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By

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Abstract

This thesis provides results on soft topological linear spaces and study of it is properties.

The main objective of this work is to create a new type of a soft topological linear space , namely soft topological linear space (which is soft locally convex space) and considered as the basis of our main definitions. Throughout this work, some important and new concepts have been illustrated including a soft topological linear space induced by a family of soft seminorms on a soft linear spaces over a soft scalar field , soft separated family, soft bounded set, soft bounded mapping , soft absolutely convex set and soft barrel set and their properties. Finally, we study another types of soft locally convex soft topological linear spaces called soft barrelled and soft bornological spaces. Moreover, we introduce some properties of this concepts.

Also, this study proves that:

($\boldsymbol{i}$) Every soft nbhd of P_0^{Θ} of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ contains a soft balanced nbhd of P_0^{Θ}.

($\boldsymbol{ii}$) Every soft nbhd of P_0^{Θ} of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft absorbing.

($\boldsymbol{iii}$) Every proper soft linear subspace of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ has a null soft interior.

($\boldsymbol{iv}$) Every soft closed and soft balanced nbhd of P_0^{Θ} of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, forms a soft local base at P_0^{Θ}.

($\boldsymbol{v}$) Every a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft Hausdorff and soft separated.

($\boldsymbol{vi}$) A $ST\ \tilde{\tau}_{\mathcal{P}}$ which is determined by a family of soft seminorms on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is $STLS$ which is soft separated.

($\boldsymbol{vii}$) Every soft normed space $(\tilde{X}_A , \|\tilde{\cdot}\| , A)$ is $STLS$, but need not to be a soft locally convex space.

($\boldsymbol{iix}$) A soft convex hull of a soft bounded set of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex space) is soft bounded.

($\boldsymbol{ix}$) Every soft nbhd of P_0^{Θ} of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}$(A) is contained in a soft nbhd of P_0^{Θ} which is soft barrel.

($\boldsymbol{x}$) A $STLS\ SL(\tilde{X}_A)$over $\mathbb{R}$(A) (which is soft locally convex and soft second category space or soft Baire space) is soft barreled space.

($\boldsymbol{xi}$) A $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex and soft first countable space or soft metrizable space) is soft bornological space.

Introduction

The concept of soft set, coined by Molodtsov [15], in his seminar paper "Soft set theory-first results " in 1999 has emerged as a fundamental and fresh idea exploring softness mathematically for first time. Soft is generalization of " Soft Topology " in classical mathematics, but it also has its own differentially marked characteristics.

It deepens the understanding of basic structures of classical new methods and significant results in this area. The art of defining soft norm one of the fundamental problems in soft mathematics. The application of the soft sets provided a natural framework for generalizing many concepts of topology which is called the soft topological space as initiated by [1], [4], [8], [11], [12], [16], [18] , [19] , [21] , [23] , [24], [30] , [31] and [32].

A lot of activity has been shown in soft set theory (see [5] , [14], [15], [17] and [25] .

The objective of this thesis is to analyze certain results in "soft topological linear space", where the soft point chosen that was previously defined in [26]. Some of his proofs were thoroughly revised. In addition, explain or more detailed descriptions and explanations were added to them. In 2014 [29], an idea of soft linear spaces and soft norm on a soft linear spaces are given and some of their properties are studied. Soft linears in soft linear spaces are introduced and their properties are studied.

Finally, we examine the properties of this soft normed space and present some investigations about soft continuous functional in the soft linear space, see [29].

In 2015, Chiney M. and Samanta S. [6], introduced a notion of a vector soft topology and studied some of its basic properties depending on the crisp point in soft set. After that, in 2016 [7], they also introduced a generalization of the concept of seminorm called soft seminorm on a soft linear space over a soft scalar field depending on soft element. But in our thesis terms are defined above depending on the concept of soft point. The thesis entitled **" On Soft Topological Linear Spaces "** consists of three Chapters. In this work, the main concentration is focused on defining different types of soft topological linear spaces and some soft topological linear spaces which are induced by the family of soft seminorms on a soft linear space over a soft scalar filed and which was defined by Intisar R. [9], it is a special type of soft topological linear spaces.

In the first chapter of the thesis preliminary definitions and results are briefly delineated. This chapter is divided into two sections, section one includes some basic algebraic operations on soft sets. We introduce equality of two soft sets, soft subsets, complement of soft sets, null soft set, absolute soft set, soft point and singleton soft set.

There are several basic properties which are not hold true in general and proved results with already defined operations on union, intersection and different soft sets, furthermore we introduced the notion of soft real sets, soft real numbers and we discuss their properties, see [25] and [28]. In section two some basic properties of soft topological spaces are studied. Chapter two, divided into two sections. In section one we introduced an important notions called soft convex, soft balanced, soft absorbing and soft symmetric set in a soft linear space over a soft scalar field and study some properties of them. Also, we introduce a notion of a linear soft topologies and studied some of its basic properties. In this section, some facts of the system of soft neighborhoods of the soft zero linear of a soft topological linear space are established. The concept of soft topological linear space is introduced in this section along with some basic properties of such spaces. Section two aims at studying some basic theorems which are needed in this chapter and the next one.

Chapter three contains three sections, in section one we study a soft linear spaces equipped with a soft topology generated by a family of soft seminorms on a soft scalar field are called soft topological linear spaces. It is possible to consider this soft topological linear space as the central theme in this thesis and some basic properties of this soft topological linear spaces were studied. In section two, we define a soft locally convex spaces in two equivalent ways, one by means of a soft local base at soft zero liear that consist of soft convex sets and one that requires soft seminorms. A soft locally convex space is a major topic and worthy of its own section. There is a soft topological linear spaces that are not soft locally convex space. It is difficult to find somewhat pathological examples in the soft analysis. Finally, in section three, we study a soft barreled and soft bornological spaces which considered the features of the soft locally convex space. Also, there is an equivalence to the definition of soft barrelled and soft bornological spaces. On the other hand, there is an example leads to is not soft barrelled space and soft bornological space.

§ (1.1) Soft sets and its properties:

In this section, we give some basic definitions of soft sets and their necessary operations.

Definition (1.1.1) [15]:

Let X be a universe set and E be a set of parameters , $P(X)$ the power set of X and $A \subseteq E$. A pair (F, A) is called soft set over X with recpect to A and F is a mapping given by $F: A \to P(X)$, $(F, A) = \{F(e) \in P(X): e \in A\}$.

Remark (1.1.2) [15] , [18]:

$(\boldsymbol{i})$ $F(e)$, $\forall e \in A$ may be arbitrary set , may be empty set.

$(\boldsymbol{ii})$ The soft set can be represented by two ways:

- $(F, A) = \{ F(e) \in P(X),\ \forall\ e \in A \}$.
- By ordered pairs : $(F, A) = \{ (e, F(e)) : \forall e \in A,\ F(e) \in P(X) \}$.

$(\boldsymbol{iii})$ $S(X)$ denote to collection of all soft sets over a universe X.

Remark (1.1.3) [21]:

$(\boldsymbol{i})$ It is clear that every set is soft set.

$(\boldsymbol{ii})$ In sense , when $|A| = 1$, a soft set (F, A) behaves similar to a set. In this case the soft set is same as the set $F(e)$, where $A = \{e\}$.

Definition (1.1.4) [26] , [25]:

$(\boldsymbol{i})$ A soft (F, A) over X is called soft point and its denoted by: $P_e^x = \{(e, F(e))\}$, if exactly one $e \in A$, $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \emptyset$ for all $e' \in A \backslash \{e\}$.

$(\boldsymbol{ii})$ A soft set (F, A) for which $F(e)$ is a singleton set , for all $e \in A$, is called singleton soft set.

Remark (1.1.5) [26]:

$(\boldsymbol{i})$ The fact that P_e^x is a soft point of (F, A) and will be denoted by $P_e^x \tilde{\in} (F, A)$, if $x \in F(e)$.

$(\boldsymbol{ii})$ For two soft points P_e^x and $P_{e'}^y$, then $P_e^x \neq P_{e'}^y$ if and only if $x \neq y$ or $e \neq e'$.

$(\boldsymbol{iii})$ The collection of all soft points P_e^x over a universe X denotes by $S_P(X)$.

Definition (1.1.6) [14]:

$(\boldsymbol{i})$ A soft set (F,A) over X is called null soft set , denoted by $\widetilde{\emptyset}_A$, if for all $e \in A$, we have $F(e) = \emptyset$.

$(\boldsymbol{ii})$ A soft set (F,A) over X is called absolute soft set and its denoted by $\tilde{X}_A$, if for all $e \in A$, we have $F(e) = X$.

Definition (1.1.7) [14] , [26]:

The soft complement of a soft set (F,A) over a universe X is denoted by $(F,A)^c$ and it is defined by $(F,A)^c = (F^c,A)$, where F^c is a mapping given by $F^c : A \to P(X)$, $F^c(e) = X \backslash F(e)$, for all $e \in A$.
i.e. $(F,A)^c = \{(e, X \backslash F(e)) : \forall\, e \in A\}$.
It is clear that:
$(\boldsymbol{i})$ ${\widetilde{\emptyset}_A}^{\,c} = \tilde{X}_A$; ${\tilde{X}_A}^{\,c} = \widetilde{\emptyset}_A$.
$(\boldsymbol{ii})$ If $P_e^x \,\tilde{\in}\, (F,A)$, then $P_e^x \notin (F,A)^c$ i.e. $x \notin F^c(e)$.

Definition (1.1.8) [14] , [5]:

Let (F,A) and (G,B) be two soft sets over X, we say that (F,A) is a soft subset of (G,B) and denoted by $(F,A) \,\tilde{\subseteq}\, (G,B)$, if:
- $A \subseteq B$.
- $F(e) \subseteq G(e)$, $\forall e \in A$.

Also, we say that (F,A) and (G,B) are soft equal is denoted by $(F,A) = (G,B)$, if $(F,A) \,\tilde{\subseteq}\, (G,B)$ and $(G,B) \,\tilde{\subseteq}\, (F,A)$.
It is clear that:
$(\boldsymbol{i})$ $\widetilde{\emptyset}_A$ is a soft subset of any soft set (F,A).
$(\boldsymbol{ii})$ Any soft set (F,A) is a soft subset of $\tilde{X}_A$.

Difinition (1.1.9) [14] , [16]:

$(\boldsymbol{i})$ The intersection of two soft sets (F,A) and (G,B) over a universe X is the soft set $(H,C) = (F,A) \widetilde{\cap} (G,B)$, where $C = A \cap B$ and for all $e \in C$, write $(H,C) = (F,A) \widetilde{\cap} (G,B)$ such that $H(e) = F(e) \cap G(e)$.
$(\boldsymbol{ii})$ The union of two soft sets (F,A) and (G,B) over X is the soft set (G,B) , where $C = A \cup B$ and $\forall\, e \in A$ we write $(H,C) = (F,A) \widetilde{\cup} (G,B)$

such that $$H(e) = \begin{cases} F(e) & , \; if\ e \in A \backslash B \\ G(e) & , \; if\ e \in B \backslash A \\ F(e) \cup G(e) & , if\ e \in A \cap B \end{cases} .$$

It is clear that every soft set can be expressed as a union of all soft points belong to it. i.e. $(F,A) = \tilde{\cup}_{P_e^x \tilde{\in} (F,A)} P_e^x$ [26].

$(\boldsymbol{iii})$ The difference of two soft sets (F,A) and (G,A) over X , denoted by $(H,C) = (F,A)\tilde{\setminus}(G,A)$ is defined as $H(e) = F(e)\setminus G(e)$, for all $e \in A$.

Definition (1.1.10) [25] , [7]:

Let $\mathbb{R}$ be the set of all real numbers , B($\mathbb{R}$) be the collection of all non-empty bounded subset of $\mathbb{R}$ and A be a set of parameters.

Then a mapping $F: A \to B(\mathbb{R})$ is called a soft real set .

If a soft real set is a singleton soft set it will be called a soft real number . We denote soft real numbers by $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ where as $\bar{r}, \bar{s}, \bar{t}$ will denote particular type of soft real numbers such that $\bar{r}(e) = r$, $\bar{0}(e) = 0$ $\bar{1}$ $(e) = 1$ for all $e \in A$. The set of all soft real numbers it is denoted by $\mathbb{R}(A)$ and the set of all non-negative soft real numbers by $\mathbb{R}(A)^*$.

Remark (1.1.11) [28]:

Let $\tilde{r}, \tilde{s}, \tilde{t} \tilde{\in} \mathbb{R}(A)$. Then the soft addition $\tilde{r} \tilde{+} \tilde{s}$ of $\tilde{r}, \tilde{s}$ and soft scaler multiplication $\tilde{t} \tilde{\cdot} \tilde{r}$ of $\tilde{t}$ and $\tilde{r}$ are defined by:

$(\boldsymbol{i})(\tilde{r} \tilde{+} \tilde{s})(e) = \tilde{r}(e) + \tilde{s}(e)$, for all $e \in A$.

$(\boldsymbol{ii})(\tilde{r} \tilde{-} \tilde{s})(e) = \tilde{r}(e) - \tilde{s}(e)$, for all $e \in A$.

$(\boldsymbol{iii})(\tilde{r} \tilde{\cdot} \tilde{s})(e) = \tilde{r}(e) \cdot \tilde{s}(e)$, for all $e \in A$.

$(\boldsymbol{iv})$ $(\tilde{r} \tilde{/} \tilde{s})(e) = \tilde{r}(e) / \tilde{s}(e)$, and $\tilde{s}(e) \neq 0$ for all $e \in A$.

Definition (1.1.12) [28]:

The inverse of any soft real number $\tilde{r}$, denoted by $\tilde{r}^{-1}$ and defined as $\tilde{r}^{-1}(e) = (\tilde{r}(e))^{-1}$, for all $e \in A$.

Theorem (1.1.13) [28]:

The set of all soft real numbers $\mathbb{R}(A)$ forms a field and called soft real field.

Definition (1.1.14) [4] , [23]:

$(\boldsymbol{i})$ For two soft sets (F,A) and (G,B) over X and Y respectively , the cartisian product of (F,A) and (G,B) is defined as:

- $(F,A)\,\tilde{\times}\,(G,B)=(H,C)$, such that $C=A\times B$ and $H:C\to P(X\times X)$, $H(e,e')=F(e)\times G(e')$, for all $(e,e')\in C$.
- $\tilde{X}_A\,\tilde{\times}\,\tilde{Y}_A\;=\widetilde{X\times Y}_{A\times A}$, denotes the absolute soft set over $X\times Y$ with the parameter set $A\times A$.

$(\boldsymbol{ii})$ A relation from (F,A) to (G,A) is a soft subset of $(F,A)\,\tilde{\times}\,(G,B)$ as follows $(R\,\tilde{\sqsubseteq}\,(F,A)\,\tilde{\times}\,(G,B)=(F\times G,A\times B))$.

In other words a relation from (F,A) to (G,B) is of the form (H_1,S), where $S\subseteq A\times B$ and $H_1(e,e')=H(e,e')$ for all $(e,e')\in S$.

Remark (1.1.15) [26]:

For two soft real numbers $\tilde{r}$, $\tilde{s}$, then:

$(\boldsymbol{i})$ $\tilde{r}\;\tilde{\leq}\;\tilde{s}$ if $\tilde{r}(e)\;\leq\;\tilde{s}(e)$; $\tilde{r}\;\tilde{<}\;\tilde{s}$ if $\tilde{r}(e)\;<\;\tilde{s}(e)$, $\forall\, e\in A$.

$(\boldsymbol{ii})$ $\tilde{r}\;\tilde{\geq}\;\tilde{s}$ if $\tilde{r}(e)\;\geq\;\tilde{s}(e)$; $\tilde{r}\;\tilde{>}\;\tilde{s}$ if $\tilde{r}(e)\;>\;\tilde{s}(e)$, $\forall\, e\in A$.

Remark (1.1.16) :

For two soft real numbers $\tilde{r}$, $\tilde{s}$, then:

$(\boldsymbol{i})$ If $\tilde{r}\;\tilde{\leq}\;\tilde{s}$, then $\tilde{r}\;\tilde{+}\;\tilde{t}\;\tilde{\leq}\;\tilde{s}\;\tilde{+}\;\tilde{t}$; for all $\tilde{t}\,\tilde{\in}\,\mathbb{R}(A)$.

$(\boldsymbol{i})$ If $\tilde{r}\;\tilde{\leq}\;\tilde{s}$, then $\tilde{r}\;\tilde{\cdot}\;\tilde{t}\;\tilde{\leq}\;\tilde{s}\;\tilde{\cdot}\;\tilde{t}$; for all $\tilde{t}\,\tilde{\in}\,\mathbb{R}(A)^{*}$.

Definition (1.1.17) [18]:

Let (F,A) be a soft set in $\tilde{R}_A$. Then (F,A) is called soft bounded above , if there is a soft number ñ so that any $\tilde{x}\,\tilde{\in}\,(F,A)$ is soft less than or soft equal to $\tilde{n}$: $\tilde{n}\;\tilde{\geq}\;\tilde{x}$. The soft number $\tilde{n}$ is called a soft upper bound for the soft set (F,A) .

Definition (1.1.18) [18]:

Let (F,A) be a soft set in $\tilde{R}_A$ the soft number $\tilde{t}$ is called the soft least upper bound (or soft supremum) of the soft set (F,A) if:

$(\boldsymbol{i})$ $\tilde{t}$ is a soft upper bound if any $\tilde{x}\,\tilde{\in}\,(F,A)$ satisfies $\tilde{x}\;\tilde{\leq}\;\tilde{t}$.

$(\boldsymbol{ii})$ $\tilde{t}$ is the smallest soft upper bound i.e. If $\tilde{k}$ is any other soft upper bound of the soft set (F,A) then $\tilde{t}\;\tilde{<}\;\tilde{k}$.

Definition (1.1.19) [18]:

Let (F, A) be a soft set in $\tilde{R}_A$ the soft set (F, A) is called soft bounded below, if there is a soft number $\tilde{m}$ so that any $\tilde{x} \tilde{\in} (F, A)$ is soft bigger than or soft equal to $\tilde{m}$: $\tilde{x} \tilde{\geq} \tilde{m}$, the soft number $\tilde{m}$ is called a soft lower bound for the soft set (F, A) .

Definition (1.1.20) [18]:

Let (F, A) be a soft set in $\tilde{R}_A$ the soft number $\tilde{s}$ is called the soft greatest lower bound (or soft infimum) of the soft set (F, A) if :

$(\boldsymbol{i})$ $\tilde{s}$ is a soft lower bound , if any $\tilde{x} \tilde{\in} (F, A)$ satisfies $\tilde{s} \tilde{\leq} \tilde{x}$.

$(\boldsymbol{ii})$ $\tilde{s}$ is the greatest soft lower bound i.e. If $\tilde{k}$ is any other soft lower bound of the soft set (F, A) then $\tilde{k} \tilde{<} \tilde{s}$.

Remark (1.1.21) [18].

$(\boldsymbol{i})$ When the soft supremum of a soft set (F, A) is a soft number that soft belong to (F, A) then it is also called soft maximum .

$(\boldsymbol{ii})$ When the soft infimum of a soft set (F, A) is a soft number that soft belong to (F, A) then it is also called soft minimum .

Definition (1.1.22) [25]:

$(\boldsymbol{i})$ For any (F, A) be a soft real set and for any positive integer n , we define (F^n, A) by $F^n(e) = \{ x^n : x \in F(e) \}$, for each $e \in A$. Obviously (F^n, A) is soft real set.

$(\boldsymbol{ii})$ For any soft real set (F, A) of $\mathbb{R}(A)^*$, we define the sequere root of by (F, A) and denoted by $(\sqrt{F}, A)$ is defined by $\sqrt{F}(e) = \{ \sqrt{x} : x \in F(e) \}$, for each $e \in A$.

$(\boldsymbol{iii})$ For any soft real set (F, A) defined $(|F|, A)$ By:

$$|F|(e) = \{|x| : x \in F(e)\} .$$

Then for any soft real set (F, A) , implies that $(|F|, A) \tilde{\subseteq} \mathbb{R}(A)^*$.

Definition (1.1.23) [17] , [25]:

Let $\mathcal{C}$ be a family of all bounded intervals of real numbers, then a mapping $F: A \to \mathcal{C}$ is known as a soft open interval. Each soft open interval may be expressed as an ordered pair of soft real numbers. That is if $F: A \to \mathcal{C}$ is defined by $F(e) = (a\,, b)$, $\forall\, e \in A$, then the soft open interval (F, A) may be expressed as an ordered pair of soft real numbers $(\tilde{r}\,, \tilde{s})$, where $\tilde{r}\,(e) = a\,, \tilde{s}(e) = b, \forall\, e \in A$.

Similarty the mapping $F: A \to \mathcal{C}$ is called soft closed interval , if it is defined by $F(e) = [a\,, b], \forall\, e \in A$.

Definition (1.1.24) [18]:

A soft real set (F, A) is called soft bounded , if there are two soft real numbers $\tilde{m}$ and $\tilde{n}$ such that $\tilde{m} \;\tilde{\leqq}\; \tilde{r} \;\tilde{\leqq}\; \tilde{n}$ for any $\tilde{r} \;\tilde{\in}\; (F, A)$. A soft set which is not soft bounded is called soft unbounded.

§ (1.2) Soft topological space and some results:

In this section some notations are introduced in soft topological space such as soft Hausdorff space , soft compact spaces , soft continuous mapping. Also, we introduce some results about soft convergence of soft net and investigate the relations between these concepts.

Definition (1.2.1) [16]:

Let $\tilde{\tau}$ be the collection of soft sets over X, then $\tilde{\tau}$ is said to be a soft topology on $\tilde{X}_A$, if :

$(\boldsymbol{i})$ $\tilde{\emptyset}_A$, $\tilde{X}_A$ $\tilde{\in}$ $\tilde{\tau}$.

$(\boldsymbol{ii})$ If (F_α, A) $\tilde{\in}$ $\tilde{\tau}$ for all $\alpha \in \Lambda$, then $\tilde{\bigcup}_{\alpha \in \Lambda}(F_\alpha, A) \tilde{\in} \tilde{\tau}$.

$(\boldsymbol{iii})$ If (F_1, A) , $(F_2, A) \tilde{\in} \tilde{\tau}$, then $(F_1, A) \tilde{\cap} (F_2, A) \tilde{\in} \tilde{\tau}$.

The triple $(\tilde{X}_A, \tilde{\tau}, A)$ is called soft topological space (for short STS) , the members of $\tilde{\tau}$ is called soft open sets . The complement of the members of $\tilde{\tau}$ is called soft closed set .
Indeed, the difference between STS and TS arise from this facts in remark (1.1.3.i) and (1.1.3.ii), [**21**].

Definition (1.2.2) [19]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and (F, A) be a soft set over X, then:

$(\boldsymbol{i})$ The soft closure of (F, A) denoted by $\widetilde{cl}((F, A))$ is the intersection of all soft closed sets which is containing (F, A).

$(\boldsymbol{ii})$ A soft point $P_e^x \tilde{\in}$ (F, A) is called a soft interior point of (F, A) , if there is a soft open set (G, A) such that $P_e^x \tilde{\in}$ $(G, A) \tilde{\subseteq} (F, A)$. The soft set (F, A) which contains all soft interior points is called soft interior set and it is denoted by $\widetilde{int}((F, A))$.

$(\boldsymbol{iii})$ A soft set (F, A) is called a soft neighborhood (for short soft nbhd) of a soft point $P_e^x \in S_P(X)$, if there is a soft open set (G, A) such that $P_e^x \tilde{\in}$ $(G, A) \tilde{\subseteq} (F, A)$. The soft nbhd system of P_e^x, denoted by $\mathcal{N}_{P_e^x}$ is the collection of all soft nbhd of P_e^x.

Proposition (1.2.3) [19] , [16]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and (F,A),(G,A) be two soft sets over a universe X. Then:

$(\boldsymbol{i})$ $\widetilde{int}((F,A)) \tilde{\subseteq} (F,A)$.

$(\boldsymbol{ii})$ If $(F,A) \tilde{\subseteq} (G,A)$, $\widetilde{int}((F,A)) \tilde{\subseteq} \widetilde{int}((G,A))$.

$(\boldsymbol{iii})$ $\widetilde{int}((F,A))\tilde{\cup}\widetilde{int}((G,A)) \tilde{\subseteq} \widetilde{int}((F,A)\tilde{\cup}(G,A))$.

$(\boldsymbol{iv})$ $\widetilde{int}((F,A))\tilde{\cap}\widetilde{int}((G,A)) = \widetilde{int}((F,A)\tilde{\cap}(G,A))$.

$(\boldsymbol{v})$ A soft set (F,A) is soft open if and only if $(F,A) = \widetilde{int}((F,A))$.

Proposition (1.2.4) [19]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS , (F,A) and (G,A) be two soft sets over a universe X. Then:

$(\boldsymbol{i})$ $(F,A) \tilde{\subseteq} \widetilde{cl}((F,A))$.

$(\boldsymbol{ii})$ If $(F,A) \tilde{\subseteq} (G,A)$, $\widetilde{cl}((F,A)) \tilde{\subseteq} \widetilde{cl}((G,A))$.

$(\boldsymbol{iii})$ $\widetilde{cl}((F,A))\tilde{\cup}\widetilde{cl}((G,A)) = \widetilde{cl}((F,A)\tilde{\cup}(G,A))$.

