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Abstract

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Since the financial crisis in 2008, clawback provisions have been implemented by several high profile banks and are also required by some regulators to mitigate the cost in case of a catastrophic shift in business, and also to deter excessive risk taking. In this thesis, we construct a model to investigate the long term effect on the bank's revenue of a trader's bonus payment scheme with escrow. We formulate the problem as an infinite-horizon discrete dynamic programming problem. With the proposed model, the trader's optimal investment and consumption strategy can be expressed by explicit analytic formulas, both with and without escrowing the bonus, which enables the calculation and comparison of the bank's total expected revenue under these two bonus payment schemes. The final conclusion of this comparison depends on the parameters describing the trader's risk appetite, the discount factor and the bank's level of patience, in addition to the market parameters. In particular, when the model parameters are such that the bank's total expected discounted revenue is finite under both types of bonus payment schemes, and the bank is sufficiently patient, it is better off when escrowing the trader's bonus, although not escrowing the trader's bonus brings better short term revenue.

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1 Introduction

A clawback provision, as the name suggests, refers to contractual clauses requiring executives of financial firms to return previously received compensation under certain circumstances, usually a significant negative shift in business. Clawback provisions are commonly expected to reduce incidences of severe managerial misconduct and to provide desirable long term incentives. Since the financial crisis in 2008, several high-profile banks, including Goldman Sachs Group Inc., Morgan Stanley, and UBS AG, have implemented clawback provisions for executive compensation, (see, e.g., [3, 9]). Meanwhile, policy makers have called for tighter linkage between bank employee compensation and long-term firm performance, and the relevant regulations are becoming increasingly stringent. For example, Section 954 of the Dodd-Frank Act added a new Section 10D to the Securities Exchange Act of 1934 (the "Exchange Act"), to require clawbacks of executive incentive compensation in circumstances to be established by the SEC. Also, major UK financial regulators have recently tightened previously established rules on bonus deferral and clawback with changes including an extension of the clawback period (see [14]). Because such regulatory changes aim to curb excessive risk taking, the requirements now apply not only to the senior management but also to the major deal makers and traders.

In this thesis, we focus on a performance-based compensation provision for a trader who trades in a financial market on behalf of a bank. The model is formulated as a principal-agent problem, where the bank (principal) optimizes the total expected revenue recognizing that the trader (agent) will optimize her total expected utility given the compensation contract. A main stream of research on principal-agent problems targets the characterization of the optimal contract, usually in terms of compensation or compensation flow for managers of financial institutions, see, e.g., static models such as [2, 12] and dynamic models such as [10, 13]. As in [13], our model has infinitely many time steps, and a risk-averse agent and a risk-neutral principal, both discounting the future. With the setup of our model, however, the bank does not optimize over the contractual parameter of the profit and loss division between the bank and the trader, as the trader's optimal strategy does not depend on this parameter. Rather, in our analysis, the optimization of the compensation contract is between two categories: first, we find the trader's optimal strategy under the two bonus payment schemes, namely escrowing or not escrowing the bonus, respectively; then we investigate which bonus payment scheme is more beneficial to the bank in terms of total expected revenue.

There is much discussion in academia and industry around bonus deferral and clawback provisions, and it is commonly accepted that such compensation schemes are helpful for reducing managerial misconduct that can result in financial restatement and misreporting. Nevertheless, analytical research on performance-based bonus clawback is relatively limited. The existing relevant literature mainly focuses on the effectiveness of clawback provisions in providing long-term incentives for executives, whose efforts may affect the firm's stock value, while little can be found on the effect of such compensation schemes on lower-level employees such as traders, who participate in the market on behalf of the bank but may not affect the bank's stock in a direct and immediate fashion. For example, deferred compensation for CEO's after employment termination that is tied to the future performance of the firm is studied under a dynamic programming setting in [11]. Escrow and clawback of a manager's bonus in a two-period setting is studied in [7, 1]. Our model is in general different from these lines of research. In particular, the agent in our model generates profit and loss by trading in the stock market on behalf of the principal rather than directly impacting the expected return of the firm's stock. Moreover, we consider an infinite-time horizon setting where the trader's bonus may be escrowed in each time period, while the papers mentioned before all consider a one-time deferral and clawback of the bonus.

As mentioned above, the effectiveness of performance-based compensation clawback policies remains an open question. Specifically, while it is expected that such a compensation scheme may deter excessive risk taking, it is not clear whether it will cause the trader to be "too conservative" and thereby reduce the bank's expected revenue. In this thesis, we investigate how escrowing trader's bonus affects the bank's total expected revenue. We formulate the problem as an infinite-horizon discrete dynamic programming problem. The infinite horizon is divided into periods with equal length. One may think of one period as one year, which is the usual bonus cycle in a bank. There are two participants in this problem, namely, the bank and the trader. Within each time period, the trader continuously trades a risky asset and a risk-free asset in a complete market on behalf of the bank, which has no direct access to the market. For simplicity, the risky asset is assumed to have geometric Brownian motion dynamics and the risk-free asset is a money market account with 0 interest rate. At the end of each time period, the trader gets a fraction of the wealth won or lost as a bonus or a penalty. Specifically, if the trader realizes positive earnings, the bank then pays a fraction of these earning as a bonus to the trader. If the portfolio turns out to have a loss, the trader has to pay out of her pocket a fraction of the loss as well. In both cases, the bank gets the rest of the gain or loss. At the end of each period, the trader consumes out of the available bonus and gets utility, described by either a power function or logarithmic function.

When the trader realizes positive earnings, there are two possible schemes for the bank to pay a bonus to the trader. The first one is to pay without escrow. With this scheme, the bonus gained at the end of each period is accumulated in the trader's bonus account and immediately becomes available for consumption. The second one is to pay with escrow. With this scheme, the bonus earned at the end of a period is not available for consumption until the end of the next period. If the trader generates a loss, it is always deducted from the trader's bonus account immediately. In other words, the bank claws back part of the bonus previously paid to the trader. With both the non-escrow and the escrow scheme, there is a constraint for the trader: once the amount in the bonus account becomes zero in any period, the trader cannot trade from that period onward. One can imagine this as a situation in which the trader loses her job. We assume the trader has negative infinite marginal utility for zero consumption.

The trader's objective is to maximize the total discounted expected utility from consumption throughout the infinite time horizon. We solve the trader's optimization problem by solving the equivalent Bellman equation. The optimal value function and the corresponding optimal strategy can be explicitly calculated.

The bank's outcome is measured by the total expected discounted revenue, which depends on the trader's trading strategy. Unlike the case in a typical principal-agent problem, the principal, i.e., the bank, does not optimize over the contractual constant which defines the fraction of gain or loss that the trader gets. In fact, it will become clear that the trader's optimal value function does not depend on this factor, since the trader will scale her optimal trading strategy accordingly. The rationale for not describing the bank's objective through utility is threefold. Firstly, the bank as a financial institution with a large amount of capital tends to be more risk neutral than a person. Secondly, it is more sensible to consider the aggregated revenue from different desks in a bank, but utilities are not additive. Lastly, given the trader's optimal strategy, the bank's total expected revenue can be calculated explicitly, which permits a tractable comparison between the two bonus schemes. The assumption of the bank being risk-neutral and the trader being risk-averse is also consistent with mainstream principal-agent problem literature, such as [2, 10, 13] as well as other relevant research on clawback provisions, see [11, 7].

The remaining part of the thesis is organized as follows. In Chapter 2 we provide the mathematical description of the model. In Chapter 3 we write down the Bellman equation formulation of the trader's infinite horizon problem and find its solution. In Chapter 4 we prove the solution of the Bellman equation is the solution of the trader's infinite horizon problem. In Chapter 5, we calculate the bank's total expected revenue with two bonus schemes and provide the conclusion of the comparison. Some final discussion is presented in Chapter 6.

2 The Model

2.1 The Market and the Two Parties

Consider an infinite time horizon $[0, +\infty)$ that is divided into intervals of equal length. Without loss of generality, suppose each interval is of unit length. Assume the bank has no direct access to the market, and the trader trades in the market on behalf of the bank. The bonus paid by the bank is accumulated into an account from which the trader consumes.

In the period from time k to $k+1$, the trader begins with certain amount of initial capital and invests in a market consisting of a stock with geometric Brownian motion dynamics

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

and a money market account with interest rate 0, where α and σ are non-negative constants, and W is a Brownian motion under the physical measure $\mathbb{P}$. Let Ξ_t^k, $k \leq t \leq k+1$ denote the value of the portfolio that the trader manages at time t during the time period $[k, k+1]$. Without losing of generality, we assume the trader's initial capital at the beginning of each period is zero. Let $X_{k+1} \triangleq \Xi_{k+1}^k$ denote the portfolio value at the end of this period, which can be either positive or negative. The gain or loss is split between the bank and the trader by a contractual multiplicative constant $\gamma \in (0, 1)$. Specifically, at time $k+1$, the trader gets γX_{k+1} and the bank gets $(1-\gamma)X_{k+1}$. To rule out unexpected behaviors such as the "doubling strategies" (see, e.g., [6]), we constrain the trading strategy to satisfy the following condition:

$$\gamma \Xi_t^k \geq -B_k, \quad k \leq t \leq k+1, \tag{2.1}$$

where B_k is the amount in the trader's bonus account at time k. An interpretation of this inequality is that the trader is not allowed to put an unlimited amount of the bank's capital at risk.

At time $k+1$, the trader consumes C_{k+1} and gets utility $U(C_{k+1})$, where $U(\cdot)$ is a utility function which will be described in detail in the next section. Then at time $k+1$ the bonus account is updated by the formula

$$B_{k+1} = B_k + \gamma X_{k+1} - C_{k+1}. \tag{2.2}$$

The bonus account must stay non-negative, meaning no consumption from borrowing is allowed for the trader. The non-negativity condition constrains the trader's investment and consumption decisions. The bonus scheme can be contracted in two ways, namely with or without escrowing, which we specify below:

1. If the bonus is not escrowed, then at the end of each period, the trader is allowed to consume all that is in the bonus account, i.e.,

$$0 \leq C_{k+1} \leq B_k + \gamma X_{k+1}. \tag{2.3}$$

2. If the bonus is escrowed, then the trader can only consume what was already in the bonus account from the previous period, and must leave the earnings of the current period, if any, to be escrowed to the next period. However, when the trading activity of a period results in a loss, the penalty is deducted from the bonus account immediately. In particular, at the end of period $k+1$, if $X_{k+1} < 0$, then the trader can consume no more than $B_k + \gamma X_{k+1}$. If $X_{k+1} \geq 0$, then the trader can consume no more than B_k and must escrow the more recent earnings to the next period. To summarize, we have

$$0 \leq C_{k+1} \leq (B_k + \gamma X_{k+1}) \wedge B_k. \tag{2.4}$$

Notice that in either case, X_{k+1} must satisfy

$$\gamma X_{k+1} \geq -B_k, \tag{2.5}$$

which is ensured by condition (2.1). Inequality (2.1), together with zero initial capital for each time period, are the only constraints on the trading strategy Ξ_{k+1}.

2.2 Utility Functions

Suppose the trader's utility from consumption c is given by a function $U(c)$, where U is defined on $(0, +\infty)$, continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions

$$\lim_{c \to 0+} U'(c) = \infty, \quad \lim_{c \to \infty} U'(c) = 0.$$

In this thesis, we will consider the family of utility functions with constant relative risk aversion (CRRA), which allows explicit solutions of the trader's problem. Specifically, the following three types of utility functions are considered:

(a) Power utility with positive values:

$$U(c) = \frac{1}{1-p} c^{1-p}, \quad 0 < p < 1; \tag{2.6}$$

(b) Power utility with negative values:

$$U(c) = \frac{1}{1-p} c^{1-p}, \quad p > 1; \tag{2.7}$$

(c) Logarithmic utility:

$$U(c) = \log(c). \tag{2.8}$$

The technical details in solving the trader's problem with these three types of utility functions are slightly different and will be discussed separately. However, it will be seen that the solutions to trader's problem with these utility functions have similar behavior in many ways, and will lead to the same conclusion for the bank's problem.

Remark 2.1. In this thesis, we adopt the convention that 0 to a negative power is $+\infty$, and that the logarithm of 0 is $-\infty$. Then the utility functions U defined as above are defined and continuous on $[0,+\infty)$, and take values in the extended set of real numbers $\mathbb{R}\cup\{-\infty\}$.

2.3 Objectives

2.3.1 Trader's Objective

We assume the trader's objective is to maximize the expected total discounted utility over the infinite time horizon by choosing a trading and consumption strategy in each time period. Specifically, the trader's optimal value given the initial bonus $B_0 = b \geq 0$ is given by

$$v^*(b) \triangleq \sup_{\{(X_{k+1},C_{k+1})\}_{k=0}^{\infty}\in\mathcal{A}(b)} \mathbb{E}\left[\sum_{k=0}^{\infty}\beta_T^k U(C_{k+1})\middle|B_0 = b\right], \tag{2.9}$$

where $\mathcal{A}(b)$ is the set of admissible strategies, which we will define later, $\mathbb{E}$ is the expectation under the physical measure and $\beta_T \in (0,1)$ is the trader's discount factor.

Remark 2.2. Here and throughout this thesis, we use the following convention for expectations. For a random variable χ and some condition $\mathcal{H}$, which can be an event, a random variable, or a σ-algebra, if $\mathbb{E}[\chi^+|\mathcal{H}] = \infty$ and $\mathbb{E}[\chi^-|\mathcal{H}] = \infty$, then we define $\mathbb{E}[\chi|\mathcal{H}] \triangleq -\infty$. In particular, this convention also applies to the unconditional expectation. Thus as long as there exists a strategy such that the expectation on the right hand side of (2.9) is finite, the strategies $\{(X_{k+1},C_{k+1})\}_{k=0}^{\infty}\in\mathcal{A}(b)$ such that $\mathbb{E}\left[\left(\sum_{k=0}^{\infty}\beta_T^k U(C_{k+1})\right)^-\middle|B_0 = b\right] = \infty$ are ruled out as optimal for the optimization problem on the right hand side of (2.9).

Now we define the feasible set $\mathcal{A}(b)$. Let $\{\mathcal{F}_t\}$ be the filtration generated by W. For

$k \in \mathbb{N}$,[1] let

$$Z_k \triangleq \exp\left\{-\theta\big(W_k - W_{k-1}\big) - \frac{1}{2}\theta^2\right\}, \tag{2.10}$$

where θ is the market price of risk, i.e.,

$$\theta = \frac{\alpha}{\sigma}.$$

In particular, for all $k \in \mathbb{N}$, Z_k has the same distribution as Z_1, which is the Radon-Nikodym derivative for the "one-period" change of measure, i.e.,

$$Z_1 = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_1},$$

where $\mathbb{P}$ and $\mathbb{Q}$ denote the physical measure and the risk-neutral measure, respectively. Notice also that for any $k \in \mathbb{N}$, Z_{k+1} is independent of $\mathcal{F}_k$.

For $k \in \mathbb{N}$ and $b \geq 0$, define the set of feasible strategies during the k-th period to be

$$\mathcal{A}(k,b) \triangleq \{(X,C) : X, C \text{ are } \mathcal{F}_k\text{- measurable}, \mathbb{E}[XZ_k|\mathcal{F}_{k-1}] = 0, (X,C) \in \mathcal{C}(b)\}, \tag{2.11}$$

where $\mathcal{C}(b)$ denotes a constraint set described below. Specifically,

1. when the bonus is not escrowed,

$$\mathcal{C}(b) \triangleq \{(X,C) : 0 \leq C \leq b + \gamma X\};$$

2. when the bonus is escrowed,

$$\mathcal{C}(b) \triangleq \{(X,C) : 0 \leq C \leq (b + \gamma X) \wedge b\}.$$

The condition $\mathbb{E}[XZ_k|\mathcal{F}_{k-1}] = 0$ in (2.11) is the budget constraint that at the end of each trading period, the expected wealth under the risk-neutral measure $\mathbb{Q}$ must be equal to the initial wealth 0, see Remark 2.3 below.

[1] In this thesis, we let $\mathbb{N}$ denote the set of natural numbers, excluding 0, i.e., $\mathbb{N} \triangleq \{1, 2, \cdots\}$.

Then, we can define the set of feasible strategies for the infinite time horizon problem to be

$$\mathcal{A}(b) \triangleq \big\{\pi = \{(X_{k+1}, C_{k+1})\}_{k=0}^{\infty} : (X_{k+1}, C_{k+1}) \in \mathcal{A}(k+1, B_k),$$
$$B_0 = b, B_{k+1} = B_k + \gamma X_{k+1} - C_{k+1}, k = 0, 1, \cdots \big\}. \quad (2.12)$$

To facilitate the analysis in the following chapters, we also define the set of feasible strategies for the first n periods to be

$$\mathcal{A}_n(b) \triangleq \big\{\pi_n = \{(X_{k+1}, C_{k+1})\}_{k=0}^{n-1} : (X_{k+1}, C_{k+1}) \in \mathcal{A}(k+1, B_k),$$
$$B_0 = b, B_{k+1} = B_k + \gamma X_{k+1} - C_{k+1}, k = 0, 1, \cdots n-1 \big\}. \quad (2.13)$$

Remark 2.3. We have the following remarks about trader's value function.

1. Since our model is complete, for any payoff X_{k+1} that satisfies $\mathbb{E}\big[|X_{k+1}Z_{k+1}|\big] < \infty$, there is an adapted process $\varphi_t, k \le t \le k+1$, representing the number of shares of the risky asset held by the trader at each time t, so that the revenue $\int_k^{k+1} \varphi_t dS_t$ earned by the trader over the time interval $[k, k+1]$ is equal to X_{k+1}. Therefore the choice of investment strategy on the infinite time horizon is equivalent to choosing a sequence of (distributions for) random variables X_{k+1}, $k \in \{0\} \cup \mathbb{N}$, satisfying the budget constraints $\mathbb{E}[X_{k+1}Z_{k+1}|\mathcal{F}_k] = 0$.

2. Although the interest rate in this market is zero, the discount factor β_T reflects the trader's preference for recent utility over utility at a future time. Alternatively, the discounting can be interpreted as incorporating the probability of the trader leaving the current job, where the exit time is a random variable τ, independent of $\mathcal{F}_\infty$, with $\mathbb{P}\{\tau = k+1\} = (1-\beta_T)\beta_T^k$, $k = 0, 1, \cdots$.

Notice that under this setting, given the constant γ, maximizing the trader's total discounted expected utility over (X_k, C_k) is equivalent to maximizing over (Y_k, C_k), where

$$Y_k \triangleq \gamma X_k.$$

Then the updating rule for the bonus can be written as

$$B_{k+1} = B_k + Y_{k+1} - C_{k+1}. \tag{2.14}$$

The no-borrowing and consumption constraints become:

- When bonus is not escrowed,

$$0 \leq C_{k+1} \leq B_k + Y_{k+1}; \tag{2.15}$$

- When bonus is escrowed,

$$0 \leq C_{k+1} \leq (B_k + Y_{k+1}) \wedge B_k. \tag{2.16}$$

Finally, the budget constraint becomes

$$\mathbb{E}[Y_k Z_k | \mathcal{F}_{k-1}] = 0.$$

Thus we can see that the trader's value function, if it exists, does not depend on the wealth distributing factor γ. Given γ, the trader will scale her investment strategy to achieve this optimal value. Therefore the bank cannot control the trader's behavior by choosing the constant γ.

For simplicity, we slightly abuse our notation and still use $\mathcal{A}(k, b)$, $\mathcal{A}(b)$ and $\mathcal{A}_n(b)$ to denote the feasible sets when the strategies are described in the form of (Y, C).

Finally, we point out that the trader's problem without escrow can be reformulated as a standard consumption and investment problem and solved by a known method, which we present in Appendix A. In this thesis, we instead pose it as an infinite-horizon dynamic programming problem and solve it using dynamic programming techniques, because this approach can be modified to solve the trader's problem with escrow. Moreover, even though with the methodology discussed in Appendix A, the trader's problem without escrow can be solved with a general utility function, the form of the solution we obtain and use to compute the bank's total expected revenue depends on the assumption that the trader's utility function is a power function or the logarithmic function. Also, the trader's problem with escrow cannot be solved with general utility functions.

2.3.2 Bank's Value Function

Given an investment strategy $\{X_k, k \in \mathbb{N}\}$ chosen by the trader, the bank's total discounted expected revenue is

$$v_B(b) = \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_B^k (1-\gamma) X_{k+1} \Big| B_0 = b\right],$$

where $\beta_B \in (0,1)$ is the bank's discount factor. In this thesis, we investigate how the bank's total discounted expected revenue is affected by escrowing the bonus for the trader.

3 Bellman Equation of Trader's Problem

3.1 Heuristics

We start with some heuristic arguments. At the beginning of an arbitrary trading period $[k-1,k], k \in \mathbb{N}$, let b be the amount in the bonus account. The trader chooses her consumption and investment strategy for this period based on the current amount in the bonus account. Let's denote this strategy by (Y, C). Then at the end of this period, the amount in the bonus account will be $b + Y - C$. Suppose the optimal value of the trader's infinite-time-horizon optimization problem exists as a function of the current bonus, which we denote by g^*. Then at the beginning of this time period, the optimal expected utility for the trader is $g^*(b)$. On the other hand, at the end of this trading period, the trader's optimal expected utility given the updated bonus should be $g^*(b + Y - C)$. Therefore the total expected utility at the beginning of this period should be $\mathbb{E}[U(C) + \beta_T g^*(b+Y-C)]$. Then the trader's optimal strategy should be the one that maximizes this quantity. In other words, at each time period, the trader is actually solving the following one-period problem:

$$\sup_{(Y,C)\in\mathcal{A}(k,b)} \mathbb{E}[U(C) + \beta_T g^*(b + Y - C)], \tag{3.1}$$

and the optimal value of this optimization problem should be $g^*(b)$. In the following chapters, we will make the above argument rigorous by showing that under certain conditions, for the power and logarithmic utilities defined in Section 2.2, there exists a unique function g^* such that $g^*(b)$ agrees with the quantity (3.1), and it is indeed the optimal value function of the trader's infinite-horizon problem. For the power and logarithmic utilities, the trader's optimal strategy and the optimal value function can be solved explicitly.

3.2 The $\mathcal{O}$ Operator: One Period Problem without Escrow

For a measurable function $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$, define the operator $\mathcal{O}$ as:

$$\mathcal{O}g(b) = \sup_{(Y,C)\in\mathcal{A}(1,b)} \mathbb{E}\left[U(C) + \beta_T g(b + Y - C)\right], \tag{3.2}$$

where the constraint $\mathcal{C}$ in the feasible set $\mathcal{A}(1, b)$ is defined by (2.15). Recall Remark 2.2 to ensure that the expectation on the right hand side of (3.2) is well-defined. It is obvious that $g_1 \le g_2$ implies $\mathcal{O}g_1 \le \mathcal{O}g_2$. For a utility function U, define the set of functions

$$\mathcal{G}_U = \{g : g = A_1 U + A_2, A_1 \ge 0, A_2 \in \mathbb{R}\}.$$

We will show that for any utility function U in the three categories described in Section 2.2, if $g \in \mathcal{G}_U$, then $\mathcal{O}g \in \mathcal{G}_U$. We prove this for power utility functions and the logarithmic utility function, respectively.

