$

where

$$a_t(t,\,i) = \frac{\phi\upsilon\left(i\right)\left(1-p_n\right)\iota D_n\left(t-i\right)}{\left(1-p_n\right)N_{t2}\left(t-1,i-1\right) + \iota D_n\left(t-i\right)N_{t1}\left(t-1,i-1\right)}$$

$$b_t(t,\,i) = p_n\phi\upsilon\left(i\right)$$

$$a_{t1}(t,\,i) = \frac{\phi\upsilon\left(i\right)\left(1-p_n\right)}{N_{t1}\left(t-1,i-1\right)} \qquad b_{t1}(t,\,i) = \phi\upsilon\left(i\right)\left(p_n - \frac{\left(1-p_n\right)N_{t2}\left(t-1,i-1\right)}{N_{t1}\left(t-1,i-1\right)}\right)$$

$$a_{t2}(t,\,i) = \frac{\phi\upsilon\left(i\right)\iota D_n\left(t-i\right)}{N_{t2}\left(t-1,i-1\right)} \qquad b_{t2}(t,\,i) = \phi\upsilon\left(i\right)\left(p_n - \frac{\iota D_n\left(t-i\right)N_{t1}\left(t-1,i-1\right)}{N_{t2}\left(t-1,i-1\right)}\right).$$

The objective of the firm is to maximize its total profit $\Pi_t(t)$, which is the sum of the profit from remanufactured products and the revenue earned from the fraction of buyback consumers who purchasing a new product. The remanufactured product selling price is p_r and m is the marginal contribution from new products to replacement consumers. Recall that γ is the repeat purchase rate (e.g., an average of fraction γ of customers replace their old product with a new product from the firm). A trade-in transaction induces a customer to replace their returned product with a new product from the firm, which generates incremental contribution $(1 - \gamma)m$, relative to the case of buy-back program. Accordingly, the profit in period t is

$$\Pi_t(t) = \sum_{i\in\Omega(t)} \left(p_r + (1-\gamma)m - c_t(t,i) - c_m(i)\right) s_t(t,i). \tag{1.69}$$

The optimal remanufactured product profit in period t is

$$\Pi_t^m(t) = \max_{s_t(t,i)\in\Omega_{s_t}(t,i), i\in\Omega(t)} \left\{\Pi_t(t) \,|\, \sum_{i\in\Omega(t)} s_t(t,i) \le d_r(t)\right\}. \tag{1.70}$$

$\Pi_t(t)$ is a piecewise concave function that is separable in the decision vector $\mathbf{s_t}(t)$ = $(s_t(t, 1), \ldots, s_t(t, t))$. The marginal profit associated with age i product under Regime $k \in \{1, 2\}$ is

$$\frac{\partial \Pi_t(t)}{\partial s_t(t,i)} = \begin{cases} p_r + (1-\gamma)m - c_m(i) - b_t(t,i) - 2a_t(t,i)s_t(t,i), & s_t(t,i) \in \left[0, A_{tk}(t,i)\right) \\ p_r + (1-\gamma)m - c_m(i) - b_{tk}(t,i) - 2a_{tk}(t,i)s_t(t,i), & s_t(t,i) \in \left(\begin{matrix} A_{tk}(t,i), \\ N_t(t-1, i-1) \end{matrix}\right] \end{cases}. \tag{1.71}$$

The problem structure follows the structure of the myopic buyback problem. The steps of an algorithm to solve problem (1.70) steps are described below. We suppress the parameter t in our description.

Optimal Algorithm for the Myopic Trade-in Problem

1. Initialize the decision vector, $\mathbf{s_t} = (0, \ldots, 0)$
2. From among the viable ages (contained in Ω), identify the age(s) with the maximum marginal profit, say age J.
3. Add volume to $s_t(J)$ in an amount that is the minimum of 5 values: (1) quantity that results in marginal profit of age J to equal the second-highest marginal profit, (2) quantity at which $s_t(J) = A_{tk}(J)$, (3) quantity at which $s_t(J) = N_t(J-1)$, (4) quantity at which the marginal profit of age J is zero, (5) quantity at which total return volume is equal to demand d_r.
4. If the quantity added to $s_t(J)$ is equal to the value given in either (4) or (5), then exit.
5. If the quantity added to $s_t(J)$ is equal to the value given in (3), then remove J from the viable age set Ω.
6. Go to step 2.

4.3 Optimal Proactive Trade-in Price Schedule

The problem is to maximize total discounted profit over the planning horizon of T periods. As in Section 3.4, we present two math programming formulations for this problem—one that is simple but with a non-smooth objective function and another that is complex but with a smooth objective function.

Math Programming Formulation 1

The problem is

$$\Pi_t^p = \max \sum_{t=1}^{T} (1-r)^t \, \Pi_t(t)$$

where

$$\Pi_t(t) = p_r x(t) - h\,I(t) - \sum_{i=1}^{t}\left(c_t(t,i) + c_m(i) - (1-\gamma)m\right)s_t(t,i) \tag{1.72}$$

subject to

$$N_{tj}(t,i) + s_{tj}(t,i) - N_{tj}(t-1,i-1) = 0,\ i \in \{1,\dots,t\},\ j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.73}$$

$$I(t) + x(t) - \sum_{i=1}^{t} s_t(t,i) - I(t-1) = 0,\ t \in \{1,\dots,T\} \tag{1.74}$$

$$x(t) \le d_r(t),\ t \in \{1,\dots,T\} \tag{1.75}$$

$$s_{t1}(t,i) - \min\left\{\left(\frac{c_t(t,i) - p_n\phi\upsilon(i)}{\phi\upsilon(i)(1-p_n)}\right)^+, 1\right\} N_{t1}(t-1,i-1) = 0,\ i \in \{1,\dots,t\}, t \in \{1,\dots,T\} \tag{1.76}$$

$$s_{t2}(t,i) - \min\left\{\left(\frac{c_t(t,i) - p_n\phi\upsilon(i)}{\phi\upsilon(i)\iota D_n(t-i)}\right)^+, 1\right\} N_{t2}(t-1,i-1) = 0,\ i \in \{1,\dots,t\}, t \in \{1,\dots,T\} \tag{1.77}$$

$$s_t(t,i), N_{tj}(t,i), x(t), I(t) \ge 0,\ i \in \{1,\dots,t\},\ j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.78}$$

Constraints defined in (1.73) are the flow-balance constraints for the install base of each age and segment over time. Constraints defined in (1.74) are flow-balance constraints for remanufactured product inventory. Constraints defined in (1.75) state that sales can be no more than demand. Constraints defined in (1.76) and (1.77) specify returns by segment given the trade-in price $c_t(t, i)$ as determined by total return volume $s_t(t, i)$. Constraints defined in (1.78) enforce nonnegativity of the decision variables. The expressions for $c_t(t, i)$ depend on the regime, and are given below (see (1.68)):

If $p_n + \iota D_n(t - i) \le 1$, then

$$c_t(t,i)=\begin{cases}\phi\upsilon(i)\left(\dfrac{(1-p_n)\iota D_n(t-i)s_t(t,i)}{(1-p_n)N_{t2}(t-1,i-1)+\iota D_n(t-i)N_{t1}(t-1,i-1)}+p_n\right), \\ \qquad \text{if } s_t(t,i)\in\left[0,\left(\dfrac{\iota D_n(t-i)}{1-p_n}\right)N_{t1}(t-1,i-1)+N_{t2}(t-1,i-1)\right] \\ \phi\upsilon(i)\left(\dfrac{(1-p_n)s_t(t,i)}{N_{t1}(t-1,i-1)}+p_n-\dfrac{(1-p_n)N_{t2}(t-1,i-1)}{N_{t1}(t-1,i-1)}\right), \\ \qquad \text{if } s_t(t,i)\in\left[\left(\dfrac{\iota D_n(t-i)}{1-p_n}\right)N_{t1}(t-1,i-1)+N_{t2}(t-1,i-1),\right. \\ \qquad\qquad \left. N_{t1}(t-1,i-1)+N_{t2}(t-1,i-1)\right]\end{cases}$$

if $p_n + \iota D_n(t-i) \geq 1$, then

$$c_t(t,i)=\begin{cases}\phi\upsilon(i)\left(\dfrac{(1-p_n)\iota D_n(t-i)s_t(t,i)}{(1-p_n)N_{t2}(t-1,i-1)+\iota D_n(t-i)N_{t1}(t-1,i-1)}+p_n\right), \\ \qquad \text{if } s_t(t,i)\in\left[0,N_{t1}(t-1,i-1)+\left(\dfrac{1-p_n}{\iota D_n(t-1)}\right)N_{t2}(t-1,i-1)\right] \\ \phi\upsilon(i)\left(\dfrac{\iota D_n(t-i)s_t(t,i)}{N_{t2}(t-1,i-1)}+p_n-\dfrac{\iota D_n(t-i)N_{b1}(t-1,i-1)}{N_{t2}(t-1,i-1)}\right), \\ \qquad \text{if } s_t(t,i)\in\left[N_{t1}(t-1,i-1)+\left(\dfrac{1-p_n}{\iota D_n(t-1)}\right)N_{t2}(t-1,i-1),\right. \\ \qquad\qquad \left. N_{t1}(t-1,i-1)+N_{t2}(t-1,i-1)\right]\end{cases}$$

Math Programming Formulation 2

For the second formulation, we use $y_-(t)$ to denote unsatisfied demand in period t and we use $y(t)$ to denote the difference between supply and demand in period t. Thus, the inventory at the end of period t is $y(t) + y_-(t)$, which is initialized at 0, i.e., $y(0) + y_-(0) = 0$. A number of other intermediate variables are computed in the constraints. The problem is

$$\Pi_t^p = \max\sum_{t=1}^{T}(1-r)^t\,\Pi_t(t) \tag{1.79}$$

where

$$\Pi_t(t) = p_r\left[d_r(t) - y_-(t)\right] - h\left[y(t) + y_-(t)\right] - \sum_{i=1}^{t}\left[c_t(t,i) + c_m(i) - (1-\gamma)m\right]\left[s_{t1}(t,i) + s_{t2}(t,i)\right]$$

subject to

$$z1(t,i) \le \frac{c_t(t,i) - p_n\phi\upsilon(i)}{\phi\upsilon(i)(1-p_n)},\ i \in \{1,\dots,t\}, t \in \{1,\dots,T\} \tag{1.80}$$

$$z2(t,i) \le \frac{c_t(t,i) - p_n\phi\upsilon(i)}{\phi\upsilon(i)\iota D_n(t-i)},\ i \in \{1,\dots,t\}, t \in \{1,\dots,T\} \tag{1.81}$$

$$zj(t,i) \ge 0,\ i \in \{1,\dots,t\}, j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.82}$$

$$zj(t,i) \le 1,\ i \in \{1,\dots,t\}, j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.83}$$

$$s_{tj}(t,i) = zj(t,i)N_{tj}(t-1,i-1),\ i \in \{1,\dots,t\}, j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.84}$$

$$N_{tj}(t,i) = N_{tj}(t-1,i-1) - s_{tj}(t,i),\ i \in \{2,\dots,t\}, j \in \{1,2\}, t \in \{1,\dots,T\} \tag{1.85}$$

$$y(1) = s_{t1}(1,1) + s_{t2}(1,1) - d_r(1) \tag{1.86}$$

$$y(t) = y(t-1) + y_-(t-1) + \sum_{i=1}^{t}\left(s_{t1}(t,i) + s_{t2}(t,i)\right) - d_r(t)\ , t \in \{2,\dots,T\} \tag{1.87}$$

$$y_-(1) \ge d_r(1) - \left(s_{t1}(1,1) + s_{t2}(1,1)\right) \tag{1.88}$$

$$y_-(t) \ge d_r(t) - \sum_{i=1}^{t}\left(s_{t1}(t,i) + s_{t2}(t,i)\right) - y(t-1) - y_-(t-1)\ , t \in \{2,\dots,T\} \tag{1.89}$$

$$y_-(t) \ge 0,\ t \in \{1,\dots,T\} \tag{1.90}$$

Constraints (1.80) – (1.83) ensure that the min{(·)+, 1} terms in (1.62) take on the proper values. Constraints (1.84) – (1.85) are implementations of (1.62) and (1.59). Constraints (1.86) and (1.87) define the difference between supply and demand at the end of periods 1 though *T*. Constraints (1.88) and (1.89) define unsatisfied demand in periods 1 through *T*; the inequality is tight when demand is more than supply because of the cost.

4.4 Numerical Illustrations

This section compares the performance of myopic and proactive solutions through a few numerical illustrations. The cost to acquire and remanufacture a unit of age i from a segment 1 customer is in the following range:

$$\phi \upsilon(i) p_n + c_m(i) < c_t(t, i) + c_m(i) \leq \phi \upsilon(i) + c_m(i). \tag{1.91}$$

Similarly, the cost to acquire and remanufacture a unit of age i from a segment 2 customer is in the following range:

$$\phi \upsilon(i) p_n + c_m(i) < c_t(t, i) + c_m(i) \leq \phi \upsilon(i)[p_n + \iota D_n(t-1)] + c_m(i). \tag{1.92}$$

The structure of (1.91) and (1.92) follows the structure observed in (1.51) and (1.52) under analysis of a buyback policy (i.e., the only difference is the inclusion of factor ϕ). Figure 2.6 illustrates the upper and lower bounds on the trade-in credits and product recovery cost.

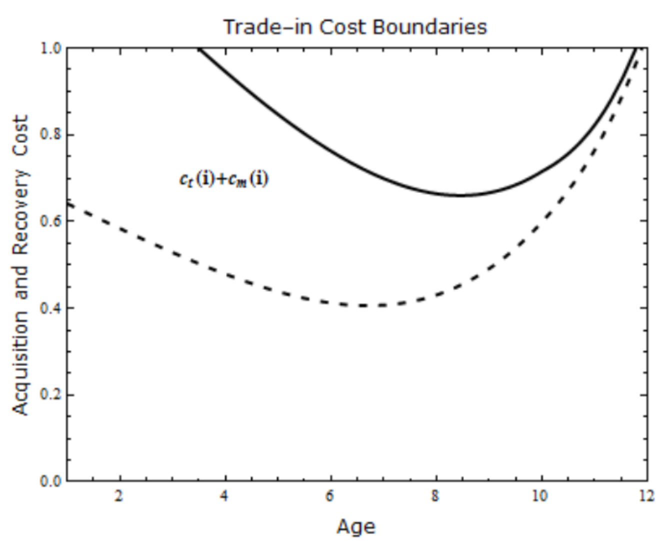

Figure 2.6. Upper and lower bounds on trade-in acquisition and recovery cost

Table 2.2 show how the differences between i^* and the time lag τ affects the relative performance of myopic and proactive solution methods.

i^*	τ	Π_M/Π_P	$\%_M$	$\%_P$
6	0	0.860	44%	34%
6	6	0.960	100%	100%
6	10	0.705	58%	69%

Table 2.2. The ratio of optimal myopic to proactive trade-in profits and the percent of remanufactured product demand satisfied by myopic and proactive trade-in policies

The results in Table 2.2 correspond to the same parameter values and diffusion process used to generate Table 2.1. The only difference is that we use the myopic algorithm outlined in Section 4.3 to generate the optimal myopic profits and we use the math programming formulation developed in Section 4.4 to facilitate search for the optimal proactive solutions. Recall, the trade-in problem requires an extra parameter, ϕ, which reflects the disutility associated with getting a store-credit versus cash. We set ϕ = 1.4. The main conclusions are the same as in Section 3.5. When τ matches the sweet spot age i^*, there is little difference between optimal, myopic trade-in profits and optimal or near optimal proactive trade-in profits. The proactive solution has a slight advantage when $\tau < i^*$ and when $\tau > i^*$. When $\tau < i^*$ for example, the proactive decision maker acquires fewer end-of-use products, and satisfies less demand for the remanufactured product. In the earlier stages of the product lifecycle, demand for the remanufactured product is low and used products are costly to acquire (i.e., the residual valuations are high). As the new product matures, demand for the remanufactured product begins to pick up. Thus, when $\tau < i$, The proactive decision maker evaluates the trade-off between When $\tau < i^*$, the proactive decision maker acquires fewer products in the earlier stages, only to acquirer them later at a lower prices. When $\tau = 10 > i^*$, the opposite is true; that is, the proactive decision maker satisfies a larger portion of the remanufactured product demand. Note that when $\tau = 10$, the proactive trade-in policy is considerably more profitable than the myopic counterpart. The proactive decision maker launches take-back and recovery activities well before observing demand for the remanufactured product.

5. Comparison of Buyback and Trade-in Policies

Key factors that influence the relative performance of a buyback and trade-in programs are the repeat purchase rate γ and the ratio of trade-in-to-buyback disutility ϕ. As discussed earlier, the value of ϕ reflects in relative increase in trade-in disutility due to the strings attached, i.e., the customer must use the credit toward the purchase of a new product from the firm. An advantage of the trade-in program is that it captures the margin on a new product sale from faction $1 - \gamma$ of customers who would not have repurchased from the firm under a buyback program. An advantage of a buyback program is that the firm can get a higher return rate for a given credit because there are no restrictions on the use of the credit.

In settings where the firm offers a wide array of products that qualify for the trade-in credit, a customer may perceive the trade-in disutility to be very close to the buyback disutility (e.g., $\phi \approx 1$) even though some customers would not purchase from the firm when receiving a buyback credit (e.g., $\gamma < 1$). If this is the case, then a trade-in program will likely be more profitable than a buyback program. This notion is formalized in the following proposition.

Proposition 1.*If $\phi = 1$ and $\gamma < 1$, then the profit from an optimal trade-in program is the same or higher than the optimal profit from a buyback program. Furthermore, if a buyback program is profitable, then a trade-in program will be more profitable.*

The effects of changing values of ϕ and γ on profits under buyback and trade-in policies are illustrated in Table 2.3 and Table 2.4. To generate the results, we used the same data that was used to generate tables 2.1 and 2.2, but with the time lag τ set to match the sweet spot, i.e., $\tau = i^* = 6$. We used the myopic algorithm to generate the profits.

Π_B/Π_T		ϕ				
		1.0	1.2	1.4	1.6	1.8
γ	1.0	1.0	6.620	∞	∞	∞
	0.8	0.337	0.638	1.83	14.2	∞
	0.6	0.195	0.273	0.441	0.899	2.448
	0.4	0.134	0.168	0.227	0.321	0.537
	0.2	0.099	0.012	0.143	0.181	0.242

Table 2.3. Ratios of buyback profit to trade-in profit.

$\%_B$ / $\%_T$		ϕ = 1.0	1.2	1.4	1.6	1.8
γ	1.0	90.3 90.3	90.3 37.5	90.3 0.0	90.3 0.0	90.3 0.0
	0.8	88.9 100.0	88.9 96.5	88.9 66.8	88.9 23.3	88.9 0.0
	0.6	86.5 100.0	86.5 100.0	86.5 100.0	86.5 85.7	86.5 54.9
	0.4	83.8 100.0	83.8 100.0	83.8 100.0	83.8 100.0	83.8 93.3
	0.2	79.7 100.0	79.7 100.0	79.7 100.0	79.7 100.0	79.7 100.0

Table 2.4. Percent of remanufactured product demand satisfied under buyback and trade-in product acquisition policies.

Column one in Table 2.3 highlights the lessons from Proposition 1. Since $\phi = 1$, consumers are indifferent between buyback cash and a trade-in credit. The cost structure of the two policies is the same and the only difference between the two objectives reduces to an extra-margin earned from the replacement purchase. Table 2.3 also shows that in most practical settings (i.e., $1 < \phi < 1.4$ and $0 < \gamma < 0.6$), trade-in programs are more profitable than buyback programs.

When repeat purchase rates are either low or high, adopting a trade-in program to facilitate the collection of end-of-use products is either very attractive or very unattractive. For instance, when repeat purchase rates are low, trade-in programs are clearly the most attractive product acquisition policy to adopt. Notice from Table 2.3 where $\phi = 1$ and $\gamma = 0.2$. In this setting, repeat purchase rates are low and the owners' disutility for the trade-in transactions is low. Both policies are profitable in this region, however, when the firm adopts a trade-in policy that allows the owner to apply the trade-in credit to an acceptable range of alternatives, the profits from doing so are exceptionally high. This is because when γ is low, the firm benefits the most from replacement purchase transactions. Since ϕ is low, the consumer has no qualms applying the trade-in credit and since γ is low the firm preserves a greater share of revenues from the replacements. The income effect from higher revenues earned on replacement purchase transactions allows the firm to increase the trade-in credit and thus acquire enough end-of-use products to completely satisfy demand for the remanufactured product. Table 2.4 show that at $\phi = 1$ and $\gamma = 0.2$, the optimal acquisition quantity under the trade-in policy satisfies 100% of the demand for the remanufactured product while the optimal acquisition quantity under the buyback policy only satisfies roughly 80% of demand for remanufactured products.

When repeat purchase rates are high, on the other hand, it is likely to be much more profitable to adopt a buyback product acquisition policy instead of a trade-in policy. Since $\gamma = 1 \Rightarrow \phi = 1$ (see (1.53)), we already know the buyback and trade-in profit expressions are equivalent when customers are very loyal to the firm (i.e., $\gamma = \phi = 1$); thus when this is the case neither policy has an advantage. However, note from Table 2.3 that when consumers perceive just a small difference between the buyback credit and the trade-in credit, the buyback to trade-in profit ratio increases significantly. We see in Table 2.3 that the relative value of adopting the buyback program is increasing in ϕ. Thus as the range of products that consumers are allowed to apply the trade-in credit to declines, the more attractive it becomes for the firm to acquire end-of-use products by offering a buyback policy instead of a trade-in policy. We see that in settings reflecting the upper right corner of Table 2.3, the buyback policy is strictly preferred. This is because, when both γ and ϕ are high, the cost of offering loyal consumers a trade-in credit is so high that remanufacturing is no longer attractive. For instance, Table 2.4 shows that when γ=0.8 and ϕ = 1.2 (i.e, profits from trade-ins are relatively more attractive), the firm acquires enough returns to satisfy 96.5% of the remanufactured product demand. When ϕ = 1.4, the firm satisfies 66.8% of the demand for the remanufactured product. When the value of ϕ reaches 1.6 the firm only acquires enough units to satisfies 23% of the demand for the remanufactured product. Finally, when consumer loyalty is considerably high, and the value of ϕ reaches 1.8, the optimal acquisition quantity under the trade-in policy is zero.

6. Summary and Conclusions

This study was motivated by extensive discussions with supply chain managers who oversee some aspect of product take-back and recovery activities for a large computer company. These companies use trade-in and buyback programs as a means to acquire a supply used products in order to meet a demand for remanufactured products. There is a genuine interest in understanding the marketing and operational merits of various product acquisition policies. Indeed, we respond to these managerial concerns by studying two buyback programs and two trade-in programs. Our analysis focused on the time dimensions of the pricing problem and our models incorporate key elements of consumer willingness-to-return and replacement purchase behavior. The results of our analysis are summarized below.

Product lifecycle dynamics influence the profitability of remanufacturing activities. The diffusion of new product sales, the consumers' depreciating valuations of the owned product, and age dependent remanufacturing cost characterizes conditions shaping product acquisition and recovery cost. We find that there exists a "sweet spot" age where acquisition and recovery cost are most economical. The sweet spot age is independent of the product lifecycle, however, the number of used products in the install base that are at the sweet spot age is affected by to product lifecycle dynamics. We find that any optimal acquisition policy will target used products that are at the sweet spot age. This is an important result because trade-in and buyback programs will often specify the model and model year of products that qualify for the take back deal. Our findings regarding the sweet spot age offers guidance to managers.

We study the relative performance between myopic and proactive acquisition policies. A common approach to setting buyback prices and trade-in credits draws on pricing algorithms which set prices to maximize profits in a single period. Such myopic pricing strategies are limiting because they ignore opportunities to exploit the dynamic condition of used product markets. We find that the relative performance between myopic and proactive product acquisition policies is tied to the time lag between new product sales and the demand for remanufactured products. Specifically, when this time lag matches the sweet-spot age, myopic pricing policies perform nearly as well as proactive policies; otherwise proactive pricing strategies will generally lead to higher profits. This finding is important considering that various remanufacturing settings are characterized by this time lag. Our findings show that myopic pricing policies are most appropriate when remanufacturing activities are targeted to satisfy demand for used products in secondhand markets; otherwise, proactive acquisition policies are likely to be more profitable.

We study the relative performance of buyback and trade-in acquisition policies. Our analysis accounts for replacement purchase rates and consumer preferences for buyback cash and trade-in credits. Our main conclusion is that in most practical settings, trade-in policies are likely to yield more profits than buyback policies. We find that the difference between the performance of trade-in and buyback programs are most pronounced when repeat purchase rates are high and when the range of products that the trade-in credits can be applied to are less appealing.

Regarding the generality and the applicability of our results, we assumed the residual valuation function for both adopter segments is the same. The residual valuation function reflects how a consumer's perception of the value of the owned good changes over time. This function is a proxy for voluntary replacement decision which could depend on several factors, such as the owner's usage rate (Raymond et. al. 1993), desires for newness, and situational factors (Jacoby et. al. 1977). This suggests that owners are, to some degree, heterogeneous with respect to their residual valuation of the used product. A model that captures heterogeneity in the residual valuation could be of potential use. We also assumed the replacement fraction is constant, where in practice the replacement fraction may depend on the age of the product. The math programming formulations can easily be extended to examine how age-dependent replacement rates influence policy preferences.

In addition to relaxing key assumptions of the model, we conclude with several suggestions for future research. We consider the company that offers either a trade-in program or a buyback program. Some companies allow the consumer to choose between a trade-in credit and buyback cash (i.e., BestBuy). The trade-in credit is generally larger than the corresponding buyback rebate. An interesting extension to this work would be a model that examines the optimal trade-in credit and buyback price when both programs are offered. Second, while our analysis focused on age-dependent policies, some companies will offer a flat credit regardless of the age or condition of the returned product. For example, in 2010 Snap On Equipment offered a fixed fee trade-in credit with only one condition, the dealer returns a wheel balancing systems – regardless of brand and working condition. A dynamic study of a family of fixed fee trade-in or buyback credits could provide new and interesting insights. Another contribution would be a rigorous examination of trade-in policies that allow consumers to return competitor equipment. A model that accounts for uncertain resale value could also provide interesting insights about the effect of uncertainty on product acquisition policy choice.

Another important issue to consider is that of competition. The number of companies offering trade-in and buyback programs are growing. A rigorous analysis of how of competition effects policy choice could add valuable insight to the growing body of literature related to competition in product recovery markets.

7. Appendix

7.1 Notation

L	=	time until all warranties expire if there is no trade-in program, i.e., length of the warranty horizon
M	=	total number of purchases over the life of the product

θ = fraction of new purchases influenced by external source, i.e., innovator segment

α = size of the remanufactured product market relative to the new product market

τ = number of periods remanufactured product demand lags behind the new product demand

p = coefficient of innovation denotes adoption due to external influences

q = coefficient of imitation denotes adoptions due to internal market influences.

$d_n(t)$ = new product demand at time t.

$d_r(t)$ = remanufactured product demand in period t

$d_{bj}(t)$, $d_{tj}(t)$ = new product sales to segment j resulting from buyback / trade-in transactions

$\theta(t)$ = the fraction of sales to the innovator segment in period t

$D_n(t-1)$ = cumulative new product sales at the start of period t

$D_r(t)$ = total remanufactured product sales in periods 1 through t

$N_{bj}(t, i)$, $N_{tj}(t, i)$ = The number of products of age i in the segment j install base at the end of period t

$s_{bj}(t, i)$, $s_t(t, i)$ = The number of products of age i returned in period t from segment j

$c_b(t, i)$ = denote the buyback price at time t for a product of age i.

$c_t(t, i)$ = trade-in price reduction on the new model of the product at time t according to policy i, i.e., trade-in discount

$c_m(i)$ = cost to remanufacture a returned product of age i

h = inventory holding cost per unit-period excluding the cost of capital (e.g., storage fees)

M	=	margin on the new model of the product
p_n	=	price of the new model of the product, i.e., $p_n = c_n + m$
p_r	=	remanufactured product selling price
r	=	net discount rate, e.g., cost of capital less inflation
γ	=	fraction of trade-in customers who purchase the new model when returning the old product (e.g., repeat purchase rate)
ϕ	=	the ratio of trade-in- to-buyback disutility
V	=	consumer's valuation of the new product
i	=	age of a product
$\upsilon(i)$	=	consumer's valuation fraction of a product of age i,

7.2 Assumptions

A1. The valuation fraction of a product of age i, $\upsilon(i)$, is the same for the innovator and imitator segments.

A2. Customers are not strategic in their return decision.

A3. The new product and the remanufactured product serve distinct markets—the products are not substitutes.

A4. The repurchase rate, γ, is independent of the age of the product when returned.

A5. Unsatisfied demand in a period results in a lost sale (i.e., no backorders).

A6. The firm remanufactures the unit in the period in which it is returned.

A7. The ratio of trade-in- to-buyback disutility, ϕ, is independent of product age.

7.3 Step 3 of the Myopic Algorithm

The logic of Step 3 of the myopic algorithm is presented at a high level. We provide additional detail on implementation in this section. We use the myopic buyback algorithm for purposes of illustration; the implementation of the myopic trade-in algorithm is identical except for changes in notation.

Recall that the profit associated with age i product under Regime $k \in \{1, 2\}$ is

$$\frac{\partial \Pi_b(t)}{\partial s_b(t,i)} = \begin{cases} p_r - c_m(i) - b_b(t,i) - 2a_b(t,i)s_b(t,i), \; s_b(t,i) \in \left[0, A_{bk}(t,i)\right) \\ p_r - c_m(i) - b_{bk}(t,i) - 2a_{bk}(t,i)s_b(t,i), \; s_b(t,i) \in \left(\begin{matrix} A_{bk}(t,i), \\ N_b(t-1,i-1) \end{matrix} \right] \end{cases}.$$

For a given volume, Δ, to allocate to ages in set J, the fraction of volume allocated to age $i \in J$ in order to maintain identical marginal profits is

$$\rho_i = \left[\sum_{j \in J} \frac{a_i}{a_j} \right]^{-1}$$

where

$$a_i = \begin{cases} a_b(t,i), \; s_b(t,i) \in \left[0, A_{bk}(t,i)\right) \\ a_{bk}(t,i), \; s_b(t,i) \in \left(\begin{matrix} A_{bk}(t,i), \\ N_b(t-1,i-1) \end{matrix} \right] \end{cases}$$

(obtained by setting marginal profits equal after allocating $\rho_i\Delta$ to each age $i \in J$ and solving for ρ_i). To simplify the notation, let

$$b_i = \begin{cases} p_r - c_m(i) - b_b(t,i), \; s_b(t,i) \in \left[0, A_{bk}(t,i)\right) \\ p_r - c_m(i) - b_{bk}(t,i), \; s_b(t,i) \in \left(\begin{matrix} A_{bk}(t,i), \\ N_b(t-1,i-1) \end{matrix} \right] \end{cases}.$$

The value of Δ is the minimum of five values, Δ_1, …, Δ_5, that are explained below.

1. Quantity that results in marginal profit of age J to equal the second-highest marginal profit:

    Let j denote the age of the second-highest marginal profit. If there is no positive second-highest marginal profit, then $\Delta_1 = \infty$; otherwise the quantity is

$$\Delta_1 = \min_{i \in J} \left\{ \frac{b_i - a_i s_b(i) - b_j}{a_i \rho_i} \right\}$$

2. Quantity at which $s_b(J) = A_{bk}(J)$:

$$\Delta_2 = \min_{i \in J} \left\{ \begin{array}{ll} \frac{A_{bk}(i) - s_b(i)}{\rho_i}, & \text{if } s_b(i) < A_{bk}(i) \\ \infty, & \text{if } s_b(i) \ge A_{bk}(i) \end{array} \right\}$$

3. Quantity at which $s_b(J) = N_b(J - 1)$:

$$\Delta_3 = \min_{i \in J} \left\{ \begin{array}{ll} \infty, & \text{if } s_b(i) < A_{bk}(i) \\ \frac{N_b(t-1, i-1) - s_b(i)}{\rho_i}, & \text{if } s_b(i) \ge A_{bk}(i) \end{array} \right\}$$

4. Quantity at which the marginal profit of age J is zero:

$$\Delta_4 = \min_{i \in J} \left\{ \frac{b_i - a_i s_b(i)}{a_i \rho_i} \right\}$$

5. Quantity at which total return volume is equal to demand d_r:

$$\Delta_5 = d_r - \sum_{i \in J} s_b(i)$$

The total quantity allocated to the ages in J is

$$\Delta = \min\{\Delta_1, \ldots, \Delta_5\}$$

and $s_b(i) = s_b(i) + \rho_i \Delta \ \forall i \in J$.

7.4 Derivations and Proofs

Proof of Proposition 1. Note that the buyback and trade-in models are identical when $\phi = \gamma = 1$ (compare (1.11), (1.18), and (1.32) with (1.58), (1.62), and (1.72)). Suppose $\phi = 1 > \gamma$. Setting trade-in prices to the optimal buyback prices, i.e.,

$$c_t(t, t) = c_b^*(t, i) \text{ for all } i \text{ and } t,$$

yields a feasible solution to the trade-in problem and the following difference in profit:

$$\sum_{t=1}^{T}(1-r)^{t}\,\Pi_{t}\left(t\,|\,c_{t}(t,i)=c_{b}^{*}(t,i)\right)-\Pi_{b}^{p}=\sum_{t=1}^{T}(1-r)^{t}\sum_{i=1}^{t}(1-\gamma)\,m\left(s_{b1}(t,i)+s_{b2}(t,i)\right)$$

(see (1.32) and (1.72)), which is positive when a buyback program is profitable (i.e., when $s_{bj}(t, i) > 0$ for some i, j, and t). ❑

CHAPTER 3: FINAL PURCHASE AND TRADE-IN PROGRAM DECISIONS IN RESPONSE TO A COMPONENT PHASE-OUT ANNOUNCEMENT

1. Introduction

In this essay, we investigate a problem faced by a durable-goods manufacturer of a product that is no longer manufactured but still under warranty. A supplier announces that a component of the product will be phased out and specifies a deadline for the final order. The manufacturer projects the component needs for the product under warranty and considers a two-stage decision problem: (1) the size of the final order and, in the event that the final order is less than actual requirements, (2) the design of a trade-in program for component harvesting.

The importance and prevalence of this problem have increased over time due to two long-term trends—shrinking product life-cycles and growth in outsourcing. These trends are especially pronounced in the computer industry where the high pace of change and technical challenges favor supply chains of independent firms with specialized expertise (e.g., AMD and Intel for processors, Seagate and Western Digital and for hard drives, Cisco and D-Link for routers, Flextronics and Selectron for assembly). Indeed, our motivation for this study comes from our discussions with management at a computer manufacturer. The following is a description of the problem by a manager at the firm.

> *What we are doing today for warranty parts is we place an end-of-life buy. The supplier will come to us and say okay, in the next three months we are going to stop producing this part forever, how many do you want? Now typically we would have three to five more years' worth of warranty life that we have to cover for that part when the buyer comes to tell us that. So we then run it through a series of parts planning tools that tells us the demand we will have for that part over the remaining service life, and we assign a service level to that. But naturally considering that it is warranty*

> *coverage and we know we can't get the part again, we have to be pretty conservative of how we place that buy. So, by definition, we over purchase on that end-of-life buy.* ***.*** *A thought is that if we had something in place such that in those situations where demand for warranty parts ends up greater than we thought, if in those situations we can go out to the install base and proactively identify those units that we would like to have back. We could offer the current customer a very good deal on an upgrade and get those systems back and then tear them down.*

We present a model of the decision problem in response to a component phase-out announcement (CPOA) by a sole source supplier. In particular, we consider the special setting where the CPOA occurs after the manufacturer has discontinued sales of the parent product. We investigate how a firm's optimal decisions and profits are influenced by industry and market characteristics. Our analysis lends insight into the determinants of the initial order quantity, the characteristics of a well-designed trade-in program to support component harvesting, and the cost of ignoring a trade-in program for component harvesting.

Our main contributions are three-fold. First, we introduce and define an important problem that, to our knowledge, has not been addressed in the literature. Second, we introduce a simple, yet rich, model of the problem. Third, we characterize optimal decisions and profits and we identify a series of insights for effective response to CPOAs.

The remainder of the essay is organized into four sections. Section 2 identifies the salient elements of the CPOA problem. Section 3 reviews the related literature. Section 4 presents the models and analyses. Section 5 provides a summary and offers suggestions for future research. A list of notation and assumptions, as well as derivations, proofs, and DP algorithms, can be found in the appendix.

2. Elements of the Component Phase-Out Announcement Problem

As a new generation of a component is introduced and the volume of the previous generation declines, a supplier eventually ceases to supply the older generation component and announces a time-line for phase out. While it is possible that a CPOA may occur when the manufacturer is still producing a product with the component, we limit consideration to the case where the product is no longer being manufactured (as is consistent with CPOA timing examples described to us by those in industry). Thus, the final component purchase decision is driven by warranty obligation considerations. Durable-goods manufacturers commonly offer a limited-time warranty to consumers. For illustrative purposes, we describe elements of Dell's warranty, which is representative of other computer manufacturer warranties. We then outline elements of the manufacturer's decision problem.

Dell provides a replacement warranty on their PCs. Each computer is sold with Dell's Basic Service Plan, which includes a minimum of 12 months of warranty coverage. The plan covers all Dell-branded component parts (e.g., motherboard, hard drive, LCD display, optical drive, graphics card, processor, power supply, fan assemblies). A consumer can extend the basic service plan for up to four additional years by paying a fee at the time of purchase. Dell's warranty is non-renewing, meaning that if a Dell-branded item fails while under the warranty period, Dell will replace any defective part with a new or refurbished part. Once the component has been repaired or replaced, however, Dell does not extend the warranty period beyond that of the service plan.

The CPOA problem can be viewed as a two-stage decision problem. The first-stage decision is the number of components in the final order. After the final order is placed, component demand is realized over time. The second-stage decision, if necessary, is the price discount to be offered on a trade-in. The second-stage decision is required if or when component supply approaches zero, in which case the firm announces a trade-in program.

In some settings, the firm may have access to customer-specific warranty data. In these settings, the trade-in offer can be targeted to specific customers based on product age and time remaining under warranty. In settings where these data are not available, a firm can announce a limited-time trade-in offer to the general public. We consider both settings in our analysis. Figure 3.1 illustrates the timing of decisions and the nature of data gathering and analysis in response to a CPOA.

Events & Actions	CPOA	1. Place final order	2. Offer trade-in program to customers, harvest & refurbish components	

Information Gathering & Analysis	• Available component stock
	• Number of units on warranty & timing of warranty expirations
	• Component failure characteristics
	• Estimation of total component needs
		• Observe component demand over time
		• If supply approaches zero, then:
		  • collect data on install base
		  • determine trade-in offer terms & distributed to install base
			• Replace faulty components until outstanding warranties expire	

Figure 3.1. Sequence of events in response to a CPOA tied to a product that is no longer manufactured.

