Republic of Iraq

Ministry of Higher Education & Scientific Research

University of AL-Qadisiyah

College of Computer Science and Mathematics

Department of Mathematics

On Some Special Topics in The Theory of Univalent and Multivalent Functions

Athesis

Submitted to the Council of the College of Computer Science and Mathematics, University of AL-Qadisiyah as a Partial Fulfilment of the Requirements for the Degree of Master of Science in Mathematics

By

Majida Hameed Majeed Alomrani

(B.Sc., University of AL-Qadisiya, 2014)

Supervised By

Assist. Prof. Dr. Waggas Galib Atshan

2016 A. D.

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بِسۡمِ ٱللَّهِ ٱلرَّحۡمَٰنِ ٱلرَّحِيمِ

ٱقۡرَأۡ بِٱسۡمِ رَبِّكَ ٱلَّذِي خَلَقَ ١ خَلَقَ ٱلۡإِنسَٰنَ مِنۡ عَلَقٍ

٢ ٱقۡرَأۡ وَرَبُّكَ ٱلۡأَكۡرَمُ ٣ ٱلَّذِي عَلَّمَ بِٱلۡقَلَمِ ٤

عَلَّمَ ٱلۡإِنسَٰنَ مَا لَمۡ يَعۡلَمۡ ٥

صدق الله العظيم

من سورة العلق

الاهداء

إلى سيد البرية سيدنا محمد صلى الله عليه وعلى آل بيته الطيبين الطاهرين . . .

إلى شهداء العراق ودمائهم الزكية وكل موتانا وموتى المسلمين . . .

إلى القوات الامنية والحشد الشعبي المقدس . . .

إلى من رباني صغيرة ولما بلغت أشدي الهماني قولاً كريماً أبي وأمي أطال الله في أعمارهم وحفظهم خيمة تضلني لحبهم وحنانهم . . .

إلى كل من أعانني في كتابة اطروحتي من أساتذة وأصدقاء وأخص بالذكر مشرفي وأستاذي (الدكتور وقاص) . . .

إلى كل من علمني حرفاً وملكني عبداً . . .

إليهم جميعاً أهدي ثمرة جهدي هذا متمثلا بقول الشاعر

هديتي لكم من قلبي ومن قلمي

إن الهدايا على مقدار مهديها

لو كان يهدي الإنسان قيمته

لكنت أهدي إليك الدنيا وما فيها

الباحثة

Acknowledgments

Thanks to ALLAH, for his help and for giving me the ability to complete this thesis.

I would like to express my deep gratitude to my supervisor Dr. Waggas Galib Atshan, for his valuable suggestions, extensive discussion, encouragement and continuous guidance.

I also wish to express my thanks to all the staff of Department of Mathematics, College of Computer Sciences and Mathematics, University of Al-Qadisiyah.

I wish to express my deep gratitude to my family for their patience during my study.

Finally, I would also like to extend my thanks to everyone who helped me to complete my study.

Also I would like to express my deep thanks to Ruaa Al-Zobaede.

Supervisor Certification

I certify that this thesis entitled " **On Some Special Topics in The Theory of Univalent and Multivalent Functions"** *was prepared by Majida Hameed Majeed Alomrani under my supervision at the Mathematics Department, College of Computer Sciences and Mathematics , Al-Qadisiyah University as a partial Fulfillment of the requirements for the degree of Master of Science in Mathematics.*

Supervisor

Signature : Waggas

*Date :*26/12/ 2016

Name: Assist. Prof. Dr. Waggas Galib Atshan.

Department of Mathematics.

College of Computer Sciences and Mathematics.

In view of the available recommendation, we forward this thesis for debate by the examining committee.

Head of the Mathematics Department

Signature :

Date : 26/12/2016

Name: Dr. Qusuay Hatim Eghaar

Dept . of Mathematics.

College of Computer Sciences and Mathematics.

Linguistic Supervisor Certification

This is to certify that I read the thesis entitled **"On Some Special Topics in The Theory of Univalent and Multivalent Functions"** *and corrected every grammatical mistakes; therefore, this thesis is qualified for debate.*

Signature:

Name: Asst. Prof. Rajaa M. Flayih

Date : 2 /3/2017

Certification of the Examining Committee

We certify that we have read this thesis " **On Some Special Topics in The Theory of Univalent and Multivalent Functions**" as an examining committee, we examined the study in its content, and that in our opinion it is adequate for the partial fulfillment of requirements for the degree of Master of Science in Mathematics.

Signature:

Name: Dr. Mushtaq Shakir A. Hussein

The Title: Asst. Prof

Date: 23 / 5 / 2017

Chairman

Signature: Kassim

Name: Dr. Kassim Abdul hameed Jassim

The Title: Asst. Prof

Date: 23 / 5 / 2017

Member

Signature: Dheia

Name: Dr. Dheia G. Salih AL-Khafaji

The Title: Asst. Prof

Date: 11 / 6 / 2017

Member

Signature: Waggas

Name: Dr. Waggas Galib Atshan

The Title: Asst. Prof

Date: 11 / 6 / 2017

Member and Supervisor

In view of the available recommendation, I forward this thesis for debate by the examining committee.

Signature:

Name: Asst. Prof. Dr. Hisham Mohammed Ali Hassan

The Title: Dean of the College of Computer Science and Mathematics,

University of Al-Qadisiyah

Date: 15 / 6 / 2017

List of symbols

$\boldsymbol{U}$	**The open unit disk.**
$\mathbb{U}$	Disk (or region).
$\boldsymbol{U}^*$	The Punctured unit disk.
$\overline{\boldsymbol{U}}$	$\{z \in \mathbb{C}: \lvert z\rvert \leq 1\}$.
$\boldsymbol{C}$	The Class of all convex function of order 0.
$\mathbb{C}$	The Complex plane.
$\mathbb{C}^*$	$\mathbb{C}/\{0\}$
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{R}$	The set of real numbers.
$\mathbb{R}^*$	$\mathbb{R} \cup \{\infty\} \cup \{-\infty\} = [-\infty, \infty]$
$\boldsymbol{S}^*$	The Class of all starlike functions of order 0.
$\boldsymbol{S}^*(\boldsymbol{\alpha})$	The Class of all starlike functions of order α in U.
$\boldsymbol{C}(\boldsymbol{\alpha})$	The Class of all convex functions of order α in U.
$\boldsymbol{\mathcal{A}}$	The class of univalent and analytic functions in $\boldsymbol{U}$.
$\boldsymbol{\mu}(\boldsymbol{\lambda}, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{k}, \boldsymbol{\alpha}_1, \boldsymbol{q}, \boldsymbol{s})$	The class of univalent $f \in \mathcal{B}$.
$\boldsymbol{S}$	The class of univalent and meromorphic functions in U^*
$\boldsymbol{\Sigma}(\boldsymbol{\alpha}, \boldsymbol{t})$	The subclass of $\mathcal{S}$.
$\boldsymbol{f} * \boldsymbol{g}$	Hadamard product of f and g.
$\boldsymbol{S}^*(\boldsymbol{\beta})$	The Class of all starlike functions of order β in U.
$\boldsymbol{C}(\boldsymbol{\beta})$	The Class of all convex functions of order β in U.
$\boldsymbol{N}_{\boldsymbol{n},\boldsymbol{\delta}}(\boldsymbol{f})$	The (n, δ) –neighborhood of a function f
$f \prec \boldsymbol{g}$	differential subordination f subordinat to g
$\mathbf{H}(\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{v}, \boldsymbol{\beta}, \boldsymbol{\mu}, \boldsymbol{q})$	The class of multivalent functions with defined by Hadamard prouduct in U
$\boldsymbol{H}(\boldsymbol{p})$	The class of harmonic multivalent functions.
$\boldsymbol{A}_{\boldsymbol{p}}$	The subclass of $H(p)$.
$\boldsymbol{N}_{\boldsymbol{\rho}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\alpha})$	New class of harmonic multivalent functions.
$\boldsymbol{\mathcal{M}p}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\alpha})$	The subclass of $N_\rho(\lambda, \mu, \alpha)$.
$\boldsymbol{I}^{\boldsymbol{n}}(\boldsymbol{g}(\boldsymbol{z}))$	Integral operator.
$\mathbf{clcoA}_{\boldsymbol{p}}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\alpha})$	The closed convex hull of $A_p(\lambda, \mu, \alpha)$.
$\boldsymbol{\vartheta}^{\boldsymbol{\sigma}} \boldsymbol{f}(\boldsymbol{z})$	The Jung – Kim – Srivastava integral operator.
$\boldsymbol{Re}(\boldsymbol{z})$	The real part of complex number z
$\boldsymbol{Im}(\boldsymbol{z})$	The imagery part of complex number z

List of Publications

1) A New Subclass of Univalent Functions Defined by Dziok-Srivastava linear Operator, European Journal of Scientific Research **(EJSR)(UK) (Impact Factor: 074)**, (Accepted for Publication).

2) Some properties of a class of Meromorphic Univalent Functions, European Journal of Scientific Research **(EJSR)(UK) (Impact Factor: 074)**, (Accepted for Publication).

3) Some Subordination properties of Univalent Functions, journal of Advances in Mathematics **(JAM)(India)(Impact Factor: 1.24)**, (Accepted for Publication).

4) On a Subclass of Multivalent Functions Defined by Hadamard product, journal of Advances in Mathematics **(JAM)(India)(Impact Factor: 1.24)**, (Accepted for Publication).

Abstract

The purpose of this thesis is to study some special topics in the theory of univalent and multivalent functions, and study a new subclass $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$ of univalent functions defined by Dziok- Srivastava linear operator. We gave some properties, like, a necessary and sufficient condition for a function f to be in the class $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$, extreme points, Hadamard product techniqes, integral mean, closure theorem, radii of starlikeness, convexity and close - to – convexity. Some properties have been considered of a class of meromorphic univalent functions $\sum(\alpha, t)$. Here, we introduced the new class $\sum(\alpha, t)$ of meromorphic univalent functions with negative coefficients. The research presented some results, like, coefficient inequality, convex linear combination, closure theorem, distortion bounds, extreme points, radius of convexity, neighborhoods of a function $f \in \sum$. we had introduced some subordination properties of univalent functions, we obtain some results. We had also discussed subclass of multivalent functions defined by Hadamard propduct $\mathrm{H}(n, p, \nu, \beta, \mu, q)$. We obtain some properties, like, a necessary and sufficient condition for a function f to be in the class $\mathrm{H}(n, p, \nu, \beta, \mu, q)$, distortion bounds, closure theorem, radius of starlikeness, and convolution properties. Also, a new class have been studied of multivalent harmonic functions defined by integral operator. We obtain some results, like, coefficient bounds, convex combination, integral operator and distortion Theorem.

Chapter One

Some Definitions and Standard Results

Introduction:

In this chapter, we have introduced a list of the definitions of the family of analytic functions, like, univalent, multivalent (p–valent) and related terms used during the study, some examples, applications of conformal map and some basic results of univalent, multivalent (p–valent) functions which are needed in subsequent chapters for research. The detailed proofs and further discussions may be found in standard texts such as Duren [14], Miller and Mocanu [27], Goodman[18] and other references in univalent function theory.

Section One

1.1 Fundamental Definitions

Definition(1.1.1)[27]: A set $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$ and of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in U). \qquad (1.1)$$

Definition(1.1.2)[14]: A function f, analytic in Domain $\mathcal{R}^3$ is said to be univalent there, if it does not take the same value twice, that is, $f(z_1) \neq f(z_2)$ for all pairs of distinct points z_1 and z_2 in $\mathcal{R}^3$. In other words, f is one – to – one (or injective) mapping of $\mathcal{R}^3$ onto another domain, and the class of all univalent functions is denoted by $\mathcal{A}$.

As examples, [4] the function $f(z) = z$ is univalent in U. Also $f(z) = z + \frac{z^n}{n}$ is univalent in U for each positive integer n.

The theory of univalent functions is so much deep, we need certain simplifying assumptions. The most obvious one in the study is to replace the arbitrary domain D by one that is convenient, and is the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Meromorphic function defined as a function f analytic in a domain $D \subset \mathbb{C}$ except for a finite number of poles in D.

The class of all meromorphic univalent functions f in a domain D is denoted by $\sum$.

Definition(1.1.3)[14]:A function f is said to be locally univalent at a point $z_0 \in \mathbb{C}$ if it is univalent in some neighborhood of z_0.

For analytic function f the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0. Both startements can be proved by appeal to Rouche's theorem [1] (Let f and g be analytic inside and on a rectifiable Jordan curve C, with $|g(z)| < |f(z)|$ on C. Then f and $(f + g)$ have the same number of zeros, counted according to multiplicity inside C).

Example(1.1.1)[22]: Consider the domain

$$D = \left\{z \in \mathbb{C}: 1 < |z| < 2, 0 < Arg\ z < \frac{3\Pi}{2}\right\},$$

and let the function $f: D \longrightarrow \mathbb{C}$ given by $f(z) = z^2$.

It is clear that f is analytic on D and locally univalent at every $z_0 \in D$, since $f'(z_0) = 2z_0 \neq 0$ for all $z_0 \in D$.

Definition(1.1.4)[14]: A set $E \subseteq \mathbb{C}$ is said to be starlike with respect to $w_0 \in E$ if the line segment joining w_0 to every other point $w \in E$ lies entirely in E. In a more picturesque language, the requirement is that every point of E be visible from w_0. The set E is said to be convex if it is starlike with respect to each of its points, that is, if the line segment joining any two points of E lies entirely in E.

Definition(1.1.5)[1]: We say that $f \in \mathcal{A}$ is normalized if f satisfies the conditions $f(0) = 0$ and $f'(0) = 1$.

Definition(1.1.6)[14]: A function f is said to be conformal at a point z_0 if it preserves the angle between oriented curves passing through z_0 in magnitude as well as in sense. Geometrically, images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0 both in magnitude and direction. A function $w = f(z)$ is said to be conformal in the domain $D \subseteq \mathbb{C}$, if it is conformal at each point of the domain. An analytic univalent function is called a conformal mapping because of its angle– preserving property.

Definition(1.1.7)[21]: A Möbius transformation, or a bilinear transformation, is a rational function $f: \mathbb{C} \to \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d} \ ,$$

where $a, b, c, d \in \mathbb{C}$ are fixed and $ad - bc \neq 0$.

Example(1.1.2)[25]: Perhaps the most important member of $\mathcal{A}$ is the Koebe function which is given by

$$K(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + \cdots,$$

and maps the unit disk to the complement of the ray $\left(-\infty, -\frac{1}{4}\right]$. This can be verified by writing

$$K(z) = \frac{1}{4}\left(\frac{1 + z}{1 - z}\right)^2 - \frac{1}{4},$$

and noting that $\frac{1+z}{1-z}$ maps the unit disk conformally onto the right half-plane $\{Re\{z\} > 0\}$ (see Fig. (1.1.1)).

