Republic of Iraq
Ministry of Higher Education & Scientific Research
University of Al-Qadisiyah
College of Computer Science and Mathematics
Department of Mathematics

A Study of Differential Subordination and Superordination Results in Geometric Function Theory

A Thesis

Submitted to Council of the College of Computer Science and Mathematics, University of Al-Qadisiyah as a Partial Fulfilment of the Requirements for the Degree of Master of Science in Mathematics

By

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2017 A.D. 1438 A.H.

بسم الله الرحمن الرحيم

((عَلَّمَ الْإِنسَانَ مَا لَمْ يَعْلَمْ))

صدق الله العلي العظيم

سورة العلق / آية : 5

To my family and …

To my son Mustafa and daughter Fatima

Acknowledgements

Firstly thanks for Allah for every thing

I would like to express my sincere appreciation to my supervisor, (Dr.Waggas Galib Atshan), for giving me the major steps to go on exploring the subject, and sharing with me the ideas related to my work.

I also would like to express my deep thanks to all staff members of Mathematics in Al-Qadisiyiah University for their help.

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We certify that we have read this thesis "**A Study of Differential Subordination and Superordination Results in Geometric Function Theory** " and as an examining committee, we examined the study in its content , and that in our opinion it meets the standards of a thesis for the degree of master of science in mathematics.

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List of publications

1. Differential Subordination for Univalent Functions, European Journal of Scientific Research (EJSR) (Impact Factor: 0.74) (UK), 145(4) (2017) 427-434.
2. Some Properties of Differential Sandwich Results of p-valent Functions Defined by Liu-Srivastava Operator, International Journal of Advances in Mathematics (IJAM) (Impact Factor: 1.67) (India), Vo.2017, No.6, pp. 101-113, (2017).
3. On Fourth-Order Differential Subordination and Superordination Results for Multivalent Analytic Functions, International Journal of Pure and Applied Mathematics (IJPAM) (Bulgaria) (2017), (Accepted for publication).
4. On Third-Order Differential Subordination Results for Meromorphic Univalent Function Associated with Linear Operator, European Jouranl of Pure and Applied Mathematics (EJPAM) (Impact Factor: 4.59) (Turkey) (2017), (Accepted for publication).

List of Symbols

Symbol	Description
$\mathbb{C}$	Complex plane
$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$
$\mathbb{N}_0$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
U	Open unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$
U^*	Punctured open unit disk $\{z \in \mathbb{C} : 0 < \lvert z\rvert < 1\}$
∂U	Boundary of open unit disk $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$
$\bar{U}$	Closed unit disk $U \cup \{z : z \in \partial U\} = \{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$
$\mathcal{A}$	Class of normalized analytic functions in U
$\mathcal{S}$	The subclass of $\mathcal{A}$ is univalent in U
$\mathcal{S}_p$	The class of all p –valent analytic functions in U
S^*	Class of starlike functions in U
$\mathcal{S}^*(\alpha)$	Class of starlike functions of order α in U
$\mathcal{C}$	Class of convex functions in U
$\mathcal{C}(\alpha)$	Class of convex functions of order α in U
K	Class of close to convex functions in U
$K(z)$	Koebe function
$\mathcal{M}$	Class of meromorphic univalent functions in U^*
$\mathcal{MS}^*(\alpha)$	Class of meromorphic starlike functions of order α
$\mathcal{MC}(\alpha)$	class of meromorphic convex functions of order α
$f * g$	Hadamard product of f and g

$f \prec F$	f subordinate to F
D	Domain
D^{n+p-1}	Ruscheweyh derivative of order $(n+p-1)$
${}_{q}^{s}F(\alpha_1, \dots, \alpha_q;\ \beta_1, \dots, \beta_s; z)$	Generalized hypergeometric functions
$(x)_n$	Pochhammer symbol
$\Re$	Real part of a complex number
$\mathbb{P}$	Class of analytic function with positive real part in D
$\mathcal{H}(U)$	Class of analytic functions in U
$\mathcal{H}[a, n]$	Class of analytic functions in U of the form: $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$
$\Psi_n[\Omega, q]$	Class of admissible functions of the form: $\psi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$
$\Psi'_n[\Omega, q]$	Class of admissible functions of the form: $\psi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$
$\mathrm{A}_n[\Omega, q]$	Class of admissible functions of the form: $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$
$\mathrm{A}'_n[\Omega, q]$	Class of admissible functions of the form: $\psi : \mathbb{C}^5 \times \overline{U} \longrightarrow \mathbb{C}$

Abstract

The purpose of this thesis is to study the differential subordination and superordination results in geometric function theory. It studies differential subordination for univalent functions. We investigate and obtain subordination results for generalized deriving function of a new class of univalent analytic functions in the open unit disk. Also, we have discussed some properties of differential sandwich results of p-valent functions defined by Liu-Srivastava operator. Results on differential subordination and superordination are obtained. Also, some sandwich theorems are derived.

We have also undertaken the study of third-order differential subordination results for meromorphic univalent functions associated with linear operator.

Here, new results for third-order differential subordination in the punctured unit disk are obtained.

We have also dealt with third-order differential superordination results for p-valent meromorphic functions involving linear operator. We derive some third-order differential superordination results for analytic functions in the punctured open unit disk by using certain classes of admissible functions.

We have also studied the fourth-order differential subordination and superordination results for multivalent analytic functions. Here, we introduce new concept that is fourth-order differential subordination and superordination associated with differential linear operator $I_p(n,\lambda)$ in open unit disk.

Introduction

The classical study of the subject of analytic univalent functions has been engaging the attention of researchers at least till as early as 1907. This has been growing vigorously with added research. This field captioned as Geometric Function Theory is found to be a mixing or an interplay of geometry and analysis. Despite the classical nature of the subject, unlike contemporary areas, this field has been fascinating researchers, with stress on the interest based on investigations by function theorists. The main ingredient motivating this line of thought is based on the famous conjecture called the Bieberbach conjecture or coefficient problem offering vast scope for development from 1916, till a positive settlement in 1985 by de Branges where innumerable results were obtained based on this problem. Since then, Geometric Function Theory was a subject in its own right. Geometric function is a classical subject. Yet it continues to find new applications in an ever-growing variety of areas such as modern mathematical physics, more traditional fields of physics such as fluid dynamics, nonlinear integrable systems theory and the theory of partial differential equations.

Detailed treatment of univalent functions are available in the standard books of Duren [23] and Goodman [27].

A function f analytic in a domain Ω of the complex plane $\mathbb{C}$ is said to be univalent or one-to-one in Ω if it never takes the same value more than once in Ω. That is, for any two distinct points z_1 and z_2 in Ω, $f(z_1) \neq f(z_2)$. The choice of the unit disc, $U = \{z : |z| < 1\}$ as a domain for the study of analytic univalent functions is a matter of convenience to make the computations simple and leads to elegant formulae. There is no loss of generality in this choice, since Riemann Mapping Theorem asserts that any simply connected proper subdomain of $\mathbb{C}$ can be mapped onto the unit disk by univalent transformation.

The class of all analytic functions in the open unit disk U with normalization $f(0) = 0$ and $f'(0) = 1$ will be denoted by $\mathcal{A}$, consisting of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (z \in U).$$

Geometrically, the normalization $f(0) = 0$ amounts to only a translation of the image domain and $f'(0) = 1$ corresponds to rotation and stretching or shrinking of the image domain.We denote the class of all analytic univalent functions with the above normalization by $\mathcal{S}$.

The function $K(z)$ called the Koebe function, is defined by

$$K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots,$$

which maps U onto the complex plane except for a slit along the negative real axis from $-\infty$ to $-\frac{1}{4}$, is a leading example of a function in $\mathcal{S}$. It plays a very important role in the study of the class $\mathcal{S}$. In fact, the Koebe function and its rotations $e^{-i\alpha}K(e^{i\alpha}z)$, $\alpha \in \mathbb{R}$ are the only extremal functions for various extremal problems in $\mathcal{S}$. The study of univalent and multivalent functions was initiated by Koebe (1907) [37]. He discovered that the range of all functions in $\mathcal{S}$ contain a common disk $|w| < \frac{1}{4}$, later named as the Koebe domain for the class $\mathcal{S}$ in honour of him.

For functions f in the class $\mathcal{S}$, [23], it is well known that the following growth and distortion estimates hold respectively as for $z = re^{i\theta}, 0 \le r < 1$

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2},$$

and

$$\frac{1-r}{(1+r)^3} \le |f'(z)| \le \frac{1+r}{(1-r)^3}.$$

Further for functions f in the class $\mathcal{S}$, [23], it is well known that the following rotation property holds:

$$|arg f'(z)| \le \begin{cases} 4\sin^{-1} r, & r \le \frac{1}{\sqrt{2}} \\ \pi + \log\frac{r^2}{1-r^2}, & r \ge \frac{1}{\sqrt{2}}, \end{cases}$$

where $|z| = r < 1$. The bound is sharp.

In [23] 1916, Bieberbach studied the second coefficient a_2 of a function $f \in \mathcal{S}$.

He has shown that $|a_2| \le 2$, with equality if and only if f is a rotation of the Koebe function and he mentioned "$|a_n| \le n$ is generally valid ". This statement is known as the Bieberbach conjecture.

In 1923 Löwner [41] proved the Bieberbach conjecture for $n = 3$, many others investigated the Bieberbach conjecture for certain values of n. Finally, 1985 de Branges [22] proved the Bieberbach conjecture for all coefficients with the help of hypergeometric functions.

Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions. Notable among them are the classes of starlike functions, convex functions and close –to–convex functions.

Subordination between analytic functions return back to Littlewood [42,43] and Lindelöf [39], where Rogosinski [60,61] introduced the term and established the basic results involving subordination. Quite recently Srivastava and Owa [67] investigated various interesting properties of the generalized hypergeometric function by applying the concept of subordination.

Ma and Minda [45] showed that many of these properties can be obtained by a unified method. For this purpose they introduced the classes $C(\phi)$ and $S^*(\phi)$ of functions $f(z) \in \mathcal{A}$, for some analytic function $\phi(z)$ with positive real part on U, with $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps the open unit disk U onto a region starlike with respect 1, symmetric with respect to the real axis, satisfying:

$$1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \text{ and } \frac{zf'(z)}{f(z)} \prec \phi(z)\,, (z \in U).$$

They developed a new method in geometric function theory known as the method of differential subordination or the method of admissible functions. This method is very effective to obtain new results.

Interest in geometric function theory has experienced resurgence in recent decades as the methods of function theory on compact Riemann surfaces and algebraic geometry.

Early string theory models depends on elements of geometric function theory for the computation of so called Veneziano amplitudes was appeared [34].

The thesis is organized as follows. In chapter one, we present a brief introduction to some background of complex concepts and the basic ideas of geometric function theory.

Chapter two consists of two sections, in the first section, we deal with the study of differential subordination for univalent functions. We obtain subordination results for generalized deriving function of a new class of univalent analytic functions in the open unit disk.

Section two is devoted for the study of some properties of differential sandwich results of p-valent functions defined by Liu-Srivasava operator. We obtain results on differential subordination and superordination. Also, we derive some sandwich theorems.

Chapter three has been divided into three sections, setion one deals with a third-order differential subordination results for meromorphic univalent functions associated with linear operator.Here, we obtain new results for third –order differential subordination in the punctured unit disk. In section two, we have introduced the third-order differential superordination results for p-valent meromorphic functions involving linear operator. We derive some third-order differential superordination results for analytic functions in the punctured open unit disk of meromorphic p-valent functions by using certain classes of adimissible functions. Section three deals with the fourth-order differential subordination and superordination results for multivalent analytic functions. Here, we introduce new concept that is fourth-order differential subordination and superordination associated with differential linear operator $I_p(n,\lambda)$ in open unit disk.

Chapter One

Complex Variable Concepts in Geometric Function Theory

Introduction:

This chapter includes three sections with some examples, the first section reviews the basic definitions that can be found in the standard text books with some examples see Churchill [18], Duren [23],Hayman [30], Kozdron [36] and Miller and Mocanu [47], where this section is about analytic functions and unvialent, multivalent (P-valent) functions, generalized hypergeometric functions, Ruscheweyh derivatives, also subordination and superordination.

Section two is about some classes of analytic functions. Some well-known the class of starlike functions, convex functions, close to convex functions α −starlike functions and α −convex functions see [5],[23],[35],[59] and [68].

In section three, basic lemmas and theorems have been mentioned they are essential and needed for the proofs of our principal results see [6], [11], [23], [47] and [71].

1.1 Basic Definitions

Definition 1.1.1 [23]: Suppose that $U = \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk in the complex plane $\mathbb{C}$. A function f of the complex variable is said to be analytic at a point z_0 if it's derivative exists not only at z_0 but also each point z in some neighborhoods of z_0. It is analytic in the unit disk U if it is analytic at every point in U. We say that f is entire function if it's analytic at every point in complex plane $\mathbb{C}$.

Example 1.1.2 [18]: The function $f(z) = {}^{i}/_{z^2}$ is analytic whenever $z \neq 0,$

and since

$$f(z) = {}^{i}/_{z^2} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2}.$$

The two functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2}, \qquad v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2},$$

the partial derivatives u_x, u_y, v_x, v_y are continuous in all the value $(x, y) \neq (0,0)$. Cauchy-Riemann equations are satisfied because

$$u_x = \frac{2y^3 - 6x^2y}{(x^2 + y^2)^3}, \qquad u_y = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}$$

$$v_x = \frac{6xy^2 - 2x^3}{(x^2 + y^2)^3}, \qquad v_y = \frac{2y^3 - 6x^2y}{(x^2 + y^2)^3}.$$

Then

$$u_x = v_y = \frac{2y^3 - 6x^2y}{(x^2 + y^2)^3}, \qquad u_y = -v_x = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3}.$$

Hence f is analytic for all $z \neq 0$.

Example 1.1.3 [18]: The function $f(z) = \sin z$ is entire function.

Definition 1.1.4 [6]: Let $\mathcal{H}(U)$ be the class of functions which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

For $n \in \mathbb{N} = \{1,2,3,\dots\}$, and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\},$$

and also let $\mathcal{H}_0 = [0,1]$ and $\mathcal{H}_1 = [1,1]$.

Definition 1.1.5 [23]: A function f analytic in domain $\mathrm{D} \subset \mathbb{C}$, is said to be univalent (schilcht), there if it does not take the same value twice, that is $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in D. In other words, f is one–to–one or (injective) mapping of D onto another domain. The theory of univalent functions is so much deep, we need certain simplifying assumptions. The most obvious one in our study is to replace the arbitrary domain D by one that is convenient, and is the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

As exmples, [23] the function $f(z) = z$ is univalent in U, while $f(z) = z^2$ is not univalent in U. Also $f(z) = z + \frac{z^{(2n+1)}}{2n+1}$ is univalent in U for all positive integer n.

We shall denote by $\mathcal{A}$ the class of all those functions f which are analytic in the open unit disk U and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$.

Definition 1.1.6 [23]: Let $\mathcal{S}$ denotes the class of all functions f in the class $\mathcal{A}$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (z \in U), \tag{1.1.1}$$

which are univalent in the open unit disk U.

We also deal with function which is meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. Meromorphic function defined as a function f analytic in a domain $\mathrm{D} \subset \mathbb{C}$ except for a finite number of poles in D.

Definition 1.1.7 [23]: Let $\mathcal{M}$ denotes the class of function f of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \qquad (z \in U), \tag{1.1.2}$$

which are meromorphic univalent in the punctured unit disk U^*.

Definition 1.1.8 [23]: A function f is said to be locally univalent at a point $z_0 \in \mathbb{C}$ if it is univalent in some neighborhood of z_0. For analytic function f ,the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0.

Example 1.1.9 [36]: Consider the domain

$$\mathrm{D} = \left\{ z \in \mathbb{C}: 1 < |z| < 2, \ \ 0 < \arg z < \frac{3\pi}{2} \right\},$$

and the function $f: \mathrm{D} \longrightarrow \mathbb{C}$ given by $f(z) = z^2$. It is clear that f is analytic on D and locally univalent at every point $z_0 \in \mathrm{D}$, since $f'(z_0) = 2z_0 \neq 0$ for all $z_0 \in \mathrm{D}$. However, f is not univalent on D, since

$$f\left(\frac{3}{2\sqrt{2}} + i\frac{3}{2\sqrt{2}}\right) = f\left(-\frac{3}{2\sqrt{2}} - i\frac{3}{2\sqrt{2}}\right) = \frac{9}{4}i.$$

Definition 1.1.10 [23]: A function f is said to be conformal at a point z_0 if it preserves the angle between oriented curves passing through z_0 in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0 both in magnitude and direction. A function $w = f(z)$ is said to be conformal in the domain $\mathrm{D} \subset \mathbb{C}$ if it is conformal at each point of the domain. Any analytic univalent function is a conformal mapping because of its angle – preserving property.

Definition 1.1.11 [23]: A Möbius transformation, or called a bilinear transformation, is a rational function $f: \mathbb{C} \longrightarrow \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$.

Example 1.1.12 [36]: Perhaps the most important member of $\mathcal{S}$ is the Koebe function which is given by

$$K(z) = \frac{z}{(1-z)^2}.$$

We can compute the Maclaurin series for K by differential the series for $\frac{1}{(1-z)^2}$ and then multiplying by z.

$$K(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots,$$

and maps the unit disk to the complement of the ray $\left(-\infty, -\frac{1}{4}\right]$. This can be verified by writing

$$K(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}.$$

Note that for this function $a_n = n$ for all n. We now show that the image of U under $K$is a slit domain that is a domain consisting of the entire complex plane except that a slit is cut out of it. To determine $K(U)$, consider the next sequence of functions:

$$u_1(z) = \frac{1+z}{1-z}, \qquad u_2(z) = \frac{1}{4}u_1^2(z), \qquad u_3(z) = u_2(z) - \frac{1}{4}.$$

Where noting that $\frac{1+z}{1-z}$ maps the unit disk conformally onto the right half-plane $\{\Re\{z\} > 0\}$; see Fig (1.1.1).

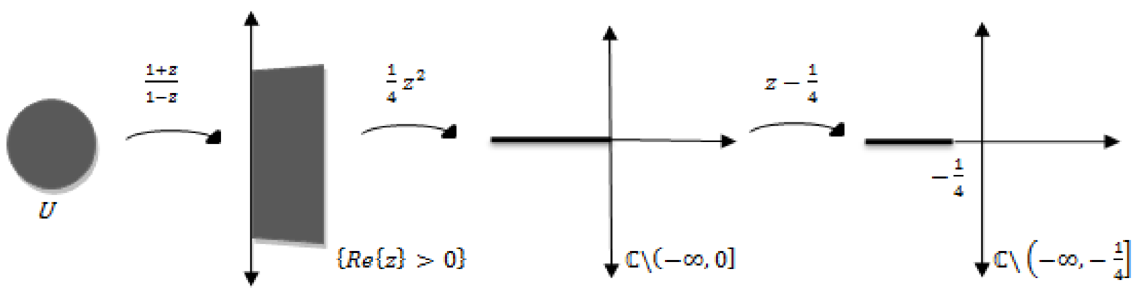

Figure (1.1.1)

The Koebe function maps U conformally onto $\mathbb{C} \setminus \left(-\infty, -\frac{1}{4}\right]$.

Now

$$u_3 \circ u_2 \circ u_1(z) = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \frac{z}{(1-z)^2}.$$

Note that u_1 is the Möbius transformation that functions maps U onto the right half-plane whose boundary is the imaginary axis. Also, u_2 is the sequaring function, while $u_3$translates the image one space to the left and then multiplies it by a factor of $\frac{1}{4}$.

