New directions in the abstract topological dynamics of Polish groups

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Introduction

The typical objects of study in topological dynamics are *flows* and *ambits*. If G is a Hausdorff topological group, a *G-flow* is a compact Hausdorff space X equipped with a continuous action $a : G \times X \to X$. Typically the action a is understood, and one writes $g \cdot x$ or gx for $a(g, x)$. A *G-ambit* is a pair (X, x_0) where X is a G-flow and $x_0 \in X$ is a distinguished point with dense orbit. Given G-flows X and Y, a map $\varphi : X \to Y$ is a *G-map* provided φ is continuous and respects the G-action; if (X, x_0) and (Y, y_0) are ambits, a *map of ambits* $\varphi : (X, x_0) \to (Y, y_0)$ is a G-map with $\varphi(x_0) = y_0$. A flow X is *minimal* if every orbit is dense.

An early theorem due to Ellis [8] asserts the existence and uniqueness of a *universal minimal flow*, a minimal flow which admits a G-map onto any other minimal flow. This flow is usually denoted by $M(G)$. To construct $M(G)$, one can use the *greatest ambit*, denoted $(S(G), 1_G)$, or just $S(G)$ when the distinguished point is understood. This is an ambit which maps onto every other ambit, and any minimal subflow of $S(G)$ is isomorphic to $M(G)$. Often $M(G)$ is large; for instance when G is a countable abelian discrete group, the underlying space of $M(G)$ is the *Gleason cover* (Stone space of the regular open algebra) of the cube $2^{\mathfrak{c}}$ [31]. For any locally compact group $M(G)$ is not metrizable, making it difficult to provide a concrete description of what $M(G)$ is.

Remarkably, there are non-trivial topological groups for which $M(G)$ is a singleton. These groups are called *extremely amenable*. Examples include the unitary group of a separable, infinite-dimensional Hilbert space with the strong operator topology [13] and the automorphism group of the rational linear order with the pointwise convergence topology [27]. Other topological groups have $M(G)$ non-trivial, but still a compact metric space. For the group G of orientation-preserving homeomorphisms of the unit circle, we have $M(G)$ the natural action of G on the circle [27]. For the group S_∞ of permutations of the natural numbers, we have $M(S_\infty) = \mathrm{LO}(\mathbb{N})$, the space of linear orders of $\mathbb{N}$ under the natural action [12].

The seminal paper by Kechris, Pestov, and Todorčević [16] considers $M(G)$ in the case that G is the automorphism group of a *Fraïssé structure*. These are countably infinite structures where every isomorphism between finite substructures extends to an automorphism of the full structure. A Fraïssé structure $\mathbf{K}$ is uniquely determined by the class of finite structures which embed into it, denoted $\mathrm{Age}(\mathbf{K})$. A classical result of Fraïssé [10] characterizes exactly which classes $\mathcal{K}$ of finite structures are $\mathrm{Age}(\mathbf{K})$ for some Fraïssé structure; if $\mathcal{K} = \mathrm{Age}(\mathbf{K})$ with $\mathbf{K}$ a Fraïssé structure, we call $\mathcal{K}$ a *Fraïssé class* and write $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$.

Examples of Fraïssé classes and their limits include finite linear orders and the rational linear order, finite graphs and the Rado graph, and finite Boolean algebras and the countable atomless Boolean algebra. Given a Fraïssé structure $\mathbf{K}$, we endow $G = \mathrm{Aut}(\mathbf{K})$ with the topology of pointwise convergence. A useful folklore result is that the automorphism groups of Fraïssé structures are exactly the *non-Archimedean* Polish groups, namely those Polish groups with a base at the identity of open subgroups. We will call such groups *automorphism groups* for short.

The main result of [16] connects the topological dynamics of an automorphism group $G = \mathrm{Aut}(\mathbf{K})$ with $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ to Ramsey-theoretic properties of the class $\mathcal{K}$. In particular, G is extremely amenable iff the class $\mathcal{K}$ has the *Ramsey property.* This means that whenever $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is $\mathbf{C} \in \mathcal{K}$ so that for any coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to 2$, there is $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ so that $\{x \circ f : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}$ is constant under γ. For example, one can restate the classical finite Ramsey theorem by saying that the class $\mathcal{K}$ of finite linear orders has the Ramsey property. This paper and later work by Nguyen Van Thé [26] also provide a mechanism for computing $M(G)$ even when $\mathcal{K}$ is not a Ramsey class. Roughly speaking, if $\mathcal{K}$ can be expanded into a Ramsey class $\mathcal{K}^*$ so that $\mathcal{K}^*$ is not too much bigger than $\mathcal{K}$, then $M(G)$ is metrizable. When such a $\mathcal{K}^*$ exists, we call the pair $(\mathcal{K}^*, \mathcal{K})$ *excellent.* A question left open was whether this phenomenon always occurred, namely if $\mathcal{K}$ is a Fraïssé class with $\mathrm{Flim}(\mathcal{K}) = \mathbf{K}$ and $M(\mathrm{Aut}(\mathbf{K}))$ metrizable, then is there an excellent expansion $\mathcal{K}^*$ of $\mathcal{K}$? A major component of this thesis is the proof of an affirmative answer to this question, first published in [34].

Theorem. *Let $\mathcal{K}$ be a Fraïssé class with $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and let $G = \mathrm{Aut}(\mathbf{K})$. If $M(G)$ is metrizable, then there is an expansion $\mathcal{K}^*$ of $\mathcal{K}$ with $(\mathcal{K}^*, \mathcal{K})$ excellent.*

This is proven by providing a combinatorial characterization of metrizable $M(G)$ much in the way that the Ramsey Property characterizes extreme amenability. This generalization is called *finite Ramsey degrees.* If $\mathcal{K}$ is a Fraïssé class and $\mathbf{A} \in \mathcal{K}$, we say that $\mathbf{A}$ has *finite Ramsey degree* if there is $k < \omega$ so that for any $\mathbf{B} \in \mathcal{K}$ and $r < \omega$, there is $\mathbf{C} \in \mathcal{K}$ so that for any coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to r$, there is $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ so that $|\{\gamma(x \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}| \leq k$. The class $\mathcal{K}$ has finite Ramsey degrees if every $\mathbf{A} \in \mathcal{K}$ has finite Ramsey degree.

Theorem. *Let $\mathcal{K}$ be a Fraïssé class with $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and let $G = \mathrm{Aut}(\mathbf{K})$. Then $M(G)$ is metrizable iff $\mathcal{K}$ has finite Ramsey degrees.*

There are several natural ways one can attempt to generalize the work in [16]. One such effort is to attempt to generalize these efforts to all Polish groups. A question along these lines was the Generic Point Problem asked by Angel, Kechris, and Lyons [1]: if G is a Polish group and $M(G)$ is metrizable, must $M(G)$ have a comeager orbit? This question arose naturally due to the examples produced by the machinery in [16]; whenever $\mathcal{K}$ is a

Fraïssé class which admits an excellent companion and $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, then $M(G)$ has a comeager orbit. Melleray, Nguyen Van Thé, and Tsankov show in [21] that if G is a Polish group so that $M(G)$ is metrizable and has a comeager orbit, then $M(G)$ is the left completion of the right coset space $\widehat{G/G_0}$ for some G_0 extremely amenable and coprecompact in G, G/G_0 is the right coset space, and the completion is taken with respect to the left uniformity on G/G_0. In particular, for $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ an automorphism group, saying that $M(G)$ is of the form $\widehat{G/G_0}$ for G_0 extremely amenable and coprecompact is exactly the statement that $\mathcal{K}$ admits an excellent expansion class. For automorphism groups, the characterization of metrizable $M(G)$ given in [34] affirmatively solves the Generic Point Problem.

Ben–Yaacov, Melleray, and Tsankov in [5] affirmatively solve the Generic Point Problem for general Polish groups. Their proof generalizes some of the key ideas from [34] by making use of new *topometric* machinery. Roughly speaking, this allows a detailed study of how a bounded left-invariant metric on the group G interacts with suitably universal dynamical systems. In recent joint work with Bartošová, we use topometric methods to produce a new characterization of when $M(G)$ is metrizable for G a Polish group. For X a G-flow, write $AP(X) = \{x \in X : \overline{xG} \text{ is minimal}\}$.

Theorem. *Let G be a Polish group. Then $M(G)$ is metrizable iff for every ambit (X, x_0), we have $AP(X) \subseteq X$ closed.*

A lingering question regarding the Generic Point Problem was the following. Suppose X is a minimal metrizable G-flow with all orbits meager; then by the Generic Point Theorem, we know $M(G)$ is non-metrizable. Could this be seen directly, i.e. can we use X to construct a non-metrizable minimal flow? It turns out that the correct object of study is the *universal highly proximal extension* of X, denoted $S_G(X)$. This object was first studied by Auslander and Glasner in [2]; for the definition, see section 2.6.

Theorem. *Let G be a Polish group, and suppose X is a minimal metrizable G-flow with all orbits meager. Then $S_G(X)$ is non-metrizable.*

The proof provides a new construction of $S_G(X)$ which generalizes the notion of universal highly proximal extension to any G-space.

Another direction one can attempt to generalize the work in [16] is to consider other combinatorial properties that a Fraïssé class $\mathcal{K}$ might have. One natural property along these lines is *big Ramsey degree*. The terminology is from [16], but the notion has been around for several decades. If $\mathcal{K}$ is a Fraïssé class and $\mathbf{A} \in \mathcal{K}$, we say that $\mathbf{A}$ has *finite big Ramsey degree* if there is $k < \omega$ so that for any $r < \omega$ and any coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to r$, there is $\eta \in \mathrm{Emb}(\mathbf{K}, \mathbf{K})$ with $|\{\gamma(\eta \circ f) : f \in \mathrm{Emb}(\mathbf{K}, \mathbf{K})\}| \leq k$. The least $k < \omega$ where this holds is called the *big Ramsey degree* of $\mathbf{A}$. The class $\mathcal{K}$ has finite big Ramsey degrees if every $\mathbf{A} \in \mathcal{K}$ has finite big Ramsey degree. Given the results of [34], one might hope that given a Fraïssé class $\mathcal{K}$ with $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$, then $\mathcal{K}$ having finite big Ramsey degrees is equivalent to the metrizability of some universal dynamical object for the group G.

In [36], a new type of dynamical object is defined, called a *completion flow*. Given this new dynamical object, it is natural to ask which topological groups admit unique universal completion flows, and when these universal objects are metrizable. To address this question for $G = \mathrm{Aut}(\mathrm{Flim}(\mathcal{K}))$ an automorphism group, a strengthening of the notion of finite big Ramsey degrees is defined. Roughly speaking, if $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, then $\mathbf{K}$ is said to admit a *big Ramsey structure* if there is an expansion $\mathbf{K}^*$ which simultaneously records the correct big Ramsey degrees of every $\mathbf{A} \in \mathcal{K}$; see section 3.8 for the precise definition.

Theorem. *Suppose $\mathcal{K}$ is a Fraïssé class with $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, and set $G = \mathrm{Aut}(\mathbf{K})$. If $\mathbf{K}$ admits a big Ramsey structure, then G admits a unique universal completion flow which is metrizable.*

The techniques in the proof of this theorem introduce other new dynamical objects which seem interesting in their own right. A G-flow X is called a *pre-ambit* if there is some point with dense orbit (so a pre-ambit is just an ambit before picking the distinguished point). Write $\mathcal{A}(X) = \{x \in X : xG \subseteq X \text{ is dense}\}$. If X and Y are pre-ambits, a G-map $\varphi : X \to Y$ is called *strong* if $\varphi``(\mathcal{A}(X)) = \mathcal{A}(Y)$. If X and Y are pre-ambits, a strong map $\psi : Y \to X$ is called *universal* if for any other pre-ambit Y_0 and strong map $\psi_0 : Y_0 \to X$, there is a G-map $\varphi : Y \to Y_0$ with $\psi = \psi_0 \circ \varphi$.

Theorem. *Let G be a topological group, and let X be a pre-ambit. Then there is a* univeral strong extension *of X, which is unique up to G-flow isomorphism over X.*

In particular, this generalizes the existence and uniqueness of the universal minimal flow $M(G)$.

Organization

This thesis is organized into three chapters. The first is on uniform spaces and is mostly background. However, the last section on topometric spaces discusses some recent new approaches to the study of the Samuel compactification of a metric space.

The second focuses on abstract topological dynamics. The key unifying feature of this chapter is the notion of a near ultrafilter. By viewing $S(G)$ as a space of near ultrafilters, we obtain an explicit formula for the semigroup operation, which allows for a detailed study of how the minimal subflows sit inside $S(G)$. Near ultrafilters also feature prominently in section 2.6 under the guise of "G-near" ultrafilters. These are used to provide a new construction of the universal highly proximal extension of a G-flow X. Abstract considerations regarding compact left-topological semigroups also feature prominently, especially in section 2.5, where we construct the universal strong extension of a pre-ambit.

The third and largest chapter focuses on automorphism groups. This chapter introduces the notion of a *diagram*, which is more or less an abstract notion of partial right action. While we will see that any diagram can be coded by a logic action, the main point of the diagram is that the particular coding doesn't matter. Therefore diagrams provide a unified approach to understanding logic actions, spaces of colorings, and $M(G)$.

Notation and conventions

Our notation follows set-theoretic standards. We write $\omega = \mathbb{N} = \{0, 1, 2, ...\}$, and we identify $k < \omega$ with the set $\{0, ..., k-1\}$. We sometimes use $:=$ for "equal by definition," though sometimes we simply use $=$ when defining new objects.

If $f : X \to Y$ is a function and $A \subseteq X$, we write $f``(A) = \{f(a) : a \in A\}$. If G is a group, X is a G-space, $U \subseteq G$, and $A \subseteq X$, we write $A \cdot U$ or just AU for the set $\{ag : a \in A \text{ and } g \in U\}$. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are structures, $K \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{B})$, and $L \subseteq \mathrm{Emb}(\mathbf{B}, \mathbf{C})$, we write $L \circ K = \{x \circ f : x \in L \text{ and } f \in K\} \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{C})$.

In this thesis, we take our G-spaces to be *right* G-spaces, i.e. with the group acting on the right. This choice is mainly due to various conventions that we will develop when G is an automorphism group in chapter 3. Much of the literature in topological dynamics takes the opposite convention and works with *left* G-spaces. When citing results from various references, we will always phrase the result to refer to right G-spaces.

Preliminaries

A good reference in general topology is [9]. For topological groups and descriptive dynamics, see [4] or [15].

Topology

All topological spaces will be assumed Hausdorff unless explicitly specified otherwise.

- Compact Hausdorff spaces are normal.

- A space X is Polish iff X is separable and *completely metrizable*, i.e. there is a compatible metric on X which is complete.

- A space X is compact iff every net $(x_i)_{i\in I}$ from X has a convergent subnet. In general, we will freely use vocabulary pertaining to nets (i.e. *eventually* $x_i \in U$, *frequently* $x_i \in U$, etc.).

- Every compact metric space is separable, hence Polish.

- If (X, d) is compact metric, Y is any space, and $f : X \to Y$ is continuous, then $f``(X)$ is metrizable. The *Hausdorff metric* defined by

$$d_H(a, b) := \sup_{x \in f^{-1}(\{a\})} \left(\inf_{y \in f^{-1}(\{b\})} (d(x, y)) \right) + \sup_{y \in f^{-1}(\{b\})} \left(\inf_{x \in f^{-1}(\{a\})} (d(x, y)) \right)$$

  is a compatible metric on $f``(X)$.

- A subset $A \subseteq X$ is called *meager* if $A \subseteq \bigcup_n F_n$ with each F_n closed and nowhere dense. $B \subseteq X$ is *comeager* if $X \setminus B$ is meager. A space X is *Baire* if every comeager set is dense. All Polish spaces and all compact spaces are Baire.

- A subset $A \subseteq X$ has the *property of Baire* or the *BP* if there is an open $U \subseteq X$ with the symmetric difference $A \triangle U$ meager.

- The following is a useful criterion for determining when a map $\varphi : X \to Y$ is continuous. I don't know of a good reference and the proof is short, so it is included here.

  Proposition 0.0.1. *Let X and Y be topological spaces, and assume that Y is regular. Let $D \subseteq X$ be dense, and let $\varphi : X \to Y$ be a map so that whenever $(d_i)_{i \in I}$ is a net from D with $d_i \to x \in X$, then $\varphi(d_i) \to \varphi(x)$. Then φ is continuous.*

  Proof. Let $U \subseteq Y$ be open with $\varphi(x) \in U$. Towards a contradiction, suppose we could find a net $(x_i)_{i \in I}$ from X with $x_i \to x$ so that frequently $\varphi(x_i) \notin U$. As Y is regular, let $V \subseteq Y$ be open with $\varphi(x) \in V$ and $\overline{V} \subseteq U$. By assumption, find open $A \subseteq X$ with $x \in A$ and $\varphi``(D \cap A) \subseteq V$. Find $i \in I$ such that $x_i \in A$ and $\varphi(x_i) \notin U$. Let $B \subseteq X$ be open so that $x_i \in B$ and $\varphi``(D \cap B) \subseteq Y \setminus \overline{V}$. Since $D \cap A \cap B \neq \emptyset$, this is a contradiction. $\square$

- We will make frequent use of spaces of ultrafilters. If X is any set, then βX is the collection of ultrafilters on X. We endow βX with the compact Hausdorff topology whose typical basic clopen set is of the form $C_A := \{p \in \beta X : A \in p\}$ for some $A \subseteq X$. We identify X as a dense subset of βX by considering principal ultrafilters. If Y is any compact space and $f : X \to Y$ is any function, then there is a unique continuous map $\tilde{f} : \beta X \to Y$ with $\tilde{f}|_X = f$.

Topological groups

- If G is a topological space and also a group, we call G a *topological group* if the evaluation map $G \times G \to G$ and the inverse map $G \xrightarrow{-1} G$ are continuous.

- A classical theorem due to Birkhoff and Kakutani says that a topological group G is metrizable iff G is first countable. Every separable metrizable group embeds densely into a Polish group.

- Every metrizable group admits a compatible bounded left-invariant metric. While every Polish group admits a compatible complete metric by virtue of being Polish, it is not always possible to find a compatible metric which is simultaneously complete and left invariant (indeed this is what makes the left completion interesting to study)

Descriptive dynamics

- A G-space X is *topologically transitive* if for every non-empty open $U \subseteq X$, $UG \subseteq X$ is dense. Any pre-ambit is topologically transitive. If X is Polish, then X is topologically transitive iff X is a pre-ambit.

- If G is a Polish group, X is a Polish G-space, and $x \in X$, then the orbit $xG \subseteq X$ is Borel, hence has the property of Baire. If furthermore X is topologically transitive, we have the *topological 0-1 law*, sometimes called *generic ergodicity*: every G-invariant set with the property of Baire is either meager or comeager. In particular, every orbit is either meager or comeager.

Chapter 1

Uniform spaces

This chapter is devoted to background on uniform spaces, which will be used frequently going forward. In particular, we introduce the notion of a near ultrafilter on a uniform space. Using near ultrafilters, we construct the Samuel compactification and discuss some of its key properties. Given a bounded metric space X, we turn the Samuel compactification $S(X)$ into a topometric space and prove a theorem of Ben–Yaacov, Melleray, and Tsankov characterizing compact metrizable subsets of $S(X)$.

1.1 Basics on uniform spaces

If X is a set, a *uniformity* on X is a collection $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ with the following properties.

1. Every $U \in \mathcal{U}$ contains the diagonal $\Delta_X := \{(x, x) : x \in X\}$.
2. $\bigcap_{U \in \mathcal{U}} U = \Delta_X$.
3. $\mathcal{U}$ is upwards closed and closed under finite intersections.
4. If $U \in \mathcal{U}$, then so is $U^{-1} := \{(y, x) : (x, y) \in U\}$.
5. Every $U \in \mathcal{U}$ admits a *square root*, which is some $V \in \mathcal{U}$ with $V \cdot V := V^2 := \{(x, y) : \exists z\, ((x, z) \in V \text{ and } (z, y) \in V)\} \subseteq U$. It follows that every $U \in \mathcal{U}$ admits *cube roots, n^{th} roots*, etc., which are defined analogously.

We call the pair $(X, \mathcal{U})$ a *uniform space*, and we sometimes call the members of $\mathcal{U}$ *entourages*. A subset $\mathcal{B} \subseteq \mathcal{U}$ is called a *base* for $\mathcal{U}$ if for every $U \in \mathcal{U}$, there is $V \in \mathcal{B}$ with $V \subseteq U$. For the remainder of the section, fix a uniform space $(X, \mathcal{U})$.

Given $A \subseteq X$, we can form the *subspace uniformity* $\mathcal{U}|_A := \{U \cap (A \times A) : U \in \mathcal{U}\}$ on A. We will always endow subsets of a uniform space with the subspace uniformity.

The *uniform topology* on X is defined as follows. If $x \in X$ and $U \in \mathcal{U}$, define $x(U) = \{y \in X : (x, y) \in U\}$. Then the neighborhood filter for this topology is the collection

$\mathcal{U}(x) := \{x(U) : U \in \mathcal{U}\}$. Equivalently, given $A \subseteq X$ and letting $A(U) = \bigcup_{x \in A} x(U)$, the uniform topology has the closure operator $\overline{A} = \bigcap_{U \in \mathcal{U}} A(U)$. Item (2) in the definition of a uniform space guarantees that the uniform topology is Hausdorff. When dealing with a uniform space, we will freely use topological notions in reference to this topology. If X is a topological space and $\mathcal{U}$ is a uniformity on X, we say that $\mathcal{U}$ is *compatible* if the topology on X coincides with the uniform topology.

If $(Y, \mathcal{V})$ is another uniform space, then a function $f : X \to Y$ is *uniformly continuous* if for every $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ with $(f \times f)``(U) \subseteq V$. If $f : X \to Y$ is uniformly continuous, then f is continuous when X and Y are given the uniform topology, but the converse is not always true.

Some spaces do not admit a compatible uniform structure.

Fact (Weil [32])**.** Let X be a topological space. Then X admits a compatible uniform structure iff X is *Tychonoff*, also called *completely regular*; if $A \subseteq X$ is closed and $x \in X \setminus A$, then A and x can be separated by a real valued function.

In practice, we often have a topological space X that we then endow with a compatible uniformity $\mathcal{U}$. When this is the case, we will often tacitly assume that entourages are *symmetric* ($U = U^{-1}$) and open in $X \times X$. It is easy to see that entourages of this type form a base for the uniformity.

The types of spaces we will most often want to uniformize are the following.

- If X is a discrete space, then the *discrete uniformity* on X is the collection $\{U \subseteq X \times X : \Delta_X \subseteq U\}$.

- If (X, d) is a metric space, then $\{\{(x, y) : d(x, y) < \epsilon\} : \epsilon > 0\}$ is the base for a compatible uniformity.

- If X is compact, then X admits a unique compatible uniform structure; a base for this uniformity is given by $\{U \subseteq X \times X : \Delta_X \subseteq U \text{ and } U \text{ is open}\}$. Any continuous function from a compact space X to another uniform space is uniformly continuous.

- If G is a topological group, then G admits several compatible uniform structures. One worth mentioning now is the *left uniformity*; if $\mathcal{N}_G$ is a base of open neighborhoods of the identity, then a base for this uniformity is given by $\{\{(x, y) : x^{-1}y \in U\} : U \in \mathcal{N}_G\}$.

A net $(x_i)_{i \in I}$ from X is called *Cauchy* if for every $U \in \mathcal{U}$, there is $i_0 \in I$ so that for any $i, j \geq i_0$ we have $(x_i, x_j) \in U$. The space $(X, \mathcal{U})$ is *complete* if every Cauchy net converges to some $x \in X$.

If $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are two Cauchy nets, we say that they are *equivalent* if for every $U \in \mathcal{U}$, we eventually have $(x_i, y_j) \in U$. Our goal is to define the *completion* $(\widehat{X}, \widehat{\mathcal{U}})$ of

$(X,\mathcal{U})$. First, we set $\widehat{X}$ to be the set of equivalence classes of Cauchy nets[1]. If $U \in \mathcal{U}$, we define

$$\begin{aligned}\widehat{U} := \{([(x_i)_{i\in I}],[(y_j)_{j\in J}]) : {} & \text{there is } V \in \mathcal{U} \text{ so that eventually, whenever} \\ & (x_i,x) \in V \text{ and } (y_j,y) \in V, \text{ we have } (x,y) \in U\}\end{aligned}$$

We can now let $\{\widehat{U} : U \in \mathcal{U}\}$ be a base for the uniformity $\widehat{\mathcal{U}}$ on $\widehat{X}$. As one would expect, the completion of a uniform space is complete. The map $i_X : X \to \widehat{X}$ sending $x \in X$ to a suitable constant net is a uniform embedding, and we often identify X as a subset of $\widehat{X}$.

The completion $(\widehat{X},\widehat{\mathcal{U}})$ of $(X,\mathcal{U})$ has the following abstract characterization. If $(Y,\mathcal{V})$ is a complete uniform space and $f : X \to Y$ is uniformly continuous, then there is a unique extension of f to a uniformly continuous $\widehat{f}$ making the following diagram commute.

$$\begin{tikzcd} \widehat{X} \arrow[dr, "\widehat{f}"] & \\ X \arrow[u, "i_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

Uniform spaces are closely related to the notion of *proximity spaces*. While we won't define those here, we will borrow some terminology. Namely, we call $A, B \subseteq X$ *apart* if there is some $U \in \mathcal{U}$ with $A(U) \cap B(U) = \emptyset$. For a more complete treatment of uniform spaces, see [9].

1.2 Near ultrafilters and the Samuel compactification

For this section, fix a uniform space $(X,\mathcal{U})$. We will simply write X when $\mathcal{U}$ is understood. Our goal is to construct the *Samuel compactification* of X. This is a compact space $S(X)$ and an embedding $j_X : X \to S(X)$ so that whenever Y is compact and $f : X \to Y$ is uniformly continuous, there is a (necessarily unique) continuous extension $\tilde{f} : S(G) \to Y$ making the following diagram commute.

$$\begin{tikzcd} S(X) \arrow[dr, "\tilde{f}"] & \\ X \arrow[u, "j_X"] \arrow[r, "f"'] & Y \end{tikzcd}$$

[1]Some care must be taken here to verify that we can limit our attention to some suitable *set* of Cauchy nets. One can check that any Cauchy net is equivalent to a Cauchy net with index set $\mathcal{U}$, which we order by reverse inclusion.

Furthermore, we want $S(X)$ to be the coarsest compactification of X with this property; namely we want that if $\varphi : S(G) \to Y$ is continuous, then $\varphi|_X$ is uniformly continuous. Like we do with completions, we simply identify X as a subset of $S(X)$.

Our construction is very similar to Samuel's original construction in [29]; see also [17].

Definition 1.2.1.

- Let $\mathcal{F} \subseteq \mathcal{P}(X)$. We say that $\mathcal{F}$ has the *near finite intersection property*, or near FIP, if whenever $A_0, ..., A_{k-1} \in \mathcal{F}$ and $U \in \mathcal{U}$, we have $\bigcap_{i<k} A_i(U) \neq \emptyset$.
- A *near ultrafilter* on X is any $p \subseteq \mathcal{P}(X)$ which is maximal with respect to having the near FIP.

Note that by Zorn's lemma, any $\mathcal{F}$ with the near finite intersection property is contained in some near ultrafilter. A key example worth keeping in mind is when X has the discrete uniformity; then near ultrafilters on X are just ultrafilters on X.

Proposition 1.2.2. *Fix p a near ultrafilter on X.*

1. *If $A_0, ..., A_{k-1} \in p$ and $U \in \mathcal{U}$, then $\bigcap_{i<k} A_i(U) \in p$.*
2. *If $A \notin p$, then there is $V \in \mathcal{U}$ with $A(V) \notin p$.*
3. *If $A \cup B \in p$, then either $A \in p$ or $B \in p$.*

Proof.