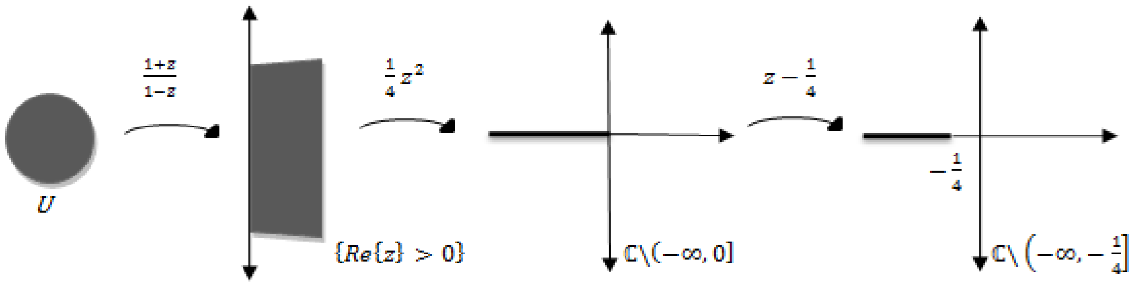

Fig. (1.1.1): The Koebe function maps U conformally onto $\mathbb{C}\backslash\left(-\infty, -\frac{1}{4}\right]$.

We note that $x_1(z) = \frac{1+z}{1-z}$, $x_2(z) = \frac{1}{4}x_1^2(z)$, $x_3(z) = x_2(z) - \frac{1}{4}$.

Now

$$x_3 \circ x_2 \circ x_1(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4} = \frac{z}{(1-z)^2},$$

and x_1 Möbius transformation that maps U onto the right half–plane whose boundary is the imaginary axis. Also, x_2 is the squaring function, while x_3 translates the image one space to the left and then multiplies it by a factor of $\frac{1}{4}$.

Definition (1.2.8)[20]: Let f be a function analytic in the unit disk U. If the equation $f(z) = W$ has never more than p-solution in U, then f is said to be p-valent in U.

The class of all p-valent analytic functions is denoted by $\mathcal{A}(p)$ expressed in one of the following forms:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p}, \qquad (p \in N = \{1,2,3,\dots\}; z \in U), \qquad (1.2)$$

or

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n\, z^n, \qquad (p \in N = \{1,2,3,\dots\}; z \in U), \qquad (1.3)$$

and, let f be a function analytic in the punctured unit disk U^*. If the equation $f(z) = w$ has never more than p-solution in U^*, then f is said to be p-valent in meromorphic U^*.

The class of all p-valent meromorphic functions is denoted by $\mathcal{A}_p^*$, and expressed in one of the following forms:

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} , (p \in N = \{1,2,3, \dots\}), \tag{1.4}$$

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n , (p \in N = \{1,2,3, \dots\}). \tag{1.5}$$

Definition(1.1.9)[14]: A function $f \in \mathcal{A}$ is said to be starlike function of order α if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ \ (f(z) \neq 0, \forall\, z \in U; \ 0 \leq \alpha < 1). \tag{1.6}$$

Denotes the class of all starlike functions of order α in U by $S^*(\alpha)$ and S^* the class of all starlike functions of order 0, $S^*(0) = S^*$. Geometrically, we can say that a starlike function is conformal mapping of the unit disk onto a domain starlike with respect to the origin. For example, the function

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} ,$$

is starlike function of order α.

Definition(1.1.10)[14]: A function $f \in \mathcal{A}$ is said to be convex function of order α if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, (0 \leq \alpha < 1; \ z \in U). \tag{1.7}$$

Denote the class of all convex functions of order α in U by $C(\alpha)$ and C for the class of all convex functions of order 0 by $C(0) = C$.

Definition(1.1.11)[14]: A function $f \in \mathcal{A}$ is said to be close – to – convex of order $\alpha (0 \leq \alpha < 1)$ if there is a convex function $g \in \mathcal{A}$ such that

$$Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \ (g'(z) \neq 0, \forall\, z \in U). \tag{1.8}$$

We denote by $K(\alpha)$, the class of close – to – convex functions of order α.

We note that $C(\alpha) \subset S^*(\alpha) \subset K(\alpha)$.

Note that the Koebe function is starlike, but not convex.

Definition(1.1.12)[10]: Let $\mathcal{A}(p)$ denote the class of analytic p–valently functions in U of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, (\, z \in U; p \in N). \tag{1.9}$$

We say that f is p–valently starlike of order α, p–valently convex of order α and p–valently close– to– convex of order $\alpha (0 \leq \alpha < p)$, respectively if:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \ \ Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ \ Re\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha.$$

Definition(1.1.13)([14],[29]): Let us denote by $\mathcal{A}_p^*$ the class of meromorphic p–valently functions f of the form:

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n, \quad (z \in U^*, p \in N), \tag{1.10}$$

which are p–valent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. We say that f is p–valently meromorphic starlike of order $\alpha (0 \leq \alpha < p)$ if

$$-Re\left(\frac{zf''(z)}{f'(z)}\right) > \alpha, \ (\, z \in U^* \,). \tag{1.11}$$

Also, f is p–valently meromorphic convex of order $\alpha (0 \leq \alpha < p)$ if

$$-Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ (\, z \in U^* \,). \tag{1.12}$$

Note that if $p = 1$, we have defined meromorphic starlike of order $\alpha(0 \le \alpha < 1)$ and meromorphic convex of order $\alpha(0 \le \alpha < 1)$ respectively.

Definition(1.1.14)[14]: Radius of starlikeness of a function f is the largest $r_1, 0 < r_1 < 1$for which it is starlike in $|z| < r_1$.

Definition(1.1.15)[14]: Radius of convexity of a function f is the largest $r_2, 0 < r_2 < 1$ for which it is convex in $|z| < r_2$.

Definition(1.1.16)([31],[33]): The convolution (or Hadamard product) for functions f and g denoted by $f * g$ is defined as following for the functions in $\mathcal{A}(p)$ and $\mathcal{A}^*(p)$ respectively:

(i) If

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n. \tag{1.13}$$

(ii) If

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n\,, \qquad g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n\,,$$

then

$$(f * g)(z) = z^{-p} + \sum_{n=p}^{\infty} a_n b_n z^n. \tag{1.14}$$

Example(1.1.3)[10]: Consider the convolution of the function

$$f(z) = \frac{1}{1-z} = \sum_{n=1}^{\infty} z^n,$$

which maps U onto the half – plane $\left\{\mathrm{Re}(z) > -\frac{1}{2}\right\}$ and the Koebe function (see Example (1.1.2)),

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n.$$

Then

$$f(z) * k(z) = \frac{1}{1-z} * \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} z^n * \sum_{n=1}^{\infty} nz^n$$

$$= (z + z^2 + z^3 + \cdots) * (z + 2z^2 + 3z^3 + \cdots)$$

$$= (z + 2z^2 + 3z^3 + \cdots) = \frac{z}{(1-z)^2}$$

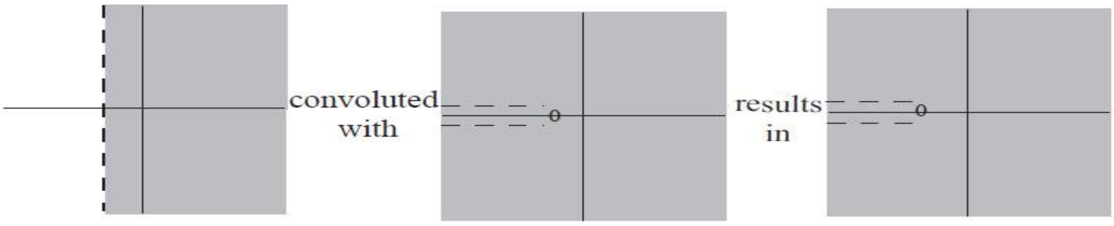

Fig.(1.1.2)

Right half-plane map convoluted with the Koebe function yields the Koebe function.

Definition (1.1.17)[27]: The weighted mean w_j of the functions f and g is defined by

$$w_j(z) = \frac{1}{2}\big((1-j)f(z) + (1+j)g(z)\big), \qquad 0 < j < 1.$$

Also

$$h(z) = \frac{1}{m}\sum_{k=1}^{\infty} f_k(z)$$

is the arithmetic mean of the functions $f_k(z), (k = 1,2,3, \dots, m)$.

Definition(1.1.18)[27]: Let f and g be analytic functions in the unit disk U. Then f is said to be subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w, which is analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g\big(w(z)\big)$ $(z \in U)$. In particular, if the function f is univalent in U, we have the following equivalence relationship holds true:

$$f(z) \prec g(z)(z \in U) \text{ if and only if } f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition(1.1.19)[27]: Let Ω and Δ be any sets in $\mathbb{C}$, let p be an analytic function in the open unit disk U with $p(0) = a$ and let $\psi(r, s, t; z): \mathbb{C}^3 \times U \to \mathbb{C}$. The heart of this monograph deals with the generalizations of the following implication:

$$\{\psi(p(z), zp'(z), z^2p''(z); z): z \in U\} \subset U \implies p(U) \subset \Delta. \qquad (1.15)$$

If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then there is a conformal mapping q of U onto Δ such that $q(0) = a$. In this case, (1.15) can be written as:

$$\{\psi(p(z), zp'(z), z^2p''(z); z): z \in U\} \subset U \implies p(U) \subset q(U).$$

If Ω is also a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of U onto Ω such that $h(0) = \psi(a, 0,0; 0)$.If in addition, the function $\psi(p(z), zp'(z), z^2p''(z); z)$ is analytic in U, then (1.15) can be written as:

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \implies p(z) \prec q(z). \qquad (1.16)$$

Definition(1.1.20)[27]: Let $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and h is univalent in U with $q \in Q$. Miller and Mocanu [28] consider the problem of determining conditions on admissible functions ψ such that

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.17}$$

implies $p(z) \prec q(z)$, for functions $p(z) \in H[a, n]$ that satisfy the differential subordination (1.7), moreover, they found conditions so that q is the smallest function with this property, called the best dominant of the subordination (1.17).

A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.17) is said to be the best dominant of (1.17).

Definition(1.1.21)[27]: Denote by Q the set of all functions q that are analytic and injective on $\overline{U} \setminus E(q)$, where $\overline{U} = U \cup \partial U$, and

$$\mathrm{E}(q) = \{\, \zeta \in \partial U : \lim_{z \to \zeta} q(z) = \infty \}, \tag{1.18}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$ and $Q(1) \equiv Q_1$.

Definition(1.1.22)[25]: A continuous function $f = u + iv$ is said to be a complex–valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in $\mathcal{R}$. If $f = u + iv$ is harmonic, then we can find analytic functions G, H such that $u = Re\ G$ and $v = Im\ H$, thus

$$f = h + \overline{g} = \frac{G + H}{2} + \frac{\overline{G} - \overline{H}}{2},$$

where h and g are analytic in D and we say that h is analytic part and g co–analytic part of f.

Definition(1.1.23)[25]: The harmonic function $f = h + \overline{g}$ is sense–preserving and locally injective if

$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \forall z \in U$, where J_f denotes the Jacobian of f. If $f = h + \overline{g}$ is harmonic, sense–preserving and injective, then we say that f is harmonic univalent.

Definition(1.1.24)[28]: Let $E \subset X$. A point $x \in E$ is called an extreme point of E if it has no representation of the form:

$$x = ty + (1 - t)z, 0 < t < 1$$

as a proper convex combination of two distinct points y and z in E.

Section Two

1.2 Some Standrd Results

In this section, we mention some results, which we have used in our research.

Lemma(1.2.1)[3]: Let $\beta \geq 0, 0 \leq \alpha < 1$ and $\gamma \in \mathbb{R}$. Then, $Re(w) > \beta|w-1| + \alpha$, if and only if $Re\left(w\left(1+\beta e^{i\gamma}\right) - \beta e^{i\gamma}\right) > \alpha$, where w be any complex number.

Lemma(1.2.2)[3]: Let $\beta \geq 0$. Then $Re(w) \geq \beta$ if and only if $|w-(1+\beta)| < |w+(1-\beta)|$, where w be any complex number.

Lemma (1.2.3)[9]: (Schwarz Lemma)

Let f be analytic function in the unit disk U with $f(0) = 0$ and $|f(z)| < 1$ in U. Then $|f'(0)| < 1$ and $|f(z)| < |z|$ in U. Strict inequality holds in both estimates unless f is a rotation of the disk $f(z) = e^{i\theta} z$.

Lemma(1.2.4)[14]: (**Caratheodory's Lemma**)

Let P the class of all functions $\emptyset$ analytic and having positive real part in U with $\emptyset(0) = 1$. If $\emptyset \in P$ and

$$\emptyset(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

then $|c_n| \leq 2, n = 1,2, \ldots$ .This inequality is sharp for each n.

Theorem(1.2.1)[14]: (Bieberbach Conjecture)

The coefficients of each $f \in \mathcal{A}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \ldots$.

The strict inequality holds for all n unless f is the Koebe function or one of its rotation.

Theorem(1.2.2)[14]: (Growth Theorem)

For each $f \in \mathcal{A}$

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, |z| = r < 1. \quad (1.19)$$

For each $z \in U, z \neq 0$ equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem(1.2.3)[14]: (Distortion Theorem)

For each $f \in \mathcal{A}$

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, |z| = r < 1. \quad (1.20)$$

For each $z \in U, z \neq 0$ equality occurs if and only if $f(z)$ is a suitable rotation of the Koebe function.

Theorem(1.2.4)[14]: (Littlewood's Theorem)

For the constant e, the coefficients of each function $f \in \mathcal{A}$ satisfy $|a_n| \leq en$ for $n = 2, 3, \ldots$.

Theorem(1.2.5)[14]: (Alexander's Theorem)

Let f be an analytic function in U with $f(0) = f'(0) - 1 = 0$. Then, f ∈ C if and only if $zf'(z) \in S^*$.

Theorem(1.2.6)[14]: (Maximum Modulus Theorem)

Suppose that a function f is continuous on boundary of $\mathbb{U}$ ($\mathbb{U}$ any disk or region). Then, the maximum value of $|f(z)|$, which is always reached , occurs somewhere on the boundary of $\mathbb{U}$ and never in the interior.

Theorem(1.2.7)[16]: If the functions f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\gamma}(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |g(z)|^{\mu} \, d\theta \tag{1.21}$$

Chapter Two

On Subclasses with Some Subordination Properties of Univalent Functions

Introduction:

The chapter two is devoted for the study of subclasses with subordination properties of univalent functions. This chapter is divided into three sections.