3.2.1 Power Utility

Lemma 3.1. *Let U be defined as in (2.6) for $0 < p < 1$, and (2.7) for $p > 1$. For $p > 0$, $p \neq 1$, let $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ be defined as*

$$g(b) = \frac{A_1}{1-p} b^{1-p} + A_2, \quad b \ge 0, \tag{3.3}$$

with constants $A_1 \ge 0$ and $A_2 \in \mathbb{R}$. Then

$$\mathcal{O}g(b) = \frac{A_1'}{1-p} b^{1-p} + A_2', \tag{3.4}$$

where

$$A_1' = \left(1 + (\beta_T A_1)^{\frac{1}{p}}\right)^p \exp\left\{\frac{\theta^2(1-p)}{2p}\right\},$$

$$A_2' = \beta_T A_2.$$

The supremum in the definition of $\mathcal{O}g$ is achieved by the strategy (Y^*, C^*), where

$$Y^* = \left[\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z_1^{-\frac{1}{p}} - 1\right] b, \tag{3.5}$$

and

$$C^* = \frac{b + Y^*}{1 + (\beta_T A_1)^{\frac{1}{p}}} \tag{3.6}$$

$$= \frac{b \exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z_1^{-\frac{1}{p}}}{1 + (\beta_T A_1)^{\frac{1}{p}}}. \tag{3.7}$$

Let $B^* = b + Y^* - C^*$. Then

$$B^* = \frac{1}{1 + (\beta_T A_1)^{-\frac{1}{p}}} \exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z_1^{-\frac{1}{p}} b. \tag{3.8}$$

Proof: For U defined in (2.6) and (2.7), and g defined in (3.3),

$$\mathcal{O}g(b) = \sup_{(Y,C)\in\mathcal{A}(1,b)} \mathbb{E}[f(C,Y)],$$

where

$$f(c,y) = \frac{1}{1-p} c^{1-p} + \frac{\beta_T A_1}{1-p}(b + y - c)^{1-p} + \beta_T A_2.$$

For any fixed y, solving

$$\frac{\partial}{\partial c} f(c,y) = 0,$$

i.e.,

$$c^{-p} - \beta_T A_1 (b + y - c)^{-p} = 0,$$

we get

$$c = \frac{b + y}{1 + (\beta_T A_1)^{\frac{1}{p}}}, \tag{3.9}$$

which is always less than or equal to $b + y$ for $\beta_T \in (0,1)$ and $A_1 \geq 0$. Since

$$\frac{\partial^2}{\partial c^2} f(c,y) = -p\left[c^{-p-1} + \beta_T A_1 (b + y - c)^{-p-1}\right] < 0$$

for $0 \le c \le b+y$, the value of c given in (3.9) maximizes $f(c,y)$ over the interval $[0,b+y]$[2]. Thus given any investment strategy Y, the optimal consumption strategy C^* as a function of Y is given by

$$C^*(Y) = \frac{b+Y}{1+(\beta_T A_1)^{\frac{1}{p}}}. \tag{3.10}$$

Substituting this into the right hand side of (3.2), we get

$$\mathcal{O}g(b) = \sup_{\mathbb{E}[YZ_1]=0, Y\ge -b} \mathbb{E}\left[h(Y)\right], \tag{3.11}$$

where

$$\begin{aligned}
h(y) &= f\left(\frac{b+y}{1+(\beta_T A_1)^{\frac{1}{p}}}, y\right) \\
&= \frac{1}{1-p}\left(\frac{b+y}{1+(\beta_T A_1)^{\frac{1}{p}}}\right)^{1-p} + \frac{\beta_T A_1}{1-p}\left(\frac{(\beta_T A_1)^{\frac{1}{p}}(b+y)}{1+(\beta_T A_1)^{\frac{1}{p}}}\right)^{1-p} + \beta_T A_2 \\
&= \frac{1}{1-p}\left(\frac{b+y}{1+(\beta_T A_1)^{\frac{1}{p}}}\right)^{1-p}\left[1+(\beta_T A_1)^{\frac{1}{p}}\right] + \beta_T A_2 \\
&= \frac{1}{1-p}\left(1+(\beta_T A_1)^{\frac{1}{p}}\right)^{p}(b+y)^{1-p} + \beta_T A_2.
\end{aligned}$$

Since $h(y)$ is strictly increasing and concave with respect to y for $y \ge -b$, by, e.g., Theorem 7.6 in Chapter 3 of [6], the optimal Y is given by

$$Y^* = I(\lambda Z_1),$$

where I is the inverse of h', and the positive constant λ is chosen so that

$$\mathbb{E}[Y^* Z_1] = 0.$$

We have

$$h'(y) = \left(1+(\beta_T A_1)^{\frac{1}{p}}\right)^{p}(b+y)^{-p}, \quad y \ge -b,$$

[2] Here we adopt the convention that zero to a negative power is $+\infty$. See Remark 2.1.

and the inverse of h' is

$$I(\psi) = \left(1 + (\beta_T A_1)^{\frac{1}{p}}\right) \psi^{-\frac{1}{p}} - b, \quad \psi \geq 0.$$

Notice that regardless of the value λ, we have

$$I(\lambda Z_1) \geq -b.$$

We solve for λ from the equation

$$\begin{aligned} 0 &= \mathbb{E}[Y^* Z_1] \\ &= \frac{1 + (\beta_T A_1)^{\frac{1}{p}}}{\lambda^{\frac{1}{p}}} \mathbb{E}\left[Z_1^{1-\frac{1}{p}}\right] - b \\ &= \frac{1 + (\beta_T A_1)^{\frac{1}{p}}}{\lambda^{\frac{1}{p}}} \exp\left\{-\frac{\theta^2(p-1)}{2p^2}\right\} - b, \end{aligned}$$

where in the last step we used the result of Lemma B.2 on page 93 with $a = 1 - \frac{1}{p}$. Thus

$$\frac{1}{\lambda^{\frac{1}{p}}} = \frac{b \exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\}}{1 + (\beta_T A_1)^{\frac{1}{p}}},$$

and therefore

$$Y^* = I(\lambda Z_1) = \left[\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z_1^{-\frac{1}{p}} - 1\right] b,$$

which is (3.5). Then from (3.10) we obtain (3.6). Substituting (3.5) into (3.6) we get (3.7).

By (3.6), we have

$$B^* = \frac{1}{1 + (\beta_T A_1)^{-\frac{1}{p}}}(b + Y^*).$$

Then by (3.5), we get (3.8).

It remains to substitute (3.7) and (3.5) into the right hand side of (3.2) to compute $\mathcal{O}g(b)$.

We compute the two terms on the right hand side of (3.2) separately. In particular,

$$\begin{aligned}
\mathbb{E}\left[U(C^*)\right] &= \frac{1}{1-p}\left(\frac{b\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\}}{1+(\beta_T A_1)^{\frac{1}{p}}}\right)^{1-p}\mathbb{E}\left[Z_1^{-\frac{1-p}{p}}\right] \\
&= \frac{b^{1-p}}{1-p}\frac{\exp\left\{-\frac{\theta^2(1-p)^2}{2p^2}\right\}}{\left(1+(\beta_T A_1)^{\frac{1}{p}}\right)^{1-p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\} \\
&= \frac{1}{\left(1+(\beta_T A_1)^{\frac{1}{p}}\right)^{1-p}}\exp\left\{\frac{\theta^2(1-p)}{2p}\right\}U(b). \qquad (3.12)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left[\beta_T g(b+Y^*-C^*)\right] &= \mathbb{E}\left[\beta_T A_1 U\left((\beta_T A_1)^{\frac{1}{p}}C^*\right)+\beta_T A_2\right] \\
&= (\beta_T A_1)(\beta_T A_1)^{\frac{1-p}{p}}\mathbb{E}\left[U(C^*)\right]+\beta_T A_2 \\
&= (\beta_T A_1)^{\frac{1}{p}}\mathbb{E}\left[U(C^*)\right]+\beta_T A_2. \qquad (3.13)
\end{aligned}$$

Summing up (3.12) and (3.13), we obtain (3.4).

□

Corollary 3.2. *Assume*

$$\beta_T \exp\left\{\frac{\theta^2(1-p)}{2p}\right\} < 1. \qquad (3.14)$$

Then for $U(c) = \frac{1}{1-p}c^{1-p}, p > 0, p \neq 1$, *there exist a unique function* $g^* = A_1^* U + A_2^* \in \mathcal{G}_U$ *such that*

$$\mathcal{O}g^*(b) = g^*(b), \quad \forall b \geq 0.$$

The constants A_1^* *and* A_2^* *are given by*

$$A_1^* = \frac{\exp\left\{\frac{\theta^2(1-p)}{2p}\right\}}{\left(1-\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}\right)^p}, \qquad (3.15)$$

$$A_2^* = 0. \qquad (3.16)$$

PROOF: By Lemma 3.1 we must have

$$A_1^* = \left(1 + (\beta_T A_1^*)^{\frac{1}{p}}\right)^p \exp\left\{\frac{\theta^2(1-p)}{2p}\right\},$$

$$A_2^* = \beta_T A_2^*,$$

which implies (3.15) and (3.16). By the assumption (3.14),

$$1 - \beta_T^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\} > 0,$$

and hence $A_1^* > 0$. □

Remark 3.3. We will prove in Chapter 4 that the fixed point g^* of the operator $\mathcal{O}$ is the trader's value function. Therefore, the assumption (3.14), which is necessary and sufficient for A_1^* to be finite, is necessary and sufficient for the trader to have a finite value function. Notice that for $p > 1$, (3.14) holds for all $\beta_T \in (0,1)$ and all θ. Therefore the assumption (3.14) is needed only for $0 < p < 1$.

To facilitate future discussions, we introduce the notation for the mapping from the initial bonus b and the state of the Radon-Nikodym derivative at the period end, Z_1, to the optimizing strategy and the updated bonus after applying this strategy. Specifically, for $p > 0$, $p \neq 1$, $A \geq 0$, $b \geq 0$ and a strictly positive random variable Z, define mappings $\mathfrak{Y}(p;\cdot,\cdot)$, $\mathfrak{C}(p,A;\cdot,\cdot)$ and $\mathfrak{B}(p,A;\cdot,\cdot)$ as

$$\mathfrak{Y}(p;b,Z) \triangleq \left[\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z^{-\frac{1}{p}} - 1\right] b, \tag{3.17}$$

$$\mathfrak{C}(p,A;b,Z) \triangleq \frac{b \exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z^{-\frac{1}{p}}}{1 + (\beta_T A)^{\frac{1}{p}}}, \tag{3.18}$$

$$\mathfrak{B}(p,A;b,Z) \triangleq \frac{1}{1 + (\beta_T A)^{-\frac{1}{p}}} \exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z^{-\frac{1}{p}} b. \tag{3.19}$$

Then if p is the parameter of the power utility function as defined in (2.6) and (2.7), A_1 is the multiplicative factor on the utility function in the definition of g as in (3.3), and Z_1 is the one-period Radon-Nikodym derivative changing the physical measure to the risk-neutral measure at the end of the first period, then the optimal strategy (Y^*, C^*) attaining the supremum in the definition of $\mathcal{O}g$ is given by

$$Y^* = \mathfrak{Y}(p; b, Z_1),$$

$$C^* = \mathfrak{C}(p, A_1; b, Z_1).$$

Moreover, the updated bonus at the end of the time period after applying this optimal strategy is

$$B^* = \mathfrak{B}(p, A_1; b, Z_1).$$

Remark 3.4. Following the notations defined above and in Lemma 3.1, by (3.13), when $A_2 = 0$, we have

$$\frac{\mathbb{E}\left[U\big(\mathfrak{C}(p, A_1; b, Z_1)\big)\right]}{\mathbb{E}\left[\beta_T g\big(b + \mathfrak{Y}(p; b, Z_1) - \mathfrak{C}(p, A_1; b, Z_1)\big)\right]} = \frac{1}{(\beta_T A_1)^{\frac{1}{p}}}. \tag{3.20}$$

In particular, this holds when g is the fixed point g^*.

We may slightly generalize (3.20) as the follows. During the $(k+1)$-th period, given a realization of the initial bonus $B_k = b$, and the Radon-Nikodym derivative at the end of the period Z_{k+1}, we have

$$\frac{\mathbb{E}\left[U\big(\mathfrak{C}(p, A_1; B_k, Z_{k+1})\big)\big|B_k = b\right]}{\mathbb{E}\left[\beta_T g\big(B_k + \mathfrak{Y}(p; B_k, Z_{k+1}) - \mathfrak{C}(p, A_1; B_k, Z_{k+1})\big)\big|B_k = b\right]} = \frac{1}{(\beta_T A_1)^{\frac{1}{p}}}. \tag{3.21}$$

Notice that the right hand side of (3.21) is a constant that does not depend on b. Therefore we have

$$\frac{\mathbb{E}\left[U\big(\mathfrak{C}(p, A_1; B_k, Z_{k+1})\big)\right]}{\mathbb{E}\left[\beta_T g\big(B_k + \mathfrak{Y}(p; B_k, Z_{k+1}) - \mathfrak{C}(p, A_1; B_k, Z_{k+1})\big)\right]} = \frac{1}{(\beta_T A_1)^{\frac{1}{p}}}. \tag{3.22}$$

3.2.2 Logarithmic Utility

Lemma 3.5. *Let U be defined as in (2.8) and $g : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ be defined as*

$$g(b) = A_1 \log(b) + A_2, \tag{3.23}$$

with constants $A_1 \geq 0$ and $A_2 \in \mathbb{R}$. Then

$$\mathcal{O}g(b) = A_1' \log(b) + A_2', \tag{3.24}$$

where

$$\begin{aligned} A_1' &= 1 + \beta_T A_1, \\ A_2' &= (1 + \beta_T A_1)\left(\frac{\theta^2}{2} - \log(1 + \beta_T A_1)\right) + \beta_T A_1 \log(\beta_T A_1) + \beta_T A_2. \end{aligned}$$

The supremum in the definition of $\mathcal{O}g$ is achieved by the optimizing strategy (Y^*, C^*), where

$$Y^* = \left(\frac{1}{Z_1} - 1\right) b, \tag{3.25}$$

and

$$\begin{aligned} C^* &= \frac{b + Y^*}{1 + \beta_T A_1} && (3.26) \\ &= \frac{b}{(1 + \beta_T A_1) Z_1}. && (3.27) \end{aligned}$$

Let $B^* = b + Y^* - C^*$. Then

$$B^* = \frac{1}{1 + (\beta_T A_1)^{-1}} Z_1^{-1} b. \tag{3.28}$$

Proof: For U defined in (2.8) and g defined in (3.23),

$$\mathcal{O}g(b) = \sup_{(Y,C) \in \mathcal{A}(1,b)} \mathbb{E}[f(C,Y)],$$

where

$$f(c,y) = \log c + \beta_T A_1 \log(b + y - c) + \beta_T A_2.$$

For any fixed $y \geq -b$ solve

$$\frac{\partial}{\partial c} f(c, y) = 0,$$

i.e.,

$$\frac{1}{c} - \frac{\beta_T A_1}{b + y - c} = 0.$$

We get

$$c = \frac{b + y}{1 + \beta_T A_1}, \tag{3.29}$$

which is always less than or equal to $b + y$ for $\beta_T \in (0, 1)$ and $A_1 \geq 0$. Since

$$\frac{\partial^2}{\partial c^2} f(c, y) = -\frac{1}{c^2} - \frac{\beta_T A_1}{(b + y - c)^2} < 0,$$

for $0 \leq c \leq b+y$, the value of c given in (3.29) maximizes $f(c, y)$ over the interval $[0, b+y]$. The end points are included due to Remark 2.1. Thus given any investment strategy Y satisfying $Y \geq -b$ almost surely, the optimal consumption strategy C^* as a function of Y is given by

$$C^*(Y) = \frac{b + Y}{1 + \beta_T A_1}, \tag{3.30}$$

which is (3.26). Substituting this into the right hand side of (3.2), we get

$$\mathcal{O}g(b) = \sup_{\mathbb{E}[Y Z_1]=0, Y \geq -b} \mathbb{E}[h(Y)], \tag{3.31}$$

where

$$\begin{aligned} h(y) &= \log\left(\frac{b + y}{1 + \beta_T A_1}\right) + \beta_T A_1 \log\left(\frac{\beta_T A_1 (b + y)}{1 + \beta_T A_1}\right) + \beta_T A_2 \\ &= (1 + \beta_T A_1) \log(b + y) - (1 + \beta_T A_1) \log(1 + \beta_T A_1) + \beta_T A_1 \log(\beta_T A_1) + \beta_T A_2. \end{aligned}$$

Since $h(y)$ is strictly increasing and concave with respect to y for $y \geq -b$, again by Theorem 7.6 in Chapter 3 of [6] the optimal Y for the problem (3.31) is given by

$$Y^* = I(\lambda Z_1), \tag{3.32}$$

where I is the inverse function of h', and the positive constant λ is such that

$$\mathbb{E}[I(\lambda Z_1)Z_1] = 0. \tag{3.33}$$

Also notice that since

$$\lim_{y\downarrow -b} h'(y) = +\infty,$$

Y^* given by (3.32) automatically satisfies $Y^* \geq -b$. We have

$$h'(y) = \frac{1+\beta_T A_1}{b+y}, \quad y \geq -b,$$

and the inverse of h' is

$$I(\psi) = \frac{1+\beta_T A_1}{\psi} - b, \quad \psi \geq 0.$$

We solve for λ from the equation

$$\begin{aligned} 0 &= \mathbb{E}[Y^* Z_1] \\ &= \frac{1+\beta_T A_1}{\lambda} - b, \end{aligned}$$

where in the last step we used the fact that $\mathbb{E}[Z_1] = 1$. Thus

$$\frac{1+\beta_T A_1}{\lambda} = b,$$

and

$$Y^* = \left(\frac{1}{Z_1} - 1\right) b,$$

which is (3.25). Substituting (3.25) into (3.26) we have (3.27).

By (3.26), we have

$$B^* = \frac{1}{1+(\beta_T A_1)^{-1}}(b+Y^*).$$

Then by (3.25), we get (3.28).

It remains to substitute (3.27) and (3.25) into the right hand side of (3.2) to compute $\mathcal{O}g$. We compute the two terms on the right hand side of (3.2) separately. In particular,

$$\begin{aligned}\mathbb{E}[U(C^*)] &= \log\left(\frac{b}{1+\beta_T A_1}\right) - \mathbb{E}[\log(Z_1)] \\ &= \log(b) + \frac{\theta^2}{2} - \log(1+\beta_T A_1). \qquad (3.34)\end{aligned}$$

$$\begin{aligned}\mathbb{E}[\beta_T g(b+Y^*-C^*)] &= \beta_T \mathbb{E}\left[A_1 \log\left(\frac{\beta_T A_1 b}{(1+\beta_T A_1)Z_1}\right) + A_2\right] \\ &= \beta_T \mathbb{E}\left[A_1 \log(\beta_T A_1 C^*) + A_2\right] \\ &= \beta_T A_1 \mathbb{E}[U(C^*)] + \beta_T A_1 \log(\beta_T A_1) + \beta_T A_2. \qquad (3.35)\end{aligned}$$

Summing up (3.34) and (3.35), we obtain (3.24). □

Corollary 3.6. *For $U(c) = \log c$, there exist a unique function $g^* = A_1^* U + A_2^* \in \mathcal{G}_U$ such that*

$$\mathcal{O}g^*(b) = g^*(b), \forall b \geq 0.$$

The constants A_1^* and A_2^* are given by

$$A_1^* = \frac{1}{1-\beta_T}, \qquad (3.36)$$

$$A_2^* = \frac{\theta^2}{2(1-\beta_T)^2} + \frac{1}{1-\beta_T}\log(1-\beta_T) + \frac{\beta_T}{(1-\beta_T)^2}\log\beta_T. \qquad (3.37)$$

Proof: Solving

$$A_1^* = 1 + \beta_T A_1^*$$

we get (3.36). Substituting (3.36) into

$$A_2^* = (1+\beta_T A_1^*)\left(\frac{\theta^2}{2} - \log(1+\beta_T A_1^*)\right) + \beta_T A_1^* \log(\beta_T A_1^*) + \beta_T A_2^*$$

and solving for A_2^*, we get (3.37). □

Remark 3.7. An interesting observation is that equations (3.25) - (3.27), (3.36) and (3.28) agree with (3.5) - (3.7), (3.15) and (3.8), respectively, if we substitute $p = 1$ into the latter equations. This follows from the fact that the derivative of the power utility function $U(c) = \frac{1}{1-p}c^{1-p}$ becomes the same as that of the logarithm utility function $U(c) = \log c$ when p converges to 1.

Similarly as for the power utility, we define the mapping from the initial bonus b and the random variable Z_1 to the optimal strategy and the updated bonus after applying this strategy. For $A \geq 0$, $b \geq 0$ and a strictly positive random variable Z, define mappings $\mathfrak{Y}(1;\cdot,\cdot)$, $\mathfrak{C}(1, A;\cdot,\cdot)$ and $\mathfrak{B}(1, A;\cdot,\cdot)$ as

$$\mathfrak{Y}(1; b, Z) \triangleq \left(\frac{1}{Z} - 1\right) b, \tag{3.38}$$

$$\mathfrak{C}(1, A; b, Z) \triangleq \frac{b}{(1 + \beta_T A)Z}, \tag{3.39}$$

$$\mathfrak{B}(1, A; b, Z) \triangleq \frac{1}{1 + (\beta_T A)^{-1}} Z^{-1} b. \tag{3.40}$$

Here, in light of Remark 3.7, we use parameter 1 in the notation of these mappings where p is used in the corresponding notation for the power utility case. Then if A_1 is the multiplicative factor on the utility function in the definition of g as in (3.23), and Z_1 is the one-period Radon-Nikodym derivative changing the physical measure to the risk-neutral measure at the end of the first period, then the optimal strategy (Y^*, C^*) attaining the supremum in the definition of $\mathcal{O}g$ is given by

$$Y^* = \mathfrak{Y}(1; b, Z_1),$$

$$C^* = \mathfrak{C}(1, A_1; b, Z_1).$$

Moreover, the updated bonus at the end of the time period after applying this optimal strategy is

$$B^* = \mathfrak{B}(1, A_1; b, Z_1).$$

3.3 The $\widetilde{\mathcal{O}}$ Operator: One Period Problem with Escrow

Now we define an operator similar to $\mathcal{O}$ with the only difference being the admissible set of the optimization problem on the right hand side of (3.2). This operator corresponds to the one-period problem of the trader when the bonus is escrowed.

For a measurable function $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$, define the operator $\widetilde{\mathcal{O}}$ as:

$$\widetilde{\mathcal{O}}g(b) = \sup_{(Y,C)\in\mathcal{A}(1,b)} \mathbb{E}\left[U(C) + \beta_T g(b + Y - C)\right], \tag{3.41}$$

where the constraint $\mathcal{C}$ in the feasible set $\mathcal{A}(1, b)$ is defined by (2.16). Remark 2.2 ensures that the expectation on the right hand side of (3.41) is well-defined. Again, $g_1 \leq g_2$ implies $\widetilde{\mathcal{O}}g_1 \leq \widetilde{\mathcal{O}}g_2$. We will show similar results for the fixed point of the operator $\widetilde{\mathcal{O}}$, in the set $\mathcal{G}_U$, for the function U being a power or logarithm utility function.

3.3.1 Power Utility

Lemma 3.8. *Let U be defined as in (2.6) for $0 < p < 1$, and (2.7) for $p > 1$. For $p > 0$, $p \neq 1$, let $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ be defined by (3.3). Then*

$$\widetilde{\mathcal{O}}g(b) = \frac{\widetilde{A}'_1}{1-p} b^{1-p} + \widetilde{A}'_2, \tag{3.42}$$

where

$$\widetilde{A}'_1 = \mathbb{E}\left[(\eta Z_1)^{1-\frac{1}{p}} \mathbb{I}_{\{\eta Z_1 \geq 1\}}\right] + \mathbb{P}\left\{\eta Z_1 \leq 1\right\} + (\beta_T A_1)^{\frac{1}{p}} \eta^{1-\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\},$$

$$\widetilde{A}'_2 = \beta_T A_2,$$

and η is the unique solution of the equation

$$\mathbb{E}\left[\left(\eta^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right] = (\beta_T A_1)^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}. \tag{3.43}$$

The supremum in the definition of $\widetilde{\mathcal{O}}g$ is achieved by the optimizing strategy $\left(\widetilde{Y}^*, \widetilde{C}^*\right)$, where

$$\widetilde{Y}^* = \begin{cases} b\left[(\eta Z_1)^{-\frac{1}{p}}\left(1 + (\beta_T A_1)^{\frac{1}{p}}\right) - 1\right], & \eta Z_1 \geq 1, \\ b(\eta Z_1)^{-\frac{1}{p}}(\beta_T A_1)^{\frac{1}{p}}, & \eta Z_1 \leq 1. \end{cases} \tag{3.44}$$

$$\widetilde{C}^* = \begin{cases} b(\eta Z_1)^{-\frac{1}{p}}, & \eta Z_1 \geq 1, \\ b, & \eta Z_1 \leq 1, \end{cases} \tag{3.45}$$

Following the notation of Lemma 3.8. Let $\widetilde{B}^* = b + \widetilde{Y}^* - \widetilde{C}^*$. Then

$$\widetilde{B}^* = b(\beta_T A_1)^{\frac{1}{p}}(\eta Z_1)^{-\frac{1}{p}}. \tag{3.46}$$

Proof: As in the proof of Lemma 3.1, set

$$f(c, y) = \frac{1}{1-p}c^{1-p} + \frac{\beta_T A_1}{1-p}(b + y - c)^{1-p} + \beta_T A_2.$$

For any fixed y, solving c from

$$\frac{\partial}{\partial c} f(c, y) = 0,$$

we get

$$c = \frac{b + y}{1 + (\beta_T A_1)^{\frac{1}{p}}},$$

which is always less than or equal to $b + y$ for $\beta_T \in (0, 1)$ and $A_1 \geq 0$. It is less than or equal to b if and only if $y \leq b(\beta_T A_1)^{\frac{1}{p}}$. For fixed y, the function $c \mapsto f(c, y)$ is increasing on the interval $\left(0, \frac{b+y}{1+(\beta_T A_1)^{\frac{1}{p}}}\right)$. Thus given any investment strategy Y, the optimal consumption strategy C^* as a function of Y is given by

$$\widetilde{C}^*(Y) = \begin{cases} \frac{b+Y}{1+(\beta_T A_1)^{\frac{1}{p}}}, & -b \leq Y \leq b(\beta_T A_1)^{\frac{1}{p}}, \\ b, & Y \geq b(\beta_T A_1)^{\frac{1}{p}}. \end{cases} \tag{3.47}$$

Substituting this into the right hand side of (3.41), we get

$$\widetilde{\mathcal{O}}g(b) = \sup_{\mathbb{E}[YZ_1]=0, Y \geq -b} \mathbb{E}\left[\widetilde{h}(Y)\right], \tag{3.48}$$

where

$$\widetilde{h}(y) = \begin{cases} \frac{1}{1-p}\left(1+(\beta_T A_1)^{\frac{1}{p}}\right)^p (b+y)^{1-p} + \beta_T A_2, & -b \leq y \leq b(\beta_T A_1)^{\frac{1}{p}}, \\ \frac{1}{1-p} b^{1-p} + \frac{\beta_T A_1}{1-p} y^{1-p} + \beta_T A_2, & y \geq b(\beta_T A_1)^{\frac{1}{p}}. \end{cases}$$

The function $\widetilde{h}$ is strictly increasing and concave for $-b \leq y \leq b(\beta_T A_1)^{\frac{1}{p}}$ and $y \geq b(\beta_T A_1)^{\frac{1}{p}}$, respectively. Notice that $\widetilde{h}$ is continuous at $b(\beta_T A_1)^{\frac{1}{p}}$. Indeed,

$$\lim_{y \uparrow b(\beta_T A_1)^{\frac{1}{p}}} \widetilde{h}(y) = \lim_{y \downarrow b(\beta_T A_1)^{\frac{1}{p}}} \widetilde{h}(y) = \frac{b^{1-p}}{1-p}\left(1+(\beta_T A_1)^{\frac{1}{p}}\right) + \beta_T A_2.$$

Moreover, we have

$$\widetilde{h}'(y) = \begin{cases} \left(1+(\beta_T A_1)^{\frac{1}{p}}\right)^p (b+y)^{-p}, & -b < y \leq b(\beta_T A_1)^{\frac{1}{p}}, \\ \beta_T A_1 y^{-p}, & y \geq b(\beta_T A_1)^{\frac{1}{p}}. \end{cases}$$

and $\widetilde{h}'$ is defined and continuous at $b(\beta_T A_1)^{\frac{1}{p}}$ with $\widetilde{h}'\left(b(\beta_T A_1)^{\frac{1}{p}}\right) = b^{-p}$.