1. This is an easy consequence of the fact that p is maximal with respect to having the near FIP.
2. Assume $A \notin p$. Then there are $B_0, ..., B_{k-1} \in p$ and $U \in \mathcal{U}$ with $A(U) \cap \bigcap_{i<k} B_i(U) = \emptyset$. Find $V \in \mathcal{U}$ a square root of U. Then $A(V)(V) \cap \bigcap_{i<k} B_i(V) = \emptyset$.
3. Suppose neither A nor B were in p. Find $C_0, ..., C_{k-1}, D_0, ..., D_{\ell-1} \in p$, and $U \in \mathcal{U}$ with $A(U) \cap \bigcap_{i<k} C_i(U) = \emptyset$ and $B(U) \cap \bigcap_{j<\ell} D_j(U) = \emptyset$. Write $C = \bigcap_{i<k} C_i(U)$ and $D = \bigcap_{j<\ell} D_j(U)$. Then $(A \cup B)(U) \cap C \cap D \neq \emptyset$. But notice that $(A \cup B)(U) = A(U) \cup B(U)$, leading to a contradiction $\square$

Corollary 1.2.3. *If p is a near ultrafilter on X, then $A \in p$ iff for every $B \in p$ and $U \in \mathcal{U}$, we have $A \cap B(U) \neq \emptyset$.*

Denote the set of near ultrafilters on X by $S(X)$. For $A \subseteq X$, let $C_A \subseteq S(X)$ denote those near ultrafilters which contain A, and write $N_A = S(X) \setminus C_A$. We endow $S(X)$ with the topology whose basic open neighborhoods are of the form N_A, where $A \subseteq X$. When X has the discrete uniformity, this is simply the space βX of all ultrafilters on X with its standard topology.

We view X as a dense subspace of $S(X)$ by associating to each $x \in X$ the principal near ultrafilter $\{A \subseteq X : x \in \overline{A}\}$. We write $j_X : X \to S(X)$ for this embedding. If $A \subseteq X$, we write $\overline{A}$ for the closure in X and $cl_{S(X)}(A)$ for the closure in $S(X)$. For sets $U \subseteq S(X)$ where there is no ambiguity, we write $\overline{U}$ for the $S(X)$-closure.

Below we collect some simple facts about this topology. Let $A, B \subseteq X$.

- $C_A = C_{\overline{A}}$ (use Proposition 1.2.2).
- $C_A \subseteq C_B$ iff $j_X``(A) \subseteq C_B$ iff $A \subseteq \overline{B}$. In particular, $j_X``(X) \cap C_{\overline{A}} = j_X``(\overline{A})$, showing that $j_X : X \to S(X)$ is a topological embedding. From now on, we will freely identify $x \in X$ with $j_X(x) \in S(X)$.
- $N_A \cap N_B = \emptyset$ iff $N_A \subseteq C_B$ iff $X = \overline{A} \cup \overline{B}$. In particular, $\overline{N_A} = C_{X \setminus \overline{A}}$ and $\text{int}(C_B) = N_{X\setminus\overline{B}}$.

Proposition 1.2.4. *The space $S(X)$ is compact Hausdorff.*

Proof. To show that $S(X)$ is Hausdorff, let $p \neq q \in S(X)$, and find $A \in p \setminus q$. Find $U \in \mathcal{U}$ with $A(U) \notin q$. Notice that $X \setminus (A(U)) \notin p$. Then $p \in N_{X\setminus(A(U))}$, $q \in N_{A(U)}$, and $N_A \cap N_{X\setminus(A(U))} = \emptyset$.

To show that $S(X)$ is compact, suppose $\{N_{A_i} : i \in I\}$ were a collection of basic open sets with no finite subcover. Then for any $i_0, ..., i_{k-1} \in I$, we can find $p \in \bigcap_{j<k} S(X) \setminus N_{A_{i_j}} = \bigcap_{j<k} C_{A_{i_j}}$. Therefore the set $\{A_i : i \in I\}$ has the near FIP, and can be extended to $q \in S(X)$. As $q \notin N_{A_i}$ for any $i \in I$, the collection $\{N_{A_i} : i \in I\}$ is not a cover. $\square$

The next two theorems will show us that $S(X)$ is the Samuel compactification of X.

Theorem 1.2.5. *Let Y be compact, and suppose $f : X \to Y$ is uniformly continuous. Then f can be continuously extended to $\tilde{f} : S(G) \to Y$.*

Proof. Given $p \in S(X)$, we define $\tilde{f}(p) = y$ iff for every open $B \ni y$, we have $f^{-1}(B) \in p$. We will check that this is well defined and continuous.

A straightforward compactness argument shows that there is at least one $y \in Y$ with the above property. Suppose $y \neq z \in Y$ both had the property. Find an open cover $\mathcal{C}$ of Y with the following property.

- If $y = y_0, y_1,...,y_k = z$ are points in Y so that for each $i < k$, there is $B_i \in \mathcal{C}$ with $y_i, y_{i+1} \in B_i$, then $k \geq 4$.

To do this, first let D_y and D_z be disjoint open neighborhoods of y and z. By normality, find open B_y and B_z with $y \in B_y$, $z \in B_z$, $\overline{B_y} \subseteq D_y$, and $\overline{B_z} \subseteq D_z$. For any $w \in Y$ with $w \notin \{y, z\}$, choose open $B_w \ni w$ as follows. If $w \in D_y$, ensure that $B_w \subseteq D_y$. Likewise for

$w \in D_z$. If $w \notin D_y \cup D_z$, then also $w \notin \overline{B_y} \cup \overline{B_z}$, so ensure $B_w \cap (\overline{B_y} \cup \overline{B_z}) = \emptyset$. Now set $\mathcal{C} = \{B_w : w \in Y\}$.

Let $V = \{(a, b) \in Y \times Y : \exists\, w \in Y \text{ with } a, b \in B_w\}$. By uniform continuity of f, find $U \in \mathcal{U}$ with $(f \times f)``(U) \subseteq V$. Now consider $f^{-1}(B_y)$ and $f^{-1}(B_z)$, which by assumption are both in p. The property of $\mathcal{C}$ is exactly what we need to conclude that $(f^{-1}(B_y))(U) \cap (f^{-1}(B_z))(U) = \emptyset$, a contradiction.

To check that $\tilde{f}$ is continuous, let $K \subseteq Y$ be closed. We show that

$$\tilde{f}^{-1}(K) = \bigcap\{C_{f^{-1}(B)} : B \supseteq K \text{ open}\}.$$

The left to right inclusion is clear. For right to left, suppose $p \in S(X)$ satisfies $\tilde{f}(p) := y \notin K$. Use normality to find open sets B_K, B_y, D_K, D_y so that $K \subseteq B_K$, $\overline{B_K} \subseteq D_K$, $y \in B_y$, $\overline{B_y} \in D_y$, and $D_K \cap D_y = \emptyset$. Now much as in the proof of uniqueness, find a suitably fine open cover of Y and use uniform continuity to conclude that $f^{-1}(B_y)$ and $f^{-1}(B_K)$ can never belong to the same near ultrafilter. $\square$

Theorem 1.2.6. *Let Y be compact and suppose $\varphi : S(X) \to Y$ is continuous. Then $\varphi|_X$ is uniformly continuous.*

Proof. Let $V \subseteq Y \times Y$ be an open neighborhood of Δ_Y. Towards a contradiction, suppose for every $U \in \mathcal{U}$, there were $x_U, y_U \in X$ with $(x_U, y_U) \in U$, but $(\varphi(x_U), \varphi(y_U)) \notin V$. By passing to a subnet, assume $x_i \to p \in S(X)$ and $y_i \to q \in S(X)$. Then $(\varphi(p), \varphi(q)) \notin V$. However, suppose $A \notin p$. Then for some $U \in \mathcal{U}$, we have $A(U) \notin p$. So eventually $A(U) \notin x_i$, implying that $x_i \in \overline{X \setminus (A(U))}$. If U' is a cube root of U, then eventually $(x_i, y_i) \in U'$, implying that eventually $y_i \in X \setminus (A(U'))$. Therefore $A \notin q$ also, and $p = q$, a contradiction. $\square$

Corollary 1.2.7 (Uniform Urysohn lemma)**.** *Suppose $A, B \subseteq X$ are apart. Then A and B may be separated by a uniformly continuous real-valued function.*

Proof. A and B are apart exactly when $C_A \cap C_B = \emptyset$. Now use Urysohn's lemma on $S(X)$. $\square$

Remark. Theorems 1.2.5 and 1.2.6 together imply that $C(S(X))$, the algebra of continuous real-valued functions on $S(X)$, is canonically isomorphic to $UC_b(X)$, the algebra of bounded uniformly continuous functions on X. One can in fact use $UC_b(X)$ and the Gelfand-Naimark theorem to provide an alternate construction of $S(X)$.

If $A \subseteq X$, then we have a map $i_A : S(A) \to S(X)$ with $i_A(p) = \{B \subseteq X : B(U) \cap A \in p \text{ for all } U \in \mathcal{U}\}$. As the inclusion of A into X preserves apartness of subsets of X, the map i_A is injective, hence an embedding of $S(A)$ onto C_A. We will often identify $S(A)$ and C_A going forward.

Corollary 1.2.8 (Uniform Tietze extension). *Suppose $A \subseteq X$, and let $f : A \to [0,1]$ be uniformly continuous. Then there is a uniformly continuous $F : X \to [0,1]$ with $F|_A = f$.*

How do the completion of X and the Samuel compactification of X interact? Consider the inclusion $j_X : X \to S(X)$. Since j_X has a continuous extension to $S(X)$, namely the identity on $S(X)$, we use Theorem 1.2.6 to conclude that j_X is uniformly continuous. As compact spaces are complete as uniform spaces, j_X has an extension to $\widehat{j_X} : \widehat{X} \to S(X)$. Lastly, if Y is compact and $\varphi : \widehat{X} \to Y$ is uniformly continuous, then so is $\varphi|_X$. Then $\varphi|_X$ has a continuous extension $\tilde{\varphi} : S(X) \to Y$. Now $\tilde{\varphi} \circ \widehat{j_X}$ and φ are two continuous functions which agree on X, so are equal. Therefore $S(\widehat{X}) \cong S(X)$, and we can identify $\widehat{X}$ as a subset of $S(X)$.

We can be much more explicit. Suppose $(x_i)_{i \in I}$ is a Cauchy net from X. For each $U \in \mathcal{U}$, choose $i_U \in I$ so that $(x_i, x_j) \in U$ whenever $i, j \geq i_U$. Now consider the collection $\mathcal{F} = \{x_{i_U}(U) : U \in \mathcal{U}\}$. Then $\mathcal{F}$ has the FIP, so in particular can be extended to some near filter. We claim $\mathcal{F}$ has a unique extension to a near filter. If p and q are distinct, we could find $A \in p$ and $U \in \mathcal{U}$ with $A(U) \notin q$. If $V \in \mathcal{U}$ is suitably small, then either $x_{i_V}(V)$ and A are apart or $x_{i_V}(V)$ or $X \setminus (A(U))$ are apart. It is routine to see that equivalent Cauchy nets produce the same near ultrafilter.

Conversely, suppose for each $U \in \mathcal{U}$ we are given $x_U \in X$ so that $\mathcal{F} = \{x_U(U) : U \in \mathcal{U}\}$ has the FIP. Then $(x_U)_{U \in \mathcal{U}}$ is a Cauchy net, and the near ultrafilter generated by $\mathcal{F}$ belongs to $\widehat{X}$.

1.3 Topological properties of $S(X)$

One of the difficulties in working with $S(X)$ is that the basic open conditions refer to *non*-membership, and this can be awkward to work with. If one is willing to work with neighborhoods that aren't necessarily open, there is a convenient fix.

Proposition 1.3.1. *Let $p \in S(X)$. Then the collection $\{C_{A(U)} : A \in p, U \in \mathcal{U}\}$ is a base of neighborhoods at p.*

Proof. First fix $A \in p$ and $U \in \mathcal{U}$. Then $N_{X \setminus (A(U))}$ is an open neighborhood of p with $N_{X \setminus (A(U))} \subseteq C_{A(U)}$. Conversely, suppose that $p \in N_B$ for some $B \subseteq X$. Find $U \in \mathcal{U}$ with $B(U) \notin p$. Then $A := X \setminus (B(U)) \in p$. If $V \in \mathcal{U}$ is a square root of U, then $C_{A(V)} \subseteq N_B$. $\square$

We can prove a similar neighborhood base theorem for closed $K \subseteq S(X)$. If $K \subseteq S(X)$ is closed, set $\mathcal{F}_K = \{A \subseteq X : K \subseteq C_A\} = \bigcap\{p : p \in K\}$.

Proposition 1.3.2. *Let $K \subseteq S(X)$ be closed. Then $\{C_{A(U)} : A \in \mathcal{F}_K, U \in \mathcal{U}\}$ is a base of neighborhoods of K.*

Proof. If $A \in \mathcal{F}_K$ and $U \in \mathcal{U}$, then by Proposition 1.3.1, $C_{A(U)}$ is a neighborhood of every $p \in K$, so is a neighborhood of K. Conversely, suppose towards a contradiction that $P \supseteq K$ is open, but for every $A \in \mathcal{F}_K$ and $U \in \mathcal{U}$, we have $C_{A(U)} \cap (S(X) \setminus P) \neq \emptyset$. Notice that if $A, B \in \mathcal{F}_K$ and $U, V \in \mathcal{U}$, then for some suitably small $W \in \mathcal{U}$ we have

$$C_{A(U)} \cap C_{B(V)} \supseteq C_{A(U) \cap B(V)} \supseteq C_{(A(W) \cap B(W))(W)}.$$

Since $A(W) \cap B(W) \in \mathcal{F}_K$, we must have

$$\bigcap \{C_{A(U)} : A \in \mathcal{F}_K, U \in \mathcal{U}\} \neq K.$$

However, we also have $C_A = \bigcap_{U \in \mathcal{U}} C_{A(U)}$, which immediately implies that $\bigcap \{C_{A(U)} : A \in \mathcal{F}_K, U \in \mathcal{U}\} = K$, a contradiction. □

It will be useful to define the following strengthening of the near FIP to talk about closed subspaces of $S(X)$.

Definition 1.3.3. A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *near filter* if it has the near FIP and in addition satisfies:

1. If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.
2. If $A_0, ..., A_{k-1} \in \mathcal{F}$ and $U \in \mathcal{U}$, then $\bigcap_{i<k} A_i(U) \in \mathcal{F}$.
3. If $A(U) \in \mathcal{F}$ for every $U \in \mathcal{U}$, then $A \in \mathcal{F}$.

Note that every near ultrafilter is a near filter. We now show that the near filters are exactly the collections $\mathcal{F}_K$ for $K \subseteq S(X)$ closed.

Proposition 1.3.4. *Suppose $K \subseteq S(X)$ is closed. Then the set $\mathcal{F}_K := \{A \subseteq X : K \subseteq C_A\}$ is a near filter. Conversely, if $\mathcal{F}$ is a near filter, then $\mathcal{F} = \mathcal{F}_K$ for some closed $K \subseteq S(X)$.*

Proof. Certainly $\mathcal{F}_K$ has the near FIP. If $A_0, ..., A_{k-1} \in \mathcal{F}_K$, then every $p \in K$ contains each of the A_i. Therefore if $U \in \mathcal{U}$, then $\bigcap_{i<k} A_i(U) \in p$ for every $p \in K$ as p is a near filter. So $\bigcap_{i<k} A_i(U) \in \mathcal{F}_K$. Now suppose $A(U) \in \mathcal{F}_K$ for every $U \in \mathcal{U}$. Then this is true for each $p \in K$. So $A \in p$ for each $p \in K$, so $A \in \mathcal{F}_K$.

Now suppose $\mathcal{F}$ is a near filter. Set $K = \bigcap \{C_A : A \in \mathcal{F}\}$. Let $A \subseteq X$ with $K \subseteq C_A$; we need to check that $A \in \mathcal{F}$. It suffices to show that $A(U) \in \mathcal{F}$ for every $U \in \mathcal{U}$. As $C_B = \bigcap_{U \in \mathcal{U}} C_{B(V)}$, it follows that $K = \bigcap \{C_{B(V)} : B \in \mathcal{F}, V \in \mathcal{U}\}$. By Proposition 1.3.2, $C_{A(U)}$ is a neighborhood of K, so find $B_0, ..., B_{k-1} \in \mathcal{F}$ and $V \in \mathcal{U}$ with $C_{B_0(V)} \cap \cdots \cap C_{B_{k-1}(V)} \subseteq C_{\overline{A(U)}}$. As $\mathcal{F}$ is a near filter, we have $B := B_0(V) \cap \cdots \cap B_{k-1}(V) \in \mathcal{F}$. But also, we know $B \subseteq \overline{A(U)}$, so $\overline{A(U)} \in \mathcal{F}$, which implies that $A(U) \in \mathcal{F}$ as desired. □

If $\mathcal{F} \subseteq \mathcal{P}(X)$ has the near FIP, we set $K_{\mathcal{F}} \subseteq S(X)$ to be the set $\bigcap\{C_A : A \in \mathcal{F}\}$. The *near filter generated by* $\mathcal{F}$ is the collection $\langle \mathcal{F} \rangle = \{A \subseteq X : \mathcal{K}_{\mathcal{F}} \subseteq C_A\}$. We conclude by noting the following explicit formula for generating a near filter.

Proposition 1.3.5. *Suppose $\mathcal{F}$ has the near FIP. Then we have*
$\langle \mathcal{F} \rangle = \{A \subseteq X : \forall U \in \mathcal{U} \exists V \in \mathcal{U} \exists A_0, ..., A_{k-1} \in \mathcal{F} \left(\bigcap_{i<k} A_i(V) \subseteq A(U)\right)\}$.

1.4 $S(X)$ as a topometric space

For this section, we let (X, d) be a bounded metric space, and we form $S(X)$ with respect to the metric uniformity. If $A \subseteq X$ and $\epsilon > 0$, we often write $A(\epsilon) := \{x \in X : d(x, A) < \epsilon\}$. Notice that if $\epsilon, \delta > 0$, then $A(\epsilon)(\delta) \subseteq A(\epsilon + \delta)$.

Our goal for this section is to introduce an extra metric structure on $S(X)$ which will interact with the topology on $S(X)$ in several key ways. In particular, we will prove a characterization of the compact metrizable subspaces of $S(X)$ due to Ben–Yaacov, Melleray, and Tsankov [5].

Definition 1.4.1. A triple (K, τ, ∂) is a *compact topometric space* if the following hold.

1. τ is a compact Hausdorff topology on K.

2. ∂ is a metric on X which is *τ-lower-semi-continuous*, or *τ-lsc*, i.e. for any $\epsilon > 0$, the set $\{(x, y) : \partial(x, y) \leq \epsilon\}$ is $(\tau \times \tau)$-closed.

If a compact topology on K is understood, we call any lsc metric on K a *topometric* and simply write (K, ∂) for the topometric space. When referring to a topometric space (K, τ, δ), we write $x_i \xrightarrow{\partial} x$ to refer to ∂-convergence, and we simply write $x_i \to x$ to refer to τ-convergence. We will always decorate ∂-topological notions with the ∂ symbol, and we will sometimes decorate τ-topological notions with the τ symbol for emphasis.

Lemma 1.4.2 (Ben–Yaacov [6])**.** *Suppose (K, τ, ∂) is a compact topometric space. Then ∂ is finer than τ and is complete.*

Proof. First suppose that $(x_i)_{i \in I}$ is a net in K with $x_i \xrightarrow{\partial} x$. We show that the net $(x_i)_{i \in I}$ is τ-convergent with limit x. By compactness, let $y \in K$ be some cluster point of $(x_i)_{i \in I}$. For any $\epsilon > 0$, the set $\{z \in K : \partial(x, z) \leq \epsilon\}$ is τ-closed, so it follows that $\partial(y, x) \leq \epsilon$. As ϵ was arbitrary, we must have $y = x$.

To show completeness, let $(x_n)_{n<\omega}$ be a ∂-Cauchy sequence. Let $y \in K$ be any τ-cluster point. For every $\epsilon > 0$, there is $N < \omega$ so that for every $m, n \geq N$, we have $\partial(x_m, x_n) \leq \epsilon$. By the τ-lsc, we eventually have $\partial(x_n, y) \leq \epsilon$, so $x_n \xrightarrow{\partial} y$. □

We now define the topometric structure on the Samuel compactification $S(X)$ of a bounded metric space. Instead of defining $\partial(p,q) = c$, it will be easier to define the inequality $\partial(p,q) \leq c$.

Definition 1.4.3. Let (X,d) be a bounded metric space, and form $S(X)$. Given $c \geq 0$, we declare $\partial(p,q) \leq c$ iff for every $\epsilon > 0$, $A \in p$, and $B \in q$, we have $A(c+\epsilon) \cap B \neq \emptyset$.

This is not the definition originally given in [5]. Their definition is the second part of the following proposition.

Proposition 1.4.4. *Suppose (X,d) has diameter 1. Let $p, q \in S(X)$, and fix $c \geq 0$. Then the following are equivalent.*

1. *$\partial(p,q) \leq c$.*
2. *For any 1-Lipschitz function $f : X \to [0,1]$, we have $|\tilde{f}(p) - \tilde{f}(q)| \leq c$.*

Proof. Suppose (2) fails, and let $f : X \to [0,1]$ be 1-Lipschitz with $|\tilde{f}(p) - \tilde{f}(q)| = r > c$. Let $\delta > 0$ be very small, and let $B_p, B_q \subseteq [0,1]$ be the δ-intervals around $\tilde{f}(p)$ and $\tilde{f}(q)$, respectively. Then $f^{-1}(B_p) \in p$ and $f^{-1}(B_q) \in q$, but $(f^{-1}(B_p))(c+\delta) \cap f^{-1}(B_q) = \emptyset$. So $\partial(p,q) > c$.

Conversely, suppose $\partial(p,q) > c$. Then there are $A \in p$, $B \in q$, and $\epsilon > 0$ with $A(c+\epsilon) \cap B = \emptyset$. Then the function $f : X \to [0,1]$ given by $f(x) = d(x,B)$ is 1-Lipschitz, and we have $\tilde{f}(q) = 0$, $\tilde{f}(p) \geq c + \epsilon$. □

Lemma 1.4.5. *Suppose $p, q \in S(X)$, and let $c \geq 0$. Then $\partial(p,q) \leq c$ iff for any $\epsilon > 0$ and $A \in p$, we have $A(c+\epsilon) \in q$.*

Proof. This is an immediate consequece of Corollary 1.2.3. □

Proposition 1.4.6. *The function ∂ from Definition 1.4.3 is a topometric on $S(X)$.*

Proof. The function ∂ is symmetric and positive definite, so to show ∂ is a metric, it remains to prove the triangle inequality. Let $p, q, r \in S(X)$, and suppose $\alpha, \beta \geq 0$ with $\partial(p,q) \leq \alpha$ and $\partial(q,r) \leq \beta$. Fix $A \in p$ and $\epsilon > 0$. Using Lemma 1.4.5, we have $A(\alpha + \epsilon/2) \in q$; using the lemma again gives $A(\alpha + \beta + \epsilon) \in r$.

To see that ∂ is lsc, suppose $c > 0$ with $\partial(p,q) > c$. Find $\epsilon > 0$ and $A \in p$, $B \in q$ with $A(c+\epsilon) \cap B = \emptyset$. Now consider $A' = A(\epsilon/4)$ and $B' = B(\epsilon/4)$. We have $p \in N_{X \setminus A'}$ and $q \in N_{X \setminus B'}$. Consider any $p' \in N(X \setminus A')$ and $q' \in N(X \setminus B')$. Then $A' \in p$, $B' \in q$, and $A'(c+\epsilon/2) \cap B' = \emptyset$, so $\partial(p', q') > c$. □

We call the topometric space $(S(X), \partial)$ given by Definition 1.4.3 the *topometric Samuel compactification* of the bounded metric space (X, d).

If $A \subseteq X$, we will want to understand how the topometric structure on $S(A)$ interacts with the topometric structure on $S(X)$.

Proposition 1.4.7. *Suppose $A \subseteq X$, and write ∂_A for the topometric distance as computed in A. Then for any $p, q \in S(A) \subseteq S(X)$, we have $\partial_A(p, q) = \partial(p, q)$.*

Proof. Suppose $\partial_A(p, q) \leq c$ for some $c > 0$. Fix $B \in p$ and $\epsilon > 0$. Then we have $B(\epsilon/2) \cap A \in p$, so $(B(\epsilon/2) \cap A)(c + \epsilon/2) \in q$. It follows that $B(c + \epsilon) \in q$.

For the other direction, suppose $\partial(p, q) \leq c$ for some $c \geq 0$. Fix $B \subseteq A$ with $B \in p$. Then $B(c + \epsilon/2) \in q$, so $B(c + \epsilon) \cap A \in q$ also. $\square$

We are now ready to characterize the compact metrizable subspaces of $S(X)$.

Theorem 1.4.8 (Theorem 2.3 from [5])**.** *Suppose $K \subseteq S(X)$ is a compact metrizable subspace. Then ∂ is a compatible metric on K.*

Lemma 1.4.9. *Suppose $K \subseteq S(X)$ is compact and ∂ is not compatible with K. Then there are $\{q_n : n < \omega\} \subseteq K$, $\{A_n : n < \omega\} \subseteq \mathcal{P}(X)$ with $A_n \in q_n$, and $c > 0$ with $\{A_n(c) : n < \omega\}$ pairwise disjoint.*

Proof. We know that ∂ is complete and finer than the topology on K. This means that if ∂ is not compatible, then it is not totally bounded. Find a $c > 0$ and points $\{p_n : n < \omega\}$ with $\partial(p_m, p_n) > 3c$ whenever $m \neq n$. Set $p_n^0 = p_n$ and $X_0 = X$. Suppose $q_0, ..., q_{k-1}$, $A_0, ..., A_{k-1}$, $\{p_n^k : n < \omega\} \subseteq \{p_n : n < \omega\}$, and $X_k \subseteq X$ with $p_n^k \in S(X_k)$ have been determined. Consider p_0^k and p_1^k. Find $B_0^k, B_1^k \subseteq X_k$ with $B_0^k \in p_0^k$, $B_1^k \in p_1^k$ with $B_0^k(c) \cap B_1^k(c) = 0$. Every other p_n^k with $n \geq 2$ must contain either $X_k \setminus B_0^k(c)$ or $X_k \setminus B_1^k(c)$, so for some $i \in \{0, 1\}$ and infinitely many $n \geq 2$, $X_k \setminus B_i^k(c) \in p_n^k$. Set $q_k = p_i^k$, $A_k = B_i^k$, and delete all the p_n^k which do not contain $X_k \setminus B_i^k(c) := X_{k+1}$. Let $\{p_n^{k+1} : n < \omega\}$ enumerate the remaining p_n^k. $\square$

Proof of Theorem 1.4.8. Let $\{q_n : n < \omega\}$ and $\{A_n : n < \omega\}$ be as in Lemma 1.4.9. Let $\varphi : \omega \to K$ be the map $\varphi(n) = q_n$, and continuously extend to $\tilde{\varphi} : \beta\omega \to K$. For $S \subseteq \omega$, set $A_S = \bigcup_{n \in S} A_n$. Then A_S and $A_{\omega \setminus S}$ are apart. It follows that $\tilde{\varphi}$ is an injection, and K cannot be metrizable. $\square$

Chapter 2

Abstract topological dynamics

In this chapter, we fix a topological group G and consider the ways in which G can act on compact spaces. We will see that this is closely connected with $S(G)$, the Samuel compactification of G equipped with its left uniformity. When G is metrizable, the topometric structure on $S(G)$ allows for a detailed understanding of when $M(G)$ is metrizable.

Sections 2.5, 2.6, and 2.7 contain mostly original results. Section 2.5 constructs the universal strong extension of any pre-ambit and shows that this object is unique, a result first appearing in [36]. Section 2.6 provides a new construction of the universal proximal extension of a G-flow and uses this to produce a new proof of the Generic Point Problem; this construction first appears in [35]. Section 2.7 is joint work with Dana Bartošová and has not appeared before; given a Polish group G, we provide a new characterization of when $M(G)$ is metrizable.

2.1 Compact left-topological semigroups

An excellent reference on compact left-topological semigroups is the first two chapters of the book by Hindman and Strauss [14]. Readers should note however the left-right switch between that reference and the presentation here.

