Republic of Iraq
Ministry of Higher Education
and Scientific Research
University of Al-Qadisiya
College of Computer Science
and Mathematics
(Department of Mathematics)

On Covering Properties By Using coc-r-Open Set

A Thesis Submitted to the Council of the College of Computer Science and Mathematics, University of Al-Qadisiya as a partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

By

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2017 *A.D.*   1438 *A. H.*

بِسمِ ٱللَّهِ الرَّحمنِ الرَّحيمِ

ن والقلمِ وما يَسْطُرون

صدق الله العلي العظيم

سُورةُ القلم (آية ١)

الاهداء

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الى من تفتح لها ابواب السماء اذا رفعت يديها بالدعاء.... والدتي العزيزة

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List of Symbols

Symbol	Definition
A°	The interior of A.
$\overline{A}$	The closure of A.
A^{c}	The complement of A.
$A^{\circ r}$	The r-interior of A.
$\overline{A}^{r}$	The r-closure of A
$A^{\circ rk}$	The coc-r-interior of A.
$\overline{A}^{rk}$	The coc-r-closure of A
A'^{rk}	The coc-r-limit point of A
$b_{rk}(A)$	The coc-r-boundary of A
$N_{rk}(x)$	The set of all coc-r-neighborhoods of x
τ^{k}	The family of all coc-open sets in X
τ^{rk}	The family of all coc-r-open sets in X
$RO(X, \tau)$	The family of all r-open sets in X
$RC(X, \tau)$	The family of all r-closed sets in X
$\beta O(X, \tau)$	The family of all β - open sets in X
$RO(X, \tau^{rk})$	The family of all coc-r-regular open sets in X

$RC(X, \tau^{rk})$	The family of all coc-r-regular closed sets in X
$\beta O(X, \tau^{rk})$	The family of all coc-r-β - open sets in X

Contents

Acknowledgments

Praise and thanks are first due to Allah Almighty for blessing me with patience and endurance to accomplish the present study.

I am very indebted to my supervisor, Dr. Raad Aziz Al- Abdulla for his valuable guidance, comments, advice and kindness which contributed a lot to the fulfillment of this work.

I wish to express my respect and think to the members of the department of mathematics for their help and encouragement in the course of this work

I would like also to extend my thanks to my family and my friends for their help and encouragement.

Finally, I would like to thank all those who had participated in one way or another in this work.

Abstract

The main objective of this thesis is to extend and study some properties of topological spaces such as compact space, lindelof space by covering properties by using coc-r-open sets .

We have formerly studied compact, lindelof, connected spaces and separation aximes. In this work we extend these concepts by using coc-r-open sets to study s-coc-r-connected, coc-r-compact, coc-r-lindelof, I-coc-r-lindelof spaces and coc-r-separation aximes

Also we studied concept (coc-r , co ć -r) function , super coc-r-open function , (coc-r , coć-r) continuous function and clarified the properties of that function. The following are among our main results :-

1. Let X be T_2-space, then the following statements are equivalent.
i) X is coc-r-compact.
ii) Every cover of X by r- open subsets has a finite subcover.

2. Let X is T_2-space, then the following statements are equivalent.
i) Every proper r- closed subset of X is coc-r-compact relative to X.
ii) X is coc-r-compact.
iii) X is r-compact.

3. Let f: X ⟶ Y be a coc-r-continuous function, onto and Y be extremally disconnected space, if X is coc-r-compact then Y is I-compact.

4. Let f: X ⟶ Y be a coc-r-open , bijective function and X be a extremally disconnected space. If Y is coc-r- lindelof then X is I- lindelof.

5. Let X is coc-r-extremally disconnected, coc′ -r-regular space, then the following statements are equivalent.

1) X is $\mathcal{S}$- coc-r - lindelof.

2) X is I-coc-r-lindelof.

3) X is coc-r-lindelof.

6. Let X is T_3, extremally disconnected space, then the following statements are equivalent.

1) X is coc-r-lindelof.

2) X is I-lindelof.

3) X is lindelof.

4) X is I-coc-r-lindelof.

7. Let f: $(X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a $\mathcal{S}$- coc-r - β -closed, super coc-r-open function, with $f^{-1}(y)$ $\mathcal{S}$- coc-r - lindelof for each $y \in Y$ and X coc-r-extremally disconnected, coc-r - P- space. If Y is I- lindelof, then X is I-coc-r-lindelof.

Introduction

This thesis introduces some concepts in general topology by using coc-r-open sets and the relationship between the spaces (compact, lindelof, I-lindelof) by using coc-r-open cover.

In the year 2011**[1]** S. Al Gore and S. Samarah provided coc-open sets in the topological spaces, where they studied continuity by using these sets. Later, some researchers have studied these sets and expanded, in1937 **[15]**, regular open sets were introduced and used to define the semiregularization space of a topological space1995**[11],**1970**[16]**, N. Bourbaki 1989**[2]** introduced the concept of compact space, in 1979**[3]** D. E. Cameron introduced the concept of I-compact space, where he studied maximal C-compact spaces, maximal QHC spaces, and maximal nearly compact spaces. He also discussed covering property which turns out to be equivalent to *S*-closed and extremally disconnected. in 1996**[9]** D. S. Jankovic and C. Konstadilaki introduced the concept of rc-compact, rc-lindelof, countably rc-compact, perfectly k-normal, Luzin space, generalized ordered space, in 2003**[17]** K. Al-Zoubi and B. Al-Nashef introduced the concept of I-lindelof spaces.

This thesis consists of three chapters. Chapter one is divided into two sections. In section one , the basic definitions have been recalled. In section two, we define coc-r-β - open and coc-r - regular open sets and we prove some properties about them.

Chapter two is divided into four sections . In section one, we recall definition of coc-r-continuous function and prove some properties about it . In section two, we recall definitions of coc-r-open function and prove some properties about it. In section three, we introduced fundamental concept of separation axioms and generalized by coc-r-open sets. In section four, we introduce the fundamental concept of connected space and generalized by coc-r-open sets.

Chapter three is divided into three sections . In section one, we recall the concept of coc-r-compact space and give some important generalizations on this concept. In section two, we recall definition, proposition and theorems of coc-r-lindelof space. In section three, we introduces the concept of I-coc-r-lindelof space and we prove some results on this concept and give the relation between I-coc-r-lindelof, coc-r-lindelof, I-lindelof, and lindelof space.

Chapter One
On Types of coc-r-open sets

This Chapter is divided into two sections. In section one , the basic definition have been recalled. In section two, we define coc-r-β-open and coc-r-regular open sets and we prove some properties about them.

1.1 On coc-r-open sets

This section present the definition of coc-r-open set , remarks ,propositions and example about the concept .

Definition (1.1.1) [1]

A subset A of a topological space (X,τ) is called cocompact open set (notation : coc-open set) if for every $x \in A$ there exists an open set $U \subseteq X$ and a compact subset K of X such that $x \in U - K \subseteq A$. The complement of coc-open set is called coc-closed set.

The family of all coc-open subsets of a space (X, τ) forms topology on (X, τ) and denoted by τ^k.

Remarks (1.1.2) [10]

1. Every open set is a coc-open set.
2. Every closed set is a coc-closed set.
3. The converse of (i, ii) is not true in general.

Definition (1.1.3) [15]

A subset A of a topological space (X,τ) is called regular open set (notation : r-open set) if $A = \overline{A}^{\circ}$. The complement of regular open set is called regular closed (r-closed) set and it is easy to see that A is regular closed if $A = \overline{A^{\circ}}$ **.**

Remarks (1.1.4) [16]

Let X be a topological space, then:

i. Every r-open set is an open set.

ii. Every r-closed set is a closed set.

iii. The converse of (i, ii) is not true in general.

Remarks (1.1.5) [11]

Let X be a topological space, then:

1) The family of all r - open sets in X is denoted by $RO(X, \tau)$.

2) The family of all r - closed sets in X is denoted by $RC(X, \tau)$.

Definition (1.1.6)

A subset A of a topological space (X, τ) is called cocompact regular open set (notation : coc -r-open set) if for every $x \in A$ there exists r-open set $U \subseteq X$ and compact subset K such that $x \in U - K \subseteq A$, the complement of coc-r-open set is called coc -r-closed set .

Remarks (1.1.7)

Let X be a topological space, then:

1- Every r-open set is coc -open.set.

2- Every r-closed.is coc - closed.set.

3- Every r-open set is coc -r-open set.

4- Every r- closed.set is coc -r- closed.set.

5- Every coc -r-open.set is coc-open.

6- Every.coc -r- closed.set is coc- closed.

Proof : It is clear

Remark (1.1.8)

The converse of Remarks (1.1.7) is not true in general as the following examples show:

Examples (1.1.9)

1- Let $X = \{1,2,3\}, \tau = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}$ be a topology on X. Notice that $\{1,2\}$ is a coc-open, coc-r-open but it is not r-open and $\{3\}$ is a coc-closed, coc-r- closed but it is not r- closed.

2- Let $X = \{1,2,3, \dots\}$, $\tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$ be a topology on X, the coc-r-open sets are $\{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, thus $\{1\}$ is a coc-open but it is not coc-r-open and $\{2,3, \dots\}$ is a coc- closed but it is not coc-r- closed.

Remark (1.1.10)

Every coc -r-open set is not necessarily to be open set, every coc-r-closed set is not necessarily to be closed set . Also every open set is not necessarily to be coc -r-open set and every closed set is not necessarily to be coc -r-closed set.

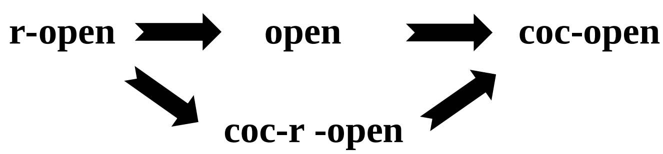
As the following examples show:

Examples (1.1.11)

1- Let $X = \{1,2,3\}, \tau = \{\emptyset, X, \{1\}, \{2\}, \{1,2\}\}$ be a topology on X, the coc-r-open sets are $\{X, \phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$ then $\{3\}$ is a coc-r-open but it is not open and $\{2\}$ is coc-r-closed but it is not closed set.

2- Let $X = \{1,2,3, \dots\}, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$ be a topology on X, the coc-r-open sets are $\{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$. Notice that $\{1\}$ is an open but is not coc-r-open and $\{2,3, \dots\}$ is a closed but it is not coc-r-closed.

The following diagram shows the relation between types of coc-r -open sets



Remark (1.1.12)

Let X be a topological space, then:

1- The intersection of two r-open.set is r-open . **[16]**

2- The.intersection of two .coc-open set is.coc-open . **[1]**

Remarks (1.1.13)

Let X be a topological space, then:

1- The intersection of r-open sets and open set is open .

2- The intersection of two coc -r -open set is coc -r -open .

3- The union of coc-r-open sets is coc-r-open set .

4- The intersection of coc-r-open sets and coc-open set is coc-open .

5- The coc-r-open sets forms topology on X denoted by τ^{rk}.

Proof :

1) It is clear.

2) Let A, B be coc-r-open, to prove $A \cap B$ is coc -r -open set. Suppose that $x \in A \cap B$, then $x \in A$ and $x \in B$, since A, B are coc-r-open, thus there exist two r-open sets $U, V \subseteq X$ and two compact subset K, L such that $x \in U - K \subseteq A$, $x \in V - L \subseteq B$, therefore $x \in (U - K) \cap (V - L) \subseteq A \cap B$ imply that $x \in (U \cap K^c) \cap (V \cap L^c) \subseteq$

$A \cap B$ then $x \in (U \cap V) \cap (K^c \cap L^c) \subseteq A \cap B$ thus we get $x \in (U \cap V) - (K \cup L) \subseteq A \cap B$, by using (1) $U \cap V$ is r-open, since $K \cup L \subseteq X$ is compact set in X. Hence $A \cap B$ is coc -r -open.

3) Let A_α , $\alpha \in \Lambda$ be coc-r-open set for each $\alpha \in \Lambda$ to prove $\cup_{\alpha\in\Lambda} A_\alpha$ is coc-r-open. Suppose $x \in \cup_{\alpha\in\Lambda} A_\alpha$, then $x \in A_\alpha$ for some $\alpha \in \Lambda$, since A_α is coc-r-open, thus there exist r-open sets $U_\alpha \subseteq X$ and compact subset K_α such that $x \in U_\alpha - K_\alpha \subseteq A_\alpha$ for some $\alpha \in \Lambda$, since $A_\alpha \subseteq \cup_{\alpha\in\Lambda} A_\alpha$. Hence $\cup_{\alpha\in\Lambda} A_\alpha$ is coc-r-open.

(4) and (5) It is clear.

Definition (1.1.14) [13]

1. Let X be a topological space and $A \subseteq X$, a point $x \in A$ is called r-interior point of A if there exists a r - open set U in X containing x such that $x \in U \subseteq A$.

The set of all r-Interior points of A is called r-Interior set of A, it is denoted by $A^{\circ r}$ and $A^{\circ r} = \cup \{B: B \text{ r-open set in } X \text{ and } B \subseteq A\}$.

Definition (1.1.15) [13]

Let X be a topological space and $B \subseteq X$. The intersection of all r - closed sets of X containing B is called the r - closure of B and is denoted by $\overline{B}^r$.

Remarks (1.1.16) [13]

Let X be a topological space and $A \subseteq X$, then:

1) $A^{\circ r} \subseteq A^\circ$.

2) $A \subseteq \overline{A}^r$.

3) If $x \in \overline{A}^r$, then for any r-open set U in X containing x we have $U \cap A \neq \phi$.

4) If A a closed set, then A° is a r-open set.

5) If A an open set, then $\overline{A}$ is a r-closed set.

6) If A a r-closed set, then A is closed set.

Definition (1.1.17) [13]

A topological space X is said to be r - compact if every r - open covering of X has a finite sub covering.

Proposition (1,1.18) [13]

Let X be a topological space, then:

1) Every compact space is r - compact space.

2) Every r-compact subset of T_2-space is r- closed set.

Theorem (1.1.19)

Let X be T_2-space, $A \subseteq X$.

1. If A is a coc-r-open in X, then $A = A^{\circ r}$.
2. if A is a coc-r-closed in X, then $A = \overline{A}^{r}$.

Proof :

1. Let A be coc -r-open in X, since $A^{\circ r} \subseteq A^{\circ} \subseteq A$, we need to prove that $A \subseteq A^{\circ r}$. Let $x \in A$, since A is coc-r-open, then there exist r-open set U and compact subset K such that $x \in U - K \subseteq A$. Since every compact is r-compact and X be T_2-space, thus K is r- closed set (by using Proposition (1.1.18), (1), (2)) , so K^c r-open subset in X and $x \in U \cap K^c \subseteq A$ and U, K^c are r-open sets in X, there fore $U \cap K^c$ is r-open in X, hence $x \in A^{\circ r}$.

2. Let $x \in \overline{A}^{r}$ and $x \notin A$, then $x \in A^c$ since A is coc-r-closed in X, thus A^c is coc-r-open in X and $x \in A^c$, there exist r-open U, compact subset K such that $x \in U - K \subseteq A^c$. Since K is compact subset in X, therefore K is r-compact , so K is r-closed (by using Proposition (1.1.18), (1), (2)), then K^c r-open, since $U \cap K^c$ is r-open, $x \in U \cap K^c \subseteq A^c$, $x \in \overline{A}^{r}$ and using by Remarks (1.1.16), (3) then $(U \cap K^c) \cap A \neq \phi$ this is contradiction with $U \cap K^c \subseteq A^c$, thus $x \in A$, since $A \subseteq \overline{A}^{r}$, hence $A = \overline{A}^{r}$.

Remarks (1.1.20)

Let X be a topological space, then:

1) If X is a finite set then τ^{rk} is a discrete topology.

2) A closed subset of compact space X is compact relative to X. **[6]**

3) In any space, the intersection of compact set with a closed set is compact. **[6]**

4) Every compact subset of T_2-space is closed set. **[6]**

5) A space X is regular space iff for every $x \in X$ and each open set U in X such that $x \in U$ there exists an open set W such that $x \in W \subseteq \overline{W} \subseteq U$. **[5]**

6) A space (X, T) is called T_3-space if X is regular space and T_1-space. **[5]**

7) Every T_3-space is T_2-space. **[5]**

Proposition (1.1.21) [13]

Let X be regular space, if $A \subseteq X$ is an open then $A \in RO(X, \tau)$.

Corollary (1.1.22)

Let X be regular space, if $F \subseteq X$ is a closed then $F \in RC(X, \tau)$.

Proof : It is clear.

Theorem (1.1.23)

Let (X, τ) be a T_2-space, then $\tau^{rk} \subseteq \tau$.

Proof :

Let $A \in \tau^{rk}$ .To prove $A \in \tau$, let $x \in A$, then there exists r-open set $U \subseteq X$ and compact subset K such that $x \in U - K \subseteq A$, thus $x \in U \cap K^C \subseteq A$. Since K is compact and X is T_2-space, therefore K is closed, so K^C is open. By using remarks (1.1.20), (5), so $U \cap K^C$ is open set in X. Hence $A \in \tau$

Remarks (1.1.24)

Let (X, τ) be a T_2-space, then

1) Every coc-r-open set is open set.

2) Every coc-r-closed set is closed set.

Proof : It is clear.

Theorem (1.1.25)

Let (X, τ) be a regular-space, then $\tau \subseteq \tau^{rk}$.

Proof : Clear, by using Proposition (1.1.21).

Corollary (1.1.26)

Let (X, τ) be a T_3-space, then $\tau = \tau^{rk}$.

Proof : It is clear.

Definition (1.1.27)

Let X be a space and A $\subseteq$ X . The intersection of all coc-r-closed sets of X containing A called coc-r-closure of A and is denoted by $\overline{A}^{rk}$, i.e $\overline{A}^{rk} = \cap \{F: F$ coc-r - closed set in X and A $\subseteq$ F $\}$.

Remark (1.1.28)

Let X be a topological space and A $\subseteq$ X, then:

$\overline{A}^{rk}$ is the smallest coc-r - closed set containing A.

Proposition (1.1.29)

Let X be a topological space and A $\subseteq$ B $\subseteq$ X, then:

i. . $\overline{A}^{rk}$ is an coc-r - closed set .

ii. A is an coc-r - closed set if and only if A = $\overline{A}^{rk}$

iii. $\overline{A}^{rk} = \overline{\overline{A}^{rk}}^{rk}$

iv. $\overline{A}^{rk} \subseteq \overline{B}^{rk}$

Proof: It is clear.

Proposition (1.1.30)

Let X be a space and A $\subseteq$ X. Then x $\in \overline{A}^{rk}$ iff for each coc-r - open set U in X contained point x we have U $\cap$ A $\neq \phi$.

Proof: It is clear.

Proposition (1.1.31)

Let X be topological space and A, B $\subseteq$ X, then:

1. $\overline{\phi}^{rk} = \phi, \overline{X}^{rk} = X$.
2. $\overline{A \cup B}^{rk} = \overline{A}^{rk} \cup \overline{B}^{rk}$.
3. $\overline{A \cap B}^{rk} \subseteq \overline{A}^{rk} \cap \overline{B}^{rk}$.

Proof: It is clear.

Definition (1.1.32)

Let X be a space and $A \subseteq X$. The union of all coc-r-open sets of X containing in A is called coc-r-Interior of A denoted by $A^{\circ rk}$, i.e $A^{\circ rk} = \cup \{U: U$ coc-r - open set in X and $U \subseteq A\}$.

Proposition (1.1.33)

Let X be a space and $A \subseteq X$, then $A^{\circ rk}$ is the largest coc-r-open set containing in A.

Proof : Clear by definition of $A^{\circ coc-r}$.

Proposition (1.1.34)

Let X be a space and $A \subseteq X$, then $x \in A^{\circ rk}$ if and only if there exists coc-r-open set U containing x such that $x \in U \subseteq A$.

Proof :

Let $x \in A^{\circ rk}$, then $x \in \cup_{\alpha\in\Lambda} V_\alpha$ such that V_α coc-r-open set and $V_\alpha \subseteq A$, $\alpha \in \Lambda$. Thus $x \in V_\alpha$ for some $\alpha \in \Lambda$, since $V_\alpha \subseteq A$ $\alpha \in \Lambda$, then $x \in U = V_\alpha \subseteq A$ for some $\alpha \in \Lambda$. Conversely, let there exists U coc-r-open set such that $x \in U \subseteq A$ then $x \in \cup U$, $U \subseteq A$ and U coc-r- open set then $x \in A^{\circ rk}$.

Proposition (1.1.35)

Let X be a space and $A, B \subseteq X$ then:

1. $A^{\circ coc-r}$ is coc-r- open set .
2. A is coc-r-open if and only if $A = A^{\circ rk}$.
3. $A^{\circ rk} = (A^{\circ rk})^{\circ rk}$.
4. if $A \subseteq B$ then $A^{\circ rk} \subseteq B^{\circ rk}$.
5. $A^{\circ rk} \cup B^{\circ rk} \subseteq (A \cup B)^{\circ rk}$.
6. $A^{\circ rk} \cap B^{\circ rk} = (A \cap B)^{\circ rk}$.

Proof : It is clear.

Remark (1.1.36)

Let (X, τ) be topological space and $A, B \subseteq X$, then:

$(A \cup B)^{\circ rk} \neq A^{\circ rk} \cup B^{\circ rk}$, as the following example shwos.

Example (1.1.37)

Let $X = \{1,2,3,\dots\}, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X: G^c$ is finite$\}$ $\cup \{\emptyset\}$, let $A = \{2\}$, $B = \{1,3,4,\dots\}$, thus $A^{\circ rk} = \emptyset$, $B^{\circ rk} = B$.
There fore $A^{\circ rk} \cup B^{\circ rk} = \emptyset \cup B = B \neq X = (A \cup B)^{\circ rk}$.

Proposition (1.1.38)

Let X be a topological space and $A \subseteq X$, then

1. $(\overline{A}^{rk})^c = (A^c)^{\circ rk}$
2. $(A^{\circ rk})^c = \overline{(A^c)}^{rk}$
3. $\overline{A}^{rk} = (A^{c\circ rk})^c$
4. $A^{\circ rk} = (\overline{A^c}^{rk})^c$

Proof :

1) since $A \subseteq \overline{A}^{rk}$, then $(\overline{A}^{rk})^c \subseteq A^c$ and $\overline{A}^{rk}$ coc-r – closed set in X, thus $(\overline{A}^{rk})^c$ is coc-r - open set in X, but $(A^c)^{\circ rk}$ is coc-r-open set in X and $(A^c)^{\circ rk} \subseteq A^c$.
By using proposition (1.1.33), then $(\overline{A}^{rk})^c \subseteq (A^c)^{\circ rk}$………(1)
Now:
Let $x \in (A^c)^{\circ rk}$, then there exist coc-r – open set U in X such that $x \in U \subseteq A^c$, to prove $x \in (\overline{A}^{rk})^c$.
Let $x \notin (\overline{A}^{rk})^c$, thus $x \in \overline{A}^{rk}$, since $x \in U$ and U coc-r - open set in X.
There fore $U \cap A \neq \phi$, this is contradiction with $U \subseteq A^c$, so $x \in (\overline{A}^{rk})^c$.
Hence $(A^c)^{\circ rk} \subseteq (\overline{A}^{rk})^c$ ………(2)
From (1), (2) we get $(A^c)^{\circ rk} = (\overline{A}^{rk})^c$.
2) By using (1), $(\overline{A^c}^{rk})^c = A^{\circ rk}$, then $\overline{A^c}^{rk} = (A^{\circ rk})^c$.
3) By using (1), $(\overline{A}^{rk})^c = (A^c)^{\circ rk}$, then $(A^{c\circ rk})^c = \overline{A}^{rk}$.
4) By using (1), $\overline{A^c}^{rk} = (A^{\circ rk})^c$, then $A^{\circ rk} = (\overline{A^c}^{rk})^c$.

