Strongly feebly Palais proper G – space

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Abstract
The main goal of this work is to create a general type of proper *G* – space , namely, strongly feebly Palais proper G – space(st – f – Palais proper *G* – space) and to explain the relation between st – f – Bourbaki proper and st – f – Palais proper *G* – space and to study some of examples and propositions of strongly feebly Palais proper G - space.

Introduction:-

Let *B* be a subset of a topological space (*X,T*). We denote the closure of *B* and the interior of *B* by $\overline{B}$ and B^o , respectively. A subset *B* of (*X*, *T*) is said to be semi – open (s. o) if there exists an open subset *O* of *X* such that $O \subseteq B \subseteq \overline{O}$. The complement of a semi – open set is defined to be semi – closed (s. c) and the intersection of all semi – closed subset of *X* containing *B* is defined to be semi – closure of *B* and denote by $\overline{B}^S$ .The subset *B* of (*X*, *T*) is called feebly open (f – open) if there is an open set *U* such that $U \subseteq B \subseteq \overline{U}^S$. The complement of a feebly open set is defined to be a feebly closed (f – closed)[2]. *B* is called $\propto$ - open if $B \subseteq \overline{B^o}^o$ and the family $T^{\propto}$ of all $\propto$ - sets in (*X,T*) is a topology on *X* larger than *T* [8]. We found in [2] that a subset *B* of *X* is f – open if and only if $B \in T^{\propto}$. Section one of this works, includes some results which are needed in section two.

Section two recalls the definition of Palais proper *G* – space, gives a new type of Palais proper *G* – space (to the best of our Knowledge), namely, strongly feebly Palais proper *G* – space, studies some of its properties and gives the relation between st – f – Bourbaki proper and st – f – Palais proper *G* – space, where *G*- space is meant T_2 – space *X* on which an f – locally f – compact, non – compact, T_2 – topological group *G* acts continuously on the left.

1. Preliminaries

1.1 Definition [12]:

A subset *B* of a space *X* is called feebly open (f – open) set if there exists an open subset *U* of *X* such that $U \subseteq B \subseteq \overline{U}^S$. The complement of a feebly open set is defined to be a feebly closed (f – closed) set. The collection of all f – open sets in a space *X* is denoted by T^f

1.2 Proposition [1]:

Let *X* and *Y* be two spaces. Then $A_1 \subseteq X$ and $A_2 \subseteq Y$ be f – open (f – closed) sets in *X* and *Y* , respectively if and only if $A_1 \times A_2$ is f – open(f – closed) in $X \times Y$.

1.3 Definition [6]:

A subset B of a space X is called feebly neighborhood (f – neighborhood) of $x\in X$ if there is an f – open subset O of X such that $x\in O\subseteq B$

1.4 Definition [10, 11,13]:

Let X and Y be spaces and $f: X\rightarrow Y$ be a function, Then:

(i) f is called feebly continuous (f – continuous) function if $f^{-1}(A)$ is an f – open set in X for every open set A in Y.

(ii) f is called feebly irresolute (f – irresolute) function if $f^{-1}(A)$ is an f – open set in X for every f- open set A in Y.

1.5 Proposition [1]:

Let $f:X\rightarrow Y$ be a function of spaces. Then f is an f - continuous function if, and only if, $f^{-1}(A)$ is an f - closed set in X for every closed set A in Y.

1.6 Proposition [1]:

Let X and Y be spaces and let $f: X\rightarrow Y$ be a continuous, open function .Then:

(i) f is f – irresolute function.

(ii) The image of any f – open subset of X is an f – open set in Y.

1.7 Definition [11,13]:

(i) A function $f: X\rightarrow Y$ is called strongly feebly closed (st – f – closed) function if the image of each f – closed subset of X is an f – closed set in Y.

(ii) A function $f: X\rightarrow Y$ is called strongly feebly open (st – f – open) function if the image of each f – open subset of X is an f – open set in Y.

1.8 Remark [6]:

(i)A function $f:(X, T)\rightarrow(Y,\tau)$ is f – tcontinuous if and only if $f:(X, T^{f})\rightarrow(Y,\tau)$ is continuous.

(ii)A function $f: (X, T) \rightarrow(Y,\tau)$ is f – irresolute if and only if $f:(X, T^{f})\rightarrow (Y, \tau^{f})$ is continuous.