$(\boldsymbol{iv})$ $\widetilde{cl}((F,A)\tilde{\cap}(G,A)) \tilde{\subseteq} \widetilde{cl}((F,A))\tilde{\cap}\widetilde{cl}((G,A))$.

$(\boldsymbol{v})$ A soft set (F,A) is soft closed if and only if $(F,A) = \widetilde{cl}((F,A))$.

Proposition (1.2.5):

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and (F,A),(G,A) be two soft sets over a universe X and $P_e^x \tilde{\in} S_P(X)$. Then:

$(\boldsymbol{i})$ If $(F,A) \tilde{\subseteq} (G,A)$ and $(F,A) \tilde{\in} \mathcal{N}_{P_e^x}$, then$(G,A) \tilde{\in} \mathcal{N}_{P_e^x}$.

$(\boldsymbol{ii})$ If $(F,A) \tilde{\in} \mathcal{N}_{P_e^x}$, then $\widetilde{int}((F,A)) \tilde{\in} \mathcal{N}_{P_e^x}$.

$(\boldsymbol{ii})$ If $(F,A) \tilde{\in} \mathcal{N}_{P_e^x}$, then $\widetilde{cl}((F,A)) \tilde{\in} \mathcal{N}_{P_e^x}$.

$(\boldsymbol{vi})$ If $(F,A), (G,A) \tilde{\in} \mathcal{N}_{P_e^x}$, then $(F,A)\tilde{\cap}(G,A) \tilde{\in} \mathcal{N}_{P_e^x}$.

Theorem (1.2.6) [8] , [19] , [30]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and (F, A),(G, A) be two soft set over X and $P_e^x \in S_P(X)$. Then:

$(\boldsymbol{i})$ $P_e^x \tilde{\in} \widetilde{cl}((F, A))$ if and only if $(F, A) \tilde{\cap} (G, A) \neq \tilde{\emptyset}_A$, for every soft open nbhd(G, A) of P_e^x .

$(\boldsymbol{ii})$ A soft set (F, A) is soft open if and only if (F, A) is soft nbhd for all its soft points.

$(\boldsymbol{iii})$ A soft set (F, A) is soft open if and only if for every $P_e^x \tilde{\in} (F, A)$, there is a soft open set (U, A) in $(\tilde{X}_A, \tilde{\tau}, A)$ such that $P_e^x \tilde{\in} (U, A) \tilde{\subseteq} (F, A)$.

Definition (1.2.7) [24] , [11]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS. Then a soft set (F, A) is called:

$(\boldsymbol{i})$ Soft dense if and only if $\widetilde{cl}(F, A) = \tilde{X}_A$.

$(\boldsymbol{ii})$ Soft nowhere dense if and only if $\widetilde{int}(\widetilde{cl}(F, A)) = \tilde{\emptyset}_A$.

Definition (1.2.8) [16] , [12] :

Let (F, A) be a soft set over universe X and Y be a non-empty subset of X. Then:

$(\boldsymbol{i})$ The soft set $(F, A)'$ over Y is denoted by $(F, A)' = \tilde{Y}_A \tilde{\cap} (F, A)$ and defined as follows $F'(e) = Y \cap F(e)$, $\forall e \in A$.

$(\boldsymbol{ii})$ Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and Y be a non-empty subset of X . Then the collection $\tilde{\tau}_Y = \{(F, A)' : (F, A)' = \tilde{Y}_A \tilde{\cap} (F, A) \,;\, (F, A) \tilde{\in} \tilde{\tau}\}$ is called a soft relative topology $\tilde{\tau}$ on $\tilde{Y}_A$. Hence , $(\tilde{Y}_A, \tilde{\tau}_Y, A)$ is called a soft topological subspace of $(\tilde{X}_A, \tilde{\tau}, A)$.

$(\boldsymbol{iii})$ Let $(\tilde{Y}_A, \tilde{\tau}_Y, A)$ be a soft topological subspace of a STS $(\tilde{X}_A, \tilde{\tau}, A)$, if (F, A) is a soft set over Y, then $\widetilde{cl}_Y\big((F, A)\big) = \tilde{Y}_A \tilde{\cap} \widetilde{cl}_X(F, A)$.

Definition (1.2.9) [1] , [31]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS . Then:

$(\boldsymbol{i})$ A sub-collection $\tilde{\mathfrak{B}}$ of $\tilde{\tau}$ is called a soft base , if every member of $\tilde{\tau}$ can be expressed as a union of some members of $\tilde{\mathfrak{B}}$.

$(\boldsymbol{ii})$ A sub-collection $\widetilde{\mathfrak{B}}_{P_e^x}$ of $\mathcal{N}_{P_e^x}$ is called soft nbhd base at P_e^x , if for each soft nbhd (U,A) of P_e^x , there is $(V,A) \in \widetilde{\mathfrak{B}}_{P_e^x}$ such that:

$$P_e^x \widetilde{\in} (V,A) \widetilde{\subseteq} (U,A) .$$

$(\boldsymbol{iii})$ Let $(\tilde{X}_A, \tilde{\tau}_1, A)$ and $(\tilde{Y}_A, \tilde{\tau}_2, A)$ be two STSs .

Let $\widetilde{\mathfrak{B}} = \{(F,A) \widetilde{\times} (G,A) : (F,A) \widetilde{\in} \tilde{\tau}_1, (G,A) \widetilde{\in} \tilde{\tau}_2\}$ be a basise of soft nbhds and $\widetilde{\mathfrak{J}}$ be a collection of all arbitrary union of elements of $\widetilde{\mathfrak{B}}$. Then $\widetilde{\mathfrak{J}}$ is a soft topology on $X \times Y$ and $(\widetilde{X \times Y}_{A\times A}, \widetilde{\mathfrak{J}}, A \times A)$ is called soft product topological space. The form of any soft point in $S_P(X \times Y)$ is $(P_e^x, P_{e'}^y)$.

Definition (1.2.10) [18]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and (F,A) , (G,A) be two soft sets over X. Then:

$(\boldsymbol{i})$ (F,A) , (G,A) are said to be soft disjoint , if $(F,A) \widetilde{\cap} (G,A) = \widetilde{\emptyset}_A$.

$(\boldsymbol{ii})$ A non-null soft (which is soft disjoint)sets $(F,A), (G,A)$ are said to be soft separated of $\tilde{X}$, if $(F,A) \widetilde{\cap} \widetilde{cl}(G,A) = \widetilde{\emptyset}_A$, $\widetilde{cl}(F,A) \widetilde{\cap} (G,A) = \widetilde{\emptyset}_A$.

Theorem (1.2.11) [18]:

Two soft closed (or soft open) subsets (F,A) and (G,A) of a STS $(\tilde{X}_A, \tilde{\tau}, A)$ are soft separated if and only if they are soft disjoint.

Definition (1.2.12) [32]:

A STS $(\tilde{X}_A, \tilde{\tau}, A)$ is soft compact if for each soft open cover of $\tilde{X}$ has finite subcover.

Also , the soft set (F,A) of a STS $(\tilde{X}_A, \tilde{\tau}, A)$ is soft compact set with respect to the soft relative topology on (F,A) , if for each soft open cover of (F,A) , there is a finite subcover.

Definition (1.2.13) [26]:

A soft set (F,A) in a STS $(\tilde{X}_A, \tilde{\tau}, A)$ is called finite (countable) soft set , if $F(e)$ finite (countable) for all $e \in A$.

Theorem (1.2.14) [18]:

($\boldsymbol{i}$) Every soft closed set of a soft compact space is soft compact.

($\boldsymbol{ii}$) The intersection of soft closed with soft compact is soft compact.

($\boldsymbol{iii}$) Every finite soft set is soft compact.

($\boldsymbol{iv}$) In STS $(\tilde{X}_A, \tilde{\tau}, A)$, if the sets A and X are finite, then $(\tilde{X}_A, \tilde{\tau}, A)$ is soft compact.

($\boldsymbol{v}$) If the soft sets (F_i, A) in a STS $(\tilde{X}_A, \tilde{\tau}, A)$ are finite (F_i are finite mapping), then $(\tilde{X}_A, \tilde{\tau}, A)$ is soft compact space.

Definition (1.2.15) [22] , [10]:

A STS $(\tilde{X}_A, \tilde{\tau}, A)$ is called :

($\boldsymbol{i}$) Soft T_1-space , if for all $P_e^x, P_{e\prime}^y \tilde{\in} S_P(X)$, with $P_e^x \neq P_{e\prime}^y$, there are two soft open sets (F, A) , (G, A) in $(\tilde{X}_A, \tilde{\tau}, A)$ such that:

$$[P_e^x \tilde{\in} (F, A) , P_{e\prime}^y \tilde{\notin} (F, A)] \text{and } [P_{e\prime}^y \tilde{\in} (G, A), \; P_e^x \tilde{\notin} (G, A)].$$

($\boldsymbol{ii}$) Soft Hausdorff space (Soft T_2-space) , if for all $P_e^x, P_{e\prime}^y \tilde{\in} S_P(X)$, with $P_e^x \neq P_{e\prime}^y$, there are two soft disjoint (soft open sets) (F, A) and (G, A) in $(\tilde{X}_A, \tilde{\tau}, A)$ such that $P_e^x \tilde{\in} (F, A)$ and $P_{e\prime}^y \tilde{\in} (G, A)$.

Theorem (1.2.16) [22]:

A soft Hausdorff space is soft T_1-space.

Theorem (1.2.17) [22] , [30]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS. Then:

($\boldsymbol{i}$) $(\tilde{X}_A, \tilde{\tau}, A)$ is soft T_1 if and only if $\{P_e^x\}$ soft closed for all $P_e^x \tilde{\in} S_P(X)$.

($\boldsymbol{ii}$) Every soft compact of a soft Hausdorff space $(\tilde{X}_A, \tilde{\tau}, A)$ is soft closed.

Definition (1.2.18) [31] , [11]:

A STS $(\tilde{X}_A, \tilde{\tau}, A)$ is called:

($\boldsymbol{i}$) soft first countable space , if each soft point P_e^x over X has a countable soft nbhd base.

$(\boldsymbol{ii})$ Soft Baire space , if the intersection of each countable family of a soft open which is soft dense sets is soft dense.

$(\boldsymbol{iii})$ Soft second category , if for any soft set (F, A) over X cannot be represented as a countable union of soft nowhere dense.

Theorem (1.2.19) [31] , [11]:

Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS . Then:

$(\boldsymbol{i})$ If $(\tilde{X}_A, \tilde{\tau}, A)$ is a soft Baire space and $\{(F_n, A) : n \in \mathbb{N}\}$ be any countable soft closed family of soft covering of $(\tilde{X}_A, \tilde{\tau}, A)$, then at least one of (F_n, A) must contains a soft open set , that is $\widetilde{int}(F_n, A) \neq \widetilde{\emptyset}_A$ for at least one n.

$(\boldsymbol{ii})$ $(\tilde{X}_A, \tilde{\tau}, A)$ is soft first countable space at P_e^x if and only if there is a countable soft open nbhd base $\{(F_n, A) : n \in \mathbb{N}\}$ at P_e^x such that :

$(F_{n+1}, A) \tilde{\subseteq} (F_n, A)$ for all $n \in \mathbb{N}$.

Example (1.2.20) :

$(\mathbb{R}(A), \tilde{\tau}_U, A)$ is soft first countable space.

Definition (1.2.21) [4] , [30]:

Let (F, A) and (G, A) be soft sets over a universe sets X, Y respectively. A soft relation

$$\tilde{f} \tilde{\subseteq} (F, A) \tilde{\times} (G, A) = (F \tilde{\times} G, A \times A)$$

is a soft set over $X \times Y$, where $F \times G: A \times A \to P(X \times Y)$ such that: $(F \times G)(e, e') = F(e) \times G(e')$, for all $(e, e') \in A \times A$, is called a soft mapping from (F, A) to (G, A) and denoted by $\tilde{f}: (F, A) \tilde{\rightarrow} (G, A)$, if for all $P_e^x \tilde{\in} (F, A)$, there is only one $P_e^y \tilde{\in} (G, B)$ such that $\tilde{f}(P_e^x) = P_e^y$.

Definition (1.2.22) [30]:

Let $\tilde{f}: (F, A) \tilde{\rightarrow} (G, A)$ be a soft mapping . Then :

$(\boldsymbol{i})$ The soft image of $(H, A) \tilde{\subseteq} (F, A)$ under $\tilde{f}$ is the soft set denoted by $\tilde{f}((H, A))$ of the form:

$$\tilde{f}((H, A)) = \begin{cases} \tilde{\bigcup} \tilde{f}(P_e^x) & ; \quad P_e^x \tilde{\in} (H, A) \\ \widetilde{\emptyset}_A & ; \quad \text{otherwise} \end{cases} .$$

• The image of any null soft set under the soft mapping is null soft set.

($\boldsymbol{ii}$) The soft inverse image of $(N,A) \tilde{\subseteq} (G,A)$ under $\tilde{f}$ is the soft set denoted by $\tilde{f}^{-1}((N,A))$ of the form:

$$\tilde{f}^{-1}((N,A)) = \begin{cases} \tilde{\cup} P_e^x & ; \quad \tilde{f}(P_e^x) \tilde{\in} (N,A) \\ \tilde{\emptyset}_A & ; \quad \text{otherwise} \end{cases}.$$

• The inverse image of any null soft set under the soft mapping is null soft set.

Proposition (1.2.23) [30]:

Let $(F,A),(G,A)$ be two soft sets over X with $(F_1,A),(F_2,A) \tilde{\subseteq} (F,A)$, $(G_1,A),(G_2,A) \tilde{\subseteq} (G,A)$ and let $\tilde{f} : (F,E) \tilde{\rightarrow} (G,E)$ be a soft mapping. Then the following hold :

($\boldsymbol{i}$) If $(F_1,A) \tilde{\subseteq} (F_2,A)$, then $\tilde{f}((F_1,A)) \tilde{\subseteq} \tilde{f}((F_2,A))$;

also if $(G_1,A) \tilde{\subseteq} (G_2,A)$, then $\tilde{f}^{-1}((G_1,A)) \tilde{\subseteq} \tilde{f}^{-1}((G_2,A))$.

($\boldsymbol{ii}$) $\tilde{f}((F_1,A) \tilde{\cup} (F_2,A)) = \tilde{f}((F_1,A)) \tilde{\cup} \tilde{f}((F_2,A))$;
and $\tilde{f}((F_1,A) \tilde{\cap} (F_2,A)) \tilde{\subseteq} \tilde{f}((F_1,A)) \tilde{\cap} \tilde{f}((F_2,A))$.

($\boldsymbol{iii}$) $(F_1,A) \tilde{\subseteq} \tilde{f}^{-1}(\tilde{f}((F_1,A)))$ and $\tilde{f}(\tilde{f}^{-1}((G_1,A))) \tilde{\subseteq} (G_1,A)$.

($\boldsymbol{iv}$) $\tilde{f}^{-1}((G_1,A) \tilde{\cup} (G_2,A)) = \tilde{f}^{-1}((G_1,A)) \tilde{\cup} \tilde{f}^{-1}((G_2,A))$ and

$\tilde{f}^{-1}((G_1,A) \tilde{\cap} (G_2,A)) = \tilde{f}^{-1}((G_1,A)) \tilde{\cap} \tilde{f}^{-1}((G_2,A))$.

Definition (1.2.24) [18]:

A soft mapping $\tilde{f} : (F,A) \tilde{\rightarrow} (G,A)$ is called:

($\boldsymbol{i}$) Soft injective , if for all $P_e^x, P_{e'}^y \tilde{\in} (F,A)$, with $\tilde{f}(P_e^x) = \tilde{f}(P_{e'}^y)$, implies that $P_e^x = P_{e'}^y$.

($\boldsymbol{ii}$) Soft surjective , if for all $P_{e'}^y \tilde{\in} (G,A)$, there is $P_e^x \tilde{\in} (F,A)$ such that $\tilde{f}(P_e^x) = P_{e'}^y$.

($\boldsymbol{iii}$) Soft bijective , if it is soft injective and soft surjective.

Propositions (1.2.25) [18].

Let $(F,A),(G,A)$ be two soft sets over X with $(F_1,A),(F_2,A) \tilde{\subseteq} (F,A)$, $(G_1,A),(G_2,A) \tilde{\subseteq} (G,A)$, let $\tilde{f}:(F,E) \tilde{\rightarrow} (G,E)$ be a soft mapping. Then:

$(\boldsymbol{i})$ $\tilde{f}((F_1,A) \tilde{\cap} (F_2,A)) = \tilde{f}((F_1,A))\tilde{\cap}\tilde{f}((F_2,A))$, if $\tilde{f}$ is soft injective.

$(\boldsymbol{ii})$ $\tilde{f}^{-1}(\tilde{f}((F_1,A))) = (F_1,A)$, if $\tilde{f}$ is soft injective.

$(\boldsymbol{iii})$ $\tilde{f}(\tilde{f}^{-1}((G_1,A))) = (G_1,A)$, if $\tilde{f}$ is soft surjective.

Definition (1.2.26) :

Let $\tilde{f}:(\tilde{X}_A,\tilde{\tau}_1,A) \tilde{\rightarrow} (\tilde{Y}_A,\tilde{\tau}_2,A)$ be a soft mapping, then $\tilde{f}$ is called soft continuous at $P_e^x \tilde{\in} S_P(X)$, if for each soft nbhd (V,A) of $\tilde{f}(P_e^x)$, in $(\tilde{Y}_A,\tilde{\tau}_2,A)$, there is a soft nbhd (U,A) of P_e^x in $(\tilde{X}_A,\tilde{\tau}_1,A)$ such that $\tilde{f}((U,A)) \tilde{\subseteq} (V,A)$. If $\tilde{f}$ is soft continuous mapping at P_e^x, then it is called soft continuous.

Theorem (1.2.27) [18]:

Let $\tilde{f}:(\tilde{X}_A,\tilde{\tau}_1,A) \tilde{\rightarrow} (\tilde{Y}_A,\tilde{\tau}_2,A)$ be a soft mapping. Then the following statements are equivalent:

$(\boldsymbol{i})$ $\tilde{f}$ is soft continuous.

$(\boldsymbol{ii})$ $\tilde{f}^{-1}((G,A))$ is soft open set over X for all soft open set (G,A) over Y.

$(\boldsymbol{iii})$ $\tilde{f}^{-1}((G,A))$ is soft closed over X for all soft closed set (G,A) over Y.

$(\boldsymbol{iv})$ For all soft set (F,A) over X, $\tilde{f}(\widetilde{cl}(F,A)) \tilde{\subseteq} \widetilde{cl}\tilde{f}((F,A))$.

Definition (1.2.28) [18]:

Let $\tilde{f}:(\tilde{X}_A,\tilde{\tau}_1,A) \tilde{\rightarrow} (\tilde{Y}_A,\tilde{\tau}_2,A)$ be a soft mapping, then $\tilde{f}$ is called:

$(\boldsymbol{i})$ Soft open (Soft closed), if $\tilde{f}((F,A))$ is soft open (soft closed) set over Y for all soft open (soft closed) (F,A) set over X.

$(\boldsymbol{ii})$ Soft homeomorphism, if $\tilde{f}$ is soft bijective, soft continuous and $\tilde{f}^{-1}$ are soft continuous mapping.

Theorem (1.2.29) [18]:

Let $\tilde{f}: (\tilde{X}_A, \tilde{\tau}_1, A) \tilde{\rightarrow} (\tilde{Y}_A, \tilde{\tau}_2, A)$ be a soft bijective and soft continuous mapping . Then the following statements are equivalent:

$(\boldsymbol{i})$ $\tilde{f}^{-1}$ is soft continuous.

$(\boldsymbol{ii})$ $\tilde{f}$ is soft open.

$(\boldsymbol{iii})$ $\tilde{f}$ is soft closed.

Theorem (1.2.30):

Let $\tilde{f}: (\tilde{X}_A, \tilde{\tau}_1, A) \tilde{\rightarrow} (\tilde{Y}_A, \tilde{\tau}_2, A)$ be a soft mapping . Then:

$(\boldsymbol{i})$ $\tilde{f}$ is soft open if and only if for each soft set (F, A) of $(\tilde{X}_A, \tilde{\tau}_1, A)$, then

$$\tilde{f}(\widetilde{int}(F, A)) \tilde{\subseteq} \widetilde{int}(\tilde{f}((F, A))).$$

$(\boldsymbol{ii})$ $\tilde{f}$ is soft closed if and only if for each soft set (F, A) of $(\tilde{X}_A, \tilde{\tau}_1, A)$, then

$$\widetilde{cl}(\tilde{f}((F, A))) \tilde{\subseteq} \tilde{f}(\widetilde{cl}(F, A)).$$

Theorem (1.2.31):

Let $\tilde{f}: (\tilde{X}_A, \tilde{\tau}_1, A) \tilde{\rightarrow} (\tilde{Y}_A, \tilde{\tau}_2, A)$ be a soft bijective mapping . Then $\tilde{f}$ is soft continuous if and only if $\widetilde{int}(\tilde{f}((F, A))) \tilde{\subseteq} \tilde{f}(\widetilde{int}((F, A)))$ for every soft set (F, A) of $(\tilde{X}_A, \tilde{\tau}_1, A)$.

Theorem (1.2.32) [30]:

A soft continuous image of a soft compact set is soft compact.

Definition (1.2.33) [10]:

$(\boldsymbol{i})$ A soft sequence in $S_P(X)$ is a soft mapping $\tilde{\chi} : \mathbb{N} \tilde{\rightarrow} S_P(X)$, where $\mathbb{N}$ is the set of all natural numbers . The soft point $\tilde{\chi}(n)$ denoted by $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$, where $F(e_n) = x_n$ such that $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $(e_n)_{n\in\mathbb{N}}$ a sequence in a parameter set A.

$(\boldsymbol{ii})$ A soft sequence $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$ is called soft converge to P_e^x in a STS $(\tilde{X}_A, \tilde{\tau}, A)$ and write $P_{e_n}^{x_n} \tilde{\rightarrow} P_e^x$, if for every soft nbhd (F, A) of P_e^x, there is $n_0 \in \mathbb{N}$ such that $P_{e_n}^{x_n} \tilde{\in} (F, A)$ for all $n \geq n_0$.

Definition (1.2.34) [23]:

A soft net in $S_P(X)$ is a soft mapping $\tilde{\chi} : D \tilde{\rightarrow} S_P(X)$, where D is directed set. A soft point $\tilde{\chi}(d)$ denoted by $(P_{e_d}^{x_d})_{\,d\in D}$ and $F(e_d) = x_d$, where $(x_d)_{\,d\in D}$ is a net in X and $(e_d)_{\,d\in D}$ be a net in a parameter set A. Every soft sequence is soft net.

Definition (1.2.35) [23]:

Let $(P_{e_d}^{x_d})_{\,d\in D}$ be a soft net in a STS $(\tilde{X}_A, \tilde{\tau}, A)$ and (F, A) be a soft set over a universe X and $P_e^x \tilde{\in} S_P(X)$. Then $(P_{e_d}^{x_d})_{\,d\in D}$ is called:

$(\boldsymbol{i})$ Eventually in (F, A) , if $\exists\, d_0 \in D$ with $P_{e_d}^{x_d} \tilde{\in} (F, A)\ \forall\ d \geq d_0$.

$(\boldsymbol{ii})$ Frequently in (F, A), if $\forall\, d_0 \in D\ \exists\, d \in D$, $d \geq d_0$, then $P_{e_d}^{x_d} \tilde{\in} (F, A)$.

$(\boldsymbol{iii})$ Soft convergence to P_e^x if $(P_{e_d}^{x_d})_{\,d\in D}$ is eventually in each soft nbhd of P_e^x (written $P_{e_d}^{x_d} \tilde{\rightarrow} P_e^x$) . The soft point P_e^x is called a limit soft point of $(P_{e_d}^{x_d})_{\,d\in D}$.

$(\boldsymbol{iv})$ Have P_e^x as a soft cluster soft point if $(P_{e_d}^{x_d})_{\,d\in D}$ is frequently in each soft nbhd of P_e^x (written $P_{e_d}^{x_d} \tilde{\propto} P_e^x$) .

Remark (1.2.36) [23]:

Let $\tilde{f} : (\tilde{X}_A, \tilde{\tau}_1, A) \tilde{\rightarrow} (\tilde{Y}_A, \tilde{\tau}_2, A)$ be a soft mapping. Then:

$(\boldsymbol{i})$ If $(P_{e_d}^{x_d})_{\,d\in D}$ is a soft net in $(\tilde{X}_A, \tilde{\tau}_1, A)$, then $\{\tilde{f}(P_{e_d}^{x_d})\}_{d\in D}$ is a soft net in $(\tilde{Y}_A, \tilde{\tau}_2, A)$.

$(\boldsymbol{ii})$ If $\tilde{f}$ is soft surjective and $(P_{e_d}^{y_d})_{\,d\in D}$ be a soft net in $(\tilde{Y}_A, \tilde{\tau}_2, A)$, there is a soft net $(P_{e_d}^{x_d})_{\,d\in D}$ in $(\tilde{X}_A, \tilde{\tau}_1, A)$ such that $\tilde{f}(P_{e_d}^{x_d}) = P_{e_d}^{y_d}$, $\forall\, d \in D$.