Definition 1.1.13 [30]: Let f be a function analytic in the unit disk. If the equation $f(z) = w$ has never more than p −solutions in U, then f is said to be p −valent in U. The class of all p −valent analytic functions is denoted by $\mathcal{S}_p$ expressed in one of the following forms:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \ , \quad (z \in U\,; p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1.3}$$

or

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ . \quad (z \in U\,; p \in \mathbb{N} = \{1,2,3,\dots\}) \tag{1.1.4}$$

And, let f be a function analytic in the punctured unit disk U^*. If the equation $f(z) = w$ has never more than p −solutions in U^*, then f is said to be p −valent in U^*. The class of all p −valent meromorphic functions is denoted by $\mathcal{M}^*{}_p$ and expressed in one of the following forms:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \ , \quad (p \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1.1.5}$$

or

$$f(z) = z^{-p} + \sum_{k=p+1}^{\infty} a_k z^k \ , \quad (p \in \mathbb{N} = \{1,2,3,\dots\}). \tag{1.1.6}$$

Definition 1.1.14 [23]: If functions f and g belonging to the class $\mathcal{A}$, given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \,, \qquad g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

then the Hadamard product or (convolution) of functions f and g denoted by $f * g$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \quad , \quad (z \in U). \tag{1.1.7}$$

Example 1.1.15 [21]: Consider the Hadamard product of the Koebe function

$$K(z) = \frac{z}{(1-z)^2},$$

and the horizontal strip map,

$$F(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

To find the Hadamard $K(z) * F(z)$, we need to compute the Maclaurin series for F. Since, see[18]

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \ , \quad (|z| < 1),$$

and by integration both sides, we have

$$\log(1-z) = -\sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}.$$

Also

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots \ , \quad (|z| < 1),$$

and by integration both sides, we have

$$\log(1+z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1}.$$

Therefore,

$$\log\left(\frac{1+z}{1-z}\right) = \log(1+z) - \log(1-z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1} + \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1}$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \cdots + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \cdots$$

$$= 2z + 2\frac{z^3}{3} + 2\frac{z^5}{5} + \cdots = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1},$$

then

$$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}.$$

Thus,

$$K(z) * F(z) = \frac{z}{(1-z)^2} * \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} nz^n * \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1}$$

$$= (z + 2z^2 + 3z^3 + 4z^4 + \cdots) * \left(z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots\right)$$

$$= z + z^3 + z^5 + z^7 + \cdots$$

$$= z(1 + z^2 + z^4 + z^6)$$

$$= \frac{z}{1-z^2}.$$

That is,

$$K(z) * F(z) = \frac{z}{(1-z)^2} * \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \frac{z}{1-z^2}.$$

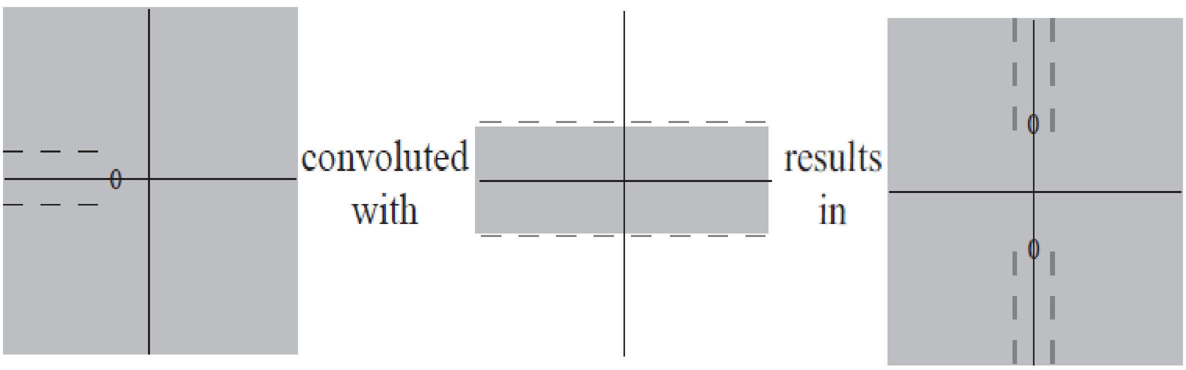

Figure (1.1.2)

The Koebe function convoluted with a horizontal strip map yields a double-slit map

Definition 1.1.16 [44]: The Pochhammer symbol or (the shifted factorial) which is denoted by $(x)_n$ is defined (in terms of the Gamma function) by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & if\ n = 0, x \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \\ x(x+1)\dots(x+n-1) & if\ n \in \mathbb{N}, x \in \mathbb{C}. \end{cases} \tag{1.1.8}$$

Definition 1.1.17 [32]: For a complex parameters α_i where $(i = 1,2,\dots, q)$ and β_i where $(i = 1,2,\dots, s)$ such that $(\beta_i \neq 0, -1, -2, \dots; i = 1, 2, \dots, s)$, the generalized hypergeometric function ${}_q^sF(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is given by, see [24,25]: as follows :

$${}_q^sF(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \leq s+1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),$$

where $(x)_n$ is the Pochhammer symbol or (shifted factorial) defined in (1.1.8).

Definition 1.1.18 [32]: For the function $f \in \mathcal{S}_p$, the Ruscheweyh derivative operator $D^{n+p-1}: \mathcal{S}_p \longrightarrow \mathcal{S}_p$ is defined by

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{n+p-1}}{(n+p-1)!} = \frac{z^p}{(1-z)^{n+p}} * f(z), \quad n > -p.$$

When $p = 1$, then it was introduced by Ruscheweyh [63], and the symbol D^{n+p-1} was introduced by Goel and Sohi [26]. Therefore, we call the symbol $D^{n+p-1}$to be the Ruscheweyh derivative of order $(n+p-1)th$.

Definition 1.1.19 [23]: A function $f: \mathbb{R} \longrightarrow \mathbb{C}$ is said to be Schwarz function , if for all $c \in \mathbb{R}, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then

$|f^{(n)}(x)| = O(|z|^C)$, where " capital O " is defined as follows:

Let $\{a_n\}$ and $\{b_n\}$ be any two sequences and $b_n \geq 0$, for all n. If there exists a constant number $\eta > 0$ suth that $a_n \leq \eta b_n$(for all n), then we write $a_n = O(b_n)$.

Definition 1.1.20 [47]: Let $f(z)$ and $F(z)$ are analytic functions in U. The function $f(z)$ is said to be subordinate to $F(z)$ or $F(z)$ is superordinate to $f(z)$

if there exists a Schwarz function $w(z)$, which is analytic in U with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in U)$, and such that $f(z) = F(w(z))$. In such case, we write

$$f \prec F, \text{ or } f(z) \prec F(z).$$

If the function $F(z)$ is univalent in U, then we have the following equivalence:

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(U) \subset F(U).$$

Definition 1.1.21 [47]: Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and $h(z)$ be univalent in U . If $p(z)$ is analytic in U and satisfies the second–order differential subordination:

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.1.9}$$

then $p(z)$ is called a solution of the differential subordination (1.1.9). A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.1.9), moreover simply dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1.9). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.1.9) is said to be the best dominant of (1.1.9).

Definition 1.1.22 [48]: Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be analytic in U. If the functions $p(z)$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and if $p(z)$ satisfies the second–order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.1.10}$$

then $p(z)$ is called a solution of the differential superordination (1.1.10). An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination (1.1.10) or more simply a subordinant, if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.1.10). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.1.10) is said to be the best subordinant.

Definition 1.1.23 [48]: Let Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} \setminus E(f)$, where $\overline{U} = U \cup \{z \in \partial U\} = \{z \in \mathbb{C}: |z| \le 1\}$, and

$$E(f) = \left\{\zeta \in \partial U \ : \lim_{z \to \zeta} f(z) \ = \infty\right\},$$

and are such that $f'(z) \neq 0$ for $\zeta \in \partial U \setminus E(f)$. Futher, let the subclass of Q for which $f(0) = a$ be denoted by $\mathrm{Q}(a)$, and $\mathrm{Q}(0) = \mathrm{Q}_0$, $\mathrm{Q}(1) = \mathrm{Q}_1$.

Definition 1.1.24 [11]: Let $\psi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in U. If the function $p(z)$ is analytic in U and satisfies the following third–order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{1.1.11}$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1.11). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.1.11) is said to be the best dominant.

Definition 1.1.25 [71]: Let $\psi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be analytic in U. If the functions $p(z)$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z),$$

are univalent in U and if $p(z)$ satisfy the following third–order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \tag{1.1.12}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.1.12). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.1.12) is said to be the best subordinant.

Definition 1.1.26 [11]: Let Q denote the set of functions q that are analytic and univalent on the set $\overline{U} \backslash \mathrm{E}(q)$, where $\overline{U} = U \cup \{z \in \partial U\} = \{z \in \mathbb{C}: |z| \leq 1\}$, and

$$\mathrm{E}(q) = \left\{\zeta \in \partial U: \lim_{z \to \zeta} q(z) = \infty\right\},$$

is such that $\min |q'(\zeta)| = \rho > 0$ for $\zeta \in \partial U \backslash \mathrm{E}(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$ and $Q(0) = Q_0$, $Q(1) = Q_1$.

Definition 1.1.27 [11]: Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N} \backslash \{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, w; z) \notin \Omega,$$

whenever

$$r = q(\zeta) \quad , \qquad s = \kappa\zeta q'(z) \quad , \qquad \Re\left(\frac{t}{s} + 1\right) \geq \kappa \Re\left(\frac{\zeta\, q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{w}{s}\right) \geq \kappa^2 \Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U, \zeta \in \partial U \backslash \mathrm{E}(q)$, and $\kappa \geq n$.

Definition 1.1.28 [71]: Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ and $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \bar{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, w; \zeta) \in \Omega,$$

whenever

$$r = q(z) \quad, \qquad s = \frac{zq'(z)}{m} \quad, \qquad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \Re\left(\frac{z\, q''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{w}{s}\right) \leq \frac{1}{m^2} \Re\left(\frac{z^2 q'''(z)}{q'(z)}\right),$$

where $z \in U, \zeta \in \partial U$, and $m \geq n \geq 2$.

1.2 Some Classes of Analytic Functions

Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric condition. Notable among them are the classes of starlike, convex, and close to convex functions. In this section, we introduce some well-known of these classes of analytic functions.

1.2.1 The class of starlike functions [23]

A set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if the linear segment joining $w_0$to every other point $w \in E$ lies entirely in E, i.e.

$$(1-\lambda)w + \lambda w_0 \in E\,, \quad 0 \leq \lambda \leq 1,$$

and a function f which maps the open unit disk U onto a starlike domain is called a starlike function, the set of all starlike functions is denoted by S^* which is analytically expressed as

$$S^* = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}. \tag{1.2.1}$$

The class S^* was first studied by Alexander [5] and the condition (1.2.1) for starlikeness is due to Nevanlinna [53].It is well-known that if analytic function f satisfies (1.2.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in U.

1.2.2 The class of convex functions [23]

A set $E \subset \mathbb{C}$ is said to be convex if it is starlike with respect to each of its points; that is, if the linear segment joining any two points of E lies entirely in E, i.e.

$$(1-\lambda)w_1 + \lambda w_2 \in E\,, \qquad \forall\ w_1, w_2 \in E, \quad 0 \leq \lambda \leq 1.$$

Let $f \in \mathcal{S}$. Then f maps U onto a convex domain, if and only if

$$\mathcal{C} = \left\{ f \in \mathcal{S} :\ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}. \tag{1.2.2}$$

Such function f is said to be convex in U or (briefly convex). The condition of (1.2.2) was first stated by Study [68].Löwner [40] also studied the class of convex functions. One can alter the condition (1.2.1) and (1.2.2) by setting other limitations on the behavior of $\frac{zf'(z)}{f(z)}$ and of $1+\frac{zf''(z)}{f'(z)}$ in U. In this way many interesting classes of analytic functions have been defined, see Hayman[30].

Thus $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}$. Note that the Koebe function see (Example 1.1.12) is starlike but not convex. There is a closely analytic connection between the convex and starlike mapping. Alexander [5] first observed this in 1915.

1.2.3 The class of α −starlike and α −convex functions [49]

Robertson [59] in 1936, introduced the class $\mathcal{S}^*(\alpha), \mathcal{C}(\alpha)$ of starlike and convex functions of order $\alpha, 0 \leq \alpha < 1$, which are defined by

$$\mathcal{S}^*(\alpha) = \left\{f \in \mathcal{A}: \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in U\right\}, \tag{1.2.3}$$

$$\mathcal{C}(\alpha) = \left\{f \in \mathcal{A}: \Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha, 0 \leq \alpha < 1, z \in U\right\}. \tag{1.2.4}$$

In particular $\mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{C}(0) = \mathcal{C}$, where $\mathcal{S}^*$ is the class of starlike functions with respect to the origin and $\mathcal{C}$ is the class of convex functions.

1.2.4 The class of close to convex functions [23]

We now turn to an interesting subclass of $\mathcal{S}$ which contains $\mathcal{S}^*$and has a simple geometric description. This is the class of close to convex functions, introduced by Kaplan [35] in 1952.

A function f analytic in the open unit disk is said to be close to convex if there is a convex function g such that

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0\,, (z \in U).$$

We shall denoted by K the class of close to convex functions f normalized by the usual conditions$f(0) = 0$ and $f'(0) = 1$. Note that f is not required a priori to be univalent. Note also that the associated function g need not be normalized. The additional condition that $g \in \mathcal{C}$ defines a proper subclass of K which will be denoted by K_0.

Every convex function is obviously close to convex. More generally, every starlike function is close to convex. Indeed, each $f \in \mathcal{S}^*$ has the form $f(z) = zg'(z)$ for some $g \in \mathcal{C}$, and

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} = \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Then from above, we conclude that

$$\mathcal{C} \subset \mathcal{S}^* \subset K_0 \subset K \subset \mathcal{S},$$

and this means that, every close to convex function is univalent.

1.2.5 The class of meromorphic starlike and meromorphic convex functions [54]

Let $f \in \mathcal{M}$ which is analytic and univalent in U^*, then f is called meromorphic starlike of order α $(0 < \alpha \leq 1)$ if $f(z) \neq 0$ in U^* and

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha\,, (z \in U^*),$$

where the class of meromorphic starlike functions of order α is denoted by $\mathcal{MS}^*(\alpha)$. Similary, a function $f \in \mathcal{M}$ which is analytic and univalent in U^*, is called meromorphic convex of order α $(0 < \alpha \leq 1)$ if $f'(z) \neq 0$ in U^* and

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha\,, (z \in U^*),$$

where the class of meromorphic convex functions of order α is denoted by $\mathcal{MC}(\alpha)$.

1.3 Fundamental Lemmas

The following lemmas are needing in the proofs of our results in this research.

Lemma 1.3.1 [23] (Schwarz Lemma): Let f be analytic function in the open unit disk U with $f(0) = 0$ and $|f(z)| < 1$ in U. Then, $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in U. Strict inequality holds in both estimates unless f is a rotation of the disk $f(z) = e^{i\theta}z$.

Lemma 1.3.2 [47]: Let $q(z)$ be univalent in U and θ be analytic function in domain D containing $q(U)$. If $zq'(z)\,\theta(q(z))$ is starlike, and

$$zr'(z)\,\theta\big(r(z)\big) \prec zq'(z)\,\theta\big(q(z)\big)\,, \tag{1.3.1}$$

then

$r(z) \prec q(z)$ and $q(z)$ is the best dominant of (1.3.1).

Lemma 1.3.3 [20]: Let $q(z)$ be a convex function in U, and let θ be analytic function in a domain D containing $q(U)$, set $h(z) = zq'(z) + \theta(q(z))$ and suppose that

$$\Re\left(\frac{h'(z)}{q'(z)}\right) = \Re\left(\theta'(q(z)) \; + \; 1 + \frac{zq''(z)}{q'(z)}\right) > 0, z \in U\,,$$

then q is univalent. Moreover, if $r(z)$ is analytic function in U with $r(0) = q(0)$ and $r(U) \subset$ D, and

$$z\,r'(z) + \theta\big(r(z)\big) \prec z\,q'(z) + \theta\big(q(z)\big), \tag{1.3.2}$$

then

$r(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.3.2).

Lemma 1.3.4 [64]: Let $q(z)$ be a convex univalent function in $U$and let $\lambda \in \mathbb{C}$,$\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0\,, -\Re\left(\frac{\lambda}{\gamma}\right)\right\}, z \in U\,.$$

If $r(z)$ is analytic function in U, and

$$\lambda r(z) + \gamma z r'(z) \prec \lambda q(z) + \gamma z q'(z), \tag{1.3.3}$$

then

$r(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.3.3).

Lemma 1.3.5 [47]: Let $q(z)$ be univalent in U, and let θ and ϕ be analytic in a domain D containing $q(U)$ with , $\phi(\text{?}) \neq 0$ when $\text{?} \in q(U)$. Set

$Q(z) = zq'(z)\phi(q(z)),\ h(z) = \theta(q(z)) + Q(z)$ and suppose that

(i) $Q(z)$ is a starlike function in U,

(ii) $\Re\left\{\frac{z\,h'(z)}{Q(z)}\right\} > 0$, for all $z \in U$.

If $r(z)$ is analytic in U, with $r(0) = q(0)$, $r(U) \subseteq$ D and

$$\theta(r(z)) + zr'(z)\phi(r(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \qquad (1.3.4)$$

then

$r(z) \prec q(z)$, and $q(z)$ is the best dominant of (1.3.4).

Lemma 1.3.6 [62]: The function $q(z) = (1-z)^{-2ab}$ where $a, b \in \mathbb{C}^*$ is univalent in U if and only if $|2ab \mp 1| \leq 1$.

Lemma 1.3.7 [48]: Let $q(z)$ be convex univalent function in U and let $\tau \in \mathbb{C}$, with $\Re(\tau) > 0$. If $r(z) \in \mathcal{H}[q(0),1]] \cap$ Q and

$r(z) + \tau zr'(z)$ is univalent in U, then

$$q(z) + \tau zq'(z) \prec r(z) + \tau zr'(z), \qquad (1.3.5)$$

which implies that $q(z) \prec r(z)$, and $q(z)$ is the best subordinant of (1.3.5).

Lemma 1.3.8 [6]: Let $q(z)$ be convex univalent in U. Let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that

(i) $Q(z) = zq'(z)\phi(q(z))$ is a starlike function in U,

(ii) $\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$

, for all $z \in U$. If $r(z) \in \mathcal{H}[q(0),1]] \cap$ Q, with $r(U) \subseteq$ D, such that

$\theta(r(z)) + zr'(z)\phi(r(z))$ is univalent in U, and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(r(z)) + zr'(z)\phi(r(z)), \qquad (1.3.6)$$

then $q(z) \prec r(z)$, and $q(z)$ is the best subordinant of (1.3.6).

Lemma 1.3.9 [23]: Let f be analytic in D, with $f(0) = f'(0) - 1 = 0$. Then $f \in S^*$ if and only if $zf'(z)/f(z) \in \mathbb{P}$ (where $\mathbb{P}$ is the class of all function φ analytic and having positive real part in D, with $\varphi(0) = 1$).

Theorem 1.3.10 [11]: Let $p \in \mathcal{H}[a, n]$ with $n \in \mathbb{N}\backslash\{1\}$, and let $q \in \mathcal{Q}(a)$ and satisfy the following conditions:

$$\Re\left(\frac{\zeta\, q''(\zeta)}{q'(\zeta)}\right) \geq 0, \qquad \left|\frac{zp'(z)}{q'(\zeta)}\right| \leq \kappa\,,$$

where $z \in U, \zeta \in \partial U\backslash \mathrm{E}(q)$ and $\kappa \geq n$. If Ω a set in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega\,,$$

then

$$p(z) \prec q(z)\,, \quad (z \in U).$$

Theorem 1.3.11 [71]: Let $q \in \mathcal{H}[a, n]$ and $\psi \in \Psi_n'[\Omega, q]$. If $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is univalent in U, and $p \in \mathcal{Q}(a)$ satisfy the following conditions:

$$\Re\left(\frac{z\, q''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{zp'(z)}{q'(z)}\right| \leq \frac{1}{m}\,,$$

where $z \in U$, and $m \geq n \geq 2$,

then

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\}\,,$$

implies that

$$q(z) \prec p(z)\,, \quad (z \in U).$$

Theorem 1.3.12 [23] (Alexander's Theorem): Let f be an analytic function in U with $f(0) = f'(0) - 1 = 0$. Then, $f \in \mathcal{C}$ if and only if $zf' \in \mathcal{S}^*$.

Chapter Two

Some Results on Differential Subordination and superordination of Univalent and Multivalent Functions

Introduction:

In [46] Miller and Mocanu extended the study of differential inequalities of real-valued functions to complex-valued functions defined in the unit disk. Following Miller and Mocanu [47,48],Bulboacâ [19] and others [6,14,50,52,64] studied differential classes of analytic functions, by means of differential subordination and superordination.

In this chapter, we concentrate in particular on the study of applications of subordination and superordination of univalent and multivalent functions. This chapter consists of two sections.

Section one deals with the study of differential subordination for univalent functions. Here, we obtain some results, like, let the function $q(z)$ be univalent in the unit disk U, $q(z) \neq 0$ and $zq'(z)\theta\big(q(z)\big) \neq 0$ is starlike function in U. If $f \in \mathrm{T}^{+}(\gamma)$ satisfies the subordination

$$-1 - \frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)} + \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)} \prec \frac{-\delta zq'\,(z)}{q(z)}, \text{ then}$$

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec q(z), \quad (z \in U, \delta \in \mathbb{C}^*),$$

and $q(z)$ is the best dominant.

Section two is devoted for the study of some properties of differential sandwich results of p-valent functions defined by Liu-Srivastava operator. We obtain results on differential subordination and superordination. Also, we derive some sandwich theorems, like, let $q(z)$ be a convex univalent in U with $q(0) = 1, 0 < \delta < 1. \beta \in \mathbb{C}^*$ and suppose that

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -\Re\left(\frac{1}{\delta\beta}\right)\right\}.$$

If $f \in W\Sigma_p$ satisfies the subordination $\Upsilon_1(z) \prec q(z) + \delta\beta zq'(z)$,

where

$$\Upsilon_1(z) = (1 - \alpha_1\beta)\left[\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} + \alpha_1\beta\left[\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \frac{F_{p,q,s}[\alpha_1 + 1, v]f(z)}{F_{p,q,s}[\alpha_1, v]f(z)},$$

then

$$\left[\frac{F_{p,q,s}[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \prec q(z),$$

and $q(z)$ is the best dominant.

2.1 Differential Subordination for Univalent Functions

Let $T^{+}(\gamma)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=1}^{\infty} a_k \, z^{k-(k/\gamma)} \quad , \qquad \gamma = \{2,3,4,\dots\}, \tag{2.1.1}$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalized condition $f(0) = f'(0) - 1 = 0$. Also,

let $T^{-}(\gamma)$ denote the subclass of the class $T^{+}(\gamma)$ of functions of the form:

$$f(z) = z - \sum_{k=1}^{\infty} a_k \, z^{k-(k/\gamma)} \quad , \qquad \gamma = \{2,3,4,\dots\}, a_k \geq 0 \ , \tag{2.1.2}$$

which are analytic in U and satisfying the normalized condition

$f(0) = f'(0) - 1 = 0.$

We note that (n) by generalized deriving.

Hints to the work done by the authors [17] , [38] when $n = 3,$ and discusse around generalize this idea with formula special in next Theorems.

Theorem 2.1.1: Let the function $q(z)$ be univalent in the unit disk $U, q(z) \neq 0$ and $zq'(z)\theta(q(z)) \neq 0$ is starlike function in U. If $f \in T^{+}(\gamma)$ satisfies the subordination

$$-1 - \frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)} + \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)} \prec \frac{-\delta z q'(z)}{q(z)}, \tag{2.1.3}$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec q(z), \quad (z \in U \ , \delta \in \mathbb{C}^{*}),$$

and $q(z)$ is the best dominant of (2.1.3).

Proof. Define the function

$$r(z) = \left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} ,(z \in U ,\delta \in \mathbb{C}^*), \tag{2.1.4}$$

then

$$zr'(z) = \frac{1}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[1 + \frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)} - \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]. \tag{2.1.5}$$

Setting $\theta(w) = \frac{-\delta}{w}$ it can easily observed that $\theta(w)$ is analytic function in $\mathbb{C}^*$, then, we have , $\theta(r(z)) = \frac{-\delta}{r(z)}$ and $\theta\big(q(z)\big) = \frac{-\delta}{q(z)}$.