In section one, we have introduced and studied a new subclass $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$ of univalent functions defined by Dziok- Srivastava linear operator of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in N),$$

and satisfying the following condition:

$$Re\left\{\frac{z^3\left(H_{q,s}(\alpha_1)f(z)\right)''' - \lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''}{z^2\left(H_{q,s}(\alpha_1)f(z)\right)'' + (\alpha+1)z\left(H_{q,s}(\alpha_1)f(z)\right)'}\right\} > \gamma,$$

where, $0 < \gamma < 1$, $0 < \lambda < 1, 0 < \alpha < 1$, $q \leq s+1$, $q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We obtain some geometric properties, like, a necessary and sufficient condition for a function f to be in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$, extreme points, Hadamard product techniqes, integral mean, closure theorem, radii of starlikeness, convexity and close - to – convexity.

The section two consists of the discussion some properties of a class of meromorphic univalent functions $\sum(\alpha,t)$. Here, we introduced the new class $\sum(\alpha,t)$ of meromorphic univalent functions with negative coefficients of the form:

$$f(z)=z^{-1}-\sum_{n=1}^{\infty}a_n\,z^n,(a_n\geq 0\,;n\in\mathbb{N}=\{1,2,\ldots\}),$$

and satisfying the following condition:

$$\left|\frac{\frac{tz^2f''(z)}{zf'(z)}+t}{(1+2t)-\frac{2tz^2f''(z)}{zf'(z)}}\right|<\alpha\,,$$

where $0<t<1, 0<\alpha<1$.

We obtained some results, like, coefficient inequality, convex linear combination, closure theorem, distortion bounds, extreme points, radius of convexity, neighborhoods of a function $f\in\sum$.

In section three, we have discussed some subordination properties of univalent functions, we obtain some properties, like, let the function q be univalent in the open unit disk U, $q'(z)\neq 0$ and $z\acute{q}(z)\theta\big(q(z)\big)\neq 0$, is starlike in U. If $f\in\mathcal{A}$ satisfies the subordination:

$$-z\left(\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right) \prec \frac{-zq'(z)}{\delta q(z)},$$ then

$\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta} \prec q(z), (z \in U, \delta \in \mathbb{C}/\{0\})$, and $q(z)$ is the best dominant.

Section One

2.1 On a New Subclass of Univalent Functions Defined by Dziok – Srivastava Linear Operator

Let $\mathcal{B}$ denote the class of functions of the form :

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, n \in \mathbb{N}), \qquad (2.1)$$

which are analytic and univalent in the unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Let the function f given by (2.1) and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n.$$

Then the Hadamard product of f and g denoted by $(f * g)(z) = (g * f)(z)$ is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \qquad (2.2)$$

For $\{\alpha_1, \alpha_2, \dots, \alpha_q\} \subseteq \mathbb{C}$ and $\{\beta_1, \beta_2, \dots, \beta_q\} \subseteq \mathbb{C} - \{0, -1, -2, \dots\}$,

$q \leq s + 1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2, \dots,\}; z \in U, j = \{1,2, \dots\}$, the generalized hypergeomtric function

$$ {}_qF_s\left(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_j,z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n,\dots,(\alpha_n)_n}{(\beta_1)_n\dots(\beta_n)_n n!}z^n, \qquad (2.3)$$

$(q\le s+1,\ q,s\in\mathbb{N}_0)$, where $(\theta)_n$ is the Pochhammer symbol defined in terms of Gamma function Γ, by

$$(\theta)_n=\frac{\Gamma(\theta+n)}{\Gamma(\theta)}=\begin{cases}1 & ,(n=0)\\ \theta(\theta+1)\dots(\theta+n-1) & ,(n\in\mathbb{N}).\end{cases}$$

Dziok – Srivastava linear operator (see [15],[16]),

$H_{q,s}\left(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_j\right):\mathcal{B}\longrightarrow\mathcal{B}$, is defined by the Hadamard product as follows :

$$H_{q,s}\left(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_j\right)=h\left(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_j,z\right)$$

$$= {}_qF_s * f(z)$$

$$=z-\sum_{n=2}^{\infty}\psi_n(\alpha_1)a_nz^n,(n\in\mathbb{N},z\in U),(2.4)$$

where

$$\psi_n(\alpha_1)=\frac{(\alpha_1)_{n-1}\dots(\alpha_s)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_s)_{n-1}(n-1)!}\ . \qquad (2.5)$$

For brevity, we write

$$H_{q,s}\left(\alpha_1,\alpha_2,\dots,\alpha_q;\beta_1,\beta_2,\dots,\beta_j\right)f(z)=H_{q,s}(\alpha_1)f(z).$$

A function $f\in\mathcal{B}$ is said to be in the class $S^*(\alpha)$ if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha\ ,\quad (z\in U,0\le\alpha<1),$$

the elements of this class are called starlike functions of order α.

A function $f \in \mathcal{B}$ is said to be in the class C(α) if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\,, \quad (z\in U, 0\leq\alpha<1),$$

the elements of this class are called convex functions of order α.

Definition (2.1.1): Let $f \in \mathcal{B}$. Then f is in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$ if it satisfies the following condition:

$$Re\left\{\frac{z^3\left(H_{q,s}(\alpha_1)f(z)\right)'''-\lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''}{z^2\left(H_{q,s}(\alpha_1)f(z)\right)''+(\alpha+1)z\left(H_{q,s}(\alpha_1)f(z)\right)'}\right\}>\gamma, \qquad (2.6)$$

where, $0<\gamma<1\,,\ 0<\lambda<1, 0<\alpha<1, q\leq s+1, q,s\in\mathbb{N}_0$.

In The following theorem, we obtain a necessary and sufficient condition for a function f to be in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$.

Theorem(2.1.1): The function $f(z)\in\mathcal{B}$ is said to be in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$ if and only if

$$\sum_{n=2}^{\infty} n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)a_n\leq(\alpha+1), \qquad (2.7)$$

where $\alpha\geq1\,,\ 0\leq\gamma<1\,,\ 0\leq\lambda<1\,,\ n\geq2,$

$$\psi_n(\alpha_1)=\frac{(\alpha_1)_{n-1},\dots,(\alpha_s)_{n-1}}{(\beta_1)_{n-1},\dots,(\beta_s)_{n-1}(n-1)!}\,.$$

Proof: Suppose that the inequality (2.7) holds true and $|z|=1$, in view of (2.6), we need to prove that

$$Re\left\{\frac{z^3\left(H_{q,s}(\alpha_1)f(z)\right)''' - \lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''}{z^2\left(H_{q,s}(\alpha_1)f(z)\right)'' + (\alpha+1)z\left(H_{q,s}(\alpha_1)f(z)\right)'}\right\} > \gamma.$$

Let $A(z) = z^3\left(H_{q,s}(\alpha_1)f(z)\right)''' - \lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''$ and $B(z) = z^2\left(H_{q,s}(\alpha_1)f(z)\right)'' + (\alpha+1)z\left(H_{q,s}(\alpha_1)f(z)\right)'$.

By Lemma (1.2.2), it suffices to show that

$|A(z) - (1+\gamma)B(z)| - |A(z) + (1-\gamma)B(z)| < 0$, for $(0 \le \gamma < 1)$.

But

$|A(z) - (1+\gamma)B(z)|$

$$= \left|\sum_{n=2}^{\infty} n(n-1)[-(n-2)+\lambda]\psi_n(\alpha_1)a_n z^n + (1+\gamma)[\sum_{n=2}^{\infty} n\big((n-1)+(\alpha+1)\big)\psi_n(\alpha_1)a_n z^n - (\alpha+1)z]\right|$$

$$= \left|\sum_{n=2}^{\infty} n\big[(n-1)\big(-(n-2)+\lambda+(1+\gamma)\big) + (1+\gamma)(\alpha+1)\big]\psi_n(\alpha_1)a_n z^n - (1+\gamma)(\alpha+1)z\right|$$

$$= \left|-\sum_{n=2}^{\infty} n[(n-1)\big((n-2)-\lambda-(1+\gamma)\big] - (1+\gamma)(\alpha+1)]\psi_n(\alpha_1)a_n z^n - (1+\gamma)(\alpha+1)z\right|$$

$$\le \sum_{n=2}^{\infty} n\big[(n-1)\big((n-2)-\lambda-(1+\gamma)\big) - (1+\gamma)(\alpha+1)\big]\psi_n(\alpha_1)a_n|z|^n - (1+\gamma)(\alpha+1)|z|,$$

also

$|A(z) + (1-\gamma)B(z)|$

$$= \left|\sum_{n=2}^{\infty} n(n-1)[-(n-2)+\lambda]\psi_n(\alpha_1)a_n z^n + (1-\gamma)[-\sum_{n=2}^{\infty} n\big((n-1)+(\alpha+1)\big)\psi_n(\alpha_1)a_n z^n + (\alpha + 1)z]\right|$$

$$= \left|\sum_{n=2}^{\infty} n[(n-1)(\,-(n-2)+\lambda-(1-\gamma)-(1-\gamma)(\alpha+1)]\psi_n(\alpha_1)a_n z^n + (1-\gamma)(\alpha+1)z\right|$$

$$\geq \sum_{n=2}^{\infty} n[(n-1)(-(n-2)+\lambda-(1-\gamma)-(1-\gamma)(\alpha+1)]\psi_n(\alpha_1)a_n|z|^n + (1-\gamma)(\alpha+1)|z|,$$

and so

$$|A(z)-(1+\gamma)B(z)| - |A(z)+(1-\gamma)B(z)|$$

$$\leq \sum_{n=2}^{\infty} n[(n-1)\;(2(n-2)-2\lambda-2\gamma)-2\gamma(\alpha+1)]\psi_n(\alpha_1)a_n - 2(\alpha+1)$$

$$= 2\sum_{n=2}^{\infty} n\big[(n-1)\,\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)a_n - 2(\alpha+1) \leq 0.$$

This is equivalent to

$$\sum_{n=2}^{\infty} n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)a_n \leq (\alpha+1),$$

by hypothesis. Then by maximum modulus Theorem, we have $f \in \mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$.

Conversely, assume that

$$Re\left\{\frac{z^3\left(H_{q,s}(\alpha_1)f(z)\right)''' - \lambda z^2\left(H_{q,s}(\alpha_1)f(z)\right)''}{z^2\left(H_{q,s}(\alpha_1)f(z)\right)'' + (\alpha+1)z\left(H_{q,s}(\alpha_1)f(z)\right)'}\right\}$$

$$= Re\left\{\frac{\sum_{n=2}^{\infty} n(n-1)[-(n-2)+\lambda]\psi_n(\alpha_1)a_n z^n}{-\sum_{n=2}^{\infty} n\big((n-1)+(\alpha+1)\big)\psi_n(\alpha_1)a_n z^n+(\alpha+1)z}\right\} > \gamma, \quad (2.8)$$

we can choose the value of z on the real axis and let $z \to 1^-$, through real values, so we can write(2.8) as

$$\sum_{n=2}^{\infty} n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)a_n \leq (\alpha+1).$$

Finally, sharpness follows if we take

$$f(z) = z - \frac{(\alpha+1)}{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)} z^n. \quad (2.9)$$

Corollary (2.1.1) : Let $f \in \mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$. Then

$$a_n \leq \frac{(\alpha+1)}{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}, \quad (n \geq 2).$$

In the next theorem, we will find extreme points for the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$.

Theorem(2.1.2): Let $f_1(z) = z$ and $f_n(z) = z - \frac{(\alpha+1)}{n[(n-1)((n-2)-\lambda-\gamma)-\gamma(\alpha+1)]\psi_n(\alpha_1)} z^n$. Then $f(z)$ in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$ if and only if it can be expressed in the form $f(z) = \sum_{n=1}^{\infty}\mu_n f_n(z)$, where μ_n=$\left(\frac{1}{2}\right)^n$, $(n \geq 1)$ and $\sum_{n=1}^{\infty}\mu_n = 1$.

Proof : Assume that $f(z) = \sum_{n=1}^{\infty}\mu_n f_n(z)$

$$= z - \sum_{n=2}^{\infty} \frac{(\alpha+1)\mu_n}{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)} z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{n[(n-1)((n-2)-\lambda-\gamma)-\gamma(\alpha+1)]\psi_n(\alpha_1)}{(\alpha+1)} \mu_n \frac{(\alpha+1)}{n[(n-1)((n-2)-\lambda-\gamma)-\gamma(\alpha+1)]\psi_n(\alpha_1)}$$

$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$

Therefore, $f \in \mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$.

Conversely, assume that $f \in \mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$, then by (2.7), we have

$$a_n \leq \frac{(\alpha+1)}{n[(n-1)((n-2)-\lambda-\gamma)-\gamma(\alpha+1)]\psi_n(\alpha_1)}, \qquad (n \geq 2).$$

Setting

$$\mu_n = \frac{n[(n-1)((n-2)-\lambda-\gamma)-\gamma(\alpha+1)]\psi_n(\alpha_1)}{(\alpha+1)} a_n$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$.

Hence, $f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z^n) = \mu_1 f(z) + \sum_{n=2}^{\infty} \mu_n f_n(z)$.

This completes the proof.

In the following theorem, we obtain the Hadamard product for the function f in the class $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$.

Theorem(2.1.3): Let $f, g \in \mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$. Then $f * g \in \mu(\lambda, \alpha, \ell, \alpha_1, q, s)$ for

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

and

$$f * g(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\ell \geq \frac{(\alpha+1)(n-1)(n-\lambda-2)-n\left[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\right]^2}{(\alpha+1)(n+\alpha)}.$$

Proof : Let $f, g \in \mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$ and so

$$\sum_{n=1}^{\infty} \frac{\left[n\Big((n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\Big)\right]}{(\alpha+1)} \psi_n(\alpha_1) a_n \leq 1 \tag{2.10}$$

and

$$\sum_{n=1}^{\infty} \frac{\left[n\Big((n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\Big)\right]}{(\alpha+1)} \psi_n(\alpha_1) b_n \leq 1. \tag{2.11}$$

We have to find the smallest number ℓ such that

$$\sum_{n=2}^{\infty} \frac{n\left[(n-1)\big((n-2)-\lambda-\ell\big)-\ell(\alpha+1)\right]}{(\alpha+1)} \psi_n(\alpha_1) a_n b_n \leq 1\,. \tag{2.12}$$

By Cauchy – Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{n\left[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\right]}{(\alpha+1)} \psi_n(\alpha_1) \sqrt{a_n b_n} \leq 1. \tag{2.13}$$

Therefore, it is enough to show

$$\frac{n\left[(n-1)\big((n-2)-\lambda-\ell\big)-\ell(\alpha+1)\right]}{(\alpha+1)} \psi_n(\alpha_1) a_n b_n \leq \frac{n\left[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\right]}{(\alpha+1)} \psi_n(\alpha_1) \sqrt{a_n b_n}.$$

That is

$$\sqrt{a_n b_n} \leq \frac{\left[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\right]}{\left[(n-1)\big((n-2)-\lambda-\ell\big)-\ell(\alpha+1)\right]}. \tag{2.14}$$

From (2.12)

$$\sqrt{a_n b_n} \leq \frac{(\alpha+1)}{n\left[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\right]}.$$

Thus, it is enough to show that

$$\frac{(\alpha+1)}{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]}\le\frac{\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]}{\big[(n-1)\big((n-2)-\lambda-\ell\big)-\ell(\alpha+1)\big]},$$

which simplifies to

$$\ell\ge\frac{(\alpha+1)(n-1)(n-\lambda-2)-n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]^2}{(\alpha+1)(n+\alpha)}.$$

Theorem(2.1.4)[23]: If the functions f and g are analytic in U with $f\prec g$, then

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^t d\theta\le\int_0^{2\pi}\left|f\big(re^{i\theta}\big)\right|^t d\theta,\ \ t>0\,,\ z=re^{i\theta},0<r<1. \quad (2.15)$$

Applying (2.7) and Theorem(2.1.4), we prove the following theorem.