Let S be a semigroup. If $x \in S$, let $\lambda_x : S \to S$ and $\rho_x : S \to S$ denote the left and right multiplication maps, respectively. A non-empty semigroup S is a *compact left-topolgical semigroup* if S is also a compact Hausdorff space so that for every $x \in S$, the map λ_x is continuous. Given $x \in S$ and a subset $T \subseteq S$, we often write $xT := \{xy : y \in T\}$ and $Tx := \{yx : y \in T\}$. A *right ideal* (respectively *left ideal*) of a semigroup S is a subset $M \subseteq S$ so that for every $x \in M$, we have $xS \subseteq M$ (respectively $Sx \subseteq M$). An *idempotent* is any element $x \in S$ with $xx = x$.

We will freely use the following facts. In the following, S denotes a compact left-topological semigroup.

Fact.

1. (Ellis) S contains an idempotent.

2. If $u \in S$ is idempotent and $x \in uS$, then $ux = x$. If $x \in Su$, then $xu = x$.

3. Every right ideal $M \subseteq S$ contains a closed right ideal; namely if $x \in M$, then $xS = \lambda_x``(S)$ is closed and a right ideal. Then by Zorn's lemma, every right ideal contains a minimal right ideal which must be closed. Every minimal right ideal is a compact left-topological semigroup, so contains an idempotent.

4. If M is a minimal right ideal and $x \in M$, then Sx is a minimal left ideal. The intersection of any minimal right ideal and any minimal left ideal is a group, hence contains exactly one idempotent.

5. If M and N are minimal right ideals and $x \in M$, then there is $y \in N$ with $yx \in N$ an idempotent.

2.2 Flows and ambits

Let G be a topological group. A *G-space* is a topological space X equipped with a continuous right action $a : X \times G \to X$. We typically suppress the notation and simply write xg for $a(x, g)$. If X and Y are G-spaces, a *G-map* from X to Y is a continuous map $\varphi : X \to Y$ which respects the G-actions. We call X a *G-flow* if X is a compact G-space.

A *G-ambit* is a pair (X, x_0) where X is a G-flow and $x_0 \in X$ is a distinguished point with dense orbit. If (X, x_0) and (Y, y_0) are ambits, then a *map of ambits* from (X, x_0) to (Y, y_0) is a G-map $\varphi : X \to Y$ with $\varphi(x_0) = y_0$. Notice that there is at most one map of ambits from (X, x_0) to (Y, y_0).

A G-flow X is *minimal* if every orbit is dense. Equivalenty, define a *subflow* of X to be a closed, G-invariant subset of X; then X is minimal iff X contains no proper subflows. Notice that if $\varphi : X \to Y$ is a G-map and Y is minimal, then φ must be surjective. If φ is surjective and X is minimal, then so is Y. The following observation is used frequently.

Proposition 2.2.1. *If X is a minimal G-flow and $U \subseteq X$ is non-empty open, then there are $g_0, ..., g_{k-1} \in G$ with $X = \bigcup_{i<k} Ug_i$.*

Proof. As X is minimal, every orbit meets U. Therefore $\{Ug : g \in G\}$ is an open cover of X, and we can appeal to compactness. □

One of the key objects of study in this thesis is the *universal minimal flow* $M(G)$ of the group G.

Fact (Ellis [8])**.** There is a minimal flow $M(G)$ which is *universal*, i.e. which admits a G-map onto any other minimal flow. The flow $M(G)$ is unique up to G-flow isomorphism.

Fix $\mathcal{N}_G$ a base of symmetric open neighborhoods of the identity 1_G. The group G admits several compatible uniform structures; recall that the *left uniformity* on G has a base given by $\{\{(x,y) : x^{-1}y \in U\} : U \in \mathcal{N}_G\}$. We will often identify $U \in \mathcal{N}_G$ with the entourage $\{(x,y) : x^{-1}y \in U\}$. Notice that if $A \subseteq G$ and $U \in \mathcal{N}_G$, we have $A(U) = AU$. When we refer to uniform notions on G, we will be referring to this uniformity unless explicitly stated otherwise. So let $S(G)$ be the Samuel compactification of G. Our first goal is to understand the dynamical properties of this space. If $p \in S(G)$ and $g \in G$, we write $pg = \{Ag : A \in p\}$.

Proposition 2.2.2. *With the above action, $S(G)$ is a G-flow.*

Proof. For each $g \in G$, the map $p \to pg$ is continuous. So suppose $p_i \to p$ and $g_i \to 1_G$, and assume $A \notin p$. Find $U \in \mathcal{N}_G$ with $AU \notin p$. Then eventually $AU \notin p_i$. Also, as $g_i \to 1_G$, we eventually have $g_i^{-1} \in U$. Whenever $Ag_i^{-1} \subseteq AU$, we have $Ag_i^{-1} \notin p_i$. So eventually $A \notin p_i g_i$ as desired. □

Proposition 2.2.3. *Let (X, x_0) be a G-ambit. Then the map $g \to x_0 g$ is uniformly continuous.*

Proof. Fix $P \subseteq X \times X$ an open neighborhood of Δ_X. Towards a contradiction, suppose that for every $U \in \mathcal{N}_G$, there were $(g_U, h_U) \in U$ with $(x_0 g_U, x_0 h_U) \notin P$. By passing to a subnet, we may assume that $x_0 g_U \to x$ and $x_0 h_U \to y$. Notice that $(x,y) \notin P$, so in particular $x \neq y$.

Let A_x and A_y be disjoint open neighborhoods of x and y respectively. By continuity of the action, find open $B_x \ni x$, $B_y \ni y$, and $U \in \mathcal{N}_G$ with $B_x U \subseteq A_x$ and $B_y U \subseteq A_y$. For some $U \in \mathcal{N}_G$, we have $x_0 g_U \in B_x$ and $x_0 h_U \in B_y$. But then $x_0 g_U (g_U^{-1} h_U) = x_0 h_U \in A_x$, a contradiction. □

Definition 2.2.4. In the setting of Proposition 2.2.3, write $\lambda_{x_0} : S(G) \to X$ for the continuous extension of the map $g \to x_0 g$.

Proposition 2.2.5. *In the setting of Proposition 2.2.3 and Definition 2.2.4, the map $\lambda_{x_0} : (S(G), 1_G) \to (X, x_0)$ is a map of ambits.*

Proof. Certainly $\lambda_{x_0}(1_G) = x_0$, so we need to check that λ_{x_0} is a G-map. If $p \in S(G)$ and $g \in G$, first find a net $(g_i)_{i \in I}$ from G with $g_i \to p$. Then $\lambda_{x_0}(pg) = \lim_i \lambda_{x_0}(g_i g) = \lim_i (x_0 g_i) g = \lambda_{x_0}(pg)$. □

Due to Proposition 2.2.5, the ambit $(S(G), 1_G)$ is called the *greatest ambit*; it admits a unique map of ambits onto any other ambit. If the base points are understood, we often suppress the notation, and write for example "let $\lambda_{x_0} : S(G) \to X$ be the map of ambits." We will be very flexible with this notation; for instance, if $Y \subseteq S(G)$ is a subflow and $x_0 \in X$ is any point, we will write $\lambda_{x_0} : Y \to X$ for the restriction of the map $\lambda_{x_0} : S(G) \to \overline{x_0 G}$.

Our next goal is to endow $S(G)$ with the structure of a compact left-topological semigroup. Fix $p \in S(G)$, and form the map $\lambda_p : S(G) \to S(G)$. We define a semigroup structure on $S(G)$ by declaring for $p, q \in S(G)$ that $pq := \lambda_p(q)$.

Proposition 2.2.6. *The operation $(p, q) \to \lambda_p(q)$ turns $S(G)$ into a compact left-topological semigroup.*

Proof. By definition, the operation is left-topological, so we only need to check associativity. Notice that $\lambda_p \circ \lambda_q$ is a G-map sending 1_G to $\lambda_p(q)$. Therefore $\lambda_p \circ \lambda_q = \lambda_{\lambda_p(q)}$. Given $r \in S(G)$ as an input, this says that $p(qr) = (pq)r$. □

Given Proposition 2.2.6, we can reinterpret Proposition 2.2.5 as follows. If X is a G-flow, $x \in X$, and $p \in S(G)$, write xp as a shorthand for $\lambda_x(p)$. Now given $x \in X$ and $p, q \in S(G)$, we have

$$(xp)q = \lim_{h_i \to q} (xp)h_i = \lim_{h_i \to q} x(ph_i) = x(pq).$$

In other words, the G-flow X can also be interpreted as an action of the semigroup $S(G)$ on X, albeit with weaker continuity properties. If $x \in X$ is fixed and $p_i \to p \in S(G)$, then we have $xp_i \to xp$. In general, we cannot deduce anything stronger.

As $S(G)$ is a space of near ultrafilters, it will be useful to phrase the semigroup operation in this language.

Definition 2.2.7. Let $p \in S(G)$ and $A \subseteq G$. We set $p^{-1}(A) := \{g \in G : A \in pg\}$.

Here are some easy facts about "p^{-1}" that will be useful to keep in mind going forward.

Lemma 2.2.8. *Fix $p \in S(G)$.*

1. $p^{-1}(Ag) = (p^{-1}(A))g$

2. $p^{-1}(A \cup B) = p^{-1}(A) \cup p^{-1}(B)$.

Proposition 2.2.9. *Let $p, q \in S(G)$, and fix $A \subseteq G$. Then $A \in pq$ iff for every $U \in \mathcal{N}_G$, we have $p^{-1}(AU) \in q$.*

Proof. Keeping in mind Propositions 1.3.1 and 1.2.5, this is immediate. The sets $\{C_{AU} : A \in pq, U \in \mathcal{N}_G\}$ form a neighborhood base for pq, and $\lambda_p : S(G) \to S(G)$ can be viewed as the continuous extension of $\lambda_p|_G$. □

2.3 Subflows of $S(G)$

Since $S(G)$ is a compact left-topological semigroup, we should understand the relationship between dynamical properties and semigroup properties of subspaces of $S(G)$.

Proposition 2.3.1. *Let $Y \subseteq S(G)$ be closed. Then Y is a subflow of $S(G)$ iff Y is a right ideal of $S(G)$.*

Proof. Certainly if Y is a right ideal, then Y is G-invariant. Conversely, if Y is G-invariant, $p \in Y$, and $q \in S(G)$ with $h_i \to q$, then $pq = \lim_i ph_i \in Y$. □

It follows that the minimal subflows of $S(G)$ correspond exactly to the minimal right ideals of $S(G)$.

Proposition 2.3.2. *Let $M, N \subseteq S(G)$ be minimal subflows, and let $\psi : M \to N$ be any G-map. Then ψ is an isomorphism.*

Proof sketch. We can sketch a proof of this using the facts from section 2.1 as follows. Suppose $M, N \subseteq G$ are minimal subflows and $\psi : M \to N$ is a G-map. Letting $u \in M$ be idempotent, we see that $\psi(p) = \psi(up) = \psi(u)p$, so $\psi = \lambda_{\psi(u)}$. Notice that since $\psi(u) = \psi(u)u$, we have that $\psi(u)$ and u belong to the same minimal left ideal. Finding $q \in N$ with $q\psi(u) := v$ an idempotent, we have $uv = u$ and $vu = v$. It follows that $\lambda_{uq} : N \to M$ is the inverse of $\lambda_{\psi(u)}$. □

Corollary 2.3.3. *There is up to isomorphism a unique universal minimal flow.*

Proof. Certainly every minimal $M \subseteq S(G)$ is universal for minimal flows. If X is any other universal minimal flow, we let $\varphi : M \to X$ and $\psi : X \to M$ be G-maps, obtaining $\psi \circ \varphi : M \to M$, which by the above must be an isomorphism. Therefore each of φ and ψ was an isomorphism. □

Viewing $S(G)$ as the space of near ultrafilters on G, we want to know which near filters correspond to subflows of $S(G)$.

Definition 2.3.4. Fix $T \subseteq G$.

1. We say that T is *thick* if the collection $\{Tg : g \in G\}$ has the FIP.
2. We say that T is *near-thick* if for every $U \in \mathcal{N}_G$, the set TU is thick.

Of course if G is a discrete group, then thick and near-thick coincide. An example to keep in mind is the case $G = \mathbb{Z}$, where $T \subseteq \mathbb{Z}$ is thick iff T contains arbitrarily long intervals.

The interaction between G as a topological group and G as a discrete group will prove fruitful. In particular, it will be helpful to understand the return times $p^{-1}(A)$ for $A \subseteq G$ and $p \in \beta G$.

Proposition 2.3.5. *Let $T \subseteq G$. Then T is thick iff there is $p \in \beta G$ with $p^{-1}(T) = G$.*

Proof. For every finite $F \subseteq G$, find $g_F \in G$ with $g_F F \subseteq T$. Viewing the finite subsets of G as a net, pass to a subnet and let $g_F \to p \in \beta G$. It follows that $T \in pg$ for every $g \in G$, i.e. that $p^{-1}(T) = G$.

For the other direction, suppose $p^{-1}(T) = G$ for some $p \in \beta G$. Let $g_i \to p$ with $g_i \in G$. For any finite $F \subseteq G$, we eventually have $g_i F \subseteq T$. So T must be thick. □

Proposition 2.3.6. *Let $T \subseteq G$. The following are equivalent.*

1. *T is near-thick.*
2. *The collection $\{Tg : g \in G\}$ has the near FIP.*
3. *For every $U \in \mathcal{N}_G$, $C_{TU} \subseteq S(G)$ contains a subflow.*
4. *$C_T \subseteq S(G)$ contains a subflow.*

Proof. (1) ⇒ (2). Suppose T is near thick. Let $g_0, ..., g_{k-1} \in G$, and fix $U \in \mathcal{N}_G$. Find suitably small $V \in \mathcal{N}_G$ so that for each $i < k$, we have $Vg_i \subseteq g_i U$. Then $\emptyset \neq \bigcap_{i<k} TVg_i \subseteq \bigcap_{i<k} Tg_i U$.

(2) ⇒ (1). Suppose $\{Tg : g \in G\}$ has the near FIP. Let $g_0, ..., g_{k-1} \in G$, and fix $U \in \mathcal{N}_G$. Find suitably small $V \in \mathcal{N}_G$ so that for each $i < k$, we have $g_i V \subseteq Ug_i$. Similar to the forward direction, we conclude that $\bigcap_{i<k} TUg_i \neq \emptyset$.

(2) ⇔ (4). Notice that $C_T \cdot g = C_{Tg}$. Also note that $\{C_{Tg} : g \in G\}$ has the FIP iff $\{Tg : g \in G\}$ is a near filter. Let $X = \bigcap_{g \in G} C_{Tg}$. Then X is G-invariant, so if non-empty is a subflow. Conversely, if $Y \subseteq C_T$ is a subflow, then $Y \subseteq C_T \cdot g$ for each $g \in G$, so $Y \subseteq X$.

(4) ⇒ (3) is clear.

(3) ⇒ (1). Fix $U \in \mathcal{N}_G$ and $g_0, ..., g_{k-1} \in G$. We want $\bigcap_{i<k} TUg_i \neq \emptyset$. Fix a small $V \in \mathcal{N}_G$. Since C_{TV} contains a minimal right ideal, we have that $\{TVg : g \in G\}$ forms a near filter. Therefore $\bigcap_{i<k} TVg_i V \neq \emptyset$. By making V small enough, we have $TVg_i V \subseteq TUg_i$ for each $i < k$. □

Remark. Notice the similarities between Proposition 2.3.6 and Proposition 2.3.5. Indeed $T \subseteq G$ is near-thick iff there is $p \in S(G)$ with $p^{-1}(G) = G$.

Corollary 2.3.7. *$M \subseteq S(G)$ is a minimal right ideal iff $\mathcal{F}_M = \{A \subseteq G : M \subseteq C_A\}$ is a near filter which is maximal with respect to the property that every member of $\mathcal{F}_M$ is near-thick.*

Now is a good time to recall one of the key concepts from the introduction.

Definition 2.3.8. A topological group G is called *extremely amenable* if $M(G)$ is the trivial flow.

As indicated in the introduction, extreme amenability and its interaction with combinatorial properties has been a fruitful object of study for the past two decades. Using Corollary 2.3.7, we can start to understand how this interaction arises.

Recall that if X is a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ is upwards closed, then $\mathcal{F}$ is called *partition regular* if for every $A \in \mathcal{F}$ an partition $A = \bigcup_{i<k} A_i$, then some $A_i \in \mathcal{F}$.

Proposition 2.3.9. *Let G be a topological group, and write $\mathcal{T} \subseteq \mathcal{P}(G)$ for the collection of near-thick subsets of G. Then G is extremely amenable iff $\mathcal{T}$ is partition regular.*

Proof. If G is extremely amenable, let $T \subseteq G$ be near thick. By Proposition 2.3.6, fix $M \subseteq C_T$ a minimal right ideal; so by assumption $M = \{p\}$ for some $p \in S(G)$. If $T = \bigcup_{i<k} T_i$, then some $T_i \in p$, so must be near-thick.

Conversely, suppose $\mathcal{T}$ is partition regular. This implies that the collection $\mathcal{I}$ of sets which are not near-thick forms an ideal. Let $\mathcal{F}$ be the dual filter, and extend $\mathcal{F}$ to some near ultrafilter p. If we can show that every member of p is near-thick, then we are done by Corollary 2.3.7. Suppose A is not near-thick. Then there is some $U \in \mathcal{N}_G$ so that AU is also not near-thick. So $G \setminus AU \in p$, implying that $A \notin p$. □

We will also be concerned with sets $B \subseteq G$ where $C_B \cap M \neq \emptyset$ for all or for some minimal right ideals M.

Definition 2.3.10. Fix $S \subseteq G$.

1. We say that S is *syndetic* if $G \setminus S$ is not thick. Equivalently, there are $g_0, ..., g_{k-1} \in G$ with $\bigcup_{i<k} Sg_i = G$.

2. We say that S is *near-syndetic* if for every $U \in \mathcal{N}_G$, we have SU syndetic.

Keeping in mind our example $G = \mathbb{Z}$, a subset $S \subseteq \mathbb{Z}$ is syndetic iff there is $k < \omega$ so that the gaps in S have size at most k.

Proposition 2.3.11. *Let $S \subseteq G$. The following are equivalent.*

1. *S is near-syndetic.*

2. *For every $U \in \mathcal{N}_G$, $C_{SU} \subseteq S(G)$ meets every minimal right ideal.*

3. *C_S meets every minimal right ideal.*

Proof. (1) ⇒ (2). Suppose $S \subseteq G$ is near syndetic. Fix $U \in \mathcal{N}_G$ and $M \subseteq S(G)$ a minimal right ideal. Find $g_0, ..., g_{k-1}$ so that $G = \bigcup_{i<k} SUg_i$. If $p \in M$, then p contains some SUg_i. So $SU \in pg_i^{-1}$, and $C_{SU} \cap M \neq \emptyset$.

(2) ⇒ (1). Fix $U \in \mathcal{N}_G$. Towards a contradiction, assume $G \setminus SU$ were thick. Then by Proposition 2.3.6, find $M \subseteq C_{G\setminus SU}$ a minimal right ideal. If V is a square root of U, then SV and $G \setminus SU$ are apart, so $C_{SV} \cap M = \emptyset$, a contradiction.

(3) ⇒ (2) is clear. For the converse, fix $M \subseteq S(G)$ a minimal right ideal. If $M \cap C_{SU} \neq \emptyset$ for every $U \in \mathcal{N}_G$, we simply note that $C_S = \bigcap_{U \in \mathcal{N}_G} C_{SU}$ and appeal to compactness. □

While we have syndetic sets in mind, the following folklore fact is a useful criterion to detect minimality of a G-flow.

Proposition 2.3.12. *Suppose X is a G-flow and $x \in X$ has dense orbit. Then $\overline{xG}$ is minimal iff for every non-empty open $U \subseteq X$ with $U \cap \overline{xG} \neq \emptyset$, we have $\{g \in G : xg \in U\}$ syndetic.*

Proof. Suppose $Y \subseteq \overline{xG}$ were a proper subflow, and find $p \in S(G)$ with $xp \in Y$. Fix $U \subseteq \overline{xG}$ relatively open with $\overline{U} \cap Y = \emptyset$. Towards a contradiction, suppose $\{g \in G : xg \in U\}$ were syndetic. Then we can find $g_0, ..., g_{k-1} \in G$ so that for any $g \in G$, we have $xgg_i \in U$ for some $i < k$. If $(g_j)_{j \in J}$ is a net with $g_j \to p$, we may assume by passing to a subnet that for some $i < k$ we have $xg_jg_i \in U$ for every $j \in J$. But then $xpg_i \in \overline{U}$, a contradiction.

Conversely, suppose $U \subseteq \overline{xG}$ is non-empty and relatively open with $T := \{g \in G : xg \in \overline{xG} \setminus U\}$ thick. For each finite $F \subseteq G$, find $g_F \in G$ with $g_F F \subseteq T$. Viewing the finite subsets of G as a directed set, find a convergent subnet with $g_F \to p \in S(G)$. But then $xpg \in \overline{xG} \setminus U$ for every $g \in G$. As xp does not have dense orbit in $\overline{xG}$, $\overline{xG}$ is not minimal. □

Definition 2.3.13. Fix $B \subseteq G$.

1. We say that B is *piecewise syndetic* if there are $g_0, ..., g_{k-1} \in G$ with $\bigcup_{i<k} Bg_i$ thick.

2. We say that B is *near-pws* if for every $U \in \mathcal{N}_G$, we have BU piecewise syndetic.

When $G = \mathbb{Z}$, a subset $B \subseteq \mathbb{Z}$ is piecewise syndetic iff there is some $k < \omega$ so that on arbitrarily long intervals $I \subseteq \mathbb{Z}$, $B \cap I$ has gaps of size at most k. When G is discrete, the following is a useful characterization of being piecewise syndetic.

Proposition 2.3.14. *Let $B \subseteq G$. Then B is piecewise syndetic iff there is $p \in \beta G$ with $p^{-1}(B)$ syndetic.*

Proof. Suppose B is piecewise syndetic, and let $g_0, ..., g_{k-1} \in G$ with $T := \bigcup_{i<k} Bg_i$ thick. Find $p \in \beta G$ with $p^{-1}(\bigcup_{i<k} Bg_i) = G$. Then $\bigcup_{i<k} p^{-1}(B)g_i = G$, so $p^{-1}(B)$ is syndetic.

Conversely, suppose there is $p \in \beta G$ with $p^{-1}(B)$ syndetic. Find $g_0, ..., g_{k-1} \in G$ with $\bigcup_{i<k} p^{-1}(B)g_i = G$. Then $p^{-1}(\bigcup_{i<k} Bg_i) = G$, so $\bigcup_{i<k} Bg_i$ is thick, and B is piecewise syndetic. □

Proposition 2.3.15. *Let $B \subseteq G$. The following are equivalent.*

1. *B is near-pws,*
2. *For every $U \in \mathcal{N}_G$, $C_{BU} \subseteq S(G)$ meets a minimal right ideal.*

Proof. (1) ⇒ (2). Suppose $B \subseteq G$ is near-pws, and fix $U \in \mathcal{N}_G$. Then find $g_0, ..., g_{k-1}$ with $T := \bigcup_{i<k} BUg_i$ thick. Then C_T contains a minimal right ideal M, so some C_{BUg_i} meets M. But as M is G-invariant, we have that $M \cap C_{BU} \neq \emptyset$.

(2) ⇒ (1). Fix $U \in \mathcal{N}_G$. We want to show that BU is piecewise syndetic. Let $V \in \mathcal{N}_G$ be suitably small, and find $M \subseteq S(G)$ a minimal right ideal with $M \cap C_{BV} \neq \emptyset$. Fix $p \in M \cap C_{BV}$. Then C_{BV^2} is a neighborhood of p, so find $g_0, ..., g_{k-1}$ with $M \subseteq \bigcup_{i<k} C_{BV^2} \cdot g_i$. It follows that $T := \bigcup_{i<k} BV^2 g_i$ is near-thick, so for any $W \in \mathcal{N}_G$, $TW := \bigcup_{i<k} BV^2 g_i W$ is thick. Choose V so that $V^3 \subseteq U$, and choose W so that $g_i W \subseteq V g_i$ for every $i < k$. We see that BU is thick as desired. □

Remark. The proof of (1) ⇒ (2) above shows that whenever $B \subseteq G$ is piecewise syndetic, then C_B meets some minimal right ideal. However, the converse of this is false. Consider $G = \mathbb{R}$ with its usual topology. Let $B = \mathbb{Z}$. Then in fact B is near syndetic. However, it is impossible for any set of the form $T := \bigcup_{i<k}(B + r_i)$ to contain every finite pattern; for instance, T could only meet k points out of an arithmetic progression with irrational gap size.

Example 2.3.16. Unlike Proposition 2.3.11, we cannot in general strengthen Proposition 2.3.15 to say that if $B \subseteq G$ is near-pws, then $C_B \cap M \neq \emptyset$ for some minimal right ideal. We sketch a construction of a counterexample. Consider $G = \mathbb{Z}^\omega$ with the product topology. If $d < \omega$, set $\mathbb{Z}^{(d)} = \{p \in \mathbb{Z}^\omega : p|_d = \vec{0} \in \mathbb{Z}^d\}$. Then $\{\mathbb{Z}^{(d)} : d < \omega\}$ is a base of clopen subgroups at the identity.

Fix $f : \mathbb{N}^+ \to \mathbb{N}^+$ where $f(k) = n + 1$ if 2^n divides k and 2^{n+1} does not. Set $\ell(d, k) = \min(d, f(k))$ Now for each $k \in \mathbb{N}^+$, set

$$B_{d,k} = \left\{(2^k, 0, ..., 0) + f(k) \cdot (N_0, ..., N_{\ell(d,k)-1}, 0, ..., 0) : 0 \leq N_i \leq \left\lfloor \frac{2^{k-1}}{f(k)} \right\rfloor \right\} \subseteq \mathbb{Z}^d$$

Set $B_d = \bigcup_k B_{d,k}$, and set $B = \varprojlim B_d$. First notice that $B\mathbb{Z}^{(d)} = \{p \in \mathbb{Z}^\omega : p|_d \in B_d\}$. As each B_d is piecewise syndetic in $\mathbb{Z}^d$, we have that B is near-pws. Suppose C_B met some minimal subflow $M \subseteq S(\mathbb{Z}^\omega)$. Then for any $d < \omega$, $C_{B\mathbb{Z}^{(d)}} \cap M$ has non-empty interior in M, so we can find $F_d \subseteq \mathbb{Z}^\omega$ finite with $M \subseteq C_{B\mathbb{Z}^{(d)}} F_d$. In particular, $B\mathbb{Z}^{(d)} F_d$ will be near-thick, and in fact it will be thick since $B\mathbb{Z}^{(d)} F_d = BF_d\mathbb{Z}^{(d)}$. We will reach a contradiction by showing that With d and F_d as above, there is a large enough $D > d$ so that whenever $F_D \subseteq \mathbb{Z}^\omega$ is finite with $B\mathbb{Z}^{(D)} F_D$ thick, we have $(B\mathbb{Z}^{(d)} F_d) \cap (B\mathbb{Z}^{(D)} F_D)$ not thick.

Letting $J_d = F_d|_d \subseteq \mathbb{Z}^d$, we have $B\mathbb{Z}^{(d)} F_d$ thick in $\mathbb{Z}^\omega$ iff $B_d J_d$ is thick in $\mathbb{Z}^d$. Consider a d-dimensional cube of side length N. For N suitably large (depending on J_d) any such cube contained in $B_d J_d$ must be contained in some $B_{d,k} J_d$. For such a k, we must have $d \leq f(k) \leq K$ for some K depending on J_d but independent of N. Now choose $D > K$. The same argument yields that a suitably large D-dimensional cube of side length N is contained in $B_{D,k} J_D$ for some k with $D \leq f(k)$. But setting $C = \{p \in \mathbb{Z}^D : p|_d \in B_d J_d\}$, any large D-dimensional cube contained in C projects onto a large d-dimensional cube in $B_d J_d$. It follows that $B_D J_D \cap C$ does not contain any large D-dimensional cube, so cannot be thick.

Recall that if S is a compact left-topological semigroup, then the *smallest ideal* $K(S)$ is the union of all of the minimal right ideals of S. We can now characterize exactly which near ultrafilters $p \in K(S(G))$ lie in the closure of the smallest ideal.