1.9 Definition [2]:Let X and Y be spaces . Then a function $f: X \rightarrow Y$ is called a st – f – homeomorphism if:

(i) f is bijective .

(ii) f is f – continuous .

(iii) f is st – f – closed (st – f – open).

1.10 Proposition[6]: Every homeomorphism is st – f – homeomorphism.

1.11 Proposition [1]:

Let X, Y be spaces and $f: X\rightarrow Y$ be a homeomorphism function. Then f is a st – f – closed function.

1.12 Definition [2]:

Let $(\chi_d)_{d\in D}$ be a net in a space X and $x\in X$. Then :

i) $(\chi_d)_{d\in D}$ f – converges to x (written $\chi_d \xrightarrow{f} x$) if $(\chi_d)_{d\in D}$ is eventually in every f – neighborhood of x . The point x is called an f – limit point of $(\chi_d)_{d\in D}$, and the notation "$\chi_d \xrightarrow{f} \infty$" is mean that $(\chi_d)_{d\in D}$ has no f – convergent subnet.

ii) $(\chi_d)_{d\in D}$ is said to have x as an f – cluster point [written $\chi_d \alpha\ x$] if $(\chi_d)_{d\in D}$ is frequently in every f - neighborhood of x .

1.13 Proposition[6]:

Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) and x_o in X. Then $\chi_d \overset{f}{\alpha}\ x_o$ if, and only if, there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \xrightarrow{f} x_o$.

1.14 Remark:

Let $(\chi_d)_{d\in D}$ be a net in a space (X, T) such that $\chi_d \overset{f}{\alpha} x$, $x\in X$ and let A be an f – open set in X which contains x. Then there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ in A such that $\chi_{dm} \xrightarrow{f} x$.

1.15 Proposition [1]: Let X be a space, $A \subseteq X$ and $x\in X$. Then $x\in \overline{A}^{f}$ if and only if there exists a net $(\chi_d)_{d\in D}$ in A and $\chi_d \xrightarrow{f} x$.

1.16 Remark [5]: Let X be a space. Then:

(i) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d \xrightarrow[f]{f} x$ then $\chi_d \to x$.

(ii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$ such that $\chi_d\ \alpha\ x$ then $\chi_d\ \alpha\ x$.

(iii) If $(\chi_d)_{d\in D}$ is a net in X, $x\in X$. Then $\chi_d \xrightarrow{f} x$ in (X, T) if, and only if, $\chi_d \to x$ in (X, T^f), and $\chi_d\ \alpha\ x$ in (X, T) if and only if $\chi_d\ \alpha\ x$ in (X, T^f).

1.17 Proposition[6]: Let f: $X\to Y$ be a function, $x\in X$. Then:

(i) f is f – continuous at x if and only if whenever a net $(\chi_d)_{d\in D}$ in X and $\chi_d \xrightarrow{f} x$ then $f(\chi_d) \longrightarrow f(x)$.

(ii) f is f – irresolute at x if and only if whenever a net $(\chi_d)_{d\in D}$ in X and $\chi_d \xrightarrow{f} x$ then $f(\chi_d) \xrightarrow{f} f(x)$.

1.18 Definition [11]:

A subset A of space X is called f – compact set if every f – open cover of A has a finite sub cover. If A=X then X is called an f – compact space.

1.19 Proposition [1]: A space (X, T) is an f – compact space if and only if every net in X has f – cluster point in X.

1.20 Proposition [1]:

Let X be a space and F be an f – closed subset of X. Then $F\cap K$ is f – compact subset of F, for every f – compact set K in X.

1.21 Definition [1]:

(i) A subset A of space X is called f - relative compact if $\overline{A}$ is f – compact.

(ii) A space X is called f – locally f – compact if every point in X has an f – relative compact f – neighborhood.

1.22 Definition [2, 13, 8]:

Let f: $X \to Y$ is a function of spaces. Then:

(i) f is called a feebly compact (f – compact) function if $f^{-1}(A)$ is a compact set in X for every f – compact set A in Y.

(ii) f is called a strongly feebly compact(st – f – compact) function if $f^{-1}(A)$ is an f – compact set in X for every f – compact set A in Y.