Therefore $\widetilde{h}(y)$ is continuous and concave on the whole interval $[-b, +\infty)$ and the optimal Y is given by

$$\widetilde{Y}^* = \widetilde{I}(\widetilde{\lambda} Z_1),$$

where $\widetilde{I}$ is the inverse of $\widetilde{h}'$, and the positive constant $\widetilde{\lambda}$ is chosen so that

$$\mathbb{E}[\widetilde{Y}^* Z_1] = 0.$$

The inverse of $\widetilde{h}'$ is

$$\widetilde{I}(\psi) = \begin{cases} \psi^{-\frac{1}{p}}\left(1+(\beta_T A_1)^{\frac{1}{p}}\right) - b, & \psi \geq b^{-p}, \\ \psi^{-\frac{1}{p}}(\beta_T A_1)^{\frac{1}{p}}, & 0 < \psi \leq b^{-p}. \end{cases} \tag{3.49}$$

Notice that regardless of the value $\widetilde{\lambda} > 0$, we have

$$\widetilde{I}(\widetilde{\lambda} Z_1) \geq -b.$$

We solve for $\widetilde{\lambda}$ from the equation

$$\begin{aligned}
0 &= \mathbb{E}[\widetilde{Y}^* Z_1] \\
&= \mathbb{E}\Big[\left((\widetilde{\lambda} Z_1)^{-\frac{1}{p}}\left(1 + (\beta_T A_1)^{\frac{1}{p}}\right) - b\right) Z_1 \mathbb{I}_{\{\widetilde{\lambda} Z_1 \geq b^{-p}\}} + (\widetilde{\lambda} Z_1)^{-\frac{1}{p}} (\beta_T A_1)^{\frac{1}{p}} Z_1 \mathbb{I}_{\{\widetilde{\lambda} Z_1 \leq b^{-p}\}}\Big] \\
&= \mathbb{E}\left[(\widetilde{\lambda} Z_1)^{-\frac{1}{p}} (\beta_T A_1)^{\frac{1}{p}} Z_1\right] + \mathbb{E}\left[\left((\widetilde{\lambda} Z_1)^{-\frac{1}{p}} - b\right) Z_1 \mathbb{I}_{\{\widetilde{\lambda} Z_1 \geq b^{-p}\}}\right].
\end{aligned}$$

We multiply by $\widetilde{\lambda}^{\frac{1}{p}}$, set $\eta = \widetilde{\lambda} b^p$ and define $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_1$ to obtain

$$(\beta_T A_1)^{\frac{1}{p}} \mathbb{E}\left[Z_1^{1-\frac{1}{p}}\right] + \mathbb{E}\left[Z_1^{1-\frac{1}{p}} \mathbb{I}_{\{\eta Z_1 \geq 1\}}\right] - \eta^{\frac{1}{p}} \mathbb{Q}\left\{\eta Z_1 \geq 1\right\} = 0$$

By Lemma B.2, we have

$$\begin{aligned}
\mathbb{E}\left[Z_1^{1-\frac{1}{p}}\right] &= \exp\left\{\frac{\theta^2}{2}\left(1 - \frac{1}{p}\right)\left(-\frac{1}{p}\right)\right\} \\
&= \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}.
\end{aligned}$$

Thus η is characterized by the equation

$$\eta^{\frac{1}{p}} \mathbb{Q}\left\{\eta Z_1 \geq 1\right\} - \mathbb{E}\left[Z_1^{1-\frac{1}{p}} \mathbb{I}_{\{\eta Z_1 \geq 1\}}\right] = (\beta_T A_1)^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}. \tag{3.50}$$

By Lemma B.3, the left hand side of (3.50) is

$$\begin{aligned}
\eta^{\frac{1}{p}} \mathbb{Q}\left\{\frac{1}{Z_1} \leq \eta\right\} - \mathbb{E}^{\mathbb{Q}}\left[Z_1^{-\frac{1}{p}} \mathbb{I}_{\left\{\frac{1}{Z_1} \leq \eta\right\}}\right] &= \eta^{\frac{1}{p}} \mathbb{P}\left\{Z_1 \leq \eta\right\} - \mathbb{E}\left[Z_1^{\frac{1}{p}} \mathbb{I}_{\{Z_1 \leq \eta\}}\right] \\
&= \mathbb{E}\left[\left(\eta^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right],
\end{aligned} \tag{3.51}$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation under measure $\mathbb{Q}$. Thus we can rewrite equation (3.50) as

$$\mathbb{E}\left[\left(\eta^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right] = (\beta_T A_1)^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\},$$

which is the equation (3.43) in the statement of Lemma 3.8. The left hand side of the above equation is 0 when $\eta = 0$, converges to ∞ as $\eta \to \infty$, and is strictly increasing. Therefore, there exists a unique η satisfying (3.43). Note that η determined by (3.43) does not depend on b. It is thus advantageous to write the optimal $\widetilde{Y}^*$ and $\widetilde{C}^*$ in terms of η rather than $\widetilde{\lambda}$. Recall that

$$\eta = \widetilde{\lambda} b^p, \quad \widetilde{\lambda} = \eta b^{-p}.$$

Substituting into (3.49) with $\psi = \widetilde{\lambda} Z_1$, we get (3.44). Note that $\widetilde{Y}^* \leq b(\beta_T A_1)^{\frac{1}{p}}$ if and only if $\eta Z_1 \geq 1$. Then by (3.47), we get

$$\widetilde{C}^* = \begin{cases} b(\eta Z_1)^{-\frac{1}{p}}, & \eta Z_1 \geq 1, \\ b, & \eta Z_1 \leq 1, \end{cases}$$

which is (3.45).

It remains to substitute (3.45) and (3.44) into the right hand side of (3.41) to compute $\widetilde{\mathcal{O}} g(b)$. We compute the two terms on the right hand side of (3.41) separately. In particular,

$$\mathbb{E}[U(\widetilde{C}^*)] = \frac{1}{1-p} b^{1-p} \left(\mathbb{E}\left[(\eta Z_1)^{-\frac{1-p}{p}} \mathbb{I}_{\{\eta Z_1 \geq 1\}} \right] + \mathbb{P}\left\{ \eta Z_1 \leq 1 \right\} \right). \tag{3.52}$$

By (3.45) and (3.44), we can derive (3.46). Indeed,

- When $\eta Z_1 \geq 1$,

$$\begin{aligned} \widetilde{B}^* &= b + \widetilde{Y}^* - \widetilde{C}^* \\ &= b(\eta Z_1)^{-\frac{1}{p}} \left(1 + (\beta_T A_1)^{\frac{1}{p}} \right) - b(\eta Z_1)^{-\frac{1}{p}} \\ &= b(\beta_T A_1)^{\frac{1}{p}} (\eta Z_1)^{-\frac{1}{p}}. \end{aligned} \tag{3.53}$$

- When $\eta Z_1 \leq 1$,

$$\begin{aligned}
\widetilde{B}^* &= b + \widetilde{Y}^* - \widetilde{C}^* \\
&= b + b(\eta Z_1)^{-\frac{1}{p}}(\beta_T A_1)^{\frac{1}{p}} - b \\
&= b(\beta_T A_1)^{\frac{1}{p}}(\eta Z_1)^{-\frac{1}{p}}. \qquad (3.54)
\end{aligned}$$

Thus the second term on the right hand side of (3.41) is

$$\begin{aligned}
\mathbb{E}\left[\beta_T g(b + \widetilde{Y}^* - \widetilde{C}^*)\right] &= \frac{\beta_T A_1}{1-p} b^{1-p}(\beta_T A_1)^{\frac{1-p}{p}} \mathbb{E}\left[(\eta Z_1)^{-\frac{1-p}{p}}\right] + \beta_T A_2 \\
&= (\beta_T A_1)^{\frac{1}{p}} \frac{b^{1-p}}{1-p} \eta^{\frac{p-1}{p}} \exp\left\{\frac{\theta^2}{2}\left(-\frac{1-p}{p}\right)\left(-\frac{1}{p}\right)\right\} + \beta_T A_2 \\
&= \frac{b^{1-p}}{1-p}(\beta_T A_1)^{\frac{1}{p}} \eta^{\frac{p-1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\} + \beta_T A_2. \qquad (3.55)
\end{aligned}$$

Summing up (3.52) and (3.55), we obtain (3.42). □

Corollary 3.9. *Assume $p > 0$, $p \neq 1$, and (3.14). Then there exists a unique function $\widetilde{g}^* = \widetilde{A}_1^* U + \widetilde{A}_2^* \in \mathcal{G}_U$ such that*

$$\widetilde{\mathcal{O}}\widetilde{g}^*(b) = \widetilde{g}^*(b), \forall b \geq 0.$$

The constant η^* is the unique solution to

$$\frac{\left(\mathbb{E}\left[\left(\eta^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right]\right)^p}{\mathbb{E}\left[(\eta - Z_1)^+\right] + 1} = \beta_T \left(\mathbb{E}\left[Z_1^{\frac{1}{p}}\right]\right)^p, \qquad (3.56)$$

and the constant $\widetilde{A}_1^*$ is determined by

$$\widetilde{A}_1^* = \eta^* \mathbb{Q}\left\{\eta^* Z_1 \geq 1\right\} + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\}. \qquad (3.57)$$

The constant $\widetilde{A}_2^*$ is 0.

PROOF: Firstly, we show that there exists a unique solution $\eta^* \in (0,\infty)$ for (3.56). By Lemma B.2 on page 93, the right hand side of (3.56) is equal to $\beta_T \exp\left\{\frac{\theta^2}{2}\left(\frac{1}{p}-1\right)\right\}$, which is obviously positive, and less than 1 given the assumption (3.14). The left hand side of (3.56) is 0 when $\eta = 0$. To see its limit as $\eta \to \infty$, we divide both the top and the bottom by η^* to rewrite it as

$$\frac{\left(\mathbb{E}\left[\left(1-\left(\frac{Z_1}{\eta}\right)^{\frac{1}{p}}\right)^+\right]\right)^p}{\mathbb{E}\left[\left(1-\frac{Z_1}{\eta}\right)^+\right]+\frac{1}{\eta}},$$

which has limit 1 as $\eta \to \infty$. Therefore, (3.56) has a solution. To see that it is unique, we show that the left hand side of (3.56) is strictly increasing with respect to η. Let

$$\begin{aligned}\nu_1(\eta) &\triangleq \mathbb{E}\left[\left(1-\left(\frac{Z_1}{\eta}\right)^{\frac{1}{p}}\right)^+\right]\\ &= \int_0^\eta \left(1-\left(\frac{z}{\eta}\right)^{\frac{1}{p}}\right) f_Z(z)dz,\end{aligned}$$

where f_Z is the probability density function of Z_1. Then

$$\nu_1'(\eta) = \frac{1}{p\eta}\int_0^\eta \left(\frac{z}{\eta}\right)^{\frac{1}{p}} f_Z(z)dz > 0.$$

Also let

$$\begin{aligned}\nu_2(\eta) &\triangleq \mathbb{E}\left[\left(1-\frac{Z_1}{\eta}\right)^+\right]+\frac{1}{\eta}\\ &= \int_0^\eta \left(1-\frac{z}{\eta}\right) f_Z(z)dz + \frac{1}{\eta},\end{aligned}$$

and compute

$$\begin{aligned}\nu_2'(\eta) &= \frac{1}{\eta}\int_0^\eta \frac{z}{\eta} f_Z(z)dz - \frac{1}{\eta^2}\\ &= \frac{1}{\eta}\left(\int_0^\eta \frac{z}{\eta} f_Z(z)dz - \int_0^\infty \frac{z}{\eta} f_Z(z)dz\right)\\ &< 0,\end{aligned}$$

where the second equality follows because $\mathbb{E}[Z_1] = 1$. Therefore the derivative of the left hand side of (3.56) with respect to η is

$$\frac{d}{d\eta}\frac{\nu_1^p(\eta)}{\nu_2(\eta)} = \frac{1}{\nu_2^2(\eta)}\left[p\nu_1^{p-1}(\eta)\nu_1'(\eta)\nu_2(\eta) - \nu_1^p(\eta)\nu_2'(\eta)\right] > 0.$$

In other words, the left hand side of (3.56) is strictly increasing in η. Therefore (3.56) has a unique solution, which we denote by η^*. It is strictly positive since the right hand side of (3.56) is positive.

From (3.57) and Lemma B.3 we have

$$\begin{aligned}
\widetilde{A}_1^* &= \eta^*\mathbb{Q}\left\{\eta^* Z_1 \geq 1\right\} + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\} \\
&= \eta^*\mathbb{Q}\left\{\eta^* Z_1 \geq 1\right\} - \mathbb{P}\left\{\eta^* Z_1 \geq 1\right\} + 1 \\
&= \eta^*\mathbb{Q}\left\{\frac{1}{Z_1} \leq \eta^*\right\} - \mathbb{P}\left\{Z_1 \geq \frac{1}{\eta^*}\right\} + 1 \\
&= \eta^*\mathbb{P}\left\{Z_1 \leq \eta^*\right\} - \mathbb{Q}\left\{\frac{1}{Z_1} \geq \frac{1}{\eta^*}\right\} + 1 \\
&= \eta^*\mathbb{P}\left\{Z_1 \leq \eta^*\right\} - \mathbb{E}\left[Z_1\mathbb{I}_{\{Z_1 \leq \eta^*\}}\right] + 1 \\
&= \mathbb{E}\left[(\eta^* - Z_1)^+\right] + 1.
\end{aligned} \tag{3.58}$$

Substituting this into (3.56) and using the fact that[3]

$$\mathbb{E}\left[Z_1^{\frac{1}{p}}\right] = \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}, \tag{3.59}$$

we obtain

$$\mathbb{E}\left[\left(\eta^{*\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right] = (\beta_T\widetilde{A}_1^*)^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}. \tag{3.60}$$

[3]For calculation of (3.59) see Lemma B.2 on page 93

From (3.51) and (3.60) we have

$$\begin{aligned}
&\eta^*\mathbb{Q}\left\{\eta^* Z_1 \geq 1\right\} + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\} && (3.61)\\
&\quad = (\eta^*)^{1-\frac{1}{p}}\eta^{*\frac{1}{p}}\mathbb{Q}\left\{\eta^* Z_1 \geq 1\right\} + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\}\\
&\quad = (\eta^*)^{1-\frac{1}{p}}\mathbb{E}^{\mathbb{Q}}\left[Z_1^{-\frac{1}{p}}\mathbb{I}_{\{\eta^* Z_1\geq 1\}}\right] + \left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}}(\eta^*)^{1-\frac{1}{p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\} + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\}\\
&\quad = \mathbb{E}\left[(\eta^* Z_1)^{1-\frac{1}{p}}\mathbb{I}_{\{\eta^* Z_1\geq 1\}}\right] + \left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}}(\eta^*)^{1-\frac{1}{p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\} + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\} && (3.62)
\end{aligned}$$

According to (3.57), the left hand side of (3.62) is $\widetilde{A}_1^*$, and hence the right hand side is as well, i.e.,

$$\widetilde{A}_1^* = \mathbb{E}\left[(\eta^* Z_1)^{1-\frac{1}{p}}\mathbb{I}_{\{\eta^* Z_1\geq 1\}}\right] + \mathbb{P}\left\{\eta^* Z_1 \leq 1\right\} + \left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}}(\eta^*)^{1-\frac{1}{p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}. \tag{3.63}$$

This last equation shows that if η^* is the unique solution of (3.56), $\widetilde{A}_1^*$ is given by (3.57), and we take $\eta = \eta^*$, $A_1 = \widetilde{A}_1^*$, and $A_2 = 0$ in Lemma 3.8, then $\widetilde{A}_1' = \widetilde{A}_1^*$ and $\widetilde{A}_2' = 0$, i.e., $g^* = \widetilde{A}_1^* U$ is a fixed point of $\widetilde{\mathcal{O}}$.

To see that this is the only fixed point of $\widetilde{\mathcal{O}}$ in $\mathcal{G}_U$, we note from Lemma 3.8 that every fixed point $\bar{g} = \bar{A}_1 U + \bar{A}_2$ must have the properties $\bar{A}_2 = \beta_T \bar{A}_2$ and $\bar{A}_1$ is a solution to (3.63), where η^* is a solution of (3.60). This implies $\bar{A}_2 = 0$ and η^*, $\bar{A}_1$ satisfy (3.62). This shows that η^*, $\bar{A}_1$ satisfy (3.57). But (3.57) implies (3.58), which combined with (3.60), yields (3.56). However, we have shown that the solution of (3.56) is unique, and $\bar{A}_1$ is then forced by (3.57) to be $\widetilde{A}_1^*$.

□

In the proof of Lemma 3.8, we have computed the updated bonus after applying the optimal strategy $b + \widetilde{Y}^* - \widetilde{C}^*$. We summarize the result in the following Corollary. Notice that although the optimal strategy $\left(\widetilde{Y}^*, \widetilde{C}^*\right)$ has different analytic formulas on the sets $\{\eta Z_1 \geq 1\}$

and $\{\eta Z_1 \leq 1\}$, the formula for the updated bonus turns out to be the same on these sets.

For operator $\widetilde{\mathcal{O}}$, we again introduce notation for the mapping from the initial bonus b and the random variable Z_1 to the optimal strategy. First, given $A_1 \geq 0$, we denote the unique η satisfying (3.43) by $\eta(p; A_1)$. Then, for $p > 0$, $p \neq 1$, $A \geq 0$, $b \geq 0$ and a strictly positive random variable Z, we define the mappings $\widetilde{\mathfrak{Y}}(p, A; \cdot, \cdot)$, $\widetilde{\mathfrak{C}}(p, A; \cdot, \cdot)$ and $\widetilde{\mathfrak{B}}(p, A; \cdot, \cdot)$ as

$$\widetilde{\mathfrak{Y}}(p, A; b, Z) \triangleq \begin{cases} b\left[\big(\eta(p; A)Z\big)^{-\frac{1}{p}}\left(1 + (\beta_T A)^{\frac{1}{p}}\right) - 1\right], & \eta(p; A)Z \geq 1, \\ b\big(\eta(p; A)Z\big)^{-\frac{1}{p}}(\beta_T A)^{\frac{1}{p}}, & \eta(p; A)Z \leq 1, \end{cases} \tag{3.64}$$

$$\widetilde{\mathfrak{C}}(p, A; b, Z) \triangleq \begin{cases} b\big(\eta(p; A)Z\big)^{-\frac{1}{p}}, & \eta(p; A)Z \geq 1, \\ b, & \eta(p; A)Z \leq 1. \end{cases} \tag{3.65}$$

$$\widetilde{\mathfrak{B}}(p, A; b, Z) \triangleq b(\beta_T A)^{\frac{1}{p}}\big(\eta(p; A)Z\big)^{-\frac{1}{p}}. \tag{3.66}$$

Then, if p is the parameter of the power utility function as defined in (2.6) and (2.7), A_1 is the multiplicative factor on the utility function in the definition of g as in (3.3), and Z_1 is the one-period Radon-Nikodym derivative changing the physical measure to the risk-neutral measure at the end of the first period, then the optimal strategy $\left(\widetilde{Y}^*, \widetilde{C}^*\right)$ attaining the supremum in the definition of $\widetilde{\mathcal{O}}g$ is given by

$$\widetilde{Y}^* = \widetilde{\mathfrak{Y}}(p, A_1; b, Z_1),$$
$$\widetilde{C}^* = \widetilde{\mathfrak{C}}(p, A_1; b, Z_1).$$

Moreover, the updated bonus at the end of the time period after applying this optimal strategy is

$$\widetilde{B}^* = \widetilde{\mathfrak{B}}(p, A_1; b, Z_1).$$

Remark 3.10. Following the notation defined above and in Lemma 3.8, from (3.52), (3.55), when $A_2 = 0$, we have

$$\begin{aligned}&\frac{\mathbb{E}\left[U\big(\widetilde{\mathfrak{C}}(p, A_1; b, Z_1)\big)\right]}{\mathbb{E}\left[\beta_T g\big(b + \widetilde{\mathfrak{Y}}(p, A_1; b, Z_1) - \widetilde{\mathfrak{C}}(p, A_1; b, Z_1)\big)\right]}\\&= \frac{\mathbb{E}\left[\big(\eta(p;A)Z_1\big)^{-\frac{1-p}{p}} \mathbb{I}_{\{\eta(p;A)Z_1 \geq 1\}}\right] + \mathbb{P}\big(\eta(p;A)Z_1 \leq 1\big)}{(\beta_T A_1)^{\frac{1}{p}}\, \eta(p;A)^{1-\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}}.\end{aligned} \tag{3.67}$$

In particular, this holds when g is the fixed point $\widetilde{g}^*$. Notice that the right hand side is a constant that does not depend on b and depends on Z_1 only through the distribution of Z_1, which is the same as the distribution of Z_{k+1} conditioned on $\mathcal{F}_k$ for $k = 1, 2, \cdots$. Therefore similarly as in Remark 3.4, in the $(k+1)$-th period, given an $\mathcal{F}_k$-measurable bonus account value B_k at the beginning of period k and the Radon-Nikodym derivative Z_{k+1} at the end of the period k, we have

$$\begin{aligned}&\frac{\mathbb{E}\left[U\big(\widetilde{\mathfrak{C}}(p, A_1; B_k, Z_{k+1})\big)\right]}{\mathbb{E}\left[\beta_T g\big(B_k + \widetilde{\mathfrak{Y}}(p, A_1; B_k, Z_{k+1}) - \widetilde{\mathfrak{C}}(p, A_1; B_k, Z_{k+1})\big)\right]}\\&= \frac{\mathbb{E}\left[\big(\eta(p;A)Z_1\big)^{-\frac{1-p}{p}} \mathbb{I}_{\{\eta(p;A)Z_1 \geq 1\}}\right] + \mathbb{P}\big(\eta(p;A)Z_1 \leq 1\big)}{(\beta_T A_1^*)^{\frac{1}{p}}\, \eta(p;A)^{1-\frac{1}{p}} \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}}.\end{aligned} \tag{3.68}$$

3.3.2 Logarithmic Utility

Lemma 3.11. *Let $g : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ be defined by (3.23). Then*

$$\widetilde{\mathcal{O}}g(b) = \widetilde{A}_1' \log(b) + \widetilde{A}_2', \tag{3.69}$$

where

$$\begin{aligned}\widetilde{A}_1' &= 1 + \beta_T A_1,\\ \widetilde{A}_2' &= \beta_T A_1 \left(\log \frac{\beta_T A_1}{\eta} + \frac{\theta^2}{2}\right) - \mathbb{E}\left[\log(\eta Z_1)\mathbb{I}_{\{\eta Z_1 \geq 1\}}\right] + \beta_T A_2,\end{aligned}$$

and η is the unique solution of the equation

$$\beta_T A_1 = \mathbb{E}\big[(\eta Z_1 - 1)^+\big]. \tag{3.70}$$

The supremum in the definition of $\widetilde{\mathcal{O}}g$ is achieved by the optimal strategy $(\widetilde{Y}^*, \widetilde{C}^*)$, where

$$\widetilde{Y}^* = \begin{cases} b\left(\frac{1+\beta_T A_1}{\eta Z_1} - 1\right), & \eta Z_1 \geq 1, \\ \frac{b\beta_T A_1}{\eta Z_1}, & \eta Z_1 \leq 1, \end{cases} \tag{3.71}$$

$$\widetilde{C}^* = \begin{cases} \frac{b}{\eta Z_1}, & \eta Z_1 \geq 1, \\ b, & \eta Z_1 \leq 1. \end{cases} \tag{3.72}$$

Let $\widetilde{B}^* = b + \widetilde{Y}^* - \widetilde{C}^*$. Then

$$\widetilde{B}^* = \frac{b\beta_T A_1}{\eta Z_1}. \tag{3.73}$$

PROOF: Define

$$f(c, y) = \log c + \beta_T A_1 \log(b + y - c) + \beta_T A_2.$$

For any fixed $y \geq -b$ solving c from

$$\frac{\partial}{\partial c} f(c, y) = 0,$$

we get

$$c = \frac{b + y}{1 + \beta_T A_1},$$

which is always less than or equal to $b+y$ for $\beta_T \in (0, 1)$ and $A_1 \geq 0$. Thus given any investment strategy Y satisfying $Y \geq -b$ almost surely, the optimal consumption strategy C^* as a function of Y is given by

$$\widetilde{C}^*(Y) = \begin{cases} \frac{b+Y}{1+\beta_T A_1}, & -b \leq Y \leq b\beta_T A_1, \\ b, & Y \geq b\beta_T A_1. \end{cases} \tag{3.74}$$