Proposition 2.3.17. *If $p \in S(G)$, then $p \in \overline{K(S(G))}$ iff every $B \in p$ is near-pws.*

Proof. If $B \in p$, then for every $U \in \mathcal{N}_G$, C_{BU} is a neighborhood of p by Proposition 1.3.1. If $p \in \overline{K(S(G))}$, then C_{BU} meets a minimal right ideal, so B is near-pws. The converse follows also by Proposition 1.3.1. □

Many of the notions and theorems here concerning near-thick, near-syndetic, and near-pws generalize existing facts known for discrete groups; see for example [14].

2.4 Completions of topological groups

We let $\widehat{G}$ denote the completion of G with respect to the left uniformity. Our first task is to endow $\widehat{G}$ with the structure of a topological semigroup. If $g \in G$ is fixed, then the right multiplication $\rho_g : G \to G$ is uniformly continuous, so extends to $\widehat{\rho_g} : \widehat{G} \to \widehat{G}$. Now for each $\eta \in \widehat{G}$, we define $\lambda_\eta : G \to \widehat{G}$ by declaring $\lambda_\eta(g) = \rho_g(\eta)$.

To show that λ_η is uniformly continuous, fix $U \in \mathcal{N}_G$, and suppose $x, y \in G$ with $x^{-1}y \in U$. We want to show that $(\eta x, \eta y) \in \widehat{U}$ (recall the definition of $\widehat{U}$ from section 1.1). Fix a Cauchy net $(g_i)_{i \in I}$ with $g_i \to \eta$. Find $V \in \mathcal{N}_G$ so that $Vx^{-1}VyV \subseteq U$. Now for suitably large $i, j \in I$, suppose $(g_i x, a) \in V$ and $(g_j y, b) \in V$. Then $a^{-1}b \in Vx^{-1}g_i^{-1}g_j yV$. As long as $g_i^{-1}g_j \in V$, we have $a^{-1}b \in U$ as desired. Therefore for $\eta, \zeta \in \widehat{G}$, we can define $\eta \cdot \zeta := \lambda_\eta(\zeta)$.

Remark. Recall the identification $S(G) = S(\widehat{G})$, and in particular the embedding $\widehat{j_G} = j_{\widehat{G}}$: $\widehat{G} \to S(G)$. For $\eta, \zeta \in \widehat{G}$, we note that $\eta \cdot \zeta = \lim_{h_j \to \zeta} \left(\lim_{g_i \to \eta} (g_i h_j) \right)$. It follows that the semigroup operation is the same whether computed in $\widehat{G}$ or in $S(G)$. As a result, we will be less careful with the "hat" notation and simply write λ_η for both the map $\widehat{G} \to \widehat{G}$ and the map $S(G) \to S(G)$. The reader is advised to review the discussion at the end of section 1.2.

Proposition 2.4.1. *The evaluation map $\widehat{G} \times \widehat{G} \to \widehat{G}$ is continuous.*

Proof. Let $\eta, \zeta \in \widehat{G} \subseteq S(G)$. Fix bases $\{x_U U : U \in \mathcal{N}_G\}$ and $\{y_U U : U \in \mathcal{N}_G\}$ for the corresponding near ultrafilters. We may assume each basis has the FIP. Keeping in mind Proposition 0.0.1, it suffices to show that the collection $\mathcal{F} = \{x_V V y_U U : U, V \in \mathcal{N}_G\}$ forms a base for the near ultrafilter $\eta \cdot \zeta$. First we note that $\mathcal{F}$ has the FIP. Second, if $U \in \mathcal{N}_G$, then there is $V \in \mathcal{N}_G$ with $V y_U \subseteq y_U U$, so $\mathcal{F}$ forms a base for some near ultrafilter corresponding to an element of $\widehat{G}$. To finish the proof, we show that every element of $\mathcal{F}$ is a member of $\eta \cdot \zeta$. Fix $U, V \in \mathcal{N}_G$, and consider $x_V V y_U U \in \mathcal{F}$. As $x_V V \in \eta$, we have $y_U U \subseteq \{g \in G : x_V V y_U U g^{-1} \in \eta\} = \eta^{-1}(x_V V y_U U)$. Since $y_U U \in \zeta$, this is more than enough to show that $x_V V y_U U \in \eta \cdot \zeta$ as desired. □

Our remaining task is to understand how $\widehat{G}$ interacts with a G-flow X. We have seen that the G-action on X extends to a semigroup action of $S(G)$ on X. By restricting our attention to $\widehat{G}$, we get far better continuity properties.

Proposition 2.4.2. *The evaluation $X \times \widehat{G} \to X$ is continuous.*

Proof. Keeping in mind Proposition 0.0.1, assume $x_i \to x$ and $g_i \to \eta$ with $g_i \in G$. By passing to a subnet, we may assume $x_i g_i \to y$ for some $y \in X$. We must show $y = x\eta$.

Let $A \ni x\eta$ be open. It is enough to show that frequently, we have $x_i g_i \in A$. By continuity of the action, there is open $B \subseteq X$ with $x\eta \in B$ and $1_G \subseteq U \subseteq G$ with $BU \subseteq A$. Note that $(g_i)_{i \in I}$ is Cauchy, so eventually we have $g_i^{-1} g_j \in U$ for all large enough $i, j \in I$. So find a suitably large index $i \in I$ with $x g_i \in B$. Then find a suitably large $j \geq i$ with $x_j g_i \in B$. It follows that $x_j g_i (g_i^{-1} g_j) = x_j g_j \in A$ as desired. □

It is reasonable to ask whether we can obtain better continuity results for larger subsets of $S(G)$. I conjecture that this is not the case.

Conjecture 2.4.3. *Suppose $q \in S(G) \setminus \widehat{G}$. Then the right multiplication $\rho_q : S(G) \to S(G)$ is not continuous.*

This is known to be true when G is locally compact [23]. In general, joint continuity properties of $S(G)$ even for G discrete seems to be quite subtle; see for instance [33].

We end the section with one more method for detecting the elements of $\widehat{G}$ in $S(G)$.

Proposition 2.4.4. *Let $q \in S(G)$. Then $q \in \widehat{G}$ iff q has a left inverse, i.e. if there is $p \in S(G)$ with $pq = 1_G$.*

Proof. Suppose $q \in \widehat{G}$, and let $\{x_u U : U \in \mathcal{N}_G\}$ be a base for q with the FIP. Form the collection $\mathcal{F} = \{U x_U^{-1} : U \in \mathcal{N}_G\}$. As $\mathcal{F}$ has the FIP, extend it to a near ultrafilter $p \in S(G)$. We claim that $pq = 1_G$. To show this, we need to show that for each $U \in \mathcal{N}_G$, we have $p^{-1}(U) \in q$. If $V \in \mathcal{N}_G$ is a square root of U, then $(V x_V^{-1}) x_V V \subseteq U$, so $x_V V \subseteq p^{-1}(U) \in q$ as desired.

Conversely, suppose there is $p \in S(G)$ with $pq = 1_G$. To show that $q \in \widehat{G}$, we need to find for each $U \in \mathcal{N}_G$ some $A \in q$ so that $A^{-1}A \subseteq U$. So fix $U \in \mathcal{N}_G$, and let $V \in \mathcal{N}_G$ be a fourth root of U. Then $p^{-1}(V) \in q$. Fix $g, h \in p^{-1}(V)$. This means that $Vg^{-1} \in p$ and $Vh^{-1} \in p$. Find suitably small $W \in \mathcal{N}_G$ so that $g^{-1}W \subseteq Vg^{-1}$ and $h^{-1}W \subseteq Vh^{-1}$. By membership in p, we must have $Vg^{-1}W \cap Vh^{-1}W \neq \emptyset$, so also $V^2 g^{-1} \cap V^2 h^{-1} \neq \emptyset$. This implies that $g^{-1}h \in V^4 \subseteq U$ as desired. □

2.5 Lifts of ambits and pre-ambits

This section introduces a generalization of the category of minimal flows and G-maps. This is used in [36] to understand dynamical issues related to big Ramsey degree; we will discuss these applications in more detail later. We introduce these ideas here since they seem of independent interest in the study of abstract dynamics.

A *pre-ambit* is a G-flow X where some point has dense orbit. In the literature, these flows are often called *topologically point transitive*. The *ambit-set* of X is the set $\mathcal{A}(X) := \{x \in X : \overline{x \cdot G} = X\}$. If X and Y are pre-ambits, then a surjective G-map $\varphi : X \to Y$ is *strong* if $\varphi``(\mathcal{A}(X)) = \mathcal{A}(Y)$. For any surjective G-map, the left-to-right inclusion holds; it is the reverse inclusion which is non-trivial. If X and Y are minimal, then any G-map $\varphi : X \to Y$ is strong. We will prove several results about pre-ambits and strong maps in this section and get results about minimal flows for free.

The following definition will be our main source of strong maps. Recall that if X is a G-flow and $x_0 \in X$, we write $\lambda_{x_0} : S(G) \to X$ for the unique G-map from $S(G)$ to X with $\lambda_{x_0}(1_G) = x_0$.

Definition 2.5.1. Let (X, x_0) be an ambit.

1. The *fixed point semigroup* of (X, x_0) is $S_{x_0} := \lambda_{x_0}^{-1}(\{x_0\}) = \{p \in S(G) : x_0 \cdot p = x_0\}$. It is a closed subsemigroup of $S(G)$, hence a compact, left-topological semigroup in its own right.

2. A *lift* of (X, x_0) is any subflow $Y \subseteq S(G)$ which is minimal subject to the property that $Y \cap S_{x_0} \neq \emptyset$.

Notice that since S_{x_0} is compact, Zorn's lemma ensures that any ambit admits a lift. The next lemma records some simple observations about lifts.

Lemma 2.5.2. *Let (X, x_0) be an ambit, and let $Y \subseteq S(G)$ be a lift of (X, x_0).*

1. *$\lambda_{x_0} : Y \to X$ is a strong G-map.*

2. *$Y \cap S_{x_0}$ is a minimal right ideal of S_{x_0}.*

Proof. For item (1), first note that $\lambda_{x_0}|_Y$ is surjective since $x_0 \in \lambda_{x_0}``(Y)$. Let $y \in Y$ be a point with $\lambda_{x_0}(y) \in \mathcal{A}(X)$. Then there is $p \in S(G)$ with $\lambda_{x_0}(y)p = x_0$. Then $yp \in S_{x_0}$, so in particular $\overline{yG} \cap S_{x_0} \neq \emptyset$. By the minimality property of lifts, we must have $\overline{yG} = Y$, so $y \in \mathcal{A}(Y)$ is a transitive point.

For item (2), certainly $Y \cap S_{x_0}$ is a right ideal of S_{x_0}, so suppose $M \subseteq Y \cap S_{x_0}$ is a minimal right ideal of S_{x_0}, and let $y \in M$. By the minimality property of lifts, we must have $yS(G) = Y$. Suppose $p \in S(G) \setminus S_{x_0}$. Then $x_0yp = x_0p \neq x_0$, so $yp \notin S_{x_0}$. It follows that $Y \cap S_{x_0} = y \cdot S_{x_0} \subseteq M$, so $M = Y \cap S_{x_0}$. □

To understand lifts, we need to understand the properties of pre-ambits which are subflows of $S(G)$. The following simple lemma is worth isolating.

Lemma 2.5.3. *Suppose $Y \subseteq S(G)$ is a pre-ambit, and assume there is an idempotent $u \in \mathcal{A}(Y)$. Let $\psi : Y \to X$ be a G-map. Then $\psi = \lambda_{\psi(u)}$.*

Proof. Notice that $\psi(u) = \psi(u \cdot u) = \psi(u) \cdot u$. Therefore ψ and $\lambda_{\psi(u)}$ are two G-maps on Y which agree on u. Since $u \in \mathcal{A}(Y)$, they must be equal. □

The next two propositions show that the choice of lift doesn't matter. The first shows that any two lifts are isomorphic, and the second limits the nature of G-maps between lifts.

Proposition 2.5.4. *Let (X, x_0) be an ambit, and let $Y_0, Y_1 \subseteq S(G)$ be two lifts of (X, x_0). Then Y_0 and Y_1 are isomorphic over (X, x_0), i.e. there is an isomorphism $\psi : Y_0 \to Y_1$ so that the following diagram commutes.*

$$\begin{tikzcd} Y_0 \arrow[rr, "\psi"] \arrow[dr, "\lambda_{x_0}"'] & & Y_1 \arrow[dl, "\lambda_{x_0}"] \\ & X & \end{tikzcd}$$

Proof. Write $M_i = Y_i \cap S_{x_0}$. Then each M_i is a minimal right ideal of S_{x_0} by item (2) of Lemma 2.5.2. Let $v \in M_1$ be an idempotent. Then the left multiplication $\lambda_v : M_0 \to M_1$ is an isomorphism of right ideals of S_{x_0}. Using Fact 2.1, let $u \in M_0$ be an idempotent in the same minimal left ideal of S_{x_0} as v. Then $uv = u$ and $vu = v$. It follows that $\psi := \lambda_v : Y_0 \to Y_1$ is an isomorphism with inverse λ_u. To check that the diagram commutes, let $y \in Y_0$. Then $y = up$ for some $p \in S(G)$. Then $\lambda_{x_0} \circ \lambda_v(up) = \lambda_{x_0}(vp) = x_0p = \lambda_{x_0}(up)$. □

Proposition 2.5.5. *Let (X, x_0) be an ambit, and let $Y_0, Y_1 \subseteq S(G)$ be two lifts of (X, x_0). Let $\psi : Y_0 \to Y_1$ be a surjective G-map making the diagram from Proposition 2.5.4 commute. Then ψ is an isomorphism.*

Proof. Once again, write $M_i = Y_i \cap S_{x_0}$. Each M_i is a minimal right ideal of S_{x_0}. Let $u \in M_0$ be an idempotent. Then $\psi = \lambda_{\psi(u)}$ by Proposition 2.5.3. As $\psi(u) \in M_1$, use Fact 2.1 to find $v \in M_0$ with $v\psi(u) \in M_0$ an idempotent. Since $v\psi(u) \in M_0$, we have $Y_0 = v\psi(u)S(G)$, so in particular, $\lambda_v \circ \lambda_{\psi(u)}$ is the identity map on Y_0. It follows that ψ is an isomorphism. □

We have shown that the lift of any ambit is canonical in the sense of Proposition 2.5.4. Remarkably, the lift of any pre-ambit is also canonical; if X is a preambit and $x_0, x_1 \in \mathcal{A}(X)$, then the lifts of the ambits (X, x_0) and (X, x_1) will be isomorphic as well. We spend the rest of this section proving this fact.

If X and Y are pre-ambits and $\psi : Y \to X$ is a strong map, we call ψ a *strong extension* of X. A strong extension $\psi_0 : Y_0 \to X$ is called *universal* if given any other strong extension $\psi_1 : Y_1 \to X$, there is a strong map $\varphi : Y_0 \to Y_1$ with $\psi_1 \circ \varphi = \psi_0$.

$$\begin{tikzcd} Y_0 \arrow[rr, "\varphi"] \arrow[dr, "\psi_0"'] & & Y_1 \arrow[dl, "\psi_1"] \\ & X & \end{tikzcd}$$

If $i < 2$ and $\psi_i : Y_i \to X$ are two strong extensions, we say that ψ_0 and ψ_1 are *isomorphic over X* if there is an isomorphism $\varphi : Y_0 \to Y_1$ with $\psi_1 \circ \varphi = \psi_0$.

Theorem 2.5.6. *Let X be a pre-ambit. Then there is a universal strong extension $\psi_X : L(X) \to X$. Any two universal strong extensions are isomorphic over X.*

We call the pre-ambit $L(X)$ given by Theorem 2.5.6 the *universal strong extension* of X. The next two propositions will prove Theorem 2.5.6. The first produces a universal strong extension of any pre-ambit, and the second shows uniqueness.

The following notation will be useful. If (X, x_0) is an ambit, we let $L(X, x_0) \subseteq S(G)$ be a lift of (X, x_0).

Proposition 2.5.7. *Let X be a pre-ambit, and let $x_0 \in \mathcal{A}(X)$. Then $\lambda_{x_0} : L(X, x_0) \to X$ is a universal strong extension.*

Proof. Let Y be a pre-ambit, and fix a strong map $\psi : Y \to X$. Pick $y_0 \in Y$ with $\psi(y_0) = x_0$. Then $y_0 \in \mathcal{A}(Y)$, and $S_{y_0} \subseteq S_{x_0}$. So we may assume that $L(X, x_0) \subseteq L(Y, y_0)$.

We will show that $L(X, x_0) = L(Y, y_0)$. Using the minimality property of lifts, it suffices to show that $\lambda_{y_0}``(L(X, x_0)) \cap \mathcal{A}(Y) \neq \emptyset$. Let $p \in L(X, x_0) \cap S_{x_0}$, towards showing that

$\lambda_{y_0}(p) \in \mathcal{A}(Y)$. First note that $\psi \circ \lambda_{y_0} = \lambda_{x_0}$. Then $\lambda_{x_0}(p) = \psi \circ \lambda_{y_0}(p) = \psi(y_0 p) = x_0$. Since ψ is strong, we have $y_0 p = \lambda_{y_0}(p) \in A(Y)$.

Putting everything together, we now have $L(X, x_0) = L(Y, y_0)$, and $\lambda_{y_0} : L(X, x_0) \to Y$ is a strong map with $\psi \circ \lambda_{y_0} = \lambda_{x_0}$. □

Proposition 2.5.8. *Let X be a pre-ambit. Then any two universal strong extensions are isomorphic over X.*

Proof. Let $x_0 \in \mathcal{A}(X)$, and form $\lambda_{x_0} : L(X, x_0) \to X$. Suppose that $\psi : Y \to X$ were another universal strong extension. By using the universal property of each map and composing, we obtain a strong map $\alpha : L(X, x_0) \to L(X, x_0)$ with $\lambda_{x_0} = \lambda_{x_0} \circ \alpha$. It is enough to note that α must be an isomorphism, and this follows by Proposition 2.5.5. □

As a last remark, notice that if X is the trivial flow, then a map $\psi : Y \to X$ is strong iff $\mathcal{A}(Y) = Y$, i.e. if Y is minimal. So $L(X) \cong M(G)$, and the notion of universal strong extension generalizes the notion of universal minimal flow.

2.6 Universal highly proximal extensions

This section provides a new construction of the *universal highly proximal extension* of a G-flow X. Throughout this section, we will write $A \subseteq_{op} X$ to mean that A is a non-empty open subset of X.

Definition 2.6.1.

1. Let X and Y be compact spaces, and let $\varphi : Y \to X$ be continuous. If $B \subseteq_{op} Y$, the *fiber image* of B is the set $\varphi_{fib}(B) := \{x \in X : \varphi^{-1}(\{x\}) \subseteq B$. Note that $\varphi_{fib}(B)$ is always open, but possibly empty.

2. Let X and Y be compact spaces. A map $\varphi : Y \to X$ is called *highly proximal* if φ is surjective and for every $B \subseteq_{op} Y$, we have $\varphi_{fib}(B) \neq \emptyset$.

3. Let X be a G-flow. The *universal highly proximal extension* of X is a G-flow $S_G(X)$ and a highly proximal G-map $\pi_X : S_G(X) \to X$ so that whenever $\varphi : Y \to X$ is another highly proximal G-map, there is a G-map $\psi : S_G(X) \to Y$ with $\pi_X = \varphi \circ \psi$.

The definition in item (1) is not standard, but it coincides with the standard one when X and Y are minimal G-flows and φ is a G-map.

Proposition 2.6.2. *Suppose X and Y are minimal G-flows, and let $\varphi : Y \to X$ be a G-map. Then the following are equivalent.*

1. *φ is highly proximal.*

2. *For every $B \subseteq_{op} Y$ and every $x \in X$, there is $g \in G$ with $\pi^{-1}(\{xg\}) \subseteq B$.*

Proof. As $\varphi_{fib}(B) \subseteq_{op} X$, find $g \in G$ with $xg \in \varphi_{fib}(B)$. □

The existence and uniqueness of the universal highly proximal extension was first proven by Auslander and Glasner [2] for minimal flows. Our new construction will generalize this to all flows. Even more, our construction of $S_G(X)$ will work even when X is just a G-space. In this generality, we will not obtain the G-map $\pi_X : S_G(X) \to X$, but we will still call $S_G(X)$ the universal highly proximal extension of X.

The key application of this new construction is from [35]. Recall that a G-space is *topologically transitive* if for every $A \subseteq_{op} X$ we have $AG \subseteq X$ dense.

Theorem 2.6.3. *Let G be a Polish group, and suppose X is a Polish, topologically transitive G-space whose orbits are all meager. Then $S_G(X)$ is not metrizable.*

Applpying Theorem 2.6.3 when X is a minimal, metrizable G-flow, this provides a new solution to the Generic Point Problem (Question 15.2 from [1]). Of course, we need to know that $S_G(X)$ is minimal if X is a minimal G-flow; we will prove this shortly.

We now proceed with the construction. Fix for now any topological group G and any G-space X. Our construction of $S_G(X)$ mimics the near ultrafilter construction of the Samuel compactification, and many proofs are the same as their counterparts from section 1.2. However, instead of working with all subsets of X, we work with $\mathrm{op}(X) := \{A : A \subseteq_{op} X\}$.

Definition 2.6.4. A *G-near filter* is any $\mathcal{F} \subseteq \mathrm{op}(X)$ so that for any $A_1, ..., A_k \in \mathcal{F}$ and any $U \in \mathcal{N}_G$, we have $A_1U \cap \cdots \cap A_kU \neq \emptyset$. A *$G$-near ultrafilter* is a maximal near filter.

Zorn's lemma shows that G-near ultrafilters exist. Let $S_G(X)$ denote the space of G-near ultrafilters on $\mathrm{op}(X)$.

Lemma 2.6.5.

1. *Let $p \in S_G(X)$, and let $A \subseteq X$ be open. If $A \notin p$, then there is some $V \in \mathcal{N}_G$ with $AV \notin p$.*

2. *Let $A \subseteq X$ be open, and let $B_1, ..., B_k \subseteq A$ be open with $B_1 \cup \cdots \cup B_k$ dense in A. If $p \in S_G(X)$ and $A \in p$, then $B_i \in p$ for some $i \leq k$.*

Proof. The proofs are nearly identical to those from Proposition 1.2.2. □

Definition 2.6.6. If $A \in \mathrm{op}(X)$, set $N_A := \{p \in S_G(X) : A \notin p\}$. We endow $S_G(X)$ with the topology whose typical basic open neighborhood is N_A for $A \in \mathrm{op}(X)$.

Proposition 2.6.7. *The topology from Definition 2.6.6 is compact Hausdorff.*

Proof. The proof is nearly identical to the one from Proposition 1.2.4. Keep in mind that we should be working with open sets, so for instance, whenever $p \in S_G(X)$ and $A \subseteq_{op} X$ with $A \notin p$, we conclude using Lemma 2.6.5 that there is $V \in \mathcal{N}_G$ with $\mathrm{int}(X \setminus AV) \in p$. □

Definition 2.6.8. If $p \in S_G(X)$ and $g \in G$, we let $pg \in S_G(X)$ be defined by declaring $A \in pg$ iff $Ag^{-1} \in p$ for each $A \in \mathrm{op}(X)$.

Proposition 2.6.9. *The action in Definition 2.6.8 is continuous.*

Proof. The proof is nearly identical to the one from Proposition 2.2.2. □

We now start to add assumptions to the G-space X in order to deduce more about $S_G(X)$. In general, a G-space X is called *minimal* if every orbit is dense. Here, we will want the following strengthening.

Definition 2.6.10. A G-space X is called *finitely minimal* if for every $A \subseteq_{op} X$, there are $g_0, ..., g_{k-1} \in G$ with $\bigcup_{i<k} Ag_i = X$.

Remark. By Proposition 2.2.1, a G-flow is finitely minimal iff it is minimal.

Proposition 2.6.11. *Suppose X is a finitely minimal G-space. Then $S_G(X)$ is a minimal G-flow.*

Proof. Let $p \in S_G(X)$, and let $A \in op(X)$ with $N_A \neq \emptyset$. Find some $V \in \mathcal{N}_G$ with $N_{AV} \neq \emptyset$. Then $B := \mathrm{int}(X \setminus AV) \neq \emptyset$. As X is finitely minimal, find $g_0, ..., g_{k-1}$ with $X = \bigcup_{i<k} Bg_i$. For some $i < k$, we must have $Bg_i \in p$. Then $B \in pg_i^{-1}$, so we must have $A \notin pg_i^{-1}$, i.e. $pg_i^{-1} \in N_A$, and the orbit of p is dense as desired. □

Proposition 2.6.12. *Let X be a G-space, and suppose there are $\{A_n : n < \omega\} \subseteq \mathrm{op}(X)$ and $V \in \mathcal{N}_G$ with $\{A_nV : n < \omega\}$ pairwise disjoint. Then $S_G(X)$ is non-metrizable.*

Proof. If $S \subseteq \omega$, let $A_S = \bigcup_{n \in S} A_n$, and let $Y = \{p \in S_G(X) : A_\omega \in p\}$. Then $Y \subseteq S_G(X)$ is a closed subspace. To show that $S_G(X)$ is non-metrizable, we will exhibit a continuous surjection $\pi : Y \to \beta\omega$. First note that if $S \subseteq \omega$, then $A_SV \cap A_{\omega \setminus S}V = \emptyset$. Therefore, if $p \in Y$, p contains exactly one of A_S or $A_{\omega \setminus S}$ for each $S \subseteq \omega$. We let $\pi : Y \to \beta\omega$ be defined so that for $S \subseteq \omega$, $S \in \pi(p)$ iff $A_S \in p$. It is immediate that π is continuous. To see that π is surjective, let $q \in \beta\omega$. Then $\{A_S : S \in q\}$ is a G-near filter; any G-near ultrafilter p extending it is a member of Y with $\pi(p) = q$. □

To prove Theorem 2.6.3, we will need the following proposition due to Rosendal; see [5] for a proof.

Proposition 2.6.13. *Let G be a Polish group, and let X be a Polish, topologically transitive G-space. Then the following are equivalent.*

1. *G has a comeager orbit.*
2. *For any $V \in \mathcal{N}_G$ and any $B \subseteq_{op} X$, there is $C \subseteq_{op} B$ so that for any $D \subseteq_{op} C$, the set $C \setminus VD$ is nowhere dense.*

The next proposition along with Proposition 2.6.12 will prove Theorem 2.6.3

Proposition 2.6.14. *Suppose G is a Polish group, and let X be a Polish, topologically transitive G-space whose orbits are all meager. Then there are $\{A_n : n < \omega\} \subseteq \mathrm{op}(X)$ and $V \in \mathcal{N}_G$ with $\{A_n V : n < \omega\}$ pairwise disjoint.*

Proof. By Proposition 2.6.13, there is $U \in \mathcal{N}_G$ and $B \subseteq_{op} X$ so that for any $C \subseteq_{op} B$, there is $D \subseteq_{op} C$ with $C \setminus DU$ somewhere dense (since C and UD are open, this is the same as $C \setminus DU$ having nonempty interior).

Let $V \in \mathcal{N}_G$ with $VV \subseteq U$. We now produce $\{A_n : n < \omega\} \subseteq \mathrm{op}(X)$ with $\{A_n V : n < \omega\}$ pairwise disjoint. First set $B_0 = B$. As $B_0 \subseteq B$, there is $A_0 \subseteq B_0$ so that $B_0 \setminus A_0 U$ has nonempty interior. Suppose open sets $B_0, ..., B_{n-1}$ and $A_0, ..., A_{n-1}$ have been produced so that $A_i \subseteq B_i$ and $\mathrm{int}(B_i \setminus A_i U) \neq \emptyset$. We continue by setting $B_n = \mathrm{int}(B_{n-1} \setminus A_{n-1} U)$. As $B_n \subseteq B$, there is $A_n \subseteq B_n$ so that $B_n \setminus A_n U$ has nonempty interior. Notice that for any $m \leq n$, we also have $A_n \subseteq B_m$. It follows that if $m < n$, we have $A_m U \cap A_n = \emptyset$. This implies that $A_m V \cap A_n V = \emptyset$ as desired. □

We return to the setting where G is any topological group, but we now consider when X is a G-flow. We want to exhibit a G-map $\pi_X : S_G(X) \to X$ and show that this is the universal highly proximal extension of X.