From (2.1.5) and simple a computation shows that

$$zr'(z)\theta\big(r(z)\big) = -1 - \frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)} + \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}, \tag{2.1.6}$$

together (2.1.3) and (2.1.6), we get

$$zr'(z)\theta\big(r(z)\big) \prec \frac{-\delta zq'(z)}{q(z)} = z\,q'(z)\theta\big(q(z)\big).$$

Thus by applying Lemma 1.3.2, we obtain $r(z) \prec q(z)$, and by using (2.1.4), we have the required result, and $q(z)$ is the best dominant of (2.1.3).

Taking $q(z) = \frac{1+Az}{1+Bz}$ where $-1 \le B < A \le 1$ in the Theorem 2.1.1, we have the next result.

Corollary 2.1.2: If $f \in T^+(\gamma)$ satisfies the subordination

$$-1 - \frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)} + \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)} \prec \frac{-\delta\,(A-B)z}{(1+Az)(1+Bz)}, \tag{2.1.7}$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz}, \qquad (z \in U ,\delta \in \mathbb{C}^*),$$

and $q(z) = \frac{1+AZ}{1+BZ}$ is the best dominant of (2.1.7).

For $A = 1$ and $B = -1$ in the above corollary, we obtain the following result.

Corollary 2.1.3: If $f \in T^{+}(\gamma)$ satisfies the subordination

$$-1 - \frac{z f^{(n)}(z)}{f^{(n-1)}(z)} + \frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)} \prec \frac{-2z\delta}{(1-z^2)} \quad , \tag{2.1.8}$$

then

$$\left[\frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec \frac{1+z}{1-z}, (z \in U, \delta \in \mathbb{C}^{*}),$$

and $q(z) = \frac{1+Z}{1-Z}$ is the best dominant of (2.1.8).

Theorem 2.1.4: Let the function $q(z)$ be a convex univalent in U and $q(0) = 0$. If $f \in T^{+}(\gamma)$ satisfies the subordination

$$\frac{1}{\delta}\left[\frac{z f^{(n)}(z)}{f^{(n-1)}(z)}\right]\left[\frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec z q'(z) + \frac{1}{\delta}[q(z)]^{1+\delta} - \frac{1}{\delta} q(z), \tag{2.1.9}$$

then

$$\left[\frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec q(z), \qquad (z \in U, \delta \in \mathbb{C}^{*}),$$

and $q(z)$ is the best dominant of (2.1.9).

Proof. If, we consider the function

$$r(z) = \left[\frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}, (z \in U, \delta \in \mathbb{C}^{*}), \tag{2.1.10}$$

then

$$z r'(z) = \frac{1}{\delta}\left[\frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[1 + \frac{z f^{(n)}(z)}{f^{(n-1)}(z)} - \frac{z f^{(n-1)}(z)}{f^{(n-2)}(z)}\right], \tag{2.1.11}$$

putting $\theta(w) = \frac{1}{\delta} w^{1+\delta} - \frac{1}{\delta} w$, it can easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, then, we have

$$\theta(r(z)) = \frac{1}{\delta}[r(z)]^{1+\delta} - \frac{1}{\delta} r(z), \;\; \theta\big(q(z)\big) = \frac{1}{\delta}[q(z)]^{1+\delta} - \frac{1}{\delta} q(z). \tag{2.1.12}$$

From (2.1.11) and (2.1.12), we have

$$zr'(z) + \theta\big(r(z)\big) = \frac{1}{\delta}\left[\frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)}\right]\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}, \tag{2.1.13}$$

together (2.1.9) with (2.1.13), we get

$$zr'(z) + \theta(r(z)) \prec z\,q'(z) + \theta\big(q(z)\big).$$

So by Lemma 1.3.3, we obtain $r(z) \prec q(z)$, and by using (2.1.10), we have the required result.

Let us consider $q(z) = ze^{\lambda Az}$ in Theorem 2.1.4, we get the following result.

Corollary 2.1.5: If $f \in T^{+}(\gamma)$ satisfies the subordination

$$\frac{1}{\delta}\left[\frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)}\right]\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec \left(z\lambda A + 1 - \frac{1}{\delta} + \frac{z^{\delta}}{\delta}e^{\delta\lambda AZ}\right)ze^{\lambda Az}, \tag{2.1.14}$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec ze^{\lambda Az}, (z \in U, \delta \in \mathbb{C}^{*}),$$

and $q(z) = ze^{\lambda Az}$ is the best dominant of (2.1.14).

Theorem 2.1.6: Let the function $q(z)$ be a convex univalent in U, $q'(z) \neq 0$ and suppose that

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{1}{\delta}\right\} > 0\,.$$

If $f \in T^{+}(\gamma)$ satisfies the subordination

$$\frac{\beta}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[\frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)} - \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right] \prec \frac{-\beta}{\delta}q(z) + \beta zq'(z), \tag{2.1.15}$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}} \prec q(z), \qquad (z \in U, \delta \in \mathbb{C}^{*})$$

and $q(z)$ is the best dominant of (2.1.15).

Proof. Let denotes

$$r(z)=\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}, \qquad (z\in U, \delta\in\mathbb{C}^*), \tag{2.1.16}$$

then

$$\beta zr'(z)=\frac{\beta}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[1+\frac{zf^{(n)}(z)}{f^{(n-1)}(z)}-\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right].$$

A simple computation, we get,

$$\frac{-\beta}{\delta}r(z)+\beta zr'(z)=\frac{\beta}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[\frac{zf^{(n)}(z)}{f^{(n-1)}(z)}-\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]. \tag{2.1.17}$$

From (2.1.16) and (2.1.18), we have

$$\frac{-\beta}{\delta}r(z)+\beta zr'(z)\prec\frac{-\beta}{\delta}q(z)+\beta zq'(z).$$

Now, using Lemma 1.3.4, where $\lambda=\frac{-\beta}{\delta}$, $\gamma=\beta$ and from (2.1.16), we obtain the required result .

Let us assume $q(z)=e^{\lambda Az}$ in the Theorem 2.1.6, we have the following result.

Corollary 2.1.7: Let $f\in T^{+}(\gamma)$ and suppose that $\Re\left\{1+\lambda Az-\frac{1}{\delta}\right\}>0$.

If f satisfies the subordination

$$\frac{\beta}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[\frac{zf^{(n)}(z)}{f^{(n-1)}(z)}-\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]\prec\left(\lambda Az-\frac{1}{\delta}\right)\beta e^{\lambda Az}, \tag{2.1.18}$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\prec e^{\lambda Az}, \qquad (z\in U, \delta\in\mathbb{C}^*),$$

and $q(z)=e^{\lambda Az}$ is the best dominant of (2.1.18).

Again by assume $q(z)=\frac{1+Az}{1+Bz}$, where $-1\leq B<A\leq 1$ in Theorem 2.1.6, we get the next result.

Corollary 2.1.8: Let $f\in T^{+}(\gamma)$ and suppose that

$$\Re\left\{\frac{1-Bz}{1+Bz}-\frac{1}{\delta}\right\}>0.$$

If f satisfies the subordination

$$\frac{\beta}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[\frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)}-\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]\prec\frac{-\beta(1+Az)}{\delta(1+Bz)}+\frac{\beta(A-B)z}{(1+Bz)^2},\quad(2.1.19)$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\prec\frac{1+Az}{1+Bz},\quad(z\in U,\delta\in\mathbb{C}^*),$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant of (2.1.19).

Theorem 2.1.9: Let $0<\delta<1$ and $\lambda,\beta\in\mathbb{C}^*$, let $q(z)$ be univalent in U and q satisfy the following condition :

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\}>\max\left\{0,\Re\left(\frac{\beta}{\lambda}\right)\right\}.\quad(2.1.20)$$

If $f\in T^+(\gamma)$ satisfies the subordination

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left\{\frac{\lambda}{\delta}\left[-1-\frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)}+\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]+\beta\right\}\prec$$

$$\beta q(z)-\lambda zq'(z),\quad(2.1.21)$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\prec q(z),\quad(z\in U,\delta\in\mathbb{C}^*),$$

and $q(z)$ is the best dominant of (2.1.21).

Proof. We begin by setting

$$r(z)=\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}},\quad(z\in U,\delta\in\mathbb{C}^*),\quad(2.1.22)$$

then by a computation shows that

$$zr'(z) = \frac{1}{\delta}\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left[1 + \frac{z\, f^{(n)}(z)}{f^{(n-1)}(z)} - \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right], \qquad (2.1.23)$$

by setting

$$\theta(w) = \beta w\,,\ \ \phi(w) = -\lambda\,, \qquad w \in \mathbb{C}\,, \qquad (2.1.24)$$

then $\theta(w)$ and $\phi(w)$ is analytic in $\mathbb{C}$. Also if, we suppose

$Q(z) = zq'(z)\phi(q(z)) = -\lambda zq'(z)$, and

$h(z) = \theta\big(q(z)\big) + Q(z) = \beta q(z) - \lambda zq'(z)$,

from assumption (2.1.20), we yield that $Q(z)$ is starlike function in U, and, we get

$$\Re\left\{\frac{z\, h'(z)}{Q(z)}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{\beta}{\lambda}\right\} > 0\ ,\quad z\in U\,.$$

A simple computation together with (2.1.23) and (2.1.24), we have ,

$$zr'(z)\phi\big(r(z)\big) + \theta\big(r(z)\big) = \left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left\{\frac{\lambda}{\delta}\left[-1 - \frac{z\, f^{(n)}(z)}{f^{(n-1)}(z)} + \frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]\beta\right\},$$

therefore the subordination (2.1.21) becomes

$$zr'(z)\phi\big(r(z)\big) + \theta\big(r(z)\big) \prec zq'(z)\phi\big(q(z)\big) + \theta\big(q(z)\big).$$

By applying Lemma 1.3.5 and using (2.1.22), we obtain our result.

Further taking $q(z) = \frac{1+Az}{1+Bz}$, where $-1 \le B < A \le 1$ and $\lambda = 1$ in Theorem 2.1.9, we obtain the following result.

Corollary 2.1.10: Let $0 < \delta < 1$, and $\Re(\beta) > 0$, with suppose that

$$\Re\left\{\frac{1 - Bz}{1 + Bz}\right\} > \max\{0, \Re\,(\beta)\}.$$

If $f \in T^{+}(\gamma)$ satisfies the subordination

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\left\{\frac{1}{\delta}\left[-1-\frac{z\,f^{(n)}(z)}{f^{(n-1)}(z)}+\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]+\beta\right\}\prec$$

$$\beta\frac{1+Az}{1+Bz}-\frac{(A-B)z}{(1+Bz)^2}\ , \qquad (2.1.25)$$

then

$$\left[\frac{zf^{(n-1)}(z)}{f^{(n-2)}(z)}\right]^{\frac{1}{\delta}}\prec\frac{1+\mathrm{Az}}{1+\mathrm{Bz}}, \qquad (z\in U\,,\delta\in\mathbb{C}^*),$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant of (2.1.25).

2.2 Some Properties of Differential Sandwich Results of p-valent Functions Defined by Liu-Srivastava Operator

Let $\mathcal{H}(U)$ denote the class of functions analytic in the open unit disk $U=\{z \in \mathbb{C}: |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1,2,3, \dots\}$, let $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U): f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, z \in U\}$, with $\mathcal{H}_1 = \mathcal{H}[1,1]$.

Let $W\Sigma_p$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ , \quad (z \in U, p \in \mathbb{N}) \ , \tag{2.2.1}$$

and $W\Sigma = W\Sigma_1$. For functions $f(z) \in W\Sigma_p$, given by (2.2.1) and g(z) given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \qquad , \qquad (z \in U, p \in \mathbb{N}) \quad , \tag{2.2.2}$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z) \quad , \quad (z \in U, p \in \mathbb{N}). \tag{2.2.3}$$

Related results on subordination can be found in [28,29,32,51,55,58].

Ali et al. [6], and Aouf et al [12], obtained sufficient conditions for certain normalized analytic functions f to satisfy:

$$q_1(z) \prec \frac{z f'(z)}{f(z)} \prec q_2(z), \tag{2.2.4}$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. So newly , Shanmugam et al.[64,65] ,and Goyal et al. [28] obtained it called sandwich results for certain classes of analytic functions . Further superordination results can be found in [1,2,6,7,12,13].

For a complex parameters α_i where $(i = 1, \dots , q)$ and β_i where $(i = 1, \dots, s)$ such that $(\beta_i \neq 0, -1, -2, \dots; i = 1, 2, \dots, s)$, the generalized hypergeometric function ${}_q^sF(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ is given by, see [24,25]: as follows :

$$
{}_q^sF(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_q)_n}{(\beta_1)_n \dots (\beta_s)_n} \frac{z^n}{n!}
$$

$$
(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in U),
$$

where $(x)_n$ is the Pochhammer symbol (or shifted factorial) defined in terms of the Gamma function by

$$
(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & if\ n = 0, x \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \\ x(x+1)\dots(x+n-1) & if\ n \in \mathbb{N}, x \in \mathbb{C}. \end{cases}
$$

Corresponding to a function $\hbar_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)$ defined by

$$
\hbar_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) = z^p\ {}_q^sF(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)
$$

Liu-Srivastava [44] consider a linear operator

$H_{p,q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) : W\Sigma_p \longrightarrow W\Sigma_p$ defined by the following Hadamard product (or convolution):

$$
H_{p,q,s}(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z)f(z) = \hbar_p(\alpha_1, \dots, \alpha_q; \beta_1, \dots, \beta_s; z) * f(z). \quad (2.2.5)
$$

This operator was encourage essentially by Dziok and Srivastava ([24,25] ; see also [44]) . The theory of differential subordination in $\mathbb{C}$ is a generalization of differential disparity in $\mathbb{R}$, and this theory of differential subordination was initiated by the works of Miller and Mocanu [46] , many important works on differential subordination were great by Miller and Mocanu, and their monograph [47] complied their huge efforts in introducing and developing the same. Newly Miller and Mocanu in [48] investigated the dual problem of differential superordination, while Bulboacâ [20] investigates both subordination and superordination.

For $v > -p$ and function $f \in W\Sigma_p$, in the form (2.2.1). The Ruscheweyh derivative of order $(v + p - 1)th$ is denoted by D^{v+p-1} and consider as following:

See[26,63],

$$D^{v+p-1}f(z) = \frac{z^p(z^{v-1}f(z))^{v+p-1}}{(v+p-1)!} = \frac{z^p}{(1-z)^{v+p}} * f(z).$$

In [32] define the linear operator $F_{p,q,s}\,[\alpha_1, v]$ on $W\Sigma_p$ as follows:

$$F_{p,q,s}\,[\alpha_1, v]f(z) = H_{p,q,s}[\alpha_1] * D^{v+p-1}f(z)$$

$$= z^p + \sum_{k=p+1}^{\infty} \Lambda\, \sigma_{k,p}(\alpha_1)\, \varphi(v+p-1,k) a_k z^k,$$

where

$$\Lambda = \frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}\,, \qquad \sigma_{k,p}(\alpha_1) = \frac{\prod_{i=1}^{q}\Gamma(\alpha_i+k-p)}{\prod_{i=1}^{s}\Gamma(\beta_i+k-p)}$$

and

$$\varphi(v+p-1,k) = \binom{v+p-1+k-1}{v+p-1}. \tag{2.2.6}$$

Then, we have

$$z(F_{p,q,s}\,[\alpha_1, v]f(z))' = \alpha_1 F_{p,q,s}\,[\alpha_1+1, v]f(z) - (\alpha_1 - p)F_{p,q,s}\,[\alpha_1, v]f(z), \tag{2.2.7}$$

that easily to verify it by applying (2.2.6), see [32].

Theorem 2.2.1: Let $q(z)$ be a convex univalent in U with $q(0) = 1, 0 < \delta < 1$. $\beta \in \mathbb{C}^*$ and suppose that

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -\Re\left(\frac{1}{\delta\beta}\right)\right\}. \tag{2.2.8}$$

If $f \in W\Sigma_p$ satisfies the subordination

$$\Upsilon_1(z) \prec q(z) + \delta\beta z q'(z), \tag{2.2.9}$$

where

$$\Upsilon_1(z) = (1 - \alpha_1\beta)\left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} +$$

$$\alpha_1\beta\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}}\frac{F_{p,q,s}\,[\alpha_1+1,v]f(z)}{F_{p,q,s}\,[\alpha_1,v]f(z)}, \qquad (2.2.10)$$

then

$$\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \prec q(z)\ , \qquad (2.2.11)$$

and $q(z)$ is the best dominant of (2.2.9).

Proof. Define the analytic function

$$r(z)=\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}}\ , z\in U, \qquad (2.2.12)$$

differentiating (2.2.12) logarithmically with respect to z, we get

$$\frac{zr'(z)}{r(z)}=\frac{1}{\delta}\left[\frac{z\left(F_{p,q,s}\,[\alpha_1,v]f(z)\right)'}{F_{p,q,s}\,[\alpha_1,v]f(z)}-p\right]$$

and using the identity (2.2.7), we have

$$\frac{zr'(z)}{r(z)}=\frac{\alpha_1}{\delta}\left[\frac{F_{p,q,s}\,[\alpha_1+1,v]f(z)}{F_{p,q,s}\,[\alpha_1,v]f(z)}-1\right].$$

Therefore

$$\delta\beta zr'(z)=\alpha_1\beta\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}}\left[\frac{F_{p,q,s}\,[\alpha_1+1,v]f(z)}{F_{p,q,s}\,[\alpha_1,v]f(z)}-1\right],$$

hence the subordination (2.2.9) and from hypothesis, yield

$$r(z)+\delta\beta zr'(z)\prec q(z)+\delta\beta q'(z).$$

By applying Lemma 1.3.4 for special case $\lambda=1$, and $\gamma=\delta\beta$, leads to (2.2.11) consequently the proof of Theorem 2.2.1 is completed.

Putting $q(z)=\frac{1+Az}{1+Bz}$, where $-1\le B<A\le 1$ in the Theorem 2.2.1, the condition (2.2.8) reduces to: (see[13,51]).

$$\Re\left\{\frac{1-Bz}{1+Bz}\right\} > \max\left\{0; -\Re\left(\frac{1}{\delta\beta}\right)\right\}, z \in U\,. \tag{2.2.13}$$

It is easy to verify that the function $\phi(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < |B|$, is convex in U , and since $\phi(\overline{\zeta}) = \overline{\phi(\zeta)}$ for all $|\zeta| < |B|$ it follows that $\phi(U)$ is a convex domain symmetric with respect to the real axis , hence

$$\inf\left\{\Re\left(\frac{1-Bz}{1+Bz}\right) : z \in U\right\} = \frac{1-|B|}{1+|B|} \tag{2.2.14}$$

Then, the inequality (2.2.13) is equivalent to

$$\Re\left(\frac{1}{\delta\beta}\right) \geq \frac{|B|-1}{|B|+1},$$

hence, we have the following result.

Corollary 2.2.2: Let $-1 \leq B < A \leq 1$, and $0 < \delta < 1$, $\beta \in \mathbb{C}^*$ with

$$\max\left\{0; -\Re\left(\frac{1}{\delta\beta}\right)\right\} \leq \frac{1-|B|}{1+|B|}.$$

If $f \in W\Sigma_p$ and $\Upsilon_1(z)$ is given by (2.2.10), satisfies the subordination

$$\Upsilon_1(z) \prec \frac{1+Az}{1+Bz} + \frac{\delta\beta(A-B)z}{(1+Bz)^2}, \tag{2.2.15}$$

then

$$\left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant of (2.2.15).

For $A = 1$ and $B = -1$, the last corollary becomes.

Corollary 2.2.3: Let $0 < \delta < 1$, and $\beta \in \mathbb{C}^*$ with $\Re\left(\frac{1}{\delta\beta}\right) \geq 0$. If $f \in W\Sigma_p$ and $\Upsilon_1(z)$ is given by (2.2.10), satisfies the subordination

$$\Upsilon_1(z) \prec \frac{1+z}{1-z} + \frac{2\delta\beta z}{(1-z)^2}, \tag{2.2.16}$$

then

$$\left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \prec \frac{1+z}{1-z}\,,$$

and $q(z)=\frac{1+Z}{1-Z}$ is the best dominant of (2.2.16).

Theorem 2.2.4: Let $q(z)$ be univalent in U with $q(0)=1$ and $q(z)\neq 0$ for all $z\in U$. Let $\delta,\gamma\in\mathbb{C}^*$ and $\alpha,x,y\in\mathbb{C}$, with $x+y\neq 0$. Let $f\in W\Sigma_p$ and suppose that f and q satisfy the following condition:

$$(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\neq 0,\ z\in U, \qquad (2.2.17)$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\}>0, \quad z\in U. \qquad (2.2.18)$$

If

$$\alpha+\frac{\gamma}{\delta}\left[\frac{xz\left(F_{p,q,s}[\alpha_1+1,v]f(z)\right)'+yz\left(F_{p,q,s}[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\right]$$

$$\prec \alpha+\gamma\frac{zq'(z)}{q(z)}, \qquad (2.2.19)$$

then

$$\left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}}\prec q(z),$$

and $q(z)$ is the best dominant of (2.2.19).

Proof. According to (2.2.17), we consider the analytic function

$$r(z)=\left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}}, \qquad (2.2.20)$$

with $r(0)=1$.

By logarithmically differentiating of (2.2.20) yields

$$\frac{zr'(z)}{r(z)}=\frac{1}{\delta}\left[\frac{xz\left(F_{p,q,s}[\alpha_1+1,v]f(z)\right)'+yz\left(F_{p,q,s}[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\right],$$

let us consider the functions

$$\theta(w)=\alpha\,, \qquad \phi(w)=\frac{\gamma}{w}\,,$$

then θ is analytic in $\mathbb{C}$ and $\phi(w) \neq 0$ is analytic in $\mathbb{C}^*$.