Definition (1.1.39)

Let X be a topological space and B any subset of a space X , a coc-r-neighborhood (coc-r-nbd) of B is any subset of X which contains an coc-r-open set containing B .The coc-r-neighborhood of subset {x} is also called coc-r-neighborhood of the point x.

Remark (1.1.40)

The family of all coc-r-neighborhoods (coc-r-nbds) of the subset B of a space X is denoted by $N_{rk}(B)$. In specific the family of all neighborhoods of x is denoted by $N_{rk}(x)$.

Proposition (1.1.41)

Let X be a topological space and for all $x \in X$, let $N_{rk}(x)$ be a family of all coc-r-nbds of x then :-

i- If $U \in N_{rk}(x)$ such that $U \subseteq V$ then $V \in N_{rk}(x)$.

ii- If $U, V \in N_{rk}(x)$ then $U \cap V \in N_{rk}(x)$ such that $U, V \subseteq X$

iii- If $U_\alpha \in N_{rk}(x)$ then $\cup\ U_\alpha \in N_{rk}(x),\ \alpha \in \Lambda$.

Proof : It is clear.

Proposition (1.1.42)

Let X be a topological space and $U \subseteq X$ then U coc-r-open set in a space X if and only if U is coc-r- nbd for all it points.

Proof :

Suppose U coc-r-open set and $x \in U$, since $x \in U \subseteq U$ then U is coc-r- nbd of x for each $x \in U$.

Conversely :

Suppose U coc-r-nbd for all it points and $x \in U$, then U is coc-r- nbd for x thus there exists coc-r-open set G_x such that $x \in G_x \subseteq U$, there fore $U = \cup\{x: x \in U\} \subseteq \cup\{G_x: x \in G_x\} \subseteq U$, so $U = \cup\{G_x: x \in G_x\}$, G_x is coc-r-open set and the union of coc-r-open sets is also coc-r-open . Hence U is coc-r-open set.

Definition (1.1.43)

Let X be a topological space and $x \in X$, $B \subseteq X$, the point x is called coc-r-limit point of B if every coc-r-open set containing x contains a point of B distinct from x. The set of all coc-r-limit point of B is called coc-r-derived set of B and denoted by B'^{rk}, then $x \in B'^{rk}$ iff for every coc-r-open set G in X, i.e $x \in G$ and $(G \cap B) - \{x\} \neq \emptyset$.

Proposition (1.1.44)

Let X be a topological space and B, $C \subseteq X$ then:

1) $\overline{B}^{rk} = B \cup B'^{rk}$.
2) B coc-r-closed set if and only if $B'^{rk} \subseteq B$.
3) If $B \subseteq C$, then $B'^{rk} \subseteq C'^{rk}$.

Proof : It is clear.

Definition (1.1.45)

Let X be a topological space and B be any subset of X . A point $x \in X$ is called coc-r-boundary point of B iff for every coc-r-open set G_x containing x, $G_x \cap B \neq \emptyset$ and $G_x \cap B^c \neq \emptyset$.

The family of every coc-r-boundary point of B is denoted by $b_{rk}(B)$

Proposition (1.1.46)

Let X be a topological space and B be any subset of X then:

1) $b_{rk}(B) = \overline{B}^{rk} \cap \overline{B^c}^{rk}$.
2) $B^{\circ rk} = B - b_{rk}(B)$.
3) $\overline{B}^{rk} = B \cup b_{rk}(B)$.
4) $\overline{B}^{rk} = B^{\circ rk} \cup b_{rk}(B)$.
5) $b_{rk}(B) = b_{rk}(B^c)$.
6) B coc-r-open set iff $b_{rk}(B) \subseteq B^c$.
7) B coc-r-closed set iff $b_{rk}(B) \subseteq B$.

Proof: It is clear.

Definition (1.1.47)

Let Y be a subspace of a space (X, τ). A subset B of a space (X, τ) is said to be an coc-r-open set in Y if for every $x \in B$ there exists a r-open set U in Y and a compact subset K in Y such that $x \in U - K \subseteq B$.

Theorem (1.1.48)

Let Y be a subspace of a space (X, τ). If Y is an open set in (X, τ) then $U \subseteq Y$ is a r-open set in Y if and only if U is a r-open set in (X, τ).

Proof :

Let $U \subseteq Y \subseteq X$, Y be an open set in X and U be a r-open set in Y then $U = \overline{U}^{Y\circ Y} = (\overline{U} \cap Y)^{\circ Y} = \overline{U}^{\circ Y} \cap Y^{\circ Y} = \overline{U}^{\circ Y} \cap Y^{\circ} = \overline{U}^{\circ}$, hence U is a r-open set in X. Conversely, let U is a r-open set in X, then $U = \overline{U}^{\circ} = \overline{U}^{\circ Y} \cap Y^{\circ} = \overline{U}^{\circ Y} \cap Y^{\circ Y} = (\overline{U} \cap Y)^{\circ Y} = \overline{U}^{Y\circ Y}$, hence U is a r-open set in Y.

Definition (1.1.49) [8]

A subset S of a topological space (X, τ) is said to be clopen if it is both open and closed in (X, τ).

Remarks (1.1.50)

Let (X, τ) be topological space, then:

1. Every clopen set is r-open set. **[8]**
2. Every clopen set is coc-r-open set.

Theorem (1.1.51)

Let Y be a subspace of a space (X, τ), $B \subseteq Y$. If Y is a clopen set in (X, τ), then B is a coc-r-open set in Y if and only if B is a coc-r-open set in (X, τ).

Proof :

Let B be a coc-r-open set in Y and $x \in B \subseteq Y$ then there exists a r-open set U_x in Y and a compact subset K_x in Y such that $x \in U_x - K_x \subseteq B$. Since Y is a clopen set

in X then Y is an open set in X, thus U_x is a r-open set in X (Theorem (1.1.48)), therefore $U_x - K_x$ is a coc-r-open set in X. Put $V = \cup_{x \in B} (U_x - K_x)$, thus V is a coc-r-open set in X. Now, we need to prove $B = V$, since $U_x - K_x \subseteq B$ for all $x \in B$ then $V \subseteq B$, let $y \in B$, thus there exists a r-open set U_y in Y and a compact subset K_y in Y such that $y \in U_y - K_y \subseteq B$, therefore $y \in \cup_{x \in B} (U_x - K_x) = V$, so that $B \subseteq V$. Hence $B = V$.
Conversely, let $x \in B$ then there exists a r-open set U in X and a compact subset K in X such that $x \in U - K \subseteq B$, since Y is a clopen set in X, then Y is a r-open set in X (Remarks (1.1.50), (1)), thus $U \cap Y$ is a r-open set in X, since $U \cap Y \subseteq Y$ and Y is an open set in X, therefore $U \cap Y$ is a r-open set in Y (Theorem (1.1.48)). Now, since K is a compact in X and Y is a closed in X, so $K \cap Y$ is a compact in X (Remarks (1.1.20), (4)) and $K \cap Y \subseteq Y$, hence $K \cap Y$ is a compact in Y. Since $x \in U - K$ then $x \in U$ but $x \notin K$, thus $x \in U \cap Y$ but $x \notin K \cap Y$, therefore $x \in (U \cap Y) - (K \cap Y) \subseteq (U - K) \cap Y \subseteq B$. Hence B is a coc-r-open set in Y.

Corollary (1.1.52)

Let Y be a clopen subspace of a space (X, τ). If G coc-r-open set in (X, τ) then $G \cap Y$ coc-r-open set in Y.

Proof :

Let Y be a clopen subspace of a space X and G be a coc-r-open set in X, since Y is a clopen set in X, then Y coc-r-open set in X (Remarks (1.1.50), (2)), thus $G \cap Y$ also coc-r-open set in X, therefore $G \cap Y$ coc-r-open set in Y(Theorem (1.1.51)).

Corollary (1.1.53)

Let Y be a subspace of a space (X, τ), $F \subseteq Y$. If Y is a clopen set in (X, τ) then F is a coc-r-closed set in Y if and only if F is a coc-r-closed set in (X, τ).
Proof :

Let F is a coc-r-closed set in Y then F^c is a coc-r-open set in Y, thus F^c is a coc-r-open in X(Theorem (1.1.51)) , therefore F is a coc-r-closed set in X.
Conversely, let F is a coc-r-closed set in X then F^c is a coc-r-open set in X, thus F^c is a coc-r-open in Y (Theorem (1.1.51)) , therefore F is a coc-r-closed set in Y.

2. On coc-r-β - open and coc-r - regular open sets.

This section present the definition of coc-r-β - open and coc-r - regular open sets, remarks ,propositions and example about them.

Definition (1.2.1) [4]

Let (X, τ) be topological space and $B \subseteq X$, then:

1) A subset B is called β - open set if $B \subseteq \overline{\overline{B}^{\circ}}$.

The complement of β - open is called to be β - closed.

2) A subset B is called β - closed set if $\overline{B^{\circ}}^{\circ} \subseteq B$.

Definition (1.2.2)

Let (X, τ) be topological space and $B \subseteq X$, then:

1) A subset B is called coc-r-β - open set if $B \subseteq \overline{\overline{B}^{rk\circ rk}}^{rk}$.

The complement of coc-r-β - open is called to be coc-r-β - closed.

2) A subset B is called coc-r-β - closed set if $\overline{B^{\circ rk}}^{rk\circ rk} \subseteq B$.

Remark (1.2.3)

Let (X, τ) be topological space, then:

1) β - open $\nrightarrow$ coc-r-β - open.

2) coc-r-β - open $\nrightarrow$ β - open.

As the following examples shows.

Examples (1.2.4)

1) Let $X = \{1,2,3, \ldots\}, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, let $A = \{1\}$, then $\overline{\overline{A}^{rk\circ rk}}^{rk} = \emptyset$, then A is not coc-r-β - open but $\overline{\overline{A}^{\circ}} = X$, then A is β - open.

2) Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}\}$, then $\tau^{rk} = \{A : A \subseteq X\}$, then $\{b\}$ is coc-r-β - open but is not β - open because $\overline{\overline{\{b\}}^{\circ}} = \emptyset$.

Remark (1.2.5)

Every coc-r-open is coc-r-β - open set in (X, τ) but the convers is not true in general, as the following example shows.

Example (1.2.6)

Let X = R with usual topology, since X is T_2 and regular space, then $\tau = \tau^{rk}$, A = (0,1], thus A is coc-r-β - open but is not coc-r-open in X.

Remark (1.2.7)

The intersection of two coc-r-β - open sets is not necessary coc-r-β - open set, as the following example show.

Example (1.2.8)

Let X = R with usual topology, since X is T_2 and regular space, then $\tau = \tau^{rk}$, A = (0,1], B = [1,2), thus A, B are coc-r-β - open but A ∩ B ={1} is not coc-r - open in X.

Remarks (1.2.9)

Let (X, τ) be topological space and B ⊆ X, then:

1) A subset B is called coc-r-β - open in (X, T) iff B is called β - open in (X, τ^{rk}).
2) The family of all coc-r-β - open sets in X is denoted by $\beta O(X, \tau^{rk})$.
3) Every r-open is coc-r-β - open set.

Definition (1.2.10)

Let X be a topological space and A ⊆ X . A is said to be coc-r - regular open set in X if $A = \overline{A}^{rk\circ rk}$ .The complement of coc-r - regular open set is called coc-r - regular closed and it is easy to see that A is coc-r - regular closed if $A = \overline{A^{\circ rk}}^{rk}$.

Remarks (1.2.11)

Let (X, τ) be topological space B ⊆ X, then:

1) A subset B is called coc-r- regular open in (X, T) iff B is called r-open in (X, τ^{rk}).
2) The family of all coc-r - regular open sets in X is denoted by $RO(X, \tau^{rk})$.
3) The family of all coc-r - regular closed sets in X is denoted by $RC(X, \tau^{rk})$.

Remarks (1.2.12)

If $A \in RO(X, \tau^{rk})$, then A is coc-r-open but the convers is not true, as the following example.

Example (1.2.13)

Let $X = \{1,2,3,\dots\}, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, let $A = \{1,3,4,5,\dots\}$ is coc-r-open in X but $\overline{A}^{rk\circ rk} = X$, hence $A \notin RO(X, \tau^{rk})$.

Proposition (1.2.14)

Let (X, τ) be topological space and $A, B \subseteq X$, then:

1) If A is coc-r-open, then $\overline{A}^{rk} \in RC(X, \tau^{rk})$.

2) If A is coc-r-closed, then $A^{\circ rk} \in RO(X, \tau^{rk})$.

3) If $A, B \in RO(X, \tau^{rk})$, then $A \cap B \in RO(X, \tau^{rk})$.

4) If $A \in \beta O(X, \tau^{rk})$, then $\overline{A}^{rk} \in RC(X, \tau^{rk})$.

Proof :

1) Let A is coc-r-open, then $A = A^{\circ rk}$. Since $A \subseteq \overline{A}^{rk}$, thus $\overline{A^{\circ rk}}^{rk} \subseteq \overline{\overline{A}^{rk\circ rk}}^{rk}$, therefore $\overline{A}^{rk} \subseteq \overline{\overline{A}^{rk\circ rk}}^{rk}$ ....(1)

Since $\overline{A}^{rk\circ rk} \subseteq \overline{A}^{rk}$, then $\overline{\overline{A}^{rk\circ rk}}^{rk} \subseteq \overline{A}^{rk}$ ....(2)

From (1), (2) we get $\overline{A}^{rk} = \overline{\overline{A}^{rk\circ rk}}^{rk}$, hence $\overline{A}^{rk} \in RC(X, \tau^{rk})$.

2) Let A is coc-r-closed, then $A = \overline{A}^{coc-r}$.

Since $\overline{A^{\circ rk}}^{rk} \subseteq \overline{A}^{rk} = A$, then $\overline{A^{\circ rk}}^{rk\circ rk} \subseteq A^{\circ rk}$.........(1)

Since $A^{\circ rk} \subseteq \overline{A^{\circ rk}}^{rk\circ rk}$ ....(2)

From (1), (2) we get $A^{\circ coc-r} = \overline{A^{\circ rk}}^{rk\circ rk}$, hence $A^{\circ rk} \in RO(X, \tau^{rk})$.

3) Let $A, B \in RO(X, \tau^{rk})$, then A, B are r-open in (X, τ^{rk}). Since the intersection of two r-open sets are r-open . Thus $A \cap B$ is r-open in (X, τ^{rk}), hence $A \cap B \in RO(X, \tau^{rk})$.

4) Since $A \in \beta O(X, \tau^{rk})$, then $A \subseteq \overline{\overline{A}^{rk\circ rk}}^{rk}$, so $\overline{A}^{rk} \subseteq \overline{\overline{A}^{rk\circ rk}}^{rk}$ .But $\overline{\overline{A}^{rk\circ rk}}^{rk} \subseteq \overline{A}^{rk}$, thus $\overline{A}^{rk} = \overline{\overline{A}^{rk\circ rk}}^{rk}$, hence $\overline{A}^{rk} \in RC(X, \tau^{rk})$.

Remarks (1.2.15)

Let (X, τ) be topological space and $A \subseteq X$, then:

1) If $A \in RO(X, \tau^{rk})$, then $A \in \beta O(X, \tau^{rk})$.

2) If $A \in RC(X, \tau^{rk})$, then $A \in \beta O(X, \tau^{rk})$.

3) If $A \in RC(X, \tau^{rk})$, then A is coc-r-closed.

Proof : It is clear.

Chapter two
On coc-r-Continuous, coc-r-open functions, coc-r-Separation Axioms, coc-r-Connected Space

Introduction

This Chapter is divided into four sections . In section one, we recall definition of coc-r-continuous function and prove some properties about it . In section two, we recall definitions of coc-r-open function and prove some properties about it. In section three, we introduced fundamental concept of separation axioms and generalized by coc-r-open sets. In section four, we introduce the fundamental concept of connected space and generalized by coc-r-open sets.

2.1 On coc-r-continuous Functions

In this section, we introduce the definition of coc-r-continuous , remarks and propositions about this concept .

Definition (2.1.1) [12]

Let $f: X \longrightarrow Y$ be a function of a space X into a space Y.Then f is called a continuous function if $f^{-1}(B)$ is an open set in X for every open set B in Y.

Theorem (2.1.2) [14]

Let $f: X \longrightarrow Y$ function of a space X into a space Y then the following statements are equivalent.

i. f is a continuous function .

ii. $f^{-1}(C)$ is a closed set in X for every closed set C in Y.

iii. $f(\overline{A}) \subseteq \overline{f(A)}$for every set A in X.

iv. $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every set B in Y.

v. $f^{-1}(B^{\circ}) \subseteq \left(f^{-1}(B)\right)^{\circ}$for every set B in Y.

Definition (2.1.3) [10]

Let $f: X \longrightarrow Y$ be a function of a space X into a space Y, then f is called coc-continuous function if $f^{-1}(B)$ is a coc-open set in X for each open set B in Y.

Definition (2.1.4)

Let $f: X \longrightarrow Y$ be a function of a space X into a space Y, then f is called coc-r-continuous function if $f^{-1}(B)$ is a coc-r-open set in X for each open set B in Y.

Proposition (2.1.5)

1. Every continuous function is coc-continuous function. **[10]**
2. Every coc-r-continuous function is coc-continuous function.

Proof:

2) Let $f: X \longrightarrow Y$ be a coc-r-continuous function and B be an open set in Y. To prove that $f^{-1}(B)$ is a coc-open set in X, since f is a coc-r-continuous function, then $f^{-1}(B)$ coc-r-open set in X and every coc-r-open set is coc-open set. Hence f is coc-continuous function.

Remark (2.1.6)

The converse of Proposition (2.1.5) is not true in general as the following examples show:

Examples (2.1.7)

1. Let $X = \{1,2\}$ and $Y = \{3,4\}$, τ_X be indiscrete topology on X and $\tau_Y = \{\phi, Y, \{3\}\}$ be a topology on Y. Let $f: X \to Y$ be a function defined by $f(1) = 3, f(2) = 4$ then f is an coc-continuous ,but is not continuous.

2. Let $X = \{1,2,3, \ldots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $Y = \{a, b, c\}$ and $\tau_Y = \{\phi, Y, \{a\}\}$, then $\tau_Y{}^{rk}$ is discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x \in \{1,2\} \\ b, & x \notin \{1,2\} \end{cases}$, since $\{a\}$ is open set in Y but $f^{-1}(\{a\}) = \{1,2\}$ is not coc-r-open set in X but $\{1,2\}$ is coc-open set in X, thus f is an coc-continuous ,but is not coc-r-continuous.

Remark (2.1.8)

1) continuous $\not\longrightarrow$ coc-r-continuous.
2) coc-r-continuous $\not\longrightarrow$ continuous.

Examples (2.1.9)

1. In Examples (2.1.7), (2) f is an continuous ,but is not coc-r-continuous.

2. Let $X = \{1,2,3, \ldots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $Y = \{a, b, c\}$ and $\tau_Y = \{\phi, Y, \{b\}\}$, then $\tau_Y{}^{rk}$discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x \in \{1,2\} \\ b, & x \notin \{1,2\} \end{cases}$, since $\{b\}$ is open set in Y but $f^{-1}(\{b\}) = \{3,4, \ldots\}$ is not open set in X but $\{3,4, \ldots\}$ is coc-r-open set in X, thus f is an coc-r-continuous ,but is not continuous.

Remarks (2.1.10)

Let f: X ⟶ Y be a function of a space X in to a space Y then

i. Every constant function is coc-r-continuous function.

ii. If(X,τ) is discrete space then f is a coc-r-continuous.

iii. If X finite set and τ any topology on X then f is a coc-r-continuous.

iv. If (Y,τ^*) is an indiscrete space then f coc-r-continuous.

v. If(X,τ) T_2 -space then every coc-r-continuous function is continuous function.

vi. If(X,τ) T_2 -space then and (Y,τ^*) indiscrete topology, then f coc-r-continuous function iff f continuous function.

vii. If(X,τ) is a discrete topology, then f is a coc-r-continuous function iff f is a continuous function

Theorem (2.1.11)

Let f: X ⟶ Y be a function of a space X into a space Y. Then the following statements are equivalent.

1. f is coc-r-continuous function.

2. $f^{-1}(B^\circ) \subseteq \left(f^{-1}(B)\right)^{\circ rk}$for every set B in Y.

3. $\overline{f^{-1}(B)}^{rk} \subseteq f^{-1}(\overline{B})$ for every set B in Y.

4. $f\left(\overline{A}^{rk}\right) \subseteq \overline{f(A)}$ for every set A in X.

5. $f^{-1}(C)$ coc-r-closed set in X for every closed set C in Y.

Proof:

(1) ⟶ (2)

Since B° is an open set in Y and f is a coc-r-continuous function then $f^{-1}(B^\circ)$ coc-r-open set in X, thus $f^{-1}(B^\circ) = (f^{-1}(B^\circ))^{\circ rk} \subseteq (f^{-1}(B))^{\circ rk}$ for every set B in Y.

$(2) \longrightarrow (3)$

Since $B \subseteq \overline{B}$ then $f^{-1}(B) \subseteq f^{-1}(\overline{B})$, we need to prove that $f^{-1}(\overline{B})$ coc-r-closed in X. Since $\overline{B}^{\,c}$ is open in Y, then $\overline{B}^{\,c\circ} = \overline{B}^{\,c}$ and $f^{-1}(\overline{B}^{\,c\circ}) \subseteq \left(f^{-1}(\overline{B}^{\,c})\right)^{\circ rk}$ thus $f^{-1}(\overline{B}^{\,c}) \subseteq \left(f^{-1}(\overline{B}^{\,c})\right)^{\circ rk}$, therefore $f^{-1}(\overline{B}^{\,c})$ coc-r-open in X and $f^{-1}(\overline{B}^{\,c}) = (f^{-1}(\overline{B}))^{c}$. So we get $f^{-1}(\overline{B})$ coc-r-closed in X, hence $\overline{f^{-1}(B)}^{rk} \subseteq f^{-1}(\overline{B})$ for every set B in Y.