Substituting this into the right hand side of (3.41), we get

$$\widetilde{\mathcal{O}}g(b) = \sup_{\mathbb{E}[YZ_1]=0, Y \geq -b} \mathbb{E}[\widetilde{h}(Y)], \tag{3.75}$$

where if $y \le b\beta_T A_1$, then

$$\widetilde{h}(y) = (1+\beta_T A_1)\log(b+y) - (1+\beta_T A_1)\log(1+\beta_T A_1) + \beta_T A_1 \log(\beta_T A_1) + \beta_T A_2;$$

and if $y \ge b\beta_T A_1$, then

$$\widetilde{h}(y) = \log b + \beta_T A_1 \log y + \beta_T A_2.$$

Observe that $\widetilde{h}(y)$ is strictly increasing and concave with respect to y for $-b \le y \le b\beta_T A_1$ and $y \ge b\beta_T A_1$, respectively. Notice also that $\widetilde{h}$ is continuous at $b\beta_T A_1$. Indeed,

$$\lim_{y \uparrow b\beta_T A_1} \widetilde{h}(y) = \lim_{y \downarrow b\beta_T A_1} \widetilde{h}(y) = (1+\beta_T A_1)\log b + \beta_T A_1 \log(\beta_T A_1) + \beta_T A_2.$$

Moreover, we have

$$\widetilde{h}'(y) = \begin{cases} \frac{1+\beta_T A_1}{b+y}, & -b \le y \le b\beta_T A_1, \\ \frac{\beta_T A_1}{y}, & y \ge b\beta_T A_1. \end{cases}$$

and $\widetilde{h}'$ is defined and continuous at $b\beta_T A_1$ with $\widetilde{h}'(b\beta_T A_1) = \frac{1}{b}$. Therefore $\widetilde{h}'$ is continuous, $\widetilde{h}$ is strictly increasing and strictly concave on the whole interval $[-b, +\infty]$, and the optimal Y is given by

$$\widetilde{Y}^* = \widetilde{I}(\widetilde{\lambda} Z_1), \tag{3.76}$$

where $\widetilde{I}$ is the inverse function of $\widetilde{h}'$, and the positive constant $\widetilde{\lambda}$ is such that

$$\mathbb{E}[\widetilde{Y}^* Z_1] = 0. \tag{3.77}$$

Also notice that since

$$\lim_{y \downarrow -b} h'(x) = +\infty,$$

Y^* given by (3.76) automatically satisfies $Y^* \ge -b$. The inverse of $\widetilde{h}'$ is

$$\widetilde{I}(\psi) = \begin{cases} \frac{1+\beta_T A_1}{\psi} - b, & \psi \ge \frac{1}{b}, \\ \frac{\beta_T A_1}{\psi}, & 0 \le \psi \le \frac{1}{b}. \end{cases} \tag{3.78}$$

Notice that regardless of the value $\widetilde{\lambda} > 0$, we have

$$\widetilde{I}(\widetilde{\lambda} Z_1) \geq -b.$$

We solve for $\widetilde{\lambda}$ from the equation

$$\begin{aligned}
0 &= \mathbb{E}[\widetilde{Y}^* Z_1] \\
&= \mathbb{E}\left[\left(\frac{1+\beta_T A_1}{\widetilde{\lambda} Z_1} - b\right) Z_1 \mathbb{I}_{\left\{\widetilde{\lambda} Z_1 \geq \frac{1}{b}\right\}} + \frac{\beta_T A_1}{\widetilde{\lambda} Z_1} Z_1 \mathbb{I}_{\left\{\widetilde{\lambda} Z_1 \leq \frac{1}{b}\right\}}\right] \\
&= \frac{\beta_T A_1}{\widetilde{\lambda}} + \frac{1}{\widetilde{\lambda}} \mathbb{E}\left[\mathbb{I}_{\left\{Z_1 \geq \frac{1}{\widetilde{\lambda} b}\right\}}\right] - b\mathbb{E}\left[Z_1 \mathbb{I}_{\left\{Z_1 \geq \frac{1}{\widetilde{\lambda} b}\right\}}\right].
\end{aligned}$$

We multiply by $\widetilde{\lambda}$ and set $\eta = \widetilde{\lambda} b$ to obtain

$$\beta_T A_1 + \mathbb{E}\left[\mathbb{I}_{\{\eta Z_1 \geq 1\}}\right] - \eta \mathbb{E}\left[Z_1 \mathbb{I}_{\{\eta Z_1 \geq 1\}}\right] = 0,$$

which can be rewritten as

$$\beta_T A_1 = \mathbb{E}\left[(\eta Z_1 - 1)^+\right]. \tag{3.79}$$

The right hand side is 0 when $\eta = 0$, converges to ∞ as $\eta \to \infty$, and is strictly increasing in η. Therefore, for a given A_1, there exists a unique η satisfying (3.79).

Notice that η determined by (3.79) does not depend on b. Similarly to Lemma 3.8, we write the optimal $\widetilde{Y}^*$ and $\widetilde{C}^*$ in terms of η rather than $\widetilde{\lambda}$. Recall that $\widetilde{\lambda} = \frac{\eta}{b}$. Substituting into (3.78) with $\psi = \widetilde{\lambda} Z_1$ we get (3.71). Then by (3.74) we get (3.72).

It remains to substitute (3.72) and (3.71) into the right hand side of (3.41) to compute $\widetilde{\mathcal{O}}g(b)$. Again, we compute the two terms separately.

$$\begin{aligned}
\mathbb{E}[U(\widetilde{C}^*)] &= \mathbb{E}\left[\log\left(\frac{b}{\eta Z_1}\right) \mathbb{I}_{\{\eta Z_1 \geq 1\}} + \log b \mathbb{I}_{\{\eta Z_1 \leq 1\}}\right] \\
&= \log b - \mathbb{E}\left[\log(\eta Z_1) \mathbb{I}_{\{\eta Z_1 \geq 1\}}\right].
\end{aligned} \tag{3.80}$$

By (3.72) and (3.71), we can derive (3.73). Indeed,

- When $\eta Z_1 \geq 1$,

$$\begin{aligned}\widetilde{B}^* &= b + \widetilde{Y}^* - \widetilde{C}^* \\ &= b + b\left(\frac{1+\beta_T A_1}{\eta Z_1} - 1\right) - \frac{b}{\eta Z_1} \\ &= \frac{b\beta_T A_1}{\eta Z_1}. \end{aligned} \tag{3.81}$$

- When $\eta Z_1 \leq 1$,

$$\begin{aligned}\widetilde{B}^* &= b + \widetilde{Y}^* - \widetilde{C}^* \\ &= b + \frac{b\beta_T A_1}{\eta Z_1} - b \\ &= \frac{b\beta_T A_1}{\eta Z_1}. \end{aligned} \tag{3.82}$$

Thus the second term on the right hand side of (3.41) is

$$\begin{aligned}\mathbb{E}\left[\beta_T g(b + \widetilde{Y}^* - \widetilde{C}^*)\right] &= \mathbb{E}\left[\beta_T\left(A_1 \log \frac{b\beta_T A_1}{\eta Z_1} + A_2\right)\right] \\ &= \beta_T\left(A_1 \log \frac{b\beta_T A_1}{\eta} + A_2\right) - \beta_T A_1 \mathbb{E}[\log Z_1] \\ &= \beta_T\left(A_1 \log \frac{b\beta_T A_1}{\eta} + A_2\right) + \beta_T A_1 \frac{\theta^2}{2}. \end{aligned} \tag{3.83}$$

Summing up (3.80) and (3.83), we obtain (3.69).

□

Corollary 3.12. *For $U(c) = \log c$, there exist a unique function $\widetilde{g}^* = \widetilde{A}_1^* U + \widetilde{A}_2^* \in \mathcal{G}_U$ such that*

$$\widetilde{\mathcal{O}}\widetilde{g}^*(b) = \widetilde{g}^*(b), \forall b \geq 0.$$

The constants $\widetilde{A}_1^*$ *and* $\widetilde{A}_2^*$ *are given by*

$$\widetilde{A}_1^* = \frac{1}{1-\beta_T}, \tag{3.84}$$

$$\widetilde{A}_2^* = \frac{\beta_T}{(1-\beta_T)^2}\left(\log\frac{\beta_T}{\eta^*(1-\beta_T)} + \frac{\theta^2}{2}\right) - \frac{1}{1-\beta_T}\mathbb{E}[\log(\eta^* Z_1)\mathbb{I}_{\{\eta^* Z_1 \geq 1\}}], \tag{3.85}$$

where η^* *is uniquely determined by*

$$\frac{\beta_T}{1-\beta_T} = \mathbb{E}\left[(\eta^* Z_1 - 1)^+\right]. \tag{3.86}$$

Proof: Solving

$$\widetilde{A}_1^* = 1 + \beta_T \widetilde{A}_1^*$$

we get (3.84). Substituting (3.84) into (3.79) we get (3.86), which uniquely determines η^*. Substituting (3.84) into

$$\widetilde{A}_2^* = \beta_T \widetilde{A}_1^* \left(\log\frac{\beta_T \widetilde{A}_1^*}{\eta^*} + \frac{\theta^2}{2}\right) - \mathbb{E}\left[\log(\eta^* Z_1)\mathbb{I}_{\{\eta^* Z_1 \geq 1\}}\right] + \beta_T \widetilde{A}_2^*$$

and solving for $\widetilde{A}_2^*$ we get (3.85).

$\square$

Similarly as in the previous section, we summarize how the bonus is updated after the trader applies the optimal strategy, for which the derivation is already performed in the proof of Lemma 3.11.

Remark 3.13. Similarly as for the $\mathcal{O}$ operator, we make the observation that (3.71), (3.72) and (3.73) agree with (3.44), (3.45) and (3.46) respectively, when $p = 1$ in the latter equations. Moreover, the value of $\widetilde{A}_1^*$ and η^* determined by (3.84) and (3.86) also agree with those determined by (3.56) and (3.57) when we take $p = 1$ in the latter equations. Indeed, when $p = 1$, (3.56) becomes

$$\frac{\mathbb{E}[(\eta^*-Z_1)^+]}{\mathbb{E}[(\eta^*-Z_1)^+]+1}=\beta_T,$$

which implies

$$\mathbb{E}\left[(\eta^*-Z_1)^+\right]=\frac{\beta_T}{1-\beta_T}. \tag{3.87}$$

By Lemma B.3,

$$\begin{aligned}\mathbb{E}\left[(\eta^*-Z_1)^+\right]&=\mathbb{E}^{\mathbb{Q}}\left[(\eta^*-\frac{1}{Z_1})^+\right]\\&=\mathbb{E}\left[Z_1(\eta^*-\frac{1}{Z_1})^+\right]\\&=\mathbb{E}\left[(\eta^*Z_1-1)^+\right].\end{aligned}$$

Thus (3.87) is equivalent to (3.86). Moreover, By (3.59), (3.57) can be written as

$$\widetilde{A}_1^*=\mathbb{E}\left[(\eta^*-Z_1)^+\right]+1=\frac{1}{1-\beta_T},$$

where the last step is by (3.87). Therefore we get (3.84).

Finally, we define the mapping from the initial bonus b and the random variable Z_1 to the optimal strategy in this case. First, given $A_1\geq 0$, we denote the unique η satisfying (3.70) by $\eta(1;A_1)$. Then for $A\geq 0$, $b\geq 0$ and a strictly positive random variable Z, define mappings $\widetilde{\mathfrak{Y}}(1,A;\cdot,\cdot)$, $\widetilde{\mathfrak{C}}(1,A;\cdot,\cdot)$ and $\widetilde{\mathfrak{B}}(1,A;\cdot,\cdot)$ as

$$\widetilde{\mathfrak{Y}}(1,A;b,Z)\triangleq\begin{cases}b\left(\frac{1+\beta_TA}{\eta(1;A)Z}-1\right), & \eta(1;A)Z\geq 1,\\ \frac{b\beta_TA}{\eta(1;A)Z}, & \eta(1;A)Z\leq 1,\end{cases} \tag{3.88}$$

$$\widetilde{\mathfrak{C}}(1,A;b,Z)\triangleq\begin{cases}\frac{b}{\eta(1;A)Z}, & \eta(1;A)Z\geq 1,\\ b, & \eta(1;A)Z\leq 1.\end{cases} \tag{3.89}$$

$$\widetilde{\mathfrak{B}}(1,A;b,Z)\triangleq\frac{b\beta_TA}{\eta(1;A)Z}. \tag{3.90}$$

Here again we use subscript 1 to be consistent with the notation for the power utility case, considering Remark 3.13. Then if A_1 is the multiplicative factor on the utility function in the definition of g as in (3.23), and Z_1 is the one-period Radon-Nikodym derivative changing the physical measure to the risk-neutral measure at the end of the first period, then the optimal strategy $\left(\widetilde{Y}^*, \widetilde{C}^*\right)$ attaining the supremum in the definition of $\widetilde{\mathcal{O}}g$ is given by

$$\widetilde{Y}^* = \widetilde{\mathfrak{Y}}(1, A_1; b, Z_1),$$
$$\widetilde{C}^* = \widetilde{\mathfrak{C}}(1, A_1; b, Z_1).$$

Moreover, the updated bonus at the end of the time period after applying this optimal strategy is

$$\widetilde{B}^* = \widetilde{\mathfrak{B}}(1, A_1; b, Z_1).$$

4 Fixed Points are Value Functions

In the previous chapter, we have shown that for the power and logarithmic utility functions, under certain conditions, the operators $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ have unique affine-type fixed points. In this chapter, we will show that these fixed points are the value functions of the trader's infinite-horizon problem. We will focus on the proof for the case without escrow. The proof for the case with escrow is similar, and we shall make proper remarks along the way articulating how the arguments should be adjusted for the case with escrow.

Recall that given $k \in \mathbb{N}$ and $b \geq 0$, $\mathcal{A}(k, b)$ denotes the set of feasible strategies at the k-th period with bonus left from the previous period being b. Moreover, given the current amount in the bonus account being b, $\mathcal{A}(b)$ and $\mathcal{A}_n(b)$ denote the sets of feasible strategies for the infinite-horizon problem and finite-horizon problem with n periods, respectively.

Given a feasible strategy $\pi = \{(X_{k+1}, C_{k+1})\}_{k=0}^{\infty} \in \mathcal{A}(b)$ for the infinite-period problem with initial bonus $B_0 = b$, let v^π be the trader's total discounted expected utility from applying this strategy:

$$v^\pi(b) = \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k U(C_{k+1})\right],$$

and let v^* be the value function of the infinite-horizon problem:

$$v^*(b) = \sup_{\pi \in \mathcal{A}(b)} v^\pi(b). \tag{4.1}$$

We are going to show that given the utility function, either power or logarithm, we have

$$g^*(b) = v^*(b), \quad \forall b \geq 0,$$

where g^* is the fixed point in $\mathcal{G}_U$ of the operator $\mathcal{O}$ (or $\widetilde{\mathcal{O}}$ in the case with escrow).

First we prove a lemma, which says that if the trader only trades a finite number of periods, then the optimal value of her finite-period problem can be obtained by iterating the operator $\mathcal{O}$ or $\widetilde{\mathcal{O}}$ in the case without escrow or with escrow, respectively. In addition, the optimal value of the trader's finite period problem is attainable.

Lemma 4.1. *Let $v_0^*(b) \equiv 0$, $\forall b \geq 0$. For any $n \in \mathbb{N}$, let v_n^* be the value function of the n-period problem:*

$$v_n^*(b) = \sup_{\left\{(Y_{k+1},C_{k+1})_{k=0}^{n-1}\right\} \in \mathcal{A}_n(b)} \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U(C_{k+1})\right]. \tag{4.2}$$

Then

$$v_{n+1}^*(b) = \mathcal{O}v_n^*(b), \quad \forall b \geq 0, \forall n = 0, 1, \cdots. \tag{4.3}$$

In particular,

$$v_n^* \in \mathcal{G}_U, \quad \forall n = 0, 1, \cdots,$$

and hence is Borel measurable so $\mathcal{O}v_n^*$ is defined. In addition, there exists an n-period strategy $\left\{\left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right)_{k=0}^{n-1}\right\} \in \mathcal{A}_n(b)$ such that

$$v_n^*(b) = \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U\left(C_{k+1}^{(n)}\right)\right]. \tag{4.4}$$

Proof: We will prove the lemma by induction. Firstly,

$$\begin{aligned} v_1^*(b) &= \sup_{\pi_1 \in \mathcal{A}_1(b)} \mathbb{E}\big[U(C_1)\big] \\ &= \sup_{(Y_1,C_1) \in \mathcal{A}(1,b)} \mathbb{E}\big[U(C_1) + \beta_T \cdot 0\big] \end{aligned}$$

Therefore (4.3) holds for $n = 0$ with $v_0^* \equiv 0$. Since $v_0^* \in \mathcal{G}_U$, by Lemma 3.1 or 3.5, depending on the definition of the utility function, we have $v_1^* \in \mathcal{G}_U$.

Let $Y_1^{(1)} = \mathfrak{Y}(p; b, Z_1)$, $C_1^{(1)} = \mathfrak{C}\,(p, 0; b, Z_1)$, where p is the parameter of the power utility function as in (2.6) and (2.7), or is 1 if the utility function is logarithmic. Then again by Lemma 3.1 or 3.5, $\left(Y_1^{(1)}, C_1^{(1)}\right) \in \mathcal{A}(1, b)$ and

$$v_1^*(b) = \mathbb{E}\left[U\left(C_1^{(1)}\right)\right] + \beta_T \cdot 0.$$

Therefore the Lemma holds for $n = 1$.

Next, suppose the Lemma holds for $n = 1, 2, \cdots, N$. We will show that the Lemma also holds for $n = N+1$. Let $\pi_{N+1} = \{(Y_{k+1}, C_{k+1})_{k=0}^{N}\} \in \mathcal{A}_{N+1}(b)$ be any $(N+1)$-period strategy with initial bonus b, and $v_{N+1}^{\pi_{N+1}}$ be the expected discounted utility associated with π_{N+1}:

$$v_{N+1}^{\pi_{N+1}}(b) = \mathbb{E}\left[\sum_{k=0}^{N} \beta_T^k U(C_{k+1})\right].$$

Then $\{(Y_{k+1}, C_{k+1})_{k=1}^{N}\}$ is an N-period strategy with initial bonus $B_1 \triangleq b + Y_1 - C_1$, and

$$\begin{aligned} v_{N+1}^{\pi_{N+1}}(b) &= \mathbb{E}\left[U(C_1) + \beta_T \mathbb{E}\left[\sum_{k=0}^{N-1} \beta_T^k U(C_{k+2}) \middle| B_1\right]\right] \\ &\leq \mathbb{E}\big[U(C_1) + \beta_T v_N^*(B_1)\big] \\ &\leq \mathcal{O} v_N^*(b). \end{aligned}$$

Notice that $v_N^* \in \mathcal{G}_U$ by the induction hypothesis, hence the expectation $\mathbb{E}[v_N^*(B_1)]$ is well defined with the convention in Remark 2.2. Maximizing over $\pi_{N+1}(b) \in \mathcal{A}_{N+1}(b)$, we obtain

$$\begin{aligned} v_{N+1}^*(b) &= \sup_{\pi_{N+1} \in \mathcal{A}_{N+1}(b)} v_{N+1}^{\pi_{N+1}}(b) \\ &\leq \mathcal{O} v_N^*(b). \end{aligned} \tag{4.5}$$

On the other hand, by definition of $\mathcal{O}$ and v_N^*, we have

$$\mathcal{O} v_N^*(b) = \sup_{(Y,C) \in \mathcal{A}(1,b)} \mathbb{E}\big[U(C) + \beta_T v_N^*(b + Y - C)\big].$$

For any $(Y, C) \in \mathcal{A}(1, b)$, given any realization (y, c) of (Y, C) at time 1, by the induction hypothesis, there exists an N-period strategy $\left\{\left(Y_{k+1}^{(N)}, C_{k+1}^{(N)}\right)\right\}_{k=0}^{N-1} \in \mathcal{A}_N(b + y - c)$ such that

$$v_N^*(b + y - c) = \mathbb{E}\left[\sum_{k=0}^{N-1} \beta_T^k U\left(C_{k+1}^{(N)}\right)\right].$$

Define the $(N+1)$-period strategy $\left\{\left(Y_{k+1}^{(N+1)}, C_{k+1}^{(N+1)}\right)\right\}_{k=0}^{N} \in \mathcal{A}_{N+1}(b)$ as the following:

$$\left(Y_{k+1}^{(N+1)}, C_{k+1}^{(N+1)}\right) = \begin{cases} (Y, C), & k = 0, \\ \left(Y_k^{(N)}, C_k^{(N)}\right), & k \geq 1. \end{cases} \tag{4.6}$$

Then

$$\begin{aligned} \mathbb{E}\big[U(C) + \beta_T v_N^*(b + Y - C)\big] =& \mathbb{E}\left[U(C) + \beta_T \mathbb{E}\left[\sum_{k=0}^{N-1} \beta_T^k U\left(C_{k+1}^{(N)}\right) \Big| (Y, C)\right]\right] \\ =& \mathbb{E}\left[\sum_{k=0}^{N} \beta_T^k U\left(C_{k+1}^{(N+1)}\right)\right] \\ \leq& v_{N+1}^*(b). \end{aligned}$$

Taking the supremum over $(Y, C) \in \mathcal{A}(1, b)$, we have

$$\mathcal{O} v_N^*(b) \leq v_{N+1}^*(b).$$

Therefore

$$v_{N+1}^*(b) = \mathcal{O} v_N^*(b).$$

In other words, (4.3) also holds for $n = N+1$. Notice that $v_N^* \in \mathcal{G}_U$ by the induction hypothesis. Thus by Lemma 3.1 or 3.5, we have $v_{N+1}^* \in \mathcal{G}_U$ and there exists $(Y^*, C^*) \in \mathcal{A}(b)$ such that

$$\mathcal{O} v_N^*(b) = \mathbb{E}\left[U(C^*) + \beta_T v_N^*(b + Y^* - C^*)\right].$$

Let $\left\{\left(Y_{k+1}^{(N+1)}, C_{k+1}^{(N+1)}\right)\right\}_{k=0}^{N} \in \mathcal{A}_{N+1}(b)$ be defined as in (4.6) with (Y, C) being (Y^*, C^*). Then

$$\begin{aligned} v_{N+1}^*(b) &= \mathcal{O} v_N^*(b) \\ &= \mathbb{E}\big[U(C^*) + \beta_T v_N^*(b + Y^* - C^*)\big] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} \beta_T^k U\left(C_{k+1}^{(N+1)}\right)\right]. \end{aligned}$$

Thus we have shown that with the induction hypothesis, the Lemma holds also for $n = N + 1$. Therefore the Lemma holds for all $n \in \mathbb{N}$. $\square$

Remark 4.2. Lemma 4.1 and its proof also apply to the case with escrow when replacing $\mathcal{O}$ with $\widetilde{\mathcal{O}}$. The difference in the constraint $\mathcal{C}$ in the admissible sets for the cases with or without escrow does not affect the statement and the proof of this lemma.

In the following sections, we will prove for each utility function defined in Section 2.2 that the fixed point of the operators $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are the optimal value function of the trader's infinite-horizon problem without and with escrowing the bonus, respectively.

4.1 Positive Power Utility

In this section, we consider the power utility function with positive values.

Theorem 4.3. *Consider the utility function U defined by (2.6). Assume the inequality (3.14) holds. For any $b \geq 0$ let*

$$v^*(b) = \sup_{\{(Y_{k+1}, C_{k+1})\}_{k=0}^{\infty} \in \mathcal{A}(b)} \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k U(C_{k+1})\right],$$

and let g^* be the unique function in $\mathcal{G}_U$ such that for any $b \geq 0$

$$g^*(b) = \sup_{(Y,C) \in \mathcal{A}(1,b)} \mathbb{E}\big[U(C) + \beta_T g^*(b + Y - C)\big].$$

Then

$$v^*(b) = g^*(b), \quad \forall b \geq 0.$$

Proof: Given $b \geq 0$, let v_n^* be defined as in (4.2). Then for the utility function U given by (2.6), we have $v_n^* \leq v_{n+1}^*$ for all $n \in \mathbb{N}$. Indeed, for any feasible n-period strategy, $\pi_n =$

$\left\{\left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right)\right\}_{k=0}^{n-1} \in \mathcal{A}_n(b)$, the $(n+1)$-period strategy $\pi_{n+1}(\pi_n) \in \mathcal{A}_{n+1}(b)$ defined as

$$\pi_{n+1}(\pi_n) = \left\{\left(Y_{k+1}^{(n+1)}, C_{k+1}^{(n+1)}\right)\right\}_{k=0}^{n} \triangleq \begin{cases} \left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right), & k \leq n-1, \\ (0,0), & k = n, \end{cases}$$

is a feasible strategy for the $(n+1)$-period problem, and

$$v_n^{\pi_n} = v_{n+1}^{\pi_{n+1}(\pi_n)},$$

where the notation $\pi_{n+1}(\pi_n)$ indicates the dependency of π_{n+1} on π_n. Taking the supremum over all $\pi_n \in \mathcal{A}_n$ we have

$$v_n^* = \sup_{\pi_n \in \mathcal{A}_n} v_n^{\pi_n} = \sup_{\pi_n \in \mathcal{A}_n} v_{n+1}^{\pi_{n+1}(\pi_n)} \leq \sup_{\pi_{n+1} \in \mathcal{A}_{n+1}} v_{n+1}^{\pi_{n+1}} = v_{n+1}^*.$$

Therefore the sequence $\{v_n^*\}$ has a limit as $n \to \infty$. We denote it by v_∞.