2 - *Strongly Feebly Bourbaki Proper Action*

2.1 Definition [4]: A topological transformation group is a triple (G,X,φ) where G is a T_2 – topological group, X is a T_2 – topological space and $\varphi : G \times X \to X$ is a continuous function such that:

(i) $\varphi(g_1,\varphi(g_2, x)) = \varphi(g_1g_2, x)$ for all $g_1,g_2 \in G$, $x \in X$.

(ii) $\varphi(e, x) = x$ for all $x \in X$, where e is the identity element of G.

We shall often use the notation $g.x$ for $\theta(g,x)$ $g.(h,x) = (gh).x$ for $\theta(g, \theta(h,x)) = \theta(gh,x)$. Similarly for $H \subseteq G$ and $A \subseteq X$ we put $HA = \{ga / a \in H, a \in A\}$ for $\varphi(H, A)$. A set A is said to be invariant under G if $GA = A$.

2.2 Remark [2]:

Let X be a G – space and $x \in X$. Then:

(i) The function φ is called an action of G on X and the space X together with φ is called a G – space (or more precisely left G – space).

(ii) A set $A \subseteq X$ is said to be invariant under G if $GA = A$.

2.3 Definition [2]: Let X and Y be two spaces. Then f: $X \to Y$ is called a strongly feebly proper (st – f - proper) function if :

(i) f is f – continuous function.

(ii) $f \times I_Z$: $X \times Z \to Y \times Z$ is a st – f – closed function, for every space Z.

2.4 Proposition [2]: Let X, Y and Z be spaces, f: $X \to Y$ and g: $Y \to Z$ be two st – f – proper function, such that f is an f – irresolute function. Then $g o f : X \to Z$ is a st – f – proper function.

2.5 Proposition [2]; Let f_1: $X_1 \to Y_1$ and f_2: $X_2 \to Y_2$ be two function. Then $f_1 \times f_2$: $X_1 \times X_2 \to Y_1 \times Y_2$ is st – f- proper function if and only if f_1 and f_2 are st – f – proper functions.

2.6 Proposition[6]: Let f: $X \to P = \{w\}$ be an f – continuous function on a space X. Then f is a st – f – proper function if and only if X is an f – compact, where w is any point which dose not belongs to X.

2.7 Lemma: Every f – continuous function from an f – compact space into a Hausdorff space is st – f- closed.

2.10 Proposition: Let f_1: $X \to Y_1$ and f_2: $X \to Y_2$ be two st – f – proper functions. If X is a Hausdorff space, then the function f: $X \to Y_1 \times Y_2$, $f(x) = (f_1(x),f_2(x))$ is a st – f- proper function.

2.11 Proposition [2]: Let X and Y be tow spaces and $f: X \to Y$ be an f – continuous, one to one function. Then the following statements are equivalent:
(i) f is a st – f – proper function.
(ii) f is a st – f – closed function.
(iii) f is a st – f – homeomorphism of X onto f – closed subset of Y.

2.12 Proposition [2]:
Let X and Y be spaces and $f: X \to Y$ be an f – continuous function, such that $f^{-1}(\{y\})$ is an f – closed subset of X, $\forall y \in Y$. Then the following statements are equivalent:
(i) f is a st – f – proper function.
(ii) f is a st – f – closed function and $f^{-1}(\{y\})$ is an f – compact set, for each $y \in Y$.
(iii) If $(\chi_d)_{d\in D}$ is a net in X and $y \in Y$ is an f – cluster point of $f(\chi_d)$, then there is an f – cluster point $x \in X$ of $(\chi_d)_{d\in D}$ such that $f(x) = y$.

2.13 Remark [2]:
(i) The function φ is called an action of G on X and the space X together with φ is called a G – space (or more precisely left G – space).
(ii) The subspace $\{g.x \,/\, g \in G\}$ is called the orbit (trajectory) of x under G, which denoted by Gx [or $\gamma(x)$], and for every $x \in X$ the stabilizer subgroup G_x of G at x is the set $\{g \in G / \, gx = x\}$.
(iii) The continuous function $l_g: G \to G$ defined by $y \to gy$ is called the left translation by g. This function has inverse l_g^{-1} which is also continuous, moreover l_g is a homeomorphism. Similarly all right translation $r_g: G \to G$ are homeomorphism for every $g \in G$.
(iv) $Ag = r_g(A) = \{ag : a \in A\}$; Ag is called the left translate of A by g, where $A \subseteq G$, $g \in G$.
(v) $gA = l_g(A) = \{ga : a \in A\}$; gA is called the right translate of A by g, where $A \subseteq G$, $g \in G$.