$(3) \longrightarrow (4)$

Let $A \subseteq X$, then $f(A) \subseteq Y$ thus $\overline{f^{-1}\big(f(A)\big)}^{rk} \subseteq f^{-1}\big(\overline{f(A)}\big)$, therefore $\overline{A}^{rk} \subseteq f^{-1}\big(\overline{f(A)}\big)$ hence $f\left(\overline{A}^{rk}\right) \subseteq f\left(f^{-1}\big(\overline{f(A)}\big)\right) \subseteq \overline{f(A)}$ for every set A in X.

$(4) \longrightarrow (5)$

Let C be closed set in Y, to prove $\overline{f^{-1}(C)}^{rk} \subseteq f^{-1}(C)$. Since $f^{-1}(C) \subseteq X$ then $f\left(\overline{f^{-1}(C)}^{rk}\right) \subseteq \overline{f\big(f^{-1}(C)\big)} \subseteq \overline{C} = C$, thus $\overline{f^{-1}(C)}^{rk} \subseteq f^{-1}(C)$, hence $f^{-1}(C)$ coc-r-closed set in X for every closed set C in Y.

$(5) \longrightarrow (1)$

Let B be open set in Y, to prove $f^{-1}(B)$ coc-r-open set in X. Since B open set in Y then B^{c} closed set in Y, thus $f^{-1}(B^{c})$ coc-r-closed set in X, there fore $f^{-1}(B)$ coc-r-open set in X, hence f is coc-r-continuous function.

Remarks (2.1.12)

From Theorem (2.1.11) we have f is a coc-r-continuous function iff the inverse image of every closed set in Y is a coc-r-closed set in X.

Proposition(2.1.13)

If $f: X \longrightarrow Y$is coc-r-continuous function and bijective then for all $y \in Y$ and for all U neighborhood of y there exists coc-r-open G in X such that $f^{-1}(y) \in G \subseteq f^{-1}(U)$ and $f^{-1}(U)$ coc-r-neighborhood of $f^{-1}(y)$.

Proof:

Let $y \in Y$ and U nbd of y, then there exists V open set V in Y such that $y \in V \subseteq U$ .Since f coc-r-continuous function then $f^{-1}(V)$ coc-r-open set in X, since f onto thus there exists $x \in X$ such that $f(x) = y$, since fis one to one so $x = f^{-1}(y)$ and $y = f(x) \in V$. There fore $f^{-1}(y) = x \in f^{-1}(V) \subseteq f^{-1}(U)$. Put $G = f^{-1}(V)$, hence $f^{-1}(y) \in G \subseteq f^{-1}(U)$ and $f^{-1}(U)$ coc-r-neighborhood of $f^{-1}(y)$.

Remarks (2.1.14)

A composition of two coc-r-continuous function is not necessary to be coc-r-continuous function.

Examples (2.1.15)

Let $X = \{1,2,3,\dots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$,then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $Y = \{a, b, c\}$ and $\tau_Y = \{\phi, Y, \{b\}\}$, then $\tau_Y{}^{rk}$ discrete, , $Z = \{d, e\}$ and $\tau_Z = \{\phi, Z, \{d\}\}$, then $\tau_Z{}^{rk}$ is also discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x \in \{1,2\} \\ b, & x \notin \{1,2\} \end{cases}$, $g: Y \to Z$ defined by $g(a) = d, g(b) = g(c) = e$. Then f, g are coc-r-continuous function, but $g \circ f: X \to Z$ is not coc-r-continuous function, since $(g \circ f)^{-1}(\{d\}) = \{1,2\}$ is not coc-r-open set in X.

Proposition (2.1.16)

Let X, Y and Z are spaces and $f: X \to Y$ be coc-r-continuous. If $g: Y \to Z$ is continuous then $g \circ f: X \to Z$ is coc-r-continuous.

Proof:

Let U be an open set in Z, since $g: Y \to Z$ is continuous, then $g^{-1}(U)$ open set in Y, since $f: X \to Y$ is coc-r-continuous, then $f^{-1}\left(g^{-1}(U)\right) = (g \circ f)^{-1}(U)$coc-r-open set in X, hence $g \circ f: X \to Z$ is coc-r-continuous.

Definition (2.1.17)

Let $f: X \longrightarrow Y$ be a function of a space X into a space Y. f is called coc-irresolute (coć-continuous) function if $f^{-1}(U)$ coc-open set in X for each coc-open set U in Y.

Definition (2.1.18)

Let $f: X \longrightarrow Y$ be a function of a space X into a space Y. Then f is called coc-r- irresolute (coć-r-continuous) function if $f^{-1}(U)$ coc-r-open set in X for each coc-r-open set U in Y.

Remarks (2.1.19)

1. Every coć-continuous function is coc-continuous function but the convers is not true in general.
2. continuous function $\nleftrightarrow$ coć-r-continuous function.
3. coc-r-continuous function $\nleftrightarrow$ coć-r-continuous function.
4. coć-continuous function $\nleftrightarrow$ coć-r-continuous function.

As the following examples show:

Examples (2.1.20)

1. Let $X = \{1,2,3,\dots\}, \tau_X = \{G \subseteq X: 1 \notin G\} \cup \{X\}$, then $\tau_X{}^{rk} = \{G \subseteq X: 1 \notin G\} \cup \{G \subseteq X: 1 \in G, G^c \text{ is finite}\}$, $Y = \{a,b,c\}$ and $\tau_Y = \{\phi, Y, \{b\}\}$, then $\tau_Y{}^{rk}$discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x = 1 \\ b, & x \neq 1 \end{cases}$, f is coc-continuous function but is not coć-continuous function since $\{a\}$ is coc-open in Y but $f^{-1}(\{a\}) = \{1\}$ is not coc-open in X.

2. (i) Let $X = \{1,2,3,\dots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$,then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $Y = \{a,b,c\}$ and $\tau_Y = \{\phi, Y, \{a\}\}$, then $\tau_Y{}^{rk}$discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x \in \{1,2\} \\ b, & x \notin \{1,2\} \end{cases}$, since $f^{-1}(\{a\}) = \{1,2\}$ is open in X but is not coc-r-open in X, hence f is continuous function but is not coć-r-continuous.

(ii) Let $X = \{1,2,3\}$ and $\tau_X = \{\phi, Y, \{2\}\}$, then $\tau_X{}^{rk}$ discrete, $Y = \{a,b,c\}$ and $\tau_Y = \{\phi, Y, \{a\}\}$, then $\tau_Y{}^{rk}$discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x = 1 \\ b, & x \neq 1 \end{cases}$, f is coć-continuous but is not coc-continuous function.

3. (i) Let $X = \{1,2,3,\dots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $Y = \{a,b,c\}$ and $\tau_Y = \{\phi, Y, \{b\}\}$, then $\tau_Y{}^{rk}$ discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x \in \{1,2\} \\ b, & x \notin \{1,2\} \end{cases}$, then f coc-r-continuous function but is not coć-r-continuous function, since $f^{-1}(\{a\}) = \{1,2\}$ is not coc-r-open set in X.

(ii) Let $X = \{1,2,3,\dots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: X \to X$ be a function defined by $(x) = x$, for all $x \in X$, then f coć-r-continuous function but is not coc-r-continuous function, since $f^{-1}(\{1\}) = \{1\}$ is not coc-r-open set in X.

4. (i) Let $X = \{1,2,3,\dots\}, \tau_1 = \{G \subseteq X: 1 \notin G\} \cup \{X\}$, then $\tau_1{}^{rk} = \{G \subseteq X: 1 \notin G\} \cup \{G \subseteq X: 1 \in G, G^c \text{ is finite}\}$, $\tau_2 = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau_1{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: (X, \tau_1) \to (X, \tau_2)$ be a function defined by $f(x) = x$, for all $x \in X$ then f coć-r-continuous function but is not coć-continuous function, since $f^{-1}(\{1\}) = \{1\}$ is not coc-open set in X.

(ii) Let $X = \{1,2,3,\dots\}, \tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then ${\tau_X}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $Y = \{a, b, c\}$ and $\tau_Y = \{\phi, Y, \{a\}\}$, then ${\tau_Y}^{rk}$discrete and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} a, & x \in \{1,2\} \\ b, & x \notin \{1,2\} \end{cases}$, then f coć-continuous function since $f^{-1}(\{a\}) = \{1,2\}$ and $f^{-1}(\{b\}) = \{2,3,4,\dots\}$ are coc-open sets in X but is not coć-r-continuous function, since $f^{-1}(\{a\}) = \{1,2\}$ is not coc-r-open set in X.

Theorem (2.1.21)

Let $f: X \longrightarrow Y$ function of a space X into a space Y then the following statements are equivalent.

1. f is coć-r-continuous function.

2. $f^{-1}(B^{\circ rk}) \subseteq \left(f^{-1}(B)\right)^{\circ rk}$for every set B in Y.

3. $\overline{f^{-1}(B)}^{rk} \subseteq f^{-1}\left(\overline{B}^{rk}\right)$ for every set B in Y.

4. $f\left(\overline{A}^{rk}\right) \subseteq \overline{f(A)}^{rk}$for every set A in X.

5. $f^{-1}(C)$ coc-r-closed set in X for every coc-r-closed set C in Y.

Proof:

(1) ⟶ (2)

Since $B^{\circ rk}$ is an coc-r-open set in Y and f is a coc-r-continuous function then

$f^{-1}(B^{\circ rk})$coc-r-open set in X,thus $f^{-1}(B^{\circ rk}) = \left(f^{-1}(B^{\circ rk})\right)^{\circ rk} \subseteq (f^{-1}(B))^{\circ rk}$

for every set B in Y.

(2) ⟶ (3)

Since $B \subseteq \overline{B}^{rk}$then $f^{-1}(B) \subseteq f^{-1}\left(\overline{B}^{rk}\right)$, we need to prove that $f^{-1}\left(\overline{B}^{rk}\right)$ coc-r-closed in X. Since ${\overline{B}^{rk}}^{c}$ is coc-r-open in Y, then $({\overline{B}^{rk}}^{c})^{\circ coc-r} = {\overline{B}^{rk}}^{c}$ and

$f^{-1}(({\overline{B}^{rk}}^{c})^{\circ rk}) \subseteq \left(f^{-1}\left({\overline{B}^{rk}}^{c}\right)\right)^{\circ rk}$ thus $f^{-1}({\overline{B}^{rk}}^{c}) \subseteq \left(f^{-1}\left({\overline{B}^{rk}}^{c}\right)\right)^{\circ rk}$, therefore $f^{-1}(\overline{B}^{rk^c})$ coc-r-open in X and $f^{-1}(\overline{B}^{rk^c}) = (f^{-1}(\overline{B}^{rk}))^c$. So we get $f^{-1}\left(\overline{B}^{rk}\right)$ coc-r-closed in X, hence $\overline{f^{-1}(B)}^{rk} \subseteq f^{-1}\left(\overline{B}^{rk}\right)$ for every set B in Y.

(3) ⟶ (4)

Let $A \subseteq X$, then $f(A) \subseteq Y$ thus $\overline{f^{-1}(f(A))}^{rk} \subseteq f^{-1}\left(\overline{f(A)}^{rk}\right)$, therefore $\overline{A}^{rk} \subseteq f^{-1}\left(\overline{f(A)}^{rk}\right)$hence $f\left(\overline{A}^{rk}\right) \subseteq f\left(f^{-1}\left(\overline{f(A)}^{rk}\right)\right) \subseteq \overline{f(A)}^{rk}$ for every set A in X.

(4) ⟶ (5)

Let C be coc-r-closed set in Y, to prove $\overline{f^{-1}(C)}^{rk} \subseteq f^{-1}(C)$. Since $f^{-1}(C) \subseteq X$ then $f\left(\overline{f^{-1}(C)}^{rk}\right) \subseteq \overline{f(f^{-1}(C))}^{rk} \subseteq \overline{C}^{rk} = C$, thus $\overline{f^{-1}(C)}^{rk} \subseteq f^{-1}(C)$, hence $f^{-1}(C)$ coc-r-closed set in X for every coc-r-closed set C in Y.

(5) ⟶ (1)

Let B be coc-r-open set in Y, to prove $f^{-1}(B)$ coc-r-open set in X. Since B coc-r-open set in Y then B^c coc-r-closed set in Y, thus $f^{-1}(B^c)$ coc-r-closed set in X, there fore $f^{-1}(B)$ coc-r-open set in X, hence f is coć-r-continuous function.

Remarks (2.1.22)

From Theorem (2.1.21) we have f coć-r-continuous function iff the inverse image of every coc-r-closed set in Y is a coc-r-closed set in X.

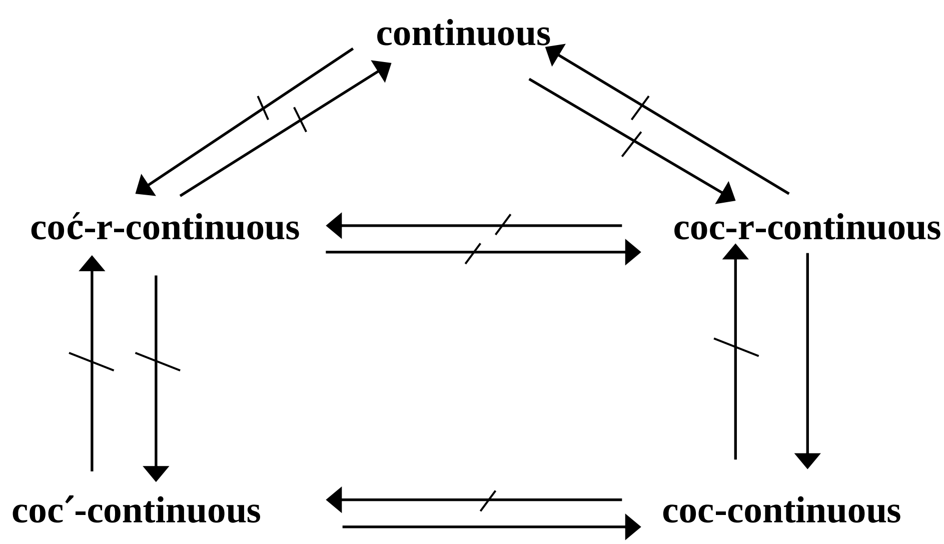
Proposition (2.1.23)

Let X, Y and Z are spaces , $f: X \to Y$ and $g: Y \to Z$ are coć-r-continuous function then $g \circ f: X \to Z$ is coć-r-continuous function.

Proof:

Let U be coc-r-open set in Z, to prove $(g \circ f)^{-1}(U)$ coc-r-open set in X, since $f: X \to Y$ and $g: Y \to Z$ are coć-r-continuous function, then $g^{-1}(U)$ coc-r-open set in Y and $f^{-1}\left(g^{-1}(U)\right)$ coc-r-open set in X, but $f^{-1}\left(g^{-1}(U)\right) = (g \circ f)^{-1}(U)$, hence $g \circ f: X \to Z$ is coć-r-continuous function.

The following diagram shows the relation among certain types of continuous functions



2.2 On coc-r-open functions

We introduce and study coc-r-open and coc-r-closed function also some properties about them.

Definition (2.2.1)

Let $f: X \longrightarrow Y$ be a function of space X into space Y then:-

1. f is called open function if f(U) is open set in Y for every open set U in X . **[2]**
2. f is called r-open function if f(U) is r-open set in Y for every open set U in X.
3. f is called coc-open function if f(U) is coc-open set in Y for every open set U in X. **[10]**
4. f is called coc-r-open function if f(U) is coc-r-open set in Y for every open set U in X.

Theorem (2.2.2) [5]

Let $f: X \longrightarrow Y$ be a function of space X into space Y then the following statements are equivalent.

1. f open function .
2. $f(A^\circ) \subseteq \left(f(A)\right)^\circ$ for every subset A of X .
3. $\left(f^{-1}(A)\right)^\circ \subseteq f^{-1}(A^\circ)$ for every subset A of Y.
4. $f^{-1}(\overline{A}) \subseteq \overline{f^{-1}(A)}$ for every subset A of Y.

Proposition (2.2.3)

1. Every r-open function is coc-r-open function.
2. Every coc-r-open function is coc-open function.

Proof: It is clear.

Remark (2.2.4)

the converse of Proposition (2.2.3) is not true in general as the following examples show:

Examples (2.2.5)

1. Let $X = \{1,2,3\}$ and $Y = \{1,2,3 \dots\}$, $\tau_X = \{\phi, X, \{1\}\}$, $\tau_Y = \{G \subseteq Y : 1 \in G\} \cup \{\emptyset\}$ then ${\tau_X}^{rk}$ discrete, ${\tau_Y}^{rk} = \{G \subseteq Y : G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: X \to Y$ be a function defined by $f(x) = \begin{cases} x, & x = 1 \\ x+1, & x \neq 1 \end{cases}$, since $f(\{1\}) = \{1\}$ is not coc-r-open set in Y, then f coc-open function but is not coc-r-open function.

2. Let $X = \{1,2,3\}$ $\tau_1 = \{\phi, X, \{1\}\}$, $\tau_2 = \{\phi, X, \{2\}\}$, then ${\tau_1}^{rk}, {\tau_2}^{rk}$ are discrete and $f: (X, \tau_1) \to (X, \tau_2)$ be a function defined by $f(x) = x$, for all $x \in X$, then f coc-r-open function but is not r-open function since $f(\{1\}) = \{1\}$ is not r-open in Y.

Remark (2.2.6)

Open function $\not\leftrightarrow$ coc-r-open function.

As the following example shows:

Example (2.2.7)

In example (2.2.5), (1) f open function but is not coc-r-open function and in example (2.2.5), (2) f coc-r-open function but is not open function.

Theorem (2.2.8)

Let $f: X \longrightarrow Y$ be a function of space X into space Y then the following statements are equivalent.

1. f coc-r-open function .
2. $f(A^\circ) \subseteq (f(A))^{\circ rk}$ for every subset A of X .
3. $\left(f^{-1}(B)\right)^\circ \subseteq f^{-1}\left(B^{\circ rk}\right)$ for every subset B of Y.
4. $f^{-1}\left(\overline{B}^{rk}\right) \subseteq \overline{f^{-1}(B)}$ for every subset B of Y.

Proof:

(1) $\longrightarrow$ (2)

Let f be coc-r-open function $A \subseteq X$, then $f(A^\circ)$ coc-r-open set in Y and $f(A^\circ) \subseteq f(A)$, thus $f(A^\circ) \subseteq (f(A))^{\circ rk}$.

(2) $\longrightarrow$ (3)

Let $B \subseteq Y$, then $f^{-1}(B) \subseteq X$ thus $f(\left(f^{-1}(B)\right)^\circ) \subseteq (f(f^{-1}(B)))^{\circ rk} \subseteq B^{\circ rk}$. therefore $\left(f^{-1}(B)\right)^\circ \subseteq f^{-1}\left(B^{\circ rk}\right)$

(3) $\longrightarrow$ (4)

Let $B \subseteq Y$ then $\left(f^{-1}(B^c)\right)^\circ \subseteq f^{-1}\left(B^{c\circ rk}\right)$ thus $((f^{-1}(B))^c)^\circ \subseteq f^{-1}\left(B^{c\circ rk}\right)$ therefore $(\overline{f^{-1}(B)})^c \subseteq f^{-1}\left((\overline{B}^{rk})^c\right) = (f^{-1}\left(\overline{B}^{rk}\right))^c$, hence $f^{-1}\left(\overline{B}^{rk}\right) \subseteq \overline{f^{-1}(B)}$.

(4) $\longrightarrow$ (1)

Let U be open set in X then $f^{-1}\left(\overline{f(U)^{c}}^{rk}\right) \subseteq \overline{f^{-1}(f(U)^{c})}$, thus $f^{-1}\left((f(U)^{\circ rk})^c\right) \subseteq \left(f^{-1}(f(U))\right)^{\circ c}$, therefore $U = U^\circ \subseteq f^{-1}\left((f(U)^{\circ rk}\right)$, so we get $f(U) \subseteq (f(U))^{\circ rk}$, hence f(U) is coc-r-open set in X.

Proposition (2.2.9)

Let X, Y and Z are spaces and $f: X \to Y$ be open function. If $g: Y \to Z$ is coc-r-open function then $g \circ f: X \to Z$ is coc-r-open function.

Proof: Clear.

Definition (2.2.10)

Let $f: X \longrightarrow Y$ be a function of space X into space Y then:-

i- f is called closed function if f(F) is closed set in Y for every closed set F in X.**[6]**

ii- f is called coc-r-closed function if f(F) is coc-r-closed set in Y for every closed set F in X.

Remark (2.2.11)

closed function $\not\longleftrightarrow$ coc-r-closed function.

As the following example shows:

Example (2.2.12)

Let $X = \{1,2,3\}$ and $Y = \{1,2,3 \dots\}$, $\tau_X = \{\phi, X, \{3\}\}$, $\tau_Y = \{G \subseteq Y: 1 \in G\} \cup \{\emptyset\}$ then ${\tau_X}^{rk}$ discrete, ${\tau_Y}^{rk} = \{G \subseteq Y: G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: X \to Y$ be a function defined by $f(x) = x$, for all $x \in X$, since $\{1,2\}$ is closed set in X but $f(\{1,2\}) = \{1,2\}$ is not closed set in Y but is coc-r-closed, then f coc-r-closed function but is not a closed function. If $X = Y = \{1,2,3 \dots\}$ $\tau_X = \{\phi, X, \{1\}\}$$\tau_Y = \{G \subseteq Y: 1 \in G\} \cup \{\emptyset\}$ then ${\tau_X}^{rk}$ discrete, ${\tau_Y}^{rk} = \{G \subseteq Y: G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: X \to Y$ be a function defined by $f(x) = x$, for all $x \in X$, since $\{2,3,4, \dots\}$ is closed set in X but $f(\{2,3,4, \dots\}) = \{2,3,4, \dots\}$ is not coc-r-closed set in Y but is closed, then f closed function but is not a coc-r-closed function.

Proposition (2.2.13)

Let $f: X \longrightarrow Y$ be a function of space X into space Y, then f is a coc-r-closed function if and only if $\overline{f(A)}^{rk} \subseteq f(\bar{A})$, for all $A \subseteq X$.

Proof:

Let f be a coc-r-closed function, $A \subseteq X$. Then $\bar{A} \subseteq X$ and $f(\bar{A})$ is a coc-r-closed set in Y, since $f(A) \subseteq f(\bar{A})$, thus $\overline{f(A)}^{rk} \subseteq f(\bar{A})$. Conversely, Let F be a closed set in X, then $\overline{f(F)}^{rk} \subseteq f(\bar{F}) = f(F)$, thus f(F) coc-r-closed set in Y. Hence f coc-r-closed function.

Theorem (2.2.14)

For bijective function $f:(X,\tau)\longrightarrow(Y,\acute{\tau})$ the following statements are equivalent .

1) f is coc-r-open .

2) f^{-1} is coc-r-continuous.

3) f is coc-r-closed .

Proof:

(1) ⟶ (2)

Let U be open set in X, then $(f^{-1})^{-1}(U)=f(U)$ is a coc-r-open set in Y(f bijective, coc-r-open function), hence f^{-1} is coc-r-continuous function.