A firm interested in component harvesting could offer a buyback program instead of a trade-in program. Buyback programs offer money for used product without the requirement that the consumer purchase a new product from the firm. In Chapter 2, we find that trade-in programs are more profitable than buyback programs in settings where the trade-in credit can be applied to a wide range of products sold by the firm (see Proposition 1). In addition, we find no interest in a buyback program in our discussions with management of the computer manufacturer. For these reasons we limit consideration to trade-in programs for component harvesting.

3. Related Literature

There are four streams of research related to our two-stage problem of how a manufacturer determines the size of the final order and designs the trade-in program for component harvesting. Stage-one of our problem is related to the literature on warranty management and the literature on spare parts inventory management. Stage-two of our problem is related to the literature on trade-in programs and the literature on product recovery operations (i.e., reverse logistics and closed-loop supply chain management).

Most consumer durables come with either a pro-rata refund or a free repair/replacement warranty policy (Blischke and Murthy 1992). Murthy et al. (2004) provide a comprehensive review of various issues associated with warranty management. There are three main issues of concern: warranty terms, warranty cost, and forecasting warranty claims. The issue of claims forecasting is relevant for our problem because we require a model of warranty claims (i.e., component requirements) over time. Warranty claims are driven by the warranty population, usage characteristics, product reliability, and warranty terms. In most of the warranty claim literature, usage characteristics, product reliability, and warranty terms are exogenous, and failure rates are characterized as a function of time of ownership. Three processes for modeling warranty claims over time are assumed: increasing failure rate (IFR), decreasing failure rate (DFR), or constant failure rate (CFR) (Hong et al. 2004). A constant failure rate implies that the time to failure is exponentially distributed. For products with IFR and DFR, failure time distributions are modeled with the Weilbull distribution (Wu et al. 2009). However, several papers develop warranty claims models where usage characteristics are endogenous. Gerner and Bryant (1980) estimate component demand by modeling consumer behavior over the warranty period. They identify a mapping from the owner's usage choice to time-to-failure. Consumers decide their usage level by considering the marginal benefits from use, the likelihood of failure during use, and the total expected cost of repair (i.e., downtime cost, unrecoverable repair expenses). In a similar vein, Murthy (1990) assumes a discrete distribution of failure time that is based on the number of uses, and each consumer weighs trade-offs to determine the usage rate. In our framework, we model the warranty claims process under the CFR assumption.

A second stream of related literature addresses of problem of spare parts inventory management. The literature on this topic is vast (e.g., see Cohen and Lee 1990 or Kennedy et al. 2002 for a review). Within this literature, a number of researchers have studied the problem managing component parts after the parent product has reached the end of its sales life-cycle. During this phase, key suppliers will usually discontinue normal production of key components and the firm has to resolve sourcing issues related to meeting warranty and service obligations. Hasselbach et al. (2001) and Inderfurth and Mukherjee (2008) discuss three sourcing options: (i) secure new components from an alternative supply source, (ii) secure new components by placing a potentially large final order, (iii) secure components by harvesting parts (i.e., remanufacturing, refurbishing, and repairing) from used product returns. The key decision concern is determining order quantities and/ or the production lot sizes. In the case of option (i), determining the number of components to acquire from an alternative supply source is not much different from traditional inventory management problem.

The final order problem, option (ii), has been addressed by a number of authors. For example, Fortuin (1980) considers a firm that manages a single service component for an obsolete machine. Fortuin (1980) introduces a model wherein the machines' remaining operating life is divided into discrete intervals and the number of components that fail in each interval is random. He proposes a method for estimating the component stock-out probability as a function of the final order quantity. Teunter and Fortuin (1999) also consider the single component setting and develop a single stage, dynamic program to determine the optimal final order quantity for a single component. In addition, they identify a closed-form expression for a near-optimal order quantity. In a follow-up paper, the authors use these closed-form expressions in a case study of a firm that produces electronic equipment (Teunter and Fortuin 1998). Teunter and Hansveld (1998) extend the single component setting of the previous papers to a multi-component ordering problem. They show the multi-component problem can be decomposed into independent single-component ordering problems. Bradley and Guerrero (2009) also consider the multi-component final buy problem, though in contrast to Teunter and Hansveld (1998) who assume all components are phased out at the same moment in time, components are phased out gradually over time. They employ a sequential newsvendor framework where the final order decision for each component is constrained by the remaining inventories of previous phased-out components. Bradley and Guerrero propose heuristics that yield near-optimal solutions. While the motivating application in these papers differs from our setting, the structure of the problem is similar to our stage-one decision when a trade-in program is not viable.

Most of above analyses assume the firm is unable to secure the phased-out component from an alternative source. Furthermore, with the exception of Teunter and Fortuin (1999), the papers discussed in the previous paragraph do not consider remanufacturing, option (iii), as a possible source for spare-parts. In their analysis, the firm passively accepts used product returns. The components harvested from these returns are shown to reduce spare-parts demand; however the supply of remanufacturable products does not affect the structure of the optimal policies. Several recent papers have also incorporated option (iii) as a source for spare-parts. For example, Minner and Kleber (2001) examine a setting where the firm produces a new component and remanufactures used components to meet a deterministic demand for spare parts. The authors develop a dynamic inventory framework and use optimal control techniques to determine an optimal production and recovery strategy for a firm who passively accepts used products. Spengler and Schröter (2003) also develop a dynamic model that integrates component harvesting. However, unlike Minner and Kleber (2001), Spengler and Schröter (2003) assume the new product is no longer produced (e.g., final service phase). Spengler and Schröter (2003) develop a systems dynamic model of a closed-loop supply chain in order to study flows of new product sales, spare parts demands, product returns, and recovery rates. Their model focuses on the behavior of the spare parts management system. They consider various policies related to acquisition of obsolete parts and redesign of new components and they observe how the system responds when the firm over and underestimates spare-parts demand. Inderfurth and Mukherjee (2008) develop a decision model where the firm uses options (i), (ii), and (iii) to meet a dynamic demand for spare-parts during the phase-out period. Their model accounts for uncertain supply of used components and uncertain demand for spares. We also consider a problem where the parent product, with outstanding warranty obligations, is no longer produced. Our model resembles that of Inderfurth and Mukherjee (2008). The decision maker chooses the final order quantity and remanufactures components from returned products to meet dynamic demand for spare parts (e.g., warranty claims). However, our model differs from their model in three ways: (1) We assume return and demand flows are deterministic rather than stochastic. We make this assumption in order to clarify the essence of a trade-off that exists between the final order quantity and the design of a trade-in program. (2) As opposed to an exogenous flow of returns, we assume that the firm proactively acquires used products from its install base. The firm sets a trade-in discount to influence the timing and quantity of the return flow. Spengler and Schröter's (2003) dynamic simulation model captures proactive acquisition of used products. In their model, the acquisition cost and returns flows are exogenous parameters that determine the stocks and flows of used products. We determine optimal trade-in credits and acquisition quantities. (3) We account for the impact of returns of product under warranty on future warranty claims.

A third stream of related literature examines the relationships between new product prices, trade-in rebates, product return volumes, and new product purchases. Ray et al. (2005) examine how a trade-in program for a product that is remanufactured can be used as a price-discrimination mechanism to increase profits. Bruce et al. (2006) study trade-in programs for expensive durables purchased with the aid of a loan (e.g., automobiles).

They examine the relationship between the magnitude of the trade-in discount and the durability of the product. Rao et al. (2009) study the value of trade-in programs for products in which used product prices are negatively affected due to information asymmetry (e.g., a positive probability of buying a “lemon”). They show that trade-in programs increase profit and are more valuable for less reliable products and for products that deteriorate more slowly. The model of our second-stage problem will draw on features of the trade-in volume models introduced in this stream.

Finally, a fourth stream of related literature investigates product recovery operations. Seitz (2007) reports that the use of recovered components to satisfy warranty claims is a common practice in the automobile and home appliance industries. The benefits of harvesting parts from product returns have been highlighted in a number or papers related to closed-loop supply chain management (e.g., Fleischmann et al. 2003, Guide and Van Wassenhove 2001, Linton 2008). Cisco, for example, began using returns to support warranty claims in 2008. The initiative increased the recovered value from returns by nine-fold, from 5% to 45% (Nidumolu et al. 2009). The papers that are most closely related to our work address the problem of procuring end-of-use products from the install base. Key decisions are the procurement quantity and the buyback price offered to the user. Guide and Van Wassenhove (2001) consider a firm that sets a buyback price to match the supply of cores with the demand for remanufactured components. Bakal and Akcali (2006) also consider a buyback price to match supply with demand and investigate the impact of supply and demand uncertainty. Galbreth and Blackburn (2006) study the interaction between procurement lot size and the firm’s sorting policies. Zikopoulos and Tagaras (2007) consider the problem of ordering used products from multiple supply sources with correlated recovery yield. Each of these papers considers a single-stage decision environment. While the purpose and method of acquiring used components in this stream differ from our setting, we draw on elements of these models in our second-stage problem.

In summary, our stage-one problem is similar to final purchase quantity problems in the literature. A key difference is the consideration of a trade-in program that leads to a two-stage decision problem. Although the final purchase quantity literature has considered active product recovery, the literature has not considered the design of trade-in programs as a mechanism for acquiring used components. Our stage-two problem is similar to the problem of designing a trade-in program that is considered in the marketing literature. As in this literature, we need to model how features of the trade-in program and other factors influence return volume. However, a key difference is that there is not an associated demand for used components that must be met, as is the case in our problem. Our stage-two problem is also similar to the problem of acquiring used product for the purposes of remanufacturing. While the mechanism for acquiring returns differs (i.e., buyback vs. trade-in), our problem shares the feature of aligning supply of a used component with demand. In the next section we outline a model of our problem, and we identify specific commonalities and differences with models from the literature.

4. Models and Analyses

We begin by presenting a general cost model of the CPOA problem. The model includes a policy-dependent second-stage cost function. Section 4.1 contains optimal decisions and cost functions for three second-stage policies. Section 4.2 characterizes the optimal first-stage decisions and costs under each of the second-stage policies.

A firm has received a CPOA for a component and must determine the final order quantity q_1 that will be received at time $t = 0$. The purchase cost per unit is c_1, the inventory holding cost rate is h, the warranty claim service cost per unit is c_w (e.g., disassembly, component replacement, reassembly, test, and shipping), and the cost per unit to dispose unused components is c_3 where $-c_3 < c_1$ (i.e., salvage value is less than purchase cost). The difference between the firm's discount rate and the rate of inflation in operating costs and margin is r. The last warranty will expire L periods after time $t = 0$.

As noted earlier, we assume return and demand flows are deterministic. Our focus is on understanding essence of a trade-off that exists between the final order quantity and the design of a trade-in program. The component demand rate at time t (i.e., due to warranty claims) is $d(t)$, the cumulative demand through period t is $D(t)$, i.e.,

$$D(t) = \int_0^t d(x)\,dx\,, \tag{2.1}$$

and $T_1(q_1)$ is the time that component inventory reaches zero, or L, whichever is smaller, i.e.,

$$T_1(q_1) = \min\left\{\min\left\{t \mid D(t) \geq q_1\right\}, L\right\} \tag{2.2}$$

The cost of ordering q_1 units is

$$C_1(q_1) = c_1 q_1 + \int_0^{T_1(q_1)} e^{-rt}\left[h\left(q_1 - D(t)\right) + c_w d(t)\right]dt + C_2\left(T_1(q_1)\right) \tag{2.3}$$

where $C_2(t)$ is the cost of satisfying warranty claims over time interval $[t, L]$ given that the final order quantity runs out at time t. (If time is discrete, then $\int$ is replaced by $\sum$ in (2.1) and (2.3).) The first term in (2.3) is the component purchase cost of the final order. The second term in (2.3) is the holding and warranty claim servicing cost, of which the parenthetical term in the integrand is the inventory at time t that is assessed a holding cost rate h..

The specification of function $C_1(q_1)$ requires assumptions on the demand process, the manner in which warranties expire over time, and the policy for satisfying warranty claims in the event of $T_1(q_1) < L$. In the remainder of this section, we introduce modeling assumptions as needed to specify cost functions and characterize optimal decisions for alternative policies.

4.1 Second-Stage Policies

Let t_1 denote the realization of $T_1(q_1)$. If $t_1 = L$, then $C_2(t_1) = 0$ and no second-stage policy is needed to satisfy remaining warranty claims. In this section, we assume $t_1 < L$ and consider three possible policies for servicing warranty claims. We use a superscript on policy-specific costs to indicate the second-stage policy.

4.1.1 No Trade-Ins – Second-Stage Policy 0

The firm acquires components, as needed, by means other than a trade-in program (e.g., component is purchased from a third party, though at a higher price compared to the OEM). For example, there is an industry with sales of $2.6 billion in 2001 (Sullivan 2002) that manufactures and sells parts that have been declared obsolete by an OEM. The cost of this policy provides a benchmark for assessing the value of trade-in programs for component harvesting.

The component purchase cost per unit is c_2^0. We assume $c_2^0 > c_1$; otherwise the firm would not purchase any units from the vendor in the first-stage. Recall that c_w is the warranty claim service cost, net of component acquisition cost. Thus, the total warranty claim service cost is c_2^0 + c_w. It is possible that the firm prefers to satisfy the claim by replacing the old model with the new model (e.g., no reasonably priced sources for the component), in which case c_2^0 + c_w represents the cost of this transaction.

The second-stage cost is

$$C_2^0(t_1) = \int_{t_1}^{L} e^{-rt}\left(c_2^0 + c_w\right) d(t)\, dt \,. \tag{2.4}$$

If r = 0, then (2.4) reduces to

$$C_2^0(t_1) = \left(c_2^0 + c_w\right)\bar{D}(t_1) \tag{2.5}$$

where $\bar{D}(t_1)$ is demand over time interval [t_1, L], i.e.,

$$\bar{D}(t_1) = D(L) - D(t_1)\,. \tag{2.6}$$

4.1.2 Relationship between Trade-in Discount, Trade-in Volume, and Trade-in Cost

A firm offering a trade-in program specifies the discount off the purchase price of a new model if the customer returns the old model. Throughout this essay, we assume that the trade-in discount is offered only to customers with product under warranty. Conceivably a firm could offer the trade-in discount to a customer with a product that is no longer under warranty. While such a customer might be willing to trade-in for a lower discount, the tactic of offering a trade-in discount for product not under warranty has two drawbacks. First, there is a risk that the component in the returned product will be faulty. This risk is low for product under warranty because, if it was faulty, the firm would have likely already received a claim. Second, the return of a product under warranty reduces the firm’s warranty liability associated with the obsolete component (i.e., the product containing the obsolete component is traded in for a new model of the product).

A1.The trade-in discount is only available to customers with product under warranty.

In this section we identify the functional relationships between trade-in discount, volume, and the cost of a component acquired through trade-in. We will make use of these relationships when we examine two alternative trade-in policies in the following sections.

We make two assumptions that allow us to define the fraction of customers who accept a trade-in offer as a function of the trade-in discount.

A2. A customer receiving a trade-in offer receives a single take-it-or-leave-it offer and accepts the offer if consumer surplus is positive.

A3. The valuation of the new model in exchange for the old model under warranty is independent of time and is uniformly distributed with range normalized to [0, 1].

An alternative to A2 is to allow multiple trade-in offers to the same customer over time. However, this promotes strategic behavior that greatly complicates the analysis and may work against the interest of the firm (e.g., customer holds out for a better offer). Uniformly distributed valuation results in a linear volume function and is common in the literature (e.g., Mussa and Rosen 1978, Purhoit and Staelin 1984).

A firm offering a trade-in program must select the trade-in discount and the rate at which customers are exposed to the trade-in offer (i.e., the trade-in offer rate), both of which may vary with time. The trade-in discount is $c_t(t)$ and the trade-in offer rate is $\nu(t)$ (i.e., $\nu(t)$ is the number of customers receiving a trade-in offer per period). The contribution margin of a new model of the product is m and the variable cost is c_n, i.e., the new model selling price is $p_n = c_n + m$. Thus, the trade-in price is $c_n + m - c_t(t)$ and, by A2 and A3, the fraction of customers who accept the trade-in offer from among those who receive it is

$$\beta(t) = P\left[V > c_n + m - c_t(t)\right] = 1 - c_n - m + c_t(t) = c_t(t) - (p_n - 1). \tag{2.7}$$

Rewriting (2.7) in terms of the trade-in credit,

$$c_t(t) = p_n - (1 - \beta(t)). \tag{2.8}$$

We see that the trade-in price is the complement of the acceptance rate $\beta(t)$, i.e.,

$$p_n - c_t(t) = 1 - \beta(t).$$

Note that the new model selling price should be more than the maximum valuation, i.e.,

$$p_n = c_n + m > 1. \tag{2.9}$$

Condition (2.9) reflects the practical reality that customers are unlikely to trade-in a product under warranty unless there is a trade-in discount. For example, $p_n < 1$ would imply that fraction $1 - p_n$ of customers would be willing to return their product (that is under warranty and functional) and pay full price for the new model.

We refer to the value of $p_n - 1$ as the *trade-in resistance*. This value is the minimum trade-in discount that is required before any customers will be willing to return their unit. The larger the value of $p_n - 1$, the greater the market resistance to a trade-in offer.

In (2.7), we see that the difference between the trade-in credit, $c_t(t)$, and the trade-in resistance, $p_n - 1$, gives the fraction of those receiving the trade-in offer who accept the offer. Thus, the product return rate is

$$s(t)=\beta(t)v(t)=\left[c_t(t)-(p_n-1)\right]v(t)\,. \tag{2.10}$$

In general, the specification of trade-in acquisition cost can be challenging due to the effect of cannibalization. We capture this effect through a single parameter γ. The interpretation of γ is straightforward when the difference between the firm's discount rate and the rate of inflation (in costs and margin) is zero (i.e., $r = 0$): γ is the fraction of trade-in customers who would have purchased the new model at full price in the future if the trade-in program was not offered, or repeat purchase rate. If $r > 0$, then the value of the full margin in the future is lower due to the time-value-of-money. All time-value-of-money effects and, more generally, all cannibalization effects are incorporated into the value of γ. Indeed, it is possible for γ to be negative in some settings, e.g., by reducing secondary market supply and thus cannibalization of new product sales.

The margin of a new product sold at the trade-in discount is $m - c_t(t)$, which takes the place of a possible full margin m in the future with probability γ. Accordingly, the cost of a component obtained through a trade-in is the reduction in margin through a trade-in sale, which is

$$c_2(t)=c_t(t)-(1-\gamma)m=\beta(t)-\tau \tag{2.11}$$

where

$$\tau = (1 - \gamma)m - (p_n - 1). \tag{2.12}$$

We refer to the value of τ as the *trade-in potential*, which is the difference between the gain from locking-in disloyal customers via the trade-in offer, $(1 - \gamma)m$, and the market resistance to the trade-in offer, $p_n - 1$. More generally, τ is the marginal profit on trade-in volume at the origin. For example, if $\tau > 0$, then trade-in potential is positive and trade-ins are profitable up to acceptance rate $\beta(t)$. On the other hand, if $\tau < 0$, then trade-in potential is negative and trade-ins are costly from the get-go.

4.1.3 Relationship between Trade-in Offers and Warranty Expirations

In the absence of a trade-in program, the rate at which warranties expire at time t is given by $n(t)$. The function $n(t)$ is known with certainty (e.g., obtained from company records). The warranty population at time $t = 0$ is N, i.e., $\int_0^L n(t)\,dt = N$.

Analysis of second-stage trade-in policies sometimes requires an expression for the number of customers who do not receive a trade-in offer before their warranty expires. Recall that trade-in units are under warranty (see A1). As a consequence, we require an assumption that allows us to determine when the warranties of trade-in units expire in the future.

A4. The warranty expiration date of a customer who accepts a trade-in offer is no later than the warranty expiration date of a customer who rejects a trade-in over.

Assumption A4 stems from the idea that customers who own product with a near-term warranty expiration date are more open to a trade-in offer than customers who own product with a more distant warranty expiration date.

Let $\Gamma(t, \nu)$ denote the number of customers during time interval $[0, t]$ who do not receive a trade-in offer before their warranty expires given trade-in offer rate function

$\nu(\cdot)$. Due to A4, the firm distributes trade-in offers to customers in order of warranty expiration. Accordingly, if there exists a point in time t such that

$$\int_0^t n(x)\,dx - \int_0^t \nu(x)\,dx > 0\,, \tag{2.13}$$

then there are some customers who do not receive a trade-in offer prior to warranty expiration. More precisely, the number of customers who do not receive a trade-in offer prior to warranty expiration is the maximum value of the left-hand side of (2.13) over time interval [0, t], i.e.,

$$\Gamma(t,\nu) = \max_{y\in[0,t]}\left(\int_0^y \left[n(x)-\nu(x)\right]dx\right)^+ . \tag{2.14}$$

Due to A2, any feasible trade-in policy must ensure that a customer does not receive a trade-in offer more than once, which implies that the sum of customers who do not receive trade-in offers and those who do is equal to the warranty population, i.e.,

$$\Gamma(\nu) + \int_0^L \nu(t)\,dt = N \tag{2.15}$$

where

$$\Gamma(\nu) = \Gamma(L,\nu) = \max_{t\in[0,L]}\left(\int_0^t \left[n(x)-\nu(x)\right]dx\right)^+ . \tag{2.16}$$

4.1.4 Full Trade-in Policy – Second-Stage Policy 1

Under the full trade-in policy, the firm launches a time-sensitive promotion to the entire warranty population. The firm only provides trade-in credits for units returned during a valid offer period. For example, HP offers featured, time-sensitive promotions on certain product categories (i.e., earn a trade-in credit for HP Designjet T1300 between

August 1 and October 31).Thus at time $t_1 < L$, when the inventory from the final order quantity reaches zero, the firm offers a trade-in opportunity to the entire warranty population. From (2.10), it follows that the number of units returned is

$$q_2 = \left(c_t^1 - \left(p_n - 1\right)\right) N\left(t_1^-\right) \tag{2.17}$$

where c_t^1 is the trade-in discount (i.e., $c_t^1 = c_t(t_1)$). From (2.7) and (2.11) , we have

$$c_t^1 = \beta\left(t_1\right) + p_n - 1 = q_2 / N\left(t_1^-\right) + p_n - 1 \tag{2.18}$$

$$c_2^1 = q_2 / N\left(t_1^-\right) - \left(1 - c_n - \gamma m\right) = \beta\left(t\right) - \tau\ . \tag{2.19}$$

We let $T_2(q_2)$ denote the time that second-stage component inventory reaches zero, or L, whichever is smaller, i.e.,

$$T_2(q_2) = \min\left\{\min\left\{t \mid D\left(t\right) - D\left(t_1\right) \geq q_2\right\}, L\right\}. \tag{2.20}$$

If $T_2(q_2) < L$, then the firm satisfies warranty claims by an alternative means at unit cost c_2^0 as discussed in Section 4.1.1. Thus, the expected cost of a full trade-in policy is

$$\begin{aligned}
C_2^1\left(t_1, q_2\right) &= e^{-rt_1} c_2^1 q_2 + \int_{t_1}^{L} e^{-rt} c_w d\left(t\right) dt + \int_{t_1}^{L} e^{-rt} h\left(q_1 + q_2 - D\left(t\right)\right) dt + \\
&\quad \int_{T_2(q_2)}^{L} e^{-rt} c_2^0 d\left(t\right) dt + e^{-rL} c_3 \left(q_2 - \bar{D}\left(t_1\right)\right)^+ \\
&= e^{-rt_1} \left[\begin{aligned} &\left(\frac{{q_2}^2}{N\left(t_1^-\right)} - \tau q_2 \right) + \int_{t_1}^{L} e^{-r(t-t_1)} c_w d\left(t\right) dt + \\ &\int_{t_1}^{T_2(q_2)} e^{-r(t-t_1)} h\left(q_1 + q_2 - D\left(t\right)\right) dt + \\ &\int_{T_2(q_2)}^{L} e^{-r(t-t_1)} c_2^0 d\left(t\right) dt + e^{-rL} c_3 \left(q_2 - \bar{D}\left(t_1\right)\right)^+ \end{aligned} \right]
\end{aligned} \tag{2.21}$$

The first term in (2.21) is the component acquisition cost through the trade-in program, the second term is warranty claim service cost, the third term is the cost of holding inventory, the fourth term is the cost to satisfy the warranty claim by alternative means if quantity q_2 is less than demand, and the fifth term is the cost to dispose unneeded components. If $r = 0$, then (2.21) reduces to

$$\begin{aligned} C_2^1(t_1,q_2) = \frac{{q_2}^2}{N\left(t_1^-\right)} - \tau q_2 + c_w\bar{D}(t_1) + \int_{t_1}^{T_2(q_2)} h\left(q_1+q_2-D(t)\right)dt + \\ c_2^0\left(\bar{D}(t_1)-q_2\right)^+ + c_3\left(q_2-\bar{D}(t_1)\right)^+ \end{aligned}. \tag{2.22}$$

The value of q_2 affects component demand over interval $[t_1, L]$ because the warranty population is reduced by quantity q_2 at time t_1. We now turn our attention to the functional forms of the warranty population function $N(t)$ and $\bar{D}(t_1)$. The warranty population at any time $t < t_1$ is $N(t) = N - \int_0^t n(x)dx$, and

$$N\left(t_1^-\right) = N - \int_0^{t_1^-} n(x)dx\,. \tag{2.23}$$

At time t_1, the warranty population is reduced by q_2, i.e., $N(t_1) = N\left(t_1^-\right) - q_2$. Due to A4, the warranty population function in general is

$$N(t) = \begin{cases} N - \int_0^t n(x)dx, & t \in [0,t_1) \\ N\left(t_1^-\right) - q_2, & t \in \left[t_1, t_2(q_2)\right) \\ N\left(t_1^-\right) - q_2 - \int_{t_2(q_2)}^{t} n(x)dx, & t \in \left[t_2(q_2), L\right) \\ 0, & t = L \end{cases} \tag{2.24}$$

where $t_2(q_2)$ is the time it takes beyond t_1 for the warranties of q_2 customers to expire, or L, whichever is smaller, i.e.,

$$t_2\left(q_2\right)=\min\left\{\min\left\{t \mid \int_{t_1}^{t} n\left(x\right)dx \geq q_2\right\}, L\right\}. \tag{2.25}$$

Note that, due to A1, the value of q_2 is limited to be no more than the warranty population, i.e.,

$$q_2 \leq N\left(t_1^-\right). \tag{2.26}$$

If $n(t) = 0$ for all $t \in [0, L]$ (i.e., all warranties expire at the end of the warranty horizon), then (2.24) reduces to

$$N\left(t\right)=\begin{cases} N, & t\in\left[0,t_1\right) \\ N-q_2, & t\in\left[t_1,L\right) \\ 0, & t=L \end{cases}. \tag{2.27}$$

If $n(t) = n$ for all $t \in [0, L]$, then

$$t_2\left(q_2\right)=\min\left\{t_1+\frac{q_2}{n}, L\right\}. \tag{2.28}$$

and (2.24) reduces to

$$N\left(t\right)=\begin{cases} N-nt, & t\in\left[0,t_1\right) \\ N-nt_1-q_2, & t\in\left[t_1,t_2\left(q_2\right)\right) \\ N-n\left[t_1+t-t_2\left(q_2\right)\right]-q_2, & t\in\left[t_2\left(q_2\right),L\right) \\ 0, & t=L \end{cases}, \tag{2.29}$$

which in turn reduces to (2.27) if $n = 0$. For the case of $n(t) = n$, it follows from the definition of L that

$$n \le N/L \tag{2.30}$$

(e.g., $n > N/L$ implies that all warranties expire prior to time L).

The demand for components is influenced by the component failure rate, which we assume to be constant at value α. The assumption of a constant failure rate facilitates tractability and is consistent with the literature (e.g., Murthy 1990, Hong et al. 2008, Wu et al. 2009, Zhou et al. 2009).

A5. The component failure rate is constant.

In the remainder of this section we analyze versions of problem $\min_{q_2 \le N(t_1^-)} C_2^1(t_1, q_2)$ that correspond to alternative sets of assumptions. Additional assumptions required for the specification of problems are listed below.

A6a. $r = 0$
A6b. $r > 0$
A7a. $n(t) = 0$ for all $t \in [0, L)$ and $n(L) = N$, by the definition of L; see (2.30))
A7b. $n(t) = n$ for all $t \in [0, L)$ (and $n(L) = N - nL \ge 0$, by the definition of L; see (2.30))
A8a. $h = 0$
A8b. $h > 0$

Throughout the essay we use the naming convention $\mathrm{X}s_i^j$ where $\mathrm{X} \in \{\mathrm{C}, \mathrm{D}\}$ indicates whether the problem is based on a continuous-time model (C) or a discrete-time model (D), $s \in \{1, 2\}$ indicates the problem stage, the superscript $j \in \{0, 1, 2\}$ indicates the second-stage policy in effect, and the subscript i indicates the set of assumptions in effect. Discrete-time models with associated solution methods are defined in the appendix.

Problem $\mathrm{C2}_1^1$: Assumptions A1 – A5, A6a, A7a, A8a

The problem is

$$\mathrm{C2}_1^1:\ \min_{q_2 \le N}\left\{C_2^1\left(t_1, q_2\right) \mid \text{A1- A5, A6a, A7a, A8a}\right\}.$$

Due to A5 and A7a, the demand rate over time interval $[t_1, L)$ is

$$d\left(t\right)=\alpha N\left(t\right)=\alpha\left(N\left(t_1^-\right)-q_2\right)=\alpha\left(N-q_2\right) \tag{2.31}$$

(see (2.27)) and the total demand is

$$\bar{D}\left(t_1\right)=\int_{t_1}^{L} d\left(t\right)dt=\ \alpha\left(L-t_1\right)\left(N\left(t_1^-\right)-q_2\right)=\alpha\left(L-t_1\right)\left(N-q_2\right). \tag{2.32}$$

Substituting (2.32) into (2.22) yields

$$C_2^1\left(t_1,q_2\right)=\begin{cases}\dfrac{{q_2}^2}{N}-\left(\left(c_w+c_2^0\right)\alpha\left(L-t_1\right)+c_2^0+\tau\right)q_2+ & \\ \qquad\left(c_w+c_2^0\right)\alpha\left(L-t_1\right)N, & q_2 \le \dfrac{\alpha\left(L-t_1\right)N}{1+\alpha\left(L-t_1\right)} \\ \dfrac{{q_2}^2}{N}-\left(\left(c_w-c_3\right)\alpha\left(L-t_1\right)-c_3+\tau\right)q_2+ & \\ \qquad\left(c_w-c_3\right)\alpha\left(L-t_1\right)N, & q_2 \ge \dfrac{\alpha\left(L-t_1\right)N}{1+\alpha\left(L-t_1\right)}\end{cases}. \tag{2.33}$$

The optimal solution is characterized below.

Proposition 1. *For* $\mathrm{C2}_1^1$, *the optimal trade-in quantity is*

$$q_2 = \begin{cases} \frac{b_1 N}{2}, & \frac{b_1(\alpha(L-t_1))}{2} \le \frac{\alpha(L-t_1)}{1+\alpha(L-t_1)} \\ \min\left\{\frac{b_2 N}{2}, N\right\}, & \frac{b_2(\alpha(L-t_1))}{2} \ge \frac{\alpha(L-t_1)}{1+\alpha(L-t_1)} \\ N\left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right), & \frac{b_2(\alpha(L-t_1))}{2} \le \frac{\alpha(L-t_1)\alpha(L-t_1)}{1+\alpha(L-t_1)} \le \frac{b_1(\alpha(L-t_1))}{2} \end{cases} \tag{2.34}$$

where

$$b_1 = \tau + c_2^0 + (c_w + c_2^0)\alpha(L-t_1)$$

$$b_2 = \tau - c_3 + (c_w - c_3)\alpha(L-t_1)\,.$$

Problem $C2_2^1$: Assumptions A1 – A5, A6a, A7b, A8a

The problem is

$$C2_2^1:\ \min_{q_2 \le N}\left\{C_2^1(t_1, q_2) \mid \text{A1-A5, A6a, A7b, A8a}\right\}.$$

Problem $C2_2^1$ generalizes $C2_1^1$ to allow $n(t) = n > 0$, i.e., if $n = 0$, then the problems are identical. We begin by identifying the possible forms of the second-stage cost function. Due to A5 and A7b, $N(t_1^-) = N - nt_1$, $t_2(q_2) = \min\left\{t_1 + \frac{q_2}{n}, L\right\}$, and the demand rate over time interval $[t_1, L]$ is

$$d(t) = \alpha N(t) = \begin{cases} \alpha\left[N(t_1^-) - q_2\right], & t \in [t_1, t_2(q_2)) \\ \alpha\left[N(t_1^-) - q_2 - n(t - t_2(q_2))\right], & t \in [t_2(q_2), L)\,. \\ 0, & t = L \end{cases} \tag{2.35}$$

(see (2.29)) and the total demand is

$$\bar{D}(t_1) = \int_{t_1}^{L} d(t)\,dt = \alpha\left[(L-t_1)\left(N(t_1^-) - q_2\right) - \frac{n}{2}\left(L - t_2(q_2)\right)^2\right]$$

$$= \begin{cases} \alpha\left[(L-t_1)\left(N(t_1^-) - \frac{n}{2}(L-t_1) \right) - \frac{q_2^{\ 2}}{2n} \right], & q_2 \le n(L-t_1) \\ \alpha\left[(L-t_1)\left(N(t_1^-) - q_2 \right) \right], & q_2 \ge n(L-t_1) \end{cases} \tag{2.36}$$

The functional form of the second-stage cost depends on the value of the warranty fall-off rate n, as shown in Proposition 2. Proposition 3 characterizes the optimal trade-in quantity.

Proposition 2. *If*

$$\frac{n}{N} \le \frac{\alpha}{1+\alpha L} \tag{2.37}$$

then

$$C_2^1(t_1,q_2) = \begin{cases} \left(\frac{1}{N(t_1^-)} - \frac{\alpha(c_w + c_2^0)}{2n} \right) q_2^{\ 2} - \left(c_2^0 + \tau\right) q_2 + \\ \qquad (c_w + c_2^0)\alpha(L-t_1)\left(N(t_1^-) - \frac{n}{2}(L-t_1) \right), & \text{if } q_2 \le n(L-t_1) \\ \frac{q_2^{\ 2}}{N(t_1^-)} - \left((c_w + c_2^0)\alpha(L-t_1) + c_2^0 + \tau \right) q_2 + \\ \qquad (c_w + c_2^0)\alpha(L-t_1)N(t_1^-), & \text{if } n(L-t_1) \le q_2 \le \frac{\alpha(L-t_1)N(t_1^-)}{1+\alpha(L-t_1)} \\ \frac{q_2^{\ 2}}{N(t_1^-)} - \left((c_w - c_3)\alpha(L-t_1) - c_3 + \tau \right) q_2 + \\ \qquad (c_w - c_3)\alpha(L-t_1)N(t_1^-), & \text{if } q_2 \ge \frac{\alpha(L-t_1)N(t_1^-)}{1+\alpha(L-t_1)} \end{cases} . \tag{2.38}$$

If

$$\frac{n}{N} \ge \frac{\alpha}{1+\alpha L} \tag{2.39}$$

then

$$C_2^1(t_1,q_2)=\begin{cases}\left(\dfrac{1}{N(t_1^-)}-\dfrac{\alpha(c_w+c_2^0)}{2n}\right){q_2}^2-(c_2^0+\tau)q_2+\\ \qquad (c_w+c_2^0)\alpha(L-t_1)\left(N(t_1^-)-\dfrac{n}{2}(L-t_1)\right),\\ \qquad\qquad \text{if }\; q_2\le\dfrac{n}{\alpha}\left[\left(1+\dfrac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}-1\right]\\ \left(\dfrac{1}{N(t_1^-)}-\dfrac{\alpha(c_w-c_3)}{2n}\right){q_2}^2-(c_3+\tau)q_2+\\ \qquad (c_w-c_3)\alpha(L-t_1)\left(N(t_1^-)-\dfrac{n}{2}(L-t_1)\right),\\ \qquad \text{if }\; \dfrac{n}{\alpha}\left[\left(1+\dfrac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}-1\right]\le q_2\le n(L-t_1)\\ \dfrac{{q_2}^2}{N(t_1^-)}-(c_w\alpha(L-t_1)+\tau)q_2+c_w\alpha(L-t_1)N(t_1^-)+\\ \qquad c_3\left(q_2-\alpha(L-t_1)\left(N(t_1^-)-q_2\right)\right), \qquad\qquad \text{if }\; q_2\ge n(L-t_1)\end{cases} \tag{2.40}$$

Proposition 3. *For* $C2_2^1$*, the optimal trade-in quantity is obtained from the following rules.*

If $\dfrac{n}{N}\le\dfrac{\alpha}{1+\alpha L}$ *, then*

for $a_3\ge 0$ *and* $n\ge\dfrac{b_1N(t_1^-)}{2(L-t_1)}$ and $\dfrac{b_2}{2a_2}\le\dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}N(t_1^-)$

$$q_2=\frac{b_3}{2a_3} \tag{2.41}$$

for $a_3\ge 0$ *and* $n\ge\dfrac{b_1N(t_1^-)}{2(L-t_1)}$ and $\dfrac{b_2}{2}N(t_1^-)\ge\dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}N(t_1^-)$

$$q_2=\arg\min\left\{f_3\left(\frac{b_3}{2a_3}\right),f_2\left(\frac{b_2}{2a_2}\right)\right\} \tag{2.42}$$

for $a_3 \le 0$ *or* $n \le \dfrac{b_1 N\left(t_1^-\right)}{2\left(L-t_1\right)}$

$$q_2 = \begin{cases} \max\left\{n\left(L-t_1\right), \dfrac{b_1}{2a_1}\right\}, & \dfrac{b_1}{2a_1} \le \dfrac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)} \\ \min\left\{\dfrac{b_2}{2a_2}, N\left(t_1^-\right)\right\}, & \dfrac{b_2}{2a_2} \ge \dfrac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)} \\ \dfrac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)}, & \dfrac{b_2}{2a_2} \le \dfrac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)} \le \dfrac{b_1}{2a_1} \end{cases} \tag{2.43}$$

if $\dfrac{n}{N} \ge \dfrac{\alpha}{1+\alpha L}$ *, then*

for $a_4 \le 0$

$$q_2 = \min\left\{\max\left\{n\left(L-t_1\right), \frac{b_2}{2a_2}\right\}, N\left(t_1^-\right)\right\} \tag{2.44}$$

for $a_4 > 0$ *and* $a_3 \le 0$

$$q_2 = \arg\min\left\{\begin{array}{l} f_4\left(\min\left\{\max\left\{\dfrac{n}{\alpha}\left[\left(1+\dfrac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{n} - \alpha^2\left(L-t_1\right)^2\right)^{1/2} - 1\right], \dfrac{b_4}{2a_4}\right\}, n\left(L-t_1\right)\right\}\right), \\ f_2\left(\min\left\{\max\left\{n\left(L-t_1\right), \dfrac{b_2}{2a_2}\right\}, N\left(t_1^-\right)\right\}\right) \end{array}\right\} \tag{2.45}$$

for $a_4 > 0$ *and* $a_3 > 0$

$$q_2 = \arg\min \left\{ \begin{array}{l} f_3\left(\min\left\{ \max\left\{0, \frac{b_3}{2a_3}\right\}, \frac{n}{\alpha}\left[\left(1 + \frac{\frac{2\alpha^2 (L-t_1) N(t_1^-)}{n}}{\alpha^2 (L-t_1)^2} - \right)^{1/2} - 1 \right] \right\} \right) \\ f_4\left(\min\left\{ \max\left\{ \frac{n}{\alpha}\left[\left(1 + \frac{\frac{2\alpha^2 (L-t_1) N(t_1^-)}{n}}{\alpha^2 (L-t_1)^2} - \right)^{1/2} - 1 \right], \frac{b_4}{2a_4} \right\}, n(L-t_1) \right\} \right), \\ f_2\left(\min\left\{ \max\left\{ n(L-t_1), \frac{b_2}{2a_2} \right\}, N(t_1^-) \right\} \right) \end{array} \right\} \quad (2.46)$$

where

$f_i(q_2) = a_i q_2^2 - b_i q_2 + e_i$

$a_1 = a_2 = 1/N(t_1^-)$

$a_3 = \frac{1}{N(t_1^-)} - \frac{\alpha(c_w + c_2^0)}{2n}$

$a_4 = \frac{1}{N(t_1^-)} - \frac{\alpha(c_w - c_3)}{2n}$

$b_1 = \tau + c_2^0 + (c_w + c_2^0)\alpha(L - t_1)$

$b_2 = \tau - c_3 + (c_w - c_3)\alpha(L - t_1)$.