$(\boldsymbol{iii})$ Let P_e^x be a soft point over X. Then $\tilde{f}$ is soft continuous at P_e^x if and only if , whenever a soft net $(P_{e_d}^{x_d})_{\,d\in D}$ in $(\tilde{X}_A, \tilde{\tau}_1, A)$ and $P_{e_d}^{x_d} \tilde{\rightarrow} P_e^x$, then $\tilde{f}(P_{e_d}^{x_d}) \tilde{\rightarrow} \tilde{f}(P_e^x)$ in $(\tilde{Y}_A, \tilde{\tau}_2, A)$.

Theorem (1.2.37) [23]:

($\boldsymbol{i}$) Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS and (F, A) be a soft set over a universe X. Then $P_e^x \tilde{\in} \widetilde{cl}(F, A)$ if and only if there is a soft net $(P_{e_d}^{x_d})_{d\in D}$ in (F, A) with $P_{e_d}^{x_d} \tilde{\rightarrow} P_e^x$.

($\boldsymbol{ii}$) Let $(\tilde{X}_A, \tilde{\tau}, A)$ be a STS , then $(\tilde{X}_A, \tilde{\tau}, A)$ is soft Hausdorff space if and only if every soft net $(P_{e_d}^{x_d})_{d\in D}$ in $(\tilde{X}_A, \tilde{\tau}, A)$ has a unique soft limite point.

Theorem (1.2.38) [23]:

Let $(\tilde{X}_A, \tilde{\tau}_1, A)$ and $(\tilde{Y}_A, \tilde{\tau}_2, A)$ be two STSs . Then:

($\boldsymbol{i}$) $(P_{e_d}^{x_d}, P_{e'_d}^{y_d})_{d\in D}$ be a soft net in $(\widetilde{X \times Y}_{A\times A}, \tilde{\mathfrak{J}}, A \times A)$, with $(P_{e_d}^{x_d})_{d\in D}$ be a soft net in $(\tilde{X}_A, \tilde{\tau}_1, A)$ and $(P_{e'_d}^{y_d})_{d\in D}$ be a soft net in $(\tilde{Y}_A, \tilde{\tau}_2, A)$.

($\boldsymbol{ii}$) A soft net $(P_{e_d}^{x_d}, P_{e'_d}^{y_d})_{d\in D}$ in $(\widetilde{X \times Y}_{A\times A}, \tilde{\mathfrak{J}}, A \times A)$ is soft converge to a soft point $(P_e^x, P_{e'}^y)$ if and only if $P_{e_d}^{x_d} \tilde{\rightarrow} P_e^x$ and $P_{e'_d}^{y_d} \tilde{\rightarrow} P_{e'}^y$ in $(\tilde{X}_A, \tilde{\tau}_1, A)$ and $(\tilde{Y}_A, \tilde{\tau}_2, A)$ respectively.

Definition (1.2.39) [26]:

A soft mapping $\tilde{d}: S_P(\tilde{X}_A) \tilde{\times} S_P(\tilde{X}_A) \tilde{\rightarrow} \mathbb{R}(A)^*$ is said to be a soft meric on a soft set $\tilde{X}_A$, if $\tilde{d}$ satisfaies the following conditions:

($\boldsymbol{i}$) $\tilde{d}(P_e^x, P_{e'}^y) \tilde{\geq} \bar{0}, \forall P_e^x, P_{e'}^y \tilde{\in} S_P(\tilde{X}_A)$.

($\boldsymbol{ii}$) $\tilde{d}(P_e^x, P_{e'}^y) = \bar{0}$ if and only if $P_e^x = P_{e'}^y$ i.e. $\tilde{d}(P_e^x, P_e^x) = \bar{0}$.

($\boldsymbol{iii}$) $\tilde{d}(P_e^x, P_{e'}^y) = \tilde{d}(P_{e'}^y, P_e^x)$, $\forall P_e^x, P_{e'}^y \tilde{\in} S_P(\tilde{X}_A)$.

($\boldsymbol{iv}$) $\tilde{d}(P_e^x, P_{e'}^y) \tilde{\leq} \tilde{d}(P_e^x, P_{e''}^z) \tilde{+} \tilde{d}(P_{e''}^z, P_{e'}^y)$, $\forall P_e^x, P_{e'}^y, P_{e''}^z \tilde{\in} S_P(\tilde{X}_A)$.

- A soft set $\tilde{X}_A$ with the soft metric $\tilde{d}$ is called a soft metric space and denoted by $(\tilde{X}_A, \tilde{d}, A)$.
- $\tilde{d}: S_P(\tilde{R}_A) \tilde{\times} S_P(\tilde{R}_A) \tilde{\rightarrow} \mathbb{R}(A)^*$ defined by $\tilde{d}(P_e^x, P_{e'}^y) = |x - y| + |e - \acute{e}|$ for all, $P_e^x, P_{e'}^y \tilde{\in} S_P(\tilde{R}_A)$. , then $\tilde{d}$ is soft metric space .

Remark (1.2.40) [26]:

A soft sequence $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$ in $(\tilde{X}_A, \tilde{d}, A)$ called soft converge to P_e^x in $(\tilde{X}_A, \tilde{d}, A)$, if $\tilde{d}(P_{e_n}^{x_n}, P_e^x) \tilde{\rightarrow} \bar{0}$ as $n \to \infty$.

This means that for every $\tilde{\epsilon} \tilde{>} \bar{0}$, there is a natural number N, such that $\bar{0} \tilde{\leq} \tilde{d}(P_{e_n}^{x_n}, P_e^x) \tilde{<} \tilde{\epsilon}$, whenever $n > N$.

Theorem (1.2.41):

Let $\tilde{f}: (\tilde{X}_A, \tilde{\tau}_1, A) \tilde{\rightarrow} (\tilde{Y}_A, \tilde{\tau}_2, A)$ be a soft mapping and $(\tilde{X}_A, \tilde{\tau}_1, A)$ be a soft first countable. Then $\tilde{f}$ is soft continuous at P_e^x if and only if, whenever a soft sequence $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$ in $(\tilde{X}_A, \tilde{\tau}_1, A)$ and $P_{e_n}^{x_n} \tilde{\rightarrow} P_e^x$, then $\tilde{f}(P_{e_n}^{x_n}) \tilde{\rightarrow} \tilde{f}(P_e^x)$ in $(\tilde{Y}_A, \tilde{\tau}_2, A)$.

Definition (1.2.42) [26]:

A soft set $B(P_e^{x_0}; \tilde{r}) = \{ P_e^x \tilde{\in} S_P(\tilde{X}_A) : \tilde{d}(P_e^x, P_e^{x_0}) \tilde{<} \tilde{r} \}$ is called a soft open ball with center $P_e^{x_0}$ and a radius $\tilde{r}$.

:

Definition (1.2.43):

A STS $(\tilde{X}_A, \tilde{\tau}, A)$ is called a soft metrizable, if $\tilde{\tau}$ generated by $B(P_e^{x_0}; \tilde{r})$, where $B(P_e^{x_0}; \tilde{r}) = \{P_e^x \tilde{\in} S_P(\tilde{X}) : \tilde{d}\left(P_e^x, P_e^{x_0}\right) \tilde{<} \tilde{r}\}$.

Theorem (1.2.44):

A soft metric space $(\tilde{X}_A, \tilde{d}, A)$ is soft first countable.

Proof:

$\tilde{\mathcal{B}} = \{B(P_e^x, \overline{(\tfrac{1}{n})}) : n \in \mathbb{N}\}$ forms countable a soft nbhd base at P_e^x.

Theorem (1.2.45) [26]:

Every soft metric space $(\tilde{X}_A, \tilde{d}, A)$ is soft Hausdorff.

Definition (1.2.46) [27] , [26]:

$(\boldsymbol{i})$ Let $(\tilde{X}_A, \tilde{d}, A)$ be a soft metric space and (F, A) be a non-null soft subset of $\tilde{X}_A$. Then (F, A) is called soft bounded if there is $P_e^x \tilde{\in} \tilde{X}_A$ and $\tilde{\epsilon} \tilde{>} \bar{0}$ such that $(F, A) \subseteq B(P_e^x, \tilde{\epsilon})$.

$(\boldsymbol{ii})$ A soft sequence $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$ in $(\tilde{X}_A, \tilde{d}, A)$ called soft bounded in $(\tilde{X}_A, \tilde{d}, A)$, if there is $\widetilde{M} \tilde{>} \bar{0}$ such that $\tilde{d}(P_{e_n}^{x_n}, P_{e_m}^{x_m}) \tilde{\leq} \widetilde{M}$ for all $n , m \in \mathbb{N}$.

Remark (1.2.47):

$(\boldsymbol{i})$ The soft compact set (F, A) of a soft metric space $(\tilde{X}_A, \tilde{d}, A)$ is soft closed and soft bounded.

$(\boldsymbol{ii})$ In($\mathbb{R}(A), \tilde{d}, A)$ the soft closed and soft bounded is soft compact.

$(\boldsymbol{iii})$ A soft sequence $(\tilde{r}_n)_{n\in\mathbb{N}}$ in $(\mathbb{R}(A), \tilde{d}, A)$ is soft converge to $\tilde{r}$ in $\mathbb{R}(A)$, if $|\tilde{r}_n \tilde{-} \tilde{r}| \tilde{<} \tilde{\epsilon}$, whenever $n > N$.

§ (2.1) Linear soft topology:

In this section, by using the concept of soft point we introduced the soft linear space in a new point of view and investigate the properties of the soft topological linear space.

Let X be a linear space over a scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$) and the parameter set A be the real number set $\mathbb{R}$.

Definition (2.1.1) [29]:

Let (F, A) be a soft set over a universe X. The soft set (F, A) is called the soft linear and denoted by P_e^x, if there is exactly one $e \in A$ such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \emptyset$, for all $e' \in A \backslash \{e\}$. The set of all soft linears of $\tilde{X}_A$ will be denoted by $SL(\tilde{X}_A)$.

Proposition (2.1.2) [29]:

The set $SL(\tilde{X}_A)$ is called soft linear space (for short SLS) over $\mathbb{R}(A)$ according to the following operations:

$(\boldsymbol{i}) P_e^x \tilde{+} P_{e'}^y = P_{e+e'}^{x+y}$; for all $P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A)$.

$(\boldsymbol{ii})\ \tilde{r} \tilde{\cdot} P_e^x = P_{re}^{rx}$; for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$ and for all $\tilde{r} \tilde{\in} \mathbb{R}(A)$.

It is clear that:

- If $\Theta \in X$ be a zero linear and $\mathrm{e} = 0 \in A$, then P_0^{Θ} is a soft zero linear in $SL(\tilde{X}_A)$.
- $-P_e^x = P_{-e}^{-x}$ be the inverse of a soft linear P_e^x.

Remark (2.1.3):

Let (F, A) and (G, A) be two soft sets of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ If $P_0^{\Theta} \tilde{\in} (F, A)$, then $(G, A) \tilde{\subseteq} (F, A) \tilde{+} (G, A)$.

$(\boldsymbol{ii})\ \tilde{\alpha}((F, A) \tilde{+} (G, A)) = \tilde{\alpha}(F, A) \tilde{+} \tilde{\alpha}(G, A)$, for all $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$.

Definition (2.1.4) [29]:

Let $SL(\tilde{X}_A)$ be a SLS over $\mathbb{R}(A)$ and $\widetilde{\mathbb{M}}_A \tilde{\subseteq} SL(\tilde{X}_A)$ be a soft subset . If $\widetilde{\mathbb{M}}_A$ is a soft linear space , then $\widetilde{\mathbb{M}}_A$ is called soft linear subspace of $SL(\tilde{X}_A)$ and denoted by $SL(\widetilde{\mathbb{M}}_A) \tilde{\subseteq} SL(\tilde{X}_A)$.

Theorem (2.1.5):

$(\boldsymbol{i})$ Let $\widetilde{\mathbb{M}}_A$ be a soft subset of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then $SL(\widetilde{\mathbb{M}}_A)$ is soft linear subspace of $SL(\tilde{X}_A)$ if and only if:

$P_0^{\Theta} \tilde{\in} SL(\widetilde{\mathbb{M}}_A)$ and $\tilde{\alpha}\, SL(\widetilde{\mathbb{M}}_A) \tilde{+} \tilde{\beta}\, SL(\widetilde{\mathbb{M}}_A) \tilde{\subseteq} SL(\widetilde{\mathbb{M}}_A)$ for all $\tilde{\alpha}, \tilde{\beta} \tilde{\in} \mathbb{R}(A)$.

$(\boldsymbol{ii})$ The soft intersection of any soft linears subspaces of $SL(\tilde{X}_A)$ is soft linear subspace of $SL(\tilde{X}_A)$.

Proof:

$(\boldsymbol{i})$ If $SL(\widetilde{\mathbb{M}}_A)$ is a soft subspace of $SL(\tilde{X}_A)$, the condition is holds.

Conversely, if $P_0^{\Theta} \tilde{\in} SL(\widetilde{\mathbb{M}}_A)$ and $\tilde{\alpha}\, SL(\widetilde{\mathbb{M}}_A) \tilde{+} \tilde{\beta}\, SL(\widetilde{\mathbb{M}}_A) \tilde{\subseteq} SL(\widetilde{\mathbb{M}}_A)$ for all $\tilde{\alpha}, \tilde{\beta} \tilde{\in} \mathbb{R}(A)$. Now, let $P_e^x, P_e^y \tilde{\in} SL(\widetilde{\mathbb{M}}_A)$ and $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$.

Then $P_e^x \tilde{+} P_e^y = \bar{1}\, P_e^x \tilde{+} \bar{1}\, P_e^y \tilde{\in} SL(\widetilde{\mathbb{M}}_A)$ and $\tilde{\alpha}\, P_e^x = \tilde{\alpha}\, P_e^x \tilde{+} \tilde{\beta}\, P_0^{\Theta} \tilde{\in} SL(\widetilde{\mathbb{M}}_A)$.

This completes the proof.

$(\boldsymbol{ii})$ It is clear.

Example (2.1.6):

Let $SL(\widetilde{\ell^P}_A) = \{ P_e^x : x \in \ell^P;\ x = (x_1, x_2, \dots); x_i \in \mathbb{R};\ \sum_{i=1}^{\infty} |x_i|^P < \infty \}$, with $1 \leq P < \infty$. Then $(SL(\widetilde{\ell^P}_A), \tilde{+}, \tilde{\cdot})$ is soft linear space over $\mathbb{R}(A)$ under the soft addition and soft scalar multiplication on $\widetilde{\ell^P}$:

$(\boldsymbol{i})$ $\left(P_e^x \tilde{+} P_{e\prime}^y\right) = P_{e+e\prime}^{x+y}$, for all $P_e^x, P_{e\prime}^y \tilde{\in} SL(\widetilde{\ell^P}_A)$.

$(\boldsymbol{ii})$ $(\tilde{r} \tilde{\cdot} P_e^x) = P_{re}^{rx}$, for all $P_e^x \tilde{\in} SL(\widetilde{\ell^P}_A)$ and $\tilde{r} \tilde{\in} \mathbb{R}(A)$.

Definition (2.1.7) [9] , [7]:

A soft set (F,A) of $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is called:

$(\boldsymbol{i})$ Soft convex if $\tilde{\lambda}(F,A)\tilde{+}(\bar{1}-\tilde{\lambda})(F,A) \tilde{\subseteq} (F,A)$, $\forall\, \tilde{\lambda} \tilde{\in} [\bar{0},\bar{1}]$.

$(\boldsymbol{ii})$ Soft balanced if $\tilde{\alpha}(F,A) \tilde{\subseteq} (F,A)$, $\forall\, \tilde{\alpha} \tilde{\in} \mathbb{R}(A)$, with $|\tilde{\alpha}| \tilde{\leq} \bar{1}$.

$(\boldsymbol{iii})$ Soft absorbing if for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$ there is $\tilde{\alpha} \tilde{>} \bar{0}$ such that $P_e^x \tilde{\in} \tilde{\alpha}(F,A)$. i.e. $\tilde{\bigcup}_{\tilde{\alpha}\tilde{>}\bar{0}}\tilde{\alpha}(\mathrm{F},\mathrm{A}) = SL(\tilde{X}_A)$ and (F,A) is soft absorb $SL(\tilde{X}_A)$ In other words, (F,A) is soft absorbs (G,A) if there is $\tilde{\alpha}_0 \tilde{>} \bar{0}$ such that $(G,A) \tilde{\subseteq} \tilde{\alpha}(F,A)$ for all $|\tilde{\alpha}| \tilde{\geq} \tilde{\alpha}_0$.

$(\boldsymbol{iv})$ Soft symmetric if $(F,A) = -\bar{1}(F,A)$

Proposition (2.1.8) [9] , [7]:

Let (F,A) and (G,A) are two soft sets of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ If (F,A) and (G,A) is two soft convex sets , then $(F,A)\tilde{\cap}(G,A)$, $\tilde{\alpha}(F,A)$ and $(F,A)\tilde{+}(G,A)$ are soft convex set ; $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$.

$(\boldsymbol{ii})$ If (F,A) and (G,A) is a soft balanced set , then $(F,A)\tilde{\cap}(G,A)$ is soft balanced set.

$(\boldsymbol{iii})$ If (F,A) is a soft balanced set and $\tilde{\alpha}_1, \tilde{\alpha}_2 \tilde{\in} \mathbb{R}(A)$, $|\tilde{\alpha}_1| \tilde{\leq} |\tilde{\alpha}_2|$ and either $\tilde{\alpha}_1, \tilde{\alpha}_2 \neq \bar{0}$ or $|\tilde{\alpha}_1|$, $|\tilde{\alpha}_2| = \bar{0}$, then $|\tilde{\alpha}_1|\, (F,A) \tilde{\subseteq} |\tilde{\alpha}_2|(F,A)$.

$(\boldsymbol{iv})$ (F,A) is a soft convex (soft balanced) set if and only if for all $e \in A$, $F(e)$ is convex (balanced) set of X.

Remark (2.1.9):

Let $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ are two SLS's over $\mathbb{R}(A)$ and $\tilde{f}: SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{Y}_A)$ is a soft linear mapping. Then:

$(\boldsymbol{i})$ If (F,A) is a soft convex (soft balanced and soft linear subspace) of $SL(\tilde{X}_A)$, then $\tilde{f}((F,A))$ is a soft convex (soft balanced and soft linear subspace) of $SL(\tilde{Y}_A)$.

If $\tilde{f}$ is a soft surjective , then the condition is valid , when (F,A) be a soft absorbing set.

$(\boldsymbol{ii})$ If (G, A) is a soft convex (soft balanced and soft linear subspace and soft absorbing) of $SL(\tilde{Y}_A)$, then $\tilde{f}^{-1}((G, A))$ is a soft convex (soft balanced and soft linear subspace and soft absorbing) of $SL(\tilde{X}_A)$.

Definition (2.1.10):

Let X be a linear space over $\mathbb{R}$, $A = \mathbb{R}$ be a parameter set . Then X with the soft topology $\tilde{\tau}$ is said to be a linear soft topology (for short LST). A triple $(\tilde{X}_A, \tilde{\tau}, A)$ is said to be a soft tological linear space (for short $STLS$) over $\mathbb{R}(A)$, if $SL(\tilde{X}_A)$ provided with $\tilde{\tau}$ having the following properties:

$(\boldsymbol{i})$ For $P_e^x \tilde{\in} SL(\tilde{X}_A)$, then $\{P_e^x\}$ is soft closed set.

$(\boldsymbol{ii})$ The soft mappings:

$\tilde{+}: SL(\tilde{X}_A) \tilde{\times} SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{X}_A)$; $\tilde{+}(P_e^x, P_{e'}^y) = P_e^x \tilde{+} P_{e'}^y$ (soft addition).

$\tilde{\cdot}: \mathbb{R}(A) \tilde{\times} SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{X}_A)$; $\tilde{\cdot}(\tilde{k}, P_e^x) = \tilde{k} \tilde{\cdot} P_e^x$

(soft scalar multiplication).

are soft continuous mappings , for all $P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A)$ and $\tilde{k} \tilde{\in} \mathbb{R}(A)$.

In the sense that , for any soft nbhd (W, A) of $P_e^x \tilde{+} P_{e'}^y$, there are soft nbhds (V_1, A) and (V_2, A) of P_e^x and $P_{e'}^y$ respectively such that :

$(V_1, A) \tilde{+} (V_2, A) \tilde{\subseteq} (W, A)$.

And for any soft nbhd (U, A) of $\tilde{k} P_e^x$, there are soft nbhds (U_1, A) of $\tilde{k}$ of $(\mathbb{R}(A), \tilde{\tau}_U, A)$ and (U_2, A) of P_e^x of $(\tilde{X}_A, \tilde{\tau}, A)$ such that:

$(U_1, A) \tilde{\cdot} (U_2, A) \tilde{\subseteq} (U, A)$.

By definition of $\tilde{\tau}_U$, there is $\tilde{r} \tilde{>} \bar{0}$ such that $\tilde{B}(\tilde{k}, \tilde{r})$ is a soft open ball centred at $\tilde{k}$.

Let $S\{\tilde{B}(\tilde{k}, \tilde{r})\} = (G, A)$ be a soft set contains of all soft real numbers which is satifay the condition: $|\tilde{s} \tilde{-} \tilde{k}| \tilde{<} \tilde{r}$ for all $\tilde{s} \tilde{\in} \mathbb{R}(A)$.

Then $(G, A) \tilde{\cdot} (U_2, A) \tilde{\subseteq} (U, A)$.

Thus $\tilde{\cdot}: \mathbb{R}(A) \tilde{\times} SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{X}_A)$ is soft continuous if and only if there is $\tilde{r} \tilde{>} \bar{0}$ such that $\tilde{s} P_e^x \tilde{\in} (U, A)$, for all $\tilde{s} \tilde{\in} \mathbb{R}(A)$ with $|\tilde{s} \tilde{-} \tilde{k}| \tilde{<} \tilde{r}$ and for all $P_e^x \tilde{\in} (U_2, A)$.

Example (2.1.11):

($\boldsymbol{i}$) The $SL(\tilde{X}_A)$ with the soft indiscrete topology is not $STLS$ (since its not satifaying ($\boldsymbol{i}$) from definition (2.1.10)).

($\boldsymbol{ii}$) The $SL(\tilde{X}_A)$ with the soft discrete topology is not $STLS$, as follows:

If $(\tilde{X}_A, \tilde{\tau}_{dis}, A)$ is a $STLS$, where $\tilde{\tau}_{dis}$ is a soft discrete topology unless $SL(\tilde{X}_A) = \{P_0^{\Theta}\}$.

Assume by a contradiction that it is a $STLS$, there is $P_0^{\Theta} \neq P_e^x$ in $SL(\tilde{X}_A)$. A soft sequence $\tilde{\alpha}_n = \overline{(\frac{1}{n})} \tilde{\to} \bar{0}$ in $\mathbb{R}(A)$, since the soft scaler multiplication is soft continuous , from definition (2.1.10.ii) , thus $\tilde{\alpha}_n\, P_e^x \tilde{\to} P_0^{\Theta}$ for every soft nbhd (U, A) of P_0^{Θ} of $(\tilde{X}_A, \tilde{\tau}, A)$.

Then there is $m \in \mathbb{N}$ such that $\tilde{\alpha}_n\, P_e^x \tilde{\in} (U, A)$ for all $n \geq m$. If we take $(U, A) = \{P_0^{\Theta}\}$ is a soft open nbhd of P_0^{Θ} of $(\tilde{X}_A, \tilde{\tau}, A)$. Hence $\tilde{\alpha}_n\, P_e^x = P_0^{\Theta}$, implies that $P_e^x = P_0^{\Theta}$, contradiction.

Theorem (2.1.12):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. For $P_e^{x_0} \tilde{\in} SL(\tilde{X}_A)$ and $\bar{0} \neq \tilde{k} \tilde{\in} \mathbb{R}(A)$. Then:

($\boldsymbol{i}$) The soft translation ; $\tilde{T}_{P_e^{x_0}} : SL(\tilde{X}_A) \tilde{\to} SL(\tilde{X}_A)$, $\tilde{T}_{P_e^{x_0}}(P_{e'}^x) = P_e^{x_0} \tilde{+} P_{e'}^x$;

($\boldsymbol{ii}$) The soft multiplication ; $\tilde{M}_{\tilde{k}} : SL(\tilde{X}_A) \tilde{\to} SL(\tilde{X}_A)$, $\tilde{M}_{\tilde{k}}(P_e^x) = \tilde{k} \tilde{.} P_e^x$;

are soft homeomrphism for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$.

Proof:

It is clearl that, $\tilde{T}_{P_e^{x_0}}$ and $\tilde{M}_{\tilde{k}}$ are soft bijective mapping on $SL(\tilde{X}_A)$, thus inverse exists. In fact $(\tilde{T}_{P_e^{x_0}})^{-1} = \tilde{T}_{-P_e^{x_0}}$ and $(\tilde{M}_{\tilde{k}})^{-1} = \tilde{M}_{\tilde{k}^{-1}}$. Consider the soft mapping $\tilde{T}_{P_e^{x_0}}$ take a soft nbhd (U, A) of $\tilde{T}_{P_e^{x_0}}(P_{e'}^x) = P_e^{x_0} \tilde{+} P_{e'}^x$. Then by soft continuouty of soft addition at $(P_{e'}^x, P_e^{x_0})$, there are soft nbhds (U_1, A) and (U_2, A) of $P_{e'}^x$ and $P_e^{x_0}$ respectively such that:

$(U_1, A) \tilde{+} (U_2, A) \tilde{\subseteq} (U, A)$.