(2) ⟶ (3)

Let F be closed set in X, then $(f^{-1})^{-1}(F)=f(F)$ is a coc-r-closed set in Y(f bijective, f^{-1} is coc-r-continuous function), hence f is coc-r-closed function.

(3) ⟶ (1)

Let U be open set in X, then U^c closed set in in X, thus $f(U^c)=(f(U))^c$ is a coc-r-closed set in Y (f bijective, coc-r-closed function), there fore f(U) is a coc-r-open set in Y. Hence f is coc-r-open function.

Definition (2.2.15)

Let X and Y be spaces. A function $f:X\longrightarrow Y$ is called coc-r-homeomorphism if:

1. f is bijective .
2. f is coc-r-continuous .
3. f^{-1} is coc-r-continuous.

Proposition (2.2.16)

Let $f: X \longrightarrow Y$ be a function of space X into space Y, then f coc-r-homeomorphism iff:

1. fis bijective .
2. f is coc-r-continuous .
3. f is coc-r-open (coc-r-closed).

Proof: It is clear.

Definition (2.2.17)

Let $f: X \longrightarrow Y$ function of a space X into a space Y then :

i. f is called coć-open function if f(U) is coc-open set in Y for every coc-open set U in X. **[10]**

ii. f is called coć-r-open function if f(U) is coc-r-open set in Y for every coc-r-open set U in X.

Remark (2.2.18)

1. Coć-r-open function $\not\longleftrightarrow$ coc-r-open function.
2. Coć-r-open function $\not\longleftrightarrow$ coć-open function.

As the following examples show:

Examples (2.2.19)

1. Let $X = \{1,2,3\}$ and $Y = \{1,2,3 \dots\}$, $\tau_X = \{\phi, X\}$, $\tau_Y = \{G \subseteq Y: 1 \in G\} \cup \{\emptyset\}$ then $\tau_Y{}^{rk} = \{G \subseteq Y: G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: X \to Y$ be a function defined by $f(x) = x$, for all $x \in X$, thus f coc-r-open and coć-open function but is not coć-r-open function, since $\{1\}$ is coc-r-open in X, but $f(\{1\})$ is not coc-r-open in Y.

2. Let $X = Y = \{1,2,3 \dots\}$ $\tau_X = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, $\tau_Y = \{G \subseteq Y: 1 \notin G\} \cup \{X\}$, then $\tau_X{}^{rk} = \{G \subseteq X: G^c \text{ is finite}\} \cup \{\emptyset\}$, $\tau_Y{}^{rk} = \{G \subseteq Y: 1 \notin G\} \cup \{G \subseteq X: 1 \in G, G^c \text{ is finite}\}$ and $f: X \to Y$ be a function defined by $f(x) = x$, for

all $x \in X$, thus f coć-r-open function but is not coc-r-open, coć-open function, since $\{1\}$ is open (coc-open) in X, but $f(\{1\})$ is not coc-r-open (coc-open) in Y.

Theorem (2.2.20)

Let $f: X \longrightarrow Y$ be a function of space X into space Y then the following statements are equivalent.

1. f coć-r-open function .
2. $f(A^{\circ rk}) \subseteq (f(A))^{\circ rk}$ for every subset A of X .
3. $(f^{-1}(B))^{\circ rk} \subseteq f^{-1}(B^{\circ rk})$ for every subset B of Y.
4. $f^{-1}\left(\overline{B}^{rk}\right) \subseteq \overline{f^{-1}(B)}^{rk}$ for every subset B of Y.

Proof:

(1) $\longrightarrow$ (2)

Let $A \subseteq X$, then $f(A^{\circ rk})$ coc-r-open set in Y and $f(A^{\circ rk}) \subseteq f(A)$, thus $f(A^{\circ rk}) \subseteq (f(A))^{\circ rk}$.

(2) $\longrightarrow$ (3)

Let $B \subseteq Y$, then $f^{-1}(B) \subseteq X$ thus $f((f^{-1}(B))^{\circ rk}) \subseteq (f(f^{-1}(B)))^{\circ rk} \subseteq B^{\circ rk}$. therefore $(f^{-1}(B))^{\circ rk} \subseteq f^{-1}(B^{\circ rk})$.

(3) $\longrightarrow$ (4)

Let $B \subseteq Y$ then $(f^{-1}(B^c))^{\circ rk} \subseteq f^{-1}(B^{c\circ rk})$ thus $(f^{-1}(B))^{c\circ rk} \subseteq f^{-1}(B^{c\circ rk})$ therefore $(\overline{f^{-1}(B)}^{rk})^c \subseteq f^{-1}\left((\overline{B}^{rk})^c\right) = (f^{-1}\left(\overline{B}^{rk}\right))^c$, hence $f^{-1}\left(\overline{B}^{rk}\right) \subseteq \overline{f^{-1}(B)}^{rk}$.

(4) ⟶ (1)

Let U be coc-r-open set in X then $f^{-1}\left(\overline{f(U)^{c}}^{rk}\right) \subseteq \overline{f^{-1}(f(U)^{c})}^{rk}$, thus $f^{-1}\left(\left(f(U)^{\circ rk}\right)^{c}\right) \subseteq \left(\left(f^{-1}(f(U))\right)^{\circ rk}\right)^{c}$,therefore $U = U^{\circ rk} \subseteq f^{-1}\left(\left(f(U)^{\circ rk}\right)\right)$, so we get $f(U) \subseteq (f(U))^{\circ rk}$, hence f(U) is coc-r-open set in X.

Proposition (2.2.21)

Let X, Y and Z are spaces and $f: X \to Y$, $g: Y \to Z$ are coć-r-open function then $g \circ f: X \to Z$ is also.

Proof: It is clear.

Definition (2.2.22)

Let $f: X \longrightarrow Y$ be a function of space X into space Y then:-

i- f is called coć-closed function if f(F) is coc-closed set in Y for every coc-closed set F in X. **[10]**

ii- f is called coć-r-closed function if f(F) is coc-r-closed set in Y for every coc-r-closed set F in X.

Proposition (2.2.23)

Let $f: X \longrightarrow Y$ be a function of space X into space Y, then f coć-r-closed function if and only if $\overline{f(A)}^{rk} \subseteq f\left(\overline{A}^{rk}\right)$, for all $A \subseteq X$.

Proof:

Let f be a coć-r-closed function, $A \subseteq X$. Then $\overline{A}^{rk} \subseteq X$ and $f\left(\overline{A}^{rk}\right)$ is a coc-r-closed set in Y. Since $f(A) \subseteq f\left(\overline{A}^{rk}\right)$, then $\overline{f(A)}^{rk} \subseteq f\left(\overline{A}^{rk}\right)$. Conversely, Let F be coc-r-closed set in X, then $\overline{f(F)}^{rk} \subseteq f\left(\overline{F}^{rk}\right) = f(F)$, thus f(F) coc-r-closed set in Y. Hence f coć-r-closed function.

Theorem (2.2.24)

For a bijective function $f: (X, \tau) \longrightarrow (Y, \acute{\tau})$ the following statements are equivalent .

1) f^{-1} is coć-r-continuous .

2) f is coć-r-open.

3) f is coć-r-closed .

Proof: It is clear.

Definition (2.2.25)

Let X , Y be spaces. A function $f: X \longrightarrow Y$ is called coć-r-homeomorphism if:

1. f is bijective .
2. f is coć-r-continuous .
3. f^{-1} is coć-r-continuous.

Proposition (2.2.26)

For bijective function $f: (X, \tau) \longrightarrow (Y, \acute{\tau})$ the following statements are equivalent:

1. f is a coć-r-homeomorphism.
2. f is coć-r-continuous, coć-r-open function .
3. f is coć-r-continuous, coć-r-closed function .
4. $f\left(\overline{A}^{rk}\right) = \overline{f(A)}^{rk}$.

Proof: It is clear.

Definition (2.2.27)

A function $f: (X, \tau) \longrightarrow (Y, \acute{\tau})$ is called

i- super coc-r-open if $f(U)$ is open in Y for each coc-r-open U in X.

ii- super coc-r-closed if $f(F)$ is closed in Y for each F coc-r-closed in X.

Remark (2.2.28)

1. super coc-r-open function $\nleftrightarrow$ coc-r-open function.
2. super coc-r-open function $\nleftrightarrow$ coć-r-open function.
3. super coc-r-open function $\nleftrightarrow$ coc-open function.
4. super coc-r-open function $\nleftrightarrow$ coć-open function.
5. super coc-r-open function $\nleftrightarrow$ r-open function.

Examples (2.2.29)

1. Let $X = \{1,2,3\}, \tau_X = \{\phi, X, \{1\}\}$, $Y = \{a, b, c\}$ and $\tau_Y = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ then $\tau_X{}^{rk}, \tau_Y{}^{rk}$ discrete and $f: X \to Y$ be a function defined by $f(1) = f(2) = a, f(3) = c$. Then f is a coc-r-open, coc-open, coć-r-open, coć-open, r-open function but is not super coc-r-open function since $\{3\}$ is a coc-r-open in X but $f(\{3\}) = \{c\}$ is not open set in Y.
2. Let $X = \{a, b, c\}$, $Y = \{1,2,3 \dots\}$, $\tau_X = \{\phi, X, \{a\}\}, \tau_Y = \{G \subseteq Y: 1 \in G\} \cup \{\emptyset\}$, then $\tau_X{}^{rk}$ discrete, $\tau_Y{}^{rk} = \{G \subseteq Y: G^c \text{ is finite}\} \cup \{\emptyset\}$ and $f: X \to Y$ be a function defined by $f(x) = 1$, for all $x \in X$, then f super coc-r-open function, but is not a coc-r-open, not coć-r-open and not r-open function since $\{a\}$ is coc-r-open (open) but $f(\{a\}) = \{1\}$ is not coc-r-open (r-open).

Remark (2.2.30)

If X is a T_2-space, then:

1. Every coc-r-open function is super coc-r-open function.
2. Every coć-r-open function is super coc-r-open function.
3. Every coc-open function is super coc-r-open function.
4. Every coć-open function is super coc-r-open function.
5. Every r-open function is super coc-r-open function.
6. Every open function is super coc-r-open function.

Proposition (2.2.31)

Let $f: X \to Y$ be bijective function then f super coc-r-open function if and only if f super coc-r-closed function.

Proof: It is clear.

Proposition (2.2.32)

Let $f: X \to Y$ be bijective and super coc-r-open function, then:

1. $f\left(A^{\circ rk}\right) = f(A)^{\circ}$ for every set A in X.

2. $f\left(\overline{A}^{rk}\right) = \overline{f(A)}$ for every set A in X.

Proof: It is clear.

Theorem (2.2.33)

Let $f: X \longrightarrow Y$ be a function of space X into space Y, if f is a super coc-r-open function then $f^{-1}\left(\overline{B}\right) \subseteq \overline{f^{-1}(B)}^{rk}$, for all $B \subseteq Y$.

Proof :

Let $f: X \longrightarrow Y$ be a super coc-r-open function, $x \in f^{-1}\left(\overline{B}\right)$ and U be a coc-r-open in X contain x, then $f(x) \in \overline{B} \cap f(U)$, since f is super coc-r-open function and U coc-r-open in X, thus f(U) is an open set in Y, there fore $f(U) \cap B \neq \emptyset$, so $f^{-1}(B) \cap U \neq \emptyset$, then $x \in \overline{f^{-1}(B)}^{rk}$, hence $f^{-1}\left(\overline{B}\right) \subseteq \overline{f^{-1}(B)}^{rk}$, for all $B \subseteq Y$.

Proposition (2.2.34)

Let X, Y , Z are spaces and $f: X \to Y$, $g: Y \to Z$ be function, then:

1. If f is super coc-r-open function and g is an open function then $g \circ f: X \to Z$ is super coc-r-open function.

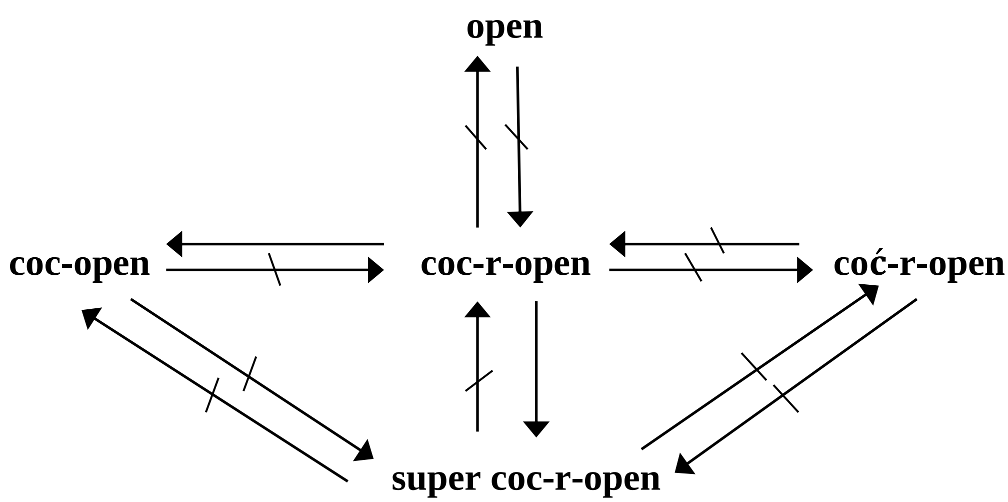
2. If f is coć-r-open function, g is super coc-r-open function then $g \circ f: X \to Z$ is super coc-r-open function.

3. If f is super coc-r-open function, g is coc-r-open function then $g \circ f: X \to Z$ is coć-r-open function.

4. If f is coć-r-conts function, bijective, $g \circ f: X \to Z$ is super coc-r-open function then g is super coc-r-open function.

Proof: It is clear.

The following diagram shows the relation among certain types of open functions



2.3 On coc-r-Separation Axioms

In this section we recall some definitions, remarks and propositions about separation properties, by using coc-r-open sets .

Definition (2.3.1)

A topological space X is called coc-r- T_0-space if and only if for each $x \neq y$ in X, there exist a coc-r-open set G such that $x \in G$, $y \notin G$ or $y \in G$, $x \notin G$.

Definition (2.3.2)

A topological space X is called coc-r-T_1-space if and only if for each $x \neq y$ in X there exist coc-r-open sets G and W such that $x \in G$, $y \notin G$ and $y \in W$, $x \notin W$.

Definition (2.3.3)

A topological space X coc-r-T_2-space (coc-r-Hausdorff) if and only if for each $x \neq y$ in X there exist disjoint coc-r-open sets G and W such that $x \in G$, $y \in W$.

Proposition (2.3.4)

Every topological space is a coc-r- T_i-space such that $i = 0, 1$.

Proof:

1. If $i = 0$

Suppose $a, b \in X$ such that $a \neq b$, since $U = X - \{b\}$ coc-r-open set in X and $a \in U$, $b \notin U$, then X is coc-r- T_0-space.

2. If $i = 1$

Suppose $a, b \in X$ such that $a \neq b$, since $U = X - \{b\}$, $V = X - \{a\}$ are coc-r-open sets in X such that $a \in U$, $b \notin U$ and $b \in V$, $a \notin V$, then X is coc-r- T_1-space.

Proposition (2.3.5)

Let X be a topological space then every clopen subspace of coc-r- T_2-space is also coc-r- T_2-space.

Proof:

Let X be coc-r- T_2-space and B be clopen subspace of a space X to prove B is coc-r- T_2-space. Suppose $a, b \in B$ such that $a \neq b$, since $B \subseteq X$ then $a, b \in X$ and X is a coc-r- T_2-space, thus there exist disjoint coc-r-open sets G and W in X such that $a \in G$, $b \in W$. Since B is a clopen set in X then $U = G \cap B, V = W \cap B$ are coc-r-open sets in B (Corollary (1.1.52)), so we get $a \in U$, $b \in V$. Now to prove $U \cap V = \emptyset$, since $U \cap V = (G \cap B) \cap (W \cap B) = B \cap (G \cap W) = B \cap \emptyset = \emptyset$, hence B is coc-r- T_2-space.

Proposition (2.3.6)

Let $f: X \longrightarrow Y$ be one to one coc-r-continuous function . If Y is T_2-space , then X is coc-r-T_2 -space.

Proof :

Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since $f: X \longrightarrow Y$ one to one function and $x_1 \neq x_2$ .Then $f(x_1) \neq f(x_2)$. Since Y is T_2-space then there exists disjoint open sets G, W in Y such that ($f(x_1) \in G$, $f(x_2) \in W$) . Since f coc-r-continuous, thus $f^{-1}(G), f^{-1}(W)$ are coc-r-open sets in X, since $f(x_1) \in G, f(x_2) \in W$ therefore $x_1 \in f^{-1}(G)$, $x_2 \in f^{-1}(W)$ and $f^{-1}(G) \cap f^{-1}(W) = f^{-1}(G \cap W) = \emptyset$, hence X is coc-r-$T_2$-space .

Proposition (2.3.7)

Let $f: X \longrightarrow Y$ be onto, coc-r-open function . If X is T_2-space , then Y is coc-r-T_2 – space.

Proof :

Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since $f: X \longrightarrow Y$ onto function, then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$, $f(x_2) = y_2$, thus $x_1 \neq x_2$. Since X is T_2-space then there exists disjoint open sets U, V in X such that ($x_1 \in U$, $x_2 \in$

V). Since f coc-r-open thus f(U), f(V) coc-r-open sets in Y, therefore $f(x_1) \in f(U), f(x_2) \in f(U)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$, hence Y is coc-r-$T_2$-space .

Proposition (2.3.8)

Let X and Y be coć-r-homeomorphism, then X coc-r-T_2-space if and only if Y is coc-r-T_2-space.

Proof: It is clear.

Proposition (2.3.9)

Let $f: X \longrightarrow Y$ be bijective, super coc-r-open function. If X is coc-r-T_2 – space , then Y is T_2 – space.

Proof: It is clear.

Definition (2.3.10)

A space X is said to be coc-r-regular space if and only if for each $x \in X$ and closed subset C of X such that $x \notin C$ there exist disjoint coc-r-open sets G, Wsuch that $x \in G$ and $C \subseteq W$.

Definition (2.3.11)

A space X is said to be coc′-r-regular space if for each $x \in X$ and coc-r-closed subset F of X such that $x \notin F$ there exist disjoint coc-r-open sets G, Wsuch that $x \in G$ and $C \subseteq W$.

Proposition (2.3.12)

A space X is coc-r-regular space if and only if for all $x \in X$ and all open set U in X such that $x \in U$ there exists coc-r-open set G such that $x \in G \subseteq \overline{G}^{rk} \subseteq U$

Proof:

Let X be coc-r-regular space and $x \in X$, U be open set in X such that $x \in U$. Then U^c is closed set in X and $x \notin U^c$ then there exist disjoint coc-r-open sets G, W such that $x \in G$, $U^c \subseteq W$, since $G \cap W = \emptyset$, thus $G \subseteq W^c$ and $W^c \subseteq U$. Hence $x \in G \subseteq \overline{G}^{rk} \subseteq \overline{W^c}^{rk} \subseteq W^c \subseteq U$.

Conversely

Let $x \in X$ and C be closed set in X such that $x \notin C$ then C^c open set in X and $x \in C^c$, thus there exist coc-r-open set G such that $x \in G \subseteq \overline{G}^{rk} \subseteq C^c$. There fore $x \in G, C \subseteq \left(\overline{G}^{rk}\right)^c$ and $G, \left(\overline{G}^{rk}\right)^c$ are disjoint coc-r-open sets. Hence X coc-r-regular space.

Proposition (2.3.13)

A space X is coc′-r-regular space if and only if for all $x \in X$ and all coc-r-open set U in X such that $x \in U$ there exists coc-r-open set G such that $x \in G \subseteq \overline{G}^{rk} \subseteq U$.

Proof

Let X be coc′-r-regular space and $x \in X$, U be coc-r-open set in X such that $x \in U$. Then U^c is coc-r-closed set in X and $x \notin U^c$ then there exist disjoint coc-r-open sets G, W such that $x \in G$, $U^c \subseteq W$, since $G \cap W = \emptyset$, thus $G \subseteq W^c$ and $W^c \subseteq U$. Hence $x \in G \subseteq \overline{G}^{rk} \subseteq \overline{W^c}^{rk} \subseteq W^c \subseteq U$.

Conversely

Let $x \in X$ and F be coc-r-closed set in X such that $x \notin F$ then F^c coc-r-open set in X and $x \in F^c$, thus there exist coc-r-open set G such that $x \in G \subseteq \overline{G}^{rk} \subseteq F^c$. There fore $x \in G, F \subseteq \left(\overline{G}^{rk}\right)^c$ and $G, \left(\overline{G}^{rk}\right)^c$ are disjoint coc-r-open sets. Hence X coc′-r-regular space.

Proposition (2.3.14)

Let X be a topological space then:

1. Every clopen subspace of coc-r-regular space is also coc-r-regular space.
2. Every clopen subspace of coc′-r-regular space is also coc′-r-regular space.

Proof:

1. Let X be coc-r-regular space and B be clopen subspace of a space X to prove B is coc-r-regular space. Suppose $a \in B$ and C be closed set in B such that $a \notin C$, since C is a closed set in B then $C = F \cap B$ where F is a closed set in X, since $a \notin C$ then $a \notin F \cap B$ thus $a \notin F$ in X and X is a coc-r-regular space, therefore there exist disjoint coc-r-open sets G, W in X such that $a \in G, F \subseteq W$. Since B is a clopen set in X then $U = G \cap B$, $V = W \cap B$ are coc-r-open sets in B (Corollary (1.1.52)), so we get $a \in U = G \cap B$, $C = F \cap B \subseteq W \cap B = V$ and $U \cap V = (G \cap B) \cap (W \cap B) = (G \cap W) \cap B = \emptyset \cap B = \emptyset$, hence B is coc-r-regular space.

2. Let X be coc′-r-regular space and B be clopen subspace of a space X to prove B is coc′-r-regular space. Suppose $a \in B$ and F be coc-r-closed set in B such that $a \notin F$, since B is a clopen set in X then F is a coc-r-closed set in X (Corollary (1.1.53)), since X is a coc′-r-regular space, then there exist disjoint coc-r-open sets G, W in X such that $a \in G, F \subseteq W$. Since B is a clopen set in X then $U = G \cap B$, $V = W \cap B$ are coc-r-open sets in B (Corollary (1.1.52)), so we get $a \in U = G \cap B$, $F = F \cap B \subseteq W \cap B = V$ and $U \cap V = (G \cap B) \cap (W \cap B) = (G \cap W) \cap B = \emptyset \cap B = \emptyset$, hence B is coc′-r-regular space.

Definition (2.3.15)

A topological space X is called coc-r-normal space iff for every disjoint closed sets C_1, C_2 there exist disjoint coc-r-open sets U_1, U_2 such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

Definition (2.3.16)

A topological space X is called coc′-r-normal space iff for every disjoint coc-r-closed sets C_1, C_2 there exist disjoint coc-r-open sets U_1, U_2 such that $C_1 \subseteq U_1$, $C_2 \subseteq U_2$.