If we suppose

$$Q(z) = zq'(z)\phi(q(z)) = \gamma \frac{zq'(z)}{q(z)}, z \in U, \qquad \text{and}$$

$$h(z) = \theta(q(z)) + Q(z) = \alpha + \gamma \frac{zq'(z)}{q(z)}, \qquad z \in U.$$

From the assumption (2.2.18), we see that $Q(z)$ is starlike function in U, and also have

$$\Re\left\{\frac{z\,h'(z)}{Q(\mathrm{z})}\right\} = \Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0, \qquad z \in U.$$

Now, by Lemma 1.3.5, we derive the subordination (2.2.19) implies $r(z) \prec q(z)$ and the function $q(z)$ is the best dominant of (2.2.19).

Letting $x = 0$, $y = \alpha = 1$ and $q(z) = \frac{1+AZ}{1+BZ}$ in the Theorem 2.2.4, it is easy to view that the assumption (2.2.18) holds whenever $-1 \leq B < A \leq 1$ which leads to the following result.

Corollary 2.2.5: Let $-1 \leq B < A \leq 1$, and $\delta, \gamma \in \mathbb{C}^*$. Let $f \in W\Sigma_p$ and suppose that $z^{-p}F_{p,q,s}[\alpha_1, v]f(z) \neq 0$, $z \in U$.
If

$$1 + \frac{\gamma}{\delta}\left[\frac{z\left(F_{p,q,s}[\alpha_1, v]f(z)\right)'}{F_{p,q,s}[\alpha_1, v]f(z)} - p\right] \prec 1 + \gamma \frac{(A-B)z}{(1+Az)(1+Bz)}, \qquad (2.2.21)$$

then

$$\left[z^{-p}F_{p,q,s}[\alpha_1, v]f(z)\right]^{\frac{1}{\delta}} \prec \frac{1+Az}{1+Bz},$$

and $q(z) = \frac{1+AZ}{1+BZ}$ is the best dominant of (2.2.21).

Taking $x = 0, p = y = \alpha = 1, \alpha_i = \beta_i (i = 1,2, \ldots s), \gamma = \frac{1}{ab}, \delta = \frac{1}{b}$, where $a, b \in \mathbb{C}^*$

and $q(z) = (1-z)^{-2ab}$ in Theorem 2.2.4, then merge this together with Lemma 1.3.6, we obtain the next result.

Corollary 2.2.6: Let $a, b \in \mathbb{C}^*$ such that $|2ab \mp 1| \leq 1$. Let $f \in W\Sigma_p$ and suppose that $z^{-1}f(z) \neq 0$ for all $z \in U$. If

$$1+\frac{1}{a}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \frac{1+z}{1-z}, \tag{2.2.22}$$

then

$$[z^{-1}f(z)]^{b} \prec (1-z)^{-2ab}.$$

and $q(z)=(1-z)^{-2ab}$ is the best dominant of (2.2.22).

Again by setting $x=0, p=y=\alpha=1, \alpha_i=\beta_i (i=1,2,\dots s), \gamma=\frac{e^{im}}{ab\cos m}$, where $a,b \in \mathbb{C}^*, |m|<\frac{\pi}{2}, \delta=\frac{1}{b}$ and $q(z)=(1-z)^{-2ab\cos m e^{-im}}$ in Theorem 2.2.4, we obtain the next result, due to Aouf et al [13].

Corollary 2.2.7: Let $a,b \in \mathbb{C}^*$ and assume that

$$\left|2ab\cos m e^{-im} \mp 1\right| \leq 1$$

such that $|m|<\frac{\pi}{2}$. Let $f \in W\Sigma_p$, and $z^{-1}f(z) \neq 0$ for all $z \in U$. If

$$1+\frac{e^{im}}{a\cos m}\left[\frac{zf'(z)}{f(z)}-1\right] \prec \frac{1+z}{1-z}, \tag{2.2.23}$$

then

$$[z^{-1}f(z)]^{b} \prec (1-z)^{-2ab\cos m e^{-im}},$$

and $q(z)=(1-z)^{-2ab\cos m e^{-im}}$ is the best dominant of (2.2.23).

Theorem 2.2.8: Let $q(z)$ be univalent in U, with $q(0)=1$, let $\delta, \gamma \in \mathbb{C}^*$ and $\alpha, x, y \in \mathbb{C}$ such that $x+y \neq 0$. Let $f \in W\Sigma_p$ and suppose that f and q satisfy the next conditions:

$$(x+y)^{-1}z^{-p}\left\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\right\} \neq 0, \quad z \in U, \tag{2.2.24}$$

and

$$\Re\left\{1+\frac{zq''(z)}{q'(z)}\right\} > \max\left\{0; -\Re\left(\frac{\alpha}{\gamma}\right)\right\}. \tag{2.2.25}$$

If

$$\Upsilon_2(z) = \left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}\,[\alpha_1+1,v]f(z) + yF_{p,q,s}\,[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}} \times$$

$$\left[\alpha + \frac{\gamma}{\delta}\left(\frac{xz\left(F_{p,q,s}\,[\alpha_1+1,v]f(z)\right)' + yz\left(F_{p,q,s}\,[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}\,[\alpha_1+1,v]f(z) + yF_{p,q,s}\,[\alpha_1,v]f(z)} - p\right)\right], \quad (2.2.26)$$

and

$$\Upsilon_2(z) \prec \alpha q(z) + \gamma z q'(z), \quad (2.2.27)$$

then

$$\left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}\,[\alpha_1+1,v]f(z) + yF_{p,q,s}\,[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}} \prec q(z)\,,$$

and $q(z)$ is the best dominant of (2.2.27)

Proof. We begin by define the function

$$r(z) = \left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}\,[\alpha_1+1,v]f(z) + yF_{p,q,s}\,[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}}, \quad (2.2.28)$$

from (2.2.24) the function $r(z)$ is analytic in U, with $r(0) = 1$, and differentiating (2.2.28) logarithmically with respect to z , we have

$$\frac{zr'(z)}{r(z)} = \frac{1}{\delta}\left[\frac{xz\left(F_{p,q,s}\,[\alpha_1+1,v]f(z)\right)' + yz\left(F_{p,q,s}\,[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}\,[\alpha_1+1,v]f(z) + yF_{p,q,s}\,[\alpha_1,v]f(z)} - p\right],$$

and hence

$$zr'(z) = \frac{r(z)}{\delta}\left[\frac{xz\left(F_{p,q,s}\,[\alpha_1+1,v]f(z)\right)' + yz\left(F_{p,q,s}\,[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}\,[\alpha_1+1,v]f(z) + yF_{p,q,s}\,[\alpha_1,v]f(z)} - p\right].$$

Setting

$\theta(w) = \alpha w\;,\;\; \phi(w) = \gamma, \quad w \in \mathbb{C}$. Then, we get

$Q(z) = zq'(z)\phi\big(q(z)\big) = \gamma z q'(z),\; z \in U,$

$h(z) = \theta\big(q(z)\big) + Q(z) = \alpha\, q(z) + \gamma z q'(z),\; z \in U.$

From the assumption (2.2.25), we see that $Q(z)$ is starlike function in U, and we also have

$$\Re\left\{\frac{z\,h'(z)}{Q(\mathrm{z})}\right\}=\Re\left\{\frac{\alpha}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right\}>0, z\in U.$$

Now, application of Lemma 1.3.5 the proof of Theorem 2.2.8 is complete .

Letting $x=\alpha=1, y=0,\ q(z)=\frac{1+AZ}{1+BZ}$ in Theorem 2.2.8, where $-1\leq B<A\leq 1$ and according to (2.2.14) the condition (2.2.25) becomes

$$\max\left\{0;-\Re\left(\frac{\alpha}{\gamma}\right)\right\}\leq\frac{1-|B|}{1+|B|}$$

we obtain the next result.

Corollary 2.2.9: Let $-1\leq B<A\leq 1$, and let $\delta,\gamma\in\mathbb{C}^*$ such that

$$\max\left\{0;-\Re\left(\frac{\alpha}{\gamma}\right)\right\}\leq\frac{1-|B|}{1+|B|}$$

Let $f\in W\Sigma_p$ and suppose that $z^{-p}F_{p,q,s}\,[\alpha_1+1,v]f(z)\neq 0,\ z\in U.$

If

$$\left[z^{-p}F_{p,q,s}\,[\alpha_1+1,v]f(z)\right]^{\frac{1}{\delta}}\left[1+\frac{\gamma}{\delta}\left(\frac{z\left(F_{p,q,s}\,[\alpha_1+1,v]f(z)\right)'}{F_{p,q,s}\,[\alpha_1+1,v]f(z)}-p\right)\right]\prec \frac{1+Az}{1+Bz}+\frac{\gamma\,(A-B)z}{(1+Bz)^2}, \qquad (2.2.29)$$

then

$$\left[z^{-p}F_{p,q,s}\,[\alpha_1+1,v]f(z)\right]^{\frac{1}{\delta}}\prec\frac{1+Az}{1+Bz}$$

and $q(z)=\frac{1+AZ}{1+BZ}$ is the best dominant of (2.2.29).

Taking $x=\gamma=p=1,\ y=0,\ \alpha_i=\beta_i(\ i=1,2,\dots,\ \ s\ )$ and $q(z)=\frac{1+Z}{1-Z}$ in Theorem 2.2.8, then, we get.

Corollary 2.2.10: Let $f \in \mathrm{W}\Sigma_{\mathrm{p}}$ such that $z^{-1}f(z) \neq 0$ for all $z \in U$, and let $\delta \in \mathbb{C}^*$. If

$$[z^{-1}f(z)]^{\frac{1}{\delta}}\left[\alpha + \frac{1}{\delta}\left(\frac{zf'(z)}{f(z)} - 1\right)\right] \prec \alpha\frac{1+z}{1-z} + \frac{2z}{(1-z)^2}, \tag{2.2.30}$$

then

$$[z^{-1}f(z)]^{\frac{1}{\delta}} \prec \frac{1+z}{1-z},$$

and

$q(z) = \frac{1+Z}{1-Z}$ is the best dominant of (2.2.30).

Theorem 2.2.11: Let $q(z)$ be a convex univalent function in U, with $q(0) = 1$, let $0 < \delta < 1$, $\beta \in \mathbb{C}^*$ with $\Re(\beta) > 0$. Let $f \in \mathrm{W}\Sigma_{\mathrm{p}}$ such that

$$\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p} \neq 0\,, z \in U,$$

and suppose that f satisfies the condition:

$$\left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \in \mathcal{H}[q(0), 1] \cap \mathrm{Q}.$$

If the function $\Upsilon_1(z)$ given by (2.2.10) is univalent in U and satisfies

$$q(z) + \delta\beta z q'(z) \prec \Upsilon_1(z), \tag{2.2.31}$$

then

$$q(z) \prec \left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}},$$

and $q(z)$ is the best subordinant of (2.2.31).

Proof. We begin by setting

$$r(z) = \left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}}, z \in U, \tag{2.2.32}$$

then $r(z)$ is analytic function in U, with $r(0) = 1$.

by differentiating (2.2.32) logarithmically with respect to z, we have

$$\frac{zr'(z)}{r(z)}=\frac{1}{\delta}\left[\frac{z\left(F_{p,q,s}\,[\alpha_1,v]f(z)\right)'}{F_{p,q,s}\,[\alpha_1,v]f(z)}-p\right].$$

A simple computation and using the identity (2.2.7), shows that

$r(z)+\delta\beta zr'(z)=$

$$(1-\alpha_1\beta)\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}}+\alpha_1\beta\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}}\frac{F_{p,q,s}\,[\alpha_1+1,v]f(z)}{F_{p,q,s}\,[\alpha_1,v]f(z)},$$

now by applying Lemma 1.3.7 ,we obtain the required result.

By taking $q(z)=\frac{1+AZ}{1+BZ}$ in Theorem 2.2.11, where $-1\le B<A\le 1$, we get the next result.

Corollary 2.2.12: Let $q(z)$ be a convex in U with $q(0)=1$, let $0<\delta<1,\beta\in\mathbb{C}^*$ with $\Re(\beta)>0$. If $f\in W\Sigma_p$ such that

$$\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\neq 0\,,z\in U,$$

and suppose that f satisfies the condition

$$\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}}\in\mathcal{H}[q(0),1]\cap Q\,.$$

If $\Upsilon_1(z)$ given by (2.2.10) is univalent in U and satisfies the superordination

$$\frac{1+Az}{1+Bz}+\frac{\delta\beta(A-B)z}{(1+Bz)^2}\prec\Upsilon_1(z), \tag{2.2.33}$$

then

$$\frac{1+Az}{1+Bz}\prec\left[\frac{F_{p,q,s}\,[\alpha_1,v]f(z)}{z^p}\right]^{\frac{1}{\delta}},$$

and $q(z)=\frac{1+AZ}{1+BZ}$ is the best subordinant of (2.2.33).

Theorem 2.2.13: Let $q(z)$ be a convex univalent in U with $q(0)=1$, let $\delta,\gamma\in\mathbb{C}^*$, and $\alpha,x,y\in\mathbb{C}$ such that $x+y\neq 0$ and $\Re\left\{\frac{\alpha q'(z)}{\gamma}\right\}>0$. Let $f\in W\Sigma_p$ and f satisfies the following condition:

$$(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\neq 0,\ z\in U, \qquad (2.2.34)$$

and

$$\left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}}\in\mathcal{H}[q(0),1]\cap Q.$$

If the function $\Upsilon_2(z)$ given by (2.2.26) is univalent in U, and

$$\alpha q(z)+\gamma zq'(z)\prec\Upsilon_2(z), \qquad (2.2.35)$$

then

$$q(z)\prec\left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}},$$

and $q(z)$ is the best subordinant of (2.2.35).

Proof. Consider the analytic function

$$r(z)=\left[(x+y)^{-1}z^{-p}\{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)\}\right]^{\frac{1}{\delta}}, \qquad (2.2.36)$$

with $r(0)=1$.

By differentiating (2.2.36) logarithmically with respect to z, yields

$$\frac{zr'(z)}{r(z)}=\frac{1}{\delta}\left[\frac{xz\left(F_{p,q,s}[\alpha_1+1,v]f(z)\right)'+yz\left(F_{p,q,s}[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\right],$$

then

$$zr'(z)=\frac{r(z)}{\delta}\left[\frac{xz\left(F_{p,q,s}[\alpha_1+1,v]f(z)\right)'+yz\left(F_{p,q,s}[\alpha_1,v]f(z)\right)'}{xF_{p,q,s}[\alpha_1+1,v]f(z)+yF_{p,q,s}[\alpha_1,v]f(z)}-p\right].$$

Setting the function

$$\theta(w)=\alpha w,\quad \phi(w)=\gamma,\qquad w\in\mathbb{C},$$

then θ and ϕ is analytic in $\mathbb{C}$, with $\phi(w)\neq 0$ for all $w\in\mathbb{C}$.

Also, we have

$Q(z)=zq'(z)\phi(q(z))=\gamma zq'(z)$, is starlike univalent function in U, and

$$\Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \Re\left\{\frac{\alpha q'(z)}{\gamma}\right\} > 0\,, z \in U,$$

by simple computation , shows that

$$\Upsilon_2(z) = \alpha r(z) + \gamma z r'(z). \tag{2.2.37}$$

From (2.2.35) and (2.2.37), with applying of Lemma 1.3.8, we have $q(z) \prec r(z)$ and using (2.2.36), we obtain the required result.

Combining results of differential subordinations and superordinations, to get at the following sandwich results .

Theorem 2.2.14: Let $q_1(z)$ and $q_2(z)$ be a convex univalent functions in U, with $q_1(0) = q_2(0) = 1$, let $0 < \delta < 1$, $\beta \in \mathbb{C}^*$ with $\Re(\beta) > 0$. Let $f \in W\Sigma_p$ such that

$$\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p} \neq 0$$

and suppose that f satisfies the condition:

$$\left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \in \mathcal{H}[q(0), 1] \cap \mathrm{Q}\,.$$

If the function $\Upsilon_1(z)$ given by (2.2.10) is univalent in U and satisfies

$$q_1(\mathrm{z}) + \delta\beta \mathrm{z} q_1'(z) \prec \Upsilon_1(z) \prec q_2(\mathrm{z}) + \delta\beta q_2'(\mathrm{z}), \tag{2.2.38}$$

then

$$q_1(\mathrm{z}) \prec \left[\frac{F_{p,q,s}\,[\alpha_1, v]f(z)}{z^p}\right]^{\frac{1}{\delta}} \prec q_2(\mathrm{z})\,,$$

and q_1 , q_2 are respectively, the best subordinant and the best dominant of (2.2.38).

Theorem 2.2.15: Let $q_1(z)$ and $q_2(z)$ be a convex univalent functions in U, with $q_1(0) = q_2(0) = 1$, let $\delta, \gamma \in \mathbb{C}^*$ and $\alpha, x, y \in \mathbb{C}$ such that $x + y \neq 0$, suppose q_1 satisfies $\Re\left\{\frac{\alpha q_1'(z)}{\gamma}\right\} > 0$ and q_2 satisfies (2.2.25). Let $f \in W\Sigma_p$ satisfy the next conditons:

$$(x + y)^{-1} z^{-p}\left\{x F_{p,q,s}\,[\alpha_1 + 1, v]f(z) + y F_{p,q,s}\,[\alpha_1, v]f(z)\right\} \neq 0,\ z \in U,$$

and

$$\left[(x+y)^{-1}z^{-p}\left\{xF_{p,q,s}\,[\alpha_1+1,v]f(z)+yF_{p,q,s}\,[\alpha_1,v]f(z)\right\}\right]^{\frac{1}{\delta}}\in\mathcal{H}[q(0),1]\cap\mathrm{Q}.$$

If the function $\Upsilon_2(z)$ given by equation (2.2.26) is univalent in U, and

$$\alpha q_1(z)+\gamma z q_1'(z)\prec\Upsilon_2(z)\prec\alpha q_2(z)+\gamma z q_2'(z), \tag{2.2.39}$$

then

$$q_1(\mathrm{z})\prec\left[(x+y)^{-1}z^{-p}\left\{xF_{p,q,s}\,[\alpha_1+1,v]f(z)+yF_{p,q,s}\,[\alpha_1,v]f(z)\right\}\right]^{\frac{1}{\delta}}\prec q_2(\mathrm{z})\,,$$

and q_1, q_2 are respectively, the best subordinant and the best dominant of (2.2.39).

Chapter Three

On (Third and Fourth)-Order Differential Subordination and Superordination Results for Multivalent and Meromorphic Functions

Introduction:

This chapter is completely devoted for the study of (Third and Fourth)-order differential subordination and superordination results for multivalent and meromorphic functions, having Taylor and Laurent series expansion containing positive and negative terms. Actually a differential subordination in the complex plane is the generalization of a differential inequality on the real line. The concept of differential subordination plays a very important role in functions of real variable. This concept also enables us to study the range of original function. In the theory of complex-valued function, there are several differential applications in which a characterization of a function is determined from a differential condition. Miller and Mocanu [47] have contributed number of papers on differential subordination. The study of differential subordination stems out from text books by Duren [23], Goodman [27] and Pommerenke [56].

This chapter is divided into three sections. The first section is concerned with the third-order differential subordination results for meromorphic univalent functions associated with linear operator, like, Let $\phi \in \Phi_I[\Omega, q]$. If the functions $f \in \Sigma_1^*$ and $q \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re\left(\frac{\zeta\, q''(\zeta)}{q'(\zeta)}\right) \geq 0\,, \qquad \left|\frac{I_1(n+1,\lambda)f(z)}{q'(\zeta)}\right| \leq \kappa\,,$$

and

$$\{\phi(I_1(n,\lambda)f(z), I_1(n+1,\lambda)f(z), I_1(n+2,\lambda)f(z), I_1(n+3,\lambda)f(z); z) : z \in U\} \subset \Omega,$$

then

$$I_1(n,\lambda)f(z) \prec q(z)\,, \qquad (z \in U).$$

The second section deals with the third-order differential superordination results for p-valent meromorphic functions involving linear operator. We derive some third-order differential superordination results for analytic functions in the punctured open unit disk of meromorphic p-valent functions by using certain classes of admissible functions, like, let $\phi \in \Phi_D[\Omega, q]$. If the function $f \in \Sigma_p^*, z^p D_{\lambda,p}^n f(z) \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the following condition:

$$\Re\left(\frac{z\,q''(z)}{q'(z)}\right) \geq 0\,, \qquad \left|\frac{z^p D_{\lambda,p}^{n+1} f(z)}{q'(z)}\right| \leq \frac{1}{m}\,,$$

and

$$\phi\left(z^p D_{\lambda,p}^n f(z),\ z^p D_{\lambda,p}^{n+1} f(z),\ z^p D_{\lambda,p}^{n+2} f(z),\ z^p D_{\lambda,p}^{n+3} f(z); z\right),$$

is univalent in U, and

$$\Omega \subset \left\{\phi\left(z^p D_{\lambda,p}^n f(z)\,, z^p D_{\lambda,p}^{n+1} f(z)\,, z^p D_{\lambda,p}^{n+2} f(z)\,, z^p D_{\lambda,p}^{n+3} f(z); z\right) : z \in U\right\},$$

then

$$q(z) \prec z^p D_{\lambda,p}^n f(z)\,, \qquad (z \in U).$$

Section three discusses the fourth – order differential subordination and superordination results for multivalent analytic functions. Here, we introduce new concept that is fourth–order differential subordination and superordination associated with differential linear operator $I_p(n,\lambda)$ in open unit disk.

3.1 On Third-Order Differential Subordination Results for Meromorphic Univalent Function Associated with Linear Operator

Let $\mathcal{H}(U)$ be the class of functions which are analytic in the open unit disk:

$$U = \{z : z \in \mathbb{C} \ : |z| < 1\}\,.$$

For $n \in \mathbb{N} = \{1,2,3,\dots\}$, and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}\,,$$

with $\mathcal{H}_1 = [1,1]$.