Theorem(2.1.5): Let $\tau>0$. If $f(z)\in\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$ and $\{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,n)\}_{n=2}^{\infty}$ are non-decreasing sequences, then for $z=re^{i\theta}$ and $0<r<1$, on has

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^t d\theta\le\int_0^{2\pi}\left|f_2\big(re^{i\theta}\big)\right|^t d\theta,\ \ t>0\,, \quad (2.16)$$

where $f_2(z)=z-\frac{(\alpha+1)}{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,2)}z^2$,

$$\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,n)=n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1).$$

Proof: Let $f(z)$ of the form (2.9) and $f_2(z)=z-\frac{(\alpha+1)}{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,2)}z^2$.

Then, we must show that

$$\int_0^{2\pi}\left|1-\sum_{n=2}^{\infty}a_nz^{n-1}\right|^t d\theta\le\int_0^{2\pi}\left|1-\frac{(\alpha+1)}{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,2)}z\right|^t d\theta.$$

By Theorem(2.1.4), it suffices to show that

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} \prec 1-\frac{(\alpha+1)}{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,2)} z.$$

Setting

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} = 1-\frac{(\alpha+1)}{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,2)} w(z). \qquad (2.17)$$

Form(2.17) and (2.7),we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,n)}{(\alpha+1)} a_n z^{n-1}\right|$$

$$\leq |z| \sum_{n=2}^{\infty} \frac{\sigma(\lambda,\alpha,\gamma,\alpha_1,q,s,k,n)}{(\alpha+1)} a_n$$

$$\leq |z| < 1.$$

This completes the proof of Theorem(2.1.5).

Theorem(2.1.6): Let $\mu_\nu \geq 0$ for $\nu = 1,2,\dots,\iota$ and $\sum_{\nu=1}^{\iota} \mu_\nu \leq 1$, if the functions $F_\nu(z)$ defined by $F_\nu(z) = z - \sum_{n=2}^{\infty} a_{n,\nu} z^n, (\, a_{n,\nu} \geq 0, \nu = 1,2,\dots,\iota),$ (2.18)

are in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$ for every $\nu = 1,2,\dots,\iota$, then the function $f(z)$ defined by

$$f(z) = z - \sum_{n=2}^{\infty}\left(\sum_{\nu=1}^{\iota} \mu_\nu a_{n,\nu}\right) z^n,$$

in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$.

Proof: Since $F_\nu \in \mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$, it follows from Theorem(2.1.1) that

$$\sum_{n=2}^{\infty} n\left[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\right]\psi_n(\alpha_1) a_{n,\nu} \leq (\alpha+1),$$

for every $\nu = 1,2, \dots, \iota$.

Hence

$$\sum_{n=2}^{\infty} n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)\left(\sum_{\nu=1}^{\iota}\mu_\nu a_{n,\nu}\right)$$

$$=\sum_{\nu=1}^{\iota}\mu_\nu\left(\sum_{n=2}^{\infty} n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\,\psi_n(\alpha_1)a_{n,\nu}\right)$$

$$\leq (\alpha+1)\sum_{\nu=1}^{\iota}\mu_\nu \leq (\alpha+1).$$

By Theorem(2.1.1), it follows that $f(z) \in \mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$.

We concentrate upon getting the radii of close – to – convexity, convexity and starlikeness.

Theorem(2.1.7): Let the function $f(z)$ defined by (2.1) be in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$. Then $f(z)$ is close – to – convex of order φ $(0 \leq \varphi < 1)$ in $|z| < r_1(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi)$, where

$$r_1(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi) = inf_{n\geq 2}\left\{\frac{(1-\varphi)n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(\alpha+1)}\right\}^{\frac{1}{n-1}}. \quad (2.19)$$

The result is sharp, with the extermal function $f(z)$ given by (2.9).

Proof: We must show that $|f'(z)-1| \leq 1-\varphi$ for $|z| < r_1(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi)$, where $r_1(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi)$ is given by (2.19). Indeed, we fined from (2.1) that

$$|f'(z)-1| \leq \sum_{n=2}^{\infty} na_n\,|z|^{n-1}.$$

Thus

$$|f'(z)-1| \le 1-\varphi \text{ if } \sum_{n=2}^{\infty}\left(\frac{n}{1-\varphi}\right)a_n\,|z|^{n-1} \le 1. \tag{2.20}$$

But by Theorem(2.1.1), we have

$$\sum_{n=2}^{\infty}\frac{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(\alpha+1)}a_n \le 1. \tag{2.21}$$

Hence (2.20) will be true if

$$\frac{n|z|^{n-1}}{1-\varphi} \le \frac{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(\alpha+1)}$$

equivalently if

$$|z| \le \left\{\frac{(1-\varphi)\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(\alpha+1)}\right\}^{\frac{1}{n-1}}, (n \ge 2). \tag{2.22}$$

The theorem follows from (2.22).

Theorem(2.1.8): Let f defined by (2.1) be in the class $\mu(\lambda,\alpha,\gamma,\alpha_1,q,s)$.

Then f is convex of order φ for $0 \le \varphi < 1$ in $|z| < r_2(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi)$, where

$$r_2(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi) = inf_{n\ge 2}\left\{\frac{(1-\varphi)\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(n-\varphi)(\alpha+1)}\right\}^{\frac{1}{n-1}}. \tag{2.23}$$

The result is sharp with extremal function f given by (2.9).

Proof: We must show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le 1-\varphi \text{ for } |z| < r_2(\lambda,\alpha,\gamma,\alpha_1,q,s,\varphi). \tag{2.24}$$

Substituting the series expansions of $f''(z)$ and $f'(z)$ in the left hand of (2.23), we have

$$\left|\frac{zf''(z)}{f'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \le \frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}.$$

The last expression above is bounded by $(1-\varphi)$ if

$$\sum_{n=2}^{\infty} \frac{n(n-\varphi)}{1-\varphi} a_n z^{n-1} \leq 1. \qquad (2.25)$$

$$\frac{n(n-\varphi)}{1-\varphi} |z|^{n-1} < \frac{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(\alpha+1)},$$

or $|z| < \left\{\frac{(1-\varphi)\left[(n-1)\left((n-2)-\lambda-\gamma\right)-\gamma(\alpha+1)\right]\psi_n(\alpha_1)}{(n-\varphi)(\alpha+1)}\right\}^{\frac{1}{n-1}}, n \geq 2.$ (2.26)

Theorem(2.1.8) follows easily from(2.26).

Theorem(2.1.9): Let f defined by (2.1) be in the class $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$.

Then f is starlike of order φ $(0 \leq \varphi < 1)$ in $|z| < r_3(\lambda, \alpha, \gamma, \alpha_1, q, s, \varphi)$, where

$$r_3(\lambda, \alpha, \gamma, \alpha_1, q, s, \varphi) = inf_{n\geq 2} \left\{\frac{(1-\varphi)n\left[(n-1)\left((n-2)-\lambda-\gamma\right)-\gamma(\alpha+1)\right]\psi_n(\alpha_1)}{(n-\varphi)(\alpha+1)}\right\}^{\frac{1}{n-1}}. \qquad (2.27)$$

The result is sharp with extremal function f given by (2.9).

Proof: It suffices to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\varphi \ \text{ for } |z| < r_3(\lambda, \alpha, \gamma, \alpha_1, q, s, \varphi).$$

We have $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} a_n z^{n-1}}$.

Thus

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\varphi \ \text{ if } \sum_{n=2}^{\infty} \frac{(n-\varphi)}{1-\varphi} a_n z^{n-1} \leq 1. \qquad (2.28)$$

Hence(2.28) will be true if

$$\frac{(n-\varphi)}{1-\varphi} |z|^{n-1} \leq \frac{n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(\alpha+1)},$$

or equivalently

$$|z| \leq \left\{\frac{(1-\varphi)n\big[(n-1)\big((n-2)-\lambda-\gamma\big)-\gamma(\alpha+1)\big]\psi_n(\alpha_1)}{(n-\varphi)(\alpha+1)}\right\}^{\frac{1}{n-1}}, (n \geq 2). \qquad (2.29)$$

Theorem follows easily from(2.29).

Section Two

2.2 Some Properties of a Class of Meromorphic Univalent Functions

Let S denote the class of functions analytic and meromorphic in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U/\{0\}$ and let $\sum$ denote the subclass of S consisting of functions of the form :-

$$f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n, (a_n \geq 0\,; n \in \mathbb{N} = \{1,2,\dots\}) \quad (2.30)$$

which are meromorphic univalent in the punctured unit disk U^*.

A function $f \in \sum$ is said to be meromorphically starlike of order β if

$$Re\left\{\frac{-zf'(z)}{f(z)}\right\} > \beta,\ (z \in U = U^* \cup \{0\}, 0 \leq \beta < 1), \quad (2.31)$$

and a function $f \in \sum$ is said to be meromorphically convex of order β if

$$Re\left\{-(1 + \frac{zf''(z)}{f'(z)})\right\} > \beta, (z \in U = U^* \cup \{0\}, 0 \leq \beta < 1). \quad (2.32)$$

We denote by $S^*(\beta), S(\beta)$, respectively, the classes of univalent meromorphic starlike functions of order β and univalent meromorphically convex functions of order β . Similar classes have been extensively studied by Clunie[13], Miller[26] and Atshan([6],[4]).

Definition(2.2.1): A function $f \in \sum$ is said to be in the class $\sum(\alpha,t)$ if the following condition is satisfied:-

$$\left|\frac{\frac{tz^2f''(z)}{zf'(z)}+t}{(1+2t)-\frac{2tz^2f''(z)}{zf'(z)}}\right| < \alpha\,,\ where\ 0<t<1, 0<\alpha<1\,. \tag{2.33}$$

The following theorem gives a necessary and sufficient condition for a function f to be in the class $\sum(\alpha,t)$.

Theorem(2.2.1): Let $f \in \sum$. Then $f \in \sum(\alpha,t)$ if and only if

$$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_n \leq \alpha+6\mathrm{t}\alpha-\mathrm{t}\,, where\ 0<t<1, 0<\alpha<1. \tag{2.34}$$

The result is sharp for the function f given by

$$f(z)=z^{-1}-\frac{\alpha+6\mathrm{t}\alpha-\mathrm{t}}{n[n-\alpha+2t\alpha n]}z^n\,, n \in \mathbb{N}. \tag{2.35}$$

Proof: Suppose that the inequality (2.34) holds true and $|z|=1$, then, we have

$$|tz^2f''(z)+tzf'(z)|-\alpha|(1+2t)zf'(z)-2tz^2f''(z)|=$$

$$\left|tz^{-1}+\sum_{n=1}^{\infty}n^2a_nz^n\right|-\alpha\left|(1+6t)z^{-1}+\sum_{n=1}^{\infty}n[1-2tn]a_nz^n\right| \leq$$

$$t+\sum_{n=1}^{\infty}n^2a_n-\alpha\,(1+6t)-\alpha\sum_{n=1}^{\infty}n[1-2tn]a_n \leq$$

$$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_n-[\alpha+6\mathrm{t}\alpha-\mathrm{t}] \leq 0\,,$$

by hypothesis. Thus by maximum modulus principle, $f \in \sum(\alpha,t)$.

Conversely, assume that

$$\left|\frac{\frac{tz^2f''(z)}{zf'(z)}+t}{(1+2t)-\frac{2tz^2f''(z)}{zf'(z)}}\right| = \left|\frac{tz^2f''(z)+tzf'(z)}{(1+2t)-2tz^2f''(z)}\right| = \left|\frac{tz^{-1}+\sum_{n=1}^{\infty}n^2a_nz^n}{-(1+6t)z^{-1}+\sum_{n=1}^{\infty}n[1-2tn]a_nz^n}\right| \le \alpha.$$

Since $Re(z) \le |z|$ for all z, we have

$$Re\left\{\frac{tz^{-1}+\sum_{n=1}^{\infty}n^2a_nz^n}{-(1+6t)z^{-1}+\sum_{n=1}^{\infty}n[1-2tn]a_nz^n}\right\} < \alpha. \qquad (2.36)$$

We choose the value of z on the real axis and $z \to 1^-$. Through real values, we obtain inequality (2.34).

Corollary(2.2.1): Let $f \in \sum(\alpha, t)$. Then

$$a_n \le \frac{\alpha+6\mathrm{t}\alpha-\mathrm{t}}{n[n-\alpha+2t\alpha n]}, \quad (n \ge 1). \qquad (2.37)$$

In the next theorem, we show that the class $\sum(\alpha, t)$ is closed under convex linear combination.

Theorem(2.2.2): The class $\sum(\alpha, t)$ is closed under convex linear combination.

Proof: Let $f_1(z) = z^{-1} - \sum_{n=1}^{\infty} a_{n,1}\, z^n$ and $f_2(z) = z^{-1} - \sum_{n=1}^{\infty} a_{n,2}\, z^n$ belong to the class $\sum(\alpha, t)$, for $0 \le \Upsilon \le 1$.

We must show that the function h defined by $h(z) = \Upsilon f_1(z) + (1-\Upsilon) f_2(z) \in \sum(\alpha, t)$.

Since f_1 and $f_2 \in \sum(\alpha, t)$, then by Theorem(2.2.1), we have $\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,1} \le \alpha + 6\mathrm{t}\alpha - \mathrm{t}$, $\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,2} \le \alpha + 6\mathrm{t}\alpha - \mathrm{t}$.

Now, $h(z) = \Upsilon f_1(z) + (1-\Upsilon) f_2(z) = z^{-1} + \sum_{n=1}^{\infty}\left[\Upsilon a_{n,1} + (1-\Upsilon)a_{n,2}\right] z^n.$

Then

$$\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]\left[\Upsilon a_{n,1} + (1-\Upsilon)a_{n,2}\right] = \Upsilon \sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,1} + (1-\Upsilon)\sum_{n=1}^{\infty} n[n-\alpha+2t\alpha n]a_{n,2}$$

$$\le \Upsilon(\alpha+6t\alpha-t) + (1-\Upsilon)(\alpha+6t\alpha-t) = \alpha+6t\alpha-t.$$

Then by Theorem(2.2.1), we have $h(z) \in \sum(\alpha, t)$ and the proof is complete.