Definition 2.6.15. Let X be a G-flow, and form $S_G(X)$. The G-map $\pi_X : S_G(X) \to X$ is defined as follows. For each $p \in S_G(X)$, there is a unique $x_p \in X$ so that every neighborhood of x_p is in p. The existence of such a point is an easy consequence of the compactness of X and the second item of 2.6.5. For uniqueness, notice that if $x \neq y \in X$, we can find open $A \ni x$, $B \ni y$ and $U \in \mathcal{N}_G$ with $AU \cap BU = \emptyset$. We set $\pi_X(p) = x_p$. This map clearly respects the G-action. To check continuity, one can check that if $K \subseteq X$ is closed, then $\pi_X^{-1}(K) = \{p \in S_G(X) : A \in p \text{ for every open } A \supseteq K\}$, and this is a closed condition.

Proposition 2.6.16. *Let X be a G-flow. Then the map $\pi_X : S_G(X) \to X$ is highly proximal.*

Proof. Let $N_A \subseteq S_G(X)$ be a nonempty basic open neighborhood. This implies that $\text{int}(X \setminus A) \neq \emptyset$. Let $x \in \text{int}(X \setminus A)$. Then there are open $B \ni x$ and $U \in \mathcal{N}_G$ with $BU \cap A = \emptyset$. It follows that any $p \in S_G(X)$ containing B cannot contain A. In particular, we have $\pi_X^{-1}(\{x\}) \subseteq N_A$. □

The next theorem generalizes the notion of universal highly proximal extension to any G-flow.

Theorem 2.6.17. *Let X be a G-flow. Then the map $\pi_X : S_G(X) \to X$ is the universal highly proximal extension of X.*

Proof. Fix a highly proximal extension $\varphi : Y \to X$. For each $y \in Y$, let $\mathcal{F}_y := \{\varphi_{fib}(B) : B \ni y \text{ open}\}$. Then $\mathcal{F}_y \subseteq \text{op}(X)$ is a filter of open sets, so in particular it is a near filter. We will show that for each $p \in S_G(X)$, there is a unique $y \in Y$ with $\mathcal{F}_y \subseteq p$. This will define the map $\psi : S_G(X) \to Y$.

We first show that for each $p \in S_G(X)$, there is at least one such $y \in Y$. To the contrary, suppose for each $y \in Y$, there were $B_y \ni y$ open so that $\varphi_{fib}(B_y) \notin p$. Find $y_0, ..., y_{k-1}$ so that $\{B_{y_0}, ..., B_{y_{k-1}}\}$ is a finite subcover. Let $A_i = \varphi_{fib}(B_{y_i})$. Each A_i is open, so we will reach a contradiction once we show that $\bigcup_{i<k} A_i$ is dense. Let $A \subseteq X$ be open. Then $C := B_{y_i} \cap \varphi^{-1}(A) \neq \emptyset$ for some $i < k$. As C is open, $\varphi_{fib}(C) \neq \emptyset$, and $\varphi_{fib}(C) \subseteq A \cap A_i$.

Now we consider uniqueness. Let $p \in S_G(X)$, and consider $y \neq z \in Y$. Find open $B \ni y$ and $C \ni z$ and some $V \in \mathcal{N}_G$ so that $BV \cap CV = \emptyset$. It follows that $\varphi_{fib}(BV) \cap \varphi_{fib}(CV) = \emptyset$. Now notice that $\varphi_{fib}(B)V \subseteq \varphi_{fib}(BV)$, and likewise for C. Hence p cannot contain both $\mathcal{F}_y$ and $\mathcal{F}_z$.

The map ψ clearly respects the G-action and satisfies $\pi_X = \varphi \circ \psi$. To show continuity, let $K \subseteq Y$ be closed. Let $\mathcal{F}_K := \{\varphi_{fib}(B) : B \supseteq K \text{ open}\}$. We will show that $\psi(p) \in K$ iff $\mathcal{F}_K \subseteq p$. From this it follows that $\psi^{-1}(K)$ is closed. One direction is clear. For the other, suppose $\psi(p) = y \notin K$. Find open sets $B \ni y$, $C \supseteq K$, and $V \in \mathcal{N}_G$ with $BV \cap CV = \emptyset$. As in the proof of uniqueness, p cannot contain both $\mathcal{F}_y$ and $\mathcal{F}_K$. □

We end the section by considering possible universal properties that $S_G(X)$ might have when X is a non-compact G-space. As a warmup, consider the following example.

Proposition 2.6.18. *Viewing G as a G-space in the natural way, we have $S_G(G) \cong S(G)$.*

Proof. If $p \in S_G(G)$, then p is a near filter. If $q \supseteq p$ is a near ultrafilter, then for any $A \in q$ and $V \in \mathcal{N}_G$, we have $AV \in p$. It follows that there is at most one extension of p to a near ultrafilter; this defines a map $\psi : S_G(G) \to S(G)$. If $A \subseteq G$, then $\psi^{-1}(N_A) = \bigcup_{V \in \mathcal{N}_G} N_{AV}$, so ψ is continuous. To check that ψ is a bijection, consider $q \in S(G)$. Then $\{AV : A \in q, V \in \mathcal{N}_G\}$ is a G-near filter which uniquely extends to a G-near ultrafilter. This defines a map $\varphi : S(G) \to S_G(G)$, and it is routine to check that $\psi \circ \varphi = 1_{S(G)}$ and $\varphi \circ \psi = 1_{S_G(G)}$. □

In this case, we have an embedding $G \hookrightarrow S_G(G)$ rather than a G-map the other way around. It is natural to ask for which other G-spaces this is the case.

Definition 2.6.19. A G-space X is called *rich* iff for any $A \subseteq_{op} X$, $x \in \overline{A}$, and $V \in \mathcal{N}_G$, we have $x \in \mathrm{int}(\overline{AV})$.

The intuition behind this definition is that rich G-spaces are complicated enough to recover much of the underlying topological information about X.

Proposition 2.6.20. *Let X be a rich G-space. Then there is an embedding $i_X^G : X \to S_G(X)$.*

Proof. Given $x \in X$, consider the collection $\mathcal{F}_x := \{A \subseteq_{op} X : x \in \overline{A}\}$. As X is rich, $\mathcal{F}_x$ is a G-near filter. Let us show that in fact, it is a G-near ultrafilter. Suppose $A \subseteq_{op} X$ with $x \notin \overline{A}$. We can find $B \ni x$ open and $U \in \mathcal{N}_G$ with $BU \cap \overline{A} = \emptyset$. As $B \in \mathcal{F}_x$, it follows that A cannot belong to any G-near ultrafilter extending $\mathcal{F}_x$.

This defines the map $i_X^G : X \to S_G(X)$. It is clearly injective and respects the G-action. To show i_X^G is continuous, fix $A \subseteq_{op} X$. Then $(i_X^G)^{-1}(N_A) = X \setminus \overline{A}$. To show that i_X^G is an embedding, fix $A \subseteq_{op} X$. Then $(i_X^G)``(A) = \mathrm{Im}(i_X^G) \cap N_{X \setminus \overline{A}}$. □

It turns out that when X is a rich G-space, this map of X into a G-flow is the largest possible in the following sense.

Definition 2.6.21. Let X be a G-space. The *maximal equivariant compactification* of X is a G-flow $\alpha_G X$ and a G-map $i_X^G : X \to \alpha_G X$ so that whenever Y is a G-flow and $\varphi : X \to Y$ is a G-map, there is $\psi : \alpha_G X \to Y$ with $\psi \circ i_X^G = \varphi$.

Theorem 2.6.22. *Let X be a rich G-space. Then $i_X^G : X \to S_G(X)$ is the maximal equivariant compactification of X.*

Proof. Let Y be a G-flow, and suppose $\varphi : X \to Y$ is a G-map. For each $y \in Y$, let $\mathcal{F}_y = \{\varphi^{-1}(B) : B \ni y \text{ open}\}$. Each $\mathcal{F}_y$ is a filter of open sets, so also a G-near filter. We will show that for each $p \in S_G(X)$, there is a unique $y \in Y$ with $\mathcal{F}_y \in p$. This will define the map $\psi : S_G(X) \to Y$.

We first show that for each $p \in S_G(X)$, there is at least one such $y \in Y$. To the contrary, suppose for each $y \in Y$, there were $B_y \ni y$ open so that $\varphi^{-1}(B_y) \notin p$. Find $\{y_0, ..., y_{k-1}\}$ so that $\{B_{y_0}, ..., B_{y_{k-1}}\}$ is a finite subcover. Then $X = \bigcup_{i<k} \varphi^{-1}(B_i)$, and this contradicts item (2) of Lemma 2.6.5.

Now we consider uniqueness. Let $p \in S_G(X)$, and consider $y \neq z \in Y$. Find open $B \ni y$ and $C \ni z$ and some $V \in \mathcal{N}_G$ so that $BV \cap CV = \emptyset$. It follows that $\varphi^{-1}(BV) \cap \varphi^{-1}(CV) = \emptyset$. Now notice that $\varphi^{-1}(B)V = \varphi^{-1}(BV)$, and likewise for C. Hence p cannot contain both $\mathcal{F}_y$ and $\mathcal{F}_z$.

The map ψ clearly respects the G-action and satisfies $\varphi = \psi \circ i_X^G$. To show continuity, let $K \subseteq Y$ be closed. Let $\mathcal{F}_K := \{\varphi^{-1}(B) : B \supseteq K \text{ open}\}$. We will show that $\psi(p) \in K$ iff $\mathcal{F}_K \subseteq p$. From this it follows that $\psi^{-1}(K)$ is closed. One direction is clear. For the other, suppose $\psi(p) = y \notin K$. Find open sets $B \ni y$, $C \supseteq K$, and $V \in \mathcal{N}_G$ with $BV \cap CV = \emptyset$. As in the proof of uniqueness, p cannot contain both $\mathcal{F}_y$ and $\mathcal{F}_K$. □

Given that our construction of $S_G(X)$ sometimes coincides with the maximal equivariant compactification, the following question seems intriguing.

Question 2.6.23. *Given a G-space X, is there a "near ultrafilter like" construction which always produces $\alpha_G X$?*

Any construction as above would necessarily be quite strange; Pestov in [28] has recently exhibited a Polish group G and a non-trivial G-space X with $\alpha_G X$ a singleton.

2.7 Ambits and metrizable $M(G)$

If X is a G-flow and $x \in X$, we say that the point x is *almost periodic* if $\overline{xG}$ is a minimal flow. Write $AP(X)$ for the collection of almost periodic points of X. It will be helpful to remember the characterization of almost periodic points given by Proposition 2.3.12. The main result in this section was obtained in joint work with Dana Bartošová.

Theorem 2.7.1. *Let G be a Polish group. The following are equivalent.*

1. *$M(G)$ is metrizable.*
2. *For every pre-ambit X, $AP(X) \subseteq X$ is closed.*

It suffices to consider the case $X = S(G)$. Recall that every compact left-topological semigroup S has a smallest two-sided ideal which is the union of the minimal right ideals. When $S = S(G)$, then $K(S(G))$ is the union of the minimal subflows of $S(G)$, and therefore $K(S(G)) = AP(S(G))$. Recall that in Proposition 2.3.17, we characterized which near ultrafilters are members of $\overline{K(S(G))}$. If (X, x_0) is an ambit and $\lambda_{x_0} : S(G) \to X$ is the map of ambits, then $\lambda_{x_0}``(K(S(G))) = AP(X)$.

Fix for the remainder of the section a Polish group G and a compatible left-invariant metric d on G with diameter one, and form the topometric space $(S(G), \partial)$. We will start with the forward direction, so assume that $M(G)$ is metrizable. Therefore by Theorem 1.4.8, ∂ is a compatible metric on any minimal subflow $M \subseteq S(G)$. If $M, N \subseteq S(G)$ are two minimal subflows, we will want to understand how $\partial|_M$ and $\partial|_N$ compare.

Lemma 2.7.2. *Let $p \in S(G)$ and $A \subseteq G$, and fix $\epsilon > 0$. Then $p^{-1}(A)(\epsilon) \subseteq p^{-1}(A(\epsilon))$.*

Proof. If $g \in p^{-1}(A)(\epsilon)$, find $h \in p^{-1}(A)$ with $d(g,h) < \epsilon$. Then $A \in ph$, so $Ah^{-1}g \in pg$, so $A(\epsilon) \in pg$. $\square$

Proposition 2.7.3. *Let $p \in S(G)$. Then the left multiplication $\lambda_p : S(G) \to S(G)$ is ∂-nonexpansive.*

Proof. Let $p, q, r \in S(G)$. Suppose $\partial(q,r) \leq c$ for some $c \geq 0$. Fix $A \in pq$ and $\epsilon > 0$. We want to show that $A(c+\epsilon) \in pr$. Since $A \in pq$, we have $p^{-1}(A(\epsilon/2)) \in q$. Then $p^{-1}(A(\epsilon/2))(c+\epsilon/2) \subseteq p^{-1}(A(c+\epsilon)) \in r$. It follows that $A(c+\epsilon) \in pr$ as desired. $\square$

Corollary 2.7.4. *If $M, N \subseteq S(G)$ are two minimal right ideals (i.e. two minimal subflows), then any G-flow isomorphism between M and N is a ∂-isometry.*

We now prove the forward direction of Theorem 2.7.1. Let $(p_i)_{i\in I}$ be a net from $K(S(G))$ with $p_i \to p$. Fix $B \in p$ and $\epsilon > 0$. Then by Proposition 1.3.1, $C_{B(\epsilon)}$ is a typical neighborhood of p. So by Proposition 2.3.12, we want to show that $p^{-1}(B(\epsilon))$ is syndetic.

Since $M(G)$ is assumed metrizable, there is a canonical compatible metric given by Corollary 2.7.4. Fix finite $F_\epsilon \subseteq G$ so that whenever $X \subseteq M(G)$ is a ball of radius $\epsilon/2$, we have $M(G) = \bigcup_{g\in F_\epsilon} Xg$.

Let $\delta > 0$ be suitably small. Notice that eventually $B(\delta) \in p_i$. Let M_i be a minimal subflow containing p_i. As $C_{B(\epsilon)}$ contains the ∂-ball of radius $\epsilon - 2\delta$ around p_i, we have $M_i \subseteq \bigcup_{g\in F_\epsilon} C_{B(\epsilon)}g$. It follows that $p_i^{-1}(\bigcup_{g\in F_\epsilon} B(\epsilon)g) = G$. Therefore $p^{-1}(\bigcup_{g\in F_\epsilon} B(\epsilon)g) = G$. So $\bigcup_{g\in F_\epsilon}(p^{-1}(B(\epsilon)))g = G$, showing that $p^{-1}(B(\epsilon))$ is syndetic. $\square$

We now turn towards the other direction. Fix for the remainder of this note $M \subseteq S(G)$ a minimal subflow which is not metrizable.

Proposition 2.7.5. *If $M(G)$ is non-metrizable, then $|M(G)| = 2^{\mathfrak{c}}$.*

Proof. We have $|M| \leq 2^{\mathfrak{c}}$ since M is a separable compact Hausdorff space. For the other inequality, recall from the proof of Theorem 1.4.8 that we can inject a copy of $\beta\omega$ into K. $\square$

Proposition 2.7.6. *In M, there is some ∂-ball which is nowhere dense.*

Proof. For $p \in M$ and $\epsilon > 0$, the set $\{q \in M : \partial(p,q) \leq \epsilon\}$ is topologically closed. So it will suffice to find a ball with empty interior.

Suppose each ∂-ball contained some open neighborhood. Fix $p \in M$. Notice that $p \cdot G$ must be ∂-dense. So M is the ∂-completion of $p \cdot G$. But this would imply that $|M| = \mathfrak{c}$, contradicting Proposition 2.7.5. $\square$

Remark. Propositions 2.7.5 and 2.7.6 are the only points in the proof of Theorem 2.7.1 where we need to assume that G is separable. Ben–Yaacov and Melleray have suggested the following alternate proof of Proposition 2.7.6 which doesn't use Proposition 2.7.5. A compact topometric space (X, ∂) is called *adequate* if for any open $U \subseteq X$ and $\epsilon > 0$, the set $B_\partial(U, \epsilon) := \{x \in X : \exists y \in U\, \partial(x, y) < \epsilon\}$ is open. One can prove that $(S(G), \partial)$ is adequate, and as $M(G)$ is a retract of $S(G)$, also $(M(G), \partial)$ is adequate. Now in the setting of Proposition 2.7.6, if every ∂-ball $B_\partial(x, \epsilon)$ had non-empty interior, it would follow that $x \in \int B(x, \epsilon)$. But this means that ∂ is a compatible metric for $M(G)$, contradicting the assumption that $M(G)$ is non-metrizable.

The group G acts on $[0, 1]^G$ on the right, where if $f \in [0, 1]^G$ and $g, h \in G$, we have $f \cdot g(h) = f(gh)$. Endow $[0, 1]^G$ with the product topology. For $0 < c \leq 1$, let $\mathrm{Lip}_c(G) \subset [0, c]^G$ be the closed metrizable subspace of 1-Lipschitz functions. Then the action $\mathrm{Lip}_c(G) \times G \to \mathrm{Lip}_c(G)$ is continuous. If $f \in \mathrm{Lip}_c(G)$ and $p \in S(G)$, we define $f \cdot p = \lim_{g \to p} f \cdot g$. Equivalently, one can first extend f to $S(G)$; then if $p, q \in S(G)$, we have $f \cdot p(q) = f(pq)$. As a word of warning, note that in general, we do not have $\lim_{g \to p} f \cdot g(q) = f \cdot p(q)$.

Our next goal is to show that for some $c > 0$, $\mathrm{Lip}_c(G)$ is topologically transitive. As $\mathrm{Lip}_c(G)$ is metrizable, this will imply that it is a pre-ambit. We will then finish the proof by showing that $AP(\mathrm{Lip}_c(G))$ is not closed.

Given subsets $\{S_i \subseteq G : i \in I\}$, call $\{S_i : i \in I\}$ *equi-syndetic* if there are $g_0, ..., g_{k-1} \in G$ so that for every $j \in I$, we have $G = \bigcup_{i<k} S_j g_i$. Notice that if $S \subseteq G$ is syndetic, then the collection $\{gS : g \in G\}$ is equi-syndetic.

Lemma 2.7.7. *If $S_n \subseteq G$ and $\{S_n : n < \omega\}$ is equi-syndetic, then the collection is not pairwise disjoint.*

Proof. Let $g_0, ..., g_{k-1} \in G$ witness that the collection $\{S_n : n < \omega\}$ is equi-syndetic. Notice that for each n, we have $g_i^{-1} \in S_n$ for some $i < k$. So we can find $m \neq n$ and some $i < k$ with $g_i^{-1} \in S_m \cap S_n$. □

Lemma 2.7.8. *There is $c > 0$ so that no $B \subseteq G$ of diameter at most c is piecewise syndetic.*

Proof. As $M(G)$ is non-metrizable, we know that G is not pre-compact (otherwise $M(G) \cong S(G) \cong \widehat{G}$). Find $c > 0$ so that for any $A \subseteq G$ of diameter at most c, there are $(g_n)|_{n<\omega}$ in G with $\{g_n A : n < \omega\}$ pairwise disjoint. To see that this can be done, fix $\{h_n : n < \omega\}$ an ϵ-discrete set for some $\epsilon > 0$. Let $c = \epsilon/3$. If $A \subseteq G$ has diameter at most c, find $g \in G$ with $1_G \in gA$. Then set $g_n = h_n g$. For every $n < \omega$, $g_n A$ has diameter at most c and $h_n \in g_n A$.

Now fix $B \subseteq G$ of diameter at most c. Towards a contradiction, assume B is piecewise syndetic. By Proposition 2.3.14, we can find $p \in \beta G$ (viewing G discretely) with $p^{-1}(B) := \{g \in G : B \in pg\}$ syndetic.

Notice that $p^{-1}(B)$ has diameter at most c. To see this, if $g, h \in p^{-1}(B)$, then Bg^{-1} and Bh^{-1} are in p, so $Bg^{-1} \cap Bh^{-1} \neq \emptyset$. Find $b, c \in B$ with $bg^{-1} = ch^{-1}$, so also $c^{-1}b = h^{-1}g$. As $d(b, c) \leq c$, it follows that $d(g, h) \leq c$. Now fix $g_n \in G$ with $\{g_n p^{-1}(B) : n < \omega\}$ a pairwise disjoint, equi-syndetic set, a contradiction. □

In order to show that $\mathrm{Lip}_c(G)$ is topologically transitive, we will need the following extension lemma for Lipschitz functions.

Lemma 2.7.9. *Let $A \subseteq G$, and let $f : A \to [0, c]$ be 1-Lipschitz. Then there is $F \in \mathrm{Lip}_c(G)$ with $F|_A = f$ and $F|_{G \setminus A(c)} \equiv 0$.*

Proof. We first define a function $\hat{F}$ as follows.

$$\hat{F}(g) := \sup_{h \in A} (f(h) - d(g, h))$$

We then set $F(g) = \max(\hat{F}(g), 0)$. To see that F is as desired, it is enough to check that $\hat{F}$ is 1-Lipschitz and satisfies $\hat{F}_A = f$. We have $\hat{F}_A = f$ because f is 1-Lipschitz on A. Now let $g_0, g_1 \in G$. For any $h \in A$, we see that $\hat{F}(g_0) \geq f(h) - d(g_0, h)$ implies $\hat{F}(g_1) \geq f(h) - d(g_0, h) - d(g_0, g_1)$, and vice versa. This shows that $\hat{F}$ is 1-Lipschitz. □

Proposition 2.7.10. *Suppose $c > 0$ is such that $2c$ works for Lemma 2.7.8. Then $\mathrm{Lip}_c(G)$ is topologically transitive.*

Proof. Let $U, V \subseteq \mathrm{Lip}_c(G)$ be two non-empty basic open neighborhoods. Then there are points $a_0, ..., a_{m-1} \in G$, $b_0, ..., b_{n-1} \in G$ and open intervals $I_0, ..., I_{m-1} \subseteq [0, c]$, $J_0, ..., J_{n-1} \subseteq [0, c]$ with $U = \{f : \forall k < m(f(a_k) \in I_k)\}$ and $V = \{f : \forall \ell < n(f(b_\ell) \in J_\ell)\}$. We need to find $g \in G$ with $Ug^{-1} \cap V \neq \emptyset$. Notice that $Ug^{-1} = \{f : \forall k < m(f(ga_k) \in I_k)\}$.

Start by fixing $f_U \in U$ and $f_V \in V$, and set $c_k = f_U(a_k)$, $d_\ell = f_V(b_\ell)$. Now set $A' = \{a_0, ..., a_{m-1}\}$ and $B = \{b_0, ..., b_{n-1}\}$. We want to find $g \in G$ so that $gA' \cap B(c) = \emptyset$. Suppose this were impossible. This implies that $\bigcup_{i<m} B(c)a_i^{-1} = G$. So $B(c)$ is syndetic, implying that some $\{b_\ell\}(c)$ is piecewise syndetic, contradicting Lemma 2.7.8.

Now set $A = gA' \cup B$. Let $f : A \to [0, c]$ be given by $f(ga_k) = c_k$ and $f(b_\ell) = d_\ell$. By choice of $g \in G$, f is 1-Lipschitz, so may be extended to $F \in \mathrm{Lip}_c(G)$. We see that $F \in Ug^{-1} \cap V$ as desired. □

The last ingredient we need is to clarify a property shared by $AP(\mathrm{Lip}_c(G))$ functions.

Lemma 2.7.11. *Suppose $f \in AP(\mathrm{Lip}_c(G))$ and $f(g) = c$ for some $g \in G$. Then for any $\epsilon > 0$, the set $f^{-1}([c - \epsilon, c])$ is syndetic.*

Proof. Towards a contradiction, assume this were false. Then for any finite $F \subseteq G$, there is $g_F \in G$ with $f(g_F h) \leq c - \epsilon$ for each $h \in F$. Viewing the finite subsets of G as a directed set, find a subnet with $fg_F \to \varphi$. Notice that φ is bounded above by $c - \epsilon$. As $f \in AP(\mathrm{Lip}_c(G))$, there is a net $h_i \in G$ with $\varphi \cdot h_i \to f$. But this is impossible as f attains the value c. □

For the rest of the section, fix $p \in M$ and $c > 0$ so that $B_\partial(p, c)$ the ∂-ball of radius c at p is nowhere dense as guaranteed by Lemma 2.7.6. We define $f \in \mathrm{Lip}_c(G)$ via $f(g) = \max(c - \partial(p, p \cdot g), 0)$. If $A \in p$, we define $f_A \in \mathrm{Lip}_c(G)$ via $f_A(g) = \max(c - \partial(A, p \cdot g), 0)$. Here $\partial(A, p \cdot g) := \inf(\epsilon > 0 : A(\epsilon) \in p \cdot g)$. Equivalently, one can define $\varphi_A \in \mathrm{Lip}_c(G)$ via $\varphi_A(g) = \max(c - d(g, A), 0)$, then set $f_A = \varphi_A \cdot p$.

Theorem 2.7.12. *For each $A \in p$, $f_A \in AP(\mathrm{Lip}_c(G))$. We have $f \in \overline{\{f_A : A \in p\}}$, and $f \notin AP(\mathrm{Lip}_c(G))$.*

Proof. Since $f_A = \varphi_A \cdot p$ and $p \in M$, we have that $f_A \in AP(\mathrm{Lip}_c(G))$. To see that $f \in \overline{\{f_A : A \in p\}}$, notice that for each $g \in G$, we have $f_A(g) \geq f(g)$. Now fix $g_0, ..., g_{k-1} \in G$ and $\epsilon > 0$. We will find $A \in p$ so that $f_A(g_i) \leq f(g_i) + \epsilon$. Let $c_i = \partial(p, p \cdot g_i)$. We may assume that for each $i < k$, we have $c_i > \epsilon$. Find $A_i \in p$ and $B_i \in p \cdot g_i$ so that $A_i(c_i - \delta) \cap B_i = \emptyset$, where δ is very small. Now form $A = \bigcap_{i<k} A_i(\delta) \in p$. Notice that $A(c_i - 2\delta) \cap B_i = \emptyset$. It follows that $\partial(A, p \cdot g) \geq c_i - 3\delta$, which implies the desired result.

We now show that $f \notin AP\mathrm{Lip}_c(G)$. Towards a contradiction, suppose it were. Then as $f(1_G) = c$, the set $S := f^{-1}([c/2, c])$ is syndetic. But notice that since S is syndetic and $p \cdot G$ is dense in M, we must have $p \cdot S$ somewhere dense in M. But $p \cdot S \subseteq B_\partial(p, c)$, which is nowhere dense. $\square$

Chapter 3

Automorphism groups

Throughout this thesis, we use the term *automorphism group* to mean a topological group of the form $G = \mathrm{Aut}(\mathbf{K})$. Here $\mathbf{K}$ is a countable first-order structure on countable underlying set K, and G is endowed with the topology of pointwise convergence. We will see that when G is an automorphism group, $S(G)$ takes on a particularly nice form. This will allow us to relate the metrizability of $M(G)$ to the combinatorial phenomenon of finite Ramsey degree. We also consider structures with finite big Ramsey degree and create new dynamical objects in an attempt to capture this notion.

One of the key notions in this chapter is that of a *diagram*, defined in section 3.4. A diagram uses embeddings between finite substructures of $\mathbf{K}$ to produce an abstract notion of partial right action. For tall enough diagrams, the ω-*diagrams*, we can construct a G-flow and relate properties of the diagram to properties of the G-flow. We will see that diagrams can always be coded by a suitable logic action, but by working with diagrams instead of expansion classes, we will never need to worry about the model-theoretic details of the expansion.

Section 3.3 constructs the *level representation* of $S(G)$, which will allow us a detailed understanding of both its semigroup structure and its metrizable subspaces. We will see that the level representation is more or less a refinement of the topometric structure on $S(G)$. Section 3.5 gives a new proof of KPT correspondence and strengthens it to characterize when $M(G)$ is metrizable. Section 3.6 provides a counterexample to the converse of the Generic Point Theorem; we show using an example of Kwiatkowska [18] that there is an automorphism group G with $M(G)$ non-metrizable, but having a comeager orbit. Section 3.8 introduces the notion of a *completion flow* and discusses some connections between this object and finite big Ramsey degree.