For any feasible n-period strategy $\pi_n = \left\{\left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right)\right\}_{k=0}^{n-1} \in \mathcal{A}_n(b)$, consider the infinite period strategy

$$\pi(\pi_n) = \left\{\left(Y_{k+1}^{(\infty)}, C_{k+1}^{(\infty)}\right)\right\}_{k=0}^{\infty} \triangleq \begin{cases} \left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right), & k \leq n-1, \\ (0,0), & k > n. \end{cases}$$

With a similar argument as above, we also have $v_n^* \leq v^*$ for all $n \in \mathbb{N}$. Then

$$v_\infty \triangleq \lim_{n \to \infty} v_n^* \leq v^*. \tag{4.7}$$

Given $\varepsilon > 0$, by the optimality of v^*, there exists a feasible strategy $\pi^\varepsilon = \left\{\left(Y_{k+1}^\varepsilon, C_{k+1}^\varepsilon\right)\right\}_{k=0}^{\infty} \in \mathcal{A}(b)$ such that

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k U\left(C_{k+1}^\varepsilon\right)\right] \geq \begin{cases} v^*(b) - \varepsilon, & \text{if } v^*(b) < \infty, \\ \frac{1}{\varepsilon}, & \text{if } v^*(b) = \infty. \end{cases}$$

- If $v^*(b) < \infty$, then $\mathbb{E}\Big[\sum_{k=0}^{\infty} \beta_T^k U\left(C_{k+1}^{\varepsilon}\right)\Big] < \infty$. Then there exists a positive integer n^{ε} such that for all $n \geq n^{\varepsilon}$,

$$\mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U\left(C_{k+1}^{\varepsilon}\right)\right] \geq \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k U\left(C_{k+1}^{\varepsilon}\right)\right] - \varepsilon$$
$$\geq v^*(b) - 2\varepsilon.$$

Notice that $\left\{\left(Y_{k+1}^{\varepsilon}, C_{k+1}^{\varepsilon}\right)\right\}_{k=0}^{n-1} \in \mathcal{A}_n(b)$. Then by the optimality of $v_n^*(b)$, we have

$$\mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U\left(C_{k+1}^{\varepsilon}\right)\right] \leq v_n^*(b). \tag{4.8}$$

Thus for any $\varepsilon > 0$, there exists a positive integer n^{ε} such that for all $n \geq n^{\varepsilon}$,

$$v^*(b) - 2\varepsilon \leq v_n^*(b).$$

Letting n converge to ∞ and then ε converge to 0 we get

$$v^*(b) \leq v_{\infty}(b). \tag{4.9}$$

- If $v^*(b) = \infty$, then there exists a positive integer n^{ε} such that for all $n > n^{\varepsilon}$,

$$\mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U\left(C_{k+1}^{\varepsilon}\right)\right] \geq \frac{1}{2\varepsilon}.$$

Meanwhile (4.8) also holds in this case. Therefore we have for all $n > n^{\varepsilon}$,

$$v_n^*(b) \geq \frac{1}{2\varepsilon}.$$

Letting n converge to ∞ and then ε converge to 0 we get

$$v_{\infty}(b) = \infty.$$

To summarize, in both cases, we have for all $b \geq 0$

$$v_{\infty}(b) = v^*(b).$$

By Corollary 3.2, the operator $\mathcal{O}$ has a unique fixed point $g^* \in \mathcal{G}_U$, which is equal to U multiplied by a positive constant. Then since $U \geq 0$ and $v_0^* = 0$, we have $g^* \geq v_0^*$. By the monotonicity of $\mathcal{O}$ and Lemma 4.1, we have

$$g^* = \mathcal{O}g^* \geq \mathcal{O}v_0^* = v_1^*.$$

Then by iterating the above argument, we have

$$g^* \geq v_n^*, \quad \forall n \in N.$$

Letting $n \to \infty$, we get

$$g^* \geq v_\infty = v^*.$$

Thus

$$v^*(b) \leq g^*(b), \quad \forall b \geq 0. \tag{4.10}$$

Moreover, since $g^*(b) < \infty$, $\forall b \geq 0$, we also have

$$v^*(b) < \infty, \quad \forall b \geq 0.$$

Now we will show the opposite inequality

$$g^*(b) \leq v^*(b), \quad \forall b \geq 0.$$

By Corollary 3.2, for any $b \geq 0$, there exists a feasible strategy $(Y_1^g, C_1^g) \in \mathcal{A}(1, b)$ that attains the supremum in the definition of $\mathcal{O}g^*(b)$. In particular, $Y_1^g = \mathfrak{Y}(p; b, Z_1)$, $C_1^g = \mathfrak{C}(p, A_1^*; b, Z_1)$, where the constant A_1^* is defined by (3.15) and the functions $\mathfrak{Y}$ and $\mathfrak{C}$ are given by (3.17) and (3.18). Then for $n \in N$, we recursively apply Corollary 3.2 and define

$$B_n^g \triangleq B_{n-1}^g + Y_n^g - C_n^g,$$

$$Y_{n+1}^g \triangleq \mathfrak{Y}(p; B_n^g, Z_{n+1}),$$

$$C^g_{n+1} \triangleq \mathfrak{C}(p, A^*_1; B^g_n, Z_{n+1}),$$

where we take B^g_0 to be b. Then we get a sequence $\left\{\left(Y^g_{n+1}, C^g_{n+1}\right)\right\}_{n=0}^{\infty}$ such that $(Y^g_n, C^g_n) \in \mathcal{A}(n, B^g_{n-1})$, for all $n \in \mathbb{N}$, and

$$\begin{aligned} g^*(B^g_n) &= \mathcal{O}g^*(B^g_n) \\ &= \mathbb{E}\left[U(C^g_{n+1}) + \beta_T g^*(B^g_n + Y^g_{n+1} - C^g_{n+1}) | B^g_n\right], \quad n = 0, 1, 2, \cdots, \end{aligned}$$

Then we can perform the following iteration:

$$\begin{aligned} g^*(b) &= \mathcal{O}g^*(b) \\ &= \mathbb{E}\left[U(C^g_1) + \beta_T g^*(b + Y^g_1 - C^g_1)\right] \\ &= \mathbb{E}\left[U(C^g_1) + \beta_T g^*(B^g_1) | B_0 = b\right] \\ &= \mathbb{E}\Big[U(C^g_1) + \beta_T \mathbb{E}\left[U(C^g_2) + \beta_T g^*(B^g_2) | B^g_1\right] \Big| B_0 = b\Big] \\ &= \mathbb{E}\left[U(C^g_1) + \beta_T U(C^g_2) + \beta_T^2 g^*(B^g_2) | B_0 = b\right] \\ &= \cdots \\ &= \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U(C^g_{k+1}) + \beta_T^n g^*(B^g_n) | B_0 = b\right] \\ &\leq v^*_n(b) + \beta_T^n \mathbb{E}\left[g^*(B^g_n) | B_0 = b\right], \end{aligned} \tag{4.11}$$

where the last step follows from the optimality of v^*_n. Letting $n \to \infty$, we have

$$\begin{aligned} g^*(b) &\leq v_\infty(b) + \limsup_{n \to \infty} \beta_T^n \mathbb{E}\left[g^*(B^g_n) | B_0 = b\right] \\ &= v^*(b) + \limsup_{n \to \infty} \beta_T^n \mathbb{E}\left[g^*(B^g_n) | B_0 = b\right]. \end{aligned} \tag{4.12}$$

By the optimality of v^*_n,

$$\mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U(C^g_{k+1})\right] \leq v^*_n(b) \leq v_\infty(b) = v^*(b) < \infty.$$

Hence

$$\lim_{n\to\infty} \beta_T^n \mathbb{E}\left[U(C_n^g)\right] = 0.$$

Recall from Remark 3.4 that

$$\mathbb{E}\left[g^*(B_n^g)\right] = \text{constant} \cdot \mathbb{E}\left[U(C_n^g)\right],$$

where the constant is independent of the bonus amount or time index n. Therefore

$$\lim_{n\to\infty} \beta_T^n \mathbb{E}\left[g^*(B_n^g)\right] = 0.$$

Then by (4.10) and (4.12), we have

$$g^*(b) = v^*(b), \quad \forall b \geq 0.$$

$\square$

4.2 Negative Power Utility

In this section, we consider the power utility function with negative values.

Theorem 4.4. *Consider the utility function U defined by (2.7). For any $b \geq 0$ let*

$$v^*(b) = \sup_{\{(Y_{k+1}, C_{k+1})\}_{k=0}^{\infty} \in \mathcal{A}(b)} \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k U(C_{k+1})\right],$$

and let g^* be the unique function in $\mathcal{G}_U$ such that for any $b \geq 0$

$$g^*(b) = \sup_{(Y,C)\in\mathcal{A}(1,b)} \mathbb{E}\big[U(C) + \beta_T g^*(b + Y - C)\big].$$

Then

$$v^*(b) = g^*(b), \quad \forall b \geq 0.$$

PROOF: When $b = 0$, by Corollary 3.2, $g^*(b) =$ a positive constant $\cdot\, U(0) = -\infty$. Meanwhile, the only feasible strategy with $b = 0$ is no trading and 0 consumption for all periods. Indeed, by constraint (2.5), we have $X_1 \geq 0$. However, due to the budget constraint $\mathbb{E}[X_1 Z_1] = 0$, we must have $X_1 = 0$ almost surely. Thus $C_1 = 0$ and $B_1 = 0$ almost surely. Iterating this argument, we have $X_k = 0$ and $C_k = 0$ almost surely for all $k \in \mathbb{N}$. Therefore $v^*(b)$ is also a positive constant $\cdot\, U(0)$ in this case. Thus we have $v^*(b) = g^*(b) = -\infty$. In the remaining part of the proof, we consider the case when $b > 0$.

Given $b > 0$, let v_n^* be defined as in (4.2). Then for the utility function U given by (2.7), we have $v_n^* \geq v_{n+1}^*$ for all $n \in \mathbb{N}$. Indeed, for any feasible $(n+1)$-period strategy, $\pi_{n+1} = \big\{(Y_{k+1}, C_{k+1})\big\}_{k=0}^{n} \in \mathcal{A}_{n+1}(b)$, the first n pairs $\big\{(Y_{k+1}, C_{k+1})\big\}_{k=0}^{n-1}$ form a feasible strategy of the n-period problem, denoted by $\pi_n(\pi_{n+1})$. Then

$$
\begin{aligned}
v_n^* &= \sup_{\pi_n \in \mathcal{A}_n} v_n^{\pi_n} \\
&\geq \sup_{\pi_n(\pi_{n+1}),\, \pi_{n+1} \in \mathcal{A}_{n+1}} v_n^{\pi_n(\pi_{n+1})} \\
&\geq \sup_{\pi_{n+1} \in \mathcal{A}_{n+1}} v_{n+1}^{\pi_{n+1}} \\
&= v_{n+1}^*,
\end{aligned}
$$

where the second inequality follows from the negativity of U. Then the sequence $\{v_n^*\}$ has a limit as $n \to \infty$. We denote it by v_∞. Similarly we have

$$v_n^* \geq v^*.$$

Thus

$$v_\infty \triangleq \lim_{n \to \infty} v_n^* \geq v^*.$$

Now we prove the opposite inequality $v_\infty \leq v^*$. Consider the feasible strategy $\hat{\pi}$ for the infinite-period problem defined as the following:

$$Y_n = 0,\ C_n = a^{n-1}(1-a)b, \quad n = 1, 2, \cdots,$$

where $a \in \left(\beta_T^{\frac{1}{p-1}}, 1\right)$. Then for $b > 0$,

$$\begin{aligned}
v^*(b) &\geq v^{\hat{\pi}}(b) \\
&= \sum_{n=0}^{\infty} \beta_T^n U\big(a^n(1-a)b\big) \\
&= \frac{1}{1-p}\big((1-a)b\big)^{1-p} \sum_{n=0}^{\infty} \big(\beta_T a^{1-p}\big)^n \\
&> -\infty.
\end{aligned}$$

Then we also have $v_\infty(b) > -\infty$ for $b > 0$. Thus given $b > 0$, for any $\varepsilon > 0$, there exists $n^\varepsilon \in \mathbb{N}$ such that for any $n_1, n_2 > n^\varepsilon$,

$$\left|v_{n_1}^*(b) - v_{n_2}^*(b)\right| < \varepsilon.$$

In particular, we have

$$v_{n_1}^*(b) < v_{n_2}^*(b) + \varepsilon.$$

For any $n > n^\varepsilon + 1$, by Lemma 4.1, there exists an n-period strategy $\pi_n = \left\{\left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right)\right\}_{k=0}^{n-1}$ that attains the optimal value of the n-period problem. Then

$$v_n^*(b) = \sum_{k=0}^{n-1} \beta_T^k \mathbb{E}\left[U\left(C_{k+1}^{(n)}\right)\right]. \tag{4.13}$$

Moreover $\pi_{n-1}(\pi_n) \triangleq \left\{\left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right)\right\}_{k=0}^{n-2}$ is an $(n-1)$-period feasible strategy. Thus

$$\begin{aligned}
\sum_{k=0}^{n-2} \beta_T^k \mathbb{E}\left[U\left(C_{k+1}^{(n)}\right)\right] &\leq v_{n-1}^*(b) \\
&< v_n^*(b) + \varepsilon. \qquad (4.14)
\end{aligned}$$

Then from (4.13) and (4.14) we get

$$\beta_T^{n-1}\mathbb{E}\left[U\left(C_n^{(n)}\right)\right] > -\varepsilon.$$

Since U is negative, we have

$$\beta_T^{n-1}\mathbb{E}\left[U\left(C_n^{(n)}\right)\right] \in (-\varepsilon, 0). \tag{4.15}$$

Consider the following ∞-period feasible strategy (for $0 < a < 1$):

$$\pi_\infty = \left\{\left(Y_{k+1}^{(\infty)}, C_{k+1}^{(\infty)}\right)\right\}_{k=0}^{\infty} = \begin{cases} \left(Y_{k+1}^{(n)}, C_{k+1}^{(n)}\right), & k = 0, 1, \cdots, n-2, \\ \left(Y_n^{(n)}, (1-a)C_n^{(n)}\right), & k = n-1, \\ \left(0, a^{k-n+1}(1-a)C_n^{(n)}\right), & k = n, n+1, \cdots \end{cases}$$

Then

$$\begin{aligned} v_\infty^{\pi_\infty}(b) &= \sum_{k=0}^{n-2}\beta_T^k\mathbb{E}\left[U\left(C_{k+1}^{(n)}\right)\right] + \sum_{k=n-1}^{\infty}\beta_T^k\mathbb{E}\left[\frac{1}{1-p}a^{(k-n+1)(1-p)}(1-a)^{1-p}{C_n^{(n)}}^{1-p}\right] \\ &= \sum_{k=0}^{n-2}\beta_T^k\mathbb{E}\left[U\left(C_{k+1}^{(n)}\right)\right] + \beta_T^{n-1}\mathbb{E}\left[U\left(C_n^{(n)}\right)\right]\sum_{k=0}^{\infty}\beta_T^k a^{k(1-p)}(1-a)^{1-p}. \end{aligned}$$

Denote

$$\begin{aligned} M_1 &\triangleq \sum_{k=0}^{\infty}\beta_T^k a^{k(1-p)}(1-a)^{1-p} \\ &= \frac{(1-a)^{1-p}}{1-\beta_T a^{1-p}}. \end{aligned}$$

For a chosen as before, M_1 is a positive finite constant. Notice that M_1 converges to ∞ as a converges to 1. Then a can be properly chosen such that $M_1 > 1$. Then by (4.15), we have

$$\begin{aligned} v_\infty^{\pi_\infty}(b) &= v_n^{\pi_n}(b) + (M_1 - 1)\beta_T^{n-1}\mathbb{E}\left[U\left(C_n^{(n)}\right)\right] \\ &> v_n^*(b) - \varepsilon(M_1 - 1). \end{aligned}$$

On the other hand,

$$v_\infty^{\pi_\infty}(b) < v^*(b).$$

Thus

$$v_n^*(b) - \varepsilon(M_1 - 1) < v^*(b).$$

Letting $n \to \infty$, we have

$$v_\infty(b) - \varepsilon(M_1 - 1) \leq v^*(b).$$

Then letting $\varepsilon \to 0$, we have

$$v_\infty(b) \leq v^*(b).$$

Therefore

$$v_\infty(b) = v^*(b).$$

Again by Remark 3.4, the fixed point $g^* \in \mathcal{G}_U$ of operator $\mathcal{O}$ is equal to U multiplied by a positive constant. Then since $U < 0$, we have $g^* < 0$. Then by the monotoniciy of $\mathcal{O}$, we have

$$g^* = \mathcal{O}g^* \leq \mathcal{O}v_0^* = v_1^*.$$

Then by iterating this argument, we have

$$g^* \leq v_n^*, \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$, we get

$$g^* \leq v_\infty = v^*.$$

We will then show that

$$g^* \geq v^*.$$

Again by iterating Corollary 3.2, given $b > 0$, we can find a sequence $\left\{\left(Y_{n+1}^g, C_{n+1}^g\right)\right\}_{n=0}^\infty$ such that for any $n \in \mathbb{N}$,

$$g^*(b) = \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U(C_{k+1}^g) + \beta_T g^*\left(B_{n-1}^g + Y_n^g - C_n^g\right)\right], \quad n = 1, 2, \cdots,$$

and the right hand side is maximized over $\mathcal{A}_n(b)$. Here $B_0^g = b$ and $B_{n+1}^g = B_n^g + Y_{n+1}^g - C_{n+1}^g$. For any $\varepsilon \geq 0$, let n^ε be chosen as before. For any $n > n^\varepsilon + 1$, let $\pi_n = \left\{ \left(Y_{k+1}^{(n)}, C_{k+1}^{(n)} \right) \right\}_{k=0}^{n-1}$ be the n-period feasible strategy that satisfies (4.13). Then

$$\begin{aligned} g^*(b) &= \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U(C_{k+1}^g) + \beta_T^n g^*(B_{n-1}^g + Y_n^g - C_n^g) \right] \\ &\geq \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k U\left(C_{k+1}^{(n)} \right) + \beta_T^n g^*\left(B_{n-1}^{(n)} + Y_n^{(n)} - C_n^{(n)} \right) \right] \\ &= v_n^*(b) + \beta_T^n \mathbb{E}\left[g^*\left(B_{n-1}^{(n)} + Y_n^{(n)} - C_n^{(n)} \right) \right]. \end{aligned}$$

By Remark 3.4,

$$\mathbb{E}\left[g^*\left(B_{n-1}^{(n)} + Y_n^{(n)} - C_n^{(n)} \right) \right] = M_2 \cdot \mathbb{E}\left[U\left(C_n^{(n)} \right) \right],$$

where M_2 is a constant independent of n. Thus (4.15) implies

$$-\beta_T M_2 \varepsilon < \beta_T^n \mathbb{E}\left[g^*\left(B_{n-1}^{(n)} + Y_n^{(n)} - C_n^{(n)} \right) \right] < 0.$$

Therefore

$$g^*(b) > v_n^*(b) - \beta_T M_2 \varepsilon.$$

Letting $n \to \infty$, we have

$$g^*(b) \geq v^*(b) - \beta_T M_2 \varepsilon.$$

Then letting $\varepsilon \to 0$, we get

$$g^*(b) \geq v^*(b).$$

We have shown

$$g^*(b) = v^*(b), \quad \forall b > 0.$$

□

Remark 4.5. The statement of Theorem 4.3 and Theorem 4.4 hold for both bonus schemes, with or without escrow. The proof given here is for the case without escrow. For the case with escrow, a similar proof applies, with $\mathcal{O}$ replaced by $\widetilde{\mathcal{O}}$, and using Corollary 3.9 and Remark 3.10 instead of Corollary 3.2 and Remark 3.4 in the argument.

4.3 Logarithmic Utility

In this section, we consider the logarithmic utility function.

Theorem 4.6. *Consider the utility function U defined as (2.8). For any $b \geq 0$ let*

$$v^*(b) = \sup_{\{(Y_{k+1}, C_{k+1})\}_{k=0}^{\infty} \in \mathcal{A}(b)} \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k U(C_{k+1})\right],$$

and let g^* be the unique function in $\mathcal{G}_U$ such that for any $b \geq 0$

$$g^*(b) = \sup_{(Y,C) \in \mathcal{A}(1,b)} \mathbb{E}\big[U(C) + \beta_T g^*(b + Y - C)\big].$$

Then

$$v^*(b) = g^*(b), \quad \forall b > 0.$$

PROOF: When $b = 0$, by Corollary 3.6, $g^*(b) = \text{constant} \cdot -\infty$. By the same argument as in the first paragraph of the proof of Theorem 4.4, we have $v^*(b) = g^*(b) = -\infty$. In the proof below, we consider the case when $b > 0$.

Let $B_0^* = b > 0$ be given. Starting from B_0^*, let $\{(Y_{k+1}^*, C_{k+1}^*)\}_{k=0}^{\infty}$ be the strategy with $B_{k+1}^* = B_k^* + Y_{k+1}^* - C_{k+1}^*$ where $Y_{k+1}^*(B_k^*)$ and $C_{k+1}^*(B_k^*)$ are the strategy such that

$$g^*(B_k^*) = \mathbb{E}\Big[U\big(C_{k+1}^*(B_k^*)\big) + \beta_T g^*\big(B_k^* + Y_{k+1}^*(B_k^*) - C_{k+1}^*(B_k^*)\big)\big|B_k^*\Big].$$

By a similar iteration argument as in previous sections, we have

$$\mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k \log C_{k+1}^* + \beta_T^n g^*(B_n^*)\right] = g^*(b). \tag{4.16}$$

We will show that

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{k=0}^{n-1} \beta_T^k \log C_{k+1}^*\right] = \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}^*\right]; \tag{4.17}$$

$$\limsup_{n\to\infty} \mathbb{E}\left[\beta_T^n g^*(B_n^*)\right] \le 0. \tag{4.18}$$

Then we let $n \to \infty$ on the left hand side of (4.16) to get

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}^*\right] \ge g^*(b).$$

By the optimality of v^*, the left hand side is less than or equal to $v^*(b)$. Then we have

$$v^*(b) \ge g^*(b).$$

To show (4.17), we first show

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k (\log C_{k+1}^*)^-\right] < \infty, \tag{4.19}$$

for which we discuss the cases with and without escrow separately.

- In the case of not escrowing the bonus, by Lemma 3.5 and Corollary 3.6,

$$C_{k+1}^* = \frac{(1-\beta_T)B_k^*}{Z_{k+1}},$$

$$B_{k+1}^* = B_k^* + Y_{k+1}^* - C_{k+1}^* = \frac{\beta_T B_k^*}{Z_{k+1}}.$$

By iteration, we have

$$B_k^* = \frac{\beta_T^k B_0^*}{\prod_{i=0}^{k-1} Z_{i+1}} = \beta_T^k B_0^* e^{\theta W_k + \frac{1}{2}\theta^2 k},$$

and

$$C_{k+1}^* = \beta_T^k (1-\beta_T) B_0^* e^{\theta W_{k+1} + \frac{1}{2}\theta^2 (k+1)}.$$

Therefore

$$\begin{aligned}\left(\log C^*_{k+1}\right)^- &= \left(\log(1-\beta_T) + k\log\beta_T + \log B^*_0 + \theta W_{k+1} + \frac{1}{2}\theta^2(k+1)\right)^- \\ &\le \Big|\log(1-\beta_T)\Big| + k\Big|\log\beta_T\Big| + \Big|\log B^*_0\Big| + \theta W^-_{k+1} + \frac{1}{2}\theta^2(k+1).\end{aligned}$$

Since

$$\begin{aligned}\mathbb{E}\left[W^-_{k+1}\right] &= \sqrt{k+1}\,\mathbb{E}\left[\left(\frac{W_{k+1}}{\sqrt{k+1}}\right)^-\right] \\ &= \sqrt{k+1}\frac{1}{\sqrt{2\pi}}\int_0^\infty xe^{-\frac{x^2}{2}}\,dx \\ &= \sqrt{\frac{k+1}{2\pi}},\end{aligned}$$

we have

$$\begin{aligned}\mathbb{E}\left[\sum_{k=0}^\infty \beta^k_T(\log C^*_{k+1})^-\right] &\le \mathbb{E}\left[\sum_{k=0}^\infty \beta^k_T\big(K_1 + K_2 k + K_3\sqrt{k+1}\big)\right] \\ &< \infty,\end{aligned}$$

where K_1, K_2 and K_3 are constants.