Thus in particular, $(U_1, A) \tilde{+} P_e^{x_0} \tilde{\subseteq} (U, A)$, i.e. , $\tilde{T}_{P_e^{x_0}}((U_1, A)) \tilde{\subseteq} (U, A)$. So, $\tilde{T}_{P_e^{x_0}}$ is soft continuous .

Similarly , $(\tilde{T}_{P_e^{x_0}})^{-1}$ is soft contiuous . Hence $\tilde{T}_{P_e^{x_0}}$ is soft homeomorphism.

By the soft continuty of soft scalar multiplication at $(\tilde{k}\ , P_e^x)$, then for any soft nbhd (W,A) of $\tilde{k}\tilde{.}P_e^x$, there is $\tilde{r}\ \tilde{>}\ \bar{0}$ and soft nbhd (U,A) of P_e^x such that $\tilde{s}\ \tilde{\cdot}\ P_e^y\ \tilde{\in}\ (W,A)$ for all $\tilde{s}\ \tilde{\in}\ \mathbb{R}(A)$ with $|\tilde{s}\tilde{-}\tilde{k}|\ \tilde{<}\ \tilde{r}$ and for all $P_e^y\ \tilde{\in}\ (U,A)$.

In particular, $\tilde{k}\tilde{.}(U,A)\ \tilde{\subseteq}\ (W,A)$ i.e. $\widetilde{M}_{\tilde{k}}(U,A)\ \tilde{\subseteq}\ (W,A)$, thus $\widetilde{M}_{\tilde{k}}$ is soft contionuous. Since $(\widetilde{M}_{\tilde{k}})^{-1} = \widetilde{M}_{\tilde{k}^{-1}}$, it follows that $\widetilde{M}_{\tilde{k}^{-1}}$ is also soft continuous. So $\widetilde{M}_{\tilde{k}}$ is soft homeomorphism.

Corollary (2.1.13):

Let (F,A) be a soft set of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. For $\bar{0} \neq \tilde{\lambda}\ \tilde{\in}\ \mathbb{R}(A)$, we have:

$(\boldsymbol{i})$ $\tilde{\lambda}\ \widetilde{int}(F,A) = \widetilde{int}\tilde{\lambda}\,(F,A)$.

$(\boldsymbol{ii})$ $\tilde{\lambda}\ \widetilde{cl}(F,A) = \widetilde{cl}\tilde{\lambda}\,(F,A)$.

Proof:

$(\boldsymbol{i})$By using theorems (2.1.12.ii)) , (1.2.30.i) and (1.2.31).

$(\boldsymbol{ii})$ By using theorems (2.1.12.ii)) , (1.2.27) and (1.2.30.ii).

Theorem (2.1.14):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. For $P_e^x\ \tilde{\in}\ SL(\tilde{X}_A)$ and $\bar{0} \neq \tilde{t}\ \tilde{\in}\ \mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ (F,A) is a soft open if and only if $P_e^x\tilde{+}(F,A)$ is a soft open.

$$\mathcal{N}_{P_e^x} = P_e^x\tilde{+}\mathcal{N}_{P_0^\Theta}\ .$$

$(\boldsymbol{ii})$$(F,A)$ is a soft open if and only if $\tilde{t}\,\tilde{\cdot}\,(F,A)$ is a soft open for any $\bar{0} \neq \tilde{t}\ \tilde{\in}\ \mathbb{R}(A)$.

$(\boldsymbol{iii})$ A soft set (F,A) is soft openif and only if for all $P_e^x\ \tilde{\in}\ (F,A)$, there a soft open nbhd (V,A) of P_0^Θ such that: $P_e^x\tilde{+}(V,A)\ \tilde{\subseteq}\ (F,A)$.

Proof:

Proof $(\boldsymbol{i})$ and $(\boldsymbol{ii})$ we get them directly from theorem (2.1.12.i) and (2.1.12.ii) respectively .

$(\boldsymbol{iii})$ Suppose that (F,A) be a soft open set of $SL(\tilde{X}_A)$. Then from theorem (1.2.6.iii) , and $(\boldsymbol{i})$, this completes the proof.

Conversely, directly deduce from a theorem (1.2.6.ii).

Definition (2.1.15) [31]:

A collection $\widetilde{\mathcal{B}}_{P_e^x}$ of soft nbhds of P_e^x of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is called a soft nbhd base (or soft local base) at P_e^x , if for any soft nbhd (G,A) of P_e^x , there is $(F,A) \tilde{\in} \widetilde{\mathcal{B}}_{P_e^x}$ such that $(F,A) \tilde{\subseteq} (G,A)$.

Theorem (2.1.16) :

In a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, every soft nbhd of P_0^Θ contains a soft balanced nbhd of P_0^Θ.

Proof:

Let (V,A) be a soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$. By using soft continuity of soft scalar multiplication $\tilde{\cdot}\,(\bar{0},P_0^\Theta) = P_0^\Theta$ at $(\bar{0},P_0^\Theta)$, for every soft nbhd (V,A) of P_0^Θ, there is a soft nbhd (W,A) of P_0^Θ and $\tilde{\delta} \tilde{>} \bar{0}$ such that $\tilde{\alpha} P_e^y \tilde{\in} (V,A)$ for all $|\tilde{\alpha}| \tilde{<} \tilde{\delta}$ and for all $P_e^y \tilde{\in} (W,A)$. Then , $\tilde{\alpha}(W,A) \tilde{\subseteq} (V,A)$ for all $|\tilde{\alpha}| \tilde{<} \tilde{\delta}$.

Let $(W',A) = \tilde{\bigcup}_{|\tilde{\alpha}| \tilde{<} \tilde{\delta}} \tilde{\alpha}(W,A)$. Then , (W',A) is soft nbhd of P_0^Θ.

Let $|\tilde{\beta}| \tilde{\leq} \bar{1}$. Then $\tilde{\delta} \tilde{>} |\tilde{\alpha}\tilde{\beta}|$. Thus $\tilde{\beta}(W',A) = \tilde{\bigcup}_{\tilde{\delta} \tilde{>} |\tilde{\alpha}|} \tilde{\alpha}\tilde{\beta}(W,A) \tilde{\subseteq} (W',A)$. So , (W',A) is soft balanced nbhd of P_0^Θ such that $(W',A) \tilde{\subseteq} (V,A)$.

Remark (2.1.17):

$(\boldsymbol{i})$ If $(F,A) \tilde{\in} \mathcal{N}_{P_0^\Theta}$, then $-\bar{1}(F,A) \tilde{\in} \mathcal{N}_{P_0^\Theta}$.

$(\boldsymbol{ii})$ In a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, every soft nbhd of P_0^Θ contains a soft open , soft balanced nbhd of P_0^Θ.

Theorem (2.1.18) :

In a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, every soft convex nbhd of P_0^Θ contains soft convex , soft balanced nbhd of P_0^Θ .

Proof:

Let (U,A) be a soft convex nbhd of P_0^Θ and let $(V,A) = \tilde{\bigcap}_{|\tilde{\delta}|=\bar{1}}\tilde{\delta}(U,A)$.

From proposition (2.1.8.i), we have $\tilde{\delta}(U,A)$ is soft convex , this implies that (V,A) is soft convex.

It is clear that (V,A) is a soft nbhd of P_0^Θ and $(V,A) \tilde{\subseteq} (U,A)$. To show that (V,A) is soft balanced . Let $|\tilde{\beta}| \tilde{\leq} \bar{1}$. If $\tilde{\beta} = \bar{0}$, then $\tilde{\beta}(V,A) = P_0^\Theta \tilde{\subseteq} (V,A)$.

For $|\tilde{\beta}| \tilde{\leq} \bar{1}$ with $\tilde{\beta} \neq \bar{0}$, then $\tilde{\beta}$ can be expressed as $\tilde{\beta} = \tilde{r}\tilde{\gamma}$, where $\tilde{r} \tilde{\in} (\bar{0},\bar{1}]$ and $|\tilde{\gamma}| = \bar{1}$.

Since $|\tilde{\delta}\tilde{\gamma}| = \bar{1}$, $\tilde{r}\tilde{\gamma}(V,A) = \tilde{\bigcap}_{|\tilde{\delta}|=\bar{1}}\tilde{\delta}\tilde{r}\tilde{\gamma}(U,A) = \tilde{\bigcap}_{|\tilde{\delta}|=\bar{1}}\tilde{r}\tilde{\delta}(U,A)$. Now , let $P_{e'}^y \tilde{\in} \tilde{r}\tilde{\delta}(U,A)$. Since $\tilde{\delta}(U,A)$be a soft convex set containing P_0^Θ, then there is $P_e^x \tilde{\in} \tilde{\delta}(U,A)$ such that:

$P_{e'}^y = \tilde{r}P_e^x = \tilde{r}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{r})P_0^\Theta \tilde{\in} \tilde{\delta}(U,A)$.

Then $\tilde{r}\tilde{\delta}(U,A) \tilde{\subseteq} \tilde{\delta}(U,A)$. Thus $\tilde{\bigcap}_{|\tilde{\delta}|=\bar{1}}\tilde{r}\tilde{\delta}(U,A) \tilde{\subseteq} \tilde{\bigcap}_{|\tilde{\delta}|=\bar{1}}\tilde{\delta}(U,A) = (V,A)$. So (V,A) is soft balanced set.

Theorem (2.1.19) :

In a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ every soft nbhd of P_0^Θ is soft absorbing.

Proof:

Let $P_e^x \tilde{\in} SL(\tilde{X}_A)$ and (V,A) be a soft nbhd of $P_0^\Theta = \bar{0} \tilde{\cdot} P_e^x$ and by soft continuity of soft scalar multiplication, there is $\tilde{r} \tilde{>} \bar{0}$ and soft nbhd (U,A) of P_e^x such that $\tilde{\lambda} \tilde{\cdot} P_{e'}^y \tilde{\in} (V,A)$ for all $\tilde{\lambda} \tilde{\in} \mathbb{R}(A)$ with $|\tilde{\lambda}| \tilde{\leq} \tilde{r}$ and for all $P_{e'}^y \tilde{\in} (U,A)$. i.e. $\tilde{\lambda} \tilde{\cdot} P_e^x \tilde{\in} (V,A)$.

Put $|\tilde{\lambda}| = \tilde{\alpha}$, then $|\tilde{\alpha}| \tilde{\leq} \tilde{r}$. This implies that $\tilde{\alpha} \tilde{\cdot} P_e^x \tilde{\in} (V,A)$ and $\tilde{\alpha} \tilde{>} \bar{0}$. So $P_e^x \tilde{\in} \tilde{\alpha}^{-1}(V,A)$, hence $P_e^x \tilde{\in} \tilde{\beta}(V,A)$ with $\tilde{\beta} = \tilde{\alpha}^{-1} \tilde{>} \bar{0}$.

i.e. $\tilde{\bigcup}_{\tilde{\beta} \tilde{>} \bar{0}}\tilde{\beta}(V,A) = SL(\tilde{X}_A)$. Therefore (V,A) is soft absorbing set.

Theorem (2.1.20):

In a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ every soft nbhd (G,A) of P_0^Θ , there is a soft symmetric soft nbhd (U,A) of P_0^Θ such that $(U,A)\tilde{+}(U,A) \tilde{\subseteq} (G,A)$.

Proof:

Let (W,A)be a soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$. Since $P_0^\Theta = P_0^\Theta \tilde{+} P_0^\Theta$ by using a soft continuity of soft addition of $SL(\tilde{X}_A)$, it follows that there are soft nbhds (F_1,A), (F_2,A) of P_0^Θ such that $(F_1,A)\tilde{+}(F_2,A) \tilde{\subseteq} (W,A)$. Let $(F,A) = (F_1,A)\tilde{\cap}(F_2,A)$. Then (F,A) is soft nbhd of P_0^Θ , also $-\tilde{1}(F,A)$ is soft nbhd of P_0^Θ.

Put $(U,A) = (F,A)\tilde{\cap}[-\tilde{1}(F,A)]$, since $(U,A) = -\tilde{1}(U,A)$, then (U,A) is soft symmetric nbhd of P_0^Θ.

So $(U,A)\tilde{+}(U,A) \tilde{\subseteq} (F,A)\tilde{+}(F,A) \tilde{\subseteq} (F_1,A)\tilde{+}(F_2,A) \tilde{\subseteq} (W,A)$.

Theorem (2.1.21):

In a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, every non-null soft compact and soft closed sets (F,A) , (G,A) respectively with $(F,A)\tilde{\cap}(G,A) = \tilde{\emptyset}_A$, there is a soft open nbhd (V',A) of P_0^Θ such that:

$$[(F,A)\tilde{+}(V',A)]\tilde{\cap}[(G,A)\tilde{+}(V',A)] = \tilde{\emptyset}_A .$$

Proof:

Since $(F,A) \neq \tilde{\emptyset}_A$, the there is $P_e^x \tilde{\in} (F,A)$. Since (G,A) be a soft closed and $P_e^x \tilde{\notin} (G,A)$ there is a soft open nbhd (W,A) of P_0^Θ of $SL(\tilde{X}_A)$ such that $P_e^x + (W,A)\tilde{\cap}(G,A) = \tilde{\emptyset}_A$ ...$(*)$. Then from theorem (2.1.20) , there a soft symmetric nbhd (U,A) of P_0^Θ such that $(U,A)\tilde{+}(U,A) \tilde{\subseteq} (W,A)$. Using theorem (2.1.20) again, there is a soft symmetric nbhd (V,A) of P_0^Θ such that $(V,A)\tilde{+}(V,A) \tilde{\subseteq} (U,A)$. Since $(V,A) \tilde{\subseteq} (U,A)$, then:

$(V,A)\tilde{+}(V,A)\tilde{+}(V,A) \tilde{\subseteq} (U,A)\tilde{+}(U,A) \tilde{\subseteq} (W,A)$. Now, from $(*)$ above, we have$[P_e^x\tilde{+}(V,A)\tilde{+}(V,A)\tilde{+}(V,A)]\tilde{\cap}(G,A) = \tilde{\emptyset}_A$.

By soft translation, we have $[P_e^x\tilde{+}(V,A)\tilde{+}(V,A)]\tilde{\cap}[(G,A)\tilde{+}(V,A)] = \tilde{\emptyset}_A$. Since (F,A) be a soft compact, then there are $P_{e_1}^{x_1}\, P_{e_2}^{x_2}$, … , $P_{e_n}^{x_n} \tilde{\in} (F,A)$ such that $(F,A) \tilde{\subseteq} \tilde{\cup}_{i=1}^{n}(P_{e_i}^{x_i}\tilde{+}(V_i,A))$, where (V_i,A) be a soft open nbhd of

P_0^Θ for all $i = 1, \dots, n$. Put $(V', A) = \widetilde{\bigcap}_{i=1}^{n}(V_i, A)$, then (V', A) is soft open nbhd of P_0^Θ.

$(F,A)\widetilde{+}(V',A) \widetilde{\sqsubseteq} \widetilde{\bigcup}_{i=1}^{n}(P_{e_i}^{x_i}\widetilde{+}(V_i,A)\widetilde{+}(V',A)) \widetilde{\sqsubseteq} \widetilde{\bigcup}_{i=1}^{n}(P_{e_i}^{x_i}\widetilde{+}(V_i,A)\widetilde{+}(V_i,A))$.

Since $[P_{e_i}^{x_i}\widetilde{+}(V_i,A)\widetilde{+}(V_i,A)]\widetilde{\sqcap}[(G,A)\widetilde{+}(V_i,A)] = \widetilde{\emptyset}_A$ for all $i = 1, \dots, n$ and $(G,A)\widetilde{+}(V',A) \widetilde{\sqsubseteq} (G,A)\widetilde{+}(V_i,A)$ for all $i = 1, \dots, n$.

Then $\widetilde{\bigcup}_{i=1}^{n}(P_{e_i}^{x_i}\widetilde{+}(V_i,A)\widetilde{+}(V_i,A))\widetilde{\sqcap}(G,A)\widetilde{+}(V',A) = \widetilde{\emptyset}_A$, this implies that:

$$[(F,A)\widetilde{+}(V',A)]\widetilde{\sqcap}[(G,A)\widetilde{+}(V',A)] = \widetilde{\emptyset}_A.$$

Theorem (2.1.22) :

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. Then:

(***i***) For any soft sets (F,A) and (G,A) the following is true:

$$\widetilde{cl}(F,A)\widetilde{+}\widetilde{cl}(G,A) \widetilde{\sqsubseteq} \widetilde{cl}[(F,A)\widetilde{+}(G,A)].$$

(***ii***) For every soft set (F,A) in $(\tilde{X}_A, \tilde{\tau}, A)$, then:

$$\widetilde{cl}(F,A) = \widetilde{\bigcap}_{(U,A)\widetilde{\in}\mathcal{N}_{P_0^\Theta}}((F,A)\widetilde{+}(U,A)).$$

Proof:

(***i***) Let $P_e^x \widetilde{\in} \widetilde{cl}(F,A)$ and $P_{e'}^y \widetilde{\in} \widetilde{cl}(G,A)$. By using theorem (1.2.37.i), there are soft nets $(P_{e_d}^{x_d})_{d\in D}$ and $(P_{e'_d}^{y_d})_{d\in D}$, such that $P_{e_d}^{x_d} \widetilde{\rightarrow} P_e^x$ and $P_{e'_d}^{y_d} \widetilde{\rightarrow} P_{e'}^y$.

By soft continuity of soft addition, it follows that $P_{e_d}^{x_d}\widetilde{+}P_{e'_d}^{y_d} \widetilde{\rightarrow} P_e^x\widetilde{+}P_{e'}^y$. Hence $P_e^x\widetilde{+}P_{e'}^y \widetilde{\in} \widetilde{cl}[(F,A)\widetilde{+}(G,A)]$, which proves (i).

(***ii***) Suppose that $P_e^x \widetilde{\in} \widetilde{cl}(F,A)$, then for every $(U,A) \widetilde{\in} \mathcal{N}_{P_0^\Theta}$, we have :

$$P_e^x\widetilde{+}(U,A)\widetilde{\sqcap}(F,A) \neq \widetilde{\emptyset}_A.$$

i.e. $P_e^x \widetilde{\in} (F,A)\widetilde{+}[-\tilde{1}(U,A)]$.

Thus $P_e^x \widetilde{\in} \widetilde{\bigcap}_{(U,A)\widetilde{\in}\mathcal{N}_{P_0^\Theta}}((F,A)\widetilde{-}(U,A)) = \widetilde{\bigcap}_{(U,A)\widetilde{\in}\mathcal{N}_{P_0^\Theta}}((F,A)\widetilde{+}(U,A))$.

Conversely, suppose that $P_e^x \widetilde{\notin} \widetilde{cl}(F,A)$. Then there is $(U,A) \widetilde{\in} \mathcal{N}_{P_0^\Theta}$ such that $P_e^x\widetilde{+}(U,A)\widetilde{\sqcap}(F,A) = \widetilde{\emptyset}_A$.

i.e. $P_e^x \widetilde{\notin} (F,A)\widetilde{+}[-\tilde{1}(U,A)]$, hence $P_e^x \widetilde{\notin} \widetilde{\bigcap}_{(U,A)\widetilde{\in}\mathcal{N}_{P_0^\Theta}}(F,A)\widetilde{+}(U,A)$.

Theorem (2.1.23) :

For any $(F,A) \tilde{\in} \mathcal{N}_{P_0^\Theta}$ of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, there is a soft nbhd (G,A) of P_0^Θ such that $\widetilde{cl}(G,A) \tilde{\subseteq} (F,A)$.

Proof:

Let $(G_1,A) = \{P_0^\Theta\}$ is a soft compact set and (H,A) is a soft open nbhd of P_0^Θ of $SL(\tilde{X}_A)$ with $(H,A) \tilde{\subseteq} (F,A)$ and $(G_2,A) = (H,A)^c$.

By using theorem (2.1.21), there is a soft open nbhd (V,A) of P_0^Θ such that

$$[(G_1,A)\tilde{\mp}(V,A)]\tilde{\cap}[(G_2,A)\tilde{\mp}(V,A)] = \tilde{\emptyset}_A.$$

It follows that $(V,A) \tilde{\subseteq} [(G_2,A)\tilde{\mp}(V,A)]^c \tilde{\subseteq} (H,A) \tilde{\subseteq} (F,A)$.

By the definition (2.1.15) , there is a soft nbhd (G,A) of P_0^Θ such that:

$$(G,A) \tilde{\subseteq} (V,A) \tilde{\subseteq} [(G_2,A)\tilde{\mp}(V,A)]^c \tilde{\subseteq} (H,A) \tilde{\subseteq} (F,A).$$

Since $[(G_2,A)\tilde{\mp}(V,A)]^c$ is soft closed ,

$$\widetilde{cl}(G,A) \tilde{\subseteq} [(G_2,A)\tilde{\mp}(V,A)]^c \tilde{\subseteq} (H,A) \tilde{\subseteq} (F,A)\ .$$

Definition (2.1.24) [2][3]:

$(\boldsymbol{i})$ Let G be a group and A be a set of parameterse and then (F,A) is called soft group over G , if $F(e)$ is subgroup of G for all $e \in A$.

$(\boldsymbol{ii})$ Let (F,A) be a soft group over G and $(H,B) \tilde{\subseteq} (F,A)$, then (H,B) is called soft subgroup of (F,A) and written as $(H,B) \tilde{<} (F,A)$, if:

- $B \tilde{\subseteq} A$.
- $H(e)$ is a subgroup of $F(e)$ for all $e \in A$.

$(\boldsymbol{iii})$ Let (F,A) be a soft group over G and $(H,B) \tilde{<} (F,A)$. Let $a \in \bigcap_{e\in A} F(e)$. Then the soft set (H_a,B) defined as ($H_a(b) = (H(b))_a$ for all $b \in B$) is called soft right coset of (H,B) in (F,A) generated by a. Similarly, the soft set $({}_aH,B)$ is called a soft left coset of (H,B) in (F,A) generated by a.

Theorem (2.1.25) [3]:

Let (F,A) be a soft group over G and $(H,B) \tilde{<} (F,A)$. Then:

$\{({}_aH,B) : a \in \bigcap_{e\in A} F(e)\ \}$ forms a partion of (F,A) over G.

Remark (2.1.26):

Let (F, A) be a soft group over G and $(H, B) \tilde{\lesssim} (F, A)$. Then:

$$\{P_e^x \tilde{+} (H, B) : P_e^x \tilde{\in} (F, A)\}$$

forms a partion of (F, A) over G. i.e. $(F, A) = \tilde{\bigcup}\{P_e^x \tilde{+} (H, B) : P_e^x \tilde{\in} (F, A)\}$.

Definition (2.1.27) [20]:

Let (F, A) be a soft group over a universe X and $\tilde{\tau}$ be a ST on $\tilde{X}_A$. Then $(\tilde{X}_A, \tilde{\tau}, A)$ is called a soft topological soft group (for short $STSG$) on X, if for all $e \in A$, $(F(e), \tau^e)$ is a topological group on $F(e)$ where is the relative topology on $F(e)$ induced from τ^e as in [19].

Theorem (2.1.28):

Every $STLS$ is $STSG$.

Proof: Clear.

Theorem (2.1.29):

Every soft convex and soft balanced subset (H, A) of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft subgroup.

Proof:

By using theorem (2.1.8.iv) and definition (2.1.24.ii).

Theorem (2.1.30):

Let (F, A) be a soft open subgroup of a $STSG$ $(\tilde{X}_A, \tilde{\tau}, A)$. Then (F, A) soft closed.

Proof:

We have that $\tilde{X}_A$ equales the union of all soft left cosets of (F, A).

i.e. $\tilde{X}_A = \tilde{\bigcup}\{P_e^x \tilde{+} (F, A) : P_e^x \tilde{\in} \tilde{X}_A\}$. Each left soft coset is $P_e^x \tilde{+} (F, A)$ soft open set in $\tilde{X}_A$, and since:

$(F, A) = \tilde{X}_A \tilde{\setminus} \tilde{\bigcup}\{P_e^x \tilde{+} (F, A) : P_e^x \tilde{\in} \tilde{X}_A \tilde{\setminus} (F, A)\}$, we see that (F, A) is soft closed set in $\tilde{X}_A$.

§ (2.2) Theories related to $STLS$:

Throughout this section, we gave some core results which were not mentioned in the sources adopted in this work and its importance in next chapter.

Theorem (2.2.1):

Let (F,A) be a soft convex subset of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. If $P_e^x \tilde{\in} \widetilde{int}(F,A)$ and $P_e^y \tilde{\in} \widetilde{cl}(F,A)$, then $\tilde{\lambda}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})P_e^y \tilde{\in} \widetilde{int}(F,A)$, for all $\bar{0} \tilde{<} \tilde{\lambda} \tilde{<} \bar{1}$.