Let Σ_1^* denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k\, z^k \ , \tag{3.1.1}$$

which are analytic and meromorphic univalent in the punctured unit disk:

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}\,.$$

We consider linear operator $I_1(n,\lambda)$ on the class Σ_1^* of meromorphic functions by the infinite series

$$I_1(n,\lambda)f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\lambda}{\lambda-1}\right)^n a_k z^k \ , \qquad (\lambda > 1), \tag{3.1.2}$$

the operator $I_p(n,\lambda)$ was studied on class of meromorphic multivalent function by [10]. It is easily verified from (3.1.2) that

$$z[I_1(n,\lambda)f(z)]' = (\lambda - 1)I_1(n+1,\lambda)f(z) - \lambda I_1(n,\lambda)f(z). \tag{3.1.3}$$

In recent years, several authors obtained many interesting results for the theory of second–order differential subordination and superordination for example [8,9,10,16,33,66], thus the aim of this section to investigate extension to the third–order differential subordination.

The first authors investigated the third order, Ponnusamy [57] published in 1992. In 2011, Antonino and Miller [11] extended the theory of second–order differential subordination in the open unit disk introduced by Miller and Mocanu [47] to the third–order case. They determined properties of functions p that satisfy the following third–order differential subordination:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\} \subset \Omega . \tag{3.1.4}$$

Recently, the only a few of authors discussed the third–order differential subordination and superordination for analytic functions in U for example [3,4,31,52,69].

We determine certain suitable classes of admissible functions and investigate some third-order differential subordination properties of analytic function. We first define the following class of admissisble functions, which are required in proving the differential subordination theorem involving the operator $I_1(n, \lambda)$ defined by (3.1.2).

Definition 3.1.1: Let Ω be a set in $\mathbb{C}, \lambda \in \mathbb{C}\backslash\{1\}$, and let $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_I[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, x, y; z) \notin \Omega ,$$

whenever

$$u = q(\zeta) \ , \ \ v = \frac{\kappa\zeta q'(\zeta) + \lambda q(\zeta)}{\lambda - 1} ,$$

$$\Re\left\{\frac{(\lambda-1)^2 x - \lambda^2 u}{(\lambda-1)v - \lambda u} - 2\lambda\right\} \geq \kappa\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\} ,$$

and

$$\Re\left\{\frac{(\lambda-1)^2[(\lambda-1)y - 3(\lambda+1)x] + (2\lambda^3 + 3\lambda^2)u}{(\lambda-1)v - \lambda u} + (3\lambda^2 + 6\lambda + 2)\right\}$$

$$\geq \kappa^2\Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U, \zeta \in \partial U\backslash \mathrm{E}(q)$ and $\kappa \geq 2$.

Theorem 3.1.2: Let $\phi \in \Phi_I[\Omega, q]$. If the functions $f \in \Sigma_1^*$ and $q \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re\left(\frac{\zeta\, q''(\zeta)}{q'(\zeta)}\right) \geq 0 , \qquad \left|\frac{I_1(n+1, \lambda)f(z)}{q'(\zeta)}\right| \leq \kappa , \tag{3.1.5}$$

and

$$\{\phi(I_1(n,\lambda)f(z), I_1(n+1,\lambda)f(z), I_1(n+2,\lambda)f(z), I_1(n+3,\lambda)f(z); z): z \in U\} \subset \Omega, \tag{3.1.6}$$

then

$$I_1(n,\lambda)f(z) \prec q(z)\,, \ (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = I_1(n,\lambda)f(z). \tag{3.1.7}$$

Then, differentiating (3.1.7) with respect to z and using (3.1.3), we have

$$I_1(n+1,\lambda)f(z) = \frac{zp'(z) + \lambda p(z)}{\lambda - 1} \tag{3.1.8}$$

Further computations show that

$$I_1(n+2,\lambda)f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda^2 p(z)}{(\lambda-1)^2}\,, \tag{3.1.9}$$

and

$$I_1(n+3,\lambda)f(z) = \frac{z^3p'''(z) + 3(\lambda+1)z^2p''(z) + (3\lambda^2+3\lambda+1)zp'(z) + \lambda^3 p(z)}{(\lambda-1)^3}\,. \tag{3.1.10}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$u(r,s,t,w) = r\,, \quad v(r,s,t,w) = \frac{s+\lambda r}{\lambda-1}\,, \quad x(r,s,t,w) = \frac{t + (2\lambda+1)s + \lambda^2 r}{(\lambda-1)^2}\,,$$

and

$$y(r,s,t,w) = \frac{w + 3(\lambda+1)t + (3\lambda^2+3\lambda+1)s + \lambda^3 r}{(\lambda-1)^3}\,.$$

Let

$$\psi(r,s,t,w;z) = \phi(u,v,x,y;z)$$

$$= \phi\left(r, \frac{s+\lambda r}{\lambda-1}, \frac{t+(2\lambda+1)s+\lambda^2 r}{(\lambda-1)^2}, \frac{w+3(\lambda+1)t+(3\lambda^2+3\lambda+1)s+\lambda^3 r}{(\lambda-1)^3}; z\right). \tag{3.1.11}$$

The proof will make use of Theorem 1.3.10. Using equations (3.1.7) to (3.1.10), and from (3.1.11), we obtain

$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) =$

$$\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ I_1(n+3,\lambda)f(z); z). \quad (3.1.12)$$

Hence (3.1.6) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega\,.$$

Note that

$$\frac{t}{s} + 1 = \frac{(\lambda-1)^2x - \lambda^2u}{(\lambda-1)v - \lambda u} - 2\lambda\,,$$

and

$$\frac{w}{s} = \frac{(\lambda-1)^2[(\lambda-1)y - 3(\lambda+1)x] + (2\lambda^3 + 3\lambda^2)u}{(\lambda-1)v - \lambda u} + (3\lambda^2 + 6\lambda + 2).$$

Thus, the admissibility condition for $\phi \in \Phi_I[\Omega, q]$ in Definition 3.1.1 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega, q]$ as given in Definition 1.1.27 with $n = 2$. Therefore, by using (3.1.5) and Theorem 1.3.10, we have

$$p(z) = I_1(n,\lambda)f(z) \prec q(z).$$

The next result is an extension of Theorem 3.1.2 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 3.1.3: Let $\Omega \subset \mathbb{C}$ and q be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \Sigma_1^*$ and q_ρ satisfy the following conditions:

$$\Re\left(\frac{\zeta\, q_\rho''(\zeta)}{q_\rho'(\zeta)}\right) \geq 0\,, \qquad \left|\frac{I_1(n+1,\lambda)f(z)}{q_\rho'(\zeta)}\right| \leq \kappa\,, \quad \left(z \in U, \zeta \in \partial U\backslash \mathrm{E}(q_\rho)\right) \quad (3.1.13)$$

and

$$\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ I_1(n+3,\lambda)f(z); z) \in \Omega\,,$$

then

$$I_1(n,\lambda)f(z) \prec q(z)\,,\ (z \in U).$$

Proof. By using Theorem 3.1.2, yields

$$I_1(n,\lambda)f(z) \prec q_\rho(z)\,,\ \ (z \in U).$$

The result asserted by Corollary 3.1.3 is now deduced from the subordination

$$q_\rho(z) \prec q(z)\ \ ,\ (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_I[h(U), q]$ is written as $\Phi_I[h\,, q]$. The following two results are immediate consequence of Theorem 3.1.2 and Corollary 3.1.3.

Theorem 3.1.4: Let $\phi \in \Phi_I[h, q]$. If the function $f \in \Sigma_1^*$ and $q \in \mathcal{Q}_1$ satisfy the condition (3.1.5) and

$$\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ I_1(n+3,\lambda)f(z); z) \prec h(z), \tag{3.1.14}$$

then

$$I_1(n,\lambda)f(z) \prec q(z)\,,\ (z \in U).$$

Corollary 3.1.5: Let $\Omega \subset \mathbb{C}$ and q be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_I[h, q_\rho\,]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \Sigma_1^*$ and q_ρ satisfy the condition (3.1.13), and

$$\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ I_1(n+3,\lambda)f(z); z) \prec h(z), \tag{3.1.15}$$

then

$$I_1(n,\lambda)f(z) \prec q(z)\,,\ (z \in U).$$

The next theorem yields the best dominant of the differential subordination (3.1.14).

Theorem 3.1.6: Let the function h be univalent in U, and let $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ and ψ be given by (3.1.11). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z), \tag{3.1.16}$$

has a solution $q(z)$ with $q(0) = 1$ and satisfies the condition (3.1.5). If the function $f \in \Sigma_1^*$ satisfies condition (3.1.14) and

$$\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ \ I_1(n+3,\lambda)f(z); z)$$

is analytic in U, then

$$I_1(n,\lambda)f(z) \prec q(z)\,,$$

and $q(z)$ is the best dominant.

Proof. From Theorem 3.1.2, we deduce that q is a dominant of (3.1.14). Since q satisfies (3.1.16), it is also a solution of (3.1.14) and therefore q will be dominated by all dominants. Hence q is the best dominant.

In the special case $q(z) = M\mathrm{z}\,,\ \ M > 0$, and in view of Definition 3.1.1, the class of admissible functions $\Phi_I[\Omega, q]$, denoted by $\Phi_I[\Omega, M]$ is expressed as follows.

Definition 3.1.7: Let Ω be a set in $\mathbb{C}$, $\lambda \in \mathbb{C}\backslash\{1\}$ and $M > 0$. The class of admissible functions $\Phi_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{\kappa+\lambda}{\lambda-1}Me^{i\theta}, \frac{L+[(2\lambda+1)\kappa+\lambda^2]Me^{i\theta}}{(\lambda-1)^2},\right.$$

$$\left.\frac{N+3(\lambda+1)L+[(3\lambda^2+3\lambda+1)\kappa+\lambda^3]Me^{i\theta}}{(\lambda-1)^3};z\right) \notin \Omega, \quad (3.1.17)$$

whenever $z \in U$, $\Re\left(Le^{-i\theta}\right) \geq (\kappa-1)\kappa M$, and $\Re\left(Ne^{-i\theta}\right) \geq 0$ for all $\theta \in \mathbb{R}$ and $\kappa \geq 2$.

Corollary 3.1.8: Let $\phi \in \Phi_I[\Omega, M]$. If the function $f \in \Sigma_1^*$ satisfies

$$|\,I_1(n+1,\lambda)f(z)| \leq \kappa M \ \ (\kappa \geq 2\,; M > 0),$$

and

$$\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ I_1(n+3,\lambda)f(z);z) \in \Omega,$$

then

$$|I_1(n,\lambda)f(z)| < M\,.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$. Corollary 3.1.8 can now be written in the following form:

Corollary 3.1.9: Let $\phi \in \Phi_I[M]$. If the function $f \in \Sigma_1^*$ satisfies the following condition:

$$|\,I_1(n+1,\lambda)f(z)| \leq \kappa M(\kappa \geq 2\,; M > 0),$$

and
$$|\phi(I_1(n,\lambda)f(z),\ I_1(n+1,\lambda)f(z),\ I_1(n+2,\lambda)f(z),\ I_1(n+3,\lambda)f(z);z)| < M,$$

then

$$|I_1(n,\lambda)f(z)| < M.$$

By taking $\phi(u,v,x,y;z) = v = \frac{\kappa+\lambda}{\lambda-1}Me^{i\theta}$ in Corollary 3.1.9, we obtain the next result.

Example 3.1.10: Let $\Re(\lambda) \geq \frac{1-\kappa}{2}$, $\kappa \geq 2$ and $M > 0$. If the function $f \in \Sigma_1^*$ satisfies

$$|I_1(n+1,\lambda)f(z)| \leq \kappa M,$$

then

$$|I_1(n,\lambda)f(z)| < M.$$

Example 3.1.11: Let $\kappa \geq 2$, $\lambda \in \mathbb{C}\backslash\{1\}$ and $M > 0$. If the function $f \in \Sigma_1^*$ satisfies

$$|I_1(n+1,\lambda)f(z)| \leq \kappa M,$$

and

$$|I_1(n+1,\lambda)f(z) - I_1(n,\lambda)f(z)| < \frac{\kappa+1}{|\lambda-1|}M,$$

then

$$|I_1(n,\lambda)f(z)| < M.$$

Proof. Let

$$\phi(u,v,x,y;z) = v-u\,, \quad \Omega = h(U),$$

where

$$h(z) = \frac{\kappa+1}{|\lambda-1|}Mz\,, \qquad M > 0.$$

In order to use Corollary 3.1.8, we need to show that $\phi \in \Phi_I[\Omega, M]$, that is, the admissibility condition (3.1.17) is satisfied. This follows easily, since

$$\left|\phi\left(Me^{i\theta}, \frac{\kappa+\lambda}{\lambda-1}Me^{i\theta}, \frac{L+[(2\lambda+1)\kappa+\lambda^2]Me^{i\theta}}{(\lambda-1)^2},\right.\right.$$

$$\left.\left.\frac{N+3(\lambda+1)L+[(3\lambda^2+3\lambda+1)\kappa+\lambda^3]Me^{i\theta}}{(\lambda-1)^3};z\right)\right|$$

$$= \left|\frac{\kappa+1}{\lambda-1} M e^{i\theta}\right|$$
$$= \frac{\kappa+1}{|\lambda-1|} M,$$
whenever $z \in U$, $\theta \in \mathbb{R}$ and $\kappa \geq 2$. The required result now follows from Corollary 3.1.8.

Definition 3.1.12: Let Ω be a set in $\mathbb{C}$, $\lambda \in \mathbb{C}\backslash\{1\}$, and let $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissisbility condition:

$$\phi(u, v, x, y; z) \notin \Omega$$

whenever

$$u = q(\zeta)\ ,\ \ v = \frac{\kappa\zeta q'\,(\zeta) + (\lambda+1)q(\zeta)}{\lambda-1},$$

$$\Re\left\{\frac{(\lambda-1)^2 x - (\lambda+1)^2 u}{(\lambda-1)v - (\lambda+1)u} - 2(\lambda+1)\right\} \geq \kappa\Re\left\{\frac{\zeta q''\,(\zeta)}{q'\,(\zeta)} + 1\right\},$$

and

$$\Re\left\{\frac{(\lambda-1)^3 y - (3\lambda+6)[(\lambda-1)^2 x - (\lambda+1)^2 u] - (\lambda+1)^3 u}{(\lambda-1)v - (\lambda+1)u} + (3\lambda^2 + 12\lambda + 11)\right\} \geq \kappa^2\Re\left\{\frac{\zeta^2 q'''(\zeta)}{q'\,(\zeta)}\right\},$$

where $z \in U$, $\zeta \in \partial U\backslash \mathrm{E}(q)$ and $\kappa \geq 2$.

Theorem 3.1.13: Let $\phi \in \Phi_{I,1}[\Omega, q]$. If the function $f \in \Sigma_1^*$ and $q \in \mathcal{Q}_1$ satisfy the following conditions:

$$\Re\left(\frac{\zeta\, q''(\zeta)}{q'(\zeta)}\right) \geq 0\,, \qquad \left|\frac{I_1(n+1,\lambda)f(z)}{zq'(\zeta)}\right| \leq \kappa\,, \tag{3.1.18}$$

and

$$\left\{\phi\left(\frac{I_1(n,\lambda)f(z)}{z}, \frac{I_1(n+1,\lambda)f(z)}{z}, \frac{I_1(n+2,\lambda)f(z)}{z}, \frac{I_1(n+3,\lambda)f(z)}{z}; z\right) : z \in U\right\} \subset \Omega, \tag{3.1.19}$$

then

$$\frac{I_1(n,\lambda)f(z)}{z} \prec q(z)\,, \quad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = \frac{I_1(n,\lambda)f(z)}{z}\,. \tag{3.1.20}$$

By using (3.1.3) and (3.1.20), we get

$$\frac{I_1(n+1,\lambda)f(z)}{z} = \frac{zp'(z) + \lambda p(z)}{\lambda - 1}. \tag{3.1.21}$$

Further computations show that

$$\frac{I_1(n+2,\lambda)f(z)}{z} = \frac{z^2p''(z) + (2\lambda+3)zp'(z) + (\lambda+1)^2p(z)}{(\lambda-1)^2}\,, \tag{3.1.22}$$

and

$$\frac{I_1(n+3,\lambda)f(z)}{z}$$
$$= \frac{z^3p'''(z) + (3\lambda+6)z^2p''(z) + (3\lambda^2+9\lambda+7)zp'(z) + (\lambda+1)^3p(z)}{(\lambda-1)^3}. \tag{3.1.23}$$

Define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$u(r,s,t,w) = r\,, \qquad v(r,s,t,w) = \frac{s + (\lambda+1)r}{\lambda - 1},$$

$$x(r,s,t,w) = \frac{t + (2\lambda+3)s + (\lambda+1)^2r}{(\lambda-1)^2},$$

and

$$y(r,s,t,w) = \frac{w + (3\lambda+6)t + (3\lambda^2+9\lambda+7)s + (\lambda+1)^3r}{(\lambda-1)^3}.$$

Let

$$\psi(r,s,t,w;z) = \phi(u,v,x,y;z) = \phi\left(r, \frac{s+(\lambda+1)r}{\lambda-1}, \frac{t+(2\lambda+3)s+(\lambda+1)^2r}{(\lambda-1)^2},\right.$$

$$\frac{w+(3\lambda+6)t+(3\lambda^2+9\lambda+7)s+(\lambda+1)^3 r}{(\lambda-1)^3};z\Bigg). \quad (3.1.24)$$

The proof will make use of Theorem 1.3.10.Using equations (3.1.20) to (3.1.23) , and from (3.1.24), we obtain

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)=$$

$$\phi\left(\frac{I_1(n,\lambda)f(z)}{z},\frac{I_1(n+1,\lambda)f(z)}{z},\frac{I_1(n+2,\lambda)f(z)}{z},\frac{I_1(n+3,\lambda)f(z)}{z};z\right).$$

Hence (3.1.19) becomes

$$\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z)\in\Omega.$$

Note that

$$\frac{t}{s}+1=\frac{(\lambda-1)^2x-(\lambda+1)^2u}{(\lambda-1)v-(\lambda+1)u}-2(\lambda+1)\,,$$

and

$$\frac{w}{s}=$$

$$\frac{(\lambda-1)^3y-(3\lambda+6)[(\lambda-1)^2x-(\lambda+1)^2u]-(\lambda+1)^3u}{(\lambda-1)v-(\lambda+1)u}+(3\lambda^2+12\lambda+11).$$

Thus, the admissibility condition for $\phi\in\Phi_{I,1}[\Omega,q]$ in Definition 3.1.12 is equivalent to the admissibility condition for $\psi\in\Psi_2[\Omega,q]$ as given in Definition 1.1.27 with $n=2$. Therefore, by using (3.1.18) and Theorem 1.3.10, we have

$$p(z)=\frac{I_1(n,\lambda)f(z)}{z}\prec q(z)\,.$$

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\Phi_{I,1}[h(U),q]$ is written as $\Phi_{I,1}[h,q]$. The next results is an immediate consequence of Theorem 3.1.13.

Theorem 3.1.14: Let $\phi\in\Phi_{I,1}[h\,,q]$. If the function $f\in\Sigma_1^*$ and $q\in\mathcal{Q}_1$ satisfy the condition (3.1.18) and

$$\phi\left(\frac{I_1(n,\lambda)f(z)}{z},\frac{I_1(n+1,\lambda)f(z)}{z},\frac{I_1(n+2,\lambda)f(z)}{z},\frac{I_1(n+3,\lambda)f(z)}{z};z\right)\prec h(z),$$

(3.1.25)

then

$$\frac{I_1(n,\lambda)f(z)}{z} \prec q(z)\,, \quad (z \in U).$$

In the special case when $q(z) = 1 + Mz,\ M > 0$, and in view of Definition 3.1.12 , the class of admissible functions $\Phi_{I,1}[\Omega\,, q]$ is denoted by $\Phi_{I,1}[\Omega\,, M]$, is described below.

Definition 3.1.15: Let Ω be a set in $\mathbb{C}$, $\lambda \in \mathbb{C}\backslash\{1\}$ and $M > 0$. The class of admissible functions $\Phi_{I,1}[\Omega\,, M]$ consists of those functions $\phi : \mathbb{C}^4 \times U \longrightarrow \mathbb{C}$ such that

$$\phi\Bigg(1 + Me^{i\theta}, \frac{(\kappa+\lambda+1)Me^{i\theta} + (\lambda+1)}{\lambda-1},$$

$$\frac{L + [(2\lambda+3)\kappa + (\lambda+1)^2]Me^{i\theta} + (\lambda+1)^2}{(\lambda-1)^2},$$

$$\frac{N + (3\lambda+6)L + [(3\lambda^2+9\lambda+7)\kappa + (\lambda+1)^3]Me^{i\theta}(\lambda+1)^3}{(\lambda-1)^3}\,; z\Bigg) \notin \Omega, \quad (3.1.26)$$

whenever $z \in U$, $\Re\left(Le^{-i\theta}\right) \geq (\kappa-1)\kappa M$, and $\Re\left(Ne^{-i\theta}\right) \geq 0$ for all $\theta \in \mathbb{R}$ and $\kappa \geq 2$.

Corollary 3.1.16: Let $\phi \in \Phi_{I,1}[h, M]$. If the function $f \in \Sigma_1^*$ satisfies

$$\left|\frac{I_1(n+1,\lambda)f(z)}{z}\right| \leq \kappa M, \quad (\kappa \geq 2\,; M > 0),$$

and

$$\phi\left(\frac{I_1(n,\lambda)f(z)}{z}, \frac{I_1(n+1,\lambda)f(z)}{z}, \frac{I_1(n+2,\lambda)f(z)}{z}, \frac{I_1(n+3,\lambda)f(z)}{z}\,; z\right) \in \Omega,$$

then

$$\left|\frac{I_1(n,\lambda)f(z)}{z} - 1\right| < M.$$

In the special case, when $\Omega = q(U) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{I,1}[\Omega, M]$ is simply denoted by $\Phi_{I,1}[M]$ and Corollary 3.1.16 has the following form:

Corollary 3.1.17: Let $\phi \in \Phi_{I,1}[M]$. If the function $f \in \Sigma_1^*$ satisfies the next condition:

$$\left|\frac{I_1(n+1,\lambda)f(z)}{z}\right| \leq \kappa M, \quad (\kappa \geq 2\,; M > 0),$$

and

$$\left|\phi\left(\frac{I_1(n,\lambda)f(z)}{z}, \frac{I_1(n+1,\lambda)f(z)}{z}, \frac{I_1(n+2,\lambda)f(z)}{z}, \frac{I_1(n+3,\lambda)f(z)}{z}; z\right) - 1\right| < M,$$

then

$$\left|\frac{I_1(n,\lambda)f(z)}{z} - 1\right| < M.$$

Example 3.1.18: Let $\Re(\lambda) \geq \frac{-\kappa}{2}$, $\lambda \in \mathbb{C}\backslash\{1\}$ $\kappa \geq 2$ and $M > 0$. If the function $f \in \Sigma_1^*$ satisfies

$$\left|\frac{I_1(n+1,\lambda)f(z)}{z}\right| \leq \kappa M,$$

and

$$\left|\frac{I_1(n+1,\lambda)f(z)}{z} - 1\right| < M,$$

then

$$\left|\frac{I_1(n,\lambda)f(z)}{z} - 1\right| < M.$$

Proof. By taking

$$\phi(u,v,x,y;z) = v - 1 = \frac{(\kappa+\lambda+1)Me^{i\theta} + (\lambda+1)}{\lambda-1} - 1,$$

in Corollary 3.1.17, the result is obtained.