$b_3 = c_2^0 + \tau$

$b_4 = \tau - c_3$

$e_1 = (c_w + c_2^0)\alpha(L - t_1)N(t_1^-)$

$e_2 = (c_w - c_3)\alpha(L - t_1)N(t_1^-)$

$e_3 = (c_w + c_2^0)\alpha(L - t_1)\left(N(t_1^-) - \frac{n}{2}(L - t_1)\right)$

$$e_4 = (c_w - c_3)\alpha(L - t_1)\left(N(t_1^-) - \frac{n}{2}(L - t_1)\right).$$

The following proposition states that if q_2 from (2.34) is such that no warranties expire until time $t = L$, then the trade-in quantity given in Proposition 1 is optimal.

Proposition 4. *a) If*

$$\tau + c_2^0 + \alpha(L - t_1)(c_w + c_2^0) \geq 2 \qquad (2.47)$$

then Proposition 1 gives the optimal solution to $\mathrm{C}2_2^1$.

b) If the trade-in quantity given in Proposition 1 is not less than warranty expirations over interval $[t_1, L)$ (*i.e.*, $n(L - t_1)$), *then Proposition 1 gives the optimal solution to* $\mathrm{C}2_2^1$.

Proposition 4 begs the question of whether a similar conclusion may hold when A7b is relaxed so that there are no restrictions on $n(t)$ (other than $\int_0^L n(t)\,dt = N$ as required by the definition of L and N). For example, if the trade-in quantity given in Proposition 1 is not less than $\int_{t_1}^{L^-} n(t)\,dt$, then is the trade-in quantity optimal? The answer is no. For example, letting q_2 denote the trade-in quantity according to Proposition 1, if $n(t_1^+) = q_2$, then it is possible that a reduction in the quantity will reduce cost.

The following proposition identifies a condition under which the optimal cost is nonincreasing in n.

Proposition 5. *If* $c_3 \leq c_w$, *then* $C_2^1(t_1, q_2^*)$ *is nonincreasing in* n.

The condition of Proposition 5 clearly holds in settings where leftover over components have a salvage value (i.e., $c_3 \le 0$) because $c_w \ge 0$. If $c_3 > c_w$, then it is possible for $C_2^1\left(t_1, q_2^*\right)$ to get larger as n increases.[9]

Problem $\mathrm{C2}_3^1$: Assumptions A1 – A5, A6b, A7a, A8b

The problem is

$$\mathrm{C2}_3^1:\ \min_{q_2 \le N}\left\{C_3^1\left(t_1, q_2\right) \mid \text{A1-A5, A6b, A7a, A8b}\right\}.$$

Problem $\mathrm{C2}_3^1$ generalizes $\mathrm{C2}_1^1$ to allow $r > 0$ and $h > 0$, i.e., if $r = h = 0$, then the problems are identical. The run-out time of the trade-in quantity q_2 given in (2.20) reduces to

$$T_2(q_2) = \min\left\{\frac{q_2}{\alpha\left(N - q_2\right)}, L\right\}. \tag{2.48}$$

Substituting (2.31), (2.32), and (2.48) into (2.21) yields

[9] This result is possible when trade-ins are profitable and q_2^* is more than the remaining warranty demand. For example, let $t_1 = 0$, $N = 1000$, $L = 10$, $\alpha = 0.05$, $c_w = 0$, $c_n = 0.1$, $m = 0.5$, $\gamma = 0$, $c_2^0 = 0.4$, $c_3 = 0.1$. At $n = 40$, we have $q_2^* = 376.5$ and $C_2^1(q_2^*) = -190.6$. At $n = 50$, we have $q_2^* = 381.0$ and $C_2^1(q_2^*) = -189.9$.

$$C_3^1\left(t_1,q_2\right)=e^{-rt_1}\left[\begin{cases}\begin{aligned}&\frac{q_2^{\ 2}}{N}-\left(\begin{aligned}&\tau-\frac{h}{r}+\left(c_w-\frac{h}{r}\right)\frac{\alpha}{r}\left(1-e^{-r(L-t_1)}\right)\\&+\left(h\left(1+\alpha\left(L-t_1\right)\right)-c_2^0\alpha\left(1-e^{-r\left(q_2/\alpha\left(N-q_2\right)-\left(L-t_1\right)\right)}\right)\right)\frac{e^{-r(L-t_1)}}{r}\end{aligned}\right)q_2\\&\quad+\left(\begin{aligned}&\left(c_w-\frac{h}{r}\right)\left(\frac{1-e^{-r(L-t_1)}}{r}\right)+c_2^0\frac{e^{-r\left(q_2/\alpha\left(N-q_2\right)\right)}}{r}\\&-\left(c_2^0-h\right)\left(L-t_1\right)\frac{e^{-r(L-t_1)}}{r}\end{aligned}\right)\alpha N,\\&\qquad\qquad\qquad\qquad\text{if } q_2\le\frac{\alpha\left(L-t_1\right)N}{1+\alpha\left(L-t_1\right)}\\&\frac{q_2^{\ 2}}{N}-\left(\begin{aligned}&\tau-\frac{h}{r}+\left(c_w-\frac{h}{r}\right)\frac{\alpha}{r}\left(1-e^{-r(L-t_1)}\right)\\&+\left(\frac{h}{r}\left(1+\alpha\left(L-t_1\right)\right)-c_3\left(1-\alpha\left(L-t_1\right)\right)\right)e^{-r(L-t_1)}\end{aligned}\right)q_2\\&\quad+\left(\left(c_w-\frac{h}{r}\right)\left(\frac{1-e^{-r(L-t_1)}}{r}\right)-\left(c_3-\frac{h}{r}\right)\left(L-t_1\right)e^{-r(L-t_1)}\right)\alpha N,\\&\qquad\qquad\qquad\qquad\text{if } q_2\ge\frac{\alpha\left(L-t_1\right)N}{1+\alpha\left(L-t_1\right)}\end{aligned}\end{cases}\right]\tag{2.49}$$

The cost function is not analytically tractable, i.e., the function that applies when $q_2\le\frac{\alpha\left(L-t_1\right)N}{1+\alpha\left(L-t_1\right)}$ does not have stationary points that can be expressed in closed form. Numerical methods are required to determine the optimal trade-in quantity and cost.

We next consider the full trade-in policy under the requirement that the trade-in quantity is equal to second-stage demand (i.e., $q_2=\bar{D}\left(t_1\right)$), and we refer to this policy as the *restricted full trade-in policy*. Substituting $q_2=\bar{D}\left(t_1\right)=\frac{\alpha\left(L-t_1\right)N}{1+\alpha\left(L-t_1\right)}$ into (2.49) yields the second-stage cost

$$C_3^1(t_1) = Ne^{-rt_1}\left[\begin{array}{l}\left(\dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)^2 - \\ \left(\dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)\left(\tau-\dfrac{h}{r}-\left(c_w-\dfrac{h}{r}\right)\left(\dfrac{1-e^{-(L-t_1)r}}{r}\right)\left(\dfrac{1}{L-t_1}\right)\right)\end{array}\right]. \tag{2.50}$$

Substituting $\beta(t) = \dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}$ into (2.8) yields the trade-in credit

$$c_t(t_1) = p_n - \frac{1}{1+\alpha(L-t_1)}. \tag{2.51}$$

If $r = 0$, then (2.50) reduces to

$$C_3^1(t_1) = N\left[\left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)^2 - \left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)\left(\tau - c_w + \frac{h(L-t_1)}{2}\right)\right] \tag{2.52}$$

Problem $\mathrm{C2}_4^1$: Assumptions A1 – A5, A6b, A7b, A8b

The problem is

$$\mathrm{C2}_4^1:\ \min_{q_2 \le N}\left\{C_4^1(t_1, q_2) \mid \text{A1-A5, A6b, A7b, A8b}\right\}.$$

Problem $\mathrm{C2}_4^1$ generalizes $\mathrm{C2}_3^1$ to allow $n(t) = n > 0$, i.e., if $n = 0$, then the problems are identical. The following proposition identifies a condition on the warranty fall-off rate n that leads to a simple expression for the second-stage cost.

Proposition 6. *If*

$$\frac{n}{N} \le \left(L + 1/\alpha\right)^{-1} \tag{2.53}$$

then the cost of the restricted full trade-in policy is

$$C_4^1(t_1) = (N - nt_1)e^{-rt_1}\left[\begin{array}{l}\left(\dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)^2 - \\ \left(\dfrac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)\left(\begin{array}{l}\tau - \dfrac{h}{r} - \\ \left(c_w - \dfrac{h}{r}\right)\left(\dfrac{1-e^{-(L-t_1)r}}{r}\right)\left(\dfrac{1}{L-t_1}\right)\end{array}\right)\end{array}\right] \tag{2.54}$$

The left-hand side of (2.53) is the warranty fall-off rate expressed as a percent of the install base. The right-hand side of (2.53) is the reciprocal of the sum of the warranty horizon and the average time between failures. If the inequality holds, then the trade-in quantity is at least as large as the total second-stage warranty fall-off, regardless of when the second-stage begins. If the inequality does not hold, then trade-in quantity will be less that the total second-stage warranty fall-off, and a more complicated cost expression applies.

4.1.5 Matching Trade-in Policy – Second-Stage Policy 2

At time $t_1 < L$, when the inventory from the final order quantity reaches zero, the firm sets the trade-in credit $c_t(t)$ and the trade-in offer rate $\nu(t)$ so that component supply matches component demand over the remainder of the warranty horizon, i.e.,

$$s(t) = \beta(t)\nu(t) = d(t) \;\; \forall t \in [t_1, L] \tag{2.55}$$

where $\beta(t) = c_t(t) - \tau$ is the trade-in acceptance rate among those customers exposed to the trade-in offer at time t (see (2.7)), or the *trade-in fraction*. Note that $\beta(t)$ must be a valid fraction, i.e.,

$$\beta(t) \in [0, 1] \; \forall t \in [t_1, L]. \tag{2.56}$$

In contrast with a full trade-in policy where the entire warranty population receives the trade-in offer at time t_1, a matching policy allows the firm to set the offer rate $\nu(t)$ (and trade-in credit) over time so as to match supply with demand. Consequently, a matching trade-in policy is only viable in settings where the firm has access to customer-specific warranty data (e.g., otherwise the firm cannot control the fraction of customers with product under warranty who are exposed to the trade-in offer). The firm's choice of customers who will receive the trade-in offer over time is influenced by A4. Recall that A4 implies that customers with a soon-to-expire warranty are more likely to accept a trade-in offer than customers with a more distant warranty expiration date. In recognition of A4, the firm sends the trade-in offer to customers in order of warranty expiration date. From (2.11) , the component acquisition cost rate for the matching trade-in policy is

$$c_2^2(t) = \beta(t) + c_n + \gamma m - 1 = \beta(t) - \tau \ . \tag{2.57}$$

The cost of the matching trade-in policy is

$$\begin{aligned} C_2^2(t_1, \beta, \nu) &= \int_{t_1}^{L} e^{-rt}\left(c_2^2(t) + c_w\right) d(t)\, dt \\ &= e^{-rt_1}\int_{t_1}^{L} e^{-r(t-t_1)}\left(\beta(t) - \tau + c_w\right) d(t)\, dt \end{aligned} \tag{2.58}$$

where functions $\beta(t)$ and $\nu(t)$ satisfy (2.55) – (2.56) and $\nu(t)$ is such that a customer receives a trade-in offer no more than once (see A2).[10] We wish to minimize (2.58). In the remainder of this section we analyze several versions of problem.

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[10] A more general form of the matching trade-in policy matches supply with demand over time interval $[t_1, t_2)$ for $t_2 \leq L$, with demand over interval $[t_2, L)$ satisfied with purchases from the third-party supplier (e.g., as in the full trade-in policy). However, as illustrated in propositions and figures below, settings in which it is optimal to set $t_2 < L$ are also settings where the full trade-in policy tends to less costly than the matching trade-in policy, e.g., corresponds to settings where the trade-in acquisition cost per unit is positive. Consideration of this general matching policy increases complexity while adding relatively little value in additional insight, and we exclude from our analysis.

Problem $C2_1^2$: Assumptions A1 – A5, A6b, A7a, A8b

Due to A7a (i.e., $n(t) = 0$ for all t), we have $N\left(t_1^-\right) = N$. The requirement that a customer receives a trade-in offer no more than once implies

$$\int_{t_1}^{L} \nu\left(t\right) dt \le N \,, \tag{2.59}$$

e.g., $\int_{t_1}^{t} \nu\left(x\right) dx$ is the number of customers who have been exposed to the trade-in offer by time t. Thus, the problem can be stated as

$$C2_1^2: \min_{\beta(t)\in[0,1],\int_{t_1}^{L}\nu(t)dt\le N,\beta(t)\nu(t)=d(t)} \left\{ C_2^2\left(t_1,\beta,\nu\right) \mid \text{A1-A5, A6b, A7a, A8b} \right\}$$

Due to A5, the demand and supply rate over time interval $[t_1, L]$ is

$$d(t) = s(t) = \alpha N(t), \tag{2.60}$$

and due to A7a the warranty population is

$$N(t) = N - \int_{t_1}^{t} s\left(x\right) dx = N - \int_{t_1}^{t} d\left(x\right) dx = N - \int_{t_1}^{t} \alpha N\left(x\right) dx \tag{2.61}$$

We obtain an explicit expression for $N(t)$ by taking the limit of a discrete-time model as the time interval goes to zero. For notational convenience, we let $t_1 = 0$. Given time interval $\Delta > 0$, the failure rate per time interval Δ is $\alpha\Delta$, we have

$$d\left(\Delta\right) = s\left(\Delta\right) = \alpha\Delta N\left(\Delta\right) = \alpha\Delta\left[N - s\left(\Delta\right)\right]$$

$$s\left(\Delta\right) = \frac{\alpha\Delta N}{1+\alpha\Delta}$$

$$N\left(\Delta\right) = N\left(0\right) - s\left(\Delta\right) = N - \frac{\alpha\Delta N}{1+\alpha\Delta} = \frac{N}{1+\alpha\Delta}$$

$$s(2\Delta) = \alpha\Delta N(2\Delta) = \frac{\alpha\Delta N(\Delta)}{1+\alpha\Delta} = \frac{\alpha\Delta N}{(1+\alpha\Delta)^2}$$

$$N(2\Delta) = N(\Delta) - s(2\Delta) = \frac{N}{1+\alpha\Delta} - \frac{\alpha\Delta N}{(1+\alpha\Delta)^2} = \frac{N}{(1+\alpha\Delta)^2}$$

and, in general, for integer i

$$N(i\Delta) = \frac{N}{(1+\alpha\Delta)^i} \, . \tag{2.62}$$

Letting $t = i\Delta$

$$N(t) \quad = \lim_{\Delta\to 0} N\left(\left(\frac{t}{\Delta}\right)\Delta\right) = \lim_{\Delta\to 0} N(1+\alpha\Delta)^{-t/\Delta} = \lim_{\Delta\to 0} e^{\ln N(1+\alpha\Delta)^{-t/\Delta}} = e^{\lim_{\Delta\to 0}\ln N(1+\alpha\Delta)^{-t/\Delta}}$$

and

$$\lim_{\Delta\to 0}\ln N(1+\alpha\Delta)^{-t/\Delta} = \ln N + \lim_{\Delta\to 0}\frac{-t\ln(1+\alpha\Delta)}{\Delta} = \ln N + \lim_{\Delta\to 0}\frac{-\alpha t}{1+\alpha\Delta} \quad \text{(by L'Hospital)}$$

$$= \ln N - \alpha t \, .$$

Therefore, for $t \in [t_1, L)$,

$$N(t) = e^{\ln N - \alpha(t-t_1)} = N\left(t_1^-\right)e^{-\alpha(t-t_1)} = Ne^{-\alpha(t-t_1)} \tag{2.63}$$

$$d(t) = s(t) = \alpha N(t) = \alpha Ne^{-\alpha(t-t_1)} . \tag{2.64}$$

Substituting (2.64) into (2.58) yields second-stage cost

$$C_2^2(t_1, \beta, \nu) = e^{-rt_1}\int_{t_1}^{L} e^{-(\alpha+r)(t-t_1)}\left(\beta(t) - \tau + c_w\right)\alpha N dt \, . \tag{2.65}$$

From (2.55) and (2.64),

$$\beta(t) = \alpha Ne^{-\alpha(t-t_1)} / \nu(t), \tag{2.66}$$

By substituting (2.66) into (2.8) we see that $c_t(t)$ is decreasing in $\nu(t)$. Therefore a necessary condition for an optimal trade-in offer rate function is

$$\int_{t_1}^{L} \nu(t)\,dt = N \tag{2.67}$$

i.e., if $\int_{t_1}^{L} \nu(t)\,dt < N$, then an increase in $\nu(t)$ will decrease the trade-in discount and reduce $C_2^2(t_1, \beta, \nu)$. Thus, $\mathrm{C2}_1^2$ can be written as

$$\min_{\beta(t),\nu(t)} \left\{ C_2^2(t_1,\beta,\nu) \middle| \int_{t_1}^{L} \nu(t)\,dt = N, \beta(t)\nu(t) = \alpha N e^{-\alpha(t-t_1)}, \beta(t) \in [0,1] \right\}. \tag{2.68}$$

Proposition 7. *For* $\mathrm{C2}_1^2$, *if*

$$\left(e^{\alpha(L-t_1)}\right)^{1+\frac{r}{2\alpha}} - \left(1+\frac{r}{2\alpha}\right)e^{\alpha(L-t_1)} \le 1 \tag{2.69}$$

then the optimal trade-in fraction is

$$\beta(t) = \frac{\alpha\left(1-e^{-(\alpha+0.5r)(L-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha+0.5r}, \tag{2.70}$$

the optimal trade-in offer rate is

$$\nu(t) = \left(\frac{(\alpha+0.5r)e^{-0.5r(t-t_1)}}{1-e^{-(\alpha+0.5r)(L-t_1)}}\right)N\left(t_1^-\right)e^{-\alpha(t-t_1)}, \tag{2.71}$$

and the optimal cost is

$$C_2^2(t_1) = e^{-rt_1}N\left(t_1^-\right)\left[\left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(L-t_1)}\right)}{\alpha+0.5r}\right)^2 - \left(\frac{\alpha\left(1-e^{-(\alpha+0.5r)(L-t_1)}\right)}{\alpha+r}\right)(\tau-c_w)\right]. \tag{2.72}$$

Note that $\alpha(L-t_1)$ is the mean number of times that a component fails over the duration of the warranty horizon. Table 1 shows the maximum value of $\alpha(L-t_1)$ that ensures condition (2.69) is satisfied for various values of the net discount rate r. For example, if the component failure rate is 0.5% per year and the annual net discount rate is 5%, then the solution given in Proposition 7 is optimal as long as the warranty horizon is no more than 76 years (i.e., $r = 10\alpha$ and $\max(L - t_1) = 0.38/\alpha = 76$). In the computer industry that is motivating this work, component failure rates tend to be low (e.g., less than 1%) and the warranty duration is on the order of three to five years. In these settings, the condition given in (2.69) is likely to hold.

r	max value of $\alpha(L - t_1)$ to satisfy (2.69)	r	max value of $\alpha(L - t_1)$ to satisfy (2.69)
0	∞	12α	0.34
2α	0.88	14α	0.31
4α	0.63	16α	0.28
6α	0.51	18α	0.26
8α	0.43	20α	0.25
10α	0.38	22α	0.23

Table 3.1. Upper limit on component failure rate per $L - t_1$ periods as a function of the net discount rate r.

The following corollary gives the optimal solution for the special case of $r = 0$.

Corollary 1. *For* $C2_1^2$ *,if* $r = 0$, *then the optimal trade-in fraction is*

$$\beta(t) = 1 - e^{-\alpha(L-t_1)}, \tag{2.73}$$

the optimal trade-in offer rate is

$$v(t) = \left(\frac{\alpha}{1 - e^{-\alpha(L-t_1)}}\right) N\left(t_1^-\right) e^{-\alpha(t-t_1)}, \tag{2.74}$$

the total number of units traded in is

$$q_2 = \left(1 - e^{-\alpha(L-t_1)}\right) N\left(t_1^-\right), \tag{2.75}$$

and the optimal cost is

$$C_2^2\left(t_1\right)=N\left(t_1^-\right)\left[\left(1-e^{-\alpha\left(L-t_1\right)}\right)^2-\left(1-e^{-\alpha\left(L-t_1\right)}\right)\left(\tau-c_w\right)\right]. \tag{2.76}$$

We see that the optimal solution has a very simple structure when $r = 0$. In particular, the optimal trade-in fraction $\beta(t)$ is independent of time. This means that the optimal trade-in discount stays constant over the warranty horizon, i.e.,

$$c_t\left(t\right)=p_n-e^{-\alpha\left(L-t_1\right)}, \tag{2.77}$$

(obtained by substituting $\beta(t)$ into (2.8)). The term in the first parentheses in the right hand side of (2.74) is the fraction of the population exposed to the trade-in offer (i.e., $N\left(t_1^-\right)e^{-\alpha\left(t-t_1\right)}$ is the warranty population), which also stays constant over the warranty horizon. In contrast, at $r > 0$, we see that $\beta(t)$ is increasing in time, and consequently, the trade-in discount is increasing in time (e.g., the firm offers higher discounts later in the horizon, which are less costly for the firm). Similarly, the fraction of the population exposed to the trade-in offer is decreasing over time.

Problem $\mathbf{C2_2^2}$: Assumptions A1 – A5, A6b, A8b

This problem is the same as the previous problem except we relax the requirement of $n(t) = 0$ for all $t \in [0, L)$. Recall that the number of customers who do not receive a trade-in offer prior to warranty expiration is

$$\Gamma\left(\nu\right)=\max_{t\in[0,L]}\left(\int_0^t\left[n\left(x\right)-\nu\left(x\right)\right]dx\right)^+. \tag{2.78}$$

and that due to A2, any feasible trade-in policy must ensure that a customer does not receive a trade-in offer more than once, i.e.,

$$\Gamma(\nu)+\int_0^L \nu(t)\,dt = N\ . \tag{2.79}$$

Thus, the problem is

$$\mathrm{C2}_2^2: \min_{\beta(t)\in[0,1],\int_{t_1}^{L}\nu(t)dt+\Gamma(\nu)=N,\beta(t)\nu(t)=d(t)} \left\{C_2^2(t_1,\beta,\nu)\,|\,\text{A1-A5, A6b, A8b}\right\}$$

Proposition 8 identifies a condition under which the solution for $\mathrm{C2}_1^2$ given in Proposition 7 is also optimal for $\mathrm{C2}_2^2$. Proposition 9 shows that Proposition 7 gives the optimal solution when *n*(*t*) is constant for all $t \in [0, L)$.

Proposition 8. *If* $\left(e^{\alpha(L-t_1)}\right)^{1+\frac{r}{2\alpha}}-\left(1+\frac{r}{2\alpha}\right)e^{\alpha(L-t_1)}\le 1$ *and*

$$\int_{t_1}^{t} n(x)\,dx \le \int_{t_1}^{t}\left(\frac{\alpha+0.5r}{1-e^{-(\alpha+0.5r)(L-t_1)}}\right)N\left(t_1^-\right)e^{-(\alpha+0.5r)(x-t_1)}dx \ \textit{ for all } t\in[t_1, L], \tag{2.80}$$

then Proposition 7 gives the optimal solution to $\mathrm{C2}_2^2$.

Proposition 9. *If* $\left(e^{\alpha(L-t_1)}\right)^{1+\frac{r}{2\alpha}}-\left(1+\frac{r}{2\alpha}\right)e^{\alpha(L-t_1)}\le 1$ *and A7b holds* (*i.e.*, $n(t) = n$ *for all* $t \in [0, L)$), *then Proposition 7 gives the optimal solution to* $\mathrm{C2}_2^2$.

Note that the condition of Proposition 9 always holds when $r = 0$ (see Table 1).

4.1.6 Comparison of Second-Stage Policy Performance

In this section we compare the cost of the second-stage policies. We investigate how failure rates, remaining warranty time, and take-back processing cost affect optimal acquisition quantities, trade-in discounts, and optimal second-stage cost. And we identify conditions where the full trade-in policy outperforms the matching trade-in policy, and vice-versa.

We limit consideration to the restricted full trade-in policy (i.e., $q_2 = \bar{D}(t_1)$). The policy captures the basic structure of the full trade-in policy while maintaining tractability and thus facilitating interpretations. It also allows a more direct comparison with the matching trade-in policy in the sense that both policies set trade-in supply equal to demand, and only differ in the timing of trade-in transactions.

The case of r = h = n(t) = 0

We initially concentrate our analysis on the case of $r = h = n(t) = 0$ (i.e., assumptions A6a, A7a, A8a)—the case for which closed-form solutions are available for the full and matching second-stage trade-in policies. Under these assumptions, the restricted full trade-in policy is identical to the optimal full trade-in policy when

$$\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)} \in \begin{bmatrix} \frac{1}{2}\left(\tau + c_w\alpha(L-t_1) - c_3\left(1+\alpha(L-t_1)\right)\right), \\ \frac{1}{2}\left(\tau + c_w\alpha(L-t_1) + c_2^0\left(1+\alpha(L-t_1)\right)\right) \end{bmatrix} \tag{2.81}$$

(see Proposition 1).

To simplify notation in the following discussion, we let $x = \alpha(L - t_1)$. The value of x can be interpreted as the *aggregate failure rate* of the component over a period of length $L - t_1$. Let q_2^1 denote the trade-in quantity under a restricted full trade-in policy, i.e.,

$$q_2^1 = \left(\frac{x}{1+x}\right)N \tag{2.82}$$

(see Proposition 1). Let q_2^2 denote the trade-in quantity under a matching trade-in policy, i.e.,

$$q_2^2 = \left(1 - e^{-x}\right) N \tag{2.83}$$

(see Corollary 1). Note that $e^{-x} < \left(1 + x\right)^{-1}$ for all $x > 0$ (i.e., the terms correspond to continuous-time and discrete-time discounting factors). Thus

$$q_2^1 < q_2^2 \, . \tag{2.84}$$

The intuition underlying (2.84) is as follows. For both policies, the total number of trade-in units is equal to the total warranty claims over time interval $[t_1, L]$. The total warranty claims are lower for the restricted full trade-in policy because the warranty population is reduced at a single time instant t_1, and of course, any unit removed from the warranty population eliminates a possible future warranty claim on the obsolete product. In contrast, the matching trade-in policy reduces the warranty population gradually over the entire time interval $[t_1, L]$. Thus, the restricted full trade-in policy has an inherent advantage over the matching trade-in policy when warranty claims are costly (i.e., due to fewer warranty claims). Indeed, when $r = h = 0$, the only settings in which the matching trade-in policy can outperform the restricted full trade-in policy is when the firm can make money on each trade-in unit.

Figure 3.2 illustrates the relationship between the restricted full trade-in quantity, the matching trade-in quantity, and the acquisition quantity under the benchmark policy of no trade-in units (i.e., components are purchased from a third party).

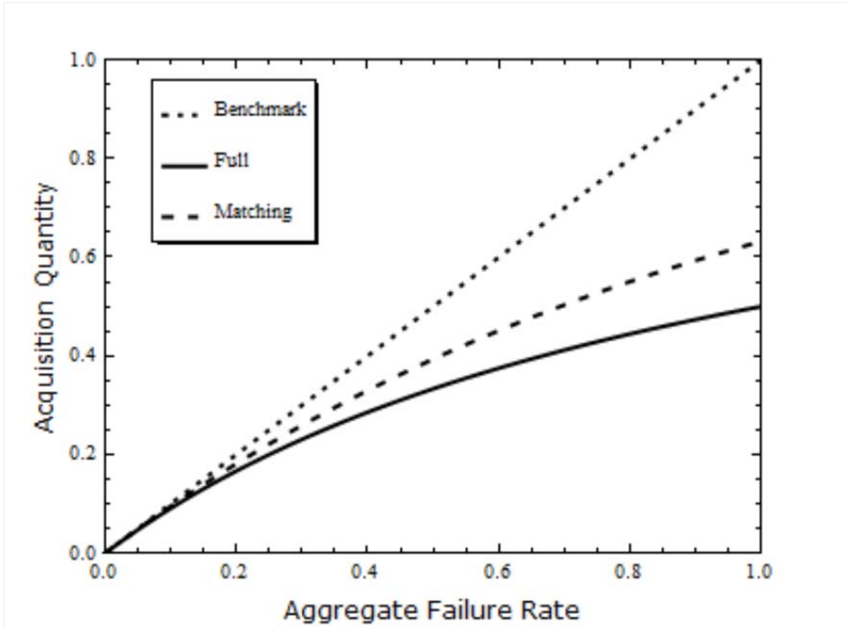


Figure 3.2. Second-stage acquisition quantities as a percent of the install base under the restricted full trade-in policy, matching trade-in policy, and the benchmark policy. The aggregate failure rate, $\alpha(L - t_1)$, is the expected number of failures per component over the warranty horizon.

As alluded to above, expression (2.84) and Figure 3.2 expose the essence of the comparative advantages and disadvantages of the two trade-in policies. The values of q_2^1 and q_2^2 are the total warranty claims under each policy. A restricted full trade-in policy has the advantage of fewer warranty claims through a single dramatic reduction of the install base at time t_1. In some settings, however, trade-in transactions are profitable, and in these settings a restricted full trade-in policy has the disadvantage of lower trade-in volume. These notions are formalized in the following proposition.

Proposition 10. $C_2^2(t_1) < C_2^1(t_1)$ *if and only if*

$$(\tau - c_w) > \frac{1+2x}{1+x} - e^{-x}. \tag{2.85}$$

Note that the right-hand-side of (2.85) is positive (follows from $e^{-x} < (1+x)^{-1}$ for all $x > 0$) and is increasing in x. Thus, conditions that favor the matching trade-in policy over the restricted full trade-in policy are a low aggregate failure rate (x), low warranty service cost (c_w), and high trade-in potential (τ)—conditions under which trade-in transactions are more likely to be profitable. Recall that the aggregate failure rate is $x = \alpha(L - t_1)$, and small values of x are associated with a low component failure rate (α) and/or a brief warranty horizon ($L - t_1$). Figure 3.3 shows the regions in which each trade-in policy is favored.

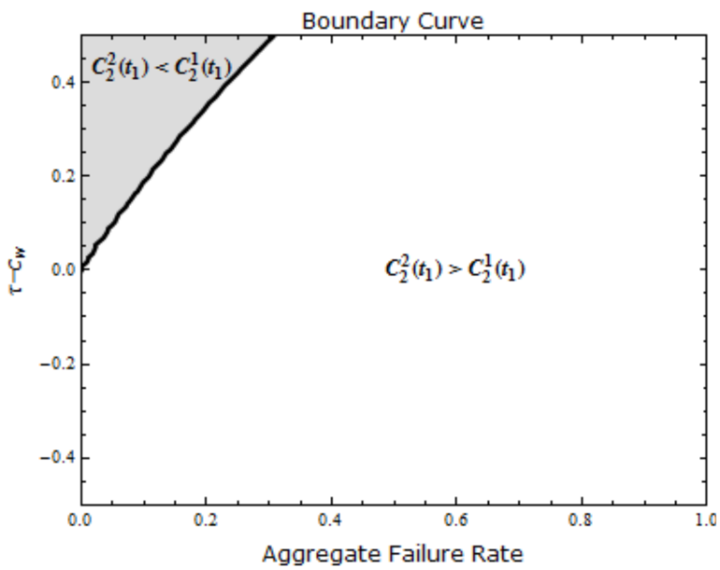


Figure 3.3. Boundary curve delineating the preference for the matching trade-in policy (to the left) and the restricted full trade-in policy (to the right). The curve is $(1 + 2x)/(1 + x) - e - x$, which defines the set of values satisfying $C_2^2(t_1) = C_2^1(t_1)$.

The fact that the processing of each warranty claim is costly (at unit cost c_w) combined with the fact that fewer warranty claims are processed under a restricted full trade-in policy than a matching trade-in policy hints that a restricted full trade-in policy may never cost too much more than a matching trade-in policy. Indeed, this is the case. The cost of each policy is

$$C_2^1(t_1) = N\left(\frac{x}{1+x}\right)\left(\frac{x}{1+x} - \tau + c_w\right) \tag{2.86}$$

$$C_2^2(t_1) = N\left(1 - e^{-x}\right)\left(1 - e^{-x} - \tau + c_w\right) \tag{2.87}$$

(follows from Proposition 1 and Corollary 1). From $1 - e^{-x} > x/(1 + x)$ (see (2.84)), it follows that

$$\tau - c_w \le 0 \Rightarrow C_2^1(t_1) \le C_2^2(t_1). \tag{2.88}$$

For positive $y = 1 - c_n - \gamma m - c_w = \tau - c_w$, the maximum percentage increase in second-stage cost when a restricted full trade-in policy is used instead of a matching trade-in policy is

$$w(y) = \max_{x \ge 0} \left\{ \frac{C_2^1(t_1) - C_2^2(t_1)}{\left|C_2^2(t_1)\right|} \right\}, \tag{2.89}$$

which is increasing in y. Furthermore, if parameters are such that $C_2^2(t_1) = 0$, then $C_2^1(t_1) < 0$ (due to $e^{-x} < (1+x)^{-1}$ for all $x > 0$). As a consequence, $w(y)$ is finite. Table 2 shows the values of $w(y)$ for various values of y (obtained via numerical search).

y	$w(y)$		y	$w(y)$
≤ 0	0%		0.6	4.9%
0.1	0.9%		0.7	5.6%
0.2	1.7%		0.8	6.3%
0.3	2.5%		0.9	7.0%
0.4	3.3%		1.0	7.7%
0.5	4.1%			

Table 3.2. Worst-case percentage increase in second-stage cost when a restricted full trade-in policy is used instead of a matching trade-in policy for various values of $y = \tau - c_w$.

The values of c_n, m, and c_w are nonnegative. Suppose γ is nonnegative, as is likely to be commonly the case in practice. Then we have $y \leq 1$ and, at worst, the restricted full trade-in policy is no more than about 8% more expensive than a matching trade-in policy. And, since the cost of restricted full trade-in policy is an upper bound on the cost of an unrestricted full trade-in policy, the same worst-case performance ratio applies. In contrast, the maximum percentage increase in second-stage cost when a matching trade-in policy is used instead of a restricted full trade-in policy is unbounded, i.e., a matching trade-in policy can be much more expensive than a restricted full trade-in policy.

Our main conclusions regarding the comparison of the restricted full trade-in policy with the matching trade-in policy for the case of $r = h = n(t) = 0$ are as follows: (i) the restricted full trade-in policy has fewer warranty claims due to upfront reduction in the warranty install base; (ii) the restricted full trade-in policy favors medium-to-high aggregate failure rates, medium-to-high warranty service cost, and low-to-moderate trade-in potential; (iii) the matching trade-in policy favors low aggregate failure rates, low warranty service cost, and high trade-in potential; (iv) at worst, the restricted full trade-in policy is no more than 8% more expensive than the matching policy.

We now consider how the trade-in policies compare with the benchmark policy. A trade-in program differs from the benchmark in two ways—volume and unit acquisition cost. We begin by examining unit acquisition cost. The average acquisition cost per unit for each policy is

$$\text{benchmark policy: } c_2^0 \tag{2.90}$$

$$\text{restricted full trade-in policy: } c_2^1 = \frac{\alpha\left(L - t_1\right)}{1 + \alpha\left(L - t_1\right)} - \tau \tag{2.91}$$

matching trade-in policy: $c_2^2 = 1 - e^{-\alpha(L-t_1)} - \tau$ (2.92)

((2.90) follows from (2.5), (2.91) follows from (2.82) and (2.86), (2.92) follows from (2.83) and (2.87)). The preceding expressions lead to the following proposition.

Proposition 11. *a) The acquisition cost per unit under the restricted full trade-in policy is less than the acquisition cost per unit under the benchmark policy if and only if*

$$\frac{q_2^1}{N} = \frac{\alpha(L-t_1)}{1+\alpha(L-t_1)} < c_2^0 + \tau \ . \tag{2.93}$$

b) The acquisition cost per unit under the matching full trade-in policy is less than the acquisition cost per unit under the benchmark policy if and only if

$$\frac{q_2^2}{N} = 1 - e^{-\alpha(L-t_1)} < c_2^0 + \tau \ . \tag{2.94}$$

Proposition 11 presents two inequalities with a volume term on the left and a cost term on the right. The left-hand side of inequality (2.93) is the number of trade-in returns (also warranty demand) as a percent of the install base under a restricted full trade-in policy (see (2.82)). Similarly, the left-hand side of inequality (2.94) is the number of trade-in returns (also warranty demand) as a percent of the install base under a matching trade-in policy (see (2.83)). The higher aggregate failure rate, $\alpha(L - t_1)$, the higher the return fraction, the larger the required trade-in credit, and the more likely that the trade-in acquisition cost will be higher than the benchmark.

The right-hand side of (2.94) is the sum of the benchmark cost term, c_2^0, and the trade-in potential $\tau = (1 - \gamma)m - (p_n - 1)$. Even without Proposition 11 it is clear that a larger value of c_2^0 makes it more likely that a trade-in acquisition cost will be lower than the benchmark acquisition cost. What is perhaps more interesting are the roles of the other parameters. Proposition 11 shows that the aggregate failure rate, trade-in potential, and third-party unit cost are the key drivers of the inequality between the trade-in and benchmark acquisition cost per unit.