Proof:

Let $\bar{0} \tilde{<} \tilde{\lambda} \tilde{<} \bar{1}$, we have to show that $\tilde{\lambda}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})P_e^y \tilde{\in} \widetilde{int}(F,A)$. By soft translation if necessary, we can arrange that:

$$\tilde{\lambda}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})P_e^y = P_0^\Theta.$$

Now, $P_e^y = \frac{-\tilde{\lambda}}{(\bar{1} \tilde{-} \tilde{\lambda})} P_e^x = \tilde{\alpha} P_e^x$ with $\tilde{\alpha} \tilde{<} \bar{0}$. Clear that $\tilde{\alpha}\widetilde{int}(F,A)$ is soft open nbhd of P_e^y. Since $P_e^y \tilde{\in} \widetilde{cl}(F,A)$, then $(F,A)\tilde{\sqcap}(\tilde{\alpha}\widetilde{int}(F,A)) \neq \tilde{\emptyset}_A$, and there is $P_e^{w_0} \tilde{\in} (F,A)\tilde{\sqcap}(\tilde{\alpha}\widetilde{int}(F,A))$, with $P_e^{w_0} = \tilde{\alpha}P_e^z$ and $P_e^z \tilde{\in} \widetilde{int}(F,A)$.

Clearl that $P_e^{w_0} \tilde{\in} (F,A)$, put $\tilde{\beta} = \frac{\tilde{\alpha}}{(\tilde{\alpha} \tilde{-} \bar{1})}$, then:

$$\tilde{\beta}P_e^z \tilde{+} (\bar{1} \tilde{-} \tilde{\beta})\tilde{\alpha}P_e^z = P_0^\Theta \text{ ......}(*), \bar{0} \tilde{<} \tilde{\beta} \tilde{<} \bar{1}.$$

Define a soft mapping $\tilde{f}: SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{X}_A)$ as follows:

$\tilde{f}(P_e^w) = \tilde{\beta}P_e^w \tilde{+} (\bar{1} \tilde{-} \tilde{\beta})\tilde{\alpha}P_e^z$ for all $P_e^w \tilde{\in} SL(\tilde{X})$ is soft homeomorphism and $\tilde{f}(P_e^z) = P_0^\Theta$, from $(*)$.

Then $(U,A) = \{\tilde{\beta}P_e^w \tilde{+} (\bar{1} \tilde{-} \tilde{\beta})\tilde{\alpha}P_e^z : P_e^w \tilde{\in} \widetilde{int}(F,A)\}$ is soft open nbhd of P_0^Θ. Since $P_e^w \tilde{\in} \widetilde{int}(F,A)$ and $\tilde{\alpha}P_e^z \tilde{\in} (F,A)$ implies that $(U,A) \tilde{\sqsubseteq} (F,A)$ (because (F,A) be a soft convex set), thus $P_0^\Theta \tilde{\in} \widetilde{int}(F,A)$ and hence: $\tilde{\lambda}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})P_e^y \tilde{\in} \widetilde{int}(F,A)$.

Theorem (2.2.2):

Let (F,A) be a soft subset of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$.

$(\boldsymbol{i})$ If (F,A) is a soft convex set , then $\widetilde{int}(F,A)$ is soft convex.

$(\boldsymbol{ii})$ If (F,A) is a soft balanced set and $P_0^{\Theta} \tilde{\in} \widetilde{int}(F,A)$, then $\widetilde{int}(F,A)$ is soft balanced.

Proof:

$(\boldsymbol{i})$ Let $\bar{0} \tilde{<} \tilde{\lambda} \tilde{<} \bar{1}$ and $P_e^x, P_e^y \tilde{\in} \widetilde{int}(F,A)$, since $\widetilde{int}(F,A) \tilde{\subseteq} \widetilde{cl}(F,A)$, by using theorem (2.2.1) , implies that:

$$\tilde{\lambda}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})P_e^y \tilde{\in} \widetilde{int}(F,A).$$

Thus $\widetilde{int}(F,A)$ is soft convex.

$(\boldsymbol{ii})$ Assume that (F,A) be a soft balanced and $\tilde{\lambda} \neq \bar{0}$, thus:

$$\tilde{\lambda}(F,A) \tilde{\subseteq} (F,A) \text{ and } \widetilde{int}\,\tilde{\lambda}(F,A) = \tilde{\lambda}\widetilde{int}\,(F,A) \tilde{\subseteq} \widetilde{int}(F,A).$$

This implies that $\widetilde{int}(F,A)$ is soft balanced.

Now, if $\tilde{\lambda} = \tilde{0}$, since $P_0^{\theta} = \bar{0}.\widetilde{int}\,(F,A) \tilde{\subseteq} \widetilde{int}(F,A)$. From hypothesis above $(P_0^{\theta} \tilde{\in} \widetilde{int}(F,A))$, we have $\widetilde{int}(F,A)$ is soft balanced.

Theorem (2.2.3):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ If (F,A) is a soft convex set , then $\widetilde{cl}(F,A)$ is soft convex .

$(\boldsymbol{ii})$ If (F,A) is a soft balanced set then $\widetilde{cl}(F,A)$ is soft balanced.

Proof:

$(\boldsymbol{i})$ Since (F,A) be a soft convex , then:

$\tilde{\lambda}(F,A) \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})(F,A) \tilde{\subseteq} (F,A)$, for all $\bar{0} \tilde{\leq} \tilde{\lambda} \tilde{\leq} \bar{1}$.

Then $\widetilde{cl}[\tilde{\lambda}(F,A) \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})(F,A)] \tilde{\subseteq} \widetilde{cl}(F,A)$ (from soft multiplication), we have:

$$\tilde{\lambda}\,\widetilde{cl}(F,A) \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})\widetilde{cl}(F,A) \tilde{\subseteq} \widetilde{cl}\,[\,\tilde{\lambda}(F,A) \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})(F,A)] \tilde{\subseteq} \widetilde{cl}(F,A)\,.$$

This implies that $\widetilde{cl}(F,A)$ is soft convex.

$(\boldsymbol{ii})$ Let $\tilde{\lambda} \tilde{\in} \mathbb{R}(A)$ such that $|\tilde{\lambda}| \tilde{\leq} \bar{1}$. Since (F, A) be a soft balanced , then:

$$\tilde{\lambda}(F, A) \tilde{\subseteq} (F, A) \Rightarrow \widetilde{cl}\tilde{\lambda}(F, A) \tilde{\subseteq} \widetilde{cl}(F, A).$$

Since $\widetilde{cl}\tilde{\lambda}(F, A) = \tilde{\lambda}\widetilde{cl}(F, A)$ (by using soft multiplication), thus $\widetilde{cl}(F, A)$ is soft balanced.

Proposition (2.2.4):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ If (F, A) is a soft absorbing set then $\tilde{\alpha}(F, A)$, $\tilde{\alpha} \tilde{>} \bar{0}$ is soft absorbing.

$(\boldsymbol{ii})$ If (F, A) is a soft absorbing set , then $\widetilde{cl}(F, A)$ is soft absorbing.

Proof: Clear.

Theorem (2.2.5):

$(\boldsymbol{i})$ Every proper soft linear subspace of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ has a null soft interior.

$(\boldsymbol{ii})$ The only soft open linear subspace of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is $SL(\tilde{X}_A)$ itself.

Proof:

$(\boldsymbol{i})$ Assume that $SL(\tilde{Y}_A)$ is a soft linear proper subspace of $SL(\tilde{X}_A)$ and $\widetilde{int}(SL(\tilde{Y}_A)) \neq \tilde{\emptyset}_A$.

Thus there is a soft open nbhd of P_0^{Θ} (U, A) of $SL(\tilde{Y}_A)$ and there is a soft point $P_e^{x_0}$ such that:

$$P_e^{x_0} \tilde{\in} P_e^{x_0} \tilde{+} (U, A) \tilde{\subseteq} SL(\tilde{Y}_A).$$

By using a soft translation , then $-\bar{1}P_e^{x_0} \tilde{+} [-\bar{1}(U, A)] \tilde{\subseteq} SL(\tilde{Y}_A)$, hence:

$$(U, A) \tilde{\subseteq} (U, A) \tilde{+} [-\bar{1}(U, A)] \tilde{\subseteq} SL(\tilde{Y}_A) \tilde{+} SL(\tilde{Y}_A) = SL(\tilde{Y}_A) .$$

Thus $SL(\tilde{Y}_A)$ is a soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$. Since every soft nbhd of P_0^{Θ} is soft absorbing , i.e. $SL(\tilde{Y}_A) = SL(\tilde{X}_A)$. This is a contradiction.

$(\boldsymbol{ii})$ Assume that $SL(\tilde{Y}_A)$ is a soft open linear subspace of $SL(\tilde{X}_A)$. Then $P_0^{\Theta} \tilde{\in} SL(\tilde{Y}_A)$. Thus $SL(\tilde{Y}_A)$ is a soft nbhd of P_0^{Θ} . By using theorem (2.1.19), $SL(\tilde{Y}_A)$ is soft absorbing. i.e. $\tilde{\bigcup}_{\tilde{\delta} \tilde{>} \bar{0}} \tilde{\delta} SL(\tilde{Y}_A) = SL(\tilde{X}_A)$.

But $SL(\tilde{Y}_A) = \tilde{\bigcup}_{\tilde{\delta} \tilde{>} \bar{0}} \tilde{\delta} SL(\tilde{Y}_A)$, this implies that $SL(\tilde{Y}_A) = SL(\tilde{X}_A)$.

Theorem (2.2.6):

If (F,A) is a soft convex and soft balanced set of $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, then $P_0^{\theta} \tilde{\in} (F,A)$ and $\tilde{\lambda}P_e^x \tilde{+} \tilde{\mu}P_e^y \tilde{\in} (F,A)$, for all $P_e^x, P_e^y \tilde{\in} (F,A)$ and for all $\tilde{\lambda}, \tilde{\mu} \tilde{\in} \mathbb{R}(A)$, where $|\tilde{\lambda}| \tilde{\leq} \bar{1}, |\tilde{\mu}| \tilde{\leq} \bar{1}$.

Proof: Clear.

Theorem (2.2.7):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$, (F,A) be a soft convex and soft balanced subset of $SL(\tilde{X}_A)$ and $\widetilde{int}(F,A) \neq \tilde{\emptyset}_A$. Then (F,A) is soft open and soft closed.

Proof:
Let (U,A) be a non-null soft open subset of (F,A) and $P_e^x \tilde{\in} (F,A)$. By using theorem (2.2.6) $P_e^x \tilde{+} (U,A) \tilde{\subseteq} (F,A)$, clearl that $P_e^x \tilde{+} (U,A)$ is soft open set . From theorem (1.2.6.iii) , (F,A), is soft open set. We conclude that by using theorems (2.1.28) , (2.1.29) and (2.1.30) , (F,A) is soft closed set.

Corollary (2.2.8):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$, (F,A) be a soft convex and soft balanced nbhd of P_0^{θ} of $SL(\tilde{X}_A)$. Then (F,A) is soft open and soft closed.

Proof: Clear.

Theorem (2.2.9):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$, (F,A) is a soft convex and soft balanced set of $SL(\tilde{X}_A)$. Then the following statements are equivalent:

(***i***) $\widetilde{int}(F,A) \neq \tilde{\emptyset}_A$.

(***ii***) $[\bar{0},\bar{1})(F,A) \tilde{\subseteq} \widetilde{int}(F,A)$ and $(F,A) \neq \tilde{\emptyset}_A$.

Proof:
(i → ii) Since $\widetilde{int}(F,A) \neq \tilde{\emptyset}_A$, there is $P_e^y \tilde{\in} \widetilde{int}(F,A)$. Let $P_e^x \tilde{\in} (F,A)$.
Since (F,A) be a soft convex and soft balanced , we have:
$(-\bar{1})P_e^y \tilde{\in} (F,A)$ and $\tilde{\lambda}P_e^x = \tilde{\lambda}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{\lambda})\overline{(\tfrac{1}{2})}P_e^y \tilde{+} \overline{(\tfrac{1}{2})}((-\bar{1})P_e^y))$; such that $\bar{0} \tilde{\leq} \tilde{\lambda} \tilde{<} \bar{1}$.

$$\tilde{\lambda}P_e^x\tilde{+}(\bar{1}\tilde{-}\tilde{\lambda})(\overline{(\tfrac{1}{2})}P_e^y\tilde{+}\overline{(\tfrac{1}{2})}((-\bar{1})P_e^y))$$

$$\tilde{\in}\ \tilde{\lambda}P_e^x\tilde{+}(\bar{1}\tilde{-}\tilde{\lambda})\overline{(\tfrac{1}{2})}((-\bar{1})P_e^y))\tilde{+}\ (\bar{1}\tilde{-}\tilde{\lambda})\overline{(\tfrac{1}{2})}\widetilde{int}(F,A)\ \tilde{\subseteq}\ (F,A)$$

It is clear that $(\bar{1}\tilde{-}\tilde{\lambda})\overline{(\frac{1}{2})}\widetilde{int}(F,A)$ is a soft open .

Since $(\bar{1}\tilde{-}\tilde{\lambda})\overline{(\frac{1}{2})}\ \tilde{>}\ \bar{0}$. By using theorems (2.1.14.i) and (2.1.14.ii), we have: $\tilde{\lambda}P_e^x\tilde{+}(\bar{1}\tilde{-}\tilde{\lambda})\overline{(\frac{1}{2})}((-\bar{1})P_e^y))\tilde{+}\ (\bar{1}\tilde{-}\tilde{\lambda})\overline{(\frac{1}{2})}\widetilde{int}(F,A)$ is soft open set.

Since $\widetilde{int}(F,A)\ \tilde{\subseteq}\ (F,A)$ and $\widetilde{int}(F,A)$ is largest soft open set in (F,A).

Then $\tilde{\lambda}P_e^x\tilde{+}(\bar{1}\tilde{-}\tilde{\lambda})\overline{(\frac{1}{2})}((-\bar{1})P_e^y))\tilde{+}\ (\bar{1}\tilde{-}\tilde{\lambda})\overline{(\frac{1}{2})}\widetilde{int}(F,A)\ \tilde{\subseteq}\ \widetilde{int}(F,A)$, thus:

$$[\bar{0}\ ,\bar{1})(F,A)\ \tilde{\subseteq}\ int(F,A).$$

(ii → i) Since $(F,A)\neq\widetilde{\emptyset}_A$, we have $\widetilde{int}(F,A)\neq\widetilde{\emptyset}_A$.

Theorem (2.2.10):

Let (F,A) be soft set of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ and $P_e^x\ \tilde{\in}\ SL(\tilde{X}_A)$. Then $P_e^x\ \tilde{\in}\ \widetilde{cl}(F,A)$ if and only if there is a soft net $(P_{e_d}^{x_d})_{\ d\in D}$ in (F,A) such that $P_{e_d}^{x_d}\ \tilde{\to}\ P_e^x$.

Proof:

Let $P_e^x\ \tilde{\in}\ \widetilde{cl}(F,A)$ and (U,A) be a soft open nbhd of P_0^{θ} of $SL(\tilde{X}_A)$, then by using theorem (2.1.14.i) , we have $P_e^x\tilde{+}(U,A)$ is a soft open nbhd of P_e^x.

Since $P_e^x\ \tilde{\in}\ \widetilde{cl}(F,A)$, then by using theorem (1.2.6.i) , implies that:

$$[P_e^x\tilde{+}(U,A)]\tilde{\cap}(F,A)\neq\widetilde{\emptyset}_A\ ......\ (*)$$

Now, from $(*)$, there is a soft point $P_{e_{(G,A)}}^{x_{(G,A)}}$ of $[P_e^x\tilde{+}(U,A)]\tilde{\cap}(F,A)$. It is clear that $(\mathcal{N}_{P_e^x},\tilde{\subseteq})$ is directed set.

Define $\tilde{\chi}:\mathcal{N}_{P_e^x}\to(F,A)$ by $\tilde{\chi}((G,A))=P_{e_{(G,A)}}^{x_{(G,A)}}$ for all $(G,A)\ \tilde{\in}\ \mathcal{N}_{P_e^x}$. Hence $(P_{e_{(G,A)}}^{x_{(G,A)}})_{\ (G,A)\tilde{\in}\mathcal{N}_{P_e^x}}$ is a soft net in (F,A).

To prove $P_{e_{(G,A)}}^{x_{(G,A)}}\ \tilde{\to}\ P_e^x$. Let $(H,A)\ \tilde{\in}\ \mathcal{N}_{P_e^x}$. Then $\tilde{\chi}((G,A))\ \tilde{\in}\ (H,A)$ for all $(G,A)\ \tilde{\subseteq}\ (H,A)$, implies that $P_{e_{(G,A)}}^{x_{(G,A)}}\ \tilde{\to}\ P_e^x$.

Conversely, suppose, there is a soft net $(P_{e_d}^{x_d})_{d\in D}$ in (F,A) such that $P_{e_d}^{x_d} \tilde{\rightarrow} P_e^x$. To prove $P_e^x \tilde{\in} \widetilde{cl}(F,A)$ let (U,A) be a soft open nbhd of P_0^{Θ}. Since $P_{e_d}^{x_d} \tilde{\rightarrow} P_e^x$ there is $d_0 \in D$ such that $P_{e_d}^{x_d} \tilde{\in} P_e^x \tilde{+} (U,A)$ for all $d \geq d_0$ But $P_{e_d}^{x_d} \tilde{\in} (F,A)$ for all $d \in D$, so that $[P_e^x \tilde{+} (U,A)] \tilde{\cap} (F,A) \neq \tilde{\emptyset}_A$ for all softopen nbhd (U,A) of P_0^{Θ}.

Thus by using theorem (1.2.6.i) , $P_e^x \tilde{\in} \widetilde{cl}(F,A)$.

Theorem (2.2.11):

A soft closed , soft balanced nbhds of P_0^{Θ} forms a soft local base at P_0^{Θ} of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$.

Proof:

Let (F,A) be a soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$, by using theorem (2.1.20) , there is a soft symmetric nbhd (G,A) of P_0^{Θ} such that:

$$(G,A) \tilde{+} (G,A) \tilde{\subseteq} (F,A).$$

Also, by using theorem (2.1.16) , there is a soft balanced nbhd (H,A) of P_0^{Θ} such that $(H,A) \tilde{\subseteq} (G,A)$, thus $(H,A) \tilde{+} (H,A) \tilde{\subseteq} (F,A)$. Then by using theorem (2.1.22.ii), implies that: $\widetilde{cl}(H,A) \tilde{\subseteq} (H,A) \tilde{+} (H,A) \tilde{\subseteq} (F,A)$, where $\widetilde{cl}(H,A)$ is a soft nbhd of P_0^{Θ} . By using theorem (2.2.3 .ii) $\widetilde{cl}(H,A)$ is a soft balanced.

Theorem (2.2.12):

Every $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft Hausdorff space.

Proof:

Let $P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A)$ with $P_e^x \neq P_{e'}^y$. Both soft sets $\{P_e^x\}$ and $\{P_{e'}^y\}$ are soft closed sets (from definition (2.1.10)) and theorem (1.2.14.iii) are soft compact sets . Then $SL(\tilde{X}_A)$ is soft Hausdorff space by theorem (2.1.21).

§(3.1) Soft topologies induced by a families of soft seminorms:

In this section we introduce the important concept which is called soft seminorm on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ and we give some results on this concept. Also, we study a new type of a soft topology by using a family of soft seminorms denoted by a soft topology which is induced by a family of soft seminorms.

Definition (3.1.1) [29]:

A soft mapping $\|\tilde{\cdot}\| : SL(\tilde{X}_A) \tilde{\rightarrow} \mathbb{R}(A)^*$ is called soft norm on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, if $\|\tilde{\cdot}\|$ satisfies the following conditions:

$(\boldsymbol{i})\|P_e^x\| \tilde{\geq} \bar{0}, \forall\, P_e^x \tilde{\in} SL(\tilde{X}_A)$.

$(\boldsymbol{ii})$ $\|P_e^x\| = \bar{0}$ if and only if $P_e^x = P_0^{\Theta}$, $\forall\, P_e^x \tilde{\in} SL(\tilde{X}_A)$.

$(\boldsymbol{iii})$ $\|\tilde{\alpha} P_e^x\| = |\tilde{\alpha}|\|P_e^x\|$, $\forall\, P_e^x \tilde{\in} SL(\tilde{X}_A)$ and $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$.

$(\boldsymbol{iv})$ $\left\|P_e^x \tilde{+} P_{e'}^y\right\| \tilde{\leq} \|P_e^x\| \tilde{+} \left\|P_{e'}^y\right\|$, $\forall\, P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A)$.

A soft linear space $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ with the soft norm $\|\tilde{\cdot}\|$ is called a soft normed space and it is denoted by $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$. It is clear that every soft normed space is soft metric space.

Remark (3.1.2)[29]:

Let X be a normed space. Then for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$, $\|P_e^x\| = |e| + \|x\|$ is soft norm.

Definition (3.1.3):

Let $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$ be a soft normed linear space and $\tilde{r} \tilde{>} \bar{0}$. We define the followings:

$B(P_e^x, \tilde{r}) = \{\, P_e^{x_0} \tilde{\in} SL(\tilde{X}_A) : \left\|P_e^x \tilde{-} P_e^{x_0}\right\| \tilde{<} \tilde{r} \,\}$ is called a soft open ball with center P_e^x and a radius $\tilde{r}$.

$\bar{B}(P_e^x, \tilde{r}) = \{\, P_e^{x_0} \tilde{\in} SL(\tilde{X}_A) : \left\|P_e^x \tilde{-} P_e^{x_0}\right\| \tilde{\leq} \tilde{r} \,\}$ is called a soft closed ball.

Remark (3.1.4):

($\boldsymbol{i}$) A soft sequence $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$ in a soft normed space $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$ is soft convergent to P_e^x ($P_{e_n}^{x_n} \tilde{\to} P_e^x$) if and only if $\|P_{e_n}^{x_n} \tilde{-} P_e^x\| \tilde{\to} \tilde{0}$, as $n \to \infty$.

($\boldsymbol{ii}$) A soft open ball of a soft normed space $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$ is soft convex, soft balanced and soft absorbing.

Theorem (3.1.5):

Every soft normed space on $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is $STLS$.

Proof:

Let $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$ be a soft normed space and $(P_{e_n}^{x_n}, P_{e_n}^{y_n}) \tilde{\to} (P_e^x, P_e^y)$ be a soft convergent sequence in $SL(\tilde{X}_A) \tilde{\times} SL(\tilde{X}_A)$. This means that $P_{e_n}^{x_n} \tilde{\to} P_e^x$ and $P_{e_n}^{y_n} \tilde{\to} P_e^y$ in $SL(\tilde{X}_A)$, by remark (3.1.4.i), we have $\|P_{e_n}^{x_n} \tilde{-} P_e^x\| \tilde{\to} \tilde{0}$ and $\|P_{e_n}^{y_n} \tilde{-} P_e^y\| \tilde{\to} \tilde{0}$, as $n \to \infty$. Now, to show that $P_{e_n}^{x_n} \tilde{+} P_{e_n}^{y_n} \tilde{\to} P_e^x \tilde{+} P_e^y$:

$\|P_{e_n}^{x_n} \tilde{+} P_{e_n}^{y_n} \tilde{-} (P_e^x \tilde{+} P_e^y)\| \tilde{\leq} \|P_{e_n}^{x_n} \tilde{-} P_e^x\| \tilde{+} \|P_{e_n}^{y_n} \tilde{-} P_e^y\|$, hence soft convergence of $(P_{e_n}^{x_n})_{n\in\mathbb{N}}$ and $(P_{e_n}^{y_n})_{n\in\mathbb{N}}$, we have the desired.

Also, let $(\tilde{t}_n, P_{e_n}^{x_n}) \tilde{\to} (\tilde{t}, P_e^x)$ in $\mathbb{R}(A) \tilde{\times} SL(\tilde{X}_A)$. Thus $\tilde{t}_n \tilde{\to} \tilde{t}$ and $P_{e_n}^{x_n} \tilde{\to} P_e^x$ in $\mathbb{R}(A)$ and $SL(\tilde{X}_A)$ respectively, so $|\tilde{t}_n \tilde{-} \tilde{t}| \tilde{\to} \bar{0}$ and $\|P_{e_n}^{x_n} \tilde{-} P_e^x\| \tilde{\to} \bar{0}$ as $n \to \infty$. To show that $\tilde{t}_n \tilde{\cdot} P_{e_n}^{x_n} \tilde{\to} \tilde{t} \tilde{\cdot} P_e^x$ in $SL(\tilde{X}_A)$, we have:

$$\|\tilde{t}_n \tilde{\cdot} P_{e_n}^{x_n} \tilde{-} \tilde{t} \tilde{\cdot} P_e^x\| = \|\tilde{t}_n \tilde{\cdot} P_{e_n}^{x_n} \tilde{-} \tilde{t}_n \tilde{\cdot} P_e^x \tilde{+} \tilde{t}_n \tilde{\cdot} P_e^x \tilde{-} \tilde{t} \tilde{\cdot} P_e^x\|$$

$\tilde{\leq} \|\tilde{t}_n \tilde{\cdot} P_{e_n}^{x_n} \tilde{-} \tilde{t}_n \tilde{\cdot} P_e^x\| \tilde{+} \|\tilde{t}_n \tilde{\cdot} P_e^x \tilde{-} \tilde{t} \tilde{\cdot} P_e^x\| = |\tilde{t}_n| \|P_{e_n}^{x_n} \tilde{-} P_e^x\| \tilde{+} \|P_e^x\| |\tilde{t}_n \tilde{-} \tilde{t}|$, it is clear that $\|\tilde{t}_n \tilde{\cdot} P_{e_n}^{x_n} \tilde{-} \tilde{t} \tilde{\cdot} P_e^x\| \tilde{\to} \bar{0}$, as $n \to \infty$.