Proposition (2.3.17)

A topological space X is coc-r-normal space if and only if for every closed set C in X and open set U in X such that $C \subseteq U$ there exists coc-r-open set G such that $\subseteq G \subseteq \overline{G}^{coc-r} \subseteq U$.

Proof

Let X be coc-r-normal space, C be closed set in X and U open set in X such that $C \subseteq U$. Then U^c is closed set in X and C, U^c are disjoint closed sets in X, since X coc-r-normal space, thus there exist disjoint coc-r-open sets G, W such that $C \subseteq G$, $U^c \subseteq W$, since $G \cap W = \emptyset$, so we get $G \subseteq W^c$ and $W^c \subseteq U$. Hence $C \subseteq G \subseteq \overline{G}^{rk} \subseteq \overline{W^c}^{rk} = W^c \subseteq U$.

Conversely:

Let C_1, C_2 are disjoint closed sets in X, then $C_1 \subseteq {C_2}^c$ and ${C_2}^c$ open set in X, thus there exist coc-r-open set G such that $C_1 \subseteq G \subseteq \overline{G}^{rk} \subseteq {C_2}^c$. There fore $C_1 \subseteq G, C_2 \subseteq \left(\overline{G}^{rk}\right)^c$ and $G, \left(\overline{G}^{rk}\right)^c$ are disjoint coc-r-open sets. Hence X coc-r- normal space.

Proposition (2.3.18)

A topological space X is coc′-r-normal space if and only if for every coc-r-closed set C in X and coc-r-open set U in X such that $C \subseteq U$ there exists coc-r-open set G such that $C \subseteq G \subseteq \overline{G}^{rk} \subseteq U$.

Proof:

Let X be coc′-r-normal space, C be coc-r-closed set in X and U coc-r-open set in X such that $C \subseteq U$. Then U^c is coc-r-closed set in X and C, U^c are disjoint coc-r-closed sets in X, since X coc′-r-normal space, thus there exist disjoint coc-r-open sets G, W such that $C \subseteq G$, $U^c \subseteq W$, since $G \cap W = \emptyset$, so we get $G \subseteq W^c$ and $W^c \subseteq U$. Hence $C \subseteq G \subseteq \overline{G}^{rk} \subseteq \overline{W^c}^{rk} = W^c \subseteq U$.

Conversely

Let C_1, C_2 are disjoint coc-r-closed sets in X, then $C_1 \subseteq {C_2}^c$ and ${C_2}^c$ coc-r-open set in X, thus there exist coc-r-open set G such that $C_1 \subseteq G \subseteq \overline{G}^{rk} \subseteq {C_2}^c$. There fore $C_1 \subseteq G, C_2 \subseteq \left(\overline{G}^{rk}\right)^c$ and $G, \left(\overline{G}^{rk}\right)^c$ are disjoint coc-r-open sets. Hence X coc′-r- normal space.

Proposition (2.3.19)

Let X be a topological space then:

1. Every clopen subspace of coc-r-normal space is also coc-r-normal space.
2. Every clopen subspace of coc′-r-normal space is also coc′-r-normal space.

Proof:

1. Let X be coc-r-normal space and B be clopen subspace of a space X to prove B is coc-r-normal space. Suppose C_1, C_2 are disjoint closed sets in B, then C_1, C_2 are disjoint closed sets in X, since X is a coc-r-normal space, thus there exist disjoint coc-r-open sets G, W in X such that $C_1 \subseteq G, C_2 \subseteq W$. Since B is a clopen set in X then $U = G \cap B, V = W \cap B$ are coc-r-open sets in B (Corollary (1.1.52)), so we get $C_1 = C_1 \cap B \subseteq G \cap B = U, C_2 = C_2 \cap B \subseteq W \cap B = V$ and $U \cap V = (G \cap B) \cap (W \cap B) = (G \cap W) \cap B = \emptyset \cap B = \emptyset$, hence B is coc-r-normal space.

2. Let X be coc′-r-normal space and B be clopen subspace of a space X to prove B is coc′-r-normal space. Suppose C_1, C_2 are disjoint coc-r-closed sets in B, since B is a clopen set in X then C_1, C_2 are disjoint coc-r-closed set in X (Corollary (1.1.53)), since X is a coc′-r-normal space, thus there exist disjoint coc-r-open sets G, W in X such that $C_1 \subseteq G, C_2 \subseteq W$, Since B is a clopen set

in X then $U = G \cap B$, $V = W \cap B$ are coc-r-open sets in B (Corollary (1.1.52)), so we get $C_1 = C_1 \cap B \subseteq G \cap B = U$, $C_2 = C_2 \cap B \subseteq W \cap B = V$ and $U \cap V = (G \cap B) \cap (W \cap B) = (G \cap W) \cap B = \emptyset \cap B = \emptyset$, hence B is coc′-r-normal space.

Proposition (2.3.20)

If a topological space X is coc-r-normal space and T_1-space, then X is coc-r-regular space.

Proof:

Let $x \in X$ and C be closed set in X such that $x \notin C$, since X is T_1-space then {x} closed set in X and $\{x\} \cap C = \emptyset$, since X is coc-r-normal space, thus there exist disjoint coc-r-open sets G, W such that $\{x\} \subseteq G, C \subseteq W$, there fore $x \in G$, $C \subseteq W$, hence X is coc-r-regular space.

2.4 On coc-r-Connected Space

we recall the concept of coc-r-connected space and give some generalization on this concept.

Definition (2.4.1) [5]

Let X be a topological space, any two subsets A and B of a space X are called τ-separated if $\overline{A} \cap B = A \cap \overline{B} = \phi$.

Remarks (2.4.2) [5]

In any topological space X, then the following statements are equivalent:

1. X is a connected space.
2. X is not union of two disjoint nonempty open sets.
3. ϕ, X are the only clopen sets in X.
4. X is not union of two nonempty separated sets.

Definition (2.4.3)

Let X be a topological space, any two subsets A and B of a space X are called coc-r-separated if $\overline{A}^{rk} \cap B = A \cap \overline{B}^{rk} = \phi$

Definition (2.4.4)

Let X be a space and $\phi \neq A \subseteq X$. Then A is called coc-r-connected set if is not union of any two coc-r-separated sets.

Remark (2.4.5)

A set B is called coc-r-clopen if it is coc-r-open and coc-r-closed.

Proposition (2.4.6)

Let X be topological space, then the following statements are equivalent:

1. X is a coc-r-connected space.

2. ϕ, X are the only coc-r-clopen sets in X.

3. X is not union of two disjoint nonempty coc-r-open sets.

Proof:

(1) $\longrightarrow$ (2)

Let X be coc-r-connected space, suppose that D is coc-r-clopen set such that $D \neq \phi$ and $D \neq X$. Let $E = X - D$, since $D \neq X$ then $E \neq \phi$. Since D is coc-r-open, then E is coc-r-closed. But $\overline{D}^{rk} \cap E = D \cap \overline{E}^{rk} = D \cap E = \phi$, thus D and E are two coc-r-separated sets and $X = D \cup E$, there fore X is not coc-r-connected space which is a contradiction. Hence the only coc-r-clopen set in the space X are X and ϕ .

(2) $\longrightarrow$ (3)

Suppose the only coc-r-clopen set in the space are X and ϕ . Assume that there exists two disjoint nonempty coc-r-open sets U and V such that $X = U \cup V$. Since $U = V^c$ then U is coc-r-clopen set. But $U \neq \phi$ and $U \neq X$ which is a contradiction. Hence X is not union of two disjoint nonempty coc-r-open sets.

(3) $\longrightarrow$ (1)

Suppose that X is not coc-r-connected space. Then there exist two coc-r-separated sets A and B such that $X = A \cup B$. Since $\overline{A}^{rk} \cap B = \phi$ and $A \cap B \subseteq \overline{A}^{rk} \cap B$ thus $A \cap B = \phi$, Since $\overline{A}^{rk} \subseteq B^c = A$, then A is coc-r-closed set. By the same way we can see that B is coc-r-closed set since $A^c = B$. Thus A and B are two disjoint coc-r-open sets such that $X = A \cup B$ which is a contradiction. Hence X is coc-r-connected space.

Remark (2.4.7)

A topological space (X, τ) is a coc-r-connected space if and only if (X, τ^{rk}) is a connected space.

Remark (2.4.8)

Every clopen set is a coc-r-clopen set.

Proposition (2.4.9)

Every coc-r-connected space is a connected space.

Proof:

Let $A \subseteq X$ be clopen set in X, then A is a coc-r-clopen set in X, since X is a coc-r-connected space, thus either $A = \phi$ or $A = X$, hence X is a connected space.

Remark (2.4.10)

The convers of proposition (2.4.9) is not true in general.

As the following example shows:

Example (2.4.11)

Let $X = \{1,2,3, \ldots\}, \tau_X = \{G \subseteq X: 1 \notin G\} \cup \{X\}$, then $\tau_X{}^{rk} = \{G \subseteq X: 1 \notin G\} \cup \{G \subseteq X: 1 \in G, G^c \text{ is finite}\}$, then (X, τ) is a connected space but (X, τ^{rk}) is not a connected space since $A = \{1,3,4,5, \ldots\}$ is clopen set in (X, τ^{rk}). Hence(X, τ) is not a coc-r-connected space.

Proposition (2.4.12)

Let A be coc-r-connected set and D, E coc-r-separated sets. If $A \subseteq D \cup E$ then either $A \subseteq D$ or $A \subseteq E$.

Proof:

Suppose A be a coc-r-connected set and D, E coc-r-separated sets and $A \subseteq D \cup E$. Suppose $A \nsubseteq D$ and $A \nsubseteq E$. assume that $A_1 = D \cap A \neq \phi$ and $A_2 = E \cap A \neq \phi$ then $A = A_1 \cup A_2$. Since $A_1 \subseteq D$, hence $\overline{A_1}^{rk} \subseteq \overline{D}^{rk}$, since$\overline{D}^{rk} \cap E = \phi$,then $\overline{A_1}^{rk} \cap A_2 = \phi$. Since $A_2 \subseteq E$, hence $\overline{A_2}^{rk} \subseteq \overline{E}^{rk}$. $\overline{E}^{rk} \cap D = \phi$, thus $\overline{A_2}^{rk} \cap A_1 = \phi$. But $A = A_1 \cup A_2$, therefor A is not coc-r-connected space which is a contradiction. There fore either $A \subseteq D$ or $A \subseteq E$.

Proposition (2.4.13)

Let X be a topological space such that any two element x and y of X are contained in some coc-r-connected subspace of X, then X is coc-r-connected space.

Proof:

Suppose X is not coc-r-connected. Then X is the union of two coc-r-separated sets A, B. since A, B are nonempty sets, thus there exists a, b such that $a \in A, b \in B$, Let D be coc-r-connected subspace of X which contains a, b. Therefore either $D \subseteq A$ or $D \subseteq B$ which is a contradiction (since $A \cap B = \phi$). Then X is coc-r- connected space.

Proposition (2.4.14)

If A is coc-r-connected set then $\overline{A}^{rk}$ is coc-r-connected.

Proof:

Suppose A is coc-r-connected and $\overline{A}^{rk}$ is not. Then there exist two coc-r-separated set D, E such that $\overline{A}^{rk} = D \cup E$. But $A \subseteq \overline{A}^{rk}$, then $A \subseteq D \cup E$ and since A is coc-r-connected set, then either $A \subseteq D$ or $A \subseteq E$.

i. If $A \subseteq D$ then $\overline{A}^{rk} \subseteq \overline{D}^{rk}$. But $\overline{D}^{rk} \cap E = \phi$, hence $\overline{A}^{rk} \cap E = \phi$ since $\overline{A}^{rk} = D \cup E$ then $E = \phi$ which is a contradiction.

ii. If $A \subseteq E$ then $\overline{A}^{rk} \subseteq \overline{E}^{rk}$. But $\overline{E}^{rk} \cap D = \phi$, hence $\overline{A}^{rk} \cap D = \phi$ since $\overline{A}^{rk} = E \cup D$ then $D = \phi$ which is a contradiction. Hence $\overline{A}^{rk}$ is coc-r-connected.

Remark (2.4.15)

Let X be a space and $A \subseteq X$, if A is coc-r-connected set in X, then $\overline{A}$ need not to be coc-r-connected set in X.

As the following example shows:

Example (2.4.16)

Let $X = \{a, b, c\}$ and $\tau = \{\phi, X\}$, then τ^{rk} discrete, assume that $A = \{a\}$, then A is a coc-r-connected set in X but $\overline{A} = X$ is not a coc-r-connected.

Proposition (2.4.17)

If D is coc-r-connected set and $D \subseteq E \subseteq \overline{D}^{rk}$, then E is coc-r-connected.

Proof:

suppose D is coc-r-connected set, $D \subseteq E \subseteq \overline{D}^{rk}$ and E is not coc-r-connected, then there exist two sets A , B such that $\overline{A}^{rk} \cap B = A \cap \overline{B}^{rk} = \phi$, $E \subseteq A \cup B$, since $D \subseteq E$, thus either $D \subseteq A$ or $D \subseteq B$. Suppose $D \subseteq A$, then $\overline{D}^{rk} \subseteq \overline{A}^{rk}$. Thus $\overline{D}^{rk} \cap B = \overline{A}^{rk} \cap B = \phi$. But $D \subseteq E \subseteq \overline{D}^{rk}$, then $\overline{D}^{rk} \cap B = B$. Therefore $B = \phi$ which is a contradiction. Hence E is coc-r-connected set. By the same way can get a contradiction if $D \subseteq B$, hence E is coc-r-connected set.

Proposition (2.4.18)

If a space X contains a coc-r-connected subspace E such that $\overline{E}^{rk} = X$, then X is coc-r-connected.

Proof:

Suppose E a coc-r-connected subspace of a space X such that $\overline{E}^{rk} = X$, since $E \subseteq X = \overline{E}^{rk}$, then by Proposition(2.4.17) X is coc-r-connected.

Proposition (2.4.19)

If every coc-r-open subset of a space X is coc-r-connected set, then every pair of nonempty coc-r-open subsets of X have a nonempty intersection.

Proof:

Suppose A, B are disjoint coc-r-open subsets of X, since $A \cup B$ is coc-r-open set and A, B are coc-r-open subsets in $A \cup B$, then $A \cup B$ is not coc-r-connected set which is a contradiction, hence $A \cap B \neq \phi$.

Proposition (2.4.20)

The coc-r-continuous image of coc-r-connected space is connected.

Proof:

Let $f: (X, \tau) \to (Y, \tau')$ be coc-r-continuous, onto function and X be coc-r-connected. To prove Y is connected , suppose Y is not connected space. So, $Y = A \cup B$ such that $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$ and $A, B \in \tau'$,hence $f^{-1}(Y) = f^{-1}(A \cup B)$, then $X = f^{-1}(A) \cup f^{-1}(B)$. Since f coc-r-continuous, then $f^{-1}(A)$ and $f^{-1}(B)$ are coc-r-open in X and since $A \neq \phi$, $B \neq \phi$ and f is onto, then $f^{-1}(A) \neq \phi$, $f^{-1}(B) \neq \phi$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$, hence X is not coc-r-connected space which is contradiction.

Proposition (2.4.21)

The coc′-r-continuous image of coc-r-connected space is coc-r-connected.

Proof:

Let $f: (X, \tau) \to (Y, \tau')$ is coc′-r-continuous, onto function and X is coc-r-connected. To prove Y is coc-r-connected , suppose Y is a not coc-r-connected space. So, $Y = A \cup B$ such that $A \neq \phi$, $B \neq \phi$ and $A \cap B = \phi$ and A, B are coc-r-open sets. $f^{-1}(Y) = f^{-1}(A \cup B)$, so $X = f^{-1}(A) \cup f^{-1}(B)$. Since that f coc′-r-continuous hence $f^{-1}(A)$ and $f^{-1}(B)$ are coc-r-open in X and since that $A \neq \phi$, $B \neq \phi$ and f is onto then $f^{-1}(A) \neq \phi$, $f^{-1}(B) \neq \phi$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$, hence X is not coc-r-connected space which is contradiction, hence Y is a coc-r-connected.

Chapter three
On Coc-r-compact, Coc-r-lindelof, I-coc-r-lindelof spaces

Introduction

This Chapter is divided into three sections . In section one, we recall the concept of coc-r-compact space and give some important generalizations on this concept. In section two, we recall definition, proposition and theorems of coc-r-lindelof space. In section three, we introduces the concept of I-coc-r-lindelof space and we prove some results on this concept and give the relation between I-coc-r-lindelof, coc-r-lindelof, I-lindelof, and lindelof space.

3.1 Coc-r-compact Space

We recall the concept of a compact space by using coc-r-open sets and give some important generalizations on this concept and also we prove some results on this concept.

Definition (3.1.1) [2]

A space X is said to be a compact if every open cover of X has finite subcover.

Definition (3.1.2)

A space X is said to be a coc-r- Compact if every coc-r - open covering of X has a finite subcovering.

Examples (3.1.3)

The following are straight forward examples of coc-r- compact spaces.

1) Any finite topological space.

2) Let $X = \{1,2,3,\dots\}, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X: G^c$ is finite$\} \cup \{\emptyset\}$, then X is coc-r- Compact space.

Remark (3.1.4)

1) Compact $\not\rightarrow$ coc-r- compact.

2) Coc-r- compact $\not\rightarrow$ compact.

Examples (3.1.5)

1) Let $X = Q$, with indiscrete topology, then $\tau^{rk} = \{A: A \subseteq X\}$, thus X is Compact but X is not coc-r- Compact.

2) Let $X = \{1,2,3,\dots\}, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X: G^c$ is finite$\} \cup \{\emptyset\}$, thus X is coc-r- Compact but X is not Compact.

Proposition (3.1.6)

If X is T_2-space, then every Compact space is coc-r- Compact space.

Proof :

It is clear to show that, since in T_2-space every coc-r- open is open set in X.

Proposition (3.1.7)

If X is regular space, then every coc-r- Compact space is Compact space.

Proof : It is clear

Definition (3.1.8)

A subset B of a topological space X is said to be coc-r-Compact relative to X if every cover of B by coc-r- open sets in X has a finite subcover of B. The subset B is coc-r- Compact iff it is coc-r- Compact as a subspace.

Remark (3.1.9)

The subset $B \subseteq X$ is coc-r- closed in (X, τ) iff B closed in (X, τ^{rk}).

Proposition (3.1.10)

1) A coc-r-closed subset of coc-r-compact space X is coc-r-compact relative to X.
2) In any space, the intersection of coc-r-compact set with a coc-r-closed set is coc-r-compact.
3) Every coc-r-compact subset of coc-r-T_2-space is coc-r-closed set.

Proof :

1) Let X be a coc-r-compact space and F be a coc-r-closed subset of X, thus F is closed in (X, τ^{rk}), since X coc-r-compact space, then (X, τ^{rk}) compact space and by using (Remarks (1.1.20), (3)) we will get F is compact relative to (X, τ^{rk}). Hence F is coc-r-compact relative to X.

2) Let F be an coc-r-closed set of X and let K be an coc-r-compact subset of X. Thus F, K are closed, compact respectively in (X, τ^{k}) then by using remarks (1.1.20), (4) $A \cap B$ is compact set in (X, τ^{rk}), hence $A \cap B$ is coc-r-compact set in X.

3) Let X be a coc-r-T_2-space and K be a coc-r-compact subset of X, thus K is compact in (X, τ^{rk}), since X coc-r-T_2-space, then (X, τ^{rk}) T_2-space and by using (Remarks (1.1.20), (5)) we will get K is closed set in (X, τ^{rk}). Hence K is coc-r-closed set in X.

Corollary (3.1.11)

Every r-closed of coc-r-compact space X is coc-r-compact relative to X.

Proof : It is clear.

Proposition (3.1.12)

If X is a topological space such that every coc-r-open subset of X is coc-r-compact relative to X, then every subset is coc-r-compact relative to X.

Proof:

Let G be an arbitrary subset of X, $\{U_\alpha : \alpha \in \Lambda\}$ be cover of G by coc-r - open subsets, then the family $\{U_\alpha : \alpha \in \Lambda\}$ is a coc-r-open cover of the coc-r-open set $\cup \{U_\alpha : \alpha \in \Lambda\}$. Thus by assumption there is a finite sub family $\{U_{\alpha i} : i = 1,2, \dots , n\}$ which covers $\cup \{U_\alpha : \alpha \in \Lambda\}$, since $G \subseteq \cup \{U_\alpha : \alpha \in \Lambda\} \subseteq \cup \{U_{\alpha i} : i = 1,2, \dots , n\}$, hence G is coc-r-compact.

Theorem (3.1.13)

Let Y be a subspace in (X, τ), X is coc-r-compact, if Y is clopen set, then Y is coc-r-compact.

Proof :

Let Y be a subspace in X, $\{U_\alpha : \alpha \in \Lambda\}$ be cover of Y by coc-r-open subsets of Y such that $Y \subseteq \cup \{U_\alpha : \alpha \in \Lambda\}$, since U_α is coc-r-open in Y, Y is clopen set in X, then U_α is coc-r-open in X for all $\alpha \in \Lambda$ (by using Theorem (1.1.51)). Thus $X = Y \cup Y^c \subseteq \cup \{U_\alpha : \alpha \in \Lambda\} \cup Y^c \subseteq \cup \{U_\alpha \cup Y^c : \alpha \in \Lambda\}$, since Y is clopen set in X, then Y is r-closed, thus Y is coc -r-closed, there fore Y^c is coc -r-open in X. Since X is coc-r-compact, then $X \subseteq \cup \{U_{\alpha i} \cup Y^c : i = 1,2, \dots , n\}$, so that $Y = X \cap Y \subseteq \cup \{U_{\alpha i} \cup Y^c : i = 1,2, \dots , n\} \cap Y = \cup \{U_{\alpha i} \cap Y : i = 1,2, \dots , n\} = \cup \{ U_{\alpha i} : i = 1, 2, \dots , n \}$, hence Y is coc-r-compact.

Theorem (3.1.14)

If X is coc-r-compact space, then every r-open covering of X has a finite sub covering.

Proof : It is clear.

Remark (3.1.15)

The convers of Theorem (3.1.14) is not true.

Example (3.1.16)

In Example (3.1.5), (1), all r-open covers are $\{\emptyset, X\}$, and it is finite cover of X, but X is not coc-r-compact space.

Theorem (3.1.17)

If X be T_2-space, then the following statements are equivalent.

i) X is coc-r-compact.

ii) Every cover of X by r- open subsets has a finite subcover.

Proof :

(i) ⟶ (ii) It is clear.

(ii) ⟶ (i)

Let $\mathcal{U}$ be coc-r-open cover of X, then $X \subseteq \cup\{U : U \in \mathcal{U}\}$, since X is T_2-space, thus U is equal to the union of r-open sets in X contained in U for each $U \in \mathcal{U}$ (Theorem (1.1.19),(1)). There fore all r-open sets in U for each $U \in \mathcal{U}$ are r-open cover of X, this r-open cover has a finite subcover. Since every element of this a finite subcover contained in U for some $U \in \mathcal{U}$, hence $\mathcal{U}$ has a finite subcover.