3.1 Fraïssé classes and structures

A *relational language* $L = \{R_i : i \in I\}$ is a set of relational symbols; each symbol R_i comes with a finite arity n_i. All languages in this paper will be relational. Given a language L, an

L-structure $\mathbf{A} = \langle A, R_i^{\mathbf{A}} \rangle$ is a set A along with an interpretation $R_i^{\mathbf{A}} \subseteq A^{n_i}$ of each symbol in L. We will use boldface for structures and lightface for the underlying set unless otherwise specified. If $\mathbf{A}$ and $\mathbf{B}$ are L-structures, an *embedding* $f : \mathbf{A} \to \mathbf{B}$ is any injective map $f : A \to B$ so that for each $i \in I$ and each n_i-tuple $a_0, ..., a_{n_i-1} \in A$, we have

$$R_i^{\mathbf{A}}(a_0, ..., a_{n_i-1}) \Leftrightarrow R_i^{\mathbf{B}}(f(a_0), ..., f(a_{n_i-1})).$$

Write $\mathrm{Emb}(\mathbf{A}, \mathbf{B})$ for the set of embeddings from $\mathbf{A}$ to $\mathbf{B}$, and write $\mathbf{A} \leq \mathbf{B}$ if $\mathrm{Emb}(\mathbf{A}, \mathbf{B}) \neq \emptyset$. If $\mathbf{A} \leq \mathbf{B}$, we say that $\mathbf{B}$ *embeds* $\mathbf{A}$. If $A \subseteq B$, then we write $\mathbf{A} \subseteq \mathbf{B}$ if the inclusion map is an embedding. An *isomorphism* is a bijective embedding, and an *automorphism* is an isomorphism from a structure to itself. We write $\mathrm{Emb}(\mathbf{A})$ for $\mathrm{Emb}(\mathbf{A}, \mathbf{A})$, and we write $\mathrm{Aut}(\mathbf{A})$ for the group of automorphisms of $\mathbf{A}$. A structure is *finite* or *countable* if the underlying set is, and we write $|\mathbf{A}| := |A|$.

If $\mathbf{K}$ is a countable L-structure, we write $\mathrm{Age}(\mathbf{K}) := \{\mathbf{A} \leq \mathbf{K} : \mathbf{A} \text{ is finite}\}$. A countable structure $\mathbf{K}$ is called *ultrahomogeneous* or a *Fraïssé structure* if for any *partial isomorphism* of $\mathbf{K}$, i.e. for any finite $\mathbf{A} \subseteq \mathbf{K}$ and embedding $f : \mathbf{A} \to \mathbf{K}$, there is $g \in \mathrm{Aut}(\mathbf{K})$ with $g|_{\mathbf{A}} = f$. Two facts are worth pointing out, both due to Fraïssé [10]. First, if $\mathbf{K}$ is a Fraïssé structure, then $\mathcal{K} := \mathrm{Age}(\mathbf{K})$ is a *Fraïssé class*; this is any class of L-structures with the following four properties.

1. $\mathcal{K}$ contains only finite structures, contains structures of arbitrarily large finite cardinality, is closed under isomorphism, and contains only countably many isomorphism types of structures.

   Remark. We will implicitly assume that (1) holds of all classes of finite structures that we consider.

2. $\mathcal{K}$ has the *Hereditary Property* (HP): if $\mathbf{B} \in \mathcal{K}$ and $\mathbf{A} \subseteq \mathbf{B}$, then $\mathbf{A} \in \mathcal{K}$.

3. $\mathcal{K}$ has the *Joint Embedding Property* (JEP): if $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, then there is $\mathbf{C} \in \mathcal{K}$ which embeds both $\mathbf{A}$ and $\mathbf{B}$.

4. $\mathcal{K}$ has the *Amalgamation Property* (AP): if $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ are embeddings, there is $\mathbf{D} \in \mathcal{K}$ and embeddings $r : \mathbf{B} \to \mathbf{D}$ and $s : \mathbf{C} \to \mathbf{D}$ with $r \circ f = s \circ g$.

Second, if $\mathcal{K}$ is a Fraïssé class, there is up to isomorphism a unique Fraïssé structure $\mathbf{K}$ with $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$. We call $\mathbf{K}$ the *Fraïssé limit* of $\mathcal{K}$ and write $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$.

The following property is nominally weaker, but actually equivalent to being Fraïssé . If $\mathbf{K}$ is a countable L-structure with age $\mathcal{K}$, we say that $\mathbf{K}$ has the *extension property* if for any $\mathbf{A} \subseteq \mathbf{B} \in \mathcal{K}$ and any embedding $f : \mathbf{A} \to \mathbf{K}$, there is some $h : \mathbf{B} \to \mathbf{K}$ with $h|_A = f$.

Fact. $\mathbf{K}$ is a Fraïssé structure iff $\mathbf{K}$ has the extension property.

It will be useful to keep some examples of Fraïssé classes and structures in mind. Many more examples can be found in [16].

- If $\mathcal{K}$ is the class of finite sets, then $\mathrm{Flim}(\mathcal{K})$ is a countably infinite set.
- If $\mathcal{K}$ is the class of finite linear orders, then $\mathrm{Flim}(\mathcal{K}) = \langle \mathbb{Q}, < \rangle$, the rational linear order.
- If $\mathcal{K}$ is the class of finite graphs, then $\mathrm{Flim}(\mathcal{K})$ is the *Rado graph*. This is sometimes called the *random graph*; it is the graph obtained with probability one by taking a countable vertex set and determining each edge with a coin toss.
- If one allows languages with functions and constant symbols, we can view the countable atomless Boolean algebra as the Fraïssé limit of the class of finite Boolean algebras, and we can view the countable infinite-dimensional vector space over a finite field F as the Fraïssé limit of the finite-dimensional F-vector spaces. These examples can be formalized relationally by considering *Fraïssé-HP* classes, which we will develop below.

For $\mathbf{K}$ a Fraïssé structure, form $G = \mathrm{Aut}(\mathbf{K})$. We endow G with the topology of pointwise convergence; a basic open neighborhood of the identity is the collection $\{g \in G : g|_A = 1|_A\}$. Notice that these basic open neighborhoods actually form subgroups; hence each is clopen. Groups with this property are called *non-Archimedean*. Letting S_∞ be the Polish group of permutations of the countably infinite set K with this topology, we see that G is a closed subgroup of S_∞, so also Polish. It turns out (see [4]) that up to isomorphism, the non-Archimedean Polish groups are exactly the closed subgroups of S_∞.

Fact. Let G be a topological group. Then G is isomorphic to a closed subgroup of S_∞ iff there is some relational Fraïssé structure $\mathbf{K}$ with $G \cong \mathrm{Aut}(\mathbf{K})$.

To understand why this is true, suppose G is a closed subgroup of S_∞. For every finite tuple $\overline{a} \in K^{<\omega}$, introduce a relational symbol $R_{\overline{a}}$ of arity $|\overline{a}|$. Using this as our language L, we build the L-structure $\mathbf{K}$ by declaring that $R_{\overline{a}}^{\mathbf{K}}(\overline{b})$ holds iff for some $g \in G$ we have $g\overline{a} = \overline{b}$. Certainly $G \subseteq \mathrm{Aut}(\mathbf{K})$, and the fact that we assumed G closed in S_∞ gives us the reverse inclusion. Therefore using the terminology in this thesis, the non-Archimedean Polish groups are exactly what we are calling *automorphism groups*.

We now proceed to develop some notational conventions regarding a countably infinite structure $\mathbf{K}$ with $\mathrm{Age}(\mathbf{K}) = \mathcal{K}$. Write $\mathrm{Fin}(\mathbf{K}) = \{\mathbf{A} \subseteq \mathbf{K} : \mathbf{A} \in \mathcal{K}\}$. An *exhaustion* of $\mathbf{K}$ is a sequence $(\mathbf{A}_n)_{n<\omega}$ with $\mathbf{A}_0 = \emptyset$, $\mathbf{A}_n \in \mathrm{Fin}(\mathbf{K})$, $\mathbf{A}_n \subseteq \mathbf{A}_{n+1}$, and $\mathbf{K} = \bigcup_n \mathbf{A}_n$. When we write $\mathbf{K} = \bigcup_n \mathbf{A}_n$, we will assume that $(\mathbf{A}_n)_{n<\omega}$ is an exhaustion unless specified otherwise. When working with an exhaustion, it is helpful to pretend that the elements of the exhaustion are the only relevant finite substructures of $\mathbf{K}$. When dealing with Fraïssé classes, we formalize this by defining a *Fraïssé-HP* class (read "Fraïssé minus HP").

Definition 3.1.1. A class $\mathcal{K}$ is *Fraïssé-HP* if it satisfies every property of being a Fraïssé class except possibly the hereditary property. Write $\mathcal{K}{\downarrow}$ for the hereditary closure of $\mathcal{K}$.

If $\mathcal{K}$ is a Fraïssé-HP class, then there is up to isomorphism a unique structure $\mathbf{K} := \mathrm{Flim}(\mathcal{K})$ with age $\mathcal{K}{\downarrow}$ which is *$\mathcal{K}$-ultrahomogeneous*, namely that partial isomorphisms with domain in $\mathcal{K}$ extend to full automorphisms of $\mathbf{K}$. This is equivalent to $\mathbf{K}$ having age $\mathcal{K}{\downarrow}$ and satisfying the *$\mathcal{K}$-extension property*, the straighforward relativization of extension property to structures in $\mathcal{K}$.

As a convention, when we write "$\mathbf{K} = \bigcup_n \mathbf{A}_n$ is a Fraïssé structure," we will take this to mean that $\mathcal{K} = \{\mathbf{A} : \exists\, n\,(\mathbf{A} \cong \mathbf{A}_n)\}$ and $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ in the Fraïssé-HP sense.

3.2 Structural Ramsey theory

If X is a set, $r < \omega$, and Y is another set with $|Y| = r$, we will use the term *r-coloring* of X to refer to any function $\gamma : X \to Y$.

Definition 3.2.1.

1. Let $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$ be L-structures, and let $r < \omega$. We write

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_r$$

   if for every coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to r$, there is $h \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $|\{\gamma(h \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}| = 1$.

2. Let $\mathcal{C}$ be a class of finite relational structures, and let $\mathbf{A} \in \mathcal{C}$. We say that $\mathbf{A}$ is a *Ramsey object in* $\mathcal{C}$ if for every $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$ and every $r < \omega$, there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{B} \leq \mathbf{C}$ so that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_r$. If the class $\mathcal{C}$ is understood, we simply sat that $\mathbf{A} \in \mathcal{C}$ is a *Ramsey object*.

3. Let $\mathcal{C}$ be a class of finite relational structures. We say that $\mathcal{C}$ has the *Ramsey Property*, or simply *RP*, if every $\mathbf{A} \in \mathcal{C}$ is a Ramsey object.

A prototypical example of a class with the RP is the class $\mathcal{L}$ of finite linear orders. In fact, the statement that $\mathcal{L}$ has the RP is equivalent to the ordinary finite Ramsey theorem. Another example is the class of linearly ordered finite graphs, a result due to Nešetřil and Rödl [22]. Many other examples can be found in [16].

It will be helpful to think about the Ramsey Property by considering what happens on a countably infinite structure.

Definition 3.2.2. Suppose $\mathbf{D}$ is a countably infinite L-structure with $\mathrm{Age}(\mathbf{D}) = \mathcal{D}$, and fix $\mathbf{A} \in \mathcal{D}$. We call a subset $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ *thick* if for every $\mathbf{B} \in \mathcal{D}$ with $\mathbf{A} \leq \mathbf{B}$, there is some $h \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $h \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}) \subseteq T$.

The terminology is deliberate, as we will soon connect this to the notions of thick and near-thick subsets of a topological group.

The next proposition seeks to understand the RP from this more infinite perspective. The role of the cofinal subclass $\mathcal{C} \subseteq \mathcal{D}$ is to allow us to freely consider Fraïssé-HP subclasses of a given Fraïssé class.

Proposition 3.2.3. *Suppose* $\mathbf{D}$ *is a countably infinite* L*-structure,* $\mathcal{D} = \mathrm{Age}(\mathbf{D})$*, and* $\mathcal{C}$ *is cofinal in* $\mathcal{D}$*. Let* $\mathbf{A} \in \mathcal{C}$*. Then the following are equivalent:*

1. $\mathbf{A}$ *is a Ramsey object in* $\mathcal{C}$*,*
2. $\mathbf{A}$ *is a Ramsey object in* $\mathcal{D}$*,*
3. *For every* $\mathbf{B} \in \mathcal{D}$ *with* $\mathbf{A} \leq \mathbf{B}$ *and every* $r < \omega$*, we have* $\mathbf{D} \to (\mathbf{B})^{\mathbf{A}}_r$*.*
4. *For any* $r < \omega$ *and* r*-coloring* γ *of* $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$*, there is some color* $\gamma^{-1}(\{i\})$ *which is thick,*
5. *For any thick* $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$*, any* $r < \omega$*, and any* r*-coloring* γ *of* T*, there is some color* $\gamma^{-1}(\{i\})$ *which is thick.*

Proof. $(1 \Leftrightarrow 2)$ and $(5 \Rightarrow 4 \Rightarrow 3)$ are straightforward.

For $(2 \Rightarrow 5)$, fix thick $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ and $\gamma : T \to r$. Say $\mathbf{A} \leq \mathbf{B} \in \mathcal{D}$, and fix $\mathbf{C} \in \mathcal{D}$ for which $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_r$ holds. Since T is thick, find $f \in \mathrm{Emb}(\mathbf{C}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq T$. Then we can find $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $f \circ x \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}) \subseteq \gamma^{-1}(\{i\})$ for some $i < r$. It remains to show that we can choose the same $i < r$ for each $\mathbf{B}$. If this weren't possible, pick a bad $\mathbf{B}_i$ for each $i < r$. As $\mathcal{D}$ is the age of a relational structure, $\mathcal{D}$ has JEP, so find $\mathbf{B}' \in \mathcal{D}$ with $\mathbf{B}_i \leq \mathbf{B}'$ for each $i < r$. We have seen that for some $f \in \mathrm{Emb}(\mathbf{B}', \mathbf{D})$ and some $i < r$, we have $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}') \subseteq \gamma^{-1}(\{i\})$. But then $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B}_i) \subseteq \gamma^{-1}(\{i\})$, contradicting the choice of $\mathbf{B}_i$.

For $(3 \Rightarrow 2)$, let $\mathbf{D} = \bigcup_n \mathbf{B}_n$ be an exhaustion with $\mathbf{A} \leq \mathbf{B}_1$. Suppose $\mathbf{B} \in \mathcal{D}$ and $r < \omega$ witnessed the fact that $\mathbf{A}$ is not a Ramsey object. Call an r-coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_n)$ *bad* if there is no $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$ with $f \circ \mathrm{Emb}(\mathbf{A}, \mathbf{B})$ monochromatic. So for each n, there is a bad r-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_n)$. Note that if γ is a bad r-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_n)$ and $m \leq n$, the restriction of γ to $\mathrm{Emb}(\mathbf{A}, \mathbf{B}_m)$ is also bad. We can now use König's lemma to find a bad r-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$. $\square$

The following lemma shows that we can limit our attention to 2 colors. The proof is a straightforward "color-fusing" argument.

Lemma 3.2.4. *Let* $\mathcal{C}$ *be a class of finite relational structures, and fix* $\mathbf{A} \in \mathcal{C}$*. Then the following are equivalent.*

1. *$\mathbf{A}$ is a Ramsey object in $\mathcal{C}$.*

2. *For every $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$, there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{B} \leq \mathbf{C}$ and $\mathbf{C} \to (\mathbf{B})_2^{\mathbf{A}}$.*

In order to describe an important corollary of Lemma 3.2.4, we need to introduce a few definitions.

Definition 3.2.5. Let $\mathbf{D}$ be a countably infinite L-structure with $\mathrm{Age}(\mathbf{D}) = \mathcal{D}$. Fix $\mathbf{A} \in \mathcal{D}$.

1. $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *syndetic* if $\mathrm{Emb}(\mathbf{A}, \mathbf{D}) \setminus S$ is not thick.

2. If $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{D}) \to Y$ is a coloring, then γ is a *syndetic r-coloring* if $|\mathrm{Im}(\gamma)| = r$, and for each $y \in Y$, we have $\gamma^{-1}(\{y\})$ either empty or syndetic. We call γ a *syndetic coloring* if γ is a syndetic $|\mathrm{Im}(\gamma)|$-coloring.

Once again, the terminology is deliberate, and we will soon connect this notion of syndetic to the corresponding group-theoretic notions.

Corollary 3.2.6. *Let $\mathbf{D}$ be a countably infinite L-structure with $\mathrm{Age}(\mathbf{D}) = \mathcal{D}$. Fix $\mathbf{A} \in \mathcal{D}$. Then if $\mathbf{A}$ is not a Ramsey object, there is a syndetic 2-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$.*

Proof. Repeat the proof of ($3 \Rightarrow 2$) in Proposition 3.2.3, but using Lemma 3.2.4 to know that we can take $r = 2$. $\square$

It will be important to know when we can find a syndetic k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ for $k \geq 2$. To do this, we need to introduce the idea of Ramsey degree.

Definition 3.2.7.

1. Let $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$ be finite L-structures, and let $k \leq r < \omega$. We write

$$\mathbf{C} \to (\mathbf{B})_{r,k}^{\mathbf{A}}$$

   if for every coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \to r$, there is $h \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$ with $|\{\gamma(h \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})\}| \leq k$.

2. Let $\mathcal{C}$ be a class of finite relational structures, and fix $\mathbf{A} \in \mathcal{C}$ and $k < \omega$. We say that $\mathbf{A}$ has *Ramsey degree k in $\mathcal{C}$* if k is least so that for every $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$ and every r with $k \leq r < \omega$, there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{B} \leq \mathbf{C}$ so that $\mathbf{C} \to (\mathbf{B})_{r,k}^{\mathbf{A}}$. If $\mathcal{C}$ is understood, we simply say $\mathbf{A} \in \mathcal{C}$ has Ramsey degree k.

3. With $\mathcal{C}$ as above, we say that $\mathcal{C}$ has *finite Ramsey degrees* if every $\mathbf{A} \in \mathcal{C}$ has finite Ramsey degree.

Notice that $\mathbf{A}$ is a Ramsey object in $\mathcal{C}$ iff $\mathbf{A}$ has Ramsey degree 1 in $\mathcal{C}$.

There are many examples of classes with finite Ramsey degrees. For $\mathcal{K}$ the class of finite sets, the Ramsey degree of the set of size n is $n!$; this follows from the ordinary Ramsey theorem, keeping in mind the $n!$ permutations of the underlying set. For $\mathcal{K}$ the class of finite graphs, a graph on n vertices also has Ramsey degree $n!$; we will soon see that this follows from the Ramsey Property for the class $\mathcal{K}^*$ of ordered finite graphs, along with a correspondence between the classes $\mathcal{K}$ and $\mathcal{K}^*$.

Much as in Lemma 3.2.4, a similar "color-fusing" argument shows that to test for Ramsey degree k, it suffices to consider $k+1$ colors.

Lemma 3.2.8. *Let $\mathcal{C}$ be a class of finite relational structures, and fix $\mathbf{A} \in \mathcal{C}$ and $k < \omega$. Then the following are equivalent.*

1. *$\mathbf{A}$ has Ramsey degree $t \leq k$ in $\mathcal{C}$.*

2. *For every $\mathbf{B} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{B}$, there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{B} \leq \mathbf{C}$ and $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{k+1,k}$.*

We also have an analogue of Proposition 3.2.3. The proof is nearly identical.

Proposition 3.2.9. *Suppose $\mathbf{D}$ is a countably infinite L-structure, $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and $\mathcal{C}$ is cofinal in $\mathcal{D}$. Let $\mathbf{A} \in \mathcal{C}$ and fix $k < \omega$. Then the following are equivalent:*

1. *$\mathbf{A}$ has Ramsey degree $t \leq k$ in $\mathcal{C}$,*

2. *$\mathbf{A}$ has Ramsey degree $t \leq k$ in $\mathcal{D}$,*

3. *For any r with $k \leq r < \omega$ and r-coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$, some k colors from γ form a thick subset.*

4. *For any thick $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$, any r with $k \leq r < \omega$, and any r-coloring γ of T, some k colors from γ form a thick subset.*

By combining Proposition 3.2.9 and Lemma 3.2.8, we obtain the following important corollary.

Corollary 3.2.10. *Let $\mathbf{D}$ be a countably infinite L-structure with $\mathrm{Age}(\mathbf{D}) = \mathcal{D}$. Fix $\mathbf{A} \in \mathcal{D}$ and $k \leq \omega$. Then if $\mathbf{A}$ does not have Ramsey degree $t < k$ in $\mathcal{D}$, there is a syndetic k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$.*

Suppose $\mathcal{D}$ is a class of finite L-structures and $\mathbf{A} \leq \mathbf{B} \in \mathcal{D}$; is it possible to compare the Ramsey degrees? In general there is surprisingly little that can be said without adding some assumptions. In the case $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, we could say something meaningful if we could "push" syndetic colorings of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$ up to $\mathrm{Emb}(\mathbf{B}, \mathbf{D})$. However, the notion of largeness we can push up happens to be thick, not syndetic.

Proposition 3.2.11. *Suppose* $\mathbf{D}$ *is a countably infinite* L*-structure with* $\mathcal{D} = \mathrm{Age}(\mathbf{D})$*, and let* $\mathbf{A} \leq \mathbf{B} \in \mathcal{D}$*. Fix some* $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$*. If* $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ *is thick, then* $\{x \in \mathrm{Emb}(\mathbf{B}, \mathbf{D}) : x \circ f \in T\}$ *is thick.*

Proof. Fix $\mathbf{C} \in \mathcal{D}$ with $\mathbf{B} \leq \mathbf{C}$. Find $y \in \mathrm{Emb}(\mathbf{C}, \mathbf{D})$ with $y \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C}) \subseteq T$. Then for every $h \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$, we have $y \circ h \circ f \in T$. So $\{x \in \mathrm{Emb}(\mathbf{B}, \mathbf{D}) : x \circ f \in T\}$ is thick as desired. □

When does the analogue of Proposition 3.2.11 hold for syndetic sets? A sufficient condition is that $\mathcal{D}$ be a Fraïssé-HP class.

Proposition 3.2.12. *Let* $\mathbf{D}$ *is a countably infinite* L*-structure with* $\mathcal{D} = \mathrm{Age}(\mathbf{D})$*. Suppose* $\mathcal{C} \subseteq \mathcal{D}$ *is a cofinal Fraïssé-HP subclass, and let* $\mathbf{A} \leq \mathbf{B} \in \mathcal{C}$*. Fix some* $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$*. If* $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ *is syndetic, then* $S_f := \{x \in \mathrm{Emb}(\mathbf{B}, \mathbf{D}) : x \circ f \in S\}$ *is syndetic.*

Proof. As S is syndetic, there is $\mathbf{C} \in \mathcal{C}$ with $\mathbf{A} \leq \mathbf{C}$ so that for every $y \in \mathrm{Emb}(\mathbf{C}, \mathbf{D})$, we have $(y \circ \mathrm{Emb}(\mathbf{A}, \mathbf{C})) \cap S \neq \emptyset$. Repeatedly use the amalgamation property to find some $\mathbf{C}' \in \mathcal{C}$ and some $z \in \mathrm{Emb}(\mathbf{C}, \mathbf{C}')$ so that for every $j \in \mathrm{Emb}(\mathbf{A}, \mathbf{C})$, there is $h_j \in \mathrm{Emb}(\mathbf{B}, \mathbf{C}')$ with $h_j \circ f = z \circ j$. Now let $s \in \mathrm{Emb}(\mathbf{C}', \mathbf{D})$. Then for some $j \in \mathrm{Emb}(\mathbf{A}, \mathbf{C})$, we have $s \circ z \circ j \in S$. Since $s \circ z \circ j = s \circ h_j \circ f$, we have $s \circ h_j \in S_f$, so S_f is syndetic as desired. □

Remark. Note that Proposition 3.2.12 did not require that $\mathbf{D} = \mathrm{Flim}(\mathcal{C})$.

3.3 Dynamics of automorphism groups

Throughout this section, we fix a Fraïssé L-structure $\mathbf{K} = \bigcup_n \mathbf{A}_n$, and we set $G = \mathrm{Aut}(\mathbf{K})$. One of our first goals is to provide a more precise representation of $S(G)$. To that end, we establish some notational conventions.

Definition 3.3.1.

1. We set $H_m := \mathrm{Emb}(\mathbf{A}_m, \mathbf{K})$ and $H_m^n := \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)$. We sometimes write $H(m, n)$ to avoid too many subscripts and superscripts. Notice that $H_m = \bigcup_{n \geq m} H_m^n$. The group G acts on H_m on the left by post-composition.

2. We let $i_m \in H_m$ be the inclusion embedding.

3. If $g \in G$, we write $g|_m$ for the embedding $g|_{A_m} = g \cdot i_m \in H_m$. We let $\pi_m : G \to H_m$ be the map given by $\pi_m(g) = g|_m$. Note that π_m is G-equivariant in the sense that $\pi_m(gh) = g \cdot \pi_m(h)$.

4. We let $G_m := \{g \in G : g|_m = i_m\}$. Notice that $\{G_n : n < \omega\}$ forms a base of clopen subgroups at 1_G, so we can take $\mathcal{N}_G = \{G_n : n < \omega\}$. Using the map π_m, we can identify the members of H_m with the left cosets of G_m.

5. For $f \in H_m$, we define $\rho_f : G \to H_m$ via $\rho_f(g) = g \cdot f$. If $f \in H_m^n$, we define $\rho_f^n : H_n \to H_m$ given by $\rho_f(x) = x \circ f$. Notice that $\rho_{i_m} = \pi_m$, and we often write π_m^n for $\rho_{i_m}^n$. Also note that ρ_f and ρ_f^n are G-equivariant.

6. Viewing H_m as discrete, form the space of ultrafilters βH_n. For $f \in H_m^n$, we let $\tilde{\rho}_f^n : \beta H_n \to \beta H_m$ denote the continuous extension of ρ_f^n.

Lemma 3.3.2. *Let $f \in H_m$. Then $\rho_f : G \to H_m$ is uniformly continuous when H_m is given the discrete uniformity.*

Proof. Find $n \geq m$ with $f \in H_m^n$. If $g^{-1}h \in G_n$, it follows that $g \cdot f = h \cdot f$. $\square$

Lemma 3.3.2 allows us to continuously extend ρ_f to the map $\tilde{\rho}_f : S(G) \to \beta H_m$. When $f = i_m$, we often write $\tilde{\pi}_m$ for this extension, and we often write $\tilde{\pi}_m^n$ for $\tilde{\rho}_{i_m}^n$. If $p \in S(G)$, we often write $p \cdot f$ for $\tilde{\rho}_f(p)$, or just $p|_m$ in the case $f = i_m$.

If $n \geq m$, notice that $\pi_m = \pi_m^n \circ \pi_n$, so upon extending to $S(G)$, we obtain $\tilde{\pi}_m = \tilde{\pi}_m^n \circ \tilde{\pi}_n$. Form the inverse limit $\varprojlim \beta H_n$ along the maps $\tilde{\pi}_m^n$. Therefore we get a continuous surjection $\tilde{\pi} : S(G) \to \varprojlim \beta H_n$.

Proposition 3.3.3. *The map $\tilde{\pi} : S(G) \to \varprojlim \beta H_n$ is injective, hence a homeomorphism.*

Proof. Suppose $p \neq q \in S(G)$. Find $A \in p$ and $B \in q$ which are apart. So for some $m < \omega$, we have $AG_m \cap BG_m = \emptyset$. But then $\pi_m(AG_m) \cap \pi_m(BG_m) = \emptyset$, so $p|_m \neq q|_m$. $\square$

When G is an automorphism group, we will identify $S(G)$ and $\varprojlim \beta H_n$, often calling the latter the *level representation* of $S(G)$.