Theorem(2.2.3): Let the function f_k defined by

$$f_k(z) = z^{-1} + \sum_{n=1}^{\infty} a_{n,k}\, z^n, \left(a_{n,k} \geq 0, n \in \mathbb{N}, k = 1,2,\dots,\ell\right)$$

be in The class $\sum(\alpha, t)$ for every $k = 1,2,\dots,\ell$. Then the function h defined by

$h(z) = z^{-1} + \sum_{n=1}^{\infty} e_n\, z^n, (e_n \geq 0, n \in \mathbb{N}, n \geq 1)$, also belongs to the class $\sum(\alpha, t)$, where $e_n = \frac{1}{\ell}\sum_{k=1}^{\ell} a_{n,k}$.

Proof: Since $f_k(z) \in \sum(\alpha, t)$, we have

$\sum_{n=1}^{\infty} n[n - \alpha + 2t\alpha n]a_k \leq \alpha + 6\text{t}\alpha - \text{t}$, for every $i = 1,2,\dots,\ell$.

Hence $\sum_{n=1}^{\infty} n[n - \alpha + 2t\alpha n]e_n = \sum_{n=1}^{\infty} n[n - \alpha + 2t\alpha n](\frac{1}{\ell}\sum_{k=1}^{\ell} a_{n,k})$

$$= \frac{1}{\ell}\sum_{k=1}^{\ell}(\sum_{n=1}^{\infty} n[n - \alpha + 2t\alpha n]a_{n,k})$$

$$= \frac{1}{\ell}\sum_{k=1}^{\infty}(\alpha + 6t\alpha - t) = \alpha + 6t - t.$$

Therefore, by Theorem(2.2.1), we have $h \in \sum(\alpha, t)$.

In the following theorem, we prove distortion bounds associated with the class introduced in (2.33).

Theorem(2.2.4): If $f \in \sum(\alpha, t)$, then

$$\frac{1}{r} - \frac{\alpha + 6t\alpha - t}{[1 - \alpha + 2t\alpha]} \leq |f(z)| \leq \frac{1}{r} + \frac{\alpha + 6t\alpha - t}{[1 - \alpha + 2t\alpha]}, (0 < |z| = r < 1). \quad (2.38)$$

The result is sharp for the function f given by (2.35).

Proof: Let $f(z) \in \sum(\alpha, t)$. Then by Theorem(2.2.1), we get

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha + 6\text{t}\alpha - \text{t}}{[1 - \alpha + 2t\alpha]}, \quad (2.39)$$

since $f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n\, z^n$, then

$$|f(z)| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n\, |z|^n \leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n \leq \frac{1}{r} + \frac{\alpha+6\mathrm{t}\alpha-\mathrm{t}}{[1-\alpha+2t\alpha]} r. \qquad (2.40)$$

Similarly

$$|f(z)| \geq \frac{1}{r} - \frac{\alpha+6\mathrm{t}\alpha-\mathrm{t}}{[1-\alpha+2t\alpha]} r. \qquad (2.41)$$

From (2.40) and (2.41), we get (2.38) and the proof is complete.

In the next theorem, we obtain extreme points for the class $\sum(\alpha, t)$.

Theorem(2.2.5): Let $f_0(z) = z^{-1}$ and $f_n(z) = z^{-1} + \frac{\alpha+6\mathrm{t}\alpha-\mathrm{t}}{n[n-\alpha+2t\alpha n]} z^n$. Then $f(z)$ is in the class $\sum(\alpha, t)$ if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} M_n f_n(z)$, where $M_n \geq 0, (n \geq 0)$ and $\sum_{n=0}^{\infty} M_n = 1$.

Proof: Assume that $f(z) = \sum_{n=0}^{\infty} M_n f_n(z) = M_0 z^{-1} + \sum_{n=1}^{\infty} \frac{(\alpha+6\mathrm{t}\alpha-\mathrm{t})M_n}{n[n-\alpha+2t\alpha n]} z^n$.

Then it follows that $\sum_{n=1}^{\infty} \frac{n[n-\alpha+2t\alpha n]}{(\alpha+6\mathrm{t}\alpha-\mathrm{t})} M_n \frac{(\alpha+6\mathrm{t}\alpha-\mathrm{t})}{n[n-\alpha+2t\alpha n]} = \sum_{n=1}^{\infty} M_n \leq 1$.

Therefore $f \in \sum(\alpha, t)$.

Conversely, assume that $f \in \sum(\alpha, t)$, then by(2.34), we have

$$a_n \leq \frac{\alpha + 6\mathrm{t}\alpha - \mathrm{t}}{n[n - \alpha + 2t\alpha n]}, \qquad (n \geq 1).$$

Setting

$M_n = \frac{n[n-\alpha+2t\alpha n]}{(\alpha+6\mathrm{t}\alpha-\mathrm{t})} a_n$ and $M_{0=1-} \sum_{n=1}^{\infty} M_n$.

Hence, $f(z) = \sum_{n=0}^{\infty} M_n f_n(z) = M_0 f(z) + \sum_{n=1}^{\infty} M_n f_n(z)$.

This completes the proof .

In the following theorem, we obtain the radius of convexity for the functions in the class $\sum(\alpha,t)$.

Theorem(2.2.6): Let $f\in\sum(\alpha,t)$. Then f is univalent meromorphic convex of order α $(0\leq\alpha<1)$ in the disk $|z|<R$, where

$$R=\inf_{n\geq1}\left\{\frac{(1-\alpha)[n-\alpha+2t\alpha n]}{(n-\alpha+2)[\propto+6t\alpha-t]}\right\}^{\frac{1}{n-1}}.$$

The result is sharp for the function f given by (2.35).

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right|\leq1-\alpha\quad \text{for}|z|<R. \tag{2.42}$$

But

$$\left|\frac{zf''(z)}{f'(z)}+2\right|=\left|\frac{zf''(z)+2f'(z)}{f'(z)}\right|\leq\frac{\sum_{n=1}^{\infty}n(n+1)a_k|z|^{n-1}}{1-\sum_{n=1}^{\infty}na_n|z|^{n-1}}.$$

Thus, (2.42) will be satisfied if

$$\frac{\sum_{n=1}^{\infty}n(n+1)a_k|z|^{n-1}}{1-\sum_{n=1}^{\infty}na_n|z|^{n-1}}\leq1-\alpha,$$

or if

$$\sum_{n=1}^{\infty}\frac{n(n-\alpha+2)}{1-\alpha}a_n|z|^{n-1}\leq1. \tag{2.43}$$

Since $f\in\sum(\alpha,t)$, we have

$$\sum_{n=1}^{\infty}\frac{n[n-\alpha+2t\alpha n]}{\alpha+6\text{t}\alpha-\text{t}}a_n\leq1.$$

Hence, (2.42) will be true if

$$\frac{n(n-\alpha+2)}{1-\alpha}|z|^{n-1} \leq \frac{n[n-\alpha+2t\alpha n]}{\alpha+6\mathrm{t}\alpha-\mathrm{t}},$$

or equivalently

$$|z| \leq \left\{\frac{(1-\alpha)[n-\alpha+2t\alpha n]}{(n-\alpha+2)[\alpha+6t\alpha-t]}\right\}^{\frac{1}{n-1}}, (n \geq 1),$$

which follows the result.

Next, we determine the inclusion relation involving (n,δ) –neighborhoods. Following the earlier works on neighborhoods of analytic functions by Goodman[19], Ruscheweyh[34] and Raina and Srivastava[32] but for meromorphic function studied by Liu and Srivastava[24] and Atshan[6].

We define the (n,δ) –neighborhoods of a function $f(z) \in \sum$ by

$$N_{n,\delta}(f) = \{g \in \textstyle\sum : g(z) = z^{-1} - \sum_{n=1}^{\infty} b_n z^n \; and \; \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \delta > 0\}. \quad (2.44)$$

Definition(2.2.2): A function $g(z) \in \sum$ is said to be in the class $\sum^{\sigma}(\alpha,t)$ if there exists a function $f(z) \in \sum(\alpha,t)$ such that

$$\left|\frac{g(z)}{f(z)} - 1\right| \leq 1-\sigma, (z \in U, 0 \leq \sigma < 1). \quad (2.45)$$

Theorem(2.2.7): Let $f(z) \in \sum(\alpha,t)$ and

$$\sigma = 1 - \frac{\delta(1-\alpha+2t\alpha)}{1-2\alpha-4t\alpha+t}. \quad (2.46)$$

Then $N_{n,\delta}(f) \subset \sum^{\sigma}(\alpha,t)$.

Proof: Let $g(z) \in N_{n,\delta}(f)$. Then, we have from (2.44) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}).$$

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \quad (n \in \mathbb{N}).$$

Also, since $f(z) \in \sum(\alpha, t)$, we have from Theorem(2.2.1)

$$\sum_{n=1}^{\infty} a_n \leq \frac{\alpha + 6\mathrm{t}\alpha - \mathrm{t}}{1 - \alpha + 2t\alpha},$$

so that

$$\left|\frac{g(z)}{f(z)} - 1\right| = \left|\frac{\sum_{n=1}^{\infty}(a_n - b_n)z^n}{z^{-1} - \sum_{n=1}^{\infty} a_n z^n}\right| \leq \frac{\sum_{n=1}^{\infty}|a_n - b_n|}{1 - \sum_{n=1}^{\infty}|a_n|} \leq \frac{\delta(1-\alpha+2t\alpha)}{1-2\alpha-4t\alpha+t} = 1 - \sigma.$$

Thus, by Definition(2.2.2), $g(z) \in \sum^{\sigma}(\alpha, t)$ for σ given by (2.46).

This completes the proof.

Section Three

2.3 Some Subordination Properties of Univalent Functions

Let $\mathcal{A}$ be denote the class of functions $f(z)$, in the open unit disk U of the form :-

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, \qquad (2.47)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying

$f(0) = f'(0) - 1 = 0.$

A function $f \in \mathcal{A}$ said to be starlike of order β if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \qquad (f(z) \neq 0, z \in U = U^* \cup \{0\}, 0 \leq \beta < 1).$$

Denote this class by $S^*(\beta)$.

A function $f \in \mathcal{A}$ is said to be convex of order β, if

$$Re\left\{\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta, (z \in U, 0 \leq \beta < 1).$$

Denote this class by $C(\beta)$.

Lemma (2.3.1)[27]: Let q be convex univalent function in the open unit disk U and $\psi, t \in \mathbb{C}/\{0\}$ with $Re\left\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{t}\right\}>0$.

If p is analytic in U and $\psi p(z)+tzp' \prec \psi q(z)+tzq'(z)$, then $p(z) \prec q(z)$, and q is the best dominant.

Lemma (2.3.2)[27]: Let q be univalent function in the open unit disk U and θ be analytic in domain D containing $q(U)$. If $zp'(z)\theta\big(p(z)\big) \prec zq'(z)\theta\big(q(z)\big)$, then $p(z) \prec q(z)$, and q is the best dominant.

Lemma (2.3.3)[30]: Let q be convex univalent function in the open unit disk U, and let θ be analytic in domain D containing $q(U)$. Assume that $Re\left\{\theta\big(q(z)\big)+1+\frac{zq''(z)}{q'(z)}\right\}>0$.

If p is analytic in U with $p(0)=q(0)$ and $p(U) \subset U$, and

$zp'(z)+\theta\big(p(z)\big) \prec zq'(z)+\theta\big(q(z)\big)$, then $p(z) \prec q(z)$, and q is the best dominant.

Theorem (2.3.1): Let the function q be univalent in the open unit disk U, $q'(z) \neq 0$ and $zq'(z)\theta\big(q(z)\big) \neq 0$, is starlike in U. If $f \in \mathcal{A}$ satisfies the subordination

$$-z\left(\frac{af'(z)+b\big(zf''(z)+f'(z)\big)}{af(z)+bzf'(z)}\right) \prec \frac{-zq'(z)}{\delta q(z)}, \qquad (2.48)$$

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta} \prec q(z), (z \in U, \delta \in \mathbb{C}/\{0\})$,

and $q(z)$ is the best dominant.

Proof: Define the function p by

$$p(z) = \left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta},\ a+b \neq 0, \tag{2.49}$$

then $zp'(z) = \delta z\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta} . \left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right]$, (2.50)

by using $\theta(w) = \frac{-1}{\delta w}$, it easy observed that $\theta(w)$ is analytic in $\mathbb{C}/\{0\}$.

Then we obtain that $\theta(p(z)) = \frac{-1}{\delta p(z)}$ and $\theta(q(z)) = \frac{-1}{\delta q(z)}$.

From (2.50), we have

$$zp'(z)\theta\big(p(z)\big) = -z\left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right], \tag{2.51}$$

and by (2.48) and (2.51), we get

$$zp'(z)\theta\big(p(z)\big) \prec \frac{-z\acute{q}(z)}{\delta q(z)} = zq'(z)\theta\big(q(z)\big).$$

Therefore, by Lemma (2.3.2), we get $p(z) \prec q(z)$ and then by using (2.49), we obtain the result.

By taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Theorem(2.3.1), we obtain the following corollary:

Corollary(2.3.1): If $f \in \mathcal{A}$ satisfies the subordination

$$-z\left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right] \prec \frac{(B-A)z}{\delta(1+Az)(1+Bz)},$$

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta} \prec \frac{1+Az}{1+Bz}$, and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.

By taking $q(z)=\frac{1+z}{1-z}$ in Theorem(2.3.1), we obtain the following corollary:

Examble(2.3.1): If $f\in\mathcal{A}$ satisfies the subordination

$$-z\left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right]\prec\frac{-2z}{\delta(1-z)(1+z)}\ ,$$

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\prec\frac{1+z}{1-z}$, and $q(z)=\frac{1+z}{1-z}$ is the best dominant.

Theorem(2.3.2): Let the function q be convex univalent in the open unit disk $U,\ q'(z)\neq 0$ and assume that

$$Re\left\{1+\frac{zq''(z)}{q'(z)}+\delta\right\}>0.(z\in U,\delta\in\mathbb{C}/\{0\}). \qquad (2.52)$$

If $f\in\mathcal{A}$ satisfies the subordination

$$t\delta\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\left[1+z\left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right]\right]\prec t\delta q(z)+tzq'(z),$$

(2.53)

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\prec q(z)$, and $q(z)$ is the best dominant.

Proof: Define the function p by

$$p(z)=\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta},(z\in U,\delta\in\mathbb{C}/\{0\}), \qquad (2.54)$$

then $zp'(z)=\delta z\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}.\left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right].$

It can easily observed that

$$t\delta p(z)+tzp'(z)=t\delta\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\left[1+z\left[\frac{af'(z)+b\big(zf''(z)+f'(z)\big)}{af(z)+bzf'(z)}\right]\right].$$

Then by (2.53) and (2.54), we get

$t\delta p(z)+tzp'(z)\prec t\delta q(z)+tzq'(z).$

By setting $\psi=t\delta$ in Lemma (2.3.1), we get $p(z)\prec q(z)$. By using (2.54), we obtain the result.