2.14 Proposition: Let G be a topological group and $(g_d)_{d\in D}$ be a net in G. Then:
(i) If $g_d \xrightarrow{f} e$, where e is identity element of G, then $gg_d \xrightarrow{f} g$ (or $g_d\, g \xrightarrow{f} g$) for each $g \in G$.
(ii) If $g_d \xrightarrow{f} \infty$, then $gg_d \xrightarrow{f} \infty$ (or $g_d\, g \xrightarrow{f} \infty$) for each $g \in G$.
(iii) If $g_d \xrightarrow{f} \infty$, then $g_d^{-1} \xrightarrow{f} \infty$.

2.15 Proposition: If (G, X, φ) is a topological transformation group, then φ is f – irresolute.

2.16 Definition [6]: A G – space X is called a strongly feebly Bourbaki proper G – space (st – f – proper G – space) if the function $\theta: G \times X \to X \times X$ which is defined by $\theta(g, x) = (x, g.x)$ is a st – f – proper function.

2.17 Example:
The topological group $Z_2 = \{-1, 1\}$ [as Z_2 with discrete topology] acts on the topological space S^n [as a subspace of R^{n+1} with usual topology] as follows:
1. $(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_{n+1})$
-1. $(x_1, x_2, \ldots, x_{n+1}) = (-x_1, -x_2, \ldots, -x_{n+1})$
Since Z_2 is f – compact, then by Proposition (2.6) the constant function $Z_2 \to P$ is a st – f – proper. Also the identity function is a st – f – proper, then by Proposition (2.4) the proper function of $Z_2 \times S^n$ into $P \times S^n$ is a st – f – proper.

Since $P \times S^n$ is homeomorphic to S^n, then by Proposition (1.10) $P\times S^n$ is a st – f – homeomorphic to S^n. Thus by Proposition (2.11) the st – f – homeomorphism of $P\times S^n$ onto S^n is st – f – proper. Since $Z_2\times S^n \to P\times S^n$ is continuous and open function, then by Proposition (1.6.i) $Z_2\times S^n \to P\times S^n$ is f – irresolute. Then by Proposition (2.4) the composition $Z_2\times S^n \to S^n$ is a st – f – proper. Let φ be the action of Z_2 on S^n. Then φ continuous, one to one function so φ is f – continuous function. Since S^n is T_2 – space, then by Lemma (2.7) φ is a st – f – closed. Then by Proposition (2.11) φ is a st – f – proper function. Thus by proposition (2.10) $Z_2\times S^n \to S^n\times S^n$ is a st – f – proper G- space.

2.18 Proposition [6]:

Let X be a G – space then the function $\theta: G\times X\to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $G\times X$ for every $(x, y)\in X\times X$.

3 – Strongly feebly Palais proper action:

From now on, in this section by G – space we mean a completely regular topological T_2 – space X on which an f – locally f – compact, non – compact, T_2 – topological group G continuously on the left (always in the sense of st – f - Palais proper G – space). Now, in this section, definitions, propositions, theorems and Examples of a strongly feebly Palais proper G - space [st – f - Palais proper G – space] are given as well as the relation between st – f – Bourbaki proper and st – f – Palais proper G – space is studied.

3.1 Definition:

Let X be a G – space. A subset A of X is said to be feebly thin (f – thin) relative to a subset B of X if the set $((A, B)) = \{g\in G \ / \ gA\cap B\neq\phi\}$ has an f – neighborhood whose closure is f – compact in G. If A is f – thin relative to itself, then it is called f – thin.

3.2 Remark:

The f – thin sets have the following properties:

(i) Since $(gA\cap B) = g(A\cap g^{-1}B)$ it follows that if A is f – thin relative to B, then B is f – thin relative to A.

(ii) Since $(gg_1A\cap g_2B) = g_2(g_2^{-1}g\, g_1A\cap B)$ it follows that if A is f – thin relative to B, then so are any translates gA and gB.

(iii) If A and B are f – relative thin and $K_1\subseteq A$ and $K_2\subseteq B$, then K_1 and K_2 are f – relatively thin.