Relative to the benchmark, a trade-in program results in lower warranty demand. Thus, the total cost of a trade-in program may be lower than the total cost of the benchmark even if both of the inequalities in Proposition 11 don't hold. Proposition 12 compares the total cost of benchmark and trade-in policies.

Proposition 12. *a) The total cost under the restricted full trade-in policy is less than the total cost unit under the benchmark policy if and only if*

$$\frac{1}{1+\alpha\left(L-t_1\right)}<\frac{c_2^0+c_w}{c_2^1+c_w}. \tag{2.95}$$

or equivalently,

$$c_w > \frac{1}{1+x}-\frac{\tau}{x}-c_2^0\left(\frac{1+x}{x}\right). \tag{2.96}$$

b) The total cost under the matching trade-in policy is less than the total cost unit under the benchmark policy if and only if

$$\frac{1-e^{-\alpha\left(L-t_1\right)}}{\alpha\left(L-t_1\right)}<\frac{c_2^0+c_w}{c_2^2+c_w}. \tag{2.97}$$

or equivalently,

$$c_w > \frac{\left(1-e^{-x}\right)^2}{x-\left(1-e^{-x}\right)}-\tau\left(\frac{1-e^{-x}}{x-\left(1-e^{-x}\right)}\right)-c_2^0\left(\frac{x}{x-\left(1-e^{-x}\right)}\right). \tag{2.98}$$

The term on the left-hand side of (2.95) and (2.97) is a ratio of trade-in-to-benchmark volumes, which as noted above, is less than 1 (e.g., Figure 3.2). The term on the right-hand side of (2.95) and (2.97) is the ratio of benchmark-to-trade-in acquisition and warranty service cost per unit. If the inequalities in Proposition 11 hold, then the inequalities in Proposition 12 also hold (i.e., the inequalities in Proposition 11 imply $c_2^1 < c_2^0$ and $c_2^2 < c_2^0$). Figure 3.4 illustrates how the least-cost policy is affected by parameter values.

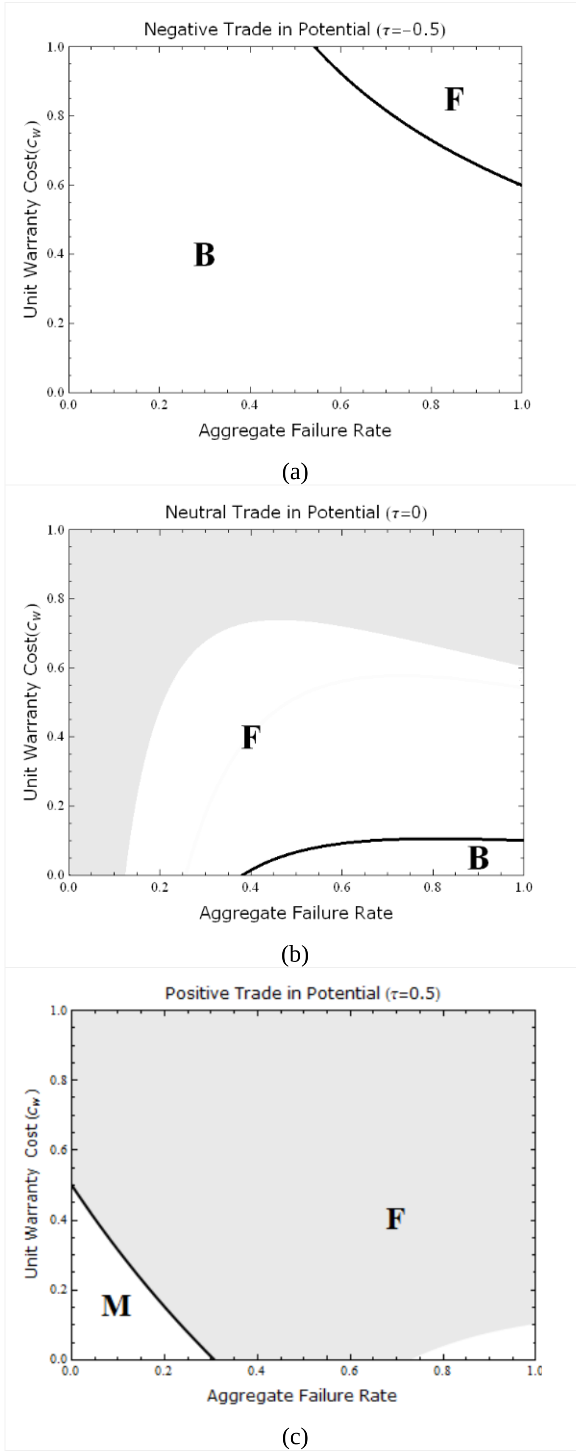


Figure 3.4. Boundary curves delineating the preference for the restricted full trade-in policy (indicated by F), the matching trade-in policy (indicated by M), and the benchmark policy of no trade-ins (indicated by B). c_2^0 = 0.2 in all three plots. The shading signifies that the cost of the restricted full trade-in policy is equal to the full trade-in policy.

The trade-in potential in the plot (a) of Figure 3.4 is $\tau = (1 - \gamma)m - (p_n - 1) = -0.5$. If $\gamma = 1$ for example (e.g., repeat purchase rate of 100%), then the setting is one in which the firm must offer a trade-in discount of at least 0.5 to trigger a return (i.e., high trade-in resistance). It is a setting where the unit acquisition cost of a trade-in program at any return volume is more than twice the benchmark cost of $c_2^0 = 0.2$ (i.e., $c_{2|x=0}^1 = c_{2|x=0}^2 = -\tau = 0.5$). We see that, even in this rather extreme setting, the restricted full trade-in policy is less expensive than the benchmark when the unit warranty service cost and the aggregate failure rate are high. For this combination of parameter values, the benefit of lower warranty demand achieved through the trade-in program more than offsets the unit acquisition cost premium.

For plot (b) in Figure 3.4 we have $\tau = (1 - \gamma)m - (p_n - 1) = 0$. In this setting, $c_2^1 \le c_2^0 = 0.2$ for $x \le 0.25$. Thus, the restricted full trade-in policy clearly dominates the benchmark policy when the aggregate failure rate is less than 0.25 (i.e., due to lower unit acquisition cost and lower return volume). The difference between the boundary curve and vertical line at $x = 0.25$ is the region in which the higher trade-in unit acquisition cost is more than offset by the reduction in warranty demand, relative to the benchmark. As the unit warranty cost (c_w) increases, the reduction in volume afforded by the trade-in program adds more value, as reflected in the widening of this region as c_w increases.

Plot (c) in Figure 3.4 illustrates a setting where the trade-in potential is positive at $\tau = (1 - \gamma)m - (p_n - 1) = 0.5$, due to a combination of low trade-in resistance ($p_n - 1$), low consumer loyalty (γ), and high margin (m). It is a setting where the firm may make a profit on trade-in volume (i.e., negative unit acquisition cost). For example, the unit acquisition cost of the restricted full trade-in policy (c_2^1) is negative over all aggregate failure rates in the figure, and the unit acquisition cost of the matching trade-in policy (c_2^2) is negative up to aggregate failure rate $x < -\ln(0.5) \approx 0.7$. The restricted full and matching trade-in policies dominate the benchmark over the entire area of the plot. The regions of trade-in policy preference reinforce the message of Figure 3.3: matching tends to be favored over restricted full when the aggregate failure rate is low, the trade-in potential is high, and the warranty service cost per unit is low.

We emphasize that the restricted full trade-in policy is a limited form of a full trade-in policy, and that the full trade-in policy dominates the benchmark policy. This is because the full trade-in policy optimally satisfies component warranty demand through a combination of trade-ins and purchases from a third party. Rewriting (2.81) in terms of the warranty service cost per unit, the cost of the restricted full trade-in policy is the same as full trade-in policy when

$$c_w \in \left[\frac{2}{1+x} - \frac{\tau}{x} - c_2^0 \left(\frac{1+x}{x} \right), \frac{2}{1+x} - \frac{\tau}{x} + c_3 \left(\frac{1+x}{x} \right) \right]. \tag{2.99}$$

The “F” regions in Figure 3.4 are shaded above the function that defines the left endpoint in (2.99). Given that it is not economical to acquire more trade-ins than warranty demand (e.g., high disposal cost), the shading indicates that the full trade-in policy sets trade-in volume to match warranty demand (i.e., restricted full trade-in and full trade-in policies are identical). In the non-shaded areas, the firm can reduce cost by acquiring some components through a trade-in program and some components from the third party.

For the remainder of this section, we examine how the relative performance among policies is impacted by relaxing various assumptions. We begin by considering a change in the way warranties expire over time.

Impact of linearly decreasing warranty population

The preceding analysis assumes that all warranties expire at time *L*. Suppose that warranties expire linearly at rate $n > 0$ (i.e., $n(t) = n$). The cost of the matching trade-in policy is unaffected by a linear decreasing warranty population (see Proposition 9). If

$$\frac{n}{N} \le \frac{\alpha}{1+\alpha L}, \tag{2.100}$$

then the cost of the restricted full trade-in policy is also unaffected by a linear decreasing warranty population (see Proposition 6), and the worst-case performance results in Table 2 continue to hold.

The worst-case performance percentages can be larger when the inequality given in (2.100) is reversed. The reason stems from the differing effects of increasing *n* on the two policies when trade-ins are profitable: a larger value of *n* translates into fewer profitable trade-ins for the restricted full trade-in policy whereas the number of trade-ins is unaffected by *n* in a matching trade-in policy.

The following two propositions identify conditions for the dominance of trade-in policies over the benchmark.

Proposition 13. *a) If*

$$\alpha\left(L-t_1\right)-\frac{\alpha n\left(L-t_1\right)^2}{2\left(N-nt_1\right)}<c_2^0+\tau \tag{2.101}$$

then the acquisition cost per unit under the restricted full trade-in policy is less than the acquisition cost per unit under the benchmark policy.

b) The acquisition cost per unit under the matching full trade-in policy is less than the acquisition cost per unit under the benchmark policy if and only if

$$\frac{q_2^2}{N} = 1 - e^{-\alpha(L-t_1)} < c_2^0 + \tau \ . \tag{2.102}$$

Proposition 14. *a) If*

$$\alpha\left(L-t_1\right) - \frac{\alpha n\left(L-t_1\right)^2}{2\left(N-nt_1\right)} < c_2^0 + \tau \tag{2.103}$$

then the total cost unit under the restricted full trade-in policy is less than the total cost under the benchmark policy.

b) The total cost under the matching trade-in policy is less than the total cost under the benchmark policy if and only if

$$\frac{1-e^{-\alpha(L-t_1)}}{\alpha\left(L-t_1\right)\left(1-0.5\alpha n\left(L-t_1\right)/\left(N-nt_1\right)\right)} < \frac{c_2^0 + c_w}{c_2^2 + c_w} \ . \tag{2.104}$$

or equivalently,

$$c_w > \frac{\left(1-e^{-x}\right)^2}{x\left(1-a\right)-\left(1-e^{-x}\right)} - \tau\left(\frac{1-e^{-x}}{x\left(1-a\right)-\left(1-e^{-x}\right)}\right) - c_2^0\left(\frac{x\left(1-a\right)}{x\left(1-a\right)-\left(1-e^{-x}\right)}\right). \tag{2.105}$$

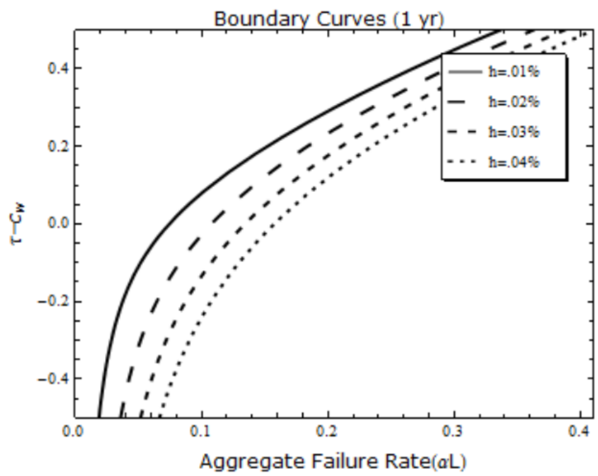
where $a = 0.5\alpha n\left(L-t_1\right)/\left(N-nt_1\right)$.

Impact of positive holding cost rate

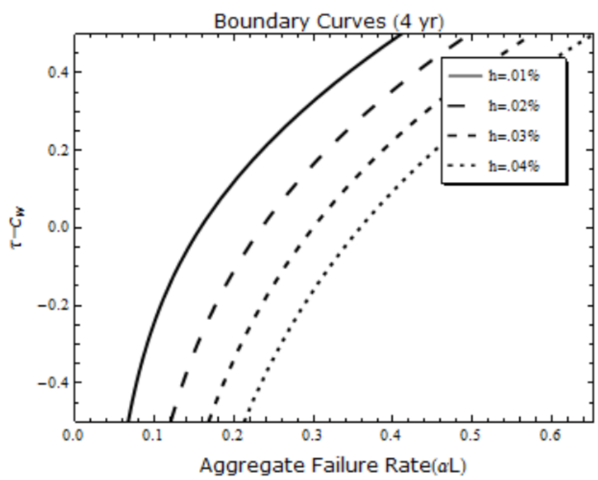
Under the restricted full trade-in policy, the firm acquires $N(t_1^-)\alpha(L-t_1)/[1+\alpha(L-t_1)]$ units at time t_1 with component inventory reaching zero at time L as the second stage ends. Since the matching policy maintains zero inventory over the duration of the second-stage, we know that the relative attractiveness of the matching policy is increasing in the holding cost rate *h*. Updating expression (2.86) for $h > 0$ (e.g., by taking the limit of (2.50) as *r* approaches zero) , yields

$$C_2^1(x) = N\left[\left(\frac{x}{x+1}\right)^2 - \left(\frac{x}{x+1}\right)(\tau - c_w) + \frac{h(L-t_1)}{2}\left(\frac{x}{x+1}\right)\right] \qquad (2.106)$$

The cost of the matching policy is unaffected by $h > 0$ and (2.87) holds. Figure 3.5 illustrates how the boundary curve in Figure 3.3 shifts to the right as *h* increases.



(a)



(b)

Figure 3.5. The curves satisfy $C_2^2\left(t_1\right)=C_2^1\left(t_1\right)$ for alternative values of *h*. Plot (a) shows the indifference curves when the duration of the second-stage is 1 period (i.e., $L - t_1 = x/\alpha = 1$). Plot (b) shows the indifference curves when the duration of the second-stage is 4 periods (i.e., $L - t_1 = x/\alpha = 4$).

We now consider how the performance of the trade-in polices compares to the benchmark. Note that since the unit acquisition cost is unaffected by $h > 0$, Proposition 11 continues to hold, i.e.,

$$c_2^1 < c_2^0 \Leftrightarrow \frac{\alpha\left(L-t_1\right)}{1+\alpha\left(L-t_1\right)} < c_2^0 + \tau \tag{2.107}$$

$$c_2^2 < c_2^0 \Leftrightarrow 1 - e^{-\alpha(L-t_1)} < c_2^0 + \tau \ . \tag{2.108}$$

Furthermore, a change from $h = 0$ to $h > 0$ results in no change in cost under the matching trade-in policy, which maintains zero inventory. And a change from $h = 0$ to $h > 0$ causes a larger absolute increase in cost under the benchmark policy than under the restricted full trade-in policy. This is because the warranty demand is higher under the benchmark policy (i.e., trade-ins lower warranty demand). Therefore the necessary and sufficient conditions for trade-in policy dominance identified in Proposition 12 for the case of $h = 0$ are sufficient conditions for trade-in policy dominance when $h > 0$.

Proposition 15. *a) If*

$$\frac{1}{1+\alpha\left(L-t_1\right)} < \frac{c_2^0 + c_w}{c_2^1 + c_w} \ . \tag{2.109}$$

or equivalently,

$$c_w > \frac{1}{1+x} - \frac{\tau}{x} - c_2^0\left(\frac{1+x}{x}\right), \tag{2.110}$$

then the total cost under the restricted full trade-in policy is less than the total cost unit under the benchmark policy.

b) If

$$\frac{1-e^{-\alpha(L-t_1)}}{\alpha\left(L-t_1\right)} < \frac{c_2^0 + c_w}{c_2^2 + c_w} \ . \tag{2.111}$$

or equivalently,

$$c_w > \frac{\left(1-e^{-x}\right)^2}{x-\left(1-e^{-x}\right)} - \tau\left(\frac{1-e^{-x}}{x-\left(1-e^{-x}\right)}\right) - c_2^0\left(\frac{x}{x-\left(1-e^{-x}\right)}\right) \tag{2.112}$$

then the total cost under the matching trade-in policy is less than the total cost unit under the benchmark policy.

Combined impact of positive discount rate and holding cost rate

We next examine the case of $r > 0$ and $h > 0$. Substituting $x = \alpha(L - t_1)$ into (2.50) and (2.72) yields

$$C_2^1(x) = Ne^{-rt_1}\left[\left(\frac{x}{x+1}\right)\left(\frac{x}{x+1} - \left(\tau - \frac{h}{r}\right) + \left(c_w - \frac{h}{r}\right)\left(\frac{1-e^{-(L-t_1)r}}{(L-t_1)r}\right)\right)\right] \tag{2.113}$$

$$C_2^2(x) = Ne^{-rt_1}\left[\left(\frac{x\left(1-e^{-0.5(L-t_1)r-x}\right)}{x+0.5(L-t_1)r}\right)^2 - \left(\frac{x\left(1-e^{-0.5r(L-t_1)-x}\right)}{x+(L-t_1)r}\right)(\tau - c_w)\right] \tag{2.114}$$

The introduction of $r > 0$ lowers the magnitude of the total cost of our two second-stage trade-in policies. However, the above expressions do not permit a simple prediction about how changes in the discount rate affect the relative performance of the restricted full and matching trade-in policies. Figure 3.6 illustrates how the boundary curves in Figure 3.5 shift as r increases when the remaining warranty life is short (i.e., $L - t_1 = 1$). (The range of $1 - c_n - \gamma m - c_w = \tau - c_w$ is the same in figures 3.5 and 3.6, from -0.5 to 0.5.) The darker regions, labeled M_1 and M_2, and the lighter regions, labeled F_1 and F_2, indicate areas where either the matching or the restricted full trade-in policy holds a relative cost advantage. The subscripts on M and F reflect the magnitude of the cost advantage where the larger subscript indicates a larger cost advantage (i.e., $C_2^2(t_1 \mid M_2) - C_2^1(t_1 \mid M_2) \leq C_2^2(t_1 \mid M_1) - C_2^1(t_1 \mid M_1) \leq 0 \leq C_2^2(t_1 \mid F_1) - C_2^1(t_1 \mid F_1) \leq C_2^2(t_1 \mid F_2) - C_2^1(t_1 \mid F_2)$).

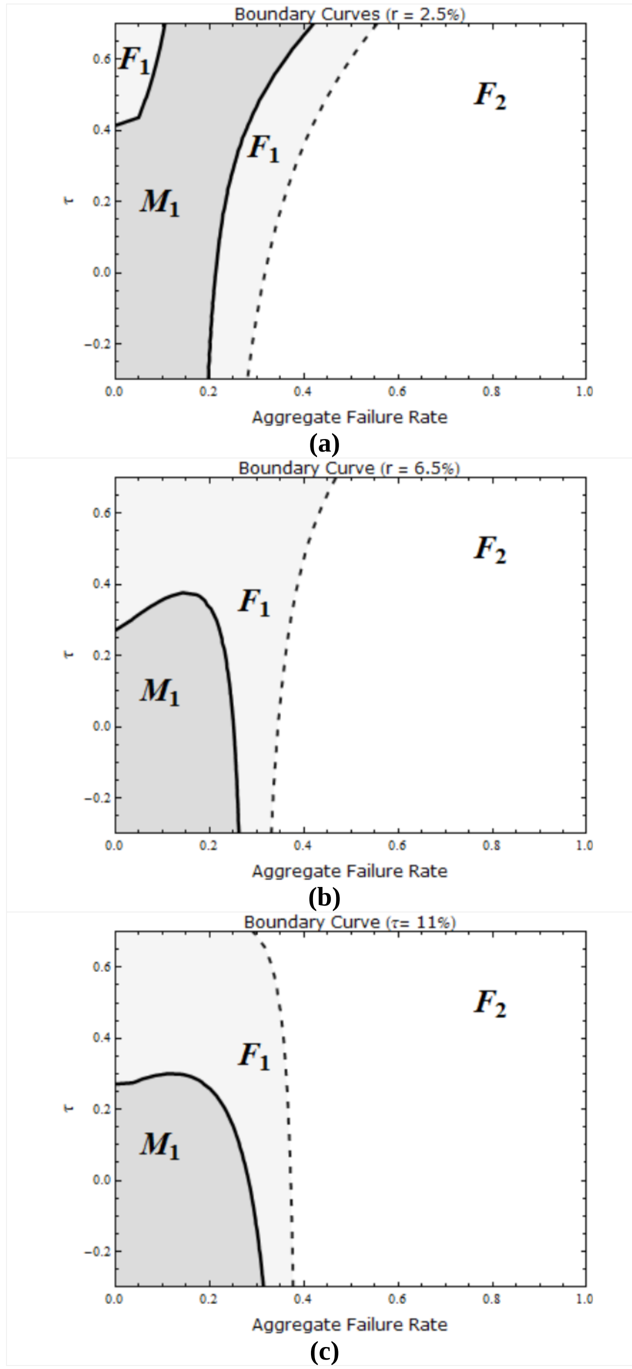
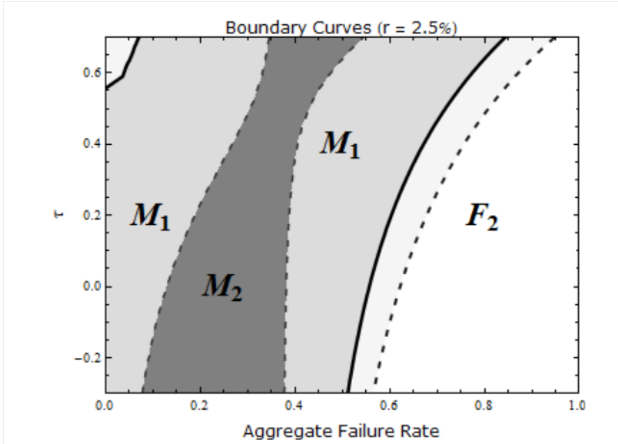


Figure 3.6. The curves separating M_1 and F_1 satisfy $C_2^2(t_1) = C_2^1(t_1)$. The plots are generated with parameter values $L - t_1 = x/\alpha = 1$= (i.e., the duration of the second-stage is one period), $h = 0.08$, and $c_w = 0.2$. The values of r are 2.5% (a), 6.5% (b), and 11% (c).

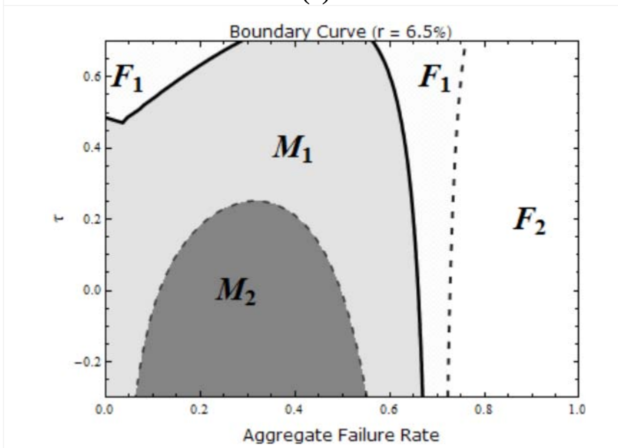
When the discount rate is low (e.g., plot (a) of Figure 3.6 where r = 2.5%), the regions of preference between our two second-stage policies are similar to case of $r = 0$ (see figures 3.3 and 3.5). That is, the restricted full trade-in policy has an advantage when phased-out component has a high failure rate and the matching trade-in policy has an advantage when component failure rate is low. When failure rates are moderate, the matching (restricted full) trade-in policy has an advantage when transactions are profitable (costly). However, note the region labeled F_1 in the upper left corner of plot (a). In this region, failure rates are low and trade-in transactions are relatively profitable. Without discounting, the matching policy has an advantage in this region; but under discounting, the restricted full trade-in policy is preferred. This shift in rankings is driven by the way discounting affects cash flows from trade-in transactions and by when these cash flows are realized under the different policies. Recall that more units are returned under the matching trade-in policy. So when trade-in transactions are profitable, the matching policy has an advantage. However, when $r > 0$, the gains from future trade-in transactions are discounted. On the other hand, return volumes and transaction gains under the restricted full trade-in policy occur at the beginning of the time horizon, and these gains are uninfluenced by discounting.

We see the effect of discounting playing a relatively larger role in the matching policy than in the restricted full policy when trade-ins are costly. Observe that the portion of the lower region in which matching is preferred moves to the right as r increases. Trade-ins are costly in this region (e.g., trade-in potential τ is negative). The high acquisition cost is discounted in the matching policy, whereas this cost is unaffected by the value of r in the restricted full policy, thus increasing the attractiveness of the matching policy.

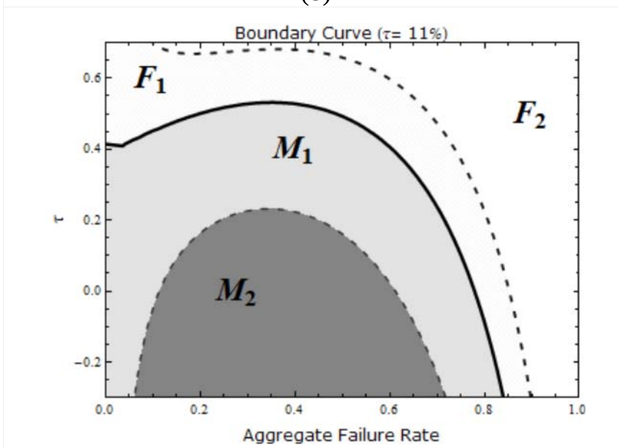
These two basic effects of increasing discount rate—restricted full displacing matching in the upper left region and matching displacing restricted full in the lower region—become more pronounced as the duration (and warranty claim volume) of the second stage increases. This is shown in Figure 3.7, where the value of $L - t_1$ is increased from 1 (in Figure 3.6) to 4.



(a)



(b)



(c)

Figure 3.7. The curves separating M_1 and F_1 satisfy $C_2^2(t_1) = C_2^1(t_1)$. The plots are generated with parameter values $L - t_1 = x/\alpha = 4$, $h = 0.08$, and $c_w = 0.2$. The values of r are 2.5% (a), 6.5% (b), and 11% (c).

We now consider how the performance of the trade-in polices compares to the benchmark. The following proposition presents sufficient conditions for trade-in policy dominance over the benchmark policy.

Proposition 16. *a) If*

$$c_2^0 > \frac{\alpha\left(L-t_1\right)}{\left(1+\alpha\left(L-t_1\right)\right)^2} - \frac{\tau}{1+\alpha\left(L-t_1\right)},\text{[11]} \tag{2.115}$$

then the total discounted cost under the restricted full trade-in policy is less than the total discounted cost unit under the benchmark policy.

b) If

$$c_2^0 > 1 - e^{-\alpha\left(L-t_1\right)} - \tau \tag{2.116}$$

then the total discounted cost under the matching trade-in policy is less than the total discounted cost unit under the benchmark policy.

4.2 First-Stage Problem

In this section we direct our attention to the manufacturer's final order decision problem. The first-stage cost of policy i given final order quantity q_1 is denoted $C_1^i\left(q_1\right)$, and the optimum is C_1^i, where

benchmark policy:	$i = 0$
restricted full trade-in policy:	$i = 1$
matching trade-in policy:	$i = 2$.

[11] Note that (2.115) is equivalent to (2.95) in Proposition 12 if $c_w = 0$.

4.2.1 First-Stage Cost Functions

We develop cost expressions for two first-stage problems that differ by the assumptions and the second-stage trade-in policy in effect.

Problem $C1_1^i$: Assumptions A1 – A5, A6a, A7a, A8a

We initially concentrate our analysis on the case of $r = h = n(t) = 0$ (i.e., assumptions A6a, A7a, A8a)—the case for which closed-form solutions are available for the full and matching second-stage trade-in policies. Due A6a – A8a, we can rewrite the first-stage cost function given in (2.3) as

$$C_1^i\left(q_1\right) = \left(c_1 + c_w\right)q_1 + C_2^i\left(t_1\left(q_1\right)\right) \tag{2.117}$$

where the superscript i denotes the second-stage policy in effect and $t_1 = q_1/(\alpha N)$ is the run-out time of the final order quantity q_1. (From $-c_3 < c_1$ it follows $q_1 \le \alpha LN$ in an optimal solution, i.e., the firm does not order more than total demand.)

As in Section 4.1.6, we consider the optimal full trade-in policy under the requirement that the trade-in quantity is equal to second-stage demand (i.e., $q_2 = \bar{D}\left(t_1\right)$), i.e., the restricted full trade-in policy. As noted in the previous section, the function captures the basic structure of the full trade-in policy while maintaining tractability and thus facilitating interpretations. It also allows a more direct comparison with the matching trade-in policy in the sense that both policies set trade-in supply equal to demand, and only differ in the timing of trade-in transactions.

The first-stage costs under the three policies are

Benchmark; no trade-ins

$$C_1^0\left(q_1\right)=\left(c_1+c_w\right)q_1+\left(c_2^0+c_w\right)\left(\alpha LN-q_1\right) \tag{2.118}$$

Restricted full trade-in policy

$$C_1^1\left(q_1\right)=\left(c_1+c_w\right)q_1+N\left[\begin{array}{l}\left(\dfrac{\alpha LN-q_1}{\alpha LN-q_1+N}\right)^2- \\ \left(\dfrac{\alpha LN-q_1}{\alpha LN-q_1+N}\right)\left(\tau-c_w\right)\end{array}\right] \tag{2.119}$$

Matching trade-in policy

$$C_1^2\left(q_1\right)=\left(c_1+c_w\right)q_1+N\left[\left(1-e^{-\frac{\alpha LN-q_1}{N}}\right)^2-\left(1-e^{-\frac{\alpha LN-q_1}{N}}\right)\left(\tau-c_w\right)\right] \tag{2.120}$$

(follows from (2.5), (2.33) with q_2 = $\dfrac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)}$, and (2.76)). The first term in each of the above cost expressions is the acquisition and warranty service cost attributed to the final order quantity, i.e., the first-stage cost. The second term in each expression is the acquisition and warranty service cost in the second stage. The following proposition characterizes the optimal solutions to the first-stage problems.

Proposition 17. *Let* $a = \tau - c_w$ and $b = c_1 + c_w$. *The optimal final order quantity given a benchmark second-stage policy is*

$$q_1^0=\alpha LN \tag{2.121}$$

and the optimal cost is

$$C_1^0=\left(c_1+c_w\right)\alpha LN\ . \tag{2.122}$$

The optimal final order quantity given a restricted full trade-in second-stage policy is:

$$\textit{for}\ \left(\frac{2-a}{3b}\right)^3-\left(\frac{1}{b}\right)^2<0:\ q_1=0 \tag{2.123}$$

$$for\ \left(\frac{2-a}{3b}\right)^3-\left(\frac{1}{b}\right)^2\geq 0:\ q_1^1=\underset{q\in\left\{\substack{\alpha LN-z_1N,\alpha LN-z_2N,\\ \alpha LN-z_3N,\alpha LN}\right\}}{\arg\min} C_1^1(q) \tag{2.124}$$

$$where\ z_i=\max\left\{0,\min\left\{2\left(\frac{2-a}{3b}\right)^{1/2}\cos\left(\frac{1}{3}\left(\cos^{-1}\left(-\frac{1}{b}\left(\frac{3b}{2-a}\right)^{3/2}\right)+2(i-1)\pi\right)\right)-1,\alpha L\right\}\right\}.$$

The optimal final order quantity given a matching trade-in second-stage policy is:

$$for\ (2-a)^2-8b\leq 0:\ q_1=0 \tag{2.125}$$

$$for\ (2-a)^2-8b>0:\ q_1^2=\underset{q\in\{0,\alpha LN-z_1N,\alpha LN-z_2N,\alpha LN\}}{\arg\min}\left\{C_1^2(q)\right\} \tag{2.126}$$

$$where\ z_i=\max\left\{0,\min\left\{\ln\left(\frac{2-a+(-1)^i\left[(2-a)^2-8b\right]^{1/2}}{2b}\right),\alpha L\right\}\right\}.$$

Problem $C1_2^i$: Assumptions A1 – A5, A6b, A7a, A8b

Now we consider the effects of $r > 0$ and $h > 0$ Discounting becomes especially important in settings characterized by rapid growth and innovation (e.g., r is a proxy for the firm's return on capital net of inflation, which tends to be larger in industries with high growth and innovation). Substituting (2.113) and (2.114) into (2.117) , we get the expressions from the first-stage discounted cost for the benchmark, matching and restricted full trade-in policies:

<u>Benchmark; no trade-ins</u>

$$C_1^0(q_1)=q_1\left(c_1+\frac{h}{r}\right)+\alpha N\left(c_w-\frac{h}{r}\right)\left(\frac{1-e^{-r(q_1/\alpha N)}}{r}\right)+\alpha N\left(c_2^0+c_w\right)\left(\frac{e^{-\frac{2q_1r}{\alpha N}}-e^{-Lr-\frac{q_1r}{\alpha N}}}{r}\right) \tag{2.127}$$

Restricted full trade-in policy

$$C_1^1(q_1) = \left(c_1 + \frac{h}{r}\right) q_1 + \alpha N \left(c_w - \frac{h}{r}\right)\left(\frac{1 - e^{-\frac{q1r}{\alpha N}}}{r}\right) +$$
$$Ne^{-\frac{q_1 r}{\alpha N}} \left(\frac{\alpha LN - q_1}{\alpha LN - q_1 + N}\right)\left(\begin{array}{l} \dfrac{\alpha LN - q_1}{\alpha LN - q_1 + N} - \tau + \dfrac{h}{r} - \\ \left(\dfrac{\alpha N}{\alpha NL - q_1}\right)\left(\dfrac{h}{r} - c_w\right)\left(\dfrac{1 - e^{-r\left(\frac{\alpha NL - q_1}{\alpha N}\right)}}{r}\right)\end{array}\right) \tag{2.128}$$

Matching trade-in policy

$$C_1^2(q_1) = \left(c_1 + \frac{h}{r}\right) q_1 + \alpha N \left(c_w - \frac{h}{r}\right)\left(\frac{1 - e^{-\frac{q_1 r}{\alpha N}}}{r}\right) +$$
$$Ne^{-\frac{q_1 r}{\alpha N}} \left[\frac{\alpha^2\left(1 - e^{-\frac{(2\alpha + r)(\alpha NL - q_1)}{2\alpha N}}\right)^2}{\left(\alpha + \frac{r}{2}\right)^2} - \frac{\alpha(\tau - c_w)\left(1 - e^{-\frac{(2\alpha + r)(\alpha NL - q_1)}{2\alpha N}}\right)}{\alpha + r}\right] \tag{2.129}$$

4.2.2 Trade-in Policy Dominance Conditions and Worst-Case Cost Savings

Proposition 18 presents a series of sufficient conditions for trade-in policy dominance over the benchmark. Proposition 19 presents lower bounds on the percent savings achieved by replacing the benchmark policy with a trade-in policy. The propositions expose key determinants of trade-in policy dominance over the benchmark, and the degree of dominance.

Proposition 18. *Sufficient conditions for trade-in policy dominance are as follows:*

Problem	i	Conditions that assure $C_1^i < C_1^0, i \in \{1,2\}$	
$n=0$, $h=0$, $r=0$	1	$c_1 > \left(\frac{1}{1+\alpha L}\right)\left(\frac{\alpha L}{1+\alpha L} - \tau - c_w \alpha L\right)$	(2.130)
	2	$c_1 > \left(\frac{1-e^{-\alpha L}}{\alpha L}\right)\left(1-e^{-\alpha L} - \tau - c_w\left(\frac{\alpha L - \left(1-e^{-\alpha L}\right)}{1-e^{-\alpha L}}\right)\right)$ $\geq 1-e^{-\alpha L} - \tau - c_w\left(\frac{\alpha L - \left(1-e^{-\alpha L}\right)}{1-e^{-\alpha L}}\right)$	(2.131)
$n>0$, $h=0$, $r=0$	1	$c_1 > \alpha L\left(1-\frac{Ln}{2N}\right) - \tau$	(2.132)
	2	$c_1 > \left(\frac{1-e^{-\alpha L}}{\alpha L\left(1-\frac{Ln}{2N}\right)}\right)\left(1-e^{-\alpha L} - \tau - c_w\left(\frac{\alpha L\left(1-\frac{Ln}{2N}\right) - \left(1-e^{-\alpha L}\right)}{1-e^{-\alpha L}}\right)\right)$	(2.133)
$n=0$, $h>0$, $r=0$	1	$c_2^0 > \frac{\alpha L}{1+\alpha L} - \tau$	(2.134)
	2	$c_2^0 > 1-e^{-\alpha L} - \tau$	(2.135)
$n=0$, $h>0$, $r>0$	1	if $\tau \geq 0$: $c_2^0 > \left(\frac{1}{1+\alpha L}\right)\left(\frac{\alpha L}{1+\alpha L} - \tau\right)$	(2.136)
	2	$c_2^0 > 1-e^{-\alpha L} - \tau$	(2.137)

Proposition 19. *a) If* $h = r = n(t) = 0$, *then*

$$1 - C_1^1 / C_1^0 \geq 1 - \left(\frac{1}{(1+\alpha L)(c_1 + c_w)}\right)\left(\frac{\alpha L}{1+\alpha L} - \tau + c_w\right) \tag{2.138}$$

$$1 - C_1^2 / C_1^0 \geq 1 - \left(\frac{1-e^{-\alpha L}}{\alpha L(c_1 + c_w)}\right)\left(1-e^{-\alpha L} - \tau + c_w\right). \tag{2.139}$$

b) If $h = r = 0$ *and* $n(t) = n$, *then*

$$1-C_1^2/C_1^0 \geq 1-\frac{\left(1-e^{-\alpha L}\right)\left(1-e^{-\alpha L}-\tau+c_w\right)}{\left(\alpha L-\frac{\alpha n L^2}{2N}\right)\left(c_1+c_w\right)}. \tag{2.140}$$

4.3 Numerical Illustrations

Section 4.1.6 provides comparative analysis of second-stage policy performance. Figure 3.4, in particular, exposes the determinants of relative performance among the policies. In this section we explore the extent to which the drivers of relative performance that are illustrated in Figure 3.4 continue to hold in the first-stage problem. We do this through numerical illustrations that correspond to the three plots in Figure 3.4.