Remark. Consider the left-invariant metric on G given by $d(g, h) = 2^{-n}$ iff n is the largest number with $g|_n = h|_n$. Form the topometric space $(S(G), \partial)$. Then $\partial(p, q) = 2^{-n}$ iff n is the largest number with $p|_n = q|_n$.

How does the right G-action on $S(G)$ look when $S(G)$ is given its level representation? Suppose $m < \omega$ and $S \subseteq H_m$. Then for $p \in S(G)$, we have $\pi_m^{-1}(S) \in p$ iff $S \in p|_m$. Now fix $g \in G$, and let $n \geq m$ be large enough so that $g|_m \in H_m^n$. Then we have

$$\pi_m^{-1}(S)g^{-1} = \pi_n^{-1}(\{x \in H_n : x \circ g|_m \in S\}).$$

In particular, we have

$$\begin{aligned} S \in (pg)|_m &\Leftrightarrow S \in p \cdot g|_m \\ &\Leftrightarrow \{x \in H_n : x \circ g|_m \in S\} \in p|_n. \end{aligned}$$

To see at a glance why the above must be true, notice that it is true when $p \in G$, so it must also be true for $p \in S(G)$ by continuous extension.

In a similar fashion, we can understand the semigroup structure on $S(G)$ under the level representation. If $m < \omega$ and $p \in S(G)$, form the map $\lambda_p^m : H_m \to \beta H_m$ given by $\lambda_p^m(f) = p \cdot f$, and let $\tilde{\lambda}_p^m : \beta H_m \to \beta H_m$ be the continuous extension. If $r \in \beta H_m$, we often write $p \cdot r$ for $\tilde{\lambda}_p^m(r)$. Now if $S \subseteq H_m$, notice that

$$\begin{aligned} p^{-1}(\pi_m^{-1}(S)) &= \{g \in G : \pi_m^{-1}(S) \in pg\} \\ &= \pi_m^{-1}(\{f \in H_m : S \in p \cdot f\}). \end{aligned}$$

In particular, by defining $p^{-1}(S) := \{f \in H_m : S \in p \cdot f\}$, we obtain the familiar formula

$$\begin{aligned} S \in (pq)|_m &\Leftrightarrow S \in p \cdot q|_m \\ &\Leftrightarrow p^{-1}(S) \in q|_m. \end{aligned}$$

Again, this can be seen at a glance by seeing that it is true for $q \in G$, then considering the continuous extension. One corollary is worth pointing out explicitly.

Corollary 3.3.4. *Let $p, q \in S(G)$ and $m < \omega$. Then $(pq)|_m = p \cdot q|_m$.*

Our next task is to understand the notions of thick, syndetic, and piecewise syndetic from chapter 2. We begin by noticing two things. First, if $A \subseteq G$ and $m < \omega$, then $AG_m = \pi_m^{-1}(\pi_m(A))$. Second, if $A = \pi_m^{-1}(S)$ for some $S \subseteq H_m$, we have $AG_m = A$. The two conclusions that we can draw from this are first, that it suffices to work with subsets of H_m for various $m < \omega$, and second, that for sets of this form, disjointness and apartness coincide, so we won't need to deal with the "near" versions of various notions.

Definition 3.3.5. Let $m < \omega$, and fix $S \subseteq H_m$. Then S is *thick, syndetic,* or *piecewise syndetic* if $\pi_m^{-1}(S)$ has the corresponding property.

The careful reader will notice that now the word *thick* has two competing definitions, the other appearing in Definition 3.2.2. Let us show that they coincide.

Proposition 3.3.6. *Let $m < \omega$, and fix $T \subseteq H_m$. The following are equivalent.*

1. *T is thick as in Definition 3.3.5.*

2. *For every $n \geq m$, there is $x \in H_n$ with $x \circ H_m^n \subseteq T$, i.e. T is thick as in Definition 3.2.2.*

Proof. (1) $\Rightarrow$ (2). Assume T is thick, i.e. the collection $\{\pi_m^{-1}(T)g : g \in G\}$ has the FIP. Fix $n \geq m$. Find $g_0, ..., g_{k-1} \in G$ so that $\{g_i|_m : i < k\} = H_m^n$. Then $\bigcap_{i<k} \pi_m^{-1}(T)g_i^{-1} \neq \emptyset$. If $h \in G$ is in this intersection, then $hg_i|_m \in T$ for each $i < k$. But this means that $h|_n \circ H_m^n \subseteq T$.

(2) $\Rightarrow$ (1). Assume (2) holds, and fix $g_0, ..., g_{k-1} \in G$. Find some $n \geq m$ large enough so that $\{g_i^{-1}|_m : i < k\} \subseteq H_m^n$. Find $x \in H_n$ with $x \circ H_m^n \subseteq T$. Now if $g \in G$ with $g|_n = x$, it follows that $g \in \pi_m^{-1}(T)g_i$ for each $i < k$, so T is thick as desired. □

An immediate corollary of this is that the two competing notions of syndetic also coincide. It will be helpful to record this explicitly.

Corollary 3.3.7. *Let $m < \omega$, and fix $S \subseteq H_m$. Then the following are equivalent.*

1. *S is syndetic as in Definition 3.3.5.*
2. *There is $n \geq m$ so that for every $x \in H_n$, we have $(x \circ H_m^n) \cap S \neq \emptyset$.*

We now seek to understand how these largeness notions move between different levels. The following is strictly weaker than Propositions 3.2.11 and 3.2.12, but this proof is more illuminating.

Proposition 3.3.8. *Suppose $f \in H_m^n$, and let $T \subseteq H_m$. Then $T \subseteq H_m$ is thick iff $(\rho_f^n)^{-1}(T) \subseteq H_n$ is thick.*

Proof. Find $g \in G$ with $g|_m = f$. Noting that $\rho_f^n \circ \pi_n = \rho_f$, we now have $\pi_n^{-1}((\rho_f^n)^{-1}(T)) = \rho_f^{-1}(T) = \pi_m^{-1}(T)g^{-1}$, and the result immediately follows. □

Corollary 3.3.9. *Let $f \in H_m^n$, and let $\gamma : H_m \to k$ be a coloring. Then γ is a syndetic k-coloring iff $\gamma \circ \rho_f^n : H_n \to k$ is a syndetic k-coloring.*

Corollary 3.3.10. *Suppose $n \geq m$ and $\mathbf{A}_m$ has Ramsey degree k in $\mathcal{K}$. Then $\mathbf{A}_n$ does not have Ramsey degree $t < k$.*

3.4 Colorings, expansions, and diagrams

We fix a Fraïssé structure $\mathbf{K} = \bigcup_n \mathbf{A}_n$ with $G = \mathrm{Aut}(\mathbf{K})$. The goal of this section is to discuss two of the most common types of G-flows, namely *spaces of colorings* and *logic actions*. We will provide a common framework for discussing both types of flows, namely that of a *diagram*. However, we spend some extra time developing notions related to colorings, as these will be used throughout the remaining sections.

3.4.1 Colorings

We first consider the *space of k-colorings of H_m*. The left G-action on H_m gives rise to a right action on k-colorings. If $\gamma \in k^{H_m}$ and $g \in G$, we define $\gamma \cdot g$ by setting $\gamma \cdot g(f) = \gamma(g \cdot f)$. The evaluation $k^{H_m} \times G \to k^{H_m}$ is continuous, giving us a G-flow. Notice that k^{H_m} is homeomorphic to Cantor space, so in particular is metrizable.

Definition 3.4.1. Suppose $\gamma \in k^{H_m}$. If $n \geq m$ and $s \in H_n$, we let $\gamma \cdot s : H_m^n \to k$ be given by $\gamma \cdot s(f) = \gamma(s \cdot f)$. If $f \in H_m$, we often blur the distinction between $\gamma \cdot f$ and $\gamma(f)$. If $m \leq n \leq N$, $\gamma \in k^{H(m,N)}$, and $s \in H_n^N$, then $\gamma \cdot s$ is similarly defined.

We can think of Definition 3.4.1 as providing a "partial action" of H_n on k^{H_m}. Indeed, note that for $\gamma \in k^{H_m}$, $s \in H_n$, and $g \in G$, we have $\gamma \cdot (g \cdot s) = (\gamma \cdot g) \cdot s$. We freely extend Definition 3.4.1 to βH_n for $n \geq m$ by noting that $k^{H(m,n)}$ is finite, so if $p \in \beta H_n$, there is a unique coloring $\gamma \cdot p : H_m^n \to k$ so that $\{s \in H_n : \gamma \cdot s = \gamma \cdot p\} \in p$. In particular, when $p \in \beta H_m$, we again blur the distinction between coloring and color and let $\gamma \cdot p$ be the color $\tilde{\gamma}(p)$, where $\tilde{\gamma} : \beta H_m \to k$ is the continuous extension of γ.

Lemma 3.4.2. *Suppose $\gamma \in k^{H_m}$, $p \in S(G)$, and $n < \omega$. Then $(\gamma \cdot p) \cdot i_n = \gamma \cdot p|_n$.*

Proof. Fix $\psi \in k^{H(m,n)}$. Then we have

$$\begin{aligned} \gamma \cdot (p|_n) = \psi &\Leftrightarrow \{s \in H_n : \gamma \cdot s = \psi\} \in p|_n \\ &\Leftrightarrow (\gamma \cdot p) \cdot i_n = \psi. \end{aligned}$$

□

Proposition 3.4.3. *Suppose $\gamma \in k^{H_m}$, $p \in S(G)$, and $r \in \beta H_n$. Then $\gamma \cdot (p \cdot r) = (\gamma \cdot p) \cdot r$*

Proof. Find some $q \in S(G)$ with $q|_n = r$. Then using Lemma 3.4.2 and Corollary 3.3.4, we have

$$\begin{aligned} (\gamma \cdot p) \cdot r &= ((\gamma \cdot p) \cdot q) \cdot i_n \\ &= (\gamma \cdot (pq)) \cdot i_n \\ &= \gamma \cdot (p \cdot r). \end{aligned}$$

□

We will often want to work with multiple spaces of colorings at the same time. The next definition introduces some useful ways of comparing different colorings.

Definition 3.4.4.

1. Let $m < \omega$, and suppose γ and δ are colorings of H_m. We say that δ *refines* γ and write $\gamma \leq \delta$ if whenever $f, j \in H_m$ with $\delta \cdot f = \delta \cdot j$, we have $\gamma \cdot f = \gamma \cdot j$. We write $\gamma \sim \delta$ for the associated equivalence relation.

2. Suppose $m \leq n < \omega$, and suppose γ is a coloring of H_m and δ is a coloring of H_n. We say that γ and δ are *coherent* and write $\gamma \preceq \delta$ if for every $f \in H_m^n$, we have $\gamma \circ \rho_f^n \leq \delta$. Equivalently, $\gamma \preceq \delta$ if whenever $s, t \in H_n$ and $\delta \cdot s = \delta \cdot t$, then $\gamma \cdot s = \gamma \cdot t$.

3. If γ_n is a coloring of H_n for each $n < r \leq \omega$, we call the sequence $(\gamma_n)_{n<r}$ *coherent* if for every $m \leq n < r$ we have $\gamma_m \preceq \gamma_n$.

Proposition 3.4.5. *Suppose $m \leq n < \omega$, and suppose $\gamma \preceq \delta$ are colorings of H_m and H_n, respectively. Then if $p \in S(G)$, we have $\gamma \cdot p \preceq \delta \cdot p$. Furthermore, suppose $|\mathrm{Im}(\gamma)| = k$ and $|\mathrm{Im}(\delta)| = \ell$. Then if $|\mathrm{Im}(\delta \cdot p)| = \ell$, then $|\mathrm{Im}(\gamma \cdot p)| = k$.*

Proof. Fix $f \in H_m^n$. Suppose $s, t \in H_n$ satisfy $\delta \cdot p(s) = \delta \cdot p(t)$. Let $(g_i)_{i \in I}$ be a net from G with $g_i \to p$. Then eventually $\gamma \cdot g_i(s \circ f) = \gamma \cdot p(s \circ f)$, $\gamma \cdot g_i(t \circ f) = \gamma \cdot p(t \circ f)$, $\delta \cdot g_i(s) = \delta \cdot p(s)$, and $\delta \cdot g_i(t) = \delta \cdot p(t)$. As $\gamma \cdot g_i \preceq \delta \cdot g_i$, we have $\gamma \cdot g_i(s \circ f) = \gamma \cdot g_i(t \circ f)$. So the same is true of $\gamma \cdot p$.

For the second claim, fix $d < k$. Find $s \in H_n$ so that $\gamma(s \circ i_m) = d$. Letting $\delta(s) = j$, we have that whenever $t \in H_n$ with $\delta(t) = j$, we have $\delta(t \circ i_m) = d$. Now suppose $\delta \cdot p(t) = j$. Letting $(g_i)_{i \in I}$ be a net from G with $g_i \to p$, we have that eventually $\delta(g_i \cdot t) = j$. So eventually $\gamma(g_i \cdot t \cdot i_m) = d$. Hence $\gamma \cdot p(t \cdot i_m) = d$. $\square$

We will want an especially detailed understanding of the dynamics of syndetic colorings and how they interact with the Ramsey degree.

Proposition 3.4.6. *Let $\gamma \in k^{H_m}$, and fix $t \leq k$. Then the following are equivalent.*

1. *γ is a syndetic t-coloring.*

2. *Every $\delta \in \overline{\gamma \cdot G} = \gamma \cdot S(G)$ has $|\mathrm{Im}(\delta)| = t$.*

3. *Every $\delta \in \overline{\gamma \cdot G} = \gamma \cdot S(G)$ is a syndetic t-coloring.*

Proof. (3) $\Rightarrow$ (2) is clear. To see (2) $\Rightarrow$ (1), suppose there were $i \in \mathrm{Im}(\gamma)$ with $\gamma^{-1}(\{i\})$ not syndetic. For each $n \geq m$, find $x_n \in H_n$ with $(x_n \circ \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)) \cap \gamma^{-1}(\{i\}) = \emptyset$. Find $g_n \in G$ with $g_n|_n = x_n$. By passing to a subsequence, we may assume $\gamma \cdot g_n \to \delta$. But then $i \notin \mathrm{Im}(\delta)$.

For (1) $\Rightarrow$ (3), suppose $(g_N)_{N<\omega}$ is a sequence from G with $\gamma \cdot g_N \to \delta$. Fix $i \in \mathrm{Im}(\gamma)$. We will show that $\delta^{-1}(\{i\})$ is syndetic. As $\gamma^{-1}(\{i\})$ is syndetic, there is $n \geq m$ so that $(x \circ \mathrm{Emb}(\mathbf{A}_m, \mathbf{A}_n)) \cap \gamma^{-1}(\{i\}) \neq \emptyset$ for every $x \in H_n$. Note that the same is true of $(\gamma \cdot g_N)^{-1}(\{i\})$. Therefore the same is true of $\delta^{-1}(\{i\})$. $\square$

Corollary 3.4.7. *Suppose $\gamma \in k^{H_m}$ is a coloring with $\gamma \cdot S(G)$ a minimal G-flow. Then γ is a syndetic coloring.*

Proof. If γ were not a syndetic coloring, find $p \in S(G)$ with $|\mathrm{Im}(\gamma \cdot p)| < |\mathrm{Im}(\gamma)|$. But then $\gamma \notin \gamma \cdot p \cdot S(G)$, so $\gamma \cdot S(G)$ cannot be minimal. $\square$

Corollary 3.4.8. *Suppose $\mathbf{A}_m$ has Ramsey degree k in $\mathcal{K}$, and let γ be a coloring of H_m. Then there is $p \in S(G)$ with $\gamma \cdot p$ a syndetic t-coloring for some $t \leq k$.*

Proof. Let $p \in S(G)$ belong to some minimal subflow. Then $\gamma \cdot p \cdot S(G)$ is a minimal flow. It follows that $\gamma \cdot p$ is a syndetic coloring. As $\mathbf{A}_m$ has Ramsey degree k, we must have $|\mathrm{Im}(\gamma \cdot p)| \leq k$. $\square$

Proposition 3.4.9.

1. *Suppose $\gamma \leq \delta$ are colorings of H_m, respectively. If δ is syndetic, then so is γ.*

2. *Suppose $m \leq n < \omega$, and let $\gamma \preceq \delta$ be colorings of H_m and H_n, respectively. If δ is a syndetic coloring, then so is γ.*

Proof. Item (1) is immediate. For item (2), notice that δ refines the coloring $\gamma \circ \rho^n_{i_m}$, so by item (1), $\gamma \circ \rho^n_{i_m}$ is a syndetic coloring. Hence so is γ by Corollary 3.3.9. $\square$

Proposition 3.4.10. *Suppose $\mathbf{A}_m$ has Ramsey degree k in $\mathcal{K}$. If $\gamma, \delta : H_m \to k$ are both syndetic k-colorings, then there is $p \in S(G)$ with $\gamma \cdot p \sim \delta \cdot p$.*

Proof. Let $\gamma \times \delta : H_m \to k \times k$ be the product coloring. Notice that $\gamma \times \delta$ refines both γ and δ. Using Corollary 3.4.8, find $p \in S(G)$ with $(\gamma \times \delta) \cdot p = \gamma \cdot p \times \delta \cdot p$ a syndetic t-coloring for some $t \leq k$. But $(\gamma \cdot \delta) \times p$ refines $\gamma \cdot p$ and $\delta \cdot p$, each of which is a syndetic k-coloring by Proposition 3.4.6. Therefore $(\gamma \cdot \delta) \times p$ is also a syndetic k-coloring, so we must have $\gamma \cdot p \sim (\gamma \times \delta) \cdot p \sim \delta \cdot p$. $\square$

3.4.2 Logic actions and diagrams

We now turn towards a more general type of G-flow, the *logic action*. Suppose $L^* \supseteq L$ is a relational language. If $\mathbf{A}^*$ is an L^*-structure, we write $\mathbf{A}^*|_L$, the *reduct* of $\mathbf{A}^*$, for the structure obtained from $\mathbf{A}^*$ by forgetting the interpretations of the relations in $L^* \setminus L$. Conversely, if $\mathbf{A}$ is an L-structure, an *expansion* of $\mathbf{A}$ is any L^*-structure $\mathbf{A}^*$ with reduct $\mathbf{A}$. The following definition provides the relevant notion of "partial action."

Definition 3.4.11. Suppose $\mathbf{A}$ is an L-structure, $\mathbf{B}^*$ is an L^*-structure, and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B}^*|_L)$. Then $\mathbf{B}^* \cdot f$ is the unique expansion of $\mathbf{A}$ so that $f \in \mathrm{Emb}(\mathbf{A}^*, \mathbf{B}^*)$.

We call a class $\mathcal{K}^*$ of finite L^*-structures an *expansion* of $\mathcal{K}$ if we have $\mathcal{K} = \{\mathbf{A}^*|_L : \mathbf{A}^* \in \mathcal{K}^*\}$. If $\mathbf{A} \in \mathcal{K}$, we let $\mathcal{K}^*(\mathbf{A})$ be the set of expansions of $\mathbf{A}$ in $\mathcal{K}^*$. We will from here on out always assume that expansion classes satisfy the following two properties.

Definition 3.4.12. Fix an expansion $\mathcal{K}^*$ of $\mathcal{K}$.

1. $(\mathcal{K}^*, \mathcal{K})$ has the *$\mathcal{K}$-Hereditary Property* or just *$\mathcal{K}$-HP*, if whenever $\mathbf{A} \in \mathcal{K}$, $\mathbf{B}^* \in \mathcal{K}^*$, and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B}^*|_L)$, we have $\mathbf{B}^* \cdot f \in \mathcal{K}^*$.

2. $(\mathcal{K}^*, \mathcal{K})$ is *reasonable* if whenever $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$, and $\mathbf{A}^* \in \mathcal{K}^*(\mathbf{A})$, then there is some $\mathbf{B}^* \in \mathcal{K}^*(\mathbf{B})$ with $\mathbf{A}^* = \mathbf{B}^* \cdot f$.

Definition 3.4.13. Let $\mathcal{K}^*$ be an expansion of $\mathcal{K}$. Then $X_{\mathcal{K}^*}$ is the collection of L^*-structures x with underlying set K so that $x|_L = \mathbf{K}$ and $\mathrm{Age}(x) \subseteq \mathcal{K}^*\!\downarrow$.

Remark.

1. We will often not use boldface when referring to structures from a space of structures.

2. When $(\mathcal{K}^*, \mathcal{K})$ is reasonable, then for any $\mathbf{A}_n^* \in \mathcal{K}^*(\mathbf{A}_n)$, there is some $x \in X_{\mathcal{K}^*}$ with $x|_{A_n} = \mathbf{A}_n^*$

We topologize $X_{\mathcal{K}^*}$ by declaring a basic open neighborhood of structures to be those of the form $\{x \in X_{\mathcal{K}^*} : x \cdot i_n = x|_{A_n} = \mathbf{A}_n^*\}$ for some expansion $\mathbf{A}_n^* \in \mathcal{K}^*(\mathbf{A}_n)$. The group G acts on $X_{\mathcal{K}^*}$ on the right via Definition 3.4.11. More explicitly, if $R \in L^* \setminus L$ is an n-ary relational symbol, $x \in X_{\mathcal{K}^*}$, $g \in G$, and $a_0, ..., a_{n-1} \in K$, we set

$$R^{x \cdot g}(a_0, ..., a_{n-1}) \Leftrightarrow R^x(ga_0, ..., ga_{n-1}).$$

This action is continuous, so $X_{\mathcal{K}^*}$ is a G-flow if it is compact. A sufficient condition for compactness is if $\mathcal{K}^*$ only contains finitely many expansions of each $\mathbf{A}_n$; in this case we say that the expansion $(\mathcal{K}^*, \mathcal{K})$ is *pre-compact*.

The similarity between spaces of colorings and logic actions is that both have a notion of “partial action” given by Definitions 3.4.1 and 3.4.11, respectively. We turn towards the definition of a *diagram*, an abstract notion of partial action, and use diagrams to create G-flows. We will see that every space of colorings and every logic action can be coded as a diagram action. As it turns out, the logic action framework is perfectly general; every G-flow arising from a diagram can be coded as a logic action, but the diagram provides exactly the information we need and nothing more.

Definition 3.4.14.

1. Suppose $m \leq n < \omega$. An *(m,n)-diagram* is a map $D : J_n \times H_m^n \to J_m$, where J_n and J_m are finite sets and so that for every $f \in H_m^n$, the map $D(-, f) : J_n \to J_m$ is surjective. When D is understood, $y \in J_n$, and $f \in H_m^n$, we often write $y \cdot f$ for $D(y, f)$. If $x \in J_m$ and $y \in J_n$, we write $\mathrm{Emb}(x, y) = \{f \in H_m^n : y \cdot f = x\}$.

2. Let $D_J : J_n \times H_m^n \to J_m$ and $D_I : I_n \times H_m^n \to I_m$ be (m,n)-diagrams. An *isomorphism of (m,n)-diagrams*, written $\sigma : D_J \Rightarrow D_I$, is a pair $\sigma := (\sigma_m, \sigma_n)$ of bijections $\sigma_m : J_m \to I_m$ and $\sigma_n : J_n \to I_n$ so that the following commutes.

$$\begin{tikzcd} J_n \times H_m^n \arrow[r, "\sigma_n \times 1"] \arrow[d, "D_J"'] & I_n \times H_m^n \arrow[d, "D_I"] \\ J_m \arrow[r, "\sigma_m"] & I_m \end{tikzcd}$$

3. Let $r \leq \omega$. An *r-diagram based on* $\{J_n : n < r\}$ is a collection $D = \{D(m,n) : m \leq n < r\}$ satisfying the following properties.

   (a) Each J_n is a finite set so that for every $m \leq n < r$, $D(m,n) : J_n \times H_m^n \to J_m$ is an (m,n)-diagram. Furthermore, $|J_0| = 1$.

   (b) If $m \leq n \leq N < r$, $f \in H_m^n$, $s \in H_n^N$, and $j \in J_N$, then $j \cdot (s \circ f) = (j \cdot s) \cdot f$.

4. Let $D_J = \{D_J(m,n) : m \leq n < r\}$ and $D_I = \{D_I(m,n) : m \leq n < r\}$ be r-diagrams based on $\{J_n : n < r\}$ and $\{I_n : n < r\}$, respectively. An *isomorphism of r-diagrams* $\sigma : D_J \Rightarrow D_I$ is a tuple $\sigma := \{\sigma_n : n < r\}$ so that for every $m \leq n < r$, $(\sigma_m, \sigma_n) : D_J(m,n) \Rightarrow D_I(m,n)$ is an isomorphism.

5. If $r \leq \omega$, $N \leq r$, and $D := \{D(m,n) : m \leq n < r\}$ is an r-diagram, then the *restriction* of D to N is the N-diagram $D|_N := \{D(m,n) : m \leq n < N\}$.

Example 3.4.15.

1. Suppose $r < \omega$, and let $(\gamma_n)_{n<r}$ be coherent colorings of H_n for $n < r$. For $m \leq n < r$, the *diagram* of $\gamma_m \preceq \gamma_n$ is the map $D(\gamma_m, \gamma_n) : \mathrm{Im}(\gamma_n) \times H_m^n \to \mathrm{Im}(\gamma_m)$ where given $f \in H_m^n$ and $j \in \mathrm{Im}(\gamma_n)$, we set $D(\gamma_m, \gamma_n)(j, f) = i$ if for any $s \in H_n$ with $\gamma_n(s) = j$, we have $\gamma_m(s \circ f) = i$. The *diagram* of $(\gamma_n)_{n<r}$ is the collection $D_\gamma := \{D(\gamma_m, \gamma_n) : m \leq n < r\}$

2. If $\mathcal{K}'$ is an expansion of $\mathcal{K}$ and $m \leq n < \omega$, then the diagram $D_{\mathcal{K}'}(m,n) : \mathcal{K}'(\mathbf{A}_n) \times H_m^n \to \mathcal{K}'(\mathbf{A}_m)$ is given by $D_{\mathcal{K}'}(m,n)(\mathbf{A}'_n, f) = \mathbf{A}'_n \cdot f$. We form an ω-diagram by setting $D_{\mathcal{K}'} = \{D_{\mathcal{K}'}(m,n) : m \leq n < \omega\}$. Keep in mind our convention that expansion classes always are reasonable and have the $\mathcal{K}$-HP.

Definition 3.4.16. Suppose $D = \{D(m,n) : m \leq n < \omega\}$ is an ω-diagram based on $\{J_n : n < \omega\}$.

1. We set $X_D = \{p \in \prod_n J_n : p(n) \cdot i_m = p(m) \text{ for } m \leq n < \omega\}$. We often write $p|_n$ for $p(n)$. This is a closed subspace of the product $\prod_n J_n$.
2. Given $p \in X_D$ and $f \in H_m^n$, we write $p \cdot f$ for $p|_n \cdot f$. If $y \in J_m$, we write $\mathrm{Emb}(y,p) := \{f \in H_m : p \cdot f = y\}$.
3. If $p \in X_D$ and $g \in G$, we define $p \cdot g \in X_D$ via $p \cdot g(m) = p \cdot g|_m$

Item 3(b) in Definition 3.4.14 is exactly what is needed to see that the action in (3) above is well defined. In particular, for any $s \in H_n$ and $f \in H_m^n$, we have $p \cdot (s \cdot f) = (p \cdot s) \cdot f$. The action is continuous, turning X_D into a G-flow. We freely extend the partial right actions to βH_m in the obvious way. In particular, the analogue of Proposition 3.4.3 holds in this more general context with an identical proof.

Proposition 3.4.17. *Suppose D_I and D_J are ω-diagrams based on $\{I_n : n < \omega\}$ and $\{J_n : n < \omega\}$, respectively. If D_I and D_J are isomorphic, then $X_{D_I} \cong X_{D_J}$.*

Proof. Let $\sigma : D_J \Rightarrow D_I$ be an isomorphism of diagrams. We define $\psi_\sigma : X_{D_J} \to X_{D_I}$ by setting $\psi_\sigma(p)|_n = \sigma_n(p|_n)$. As σ is an isomorphism of diagrams, this is well defined, and therefore is a G-flow isomorphism. □

We spend the remainder of the section showing that diagrams do nothing more than provide a flexible way for building logic actions.