Theorem (3.1.18)

If X is T_2-space, then the following statements are equivalent.

i) Every proper r- closed subset of X is coc-r-compact relative to X.

ii) X is coc-r-compact.

iii) X is r-compact.

Proof :

(i) ⟶ (ii)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of X by r - open subsets of X such that $X \subseteq \cup \{U_\alpha : \alpha \in \Lambda\}$. If $U_\lambda = X$, $\lambda \in \Lambda$ then the proof is end, if $U_\lambda \neq X$, $\lambda \in \Lambda$ then ${U_\lambda}^c$ is proper r- closed subset and ${U_\lambda}^c \subseteq \cup\{U_\alpha : \alpha \in \Lambda - \{\lambda\}\}$, by the hypothesis there exist a finite subfamily $\{U_{\alpha i} : \alpha i \in \Lambda - \{\lambda\}, i = 1,2,\dots,n\}$, such that ${U_\lambda}^c \subseteq \cup \{U_{\alpha i} : \alpha i \in \Lambda - \{\lambda\}, i = 1,2,\dots,n\}$, thus $X \subseteq \cup\{U_{\alpha i} \cup U_\lambda : \alpha i \in \Lambda - \{\lambda\}, i = 1,2,\dots,n\}$, hence X is coc-r-compact.

(ii) ⟶ (iii)

Clear, by using Theorem (3.1.17), Definition (1.1.17).

(iii) ⟶ (i)

Suppose F be proper r- closed subset of X, then $F \neq X$, let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of F by r - open subsets of X, since F is r- closed subset of X, thus F^c is r-open, since $F \cup F^c \subseteq \cup\{U_\alpha : \alpha \in \Lambda\} \cup F^c$, there fore $\{U_\alpha, F^c : \alpha \in \Lambda\}$ is r-open cover of X and X is r-compact, so $X \subseteq \cup\{U_{\alpha i} : i = 1,2,\dots,n\} \cup F^c$, hence $F \subseteq \cup\{U_{\alpha i} : i = 1,2,\dots,n\}$.

Definition (3.1.19)[12]

Let $\mathcal{U}$ is family of subset of X, then $\mathcal{U}$ has a finite intersection property if for all $U_1, U_2, \dots, U_n \in \mathcal{U}$, $n \in N$ then $\cap \{U_i : i = 1,2,\dots,n\} \neq \phi$.

Theorem (3.1.20)

If X is T_2-space, then the following statements are equivalent.

1) X is coc-r-compact.

2) Every family $\{F_\alpha : \alpha \in \Lambda\}$ of r-closed subsets of X with finite intersection property then $\cap \{F_\alpha : \alpha \in \Lambda\} \neq \phi$.

3) Every family $\mathcal{F}$ of coc-r-closed subsets of X with $\cap \{F : F \in \mathcal{F}\} = \phi$ contains a finite subfamily $\mathcal{L}$ such that $\cap \{F : F \in \mathcal{L}\} = \phi$.

4) Every family $\mathcal{F}$ of r-closed subsets of X with $\cap \{F : F \in \mathcal{F}\} = \phi$ contains a finite subfamily $\mathcal{L}$ such that $\cap \{F : F \in \mathcal{L}\} = \phi$.

Proof :

(1) ⟶ (2)

Let $\{F_\alpha : \alpha \in \Lambda\}$ be family of r-closed subsets of X with finite intersection property, Suppose that $\cap \{F_\alpha : \alpha \in \Lambda\} = \phi$. Put $U_\alpha = {F_\alpha}^c$, then U_α is r-open subsets of X, thus the family $\{U_\alpha : \alpha \in \Lambda\}$ is a r-open cover of X.

Since X is coc-r-compact, there fore $\{U_\alpha : \alpha \in \Lambda\}$ has a finite subcover $\{U_{\alpha i} : i = 1,2,\dots,n\}$ such that $X = \cup \{U_{\alpha i} : i = 1,2,\dots,n\}$ (by using Theorem (3.1.17)), then $X = \cup \{{F_{\alpha i}}^c : i = 1,2,\dots,n\} = (\cap \{F_{\alpha i} : i = 1,2,\dots,n\})^c$, thus $\cap \{F_{\alpha i} : i = 1,2,\dots,n\} = \phi$, this is contradiction with a finite intersection property. Hence $\cap \{F_\alpha : \alpha \in \Lambda\} \neq \phi$.

(2) ⟶ (3)

Let $\mathcal{F}$ be a family of coc-r-closed subsets of X with an empty intersection, since X is T_2-space, then $F = \overline{F}^r$ for each $F \in \mathcal{F}$ (by using Theorem (1.1.19), (2)). Thus F equal to the intersection of r-closed sets in X containing F for each $F \in \mathcal{F}$, there fore the intersection of all r-closed sets in X containing F for each $F \in \mathcal{F}$ is an empty intersection. By using the hypothesis this r-closed family has a finite subfamily with an empty intersection , since every element of this finite subfamily containing F for some $F \in \mathcal{F}$, hence $\mathcal{F}$ has a finite subfamily with an empty intersection.

(3) ⟶ (4)

It is clear.

(4) $\longrightarrow$ (1)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of X by r - open subsets of X such that $X = \cup \{U_\alpha : \alpha \in \Lambda\}$, then $\{U_\alpha{}^c : \alpha \in \Lambda\}$ is family of r-closed subsets of X with an empty intersection. By assumption there exist a finite subfamily such that $\cap \{U_{\alpha i}{}^c : i = 1,2,\dots,n\} = \phi$, so $X = (\cap \{U_{\alpha i}{}^c : i = 1,2,\dots,n\})^c = \cup \{U_{\alpha i} : i = 1,2,\dots,n\}$. Hence X is coc-r-compact.

Definition (3.1.21)[3]

A space (X, T) is called I-compact if every cover $\mathcal{F}$ of X by r - closed subsets of the space (X, T) contains a finite subcover $\mathcal{L}$ such that $X = \cup \{F^\circ : F \in \mathcal{L}\}$.

Remark (3.1.22)

Coc-r-compact $\not\longleftrightarrow$ I-compact.

Examples (3.1.23)

1) Let $X = R$, with indiscrete topology, then $\tau^{rk} = \{A : A \subseteq X\}$, thus X is
I-compact but X is not coc-r-compact.

2) Let $X = \{1,2,3,\dots\}$, $\tau = \{G \subseteq X : 1 \notin G\} \cup \{X\}$, then
$\tau^{rk} = \{G \subseteq X : 1 \notin G\} \cup \{G \subseteq X : 1 \in G, G^c \text{ is finite}\}$, thus X is coc-r-compact but is not I-compact because $\{\{1, x\} : 1 \in X, x \neq 1\}$ is r-closed cover of X but has not a finite subcover and $\{1, x\}^\circ = \{x\}$, $x \neq 1$.

Definition (3.1.24) [7]

A space (X, T) is called extremally disconnected if $\overline{U}$ is open for each open set U in X.

Remarks (3.1.25)[7]

1. A space (X, τ) is extremally disconnected iff for all $U, V \in RO(X, \tau)$ with $U \cap V = \emptyset$, then $\overline{U} \cap \overline{V} = \emptyset$.

2. If a topological space X is extremally disconnected, then every r-open, r-closed in X is open set.

Theorem (3.1.26)

If a topological space X is extremally disconnected space, then every coc-r-compact is I-compact.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be r-closed cover of X, then F_α is closed for each $\alpha \in \Lambda$, thus $F_\alpha{}^\circ$ is r-open for each $\alpha \in \Lambda$ (by using Remarks (1.1.4), (2) and Remarks (1.1.16), (4)). Since F_α is r-closed for each $\alpha \in \Lambda$ and X is extremally disconnected space, there fore F_α is open set in X for each $\alpha \in \Lambda$ (by using Remarks (3.1.25), (2)), so F_α is r-open, then F_α is coc- r-open set in X for each $\alpha \in \Lambda$. Since X is coc-r-compact, thus the cover $\{F_\alpha : \alpha \in \Lambda\}$ has a finite subcover such that $X = \cup \{F_{\alpha i} : i = 1,2, \dots, n\} = \cup \{F_{\alpha i}{}^\circ : i = 1,2, \dots, n\}$. Hence X is I-compact.

Theorem (3.1.27)

If a topological space X is T_2-space, then every I-compact is coc-r-compact.

Proof :

Let $\{U_\alpha : \alpha \in \Lambda\}$ be r-open cover of X, then U_α is open and $\overline{U_\alpha}$ is a r-closed set in X for each $\alpha \in \Lambda$ (by using Remarks (1.1.16), (5)), thus $\{\overline{U_\alpha} : \alpha \in \Lambda\}$ is r-closed cover of X and X I-compact, therefor this cover has a finite subcover such that $X = \cup \left\{\overline{U_{\alpha i}}^\circ : i = 1,2, \dots, n\right\} = \cup \{U_{\alpha i} : i = 1,2, \dots, n\}$. Hence X is coc-r-compact.

Proposition (3.1.28)

If a topological space X is T_2-space, then every r-closed set of I-compact space is coc-r-compact relative.

Proof :

It is clear by using theorem (3.1.27), Corollary (3.1.12).

Definition (3.1.29) [3]

A subset B of a topological space X is said to be I-compact relative to X if every cover $\mathcal{F}$ of B by r- closed sets in X has a finite subcover $\mathcal{L}$ such that $B \subseteq \cup \{F^\circ : F \in \mathcal{L}\}$.

Proposition (3.1.30)

If a topological space X is extremally disconnected space, then every r-open set of I-compact space is I-compact relative.

Proof :

Let X be extremally disconnected space, U be r-open in X and $\{F_\alpha : \alpha \in \Lambda\}$ cover of U by r-closed subsets of X such that $U \subseteq \cup\{F_\alpha : \alpha \in \Lambda\}$, then $U \cup U^c \subseteq \cup \{F_\alpha \cup U^c : \alpha \in \Lambda\}$, thus $X \subseteq \cup\{F_\alpha \cup U^c : \alpha \in \Lambda\}$, U^c is r-closed. Since X is I-compact space, there fore the cover $\{F_\alpha \cup U^c : \alpha \in \Lambda\}$ has a finite sub cover such that $X \subseteq \cup\{(F_{\alpha i} \cup U^c)^\circ : i = 1,2,\dots,n\}$, since X be e.d space, so $F_{\alpha i}$, U^c is open set (Remarks (3.1.25), (2)) for each $i = 1,2,\dots,n$, then $X \subseteq \cup\{F_{\alpha i} \cup U^c : i = 1,2,\dots,n\}$, thus $U \subseteq \cup\{F_{\alpha i} \cap U : i = 1,2,\dots,n\} \subseteq \cup\{F_{\alpha i} : i = 1,2,\dots,n\} = \cup\{F_{\alpha i}{}^\circ : i = 1,2,\dots,n\}$. Hence U is I-compact relative.

Corollary (3.1.31)

If a topological space X is extremally disconnected space, then every r-open set of coc-r-compact space is coc-r-compact relative.

Proof :

It is clear by using theorem (3.1.26), Proposition (3.1.27).

Theorem (3.1.32)

Let $f: X \longrightarrow Y$ be a coć-r-continuous function, onto, if X is coc-r-compact then Y coc-r-compact.

Proof :

Let $\{U_\alpha : \alpha \in \Lambda\}$ be coc-r-open cover of Y, since f is a coć-r-continuous function, then $f^{-1}(U_\alpha)$ is coc-r - open in X for each $\alpha \in \Lambda$, but $Y \subseteq \cup_{\alpha\in\Lambda} U_\alpha$, thus $X = f^{-1}(Y) \subseteq \cup_{\alpha\in\Lambda} f^{-1}(U_\alpha)$, since X is coc-r-compact and $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ forms a cover of X, there fore the cover $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ has a finite subcover such that $X \subseteq \cup\{f^{-1}(U_{\alpha i}), : i = 1,2,\dots,n\}$,since f onto, so $f(X) = Y \subseteq \cup\{f(f^{-1}(U_{\alpha i})) : i = 1,2,\dots,n\} \subseteq \cup\{U_{\alpha i} : i = 1,2,\dots,n\}$. Hence Y coc-r-compact.

Theorem (3.1.33)

Let $f: X \longrightarrow Y$ be a coć-r-open function, bijective, if Y is coc-r-compact then X coc-r-compact.

Proof :

Let $\{U_\alpha : \alpha \in \Lambda\}$ be coc-r-open cover of X, since f is a coć-r-open function, then $f(U_\alpha)$ is coc-r - open in Y for each $\alpha \in \Lambda$, but $X \subseteq \cup_{\alpha\in\Lambda} U_\alpha$, there fore $Y = f(X) \subseteq \cup_{\alpha\in\Lambda} f(U_\alpha)$, so $\{f(U_\alpha) : \alpha \in \Lambda\}$ forms a cover of Y, since Y is coc-r-compact, then the cover $\{f(U_\alpha) : \alpha \in \Lambda\}$ has a finite subcover such that $Y \subseteq \cup\{f(U_{\alpha i}) : i = 1,2,\dots,n\}$, thus $X = f^{-1}(Y) \subseteq \cup\{f^{-1}(f(U_{\alpha i})) : i = 1,2,\dots,n\} = \cup\{U_{\alpha i} : i = 1,2,\dots,n\}$. Hence X coc-r-compact.

Theorem (3.1.34)

Let $f: X \longrightarrow Y$ be a coc-r-continuous function, onto and Y be extremally disconnected space, if X is coc-r-compact then Y I-compact.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be r-closed cover of Y and Y be extremally disconnected, then F_α is open in Y for each $\alpha \in \Lambda$ (Remarks (3.1.25), (2)), since f is a coc-r-continuous function, thus $f^{-1}(F_\alpha)$ is coc-r - open in X for each $\alpha \in \Lambda$, but $Y \subseteq \cup_{\alpha\in\Lambda} F_\alpha$, thus $X = f^{-1}(Y) \subseteq \cup_{\alpha\in\Lambda} f^{-1}(F_\alpha)$, since X is coc-r-compact and $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ forms a cover of X, there fore the cover $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ has a finite subcover such that $X \subseteq \cup\{f^{-1}(F_{\alpha i}), : i = 1,2,\dots,n\}$, since f onto, so $f(X) = Y \subseteq \cup\{f(f^{-1}(F_{\alpha i})) : i = 1,2,\dots,n\} \subseteq \cup\{F_{\alpha i} : i = 1,2,\dots,n\} = \cup \{F_{\alpha i}^{\circ} : i = 1,2,\dots,n\}$. Hence Y I-compact.

3.2 Coc-r-lindelof Space

We recall the concept of a lindelof space by using coc-r-open sets and give some important generalizations on this concept and also we prove some results on this concept.

Definition (3.2.1) [2]

A space X is said to be a lindelof if every open cover of X has a countable sub cover.

Definition (3.2.2)

A space X is said to be a coc-r- lindelof if every coc-r - open covering of X has a countable subcovering.

Examples (3.2.3)

The following are straight forward examples of coc-r- lindelof spaces.

1) Let $X = R, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X : G^c \text{ is finite}\} \cup \{\emptyset\}$, thus X is coc-r-lindelof.

2) (The sorgenfrey line S) is R with the topology generated by base $B = \{[a, b) : a < b\}$ is lindelof, T_2-space and regular space, then $\tau = \tau^{rk}$, thus $X = R$ is coc-r- lindelof (lindelof).

Remark (3.2.4)

1) Lindelof $\not\rightarrow$ coc-r- lindelof.

2) Coc-r- lindelof $\not\rightarrow$ lindelof.

Examples (3.2.5)

1) Let $X = R$, with indiscrete topology, then $\tau^{rk} = \{G: G \subseteq X\}$, thus X is lindelof but X is not coc-r- lindelof.

2) Let $X = R, \tau = \{G \subseteq X: 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X : G^c \text{ is finite}\} \cup \{\emptyset\}$, thus X is coc-r- lindelof but X is not lindelof.

Proposition (3.2.6)

If X is T_2-space, then every lindelof space is coc-r- lindelof space.

Proof : It is clear.

Proposition (3.2.7)

If X is regular space, then every coc-r- lindelof space is lindelof space.

Proof : It is clear

Proposition (3.2.8)

Every coc-r- compact space is coc-r- lindelof space.

Proof : It is clear.

Remark (3.2.9)

The convers of Proposition (3.2.8) is not true.

Example (3.2.10)

The sorgenfrey line S T_2-space and regular space, then $\tau = \tau^{rk}$, thus S is coc-r- lindelof (lindelof) but is not compact (coc-r- compact).

Definition (3.2.11)

A space X is said to be countably coc-r- compact if every countable coc-r-open cover of X has a finite subcover.

Theorem (3.2.12)

A space X is coc-r- compact if and only if X is coc-r- lindelof and countably coc-r- compact.

Proof : It is clear.

Definition (3.2.13)

A subset B of a topological space X is said to be coc-r- lindelof relative to X if every cover of B by coc-r- open sets in X has a countable subcover of B.

The subset B is coc-r- lindelof iff it is coc-r- lindelof as a subspace.

Proposition (3.2.14)

A coc-r-closed subset of coc-r- lindelof space X is coc-r- lindelof relative to X.

Proof :

Let F be coc-r-closed and $\{U_\alpha : \alpha \in \Lambda\}$ be coc-r-open cover of F such that $F \subseteq \cup \{U_\alpha : \alpha \in \Lambda\}$, then $F \cup F^c \subseteq \cup \{U_\alpha \cup F^c : \alpha \in \Lambda\}$ and F^c is coc-r-open in X, thus $\{U_\alpha \cup F^c : \alpha \in \Lambda\}$ forms cover of X and X is coc-r- lindelof, there fore this cover has a countable subcover such that $X \subseteq \cup \{U_{\alpha n} \cup F^c : n \in N\}$, so $F \subseteq \cup \{U_{\alpha n} \cap F : n \in N\} \subseteq \cup \{U_{\alpha n} : n \in N\}$. Hence F is coc-r- lindelof relative to X.

Corollary (3.2.15)

Every r-closed of coc-r- lindelof space X is coc-r- lindelof relative to X.

Proof : It is clear.

Theorem (3.2.16)

Let Y be subspace of (X, τ) and X be coc-r- lindelof space, if Y clopen set in X, then Y coc-r- lindelof.

Proof :

Let Y be a subspace in X, $\{U_\alpha : \alpha \in \Lambda\}$ be cover of Y by coc-r-open subsets of Y such that $Y \subseteq \cup \{U_\alpha : \alpha \in \Lambda\}$, since U_α is coc-r-open in Y, Y is clopen set in X, then U_α is coc-r-open in X for all $\alpha \in \Lambda$ (by using Theorem (1.1.51)). Thus $X = Y \cup Y^c \subseteq \cup \{U_\alpha : \alpha \in \Lambda\} \cup Y^c \subseteq \cup \{U_\alpha \cup Y^c : \alpha \in \Lambda\}$, since Y is clopen set in X, then Y is r-closed, thus Y is coc -r-closed, there fore Y^c is coc -r-open in X. Since X is coc-r-lindelof, then $X \subseteq \cup \{U_{\alpha n} \cup Y^c : n \in N\}$, so that $Y = X \cap Y \subseteq \cup \{U_{\alpha n} \cup Y^c : n \in N\} \cap Y = \cup \{U_{\alpha n} \cap Y : n \in N\} = \cup \{U_{\alpha n} : n \in N\}$, hence Y is coc-r-lindelof.

Theorem (3.2.17)

If a topological space X is a countable union of clopen coc-r- lindelof subspace then X is a coc-r- Lindelof space.

Proof :

Suppose $X = \cup \{U_n : n \in N\}$ when U_n is a clopen coc-r- lindelof subspace for each $n \in N$. Let $\mathcal{V}$ be a cover of X by coc-r-open subsets, since U_n clopen set in X for each $n \in N$. Then for each $n \in N$ the family $\{V \cap U_n : V \in \mathcal{V}\}$ is a cover of U_n by coc-r-open subsets of U_n, thus we find a countable subfamily $\mathcal{V}_n$ of $\mathcal{V}$ such that $U_n \subseteq \cup \{V \cap U_n : V \in \mathcal{V}_n\}$ for each $n \in N$. Put $\mathcal{W} = \cup \{\mathcal{V}_n : n \in N\}$, there fore $\mathcal{W}$ is a countable subfamily of $\mathcal{V}$ such that $X = \cup \{U_n : n \in N\} \subseteq \cup \{\cup \{V \cap U_n : V \in \mathcal{V}_n\} : n \in N\} \subseteq \cup \{V : V \in \mathcal{W}\}$. Hence X is a coc-r- lindelof space.

Theorem (3.2.18)

If X be T_2-space, then the following statements are equivalent.

i) X is coc-r- lindelof.

ii) Every cover by r- open subsets has a countable subcover.

Proof :

(i) ⟶ (ii) It is clear.

(ii) ⟶ (i)

Let $\mathcal{U}$ be coc-r-open cover of X, then $X \subseteq \cup \{U : U \in \mathcal{U}\}$, since X is T_2-space, thus U is equal to the union of r-open sets in X contained in U for each $U \in \mathcal{U}$ (Theorem (1.1.19), (1)). There fore all r-open sets in U for each $U \in \mathcal{U}$ are r-open cover of X, this r-open cover has a countable subcover. Since every element of this a countable subcover contained in U for some $U \in \mathcal{U}$, hence $\mathcal{U}$ has a countable subcover.

Definition (3.2.19)

A space X is said to be r - lindelof if every r - open covering of X has a countable sub covering.

Theorem (3.2.20)

If X is T_2-space, then the following statements are equivalent.

i) Every proper r- closed subset of X is coc-r- lindelof relative to X.

ii) X is coc-r- lindelof.

iii) X is r- lindelof.

Proof :

(i) ⟶ (ii)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of X by r - open subsets of X such that $X \subseteq \cup \{U_\alpha : \alpha \in \Lambda\}$. If $U_\lambda = X, \lambda \in \Lambda$ then the proof is end, if $U_\lambda \neq X, \lambda \in \Lambda$ then ${U_\lambda}^c$ is proper r- closed subset and ${U_\lambda}^c \subseteq \cup \{U_\alpha : \alpha \in \Lambda - \{\lambda\}\}$, by the hypothesis there exist a countable subfamily $\{U_{\alpha n} : \alpha n \in \Lambda - \{\lambda\}, n \in N\}$, such that ${U_\lambda}^c \subseteq \cup \{U_{\alpha n} : \alpha n \in \Lambda - \{\lambda\}, n \in N\}$, thus $X \subseteq \cup \{U_{\alpha n} \cup U_\lambda : \alpha n \in \Lambda - \{\lambda\}, n \in N\}$, hence X is coc-r- Lindelof.

(ii) ⟶ (iii)

Clear, by using Theorem (3.2.18), Definition (3.2.19).

(iii) ⟶ (i)

Suppose F be proper r- closed subset of X, then $F \neq X$, let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of F by r - open subsets of X, since F is r- closed subset of X, thus F^c is r-open, since $F \cup F^c \subseteq \cup \{U_\alpha : \alpha \in \Lambda\} \cup F^c$, there fore $\{U_\alpha, F^c : \alpha \in \Lambda\}$ is r-open cover of X and X is r-lindelof, so $X \subseteq \cup \{U_{\alpha n} : n \in N\} \cup F^c$, hence $F \subseteq \cup \{U_{\alpha n} : n \in N\}$.