(iv) Let X be a G – space and K_1, K_2 be f – compact subset of X, then $((K_1, K_2))$ is f – closed in G.

(v) If K_1 and K_2 are f – compact subset of G – space X such that K_1 and K_2 are f – relatively thin, then $((K_1, K_2))$ is an f – compact subset of G.

Proof:

The prove of (i), (ii), (iii) and (v) are obvious.

(iv) Let $g\in \overline{((K_1,K_2))}^f$. Then by Proposition (1.16) there is a net $(g_d)_{d\in D}$ in $((K_1, K_2))$ such that $g_d \xrightarrow{f} g$. Then we have net $(k_d^1)_{d\in D}$ in K_1, such that $g_d k_d^1 \in K_2$, since K_2 is f – compact, then by Theorem (1.14) there exists a subnet $(g_{d_m} k_{d_m}^1)$ of $(g_d k_d^1)$ such that $g_{d_m} k_{d_m}^1 \xrightarrow{f} k_o^2$, where $k_o^2 \in K_2$. But $(k_{d_m}^1)$ in K_1 and K_1 is f – compact , thus there is a point $k_o^1 \in K_1$ and a subnet of $k_{d_m}^1$ say itself such that $k_{d_m}^1 \xrightarrow{f} k_o^1$, Theorem (1.14).

Then by Proposition (1.18.ii) $g_{d_m} k^1_{d_m} \xrightarrow{f} gk^1_o = k^2_o$, which mean that g∈(($K_1$, K_2)), there fore ((K_1, K_2)) is f – closed in *G*.

3.3 Definition:

A subset *S* of a *G* – space *X* is a feebly small (f – small) subset of *X* if each point of *X* has f – neighborhood which f – thin relative to *S*.

3.4 Theorem:

Let *X* be a *G* – space. Then:

(i) Each f – small neighborhood of a point *x* contains an f – thin neighborhood of *x*.
(ii) A subset of an f – small set is f – small.
(iii) A finite union of an f – small sets is f – small.
(iv) If *S* is an f – small subset of *X* and *K* is an f – compact subset of *X* then *K* is f – thin relative to *S*.

Proof:

i) Let *S* is an f – small neighborhood of *x*. Then there is an f – neighborhood *U* of *x* which is f – thin relative to *S*. Then ((*U*, *S*)) has f – neighborhood whose closure is f – compact. Let *V*= $U \cap S$, then *V* is f – neighborhood of *x* and ((*V*, *V*)) ⊆ ((*U*, *S*)), therefore *V* is *V* is f – thin neighborhood of *x*.

ii) Let *S* be an f – small set and *K* ⊆ *S*. Let *x*∈*X*, then there exists an f – neighborhood *U* of *x*, which is f – thin relative to *S*. Then ((*U*, *K*)) ⊆ ((*U*, *S*)), thus ((*U*, *K*)) has f – neighborhood whose closure is f – compact. Then *K* is f – small.

iii) Let $\{S_i\}_{i=1}^n$ be a finite collection of f – small sets and *y*∈*X*. Then for each *i* there is f – neighborhood K_i of *y* such that the set ((S_i , K_i)) has f – neighborhood whose closure is f – compact. Then $\bigcup_{i=1}^{n}$ ((S_i , K_i)) has f – neighborhood whose closure is f – compact. But (($\bigcup_{i=1}^{n} S_i$, $\bigcap_{i=1}^{n} K_i$)) ⊆ $\bigcup_{i=1}^{n}$ ((S_i , K_i)), thus $\bigcup_{i=1}^{n} S_i$ is an f – small set.

iv) Let *S* be an f – small set and *K* be f – compact. Then there is an f – neighborhood U_k of *K*, ∀*k*∈*K*, such that U_k is f – thin relative to *S*. Since $K \subseteq \bigcup_{k\in K} U_k$ .i.e., $\{U_k\}_{k\in K}$ is f – open cover of *K* , which is f – compact, so there is a finite sub cover $\{U_{k_i}\}_{i=i}^n$ of $\{U_k\}_{k\in K}$, since ((U_{k_i} ,*S*)) has f – neighborhood whose closure is f – compact , thus $((\bigcup_{i=1}^{n} U_{k_i}, S))$ so is . But ((*K*,*S*)) ⊆ $((\bigcup_{i=1}^{n} U_{k_i}, S))$ therefore *K* is f – thin relative to *S*.