By taking $q(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$ in Theorem(2.3.2), we obtain the result.

Corollary(2.3.3): If $f\in\mathcal{A}$ and assume that $Re\left\{\frac{1-Bz}{1+Bz}+2\delta\right\}>0$. If f satisfies the subordination

$$t\delta\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\left[1+z\left[\frac{af'(z)+b\big(zf''(z)+f'(z)\big)}{af(z)+bzf'(z)}\right]\right]\prec t\delta\frac{1+Az}{1+Bz}+\frac{t(A-B)z}{(1+Bz)^2}\ ,$$

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\prec\frac{1+Az}{1+Bz}$, and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant.

Corollary(2.3.4): If $f\in\mathcal{A}$ and assume that $Re\left\{\frac{1-Bz}{1+Bz}+2\delta\right\}>0$. If f satisfies the subordination

$$t\delta\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\left[1+z\left[\frac{af'(z)+b\big(zf''(z)+f'(z)\big)}{af(z)+bzf'(z)}\right]\right]\prec$$

$$\frac{\propto\left[t\delta(1-z)(\alpha-z)-\big(t(\alpha-z)+(1-z)\big)z\right]}{(\alpha-z)^2},$$

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\prec\frac{\propto(1-z)}{(\propto-z)}$, and $q(z)=\frac{\propto(1-z)}{(\propto-z)}$ is the best dominant.

By taking $q(z)=e^{\gamma Az}$ in Theorem(2.3.2), we obtain the following corollary:

Corollary(2.3.5): If $f \in \mathcal{A}$ and assume that $Re\{1+\gamma Az+2\delta\}>0$. If f satisfies the subordination.

$$t\delta\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\left[1+z\left[\frac{af'(z)+b\left(zf''(z)+f'(z)\right)}{af(z)+bzf'(z)}\right]\right]\prec(\delta+\gamma Az)te^{\gamma Az},$$

then $\left[\frac{af(z)+bzf'(z)}{a+b}\right]^{\delta}\prec e^{\gamma Az}$, and $q(z)=e^{\gamma Az}$ is the best dominant.

Chapter Three

Some Geometric Properties of New Subclasses of Multivalent and Multivalent Harmonic Functions

Introduction:

Chapter three is fully devoted for the study of some geometric properties of new subclasses of multivalent and multivalent harmonic functions.

This chapter is divided into two sections. In section one, we discuss a subclass of multivalent functions defined by Hadamard product $H(n,p,\nu,\beta,\mu,q)$ of the form :

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k\, z^k, (a_k \geq 0\,; k \geq n+p; n,p \in \mathbb{N} = \{1,2,\dots\})$$

and satisfying the condition:

$$Re\left\{\frac{z((f*g)(z))^{q+2} + \nu z^2((f*g)(z))^{q+3}}{(1-\nu)((f*g)(z))^{q+1} + \nu z((f*g)(z))^{q+2}}\right\}$$
$$\geq \beta\left|\frac{z((f*g)(z))^{q+2} + \nu z^2((f*g)(z))^{q+3}}{(1-\nu)((f*g)(z))^{q+1} + \nu z((f*g)(z))^{q+2}} - 1\right| + \mu,$$

where $0 \leq \mu < 1, p > q, n \in \mathbb{N}, 0 < \nu < 1, q \in \mathbb{N}_0 = \{0,1,2,\dots\}, \beta \geq 0.$

We obtain some properties, like, a necessary and sufficient condition for a function f to be in the class $H(n,p,\nu,\beta,\mu,q)$, distortion bounds, closure theorem, radius of starlikeness, and convolution properties. In section two, we have introduced a new class of multivalent harmonic functions defined by integral operator. We obtain some results, like, coefficient bounds, convex combination, integral operator and distortion theorem. Several authors studied multivalently harmonic functions, like, Atshan, Kulkami and Raina[5] studied a class of multivalent harmonic functions involving a generalized Ruscheweyh type operator. Ahuja and Jahangiri[2] studied linear combinations of a class of multivalently harmonic functions.

Section One

3.1 On a Subclass of Multivalent Functions Defined by Hadamard product

Let $W(p,n)$ denote the class of functions of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, (a_k \geq 0\, ; k \geq n+p; n, p \in \mathbb{N} = \{1,2,\dots\}), \tag{3.1}$$

which are analytic and multivalent in the open unit disk U=$\{z \in \mathbb{C}: |z| < 1\}$. If $f \in W(p,n)$ is given by (3.1) and $g \in W(p,n)$ given by

$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$,$(b_k \geq 0)$, then the Hadamard product $f * g$ of f and g is defined by

$$(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k. \tag{3.2}$$

Definition(3.1.1): Let f and g be given by (3.1) , is said to be in the class $H(n,p,\nu,\beta,\mu,q)$ if and only if satisfies the inequality:-

$$Re\left\{\frac{z((f*g)(z))^{q+2}+\nu z^2((f*g)(z))^{q+3}}{(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}}\right\} \geq \beta \left|\frac{z((f*g)(z))^{q+2}+\nu z^2((f*g)(z))^{q+3}}{(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}} - 1\right| + \mu \text{ ,} \tag{3.3}$$

where $0 \leq \mu < 1, p > q, n \in \mathbb{N}, 0 < \nu < 1, q \in \mathbb{N}_0 = \{0,1,2,\dots\}, \beta \geq 0$ and for each $f \in W(p,n)$, we have

$$f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q) a_k z^{k-q}, \tag{3.4}$$

$$\delta(i,j)=\frac{i!}{(i-j)}=\begin{cases}1 & ,\ (j=0)\\ i(i-1)\ldots(i-j+1), & (j\neq 0)\end{cases} \qquad (3.5)$$

The following theorem gives a necessary and sufficient condition for a function f to be in the class $H(n,p,\nu,\beta,\mu,q)$.

Theorem(3.1.1): Let the function f be in the form (3.1), then f is in the class $H(n,p,\nu,\beta,\mu,q)$ if and only if

$$\sum_{k=n+p}^{\infty}\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]a_k b_k \leq \delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu], \qquad (3.6)$$

where $p,n\in\mathbb{N}, q\in\mathbb{N}_0, k\geq n+p, 0\leq\mu<1, p>q, 0<\nu<1, \beta\geq 0$.

The result is sharp for the function f given by

$$f(z)=z^p-\frac{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k}z^k, (p,n\in\mathbb{N}, q\in\mathbb{N}_0, p>q, z\in U). \qquad (3.7)$$

Proof: Let $f\in H(n,p,\nu,\beta,\mu,q)$. Then f satisfies the inequality (3.3) which is equivalent to

$$Re\left\{\frac{z((f*g)(z))^{q+2}+\nu z^2((f*g)(z))^{q+3}}{(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}}\left(1+\beta e^{i\varphi}\right)-\beta e^{i\varphi}\right\}\geq\mu,$$

by using Lemma (1.2.2)

$$Re\left\{\frac{\left[z((f*g)(z))^{q+2}+\nu z^2\left((f*g)(z)\right)^{q+3}\right]\left(1+\theta e^{i\varphi}\right)-\theta e^{i\varphi}\left[(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}\right]}{(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}}\right\}\geq\mu. \qquad (3.8)$$

Let $g(z)=[z((f*g)(z))^{q+2}+\nu z^2((f*g)(z))^{q+3}]\left(1+\theta e^{i\varphi}\right)-\theta e^{i\varphi}[(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}]$

$$h(z)=(1-\nu)((f*g)(z))^{q+1}+\nu z((f*g)(z))^{q+2}\ .$$

Then (3.8) is equivalent to

$|g(z)+(1-\mu)h(z)| \geq \ |g(z)-(1+\mu)h(z)|$ for $0 \leq \mu < p-q$.

$$|g(z)+(1-\mu)h(z)| = \left|\left[\frac{p!\,(p-q)(p-q-1)}{(p-q)!}z^{p-q-1} - \right.\right.$$

$$\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)}{(k-q)!}a_k b_k z^{k-q-1} + \frac{p!\,(p-q)(p-q-1)(p-q-2)}{(p-q)!}\nu z^{p-q-1}$$

$$\left. - \sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)(k-q-2)}{(k-q)!}\nu a_k b_k z^{k-q-1}\right]\left(1+\theta e^{i\varphi}\right)$$

$$-\theta e^{i\varphi}\left[\frac{p!\,(p-q)}{(p-q)!}(1-\nu)z^{p-q-1} - \sum_{k=n+p}^{\infty}\frac{k!\,(k-q)}{(k-q)!}(1-\nu)a_k b_k z^{k-q-1}\right.$$

$$\left. + \frac{p!\,(p-q)(p-q-1)}{(p-q)!}\nu z^{p-q-1} - \sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)}{(k-q)!}a_k b_k z^{k-q-1}\right]$$

$$+(1\ -\mu)\left[\tfrac{p!(p-q)}{(p-q)!}(1-\nu)z^{p-q-1} - \textstyle\sum_{k=n+p}^{\infty}\tfrac{k!(k-q)}{(k-q)!}(1-\nu)a_k b_k z^{k-q-1}\right.$$

$$\left.\left. + \tfrac{p!(p-q)(p-q-1)}{(p-q)!}\nu z^{p-q-1} - \textstyle\sum_{k=n+p}^{\infty}\tfrac{k!(k-q)(k-q-1)}{(k-q)!}\nu a_k b_k z^{k-q-1}\right]\right|$$

$$\geq \tfrac{p!(p-q)}{(p-q)!}\big[(p-q-1)\left(1+\theta e^{i\varphi}\right) + (p-q-1)(p-q-2)\nu\left(1+\theta e^{i\varphi}\right)$$

$$-(1-\nu)\theta e^{i\varphi} - (p-q-1)\nu\theta e^{i\varphi} + (1-\nu)(1-\mu) + (p-q-1)\nu(1-\mu)\big]|z|^{p-q-1}$$

$$-\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)}{(k-q)!}\big[(k-q-1)(1+\theta e^{i\varphi}) + (k-q-1)(k-q-2)\nu(1+\theta e^{i\varphi})$$

$$-(1-\nu)\theta e^{i\varphi} - (k-q-1)\nu\theta e^{i\varphi} + (1-\nu)(1-\mu) + (k-q-1)\nu(1-\mu)\big]a_k b_k |z|^{k-q-1}.$$

Similarity,

$$|g(z)-(1+\mu)h(z)| = \left|\left[\frac{p!\,(p-q)(p-q-1)}{(p-q)!}z^{p-q-1}\right.\right.$$

$$-\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)}{(k-q)!}a_k b_k z^{k-q-1}+\frac{p!\,(p-q)(p-q-1)(p-q-2)}{(p-q)!}\Bigg]vz^{p-q-1}$$

$$-\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)(k-q-2)}{(k-q)!}va_k b_k z^{k-q-1}]\left(1+\theta e^{i\varphi}\right)$$

$$-\theta e^{i\varphi}\left[\frac{p!\,(p-q)}{(p-q)!}(1-v)z^{p-q-1}-\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)}{(k-q)!}a_k b_k z^{k-q-1}\right]$$

$$-(1+\mu)\Bigg[\frac{p!\,(p-q)(p-q-1)}{(p-q)!}vz^{p-q-1}-\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)}{(k-q)!}(1-v)a_k b_k z^{k-q-1}$$

$$+\frac{p!\,(p-q)(p-q-1)}{(p-q)!}vz^{p-q-1}-\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)(k-q-1)}{(k-q)!}a_k b_k z^{k-q-1}\Bigg]\Bigg|$$

$$\leq\frac{p!\,(p-q)}{(p-q)!}[(p-q-1)(1+\theta e^{i\varphi})+(p-q-1)(p-q-2)v(1+\theta e^{i\varphi})-(1-v)\theta e^{i\varphi}$$

$$-(p-q-1)v\theta e^{i\varphi}-(1-v)(1+\mu)-(p-q-1)v(1+\mu)]|z|^{p-q-1}$$

$$+\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)}{(k-q)!}[(k-q-1)(1+\theta e^{i\varphi})+(k-q-1)(k-q-2)v(1+\theta e^{i\varphi})$$

$$-(1-v)\theta e^{i\varphi}-(k-q-1)v\theta e^{i\varphi}-(1-v)(1+\mu)-(k-q-1)v(1+\mu)]a_k b_k|z|^{k-q-1}.$$

Therefore,

$$|g(z)+(1-\mu)h(z)|-|g(z)-(1+\mu)h(z)|=$$

$$\frac{2p!\,(p-q)}{(p-q)!}[(1-v)+(p-q-1)v]-2\sum_{k=n+p}^{\infty}\frac{k!\,(k-q)}{(k-q)!}[(1-v)+(k-q-1)2\,\mu a_k b_k\geq 0.$$

Hence

$$\sum_{k=n+p}^{\infty}\delta(k,q)(k-q)[(1-v)+(k-q-1)v\mu]a_k b_k\leq\delta(p,q)(p-q)[(1-v)+(p-q-1)v].$$

Conversely, by considering (3.6), we must show that

$$Re\left\{\frac{\left[z((f*g)(z))^{q+2}+vz^2\left((f*g)(z)\right)^{q+3}\right]\left(1+\theta e^{i\varphi}\right)-\theta e^{i\varphi}\left[(1-v)((f*g)(z))^{q+1}+vz((f*g)(z))^{q+2}\right]}{(1-v)((f*g)(z))^{q+1}+vz((f*g)(z))^{q+2}}\right\}\geq\mu. \quad (3.9)$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, $Re(-e^{i\theta}) \geq -\left|e^{i\theta}\right| = -1$ and letting $r \to 1$, we conclude to(3.9) by using (3.6) in the left hard of (3.7).

Corollary(3.1.1): Let $f \in H(n,p,v,\beta,\mu,q)$. Then

$$a_k \leq \frac{\delta(p,q)(p-q)[(1-v)+(p-q-1)v]}{\delta(k,q)(k-q)[(1-v)+(k-q-1)v\mu]b_k}, \quad (3.10)$$

where $p,n \in \mathbb{N}, q \in \mathbb{N}_0, p > q, k \geq n+p, 0 < v < 1, 0 \leq \mu < 1$.

In the following theorem, we obtain the distortion bounds for the function $f \in H(n,p,v,\beta,\mu,q)$.