Example 3.1.19: Let $\kappa \geq 2$, $\lambda \in \mathbb{C}$ and $M > 0$. If the function $f \in \Sigma_1^*$ satisfies

$$\left|\frac{I_1(n+1,\lambda)f(z)}{z}\right| \leq \kappa M,$$

and

$$\left|(\lambda-1)^3\frac{I_1(n+3,\lambda)f(z)}{z}-(\lambda-1)^2(\lambda+1)\frac{I_1(n+2,\lambda)f(z)}{z}\right|$$
$$< 2(|2\lambda+5|+|\lambda+2|^2)M,$$

then

$$\left|\frac{I_1(n,\lambda)f(z)}{z}-1\right|<M.$$

Proof. By taking

$$\phi(u,v,x,y;z)=(\lambda-1)^3y-(\lambda-1)^2(\lambda+1)x,$$

and $\Omega=h(U)$, where $h(z)=2(|2\lambda+5|+|\lambda+2|^2)Mz,\ \ M>0.$

Using Corollary 3.1.16, we need to show that $\phi\in\Phi_{I,1}[\Omega\,,M]$.

Since

$$\left|\phi\left(1+Me^{i\theta},\frac{(\kappa+\lambda+1)Me^{i\theta}+(\lambda+1)}{\lambda-1},\right.\right.$$
$$\frac{L+[(2\lambda+3)\kappa+(\lambda+1)^2]Me^{i\theta}+(\lambda+1)^2}{(\lambda-1)^2},$$
$$\left.\left.\frac{N+(3\lambda+6)L+[(3\lambda^2+9\lambda+7)\kappa+(\lambda+1)^3]Me^{i\theta}+(\lambda+1)^3}{(\lambda-1)^3};z\right)\right|$$

$$=\left|N+(2\lambda+5)L+\kappa(\lambda+2)^2Me^{i\theta}\right|$$

$$=\left|\frac{Ne^{-i\theta}+(2\lambda+5)Le^{-i\theta}+\kappa(\lambda+2)^2M}{e^{-i\theta}}\right|$$

$$\geq\ \Re\left(Ne^{-i\theta}\right)+|2\lambda+5|\Re\left(Le^{-i\theta}\right)+\kappa|\lambda+2|^2M$$

$$\geq|2\lambda+5|(\kappa-1)\kappa M+\kappa|\lambda+2|^2M\geq 2(|2\lambda+5|+|\lambda+2|^2)M,$$

whenever $z\in U,\ \Re\left(Le^{-i\theta}\right)\geq(\kappa-1)\kappa M,$ and $\Re\left(Ne^{-i\theta}\right)\geq 0$ for all $\theta\in\mathbb{R}$ and $\kappa\geq 2$. The proof is complete.

3.2 On Third-Order Differential Superordination Results for P-valent Meromorphic Functions Involving Linear Operator

Let Σ_p^* denote the class of functions of the form :

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad , \quad (p \in \mathbb{N} = \{1,2,3, \dots \}), \tag{3.2.1}$$

which are analytic and p-valent in the punctured unit disk:

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \backslash \{0\}.$$

For functions $f \in \Sigma_p^*$, we define the linear operator:

$$D_{\lambda,p}^n f(z) \colon \Sigma_p^* \longrightarrow \Sigma_p^* , \quad (\lambda \geq 0; \quad p \in \mathbb{N}; \quad n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \text{ by}$$

$$D_{\lambda,p}^0 f(z) = f(z),$$

$$D_{\lambda,p}^1 f(z) = D_{\lambda,p} f(z) = (1-\lambda) f(z) + \lambda z^{-p} \left(z^{p+1} f(z)\right)'$$

$$= z^{-p} + \sum_{k=0}^{\infty} [1 + \lambda(k+p)] a_k z^k \quad , (\lambda \geq 0; \quad p \in \mathbb{N}),$$

$$D_{\lambda,p}^2 f(z) = D_{\lambda,p}\left(D_{\lambda,p} f(z)\right) = z^{-p} + \sum_{k=0}^{\infty} [1 + \lambda(k+p)]^2 a_k z^k \quad , (\lambda \geq 0; \; p \in \mathbb{N}),$$

and (in general)

$$D_{\lambda,p}^n f(z) = z^{-p} + \sum_{k=0}^{\infty} [1 + \lambda(k+p)]^n a_k z^k \quad , (\lambda \geq 0; \quad p \in \mathbb{N} \, ; \; n \in \mathbb{N}_0) \, , \tag{3.2.2}$$

where the operator $D_{\lambda,p}^n (f * g)(z)$ was studied on class of meromorphic p-valent function by [15]. From (3.2.2) it is easy to verify that

$$z\left[D_{\lambda,p}^{n} f(z)\right]' = \frac{1}{\lambda} D_{\lambda,p}^{n+1} f(z) - \left(\frac{1}{\lambda} + p\right) D_{\lambda,p}^{n} f(z)\,, \quad (\lambda > 0). \tag{3.2.3}$$

Let $\mathcal{H}(U)$ be the class of functions which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} : |z| < 1\}\,.$$

For $n \in \mathbb{N} = \{1,2,3, \dots\}$, and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}\,,$$

with $\mathcal{H}_1 = [1,1]$.

In recently years, there are many researchers dealing with the second–order subordination and superodination problems for analytic function for example [8,9,10, 16, 33, 66], therefore in this section we investigate extend to the third–order differential superordination. The first authors investigated the third order, Ponnusamy [57] published in 1992. In 2014, [71] extended the theory of second–order differential superordination in the open unit disk introduced by Miller and Mocanu [48] to the third–order case.They determined properties of functions p that satisfy the following third–order differential superordination:

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in U\}.$$

Recently, the only a few authors are dealing with the third–order differential subordination and superordination for analytic functions in U for example [3, 4,11, 31, 70, 71].

By using the third–order differential superordination results by Tang et al [71] , we define certain classes of admissisble functions and investigate some superordination properties of meromorphic p-valent functions associated with the operator $D_{\lambda,p}^{n}$ defined by (3.2.2).

we consider the class of admissible functions is given in the next definition.

Definition 3.2.1: Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_D[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, x, y; \zeta) \in \Omega\,,$$

whenever

$$u = q(z)\ , \quad v = \frac{zq'(z) + \frac{m}{\lambda} q(z)}{\frac{m}{\lambda}}, \qquad \Re\left\{\frac{x - 2v + u}{\lambda(v - u)}\right\} \leq \frac{1}{m} \Re\left\{\frac{z\, q''(z)}{q'(z)} + 1\right\},$$

and

$$\Re\left\{\frac{y-u-3(x-v)-3\lambda(x-2v+u)}{\lambda^2(v-u)}+2\right\} \leq \frac{1}{m^2}\Re\left\{\frac{z^2q'''(\zeta)}{q'(z)}\right\},$$

where $z \in U,\ \zeta \in \partial U,\ \lambda \in \mathbb{C}\backslash\{0\}$ and $m \geq 2$.

Theorem 3.2.2: Let $\phi \in \Phi_D[\Omega, q]$. If the function $f \in \Sigma_p^*, z^p D_{\lambda,p}^n f(z) \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the following condition:

$$\Re\left(\frac{z\, q''(z)}{q'(z)}\right) \geq 0\,, \qquad \left|\frac{z^p D_{\lambda,p}^{n+1} f(z)}{q'(z)}\right| \leq \frac{1}{m} \tag{3.2.4}$$

and

$$\phi\left(z^p D_{\lambda,p}^n f(z),\ z^p D_{\lambda,p}^{n+1} f(z),\ z^p D_{\lambda,p}^{n+2} f(z),\ z^p D_{\lambda,p}^{n+3} f(z); z\right),$$

is univalent in U, and

$$\Omega \subset \left\{\phi\left(z^p D_{\lambda,p}^n f(z)\,, z^p D_{\lambda,p}^{n+1} f(z)\,, z^p D_{\lambda,p}^{n+2} f(z)\,, z^p D_{\lambda,p}^{n+3} f(z); z\right): z \in U\right\}, \tag{3.2.5}$$

then

$$q(z) \prec z^p D_{\lambda,p}^n f(z)\,, \qquad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = z^p D_{\lambda,p}^n f(z)\,. \tag{3.2.6}$$

In view of the relation (3.2.3), and differentiating (3.2.6) with respect to z, we have

$$\frac{1}{\lambda} z^p D_{\lambda,p}^{n+1} f(z) = \frac{1}{\lambda} p(z) + zp'(z). \tag{3.2.7}$$

Further computations show that

$$\frac{1}{\lambda^2} z^p D_{\lambda,p}^{n+2} f(z) = \frac{1}{\lambda^2} p(z) + \left(\frac{2}{\lambda}+1\right) zp'(z) + z^2p''(z), \tag{3.2.8}$$

and

$$\frac{1}{\lambda^3} z^p D_{\lambda,p}^{n+3} f(z) = \frac{1}{\lambda^3} p(z) + \left(\frac{3}{\lambda^3}+\frac{3}{\lambda}+1\right) zp'(z) + \left(\frac{3}{\lambda}+3\right) z^2p''(z) + z^3p'''(z). \tag{3.2.9}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$u(r,s,t,w)=r, \qquad v(r,s,t,w)=\frac{\frac{r}{\lambda}+s}{\frac{1}{\lambda}}, \qquad x(r,s,t,w)=\frac{\frac{r}{\lambda^2}+\left(\frac{2}{\lambda}+1\right)s+t}{\frac{1}{\lambda^2}} \tag{3.2.10}$$

and

$$y(r,s,t,w)=\frac{\frac{r}{\lambda^3}+\left(\frac{3}{\lambda^3}+\frac{3}{\lambda}+1\right)s+\left(\frac{3}{\lambda}+3\right)t+w}{\frac{1}{\lambda^3}} \tag{3.2.11}$$

Let

$$\begin{aligned}\psi(r,s,t,w;z)&=\phi(u,v,x,y;z)\\ &=\phi\left(r,\frac{\frac{r}{\lambda}+s}{\frac{1}{\lambda}},\frac{\frac{r}{\lambda^2}+\left(\frac{2}{\lambda}+1\right)s+t}{\frac{1}{\lambda^2}},\frac{\frac{r}{\lambda^3}+\left(\frac{3}{\lambda^3}+\frac{3}{\lambda}+1\right)s+\left(\frac{3}{\lambda}+3\right)t+w}{\frac{1}{\lambda^3}};z\right).\end{aligned} \tag{3.2.12}$$

The proof will make use of Theorem 1.3.11. Using equations (3.2.6) to (3.2.9), we find from (3.2.12) that

$$\begin{aligned}&\psi(p(z),\ zp'(z),\ z^2p''(z),\ z^3p'''(z);z)=\\ &\quad\phi\left(z^pD_{\lambda,p}^{n}f(z),z^pD_{\lambda,p}^{n+1}f(z),z^pD_{\lambda,p}^{n+2}f(z),z^pD_{\lambda,p}^{n+3}f(z);z\right).\end{aligned} \tag{3.2.13}$$

Since $\phi\in\Phi_D[\Omega,q]$, from (3.2.13) and (3.2.5) yield

$$\Omega\subset\{\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in U\}.$$

From (3.2.10) and (3.2.11), we have

$$\frac{t}{s}+1=\frac{x-2v+u}{\lambda(v-u)}\ , \qquad \frac{w}{s}=\frac{y-u-3(x-v)-3\lambda(x-2v+u)}{\lambda^2(v-u)}+2\,.$$

Now, we see that the admissible condition for $\phi\in\Phi_D[\Omega,q]$ in Definition 3.2.1 is equivalent to the admissible condition for ψ as given in Definition 1.1.28 with $n=2$. Hence $\psi\in\Psi_2'[\Omega,q]$, and by using (3.2.4) and Theorem 1.3.11, we obtain

$$q(z) \prec p(z) = z^p D_{\lambda,p}^n f(z)\,, \qquad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω, then the class $\Phi_D[h(U), q]$ is written simply as $\Phi_D[h\,, q]$. With proceedings similar as in the previous section, the next result is an immediate consequence of Theorem 3.2.2.

Theorem 3.2.3: Let $\phi \in \Phi_D[h, q]$ and the function h be analytic in U. If the function $f \in \Sigma_p^*, z^p D_{\lambda,p}^n f(z) \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$with $q'(z) \neq 0$ satisfy the condition (3.2.4), and

$$\phi\left(z^p D_{\lambda,p}^n f(z)\,, z^p D_{\lambda,p}^{n+1} f(z)\,, z^p D_{\lambda,p}^{n+2} f(z)\,, z^p D_{\lambda,p}^{n+3} f(z)\,; z\right)$$

is univalent in U,

then

$$h(z) \prec \phi\left(z^p D_{\lambda,p}^n f(z)\,, z^p D_{\lambda,p}^{n+1} f(z)\,, z^p D_{\lambda,p}^{n+2} f(z)\,, z^p D_{\lambda,p}^{n+3} f(z)\,; z\right), \qquad (3.2.14)$$

implies that

$$q(z) \prec z^p D_{\lambda,p}^n f(z)\,, \qquad (z \in U).$$

Theorem 3.2.2 and Theorem 3.2.3 can only be used to obtain subordinations of the third−order differential superordination of the forms (3.2.5) or (3.2.14). The following Theorem proves the existence of the best subordinant of (3.2.14) for a suitable chosen.

Theorem 3.2.4: Let the function h be analytic in U, and let $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ and ψ be given by (3.2.12). Suppose that the differential equation

$$\psi(q(z)\,, zq'(z)\,, z^2 q''(z)\,, z^3 q'''(z); z\,) = h(z)\,, \qquad (3.2.15)$$

has a solution $q(z) \in \mathcal{Q}_1$. If the functions $f \in \Sigma_p^*$, $z^p D_{\lambda,p}^n f(z) \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the condition (3.2.4) and

$$\phi\left(z^p D_{\lambda,p}^n f(z)\,, z^p D_{\lambda,p}^{n+1} f(z)\,, z^p D_{\lambda,p}^{n+2} f(z)\,, z^p D_{\lambda,p}^{n+3} f(z)\,; z\right),$$

is univalent in U, then

$$h(z) \prec \phi\left(z^p D_{\lambda,p}^n f(z)\,, z^p D_{\lambda,p}^{n+1} f(z)\,, z^p D_{\lambda,p}^{n+2} f(z)\,, z^p D_{\lambda,p}^{n+3} f(z)\,; z\right),$$

implies that

$$q(z) \prec z^p D_{\lambda,p}^n f(z)\,, \qquad (z \in U).$$

and $q(z)$ is the best subordinant.

Proof. By applying Theorem 3.2.2, we deduce that q is a subordinant of (3.2.14) Since q satisfies (3.2.15), it is also a solution of (3.2.14). and therefore, q will be subordinanted by all subordinants. Hence $q(z)$ is the best subordinant.

In view of Definition 3.2.1, in the special case when $q(z) = Mz, M > 0$, the class $\Phi_D[\Omega, q]$ of admissible functions, denoted simply by $\Phi_D[\Omega, M]$, is expressed as follows.

Definition 3.2.5: Let Ω be a set in $\mathbb{C}$, $\lambda \in \mathbb{C}\backslash\{0\}$, and $M > 0$. The class of admissible functions $\Phi_D[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{\frac{1}{m}+\frac{1}{\lambda}}{\frac{1}{\lambda}}Me^{i\theta}, \frac{L+\left[\left(\frac{2}{\lambda}+1\right)\frac{1}{m}+\frac{1}{\lambda^2}\right]Me^{i\theta}}{\frac{1}{\lambda^2}},\right.$$
$$\left.\frac{N+3\left(\frac{1}{\lambda}+1\right)L+\left[\left(\frac{3}{\lambda^2}+\frac{3}{\lambda}+1\right)\frac{1}{m}+\frac{1}{\lambda^3}\right]Me^{i\theta}}{\frac{1}{\lambda^3}}; z\right) \in \Omega, \qquad (3.2.16)$$

whenever $z \in U$, $\Re\left(Le^{-i\theta}\right) \geq \left(\frac{1}{m}-1\right)\frac{M}{m}$ and $\Re\left(Ne^{-i\theta}\right) \geq 0$, for all $\theta \in \mathbb{R}$ and $m \geq 2$.

Corollary 3.2.6: Let $\phi \in \Phi_D[\Omega, M]$. If the function $f \in \Sigma_p^*$ satisfies

$$\left|z^p D_{\lambda,p}^{n+1} f(z)\right| \leq \frac{M}{m}, \qquad (m \geq 2, \qquad M > 0),$$

and

$$\Omega \subset \left\{\phi\left(z^p D_{\lambda,p}^{n} f(z), z^p D_{\lambda,p}^{n+1} f(z), z^p D_{\lambda,p}^{n+2} f(z), z^p D_{\lambda,p}^{n+3} f(z); z\right): z \in U\right\},$$

then

$$\left|z^p D_{\lambda,p}^{n} f(z)\right| < M, \qquad M > 0.$$

In the special case when $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\Phi_D[\Omega, M]$ is simply denoted by $\Phi_D[M]$.

Eaxmple 3.2.7: Let $m \geq 2$, $\lambda \in \mathbb{C}\backslash\{0\}$ and $M > 0$. If the function $f \in \Sigma_p^*$ satisfies

$$\left|z^p D_{\lambda,p}^{n+1} f(z)\right| \le \frac{M}{m},$$

and

$$\left|z^p \left(D_{\lambda,p}^{n+3} f(z) - D_{\lambda,p}^{n+2} f(z)\right)\right| \le \left(\left|\frac{1}{\lambda}+1\right|^2 - \frac{1}{2}\left|\frac{2}{\lambda}+3\right|\right)\frac{M}{2},$$

then

$$\left|z^p D_{\lambda,p}^{n} f(z)\right| < M\,, \qquad M > 0.$$

Proof. We define

$$\phi(u,v,x,y;z) = \frac{1}{\lambda^3}y - \frac{1}{\lambda^3}x.$$

Using Corollary 3.2.6 with $\Omega = h(U)$, where,

$$h(z) = \left(\left|\frac{1}{\lambda}+1\right|^2 - \frac{1}{2}\left|\frac{2}{\lambda}+3\right|\right)\frac{M}{2}z.$$

Now we show that $\phi \in \Phi_D[\Omega, M]$. Since

$$\left|\phi\left(Me^{i\theta}, \frac{\frac{1}{m}+\frac{1}{\lambda}}{\frac{1}{\lambda}}Me^{i\theta}, \frac{L+\left[\left(\frac{2}{\lambda}+1\right)\frac{1}{m}+\frac{1}{\lambda^2}\right]Me^{i\theta}}{\frac{1}{\lambda^2}},\right.\right.$$

$$\left.\left.\frac{N+3\left(\frac{1}{\lambda}+1\right)L+\left[\left(\frac{3}{\lambda^2}+\frac{3}{\lambda}+1\right)\frac{1}{m}+\frac{1}{\lambda^3}\right]Me^{i\theta}}{\frac{1}{\lambda^3}};z\right)\right|$$

$$= \left|N + \left(\frac{2}{\lambda}+3\right)L + \left(\frac{1}{\lambda}+1\right)^2\frac{Me^{i\theta}}{m}\right|$$

$$\ge \Re\left(Ne^{-i\theta}\right) + \left|\frac{2}{\lambda}+3\right|\Re\left(Le^{-i\theta}\right) + \left|\frac{1}{\lambda}+1\right|^2\frac{M}{m}$$

$$\ge \left(\left|\frac{1}{\lambda}+1\right|^2 - \frac{1}{2}\left|\frac{2}{\lambda}+3\right|\right)\frac{M}{2},$$

whenever $z \in U$, $\Re\left(Le^{-i\theta}\right) \ge \left(\frac{1}{m}-1\right)\frac{M}{m}$ and $\Re\left(Ne^{-i\theta}\right) \ge 0$, for all $\theta \in \mathbb{R}$ and $m \ge 2$.

Definition 3.2.8: Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_{D,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, x, y; \zeta) \in \Omega\,,$$

whenever

$$u = q(z), v = \lambda\left[\frac{1}{\lambda}q(z) + \frac{zq'(z)}{mq(z)}\right], \Re\left\{\frac{\frac{1}{\lambda}[(x-3u)v + 2u^2]}{v-u}\right\} \leq \frac{1}{m}\Re\left\{\frac{z\,q''(z)}{q'(z)} + 1\right\},$$

and

$$\Re\left\{\left[\frac{\frac{y}{\lambda^4}(xv^2 - xuv - 3uv^2 + 5u^2v - 2u^3) + \frac{y}{\lambda^3}(xuv + 2uv^2 - 4u^2v + 2uv - v^2}{\frac{3u(v-u)^3}{\lambda^3} - \frac{(v-u)}{u}}\right.\right.$$

$$\frac{-u^2 + u^3) + \frac{3u}{\lambda^3}\left(xuv - xv^2 - xv - 2u^2 + u^3 + uv^2 - 2u^2v + 3uv\right) + \frac{y}{\lambda^2}(v-u)^2}{\frac{3u(v-u)^3}{\lambda^3} - \frac{(v-u)}{u}}$$

$$\left.\left.\frac{+\frac{3u}{\lambda^2}(v - 3uv - u + u^2 + 3v^2) - \frac{3xv^2}{\lambda^2} + \frac{2v}{\lambda}(u-v)}{\frac{3u(v-u)^3}{\lambda^3} - \frac{(v-u)}{u}}\right] \times \left[\frac{u}{\lambda}(v-u)\right]^{-1}\right\}$$

$$\leq \frac{1}{m^2}\Re\left\{\frac{z^2q'''(\zeta)}{q'(z)}\right\},$$

where $z \in U, \zeta \in \partial U, \lambda \in \mathbb{C}\backslash\{0\}$ and $m \geq 2$.