3.5 Definition:

A *G* – space *X* is said to be a strongly feebly Palais proper *G* - space (st – f – Palais proper *G* – space) if every point *x* in *X* has an f – neighborhood which is f – small set.

3.6 Examples:

(i) The topological group Z_2={-1, 1} act on itself (as Z_2 with discrete topology) as follows:

$r_1.r_2 = r_1 r_2 \quad \forall r_1, r_2 \in Z_2$.

for each point $x \in Z_2$, there is an f – neighborhood which is f – small U of x where $U=\{x\}$, i.e., for any point y of Z_2, there exists an f – neighborhood V of y such that $V=\{y\}$ and $((U, V)) = \{r\in Z_2 \ / \ rU\cap V\neq\phi \} = Z_2$, then $((U, V))$ has f – neighborhood whose closure is compact.

(ii) R – {0} be f – locally f – compact topological group (as R – {0} with discrete topology) acts on the completely regular Hausdorff space R^2 as follows:

$r.(x_1, x_2) = (rx_1, rx_2)$, for every $r\in R - \{0\}$ and $(x_1, x_2)\in R^2$.

Clear R^2 is (R – {0}) – space. But $(0,0)\in R^2$ has no f – neighborhood which is an f – small. Since for any two f – neighborhoods U *and* V of (0,0) then $((U,V)) = R - \{0\}$. Since R is not compact, then R is not f – compact. Thus R^2 is not a st – f – Palais proper (R – {0}) – space.

<u>3.7 Proposition:</u>

Let X be a G – space . Then:

(i) If X is st – f – Palais proper G – space , then every f – compact subset of X is an f – small set.

(ii) If X is a st – f – Palais proper G – space and K is an f – compact subset of X , then $((K,K))$ is an f – compact subset of G.

<u>Proof:</u>

i) Let A be a subset of X such that A is f – compact. Let $x\in X$, since X is a st – f – proper G – space then there is an f – neighborhood of x U which is f – small. Then for every $a\in A$ there exists a neighborhood U_a which is f – small , then $A \subseteq \bigcup_{a\in A} U_a$, since A is f – compact , then there exists $a_1, a_2, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} U_{a_i}$, Thus by Theorem (3.4.iii.ii) A is an f – small set in X.

ii) Let X be a st – f – proper G – space and K is f – compact , then by (i) K is an f – small subset of X , and by Theorem (3.4.iv) K is f – thin , so $((K,K))$ has f – neighborhood whose closure is f – compact . Then by Remark (3.2.ix) $((K,K))$ is f – closed in G. Thus $((K,K))$ is f – compact .

<u>*3.8 Definition*</u>: Let X be a G – space and $x\in X$. Then $J^f(x)=\{y\in X$: there is a net $(g_d)_{d\in D}$ in G and there is a net $(\chi_d)_{d\in D}$ in X with $g_d \xrightarrow{f} \infty$ and $\chi_d \xrightarrow{f} x$ such that $g_d x \xrightarrow{f} y\}$ is called feebly first prolongation limit set of x.

<u>*3.9 Proposition:*</u>

Let X be a G – space. Then X is a st – f – Bourbaki proper G – space if and only if $J^f(x)=\phi$ for each $x\in X$.

<u>Proof:</u> $\Rightarrow$ Suppose that $y\in J^f(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{f} \infty$ and there is a net $(\chi_d)_{d\in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} y$, so $\theta((g_d,\chi_d))=(x_d, g_d\chi_d) \xrightarrow{f} (x, y)$. But X is a st – f – Bourbaki proper, then by Proposition (2.12) there is $(g, x_1)\in G\times X$ such that $(g_d, x_d) \underset{f}{\alpha} (g, x_1)$. Thus $(g_d)_{d\in D}$ has a sub net (say itself). such that $g_d \xrightarrow{f} g$, which is contradiction, thus $J^f(x)=\phi$.