It clear that $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$ is soft Hausdorff space, then $\{P_e^x\}$ is soft closed set for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$. Thus, we have completed the proof.

Definition (3.1.6):

Let $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ be two SLS's over $\mathbb{R}(A)$.

A soft mapping $\tilde{f} : SL(\tilde{X}_A) \tilde{\to} SL(\tilde{Y}_A)$ is called a soft linear, if it is having the following:

$\tilde{f}(\tilde{\alpha} P_e^x \tilde{+} \tilde{\delta} P_{e'}^y) = \tilde{\alpha} \tilde{f}(P_e^x) \tilde{+} \tilde{\delta} \tilde{f}(P_{e'}^y)$; $\forall\, P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A), \forall\, \tilde{\alpha}, \tilde{\delta} \in \mathbb{R}(A)$.

Theorem (3.1.7):

Let $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ be two SLS's over $\mathbb{R}(A)$. If $\tilde{f} : SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{Y}_A)$ be a soft linear, then:

$(\boldsymbol{i})$ $\tilde{f}(P_e^x \tilde{-} P_{e'}^y) = \tilde{f}(P_e^x) \tilde{-} \tilde{f}(P_{e'}^y)$; $\forall$ $P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A)$.

$(\boldsymbol{ii})$ $\tilde{f}(P_0^\Theta) = P_0^\Theta$.

Proof: Is clear.

Definition (3.1.8) [9]:

A soft mapping $\tilde{p}: SL(\tilde{X}_A) \tilde{\rightarrow} \mathbb{R}(A)^*$ is called soft seminorm on $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, if it is having the following:

$(\boldsymbol{i})$ $\tilde{p}(P_e^x \tilde{+} P_{e'}^y) \leq \tilde{p}(P_e^x) \tilde{+} \tilde{p}(P_{e'}^y)$ for all $P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X})$;

$(\boldsymbol{ii})$ $\tilde{p}(\tilde{\alpha} P_e^x) = |\tilde{\alpha}| \tilde{p}(P_e^x)$ for all $P_e^x \tilde{\in} SL(\tilde{X})$ and for all $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$.

Remark (3.1.9) [9]

Let $\tilde{p}$ be a soft seminorm on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ If $P_e^x = P_0^\Theta$, then $\tilde{p}(P_e^x) = \bar{0}$.

$(\boldsymbol{ii})$ $|\tilde{p}(P_e^x) \tilde{-} \tilde{p}(P_{e'}^y)| \tilde{\leq} \tilde{p}(P_e^x \tilde{-} P_{e'}^y)$, for all $P_e^x, P_{e'}^y \tilde{\in} SL(\tilde{X}_A)$.

$(\boldsymbol{iii})$ $\tilde{p}(P_e^x) \geq \bar{0}$; for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$.

Theorem (3.1.10) [9]:

If $\tilde{p}: SL(\tilde{X}_A) \tilde{\rightarrow} \mathbb{R}(A)^*$ be a soft seminorm on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, then:

$(\boldsymbol{i})$ $ker\tilde{p} = \{ P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}(P_e^x) = \bar{0}\}$ is soft linear subspace of $SL(\tilde{X}_A)$.

$(\boldsymbol{ii})$ A soft open unit semiball $B(P_0^\Theta, \tilde{p}; \bar{1}) = \{ P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}(P_e^x) \tilde{<} \bar{1}\}$, is soft convex , soft balanced and soft absorbing.

Theorem (3.1.11):

A soft closed unit semiball $\bar{B}(P_0^\Theta, \tilde{p}; \bar{1}) = \{P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}(P_e^x) \tilde{\leq} \bar{1}\}$, is soft convex, soft balanced and soft absorbing.

Proof: Is clear.

Theorem (3.1.12):

Let $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ be a family of soft seminorms on a $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. For $P_e^x \tilde{\in} SL(\tilde{X}_A)$, let $\mathcal{S}_{P_e^x}$ denote the collection of all soft open semiball of $SL(\tilde{X}_A)$.

Let $\tilde{\tau}_{\mathcal{P}}$ be the collection of all soft sets of $SL(\tilde{X}_A)$ consisting of $\tilde{\emptyset}_A$ together with all those soft sets $(H,A) \tilde{\subseteq} \tilde{X}_A$ such that for any $P_e^x \tilde{\in} (H,A)$, there is $B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$ such that $B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}) \tilde{\subseteq} (H,A)$. Then $\tilde{\tau}_{\mathcal{P}}$ is a soft topology on $\tilde{X}_A$ compatible with $SL(\tilde{X}_A)$.

Proof:

Evidently $\tilde{X}_A \tilde{\in} \tilde{\tau}_{\mathcal{P}}$, and it is clear that the union of any family soft sets of $\tilde{\tau}_{\mathcal{P}}$ is also a soft member of $\tilde{\tau}_{\mathcal{P}}$. We showthat:

if $(G,A)\,,(H,A) \tilde{\in} \tilde{\tau}_{\mathcal{P}}$, then $(G,A)\tilde{\cap}(H,A) \tilde{\in} \tilde{\tau}_{\mathcal{P}}$. If $(G,A)\tilde{\cap}(H,A) = \tilde{\emptyset}_A$, there is no more to be done and so suppose that $P_e^x \tilde{\in} (G,A)\tilde{\cap}(H,A)$. Then $P_e^x \tilde{\in} (G,A)$, $P_e^x \tilde{\in} (H,A)$, and so there are $B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$, $B(P_e^x, \tilde{p'}_1, \dots, \tilde{p'}_m; \tilde{s})$ such that:

$B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}) \tilde{\subseteq} (G,A)$ and $B(P_e^x, \tilde{p'}_1, \dots, \tilde{p'}_m; \tilde{s}) \tilde{\subseteq} (H,A)$.

Put $B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n, \tilde{p'}_1, \dots, \tilde{p'}_m; \tilde{t})$, where $\tilde{t} = \widetilde{min}\{\tilde{r}, \tilde{s}\}$, clearly that:

$$B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n, \tilde{p'}_1, \dots, \tilde{p'}_m; \tilde{t}) \tilde{\subseteq}$$

$$B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}) \tilde{\cap} B(P_e^x, \tilde{p'}_1, \dots, \tilde{p'}_m; \tilde{s}) \tilde{\subseteq} (G,A)\tilde{\cap}(H,A).$$

It follows that $\tilde{\tau}_{\mathcal{P}}$ is soft topology on $\tilde{X}_A$.

Theorem (3.1.13):

A soft topology $\tilde{\tau}_{\mathcal{P}}$ on $SL(\tilde{X}_A)$ is linear soft topology determined by a family $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ of soft seminorms on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$, then $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ is a $STLS$.

Proof:

Suppose $(P_{e_d}^{x_d}, P_{e'_d}^{y_d}) \tilde{\rightrightarrows} (P_e^x, P_{e'}^y)$in $SL(\tilde{X}_A) \tilde{\times} SL(\tilde{X}_A)$. We want to show that

$$P_{e_d}^{x_d} \tilde{+} P_{e'_d}^{y_d} \tilde{\rightrightarrows} P_e^x \tilde{+} P_{e'}^y .$$

Let $B(P_e^x \tilde{+} P_{e'}^y, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m; \tilde{r})$ be a soft basic soft nbhd of $P_e^x \tilde{+} P_{e'}^y$. Since $(P_{e_d}^{x_d}, P_{e'_d}^{y_d}) \tilde{\rightrightarrows} (P_e^x, P_{e'}^y)$, there is d_0 such that:

$$(P_{e_d}^{x_d}, P_{e'_d}^{y_d}) \tilde{\in} B(P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m; \tfrac{\tilde{r}}{\tilde{2}}) \tilde{\times} B(P_{e'}^y, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m; \tfrac{\tilde{r}}{2})$$

whenever $d \geq d_0$. For $i = 1, 2, \dots, m$ and $d \geq d_0$, this implies that:

$\tilde{p}_i\ (P_e^x \tilde{+} P_{e'}^y \tilde{-} (P_{e_d}^{x_d} \tilde{+} P_{e'_d}^{y_d})\ ) \tilde{\leq}\ \tilde{p}_i\ (P_e^x \tilde{-} P_{e_d}^{x_d}) \tilde{+} \tilde{p}_i(P_{e'}^y \tilde{-} P_{e'_d}^{y_d}) \tilde{<} \frac{\tilde{r}}{2} \tilde{+} \frac{\tilde{r}}{2}$; and so $P_{e_d}^{x_d} \tilde{+} P_{e'_d}^{y_d} \tilde{\in} B(P_e^x \tilde{+} P_{e'}^y, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m\ ; \tilde{r})$. Thus we get the required.

Now, suppose that $(\tilde{t}_n, P_{e_d}^{x_d}) \tilde{\rightarrow} (\tilde{t}, P_e^x)$ in $\mathbb{R}(A) \tilde{\times} SL(\tilde{X}_A)$.

Let $B(\tilde{t}\ \tilde{\cdot}\ P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m; \tilde{r})$ be a soft basic nbhd of $\tilde{t}\ \tilde{\cdot}\ P_e^x$. For all $\tilde{\epsilon} \tilde{>} \bar{0}$ and $\tilde{\delta} \tilde{>} \bar{0}$, there is d_0 such that:

$$(\tilde{t}_n, P_{e_d}^{x_d}) \tilde{\in} \{\tilde{t} \tilde{\in} \mathbb{R}(A) : |\tilde{t}_n \tilde{-} \tilde{t}| \tilde{<} \tilde{\epsilon}\} \tilde{\times} B(P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m; \tilde{\delta}).$$

Hence for all $i = 1,2, \dots, m$ and $d \geq d_0$, we have:

$$\tilde{p}_i(\tilde{t}\ \tilde{\cdot}\ P_e^x \tilde{-} \tilde{t}_n\ \tilde{\cdot}\ P_{e_d}^{x_d}) \tilde{\leq} \tilde{p}_i(\tilde{t}\ \tilde{\cdot}\ P_e^x \tilde{-} \tilde{t}_n . P_e^x) \tilde{+} \tilde{p}_i(\tilde{t}_n\ \tilde{\cdot}\ P_e^x \tilde{-} \tilde{t}_n\ \tilde{\cdot}\ P_{e_d}^{x_d})$$

$$\tilde{\leq} |\tilde{t} - \tilde{t}_n|\ \tilde{\cdot}\ \tilde{p}_i(P_e^x) \tilde{+} |\tilde{t}_n|\ \tilde{\cdot}\ \tilde{p}_i(P_e^x \tilde{-} P_{e_d}^{x_d}) \tilde{<} \tilde{\epsilon}\ \tilde{\cdot}\ \tilde{p}_i(P_e^x) + (|\tilde{t}| \tilde{+} \tilde{\varepsilon})\tilde{\delta} \tilde{<} \tilde{r}$$

; if we choose $\tilde{\epsilon} \tilde{>} \bar{0}$ such that $\tilde{\epsilon}\ \tilde{\cdot}\ \tilde{p}_i(P_e^x) \tilde{<} \frac{\tilde{r}}{2}$ and then $\tilde{\delta} \tilde{>} \tilde{0}$ such that $(|\tilde{t}| \tilde{+} \tilde{\varepsilon})\ \tilde{\cdot}\ \tilde{\delta} \tilde{<} \frac{\tilde{r}}{2}$. So that $\tilde{t}_n\ \tilde{\cdot}\ P_{e_d}^{x_d} \tilde{\in} B(\tilde{t}\ \tilde{\cdot}\ P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_m; \tilde{r})$, whenever $d \geq d_0$. We conclude that $\tilde{t}_d\ \tilde{\cdot}\ P_{e_d}^{x_d} \tilde{\rightarrow} \tilde{t}\ \tilde{\cdot}\ P_e^x$.

Now , to show that $\{P_e^x\}$ is soft closed set for all $P_e^x \tilde{\in} SL(\tilde{X}_A)$. Sufficient to show that $\{P_e^x\}^c$ is soft open. From theorem (3.1.12) , for all $P_{e'}^y \tilde{\in} \{P_e^x\}^c$ and $P_e^x \neq P_{e'}^y$ (i.e. $P_e^x \tilde{-} P_{e'}^y \neq P_0^\Theta$). Then $\tilde{p}_i(P_e^x \tilde{-} P_{e'}^y) \tilde{>} \bar{0}$, $i = 1, 2, \dots, n$, let $\widetilde{max}\{\tilde{p}_i(P_e^x \tilde{-} P_{e'}^y)\} = \tilde{r} \tilde{>} \bar{0}$.

Thus $P_e^x \tilde{\notin} B(P_{e'}^y, \tilde{p}_i\ ; \tilde{r})$, i.e. $\{P_e^x\} \tilde{\cap} B(P_{e'}^y, \tilde{p}_i\ ; \tilde{r}) = \tilde{\emptyset}_A$, this implies that $B(P_{e'}^y, \tilde{p}_i\ ; \tilde{r}) \tilde{\subseteq} \{P_e^x\}^c$, and $\{P_e^x\}^c$ is soft open set.

Remark (3.1.14):

Let $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ be a family of soft seminorms on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. We consider the soft open semiball:

$B(P_e^{x_o},\ \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n\ ; \tilde{r}) = \{ P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_i(P_e^x \tilde{-} P_e^{x_o}) \tilde{<} \tilde{r}, i = 1, \dots, n$; $P_e^{x_o} \tilde{\in} SL(\tilde{X}_A), \tilde{r} \tilde{>} \bar{0}\}$.

$\bar{B}(P_e^{x_o},\ \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n\ ; \tilde{r}) = \{ P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_i(P_e^x \tilde{-} P_e^{x_o}) \tilde{\leq} \tilde{r}, i = 1, \dots, n$; $P_e^{x_o} \tilde{\in} SL(\tilde{X}_A), \tilde{r} \tilde{>} \bar{0}\}$.

Notice that:

(***i***) $B(P_e^{x_o}, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n\ ; \tilde{r}) = P_e^{x_o} \tilde{+} B(P_0^\Theta, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n\ ; \tilde{r})$

$= P_e^{x_o} \tilde{+} \tilde{r} B(P_0^\Theta, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n\ ; \bar{1})$.

$(\boldsymbol{ii})$ $\bar{B}(P_e^{x_o}, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r}) = P_e^{x_o} \tilde{+} \bar{B}(P_0^{\Theta}, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r})$

$$= P_e^{x_o} \tilde{+} \tilde{r}\bar{B}(P_0^{\Theta}, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \bar{1}).$$

$(\boldsymbol{iii})$ $B(P_e^{x_o}, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r})$ and $\bar{B}(P_e^{x_o}, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r})$ are soft convex, soft balanced and soft absorbing set.

Theorem (3.1.15):

In a $STLS$ $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$, then:

$(\boldsymbol{i})$ Every soft open semiball is soft open set.

$(\boldsymbol{ii})$ Every soft closed semiball is soft closed set.

Proof:

$(\boldsymbol{i})$ We prove that $B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$ is soft open set.

Let $P_e^z \tilde{\in} B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$, then $\tilde{p}_i(P_e^z \tilde{-} P_e^x) \tilde{<} \tilde{r}$, so $\tilde{r} \tilde{-} \tilde{p}_i(P_e^z \tilde{-} P_e^x) \tilde{>} \bar{0}$.

Put $\tilde{r} \tilde{-} \widetilde{max}\{\tilde{p}_i(P_e^z \tilde{-} P_e^x)\} = \tilde{\delta} \tilde{>} \bar{0}$; $i = 1, 2, \dots, n$.

We must prove that $B(P_e^z, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{\delta}) \tilde{\subseteq} B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$.

Let $P_e^y \tilde{\in} B(P_e^z, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{\delta})$. So $\tilde{p}_i(P_e^y \tilde{-} P_e^z) \tilde{<} \tilde{\delta} = \tilde{r} \tilde{-} \widetilde{max}\{\tilde{p}_i(P_e^z \tilde{-} P_e^x)\}$, and then $\tilde{p}_i(P_e^y \tilde{-} P_e^z) \tilde{+} \widetilde{max}\{\tilde{p}_i(P_e^z \tilde{-} P_e^x)\} \tilde{<} \tilde{r}$. This implies that:

$$\tilde{p}_i(P_e^y \tilde{-} P_e^x) \tilde{\leq} \tilde{p}_i(P_e^y \tilde{-} P_e^z) \tilde{+} \tilde{p}_i(P_e^z \tilde{-} P_e^x) \tilde{<} \tilde{r}.$$

Hence $P_e^y \tilde{\in} B(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$ and its soft open set.

$(\boldsymbol{ii})$ We prove that $\bar{B}(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r})$ is soft closed set. Enough to prove that $(F, A) = (\bar{B}(P_e^x, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}))^c$ is soft open set.

Since $\bar{B}(P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r}) = \{ P_e^{x_o} \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_i(P_e^x \tilde{-} P_e^{x_o}) \tilde{\leq} \tilde{r} \}$.

Then $(F, A) = \{P_e^{x_o} \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_i(P_e^x \tilde{-} P_e^{x_o}) \tilde{>} \tilde{r}\}$. Let $P_e^z \tilde{\in} (F, A)$, then $\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{>} \tilde{r}$, and hence $\widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{-} \tilde{r} \tilde{>} \bar{0}$.

$$\text{Put } \widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{-} \tilde{r} = \tilde{\delta} \tilde{>} \bar{0}.$$

We must prove that $B(P_e^z, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{\delta}) \tilde{\subseteq} (F, A)$.

Let $P_e^y \tilde{\in} B(P_e^z, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{\delta})$. Then $\tilde{p}_i(P_e^y \tilde{-} P_e^z) \tilde{<} \tilde{\delta} = \widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{-} \tilde{r}$ and so $\widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{-} \tilde{p}_i(P_e^y \tilde{-} P_e^z) \tilde{>} \tilde{r}$.

This implies that: $\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{\leq} \tilde{p}_i(P_e^z \tilde{-} P_e^y) \tilde{+} \tilde{p}_i(P_e^y \tilde{-} P_e^{x_o})$

$$\Rightarrow \widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{\leq} \tilde{p}_i(P_e^z \tilde{-} P_e^y) \tilde{+} \tilde{p}_i(P_e^y \tilde{-} P_e^{x_o})$$

$$\Rightarrow \widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{-} \tilde{p}_i(P_e^z \tilde{-} P_e^y) \tilde{\leq} \tilde{p}_i(P_e^y \tilde{-} P_e^{x_o}).$$

But

$$\tilde{r} \tilde{<} \widetilde{min}\tilde{p}_i(P_e^z \tilde{-} P_e^{x_o}) \tilde{-} \tilde{p}_i(P_e^z \tilde{-} P_e^y) \tilde{\leq} \tilde{p}_i(P_e^y \tilde{-} P_e^{x_o}) \Rightarrow \tilde{p}_i(P_e^y \tilde{-} P_e^{x_o}) \tilde{>} \tilde{r}.$$

Thus we have completed the proof.

Remark (3.1.16):

(***i***) $B(P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r})$ and $\bar{B}(P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r})$ are non-null soft sets.

(***ii***) $P_e^x \tilde{\in} \widetilde{int}\,(B(P_e^x, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r}))$.

Theorem (3.1.17):

Let $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ be a family of soft seminorms on a $SLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. For $P_e^x \tilde{\in} SL(\tilde{X}_A)$ a collection $\mathcal{S}_{P_e^x}$, constitute of a soft local base at P_e^x for $\tilde{\tau}_{\mathcal{P}}$ compatible with $SL(\tilde{X}_A)$.

Proof: Clear.

Definition (3.1.18):

A family $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ of soft seminorms on a $SLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is said to be soft separated, if for each $P_0^\Theta \neq P_e^x \tilde{\in} SL(\tilde{X}_A)$ there is $\alpha \in I$ such that $\tilde{p}_\alpha(P_e^x) \neq \bar{0}$.

Theorem (3.1.19):

In a $STLS\ (\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ a family $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ of soft seminorms on a $SLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft separating family.

Proof:

For $P_e^x \tilde{\in} SL(\tilde{X}_A)$ with $P_e^x \neq P_0^\Theta$, there is a soft open set (G, A) such that $P_0^\Theta \tilde{\in} (G, A)$ and $P_e^x \tilde{\notin} (G, A)$ (from soft Hausdorffness of $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$). From theorem (3.1.12) there is soft nbhd $B(P_0^\Theta, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r}) \tilde{\subseteq} (G, A)$. It is clear that $P_e^x \tilde{\notin} B(P_0^\Theta, \tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n; \tilde{r})$.

This implies that there is at least i and $\tilde{p}_i \in \mathcal{P}, i = 1, \dots, n$ and $\tilde{r} \tilde{>} \tilde{0}$ such that $\tilde{p}_i(P_e^x \tilde{-} P_0^\Theta) = \tilde{p}_i(P_e^x) \tilde{\geq} \tilde{r}$ and so certainly $\tilde{p}_i(P_e^x) \tilde{>} \bar{0}$, we see that $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ is soft separating family.

Theorem (3.1.20):

In a $STLS\ (\tilde{X}_A, \tilde{\tau}_{\tilde{p}}, A)$ the soft closure of $\{P_0^\Theta\}$ is the soft intersection of the soft kernels of all soft seminorms of the family $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$.

$\{P_0^\Theta\} = \tilde{\bigcap}_{\alpha \in I}\{P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_\alpha(P_e^x) = P_0^\Theta\ ;\ \tilde{p}_\alpha \in \mathcal{P}\ ; \alpha \in I\} = \tilde{\bigcap}_{\alpha \in I} ker(\tilde{p}_\alpha)$.

Proof:

Since $P_0^{\Theta} \tilde{\in} \tilde{p}_{\alpha}{}^{-1}(\{\bar{0}\}) = ker(\tilde{p}_{\alpha})$ for all $\alpha \in I$, then $P_0^{\Theta} \tilde{\in} \tilde{\bigcap}_{\alpha \in I} ker(\tilde{p}_{\alpha})$.

We have $\{P_0^{\Theta}\} \tilde{\subseteq} \tilde{\bigcap}_{\alpha \in I} ker(\tilde{p}_{\alpha})$ ... (*).

Now , to show that $\tilde{\bigcap}_{\alpha \in I} ker(\tilde{p}_{\alpha}) \tilde{\subseteq} \{P_0^{\Theta}\}$.

Suppose that $P_0^{\Theta} \neq P_e^x \tilde{\in} \tilde{\bigcap}_{\alpha \in I} ker(\tilde{p}_{\alpha})$, then $P_e^x \tilde{\in} ker(\tilde{p}_{\alpha})$ for all $\alpha \in I$. Since $\mathcal{P} = \{\tilde{p}_{\alpha} : \alpha \in I\}$ be a soft separated , theorem (3.1.19), then from definition (3.1.18) , there is $\alpha \in I$ such that $\tilde{p}_{\alpha}(P_e^x) \neq \bar{0}$. This implies that $P_e^x \tilde{\notin} ker(\tilde{p}_{\alpha})$, which is a contradiction , then $\tilde{\bigcap}_{\alpha \in I} ker(\tilde{p}_{\alpha}) \tilde{\subseteq} \{P_0^{\Theta}\}$ ... (**).

Corollary (3.1.21):

In a $STLS$ $(\tilde{X}_A, \tilde{\tau}_{\tilde{p}}, A)$, the soft set $\{P_0^{\Theta}\}$ is the soft linear subspace of $SL(\tilde{X}_A)$.

Proof:

By using theorems (3.1.20) , (3.1.10.i) and (2.1.5.ii).

Theorem (3.1.22):

A soft net $(P_{e_d}^{x_d})_{d \in D}$ in a $STLS$ $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ is soft converging to P_0^{Θ} in a SLS $SL(\tilde{X}_A)$ if and only if $\tilde{p}(P_{e_d}^{x_d}) \tilde{\rightarrow} \bar{0}$, for each $\tilde{p}_{\alpha} \in \mathcal{P} = \{\tilde{p}_{\alpha} : \alpha \in I\}$ where $\mathcal{P}$ be a family of soft continuous soft seminorm on $SL(\tilde{X}_A)$.

Proof:

Assume that $P_{e_d}^{x_d} \tilde{\rightarrow} P_0^{\Theta}$ in $SL(\tilde{X}_A)$, by soft continuity of $\tilde{p}_{\alpha} \in \mathcal{P}$, then we have $\tilde{p}_{\alpha}(P_{e_d}^{x_d}) \tilde{\rightarrow} \tilde{p}_{\alpha}(P_0^{\Theta}) = \bar{0}$.

Conversely, suppose that the condition is holds. Let $\tilde{p}_1, \tilde{p}_2 , \dots , \tilde{p}_n \in \mathcal{P}$ and $\tilde{r} \tilde{>} \bar{0}$. There is d_0 such that $\tilde{p}_i(P_{e_d}^{x_d}) \tilde{<} \tilde{r}$, whenever $d \geq d_0$ for all $i = 1 ,2 , \dots , n$. Hence $P_{e_d}^{x_d} \tilde{\in} B(P_0^{\Theta}, \tilde{p}_1, \tilde{p}_2 , \dots , \tilde{p}_n ; \tilde{r})$, whenever $d \geq d_0$, for all $B(P_0^{\Theta}, \tilde{p}_1, \tilde{p}_2 , \dots , \tilde{p}_n; \tilde{r})$ be a soft nbhd of P_0^{Θ}. It follows the result is obtain.