Figure 3.4 displays behavior in a single-stage problem (i.e., stage two) with c_2^0 = 0.2. In our numerical illustrations, we set c_1 = 0.2. This allows us to capture how the introduction of a multistage element of the problem affects insights from our single-stage analysis. In particular, the benchmark policy covers all warranty demand with the final order quantity (due to $r = h = 0$), and consequently, there is no second stage under this policy. The cost of the benchmark policy for the first-stage problem is identical to the cost of the benchmark policy for the second-stage problem that underlies the results in Figure 3.4. Thus, any differences in behavior between Figure 3.4 and Figure 3.8 below are solely due to the additional flexibility afforded to the trade-in policies through a two-stage decision (i.e., acquire some units from the vendor and some units from trade-ins as opposed to all units from trade-ins).

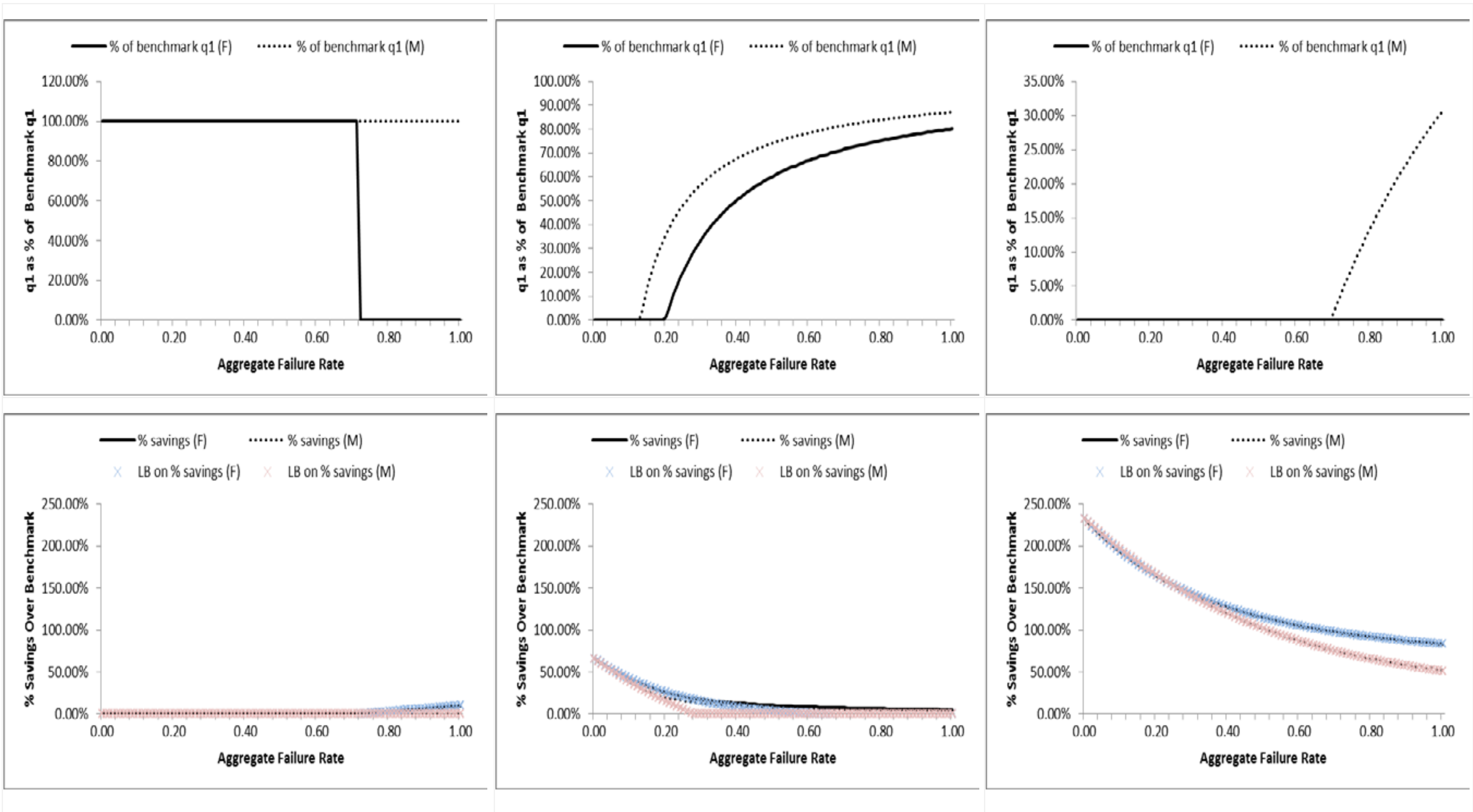


Figure 3.8. First-stage order quantity as a percent of the benchmark first-stage order quantity, percent cost savings over the benchmark, and lower bound on the percent costs savings over the benchmark. The parameter values are $c_1 = 0.2, \tau = -0.5$, $c_w = 0.8$ (left plot), $c_1 = 0.2, \tau = 0$, $c_w = 0.1$ (middle plot), and $c_1 = 0.2, \tau = 0.5$, $c_w = 0.1$ (right plot).

The left plot of Figure 3.4 shows the regions of policy preference in a setting where the trade-in potential is negative, i.e., $\tau = -0.5$. In this setting, the unit acquisition cost of a trade-in program at any return volume is more than twice the benchmark acquisition cost. It is a setting that is very unfavorable to a trade-in policy. Even in this rather extreme setting, the restricted full trade-in policy is less expensive than the benchmark when the unit warranty service cost and the aggregate failure rate are high; the benefit of lower warranty demand achieved through the trade-in program more than offsets the unit acquisition cost premium. The left plots in Figure 3.8 correspond to parameter values associated with a horizontal line drawn at $c_w = 0.8$ in Figure 3.4. If we study Figure 3.4 at $c_w = 0.8$, we see that the benchmark is preferred up to an aggregate failure rate of 0.71, with the restricted full trade-in policy preferred for larger values. This behavior is replicated in Figure 3.8. In fact, this is a scenario where our second-stage analysis completely informs first-stage policy performance. We see that the restricted full trade-in policy sets q_1 to cover entire demand up to an aggregate failure rate of 0.71 (i.e., reduces to the benchmark policy), and sets $q_1 = 0$ thereafter. The two-stage problem reduces to single-stage problem over all aggregate failure rates; all units are either acquired from the vendor or from trade-ins. The optimal matching trade-in policy sets q_1 to cover all warranty demand over all aggregate failure rates (i.e., trade-ins are never profitable under this policy).

The bottom-left plot in Figure 3.8 shows that the percent savings due to the restricted full trade-in policy is relatively small, hitting a maximum of 10% at an aggregate failure rate of 100%. We also see that the lower bound on percent savings is tight. The lower bound on percent savings given in Proposition 19 derives from propositions that apply to second-stage problem, and is tight whenever the solution to the first-stage problem results in a single-stage decision (i.e., the second stage is not activated), as is the case here.

The middle plot of Figure 3.4 shows the region of policy preference in a setting where the trade-in potential is neutral, i.e., $\tau = 0$. The middle plots in Figure 3.8 correspond to parameter values associated with a horizontal line drawn at $c_w = 0.1$ in Figure 3.4. If we study Figure 3.4 at $c_w = 0.1$, we see that the restricted full trade-in policy is preferred up to an aggregate failure rate of 0.67, with the benchmark policy preferred for larger values. In contrast with the left plots in Figure 3.8, we now see a difference in behavior that stems from the multistage element of the problem. In particular, the benchmark policy is never preferred. For aggregate failure rates larger than 0.67, which is where the benchmark policy is preferred under second-stage analysis, both trade-in policies exploit the flexibility of dividing the source of components between the final order quantity and trade-ins from customers. We see that the fraction acquired from the vendor increases as the aggregate failure rate increases. This reflects the fact that the trade-in acquisition cost per unit is increasing in the trade-in quantity (see (2.11)), so more volume is shifted to the vendor as warranty demand increases.

The bottom-middle plot in Figure 3.8 shows that the percent savings due to the trade-in policies is significant when the aggregate failure rate is small (e.g., 64% at a 1% aggregate failure rate). The savings diminish as the aggregate failure rate increases, reaching 5% and 4% at an aggregate failure rate of 100% for the restricted full and matching trade-in policies, respectively. We also see that a gap between the actual percent savings and the lower bound on percent savings arises at the point where the trade-in policies begin dividing the source of components between the vendor and customers. This is as expected because, as noted above, the lower bound is derived from the case where the optimal decision reduces to a single-stage form. Finally, we note that the restricted full trade-in policy dominates the matching trade-in policy over the entire region of the plot, which is consistent with the results in Figure 3.4.

The right plot of Figure 3.4 shows the region of policy preference in a setting where the trade-in potential is positive, i.e., $\tau = 0.5$. As in the middle plots, the right plots in Figure 3.8 correspond to parameter values associated with a horizontal line drawn at $c_w = 0.1$ in Figure 3.4. If we study Figure 3.4 at $c_w = 0.1$, we see that the matching trade-in policy is preferred up to an aggregate failure rate of 0.23, with the restricted full trade-in policy preferred for larger values. This boundary point of preference between the two trade-in policies is unchanged in right plots of Figure 3.8. In this example, the regions of preference are unaffected by the multistage element of the problem. In fact, we can see that the restricted full trade-in policy reduces to a single-stage problem over all aggregate failure rates (i.e., no units are ordered from the vendor). The matching trade-in policy, on the other hand, begins to divide the sourcing of components between the vendor and customers once the aggregate failure rate reaches 70%. As we saw in our second-stage analysis, the matching trade-in policy results in larger warranty demand than the restricted full trade-in policy. And it is this difference, with the consequent pressure on the acquisition cost per unit, that drives the more liberal use of sourcing flexibility in the matching trade-in policy.

The main lesson in the bottom-right plot in Figure 3.8 pertains to the significance of the savings achieved through the use of a trade-in policy. The restricted full trade-in policy saves more than 100% of the benchmark cost when the aggregate failure rate is less than 66% (51% for the matching trade-in policy), and is as high as 230% for low aggregate failure rates. Over this range, the profit from trade-in volume more than offsets the cost of servicing warranty claims. The plot also reinforces the insight from second-stage analysis that the restricted full trade-in policy is not a bad choice when the discount and holding cost rates are close to zero. For the data in the example, the profit from the restricted full trade-in policy is, at most, 3.3% less than the profit from the matching trade-in policy.

We conclude by summarizing the preceding discussion in the form of two observations. First, insights relating to regions of policy preference from our second-stage analysis largely transfer to the first-stage problem with one clarification: second-stage analysis tends to understate the value of trade-in policies relative to the benchmark. For some parameter values, the trade-in policies are able to exploit the flexibility of dual sourcing of components—a flexibility that is not present in the second-stage problem. This flexibility is relatively more valuable for trade-in policies because the firm may be able to make money from trade-in transactions when the trade-in volume is low. In contrast, acquiring units from a third-party vendor, which is a sourcing option under the benchmark policy, is always costly. Second, the savings obtained by supplementing the final order quantity with a trade-in program can be significant, even to the point where the firm is able to make money servicing warranty claims. The single most significant indicator of a savings opportunity is the trade-in potential. Trade-in potential is the difference between the increase in profit from locking in disloyal customers and the minimum markdown required for a customer to accept a trade-in offer (i.e., trade-in resistance). If trade-in potential is positive, it is a strong positive signal for the use of a trade-in policy.

5. Summary and Conclusions

In this essay, we consider the final order problem that arises when a vendor announces that production of a component will cease in the near future. Our interest in this problem comes from discussions with management at a large computer manufacturer. While the firm is typically not producing the product when the supplier issues a component phase-out announcement (CPOA), there is an install base of products under warranty that must be serviced in the event of component failure. The firm is interested in understanding the merits supplementing the final order quantity with a trade-in program that is offered to owners of obsolete product under warranty. Components harvested from returned products can be used to satisfy future warranty claims while simultaneously reducing the install base (and future warranty claims) and potentially generating profits from sales that would not have otherwise occurred.

While there has been research on the final order quantity problem and there has been research on the design and merits of trade-in programs, the combination of these two elements has, to our knowledge, not been studied. There is a pressing need for research on this problem for two reasons. First, product lifecycles are shrinking and outsourcing is expanding, which are increasing the incidence of CPOAs. Second, the prevalence of trade-in programs is increasing. This is in part due to economic reasons as more firms are discovering opportunities for value recovery in end-of-use product. And in part due to environmental reasons, whether driven by take-back legislation or the desire to enhance a firm's reputation of environmental stewardship. Interestingly, the firm motivating this work has an active trade-in program located in a division that is separate from the warranty group. The presence of this activity has prompted the warranty group to think about the possibility of targeted trade-in programs that offer both marketing benefits (e.g., locking in disloyal customers) while providing a source of obsolete components for warranty claims. The purpose of this essay is to help understand when the use of a trade-in program is likely to be a high value opportunity and, in settings of high value, provide guidance on trade-in program design.

Under the benchmark policy, which reflects the firm's current policy for responding to a CPOA, the firm places a final order with the vendor that is large enough to cover warranty demand. In addition to a benchmark policy, we consider two simple trade-in policies—a full trade-in policy and a matching trade-in policy. The full trade-in policy offers a trade-in discount to all owners with product under warranty shortly before the component supply reaches zero. The matching trade-in policy also goes into effect as component supply reaches zero, but makes the trade-in offer to select owners over time so as to match supply with demand. Both trade-in policies dominate the benchmark policy (i.e., feasible region includes the benchmark solution). Our analysis illuminates the relationship between parameter values and the degree of dominance over the benchmark (e.g., magnitude of cost savings) and conditions under which each trade-in policy is preferred. We identify sufficient conditions for trade-in policy dominance over the benchmark and we obtain lower bounds on the percent savings obtained by replacing the benchmark policy with a trade-in policy.

We find that the single most important driver of trade-in policy value is the trade-in potential—a simple measure that is the difference between the increase in profit from locking in disloyal customers by means of a trade-in transaction and the minimum markdown required for a customer to accept a trade-in offer. If trade-in potential is positive, it is a strong positive signal for the use of a trade-in policy. Indeed, the use of a trade-in program in these settings can drive warranty service costs below zero, especially when the failure rate over the warranty horizon is moderate (e.g., 20% or less).

We also find that the use of a full trade-in policy is relatively robust—it will not perform much worse than the matching trade-in policy as long as the discount rate (net of inflation) and the holding cost rate are not too high, and will potentially lead to much greater savings. The reason is that a full trade-in policy benefits from a single large reduction in the install base that maximizes the reduction in future claims, and consequently, warranty service costs. This is a fortuitous result in the sense that a matching trade-in policy may not even be an option for some firms. A matching trade-in policy requires that the firm have a warranty database that includes owner contact information. Firms that sell directly to customers will often have access to this level of detail, but this is less likely to be the case for firms that sell through distributors or retailers.

While a full trade-in policy is generally a safe choice, there are three key indicators of when a matching trade-in policy is likely to be preferred: (1) positive trade-in potential, (2) low warranty service cost per unit, and (3) a high inventory holding cost rate. The reason for (1) and (2) is that, relative to the full trade-in policy, trade-in volume (and warranty demand) is higher. Higher volume can lead to higher profit when trade-ins are profitable (positive trade-in potential) and the cost to service a claim is low (low warranty service cost per unit). The reason for (3) is clear—by aligning supply with demand, the matching trade-in policy maintains no inventory. In contrast, component inventory can be significant under the full trade-in policy.

In summary, this essay offers three broad contributions. First, we introduce an important problem that has not previously been studied. It is a problem that draws on two major branches in the literature—literature on the final order problem and literature on the design and merits of trade-in programs. It is an area ripe for additional research. Second, we introduce a parsimonious, yet rich, model that is defined by five basic assumptions (A1 – A5). All warranty service cost expressions and optimal policy decisions flow from these assumptions. Our investigation has touched on a small set of questions and results, and we believe there is ample opportunity for further study using this model as is, or as a foundation for a richer model. Third, we provide insight into the merits and effective use of trade-in policies as discussed above.

We offer two directions for future research. First, there is a need for a broader assessment of trade-in policy designs. The full trade-in policy and the matching trade-in policy are just two of many possibilities. There is a question of how good or bad these policies perform relative to a wider set of alternatives. In our appendix we introduce discrete dynamic programming algorithms that can be used to investigate this question. One algorithm returns the optimal cost under a full trade-in policy, one algorithm returns the optimal cost under a matching trade-in policy, and a third algorithm returns the optimal cost with no restrictions on the trade-in policy. By comparing solutions returned by these algorithms, one will be able to develop a sense of how the optimal policy compares with the full and matching trade-in policies.

Second, there is a need to consider the impact of uncertainty in warranty demand. Compared to the benchmark of placing a very large final order up-front, we know that the introduction of uncertainty will generally increase the value of a trade-in program (e.g., by virtue of a sourcing alternative if realized demand is greater than the final order quantity). However, there are important unanswered questions on the degree of value-added as related to the nature of uncertainty and how uncertainty may shape the design of an effective trade-in program.

6. Appendix

6.1 Notation

L	=	time until all warranties expire if there is no trade-in program, i.e., length of the warranty horizon
$n(t)$	=	rate of reduction in components under warranty at time t due to warranty expiration
$N(t)$	=	number of working components under warranty at time t, i.e., warranty population
N	=	number of components under warranty at time 0, i.e., $N = N(0)$
α	=	component failure rate
x	=	aggregate failure rate at time t_1, i.e., $x = \alpha(L - t_1)$
$d(t)$	=	component demand rate at time t, i.e., due to warranty claims
$D(t)$	=	cumulative component demand through time t
$\bar{D}(t)$	=	total component demand from time t to time L, i.e., $\bar{D}(t) = D(L) - D(t)$
$T_1(q_1)$	=	run-out time of final order quantity q_1
$T_2(q_2)$	=	run-out time of second-stage full trade-in quantity q_2 given realization t_1 of $T_1(q_1)$
$t_2(q_2)$	=	time after realization t_1 of $T_1(q_1)$ for the warranties of q_2 trade-in customers to expire, or L, whichever is smaller for a full trade-in policy
r	=	net discount rate, e.g., cost of capital less inflation
h	=	inventory holding cost per unit-period excluding the cost of capital (e.g., storage fees)

c_1	=	component cost/unit in the final order
$c_2^i(t)$	=	total cost per unit to acquire and refurbish a component via policy i at time t (e.g., accounts for trade-in price discount, cannibalization, and refurbishment)
c_3	=	component disposal cost per unit (e.g., $c_3 < 0$ implies a salvage value, which is less than the final order cost/unit, i.e., $-c_3 < c_1$)
c_w	=	warranty claim service cost, net of component acquisition cost (e.g., cost per unit for disassembly, component replacement, reassembly, test, and shipping)
$c_t^i(t)$	=	trade-in price reduction on the new model of the product at time t according to policy i, i.e., trade-in discount
c_n	=	unit cost of the new model of the product
M	=	margin on the new model of the product
p_n	=	price of the new model of the product, i.e., $p_n = c_n + m$
γ	=	fraction of trade-in customers who would have purchased the new model at full price in the future if the trade-in program was not offered (e.g., repeat purchase rate)
τ	=	trade-in potential, i.e., $\tau = (1 - \gamma)m - (p_n - 1)$
V	=	valuation of the new model when exchanged for the old model under warranty
$s(t)$	=	component trade-in volume at time t
$\nu(t)$	=	rate at which customers are exposed to trade-in offer at time t, i.e., trade-in offer rate
$\beta(t)$	=	fraction of customers who elect to trade-in their product from among those receiving a trade-in offer at time t, i.e., trade-in acceptance rate
q_1	=	final order quantity of the component

q_2 = trade-in quantity under a full trade-in policy

6.2 Assumptions

A1. The trade-in discount is only available to customers with product under warranty

A2. A customer receiving a trade-in offer receives a single take-it-or-leave-it offer and accepts the offer if consumer surplus is positive

A3. The valuation of the new model in exchange for the old model under warranty is independent of time and is uniformly distributed with range normalized to [0, 1]

A4. The warranty expiration date of a customer who accepts a trade-in offer is no later than the warranty expiration date of a customer who rejects a trade-in over

A5. The component failure rate is constant

A6a. $r = 0$

A6b. $r \geq 0$

A7a. $n(t) = 0$ for all $t \in [0, L)$ and $n(L) = N$, by the definition of L; see (2.30))

A7b $n(t) = n$ for all $t \in [0, L)$ (and $n(L) = N - nL \geq 0$, by the definition of L; see (2.30))

A8a. $h = 0$

A8b. $h \geq 0$

6.3 Derivations and Proofs

Proof of Proposition 1. The second-stage cost is

$$C_2^1(t_1, q_2) = \begin{cases} f_1(q_2), \; q_2 \leq \dfrac{e_1 - e_2}{b_1 - b_2} \\ f_2(q_2), \; q_2 \geq \dfrac{e_1 - e_2}{b_1 - b_2} \end{cases} \tag{2.141}$$

where

$f_i(q_2) = aq_2^2 - b_i q_2 + e_i$

$a = 1/N$

$b_1 = 1 + c_w\alpha\left(L - t_1\right) - c_n - \gamma m + c_2^0\left(1 + \alpha\left(L - t_1\right)\right)$

$b_2 = 1 + c_w\alpha\left(L - t_1\right) - c_n - \gamma m - c_3\left(1 + \alpha\left(L - t_1\right)\right)$

$e_1 = \left(c_w + c_2^0\right)\alpha\left(L - t_1\right)N$

$e_2 = \left(c_w - c_3\right)\alpha\left(L - t_1\right)N$.

(see (2.33)). The function $f_i(q_2)$ is convex and thus is minimized at stationary point

$$q_2 = \frac{b_i}{2a}. \tag{2.142}$$

Note that $-c_3 < c_2^0$ (e.g., $-c_3 \geq c_2^0$ implies that firm can sell a leftover component for more than the cost to buy the component from a third party). Note that $1 - c_n - m$ is the fraction of the population with positive utility from purchasing a new product, which cannot be negative (i.e., $1 - c_n - m \geq 0$). Thus, $b_1 \geq 0$ and $a > 0$. In addition, $b_1 > b_2$, $e_1 > e_2$, and thus

$$\frac{b_2}{2a} < \frac{b_1}{2a} \tag{2.143}$$

$$f_2(0) < f_1(0) \tag{2.144}$$

and the two curves intersect at

$$q_2 = \frac{e_1 - e_2}{b_1 - b_2}. \tag{2.145}$$

The optimal trade-in quantity q_2 follows from (2.142) – (2.145):

$$\text{if } \frac{b_1}{2a} \leq \frac{e_1 - e_2}{b_1 - b_2}\text{, then } \frac{b_2}{2a} \leq \frac{e_1 - e_2}{b_1 - b_2} \text{ \& thus } q_2 = \frac{b_1}{2a} \text{ \& } C_2^1\left(t_1, q_2\right) = f_1\left(\frac{b_1}{2a}\right) \tag{2.146}$$

if $\frac{b_2}{2a} \geq \frac{e_1 - e_2}{b_1 - b_2}$, then $\frac{b_1}{2a} \geq \frac{e_1 - e_2}{b_1 - b_2}$ & thus $q_2 = \min\left\{\frac{b_2}{2a}, N\right\}$ & $C_2^1(t_1, q_2) = f_2\left(\min\left\{\frac{b_2}{2a}, N\right\}\right)$

(2.147)

if $\frac{b_2}{2a} \leq \frac{e_1 - e_2}{b_1 - b_2} \leq \frac{b_1}{2a}$, then $q_2 = \frac{e_1 - e_2}{b_1 - b_2}$ & $C_2^1(t_1, q_2) = f_1\left(\frac{e_1 - e_2}{b_1 - b_2}\right) = f_2\left(\frac{e_1 - e_2}{b_1 - b_2}\right)$. ❑ (2.148)

Proof of Proposition 2. Setting $\bar{D}(t_1) = q_2$ and solving for q_2 yields

$$q_2 = \begin{cases} \dfrac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)}, & \text{if } n \leq \dfrac{\alpha N(t_1^-)}{1 + \alpha(L - t_1)} \\ \dfrac{n}{\alpha}\left[\left(1 + \dfrac{2\alpha^2(L - t_1)N(t_1^-)}{n} - \alpha^2(L - t_1)^2\right)^{1/2} - 1\right], & \text{if } n \geq \dfrac{\alpha N(t_1^-)}{1 + \alpha(L - t_1)} \end{cases}. \tag{2.149}$$

Note that

$$\alpha\left[(L - t_1)\left(N(t_1^-) - \frac{n}{2}(L - t_1)\right) - \frac{{q_2}^2}{2n}\right] \leq \alpha\left[(L - t_1)\left(N(t_1^-) - q_2\right)\right] \tag{2.150}$$

(follows from $(L - t_2(q_2))^2 \geq 0$) with equality at $q_2 = n(L - t_1)$. Therefore

$$\frac{n}{\alpha}\left[\left(1 + \frac{2\alpha^2(L - t_1)N(t_1^-)}{n} - \alpha^2(L - t_1)^2\right)^{1/2} - 1\right] \leq \frac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)} \tag{2.151}$$

with equality if any only if

$$n = \left(\frac{\alpha(L - t_1)}{1 + \alpha(L - t_1)}\right)\frac{N(t_1^-)}{L - t_1}. \tag{2.152}$$

Since $h = 0$, the second-stage cost is

$$C_2^1(t_1, q_2) = \frac{{q_2}^2}{N(t_1^-)} - (1 - c_n - \gamma m)q_2 + c_w\bar{D}(t_1) + c_2^0\left(\bar{D}(t_1) - q_2\right)^+ + c_3\left(q_2 - \bar{D}(t_1)\right)^+ \tag{2.153}$$

(see (2.22)). We consider two cases.

<u>Case 1</u>: $n \le \left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)\frac{N(t_1^-)}{L-t_1}$. Rearranging the inequality, we have

$$n(L-t_1) \le \left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)N(t_1^-). \tag{2.154}$$

Substituting (2.36) into (2.153) while accounting for (2.154) and noting that

$$q_2 \ge \alpha\left[(L-t_1)\left(N(t_1^-)-q_2\right)\right] \text{ iff } q_2 \ge \frac{\alpha(L-t_1)N(t_1^-)}{1+\alpha(L-t_1)}$$

yields (2.38).

<u>Case 2</u>: $n \ge \left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)\frac{N(t_1^-)}{L-t_1}$. Rearranging the inequality while accounting for (2.151)

, we have

$$\frac{n}{\alpha}\left[\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}-1\right] \le \left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)N(t_1^-) \le n(L-t_1). \tag{2.155}$$

From (2.149) and (2.155) it follows that that the trade-in quantity that matches supply with demand is given by the left-most term in (2.155), i.e.,

$$\Lambda = \frac{n}{\alpha}\left[\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}-1\right].$$

Thus,

$$q_2 < \bar{D}(t_1) \text{ iff } q_2 < \Lambda. \tag{2.156}$$

Substituting (2.36) into (2.153) while accounting for (2.155) and (2.156) yields (2.40).

We simplify the inequality that defines the two cases by substituting $N\left(t_1^-\right)=N-nt_1$ into

$$n\le\left(\frac{\alpha\left(L-t_1\right)}{1+\alpha\left(L-t_1\right)}\right)\frac{N\left(t_1^-\right)}{L-t_1}$$

and consolidating terms to get

$$\frac{n}{N}\le\frac{\alpha}{1+\alpha L}\,.\ \square$$

Proof of Proposition 3. Define

$f_i(q_2) = a_i q_2^2 - b_i q_2 + e_i$

$a_1 = a_2 = 1/N\left(t_1^-\right)$

$$a_3=\frac{1}{N\left(t_1^-\right)}-\frac{\alpha\left(c_w+c_2^0\right)}{2n}$$

$$a_4=\frac{1}{N\left(t_1^-\right)}-\frac{\alpha\left(c_w-c_3\right)}{2n}$$

$b_1 = 1+\left(c_w+c_2^0\right)\alpha\left(L-t_1\right)+c_2^0-c_n-\gamma m$

$b_2 = 1+\left(c_w-c_3\right)\alpha\left(L-t_1\right)-c_3-c_n-\gamma m$

$b_3 = 1+c_2^0-c_n-\gamma m$

$b_4 = 1-c_3-c_n-\gamma m$

$e_1 = \left(c_w+c_2^0\right)\alpha\left(L-t_1\right)N\left(t_1^-\right)$

$e_2 = \left(c_w-c_3\right)\alpha\left(L-t_1\right)N\left(t_1^-\right)$

$$e_3=\left(c_w+c_2^0\right)\alpha\left(L-t_1\right)\left(N\left(t_1^-\right)-\frac{n}{2}\left(L-t_1\right)\right)$$

$$e_4 = (c_w - c_3)\alpha(L - t_1)\left(N(t_1^-) - \frac{n}{2}(L - t_1)\right).$$

Case 1: $n \le \alpha N(t_1^-) / [1 + \alpha(L - t_1)]$. If $n \le \dfrac{\alpha N(t_1^-)}{1 + \alpha(L - t_1)}$, then the second-stage cost is

$$C_2^1(t_1, q_2) = \begin{cases} f_3(q_2),\ q_2 \le n(L - t_1) \\ f_1(q_2),\ n(L - t_1) \le q_2 \le \dfrac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)} \\ f_2(q_2),\ q_2 \ge \dfrac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)} \end{cases} \tag{2.157}$$

(see (2.38)). In general, the optimal trade-in quantity is

$$q_2 = \arg\min \left\{ \begin{array}{l} \min\limits_{q_2 \in [0, n(L - t_1)]} f_3(q_2), \\ \min\limits_{q_2 \in \left[n(L - t_1), \frac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)}\right]} f_1(q_2), \\ \min\limits_{q_2 \in \left[\frac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)}, N(t_1^-)\right]} f_2(q_2) \end{array} \right\}. \tag{2.158}$$

We identify alternative closed-form expressions for the optimal trade-in quantity that are contingent on the parameter values.

Note that $f_i(q_2)$ for $i \in \{1, 2\}$ is convex and thus is minimized at stationary point

$$q_2 = \frac{b_i}{2a_i}. \tag{2.159}$$

If $a_3 > 0$, then $f_3(q_2)$ is convex and minimized at the stationary point (2.159) with $i = 3$. If $a_3 \le 0$, then $f_3(q_2)$ is concave and minimized at

$$q_2 = \infty. \tag{2.160}$$

Note that $-c_3 < c_2^0$ (e.g., $-c_3 \geq c_2^0$ implies that firm can sell a leftover component for more than the cost to buy the component from a third party). Note that $1 - c_n - m$ is the fraction of the population with positive utility from purchasing a new product, which cannot be negative (i.e., $1 - c_n - m \geq 0$). Thus, $b_1 \geq 0$, $a_1 > 0$, and $a_2 > 0$. In addition, $b_1 > b_2$, $e_1 > e_2$, and thus

$$\frac{b_2}{2a_2} < \frac{b_1}{2a_1} \tag{2.161}$$

$$f_2(0) < f_1(0) \tag{2.162}$$

$f_1(q_2)$ and $f_2(q_2)$ intersect at

$$q_2 = \frac{\alpha(L - t_1)N(t_1^-)}{1 + \alpha(L - t_1)}, \tag{2.163}$$

and $f_3(q_2)$ and $f_1(q_2)$ intersect at

$$q_2 = n(L - t_1). \tag{2.164}$$

Note that functions $f_1(q_2)$ and $f_3(q_2)$ have the same form, i.e.,

$$f_i(q_2) = \frac{q_2^{\;2}}{N(t_1^-)} - (1 - c_n - \gamma m)q_2 + c_w\bar{D}(t_1) + c_2^0\left(\bar{D}(t_1) - q_2\right), \tag{2.165}$$

and differ only by the remaining demand function, i.e.,

$$\text{for } f_1,\ \bar{D}(t_1) = \alpha\left[(L - t_1)\left(N(t_1^-) - q_2\right)\right] \tag{2.166}$$

$$\text{for } f_3,\ \bar{D}(t_1) = \alpha\left[(L - t_1)\left(N(t_1^-) - q_2\right) - \frac{n}{2}\left(L - t_1 - \frac{q_2}{n}\right)^2\right] \leq \alpha\left[(L - t_1)\left(N(t_1^-) - q_2\right)\right]. \tag{2.167}$$

Therefore

$$f_3(q_2) \leq f_1(q_2) \text{ for all } q_2 \geq 0. \tag{2.168}$$

If $a_3 \le 0$, then from (2.160) it follows that the optimal trade-in quantity subject to $q_2 \le n(L - t_1)$ (i.e., the range of the cost function over which $f_3(q_2)$ applies) is $q_2 = n(L - t_1)$. Thus the optimal trade-in quantity can be obtained by minimizing (2.157) over the range $\left[n\left(L-t_1\right), N\left(t_1^-\right)\right]$ for which the second-stage cost is defined by functions $f_1(q_2)$ and $f_2(q_2)$. Accordingly, the optimal trade-in quantity follows from (2.159) – (2.163):

if $\dfrac{b_1}{2a_1} \le \dfrac{\alpha(L-t_1)N\left(t_1^-\right)}{1+\alpha(L-t_1)}$,

then $\dfrac{b_2}{2a_2} \le \dfrac{\alpha(L-t_1)N\left(t_1^-\right)}{1+\alpha(L-t_1)}$ & thus $q_2 = \max\left\{n\left(L-t_1\right), \dfrac{b_1}{2a_1}\right\}$. (2.169)

if $\dfrac{b_2}{2a_2} \ge \dfrac{\alpha(L-t_1)N\left(t_1^-\right)}{1+\alpha(L-t_1)}$,

then $\dfrac{b_1}{2a_1} \ge \dfrac{\alpha(L-t_1)N\left(t_1^-\right)}{1+\alpha(L-t_1)}$ & thus $q_2 = \min\left\{\dfrac{b_2}{2a_2}, N\left(t_1^-\right)\right\}$ (2.170)

if $\dfrac{b_2}{2a_2} \le \dfrac{\alpha(L-t_1)N\left(t_1^-\right)}{1+\alpha(L-t_1)} \le \dfrac{b_1}{2a_1}$,

then $q_2 = \dfrac{\alpha(L-t_1)N\left(t_1^-\right)}{1+\alpha(L-t_1)}$. (2.171)

If $a_3 \ge 0$, then from (2.159) it follows that the optimal trade-in quantity subject to $q_2 \le n(L - t_1)$ (i.e., the range of the cost function over which $f_3(q_2)$ applies) is $q_2 = \min\{b_3/(2a_3), n(L - t_1)\}$. If $n(L - t_1) \le b_3/(2a_3)$, then the optimal trade-in quantity can be obtained by minimizing (2.157) over the range $\left[n\left(L-t_1\right), N\left(t_1^-\right)\right]$, and the optimal trade-in quantity is given in (2.169) – (2.171). If $n(L - t_1) \ge b_3/(2a_3)$, then we need to compare the cost at $q_2 = b_3/(2a_3)$ (i.e., $f_3(b_3/(2a_3))$) with the least cost over the range

$\left[n\left(L-t_1\right), N\left(t_1^-\right)\right]$ that is obtained from (2.169) – (2.171). Note that if conditions defined in either (2.169) or (2.171) apply, then the optimal cost over range $\left[n\left(L-t_1\right), N\left(t_1^-\right)\right]$ can be expressed in terms of the function $f_1(q_2)$, and thus from (2.168) it follows that the optimal trade-in quantity is $q_2 = b_3/(2a_3)$. Otherwise, the optimal trade-in quantity is based lowest cost among the pair $f_3(b_3/(2a_3))$ and $f_2(b_2/(2a_2))$.

Note that

$$n(L-t_1) \le b_3/(2a_3) \Leftrightarrow n \le \frac{b_3}{2a_3\left(L-t_1\right)}$$

$$\Leftrightarrow \frac{1}{n} \ge \frac{\left(2a_1-\alpha\left(c_w+c_2^0\right)/n\right)\left(L-t_1\right)}{b_1-\alpha\left(c_w+c_2^0\right)\left(L-t_1\right)}$$

$$\Leftrightarrow \frac{1}{n}\left(1+\frac{\alpha\left(c_w+c_2^0\right)\left(L-t_1\right)}{b_1-\alpha\left(c_w+c_2^0\right)\left(L-t_1\right)}\right) \ge \frac{2a_1\left(L-t_1\right)}{b_1-\alpha\left(c_w+c_2^0\right)\left(L-t_1\right)}$$

$$\Leftrightarrow \frac{1}{n}\left(\frac{b_1}{b_3}\right) \ge \frac{2a_1\left(L-t_1\right)}{b_3}$$

$$\Leftrightarrow n \le \frac{b_1}{2a_1\left(L-t_1\right)} = \frac{b_1 N\left(t_1^-\right)}{2\left(L-t_1\right)}$$

Therefore, in summary, the optimal trade-in quantity for the case of $n \le \alpha N\left(t_1^-\right)/\left[1+\alpha\left(L-t_1\right)\right]$ is obtained from the following rules:

$$\text{if } a_3 \le 0 \text{ or } n \le \frac{b_1 N\left(t_1^-\right)}{2\left(L-t_1\right)}, \text{ then } q_2 \text{ is obtained from } (2.169)-(2.171) \tag{2.172}$$

$$\text{if } a_3 \ge 0 \text{ and } n \ge \frac{b_1 N\left(t_1^-\right)}{2\left(L-t_1\right)} \ \& \ \frac{b_2}{2a_2} \le \frac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)}, \text{ then } q_2 = \frac{b_3}{2a_3} \tag{2.173}$$

if $a_3 \geq 0$ and $n \geq \frac{b_1 N\left(t_1^-\right)}{2\left(L-t_1\right)}$ & $\frac{b_2}{2a_2} \geq \frac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)}$,

then $q_2 = \arg\min\left\{ f_3\left(\frac{b_3}{2a_3}\right), f_2\left(\frac{b_2}{2a_2}\right)\right\}$. (2.174)

Case 2: $n \geq \alpha N\left(t_1^-\right)/\left[1+\alpha\left(L-t_1\right)\right]$. If $n \geq \frac{\alpha N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)}$, then the second-stage cost is

$$C_2^1\left(t_1, q_2\right) = \begin{cases} f_3\left(q_2\right),\ q_2 \leq \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{n} - \alpha^2\left(L-t_1\right)^2\right)^{1/2}\right] \\ f_4\left(q_2\right),\ \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{n} - \alpha^2\left(L-t_1\right)^2\right)^{1/2}\right] \leq q_2 \leq n\left(L-t_1\right). \\ f_2\left(q_2\right),\ q_2 \geq n\left(L-t_1\right) \end{cases} \quad (2.175)$$

(see (2.40)). In general, the optimal trade-in quantity is

$$q_2 = \arg\min\left\{ \begin{array}{l} \min\limits_{q_2 \in \left[0, \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{n} - \alpha^2\left(L-t_1\right)^2\right)^{1/2}\right]\right]} f_3\left(q_2\right), \\ \min\limits_{q_2 \in \left[\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{n} - \alpha^2\left(L-t_1\right)^2\right)^{1/2}\right], n\left(L-t_1\right)\right]} f_4\left(q_2\right), \\ \min\limits_{q_2 \in \left[n\left(L-t_1\right), N\left(t_1^-\right)\right]} f_2\left(q_2\right) \end{array} \right\}. \quad (2.176)$$

We identify a more precise characterization of (2.176), but due to the many combinations of positive and negative parameter values, we do not write closed-form optimal expressions for every possible combination. Note that $a_2 > 0$, and thus f_2 is convex and minimized at its stationary point $b_2/(2a_2)$.