Theorem 3.2.9: Let $\phi \in \Phi_{D,1}[\Omega, q]$. If the function $f \in \Sigma_p^*$, $\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)} \in Q_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the following conditions:

$$\Re\left(\frac{z\,q''(z)}{q'(z)}\right) \geq 0\,, \qquad \left|\frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)q'(z)}\right| \leq \frac{1}{m}\,, \tag{3.2.17}$$

and

$$\phi\left(\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}, \frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)}, \frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)}; z\right),$$

is univalent in U, and

$$\Omega \subset \left\{ \phi\left(\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}, \frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)}, \frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)}; z \right) : z \in U \right\}, \tag{3.2.18}$$

then

$$q(z) \prec \frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \qquad (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = \frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}. \tag{3.2.19}$$

Using equation (3.2.3) and differentiating (3.2.19) with respect to z, we have

$$\frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)} = \lambda\left[\frac{1}{\lambda}p(z) + \frac{zp'(z)}{p(z)}\right]. \tag{3.2.20}$$

Further computations show that

$$\frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)} = \lambda\left[\frac{zp'(z)}{p(z)} + \frac{\frac{1}{\lambda}zp'(z) + \frac{zp'(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{z^2p''(z)}{p(z)}}{\frac{1}{\lambda}p(z) + \frac{zp'(z)}{p(z)}}\right] + p(z), \tag{3.2.21}$$

and

$$\frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)}$$

$$= \frac{6(zp'(z))^2 + 3zp'(z)p(z) + 3z^2p''(z) + \frac{3zp'(z)p^2(z)}{\lambda} + \lambda\left[\frac{3(zp'(z))^2}{p(z)} + \right.}{\frac{2zp'(z)p(z)}{\lambda} + zp'(z) + z^2p''(z) + \frac{2(zp'(z))^2}{p(z)} + \lambda\left(\frac{zp'(z)}{p(z)}\right)^2 + \frac{z^3p'(z)p''(z)}{p^2(z)}}$$

$$\frac{\frac{3(zp'(z))^3}{p^2(z)}+\frac{3z^3p'(z)p''(z)}{p(z)}+zp'(z)+3z^2p''(z)+z^3p'''(z)\Big]+\lambda^2\left[\left(\frac{zp'(z)}{p(z)}\right)^2+\right.}{\frac{2zp'(z)p(z)}{\lambda}+zp'(z)+z^2p''(z)+\frac{2(zp'(z))^2}{p(z)}+\lambda\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^3p'(z)p''(z)}{p^2(z)}}$$

$$\frac{\left.\frac{3z^3p'(z)p''(z)}{p^2(z)}+\frac{z^4p'(z)p'''(z)}{p^2(z)}\right]}{\frac{2zp'(z)p(z)}{\lambda}+zp'(z)+z^2p''(z)+\frac{2\left(zp'(z)\right)^2}{p(z)}+\lambda\left(\frac{zp'(z)}{p(z)}\right)^2+\frac{z^3p'(z)p''(z)}{p^2(z)}}.$$

(3.2.22)

We define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$u(r,s,t,w)=r\ ,\qquad v(r,s,t,w)=r+\lambda\left(\frac{s}{r}\right)\ ,$$

$$x(r,s,t,w)=r+\lambda\left[\frac{s}{r}+\frac{\frac{s}{\lambda}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{t}{r}}{\frac{r}{\lambda}+\frac{s}{r}}\right],\quad (3.2.23)$$

and

$y(r,s,t,w)$

$$=\frac{6s^2+3sr+3t+\frac{3sr^2}{\lambda}+\lambda\left[\frac{3s^2}{r}+\frac{3s^3}{r^2}+\frac{3st}{r}+s+3t+w\right]+}{\frac{2sr}{\lambda}+s+t+\frac{2s^2}{r}+\lambda\left(\frac{s}{r}\right)^2+\frac{st}{r^2}}$$

$$\frac{\lambda^2\left[\left(\frac{s}{r}\right)^2+\frac{3st}{r^2}+\frac{sw}{r^2}\right]}{\frac{2sr}{\lambda}+s+t+\frac{2s^2}{r}+\lambda\left(\frac{s}{r}\right)^2+\frac{st}{r^2}}.$$

(3.2.24)

Let

$\psi(r,s,t,w;z)=\phi(u,v,x,y;z)$

$$= \phi\left(r, r+\lambda\left(\frac{s}{r}\right), r+\lambda\left[\frac{s}{r}+\frac{\frac{s}{\lambda}+\frac{s}{r}-\left(\frac{s}{r}\right)^2+\frac{t}{r}}{\frac{r}{\lambda}+\frac{s}{r}}\right], \frac{6s^2+3sr+3t+\frac{3sr^2}{\lambda}+}{\frac{2sr}{\lambda}+s+t+\frac{2s^2}{r}+\lambda\left(\frac{s}{r}\right)^2+\frac{st}{r^2}}\right.$$

$$\left.\frac{\lambda\left[\frac{3s^2}{r}+\frac{3s^3}{r^2}+\frac{3st}{r}+s+3t+w\right]+\lambda^2\left[\left(\frac{s}{r}\right)^2+\frac{3st}{r^2}+\frac{sw}{r^2}\right]}{\frac{2sr}{\lambda}+s+t+\frac{2s^2}{r}+\lambda\left(\frac{s}{r}\right)^2+\frac{st}{r^2}}; z\right). \tag{3.2.25}$$

The proof will make use of Theorem 1.3.11.Using equaitions (3.2.19) to (3.2.22) , and from (3.2.25), we have

$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) =$

$$\phi\left(\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}, \frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)}, \frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)}; z\right). \tag{3.2.26}$$

Since $\phi \in \Phi_{D,1}[\Omega, q]$, it follows from (3.2.26) and (3.2.18) yield

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in U\}.$$

From (3.2.23) and (3.2.24), we see that the admissible condition for $\phi \in \Phi_{D,1}[\Omega, q]$ in Definition 3.2.8 is equivalent to the admissible condition for ψ as given in Definintion 1.1.28 with $n = 2$. Hence $\psi \in \Psi_2'[\Omega, q]$, and by using (3.2.17) and Theorem 1.3.11, we get

$$q(z) \prec p(z) = \frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \qquad (z \in U).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω, then the class $\Phi_{D,1}[h(U), q]$ is written simply as $\Phi_{D,1}[h, q]$. With proceedings similar as in the previous section, the following result is an immediate consequence of Theorem 3.2.9.

Theorem 3.2.10: Let $\phi \in \Phi_{D,1}[h, q]$ and the function h be analytic in U. If the function $f \in \Sigma_p^*$, $\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)} \in \mathcal{Q}_1$ and $q \in \mathcal{H}_1$ with $q'(z) \neq 0$ satisfy the condition (3.2.17),

and

$$\phi\left(\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}, \frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)}, \frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)}; z\right),$$

is univalent in U,

then

$$h(z) \prec \phi\left(\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}, \frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)}, \frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)}; z\right), \tag{3.2.27}$$

implies that

$$q(z) \prec \frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}, \qquad (z \in U).$$

In the particular case $q(z) = 1 + Mz$, $M > 0$, the class $\Phi_{D,1}[\Omega, q]$ of admissible functions in Definition 3.2.8 is simply denoted by $\Phi_{D,1}[\Omega, M]$, is expressed as follows.

Definition 3.2.11: Let Ω be a set in $\mathbb{C}$, $\lambda \in \mathbb{C}\backslash\{0\}$, and $M > 0$. The class of admissible functions $\Phi_{D,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^4 \times \overline{U} \longrightarrow \mathbb{C}$ such that

$$\phi\Bigg(1 + Me^{i\theta}, 1 + Me^{i\theta} + \frac{\lambda Me^{i\theta}}{m(1 + Me^{i\theta})},$$

$$1 + Me^{i\theta} + \lambda\left[\left(\frac{1}{m(1 + Me^{i\theta})} + \frac{1 + Me^{i\theta} + \lambda - \frac{\lambda Me^{i\theta}}{m(1 + Me^{i\theta})}}{m(1 + Me^{i\theta})^2 + \lambda Me^{i\theta}}\right)Me^{i\theta} + \right.$$

$$\left.\frac{\lambda mL}{m(1 + Me^{i\theta})^2 + \lambda Me^{i\theta}}\right], \frac{\left[\frac{6Me^{i\theta}}{m} + 3\left(1 + Me^{i\theta}\right) + \frac{3\left(1 + Me^{i\theta}\right)^2}{\lambda} + \right.}{\left[1 + \frac{2(1 + Me^{i\theta})}{\lambda} + \frac{2Me^{i\theta}}{m(1 + Me^{i\theta})}\right.}$$

$$\frac{\left.\frac{\lambda^2 Me^{i\theta}}{m(1 + Me^{i\theta})^2} + \lambda\left(1 + \frac{3Me^{i\theta}}{m(1 + Me^{i\theta})} + 3\left(\frac{Me^{i\theta}}{m(1 + Me^{i\theta})}\right)^2\right)\right]\frac{Me^{i\theta}}{m} +}{\left. + \frac{\lambda Me^{i\theta}}{m(1 + Me^{i\theta})^2}\right]\frac{Me^{i\theta}}{m} + \left[1 + \frac{Me^{i\theta}}{m(1 + Me^{i\theta})^2}\right]L}$$

$$\frac{\left[1+\lambda+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})}+\frac{\lambda^2 Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]3L+\lambda\left[1+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]N}{\left[1+\frac{2(1+Me^{i\theta})}{\lambda}+\frac{2Me^{i\theta}}{m(1+Me^{i\theta})}+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]\frac{Me^{i\theta}}{m}+\left[1+\frac{Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]L}$$

$$,z)\in\Omega\,,$$

whenever $z\in U$, $\Re\left(Le^{-i\theta}\right)\geq\left(\frac{1}{m}-1\right)\frac{M}{m}$ and $\Re\left(Ne^{-i\theta}\right)\geq 0$, for all $\theta\in\mathbb{R}$ and $m\geq 2$.

Corollary 3.2.12: Let $\phi\in\Phi_{D,1}[\Omega,M]$. If the function $f\in\Sigma_p^*$ satisfies

$$\left|\frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}\right|\leq\frac{M}{m}\,,\qquad(m\geq 2,\qquad M>0),$$

and

$$\Omega\subset\left\{\phi\left(\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)},\frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)},\frac{D_{\lambda,p}^{n+3}f(z)}{D_{\lambda,p}^{n+2}f(z)},\frac{D_{\lambda,p}^{n+4}f(z)}{D_{\lambda,p}^{n+3}f(z)};z\right):z\in U\right\},$$

then

$$\left|\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}-1\right|<M.$$

In the special case $\Omega=q(U)=\{\omega:|\omega-1|<M\}$, the class $\Phi_{D,1}[\Omega,M]$ is simply denoted by $\Phi_{D,1}[M]$.

Example 3.2.13: Let $m\geq 2$, $\lambda\in\mathbb{C}\backslash\{0\}$, and $M>0$. If the function $f\in\Sigma_p^*$ satisfies the following conditions:

$$\left|\frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}\right|\leq\frac{M}{m}\,,$$

and

$$\left|\frac{D_{\lambda,p}^{n+2}f(z)}{D_{\lambda,p}^{n+1}f(z)}-\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}\right|\leq\frac{|\lambda|M}{2(1+M)},$$

then

$$\left|\frac{D_{\lambda,p}^{n+1}f(z)}{D_{\lambda,p}^{n}f(z)}-1\right|<M.$$

Proof. By taking $\phi(u,v,x,y;z)=v-u$, and $\Omega=h(U)$,

where

$$h(z)=\frac{|\lambda|Mz}{2(1+Mz)}\ ,M>0.$$

Using Corollary 3.2.12, we need to show that $\phi\in\Phi_{D,1}[\Omega,M]$.

Since

$$\Bigg|\phi\Big(1+Me^{i\theta},1+Me^{i\theta}+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})},1+Me^{i\theta}+$$

$$\lambda\left[\left(\frac{1}{m(1+Me^{i\theta})}+\frac{1+Me^{i\theta}+\lambda-\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})}}{m(1+Me^{i\theta})^2+\lambda Me^{i\theta}}\right)Me^{i\theta}+\right.$$

$$\left.\frac{\lambda mL}{m(1+Me^{i\theta})^2+\lambda Me^{i\theta}}\right],\frac{\left[\frac{6Me^{i\theta}}{m}+3\left(1+Me^{i\theta}\right)+\frac{3\left(1+Me^{i\theta}\right)^2}{\lambda}\right.}{\left[1+\frac{2(1+Me^{i\theta})}{\lambda}+\frac{2Me^{i\theta}}{m(1+Me^{i\theta})}\right.}$$

$$\frac{\left.+\frac{\lambda^2Me^{i\theta}}{m(1+Me^{i\theta})^2}+\lambda\left(1+\frac{3Me^{i\theta}}{m(1+Me^{i\theta})}+3\left(\frac{Me^{i\theta}}{m(1+Me^{i\theta})}\right)^2\right)\right]\frac{Me^{i\theta}}{m}+}{\left.+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]\frac{Me^{i\theta}}{m}+\left[1+\frac{Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]L}$$

$$\frac{\left[1+\lambda+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})}+\frac{\lambda^2Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]3L+\lambda\left[1+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]N}{\left[1+\frac{2(1+Me^{i\theta})}{\lambda}+\frac{2Me^{i\theta}}{m(1+Me^{i\theta})}+\frac{\lambda Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]\frac{Me^{i\theta}}{m}+\left[1+\frac{Me^{i\theta}}{m(1+Me^{i\theta})^2}\right]L},z\Big)\Bigg|$$

$$= \left|\frac{\lambda M e^{i\theta}}{m(1+Me^{i\theta})}\right|$$

$$\leq \frac{|\lambda| M}{2(1+M)}.$$

Whenever $z \in U$, $\theta \in \mathbb{R}$ and $m \geq 2$. The proof is complete.

3.3 On Fourth-Order Differential Subordination and Superordination Results for Multivalent Analytic Functions

Let $\mathcal{H}(U)$ be the class of functions which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} : |z| < 1\}.$$

For $n \in \mathbb{N} = \{1,2,3,\dots\}$, and $a \in \mathbb{C}$, let $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots\}$, and also let $\mathcal{H}_0 = [0,1]$.

Let $\Sigma_p$denote the class of all analytic functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1,2,3,\dots\}\ . \tag{3.3.1}$$

We consider a linear operator $I_p(n,\lambda)$ on the class Σ_p of multivalent functions by the infinite series

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k\,, \qquad (\lambda > -p). \tag{3.3.2}$$

The operator $I_p(n,\lambda)$ was studied by [9]. It is easily verified from (1.2) that

$$z\left[I_p(n,\lambda)f(z)\right]' = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z)\,. \tag{3.3.3}$$

For several past years, there are many authors introduce and dealing with the theory of second–order differential subordination and superordination for example [8,9,10,16,33,66] recently years, the many authors discussed the theory of third–order differential subordination and superordination for example [11, 3,4,31,69,70,71]. In the present section, we investigate extend to the fourth–order. In 2011, Antonino and Miller [11] extended the theory of second–order differential subordination in the open unit disk introduced by Miller and Mocanu [47] to the third–order case, now, we extend this to fourth–order differential subordination. They determined properties of functions p that satisfy following the fourth–order differential subordination:

$$\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) : z \in U\} \subset \Omega\,.$$

In 2014, Tang et al [71] extended the theory of second−order differential superordination in the open unit disk introduced by Miller and Mocanu [48] to third−order case, now, we extend this to fourth−order differential superordination. They determined properties of functions p that satisfy the following fourth−order differential superordination:

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z): z \in U\}.$$

To prove our main results, we need the basis concepts in theory of the fourth−order.

Definition 3.3.1: Let $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be univalent in U. If the function $p(z)$ is analytic in U and satisfies the following fourth−order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \prec h(z), \qquad (3.3.4)$$

then $p(z)$ is called a solution of the differential subordination. A univalent function $q(z)$ is called a dominant of the solutions of the differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (3.3.4). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (3.3.4) is said to be the best dominant.

Definition 3.3.2: Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}\backslash\{2\}$. The class of admissible functions $\mathrm{A}_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, w, b; z) \notin \Omega,$$

whenever

$$r = q(\zeta) \ , \qquad s = \kappa\zeta q'(\zeta) \ , \qquad \Re\left(\frac{t}{s} + 1\right) \geq \kappa\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

and

$$\Re\left(\frac{w}{s}\right) \geq \kappa^2\Re\left(\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right), \qquad \Re\left(\frac{b}{s}\right) \geq \kappa^3\Re\left(\frac{\zeta^3 q''''(\zeta)}{q'(\zeta)}\right),$$

where $z \in U, \zeta \in \partial U\backslash \mathrm{E}(q)$, and $\kappa \geq n$.

Definition 3.3.3: Let $p \in \mathcal{H}[a, n]$ with $n \in \mathbb{N}\backslash\{2\}$. Also, let $q \in Q(a)$ and satisfy the following conditions:

$$\Re\left(\frac{\zeta^2\, q'''(\zeta)}{q'(\zeta)}\right) \geq 0, \qquad \left|\frac{z^2p''(z)}{q'(\zeta)}\right| \leq \kappa^2 ,$$

where $z \in U, \zeta \in \partial U\backslash \mathrm{E}(q)$ and $\kappa \geq n$. If Ω a set in $\mathbb{C}$, $\psi \in \mathrm{A}_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \in \Omega\,,$$

then

$$p(z) \prec q(z)\,, \quad (z \in U).$$

Definition 3.3.4: Let $\psi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ and the function $h(z)$ be analytic in U. If the functions $p(z)$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z),$$

are univalent in U and if $p(z)$ satisfy the following fourth–order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z), \tag{3.3.5}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (3.3.5). A univalent subordinant $\tilde{q}(z)$ that satisfies the condition $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (3.3.5) is said to be the best subordinant. We note that the best subordinat is unique up to a rolation of U.

Definition 3.3.5: Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{H}[a, n]$ and $q'(z) \neq 0$. The class of admissible functions $\mathrm{A}'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^5 \times \overline{U} \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\psi(r, s, t, w, b; \zeta) \in \Omega\,,$$

whenever

$$r = q(z)\,, \qquad s = \frac{zq'(z)}{m}\,, \qquad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\Re\left(\frac{z\,q''(z)}{q'(z)} + 1\right),$$

and

$$\Re\left(\frac{w}{s}\right) \leq \frac{1}{m^2}\Re\left(\frac{z^2q'''(z)}{q'(z)}\right), \qquad \Re\left(\frac{b}{s}\right) \leq \frac{1}{m^3}\Re\left(\frac{z^3q''''(z)}{q'(z)}\right),$$

where $z \in U, \zeta \in \partial U$, and $m \geq n \geq 3$.

Definitions 3.3.6: Let $q \in \mathcal{H}[a, n]$ and $\psi \in \mathrm{A}'_n[\Omega, q]$. If $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z)$ is univalent in U, and $p \in \mathcal{Q}(a)$ satisfy the following conditions:

$$\Re\left(\frac{z^2q'''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{z^2p''(z)}{q'(z)}\right| \leq \frac{1}{m^2},$$

where $z \in U, \zeta \in \partial U$ and $m \geq n \geq 3$,

then

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) : z \in U\},$$

implies that

$$q(z) \prec p(z), \quad (z \in U).$$

We first define the following class of admissible functions, which are required in proving the differential subordination theorem involving the operator $I_p(n,\lambda)$ defined by (3.3.2).

Definition 3.3.7: Let Ω be a set in $\mathbb{C}$, and let $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\mathrm{B}_I[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u, v, x, y, g; z) \notin \Omega,$$

whenever

$$u = q(\zeta), \;\; v = \frac{\kappa\zeta q'(\zeta) + \lambda q(\zeta)}{p+\lambda}, \;\; \Re\left\{\frac{(p+\lambda)^2x - \lambda^2u}{(p+\lambda)v - \lambda u} - 2\lambda\right\} \geq \kappa\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\},$$

$$\Re\left\{\frac{(p+\lambda)^2[(p+\lambda)y - (3\lambda+3)x] + (2\lambda^3 + 3\lambda^2)u}{(p+\lambda)v - \lambda u} + (3\lambda^2 + 6\lambda + 2)\right\}$$

$$\geq \kappa^2\Re\left\{\frac{\zeta^2q'''(\zeta)}{q'(\zeta)}\right\},$$

and

$$\Re\left\{\frac{(p+\lambda)[(p+\lambda)^3g - (p+\lambda)^2(4\lambda+6)y + (p+\lambda)(8\lambda^2+18\lambda+11)x -}{(p+\lambda)v - \lambda u}\right.$$

$$\left.\frac{(8\lambda^3 + 18\lambda^2 + 22\lambda + 6)v] + (3\lambda^4 + 6\lambda^3 + 11\lambda^2 + 6\lambda)u}{(p+\lambda)v - \lambda u}\right\} \geq \kappa^3\Re\left\{\frac{\zeta^3q''''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in U, \lambda > -p, \zeta \in \partial U \backslash \mathrm{E}(q)$ and $\kappa \geq 3$.