Theorem(3.1.2): Let the function$f \in H(n,p,v,\beta,\mu,q)$. Then

$$\left[1-\frac{(p-q)[(1-v)+(p-q-1)v]}{(n+p-q)[(1-v)+(n+p-q-1)v\mu]b_k}|z|^n\right]\delta(p,q)|z|^{p-q} \leq \left|f^{(q)}(z)\right| \leq$$
$$\left[1+\frac{(p-q)[(1-v)+(p-q-1)v]}{(n+p-q)[(1-v)+(n+p-q-1)v\mu]b_k}|z|^n\right]\delta(p,q)|z|^{p-q}. \quad (3.11)$$

The result is sharp for the function$f$ given by (3.7).

Proof: Let $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$, then

$$f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q)a_k z^{k-q},$$

where $\delta(i,j) = \frac{i!}{(i-j)} = \begin{cases} 1 & , \ (j=0) \\ i(i-1)\dots(i-j+1), & (j \neq 0) \end{cases}.$

By (3.10), we get

$$\left|f^{(q)}(z)\right| \le \delta(p,q)|z|^{p-q} + \frac{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}{(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k}|z|^{k-q}$$

$$\le \delta(p,q)|z|^{p-q} + \frac{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}{(n+p-q)[(1-\nu)+(n+p-q-1)\nu\mu]b_{n+p}}|z|^{n+p-q}$$

$$\left|f^{(q)}(z)\right| \le \left[1 + \frac{(p-q)[(1-\nu)+(p-q-1)\nu]}{(n+p-q)[(1-\nu)+(n+p-q-1)\nu\mu]b_k}|z|^n\right]\delta(p,q)|z|^{p-q}, \qquad (3.12)$$

and similarly, we can get

$$\left|f^{(q)}(z)\right| \ge \left[1 - \frac{(p-q)[(1-\nu)+(p-q-1)\nu]}{(n+p-q)[(1-\nu)+(n+p-q-1)\nu\mu]b_k}|z|^n\right]\delta(p,q)|z|^{p-q}. \qquad (3.13)$$

From (3.12) and (3.13), we get (3.11) and the proof is complete.

If $q = 0$, Theorem(3.1.2) would provide the growth property of function in the class $H(n,p,\nu,\beta,\mu,q)$. For $q \in \mathbb{N}$, the results may be looked upon as the distortion properties for the class $H(n,p,\nu,\beta,\mu,q)$.

Let the functions $f_i(z)$ $(i = 1,2,\dots,\nu)$ be defined by

$$f_i(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,i}z^k, \ (a_{k,i} \ge 0). \qquad (3.14)$$

We shall prove the closure property of the functions in the class $H(n,p,\nu,\beta,\mu,q)$.

Theorem (3.1.3): Let the functions $f_i(z)$ $(i = 1,2,\dots,\nu)$ be defined by (3.14) be in the class $H(n,p,\nu,\beta,\mu,q)$. Then the function $h(z)$ defined by $h(z) = \sum_{i=1}^{\infty} c_i f_i(z)$, $(c_i \ge 0)$, is also in the $H(n,p,\nu,\beta,\mu,q)$, where $\sum_{i=1}^{\infty} c_i = 1$.

Proof: According to the definition of $h(z)$, it can be written as

$$h(z) = \sum_{i=1}^{\infty} c_i \left(z^p - \sum_{k=n+p}^{\infty} a_{k,i} z^k \right)$$

$$= \sum_{i=1}^{\infty} c_i z^p - \sum_{i=1}^{\infty} \sum_{k=n+p}^{\infty} c_i a_{k,i} z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \sum_{i=1}^{\infty} c_i a_{k,i} z^k.$$

Furthermore, since the functions $f_i(z)\ (i = 1,2, \ldots, \nu)$ are in the class $H(n, p, \nu, \beta, \mu, q)$, then

$$\sum_{k=n+p}^{\infty} \delta\,(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]a_k b_k \leq \delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu].$$

Hence

$$\sum_{k=n+p}^{\infty} \delta\,(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k \left(\sum_{i=1}^{\infty} c_i a_{k,i}\right)$$

$$= \sum_{i=1}^{\infty} c_i [\sum_{k=n+p}^{\infty} \delta\,(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k a_{k,i}$$

$$\leq \delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu],$$

which implies that $h(z)$ be in the class $H(n, p, \nu, \beta, \mu, q)$.

In the following theorem, we obtain the radius of starlikeness for the function in the class $H(n, p, \nu, \beta, \mu, q)$.

Theorem (3.1.4): Let $f \in H(n, p, \nu, \beta, \mu, q)$. Then f is p-valent starlike of order $\delta(0 \leq \delta < p)$ in the disk $|z| < R$, where

$$R = inf_k \left\{ \left[\frac{(p-\delta)[\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]]b_k}{(k-\delta)[\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]]}\right]^{\frac{1}{k-q}} \right\}, \quad k \geq n+p. \qquad (3.15)$$

The result is sharp for the function $f(z) = z^p - \frac{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k} z^k$.

Proof: It is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-p\right| \le p-\delta$ for $|z|<R$,

we have $\left|\frac{zf'(z)}{f(z)}-p\right| \le \frac{\sum_{k=n+p}^{\infty}(k-q)a_k|z|^{k-q}}{1-\sum_{k=n+p}^{\infty}a_k|z|^{k-q}}$.

Thus

$$\left|\frac{zf'(z)}{f(z)}-p\right| \le p-\delta \text{ if } \sum_{k=n+p}^{\infty}\left(\frac{k-\delta}{p-\delta}\right)a_k|z|^{k-q} \le 1\,. \tag{3.16}$$

By using Theorem(3.1.1), (3.16) will be true if

$$\frac{k-\delta}{p-\delta}|z|^{k-q} \le \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}$$

or equivalently

$$|z| \le \left\{\left[\frac{(p-\delta)[\delta(k,q)(k-q)[(1-\nu)+(k-q)\nu]\mu]b_k}{(k-\delta)[\delta(p,q)(p-q)[(1-\nu)+(p-q)\nu]}\right]^{\frac{1}{k-q}}\right\}, \quad k \ge n+p\,. \tag{3.17}$$

The theorem follows easily from (3.17).

In the following theorem, we obtain convolution properties of the class $H(n,p,\nu,\beta,\mu,q)$.

Theorem (3.1.5): Let the function $f_j(z)$ $(j=1,2)$ defined by $f_j(z)=z^p-\sum_{k=n+p}^{\infty}a_{k,j}z^k$ be in the class $H(n,p,\nu,\beta,\mu,q)$. Then the function h defined by

$$h(z)=z^p-\sum_{k=n+p}^{\infty}\left(a_{k,1}^2+a_{k,2}^2\right)z^k, \tag{3.18}$$

belongs to the class $H(n,p,\nu,\beta,\tau,q)$, where

$$\tau \le \frac{1}{(k-q-1)}\left[1-\frac{1}{\nu}+\frac{\delta(k,q)(k-q)[(1-\nu)+=(k-q-1)\nu\mu]^2}{2\nu\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}\right].$$

Proof: We must find the largest τ such that

$\sum_{k=n+p}^{\infty} \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\tau]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1.$

Since $f_j(z)\ \ (j=1,2) \in H(n,p,\nu,\beta,\mu,q)$, then

$\sum_{k=n+p}^{\infty} \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} a_{k,1} b_k \leq 1$

and $\sum_{k=n+p}^{\infty} \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} a_{k,2} b_k \leq 1.$

Therefore

$$\sum_{k=n+p}^{\infty} \left\{\frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} a_{k,1} b_k\right\}^2 \leq$$

$$\left\{\sum_{k=n+p}^{\infty} \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} a_{k,1} b_k\right\}^2 \leq 1, \tag{3.19}$$

and $\sum_{k=n+p}^{\infty} \left\{\frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} a_{k,2} b_k\right\}^2 \leq$

$$\left\{\sum_{k=n+p}^{\infty} \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} a_{k,2} b_k\right\}^2 \leq 1. \tag{3.20}$$

Combining the inequalities (3.19) and (3.20), gives

$$\sum_{k=n+p}^{\infty} \frac{1}{2}\left\{\frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]b_k}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}\right\}^2 \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1.$$

But $h \in H(n,p,\nu,\beta,\tau,q)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\tau]b_k}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} \left(a_{k,1}^2 + a_{k,2}^2\right) \leq 1. \tag{3.21}$$

The inequality(3.21) will be satisfied if

$$\frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\tau]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]} \leq \frac{1}{2}\left\{\frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]}{\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}\right\}^2.$$

This is,

$$\tau \leq \frac{1}{(k-q-1)}\left[1-\frac{1}{\nu}+\frac{\delta(k,q)(k-q)[(1-\nu)+(k-q-1)\nu\mu]^2}{2\nu\delta(p,q)(p-q)[(1-\nu)+(p-q-1)\nu]}\right].$$

Section Two

3.2 A New Class of Multivalent Harmonic Functions Defined by Integral Operator

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co- analytic part of f .

A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D, see Clunie and Sheil-Small [12].

Denote by $A(p)$ the class of function $f = h + \bar{g}$ that are harmonic multivalent and sense- preserving in the unit disk $U = \{z : |z| < 1\}$. The class $A(p)$ was studied by Ahuja and Jahangiri [1].

For $f = h + \bar{g} \in A(p)$, we may express the analytic function h and g as

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, |b_p| < 1. \qquad (3.22)$$

Let $\mathcal{M}_p$ denote the subclass of $A(p)$ consisting of function $f = h + \bar{g}$,where h and g are given by

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} \quad, g(z) = -\sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \qquad (3.23)$$

where

$a_{k+p-1} \geq 0$, $b_{k+p-1} \geq 0$ and $|b_p| < 1$.

An integral operator I^n was introduced by Salagean [11] which is given below in a slightly modified form as Stated by [12]

(i) $I^0 f(z) = f(z)$;

(ii) $I^1 f(z) = If(z) = p\int_0^z f(t)t^{-1}\,dt$;

(iii) $I^n f(z) = I\big(I^{n-1}f(z)\big), n \in \mathbb{N}, f \in M,$

where $M = \{f \in A, f(z) = z + a_2 z^2 + \cdots\}$ and $A = A(U)$, The class of analytic functions in U. The modified Salagean integral operator of $f = h + \bar{g}$ given by (3.22) is defined [4] as

$$I^n\big(f(z)\big) = I^n\big(h(z)\big) + \overline{I^n\big(g\ (z)\big)}. \tag{3.24}$$

where

$$I^n\big(h(z)\big) = z^p + \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n a_{k+p-1}z^{k+p-1} \text{ and}$$

$$I^n(g(z)) = z^p + \sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n b_{k+p-1}z^{k+p-1}, \tag{3.25}$$

for $p, n \in \mathbb{N}$, $z \in U$.

Definition(3.2.1): We define a new class $N_p(A, B, \alpha, \lambda)$ of harmonic functions of the form (3.22) that satisfy the inequality**:**

$$\left|\frac{z^2(I^n f(z))'' - z(I^n f(z))'}{\lambda z^2(I^n f(z))'' + (A+B)z(I^n f(z))'}\right| < \alpha, \tag{3.26}$$

where $p, n \in \mathbb{N} = \{1,2,\cdots\}, 0 < A < 1,\ 0 < B < 1,\ 0 < \lambda < 1,\ 0 < \alpha < 1.$

We further denote by (λ, μ, α) the subclass of $N_p(A, B, \lambda, \alpha)$ that satisfies the relation

$$\mathcal{M}_p(A, B, \lambda, \alpha) = \mathcal{M}_p \cap N_p(A, B, \lambda, \alpha). \tag{3.27}$$

In the following theorem , we determine the sufficient condition for the function $f = h + \bar{g}$ to be in the class $N_p(A, B, \lambda, \alpha)$.

Theorem(3.2.1): Let $f = h + \overline{g}$ (h and g being given by (3.22)). If

$$\sum_{k=2}^{\infty} (k+p-1)\big[(k+p-3) - \alpha\big(\lambda(k+p-2) + (A+B)\big)\big]\left(\frac{p}{k+p-1}\right)^n |a_{k+p-1}|$$
$$-\sum_{k=1}^{\infty} (k+p-1)[(k+p-3) - \alpha(\lambda(k+p-2) + (A+B)]\left(\frac{p}{k+p-1}\right)^n |b_{k+p-1}|$$
$$\leq p[\alpha(\lambda(p-1) + (A+B) - p + 1)], \tag{3.28}$$

where $p, n \in \mathbb{N} = \{1,2,\cdots\}$, $0 < A < 1,\ 0 < B < 1,\ 0 < \lambda < 1, 0 < \alpha < 1,$

then f is harmonic p- valent sense – preserving in U and $f \in N_p(A, B, \lambda, \alpha)$.

Proof: When the condition (3.28) holds for the Coefficients of $f = h + \bar{g}$, it is shown that the inequality (3.26) is satisfied. Write the left side of inequality (3.26) as

$$|z^2(I^n f(z))'' - z(I^n f(z))'| - \alpha \left|\lambda z^2\big(I^n f(z)\big)'' + (A+B)z(I^n f(z))'\right| \leq$$
$$p(p-1)|z|^p + \sum_{k=2}^{\infty} (k+p-1)(k+p-3)\left(\frac{p}{k+p-1}\right)^n |a_{k+p-1}||z|^{k+p-1}$$
$$-\sum_{k=1}^{\infty}(k+p-1)(k+p-3)\left(\frac{p}{k+p-1}\right)^n |b_{k+p-1}||z|^{k+p-1} - \alpha p\big(\lambda(p-1) + (A+B)\big)|z|^p$$
$$-\alpha\sum_{k=2}^{\infty}(k+p-1)(\lambda(k+p-2) + (A+B))\left(\frac{p}{k+p-1}\right)^n |a_{k+p-1}||z|^{k+p-1}$$

$$-\alpha\sum_{k=1}^{\infty}(k+p-1)(\lambda(k+p-2)+(A+B)\left(\frac{p}{k+p-1}\right)^{n}|b_{k+p-1}||z|^{k+p-1}$$

$$=\sum_{k=2}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B))]\left(\frac{p}{k+p-1}\right)^{n}|a_{k+p-1}|$$

$$-\sum_{k=1}^{\infty}(k+p-1)[(k+p-3)+\alpha(\lambda(k+p-2)+A+B)\left(\frac{p}{k+p-1}\right)^{n}|b_{k+p-1}|$$

$$+p[-(\alpha(\lambda(p-1)+(A+B)-p+1)]\leq 0.$$

The harmonic functions

$$f(z)=z^{p}$$

$$+\sum_{k=2}^{\infty}\frac{p[-(\alpha(\lambda(p-1)+(A+B)-p+1)]\,x_k}{(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B))]\left(\frac{p}{k+p-1}\right)^{n}a_{k+p-1}}z^{k+p-1}$$

$$-\sum_{k=1}^{\infty}\frac{p[-(\alpha(\lambda(p-1)+(A+B)-p+2)]\,\overline{y_k}}{(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+A+B)\left(\frac{p}{k+p-1}\right)^{n}b_{k+p-1}}(\overline{z})^{k+p-1}\quad , \qquad (3.29)$$

where

$$\left(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|\overline{y_k}|=p[-(\alpha(\lambda(p-1)+(A+B)-p+1)]\right),$$

show that the coefficients bounds given by (3.28) is sharp.