Remark (3.1.23):

The soft converge of a soft net $(P_{e_d}^{x_d})_{d \in D}$ to P_e^x is not necessarily implied by a soft convergence of $\tilde{p}(P_{e_d}^{x_d}) \tilde{\rightarrow} \tilde{p}(P_e^x)$ in $\mathbb{R}(A)^*$ for all $\tilde{p} \in \mathcal{P}$.

Indeed , for all $P_e^x \neq P_0^\Theta$ and any $\tilde{p} \in \mathcal{P}$, $\tilde{p}((-\bar{1})^n P_e^x) \tilde{\rightarrow} \tilde{p}(P_e^x)$, as $n \rightarrow \infty$, but it is not true that $((-\bar{1})^n P_e^x) \tilde{\rightarrow} P_e^x$ if $SL(\tilde{X}_A)$ soft separated.

Definition (3.1.24):

Let (F, A) be a soft subset of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then (F, A) is said to be a soft bounded, if for every soft nbhd (G, A) of P_0^Θ, there is $\tilde{\gamma} \tilde{>} \bar{0}$ such that $(F, A) \tilde{\subseteq} \tilde{\gamma}(G, A)$.

In other words :

The soft set in a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft bounded, if it is soft absorbed by every soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$.

Proposition (3.1.25):

Let (F, A) and (G, A) be two soft sets in a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ and $P_e^x \tilde{\in} SL(\tilde{X}_A)$. Then:

$(\boldsymbol{i})$ A soft set $\{P_e^x\}$ is soft bounded .

$(\boldsymbol{ii})$ If (F, A) is a soft bounded set and $(G, A) \tilde{\subseteq} (F, A)$, then (G, A) is soft bounded.

$(\boldsymbol{iii})$ If (F, A) is a soft bounded set , then $\tilde{\alpha}(F, A)$ is also for all $\tilde{\alpha} \tilde{\in} \mathbb{R}(A)$.

$(\boldsymbol{iv})$ If (F, A) is a soft bounded set , then $\widetilde{cl}(F, A)$ is soft bounded.

Proof:

$(\boldsymbol{i})$ Let (U, A) be a soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$. Then (U, A) is soft absorbing set , i.e. there is $\tilde{\gamma} \tilde{>} \bar{0}$ such that $P_e^x \tilde{\in} \tilde{\gamma}(U, A)$, that is $\{P_e^x\}$ is soft bounded.

$(\boldsymbol{ii})$ Let (H, A) be a soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$. Since (F, A) is a soft bounded set , then there is $\tilde{\gamma} \tilde{>} \bar{0}$ such that $(F, A) \tilde{\subseteq} \tilde{\gamma}(H, A)$ for every soft nbhd (H, A) of P_0^Θ. Since $(G, A) \tilde{\subseteq} (F, A)$, we have $(G, A) \tilde{\subseteq} \tilde{\gamma}(H, A)$. We easily get that (G, A) is soft bounded.

$(\boldsymbol{iii})$ If $\tilde{\alpha} = \bar{0}$, this follows immediately from $P_0^\Theta = \tilde{\alpha}(F, A) \tilde{\subseteq} \tilde{\gamma}(U, A)$. If $\tilde{\alpha} \neq \bar{0}$, and (H, A) be a soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$. By using theorem (2.1.16) , then there is a soft balanced nbhd (W, A) of P_0^Θ such that:

$$(W, A) \tilde{\subseteq} (H, A).$$

Since (F, A) is a soft bounded set , there is $\tilde{\gamma} \tilde{>} \tilde{0}$ such that:

$$(F, A) \tilde{\subseteq} \tilde{\gamma}(W, A).$$

Put $\tilde{r} = \tilde{\gamma}|\tilde{\alpha}| \Longrightarrow \tilde{r} \tilde{>} \bar{0}$, since (W, A) is a soft balanced, we have:

$$\tilde{\alpha}(W, A) \tilde{\subseteq} |\tilde{\alpha}|(W, A).$$

Implies that $\tilde{\gamma}\tilde{\alpha}(W, A) \tilde{\subseteq} \tilde{\gamma}|\tilde{\alpha}|(W, A)$.

Since $(F,A) \tilde{\subseteq} \tilde{\gamma}(W,A)$, so $\tilde{\alpha}(F,A) \tilde{\subseteq} \tilde{\gamma}\tilde{\alpha}(W,A) \tilde{\subseteq} \tilde{\gamma}|\tilde{\alpha}|(W,A) = \tilde{r}(W,A)$. i.e. $\tilde{r}(W,A) \tilde{\subseteq} \tilde{r}(H,A)$, then $\tilde{\alpha}(F,A) \tilde{\subseteq} \tilde{r}(H,A)$, proving $\tilde{\alpha}(F,A)$ is soft bounded.

$(\boldsymbol{iv})$ Let (H,A) is a soft nbhd of P_0^{Θ} in $SL(\tilde{X}_A)$. Then there is a soft nbhd (W,A) of P_0^{Θ} , and by using theorem (2.1.23) , we get:

$$\widetilde{cl}(W,A) \tilde{\subseteq} (H,A).$$

Since (F,A) is a soft bounded , there is $\tilde{\gamma} \tilde{>} \bar{0}$ such that $(F,A) \tilde{\subseteq} \tilde{\gamma}(W,A)$. By using corollary (2.1.13.ii), we have:

$$\widetilde{cl}(F,A) \tilde{\subseteq} \widetilde{cl}\tilde{\gamma}(W,A) = \tilde{\gamma}\widetilde{cl}(W,A) \tilde{\subseteq} \tilde{\gamma}(H,A).$$

We end up we have reached that $\widetilde{cl}(F,A)$ is soft bounded.

Theorem (3.1.26):

Let (F,A),(G,A) be two soft bounded sets in a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ $(F,A)\tilde{\cap}(G,A)$ is soft bounded.

$(\boldsymbol{ii})$ $(F,A)\tilde{\cup}(G,A)$ is soft bounded.

$(\boldsymbol{iii})$ $(F,A)\tilde{+}(G,A)$ is soft bounded.

Proof:

$(\boldsymbol{i})$ $(F,A)\tilde{\cap}(G,A)$ Produced directly from proposition (3.1.25.ii).

$(\boldsymbol{ii})$ let (H,A) is a soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$, then by using theorem (2.1.16) there is a soft balanced nbhd (W,A) of P_0^{Θ} such that $(W,A) \tilde{\subseteq} (H,A)$. Since (F,A) and (G,A) be two soft bounded, then there are $\tilde{\gamma}_1 , \tilde{\gamma}_2 \tilde{>} \bar{0}$ such that $(F,A) \tilde{\subseteq} \tilde{\gamma}_1(W,A)$ and $(G,A) \tilde{\subseteq} \tilde{\gamma}_2(W,A)$. Take $\tilde{\gamma} \tilde{>} \widetilde{max}\,\{\tilde{\gamma}_1 , \tilde{\gamma}_2\}$. Since (W,A) be a soft balanced , from proposition (2.1.8.iii), $(F,A)\tilde{\cup}(G,A) \tilde{\subseteq} \tilde{\gamma}(W,A) \tilde{\subseteq} \tilde{\gamma}(H,A)$. And so $(F,A)\tilde{\cup}(G,A)$ is soft bounded.

$(\boldsymbol{iii})$ let (H,A) is a soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$, then by using theorem (2.1.20) , there is a soft symmetric nbhd (W,A) of P_0^{Θ} such that:

$$(W,A)\tilde{+}(W,A) \tilde{\subseteq} (H,A).$$

Then there a soft balanced nbhd (U,A) of P_0^{Θ} such that $(U,A) \tilde{\subseteq} (W,A)$. Since (F,A) and (G,A) is two soft bounded sets , then there are $\tilde{\gamma}_1 , \tilde{\gamma}_2 \tilde{>} \bar{0}$ such that $(F,A) \tilde{\subseteq} \tilde{\gamma}_1(U,A)$ and $(G,A) \tilde{\subseteq} \tilde{\gamma}_2(U,A)$.

Take $\tilde{\gamma} \tilde{>} \widetilde{max}\,\{\tilde{\gamma}_1 , \tilde{\gamma}_2\}$. Since (U,A) is a soft balanced , then:

$$\tilde{\gamma}_1(U,A)\tilde{+}\tilde{\gamma}_2(U,A) \tilde{\subseteq} \tilde{\gamma}\,[(U,A)\tilde{+}(U,A)] \tilde{\subseteq} \tilde{\gamma}\,[(W,A)\tilde{+}(W,A)] \tilde{\subseteq} \tilde{\gamma}(H,A).$$

Thus $(F,A)\tilde{+}(G,A)$ is soft bounded.

Theorem (3.1.27):

Every soft compact set of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is soft bounded.

Proof:

Suppose that (F,A) is a soft compact set of $SL(\tilde{X}_A)$ and (V,A) is a soft nbhd of P_0^{Θ} from remark(2.1.17ii) there a soft balanced , soft open nbhd (W,A) of P_0^{Θ} of $SL(\tilde{X}_A)$ for which $(W,A) \tilde{\subseteq} (V,A)$. Then for all $P_e^x \tilde{\in} (F,A)$, $\overline{(\frac{1}{n})}P_e^x \tilde{\rightarrow} P_0^{\Theta}$, since $\overline{(\frac{1}{n})} \tilde{\rightarrow} \bar{0}$ in $\mathbb{R}(A)$, so $\overline{(\frac{1}{n})}P_e^x \tilde{\in} (W,A)$ for all $n \in \mathbb{N}$. That soft compactness, there are $\bar{n}_1 < \bar{n}_2 < \cdots .. < \bar{n}_r$ such that:

$$(F,A) \tilde{\subseteq} \tilde{\cup}_{n=1}^{r} \bar{n}_i (W,A) = \bar{n}_r (W,A).$$

If $\tilde{\lambda} \tilde{>} \bar{n}_r$, This implies that $(F,A) \tilde{\subseteq} \tilde{\lambda}(W,A) \tilde{\subseteq} \tilde{\lambda}(V,A)$. Thus (F,A) is soft bounded set.

Theorem (3.1.28):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. For every $P_0^{\Theta} \neq P_e^x \tilde{\in} SL(\tilde{X}_A)$. The soft set $(F,A) = \{ \tilde{n}P_e^x : \tilde{n} \tilde{>} \bar{0} \}$ is not soft bounded.

Proof:

Since $P_e^x \neq P_0^{\Theta}$. Then by soft Hausdorffness, there is a soft open nbhd (G,A) of P_0^{Θ} such that $P_e^x \tilde{\notin} (G,A)$. Hence $\tilde{n}P_e^x \tilde{\notin} \tilde{n}(G,A)$ for all $\tilde{n} \tilde{>} \bar{0}$, this means that $(F,A) \tilde{\nsubseteq} \tilde{n}(G,A)$.

Remark (3.1.29):

Only soft bounded soft linear subspace of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is $\{P_0^{\Theta}\}$.

Definition (3.1.30):

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ has a soft Hine Borel property, if every soft closed and soft bounded set is a soft compact set.

Definition (3.1.31):

For two SLS's $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ over $\mathbb{R}(A)$.

A soft linear mapping $\tilde{f} : SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{Y}_A)$ is said to be a soft bounded , if $\tilde{f}((F,A))$ is soft bounded set in $SL(\tilde{Y}_A)$ for all soft bounded set (F,A) of $SL(\tilde{X}_A)$.

Proposition (3.1.32):

Let $SL(\tilde{X}_A)$, $SL(\tilde{Y}_A)$ be two $STLS$'s over $\mathbb{R}(A)$ and $\tilde{f} : SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{Y}_A)$ is a soft linear mapping . If $\tilde{f}$ is a soft continuous , then it is soft bounded.

Proof:

Suppose that $\tilde{f}$ is soft continuous. Let (F,A) is a soft bounded set of $SL(\tilde{X}_A)$. Let (W,A) be a soft nbhd of $\tilde{f}(P_0^{\Theta})$ of $SL(\tilde{Y}_A)$. Since $\tilde{f}(P_0^{\Theta}) = P_0^{\Theta}$, then there is a soft nbhd (U,A) of P_0^{Θ} of $SL(\tilde{X}_A)$ with $\tilde{f}((U,A)) \tilde{\subseteq} (W,A)$. Since (F,A) is soft bounded, there is $\tilde{\delta} \tilde{>} \overline{0}$, such that $(F,A) \tilde{\subseteq} \tilde{\delta}(U,A)$. By soft linearity , for all $\tilde{\delta} \tilde{>} \overline{0}$, we have $\tilde{f}((F,A)) \tilde{\subseteq} \tilde{f}(\tilde{\delta}(U,A)) = \tilde{\delta}\tilde{f}((U,A))$, hence $\tilde{f}((F,A))$ is soft bounded , which implies that $\tilde{f}$ is soft bounded.

Example (3.1.33):

$(\boldsymbol{i})$ For any soft identity linear mapping from $(\tilde{X}_A, \|\tilde{\cdot}\|, A)$ is soft bounded.

$(\boldsymbol{ii})$ A soft linear mapping $\tilde{f}: SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{X}_A)$, defined by $\tilde{f}(P_e^x) = \overline{2}P_e^x$, for every $P_e^x \tilde{\in} SL(\tilde{X})$ is soft bounded.

$(\boldsymbol{iii})$ A soft linear mapping $\tilde{f}: SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{X}_A)$, defined by $\tilde{f}(P_e^x) = P_0^{\Theta}$, for all $P_e^x \tilde{\in} SL(\tilde{X})$ is soft bounded.

§(3.2) Soft locally convex spaces:

In applications, it is often useful to define a soft locally convex space by means of a family of soft seminorms on a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. In this section we will investigate the relation between soft locally convex soft topological linear space and soft seminorms.

Definition (3.2.1):

Let $SL(\tilde{X}_A)$ be a SLS over $\mathbb{R}(A)$. A soft linear P_e^x of $SL(\tilde{X}_A)$ is a soft linear combination of the soft linears $P_e^{x_1}, P_e^{x_2}, \dots, P_e^{x_n}$ of $SL(\tilde{X}_A)$, if P_e^x can be expressed as:

$$P_e^x = \tilde{r}_1 P_{e_1}^{x_1} \tilde{+} \tilde{r}_2 P_{e_2}^{x_2} \tilde{+} \dots \tilde{+} \tilde{r}_n P_{e_n}^{x_n} \text{ for some } \tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n \tilde{\in} \mathbb{R}(A) .$$

Definition (3.2.2)[13]:

The soft convex hull of a soft set (F, A) of $SL(\tilde{X}_A)$, which is denoted by $\widetilde{conv}(F, A)$ is the smallest soft convex set which is containing (F, A).

In other words:

$\widetilde{conv}(F, A)$ is the intersection of all soft convex sets containing (F, A).

Theorem (3.2.3):

Let (F, A) be a soft set of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

$\widetilde{conv}(F, A) = \{ \sum_{i=1}^{n} \tilde{\lambda}_i P_{e_i}^{x_i} : \tilde{\lambda}_i \tilde{\in} [\bar{0}, \bar{1}] , P_{e_i}^{x_i} \tilde{\in} (F, A) , \sum_{i=1}^{n} \tilde{\lambda}_i = \bar{1} ; n \in \mathbb{N} \}$.

Proposition (3.2.4):

Let (F, A) and (G, A) be two soft sets of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

$(\boldsymbol{i})$ $(F, A) \tilde{\subseteq} \widetilde{conv}(F, A)$.

$(\boldsymbol{ii})$ $\widetilde{conv}(F, A)$ is soft convex.

$(\boldsymbol{iii})$ (F, A) is soft convex if and only if $(F, A) = \widetilde{conv}(F, A)$.

$(\boldsymbol{iv})$ If $(G, A) \tilde{\subseteq} (F, A)$, then $\widetilde{conv}(G, A) \tilde{\subseteq} \widetilde{conv}(F, A)$.

$(\boldsymbol{v})$ $\widetilde{conv}\big(\widetilde{conv}(F, A)\big) = \widetilde{conv}(F, A)$.

The proof of this proposition is follows immediately from the definition (3.2.2).

Definition (3.2.5):

Let $SL(\tilde{X}_A)$ be a $STLS$ over $\mathbb{R}(A)$. The soft closed convex hull of a soft set (F, A), denoted by $\widetilde{cl}\widetilde{conv}(F, A)$ is the smallest soft closed soft convex set containing (F, A).

In other words:

$\widetilde{cl}\widetilde{conv}(F, A)$ is the intersection of all soft closed, soft convex sets which containing (F, A).

Theorem (3.2.6):

Let (F, A) is a soft set of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$. Then:

($\boldsymbol{i}$) If (F, A) is a soft open , then $\widetilde{conv}(F, A)$ is soft open.

($\boldsymbol{ii}$) If (F, A) is a soft balanced , then $\widetilde{conv}(F, A)$ is soft balanced.

($\boldsymbol{iii}$) If (F, A) is a soft symmetric , then $\widetilde{conv}(F, A)$ is soft symmetric.

($\boldsymbol{iv}$) If (F, A) is a soft absorbing , then $\widetilde{conv}(F, A)$ is soft absorbing.

($\boldsymbol{v}$) If (F, A) is a soft nbhd of P_0^{Θ}, then $\widetilde{conv}(F, A)$ is soft nbhd of P_0^{Θ} .

Proof:

($\boldsymbol{i}$) Since (F, A) is a soft open set of $SL(\tilde{X}_A)$, then $(F, A) = \widetilde{int}(F, A)$, and $(F, A) \tilde{\subseteq} \widetilde{conv}(F, A)$, we have:

$$(F, A) = \widetilde{int}(F, A) \tilde{\subseteq} \widetilde{int}\widetilde{conv}(F, A).$$

By using theorem (2.2.2.i), implies that $\widetilde{int}\widetilde{conv}(F, A)$ is soft convex containing (F, A).

But $\widetilde{conv}(F, A)$ is the smallest soft convex set containing (F, A), so $\widetilde{conv}(F, A) \tilde{\subseteq} \widetilde{int}\widetilde{conv}(F, A)$, but $\widetilde{int}\widetilde{conv}(F, A) \tilde{\subseteq} \widetilde{conv}(F, A)$. And so we get the required.

($\boldsymbol{ii}$) Since (F, A) is a soft balanced, then $\tilde{\lambda}(F, A) \tilde{\subseteq} (F, A)$, for all $\tilde{\lambda} \tilde{\in} \mathbb{R}(A)$ with $|\tilde{\lambda}| \tilde{\leq} \bar{1}$. From proposition (3.2.4.iv) , we have:

$$\widetilde{conv}\tilde{\lambda}(F, A) \tilde{\subseteq} \widetilde{conv}(F, A).$$

Since $\widetilde{conv}\tilde{\lambda}(F, A) = \tilde{\lambda}\widetilde{conv}(F, A)$, for all $\tilde{\lambda} \tilde{\in} \mathbb{R}(A)$ such that $|\tilde{\lambda}| \tilde{\leq} \bar{1}$. Thus $\widetilde{conv}(F, A)$ is soft balanced.

($\boldsymbol{iii}$) Since (F, A) is a soft symmetric, then $(F, A) = -\bar{1}(F, A)$ and we have $\widetilde{conv}(F, A) = -\bar{1}\widetilde{conv}(F, A)$. Thus $\widetilde{conv}(F, A)$ is soft symmetric.

($\boldsymbol{iv}$) Since (F, A) is a soft absorbing set , then $\tilde{\bigcup}_{\tilde{\alpha} \tilde{>} \bar{0}} \tilde{\alpha}(F, A) = \tilde{X}_A$.

In general $(F, A) \tilde{\subseteq} \widetilde{conv}(F, A)$, then $\tilde{\alpha}(F, A) \tilde{\subseteq} \tilde{\alpha}\widetilde{conv}(F, A)$. Implies that $SL(\tilde{X}_A) = \tilde{\bigcup}_{\tilde{\alpha} \tilde{>} \bar{0}} \tilde{\alpha}(F, A) \tilde{\subseteq} \tilde{\bigcup}_{\tilde{\alpha} \tilde{>} \bar{0}} \tilde{\alpha}\widetilde{conv}(F, A) \tilde{\subseteq} SL(\tilde{X}_A)$, hence $\widetilde{conv}(F, A)$ is soft absorbing set.

($\boldsymbol{v}$) By using proposition (1.2.5.i).

Definition (3.2.7) [6]:

A soft set (F, A) of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is called an absolutely soft convex, if it is soft convex and soft balanced.

Example (3.2.8):

A soft open unit semiball $B(P_0^\Theta, \tilde{p}\,; \bar{1}) = \{ P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}(P_e^x) \tilde{<} \bar{1}\}$ is an absolutely soft convex.

Definition (3.2.9):

An absolutely soft convex hull of a soft set (F, A) of a SLS $SL(\tilde{X}_A)$ over $\mathbb{R}$(A), denoted by $a\widetilde{conv}(F, A)$ is the intersection of all absolutely soft convex sets which is containing (F, A).

In other words:

$a\widetilde{conv}(F, A) = \{ \sum_{i=1}^{n} \tilde{\lambda}_i P_{e_i}^{x_i} : P_{e_i}^{x_i} \tilde{\in} (F, A), |\tilde{\lambda}_i| \tilde{\leq} \bar{1}\,; \tilde{\lambda}_i \tilde{\in} \mathbb{R}(A)\,; n \in \mathbb{N} \}$.

Definition (3.2.10):

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) is called a soft locally convex space , if there is a soft local base at P_0^Θ, whose every members are soft convex.

Example (3.2.11):

$(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ is soft locally convex.

By using the remark (3.1.14.iii) and theorem (3.1.17) .

Theorem (3.2.12):

A soft set (H, A) of a $STLS$ $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ (which is soft locally convex space) is soft bounded if and only if $\tilde{p}_\alpha((H, A))$ is soft bounded of $\mathbb{R}(A)^*$ for all $\tilde{p}_\alpha$ in $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$, where $\mathcal{P}$ be a family of soft continuous soft seminorm on $SL(\tilde{X}_A)$.

Proof:

Suppose that (H, A) is a soft bounded set and $\tilde{p}_\alpha \in \mathcal{P}$. It is clear that:

$$B(P_0^\Theta, \tilde{p}_\alpha\,; \bar{1}) = \{P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_\alpha(P_e^x) \tilde{<} \bar{1}\}$$

is soft nbhd of P_0^Θ of $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$. Since (H, A) is a soft bounded, then there is $\tilde{\gamma} \tilde{>} \bar{0}$ such that $(H, A) \tilde{\subseteq} \tilde{\gamma} B(P_0^\Theta, \tilde{p}_\alpha\,; \bar{1})$.

If $P_e^x \tilde{\in} (H, A)$, then $\tilde{\gamma}^{-1} P_e^x \tilde{\in} B(P_0^\Theta, \tilde{p}_\alpha\,; \bar{1})$, so $\tilde{p}_\alpha(\tilde{\gamma}^{-1} P_e^x) \tilde{<} \bar{1}$.

Thus $\tilde{p}_\alpha(P_e^x) \tilde{<} \tilde{\gamma}$ for all $P_e^x \tilde{\in} (H, A)$, by definition (1.1.24), implies that $\tilde{p}_\alpha((H, A))$ is soft bounded.

Conversely, suppose that $\tilde{p}_\alpha((H,A))$ is a soft bounded for each soft seminorms $\tilde{p}_\alpha$ on $SL(\tilde{X}_A)$. Let (F,A) be a soft nbhd of P_0^Θ.
Since $(\tilde{X}_A, \tilde{\tau}_\mathcal{P}, A)$ is a $STLS$ (soft locally convex space), there is a family $\mathcal{P} = \{\tilde{p}_\alpha : \alpha \in I\}$ of soft seminorms on $SL(\tilde{X}_A)$, which determine the soft topology $\tilde{\tau}_\mathcal{P}$. Thus there is $\tilde{r} \tilde{>} \bar{0}$ and soft seminorms $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n \in \mathcal{P}$ such that:
$B(P_0^\Theta, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}) \tilde{\subseteq} (F,A)$. From hypothesis above, each $\tilde{p}_i((H,A))$ is soft bounded for all $i = 1, \dots, n$, so there is $\widetilde{M} \tilde{\geq} \bar{0}$ such that $\tilde{p}_i(P_e^x) \tilde{\leq} \widetilde{M}$ for all $P_e^x \in (H,A)$ and $i = 1, \dots, n$.
So, for all $P_e^x \in (H,A)$, $\tilde{p}_i(P_e^x \tilde{s}^{-1}) \tilde{\leq} \widetilde{M}\tilde{s}^{-1} \tilde{<} \tilde{r}$, whenever $\tilde{s} \tilde{\geq} \widetilde{M}\tilde{r}^{-1}$.
i.e.
$P_e^x \tilde{s}^{-1} \tilde{\in} B(P_0^\Theta, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}) \tilde{\subseteq} (F,A)$, whenever $\tilde{s} \tilde{\geq} \widetilde{M}\tilde{r}^{-1}$. It follows that $(H,A) \tilde{\subseteq} \tilde{s}B(P_0^\Theta, \tilde{p}_1, \dots, \tilde{p}_n; \tilde{r}) \tilde{\subseteq} \tilde{s}(F,A)$ for all $\tilde{s} \tilde{\geq} \widetilde{M}\tilde{r}^{-1}$ and we conclude that (H,A) is soft bounded, as required.

Theorem (3. 2.13):

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ (which is soft locally convex) has a soft local base at P_0^Θ consisting every members are soft open, soft absorbing and absolutely soft convex.