Theorem 3.3.8: Let $\phi \in \mathrm{B}_I[\Omega, q]$. If the functions $f \in \Sigma_p$ and $q \in \mathcal{Q}_0$ satisfy the following conditons:

$$\Re\left(\frac{\zeta^2\, q'''(\zeta)}{q'(\zeta)}\right) \geq 0\,, \qquad \left|\frac{I_p(n+2,\lambda)f(z)}{q'(\zeta)}\right| \leq \kappa^2\,, \tag{3.3.6}$$

and

$$\{\phi\big(I_p(n,\lambda)f(z),\ I_p(n+1,\lambda)f(z),\ I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big): z \in U\} \subset \Omega\,, \tag{3.3.7}$$

then

$$I_p(n,\lambda)f(z) \prec q(z)\,,\ (z \in U).$$

Proof. Define the analytic function $p(z)$ in U by

$$p(z) = I_p(n,\lambda)f(z). \tag{3.3.8}$$

Then, differentiating (3.3.6) with respect to z and using (3.3.3), we have

$$I_p(n+1,\lambda)f(z) = \frac{zp'(z) + \lambda p(z)}{p+\lambda}. \tag{3.3.9}$$

Further computations show that

$$I_p(n+2,\lambda)f(z) = \frac{z^2p''(z) + (2\lambda+1)zp'(z) + \lambda^2 p(z)}{(p+\lambda)^2}\,, \tag{3.3.10}$$

$$I_p(n+3,\lambda)f(z) = \frac{z^3p'''(z) + (3\lambda+3)z^2p''(z) + (3\lambda^2+3\lambda+1)zp'(z) + \lambda^3 p(z)}{(p+\lambda)^3}\,, \tag{3.3.11}$$

and

$$I_p(n+4,\lambda)f(z) = \frac{z^4p''''(z) + (4\lambda+6)z^3p'''(z) + (4\lambda^2+12\lambda+7)z^2p''(z) +}{(p+\lambda)^4}$$
$$\frac{(4\lambda^3+4\lambda^2+4\lambda+1)zp'(z) + \lambda^4 p(z)}{(p+\lambda)^4}. \tag{3.3.12}$$

Define the transformation from $\mathbb{C}^5$ to $\mathbb{C}$ by

$$u(r,s,t,w,b) = r\,,\ \ v(r,s,t,w,b) = \frac{s+\lambda r}{p+\lambda},\ \ x(r,s,t,w,b) = \frac{t + (2\lambda+1)s + \lambda^2 r}{(p+\lambda)^2},$$

$$y(r,s,t,w,b) = \frac{w + (3\lambda + 3)t + (3\lambda^2 + 3\lambda + 1)s + \lambda^3 r}{(p + \lambda)^3},$$

and

$$g(r,s,t,w,b)$$
$$= \frac{b + (4\lambda + 6)w + (4\lambda^2 + 12\lambda + 7)t + (4\lambda^3 + 4\lambda^2 + 4\lambda + 1)s + \lambda^4 r}{(p + \lambda)^4}. \qquad (3.3.13)$$

Let

$$\psi(r,s,t,w,b;z) = \phi(u,v,x,y,g;z)$$
$$= \phi\left(r, \frac{s + \lambda r}{p + \lambda}, \frac{t + (2\lambda + 1)s + \lambda^2 r}{(p + \lambda)^2}, \frac{w + (3\lambda + 3)t + (3\lambda^2 + 3\lambda + 1)s + \lambda^3 r}{(p + \lambda)^3},\right.$$
$$\left.\frac{b + (4\lambda + 6)w + (4\lambda^2 + 12\lambda + 7)t + (4\lambda^3 + 4\lambda^2 + 4\lambda + 1)s + \lambda^4 r}{(p + \lambda)^4}; z\right). \qquad (3.3.14)$$

The proof will make use of Definition 3.3.3. Using equations (3.3.8) to (3.3.12), we have from (3.3.14) that

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) = \phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z),$$
$$I_p(n+2,\lambda)f(z), I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big). \qquad (3.3.15)$$

Hence (3.3.7) becomes

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) \in \Omega.$$

We note that

$$\frac{t}{s} + 1 = \frac{(p + \lambda)^2 x - \lambda^2 u}{(p + \lambda)v - \lambda u} - 2\lambda,$$

$$\frac{w}{s} = \frac{(p + \lambda)^2[(p + \lambda)y - (3\lambda + 3)x] + (2\lambda^3 + 3\lambda^2)u}{(p + \lambda)v - \lambda u} + (3\lambda^2 + 6\lambda + 2),$$

and

$$\frac{b}{s} = \frac{(p + \lambda)[(p + \lambda)^3 g - (p + \lambda)^2(4\lambda + 6)y + (p + \lambda)(8\lambda^2 + 18\lambda + 11)x -}{(p + \lambda)v - \lambda u}$$

$$\frac{(8\lambda^3+18\lambda^2+22\lambda+6)v]+(3\lambda^4+6\lambda^3+11\lambda^2+6\lambda)u}{(p+\lambda)v-\lambda u}.$$

Therefore, the admissibility condition for $\phi \in \mathrm{B}_I[\Omega, q]$ in Definition 3.3.7 is equivalent to the admissibility condition for $\psi \in \mathrm{A}_3[\Omega, q]$ as given in Definition 3.3.2 with $n = 3$. Therefore, by using (3.3.6) and Definition 3.3.3, we obtain

$$p(z) = I_p(n, \lambda) f(z) \prec q(z).$$

The next Corollary is an extension of Theorem 3.3.8 to the case where the behavior of $q(z)$ on ∂U is not known.

Corollary 3.3.9: Let $\Omega \subset \mathbb{C}$, and let the function $q(z)$ be univalent in U with $q(0) = 0$. Let $\phi \in \mathrm{B}_I[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f(z) \in \Sigma_p$ and $q_\rho(z)$ satisfy the following conditions:

$$\Re\left(\frac{\zeta^2 q_\rho'''(\zeta)}{q_\rho'(\zeta)}\right) \geq 0, \qquad \left|\frac{I_p(n+2,\lambda)f(z)}{q_\rho'(\zeta)}\right| \leq \kappa^2, \quad \left(z \in U, \zeta \in \partial U \backslash \mathrm{E}(q_\rho)\right) \quad (3.3.16)$$

and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \in \Omega,$$

then

$$I_p(n, \lambda) f(z) \prec q(z), \qquad (z \in U).$$

Proof. By using Theorem 3.3.8, yields $I_p(n, \lambda) f(z) \prec q_\rho(z)$. Then we obtain the result from $q_\rho(z) \prec q(z)$, $(z \in U)$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω. In this case, the class $\mathrm{B}_I[h(U), q]$ is written as $\mathrm{B}_I[h, q]$. The following two results are immediate consequence of Theorem 3.3.8 and Corollary 3.3.9.

Theorem 3.3.10: Let $\phi \in \mathrm{B}_I[h, q]$. If the function $f \in \Sigma_p$ and $q \in \mathcal{Q}_0$ satisfy the condition (3.3.6) and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \prec h(z), \qquad (3.3.17)$$

then

$$I_p(n,\lambda)f(z) \prec q(z)\,, \qquad (z \in U).$$

Corollary 3.3.11: Let $\Omega \subset \mathbb{C}$ and q be univalent in U with $q(0) = 0$. Let $\phi \in \mathrm{B}_I[h, q_\rho\,]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \Sigma_p$ and q_ρ satisfy the condition (3.3.16), and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \prec h(z)\,, \qquad (3.3.18)$$

then

$$I_p(n,\lambda)f(z) \prec q(z)\,,\ (z \in U).$$

Our next theorem yields the best dominant of the differential subordination (3.3.17).

Theorem 3.3.12: Let the function h be univalent in U. Also let $\phi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ and suppose that the differential equation

$$\phi\left(q(z), \frac{zq'(z) + \lambda q(z)}{p+\lambda}, \frac{z^2q''(z) + (2\lambda+1)zq'(z) + \lambda^2 q(z)}{(p+\lambda)^2},\right.$$
$$\frac{z^3q'''(z) + (3\lambda+3)z^2q''(z) + (3\lambda^2+3\lambda+1)zq'(z) + \lambda^3 q(z)}{(p+\lambda)^3},$$
$$\frac{z^4q''''(z) + (4\lambda+6)z^3q'''(z) + (4\lambda^2+12\lambda+7)z^2q''(z) +}{(p+\lambda)^4}$$
$$\left.\frac{(4\lambda^3+4\lambda^2+4\lambda+1)zq'(z) + \lambda^4 q(z)}{(p+\lambda)^4}.; z\right) = h(z), \quad (3.3.19)$$

has a solution $q(z)$ with $q(0) = 0$ and satisfies the condition (3.3.6). If the function $f \in \Sigma_p$ satisfies condition (3.3.17) and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big),$$

is analytic in U, then

$$I_p(n,\lambda)f(z) \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. By using Theorem 3.3.8, that $q(z)$ is a dominant of (3.3.17). Since $q(z)$ satisfies (3.3.19), it is also a solution of (3.3.17) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the special case $q(z) = Mz$, $M > 0$, and in view of Definition 3.3.7, the class of admissible functions $B_I[\Omega, q]$, denoted by $B_I[\Omega, M]$ is defined below.

Definition 3.3.13: Let Ω be a set in $\mathbb{C}$, and $M > 0$. the class of admissible functions $B_I[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^5 \times U \longrightarrow \mathbb{C}$ that satisfy the admissibility condition:

$$\phi\left(Me^{i\theta}, \frac{\kappa+\lambda}{p+\lambda}Me^{i\theta}, \frac{L+[(2\lambda+1)\kappa+\lambda^2]Me^{i\theta}}{(p+\lambda)^2}, \frac{N+(3\lambda+3)L+}{(p+\lambda)^3}\right.$$

$$\frac{[(3\lambda^2+3\lambda+1)\kappa+\lambda^3]Me^{i\theta}}{(p+\lambda)^3}, \frac{A+(4\lambda+6)N+(4\lambda^2+12\lambda+7)L+}{(p+\lambda)^4}$$

$$\left.\frac{[(4\lambda^3+4\lambda^2+4\lambda+1)\kappa+\lambda^4]Me^{i\theta}}{(p+\lambda)^4}; z\right) \notin \Omega, \quad (3.3.20)$$

where $p > -\lambda, z \in U$, $\Re\left(Le^{-i\theta}\right) \geq (\kappa-1)\kappa M$, $\Re\left(Ne^{-i\theta}\right) \geq 0$ and $\Re\left(Ae^{-i\theta}\right) \geq 0$ for all $\theta \in \mathbb{R}$ and $\kappa \geq 3$.

Corollary 3.3.14: Let $\phi \in B_I[\Omega, M]$. If the function $f \in \Sigma_p$ satisfies the next conditions:

$$\left|I_p(n+2,\lambda)f(z)\right| \leq \kappa^2 M, \quad (\kappa \geq 3\,; M > 0),$$

and

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$

$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \in \Omega,$$

then

$$\left|I_p(n,\lambda)f(z)\right| < M.$$

In the special case $\Omega = q(U) = \{\omega : |\omega| < M\}$, the class $\mathrm{B}_I[\Omega, M]$ is simply denoted by $\Phi_I[M]$.

Example 3.3.15: Let $\kappa \geq 3, \lambda > -p$ and $M > 0$. If the function $f \in \Sigma_p$ satisfies

$$\left|I_p(n+2,\lambda)f(z)\right| \leq \kappa^2 M,$$

and

$$\left|(p+\lambda)^4 I_p(n+4,\lambda)f(z) - \lambda(p+\lambda)^3 I_p(n+3,\lambda)f(z)\right| < (|\lambda^3+\lambda^2+3\lambda+1| + 2|\lambda^2+9\lambda+7|)3M$$

then

$$\left|I_p(n,\lambda)f(z)\right| < M.$$

Proof. Let

$$\phi(u,v,x,y,g;z) = (p+\lambda)^4 g - \lambda(p+\lambda)^3 y, \quad \Omega = h(U),$$

where

$$h(z) = (|\lambda^3+\lambda^2+3\lambda+1| + 2|\lambda^2+9\lambda+7|)3Mz\,, \qquad M > 0.$$

Using Corollary 3.3.14, we need to show that $\phi \in \mathrm{B}_{I,1}[\Omega\,, M]$.
Snice

$$\left|\phi\left(Me^{i\theta}, \frac{\kappa+\lambda}{p+\lambda}Me^{i\theta}, \frac{L+[(2\lambda+1)\kappa+\lambda^2]Me^{i\theta}}{(p+\lambda)^2}, \frac{N+(3\lambda+3)L+[(3\lambda^2+3\lambda+1)\kappa+\lambda^3]Me^{i\theta}}{(p+\lambda)^3},\right.\right.$$
$$\left.\left.\frac{A+(4\lambda+6)N+(4\lambda^2+12\lambda+7)L+[(4\lambda^3+4\lambda^2+4\lambda+1)\kappa+\lambda^4]Me^{i\theta}}{(p+\lambda)^4}; z\right)\right|$$

$$= \left|A+(3\lambda+6)N+(\lambda^2+9\lambda+7)L+(\lambda^3+\lambda^2+3\lambda+1)\kappa Me^{i\theta}\right|$$

$$= \left|Ae^{-i\theta}+(3\lambda+6)Ne^{-i\theta}+(\lambda^2+9\lambda+7)Le^{-i\theta}+(\lambda^3+\lambda^2+3\lambda+1)\kappa M\right|$$

$$\geq$$

$$\Re\left(Ae^{-i\theta}\right) + |3\lambda+6|\Re\left(Ne^{-i\theta}\right) + |\lambda^2+9\lambda+7|\Re\left(Le^{-i\theta}\right) + |\lambda^3+\lambda^2+3\lambda+1|\kappa M$$

$$\geq |\lambda^3+\lambda^2+3\lambda+1|\kappa M+|\lambda^2+9\lambda+7|\kappa(\kappa-1)M$$

$$\geq (|\lambda^3+\lambda^2+3\lambda+1|+2|\lambda^2+9\lambda+7|)3M$$

whenever $z \in U$, $\Re\left(Le^{-i\theta}\right) \geq (\kappa-1)\kappa M$, $\Re\left(Ne^{-i\theta}\right) \geq 0$ and $\Re\left(Ae^{-i\theta}\right) \geq 0$ for all $\theta \in \mathbb{R}$ and $\kappa \geq 3$. The proof is complete.

We obtain fourth–order differential superordination and sandwich–type results for multivalent functions associated with the operator $I_p(n,\lambda)$ defined by (3.3.2). For this aim,the class of admissible functins is given in the following definition.

Definition 3.3.16: Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$. The class of admissible functions $\mathrm{B}_I'[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^5\times\overline{U}\longrightarrow\mathbb{C}$ that satisfy the following admissibility condition:

$$\phi(u,v,x,y,g;\zeta)\in\Omega\,,$$

whenever

$$u=q(z),\quad v=\frac{zq'(z)+mq(z)}{(p+\lambda)m},\quad \Re\left\{\frac{(p+\lambda)^2x-\lambda^2u}{(p+\lambda)v-\lambda u}-2\lambda\right\}\leq\frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)}+1\right\},$$

$$\Re\left\{\frac{(p+\lambda)^2[(p+\lambda)y-(3\lambda+3)x]+(2\lambda^3+3\lambda^2)u}{(p+\lambda)v-\lambda u}+(3\lambda^2+6\lambda+2)\right\}$$

$$\leq\frac{1}{m^2}\Re\left\{\frac{z^2q'''(z)}{q'(z)}\right\},$$

and

$$\Re\left\{\frac{(p+\lambda)[(p+\lambda)^3g-(p+\lambda)^2(4\lambda+6)y+(p+\lambda)(8\lambda^2+18\lambda+11)x-}{(p+\lambda)v-\lambda u}\right.$$

$$\left.\frac{(8\lambda^3+18\lambda^2+22\lambda+6)v]+(3\lambda^4+6\lambda^3+11\lambda^2+6\lambda)u}{(p+\lambda)v-\lambda u}\right\}\leq\frac{1}{m^3}\Re\left\{\frac{z^3q''''(z)}{q'(z)}\right\},$$

where $z\in U,\zeta\in\partial U$, $\lambda>-p$, and $m\geq 3$.

Theorem 3.3.17: Let $\phi\in\mathrm{B}_I'[\Omega,q]$. If the functions $f(z)\in\Sigma_p$ and $I_p(n,\lambda)f(z)$ $\in\mathcal{Q}_0$ satisfy the following conditions:

$$\Re\left(\frac{z^2 q'''(z)}{q'(z)}\right) \geq 0, \qquad \left|\frac{I_p(n+2,\lambda)f(z)}{q'(z)}\right| \leq \frac{1}{m^2}, \tag{3.3.21}$$

$$\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big)$$

is univalent in U, and

$$\Omega \subset \big\{\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) : z \in U\big\}, \tag{3.3.22}$$

then

$$q(z) \prec I_p(n,\lambda)f(z).$$

Proof. Let the functions $p(z)$ be defined by (3.3.8) and ψ by (3.3.14). Since $\phi \in \mathrm{B}'_I[\Omega, q]$. Thus from (3.3.15) and (3.3.22) yield

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z), z^4p''''(z); z) : z \in U\}.$$

In view from (3.3.13) that the admissible condition for $\phi \in \mathrm{B}'_I[\Omega, q]$ in Definiton 3.3.16 is equivalent the admissible condition for ψ as given in Definition 3.3.5 with $n = 3$. Hence $\psi \in \mathrm{A}'_3[\Omega, q]$, and by using (3.3.21) and Definition 3.3.6, we have

$$q(z) \prec p(z) = I_p(n,\lambda)f(z).$$

Therefore, we completes the proof of Theorem 3.3.17.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(U)$ for some conformal mapping $h(z)$ of U onto Ω, in this case the class $\mathrm{B}'_I[h(U), q]$ is written as $\mathrm{B}'_I[h, q]$. The next Theorem is directly consequence of Theorem 3.3.17.

Theorem 3.3.18: Let $\phi \in \mathrm{B}'_I[h, q]$. Also, let the function $h(z)$ be analytic in U. If the function $f \in \Sigma_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ satisfies the condition (3.3.21),

$$\big\{\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) : z \in U\big\},$$

is univalent in U, and

$$h(z) \prec \phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big), \qquad (3.3.23)$$

then

$$q(z) \prec I_p(n,\lambda)f(z).$$

Theorem 3.3.19: Let the function h be analytic in U, and let $\phi : \mathbb{C}^5 \times \overline{U} \longrightarrow \mathbb{C}$ and ψ be given by (3.3.14). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z), z^4q''''(z); z) = h(z), \qquad (3.3.24)$$

has a solution $q(z) \in \mathcal{Q}_0$. If the functions $f \in \Sigma_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_0$ and $q \in \mathcal{H}_0$ with $q'(z) \neq 0$ satisfy the condition (3.3.21),

$$\{\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big): z \in U\},$$

is univalent in U, and

$$h(z) \prec \phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big),$$

then

$$q(z) \prec I_p(n,\lambda)f(z),$$

and $q(z)$ is the best subordinant of (3.3.23).

Proof. The proof is similar to that of Theorem 3.3.12 and it is being omitted here.

By Combining Theorem 3.3.10 and Theorem 3.3.18, we obtain the following sandwich type result.

Corollary 3.3.20: Let the functions $h_1(z)$, $q_1(z)$ be analytic in U and let the function $h_2(z)$ be univalent in U, $q_2(z) \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \mathrm{B}_I[h_2, q_2] \cap \mathrm{B}'_I[h_1, q_1]$. If the function $f \in \Sigma_p$, $I_p(n,\lambda)f(z) \in \mathcal{Q}_0 \cap \mathcal{H}_0$,

$$\{\phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big): z \in U\},$$

is univalent in U, and the conditions (3.3.6) and (3.3.21) are satisfied,

$$h_1(z) \prec \phi\big(I_p(n,\lambda)f(z), I_p(n+1,\lambda)f(z), I_p(n+2,\lambda)f(z),$$
$$I_p(n+3,\lambda)f(z), I_p(n+4,\lambda)f(z); z\big) \prec h_2(z),$$

then

$$q_1(z) \prec I_p(n,\lambda)f(z) \prec q_2(z).$$

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المستخلص

الغرض من هذه الرسالة هو دراسة نتائج التابعية التفاضلية والتابعية التفاضلية العليا في نظرية الدالة الهندسية. هي دراسة التابعية التفاضلية للدوال احادية التكافؤ حيث حصلنا على نتائج التابعية لدالة مشتقة معممة لصنف جديد من الدوال احادية التكافؤ التحليلية في قرص الوحدة المفتوح. ناقشنا أيضاً بعض خواص نتائج الساندوج التفاضلية للدوال متعددة التكافؤ المعرفة بواسطة مؤثر ليو- سرفستافا. تم الحصول على نتائج حول التابعية التفاضلية و التابعية التفاضلية العليا و اشتقينا ايضاً بعض مبرهنات الساندوج. تم دراسة نتائج التابعية من الرتبة الثالثة للدوال احادية التكافؤ الميرمورفية و المعرفة بواسطة المؤثر الخطي و هنا تم الحصول على نتائج جديدة للتابعية التفاضلية من الرتبة الثالثة في قرص الوحدة المثقوب.

تعاملنا ايضاً مع نتائج التابعية التفاضلية العليا من الرتبة الثالثة للدوال متعددة التكافؤ الميرمورفية والمعرفة بواسطة المؤثر الخطي. اشتقينا بعض نتائج التابعية التفاضلية العليا للدوال التحليلية في قرص الوحدة المثقوب من خلال استخدام اصناف اكيدة من الدوال المقبولة (المسموح بها). درسنا ايضاً نتائج التابعية التفاضلية والتابعية التفاضلية العليا من الرتبة الرابعة للدوال التحليلية المتعددة التكافؤ. هنا قدمنا مفهوم جديد وهو التابعية التفاضلية و التابعية التفاضلية العليا من الرتبة الرابعة والمعرفة بواسطة المؤثر الخطي التفاضلي $I_p(n,\lambda)$ في قرص الوحدة المفتوح.

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قسم الرياضيات

دراسة نتائج التابعية التفاضلية والتابعية التفاضلية العليا في نظرية الدالة الهندسية

رسالة

مقدمة إلى مجلس كلية علوم الحاسوب والرياضيات في جامعة القادسية كجزء من متطلبات نيل درجة ماجستيرعلوم في الرياضيات

من قبل

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1438هـ 2017م