If $a_4 \le 0$, then from $a_3 \le a_4$ (follows from $-c_3 < c_2^0$), we must have $a_3 \le 0$, and thus f_3 and f_4 are concave and minimized at $q_2 = \infty$. Thus,

$$\arg\min\left\{ \min_{q_2 \in \left[0, \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right]\right]} f_3(q_2) \right\} = \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right],$$

which is on f_4,

$$\arg\min\left\{ \min_{q_2 \in \left[\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right], n(L-t_1)\right]} f_4(q_2) \right\} = n(L-t_1),$$

and thus

$$\arg\min\left\{ \begin{array}{l} \min\limits_{q_2 \in \left[0, \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right]\right]} f_3(q_2), \\ \min\limits_{q_2 \in \left[\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right], n(L-t_1)\right]} f_4(q_2) \end{array} \right\} = n(L-t_1),$$

which is on f_2. Therefore

$$\arg\min\left\{ \begin{array}{l} \min\limits_{q_2 \in \left[0, \frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right]\right]} f_3(q_2), \\ \min\limits_{q_2 \in \left[\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right], n(L-t_1)\right]} f_4(q_2), \\ \min\limits_{q_2 \in \left[n(L-t_1), N(t_1^-)\right]} f_2(q_2) \end{array} \right\} = \arg\min\left\{ \min_{q_2 \in \left[n(L-t_1), N(t_1^-)\right]} f_2(q_2) \right\}$$

$$= \min\left\{ \max\left\{ n(L-t_1), \frac{b_2}{2a_2} \right\}, N(t_1^-) \right\},$$

i.e., the optimal trade-in quantity is

$$q_2 = \min\left\{\max\left\{n\left(L-t_1\right),\frac{b_2}{2a_2}\right\},N\left(t_1^-\right)\right\}. \tag{2.177}$$

If $a_3 \le 0$ and $a_4 > 0$, then f_3 is concave and minimized at $q_2 = \infty$ and f_4 is convex and minimized at its stationary point $a_4/(2b_4)$. Thus,

$$\arg\min\left\{\begin{array}{l}\min\limits_{q_2\in\left[0,\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{\substack{n\\ \alpha^2(L-t_1)^2}}-\right)^{1/2}\right]\right]} f_3\left(q_2\right),\\ \min\limits_{q_2\in\left[\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{\substack{n\\ \alpha^2(L-t_1)^2}}-\right)^{1/2}\right],n(L-t_1)\right]} f_4\left(q_2\right)\end{array}\right\} = \min\left\{\begin{array}{l}\max\left\{\begin{array}{l}\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{\substack{n\\ \alpha^2\left(L-t_1\right)^2}}-\right)^{1/2}\right],\\ \frac{b_4}{2a_4}\end{array}\right\},\\ n\left(L-t_1\right)\end{array}\right\},$$

$$\arg\min\left\{\min_{q_2\in\left[n(L-t_1),N(t_1^-)\right]} f_2\left(q_2\right)\right\} = \min\left\{\max\left\{n\left(L-t_1\right),\frac{b_2}{2a_2}\right\},N\left(t_1^-\right)\right\},$$

and the optimal trade-in quantity is

$$q_2 = \arg\min\left\{\begin{array}{l}f_4\left(\min\left\{\max\left\{\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{\substack{n\\ \alpha^2\left(L-t_1\right)^2}}-\right)^{1/2}\right],\frac{b_4}{2a_4}\right\},n\left(L-t_1\right)\right\}\right),\\ f_2\left(\min\left\{\max\left\{n\left(L-t_1\right),\frac{b_2}{2a_2}\right\},N\left(t_1^-\right)\right\}\right)\end{array}\right\}. \tag{2.178}$$

If $a_3 > 0$ and $a_4 > 0$, then f_3 and f_4 are convex and minimized at stationary points $a_3/(2b_3)$ and $a_4/(2b_4)$, respectively. Thus,

$$\arg\min\left\{\min_{q_2\in\left[0,\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{\substack{n\\ \alpha^2(L-t_1)^2}}-\right)^{1/2}\right]\right]} f_3\left(q_2\right)\right\} = \min\left\{\max\left\{0,\frac{b_3}{2a_3}\right\},\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2\left(L-t_1\right)N\left(t_1^-\right)}{\substack{n\\ \alpha^2\left(L-t_1\right)^2}}-\right)^{1/2}\right]\right\},$$

$$\arg\min\left\{\min_{q_2\in\left[\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right],\,n(L-t_1)\right]} f_4(q_2)\right\} = \min\left\{\max\left\{\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right], \frac{b_4}{2a_4}\right\}, n(L-t_1)\right\},$$

$$\arg\min\left\{\min_{q_2\in\left[n(L-t_1),N(t_1^-)\right]} f_2(q_2)\right\} = \min\left\{\max\left\{n(L-t_1),\frac{b_2}{2a_2}\right\},N(t_1^-)\right\},$$

and the optimal trade-in quantity is

$$q_2 = \arg\min\left\{\begin{array}{l} f_3\left(\min\left\{\max\left\{0,\frac{b_3}{2a_3}\right\},\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right]\right\}\right) \\ f_4\left(\min\left\{\max\left\{\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{n}-\alpha^2(L-t_1)^2\right)^{1/2}\right],\frac{b_4}{2a_4}\right\},n(L-t_1)\right\}\right), \\ f_2\left(\min\left\{\max\left\{n(L-t_1),\frac{b_2}{2a_2}\right\},N(t_1^-)\right\}\right) \end{array}\right\}. \qquad (2.179)$$

In summary, the optimal trade-in quantity for the case of $n \geq \alpha N(t_1^-)/\left[1+\alpha(L-t_1)\right]$ is obtained from the following rules:

if $a_4 \leq 0$, then $a_3 \leq 0$ and

$$q_2 = \min\left\{\max\left\{n(L-t_1),\frac{b_2}{2a_2}\right\},N(t_1^-)\right\} \qquad (2.180)$$

if $a_4 > 0$ and $a_3 \leq 0$, then

$$q_2 = \arg\min\left\{\begin{array}{l} f_4\left(\min\left\{\max\left\{\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{\alpha^2(L-t_1)^2}-\frac{n}{}\right)^{1/2}\right],\frac{b_4}{2a_4}\right\},n(L-t_1)\right\}\right), \\ f_2\left(\min\left\{\max\left\{n(L-t_1),\frac{b_2}{2a_2}\right\},N(t_1^-)\right\}\right) \end{array}\right\} \quad (2.181)$$

if $a_4 > 0$ and $a_3 > 0$, then

$$q_2 = \arg\min\left\{\begin{array}{l} f_3\left(\min\left\{\max\left\{0,\frac{b_3}{2a_3}\right\},\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{\alpha^2(L-t_1)^2}-\frac{n}{}\right)^{1/2}\right]\right\}\right) \\ f_4\left(\min\left\{\begin{array}{l}\max\left\{\frac{n}{\alpha}\left[1+\left(1+\frac{2\alpha^2(L-t_1)N(t_1^-)}{\alpha^2(L-t_1)^2}-\frac{n}{}\right)^{1/2}\right],\frac{b_4}{2a_4}\right\}, \\ n(L-t_1)\end{array}\right\}\right), \\ f_2\left(\min\left\{\max\left\{n(L-t_1),\frac{b_2}{2a_2}\right\},N(t_1^-)\right\}\right) \end{array}\right\}. \ \square \quad (2.182)$$

Proof of Proposition 4. If $\frac{b_1 N(t_1^-)}{2} \geq n(L-t_1)$, then the optimal trade-in quantity is given by (2.43), which is the same as (2.34) in Proposition 1. Rewriting $\frac{b_1 N(t_1^-)}{2} \geq n(L-t_1)$, we get

$$\frac{b_1 N(t_1^-)}{2} \geq n(L-t_1) \Rightarrow n \leq \left(\frac{N(t_1^-)}{L-t_1}\right)\left(\frac{1+c_2^0+\alpha(L-t_1)(c_w+c_2^0)-c_n-\gamma m}{2}\right)$$

$$\Rightarrow n \leq \left(\frac{N-nt_1}{L-t_1}\right)\left(\frac{1+c_2^0+\alpha(L-t_1)(c_w+c_2^0)-c_n-\gamma m}{2}\right)$$

$$\Rightarrow n \le \frac{N}{L}\left(\frac{1+c_2^0+\alpha(L-t_1)(c_w+c_2^0)-c_n-\gamma m}{2}\right), \tag{2.183}$$

and, by the definition of L, $n \le N/L$ (i.e., all warranties cannot expire before time L). Therefore (2.183) holds if the parenthetical term is not less than 1, i.e.,

$$1+c_2^0+\alpha(L-t_1)(c_w+c_2^0)-c_n-\gamma m \ge 2\,.$$

For the second part of Proposition 4, note that $\frac{b_1}{2a_1}=\frac{b_1 N(t_1^-)}{2}$. From the proof of Proposition 3, $\frac{b_2}{2a_2}<\frac{b_1}{2a_1}$ (see (2.161)). Letting q_2^* denote the trade-in quantity according to (2.34) (or equivalently (2.43)), we have $q_2^* \le \frac{b_1 N(t_1^-)}{2}$. Thus,

$$q_2^* \ge n(L-t_1) \tag{2.184}$$

implies $\frac{b_1 N(t_1^-)}{2} \ge n(L-t_1)$, i.e., if q_2^* from Proposition 1 satisfies (2.184), then it is optimal for $\mathrm{C2}_2^1$. ❑

Proof of Proposition 5. *If* $q_2 \le \alpha\left[(L-t_1)\left(N(t_1^-)-q_2\right)-\frac{n}{2}\left(L-t_2(q_2)\right)^2\right]$, then

$$C_2^1(t_1,q_2)=\frac{{q_2}^2}{N(t_1^-)}-\left(1-c_n-\gamma m-c_2^0\right)q_2+\left(c_w+c_2^0\right)\alpha\left[(L-t_1)\left(N(t_1^-)-q_2\right)-\frac{n}{2}\left(L-t_2(q_2)\right)^2\right].$$

if $q_2 \ge \alpha\left[(L-t_1)\left(N(t_1^-)-q_2\right)-\frac{n}{2}\left(L-t_2(q_2)\right)^2\right]$, then

$$C_2^1(t_1,q_2)=\frac{{q_2}^2}{N(t_1^-)}-\left(1-c_n-\gamma m+c_3\right)q_2+\left(c_w-c_3\right)\alpha\left[(L-t_1)\left(N(t_1^-)-q_2\right)-\frac{n}{2}\left(L-t_2(q_2)\right)^2\right].$$

Recall that $t_2\left(q_2\right)=\min\left\{t_1+\frac{q_2}{n},L\right\}$, which is nonincreasing in n. Thus, for any given q_2, $C_2^1\left(t_1,q_2\right)$ is nonincreasing in n. ❑

Proof of Proposition 6. Inequality (2.53) corresponds to Case 1 in the proof of Proposition 2, which is the case where the total warranty fall-off, $n(L-t_1)$, is less than the trade-in quantity that matches supply with demand

$$q_2=\frac{\alpha\left(L-t_1\right)N\left(t_1^-\right)}{1+\alpha\left(L-t_1\right)}$$

(see (2.149)), i.e., $q_2 \geq n(L-t_1)$. Consequently, there is no reduction in the warranty install base after the q_2 trade-in units are received. Thus, and the cost is identical to the case of $n = 0$ except that $N\left(t_1^-\right)=N-nt_1$ replaces N in (2.50) and (2.52). ❑

Proof of Proposition 7. The problem is

$$\min_{\beta(t),\nu(t)}\left\{C_2^2\left(t_1,\beta,\nu\right)\left|\int_{t_1}^{L}\nu\left(t\right)dt=N,\beta\left(t\right)\nu\left(t\right)=\alpha Ne^{-\alpha\left(t-t_1\right)},\beta\left(t\right)\in\left[0,1\right]\right.\right\} \tag{2.185}$$

where $C_2^2\left(t_1,\beta,\nu\right) = e^{-rt_1}\int_{t_1}^{L}e^{-(\alpha+r)(t-t_1)}\left(\beta\left(t\right)+c_w+c_n+\gamma m-1\right)\alpha Ndt$. We see that $C_2^2\left(t_1,\beta,\nu\right)$ is minimized by minimizing $\int_{t_1}^{L}e^{-(r+\alpha)(t-t_1)}\beta\left(t\right)dt$. Note that $\beta(t)\,\nu(t) = \alpha Ne^{-\alpha\left(t-t_1\right)}$ can be rewritten as $\nu(t) = \alpha Ne^{-\alpha\left(t-t_1\right)}/\beta\left(t\right)$. Thus,

$$\int_{t_1}^{L}\nu\left(t\right)dt=N \Leftrightarrow \int_{t_1}^{L}\frac{e^{-\alpha\left(t-t_1\right)}}{\beta\left(t\right)}dt=\frac{1}{\alpha}. \tag{2.186}$$

We solve the following equivalent problem

$$\min_{\beta(t)}\left\{\int_{t_1}^{L} e^{-(\alpha+r)(t-t_1)}\beta(t)\,dt \,\middle|\, \int_{t_1}^{L}\frac{e^{-\alpha(t-t_1)}}{\beta(t)}\,dt=\frac{1}{\alpha},\beta(t)\in[0,1]\right\}, \tag{2.187}$$

but with the bound constraint $\beta(t) \in [0, 1]$ ignored (i.e., unrestricted problem), i.e., we solve

$$\min_{\beta(t)}\left\{\int_{t_1}^{L} e^{-(\alpha+r)(t-t_1)}\beta(t)\,dt \,\middle|\, \int_{t_1}^{L}\frac{e^{-\alpha(t-t_1)}}{\beta(t)}\,dt=\frac{1}{\alpha}\right\}. \tag{2.188}$$

After solving the unrestricted problem (2.188), we identify conditions on parameter values that ensure the solution is also optimal for the restricted problem.

To simplify notation and without loss of generality, we let $t_1 = 0$. We define

$$y(t) = \int_0^t \frac{e^{-\alpha x}}{\beta(x)}\,dx, \tag{2.189}$$

which implies

$$y'(t) = \frac{e^{-\alpha t}}{\beta(t)} \tag{2.190}$$

$$\beta(t)=e^{-\alpha t}y'(t)^{-1}. \tag{2.191}$$

Thus, problem (2.188) is

$$\min_{y(t)}\left\{\int_0^L e^{-(2\alpha+r)t}y'(t)^{-1}\,dt \,\middle|\, y(0)=0, y(L)=\frac{1}{\alpha}\right\}, \tag{2.192}$$

which can be solved using calculus of variations methods. Let $y^*(t)$ denote the optimal function. We express $y(t)$ in terms of parameter a, $y^*(t)$, and difference function $h(t)$, i.e.,

$$y(t) = y^*(t) + ah(t) \tag{2.193}$$

and thus $y'(t) = y^{*\prime}(t) + ah'(t)$. For any feasible $y(t)$, we must have $h(0) = h(L) = 0$ (i.e., in order to satisfy the boundary conditions, which are clearly satisfied by the function $y^*(t)$). Let

$$g(a) = \int_0^L e^{-(2\alpha+r)t} y'(t)^{-1}\,dt = \int_0^L e^{-(2\alpha+r)t}\left[y^{*}{}'(t) + ah'(t)\right]^{-1} dt\,. \tag{2.194}$$

Note that

$$g'(a) = -\int_0^L e^{-(2\alpha+r)t}\left[y^{*}{}'(t) + ah'(t)\right]^{-2} h'(t)\,dt\,. \tag{2.195}$$

Applying integration by parts and recognizing $uv\big|_0^L = 0$ due to $h(0) = h(L) = 0$, we have

$$g'(a) = -\int_0^L e^{-(2\alpha+r)t}\begin{bmatrix}(2\alpha+r)\left(y^{*}{}'(t)+ah'(t)\right)^{-2}+2\\ \left(y^{*}{}'(t)+ah'(t)\right)^{-3}\left(y^{*}{}''(t)+ah''(t)\right)\end{bmatrix} h(t)\,dt\,. \tag{2.196}$$

Since $y^*(t)$ is optimal, we can conclude that for any $h(t)$ (with $h(0) = h(L) = 0$), we must have $g'(0) = 0$. This implies that the integrand of the above (with $a = 0$) must be equal to zero at all values of t, i.e., we must have

$$e^{-(2\alpha+r)t}\left[(2\alpha+r)\left(y^{*}{}'(t)\right)^{-2} + 2\left(y^{*}{}'(t)\right)^{-3} y^{*}{}''(t)\right]h(t) = 0$$

$$\Leftrightarrow\ (2\alpha+r)\,y^{*}{}'(t) + 2y^{*}{}''(t) = 0$$

$$\Leftrightarrow\ y^{*}{}''(t) = -\left(\frac{2\alpha+r}{2}\right)y^{*}{}'(t)$$

Solving the differential equation, we get

$$y^{*}{}'(t) = Ae^{-(\alpha+0.5r)t}\,, \tag{2.197}$$

where A is obtained from the boundary condition, i.e.,

$$y^*(L) = \int_0^L y^*{}'(t)\,dt = \frac{1}{\alpha} \Leftrightarrow A = \frac{\alpha + 0.5r}{\alpha\left(1 - e^{-(\alpha+0.5r)L}\right)} \tag{2.198}$$

Thus

$$y^*{}'(t) = \frac{(\alpha + 0.5r)e^{-(\alpha+0.5r)t}}{\alpha\left(1 - e^{-(\alpha+0.5r)L}\right)}\,. \tag{2.199}$$

The function $y^*{}'(t)$ is a unique extremal (i.e., no other function yields $g'(0) = 0$). Taking the derivative of (2.195) and evaluating at $a = 0$, we get

$$g''(0) = \int_0^L 2e^{-(2\alpha+r)t}\left[y^*{}'(t)\right]^{-3}\left[h'(t)\right]^2 dt\,. \tag{2.200}$$

Since $y^*{}'(t) > 0$ for all $t \in [0, L]$, it follows that $g''(0) > 0$, and thus $y^*{}'(t)$ minimizes (2.192).

Substituting (2.199) into (2.191) yields optimal trade-in fraction

$$\beta(t) = e^{-\alpha t} y'(t)^{-1} = \frac{e^{-\alpha t}\alpha\left(1 - e^{-(\alpha+0.5r)L}\right)e^{(\alpha+0.5r)t}}{\alpha + 0.5r} = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)L}\right)e^{0.5rt}}{\alpha + 0.5r}$$

and accounting for $t_1 > 0$ yields optimal trade-in fraction for the unrestricted problem

$$\beta(t) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)(L-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha + 0.5r}\,. \tag{2.201}$$

Recall that the difference between the unrestricted problem and the restricted problem is that the restricted problem includes the constraint $\beta(t) \in [0, 1]$. From (2.201) we see that $\beta(t)$ is increasing in t and $\beta(t_1) \in [0, 1]$. Thus, if $\beta(L) \le 1$, then the optimal solution to the unrestricted problem is also optimal for the restricted problem. Note that

$$\beta(L) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)(L-t_1)}\right)e^{0.5r(L-t_1)}}{\alpha + 0.5r} \le 1 \Leftrightarrow \left(e^{\alpha(L-t_1)}\right)^{1+\frac{r}{2\alpha}} - \left(1 + \frac{r}{2\alpha}\right)e^{\alpha(L-t_1)} \le 1\,. \tag{2.202}$$

Thus, $\beta(t) \in [0, 1]$ if and only if (2.202) holds. The optimal trade-in offer rate is obtained by substituting (2.201) into (2.66) and solving for $\nu(t)$, and the optimal cost is obtained by substituting (2.201) into (2.65). ❑

Proof of Corollary 1. Expressions (2.73), (2.74), and (2.76) follow directly from substituting $r = 0$ into (2.70) – (2.72). The trade-in quantity expression (2.75) follows from two observations: (1) 100% of the install base receives the trade-in offer over the warranty horizon (i.e., $\int_{t_1}^{L} \nu(t)\,dt = N\left(t_1^-\right)$) and (2) the fraction of those receiving the offer who trade-in is the constant $\beta(t) = 1 - e^{-\alpha(L-t_1)}$. ❑

Proof of Proposition 8. Assume that $\left(e^{\alpha(L-t_1)}\right)^{1+\frac{r}{2\alpha}} - \left(1+\dfrac{r}{2\alpha}\right)e^{\alpha(L-t_1)} \leq 1$ holds. Note that

$$\int_{t_1}^{L} \nu(t)\,dt + \Gamma(\nu) = \int_{t_1}^{L} \nu(t)\,dt + \max_{t\in[t_1,L]}\left(\int_{t_1}^{t}\left[n(x) - \nu(x)\right]dx\right)^{+} + \int_{0}^{t_1^-} n(x)\,dx$$

and thus constraint $\int_{t_1}^{L} \nu(t)\,dt + \Gamma(\nu) = N$ can be rewritten as

$$\int_{t_1}^{L} \nu(t)\,dt + \max_{t\in[t_1,L]}\left(\int_{t_1}^{t}\left[n(x) - \nu(x)\right]dx\right)^{+} = N\left(t_1^-\right). \tag{2.203}$$

If (2.80) holds, then setting

$$\beta(t) = \frac{\alpha\left(1 - e^{-(\alpha+0.5r)(L-t_1)}\right)e^{0.5r(t-t_1)}}{\alpha + 0.5r} \tag{2.204}$$

$$\nu(t) = \left(\frac{(\alpha + 0.5r)e^{-0.5r(t-t_1)}}{1 - e^{-(\alpha+0.5r)(L-t_1)}}\right)N\left(t_1^-\right)e^{-\alpha(t-t_1)} \tag{2.205}$$

yields a feasible solution for problem $\mathrm{C2}_2^2$, i.e.,

$\beta(t) \in [0, 1]$

$$\int_{t_1}^{L} \nu(t)\,dt + \max_{t\in[t_1,L]}\left(\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx\right)^{+} = \int_{t_1}^{L}\nu(t)\,dt = N\left(t_1^-\right)$$

$$\beta(t)\,\nu(t) = d(t) = \alpha N\left(t_1^-\right)e^{-\alpha(t-t_1)}$$

and, by Proposition 7, (2.204) and (2.205) uniquely solve

$$\min_{\beta(t),\nu(t)}\left\{C_2^2\left(t_1,\beta,\nu\right)\middle|\begin{array}{l}\int_{t_1}^{L}\nu(t)\,dt+\max_{t\in[t_1,L]}\left(\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx\right)^{+}=N\left(t_1^-\right),\beta(t)\nu(t)=\alpha N\left(t_1^-\right)e^{-\alpha(t-t_1)},\\ \beta(t)\in[0,1],\max_{t\in[t_1,L]}\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx\le 0\end{array}\right\},$$

which is the same as $\mathrm{C2}_2^2$ except with the additional constraint $\max_{t\in[t_1,L]}\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx \le 0$.

What remains is to consider whether cost can decrease by allowing $\max_{t\in[t_1,L]}\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx > 0$. The answer is no. The reason is that $\max_{t\in[t_1,L]}\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx > 0$ implies that $\max_{t\in[t_1,L]}\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx$ customers did not receive a trade-in offer during time interval $[t_1, L]$. And increasing $\nu(t)$ just enough so that $\max_{t\in[t_1,L]}\int_{t_1}^{t}\left[n(x)-\nu(x)\right]dx = 0$ results in a lower trade-in discount (for the same trade-in rate) and results in lower cost. ❑

Proof of Proposition 9. From A7b, the left-hand side of (2.80)

$$f(t) = \int_{t_1}^{t} n(x)\,dx = n\left(t-t_1\right) \tag{2.206}$$

is linear increasing in t with $f(t_1) = 0$ and $f(L) = N\left(t_1^-\right)$. The right-hand side of (2.80)

$$g(t)=\int_{t_1}^{t}\left(\frac{\alpha+0.5r}{1-e^{-(\alpha+0.5r)(L-t_1)}}\right)N\left(t_1^-\right)e^{-(\alpha+0.5r)(x-t_1)}dx=\left(\frac{1-e^{-(\alpha+0.5r)(t-t_1)}}{1-e^{-(\alpha+0.5r)(L-t_1)}}\right)N\left(t_1^-\right) \tag{2.207}$$

is concave increasing in t with $g(t_1) = 0$ and $g(L) = N\left(t_1^-\right)$. Therefore $f(t) \leq g(t)$ for all $t \in [t_1, L]$. ❑

Proof of Proposition 10. Letting $y = 1 - c_n - \gamma m - c_w$, the cost of each policy is

$$C_2^1(t_1)=q_2^1\left(\frac{q_2^1}{N\left(t_1^-\right)}-y\right)=N\left(t_1^-\right)\left(1-\frac{1}{1+x}\right)\left(1-\frac{1}{1+x}-y\right) \tag{2.208}$$

$$C_2^2(t_1)=q_2^2\left(\frac{q_2^2}{N\left(t_1^-\right)}-y\right)=N\left(t_1^-\right)\left(1-e^{-x}\right)\left(1-e^{-x}-y\right) \tag{2.209}$$

(follows from Proposition 1 and Corollary 1; see also (2.86) and (2.87)). Letting

$$\Delta = \frac{q_2^2-q_2^1}{N\left(t_1^-\right)}=\frac{1}{1+x}-e^{-x}>0, \tag{2.210}$$

we have

$$\frac{C_2^1(t_1)-C_2^2(t_1)}{N\left(t_1^-\right)}=\left(1-\frac{1}{1+x}\right)\left(1-\frac{1}{1+x}-y\right)-\left(1-\frac{1}{1+x}+\Delta\right)\left(1-\frac{1}{1+x}+\Delta-y\right)$$

$$=\Delta y-\Delta\left(\frac{2x}{1+x}+\Delta\right)$$

$$=\Delta\left(y-\left(\frac{1+2x}{1+x}-e^{-x}\right)\right).$$

Therefore $C_2^1(t_1)>C_2^2(t_1)$ if and only if $\left(1-c_n-\gamma m-c_w\right)>\frac{1+2x}{1+x}-e^{-x}$. ❑

Proof of Proposition 11. Follows directly from (2.90) – (2.92) and $p_n = c_n + m$. ❑

Proof of Proposition 12. Follows directly from (2.5), (2.86), and (2.87). ❑

Proof of Proposition 13. a) The unit acquisition cost under a full trade policy as a function of the trade-in quantity is the difference between the trade-in quantity as a percent of the install base and the trade-in potential $\tau = (1-\gamma)m - (p_n - 1)$, i.e.,

$$c_2^1 = \frac{q_2}{N\left(t_1^-\right)} - \left[(1-\gamma)m - \left(p_n - 1\right)\right].$$

(see (2.19)). Recall that the warranty demand under a trade-in policy is always less than the warranty demand under the benchmark policy (because trade-ins reduce the install base, and consequently warranty demand), i.e.,

$$q_2^1 < q_2^0. \tag{2.211}$$

The warranty demand as a percent of the install base under the benchmark policy is

$$\frac{q_2^0}{N\left(t_1^-\right)} = \frac{\bar{D}\left(t_1\right)}{N\left(t_1^-\right)} = \frac{1}{N\left(t_1^-\right)}\int_0^{L-t_1}\left(N\left(t_1^-\right) - ns\right)dt = \alpha\left(L - t_1\right) - \frac{\alpha n\left(L-t_1\right)^2}{2\left(N - nt_1\right)}. \tag{2.212}$$

Therefore,

$$\begin{aligned}
c_2^1 &= \frac{q_2^1}{N\left(t_1^-\right)} - \left[(1-\gamma)m - \left(p_n - 1\right)\right] \\
&< \frac{q_2^0}{N\left(t_1^-\right)} - \left[(1-\gamma)m - \left(p_n - 1\right)\right] && \text{(due to (2.211))} \\
&= \alpha\left(L - t_1\right) - \frac{\alpha n\left(L-t_1\right)^2}{2\left(N - nt_1\right)} - \left[(1-\gamma)m - \left(p_n - 1\right)\right] && \text{(due to (2.212))} \\
&< c_2^0 && \text{(due to (2.101).}
\end{aligned}$$

We next consider Proposition 13b. From Corollary 1 it follows that

$$c_2^2 = \frac{q_2^2}{N\left(t_1^-\right)} - \left[(1-\gamma)m - \left(p_n - 1\right)\right] = 1 - e^{-\alpha(L-t_1)} - \left[(1-\gamma)m - \left(p_n - 1\right)\right] \tag{2.213}$$

and $c_2^2 < c_2^0$ if and only if (2.102) holds. ❑

Proof of Proposition 14. a) From Proposition 13, the restricted full trade-in policy results in lower unit acquisition cost when (2.103). And warranty claim volume is lower under the trade-in policy, and thus total cost is lower.

b) The cost of the benchmark policy as a percent of the install base is

$$\frac{C_2^0\left(t_1\right)}{N\left(t_1^-\right)} = \frac{q_2^0\left(c_2^0 + c_w\right)}{N\left(t_1^-\right)} = \left(\frac{1}{N\left(t_1^-\right)}\int_0^{L-t_1}\left(N\left(t_1^-\right) - ns\right)dt\right)\left(c_2^0 + c_w\right) = \left(\alpha\left(L - t_1\right) - \frac{\alpha n\left(L - t_1\right)^2}{2\left(N - nt_1\right)}\right)\left(c_2^0 + c_w\right).$$

The cost of the matching trade-in policy as a percent of the install base is

$$\frac{C_2^2\left(t_1\right)}{N\left(t_1^-\right)} = \frac{q_2^2\left(c_2^2 + c_w\right)}{N\left(t_1^-\right)} = \left(1 - e^{-x}\right)\left(c_2^2 + c_w\right) = \left(1 - e^{-x}\right)\left(1 - e^{-x} - \left(1 - c_n - \gamma m\right) + c_w\right)$$

(see Corollary 1). Eqs. (2.104) and (2.105) follow from substitution. ❑

Proof of Proposition 15. Follows from Proposition 11 and the fact that the benchmark policy results in higher inventory than either of the trade-in policies and thus is more negatively affected by increasing *h*. ❑

Proof of Proposition 16. a) Since warranty claim volume is lower under the trade-in policy, the discounted cost of servicing warranty claims under the restricted full trade-in policy is lower than under the benchmark policy. Let Δ denote the positive savings is discounted warranty claim service costs and a percent of the install base. Thus, if (2.115) holds, then

$$\frac{C_2^0(t_1)-C_2^1(t_1)}{N} = \frac{c_2^0q_2^0-c_2^1q_2^1}{N}+\Delta$$

$$= c_2^0\alpha(L-t_1)-\left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}-\left[(1-\gamma)m-(p_n-1)\right]\right)\left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)+\Delta$$

$$=\left(\frac{\alpha(L-t_1)}{1+\alpha(L-t_1)}\right)\left(c_2^0-\left(\frac{1}{\left(1+\alpha(L-t_1)\right)^2}-\frac{\left[(1-\gamma)m-(p_n-1)\right]}{\alpha(L-t_1)}\right)\right)+\Delta$$

$> 0.$ (due to $\Delta > 0$ and (2.115).

For b), we begin by noting that for the case of r = 0, (2.116) implies

$$c_{2|r=0}^2 < c_2^0 \tag{2.214}$$

(see Proposition 11). The magnitude of the present value of the average acquisition cost per unit under the matching trade-in policy is decreasing in r (due to the effect of discounting), i.e.,

$$\left|c_{2|r>0}^2\right| < \left|c_{2|r=0}^2\right|. \tag{2.215}$$

From (2.214), (2.215), and the fact that $c_2^0 > 0$, it follows that $c_{2|r>0}^2 < c_2^0$. Since the matching trade-in policy results in lower unit acquisition cost and lower warranty claim demand, it follows that $C_2^2(t_1) < C_2^0(t_1)$. ❑

Proof of Proposition 17. Let x denote the aggregate failure rate in the second stage, i.e., $x = \alpha(L - t_1)$. Note that $q_1 = \alpha LN - xN$. We write and analyze the first-stage cost under each policy as a function of x (see (2.86) and (2.87) for the second-stage cost of the trade-in policies as a function of x). For the benchmark policy, due to $c_2^0 > c_1$,

$x_0 = \arg\min_{x\in[0,\alpha L]} C_1^0(x) = 0$ and $C_1^0(x_0) = N\left[(c_1 + c_w)\alpha L\right]$, i.e., q_1 is set to exactly match total demand.

The problem $\min_{x\in[0,\alpha L]} C_1^1(x)$ is equivalent to

$$\min_{x\in[0,\alpha L]}\left\{f(x) = \left(\frac{x}{1+x}\right)^2 - a\left(\frac{x}{1+x}\right) - bx\right\}$$

where $a = 1 - c_n - \gamma m - c_w$ and $b = c_1 + c_w$. The first-order condition is

$$f'(x) = -b(1+x)^3 + (2-a)(1+x) - 2 = 0 \qquad (2.216)$$

and

$$x_1 = \arg\min_{x\in[0,\alpha L]} C_1^1(x) = \arg\min_{x\in\{0,z_1,z_2,z_3,\alpha L\}} C_1^1(x).$$

where $z_i = \max\left\{0, \min\left\{2\left(\frac{2-a}{3b}\right)^{1/2}\cos\left(\frac{1}{3}\left(\cos^{-1}\left(-\frac{1}{b}\left(\frac{3b}{2-a}\right)^{3/2}\right) + 2(i-1)\pi\right)\right) - 1, \alpha L\right\}\right\}$.

The problem $\min_{x\in[0,\alpha L]} C_1^2(x)$ is equivalent to

$$\min_{x\in[0,\alpha L]}\left\{f(x) = \left(1 - e^{-x}\right)^2 - a\left(1 - e^{-x}\right) - bx\right\}$$

where $a = 1 - c_n - \gamma m - c_w$ and $b = c_1 + c_w$. The first-order condition is

$$f'(x) = -b\left(e^x\right)^2 + (2-a)e^x - 2 = 0. \qquad (2.217)$$

The values of e^x that solve (2.217) are roots of a quadratic, i.e., the stationary points of $f(x)$ are

$$x = \ln\left(\frac{2 - a \pm \left[(2-a)^2 - 8b\right]^{1/2}}{2b}\right)$$

and

$$x_2 = \underset{x\in[0,\alpha L]}{\arg\min}\, C_1^2(x) = \underset{x\in\{0,z_1,z_2,\alpha L\}}{\arg\min} \left\{C_1^2(x)\right\}$$

where $z_i = \ln\left(\dfrac{2-a+(-1)^i\left[(2-a)^2-8b\right]^{1/2}}{2b}\right)$. ❑

Proof of Proposition 18. Suppose that t_1^* is the optimal start time of the second stage under the benchmark policy (i.e., the point in time when components to cover remaining warranty demand are ordered from a third party). If a trade-in policy dominates the benchmark policy in the second stage that begins at time t_1^*, then the trade-in policy dominates the benchmark policy in the first-stage problem. This is because the cost of the first stage is the same for both policies, and the trade-in policy is less costly in the second stage. By similar reasoning, if a trade-in policy dominates the benchmark policy in the second stage for all possible $t_1 \in [0, \alpha L)$, then the trade-in policy dominates the benchmark policy in the first-stage problem. We use these observations to reinterpret sufficient conditions from the second-stage problem for the first-stage problem.

If $h = r = 0$, then $t_1^* = 0$. This is because $c_1 < c_2^0$, and without the presence of holding cost or discounting, the firm will never elect to purchase from the third party (i.e., it is less expensive to purchase all units from the vendor at time 0). Since $t_1^* = 0$, the firm purchases q_1 units at cost c_1, and thus we can make use of earlier second-stage propositions but with c_1 in place of c_2^0. Substituting $t_1^* = 0$ into (2.96) and (2.98) of Proposition 12 and replacing c_2^0 with c_1 and solving for c_1 yields (2.130) and (2.131). Substituting $t_1^* = 0$ into (2.103) and (2.105) of Proposition 14 and replacing c_2^0 with c_1 and solving for c_1 yields (2.132) and (2.133).

If $h > 0$ and $n = r = 0$, then $t_1^* > 0$ is possible. Note that

$$c_2^1 < c_2^0 \Leftrightarrow \frac{\alpha\left(L-t_1\right)}{1+\alpha\left(L-t_1\right)} < c_2^0 + \left[\left(1-\gamma\right)m - \left(p_n - 1\right)\right]$$

$$c_2^2 < c_2^0 \Leftrightarrow 1-e^{-\alpha\left(L-t_1\right)} < c_2^0 + \left[\left(1-\gamma\right)m - \left(p_n - 1\right)\right].$$

(see (2.107) and (2.108)). The above inequalities holds for all $t_1 \in [0, \alpha L)$ if they hold at $t_1 = 0$ (i.e., the left-hand sides are maximized at $t_1 = 0$). Substituting $t_1 = 0$ into the above yields (2.134) and (2.135). And if the unit acquisition cost is lower under a trade-in policy, then total cost is lower (due to lower warranty claim volume).

If $h > 0$, $r > 0$, and $n = 0$, then $t_1^* > 0$ is possible. Note that if $\tau \geq 0$, then the right-hand side of (2.115) is decreasing in t_1, and thus is maximized at $t_1 = 0$. Substituting $t_1 = 0$ into (2.115) yields (2.136). Thus condition (2.136) ensures that the restricted full trade-in policy dominates that benchmark for any t_1^*. Note that the left-hand side (2.116) is maximized at $t_1 = 0$, which when substituted into (2.116) yields (2.137). Thus, if (2.137) holds, the matching trade-in policy dominates that benchmark for any t_1^*. ❑

Proof of Proposition 19. If $h = r = 0$, then under the benchmark policy, $t_1^* = 0$ (see the proof of Proposition 18 for details on the support for this conclusion).

a) If $n(t) = 0$, then the optimal order quantity under the benchmark policy is equal to the remaining demand (i.e., $q_1^* = \alpha LN$) and

$$C_1^0 = \alpha LN\left(c_1 + c_w\right).$$

Therefore

$$1-C_1^1/C_1^0 \geq 1-C_1^1(0)/C_1^0 = \left(\frac{1}{(1+\alpha L)(c_1+c_w)}\right)\left(\frac{\alpha L}{1+\alpha L}-\tau+c_w\right) \text{ (see (2.86))}$$

$$1-C_1^2/C_1^0 \geq 1-C_1^2(0)/C_1^0 = 1-\left(\frac{1-e^{-\alpha L}}{\alpha L(c_1+c_w)}\right)\left(1-e^{-\alpha L}-\tau+c_w\right) \text{ (see (2.87)).}$$

b) If $n(t) = n$, then the optimal order quantity under the benchmark policy is equal to the remaining demand and

$$C_1^0 = N\left(\alpha L - \frac{\alpha n L^2}{2N}\right)(c_1+c_w)$$

(see the proof of Proposition 14). Recall that the matching trade-in policy is unaffected by the value of n (see Corollary 1). Therefore

$$1-C_1^2/C_1^0 \geq 1-C_1^2(0)/C_1^0 = 1-\frac{\left(1-e^{-\alpha L}\right)\left(1-e^{-\alpha L}-\tau+c_w\right)}{\left(\alpha L-\frac{\alpha n L^2}{2N}\right)(c_1+c_w)} \text{ (see (2.76)).} \ \square$$

6.4 Discrete-Time Problems

Sections 4.1 and 4.2 contain analyses of problems when time is continuous. We use the analytical framework in these sections as a means to understand the relationships between environmental characteristics (e.g., as reflected in the values of parameters), relative effectiveness of alternative trade-in policies, and optimal decisions/costs. However, while continuous-time models offer the advantage of tractability and consequent characterizations of behavior, the models rely on simplifying assumptions. In this section, we define problems based on discrete-time models and we specify solution algorithms.