The functions of the form (3.29) are in $N_p(A,B,\lambda,\alpha)$ because in view of (3.29), we infer that

$$\sum_{k=2}^{\infty}(k+p-1)[(k+p-3)-\lambda(k+p-2)+(A+B)]\left(\frac{p}{k+p-1}\right)^{n}|a_{k+p-1}|$$

$$-\sum_{k=1}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+A+B)\left(\frac{p}{k+p-1}\right)^{n}|b_{k+p-1}|$$

$$=\sum_{k=2}^{\infty}|x_n|+\sum_{k=1}^{\infty}|\overline{y_n}|=p[-(\alpha(\lambda(p-1)+(A+B)-p+1)]\,.$$

The restriction placed in Theorem (3.2.1) on the moduli of coefficients of $f = h + \overline{g}$ implies that for arbitrary rotation of the coefficients of f , the resulting functions would still be harmonic multivalent and $f \in N_p(A, B, \lambda, \alpha)$.

The following theorem shows that the condition (3.28) is also necessary for function f to belong to the class $\mathcal{M}_p(A, B, \lambda, \alpha)$.

Theorem(3.2.2): Let $f = h + \overline{g}$ with h and g are given by (3.23). Then $f \in \mathcal{M}_p(A, B, \lambda, \alpha)$ if and only if

$$\sum_{k=2}^{\infty}(k+p-1)\big[(k+p-3)-\alpha\big(\lambda(k+p-2)+(A+B)\big)\big]\left(\frac{p}{k+p-1}\right)^n a_{k+p-1}$$

$$+\sum_{k=1}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B)]\left(\frac{p}{k+p-1}\right)^n b_{k+p-1}$$

$$\leq p[\alpha(\lambda(p-1)+(A+B)-p+1)], \tag{3.30}$$

where $p, n \in \mathbb{N} = \{1,2,\cdots\}$, $0 < A < 1$, $0 < B < 1$, $0 < \lambda < 1$,$0 < \alpha <1$.

Proof: By noting that $\mathcal{M}_p(A, B, \lambda, \alpha) \subset N_p(A, B, \lambda, \alpha)$, the sufficiency part of Theorem (3.2.2) follows at once from Theorem (3.2.1). To prove the necessary part, let us assume that $f \in \mathcal{M}_p(A, B, \lambda, \alpha)$. Using (3.26), we get

$$Re\left\{\frac{z^2(I^n f(z))''-z(I^n f(z))'}{\lambda z^2(I^n f(z))''+(A+B)z(I^n f(z))'}\right\}$$

$$= Re\left\{\frac{p(p-1)z^p+\sum_{k=2}^{\infty}Ra_{k+p-1}z^{k+p-1}-\sum_{k=1}^{\infty}Rb_{k+p-1}z^{k+p-1}}{p\big(\lambda(p-1)+(A+B)\big)z^p+\sum_{k=2}^{\infty}La_{k+p-1}z^{k+p-1}-\sum_{k=1}^{\infty}Lb_{k+p-1}z^{k+p-1}}\right\}$$

$\leq \alpha$.

Such that: $R = (k+p-1)(k+p-3)\left(\frac{p}{k+\rho-1}\right)^n$

L=$(k+p-1)(\lambda(k+p-2)+(A+B))\left(\frac{p}{k+p-1}\right)^n$.

If we choose z to be real and let $z \to 1^-$, we obtain the condition (3.30).

Theorem(3.2.3): Let $f \in \mathcal{M}_p(A,B,\lambda,\alpha)$. Then

$$(1-b_p)|z|^p - \frac{p[\alpha(\lambda(p-1)+(A+B)-p+1)]-b_p}{(p+1)\big[(p-1)-\alpha\big(\lambda p+(A+B)\big)\big]}|z|^{k+p-1} \le |I^n f(z)| \le$$

$$(1-b_p)|z|^p + \frac{p[\alpha(\lambda(p-1)+(A+B)-p+1)]-b_p}{(p+1)[(p-1)-\alpha(\lambda p+(A+B))]}|z|^{k+p-1}.$$

.

Proof: Let $f \in \mathcal{M}_p(A,B,\lambda,\alpha)$. Then, we have

$$(p+1)[(p-1)-\alpha(\lambda p+(A+B)]\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n(a_{k+p-1}-b_{k+p-1})$$

$$\le p[\alpha(\lambda(p-1)+(A+B)-p+1)]+b_p\,,$$

which implies that

$$\sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n(a_{k+p-1}-b_{k+p-1}) \le \frac{p[\alpha(\lambda(p-1)+(A+B)-p+1)]+b_p}{(p+1)[(p-1)-\alpha(\lambda p+(A+B)]}.$$

Applying this inequality in the following assertion , we obtain

$$|I^n f(z)| = \left|z^p - \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n a_{k+p-1}z^{k+p-1} - \sum_{k=1}^{\infty}\left(\frac{p}{k+p-1}\right)^n b_{k+p-1}(\overline{z})^{k+p-1}\right|$$

$$\le (1-b_p)|z|^p + \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n(a_{k+p-1}+b_{k+p-1})\,|z|^{k+p-1}$$

$$\le (1-b_p)|z|^p + \frac{p[\alpha(\lambda(p-1)+(A+B)-p+1)]+b_p}{(p+1)[(p-1)-\alpha(\lambda p+(A+B)]}|z|^{k+p-1}.$$

Also , on the other hand, we obtain

$$|I^n f(z)| \ge (1-b_p)|z|^p - \sum_{k=2}^{\infty}\left(\frac{p}{k+p-1}\right)^n(a_{k+p-1}+b_{k+p-1})\,|z|^{k+p-1}$$

$$\ge (1-b_p)|z|^p - \frac{p[\alpha(\lambda(p-1)+(A+B)-p+1)]+b_p}{(p+1)[(p-1)-\alpha(\lambda p+(A+B)]}|z|^{k+p-1}.$$

Now, we prove the class $WR_{\overline{H}}(a,b;\lambda,\gamma;v,\mu)$ is closed under convex combination of these members .

Theorem(3.2.4): The class in $\mathcal{M}_p(A,B,\lambda,\alpha)$ is closed under convex combination.

Proof: For $j=1,2,3,\ldots$, let $f_j \in \mathcal{M}_p(A,B,\lambda,\alpha)$, where f_j is given

$$f_j(z)=z^p+\sum_{k=2}^{\infty} a_{k+p-1,j}z^{k+p-1}-\sum_{n=1}^{\infty} b_{k+p-1,j}(\bar{z})^{k+p-1}.$$

Then by (3.28) , we have

$$\sum_{k=2}^{\infty}(k+p-1)\big[(k+p-3)-\alpha\big(\lambda(k+p-2)+(A+B)\big)\big]\left(\frac{p}{k+p-1}\right)^n a_{k+p-1,\mathrm{j}}$$
$$-\sum_{k=1}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B)]\left(\frac{p}{k+p-1}\right)^n b_{k+p-1,j}$$
$$\leq p[\alpha(\lambda(p-1)+(A+B)-p+1)]. \tag{3.31}$$

For $\sum_{j=1}^{\infty} t_j=1$, $0\leq t_j\leq 1$, the convex combination of f_j may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z)=z^p-\sum_{n=2}^{\infty}\left(\sum_{j=1}^{\infty} t_j a_{k+p-1,j}\right)z^{k+p-1}-\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} t_j b_{k+p-1,j}\right)(\bar{z})^{k+p-1}.$$

Then , by (3.28), we have

$$\sum_{n=2}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B))]\left(\frac{p}{k+p-1}\right)^n\left(\sum_{j=1}^{\infty} t_j a_{k+p-1,j}\right)$$
$$-\sum_{n=1}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B))]\left(\frac{p}{k+p-1}\right)^n\left(\sum_{j=1}^{\infty} t_j b_{k+p-1,j}\right)$$
$$=\sum_{j=1}^{\infty} t_j\left\{\sum_{n=2}^{\infty}(k+p-1)\big[(k+p-3)-\alpha\big(\lambda(k+p-2)+(A+B)\big)\big]\left(\frac{p}{k+p-1}\right)^n a_{k+p-1.j}\right.$$
$$\left.-\sum_{n=1}^{\infty}(k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B))]\left(\frac{p}{k+p-1}\right)^n b_{k+p-1.j}\right\}$$
$$\leq\sum_{j=1}^{\infty} t_j\, p\big[\alpha\big(\lambda(p-1)+(A+B)\big)\big]=p[\alpha(\lambda(p-1)+(A+B)-p+1)],$$

therefore

$$\sum_{j=1}^{\infty} t_j f_j(z) \in \mathcal{M}_p(A,B,\lambda,\alpha).$$

This complete this proof .

Definition(3.2.2)[17]: The Jung – Kim – Srivastava integral operator is defined by

$$\vartheta^{\sigma} K(z) = \frac{(p+1)^{\sigma}}{z\Gamma(\sigma)} \int_0^z \left(log\frac{z}{t}\right)^{\sigma} K(t)dt\,, \qquad \sigma > 0\ . \tag{3.32}$$

If $K(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}$,

then $\vartheta^{\sigma} K(z) = z^p - \sum_{k=2}^{\infty} \left(\frac{p+1}{n+1}\right)^{\sigma} a_{k+p-1} z^{k+p-1}$, (3.33)

also ϑ^{σ} is a linear operator.

Remark(3.2.1): If $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}\,,\ \ g(z) = -\sum_{k=1}^{\infty} b_{k+\rho-1}\, z^{k+p-1}\,, \qquad |b_{k+\rho-1}| < 1\,,$$

then

$$\vartheta^{\sigma} f(z) = \vartheta^{\sigma}\, h(z) + \overline{\vartheta^{\sigma}\, g(z)}\,. \tag{3.34}$$

Theorem(3.2.5): If $f \in \mathcal{M}_p(A,B,\lambda,\alpha)$, then $\vartheta^{\sigma} f$ is also in $\mathcal{M}_p(A,B,\lambda,\alpha)$.

Proof: By (3.33) and (3.34) , we obtain

$$\vartheta^{\sigma} f(z) = \vartheta^{\sigma}\left(z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} - \sum_{k=1}^{\infty} b_{k+p-1}\ (\overline{z}\,)^{k+p-1}\right)$$

$$= z^p - \sum_{k=2}^{\infty} \left(\frac{p+1}{n+1}\right)^{\sigma} a_{k+p-1} z^{k+p-1} - \sum_{k=1}^{\infty} \left(\frac{p+1}{n+1}\right)^{\sigma} b_{k+p-1}\ (\overline{z}\,)^{k+p-1}\,,$$

since $f \in \mathcal{M}_p(A,B,\lambda,\alpha)$, then by Theorem (3.2.2), we have

$$\sum_{k=2}^{\infty} (k+p-1)\big[(k+p-3) - \alpha\big(\lambda(k+p-2) + (A+B)\big)\big]\left(\frac{p}{k+p-1}\right)^{n} a_{k+p-1}$$

$$-\sum_{k=1}^{\infty} (k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B)]\left(\frac{p}{k+p-1}\right)^{n} b_{k+p-1}$$

$$\leq p[\alpha(\lambda(p-1)+(A+B)-p+1)], \tag{3.35}$$

we must show

$$\sum_{k=2}^{\infty} (k+p-1)\big[(k+p-3)-\alpha\big(\lambda(k+p-2)+(A+B)\big)\big]\left(\frac{p}{k+p-1}\right)^{n}\left(\frac{p+1}{n+1}\right)^{\sigma} a_{k+p-1}$$

$$-\sum_{k=1}^{\infty} (k+p-1)[(k+p-3)-\alpha(\lambda(k+p-2)+(A+B)]\left(\frac{p}{k+p-1}\right)^{n}\left(\frac{p+1}{n+1}\right)^{\sigma} b_{k+\rho-1}$$

$$\leq p\big[\alpha\big(\lambda(p-1)+(A+B)\big)\big]. \tag{3.36}$$

But in view of (3.35) the inequality in (3.36) holds true if

$$\left(\frac{p+1}{n+1}\right)^{\sigma} \leq 1,$$

since $\sigma > 0$ and $p \leq n$,therefore (3.36) holds true and this gives the result.

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المستخلص

الغرض من هذه الرسالة هو دراسة بعض المواضيع الخاصة في نظرية الدالة احادية التكافؤ والمتعددة التكافؤ ودراسة صنف جزئي جديد $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$ من الدوال احادية التكافؤ المعرفة بواسطة مؤثر ديزايوك- سرفيستافا الخطي . اعطينا بعض الخواص ، مثل ، الشرط الضروري والكافي للدالة f كي تكون في الصنف $\mu(\lambda, \alpha, \gamma, \alpha_1, q, s)$ ، النقاط المتطرفة ، تكنيكات ضرب هادمارد ، متوسط التكامل ، مبرهنة الانقلاب ، انصاف اقطار النجمية ، التحدب والقريبة الى التحدب . اعتبرنا ايضآ بعض الخواص لنصف من الدوال احادية التكافؤ من الدوال احادية التكافؤ الميرومورفية $\sum(\alpha, t)$. هنا قدمنا صنف جديد$\sum(\alpha, t)$ من الدوال احادية التكافؤ الميرومورفية ذات المعاملات السالبة . حضرنا بعض النتائج ، متل ، متراجحة المعامل ، التركيب الخطي المحدب ، مبرهنة الانقلاب، حدود التشويه ، النقاط المتطرفة ، نصف قطر التحدب ، الجوارات للدالة $f \in \sum$. قدمنا ايضا بعض خواص التبعية التفاضلية للدوال احادية التكافؤ . حيث حصلنا على بعض النتائج . ناقشنا ايضا صنف جزئي من الدوال متعددة التكافؤ المعرفة بواسطة ضرب هادمرد $H(n, p, r, \beta, \mu, q)$. حصلنا على بعض الخواص ، مثل الشرط الضروري والكافي للدالة f كي تكون في الصنف $H(n, p, r, \beta, \mu, q)$ ، حدود التشويه ، مبرهنة الانغلاق ، نصف قطر النجمية و خواص ضرب الالتواء . درسنا ايضا صنف جديد من الدوال متعددة التكافؤ التوافقية المعرفة بواسطة المؤثر التكاملي حيث حصلنا على بعض النتائج ، مثل ، حدود المعامل ، التركيب المحدب ، مؤثر تكاملي و مبرهنة التشويه .

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رسالة مقدمة الى
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درجة ماجستير علوم في الرياضيات

من الطالبة
ماجدة حميد مجيد

أشراف
أ.م.د. وقاص غالب عطشان

١٤٣٨هـ ٢٠١٦م

رقم الايداع في دار الكتب و الوثائق ببغداد (٥٠١٧) لسنة ٢٠١٦