Definition (3.2.21) [12]

Let $\mathcal{U}$ is family of subset of X, then $\mathcal{U}$ has a countable intersection property if $\cap \{U_n : n \in N\} \neq \phi$ such that $U_n \in \mathcal{U}$ for each $n \in N$.

Theorem (3.2.22)

If X is T_2-space, then the following statements are equivalent.

1) X is coc-r- lindelof.

2) Every family $\{F_\alpha : \alpha \in \Lambda\}$ of r-closed subsets of X with a countable intersection property then $\cap \{F_\alpha : \alpha \in \Lambda\} \neq \phi$.

3) Every family $\mathcal{F}$ of coc-r-closed subsets of X with $\cap \{F : F \in \mathcal{F}\} = \phi$ contains a countable subfamily $\mathcal{L}$ such that $\cap \{F : F \in \mathcal{L}\} = \phi$.

4) Every family $\mathcal{F}$ of r-closed subsets of X with $\cap \{F : F \in \mathcal{F}\} = \phi$ contains a countable subfamily $\mathcal{L}$ such that $\cap \{F : F \in \mathcal{L}\} = \phi$.

Proof :

(1) $\longrightarrow$ (2)

Suppose that $\{F_\alpha : \alpha \in \Lambda\}$ family of r-closed subsets of X with a countable intersection property, let $\cap \{F_\alpha : \alpha \in \Lambda\} = \phi$. Put $U_\alpha = {F_\alpha}^c$, then U_α is r-open subsets of X, thus the family $\{U_\alpha : \alpha \in \Lambda\}$ is a r-open cover of X. Since X is coc-r-Lindelof, there fore $\{U_\alpha : \alpha \in \Lambda\}$ has a countable subcover $\{U_{\alpha n} : n \in N\}$ such that $X = \cup \{U_{\alpha n} : n \in N\}$ (by using Theorem (3.2.18)), then $X = \cup \{{F_{\alpha n}}^c : n \in N\} = (\cap \{F_{\alpha n} : n \in N\})^c$, thus $\cap \{F_{\alpha n} : n \in N\} = \phi$, this is contradiction with a countable intersection property. Hence $\cap \{F_\alpha : \alpha \in \Lambda\} \neq \phi$.

(2) $\longrightarrow$ (3)

Let $\mathcal{F}$ be a family of coc-r-closed subsets of X with an empty intersection, since X is T_2-space, then $F = \overline{F}^r$ for each $F \in \mathcal{F}$ (by using (Theorem (1.1.19),(2))). Thus F equal to the intersection of r-closed sets in X containing F for each $F \in \mathcal{F}$, there fore the intersection of all r-closed sets in X containing F for each $F \in \mathcal{F}$ is an empty intersection. By using the hypothesis this r-closed family has a countable subfamily with an empty intersection , since every element of this a countable subfamily containing F for some $F \in \mathcal{F}$, hence $\mathcal{F}$ has a countable subfamily with an empty intersection.

(3) $\longrightarrow$ (4) It is clear.

(4) $\longrightarrow$ (1)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of X by r - open subsets of X such that $X = \cup \{U_\alpha : \alpha \in \Lambda\}$, then $\{U_\alpha{}^c : \alpha \in \Lambda\}$ is family of r-closed subsets of X with an empty intersection. By assumption there exist a countable subfamily such that $\cap \{U_{\alpha n}{}^c : n \in N\} = \phi$, so $X = (\cap \{U_{\alpha n}{}^c : n \in N\})^c = \cup \{U_{\alpha n} : n \in N\}$. Hence X is coc-r- lindelof.

Definition (3.2.23) [17]

A space (X, T) is called I-lindelof if every cover $\mathcal{F}$ of X by r - closed subsets of the space (X, T) contains a countable subcover $\mathcal{L}$ such that $X = \cup \{F^\circ : F \in \mathcal{L}\}$.

Remarks (3.2.24)

i. I- compact $\longrightarrow$ I- lindelof. **[17]**

ii. I- lindelof $\not\longrightarrow$ I- compact. **[17]**

iii. coc-r- lindelof $\not\longleftrightarrow$ I- lindelof.

Examples (3.2.25)

1) Let $X = R$, with indiscrete topology, then $\tau^{rk} = \{A : A \subseteq X\}$, thus X is I- Lindelof but X is not coc-r- Lindelof.

2) Let $X = R$, $\tau = \{G \subseteq X : 1 \notin G\} \cup \{X\}$, then $\tau^{rk} = \{G \subseteq X : 1 \notin G\} \cup \{G \subseteq X : 1 \in G, G^c \text{ is finite}\}$, thus X is coc-r- Lindelof but is not I-Lindelof because $\{\{1, x\} : 1 \in X, x \neq 1\}$ is r-closed cover of X but has not a countable subcover and $\{1, x\}^\circ = \{x\}, x \neq 1$.

Theorem (3.2.26)

If a topological space X is extremally disconnected space, then every coc-r-lindelof is I- lindelof.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be r-closed cover of X, then F_α is closed for each $\alpha \in \Lambda$, thus $F_\alpha{}^\circ$ is r-open for each $\alpha \in \Lambda$ (by using Remarks (1.1.4), (2) and Remarks (1.1.16), (4)). Since F_α is r-closed for each $\alpha \in \Lambda$ and X is extremally disconnected space, there fore F_α is open set in X for each $\alpha \in \Lambda$ (by using Remarks (3.1.25), (2)), so F_α is r-open, then F_α is coc- r-open set in X for each $\alpha \in \Lambda$. Since X is coc-r-lindelof, thus the cover $\{F_\alpha : \alpha \in \Lambda\}$ has a countable subcover such that $X = \cup \{F_{\alpha i} : i = 1,2, \dots, n\} = \cup \{F_{\alpha i}{}^{\circ} : i = 1,2, \dots, n\}$. Hence X is I- lindelof.

Theorem (3.2.27)

If a topological space X is T_2-space, then every I-lindelof is coc-r-lindelof.

Proof :

Let $\{U_\alpha : \alpha \in \Lambda\}$ be r-open cover of X, then U_α is open and $\overline{U_\alpha}$ is a r-closed set in X for each $\alpha \in \Lambda$ (by using remarks (1.1.16),(5)), thus $\{\overline{U_\alpha} : \alpha \in \Lambda\}$ is r-closed cover of X and X I-lindelof, there for this cover has a countable sub cover such that $X = \cup \left\{\overline{U_{\alpha n}}^{\circ} : n \in N\right\} = \cup \{U_{\alpha n} : n \in N\}$. Hence X is coc-r-lindelof .

Theorem (3.2.28)

Let $f: X \longrightarrow Y$ be a coć-r-continuous function, onto, if X is lindelof, T_2-space then Y coc-r- lindelof.

Proof :

Let $\{U_\alpha : \alpha \in \Lambda\}$ be coc-r-open cover of Y, since f is a coć-r-continuous function, then $f^{-1}(U_\alpha)$ is coc-r - open in X for each $\alpha \in \Lambda$, but $Y \subseteq \cup_{\alpha \in \Lambda} U_\alpha$, thus $X = f^{-1}(Y) \subseteq \cup_{\alpha \in \Lambda} f^{-1}(U_\alpha)$, since X is lindelof, T_2-space and $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ forms a cover of X, then the cover $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ has a countable subcover such that $X \subseteq \cup \{f^{-1}(U_{\alpha n}), : n \in N \}$, since f onto, so $f(X) = Y \subseteq \cup \{f(f^{-1}(U_{\alpha n})) : n \in N\} \subseteq \cup \{U_{\alpha n} : n \in N \}$. Hence Y coc-r- lindelof.

Theorem (3.2.29)

Let $f: X \longrightarrow Y$ be a coc-r-open function, bijective, if Y is coc-r-lindelof then X lindelof.

Proof :

Let $\{U_\alpha : \alpha \in \Lambda\}$ be open cover of X, since f is a coc-r-open function, then $f(U_\alpha)$ is coc-r - open in Y for each $\alpha \in \Lambda$, but $X \subseteq \cup_{\alpha \in \Lambda} U_\alpha$, there fore $Y = f(X) \subseteq \cup_{\alpha \in \Lambda} f(U_\alpha)$, so $\{f(U_\alpha) : \alpha \in \Lambda\}$ forms a cover of Y, since Y is coc-r- lindelof, then the cover $\{f(U_\alpha) : \alpha \in \Lambda\}$ has a countable subcover such that $Y \subseteq \cup \{f(U_{\alpha n}) : n \in N\}$, thus $X = f^{-1}(Y) \subseteq \cup \left\{f^{-1}\left(f(U_{\alpha n})\right) : n \in N \right\} = \cup \{U_{\alpha n} : n \in N \}$. Hence X lindelof.

Theorem (3.2.30)

Let $f: X \longrightarrow Y$ be a coc-r-open function, bijective and X be extremally disconnected space, if Y is coc-r- lindelof then X I- lindelof.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be r-closed cover of X and X be extremally disconnected, then F_α is open in Y for each $\alpha \in \Lambda$ (Remarks (3.1.25), (2)), since f is a coc-r-open function, then $f(F_\alpha)$ is coc-r - open in Y for each $\alpha \in \Lambda$, but $X \subseteq \cup_{\alpha \in \Lambda} F_\alpha$, there fore $Y = f(X) \subseteq \cup_{\alpha \in \Lambda} f(F_\alpha)$, so $\{f(F_\alpha) : \alpha \in \Lambda\}$ forms a cover of Y, since Y is coc-r- lindelof, then the cover $\{f(F_\alpha) : \alpha \in \Lambda\}$ has a countable subcover such that $Y \subseteq \cup \{f(F_{\alpha n}) : n \in N\}$, thus $X = f^{-1}(Y) \subseteq \cup \left\{f^{-1}\left(f(F_{\alpha n})\right): n \in N \right\} = \cup \{F_{\alpha n}: n \in N \} = \cup \{{F_{\alpha n}}^{\circ}: n \in N \}$. Hence X I- lindelof.

3.3 On I-coc-r-lindelof spaces

We recall the concept of a I-lindelof space by using coc-r-open sets and give some important generalizations on this concept and also we prove some results on this concept.

Definition (3.3.1)

A space (X, T) is called a I- coc-r - lindelof if every cover $\mathcal{F}$ of X by coc-r - regular closed subsets of the space (X, T) contains a countable subcover $\mathcal{L}$ such that $X = \cup \{F^{\circ rk} : F \in \mathcal{L}\}$.

Examples (3.3.2)

The following are straight forward examples of I- coc-r - lindelof spaces.

1) Let $X = \{1,2,3,\ldots\}, \tau = \{G \subseteq X : 1 \in G\} \cup \{\emptyset\}$, then $\tau^{rk} = \{G \subseteq X : G^c \text{ is finite}\} \cup \{\emptyset\}$, since all coc-r - regular closed subsets of the space (X, T) are $\emptyset$, X, thus X is I- coc-r – Lindelof.

2) Let $X = \{1,2,3,4\}, \tau = \{\emptyset, X, \{4\}, \{2,3\}, \{2,3,4\}\}$, then $\tau^{rk} = \{G : G \subseteq X\}$. Thus X is I- coc-r - lindelof.

Theorem (3.3.3)

The following statements are equivalent for a space (X, T).

i) X is a I- coc-r - lindelof.

ii) Every cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by coc-r-β - open subsets contains a countable subcover such that

$X = \cup_{\alpha \in N} \overline{U_\alpha}^{rk^{\circ rk}}$.

iii) Every family $\{U_\alpha : \alpha \in \Lambda\}$ of X by coc-r - regular open subsets with empty intersection contains a countable subfamily such that

$\cap_{n \in N} \overline{U_{\alpha n}}^{rk} = \emptyset$.

Proof :

(i) $\longrightarrow$ (ii)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be cover of X by coc-r-β - open subsets, then $\overline{U_\alpha}^{rk} \in RC(X, \tau^{rk})$, (by using Proposition (1.2.14), (4)) for all $\alpha \in \Lambda$. Thus $\{\overline{U_\alpha}^{rk} : \alpha \in \Lambda\}$ forms cover of X, since X is I- coc-r - lindelof, there fore $\{ \overline{U_\alpha}^{rk} : \alpha \in \Lambda\}$ has a countable subcover such that $X = \cup_{n\in N} \overline{U_{\alpha n}}^{rk^{\circ rk}}$.

(ii) ⟶ (iii)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be a family of coc-r - regular open subsets of X with empty intersection. Since $U_\alpha{}^c \in RC(X, \tau^{rk})$ for all $\alpha \in \Lambda$, by using (Remarks (1.2.15), (2)) we get $U_\alpha{}^c \in \beta O(X, \tau^{rk})$ for all $\alpha \in \Lambda$. Since $\cap_{\alpha\in\Lambda} U_\alpha = \emptyset$, then $X = \cup_{\alpha\in\Lambda} U_\alpha{}^c$, thus $\{U_\alpha{}^c : \alpha \in \Lambda\}$ is cover of X. By assumption, $\{U_\alpha{}^c : \alpha \in \Lambda\}$ has a countable subcover such that $X = \cup_{n\in N} \overline{U_{\alpha n}{}^c}^{rk^{\circ rk}} = \cup_{n\in N} (U_{\alpha n}{}^{\circ rk^c})^{\circ rk}$. There fore $\emptyset = \cap_{n\in N} \overline{U_{\alpha n}{}^{\circ rk}}^{rk} = \cap_{n\in N} \overline{U_{\alpha n}}^{rk}$.

(iii) ⟶ (i)

Let $\{F_\alpha : \alpha \in \Lambda\}$ be cover of X by coc-r - regular closed subsets of X, then $X = \cup_{\alpha\in\Lambda} F_\alpha$, thus $\emptyset = \cap_{n\in\Lambda} F_\alpha{}^c$. Since $F_\alpha \in RC(X, \tau^{rk})$, there fore $F_\alpha{}^c$ $RO(X, \tau^{rk})$ for all $\alpha \in \Lambda$, by assumption The family $\{F_\alpha{}^c : \alpha \in \Lambda\}$ has a countable subfamily such that $\emptyset = \cap_{n\in N} \overline{F_{\alpha n}{}^c}^{rk}$.Then $X = \cup_{n\in N} (\overline{F_{\alpha n}{}^c}^{rk})^c = \cup_{n\in N} F_{\alpha n}{}^{\circ rk}$, hence X is I- coc-r - lindelof.

Definition (3.3.4)

A space (X, T) is called I- coc-r - Compact if every cover $\mathcal{U}$ of X by coc-r - regular closed subsets of the space (X, T) contains a finite subcover $\mathcal{V}$ such that
$X = \cup \{U^{\circ rk} : U \in \mathcal{V}\}$.

Remark (3.3.5)

Every I- coc-r - Compact space is I-coc-r-Lindelof but the convers is not true in general, as the following example shows.

Example (3.3.6)

Let $X = \{1,2,3,\dots\}, \tau = \{A : A \subseteq X\}$, then $\tau^{rk} = \{A : A \subseteq X\}$, since (X, τ^{rk}) is desecrate topology, then $A \in RC(X, \tau^{rk})$ for every $A \subseteq X$, X is a countable set, thus X is I-coc-r-Lindelof, but $\{\{x\} : x \in X\}$ is a cover of X such that $\{x\} \in RO(X, \tau^{rk})$, but it has not a finite subcover, hence X is not I- coc-r - Compact.

Remark (3.3.7)

Every coc-r-lindelof space is not necessary to be I-coc-r-lindelof, as the following example.

Example (3.3.8)

Let $X = \{1,2,3, \ldots\}, \tau = \{G \subseteq X: 1 \notin G\} \cup \{X\}$, then $\tau^{rk} = \{G \subseteq X: 1 \notin G\} \cup \{G \subseteq X: 1 \in G, G^c \text{ is finite}\}$, thus X is coc-r-lindelof space but the cover $\{A = \{1,2,4,6, \ldots\}\} \cup \{\{x\}: x \notin A\}$ of X by coc-r - regular closed subsets is a countable cover but $X \neq A^{\circ rk} \cup_{x \notin A} \{x\}^{\circ rk} = \{2,4,6, \ldots\} \cup_{x \notin A} \{x\}$.

Definition (3.3.9)

A space (X, T) is called coc-r - extremally disconnected (coc-r-e.d) if $\overline{U}^{rk}$ is coc-r-open for each coc-r-open U in X.

Proposition (3.3.10)

If $A \cap B = \emptyset$, A is coc-r-open, then $A \cap \overline{B}^{rk} = \emptyset$.

Proof :

Let $A \cap \overline{B}^{rk} \neq \emptyset$, then there exist $x \in A \cap \overline{B}^{rk}$, since $A \cap B = \emptyset$ and A coc-r-open, thus $x \notin B$, so $x \in \acute{B}^{rk}$ and $A \cap B - \{x\} \neq \emptyset$. There fore $A \cap B \neq \emptyset$, this is contradiction, hence $A \cap \overline{B}^{rk} = \emptyset$.

Proposition (3.3.11)

A space X is coc-r- extremally disconnected iff for all $U, V \in RO(X, \tau^{rk})$ with $U \cap V = \emptyset$, then

$\overline{U}^{rk} \cap \overline{V}^{rk} = \emptyset$.

Proof :

Let X be coc-r- extremally disconnected and $U, V \in RO(X, \tau^{rk})$ with $U \cap V = \emptyset$. Since $U \in RO(X, \tau^{rk})$, then U is coc-r-open, thus $U \cap \overline{V}^{rk} = \emptyset$, (By using Proposition (3.3.10)). Since X is coc-r-extremally disconnected and $V \in RO(X, \tau^{rk})$, there fore $\overline{V}^{rk}$ is coc-r-open in X ,so $\overline{V}^{rk} \cap \overline{U}^{rk} = \emptyset$, by using Proposition (3.3.10). Conversely : Let U be coc-r-open, then $\overline{U}^{rk} \in RC(X, \tau^{rk})$(By using Proposition (1.2.14), (1)), thus $\overline{U}^{rk^c} \in RO(X, \tau^{rk})$, since $\overline{U}^{rk}$ is coc-r-closed , there fore $\overline{U}^{rk^{\circ rk}} \in RO(X, \tau^{rk})$, (Proposition (1.2.14), (2)), since $\overline{U}^{rk^c} \cap \overline{U}^{rk^{\circ rk}} = \emptyset$, by using assumption we get $\overline{\overline{U}^{rk^c}}^{rk} \cap \overline{\overline{U}^{rk^{\circ rk}}}^{rk} = \emptyset$, then $\overline{U}^{rk} \cap \overline{\overline{U}^{rk^c}}^{rk} = \emptyset$, so $\overline{U}^{rk} \subseteq \overline{U}^{rk^{\circ rk}}$, then $\overline{U}^{rk}$ is coc-r-open. Hence X is coc-r-extremally disconnected.

Proposition (3.3.12)

If X is coc-r-extremally disconnected, $F \subseteq X$ such that $F \in RC(X, \tau^{rk})$, then F coc-r-open.

Proof :

Let $F \in RC(X, \tau^{rk})$, then $F = \overline{F^{\circ rk}}^{rk}$, since X is coc-r-extremally disconnected, thus $\overline{F^{\circ rk}}^{rk}$ is coc-r-open, hence F coc-r-open.

Proposition (3.3.13)

Every I-coc-r-lindelof space is coc-r- extremally disconnected.

Proof :

Let X be I-coc-r-lindelof space. Suppose that X is not coc-r-extremally disconnected, then there is $U, V \in RO(X, \tau^{rk})$ such that $U \cap V = \emptyset$, but $\overline{U}^{rk} \cap \overline{V}^{rk} \neq \emptyset$. Then there is $x \in \overline{U}^{rk} \cap \overline{V}^{rk}$, since $U, V \in RO(X, \tau^{rk})$, thus $U^c, V^c \in RC(X, \tau^{rk})$, there fore $\{U^c, V^c\}$ forms a cover of X, since X is I-coc-r-lindelof space, so $X = U^{c\circ rk} \cup V^{c\circ rk}$. Let $x \in U^{c\circ rk}$, but $x \in \overline{U}^{rk}$, then $U^{c\circ rk} \cap U \neq \emptyset$. Since $U^{c\circ rk} \cap U \subseteq U^c \cap U$, this is contradiction, thus $\overline{U}^{rk} \cap \overline{V}^{rk} = \emptyset$. Hence X is coc-r-extremally disconnected (By using Proposition (3.3.11))

Remark (3.3.14)

The convers of Proposition (3.3.13) is not true in general, as the following example shows.

Example (3.3.15)

Let $X = R$, with indiscrete topology, then $\tau^{rk} = \{A: A \subseteq X\}$, then X is coc-r-extremally disconnected. Since $\{x\} \in RC(X, \tau^{rk})$ for each $x \in X$, since the cover $\{\{x\}: x \in X\}$ of X has not a countable subcover such that $X = \cup_{x\in X} \{x\}^{\circ rk}$.

Theorem (3.3.16)

Every coc-r-lindelof, coc-r-extremally disconnected space is I-coc-r-lindelof space.

Proof :

Let $\{ F_\alpha: \alpha \in \Lambda\}$ be cover of X, $F_\alpha \in RC(X, \tau^{rk})$ for all $\alpha \in \Lambda$, then F_α is coc-r-open for all $\alpha \in \Lambda$ (By using Proposition (3.3.12)). Thus $\{ F_\alpha: \alpha \in \Lambda\}$ is cover of X by coc-r-open subsets, since X is coc-r-lindelof Space, there fore $\{ F_\alpha: \alpha \in \Lambda\}$ has a countable subcover such that $X = \cup_{n\in N} F_{\alpha n} = \cup_{n\in N} F_{\alpha n}{}^{\circ rk}$. Hence X is I-coc-r-lindelof space.

Remark (3.3.17)

1) I-coc-r-lindelof $\not\rightarrow$ I-lindelof.

2) I-lindelof $\not\rightarrow$ I-coc-r-lindelof.

As the following examples show.

Examples (3.3.18)

1) Let $X = \{1,2,3,4\}, \tau = \{\emptyset, X, \{4\}, \{2,3\}, \{2,3,4\}\}$, then $\tau^{rk} = \{G: G \subseteq X\}$.
Thus X is I- coc-r - lindelof, since $\{1,2,3\}, \{1,4\}$ are r-closed cover of X, but
$X \neq \{1,2,3\}^{\circ rk} \cup \{1,4\}^{\circ rk} = \{2,3\} \cup \{4\} = \{2,3,4\}$, then X is not I-lindelof.

2) Let $X = R$, with indiscrete topology, then $\tau^{rk} = \{A: A \subseteq X\}$, thus X is I-lindelof but X is not I-coc-r-lindelof.

Definition (3.3.19)

A space (X, T) is called $\mathcal{S}$- coc-r - lindelof if every cover $\mathcal{F}$ of X by coc-r - regular closed subsets of the space (X, T) contains a countable subcover $\mathcal{L}$ such that $X = \cup \{F: F \in \mathcal{L}\}$.