The number of components in stock at the beginning of the first period is 0 and the number of units under warranty at the beginning of the first period is N. The sequence of events at the start and end of a time period are as follows.

Start of period t

1. Place final order (t = 1 only)
2. Send trade-in offer to selected customers

End of period t

5. Receive final order and incur cost c_1 per unit (t = 1 only)
6. Receive trade-in units and incur cost c_2^i per unit where $i \in \{1, 2\}$ denotes the trade-in policy
7. Warranty population reduced by trade-in quantity and/or warranty expirations
8. Observe demand from customers with product under warranty ($t < L$ only because remaining warranties expire in period L)
9. If demand is more than supply, then acquire the shortfall at cost c_2^0 per unit
10. Process demand and incur service cost c_w per unit
11. Incur cost h per unit on inventory
12. If inventory is positive, dispose/salvage desired number of units at cost c_3, or all units if $t = L$

6.4.1 Full Trade-in Policy

We define several problems for the case where the firm uses a full trade-in policy in the second-stage and we specify dynamic program (DP) algorithms that return the minimum cost. Under a full trade-in policy, the trade-in discount is disseminated to all customers with product under warranty in a single period. The number of products under warranty at the beginning of period t is $N(t)$. From the continuous-time relationships established in sections 4.1.2 and 4.1.4, it follows that the number of units returned is

$$q_2 = \left(1 - c_n - m + c_t^1\right) N(t) \tag{2.218}$$

where c_t^1 is the trade-in discount. Solving (2.218) for c_t^1 yields

$$c_t^1 = q_2 / N(t) - 1 + c_n + m \tag{2.219}$$

which when substituted into (2.11) yields trade-in cost

$$c_2^1 = q_2 / N(t) - \left(1 - c_n - \gamma m\right). \tag{2.220}$$

Problem $\mathbf{D1_1^1}$: Assumptions A1 – A5, A6b, A7a, A8b

Problem $\mathrm{D1}_1^1$ assumes that all warranties expire at the end of period L (i.e., $n(L) = N$. Let

$g_1^1(t, x_1, x_2, 1) =$ minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, and customers received the trade-in offer before period t

$g_1^1(t, x_1, x_2, 0) =$ minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, and no trade-in offer to-date

where (x_1, x_2) is the state at the beginning of period t.

For some sets of parameter values it can be advantageous to acquire more trade-ins than what is necessary to satisfy demand. In these cases, it may be cost effective for the firm to dispose/salvage the unneeded components immediately after acquisition rather than carrying the inventory until the end of the warranty horizon. However, due to deterministic demand (A9a) and $c_1 > -c_3$, it is never optimal to dispose of inventory prior to the trade-in offer (i.e., the firm would never set q_1 more than total demand). Consequently, the alternative of disposal is considered in the trade-in period and every period thereafter in the following expressions.

For period L,

$$g_1^1(L, x_1, x_2, 0) = g_1^1(L, x_1, x_2, 1) = (1-r)(h + c_3)x_1 \tag{2.221}$$

and for $t \in [2, L-1]$

$$g_1^1(t, x_1, x_2, 1) = (1-r)\left[\begin{array}{l} c_w \alpha x_2 + c_2^0 (\alpha x_2 - x_1)^+ + h(x_1 - \alpha x_2)^+ + \\ \min\limits_{q \le (x_1 - \alpha x_2)^+} \left\{ c_3 q + g_1^1\left(t+1, \left((x_1 - q) - \alpha x_2\right)^+, x_2, 1\right)\right\} \end{array}\right] \tag{2.222}$$

$$g_1^1(t, x_1, x_2, 0) =$$

$$(1-r)\min\left\{\begin{array}{l} c_w \alpha x_2 + c_2^0(\alpha x_2 - x_1)^+ + h(x_1 - \alpha x_2)^+ + g_1^1\left(t+1, (x_1 - \alpha x_2)^+, x_2, 0\right), \\ \min\limits_{q_2 \ge 1}\left\{\begin{array}{l} c_w \alpha (x_2 - q_2) + \dfrac{q_2{}^2}{x_2} - (1 - c_n - \gamma m) q_2 + c_2^0\left(\alpha(x_2 - q_2) - x_1 - q_2\right)^+ \\ \min\limits_{q \le \left(x_1 + q_2 - \alpha(x_2 - q_2)\right)^+}\left\{\begin{array}{l} h\left(x_1 + q_2 - \alpha(x_2 - q_2)\right)^+ + \\ c_3 q + g_1^1\left(\begin{array}{l} t+1, \left(x_1 + q_2 - q - \alpha(x_2 - q_2)\right)^+, \\ x_2 - q_2, 1 \end{array}\right) \end{array}\right\} \end{array}\right\} \end{array}\right\} \tag{2.223}$$

The optimum is

$$g_1^1(1,0,N,0)=(1-r)\min\left\{\begin{array}{l}\min\limits_{q_1\ge 0}\left\{\begin{array}{l}c_1q_1+c_w\alpha N+c_2^0(\alpha N-q_1)^+ + h(q_1-\alpha N)^+ + \\ g_1^1\left(2,(q_1-\alpha N)^+,N,0\right)\end{array}\right\}, \\ \min\limits_{\substack{q_1\ge 0\\ q_2\ge 0}}\left\{\begin{array}{l}c_1q_1+c_w\alpha(N-q_2)+ \\ \dfrac{{q_2}^2}{N}-(1-c_n-\gamma m)q_2 + \\ c_2^0\left(\alpha(N-q_2)-q_1-q_2\right)^+ + \\ \min\limits_{q\le(q_1+q_2-\alpha(N-q_2))^+}\left\{\begin{array}{l}h\left(q_1+q_2-\alpha(N-q_2)\right)^+ + \\ c_3q+g_1^1\left(\begin{array}{l}2,\left(\begin{array}{l}q_1+q_2-q- \\ \alpha(N-q_2)\end{array}\right)^+, \\ N-q_2,1\end{array}\right)\end{array}\right\}\end{array}\right\}\end{array}\right\} \quad (2.224)$$

The top expression in (2.224) returns the minimum cost given that no trade-in offers are disseminated in the first period. Since it is never optimal to dispose f inventory prior to the trade-in offer, there is no need to search over alternative values for the disposal quantity. Such a search is necessary in the bottom expression. The bottom expression in (2.224) returns the minimum cost given that trade-in offers are disseminated in the first period (though the firm can set $q_2 = 0$, which essentially prohibits trade-ins for the duration of the horizon). Pseudocode for a DP algorithm based on the above recursion is given in Section 7.4 of the appendix.

Problem $\mathbf{D1_2^1}$: Assumptions A1 – A5, A6b, A7b, A8b, A9a

Problem $\mathrm{D1}_2^1$ generalizes $\mathrm{D1}_1^1$ to allow nonzero reductions in the warranty population over time. If no trade-in program is offered, then the number of warranties that expire at the end of each period are $n(1)$, $n(2)$, …, $n(L)$, and the sum of these values is the warranty population at the beginning of the first period, i.e., $N = \sum_{t=1}^{L} n(t)$. If q_2 trade-in units are received in period t, then the warranty population is reduced by q_2 units. Since the trade-in units are from customers with warranties that expire no later than units not traded in (see A4), the number of warranties that expire in period $t + i$ is

$$\left(n(t+i) - \left(q_2 - \sum_{j=t}^{t+i-1} n(j)\right)^+\right)^+ \text{ for } i \in [0, L-t]. \tag{2.225}$$

The term $\left(q_2 - \sum_{j=t}^{t+i-1} n(j)\right)^+$ is the number of consumers who traded-in their product in period t and whose warranty would have expired in period $t + i$ or later. We refer to this number as the *warranty expiration carryover in period* $t + i$. For example, if the warranty expiration carryover in period $t + i$ is more than $n(t + i)$, then no warranties expire in period $t + i$.

Let

$g_2^1(t, x_1, x_2, x_3, 1) =$ minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, warranty expiration carryover z, and customers received the trade-in offer before period t

$g_2^1(t, x_1, x_2, 0) =$ minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, and no trade-in offer to-date

where (x_1, x_2, x_3) is the state at the beginning of period t. For period L,

$$g_2^1(L, x_1, x_2, 0) = g_2^1(L, x_1, x_2, x_3, 1) = (1-r)(h + c_3)x_1 \tag{2.226}$$

and for $t \in [2, L-1]$

$$g_2^1\left(t,x_1,x_2,x_3,1\right)=\left(1-r\right)\left[\begin{array}{l} c_w\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)+c_2^0\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)-x_1\right)^+ + \\ \min\limits_{q\le\left(x_1-\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right)^+}\left\{\begin{array}{l} h\left(x_1-\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)-q\right)^+ + \\ c_3q+g_2^1\left(\begin{array}{l} t+1,\left(\begin{array}{l} x_1-q- \\ \alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\end{array}\right)^+, \\ x_2-\left(n(t)-x_3\right)^+,\left(x_3-n(t)\right)^+,1\end{array}\right)\end{array}\right\}\end{array}\right] \quad (2.227)$$

$$g_2^1\left(t,x_1,x_2,0\right)=\left(1-r\right)\min\left\{\begin{array}{l} c_w\alpha\left(x_2-n(t)\right)+c_2^0\left(\alpha\left(x_2-n(t)\right)-x_1\right)^+ + \\ h\left(x_1-\alpha\left(x_2-n(t)\right)\right)^+ + \\ g_2^1\left(t+1,\left(x_1-\alpha\left(x_2-n(t)\right)\right)^+,x_2-n(t),0\right), \\ \min\limits_{q_2\ge 0}\left\{\begin{array}{l} c_w\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)+\dfrac{{q_2}^2}{x_2}-\left(1-c_n-\gamma m\right)q_2 + \\ c_2^0\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)-x_1-q_2\right)^+ + \\ \min\limits_{q\le\left(\begin{array}{l} x_1+q_2- \\ \alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\end{array}\right)^+}\left\{\begin{array}{l} c_3q+ \\ h\left(\begin{array}{l} x_1+q_2- \\ \alpha\left(x_2-\max\left\{n(t),q_2\right\}\right) \\ -q\end{array}\right)^+ + \\ g_2^1\left(\begin{array}{l} t+1, \\ \left(\begin{array}{l} x_1+q_2-q- \\ \alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\end{array}\right)^+, \\ x_2-\max\left\{n(t),q_2\right\}, \\ \left(q_2-n(t)\right)^+,1\end{array}\right)\end{array}\right\}\end{array}\right\}\end{array}\right\}$$

(2.228)

The optimum is

$$g_2^1\left(1,0,N,0\right)=\left(1-r\right)\min\left\{\begin{array}{l}\min\limits_{q_1\geq 0}\left\{\begin{array}{l}c_1q_1+c_w\alpha\left(N-n\left(1\right)\right)+c_2^0\left(\alpha\left(N-n\left(1\right)\right)-q_1\right)^++\\ h\left(q_1-\alpha\left(N-n\left(1\right)\right)\right)^++\\ g_2^1\left(2,\left(q_1-\alpha\left(N-n\left(1\right)\right)\right)^+,N-n\left(1\right),0\right)\end{array}\right\},\\ \min\limits_{\substack{q_1\geq 0\\ q_2\geq 0}}\left\{\begin{array}{l}c_1q_1+c_w\alpha\left(N-\max\left\{n\left(1\right),q_2\right\}\right)+\dfrac{{q_2}^2}{N}-\\ \left(1-c_n-\gamma m\right)q_2+\\ c_2^0\left(\alpha\left(N-\max\left\{n\left(1\right),q_2\right\}\right)-q_1-q_2\right)^++\\ \min\limits_{q\leq\left(\begin{array}{l}q_1+q_2-\\ \alpha\left(N-\max\left\{n\left(1\right),q_2\right\}\right)\end{array}\right)^+}\left\{\begin{array}{l}c_3q+h\left(\begin{array}{l}q_1+q_2-\\ \alpha\left(\begin{array}{l}N-\\ \max\left\{n\left(1\right),q_2\right\}\end{array}\right)\end{array}\right)^++\\ g_2^1\left(\begin{array}{l}2,\\ \left(\begin{array}{l}q_1+q_2-q-\\ \alpha\left(N-\max\left\{n\left(1\right),q_2\right\}\right)\end{array}\right)^+,\\ N-\max\left\{n\left(1\right),q_2\right\},\\ \left(q_2-n\left(1\right)\right)^+,1\end{array}\right)\end{array}\right\}\end{array}\right\}\end{array}\right\}.(2.229)$$

Pseudocode for a DP algorithm based on the above recursion is given in Section 7.4 of the appendix.

Problem $\mathrm{D1}_3^1$: Assumptions A1 – A5, A6b, A7b, A8b

Problem $\mathrm{D1}_3^1$ generalizes $\mathrm{D1}_2^1$ to allow for stochastic demand. Recall that *d*(*t*) is the demand in period *t*, which for this problem, is stochastic. In the recursion we suppress the time parameter and add a parameter for the expected demand. We do this in order to clarify the relationship between decisions and the nature of random demand. Accordingly, letting *x* denote the warranty population at the point in time when demand is realized, the random demand in the period is

$$d(\alpha x) \tag{2.230}$$

and

$$E[d(\alpha x)] = \alpha x. \tag{2.231}$$

The recursion closely follows the recursion for $\mathrm{D}1_2^1$. Let

$g_3^1(t, x_1, x_2, x_3, 1) =$ minimum expected cost from period t through the end of the horizon given inventory x_1, warranty population x_2, warranty expiration carryover x_3, and customers received the trade-in offer before period t

$g_3^1(t, x_1, x_2, 0) =$ minimum expected cost from period t through the end of the horizon given inventory x_1, warranty population x_2, and no trade-in offer to-date

where (x_1, x_2, x_3) is the state at the beginning of period t. For period L,

$$g_3^1(L, x_1, x_2, 0) = g_3^1(L, x_1, x_2, x_3, 1) = (1-r)(h+c_3)x_1 \tag{2.232}$$

and for $t \in [2, L-1]$

$$g_3^1(t,x_1,x_2,x_3,1)=(1-r)\left[\begin{array}{l} c_w d\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right)+ \\ c_2^0\left(d\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right)-x_1\right)^+ + \\ h\left(x_1-d\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right)\right)^+ + \\ c_3 q^*\left(d\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right)\right)+ \\ g_3^1\left(\begin{array}{l} t+1, \\ \left(\begin{array}{l} x_1-q^*\left(d\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right)\right)- \\ d\left(\alpha\left(x_2-\left(n(t)-x_3\right)^+\right)\right) \end{array}\right)^+, \\ x_2-\left(n(t)-x_3\right)^+,\left(x_3-n(t)\right)^+,1 \end{array}\right) \end{array}\right] \tag{2.233}$$

$$g_3^1(t,x_1,x_2,0)=(1-r)\min\left\{\begin{array}{l} E\left[\begin{array}{l} c_w d\left(\alpha\left(x_2-n(t)\right)\right)+c_2^0\left(d\left(\alpha\left(x_2-n(t)\right)\right)-x_1\right)^+ + \\ h\left(x_1-d\left(\alpha\left(x_2-n(t)\right)\right)\right)^+ + \\ g_3^1\left(t+1,\left(x_1-d\left(\alpha\left(x_2-n(t)\right)\right)\right)^+,x_2-n(t),0\right) \end{array}\right], \\ \min\limits_{q_2\geq 0} E\left[\begin{array}{l} c_w d\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\right)+ \\ \dfrac{{q_2}^2}{x_2}-\left(1-c_n-\gamma m\right)q_2+ \\ c_2^0\left(d\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\right)-x_1-q_2\right)^+ + \\ h\left(x_1+q_2-d\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\right)\right)^+ + \\ c_3 q^*\left(d\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\right)\right)+ \\ g_3^1\left(\begin{array}{l} t+1, \\ \left(\begin{array}{l} x_1+q_2-q^*\left(d\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\right)\right)- \\ d\left(\alpha\left(x_2-\max\left\{n(t),q_2\right\}\right)\right) \end{array}\right)^+, \\ x_2-\max\left\{n(t),q_2\right\},\left(q_2-n(t)\right)^+,1 \end{array}\right) \end{array}\right] \end{array}\right\} \tag{2.234}$$

where $q^*\left(\tilde{d}\right)$ is the disposal quantity that minimize expected cost over the remainder of the horizon given any realization $\tilde{d}$ of random demand, i.e., for (2.233)

$$q^*\left(\tilde{d}\right) = \underset{q \le \left(x_1 - \tilde{d}\right)^+}{\arg\min} \left\{ c_3 q + g_3^1\left(t+1, \left(x_1 - q - \tilde{d}\right)^+, x_2 - \left(n(t) - x_3\right)^+, \left(x_3 - n(t)\right)^+, 1\right) \right\} \tag{2.235}$$

and for (2.234)

$$q^*\left(\tilde{d}\right) = \underset{q \le \left(x_1 + q_2 - \tilde{d}\right)^+}{\arg\min} \left\{ c_3 q + g_3^1\left(t+1, \left(x_1 + q_2 - q - \tilde{d}\right)^+, x_2 - \max\left\{n(t), q_2\right\}, \left(q_2 - n(t)\right)^+, 1\right) \right\}. \tag{2.236}$$

The optimum is

$$g_3^1(1,0,N,0) = (1-r)\min \left\{ \begin{array}{l} \min\limits_{q_1 \ge 0} E\left[\begin{array}{l} c_1 q_1 + c_w d\left(\alpha\left(N - n(1)\right)\right) + c_2^0 \begin{pmatrix} d\left(\alpha\left(N - n(1)\right)\right) \\ -q_1 \end{pmatrix}^+ + \\ h\left(q_1 - d\left(\alpha\left(N - n(1)\right)\right)\right)^+ + \\ g_3^1\left(2, \left(q_1 - d\left(\alpha\left(N - n(1)\right)\right)\right)^+, N - n(1), 0\right) \end{array} \right], \\ \min\limits_{\substack{q_1 \ge 0 \\ q_2 \ge 0}} E\left[\begin{array}{l} c_1 q_1 + c_w d\left(\alpha\left(N - \max\left\{n(1), q_2\right\}\right)\right) + \\ \dfrac{q_2{}^2}{N} - \left(1 - c_n - \gamma m\right) q_2 + \\ c_2^0\left(d\left(\alpha\left(N - \max\left\{n(1), q_2\right\}\right)\right) - q_1 - q_2\right)^+ + \\ h\left(q_1 + q_2 - d\left(\alpha\left(N - \max\left\{n(1), q_2\right\}\right)\right)\right)^+ + \\ c_3 q^*\left(d\left(\alpha\left(N - \max\left\{n(1), q_2\right\}\right)\right)\right) + \\ g_3^1 \left(\begin{array}{l} 2, \\ \begin{pmatrix} q_1 + q_2 - q^*\left(d\left(\alpha\left(N - \max\left\{n(1), q_2\right\}\right)\right)\right) - \\ d\left(\alpha\left(N - \max\left\{n(1), q_2\right\}\right)\right) \end{pmatrix}^+, \\ N - \max\left\{n(1), q_2\right\}, \left(q_2 - n(1)\right)^+, 1 \end{array} \right) \end{array} \right] \end{array} \right\} \tag{2.237}$$

where

$$q^*\left(\tilde{d}\right) = \underset{q \le \left(q_1+q_2-\tilde{d}\right)^+}{\arg\min} \left\{ c_3 q + g_3^1\left(2, \left(q_1+q_2-q-\tilde{d}\right)^+, N - \max\left\{n(1), q_2\right\}, \left(q_2 - n(1)\right)^+, 1\right)\right\}. \quad (2.238)$$

6.4.2 Matching Trade-in Policy

Under a matching trade-in policy, the trade-in discounts begin to be to customers with product under warranty during the period where there is insufficient inventory to cover demand. Trade-in volume is set to match demand in each subsequent period. No customer receives a trade-in offer more than once (assumption A2).

For a given final order quantity q_1, trade-in discounts begin to be offered in period

$$T_1(q_1) = \min\left\{\min\left\{t \mid D(t) > q_1\right\}, L\right\} = \min\left\{\min\left\{t \mid \sum_{i=1}^{t} \alpha\left(N - \sum_{j=1}^{i} n(j)\right) > q_1\right\}, L\right\}, \quad (2.239)$$

which is also known as the run-out time of the final order quantity. Recall that $c_t(t)$ is the trade-in discount off the price of a new model, $\nu(t)$ is the number of customers who receive the trade-in offer, $\beta(t)$ is the fraction of customers who accept the trade-in offer, $s(t)$ is the number of trade-in units, and $c_2^2(t)$ is the per-unit acquisition cost of trade-in units. From the continuous-time relationships established in sections 4.1.2 and 4.1.5, the relationships among these terms is as follows:

$$\beta(t) = 1 - c_n - m - c_t(t) \quad (2.240)$$

$$s(t) = \beta(t)\nu(t) = [1 - c_n - m - c_t(t)]\nu(t) \quad (2.241)$$

$$c_2^2(t) = c_t(t) - (1-\gamma)m = \beta(t) + c_n + \gamma m - 1 = s(t)/\nu(t) + c_n + \gamma m - 1 \quad (2.242)$$

The firm makes three types of decisions over time: (1) the value of the final order quantity q_1 in period 1, then in periods $T_1(q_1), \ldots, L-1$ the firm determines (2) the number of customers who receive the trade-in offer $\nu(t)$ and (3) the trade-in discount $c_t(t)$ (or equivalently, the trade-in fraction $\beta(t)$).

Due to the matching policy, the number of trade-ins plus any initial inventory is set to match demand. Let I denote the inventory at the beginning of period $t = T_1(q_1)$. The trade-in quantity in period $t = T_1(q_1)$ satisfies

$$I + s(t) = d(t) = \alpha[N(t) - \max\{n(t), s(t)\}]. \tag{2.243}$$

The value of $\max\{n(t), s(t)\}$ is the reduction in the warranty population during the period, which is either the number of warranties that expire or the number of trade-ins, whichever is larger. Solving (2.243) for the trade-in quantity yields

$$t = T_1(q_1):\ \ s(t) = \begin{cases} \dfrac{\alpha N(t) - I}{1+\alpha} & , n(t) \le \dfrac{\alpha N(t) - I}{1+\alpha} \\ \alpha\left[N(t) - n(t)\right] - I & , n(t) \ge \dfrac{\alpha N(t) - I}{1+\alpha} \end{cases}. \tag{2.244}$$

Let O denote the warranty expiration carryover at the beginning of period $t > T_1(q_1)$. By virtue of the matching policy, the inventory at the beginning of period t is zero. The trade-in quantity in period $t > T_1(q_1)$ satisfies

$$s(t) = d(t) = \alpha[N(t) - \max\{(n(t) - O)^+, s(t)\}]. \tag{2.245}$$

The value of $(n(t) - O)^+$ is the number of warranties that expire in period t if there were no trade-ins in the period (see (2.225)). Thus, the value of $\max\{(n(t) - O)^+, s(t)\}$ is the reduction in the warranty population during the period, which is either the number of warranties that expire or the number of trade-ins, whichever is larger. Solving (2.245) for the trade-in quantity yields

$$t > T_1(q_1):\ \ s(t) = \begin{cases} \dfrac{\alpha N(t)}{1+\alpha} & ,\max\left\{\left(n(t)-O\right)^+\right\} \le \dfrac{\alpha N(t)-I}{1+\alpha} \\ \alpha\left[N(t)-\left(n(t)-O\right)^+\right] & ,\max\left\{\left(n(t)-O\right)^+\right\} \ge \dfrac{\alpha N(t)-I}{1+\alpha} \end{cases}. \qquad (2.246)$$

Consolidating (2.244) and (2.246), we have

$$t \ge T_1(q_1):\ \ s(t) = \min\left\{\frac{\alpha N(t)-I}{1+\alpha}, \alpha\left[N(t)-\left(n(t)-O\right)^+\right]-I\right\} \qquad (2.247)$$

which reduces to

$$t \ge T_1(q_1):\ \ s(t) = \frac{\alpha N(t)-I}{1+\alpha}. \qquad (2.248)$$

when $n(t) = 0$ for all t.

Problem $\mathrm{D1}_1^2$: Assumptions A1 – A5, A6b, A7a, A8b

Problem $\mathrm{D1}_1^2$ assumes that all warranties expire at the end of period L (i.e., $n(L) = N$ (see A7a). Due to A7a, the warranty population at the beginning of period t is

$$N(t) = N - \sum_{i=1}^{t-1} s(i). \qquad (2.249)$$

Due to A2, we require

$$\sum_{t=T_1(q_1)}^{L-1} \nu(t) \le N \qquad (2.250)$$

(i.e., a customer receives a trade-in offer no more than once). We are now ready to specify the DP relationships. From (2.250) it follows that in any given period where trade-ins occur, the number of trade-in offers must be between the required supply and the number of customers who have not received a trade-in offer, i.e.,

$$s(t) \le \nu(t) \le N - \sum_{t=T_1(q_1)}^{t-1} \nu(t). \tag{2.251}$$

Let $\mathbf{x} = (x_1, x_2, x_3)$ and

$g_1^2(t,\mathbf{x})$ = minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, and number of customers in the warranty population who have not received a trade-in offer x_3.

where $\mathbf{x}$ is the state at the beginning of period t. Due to the matching policy and the fact that demand is deterministic (A9a), inventory at the start of period L is assured to be zero (i.e., the firm will never set the initial order quantity to more than total demand). Consequently, for period L,

$$g_1^2(L,0,x_2,x_3) = 0\,. \tag{2.252}$$

Inequality (2.251) is incorporated into the following recursion that applies for $t \in [2, L - 1]$:

$$g_1^2(t,\mathbf{x}) = (1-r)\begin{cases} c_w \alpha x_2 + h(x_1 - \alpha x_2) + g_1^2(t+1, x_1 - \alpha x_2, x_2, x_3) & , x_1 \ge \alpha x_2 \\ c_w(s + x_1) + \begin{cases} \min\limits_{\nu \in [s, x_3]} \left\{ \begin{array}{l} (s/\nu + c_n + \gamma m - 1)s + \\ g_1^2(t+1, 0, x_2 - s, x_3 - \nu) \end{array} \right\} & , s \le x_3 \\ \infty & , s > x_3 \end{cases} & , x_1 < \alpha x_2 \end{cases} \tag{2.253}$$

where

$$s = \frac{\alpha x_2 - x_1}{1+\alpha} \tag{2.254}$$

(see (2.248)). The top expression in (2.253) applies when there is enough inventory to cover demand in the period. The bottom expression applies when trade-ins are necessary to cover demand. If $s > x_3$, then the number of customers who have not yet received a trade-in offer is less than the supply needed to match demand. In this case we have an infeasible implementation of the matching policy and we set the cost to ∞.

In period 1, if $q_1 < \alpha N$ (i.e., the order quantity is less than demand in period 1), then the trade-in quantity to match total supply with demand is

$$s(q_1) = \frac{\alpha N - q_1}{1+\alpha} \tag{2.255}$$

(see (2.248)). In period 1, $\mathbf{x}$ = (0, N, N) and the optimum is

$$g_1^2(1,\mathbf{x}) = (1-r)\min_{q_1 \ge 0} \begin{cases} \begin{Bmatrix} c_1 q_1 + c_w \alpha N + h(q_1 - \alpha N) + \\ g_1^2(2, q_1 - \alpha N, N, N) \end{Bmatrix} & , q_1 \ge \alpha N \\ \begin{Bmatrix} c_1 q_1 + c_w\left(s(q_1) + q_1\right) + \\ \min\limits_{\nu \in [s(q_1), N]} \begin{Bmatrix} \left(s(q_1)/\nu + c_n + \gamma m - 1\right) s(q_1) + \\ g_1^2\left(2, 0, N - s(q_1), N - \nu\right) \end{Bmatrix} \end{Bmatrix} & , q_1 < \alpha N \end{cases}. \tag{2.256}$$

Pseudocode for a DP algorithm based on the above recursion is given in Section 7.4 of the appendix.

Problem $\mathrm{D1}_2^2$: Assumptions A1 – A5, A6b, A7b, A8b

Problem $\mathrm{D1}_2^2$ generalizes $\mathrm{D1}_1^2$ to allow nonzero reductions in the warranty population over time. If there are no trade-ins, then the number of warranties that expire at the end of each period are $n(1)$, $n(2)$, …, $n(L)$, and the sum of these values is the warranty population at the beginning of the first period, i.e., $N = \sum_{t=1}^{L} n(t)$. As in problem $\mathrm{D1}_2^1$, we keep track of the warranty expiration carryover.

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and

$g_2^2(t,\mathbf{x})$ = minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, number of customers in the warranty population who have not received a trade-in offer x_3, and warranty expiration carryover x_4

where **x** is the state at the beginning of period t. Due to the matching policy and the fact that demand is deterministic (A9a), inventory at the start of period L is assured to be zero (i.e., the firm will never set the initial order quantity to more than total demand). Consequently, for period L,

$$g_2^2\left(L,0,x_2,x_3,x_4\right)=0\,. \tag{2.257}$$

For $t \in [2, L-1]$

$$g_2^2\left(t,\mathbf{x}\right)=\left(1-r\right)\begin{cases}\left\{\begin{array}{l}c_w\alpha\left(x_2-n(t)\right)+h\left(x_1-\alpha\left(x_2-n(t)\right)\right)+\\ g_2^2\left(t+1,x_1-\alpha\left(x_2-n(t)\right),x_2-n(t),x_3-n(t),0\right)\end{array}\right\} & ,x_1\geq\alpha\left(x_2-n(t)\right)\\ \begin{cases}\begin{array}{l}c_w\left(s+x_1\right)+\\ \min\limits_{v\in[s,x_3]}\left\{\begin{array}{l}\left(s/v+c_n+\gamma m-1\right)s+\\ g_2^2\left(\begin{array}{l}t+1,0,\\ x_2-\max\left\{\begin{array}{l}\left(n(t)-x_4\right)^+,\\ s\end{array}\right\},\\ x_3-\max\left\{\begin{array}{l}\left(n(t)-x_4\right)^+,\\ v\end{array}\right\},\\ \left(x_4+s-n(t)\right)^+\end{array}\right)\end{array}\right\}\end{array} & ,s\leq x_3\\ \infty & ,s>x_3\end{cases} & ,x_1<\alpha\left(x_2-n(t)\right)\end{cases} \tag{2.258}$$

where

$$s=\min\left\{\frac{\alpha x_2-x_1}{1+\alpha},\alpha\left(x_2-\left(n(t)-x_4\right)^+\right)-x_1\right\} \tag{2.259}$$

(see (2.247)). The top expression in (2.258) applies when there is enough inventory to cover demand in the period (e.g., trade-in offers have yet been disseminated). The bottom expression applies when trade-ins are necessary to cover demand. If $s > x_3$, then the number of customers who have not yet received a trade-in offer is less than the supply needed to match demand. In this case we have an infeasible implementation of the matching policy and we set the cost to ∞.

The expressions for the elements of **x** in $g_2^2(t,\mathbf{x})$ in the bottom expression in (2.258) may benefit from explanations. The inventory going into the next period is zero by virtue of the matching trade-in policy (i.e., supply matches demand). The second term in **x**, which is the warranty population going into the next period, is the current population x_2 reduced by $\max\{(n(t) - x_4)^+, s\}$. The value of $(n(t) - x_4)^+$ is the number of warranties that would expire in the period if there were no trade-ins, so the reduction in the warranty population is the larger of this value and the number of trade-ins (s). The third term in **x**, which is the number of customers in the warranty population who have not received a trade-in offer as of the start of the next period, is the number at the start of the current period (x_3) reduced by $\max\{(n(t) - x_4)^+, \nu\}$. The number of customers who are in the warranty population and who have not received a trade-in is reduced by at least the number of trade-in offers (ν). However, if $(n(t) - x_4)^+ > \nu$, then $((n(t) - x_4)^+ - \nu)$ customers with warranties that expired in the period did not receive a trade-in offer, and these customers are no longer in the warranty population at the start of the next period. The fourth term in **x** is warranty expiration carryover, which is the warranty expiration carryover at the start of the period (x_4) plus the net change in warranty expiration carryover, adjusted to zero if necessary to account for the fact that carryover cannot be negative. The net change in warranty expiration carryover is the difference between the number of trade-ins during the period (s) and the number of warranties that would have expired in the period if there were no trade-ins ($n(t)$).

In period 1, if $q_1 < \alpha(N - n(1))$ (i.e., the order quantity is less than demand in period 1), then the trade-in quantity to match supply with demand is

$$s\left(q_1\right) = \min\left\{\frac{\alpha N - q_1}{1+\alpha}, \alpha\left(N - n(1)\right) - q_1\right\} \tag{2.260}$$

(see (2.247)). Thus, $\mathbf{x} = (0, N, N, 0)$ and the optimum is

$$g_2^2\left(1,\mathbf{x}\right) = \left(1-r\right)\min_{q_1 \geq 0}\begin{cases} \left\{\begin{matrix} c_1 q_1 + c_w \alpha\left(N - n(1)\right) + h\left(q_1 - \alpha\left(N - n(1)\right)\right) + \\ g_2^2\left(2, q_1 - \alpha\left(N - n(1)\right), N - n(1), N - n(1), 0\right) \end{matrix}\right\} & , q_1 \geq \alpha\left(N - n(1)\right) \\ \left\{\begin{matrix} c_1 q_1 + c_w\left(s\left(q_1\right) + q_1\right) + \\ \min\limits_{\nu \in \left[s(q_1), N\right]}\left\{\begin{matrix} \left(s\left(q_1\right)/\nu + c_n + \gamma m - 1\right)s\left(q_1\right) + \\ g_2^2\left(\begin{matrix} 2, 0, \\ N - \max\left\{n(1), s\left(q_1\right)\right\}, \\ N - \max\left\{n(1), \nu\right\}, \\ \left(s\left(q_1\right) - n(1)\right)^+ \end{matrix}\right) \end{matrix}\right\} \end{matrix}\right\} & , q_1 < \alpha\left(N - n(1)\right) \end{cases}$$

(2.261)

Pseudocode for a DP algorithm based on the above recursion is given in Section 7.4 of the appendix.

Problem $\mathrm{D1}_3^2$: Assumptions A1 – A5, A6b, A7a, A8b

Problem $\mathrm{D1}_3^2$ generalizes $\mathrm{D1}_2^2$ to allow for stochastic demand. In contrast to $\mathrm{D1}_2^2$, the firm cannot exactly match supply with demand due to randomness, i.e., demand is realized after trade-ins are received. The matching policy in the following recursion sets the number of trade-ins so that supply matches expected demand. In the next section we identify a recursion where this restriction is relaxed (see problem $\mathrm{D1}_2^3$).

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and

$g_3^2(t,\mathbf{x})$ = minimum expected cost from period t through the end of the horizon given inventory x_1, warranty population x_2, number of customers in the warranty population who have not received a trade-in offer x_3, and warranty expiration carryover x_4

where $\mathbf{x}$ is the state at the beginning of period t. For period L,

$$g_3^2(L,\mathbf{x}) = (1-r)(h+c_3)x_1 . \tag{2.262}$$

For $t \in [2, L-1]$

$$g_3^2(t,\mathbf{x}) = (1-r)E\left[\begin{array}{l} c_w d\left(\alpha\left(x_2-\left(n(t)-x_4\right)^+\right)\right)+ \\ h\left(x_1-d\left(\alpha\left(x_2-\left(n(t)-x_4\right)^+\right)\right)\right)^+ + \\ c_2^0\left(d\left(\alpha\left(x_2-\left(n(t)-x_4\right)^+\right)\right)-x_1\right)^+ + \\ g_3^2\left(\begin{array}{l} t+1, \\ \left(x_1-d\left(\alpha\left(x_2-\left(n(t)-x_4\right)^+\right)\right)\right)^+, \\ x_2-\left(n(t)-x_4\right)^+, x_3-\left(n(t)-x_4\right)^+, \\ \left(x_4-n(t)\right)^+ \end{array}\right) \end{array}\right], x_1 \geq \alpha\left(x_2-n(t)\right) \tag{2.263}$$

$$g_3^2(t,\mathbf{x}) = (1-r)\left[\begin{array}{ll} \min\limits_{\nu\in[s,x_3]} E\left[\begin{array}{l} c_w d\left(\alpha\left(x_2 - \max\left\{\left(n(t)-x_4\right)^+, s\right\}\right)\right) + \\ h\left(x_1 - d\left(\alpha\left(x_2 - \max\left\{\left(n(t)-x_4\right)^+, s\right\}\right)\right)\right)^+ + \\ c_2^0\left(d\left(\alpha\left(x_2 - \max\left\{\left(n(t)-x_4\right)^+, s\right\}\right)\right) - x_1\right)^+ + \\ \left(s/\nu + c_n + \gamma m - 1\right)s + \\ g_3^2\left(\begin{array}{l} t+1, \\ \left(x_1 - d\left(\alpha\left(x_2 - \max\left\{\left(n(t)-x_4\right)^+, s\right\}\right)\right)\right)^+, \\ x_2 - \max\left\{\left(n(t)-x_4\right)^+, s\right\}, \\ x_3 - \max\left\{\left(n(t)-x_4\right)^+, \nu\right\}, \\ \left(x_4 + s - n(t)\right)^+ \end{array}\right) \end{array}\right] & , s \le x_3 \\ \infty & , s > x_3 \end{array}\right] \quad , x_1 < \alpha\left(x_2 - n(t)\right)$$

(2.264)

where

$$s = \min\left\{\frac{\alpha x_2 - x_1}{1+\alpha}, \alpha\left(x_2 - \left(n(t)-x_4\right)^+\right) - x_1\right\}. \qquad (2.265)$$

In period 1, if $q_1 < \alpha(N - n(1))$ (i.e., the order quantity is less than demand in period 1), then the trade-in quantity to match supply with expected demand is

$$s(q_1) = \min\left\{\frac{\alpha N - q_1}{1+\alpha}, \alpha\left(N - n(1)\right) - q_1\right\} \tag{2.266}$$

(see (2.260)). Thus, **x** = (0, *N*, *N*, 0) and the optimum is

$$g_3^2(1,\mathbf{x}) = (1-r)\min_{q_1 \ge 0}\begin{cases} E\left[\begin{array}{l} c_1 q_1 + c_w d\left(\alpha\left(N-n(1)\right)\right) + \\ h\left(q_1 - d\left(\alpha\left(N-n(1)\right)\right)\right)^+ + \\ c_2^0\left(d\left(\alpha\left(N-n(1)\right)\right) - q_1\right)^+ + \\ g_3^2\left(\begin{array}{l} 2, \left(q_1 - d\left(\alpha\left(N-n(1)\right)\right)\right)^+, \\ N-n(1), N-n(1), 0 \end{array}\right) \end{array}\right] & , q_1 \ge \alpha\left(N-n(1)\right) \\ \min\limits_{\nu \in [s(q_1), N]} E\left[\begin{array}{l} c_1 q_1 + \\ c_w d\left(\alpha\left(N - \max\{n(1), s(q_1)\}\right)\right) + \\ h\left(q_1 - d\left(\alpha\left(N - \max\{n(1), s(q_1)\}\right)\right)\right)^+ + \\ c_2^0\left(d\left(\alpha\left(N - \max\{n(1), s(q_1)\}\right)\right) - q_1\right)^+ + \\ \left(s(q_1)/\nu + c_n + \gamma m - 1\right)s(q_1) + \\ g_3^2\left(\begin{array}{l} 2, \left(q_1 - d\left(\alpha\left(N - \max\{n(1), s(q_1)\}\right)\right)\right)^+, \\ N - \max\{n(1), s(q_1)\}, \\ N - \max\{n(1), \nu\}, \\ \left(s(q_1) - n(1)\right)^+ \end{array}\right) \end{array}\right] & , q_1 < \alpha\left(N-n(1)\right) \end{cases}$$

(2.267)

Recursions (2.263) and (2.267) mirror recursion (2.258) and (2.261), respectively, but include the costs of mismatches between supply and demand due to randomness.