- In the case of escrowing the bonus, by Lemma 3.11 and Corollary 3.12,

$$\begin{aligned}B^*_k &= B^*_{k-1} + Y^*_k - C^*_k \\ &= \frac{\beta_T B^*_{k-1}}{(1-\beta_T)\eta^* Z_k} \\ &= \left(\frac{\beta_T}{(1-\beta_T)\eta^*}\right)^k B^*_0 e^{\theta W_k + \frac{1}{2}\theta^2 k},\end{aligned}$$

$$C^*_{k+1} = \begin{cases} \frac{B^*_k}{\eta^* Z_{k+1}}, & \eta^* Z_{k+1} \ge 1, \\ B^*_k, & \eta^* Z_{k+1} \le 1. \end{cases}$$

Then

$$\log C^*_{k+1} = k\log\frac{\beta_T}{(1-\beta_T)\eta^*} + \log B^*_0 - \big(\log(\eta^* Z_{k+1})\big)\mathbb{I}_{\{\eta^* Z_{k+1}\ge 1\}} + \theta W_k + \frac{1}{2}\theta^2 k.$$

Notice that

$$\begin{aligned}\mathbb{E}\Big[\big| - \big(\log(\eta^* Z_{k+1})\big)\mathbb{I}_{\{\eta^* Z_{k+1} \geq 1\}}\big|\Big] &\leq \mathbb{E}\Big[\big|\log(\eta^* Z_{k+1})\big|\Big] \\ &= \mathbb{E}\left[\left| - \log \eta^* + \theta(W_{k+1} - W_k) + \frac{1}{2}\theta^2 \right|\right],\end{aligned}$$

which is a constant not depending on k. Therefore

$$\begin{aligned}\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k (\log C_{k+1}^*)^-\right] &\leq \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \big(\widetilde{K_1} + \widetilde{K_2} k + \widetilde{K_3}\sqrt{k}\big)\right] \\ &< \infty,\end{aligned}$$

where $\widetilde{K_1}$, $\widetilde{K_2}$ and $\widetilde{K_3}$ are constants.

We have shown (4.19) holds in both cases. Thus if

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \left(\log C_{k+1}^*\right)^+\right] = \infty,$$

then

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}^*\right] = \infty.$$

But

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}^*\right] \leq \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \frac{1}{1-p}\left(C_{k+1}^*\right)^{1-p}\right], \quad \forall p \in (0,1).$$

The right hand side is bounded above by the trader's value function with power utility with positive values, which we have shown to be finite in the proof of Theorem 4.3, when the inequality (3.14) holds. Notice that for any $\beta_T \in (0,1)$, there exists $p \in (0,1)$ such that (3.14) holds. Therefore

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}^*\right] < \infty,$$

and hence

$$\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \left(\log C_{k+1}^*\right)^+\right] < \infty.$$

Then by the Dominated Convergence Theorem, we have (4.17).

To show (4.18), we will show a more general result: for any feasible strategy $\{(Y_{k+1}, C_{k+1})\}_{k=0}^{\infty} \in \mathcal{A}(b)$ with $B_{k+1} = B_k + Y_{k+1} - C_{k+1}$,

$$\limsup_{n\to\infty} \mathbb{E}\left[\beta_T^n g^*(B_n)\right] \leq 0.$$

Since $C_k \geq 0$ for all $k \in \mathbb{N}$,

$$B_{k+1} \leq B_k + Y_{k+1}.$$

Then by iteration, we have for $n \in \mathbb{N}$,

$$B_n \leq b + \sum_{j=1}^{n} Y_j \triangleq \widehat{Y}_n.$$

In addition, we define $\widehat{Y}_0 \triangleq b$ and let $Z_0 = 1$. Then $\left\{\left(\prod_{k=0}^{n} Z_k\right)\widehat{Y}_n\right\}_n$ is an $\mathcal{F}_n$-martingale. Indeed,

$$\begin{aligned}
\mathbb{E}\left[\left(\prod_{k=0}^{n+1} Z_k\right)\widehat{Y}_{n+1}\Big|\mathcal{F}_n\right] &= \prod_{k=0}^{n} Z_k \mathbb{E}\big[Z_{n+1}\widehat{Y}_{n+1}\big|\mathcal{F}_n\big] \\
&= \prod_{k=0}^{n} Z_k\Big(\widehat{Y}_n \mathbb{E}[Z_{n+1}|\mathcal{F}_n] + \mathbb{E}\big[Z_{n+1}Y_{n+1}\big|\mathcal{F}_n\big]\Big) \\
&= \left(\prod_{k=0}^{n} Z_k\right)\widehat{Y}_n.
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left(\prod_{k=0}^{n} Z_k\right)\widehat{Y}_n\right] = b. \tag{4.20}$$

By Corollaries 3.6 and 3.12,

$$g^*(b) = \frac{1}{1-\beta_T}\log b + A_2, \tag{4.21}$$

where A_2 is a constant[4]. Then the derivative is

$$\frac{d}{db}g^*(b) = \frac{1}{(1-\beta_T)b},$$

[4]This constant is equal to A_2^* given by (3.37) in the case of not escrowing the bonus, and is equal to $\widetilde{A}_2^*$ given by (3.85) in the case of escrowing the bonus.

and its inverse is

$$I_g(\mu) = \frac{1}{(1-\beta_T)\mu}.$$

Denote

$$\zeta_n \triangleq \prod_{k=0}^{n-1} Z_{k+1} = e^{-\theta W_n - \frac{1}{2}\theta^2 n}.$$

Then the optimal $\widehat{Y}_n$ that maximizes $\mathbb{E}\Big[g^*(\widehat{Y}_n)\Big]$ and satisfies (4.20) is

$$\begin{aligned}\widehat{Y}_n^* &= I_g(\lambda_g \zeta_n) \\ &= \frac{1}{(1-\beta_T)\lambda_g \zeta_n},\end{aligned}$$

where the constant λ_g is chosen such that (4.20) holds. It is straightforward to get that $\lambda_g = \frac{1}{1-\beta_T}$ and hence

$$\widehat{Y}_n^* = \frac{b}{\zeta_n}.$$

Therefore we have

$$\begin{aligned}\mathbb{E}\left[g^*(B_n)\right] &\leq \mathbb{E}\left[g^*\left(\widehat{Y}_n^*\right)\right] \\ &= \frac{1}{1-\beta_T}\mathbb{E}\big[\log b - \log \zeta_n\big] + A_2 \\ &= \frac{1}{1-\beta_T}\left(\log b + \frac{1}{2}\theta^2 n\right) + A_2.\end{aligned}$$

Then

$$\limsup_{n\to\infty} \mathbb{E}\left[\beta_T^n g^*(B_n)\right] \leq 0.$$

We have shown $v^*(b) \geq g^*(b)$. Now we will show the opposite inequality. Let $\{(Y_{k+1}, C_{k+1})\}_{k=0}^{\infty} \in \mathcal{A}(b)$ be any feasible strategy, and $B_{k+1} = B_k + Y_{k+1} - C_{k+1}$. Notice that for any $n \in \mathbb{N}$,

$$C_n \leq B_{n-1} + Y_n \leq \widehat{Y}_{n-1} + Y_n = \widehat{Y}_n.$$

Then

$$
\begin{aligned}
\mathbb{E}[\log C_n] &\le \mathbb{E}\left[\log \widehat{Y}_n^*\right] \\
&= \mathbb{E}[\log b - \log \zeta_n] \\
&= \log b + \frac{1}{2}\theta^2 n.
\end{aligned}
$$

Therefore given the initial bonus b,

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}\right] &\le \sum_{k=0}^{\infty} \beta_T^k \big(\log b + \frac{1}{2}\theta^2 (k+1)\big) \\
&= \frac{1}{1-\beta_T}\log b + \frac{1}{2}\theta^2 \sum_{k=0}^{\infty} \beta_T^k (k+1) \\
&\triangleq \frac{1}{1-\beta_T}\log b + M, \qquad (4.22)
\end{aligned}
$$

where M is a constant that does not depend on b. Then for any $K \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}\right] &= \mathbb{E}\left[\sum_{k=0}^{K} \beta_T^k \log C_{k+1} + \beta_T^{K+1} g^*(B_{K+1})\right] + \mathbb{E}\left[\sum_{k=K+1}^{\infty} \beta_T^k \log C_{k+1} - \beta_T^{K+1} g^*(B_{K+1})\right] \\
&\le g^*(b) + \beta_T^{K+1}\mathbb{E}\left[\frac{1}{1-\beta_T}\log B_{K+1} + M - \frac{1}{1-\beta_T}\log B_{K+1} - A_2\right] \\
&= g^*(b) + \beta_T^{K+1}(M - A_2),
\end{aligned}
$$

where the inequality is due to the optimality of g^*, (4.21) and (4.22). Letting $K \to \infty$, we get

$$
\mathbb{E}\left[\sum_{k=0}^{\infty} \beta_T^k \log C_{k+1}\right] \le g^*(b).
$$

Taking the supremum over all feasible strategies in $\mathcal{A}(b)$ we get

$$
v^*(b) \le g^*(b).
$$

Therefore we have proved

$$
v^*(b) = g^*(b).
$$

$\square$

5 Bank's Total Expected Revenue

In the previous chapter, we have shown that the trader's optimal value for the infinite-time-horizon problem is the same as the fixed point of the operator $\mathcal{O}$ when the bonus is not escrowed, or $\widetilde{\mathcal{O}}$ when the bonus is escrowed. The optimal investment and consumption strategy that achieves this optimal total expected utility exists, and has an explicit expression. In this chapter, we will calculate and compare the bank's total expected revenue with or without escrowing the trader's bonus, given that the trader adopts the optimal strategy in these two scenarios.

Recall that given an investment strategy sequence $\{X_k, k \in \mathbb{N}\}$ with initial bonus $b > 0$, the bank's total expected revenue is given by

$$\begin{aligned} v_B(b) &= \mathbb{E}\left[\sum_{k=0}^{\infty} \beta_B^{k+1}(1-\gamma)X_{k+1}\Big| B_0 = b\right] \\ &= (1-\gamma)\sum_{k=0}^{\infty} \beta_B^{k+1}\mathbb{E}\left[X_{k+1}\Big| B_0 = b\right], \end{aligned} \tag{5.1}$$

where γ is the contractual constant as described in Section 2.1. During each time period, the trader chooses the optimal investment strategy based on the bonus b at the beginning of this period. To calculate the bank's total expected revenue, we need to switch from the "Y-notation" for the trader's investment strategy back to the "X-notation". Specifically, at time k, we denote the optimal investment by $X_k^*(b) \triangleq \frac{1}{\gamma}Y_k^*(b)$ in the case of not escrowing the bonus, and by $\widetilde{X}_k^*(b) \triangleq \frac{1}{\gamma}\widetilde{Y}_k^*(b)$ in the case when the bonus is escrowed. We also introduce the "X-notation" for the mapping from the initial bonus and the Radon-Nikodym derivative changing the physical measure to the risk-neutral measure at the end of a period to the optimal investment strategy: denote $\mathfrak{X}(p; b, Z) \triangleq \frac{1}{\gamma}\mathfrak{Y}(p; b, Z)$ in the case of not escrowing the bonus, and denote $\widetilde{\mathfrak{X}}(p, A; b, Z) = \frac{1}{\gamma}\widetilde{\mathfrak{Y}}(p, A; b, Z)$ in the case of escrowing the bonus. Recall that in this notation, p takes values in $(0, \infty)$, and $p = 1$ corresponds to the case when the trader's utility function is logarithmic. Given this consistency between the cases when the trader's utility function is a power function and the logarithm function, and the fact that the stochasticity of the bank's revenue only comes from the trader's investment strategy, in this chapter, a separate discussion of the two cases is not needed, and the same set of conclusions apply to both utility functions.

First, we observe a separable property of the optimal investment strategy that will be useful for later discussions.

Lemma 5.1. *For any given time period $[k, k+1]$, let B_k be the bonus at the beginning of the period. Then for $p > 0$ and $A \geq 0$, the optimal investment strategy satisfies the following equations:*

- *In the case when the trader's bonus is not escrowed,*

$$\mathbb{E}\big[\mathfrak{X}(p; B_k, Z_{k+1})\big] = \mathbb{E}[B_k]\mathbb{E}\big[\mathfrak{X}(p; 1, Z_1)\big]; \tag{5.2}$$

- *In the case when the trader's bonus is escrowed,*

$$\mathbb{E}\left[\widetilde{\mathfrak{X}}(p, A; B_k, Z_{k+1})\right] = \mathbb{E}[B_k]\mathbb{E}\left[\widetilde{\mathfrak{X}}(p, A; 1, Z_1)\right]. \tag{5.3}$$

Proof:

- When the trader's bonus is not escrowed, by (3.17) and (3.38), we have:

$$\begin{aligned}
\mathbb{E}\big[\mathfrak{X}(p; B_k, Z_{k+1})\big] &= \mathbb{E}\left[\frac{1}{\gamma}\mathfrak{Y}(p; B_k, Z_{k+1})\right] \\
&= \mathbb{E}\left[\frac{1}{\gamma}\left(\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} Z_{k+1}^{-\frac{1}{p}} - 1\right) B_k\right] \\
&= \mathbb{E}[B_k]\mathbb{E}\big[\mathfrak{X}(p; 1, Z_{k+1})\big] \\
&= \mathbb{E}[B_k]\mathbb{E}\big[\mathfrak{X}(p; 1, Z_1)\big],
\end{aligned}$$

where in the last two steps, we used the fact that B_k is independent with Z_{k+1} and that Z_{k+1} has the same distribution as Z_1.

- When the trader's bonus is escrowed, by (3.64) and (3.88), we have:

$$\begin{aligned}
\mathbb{E}\left[\widetilde{\mathfrak{X}}(p, A; B_k, Z_{k+1})\right] &= \mathbb{E}\left[\frac{1}{\gamma}\widetilde{\mathfrak{Y}}(p, A; B_k, Z_{k+1})\right] \\
&= \mathbb{E}\Bigg[\frac{1}{\gamma}B_k\big(\eta(p; A)Z_{k+1}\big)^{-\frac{1}{p}}(\beta_T A)^{\frac{1}{p}}\mathbb{I}_{\{\eta(p;A)Z_{k+1}\leq 1\}} \\
&\quad + \frac{1}{\gamma}B_k\left[(\eta(p; A)Z_{k+1})^{-\frac{1}{p}}\left(1 + (\beta_T A)^{\frac{1}{p}}\right) - 1\right]\mathbb{I}_{\{\eta(p;A)Z_{k+1}\geq 1\}}\Bigg] \\
&= \mathbb{E}[B_k]\mathbb{E}\left[\widetilde{\mathfrak{X}}(p, A; 1, Z_{k+1})\right] \\
&= \mathbb{E}[B_k]\mathbb{E}\left[\widetilde{\mathfrak{X}}(p, A; 1, Z_1)\right],
\end{aligned}$$

where in the last two steps, we again used the fact that B_k is independent with Z_{k+1}, that $\eta(p, A)$ does not depend on B_k and that Z_{k+1} has the same distribution as Z_1.

$\square$

With the help of the above lemma, we can calculate the bank's total expected revenue for both cases, namely escrowing the trader's bonus or not, given that the trader chooses her optimal investment and consumption strategy during each time period.

Lemma 5.2. *Let $p > 0$ be given. Suppose the initial bonus is $b \geq 0$ at time 0 and the trader chooses the optimal investment and consumption strategy during all time periods. Let ${v_B}^*(b)$ and $\widetilde{v}_B^*(b)$ denote the bank's total expected revenue when the trader's bonus is not escrowed and is escrowed, respectively. Assume (3.14). Let A_1^* be the constant defined by (3.15), and η^* and $\widetilde{A}_1^*$ be the constants defined by (3.56) and (3.57)[5]. Then*

[5]In light of Remarks 3.7 and 3.13, the equations defining A_1^* and $\widetilde{A}_1^*$ when the trader's utility function is logarithmic are special cases of their counterparts when the trader's utility function is a power function, and hence are not explicitly listed here.

- *when the bonus is not escrowed:*

$$v_B{}^*(b) = \frac{(1-\gamma)\beta_B b}{1-\beta_B\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}}\mathbb{E}\left[\mathfrak{X}(p;1,Z_1)\right]; \tag{5.4}$$

when $\beta_B\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\} \in (0,1)$*, and* $v_B{}^*(b) = +\infty$ *otherwise;*

- *when the bonus is escrowed:*

$$\widetilde{v_B{}^*}(b) = \frac{(1-\gamma)\beta_B b}{1-\beta_B\left(\frac{\beta_T\widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}}\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_1)\right]. \tag{5.5}$$

when $\beta_B\left(\frac{\beta_T\widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\} \in (0,1)$*, and* $\widetilde{v_B{}^*}(b) = +\infty$ *otherwise.*

Proof:

- When trader's bonus is not escrowed, by (3.8), given the bonus B_{k-1}^* at the beginning of the k-th time period, the bonus at the beginning of the next time period is

$$\begin{aligned} B_k^* &= \mathfrak{B}(p,A_1^*;B_{k-1}^*,Z_k) \\ &= \frac{1}{1+(\beta_T A_1^*)^{-\frac{1}{p}}}\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\}Z_k^{-\frac{1}{p}}B_{k-1}^* \end{aligned}$$

Then by iteration and the fact that $Z_1, Z_2, \cdots, Z_k$ are independent, we have

$$\begin{aligned} \mathbb{E}\left[B_k^*|B_0^*=b\right] &= \mathbb{E}\left[\frac{1}{\left(1+(\beta_T A_1^*)^{-\frac{1}{p}}\right)^k}\exp\left\{\frac{\theta^2(p-1)k}{2p^2}\right\}\prod_{i=1}^k Z_i^{-\frac{1}{p}}B_0^*\middle|B_0^*=b\right] \\ &= \frac{\exp\left\{\frac{\theta^2 k}{p}\right\}b}{\left(1+(\beta_T A_1^*)^{-\frac{1}{p}}\right)^k}. \end{aligned}$$

Therefore given the initial bonus b, by Lemma 5.1 the bank's (undiscounted) expected revenue at time $k+1$ is given by

$$
\begin{aligned}
&(1-\gamma)\mathbb{E}\left[\mathfrak{X}(p;B_k,Z_{k+1})\middle|B_0^*=b\right]\\
=&(1-\gamma)\mathbb{E}[B_k|B_0^*=b]\mathbb{E}\left[\mathfrak{X}(p;1,Z_1)\right]\\
=&(1-\gamma)\frac{\exp\left\{\frac{\theta^2 k}{p}\right\}b}{\left(1+(\beta_T A_1^*)^{-\frac{1}{p}}\right)^k}\mathbb{E}\left[\mathfrak{X}(p;1,Z_1)\right]\\
=&(1-\gamma)\beta_T^{\frac{k}{p}}\exp\left\{\frac{\theta^2 k(1+p)}{2p^2}\right\}b\mathbb{E}\left[\mathfrak{X}(p;1,Z_1)\right],
\end{aligned}
\tag{5.6}
$$

where in the last step we used the value of A_1^* given in (3.15). Hence the bank's total expected revenue is given by

$$
\begin{aligned}
v_B{}^*(b) &= (1-\gamma)\sum_{k=0}^{\infty}\beta_B^{k+1}\mathbb{E}\left[\mathfrak{X}(p;B_k,Z_{k+1})\middle|B_0^*=b\right]\\
&= (1-\gamma)\sum_{k=0}^{\infty}\beta_B^{k+1}\beta_T^{\frac{k}{p}}\exp\left\{\frac{\theta^2 k(1+p)}{2p^2}\right\}b\mathbb{E}\left[\mathfrak{X}(p;1,Z_1)\right]\\
&= \frac{(1-\gamma)\beta_B b}{1-\beta_B\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}}\mathbb{E}\left[\mathfrak{X}(p;1,Z_1)\right],
\end{aligned}
$$

when $\beta_B\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}\in(0,1)$, and $v_B{}^*(b)=+\infty$ when $\beta_B\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}\geq 1$.

- When trader's bonus is escrowed, by (3.46), given the bonus B_{k-1}^* at the beginning of the k-th time period, the bonus at the beginning of the next time period is

$$
\begin{aligned}
B_k^* &= \widetilde{\mathfrak{B}}(p,A_1^*;B_{k-1}^*,Z_k)\\
&= (\eta^* Z_k)^{-\frac{1}{p}}\left(\beta_T\widetilde{A}_1^*\right)^{\frac{1}{p}}B_{k-1}^*.
\end{aligned}
$$

Then again by iteration and the fact that $Z_1,Z_2,\cdots,Z_k$ are independent, we have

$$
\begin{aligned}
\mathbb{E}[B_k^*|B_0^*=b] &= \mathbb{E}\left[\left(\frac{\eta^*}{\beta_T\widetilde{A}_1^*}\right)^{-\frac{k}{p}}\prod_{i=1}^{k}Z_i^{-\frac{1}{p}}B_0^*\middle|B_0^*=b\right]\\
&= \left(\frac{\eta^*}{\beta_T\widetilde{A}_1^*}\right)^{-\frac{k}{p}}\exp\left\{\frac{\theta^2(1+p)k}{2p^2}\right\}b.
\end{aligned}
$$

Therefore given the initial bonus b, by Lemma 5.1, the bank's (undiscounted) expected revenue at time $k+1$ is

$$
\begin{aligned}
&(1-\gamma)\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_{k+1})\middle|B_0^*=b\right]\\
=&(1-\gamma)\mathbb{E}[B_k|B_0^*=b]\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_1)\right]\\
=&(1-\gamma)\left(\frac{\eta^*}{\beta_T\widetilde{A}_1^*}\right)^{-\frac{k}{p}}\exp\left\{\frac{\theta^2(1+p)k}{2p^2}\right\}b\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_1)\right].
\end{aligned}
\tag{5.7}
$$

Hence the bank's total expected revenue is

$$
\begin{aligned}
\widetilde{v_B}^*(b)&=(1-\gamma)\sum_{k=0}^{\infty}\beta_B^{k+1}\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_{k+1})\middle|B_0^*=b\right]\\
&=(1-\gamma)\sum_{k=0}^{\infty}\beta_B^{k+1}\left(\frac{\eta^*}{\beta_T\widetilde{A}_1^*}\right)^{-\frac{k}{p}}\exp\left\{\frac{\theta^2(1+p)k}{2p^2}\right\}b\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_1)\right]\\
&=\frac{(1-\gamma)\beta_B b}{1-\beta_B\left(\frac{\beta_T\widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}}\mathbb{E}\left[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_1)\right],
\end{aligned}
$$

when $\beta_B\left(\frac{\beta_T\widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}\in(0,1)$. Otherwise, when $\beta_B\left(\frac{\beta_T\widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}\geq 1$, the bank's total expected revenue is $+\infty$.

□

Now we consider the more interesting case when the bank's total expected revenue is finite both without and with escrowing the bonus. From (5.4) and (5.5) we can see that in order to compare the total expected revenue for the bank under these two bonus schemes, we need to make the following two comparisons:

1. η^* and $\widetilde{A}_1^*$;

2. $\mathbb{E}[\mathfrak{X}(p;1,Z_1)]$ and $\mathbb{E}[\widetilde{\mathfrak{X}}(p,\widetilde{A}_1^*;1,Z_1)]$.

In the following Lemmas we present the result of these comparisons.

Lemma 5.3. *Let η^* and $\widetilde{A}_1^*$ be the constants defined by (3.56) and (3.57). Then*

$$\eta^* < \widetilde{A}_1^*. \tag{5.8}$$

Proof: From (3.57),

$$\begin{aligned}\widetilde{A}_1^* - \eta^* &= \mathbb{P}(\eta^* Z_1 \leq 1) - \eta^* \mathbb{Q}(\eta^* Z_1 \leq 1) \\ &= N\left(-\frac{1}{\theta}\ln\eta^* + \frac{\theta}{2}\right) - \eta^* N\left(-\frac{1}{\theta}\ln\eta^* - \frac{\theta}{2}\right),\end{aligned} \tag{5.9}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. Taking the derivative with respect to η^*, we have

$$\frac{\partial}{\partial\eta^*}(\widetilde{A}_1^* - \eta^*) = -N\left(-\frac{1}{\theta}\ln\eta^* - \frac{\theta}{2}\right),$$

which is negative for $\eta^* < +\infty$. Meanwhile, by (5.9), when η^* converges to $+\infty$, the limit of $\widetilde{A}_1^* - \eta^*$ is 0. Therefore

$$\widetilde{A}_1^* - \eta^* > 0$$

for all finite η^*.

$\square$

Lemma 5.4. *Let η^* and $\widetilde{A}_1^*$ be the constants defined by (3.56) and (3.57). Then*

$$\mathbb{E}[\mathfrak{X}(p; 1, Z_1)] \geq \mathbb{E}[\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)]. \tag{5.10}$$

Proof: First we prove the following inequality:

$$\eta^{*\frac{1}{p}} \exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} < \left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}} + 1. \tag{5.11}$$

From (3.58) and (3.56) we have

$$\left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}} \mathbb{E}\left[Z_1^{\frac{1}{p}}\right] = \mathbb{E}\left[\left((\eta^*)^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right].$$

Therefore

$$
\begin{aligned}
\eta^{*\frac{1}{p}} &= \mathbb{E}\left[\left((\eta^*)^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right] - \mathbb{E}\left[\left(Z_1^{\frac{1}{p}} - (\eta^*)^{\frac{1}{p}}\right)^+\right] + \mathbb{E}\left[Z_1^{\frac{1}{p}}\right] \\
&< \mathbb{E}\left[\left((\eta^*)^{\frac{1}{p}} - Z_1^{\frac{1}{p}}\right)^+\right] + \mathbb{E}\left[Z_1^{\frac{1}{p}}\right] \\
&= \mathbb{E}\left[Z_1^{\frac{1}{p}}\right]\left(\left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}} + 1\right) \\
&= \exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}\left(\left(\beta_T \widetilde{A}_1^*\right)^{\frac{1}{p}} + 1\right),
\end{aligned}
$$

where the last step is by Lemma B.2 on page 93.