Remark (3.3.20)

Every I- coc-r - lindelof space is $\mathcal{S}$- coc-r - lindelof but the convers is not true in general, as the following example shows.

Example (3.3.21)

Let $X = \{1,2,3,\dots\}, \tau = \{G \subseteq X: 1 \notin G\} \cup \{X\}$, then $\tau^{rk} = \{G \subseteq X: 1 \notin G\} \cup \{G \subseteq X: 1 \in G, G^c \text{ is finite}\}$, thus X is $\mathcal{S}$- coc-r - lindelof, since $A = \{3,5,7,\dots\}$ is coc-r-open in X but $\overline{A}^{rk} = \{1,3,5,7,\dots\}$ is not coc-r-open in X, there fore X is not coc-r-extremally disconnected, hence X is not I- coc-r - lindelof (By using the convers of Proposition (3.3.13)).

Theorem (3.3.22)

A space X is I-coc-r-lindelof if and only if it is a coc-r-extremally disconnected and $\mathcal{S}$- coc-r - lindelof space .

Proof :

As necessity is clear, we prove only sufficiency.

Let $\{F_\alpha: \alpha \in \Lambda\}$ be cover of X by coc-r - regular closed subsets, since X is a coc-r-extremally disconnected, then F_α is coc-r-open for each $\alpha \in \Lambda$ (By using Proposition (3.3.12)), thus $F_\alpha = F_\alpha{}^{\circ coc-r}$. Since X is $\mathcal{S}$- coc-r-lindelof space, there fore $\{F_\alpha: \alpha \in \Lambda\}$ has a countable subcover such that X = $\cup_{n \in N} F_{\alpha n}$= $\cup_{n \in N} F_{\alpha n}{}^{\circ coc-r}$, hence X is I-coc-r-lindelof.

Remark (3.3.23)

A space X is said to be coc′-r-regular space if and only if (X, τ^{rk}) is regular space.

Proposition (3.3.24)

Let X be coc′-r-regular space, if G is coc-r-open then $G \in RO(X, \tau^{rk})$.

Proof :

Let G is coc-r-open in X, since X is coc′-r-regular space, then for each $x \in G$ there exists an coc-r-open set W_x such that $x \in W_x \subseteq \overline{W_x}^{rk} \subseteq G$ (By using

Proposition (2.3.13)) . Thus $G = \cup\left\{\overline{W_x}^{rk} : x \in G\right\}$, there fore $\overline{G}^{rk^{\circ rk}} = \overline{(\cup_{x\in G} \overline{W_x}^{rk})}^{rk^{\circ rk}} = (\cup_{x\in G} \overline{W_x}^{rk})^{\circ rk} = G^{\circ rk} = G$. Hence $G \in RO(X, \tau^{rk})$.

Remark (3.3.25)

If X is coc′-r-regular space, C is coc-r-closed then $C \in RC(X, \tau^{rk})$.

Proof : It is clear.

Theorem (3.3.26)

If X is coc-r-extremally disconnected, coc′-r-regular space, then the following statements are equivalent.

1) X is $\mathcal{S}$- coc-r - lindelof.

2) X is I-coc-r-lindelof.

3) X is coc-r-lindelof.

Proof :

(1)⟶ (2)

It is clear, by using Theorem (3.3.22).

(2)⟶ (3)

Let $\{ U_\alpha : \alpha \in \Lambda\}$ be cover of X by coc-r-open subsets, since X is coc′-r-regular space, then by using Proposition (3.3.24) we get $U_\alpha \in RO(X, \tau^{rk})$ for each $\alpha \in \Lambda$. Thus $\overline{U_\alpha}^{rk} \in RC(X, \tau^{rk})$ for each $\alpha \in \Lambda$, there fore $\{ \overline{U_\alpha}^{rk} : \alpha \in \Lambda\}$ forms a cover of X, since X is I-coc-r-lindelof, there fore $\{ \overline{U_\alpha}^{rk} : \alpha \in \Lambda\}$ has a countable subcover such that $X = \cup_{n\in N} \overline{U_{\alpha n}}^{rk^{\circ rk}}$, since $U_\alpha \in RO(X, \tau^{rk})$ for each $\alpha \in \Lambda$, so $X = \cup_{n\in N} U_{\alpha n}$. Hence X is coc-r-lindelof.

(3)⟶ (1)

It is clear by using Theorem(3.3.16) and Remark (3.3.20).

Remarks (3.3.27)

1) If X is T_3-space (T_1 + regular space), then every coc-r-closed is r-closed.

2) If X is T_3-space, then X is extremally disconnected if and only if X is coc-r-extremally disconnected.

3) If X is T_3-space, then X is regular space if and only if X is coc′-r-regular space.

Proof :

1) Let $A \subseteq X$ be coc-r-closed, since X is T_3-space, then X is T_2-space, thus A is closed set in X, there fore A is r-closed (X is regular space).

(2), (3) It is clear, since X is T_3-space, then $\tau = \tau^{rk}$.

Theorem (3.3.28)

If X is T_3, extremally disconnected space, then the following statements are equivalent.

1) X is coc-r-lindelof.

2) X is I-lindelof.

3) X is lindelof.

4) X is I-coc-r-lindelof.

Proof :

(1) $\longrightarrow$ (2)

Since X is extremally disconnected, then X is I-Lindelof (Theorem (3.2.26))

(2) $\longrightarrow$ (3)

Let $\{U_\alpha : \alpha \in \Lambda\}$ be open cover of X, since X is T_3-space, then X is regular space, thus U_α is r-open in X (Proposition (1.1.21)) for each $\alpha \in \Lambda$. Since U_α is open in X, there fore $\overline{U_\alpha}$ is r-closed (Remarks (1.1.16), (5)) for each $\alpha \in \Lambda$. Then $\{\overline{U_\alpha} : \alpha \in \Lambda\}$ forms a cover of X, since X is I-lindelof, there fore $\{\overline{U_\alpha} : \alpha \in \Lambda\}$ } has a countable subcover such that $X = \cup_{n \in N} \overline{U_{\alpha n}}^{\circ} = \cup_{n \in N} U_{\alpha n}$,hence X is lindelof space.

(3) $\longrightarrow$ (4)

Let $\{ F_\alpha : \alpha \in \Lambda\}$ be cover of X by coc-r - regular closed subsets, then F_α is coc-r-closed in X, then F_α is r-closed for each $\alpha \in \Lambda$ (By using Remarks (3.3.27), (1)), thus F_α is open for each $\alpha \in \Lambda$ (By using Remarks (3.1.25), (2)), there fore $\{ F_\alpha : \alpha \in \Lambda\}$ forms a cover of X, since X is lindelof, so $\{ F_\alpha : \alpha \in \Lambda\}$ has a countable subcover such that $X = \cup_{n \in N} F_{\alpha n}$,since X is T_3-space, then F_α coc-r-open in X for each $\alpha \in \Lambda$, thus $X = \cup_{n \in N} F_{\alpha n}^{\circ coc-r}$, hence X is I-coc-r-lindelof.

(4) $\longrightarrow$ (1)

It is clear by using (Remarks (3.3.27), (2), (3)) and Theorem (3.3.26).

Theorem (3.3.29)

Let $f: X \longrightarrow Y$ be a coć-r-continuous function, onto and (Y, τ_Y) be coc-r-extremally disconnected space, if X is coc-r-lindelof, then Y is I-coc-r-lindelof.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be cover of Y by coc-r - regular closed subsets, since Y is a coc-r-extremally disconnected, then F_α is coc-r - open in Y for each $\alpha \in \Lambda$ (By using Proposition (3.3.12)), since f is a coć-r-continuous function, thus $f^{-1}(F_\alpha)$ is coc-r - open in X, but $Y \subseteq \cup_{\alpha\in\Lambda} F_\alpha$, there fore $X = f^{-1}(Y) \subseteq \cup_{\alpha\in\Lambda} f^{-1}(F_\alpha)$, so $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ forms a cover of X, since X is coc-r-lindelof, then $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ has a countable subcover such that $X \subseteq \cup_{n\in N} f^{-1}(F_{\alpha n})$, since f onto, thus $f(X) = Y \subseteq \cup_{n\in N} f(f^{-1}(F_{\alpha n})) = \cup_{n\in N} F_{\alpha n} = \cup_{n\in N} {F_{\alpha n}}^{\circ rk}$, hence Y is I-coc-r-lindelof.

Theorem (3.3.30)

Let $f: X \longrightarrow Y$ be a coć-r-open function, bijective and (X, τ_X) be coc-r-extremally disconnected space, if Y is coc-r-lindelof, then X is I-coc-r-lindelof.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be cover of X by coc-r - regular closed subsets, since X is a coc-r-extremally disconnected, then F_α is coc-r - open in X for each $\alpha \in \Lambda$ (By using Proposition (3.3.15)), since f is a coć-r-open function, thus $f(F_\alpha)$ is coc-r - open in Y, but $X \subseteq \cup_{\alpha\in\Lambda} F_\alpha$, there fore $Y = f(X) \subseteq \cup_{\alpha\in\Lambda} f(F_\alpha)$, so $\{f(F_\alpha) : \alpha \in \Lambda\}$ forms a cover of Y, since Y is coc-r-lindelof, then $\{f(F_\alpha) : \alpha \in \Lambda\}$ has a countable subcover such that $Y \subseteq \cup_{n\in N} f(F_{\alpha n})$, thus $X = f^{-1}(Y) \subseteq \cup_{n\in N} f^{-1}(f(F_{\alpha n})) = \cup_{n\in N} F_{\alpha n} = \cup_{n\in N} {F_{\alpha n}}^{\circ coc-r}$, hence X is I-coc-r-lindelof.

Theorem (3.3.31)

Let $f: X \longrightarrow Y$ be a coć-r-continuous, coć-r-open function, onto and X coc'-r-regular space, if X is I-coc-r-lindelof, then Y is also.

Proof :

Let $\{F_\alpha : \alpha \in \Lambda\}$ be cover of Y by coc-r - regular closed subsets, then ${F_\alpha}^c \in RO(X, \tau^{rk})$, thus ${F_\alpha}^c$ coc-r - open in Y for each $\alpha \in \Lambda$, since f is a coć-r-continuous function, there fore $f^{-1}({F_\alpha}^c)$ is coc-r - open in X for each $\alpha \in \Lambda$, so $(f^{-1}({F_\alpha}^c))^c = f^{-1}(F_\alpha)$ is coc-r - closed in X for each $\alpha \in \Lambda$, since X coc'-r-regular space, then$f^{-1}(F_\alpha) \in RC(X, \tau^{rk})$ (By using Remark (3.3.25)), since $Y \subseteq \cup_{\alpha\in\Lambda} F_\alpha$, thus $X = f^{-1}(Y) \subseteq \cup_{\alpha\in\Lambda} f^{-1}(F_\alpha)$, there fore $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ forms a cover of X, since X is I-coc-r-lindelof space, then $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ has a countable subcover such that $X \subseteq \cup_{n\in N} (f^{-1}(F_{\alpha n}))^{\circ rk}$, thus $Y = f(X) \subseteq$

$\cup_{n\in N} f((f^{-1}(F_{\alpha n}))^{\circ rk}) \subseteq \cup_{\alpha\in\Lambda} (f(f^{-1}(F_{\alpha n})))^{\circ rk} = \cup_{n\in N} F_{\alpha n}{}^{\circ rk}$, hence Y is I-coc-r-lindelof.

Definition (3.3.32)

Let $f: X \longrightarrow Y$ be a function of space X into space Y, then f is called $\mathcal{S}$- coc-r -β -closed function if for each $y \in Y$ and for each $U \in \tau^{rk}$ with $f^{-1}(y) \subseteq U$, there exist β-open set V such that $y \in V$, $f^{-1}(V) \subseteq U$.

Definition (3.3.33)

A space (X, T) is called coc-r - P- space if the countable union of coc-r - closed subsets is coc-r - closed.

Definitions (3.3.34) [9]

i. A space (X, T) is called rc- lindelof if every cover of X by regular closed subsets of the space (X, T) contains a countable subcover.

ii. A space X is said to be countably nearly compact if every countable open cover $\mathcal{U}$ of (X ,T) contains a finite subfamily $\mathcal{V}$ such that $X = \cup\{int(cl(U)) : U \in \mathcal{V}\}$.

Proposition (3.3.35) [9]

A space (X, T) is called rc- lindelof iff every cover $\mathcal{U}$ of X by β - open subsets contains a countable subcover $\mathcal{V}$ such that

$X = \cup\{\overline{U} : U \in \mathcal{V}\}$.

Proposition (3.3.36) [17]

Every I- lindelof space is rc- lindelof space.

Theorem (3.3.37)

Let $f: (X, \tau_X) \longrightarrow (Y, \tau_Y)$ be a $\mathcal{S}$- coc-r - β -closed, super coc-r-open function, with $f^{-1}(y)$ $\mathcal{S}$-coc-r-lindelof for each $y \in Y$ and X coc-r-extremally disconnected, coc-r - P- space , if Y is I-lindelof, then X is I-coc-r-lindelof.

Proof :

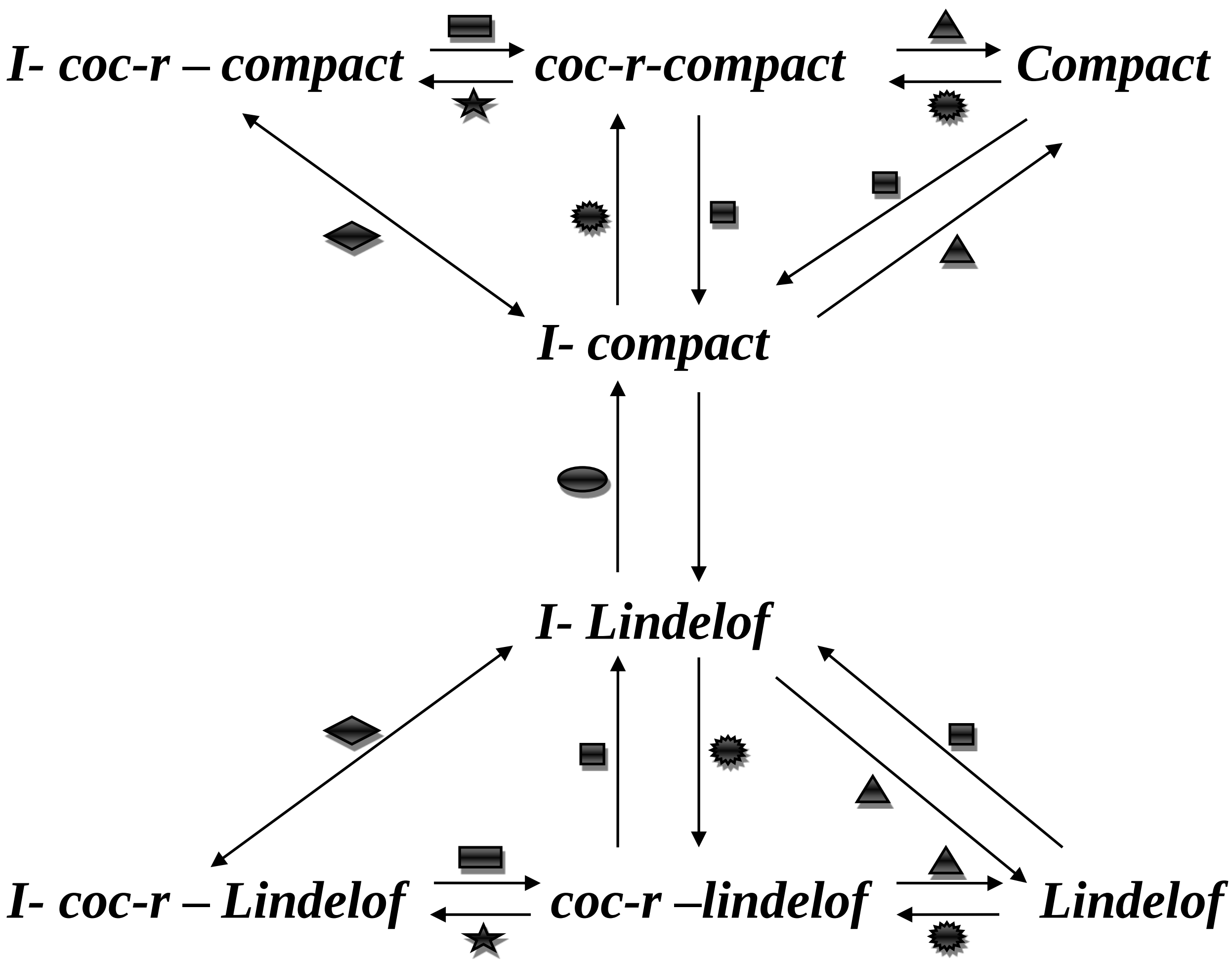
We need to show that X is $\mathcal{S}$- coc-r - lindelof. Let $\mathcal{F}$ be a cover of X by coc-r-regular closed subsets, then for all $y \in Y$, $\mathcal{F}$ forms a cover of $f^{-1}(y)$, since $f^{-1}(y)$

$\mathcal{S}$- coc-r-lindelof, thus we find a countable subcover $\mathcal{F}_y$ of $\mathcal{F}$ such that $f^{-1}(y) \subseteq \cup \{F: F \in \mathcal{F}_y\}$. Put $G_y = \cup \{F: F \in \mathcal{F}_y\}$, there fore G_y is union of coc-r-regular closed subsets, since $F \in RC(X, \tau^{rk})$ for each $F \in \mathcal{F}_y$ and since X coc-r-extremally disconnected, so we get G_y is coc-r - open (By using Proposition (3.3.12)) and $f^{-1}(y) \subseteq G_y$ for each $y \in Y$. Since f is $\mathcal{S}$-coc-r - β -closed, then there is β - open V_y such that $y \in V_y$, $f^{-1}(V_y) \subseteq G_y$.

Now:

The family $\{V_y: y \in Y\}$ forms a cover of Y by β - open subsets, since Y is I-lindelof and By using Proposition (3.3.35), (3.3.36) then the cover $\{V_y: y \in Y\}$ contains a countable subcover such that $Y = \cup \{\overline{V_{yn}}: n \in N\}$. Put $\mathcal{L} = \cup \{\mathcal{F}_{yn}: n \in N\}$, it is clear that $\mathcal{L}$ is a countable family. Let $x \in X$ and $y = f(x)$, thus $y \in \overline{V_{yk}}$, $k \in N$, there fore $x \in f^{-1}(\overline{V_{yk}})$, since f super coc-r-open function and by using Theorem (2.2.33) we get $x \in f^{-1}(\overline{V_{yk}}) \subseteq \overline{f^{-1}(V_{yk})}^{rk} \subseteq \overline{G_{yk}}^{rk}$, by using remark ((1.2.15),(3)) and since X coc-r - P- space, so we get $x \in \overline{G_{yk}}^{rk} = G_{yk} = \cup \{F: F \in \mathcal{F}_{yk}\} \subseteq \cup \{F: F \in \mathcal{L}\}$, then X is $\mathcal{S}$- coc-r - lindelof, hence X is I-coc-r-lindelof (By using Theorem (3.3.22)).

The following diagram explains the relationship among these types of compact and lindelof spaces.



Regular space

T_2-space

extremally disconnected space

T_3-space

coc-r-extremally disconnected space

coc′-r-regular space

Countably nearly compact

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الخلاصة

الهدف الأساسي في هذا البحث هو توسيع ودراسة بعض أنواع الفضاءات التبولوجية مثل الفضاءات المرصوصة والليندلوف عن طريق خواص الغطاءات بأستخدام المجموعات المفتوحة من النمط coc-r .

لقد درسنا سابقا المفاهيم المرصوص واللندلوف والمتصل وبديهيات الفصل للفضاءات وفي هذا العمل سوف نوسع هذه المفاهيم بأستخدام المجموعات المفتوحة من النمط coc-r لتعريف ودراسة المتصل من النمط coc-r و وبديهيات الفصل من النمط coc-r والمرصوص من النمط coc-r واللندلوف من النمط coc-r و I-coc-r.

ايضا تناولنا خلال البحث مفهوم الدوال(المفتوحة والمغلقة)من النمط coc-r والنمط coc´-r والنمط Super coc-r و المستمرة من النمط (coc´-r، coc-r) ووضحنا خواص تلك الدوال.

وما ياتي اهم النتائج الرئيسية :

١- ليكن X فضاء هاوزدورف، فأن العبارات التالية تكون متكافئة:-

أ- X يكون مرصوص من النمط coc-r.

ب- إذا كان لكل غطاء مفتوح منتظم للفضاء X يمتلك غطاء منته.

٢- ليكن X فضاء هاوزدورف، فأن العبارات التالية تكون متكافئة:-

أ- كل مجموعة جزئية فعلية مغلقة منتظمة تكون مرصوصة من النمط coc-r في X.

ب- X يكون مرصوص من النمط coc-r.

ج- X يكون مرصوص من النمط r.

٣- اذا كانت $f: X \longrightarrow Y$ دالة مستمرة من النمط coc-r وشاملة و Y غير متصل جدا فانه اذا كان X مرصوص من النمط coc-r فان Y فضاء مرصوص من النمط I.

٤- اذا كانت $f: X \longrightarrow Y$ دالة مفتوحة من النمط coc-r ومتقابلة و X غير متصل جدا فانه اذا كان Y لندلوف من النمط coc-r فان X فضاء لندلوف من النمط I.

٥- ليكن X فضاء غير متصل جدا من النمط coc-r ومنتظم من النمط coc´-r ، فأن العبارات التالية تكون متكافئة:-

أ- X يكون فضاء لندلوف من النمط $\mathcal{S}$- coc-r .

ب- X يكون فضاء لندلوف من النمط I-coc-r .

ج- X يكون فضاء لندلوف من النمط coc-r.

٦- ليكن X فضاء T_3 وغير متصل جدا، فأن العبارات التالية تكون متكافئة:-

أ- X يكون فضاء لندلوف من النمط coc-r.

ب- X يكون فضاء لندلوف من النمط I.

ج- X يكون فضاء لندلوف.

د- X يكون فضاء لندلوف من النمط I-coc-r.

٧- اذا كانت $f: X \longrightarrow Y$ دالة مغلقة من النمط β - $\mathcal{S}$- coc-r ومفتوحة من النمط Super coc-r, و $f^{-1}(y)$ لندلوف من النمط $\mathcal{S}$- coc-r في X لكل $y \in Y$ و X فضاء غير متصل جدا و فضاءP من النمط coc-r، فانه اذا كان Y لندلوف من النمط I فان X فضاء لندلوف من النمط I-coc-r.

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قسم الرياضيات

حول بعض خصائص الغطائات بأستخدام المجموعةالمفتوحة -coc-r

رسالة

مقدمة إلى كلية علوم الحاسوب والرياضيات - جامعة القادسية وهي جزء من متطلبات نيل درجة ماجستير علوم في الرياضيات

من قبل

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بإشراف

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١٤٣٨ هـ ٢٠١٧ م