$\Leftarrow$ Let $(g_d, \chi_d)_{d\in D}$ be a net in $G\times X$ and $(x, y)\in X\times X$ such that $\theta((g_d, \chi_d))=(\chi_d, g_d\chi_d) \overset{f}{\alpha} (x, y)$, so $(\chi_d, g_d\chi_d)_{d\in D}$ has a sub net, say itself, such that $(\chi_d, g_d\chi_d) \xrightarrow{f} (x, y)$, then $\chi_d \xrightarrow{f} x$ and $g_d\chi_d \xrightarrow{f} y$. Suppose that $g_d \xrightarrow{f} \infty$ then $y\in J^f(x)$, which is contradiction . Then there is $g \in G$ such that $g_d \xrightarrow{f} g$, then $(g_d, \chi_d) \xrightarrow{f} (g, x)$ and $\theta(g, x) = (x, y)$. Thus by Proposition (2.18) and Proposition (2.12) X is a st – f – Bourbaki proper G – space.

3.10 Proposition:

Let X be a G – space and y be a point in X . Then y has no f – small whenever $y \in J^f(x)$ for some point $x \in X$.

Proof:

Let $y \in J^f(x)$, then there is a net $(g_d)_{d \in D}$ in G with $g_d \xrightarrow{f} \infty$ and a net $(\chi_d)_{d \in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} y$. Now, for each f – neighborhood S of y and every f – neighborhood U of x there is $d_o \in D$ such that $\chi_d \in U$ and $g_d\chi_d \in S$ for each $d \geq d_o$, thus $g_d \in ((U,S))$, but $g_d \xrightarrow{f} \infty$, thus $((U,S))$ has no f – compact closure . i.e., S is not an f – small neighborhood.

3.11 Proposition:

Let X be a st – f – Palais proper G – space. Then $J^f(x) = \phi$ for each $x \in X$.

Proof:

Suppose that there exists $x \in X$ such that $J^f(x) \neq \phi$, then there exists $y \in J^f(x)$. Thus there is a net $(g_d)_{d \in D}$ in G with $g_d \xrightarrow{f} \infty$ and a net $(\chi_d)_{d \in D}$ in X with $\chi_d \xrightarrow{f} x$ such that $g_d\chi_d \xrightarrow{f} y$. Since X be a st – f – proper G – space, then there is an f – small (f – thin) f – neighborhood U of x. Thus there is $d_o \in D$ such that $g_d\chi_d \in U$ and $\chi_d \in U$ for each $d \geq d_o$, so $g_d \in ((U,U))$, which has an f – compact closure , therefore $(g_d)_{d \in D}$ must have an f – convergent subnet , which is a contradiction. Thus $J^f(x) = \phi$ for each $x \in X$.

In general, the definition of a st – f – Palais proper G – space implies that st – f – Bourbaki proper G – space, which is review in following proposition.

3.12 Proposition:

Every st – f – Palais proper G – space is st – f – Bourbaki proper G – space.

Proof:

By Propositions (3.11) and (3.9).

The converse of Propositions (3.12), is not true in general as the following example shows.

3.13 Example:

Let G be a topological group where G is not f – locally f – compact, then G is acts on itself translation. The map $\theta: G \times G \to G \times G$, which is defined by $\theta(g_1, g_2) = (g_2, g_1g_2)$, $\forall (g_1, g_2) \in G \times G$ is a st – f – homeomorphism , hence it is st – f – Bourbaki proper G – space . But it is not st – f – Palais proper G – space, because G is not f – locally f – compact.

3.14 Lemma: Let X be an f – locally f – compact G – space . Then $J^f(x) = \phi$ for each $x \in X$ if and only if every pair of points of X has f – relatively thin f – neighborhood.

3.15 Proposition:

Let X be an f – locally f – compact G – space. Then the definition of st – f – Palais proper G – space and the definition st – f – Bourbaki proper G – space are equivalent.

Proof:

The definition of st – f – Palais proper G – space implies to the definition st – f – Bourbaki proper G – space. by Propositions (3.12).

Conversely, let X be a st – f – Bourbaki proper G – space, then by Proposition (3.9) $J^f(x)=\phi$ for each $x\in X$. Let $x\in X$, we will how that x has a f – small f – neighborhood. Since X is f – locally f – compact, then there is a f – compact f – neighborhood U_x of X, we claim that U_x is an f – small f – neighborhood of x. Let $y\in X$, we may assume without loss of generality , that U_y is an f – compact f – neighborhood of y such that U_x and U_y are f – relative thin i.e., $((U_x, U_y))$ has f – compact closure, by Lemma(3.14). therefore U_x is an f – small f – neighborhood of x. Thus X is st – f – Palais proper G – space.

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