Proof:

Let $\widetilde{\mathcal{B}}$ be a soft local base at P_0^Θ and (F,A) be a soft nbhd of P_0^Θ of $SL(\tilde{X}_A)$, which is soft locally convex. Then there is a soft convex nbhd (W,A) of P_0^Θ in $\widetilde{\mathcal{B}}$ such that $(W,A) \tilde{\subseteq} (F,A)$. By using theorem (2.1.16) , there is a soft balanced nbhd (U,A) of P_0^Θ of $SL(\tilde{X}_A)$ such that $(U,A) \tilde{\subseteq} (W,A)$.
Then using that is a soft convex set containing (U,A) , we get:
$\widetilde{conv}(U,A) \tilde{\subseteq} (W,A) \tilde{\subseteq} (F,A)$.
Assume that:

$$(O,A) = \widetilde{conv}(U,A) \tilde{\subseteq} (W,A) \tilde{\subseteq} (F,A).$$

Clearly that: $\widetilde{int}(O,A) \tilde{\subseteq} (O,A) \tilde{\subseteq} (W,A) \tilde{\subseteq} (F,A)$.
From the soft conclusion , implies that $\widetilde{int}(O,A)$ is soft open, soft balanced (because $P_0^\Theta \tilde{\in} \widetilde{int}(O,A)$, see theorem (2.2.3.ii)) and soft convex nbhd of P_0^Θ of $SL(\tilde{X}_A)$.

Example (3.2.14):

$SL(\widetilde{\ell^2}_A)$ is not soft locally convex space.

Proof:

It is easily that $\left\| P_e^{(x_1,x_2,x_3,\dots)} \right\| = |e| + \sqrt{\sum_{i=1}^{n} |x_i|^2}$ is a soft norm on $SL(\widetilde{\ell^2}_A)$ over $\mathbb{R}(A)$, for all $P_e^{(x_1,x_2,x_3,\dots)} \tilde{\in} SL(\widetilde{\ell^2}_A)$. This soft norm induces a soft topology $\tilde{\tau}$ on $\widetilde{\ell^2}_A$ making $SL(\widetilde{\ell^2}_A)$ into a $STLS$.

We now show that $\tilde{\tau}$ is not soft locally convex. Suppose it was , let:

$$\bar{B}(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1}) = \{P_e^{(x_1,x_2,x_3,\dots)} \tilde{\in} SL(\widetilde{\ell^2}_A) : \left\| P_e^{(x_1,x_2,x_3,\dots)} \right\| \tilde{\leq} \bar{1}\},$$

is a soft open unit ball . Then there is a soft convex , soft balanced set:

$$B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1}) = \{ P_e^{(x_1,x_2,x_3,\dots)} \tilde{\in} SL(\widetilde{\ell^2}_A) : \left\| P_e^{(x_1,x_2,x_3,\dots)} \right\| \tilde{<} \bar{1}\}$$

in $SL(\widetilde{\ell^2}_A)$, which is a soft nbhd of $P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}$ such that:

$$B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1}) \tilde{\subseteq} \bar{B}(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1}).$$

Also, there is a soft open ball $B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, |\tilde{\delta}|)$, $\bar{0} \tilde{<} |\tilde{\delta}| \tilde{<} \bar{1}$ such that:

$\bar{B}(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, |\tilde{\delta}|) \tilde{\subseteq} B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1})$.

Thus:

$$\bar{B}(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, |\tilde{\delta}|) \tilde{\subseteq} B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1}) \tilde{\subseteq} \bar{B}(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1}).$$

The soft linears:

$P_0^{(\delta,\Theta,\Theta,\dots,\Theta,\dots)}$; $P_0^{(\Theta,\delta,\Theta,\dots,\Theta,\dots)}$,..., $P_0^{(\Theta,\Theta,\Theta,\dots,\delta,\dots)}$ $\tilde{\in} B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1})$,

then by an absolutely soft convexity:

$P_0^{(\delta,\Theta,\Theta,\dots,\Theta,\dots)} \tilde{\mp} P_0^{(\Theta,\delta,\Theta,\dots,\Theta,\dots)} \tilde{\mp} \dots \tilde{\mp} P_0^{(\Theta,\Theta,\Theta,\dots,\delta,\dots)} \tilde{\in} B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1})$.

But $\left\| P_0^{(\delta,\Theta,\Theta,\dots,\Theta,\dots)} \tilde{\mp} P_0^{(\Theta,\delta,\Theta,\dots,\Theta,\dots)} \tilde{\mp} \dots \tilde{\mp} P_0^{(\Theta,\Theta,\Theta,\dots,\delta,\dots)} \right\| = \left\| P_0^{(\delta,\delta,\delta,\dots,\delta,\dots)} \right\|$

$$= \sqrt{n|\delta|^2} = \sqrt{n}|\delta|$$

conflicting the soft boundedness of $B(P_0^{(\Theta,\Theta,\Theta,\dots,\Theta,\dots)}, \bar{1})$.

Remark (3.2.15):

It is known that there is a $STLS$ $SL(\tilde{X}_A)$ which is not soft locally convex space, see example (3.2.14).

Theorem (3.2.16):

A soft convex hull of a soft bounded set (F, A) of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ (which is soft locally convex) is soft bounded.

Proof:

Let (U, A) be a soft nbhd of P_0^{Θ} of a soft locally convex space $SL(\tilde{X}_A)$, then there is a soft convex nbhd (V, A) of P_0^{Θ} such that $(V, A) \tilde{\subseteq} (U, A)$.

Since (F, A) be a soft bounded , then there is $\tilde{\gamma} \tilde{>} \bar{0}$ such that:

$$(F, A) \tilde{\subseteq} \tilde{\gamma}(V, A).$$

From proposition (3.2.4.iv) , we have:

$$\widetilde{conv}(F, A) \tilde{\subseteq} \widetilde{conv}(\tilde{\gamma}(V, A)) = \tilde{\gamma}\widetilde{conv}(V, A).$$

Also, by using proposition (3.2.4.iii) , then $\tilde{\gamma}\widetilde{conv}(V, A) = \tilde{\gamma}(V, A)$ and so $\widetilde{conv}(F, A) \tilde{\subseteq} \tilde{\gamma}(V, A) \tilde{\subseteq} \tilde{\gamma}(U, A)$.

This means that $\widetilde{conv}(F, A)$ is soft bounded.

Remark (3.2.17):

($\boldsymbol{i}$) In a $STLS$ $SL(\tilde{X}_A)$ (which is not soft locally convex space) a soft convex hull of a soft bounded set, need not to be a soft bounded set, as the example (3.2.14).

($\boldsymbol{ii}$) A soft convex hull of a soft closed set need not to be a soft closed set.

($\boldsymbol{iii}$) It does not have to be $\widetilde{conv}\left(\widetilde{cl}(F, A)\right) \neq \widetilde{cl}(\widetilde{conv}(F, A))$; and although $\widetilde{conv}\left(\widetilde{cl}(F, A)\right) \tilde{\subseteq} \widetilde{cl}(\widetilde{conv}(F, A))$.

§(3.3): Other types of soft locally convex spaces:

In this section, we discuss the elementary properties of two types of a soft locally convex spaces that occur frequently in applications.

Definition (3. 3.1):

A soft set (F,A) of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is said to be a soft barrel set, if it has the following properties:

$(\boldsymbol{i})$ Soft absorbing.

$(\boldsymbol{ii})$ Absolutely soft convex .

$(\boldsymbol{iii})$ Soft closed.

Example (3.3.2):

In a $STLS\ (\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$, then:

A soft closed unit semiball $\bar{B}(P_0^{\Theta}, \tilde{p}_\alpha; \bar{1}) = \{P_e^x \tilde{\in} SL(\tilde{X}_A): \tilde{p}_\alpha(P_e^x) \tilde{\leq} \bar{1}\}$ is soft barrel.

Theorem (3.3.3):

Every soft nbhd of P_0^{Θ} of a $STLS\ SL(\tilde{X}_A)$ over $\mathbb{R}(A)$ is contained in a soft nbhd of P_0^{Θ} which is soft barrel.

Proof:

Let (F,A) be a soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$.

$$(G,A) = \widetilde{cl}\,\widetilde{conv}\,\tilde{\bigcup}_{\tilde{\lambda}\tilde{\in}\mathbb{R}(A),|\tilde{\lambda}|\tilde{\leq}\bar{1}} \tilde{\lambda}(F,A).$$

It is clear that $(F,A) \tilde{\subseteq} (G,A)$. Thus (G,A) is soft nbhd of P_0^{Θ} and so by theorem (2.1.19), it is soft absorbing. By construction (G,A) is also soft closed and soft convex. To prove that (G,A) is soft barrel it remains to show that it is soft balanced.

It is easy to see that any $P_e^z \tilde{\in} \widetilde{conv}\,\tilde{\bigcup}_{\tilde{\lambda}\tilde{\in}\mathbb{R}(A),|\tilde{\lambda}|\tilde{\leq}\bar{1}} \tilde{\lambda}(F,A)$ can be written as:

$P_e^z = \tilde{t}P_e^x \tilde{+} (\bar{1} \tilde{-} \tilde{t})\, P_e^y$, with $\tilde{t} \tilde{\in} [\bar{0}, \bar{1}]$ and $P_e^x \tilde{\in} \tilde{\lambda}(F,A)$, $P_e^y \tilde{\in} \tilde{\delta}(F,A)$for some $\tilde{\lambda}$, $\tilde{\delta} \tilde{\in} \mathbb{R}(A)$ such that $|\tilde{\lambda}| \tilde{\leq} \bar{1}$ and $|\tilde{\delta}| \tilde{\leq} \bar{1}$. Then for any $\tilde{\gamma} \tilde{\in} \mathbb{R}(A)$ with $|\tilde{\gamma}| \tilde{\leq} \bar{1}$, we have:

$\tilde{\gamma}P_e^z = \tilde{\gamma}(\tilde{t}P_e^x) \tilde{+} \tilde{\gamma}((\bar{1} \tilde{-} \tilde{t})P_e^y) \tilde{\in} \widetilde{conv}\,\tilde{\bigcup}_{\tilde{\lambda}\tilde{\in}\mathbb{R}(A),|\tilde{\lambda}|\tilde{\leq}\bar{1}} \tilde{\lambda}(F,A)$,since $|\tilde{\gamma}\tilde{\lambda}| \tilde{\leq} \bar{1}$ and $|\tilde{\gamma}\tilde{\delta}| \tilde{\leq} \bar{1}$. This proves that $\widetilde{conv}\,\tilde{\bigcup}_{\tilde{\lambda}\tilde{\in}\mathbb{R}(A),|\tilde{\lambda}|\tilde{\leq}\bar{1}} \tilde{\lambda}(F,A)$ is soft balanced. Now, by using theorem (2.2.3.ii), we have (G,A) is soft balanced.

Corollary (3.3.4):

Every soft nbhd of P_0^θ of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) is contained in a soft nbhd of P_0^θ which is absolutely soft convex.

Proof: Clear.

Theorem (3.3.5) :

An absolutely soft convex (it is soft nbhd of P_0^θ) of a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) is soft barrels.

Proof:

Let (F,A) an absolutely soft convex, soft nbhd of P_0^θ of $SL(\tilde{X}_A)$. By using theorem (2.1.19), (F,A) is soft absorbing set . Now, since (F,A) be a soft nbhd of P_0^θ, then from definition (1.2.2.iii), we have $P_0^\theta \tilde{\in} \widetilde{int}(F,A)$, this implies that $\widetilde{int}(F,A) \neq \widetilde{\emptyset}_A$. Then from theorem (2.2.7), we have, (F,A) is soft closed set.

Corollary (3.3.6) :

In a $STLS$ $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$, then:

A soft open unit semiball $B(P_0^\theta, \tilde{p}_\alpha; \bar{1}) = \{P_e^x \tilde{\in} SL(\tilde{X}_A) : \tilde{p}_\alpha(P_e^x) \tilde{<} \bar{1}\}$ is soft barrel.

Proof: Clear.

Theorem (3.3.7) :

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex space) has a soft local base at P_0^θ consisting of soft barrels.

Proof:

Let $\widetilde{\mathcal{B}}$ be a soft local base at P_0^θ and (F,A) a soft nbhd of P_0^θ of $SL(\tilde{X}_A)$. By using theorem (2.1.23), there is a soft nbhd (W,A) of P_0^θ such that:

$$\widetilde{cl}(W,A) \tilde{\subseteq} (F,A) .$$

Put $(G,A) = \widetilde{cl}(W,A)$, clearly that (G,A) is soft closed nbhd of P_0^θ. Since $SL(\tilde{X}_A)$ be a soft locally convex , then there is a soft convex nbhd (U,A) of P_0^θ such that $(U,A) \tilde{\subseteq} (G,A)$. By using theorem (2.1.16), there is a soft balanced nbhd (V,A) of P_0^θ of $SL(\tilde{X}_A)$ such that $(V,A) \tilde{\subseteq} (U,A)$.

Summing up, we have:

$(V,A) \tilde{\subseteq} (U,A) \tilde{\subseteq} (G,A) \tilde{\subseteq} (F,A)$ for some (V,A) ,(U,A) and (G,A)soft closed nbhds of P_0^θ such that (V,A) soft balanced (U,A) soft convex.

Then using that (U,A) is a soft convex set containing (V,A), we get:

$$\widetilde{conv}(V,A) \tilde{\subseteq} (U,A).$$

Passing to the soft closure and using that (G,A), we get:

$$\widetilde{cl}\widetilde{conv}(V,A) \tilde{\subseteq} \widetilde{cl}(U,A) \tilde{\subseteq} \widetilde{cl}(G,A) = (G,A) \tilde{\subseteq} (F,A)$$

Hence, the soft conclusion holds , because we have already showed in theorems (3.2.6.ii) , (3.2.6.v) and (2.2.3), implies that $\widetilde{cl}\widetilde{conv}(V,A)$ is soft barrel set.

Definition (3.3.8):

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex space) is called a soft barrelled, if each soft barrel set of $SL(\tilde{X}_A)$ is soft nbhd of P_0^{Θ}. Equivalently, a soft barrelled space is a $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex space) in which the family of all soft barrels form a soft nbhd of P_0^{Θ}.

Example (3.3.9):

A soft locally convex space $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ is not soft barrelled space.

Theorem (3.3.10) :

Let $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ be two $STLS$'s over $\mathbb{R}$(A) (both are soft locally convex spaces), $\tilde{f} : SL(\tilde{X}_A) \rightrightarrows SL(\tilde{Y}_A)$ be a soft continuous, soft linear, soft open and soft surjective mapping. If $SL(\tilde{X}_A)$ be a soft barrelled space, then $SL(\tilde{Y}_A)$ is soft barrelled.

Proof:

Let (F,A) be a soft barrel set of $SL(\tilde{Y}_A)$, we prove that (F,A) is soft nbhd P_0^{Θ} of of $SL(\tilde{Y}_A)$. Since $\tilde{f}$ be a soft continuous and soft linear mapping, then $\tilde{f}^{-1}((F,A))$ is soft barrel set of $SL(\tilde{X}_A)$. Since $SL(\tilde{X}_A)$ be a soft barrelled space , then $\tilde{f}^{-1}((F,A))$ is a soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$.
Thus $\tilde{f}(\tilde{f}^{-1}((F,A))) = (F,A)$ is soft nbhd of P_0^{Θ} of $SL(\tilde{Y}_A)$, since $\tilde{f}$ is soft open and soft surjective mapping. Hence $SL(\tilde{Y}_A)$ is soft barrelled space.

Theorem (3.3.11):

Let $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ be two $STLS$'s over $\mathbb{R}$(A) (both are soft locally convex spaces), $\tilde{f} : SL(\tilde{X}_A) \rightrightarrows SL(\tilde{Y}_A)$ be a soft linear, soft closed, soft continuous and soft bijective mapping. If $SL(\tilde{Y}_A)$ be a soft barrelled space, then $SL(\tilde{X}_A)$ is soft barrelled.

Proof:

Let (F,A) be a soft barrel set of $SL(\tilde{X}_A)$, we prove that (F,A) is soft nbhd P_0^{Θ} of of $SL(\tilde{X}_A)$. Since $\tilde{f}$ be a soft linear , soft closed and soft surjective mapping , then $\tilde{f}((F,A))$ is soft barrel set of $SL(\tilde{Y}_A)$. Since $SL(\tilde{Y}_A)$ be a soft barrelled space, then $\tilde{f}((F,A))$ is a soft nbhd of P_0^{Θ} of $SL(\tilde{Y}_A)$.

Thus $\tilde{f}^{-1}(\tilde{f}((F,A))) = (F,A)$ is soft nbhd of P_0^{Θ} of $SL(\tilde{X}_A)$, (because $\tilde{f}$ is soft continuous and soft injective mapping). Hence $SL(\tilde{X}_A)$ is soft barrelled space.

Theorem (3.3.12) :

Let $SL(\tilde{X}_A)$ and $SL(\tilde{Y}_A)$ be two $STLS$'s over $\mathbb{R}$(A) (both are soft locally convex spaces), $\tilde{f} : SL(\tilde{X}_A) \tilde{\rightarrow} SL(\tilde{Y}_A)$ be a soft linear, homeomorphism mapping. Then $SL(\tilde{X}_A)$ is soft barrelled space if and only if $SL(\tilde{Y}_A)$ is soft barrelled.

Proof: By using theorems (1.2.29) , (3.3.10) and (3.3.11) .

Theorem (3.3.13) :

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex and soft second category space) is soft barreled space.

Proof:

Let (F,A) be a soft barrel set of $SL(\tilde{X}_A)$. Since (F,A) be a soft absorbing, we have:

$$\tilde{\bigcup}_{\tilde{n}_i \tilde{\geq} \bar{0}} \tilde{n}_i(F,A) = SL(\tilde{X}_A)\ ,\ i = 1\ ,2\ ,3\ ,\ldots.$$

Hence $\widetilde{int}(\tilde{n}_i(F,A)) \neq \tilde{\emptyset}_A$ for some $\tilde{n}_i$ (soft second category), see theorem (1.2.18.iii).

But $\widetilde{int}(\tilde{n}_i(F,A)) = \tilde{n}_i\widetilde{int}\ ((F,A))$, since a soft multiplication by $\tilde{n}_i$ is soft homeomorphism. By using theorem (2.2.9), implies that (F,A) is soft nbhd of P_0^{Θ}.

Theorem (3.3.14):

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (which is soft locally convex and soft Baire space) is soft barreled space.

Proof:

Let (F,A) be a soft barrel set of $SL(\tilde{X}_A)$. Since (F,A) be a soft absorbing , we have $\tilde{\bigcup}_{\tilde{n}_i \tilde{\geq} \bar{0}} \tilde{n}_i(F,A) = SL(\tilde{X}_A)$, $i = 1\ ,2\ ,3\ ,\ldots$. Since $SL(\tilde{X}_A)$ be a soft Baire space , then there is $\tilde{n}_i \tilde{>} \bar{0}$ such that $\tilde{n}_i(F,A)$ (which is soft closed) has soft interior point, see definition (1.2.19.i). Hence (F, A) has a soft interior point P_e^x. Since (F, A) be a soft balanced, then $-\bar{1}\tilde{.} P_e^x \tilde{\in} (F, A)$. Implies that:

$$P_0^{\Theta} = \overline{(\tfrac{1}{2})}\, P_e^x \tilde{+} \overline{(\tfrac{1}{2})}(-\bar{1}\tilde{.} P_e^x)$$

, is soft interior to (F, A) from theorem (2.2.1), because (F, A) is soft convex. Therefore (F, A) is soft nbhd of P_0^{Θ}.

Definition (3.3.15):

A $STLS$ $SL(\tilde{X}_A)$ over ℝ(A) (which is soft locally convex space) is called soft bornological space, if each absolutely soft convex set of $SL(\tilde{X}_A)$ that soft absorbs all soft bounded sets is a soft nbhd of P_0^{Θ}.

Theorem (3.3.16):

Let $SL(\tilde{X}_A)$ be a $STLS$ over ℝ(A) (which is soft locally convex space). Then the following conditions are equivalent:

(***i***) $SL(\tilde{X}_A)$ is soft bornological space .

(***ii***) every soft semi norm on $SL(\tilde{X}_A)$ which is soft bounded on a soft bounded subsets of $SL(\tilde{X}_A)$ is soft continuous.

Proof:

It is immediate that (i) and (ii) are equivalent in view of the correspondence between soft semi norms and soft convex , soft balanced and soft absorbing subsets. If $\tilde{p}$ be a soft semi norm on $SL(\tilde{X}_A)$, which is soft continuous on each soft subset of $SL(\tilde{X}_A)$, then it is soft bounded on a soft bounded subset of $SL(\tilde{X}_A)$.

Theorem (3.3.17):

A $STLS$ $SL(\tilde{X}_A)$ over ℝ(A) (which is soft locally convex and soft first countable space) is soft bornological space.

Proof:

Let $\tilde{\mathcal{B}} = \{(V_1, A), (V_2, A), (V_3, A), \dots\}$ be countable soft local base at P_0^{Θ} such that $(V_1, A) \tilde{\supseteq} (V_2, A) \tilde{\supseteq} (V_3, A) \tilde{\supseteq}$ .... Suppose (F, A) be an absolutely soft convex set of $SL(\tilde{X}_A)$ that soft absorbs all soft bounded sets, then proving that (F, A) is soft nbhd of P_0^{Θ}.

Assume that (F, A) is not soft nbhd of P_0^{Θ} and construct a soft bounded (in fact, soft compact) set that (F, A) does not soft absorb. Then for all $n \in \mathbb{N}$, we have:

$$\overline{(\tfrac{1}{n})}(V_n, A) \,\widetilde{\nsubseteq}\, (F, A) \Rightarrow (V_n, A) \,\widetilde{\nsubseteq}\, \bar{n}(F, A).$$

If $P_{e_n}^{x_n} \,\widetilde{\in}\, (V_n, A) \simeq \bar{n}(F, A)$.

Since $P_{e_n}^{x_n} \,\widetilde{\in}\, (V_n, A)$, $(V_1, A) \,\widetilde{\supseteq}\, (V_2, A) \,\widetilde{\supseteq}\, (V_3, A) \,\widetilde{\supseteq}\, \dots$, then $P_{e_n}^{x_n} \,\widetilde{\to}\, P_0^{\Theta}$, so by using theorem (1.2.14.iii), $\{ P_{e_n}^{x_n}\} \widetilde{\cup} \{ P_0^{\Theta}\}$ is soft compact, by using theorem (3.1.27) it is soft bounded.

But $P_{e_n}^{x_n} \,\widetilde{\notin}\, \bar{n}(F, A)$, says that (F, A) cannot soft absorbs $\{ P_{e_n}^{x_n}\} \widetilde{\cup} \{ P_0^{\Theta}\}$.

Example (3.3.18):

A soft locally convex space $(\tilde{X}_A, \tilde{\tau}_{\mathcal{P}}, A)$ is soft bornological space.

Theorem (3.319) :

A $STLS$ $SL(\tilde{X}_A)$ over $\mathbb{R}$(A) (is soft locally convex and soft metrizable) is soft bornological space.

Proof:

Since $SL(\tilde{X}_A)$ be a soft metrizable from definition (1.2.43)and theorem (1,2,44) , there is a countable soft local base $\widetilde{\mathcal{B}} = \{B(P_0^{\Theta}, \overline{(\tfrac{1}{n})}) : n \in \mathbb{N}\}$ at P_0^{Θ}. Suppose that (F, A) be an absolutely soft convex of $SL(\tilde{X}_A)$ that absorbs all soft bounded sets , then proving that (F, A) is soft nbhd of P_0^{Θ}. Assume that (F, A) is not soft nbhd of P_0^{Θ} and construct a soft bounded (in fact, soft compact) set that (F, A) does not absorb.

Then for all $\bar{n} \,\widetilde{>}\, \bar{0}$, we have:

$$\overline{(\tfrac{1}{n})} B\big(P_0^{\Theta}, \overline{(\tfrac{1}{n})}\big) \,\widetilde{\nsubseteq}\, (F, A) \text{ and so } B\big(P_0^{\Theta}, \overline{(\tfrac{1}{n})}\big) \,\widetilde{\nsubseteq}\, \bar{n}(F, A).$$

Choose $P_{e_n}^{x_n} \,\widetilde{\in}\, B\big(P_0^{\Theta}, \overline{(\tfrac{1}{n})}\big) \simeq \bar{n}(F, A)$.

Since $P_{e_n}^{x_n} \,\widetilde{\in}\, B\big(P_0^{\Theta}, \overline{(\tfrac{1}{n})}\big)$, then $P_{e_n}^{x_n} \,\widetilde{\to}\, P_0^{\Theta}$. Since $\{ P_{e_n}^{x_n}\} \widetilde{\cup} \{ P_0^{\Theta}\}$ be a soft compact, by using theorem (3.1.27) , then $\{ P_{e_n}^{x_n}\} \widetilde{\cup} \{ P_0^{\Theta}\}$ is soft bounded.

But $P_{e_n}^{x_n} \,\widetilde{\notin}\, \bar{n}(F, A)$, says that (F, A) cannot soft absorbs $\{ P_{e_n}^{x_n}\} \widetilde{\cup} \{ P_0^{\Theta}\}$.

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إلى

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جامعة القادسية

كجزء من متطلبات نيل درجة ماجستير في علوم الرياضيات من قبل

خلود محمد حسن عباس

باشراف

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١٤٣٩ هـ **٢٠١٨ م**