6.4.3 General Trade-in Policy

In this section we define DP algorithms that solve problems with no restrictions on the trade-in policy. Trade-ins can be offered in any period. The only requirement is that a customer receives a trade-in offer no more than a once (assumption A2). By comparing solutions obtained from this algorithm with solutions under the simple full and matching trade-in policies, we are able to assess to potential cost premiums associated with these simple policies.

Problem $\mathbf{D1}_1^3$: Assumptions A1 – A5, A6b, A7b, A8b

Recall that for a given number of trade-in units $s(t)$, the acquisition cost per unit is

$$c_2^2(t) = s(t)/\nu(t) + c_n + \gamma m - 1 = \beta(t) + c_n + \gamma m - 1. \qquad (2.268)$$

(see (2.241)). In contrast with a matching policy, there is no requirement that supply match demand.

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and

$g_1^3(t,\mathbf{x})$ = minimum cost from period t through the end of the horizon given inventory x_1, warranty population x_2, number of customers in the warranty population who have not received a trade-in offer x_3, and warranty expiration carryover x_4

where $\mathbf{x}$ is the state at the beginning of period t. For period L,

$$g_1^3(L,\mathbf{x}) = (1-r)(h+c_3)x_1 \qquad (2.269)$$

and for $t \in [2, L-1]$

$$g_1^3(t,\mathbf{x}) = (1-r)\min_{\substack{s\geq 0\\ v\in[s,x_3]}}\left\{\begin{array}{l}
c_w\alpha\left(x_2-\max\left\{\begin{array}{l}(n(t)-x_4)^+,\\ s\end{array}\right\}\right)+\dfrac{s^2}{v}-(1-c_n-\gamma m)s+\\
c_2^0\left(\alpha\left(x_2-\max\left\{\begin{array}{l}(n(t)-x_4)^+,\\ s\end{array}\right\}\right)-x_1-s\right)^+ +\\
h\left(x_1+s-\alpha\left(x_2-\max\left\{\begin{array}{l}(n(t)-x_4)^+,\\ s\end{array}\right\}\right)\right)^+ +\\
\min\limits_{q\leq\left(\begin{array}{l}x_1+s-\\ \alpha\left(x_2-\max\left\{\begin{array}{l}(n(t)-x_4)^+,\\ s\end{array}\right\}\right)\end{array}\right)^+}\left\{\begin{array}{l}
c_3q+\\
g_1^3\left(\begin{array}{l}
t+1,\\
\left(\begin{array}{l}x_1+s-q-\\ \alpha\left(x_2-\max\left\{\begin{array}{l}(n(t)-x_4)^+,\\ s\end{array}\right\}\right)\end{array}\right)^+,\\
x_2-\max\left\{\begin{array}{l}(n(t)-x_4)^+,\\ s\end{array}\right\},\\
x_3-\max\left\{(n(t)-x_4)^+,v\right\},\\
(x_4+s-n(t))^+
\end{array}\right)
\end{array}\right\}
\end{array}\right\} \tag{2.270}$$

The terms in (2.270) draw on the terms in (2.228) and (2.258). The demand term, which is based on the warranty population after trade-ins and warranty expirations, is the same as in (2.228) except that warranty expiration carryover is considered (there was no such adjustment in (2.228) because trade-ins, if offered, are offered in only one period). The trade-in acquisition cost is $s^2/v - (1 - c_n - \gamma m)s$, which is the same as in (2.228) except s replaces q_2. The inventory going into the next period is the same as in (2.228) but with the added element to account for warranty expiration carryover. The warranty population going into the next period is the same as in (2.228) but with the added element to account for warranty expiration carryover. The number of customers in the warranty population who have not received a trade-in offer at the start of the next period is the same as in

(2.258). The warranty expiration carryover going into the next period is the same as in (2.258).

In period 1, **x** = (0, N, N, 0) and the optimum is

$$g_1^3(1,\mathbf{x}) = (1-r)\min_{\substack{q_1\geq 0\\ s\geq 0\\ v\in[s,x_3]}}\left\{\begin{array}{l} c_1q_1 + \\ c_w\alpha\left(N-\max\{n(1),s\}\right) + \\ \dfrac{s^2}{v} - (1-c_n-\gamma m)s + \\ c_2^0\left(\alpha\left(N-\max\{n(1),s\}\right)-q_1-s\right)^+ + \\ h\left(q_1+s-\alpha\left(N-\max\{n(1),s\}\right)\right)^+ + \\ \min\limits_{q\leq\left(\begin{array}{l} q_1+s- \\ \alpha\left(N-\max\{n(1),s\}\right)\end{array}\right)^+}\left\{c_3q + g_1^3\left(\begin{array}{l} 2,\left(\begin{array}{l} q_1+s-q- \\ \alpha\left(N-\max\{n(1),s\}\right)\end{array}\right)^+, \\ N-\max\{n(1),s\}, \\ N-\max\{n(1),v\}, \\ \left(s-n(1)\right)^+ \end{array}\right)\right\} \end{array}\right\} \qquad (2.271)$$

Pseudocode for a DP algorithm based on the above recursion is given in Section 7.4 of the appendix.

Problem $\mathrm{D1}_2^3$: Assumptions A1 – A5, A6b, A7a, A8b

Problem $\mathrm{D1}_2^3$ generalizes $\mathrm{D1}_1^3$ to allow for stochastic demand. Let **x** = (x_1, x_2, x_3, x_4) and

$g_2^3(t,\mathbf{x})$ = minimum expected cost from period t through the end of the horizon given inventory x_1, warranty population x_2, number of customers in the warranty population who have not received a trade-in offer x_3, and warranty expiration carryover x_4

where **x** is the state at the beginning of period t. For period L,

$$g_2^3\left(L,\mathbf{x}\right)=\left(1-r\right)\left(h+c_3\right)x_1 \tag{2.272}$$

and for $t \in [2, L-1]$

$$g_2^3\left(t,\mathbf{x}\right)=\left(1-r\right)\min_{\substack{s\geq 0\\ \nu\in[s,x_3]}} E\left[\begin{array}{l} c_w d\left(\alpha\left(x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}\right)\right)+ \\ \dfrac{s^2}{\nu}-\left(1-c_n-\gamma m\right)s+ \\ c_2^0\left(d\left(\alpha\left(x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}\right)\right)-x_1-s\right)^+ + \\ h\left(x_1+s-d\left(\alpha\left(x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}\right)\right)\right)^+ + \\ c_3 q^*\left(d\left(\alpha\left(x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}\right)\right)\right)+ \\ g_2^3\left(\begin{array}{l} t+1, \\ \left(\begin{array}{l} x_1+s- \\ q^*\left(d\left(\alpha\left(x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}\right)\right)\right)- \\ d\left(\alpha\left(x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}\right)\right) \end{array}\right)^+, \\ x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}, \\ x_3-\max\left\{\left(n(t)-x_4\right)^+,\nu\right\},\left(x_4+s-n(t)\right)^+ \end{array}\right) \end{array}\right] \tag{2.273}$$

where $q^*\left(\tilde{d}\right)$ is the disposal quantity that minimize expected cost over the remainder of the horizon given any realization $\tilde{d}$ of random demand, i.e.,

$$q^*\left(\tilde{d}\right)=\underset{q\leq\left(x_1+s-\tilde{d}\right)^+}{\arg\min}\left\{c_3 q+g_1^3\left(\begin{array}{l} t+1,\left(x_1+s-q-\tilde{d}\right)^+,x_2-\max\left\{\left(n(t)-x_4\right)^+,s\right\}, \\ x_3-\max\left\{\left(n(t)-x_4\right)^+,\nu\right\},\left(x_4+s-n(t)\right)^+ \end{array}\right)\right\}. \tag{2.274}$$

In period 1, $\mathbf{x}$ = (0, N, N, 0) and the optimum is

$$g_2^3(1,\mathbf{x}) = (1-r)\min_{\substack{q_1\ge 0\\ s\ge 0\\ \nu\in[s,x_3]}} E\left[\begin{array}{l} c_1 q_1 + c_w d\Big(\alpha\big(N-\max\{n(1),s\}\big)\Big)+ \\ \dfrac{s^2}{\nu} - (1-c_n-\gamma m)s + \\ c_2^0\Big(d\Big(\alpha\big(N-\max\{n(1),s\}\big)\Big)-q_1-s\Big)^+ + \\ h\Big(q_1+s-d\Big(\alpha\big(N-\max\{n(1),s\}\big)\Big)\Big)^+ + \\ c_3 q^*\Big(d\Big(\alpha\big(N-\max\{n(1),s\}\big)\Big)\Big)+ \\ g_2^3\left(\begin{array}{l} t+1, \\ \left(\begin{array}{l} q_1+s- \\ q^*\Big(d\Big(\alpha\big(N-\max\{n(1),s\}\big)\Big)\Big)- \\ d\Big(\alpha\big(N-\max\{n(1),s\}\big)\Big) \end{array}\right)^+, \\ N-\max\{n(1),s\}, \\ N-\max\{n(1),\nu\},\big(s-n(1)\big)^+ \end{array}\right) \end{array}\right] \tag{2.275}$$

where

$$q^*\big(\tilde{d}\big) \operatorname*{arg\,min}_{q\le\left(q_1+s-\tilde{d}\right)^+}\left\{c_3 q + g_1^3\left(\begin{array}{l} 2,\big(q_1+s-q-\tilde{d}\big)^+, N-\max\{n(1),s\}, \\ N-\max\{n(1),\nu\},\big(s-n(1)\big)^+ \end{array}\right)\right\}. \tag{2.276}$$

6.5 Pseudocode for DP Algorithms

6.5.1 Algorithm for Problem $\mathbf{D1}_1^1$ (see (2.221) – (2.224))

Notes:

- $R(\cdot)$ = round function
- Global variables are c_w, c_1, c_2^0, c_3, h, r, α, L, N, $g(\cdot,\cdot,\cdot,\cdot)$. All other variables are local, e.g., defining z_1 in a subroutine does not affect a value of z_1 in the main algorithm or a different subroutine

Main Algorithm

S1: Initialization

Input c_w, c_1, c_2^0, c_3, c_n, γ, m, h, r, α, L, N, $n(1)$, …, $n(L)$

S2: Compute $g(L, \cdot, \cdot, \cdot,\cdot)$

```
Do x_1 = 0 to N
    Do x_2 = 1 to n(L)
        Do x_3 = 0 to n(L)
            g(L, x_1, x_2, x_3,1) = g(L, x_1, x_2, 0) = (1 – r)(h + c_3)x_1
        End
    End
End
```

S2: Compute $g(t, \cdot, \cdot, \cdot,\cdot)$ for $t \in \{2, …, L – 1\}$

```
l = n(L)
Do t = L – 1 to 2
    l = l + n(t) (l is an upper limit on the warranty population at the start of period t)
    Do x_1 = 0 to N
        Do x_2 = 1 to l
            Do x_3 = 0 to l
                d_1 = R(α(x_2 – (n(t) – x_3)^+)
                d_2 = R(α(x_2 – n(t))
                g(t, x_1, x_2, x_3, 1) = (1 – r)[c_w d_1 + c_2^0 (d_1 – x_1)^+ + h(x_1 – d_1)^+ +
                    Sub1(t + 1, (x_1 – d_1)^+, x_2 – (n(t) – x_3)^+, (x_3 – n(t)^+)]
                z_1 = c_w d_2 + c_2^0 (d_2 – x_1)^+ + h(x_1 – d_2)^+ + g(t + 1, (x_1 – d_2)^+, x_2 – n(t), 0)
                g(t, x_1, x_2, 0) = (1 – r)min{z_1, Sub2(t, x_1, x_2)}
            End
        End
    End
    g(t + 1, ·, ·,·,·) = ∅ (clear out array g(t + 1, ·, ·, ·, ·) because these values are no
longer needed)
End
```

S3: Compute and return optimal cost $g(1, 0, N, 0)$

```
z_1 = Sub3(1, 0, N) (minimum cost if there is no trade-in program in period 1)
z_2 = Sub4(1, 0, N) (minimum cost if there is a trade-in program in period 1)
Return (1 – r)min{z_1, z_2}
```

Algorithm Sub1(t, x1, x2, x3)

S1: Search over disposal quantity to find minimum cost

```
C = g(t, x_1, x_2, x_3, 1)
Do q = 1 to x_1
    z_1 = c_3 q + g(t, x_1 – q, x_2, x_3, 1)
    If z_1 < C, then C = z_1
End
Return C
```

Algorithm Sub2(t, x_1, x_2)

S1: Search over trade-in quantity to find minimum cost

$C = \infty$

Do $q_2 = 0$ to $\sum_{i=t}^{L} n(i)$

$d = R(\alpha(x_2 - \max\{n(t), q_2\}))$

$z_1 = c_w d + q_2^2/x_2 - (1 - c_n - \gamma m)q_2 + c_2^0(d - x_1 - q_2)^+ + h(x_1 + q_2 - d)^+ +$ Sub1$(t + 1, (x_1 + q_2 - d)^+, x_2 - \max\{n(t), q_2\}, (q_2 - n(t))^+)$

If $z_1 < C$, then $C = z_1$

End

Return C

Algorithm Sub3(t, x_1, x_2)

S1: Search over final order quantity to find minimum cost given no trade-in offer in period 1

$C = \infty$

$d = R(\alpha(x_2 - n(1)))$

Do $q_1 = 0$ to $R(\alpha L x_2)$ ($R(\alpha L x_2)$ is max demand in period 1 through L)

$z_1 = c_1 q_1 + c_w d + c_2^0(d - x_1 - q_1)^+ + h(x_1 + q_1 - d)^+ + g(2, (x_1 + q_1 - d)^+, x_2 - n(1), 0)$

If $z_1 < C$, then $C = z_1$

End

Return C

Algorithm Sub4(t, x_1, x_2)

S1: Search over final order quantity and trade-in quantity to find minimum cost

$C = \infty$

Do $q_1 = 0$ to $R(\alpha L x_2)$ ($R(\alpha L x_2)$ is max demand in period 1 through L)

$z_1 = $ Sub2$(t, x_1 + q_1, x_2)$ (returns cost based on optimal trade-in quantity given initial order q_1)

If $c_1 q_1 + z_1 < C$, then $C = c_1 q_1 + z_1$

End

Return C

6.5.2 Algorithm for Problem $\mathbf{D1}_2^1$ (see (2.226) – (2.229))

Notes:

- Append $n(1), \ldots, n(L)$ to the global variables where $\sum_{t=1}^{L} n(t) = N$

<u>Main Algorithm</u>

S1: Initialization

Input c_w, c_1, c_2^0, c_3, c_n, γ, m, h, r, α, L, N, $n(1)$, …, $n(L)$

S2: Compute $g(L, \cdot, \cdot, \cdot,\cdot)$

```
Do x = 0 to N
    Do y = 1 to n(L)
        Do z = 0 to n(L)
            g(L, x1, x2, x3,1) = g(L, x1, x2, 0) = (1 – r)(h + c3) x1
        End
    End
End
```

S2: Compute $g(t, \cdot, \cdot, \cdot,\cdot)$ for $t \in \{2, \ldots, L - 1\}$

```
x1 = n(L)
Do t = L – 1 to 2
    x1 = x1 + n(t)        (x1 is an upper limit on the warranty population at the start of
period t)
    Do x1 = 0 to N
        Do x2 = 1 to x1
            Do x3 = 0 to x1
                d1 = R(α(x2 – (n(t) – x3)^+)
                d2 = R(α(x2 – n(t))
                g(t, x1, x2, x3, 1) = (1 – r)[cw d1 + c2^0 (d1 – x1)^+ + h(x1 – d1)^+ +
                  Sub1(t + 1, (x1– d1)^+, x2 – (n(t) – x3)^+, (x3 – n(t)^+)]
                z1 = cw d2+ c2^0 (d2 – x1)^+ + h(x1 – d2)^+ + g(t + 1, (x1 – d2)^+, x2 – n(t), 0)
                g(t, x1, x2, 0) = (1 – r)min{z1, Sub2(t, x1, x2)}
            End
        End
    End
    g(t + 1, ·, ·,·,·) = ∅  (clear out array g(t + 1, ·, ·, ·, ·) because these values are no
    longer needed)
End
```

S3: Compute and return optimal cost $g(1, 0, N, 0)$

z_1 = Sub3(1, 0, N) (minimum cost if there is no trade-in program in period 1)

z_2 = Sub4(1, 0, N) (minimum cost if there is a trade-in program in period 1)

Return $(1 - r)\min\{z_1, z_2\}$

<u>Algorithm Sub1(t, x_1, x_2, x_3)</u>

S1: Search over disposal quantity to find minimum cost

```
C = g(t, x1, x2, x3, 1)
Do q = 1 to x1
    z1 = c3 q + g(t, x1 – q, x2, x3, 1)
    If z1 < C, then C = z1
End
Return C
```

Algorithm Sub2(t, x_1, x_2)

S1: Search over trade-in quantity to find minimum cost

$C = \infty$

Do $q_2 = 0$ to $\sum_{i=t}^{L} n(i)$

$d = R(\alpha(x_2 - \max\{n(t), q_2\}))$

$z_1 = c_w d + q_2^2 / x_2 - (1 - c_n - \gamma m)q_2 + c_2^0 (d - x_1 - q_2)^+ + h(x_1 + q_2 - d)^+ +$ $\text{Sub1}(t + 1, (x_1 + q_2 - d)^+, x_2 - \max\{n(t), q_2\}, (q_2 - n(t))^+)$

If $z_1 < C$, then $C = z_1$

End

Return C

Algorithm Sub3(t, x_1, x_2)

S1: Search over final order quantity to find minimum cost given no trade-in offer in period 1

$C = \infty$

$d = R(\alpha(x_2 - n(1)))$

Do $q_1 = 0$ to $R(\alpha L\, x_2)$ ($R(\alpha Ly)$ is max demand in period 1 through L)

$z_1 = c_1 q_1 + c_w d + c_2^0 (d - x_1 - q_1)^+ + h(x_1 + q_1 - d)^+ + g(2, (x_1 + q_1 - d)^+, x_2 - n(1), 0)$

If $z_1 < C$, then $C = z_1$

End

Return C

Algorithm Sub4(t, x_1, x_2)

S1: Search over final order quantity and trade-in quantity to find minimum cost

$C = \infty$

Do $q_1 = 0$ to $R(\alpha L\, x_2)$ ($R(\alpha Ly)$ is max demand in period 1 through L)

$z_1 = \text{Sub2}(t, x_1 + q_1, x_2)$ (returns cost based on optimal trade-in quantity given initial order q_1)

If $c_1 q_1 + z_1 < C$, then $C = c_1 q_1 + z_1$

End

Return C

6.5.3 Algorithm for Problem $\mathbf{D1}_1^2$ (see (2.252) – (2.256))

Main Algorithm

S1: Initialization

Input c_w, c_1, c_2^0, c_3, c_n, γ, m, h, r, α, L, N

S2: Compute $g(L, \cdot, \cdot, \cdot)$

Do $x_2 = 1$ to N

Do $x_3 = 1$ to N

$g(L, 0, x_2, x_3) = 0$
End
End
S2: Compute $g(t, \cdot, \cdot, \cdot)$ for $t \in \{2, \ldots, L\}$
Do $t = L - 1$ to 2
Do $x_1 = 0$ to N
Do $x_2 = 1$ to N
$d = R(\alpha x_2)$
$s = R((\alpha x_2 - x_1)/(1+\alpha))$
Do $x_3 = 1$ to N
If $x_1 \geq d$ then $g(t, x_1, x_2, x_3) = (1- r)[c_w d + h(x_1 - d) + g(t + 1, x_1 - d, x_2, x_3)]$
If $x_1 < d$ then $g(t, x_1, x_2, x_3) = (1- r)[c_w(s + x_1) + \text{Sub1}(t, x_1, x_2, x_3, s)]$
End
End
End
End
$g(t + 1, \cdot, \cdot, \cdot) = \varnothing$ (clear out array $g(t + 1, \cdot, \cdot, \cdot)$ because these values are no longer needed)
S3: Compute and return optimal cost $g(1, 0, N, N)$
$C = \infty$
$d = R(\alpha N)$
Do $q_1 = 0$ to $R(\alpha LN)$ ($R(\alpha LN)$ is max demand in period 1 through L)
If $q_1 \geq d$, then $z_1 = (1- r)[c_1 q_1 + c_w d + h(q_1 - d) + g(2, q_1 - d, N, N)]$
If $q_1 < d$, then
$s = R((\alpha N - q_1)/(1+\alpha))$
$z_1 = (1- r)[c_1 q_1 + c_w(s + q_1) + \text{Sub1}(1, q_1, N, N, s))]$
Endif
End
Return C

Algorithm Sub1(t, x_1, x_2, x_3, s)
S1: Search over the number of trade-in offers to find minimum cost
$C = \infty$
If $s \leq x_3$ then
Do $\nu = s$ to x_3
$z_1 = (s/\nu + c_n + \gamma m - 1)s + g(t + 1, 0, x_2 - s, x_3 - \nu))$
If $z_1 < C$, then $C = z_1$
End
Endif
Return C

6.5.4 Algorithm for Problem $\mathbf{D1_2^2}$ (see (2.257) – (2.261))

<u>Main Algorithm</u>

S1: Initialization

Input c_w, c_1, c_2^0, c_3, c_n, γ, m, h, r, α, L, N, $n(1)$, …, $n(L)$

S2: Compute $g(L, \cdot, \cdot, \cdot)$

Do $x_2 = 1$ to N

Do $x_3 = 1$ to N

$g(L, 0, x_2, x_3) = 0$

End

End

S2: Compute $g(t, \cdot, \cdot, \cdot)$ for $t \in \{2, …, L\}$

Do $t = L - 1$ to 2 (for each period)

$l = n(L)$

Do $x_1 = 0$ to N (for each possible inventory carryover)

$l = l + n(t)$

Do $x_2 = 1$ to l (for each possible member in the warranty population at period t)

Do $x_3 = 1$ to l (for customer who have not received an offer by period t)

Do $x_4 = 1$ to N (for each warranty expiration carry over)

$d = R(\alpha(x_2 - n(t)))$

$s = \min\{R((\alpha x_2 - x_1)/(1+\alpha)), \alpha(x_2 - (n(t) - x_4)^+ - x_1)\}$

If $x_1 \geq d$ then $g(t, x_1, x_2, x_3, x_4) = (1- r)[c_w d + h(x_1 - d) +$
$g(t + 1, x_1 - d, x_2 - n(t), x_3 - n(t), 0)]$

If $x_1 < d$ then $g(t, x_1, x_2, x_3, x_4) = (1- r)[c_w(s + x_1) +$
$\text{Sub1}(t, x_1, x_2, x_3, x_4, s)]$

End

End

End

End

End

$g(t + 1, \cdot, \cdot, \cdot) = \varnothing$ (clear out array $g(t + 1, \cdot, \cdot, \cdot)$ because these values are no longer needed)

S3: Compute and return optimal cost $g(1, 0, N, N,0)$

$C = \infty$

$d = R(\alpha(N - n(t)))$

Do $q_1 = 0$ to $R(\alpha LN)$ ($R(\alpha LN)$ is max demand in period 1 through L)

If $q_1 \geq d$, then $z_1 = (1- r)[c_1 q_1 + c_w d + h(q_1 - d) + g(2, q_1 - d, N - n(1), N - n(1),0)]$

If $q_1 < d$, then

$s = \min\{R((\alpha N - q_1)/(1+\alpha)), \alpha(N - (N - n(1)) - q_1)\}$

$z_1 = (1- r)[c_1 q_1 + c_w(s + q_1) + \text{Sub1}(1, 0, N, N,0,s))]$

Endif

End

Return C

Algorithm Sub1(t, x_1, x_2, x_3, x_4, s)
S1: Search over the number of trade-in offers to find minimum cost
$C = \infty$
If $s \le x_3$ then
Do $\nu = s$ to x_3
$z_1 = (s/\nu + c_n + \gamma m - 1)s +$
$g(t + 1, 0, x_2 - \max\{s,(n(t) - x_4)^+\}, x_3 - \max\{\nu, (n(t) - x_4)^+\}, (x_4 + s - n(t))^+))$
If $z_1 < C$, then $C = z_1$
End
Endif
Return C

Algorithm Sub2(t, x_1, x_2, x_3, x_4, s)
S1: Search over the number of trade-in offers to find minimum cost
$C = \infty$
Do $\nu = s$ to x_2
$z_1 = (s/\nu + c_n + \gamma m - 1)s + g(t + 1, 0, x_2 - \max\{s, n(t)\}, x_2 - \max\{\nu, (n(t)\}, (s - n(t))^+))$
If $z_1 < C$, then $C = z_1$
End
Return C

6.5.5 Algorithm for Problem $\mathbf{D1}_1^3$ (see (2.269) – (2.271))

Main Algorithm
S1: Initialization
Input c_w, c_1, c_2^0, c_3, c_n, γ, m, h, r, α, L, N, $n(1)$, …, $n(L)$
S2: Compute $g(L, \cdot, \cdot, \cdot, \cdot)$
Do $x_2 = 1$ to N
Do $x_3 = 1$ to N
$g(L, 0, x_2, x_3, x_4) = (1 - r)(h + c_3)x$
End
End
S2: Compute $g(t, \cdot, \cdot, \cdot, \cdot)$ for $t \in \{2, \ldots, L\}$
Do $t = L - 1$ to 2 (for each period)
$l = n(L)$
Do $x_1 = 0$ to N (for each possible inventory carryover)
$l = l + n(t)$
Do $\nu = 1$ to N
Do $s = 1$ to ν

Do x_2 = 1 to l (for each possible member in the warranty population at period t)
Do x_3 = 1 to l (for customer who have not received an offer by period t)
Do x_4 = 1 to N (for each warranty expiration carry over)
$d = R(\alpha(x_2 - \max\{(n(t) - x_4)^+, s\}))$
$g(t, x_1, x_2, x_3, x_4) = (1- r)[c_w d + s^2/\nu - (1 - c_n - \gamma m)s + c_2^0(d - x_1 - s)^+ + h(x_1 + s - d) +$
$\text{Sub1}(t + 1, x_1 + s - d, x_2 - \max\{(n(t) - x_4)^+, s\},$
$x_3 - \max\{(n(t) - x_4)^+, \nu\},(x_4 + s - n(t))^+)$
End
End
End
End
End
End
$g(t + 1, \cdot, \cdot, \cdot, \cdot) = \varnothing$ (clear out array $g(t + 1, \cdot, \cdot, \cdot)$ because these values are no longer needed)
S3: Compute and return optimal cost $g(1, 0, N, N, 0)$
$C = \infty$
$d = R(\alpha(N - \max\{ n(1),s\}))$
Do q_1 = 0 to $R(\alpha LN)$ ($R(\alpha LN)$ is max demand in period 1 through L)
Do s = 1 to N
Do ν =1 to N
$z_1 = (1- r)[c_1 q_1 + c_w d + s^2/\nu - (1 - c_n - \gamma m)s + c_2^0(d - x_1 - s)^+ +$
$h(q_1 + s - d) +$
$\text{Sub1}(2, q_1 + s - d, N - \max\{n(t), s\},N - \max\{n(t),\nu\},(s - n(1))^+)$
End
End
End
Return C

Algorithm Sub1(t, x_1, x_2, x_3, x_4)
S1: Search over disposal quantity to find minimum cost
$C = g(t, x_1, x_2, x_3, x_4)$
Do q = 1 to x_1
$z_1 = c_3 q + g(t, x_1 - q, x_2, x_3, x_4)$
If $z_1 < C$, then $C = z_1$
End
Return C

6.6 Stochastic DP Implementation

The recursions for stochastic demand given in Section 7.4 allow for disposal of components at the end of any period. The inclusion of this possibility significantly increases the computational complexity of the stochastic DP algorithms. Consequently, in our implementations of stochastic DPs, we limit consideration to the case where components can only be disposed in the last period L. This means that $q^* = 0$ in the above recursions.

To implement the stochastic DPs, we make use of the following identities that pertain to nonnegative discrete random demand $d(\alpha x)$:

$$E\left[\left(d(\alpha x)-y\right)^+\right]=E\left[\left(y-d(\alpha x)\right)^+\right]+\alpha x-y \tag{2.277}$$

$$E\left[\left(y-d(\alpha x)\right)^+\right]=2\alpha x-y-\sum_{i=0}^{y-1}P\left[d(\alpha x)>i\right] \tag{2.278}$$

$$E\left[g\left(t,\left(y-d(\alpha x)\right)^+,\ldots\right)\right]=P\left[d(\alpha x)\geq y\right]g(t,0,\ldots)+\sum_{i=0}^{y-1}P\left[d(\alpha x)=i\right]g(t,y-i,\ldots) \tag{2.279}$$

If $d(\alpha x)$ is a Poisson random variable, then

$$P\left[d(\alpha x)=i\right]=\frac{(\alpha x)^i e^{-\alpha x}}{i!}=\left(\frac{\alpha x}{i}\right)P\left[d(\alpha x)=i-1\right] \tag{2.280}$$

and (2.277) – (2.278) can be written as

$$E\left[\left(d(\alpha x)-y\right)^+\right]=(\alpha x-y)P\left[d(\alpha x)>y\right]+\alpha xP\left[d(\alpha x)=y\right] \tag{2.281}$$

$$E\left[\left(y-d(\alpha x)\right)^+\right]=(y-\alpha x)P\left[d(\alpha x)\leq y\right]+\alpha xP\left[d(\alpha x)=y\right]. \tag{2.282}$$

CHAPTER 4: CONCLUSION

1. Contributions

The economic and environmental benefits of taking back and recovering end-of-use products have captured the attention of a growing number large OEMs and retailers. End-of-use products will often have residual economic value. For example, the returned products can be remanufactured and then sold to satisfy a growing appetite for gently used products. Alternatively, companies can harvest the returned products for reusable components and spare parts. Nevertheless, taking advantage of these economic benefits is not without managerial complexity: capacity to satisfy demand for remanufactured products is linked to past sales of the new product, access to used products depends on consumer's willingness to return and their replacement purchase behaviors, and remanufacturing cost depends on the age-condition profile of the returned products. These are all special features of product take back and recovery channels. Each of these factors can be controlled by designing an effective product acquisition management system. Indeed, the analysis presented in this dissertation provides practitioners and researchers with insight into some of the main economic trade-offs related to used product acquisition decisions.

Our focus on trade-in and buyback acquisition policies is motivated by extensive talks with a handful of managers who oversee take back and recovery operations for leading companies in the computer and electronics industry. Although all of these managers work for divisions responsible for collection, recovery, and remarketing of used product, they all emphasized a clear need to coordinate acquisition pricing decisions with the companies' marketing groups. For example, two of the managers worked for companies that take back and recover end-of-use products in order to satisfy demand for remanufactured products. One of the companies offered a trade-in program, while the other offered a buyback program. Both managers emphasized a need to incorporate product lifecycle considerations into decisions regarding take back policies and recovery strategies, and both managers mentioned elements relating to the current owner's replacement purchase rate or preference for buyback cash in lieu trade-in credits. A key objective of this dissertation is to provide guidance to managers on the design and implementation of alternative acquisition policies. To this end, our analysis provides insights and makes the following contributions with respect to designing trade-in and buyback programs to support value added product recovery.

Product Acquisition Policies for the Purpose of Remanufacturing

As it relates to whether to implement a trade-in or a buyback policy, we introduced an algorithm to determine the optimal myopic buyback price and an algorithm to determine the optimal myopic trade-in credit. Both algorithms incorporate elements to account for lifecycle effects, repeat purchase rates, valuations of the owned product, and consumer's preference to take the money and run. We find that if repeat purchase rates are expected to be less than 100%, and owners are satisfied with the range of alternatives for which a trade-in credit can be applied, then the profit from an optimal trade-in program is the same or higher than the optimal profit from a buyback program. However, if repeat purchase rates are either very low or very high, adopting a trade-in program can either be very attractive or very unattractive.

As it relates to whether to implement a myopic policy that aims to maximize profits in each period, versus a proactive policy which maximizes profits over the product life cycle; we propose four math programming formulations. Solutions to two of the formulations to determine the optimal (or near optimal) buyback prices and take-back quantities for a product of a given age in a given period. Solutions to the other two formulations determine the optimal (or near optimal) trade-in credits and take-back quantities. We find that the relative performance between myopic and proactive policies depends on the relationship between the sweet-spot age, which is a measure of the residual economic value of used product, and lag between new product sales and demand for the remanufactured product. In general, unless the sweet spot age is close to the time lag, the company can do better by pursuing a proactive acquisition policy.

Product Acquisition Policies for the Purpose of Component Harvesting

The above findings and contributions provide direction to managers on the design and management of acquisition policies that support product take back and recovery for the purpose of remarketing. The dissertation also provides guidance to managers who design a trade-in program where products are acquired for the purpose of component harvesting to support warranty claims. One of the managers who helped motivate this dissertation worked for a company already recognized as leader for its product take back and recovery activities. The company was contemplating designing a trade-in program to acquire spare-parts to service warranty claims in the face of a growing number of component phase out announcements. Management was concerned about how consumers would respond to the trade-in deal. They were also concerned about how a take-back program interacts with replacement purchase rates. To this end, the dissertation introduces a decision framework to account for marketing concerns regarding cannibalization and repeat purchase rates as well as operational concerns regarding holding cost, warranty service cost, and disposal cost. We identify two acquisition policies (i.e., full trade-in, and matching trade-in) and examine different settings where these policies do and do not work, and key factors under management control that drive performance.

Our analysis provides insights and makes the following contributions with respect to designing a trade-in program to support warranty claims. First, we introduce an important problem that has not previously been studied. It is a problem that draws on two major branches in the literature—literature on the final order problem and literature on the design and merits of trade-in programs. It is an area that is ripe for additional research. Second, we introduce a parsimonious, yet rich, model that is defined by five basic assumptions. All warranty service cost expressions and optimal policy decisions flow from these assumptions. Our investigation has touched on a small set of questions and results, and we believe there is ample opportunity for further study using this model as is, or as a foundation for a richer model. Third, we provide insight into the merits and effective use of trade-in policies as discussed above.

2. Directions for Future Research

This dissertation has considered only a few of the growing challenges related to designing product acquisition policies to support product take back and recovery systems. Our focus on designing product acquisition policies for a monopoly producer and a single product recovery agent leaves many questions for future research. To this end we offer the following directions for future research.

We begin with the case of product acquisition for the purpose of remarketing remanufacturing. First there is a question as to how to design a product acquisition strategy when both buyback and trade-in credits are offered together. We limited our analysis to the company that offers either a trade-in program or a buyback program. Some companies allow the consumer to choose between a trade-in credit and buyback cash (i.e., BestBuy). The trade-in credit is generally larger than the corresponding buyback rebate. Second, a rigorous examination of trade-in policies that allow consumers to return competitor equipment would help address another important managerial concern that has not be covered in the literature. It is a common practice for consumers to return competitor equipment. Third, a model that accounts for uncertain resale value could provide interesting insights about the effect of uncertainty on product acquisition policy choice. Finally, the increasing popularity take-back and recovery programs raises question about competition. Our dynamic product acquisition framework can serve as a building block to conduct a rigorous analysis of how of competition and product lifecycle factors influence policy choice, could add valuable insight to the growing body of literature related to competition in product recovery markets.

For the case of product acquisition for the purpose of component harvesting to support the final order decisions, we offer two directions for future research. First, there is a need for a broader assessment of trade-in policy designs. The full trade-in policy and the matching trade-in policy are just two of many possibilities. There is a question of how good or bad these policies perform relative to a wider set of alternatives. In our appendix, we introduce discrete dynamic programming algorithms that can be used to investigate this question. One algorithm returns the optimal cost under a full trade-in policy, one algorithm returns the optimal cost under a matching trade-in policy, and a third algorithm returns the optimal cost with no restrictions on the trade-in policy. By comparing solutions returned by these algorithms, one will be able to develop a sense of how the optimal policy compares with the full and matching trade-in policies.

Second, there is a need to consider the impact of uncertainty in warranty demand. Compared to the benchmark of placing a very large final order up-front, we know that the introduction of uncertainty will generally increase the value of a trade-in program (e.g., by virtue of a sourcing alternative if realized demand is greater than the final order quantity). However, there are important unanswered questions on the degree of value-added as related to the nature of uncertainty and how uncertainty may shape the design of an effective trade-in program.

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Biographical Data

Name of Author: Dwayne D. Cole

Place of Birth: Pontiac, MI

Date of Birth: October 1, 1968

Graduate Schools and Under Graduate Schools Attended
Florida Agriculture & Mechanical University, Tallahassee FL
Florid State University, Tallahassee FL
Florida Agriculture & Mechanical University, Tallahassee FL

Degrees Awarded
Bachelors of Science, 1991, Florida Agriculture & Mechanical University
Masters of Science in MIS, 1999, Florida State University
Masters of Business Administration, 2000, Florida A & M University

Professional Experience
Lecturer, University of Central Florida, 2011 - Present
Teaching Assistant, Syracuse University, 2007-2011
Visiting Professor, Florida A&M University, 2000-2007

Publications

Benjamin, C.O., D. Cole and A. Bradford, “A QFD Framework for Developing Campus Wide Entrepreneurship Programs”, International Journal of Business Strategy, 2007