By the inequality (5.11), comparing (3.17) and (3.64), when $\eta^* Z_1 \geq 1$ we have

$$\mathfrak{X}(p; b, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; b, Z_1) < 0.$$

Then in order to satisfy the budget constraint

$$\mathbb{E}[\mathfrak{X}(p; b, Z_1) Z_1] = \mathbb{E}[\widetilde{\mathfrak{X}}(p, A_1^*; b, Z_1) Z_1] = 0, \tag{5.12}$$

we must have $\mathfrak{X}(p; b, Z_1) > \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; b, Z_1)$ on a subset of $\left\{Z_1 \leq \frac{1}{\eta^*}\right\}$ which is not a null set. When

$$Z_1 \leq \frac{1}{\eta^*}, \tag{5.13}$$

$\mathfrak{X}(p; b, Z_1) > \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; b, Z_1)$ is equivalent to

$$\frac{b}{\gamma} Z_1^{-\frac{1}{p}} \left[\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} - \eta^{*-\frac{1}{p}} (\beta_T \widetilde{A}_1^*)^{\frac{1}{p}}\right] - \frac{b}{\gamma} > 0,$$

i.e.,

$$Z_1 < \left[\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} - \eta^{*-\frac{1}{p}} (\beta_T \widetilde{A}_1^*)^{\frac{1}{p}}\right]^p. \tag{5.14}$$

Notice that due to inequality (5.11),

$$\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} - \eta^{*-\frac{1}{p}} (\beta_T \widetilde{A}_1^*)^{\frac{1}{p}} < \eta^{*-\frac{1}{p}}.$$

Therefore (5.14) is a stricter inequality than (5.13), and hence is a sufficient and necessary condition for $\mathfrak{X}(p; b, Z_1) > \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; b, Z_1)$. Then by (5.12) with $b = 1$,

$$
\begin{aligned}
0 &= \mathbb{E}\left[\left(\mathfrak{X}(p; 1, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)\right) Z_1\right] \\
&= \mathbb{E}\left[\left(\mathfrak{X}(p; 1, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)\right) Z_1 \mathbb{I}_{\{Z_1 \leq z^*\}}\right] + \mathbb{E}\left[\left(\mathfrak{X}(p; 1, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)\right) Z_1 \mathbb{I}_{\{Z_1 \geq z^*\}}\right],
\end{aligned}
$$

for any $z^* > 0$. Let

$$
z^* \triangleq \left[\exp\left\{\frac{\theta^2(p-1)}{2p^2}\right\} - \eta^{*-\frac{1}{p}} (\beta_T \widetilde{A}_1^*)^{\frac{1}{p}}\right]^p.
$$

Then since $\mathfrak{X}(p; 1, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1) \geq 0$ on the set $\{Z_1 \leq z^*\}$, and $\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1) - \mathfrak{X}(p; 1, Z_1) \geq 0$ on the set $\{Z_1 \geq z^*\}$, we have

$$
\begin{aligned}
&\mathbb{E}[(\mathfrak{X}(p; 1, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)) z^* \mathbb{I}_{\{Z_1 \leq z^*\}}] \\
\geq &\mathbb{E}[(\mathfrak{X}(p; 1, Z_1) - \widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)) Z_1 \mathbb{I}_{\{Z_1 \leq z^*\}}] \\
= &\mathbb{E}[(\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1) - \mathfrak{X}(p; 1, Z_1)) Z_1 \mathbb{I}_{\{Z_1 \geq z^*\}}] \\
\geq &\mathbb{E}[(\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1) - \mathfrak{X}(p; 1, Z_1)) z^* \mathbb{I}_{\{Z_1 \geq z^*\}}].
\end{aligned}
$$

Therefore,

$$
\mathbb{E}[\mathfrak{X}(p; 1, Z_1)] \geq \mathbb{E}[\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)]. \tag{5.15}
$$

□

Remark 5.5. From the above lemmas, one may compare the period-wise expected revenue under the two bonus schemes. In particular, if starting with the same initial bonus, by Lemma 5.4, the bank has larger expected revenue in the first period without escrowing the trader's bonus. However, by comparing (5.6) and (5.7), we can see that as time goes by, the period-wise expected revenue with escrowing the trader's bonus catches up, and wins over from period k

on for

$$k = \inf \left\{ i \in \mathbb{N} : \left(\frac{\widetilde{A}_1^*}{\eta^*} \right)^{\frac{i}{p}} \geq \frac{\mathbb{E}[\mathfrak{X}(p; 1, Z_1)]}{\mathbb{E}[\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)]} \right\}.$$

Recall from Lemma 5.3 that $\frac{\widetilde{A}_1^*}{\eta^*} > 1$. Therefore the set in the right hand side of the above equation is not empty and such a number $k \in \mathbb{N}$ exists.

The comparison of the bank's total expected revenue under the two bonus schemes depends on the relative relationship among the parameters describing the trader's risk appetite and the bank's patience, in other words, the preference for future revenue over the near-term revenue. Specifically, we summarize the comparison of the bank's total expected revenue with or without escrowing the bonus to the trader in the following Theorem:

Theorem 5.6. *Let η^* and $\widetilde{A}_1^*$ be the constants defined by (3.56) and (3.57). Define*

$$\overline{\beta_B} \triangleq \exp\left\{ -\frac{\theta^2(1+p)}{2p^2} \right\} \beta_T^{-\frac{1}{p}},$$

$$\underline{\beta_B} \triangleq \exp\left\{ -\frac{\theta^2(1+p)}{2p^2} \right\} \left(\frac{\beta_T \widetilde{A}_1^*}{\eta^*} \right)^{-\frac{1}{p}},$$

$$\beta_B^* \triangleq \frac{\mathbb{E}[\mathfrak{X}(p; 1, Z_1)] - \mathbb{E}[\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)]}{\beta_T^{\frac{1}{p}} \exp\left\{ \frac{\theta^2(1+p)}{2p^2} \right\} \left(\left(\frac{\widetilde{A}_1^*}{\eta^*} \right)^{\frac{1}{p}} \mathbb{E}[\mathfrak{X}(p; 1, Z_1)] - \mathbb{E}[\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)] \right)}.$$

Then given the same initial bonus b at time 0,

1. *If $\beta_B \geq \overline{\beta_B}$, then the bank has infinite total expected discounted revenue whether escrowing the trader's bonus or not;*

2. *If $\underline{\beta_B} \leq \beta_B < \overline{\beta_B}$, then the bank is better off in terms of total expected discounted revenue when escrowing the trader's bonus;*

3. *If $0 < \beta_B < \underline{\beta_B}$, then the bank is not worse off in terms of total expected discounted revenue when escrowing the trader's bonus if and only if $\beta_B > \beta_B^*$.*

In particular, the only situation in which the bank should not escrow the trader's bonus is when $0 < \beta_B < \beta_B^*$.

PROOF:

1. If $\beta_B \geq \overline{\beta_B}$, then by Lemma 5.2, $v_B^* = +\infty$. By (5.8), we also have $\beta_B \geq \underline{\beta_B}$. Then by Lemma 5.2, $\widetilde{v_B{}^*} = +\infty$. In other words, the bank has infinite total expected revenue in both cases, whether the trader's bonus is escrowed or not.

2. If $\underline{\beta_B} \leq \beta_B < \overline{\beta_B}$, then by Lemma 5.2, v_B^* is finite while $\widetilde{v_B{}^*} = +\infty$. Therefore the bank is better off when the trader's bonus is escrowed.

3. If $0 \leq \beta_B < \underline{\beta_B}$, then the bank has finite total expected revenue in both cases. Due to Lemma 5.3 and Lemma 5.4, the comparison between v_B^* and $\widetilde{v_B{}^*}$ depends on additional information. Specifically, by Lemma 5.2, $\widetilde{v_B{}^*} > v_B^*$ if and only if the following inequality holds:
$$\frac{1 - \beta_B \beta_T^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}}{1 - \beta_B \left(\frac{\beta_T \widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}} > \frac{\mathbb{E}[\mathfrak{X}(p; 1, Z_1)]}{\mathbb{E}[\widetilde{\mathfrak{X}}(p, \widetilde{A}_1^*; 1, Z_1)]}, \tag{5.16}$$
which is equivalent to $\beta_B > \beta_B^*$.

$\square$

6 Conclusion and Discussion

6.1 Conclusion

From Theorem 5.6, we can see that given the trader's risk appetite, described by parameter p, the discount factor β_T, and the market price of risk θ, we have the following.

Case 1 When the bank is impatient, in other words, β_B is close to 0 so that we are in case 3 of Theorem 5.6 and the left hand side of (5.16) is close to 1 and is less than the right hand side of (5.16), the bank prefers not to escrow the trader's bonus.

Case 2 When the bank is patient, in other words, β_B is moderately large such that case 2 in Theorem 5.6 applies or case 3 applies with (5.16), the bank prefers to escrow the trader's bonus.

Case 3 When the bank is extremely patient and β_B is close to 1, the bank is indifferent about escrowing the trader's bonus or not.

As mentioned in several places throughout this thesis, we can see that although the solution for the trader's optimization problem is derived separately for different utility functions of the trader, the above conclusion about the bank's total expected revenue turns out to be qualitatively consistent whether the trader's utility is power or logarithmic.

6.2 Numerical Example

To gain further insights about the conclusion, it is worthwhile to visualize the relationship of the three critical values of β_B, namely $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* with reasonable numerical values for the model parameters. For this purpose, we consider several combinations of values for the market price of risk θ and the trader's relative risk aversion p, and investigated how $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* change with respect to the trader's discount factor β_T. In particular, we let θ take values in $\{1, 2, 3\}$, which are possible values of the annual Sharpe Ratio of hedge funds (see, e.g., [8]). For the value of p we consider the range $[2, 10]$. In the relevant empirical studies, the conclusions about typical values of the relative risk aversion of an individual investor vary a lot, ranging from around 2 to the order of 20 or even higher (see, e.g., [4, 5]). However, it seems to be a general consensus that $p > 1$ for an individual investor. Notice that the values of the parameters θ and p depend on the assumptions and methodologies used to estimate them, and the views in relevant literature vary. The values used in this numerical example are not meant to be comprehensive, but are selected to be within a reasonable range to demonstrate qualitative trends of the outcomes of our model.

For each pair of values for θ and p, the plots of $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* against β_T are shown in Figures 1 to 5, where the range for β_T is taken to be $[0.5, 0.95]$. Recall that the region above (below) the curve of β_B^* corresponds to the case where the bank is better off with (without) escrowing the trader's bonus. We now discuss some key observations about the behavior of β_B^*. Firstly, for a given value of θ, the plot of β_B^* with respect to β_T shifts up when p increases. This means that given the market price of risk, as the trader's risk aversion increases, there is less need for the bank to escrow her bonus. Secondly, for a given value of p, the plot of β_B^* with respect to β_T shifts down when θ increases. This means that given the trader's risk aversion, as the performance of the risky asset increases, there is more need for the bank to escrow the trader's bonus. These observations are in line with our intuition as escrowing is expected to be a mitigation mechanism to prevent the trader from being too risk prone. Thirdly, given the values of θ and p, β_B^* is decreasing with respect to β_T. This means that as the trader becomes more patient, there is more need for the bank to escrow her bonus. This observation may not seem as intuitive as the previous two. Heuristically, the reason for this behavior is the following. As β_T decreases, the trader wants to shift consumption forward in time because the future is more heavily discounted. In the extreme case, as β_T goes down to 0, the trader gets benefit from consuming only after one round of trading. In this case, the optimal strategy for the trader would be no trading, because losses will reduce her ability to consume after this round of trading and gains will not be realized. For the bank, it is the high β_B values that makes it advantageous to escrow the trader's profits. A bank that heavily discounts the future and thus seeks quick profits will encourage risk-taking by the trader. The trader facing escrow becomes more conservative as β_T decreases, so much so that it becomes detrimental to the bank, and hence the bank needs a higher degree of patience in order for it to be advantageous to escrow the trader's profits. Therefore β_B^* increases as β_T decreases. Additional analysis illustrating the extreme case of $\beta_T \downarrow 0$ is provided in Appendix C.

For the other critical values $\overline{\beta_B}$ and $\underline{\beta_B}$, we also have similar observations. Specifically, for a given value of θ, the plots of $\overline{\beta_B}$ and $\underline{\beta_B}$ with respect to β_T shift up when p increases. In other words, given the market price of risk, the more risk averse the trader is, the higher the level of patience needed for the bank to get infinite total expected revenue and to get higher total expected revenue escrowing the trader's bonus. In addition, the three curves ($\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^*) move closer as p increases. As illustrated in Figures 1, 3 and 5, the distances between these curves can become invisible when p is sufficiently large.[6] In other words, given the market price of risk, when the trader is sufficiently risk averse, the bank's total expected revenues with or without escrowing trader's bonus are either both finite or both infinite, depending on the bank's discount factor. Moreover, For a given value of p, plots of $\overline{\beta_B}$ and $\underline{\beta_B}$ with respect to β_T shift down when θ increases. In other words, given the trader's risk aversion, the better the

[6]This is also why we didn't pursue the cases where p takes value greater than 10 in this numerical example.

investment opportunity is, the lower the level of patience needed for the bank to get infinite total expected revenue and to get higher total expected revenue escrowing the trader's bonus. In particular, as shown in Figure 4, when the performance of the risky asset is exceptionally good ($\theta = 3$), the bank can get infinite total expected revenue with both bonus schemes even when the future revenue is discounted very heavily. Finally, in all cases, $\overline{\beta_B}$ and $\underline{\beta_B}$ are decreasing with respect to β_T. This means that the more patient the trader is, the lower the level of patience needed for the bank to get infinite total expected revenue, and this is the case for both escrowing the trader's bonus or not.

In general, the distance between $\underline{\beta_B}$ and β_B^* is narrow, implying that **Case 2** described in Section 6.1 is not very typical. On the other hand, depending on the value of θ and p, either **Case 1** or **Case 3** can be the most typical. It is interesting to notice in Figure 1 that when $\theta = 1$, $p = 3$ or 4, and $\beta_T = 0.5$, both $\overline{\beta_B}$ and $\underline{\beta_B}$ are greater than 1. This means that in this case, as long as the bank discounts the future revenue, the total expected revenue will always be finite, both with or without escrowing the trader's bonus. Also in Figure 1, when $p = 4$ and $\beta_T = 0.5$, $\beta_B^* > 1$. This means that the bank always gets higher total expected revenue without escrowing the trader's bonus.

6.3 Challenges and Potential Areas of Improvement

One of the major challenges of this work is to find a formulation of the problem that permits an explicit solution of the trader's optimization problem that allows the calculation and comparison of the bank's benefit (or loss) under the two bonus payment schemes. Many simplifying assumptions are made as a trade-off for tractability. For example, in our model, the portfolio that the trader manages restarts with zero initial capital at the beginning of each time period. However, in practice, the portfolio may continue to vest for a longer time. Moreover, our model does not explicitly consider the event of employment termination, although it is partially captured in the trader's discount factor. Features associated with the trader's employment termination, such as clawback of bonus after termination, bank's replacement cost, etc, are not considered. Lastly, our conclusion of whether the bank is better off with or without escrowing the trader's bonus is based on total expected revenue, while other risk-related metrics may also be interesting to consider, given one of the intentions for a clawback provision is to curb excessive risk taking. For example, it may be worthwhile to investigate whether a clawback provision can help reduce the probability of bank default. These considerations provide directions for future work.

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Appendices

A A Classical Solution for the Trader's Problem When Bonus is Not Escrowed

We notice that when the bonus is not escrowed, the trader's problem can be formulated as a standard consumption and investment problem, for which a classical solution exists. In this section, we present this standard argument for optimal consumption and investment. This provides a shorter derivation of Theorems 4.3, 4.4 and 4.6.

Consider an agent who begins with initial endowment $B_0 \geq 0$ and trades in the market as described in Section 2.1. At each time t, the agent holds Δ_t shares of stock, and at times $k = 1, 2, \cdots$, she consumes a lump sum C_k. Then at each time t, the agent's capital evolves as

$$B_t = B_0 + \int_0^t \Delta_u dS_u - \sum_{k=0}^{\lfloor t \rfloor - 1} C_{k+1}, \quad t \geq 0, \tag{A.1}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. The agent is required to choose Δ_t, $t \geq 0$, and $C_k, k = 1, 2, \cdots$, so that $B_t \geq 0$ for all t. In addition, the agent also has the budget constraint

$$\mathbb{E}\left[\frac{\zeta_t}{\zeta_s} \int_s^t \Delta_u dS_u \middle| \mathcal{F}_s \right] = 0, \tag{A.2}$$

where

$$\zeta_t \triangleq \exp\left\{ -\theta W_t - \frac{1}{2}\theta^2 t \right\}, \tag{A.3}$$

and $\{\mathcal{F}_t\}$ is the filtration generated by W_t. Subject to these constraints, the agent maximizes total expected utility from consumption

$$\mathbb{E} \sum_{k=0}^{\infty} \beta_T^k U(C_{k+1}).$$

The utility function U is assumed to be strictly increasing, strictly concave, and to satisfy the Inada conditions

$$\lim_{c\to 0+} U'(c) = \infty, \quad \lim_{c\to\infty} U'(c) = 0.$$

If we identify Y_{k+1} in the trader's problem with $\int_k^{k+1} \Delta_u dS_u$, we see that this problem is the same as the trader's problem in the model without escrowing the bonus.

The solution to this problem follows a classical argument, which we present.

Theorem A.1. *Let I_U be the inverse of U'. The optimal consumptions for the agent's problem described above are*

$$C^*_{k+1} = I_U\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right), \quad k = 0, 1, 2, \cdots, \tag{A.4}$$

where $\lambda > 0$ is chosen so that

$$\mathbb{E}\sum_{k=0}^{\infty} \zeta_{k+1} I_U\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right) = B_0. \tag{A.5}$$

Proof: First we show that

$$\zeta_t B_t + \sum_{k=0}^{\lfloor t\rfloor -1} \zeta_{k+1} C_{k+1}$$

is an $\mathcal{F}_t$-martingale.Indeed, for $0 \le s \le t$ such that $\lfloor s\rfloor \le \lfloor t\rfloor - 1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\zeta_t B_t + \sum_{k=0}^{\lfloor t\rfloor -1} \zeta_{k+1} C_{k+1}\middle|\mathcal{F}_s\right]\\
=&\mathbb{E}\left[\zeta_t B_s + \zeta_t(B_t - B_s) + \sum_{k=0}^{\lfloor s\rfloor -1} \zeta_{k+1} C_{k+1} + \sum_{k=\lfloor s\rfloor}^{\lfloor t\rfloor -1} \zeta_{k+1} C_{k+1}\middle|\mathcal{F}_s\right]\\
=&\zeta_s B_s + \sum_{k=0}^{\lfloor s\rfloor -1} \zeta_{k+1} C_{k+1} + \mathbb{E}\left[\zeta_t(B_t - B_s) + \sum_{k=\lfloor s\rfloor}^{\lfloor t\rfloor -1} \zeta_{k+1} C_{k+1}\middle|\mathcal{F}_s\right].\\
=&\zeta_s B_s + \sum_{k=0}^{\lfloor s\rfloor -1} \zeta_{k+1} C_{k+1} + \mathbb{E}\left[\zeta_t \int_s^t \Delta_u dS_u - \zeta_t \sum_{k=\lfloor s\rfloor}^{\lfloor t\rfloor -1} C_{k+1} + \sum_{k=\lfloor s\rfloor}^{\lfloor t\rfloor -1} \zeta_{k+1} C_{k+1}\middle|\mathcal{F}_s\right],
\end{aligned} \tag{A.6}$$

where we used (A.1) in the last step. Due to the budget constraint for the agent's trading strategy,

$$\begin{aligned}
&\mathbb{E}\left[\zeta_t \int_s^t \Delta_u dS_u \middle| \mathcal{F}_s\right] \\
=&\zeta_s \mathbb{E}\left[\frac{\zeta_t}{\zeta_s} \int_s^t \Delta_u dS_u \middle| \mathcal{F}_s\right] \\
=&0. \qquad \text{(A.7)}
\end{aligned}$$

Moreover, for any $\lfloor s \rfloor \leq k \leq \lfloor t \rfloor - 1$,

$$\begin{aligned}
&\mathbb{E}\Big[\zeta_t C_{k+1} - \zeta_{k+1} C_{k+1} \Big| \mathcal{F}_s\Big] \\
=&\mathbb{E}\Big[\mathbb{E}[\zeta_t C_{k+1} - \zeta_{k+1} C_{k+1} | \mathcal{F}_{k+1}] \Big| \mathcal{F}_s\Big] \\
=&\mathbb{E}\Big[C_{k+1} \mathbb{E}[\zeta_t - \zeta_{k+1} | \mathcal{F}_{k+1}] \Big| \mathcal{F}_s\Big] \\
=&0. \qquad \text{(A.8)}
\end{aligned}$$

Then by (A.6), (A.7) and (A.8) we have

$$\mathbb{E}\left[\zeta_t B_t + \sum_{k=0}^{\lfloor t \rfloor - 1} \zeta_{k+1} C_{k+1} \middle| \mathcal{F}_s\right] = \zeta_s B_s + \sum_{k=0}^{\lfloor s \rfloor - 1} \zeta_{k+1} C_{k+1}. \qquad \text{(A.9)}$$

For $0 \leq s \leq t$ such that $\lfloor s \rfloor = \lfloor t \rfloor$, we have

$$\begin{aligned}
&\mathbb{E}\left[\zeta_t B_t + \sum_{k=0}^{\lfloor t \rfloor - 1} \zeta_{k+1} C_{k+1} \middle| \mathcal{F}_s\right] \\
=&\zeta_s B_s + \sum_{k=0}^{\lfloor s \rfloor - 1} \zeta_{k+1} C_{k+1} + \mathbb{E}[\zeta_t (B_t - B_s) | \mathcal{F}_s] \\
=&\zeta_s B_s + \sum_{k=0}^{\lfloor s \rfloor - 1} \zeta_{k+1} C_{k+1} + \mathbb{E}\left[\zeta_t \int_u^t \Delta_u dS_u \middle| \mathcal{F}_s\right] \\
=&\zeta_s B_s + \sum_{k=0}^{\lfloor s \rfloor - 1} \zeta_{k+1} C_{k+1}, \qquad \text{(A.10)}
\end{aligned}$$

where the last step is due to (A.7).

We have shown that $\zeta_t B_t + \sum_{k=0}^{\lfloor t\rfloor -1} \zeta_{k+1} C_{k+1}$ is an $\mathcal{F}_t$-martingale. Then since $B_t \geq 0$ for all t, we must have for any consumption sequence $\{C_{k+1}\}_{k=1}^{\infty}$ that

$$\mathbb{E}\sum_{k=0}^{n} \zeta_{k+1} C_{k+1} \leq B_0$$

for all $n \in \mathbb{N}$. Thus, any consumption sequence must satisfy

$$\mathbb{E}\sum_{k=0}^{\infty} \zeta_{k+1} C_{k+1} \leq B_0. \tag{A.11}$$

In addition, because the market in our model is complete, any sequence of consumptions satisfying (A.11) can be financed by trading, i.e., there exists an accompanying process Δ_t, $t \geq 0$, so that $B_t \geq 0$ for all t. Therefore, to prove optimality of $\left\{C^*_{k+1}\right\}_{k=0}^{\infty}$, we only need to prove that the agent gets more utility from $\left\{C^*_{k+1}\right\}_{k=0}^{\infty}$ than any other sequence $\{C_{k+1}\}_{k=0}^{\infty}$ satisfying (A.11).

It is straightforward to verify that

$$U\big(I_U(y)\big) - yI(y) \geq U(x) - xy, \quad \forall x \geq 0, y > 0. \tag{A.12}$$

Indeed, given any $y > 0$, from the first and second order derivatives of the function $f_y(x) \triangleq U(x) - xy$, one can see that $f_y(x)$ achieves its maximum over $x \geq 0$ when $x = I_U(y)$, which is the left hand side of (A.12). Let $\{C_{k+1}\}_{k=0}^{\infty}$ satisfy (A.11). Relations (A.4), (A.5), (A.12) and

(A.11), used in that order, imply

$$
\begin{aligned}
&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k U(C_{k+1}^*)\\
=&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k U\left(I_U\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right)\right)+\lambda\left\{B_0-\mathbb{E}\sum_{k=0}^{\infty}\zeta_{k+1}I_U\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right)\right\}\\
=&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k\left\{U\left(I_U\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right)\right)-\frac{\lambda}{\beta_T^k}\zeta_{k+1}I_U\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right)\right\}+\lambda B_0\\
\geq&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k\left\{U(C_{k+1})-\frac{\lambda}{\beta_T^k}\zeta_{k+1}C_{k+1}\right\}+\lambda B_0\\
=&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k U(C_{k+1})+\lambda\left\{B_0-\sum_{k=0}^{\infty}\zeta_{k+1}C_{k+1}\right\}\\
\geq&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k U(C_{k+1}).
\end{aligned}
$$

□

A.1 Power Utility

We apply Theorem A.1 to the case when the trader's utility is given by the power function, namely (2.6) or (2.7). Then

$$I_U(z)=z^{-\frac{1}{p}},$$

and (A.5) becomes

$$
\begin{aligned}
B_0&=\mathbb{E}\sum_{k=0}^{\infty}\zeta_{k+1}\left(\frac{\lambda}{\beta_T^k}\zeta_{k+1}\right)^{-\frac{1}{p}}\\
&=\lambda^{-\frac{1}{p}}\sum_{k=0}^{\infty}\beta_T^{\frac{k}{p}}\mathbb{E}\zeta_{k+1}^{1-\frac{1}{p}} \qquad\qquad (A.13)\\
&=\lambda^{-\frac{1}{p}}\sum_{k=0}^{\infty}\beta_T^{\frac{k}{p}}\exp\left\{\frac{\theta^2(k+1)}{2}\left(1-\frac{1}{p}\right)\left(-\frac{1}{p}\right)\right\}\\
&=\lambda^{-\frac{1}{p}}\frac{\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}}{1-\beta_T^{\frac{1}{p}}\exp\left\{\frac{\theta^2(1-p)}{2p^2}\right\}},
\end{aligned}
$$

where we have used Lemma B.1 and Assumption (3.14). From (A.13) We conclude that

$$\lambda^{-\frac{1}{p}} = \frac{B_0}{\sum_{k=0}^{\infty} \beta_T^{\frac{k}{p}} \mathbb{E}\zeta_{k+1}^{1-\frac{1}{p}}}.$$

Returning to (A.4), we see that

$$\begin{aligned} C_{k+1}^* &= \left(\frac{\lambda}{\beta_T^k} \zeta_{k+1} \right)^{-\frac{1}{p}} \\ &= \frac{B_0}{\sum_{k=0}^{\infty} \beta_T^{\frac{k}{p}} \mathbb{E}\zeta_{k+1}^{1-\frac{1}{p}}} \beta_T^{\frac{k}{p}} \zeta_{k+1}^{-\frac{1}{p}}. \end{aligned}$$

Therefore the optimal value for the trader is

$$\begin{aligned} &\mathbb{E} \sum_{k=0}^{\infty} \beta_T^k U(C_{k+1}^*) \\ =&\mathbb{E} \sum_{k=0}^{\infty} \frac{\beta_T^k}{1-p} {C_{k+1}^*}^{1-p} \\ =&\mathbb{E} \sum_{k=0}^{\infty} \frac{\beta_T^k}{1-p} \frac{B_0^{1-p}}{\left(\sum_{j=0}^{\infty} \beta_T^{\frac{j}{p}} \mathbb{E}\zeta_{j+1}^{1-\frac{1}{p}} \right)^{1-p}} \beta_T^{\frac{k(1-p)}{p}} \zeta_{k+1}^{-\frac{1-p}{p}} \\ =&\frac{B_0^{1-p}}{1-p} \left(\sum_{j=0}^{\infty} \beta_T^{\frac{j}{p}} \mathbb{E}\zeta_{j+1}^{1-\frac{1}{p}} \right)^{p-1} \mathbb{E} \sum_{k=0}^{\infty} \beta_T^{\frac{k}{p}} \mathbb{E}\zeta_{k+1}^{1-\frac{1}{p}} \\ =&\frac{B_0^{1-p}}{1-p} \left(\sum_{k=0}^{\infty} \beta_T^{\frac{k}{p}} \mathbb{E}\zeta_{k+1}^{1-\frac{1}{p}} \right)^{p} \\ =&\frac{B_0^{1-p}}{1-p} \frac{\exp\left\{ \frac{\theta^2(1-p)}{2p} \right\}}{\left(1 - \beta_T^{\frac{1}{p}} \exp\left\{ \frac{\theta^2(1-p)}{2p^2} \right\} \right)^p} \\ =&\frac{B_0^{1-p}}{1-p} A_1^*, \end{aligned}$$

where

$$A_1^* = \frac{\exp\left\{ \frac{\theta^2(1-p)}{2p} \right\}}{\left(1 - \beta_T^{\frac{1}{p}} \exp\left\{ \frac{\theta^2(1-p)}{2p^2} \right\} \right)^p}$$

is given by (3.15). With this we have provided a simpler derivation of Theorems 4.3 and 4.4.

A.2 Logarithmic Utility

In this section, we give the simpler derivation of Theorem 4.6 similarly as in the previous section. We apply Theorem A.1 to the case when the trader's utility is given by the power function, namely (2.8). Then

$$I_U(z) = \frac{1}{z},$$

and (A.5) becomes

$$\begin{aligned} B_0 &= \mathbb{E}\sum_{k=0}^{\infty} \zeta_{k+1} \cdot \frac{1}{\frac{\lambda}{\beta_T^k}\zeta_{k+1}} \\ &= \frac{1}{\lambda}\sum_{k=0}^{\infty} \beta_T^k \\ &= \frac{1}{\lambda(1-\beta_T)}. \end{aligned}$$

Then we have

$$\lambda = \frac{1}{B_0(1-\beta_T)}.$$

Returning to (A.4), we see that

$$\begin{aligned} C_{k+1}^* &= \frac{1}{\frac{\lambda}{\beta_T^k}\zeta_{k+1}} \\ &= \frac{B_0(1-\beta_T)\beta_T^k}{\zeta_{k+1}}. \end{aligned}$$

Here we again make the observation that the optimal consumption for logarithmic utility is the same as the one for power utility with $p = 1$.

Now we calculate the optimal value of the trader with logarithmic utility.

$$
\begin{aligned}
&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k U(C_{k+1}^*)\\
=&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k \log\left(\frac{B_0(1-\beta_T)\beta_T^k}{\zeta_{k+1}}\right)\\
=&\mathbb{E}\sum_{k=0}^{\infty}\beta_T^k \left[\log\left(B_0(1-\beta_T)\beta_T^k\right)-\left(-\theta W_{k+1}-\frac{1}{2}\theta^2(k+1)\right)\right]\\
=&\left(\log B_0(1-\beta_T)+\frac{\theta^2}{2}\right)\sum_{k=0}^{\infty}\beta_T^k+\left(\log\beta_T+\frac{\theta^2}{2}\right)\sum_{k=0}^{\infty}k\beta_T^k\\
=&\left(\log B_0(1-\beta_T)+\frac{\theta^2}{2}\right)\frac{1}{1-\beta_T}+\left(\log\beta_T+\frac{\theta^2}{2}\right)\frac{\beta_T}{(1-\beta_T)^2}\\
=&A_1^*\log B_0+A_2^*,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^* &= \frac{1}{1-\beta_T}\\
A_2^* &= \frac{\theta^2}{2(1-\beta_T)^2}+\frac{1}{1-\beta_T}\log(1-\beta_T)+\frac{\beta_T}{(1-\beta_T)^2}\log\beta_T
\end{aligned}
$$

are given by (3.36) and (3.37), respectively. With this we have provided a simpler derivation of Theorem 4.6.

B Some Useful Results

In this section, we present some technical results that are useful in the calculations and proofs in this thesis.

Lemma B.1. *For ζ_t as defined in (A.3), and any constant a,*

$$\mathbb{E}\left[\zeta_t^a\right]=\exp\left\{\frac{\theta^2 t}{2}a(a-1)\right\}. \tag{B.14}$$

PROOF:

$$
\begin{aligned}
\mathbb{E}\left[\zeta_t^a\right] &= \mathbb{E}\left[\exp\left\{-a\theta W_t - \frac{a}{2}\theta^2 t\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{-a\theta W_t - \frac{1}{2}(a\theta)^2 t\right\} + \frac{\theta^2 t}{2}a(a-1)\right] \\
&= \exp\left\{\frac{\theta^2 t}{2}a(a-1)\right\}.
\end{aligned}
$$

□

Notice that $\zeta_1 = Z_1$, where Z_1 is defined in (2.10) with $k = 1$. Then we have the following Corollary:

Corollary B.2. *For Z_1 as defined in (2.10) with $k = 1$, and any constant a,*

$$
\mathbb{E}\left[Z_1^a\right] = \exp\left\{\frac{\theta^2}{2}a(a-1)\right\}. \tag{B.15}
$$

Lemma B.3. *Let Z_1 be as defined in (2.10) with $k = 1$. Let $\mathbb{P}$ and $\mathbb{Q}$ be the physical measure and the risk-neutral measure, respectively with $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_1$. Then $\frac{1}{Z_1}$ has the same distribution under $\mathbb{Q}$ as Z_1 under $\mathbb{P}$.*

PROOF:

$$
\begin{aligned}
\frac{1}{Z_1} &= \exp\left\{\theta W_1 + \frac{1}{2}\theta^2\right\} \\
&= \exp\left\{\theta W_1^{\mathbb{Q}} - \frac{1}{2}\theta^2\right\},
\end{aligned}
$$

where $W_t^{\mathbb{Q}} \triangleq W_t + \theta t, 0 \leq t \leq 1$ is a Brownian motion under $\mathbb{Q}$. Therefore $\frac{1}{Z_1}$ has the same distribution under $\mathbb{Q}$ as Z_1 under $\mathbb{P}$.

□

C When the trader becomes impatient

In Section 6.2, we observe that β_B^* decreases with respect to β_T, which implies that as the trader becomes more patient, there is more need for the bank to escrow. This may not seem intuitive since it is not obvious what incentive the escrowing provides when the trader is already patient. We have discussed heuristically the reasons driving this behavior in Section 6.2. In this Appendix, we present additional analysis illustrating the extreme case when β_T goes down to 0 and all other parameter values remain fixed.

Consider the case of escrowing the bonus with $p \neq 1$[7]. Equation (3.56), which characterizes the constant η^*, is equivalent to

$$\frac{\left(\mathbb{E}\left[\left(1-\left(\frac{Z_1}{\eta^*}\right)^{\frac{1}{p}}\right)^+\right]\right)^p}{\mathbb{E}\left[\left(1-\frac{Z_1}{\eta^*}\right)^+\right]+\frac{1}{\eta^*}} = \beta_T\left(\mathbb{E}\left[Z_1^{\frac{1}{p}}\right]\right)^p. \tag{C.16}$$

As $\beta_T \downarrow 0$, the right-hand side of (C.16) converges to 0. In the proof of Corollary 3.9, we have shown that the left-hand side of (C.16) is strictly increasing with respect to η^*, and goes to 0 when $\eta^* \to 0$. Therefore we have

$$\lim_{\beta_T \downarrow 0} \eta^* = 0. \tag{C.17}$$

[7]Similarly as in Chapter 5, in this section, we only reference the optimal trading strategy with power utility. The optimal trading strategy with logarithmic utility will have the similar behavior.

Moreover, (3.56) also implies that

$$\begin{aligned}
\frac{\beta_T}{\eta^*} &= \frac{1}{\left(\mathbb{E}\left[Z_1^{\frac{1}{p}}\right]\right)^p} \cdot \frac{\left(\mathbb{E}\left[\left(1-\left(\frac{Z_1}{\eta^*}\right)^{\frac{1}{p}}\right)^+\right]\right)^p}{\mathbb{E}\left[(\eta - Z_1)^+\right]+1} \\
&\le \frac{\left(\mathbb{E}\left[\left(1-\left(\frac{Z_1}{\eta^*}\right)^{\frac{1}{p}}\right)^+\right]\right)^p}{\left(\mathbb{E}\left[Z_1^{\frac{1}{p}}\right]\right)^p}.
\end{aligned}$$

As $\eta^* \to 0$, the upper bound converges to 0. Therefore

$$\lim_{\beta_T \downarrow 0} \frac{\beta_T}{\eta^*} = 0. \tag{C.18}$$

From (3.57) and (C.17) we have

$$\lim_{\beta_T \downarrow 0} \widetilde{A}_1^* = 1. \tag{C.19}$$

If the trader begins with initial bonus $B_0 = b$, according to Lemma 3.8, in the first period,

$$\begin{aligned}
\widetilde{Y}^* &= b\left[(\eta^* Z_1)^{-\frac{1}{p}}\left(1+(\beta_T \widetilde{A}_1^*)^{\frac{1}{p}}\right)-1\right]\mathbb{I}_{\{\eta^* Z_1 \ge 1\}} + b(\eta^* Z_1)^{-\frac{1}{p}}(\beta_T \widetilde{A}_1^*)^{\frac{1}{p}}\mathbb{I}_{\{\eta^* Z_1 \le 1\}} \\
&\le b(\beta_T \widetilde{A}_1^*)^{\frac{1}{p}} + b\left(\frac{\beta_T \widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}} Z_1^{-\frac{1}{p}}, \\
\widetilde{C}^* &= b(\eta^* Z_1)^{-\frac{1}{p}}\mathbb{I}_{\{\eta^* Z_1 \ge 1\}} + b\mathbb{I}_{\{\eta^* Z_1 \le 1\}} \\
&\le b, \\
\widetilde{B}^* &= b\left(\frac{\beta_T \widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}} Z_1^{-\frac{1}{p}}.
\end{aligned}$$

From (C.17), (C.18) and (C.19), we have the almost sure convergences

$$\lim_{\beta_T \downarrow 0} \widetilde{Y}^* = 0, \quad \lim_{\beta_T \downarrow 0} \widetilde{C}^* = b, \quad \lim_{\beta_T \downarrow 0} \widetilde{B}^* = 0. \tag{C.20}$$

Because $\mathbb{E}\left[Z_1^{-\frac{1}{p}}\right] < \infty$, these convergences are also in L_1. This shows that in the extreme case, as $\beta_T \downarrow 0$, the optimal strategy for the trader would be no trading and consuming all the initial escrow balance in one period.

Recall that the bank's total expected discounted revenue when bonuses are escrowed, given by (5.5), is

$$\widetilde{v_B{}^*}(b) = \frac{(1-\gamma)\beta_B b}{1 - \beta_B \left(\frac{\beta_T \widetilde{A}_1^*}{\eta^*}\right)^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}} \mathbb{E}\left[\frac{1}{\gamma}\widetilde{Y}^*\right].$$

From (C.17) - (C.20), we conclude that

$$\lim_{\beta_T \downarrow 0} \widetilde{v_B{}^*}(b) = 0. \tag{C.21}$$

On the other hand, when bonuses are not escrowed, the bank's total expected discounted revenue, given by (5.4), is

$$v_B{}^*(b) = \frac{(1-\gamma)\beta_B b}{1 - \beta_B \beta_T^{\frac{1}{p}} \exp\left\{\frac{\theta^2(1+p)}{2p^2}\right\}} \mathbb{E}\left[\frac{1}{\gamma}Y^*\right],$$

where by Lemma 3.1, Y^* does not depend on β_T. Therefore,

$$\lim_{\beta_T \downarrow 0} v_B{}^*(b) = (1-\gamma)\beta_B b \mathbb{E}\left[\frac{1}{\gamma}Y^*\right] > 0. \tag{C.22}$$

Thus for sufficiently small β_T, in other words, when the trader is extremely impatient, the bank has higher expected revenue if it does not escrow the trader's bonuses.

D Figures

Figure 1: Plot of $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* against β_T with $\theta = 1$ and $p = 2, 3, 4$. The curves produced with the same value of θ are shown in the same color (blue, red, or green). The curves corresponding to the three quantities $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* are plotted with the solid line, dashed line, and solid line with circle, respectively. For detailed discussion see Section 6.2

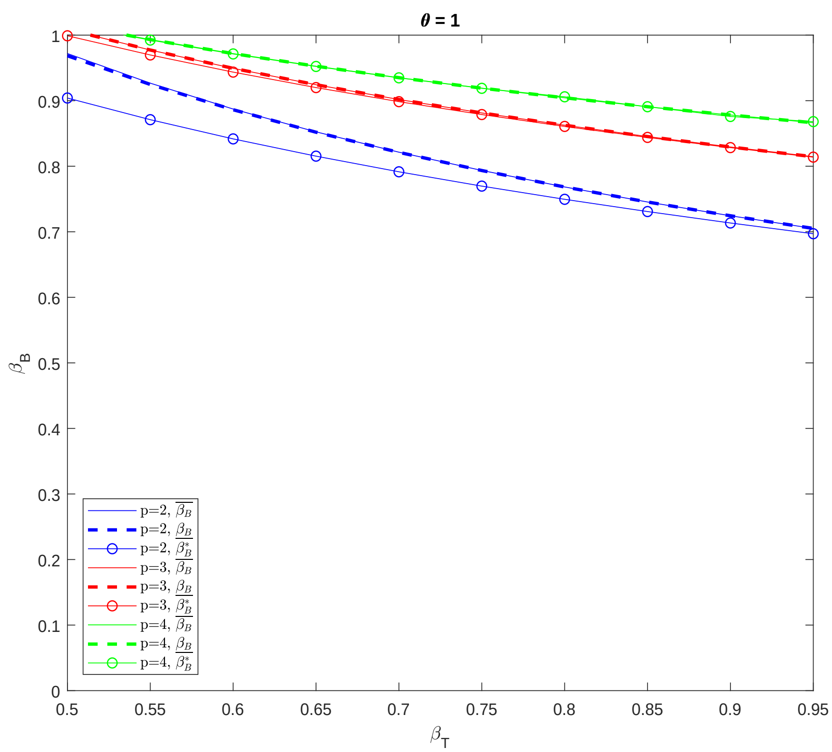

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Figure 2: Plot of $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* against β_T with $\theta = 2$ and $p = 2, 3, 4$. The curves produced with the same value of θ are shown in the same color (blue, red, or green). The curves corresponding to the three quantities $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* are plotted with the solid line, dashed line, and solid line with circle, respectively. For detailed discussion see Section 6.2

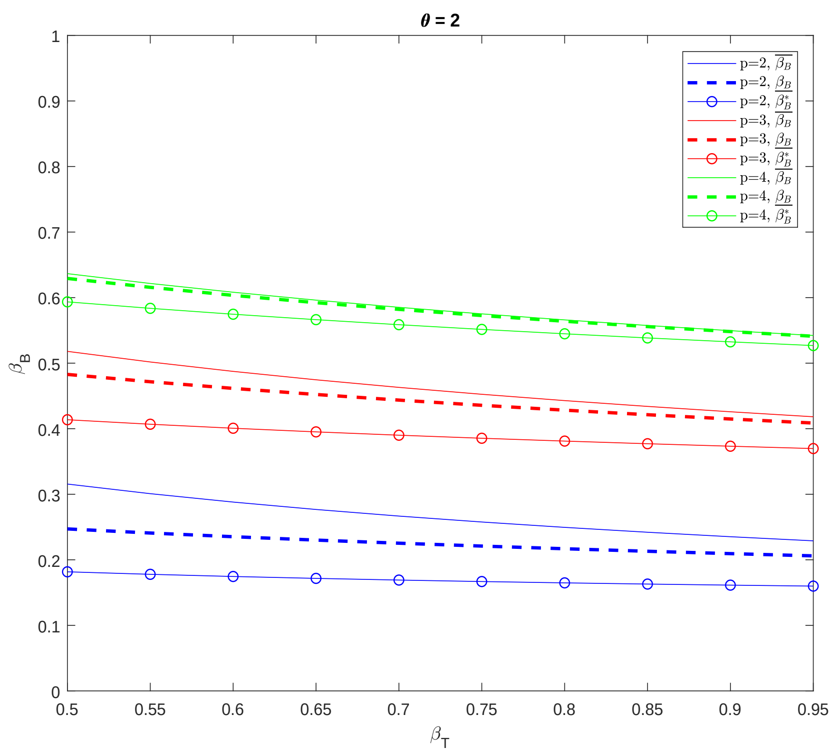

Figure 3: Plot of $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* against β_T with $\theta = 2$ and $p = 5, 8, 10$. The curves produced with the same value of θ are shown in the same color (blue, red, or green). The curves corresponding to the three quantities $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* are plotted with the solid line, dashed line, and solid line with circle, respectively. For detailed discussion see Section 6.2

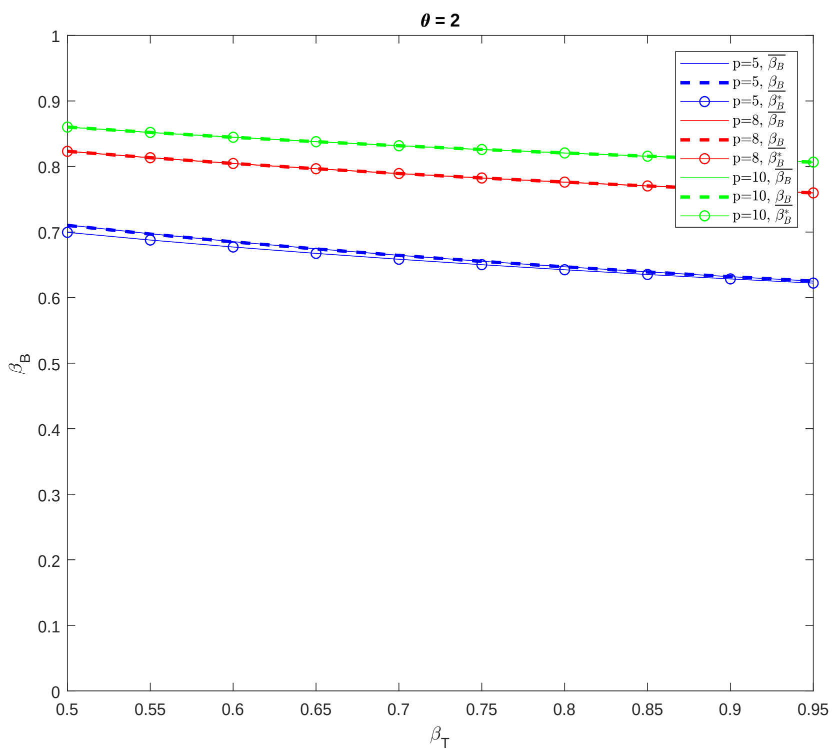

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Figure 4: Plot of $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* against β_T with $\theta = 3$ and $p = 2, 3, 4$. The curves produced with the same value of θ are shown in the same color (blue, red, or green). The curves corresponding to the three quantities $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* are plotted with the solid line, dashed line, and solid line with circle, respectively. For detailed discussion see Section 6.2

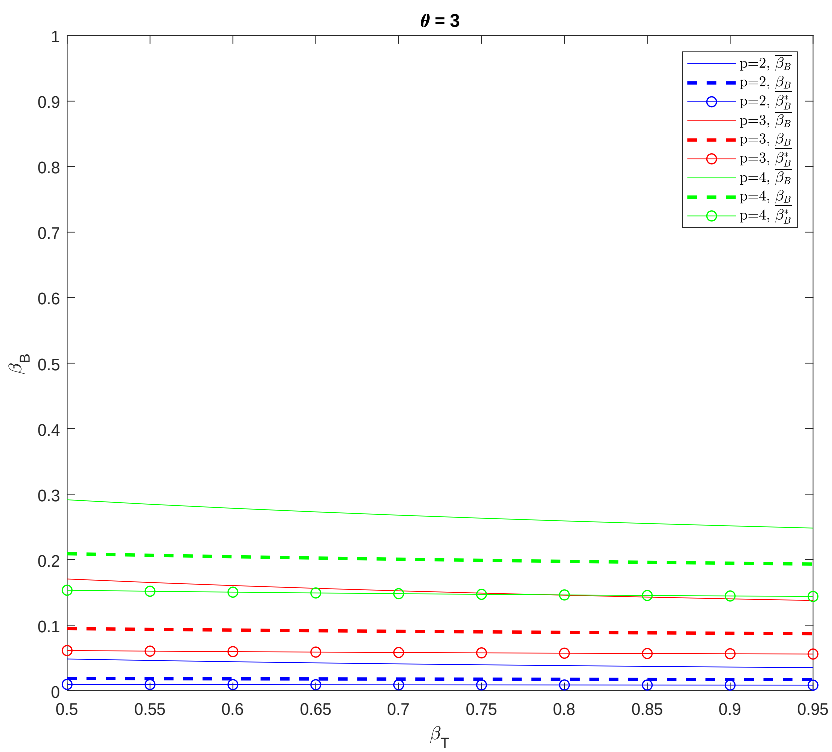

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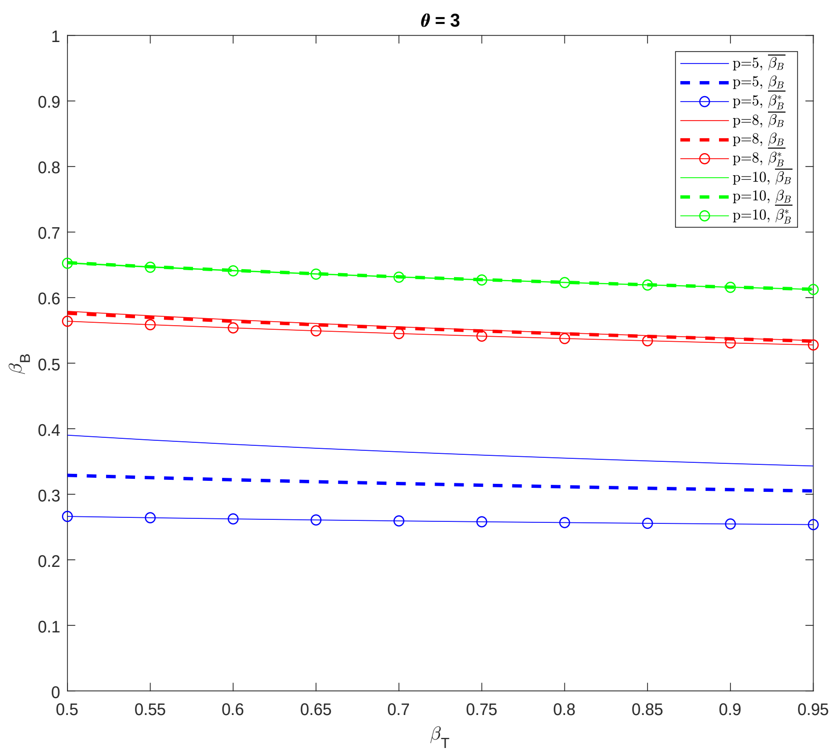
Figure 5: Plot of $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* against β_T with $\theta = 3$ and $p = 5, 8, 10$. The curves produced with the same value of θ are shown in the same color (blue, red, or green). The curves corresponding to the three quantities $\overline{\beta_B}$, $\underline{\beta_B}$ and β_B^* are plotted with the solid line, dashed line, and solid line with circle, respectively. For detailed discussion see Section 6.2



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