Proposition 3.4.18. *Let $\mathcal{K}'$ be an expansion of $\mathcal{K}$, and form the diagram $D_{\mathcal{K}'}$. Then $X_{D_{\mathcal{K}'}} \cong X_{\mathcal{K}'}$.*

Proof. Given $x \in X_{D_{\mathcal{K}'}}$, we define $\varphi(x) \in X_{\mathcal{K}'}$ by setting $\varphi(x) = \bigcup_n x|_n$. This is a bijective G-map. □

Proposition 3.4.19. *Let D be an ω-diagram based on $\{J_n : n < \omega\}$. Then there is a language $L' \supseteq L$ and an L'-expansion $\mathcal{K}'$ of $\mathcal{K}$ so that $D \cong D_{\mathcal{K}'}$.*

Proof. Let $r_n = |A_n|$, and fix an enumeration of K so that $A_n = \{a_0, ..., a_{r_n - 1}\}$. For each $y \in J_n$, let R_y be a new r_n-ary relation symbol. Set $L' = L \cup \{R_y : y \in \bigcup_n J_n\}$. Now for each $y \in J_n$, we define $\mathbf{A}^y$ an L'-expansion of $\mathbf{A}_n$ as follows. For every $m \leq n$ and embedding $f \in H_m^n$, we declare that $R_{y \cdot f}^{\mathbf{A}^y}(f(a_0), ..., f(a_{r_m - 1}))$ holds. We then set $\mathcal{K}' = \{\mathbf{A}' : \exists y \in \bigcup_n J_n\, (\mathbf{A}' \cong \mathbf{A}^y)\}$.

Let $\sigma_n : J_n \to \mathcal{K}'(\mathbf{A}_n)$ be the map $\sigma_n(y) = \mathbf{A}^y$. To verify that $\sigma = (\sigma_n)_{n<\omega}$ is an isomorphism from D to $D_{\mathcal{K}'}$, suppose $y \in J_N$ and $s \in H_n^N$. We need to show that $\mathbf{A}^{y \cdot s} = \mathbf{A}^y \cdot s$.

So let $m \leq n$. If $\vec{b}$ is a tuple of length r_m from A_n and $z \in J_m$ so that $R_z^{\mathbf{A}^{y\cdot s}}(\vec{b})$ holds, this means that $\vec{b}$ enumerates some embedding $f \in H_m^n$ and $z = (y \cdot s) \cdot f$. On the other hand, $R_z^{\mathbf{A}^y \cdot s}(\vec{b})$ holds iff $R_z^{\mathbf{A}^y}(s(\vec{b}))$ holds. This means that $s(\vec{b})$ enumerates some $h \in H_m^N$ and $z = y \cdot h$. By letting $h = s \cdot f$, we see that $R_z^{\mathbf{A}^{y\cdot s}}(\vec{b})$ implies $R_z^{\mathbf{A}^y \cdot s}(\vec{b})$. However, if $s(\vec{b})$ enumerates $h \in H_m^N$, then it must be the case that $\vec{b}$ enumerates some $f \in H_m^n$ with $h = s \circ f$. So $R_z^{\mathbf{A}^y \cdot s}(\vec{b})$ implies $R_z^{\mathbf{A}^{y\cdot s}}(\vec{b})$, and $\mathbf{A}^{y\cdot s} = \mathbf{A}^y \cdot s$.

Lastly, we check that $\mathcal{K}'$ has the $\mathcal{K}$-HP and is reasonable. The equality $\mathbf{A}^{y\cdot s} = \mathbf{A}^y \cdot s$ verifies the $\mathcal{K}$-HP, and the fact that the maps $D(m,n)(-,f)$ are surjective shows that $\mathcal{K}'$ is reasonable. □

We often write $\mathcal{K}^D$ for the expansion of $\mathcal{K}$ constructed from an ω-diagram D as in Proposition 3.4.19. Using this construction, we can take notions about expansion classes and rephrase all of them in terms of diagrams. In particular, we can say that an ω-diagram D has the JEP, AP, Ramsey Property, etc. iff the class $\mathcal{K}^D$ does. It will be helpful to write down what this means in terms of D itself. For the next few items, fix an ω-diagram D based on $\{J_n : n < \omega\}$.

- D has the JEP if for any $x \in J_m$ and $y \in J_n$, there are $N \geq n$, $z \in J_N$, $f \in H_m^N$ and $s \in H_n^N$ with $z \cdot f = x$ and $z \cdot s = y$.

- D has the AP if for any $x \in J_m$, any $n, \ell \geq m$, and any $y_0 \in J_n$, $y_1 \in J_\ell$, $f_0 \in H_m^n$, and $f_1 \in H_m^\ell$ with $y_0 \cdot f_0 = y_1 \cdot f_1 = x$, there are $N \geq m$, $z \in J_N$, $s_0 \in H_n^N$, and $s_1 \in H_\ell^N$ with $z \cdot s_0 = y_0$, $z \cdot s_1 = y_1$, and $s_0 \circ f_0 = s_1 \circ f_1$.

- D has the Ramsey Property if for any $x \in J_m$ and $y \in J_n$ with $\mathrm{Emb}(x,y) \neq \emptyset$, there is $N \geq n$ and $z \in J_N$ with $\mathrm{Emb}(y,z) \neq \emptyset$ so that $z \to (y)_2^x$.

3.5 Strengthenings of KPT correspondence

We continue with the notational conventions established at the beginning of section 3.3. So let $G = \mathrm{Aut}(\mathbf{K})$, where $\mathbf{K} = \bigcup_n \mathbf{A}_n$ is a Fraïssé L-structure. We proceed to extend some notation from section 2. If $S \subseteq H_m$, we let $C_S = \{p \in S(G) : S \in p|_m\}$. Notice that each C_S is clopen, and sets of this form form a basis for $S(G)$. Combining Propositions 2.3.6 and 3.3.6, we immediately obtain the following.

Corollary 3.5.1. *Let $m < \omega$, and fix $T \subseteq H_m$. Then T is thick iff $C_T \subseteq S(G)$ contains a minimal subflow of $S(G)$.*

Combining Propositions 3.2.3 and 2.3.9, we recover one of the major theorems from [16].

Theorem 3.5.2 (Theorem 4.7 from [16]). *G is extremely amenable iff $\mathcal{K}$ has the Ramsey Property.*

Remark. The statement of Theorem 4.7 from [16] uses a slightly different notion of Ramsey Property involving substructures instead of embeddings, so must also demand that the structures in $\mathcal{K}$ are rigid, i.e. have no non-trivial automorphisms. Our notion of Ramsey Property is an "embedding" notion, so is equivalent to "substructure" Ramsey plus rigid structures.

Proof. If G is extremely amenable, then by Proposition 2.3.9, the collection of near-thick subsets of G is partition regular. If $T \subseteq H_m$ is thick and $T = T_0 \cup T_1$, then one of $\pi_m^{-1}(T_0)$ or $\pi_m^{-1}(T_1)$ must be near-thick, hence thick, so either T_0 or T_1 is thick. This shows that $\mathbf{A}_m$ is a Ramsey object by Proposition 3.2.3.

Conversely, suppose $\mathcal{K}$ has the Ramsey Property. Fix $T \subseteq G$ a near-thick subset, and write $T = T_0 \cup T_1$. To show that one of T_0 or T_1 is near-thick, fix $m < \omega$. Then $\pi_m(T)$ is thick, and $\pi_m(T) = \pi_m(T_0) \cup \pi_m(T_1)$. By the Ramsey Property and Proposition 3.2.3, some $\pi_m(T_i)$ is thick. Since this is true for each $m < \omega$, for some $i \in \{0, 1\}$ we have $\pi_m(T_i)$ thick for cofinally many $m < \omega$, hence for all $m < \omega$. Hence the near thick subsets of G are partition regular, so G is extremely amenable. $\square$

The major goal for the rest of this section is to similarly connect metrizability of $M(G)$ with the phenomenon of $\mathcal{K}$ having finite Ramsey degrees.

We first need to understand what metrizable subspaces of $S(G)$ can look like. If $Y \subseteq S(G)$ is compact, we often write $Y_m = \tilde{\pi}_m``(Y) \subseteq \beta H_m$. note that $Y = \varprojlim Y_m$. Now if $Y \subseteq S(G)$ is compact metrizable, then so is Y_m. Conversely, if each Y_m is metrizable, then so is $Y \cong \varprojlim Y_m$. So it suffices to understand compact metrizable subspaces of βH_m. The following folklore theorem can be found in [14], but we have already proven a stronger theorem in Theorem 1.4.8.

Proposition 3.5.3. *Suppose X is a set and $Y \subseteq \beta X$ is compact metrizable. Then Y is finite.*

Proof. Endow X with the discrete metric, i.e. $d(x, y) = 1$ whenever $x \neq y$. Then we have $\beta X = S(X)$ with respect to this metric. If ∂ is the discrete metric on βX, then $(S(X), \partial)$ is the topometric Samuel compactification of X, and we appeal to Theorem 1.4.8. $\square$

We can now prove one of the main theorems from [34].

Lemma 3.5.4. *Suppose $Y \subseteq S(G)$ is a subflow, and let $p \in Y$. Then if $f \in H_m$, we have $p \cdot f \in Y_m$.*

Proof. Find $g \in G$ with $g|_m = f$. Then $p \cdot g \in Y$, so $\tilde{\pi}_m(p \cdot g) = p \cdot f \in Y_m$. $\square$

Theorem 3.5.5. *$M(G)$ is metrizable iff $\mathcal{K}$ has finite Ramsey degrees. In particular, if $Y \subseteq S(G)$ is a minimal, metrizable subflow, then $\mathbf{A}_m$ has Ramsey degree $|Y_m|$ in $\mathcal{K}$.*

Proof. First assume Y is metrizable, and fix $p \in Y$. Form the coloring $\lambda_p^m : H_m \to Y_m$ given by $\lambda_p^m(f) = p \cdot f$. Since Y is a minimal subflow, we have by Proposition 2.3.12 that λ_p^m is a syndetic $|Y_m|$-coloring.

Suppose $\gamma : H_m \to k$ is a syndetic k-coloring; we want to show that $k \leq |Y_m|$. Let $\lambda_\gamma : S(G) \to X$ be the map of ambits. By Proposition 3.4.3, we have $\gamma \cdot (p \cdot f) = (\gamma \cdot p) \cdot f$. It follows that the coloring λ_p^m refines the coloring $\gamma \cdot p$. As $\gamma \cdot p$ is a syndetic k-coloring by Proposition 3.4.6, we have $k \leq |Y_m|$ as desired.

In the other direction, suppose $Y \subseteq S(G)$ is a minimal subflow which is not metrizable. Then Y_m is infinite for some $m < \omega$. If $k < \omega$, let $\psi_k : Y_m \to k$ be a partition of Y_m into k clopen pieces. Then if $p \in Y$, the coloring $\psi_k \circ \lambda_p^m$ is a syndetic k-coloring. As k was arbitrary, $\mathbf{A}_m$ cannot have finite Ramsey degree in $\mathcal{K}$. $\square$

When $M(G)$ is metrizable, we can represent $M(G)$ as X_D for a suitable diagram.

Proposition 3.5.6. *Suppose $Y \subseteq S(G)$ is a minimal, metrizable subflow. Then letting $D_Y(m,n) : Y_n \times H_m^n \to Y_m$ be given by $D(m,n)(p,f) = p \cdot f$, we have that $D_Y = \{D_Y(m,n) : m \leq n < \omega\}$ is an ω-diagram based on $\{Y_n : n < \omega\}$, and $Y \cong X_{D_Y}$.*

Proof. To verify that D_Y is a diagram, the only thing we need to check is that for $f \in H_m^n$, the map $D(m,n)(-,f)$ is surjective. Suppose $r \in Y_m$. Let $p \in Y$ with $p|_m = r$. Find $g \in G$ with $g|_m = f$, and then $(pg^{-1})|_n \cdot f = pg^{-1} \cdot g|_m = p|_m = r$ as desired. The map $\psi : Y \to X_{D_Y}$ given by $\psi(p) = (p|_n)_{n<\omega}$ is a G-flow isomorphism. $\square$

We now seek to understand which properties of a diagram D ensure that $X_D \cong M(G)$.

Proposition 3.5.7. *Let D be an ω-diagram based on $\{J_n : n < \omega\}$. Then X_D is a pre-ambit iff D has the JEP.*

Proof. Suppose D has the JEP, and write $\{x_i : i < \omega\} = \bigcup_n J_n$. Let $n_0 < \omega$ so that $x_0 \in \mathcal{K}^*(\mathbf{A}_{n_0})$. Suppose some $y_{n_k} \in J_{n_k}$ has been determined which embeds each x_i for $i \leq k$. Use JEP to find some $n \geq n_k$ and $z_n \in J_n$ which embeds both y_{n_k} and x_{k+1}. Suppose $s \in H(n_k, n)$ satisfies $z_n \cdot s = y_{n_k}$. Using the Extension Property for $\mathbf{K}$, we can find $n_{k+1} \geq n$ and $t \in H(n, n_{k+1})$ so that $t \circ s = i_{n_k}$. Then find any $y_{n_{k+1}} \in J_{n_{k+1}}$ with $y_{n_{k+1}} \cdot t = z_n$. Notice that $y_{n_{k+1}} \cdot i_{n_k} = y_{n_k}$. We then let $y \in X_D$ be the unique element with $y|_{n_k} = y_{n_k}$ for each $k < \omega$. Since y embeds every member of $\bigcup_n J_n$, we see that $y \cdot G$ is dense in X_D.

Conversely, suppose D does not have the JEP, as witnessed by some $y_m \in J_m$ and $y_n \in J_n$. Suppose $x \in X_D$ has $x|_m = y_m$. Then $x \cdot G$ does not meet the non-empty open neighborhood $\{z \in X_D : z|_n = y_n\}$. Since x was an arbitrary member of the non-empty open set $\{x \in X_D : x|_m = y_m\}$, it follows that X_D cannot be a pre-ambit. $\square$

Next, we define a combinatorial property which captures minimality of the flow $X_{\mathcal{K}^*}$.

Definition 3.5.8. Let D be an ω-diagram based on $\{J_n : n < \omega\}$. We say that D has the *Minimal Property* if whenever $m < \omega$ and $x \in J_m$, there is some $n \geq m$ so that for any $y \in J_n$, we have $\mathrm{Emb}(x, y) \neq \emptyset$.

Remark.

1. When referring to expansion classes, the Minimal Property is often called the *Expansion Property*, or the *Order Property* when $\mathcal{K}^*$ is an expansion of $\mathcal{K}$ by linear orders.

2. Since J_m is finite, we can find an $n \geq m$ which works for every element of J_m simultaneously. However, we will soon define *weak diagrams* where we allow the J_m to be countable, and in this context Definition 3.5.8 is the correct definition to use.

Proposition 3.5.9. *Let D be an ω-diagram based on $\{J_n : n < \omega\}$. Then X_D is a minimal G-flow iff D has the Minimal Property.*

Proof. Suppose D has the Minimal Property. Let $x \in X_D$, and fix $y \in J_m$. We want to find $g \in G$ with $(x \cdot g)|_m = y$. The Minimal Property ensures that there is some $n \geq m$ so that every $z \in J_n$ embeds y. In particular, there is $f \in H_m^n$ with $x|_n \cdot f = y$. Finding a $g \in G$ with $g|_m = f$, we have $x \cdot g = y$ as desired.

Conversely, suppose D does not have the Minimal Property as witnessed by $y \in J_m$. For each $n < \omega$, we can find $x_n \in X_D$ so that $x_n|_n$ does not embed y. By passing to a convergent subsequence, we obtain $x \in X_D$ with $\mathrm{Emb}(y, x) = \emptyset$. Then the orbit $x \cdot G$ avoids the non-empty open set $\{z \in X_D : z|_m = y\}$, so X_D is not minimal □

Remark. Since minimal flows are pre-ambits, this shows that the Minimal Property implies the JEP.

We can now state another of the major theorems from [16]; we phrase it here in terms of expansions.

Theorem 3.5.10 (Theorem 7.5 and 10.8 in [16], see also Theorem 5 from [26])**.** *Suppose $(\mathcal{K}^*, \mathcal{K})$ is pre-compact, and form $X_{\mathcal{K}^*}$. Then the following are equivalent.*

1. *$X_{\mathcal{K}^*} \cong M(G)$.*

2. *$\mathcal{K}^*$ is a Fraïssé class with the Ramsey Property and $(\mathcal{K}^*, \mathcal{K})$ has the Minimal Property.*

Remark. Expansions $(\mathcal{K}^*, \mathcal{K})$ satisfying item (2) are often called *excellent*.

We turn our attention towards the proof of this result. Along the way, we will strengthen it and recover another major result from [34].

Theorem 3.5.11 (Theorem 8.14 in [34])**.** *The following are equivalent.*

1. *$M(G)$ is metrizable.*

2. *Every $\mathbf{A}$ has finite Ramsey degree in $\mathcal{K}$.*

3. *There is a pre-compact expansion $(\mathcal{K}^*, \mathcal{K})$ satisfying item (2) from Theorem 3.5.10.*

This theorem also provided the first proof of the Generic Point Problem for G an automorphism group. Of course, we have already seen a proof of the full Generic Point Problem in section 2.6, but the original proof follows from the above two theorems and the following observation.

Proposition 3.5.12. *Suppose D is a Fraïssé diagram based on $\{J_n : n < \omega\}$. Then X_D has a comeager orbit.*

Proof. Let $x \in X_D$ be a Fraïssé limit of the diagram D. First note that if $y \in X_D$ is another Fraïssé limit, then there is $g \in G$ with $xg = y$; this is because any two Fraïssé limits are isomorphic, and any isomorphism is necessarily an automorphism of the reduct $\mathbf{K}$. Second, note that $x \in X_D$ is a Fraïssé limit iff the following form of the Extension Property holds: whenever $z_0 \in J_m$, $z_1 \in J_n$, $f \in \mathrm{Emb}(z_0, z_1)$, and $j \in \mathrm{Emb}(z_0, x)$, then there is $s \in \mathrm{Emb}(z_1, x)$ with $s \circ f = j$. The AP for D tells us that fixing z_0, z_1, $f \in \mathrm{Emb}(z_0, z_1)$, and $j \in H_m$ as above, the set

$$\{x \in X_D : j \notin \mathrm{Emb}(z_0, x) \text{ or } \exists s \in \mathrm{Emb}(z_1, x) \text{ with } s \circ f = j\}$$

is open dense in X_D. The Expansion Property is the demand that $x \in X_D$ belong to the above set for every choice of z_0, z_1, f, and j, and there are countably many such choices. Therefore the set of $x \in X_D$ satisfying the Extension Property is comeager. □

We have already shown (1) ⇔ (2) in Theorem 3.5.5. We will phrase the remaining proofs in terms of diagrams.

Definition 3.5.13. Let D be an ω-diagram. Then D is called *excellent* if D is Fraïssé and has the Minimal and Ramsey Properties.

We first address (1) ⇒ (3). For the next few propositions, fix $Y \subseteq S(G)$ a minimal metrizable subflow, and form the diagram D_Y as in Proposition 3.5.6. We will show that D_Y is excellent. Since $Y \cong X_{D_Y}$ is minimal, we know that D_Y has the Minimal Property, which also implies that D_Y has the JEP. It remains to show that D_Y has the AP and the Ramsey Property. To simplify our task, we note the following theorem of Nešetřil and Rödl [22].

Fact. Suppose $\mathcal{D}$ is a class of finite structures with the JEP and the Ramsey Property. Then D has the AP.

So we only need to show that $D := D_Y$ has the Ramsey Property. The idea will be to view $x \in Y$ as an element of $X_{\mathcal{K}^D}$ and use Proposition 3.2.3 in tandem with Proposition 3.2.9 for the reduct $\mathbf{K}$. We will need to be a little careful with our terminology, as we will be dealing with two different notions of thickness; we will emphasize this by writing x-thick or $\mathbf{K}$-thick as needed.

Lemma 3.5.14. *Suppose D is an ω-diagram based on $\{J_n : n < \omega\}$, and fix $x \in X_D$ and $y \in J_m$.*

1. *If $T(y,x) \subseteq \mathrm{Emb}(y,x)$ is x-thick, then there is some $\mathbf{K}$-thick $T \subseteq H_m$ with $T \cap \mathrm{Emb}(y,x) = T(y,x)$.*
2. *If D has the Minimal Property, the converse is true; if $T \subseteq H_m$ is $\mathbf{K}$-thick, then $T(y,x) := T \cap \mathrm{Emb}(y,x)$ is thick. Further*

Proof. Suppose $T(y,x) \subseteq \mathrm{Emb}(y,x)$ is x-thick. For each $n \geq m$ and $z \in J_n$ with $\mathrm{Emb}(z,x) \neq \emptyset$, find $s_z \in \mathrm{Emb}(z,x)$ with $s_z \circ \mathrm{Emb}(y,z) \subseteq T(y,x)$. Then set $T' = \bigcup_n \bigcup_{z \in J_n} s_z \circ H^n_m$. Then $T' \subseteq H_m$ is $\mathbf{K}$-thick, as is $T := T' \cup T(y,x)$. Then $T \cap \mathrm{Emb}(y,x) = T(y,x)$ as desired.

Now assume D has the Minimal Property, and suppose $T \subseteq H_m$ is $\mathbf{K}$-thick. Then consider $T(y,x) := T \cap \mathrm{Emb}(y,x)$. Let $z \in J_n$; we want to find $h \in \mathrm{Emb}(z,x)$ with $h \circ \mathrm{Emb}(y,z) \subseteq T(x,y)$. First find $N \geq n$ witnessing the Minimal Property for z. Then find $s \in H_N$ with $s \circ H^N_m \subseteq T$. As $x \cdot s \in J_N$, find $t \in H^N_n$ with $x \cdot s \cdot t = z$. So $s \circ t \in \mathrm{Emb}(z,x)$ and $s \circ t \circ \mathrm{Emb}(y,z) \subseteq T(x,y)$ as desired. □

Proposition 3.5.15. *The diagram $D := D_Y$ has the Ramsey Property.*

Proof. Fix $x \in X_D$ and $y \in Y_m$. Let $\gamma : \mathrm{Emb}(y,x) \to 2$ be a coloring; we need to show that some $\gamma^{-1}(\{i\})$ is x-thick. We use γ to form a coloring $\delta : H_m \to (Y_m \setminus \{y\}) \cup \{0,1\}$ as follows.

$$\delta(f) = \begin{cases} x \cdot f & \text{if } f \notin \mathrm{Emb}(y,x) \\ \gamma(f) & \text{if } f \in \mathrm{Emb}(y,x) \end{cases}$$

By Theorem 3.5.5, $\mathbf{A}_m$ has Ramsey degree $|Y_m|$ in $\mathcal{K}$. So for some $\mathbf{K}$-thick $T \subseteq H_m$, we have $|\delta``(T)| = |Y_m|$. Notice that $\delta^{-1}(\{z\})$ is $\mathbf{K}$-syndetic for each $z \in Y_m \setminus \{y\}$ by Proposition 2.3.12. Therefore the color missing from $\delta``(T)$ is either 0 or 1; we may assume $\delta``(T) = (Y_m \setminus \{y\}) \cup \{0\}$.

By Lemma 3.5.14, $T(y,x) := T \cap \mathrm{Emb}(y,x)$ is x-thick. Now observe that $\gamma``(T(y,x)) = \{0\}$, so $\gamma^{-1}(\{0\})$ is x-thick and we are done. □

The direction (3) → (1) in Theorem 3.5.11 is addressed by (2) → (1) in Theorem 3.5.10. For completeness, we provide a proof; given D an excellent ω-diagram, we need to show that $X_D \cong M(G)$. One of the key steps along the way is the following.

Proposition 3.5.16. *If D is an excellent diagram based on $\{J_n : n < \omega\}$, then each $\mathbf{A}_m \in \mathcal{K}$ has Ramsey degree $|J_m|$ in $\mathcal{K}$.*

Remark. As promised earlier, we can now easily explain Ramsey degrees in the class $\mathcal{K}$ of finite graphs. It is shown in [16] that if $\mathcal{K}^*$ is the expansion class of finite linearly ordered graphs, then $(\mathcal{K}^*, \mathcal{K})$ is excellent. As a graph on n vertices can be expanded by $n!$ different linear orders, we see that the Ramsey degree of a graph on n vertices is $n!$.

Lemma 3.5.17. *Let D be a diagram based on $\{J_n : n < \omega\}$, and fix $x \in X_D$ and $y \in J_m$. Suppose $\gamma : H_m \to r$ is a coloring which is constant on $\mathrm{Emb}(y, x)$. Then for any $p \in S(G)$, the coloring $\gamma \cdot p$ is constant on $\mathrm{Emb}(y, x \cdot p)$.*

Proof. Fix $f_0, f_1 \in \mathrm{Emb}(y, x \cdot p)$. Then $x \cdot p \cdot f_0 = x \cdot p \cdot f_1 = y$. Letting $g_i \to p$, we eventually have $g_i \cdot f_0$ and $g_i \cdot f_1 \in \mathrm{Emb}(y, x)$. So $\gamma(g_i \cdot f_0) = \gamma(g_i \cdot f_1)$, and therefore $\gamma \cdot p(f_0) = \gamma \cdot p(f_1)$. □

Proof of Proposition 3.5.16. Write $k = |J_m|$, and let $\gamma : H_m \to r$ be a coloring. It suffices to find $p \in S(G)$ so that $|\mathrm{Im}(\gamma \cdot p)| \leq k$.

Write $J_m = \{y_0, ..., y_{k-1}\}$. We inductively construct $p_0, ..., p_{k-1} \in S(G)$. Suppose $p_0, ..., p_{i-1}$ have been determined, and set $q_i = p_0 \cdots p_{i-1}$; for $i = 0$ set $q_0 = 1_G$. Consider the coloring $\gamma \cdot q_i$ on $\mathrm{Emb}(y_i, x \cdot q_i)$. Inductively assume that $\gamma \cdot q_i$ is constant on $\mathrm{Emb}(y_j, x \cdot q_i)$ for each $j < i$. Find an x-thick $T_i \subseteq \mathrm{Emb}(y_i, x \cdot q_i)$ with $(\gamma \cdot q_i)``(T_i)$ a singleton. For each $n < \omega$, fix $s_n \in H_n$ with $s_n \circ \mathrm{Emb}(y, x|_n) \subseteq T_i$. Find $g_n \in G$ with $g_n|_n = s_n$, and let $p_i \in S(G)$ be a cluster point. Then $\gamma \cdot q_i \cdot p_i = \gamma \cdot q_{i+1}$ is monochromatic on $\mathrm{Emb}(y_i, x \cdot q_{i+1})$; by Lemma 3.5.17, $\gamma \cdot q_{i+1}$ is monochromatic on $\mathrm{Emb}(y_j, x \cdot q_{i+1})$ for every $j \leq i$.

Now set $p = q_k$. The coloring $\gamma \cdot p$ is monochromatic on $\mathrm{Emb}(y_i, x \cdot p)$ for every $i < k$. As $H_m = \bigcup_{i<k} \mathrm{Emb}(y_i, x \cdot p)$, we have that $|\mathrm{Im}(\gamma \cdot p)| \leq k$ as desired. □

We can now complete the proofs of (3) $\Rightarrow$ (1) in Theorem 3.5.11 and (2) $\Rightarrow$ (1) in Theorem 3.5.10. Let D be an excellent diagram based on $\{J_n : n < \omega\}$. Let $Y \subseteq S(G)$ be a minimal subflow. Then $Y = \varprojlim Y_m$ with $Y_m \subseteq \beta H_m$ and $|Y_m| = |J_m|$. Fix $x \in X_D$, and let $\lambda_x : Y \to X_D$ be the G-map $\lambda_x(p) = x \cdot p$. Now if $p_0, p_1 \in Y$ have $x \cdot p_0 = x \cdot p_1$, then for every $m < \omega$, we have $x \cdot p_0 \cdot i_m = x \cdot p_1 \cdot i_m$. But then since $|J_m| = |Y_m|$ and λ_x is surjective, we must have $p_0|_m = p_1|_m$, hence $p_0 = p_1$.

3.6 Weak Diagrams

In this section, we negatively answer a question due to Ben–Yaacov, Melleray, and Tsankov [5] asking about the converse to the Generic Point Problem.

Theorem 3.6.1. *There is a Polish group G with $M(G)$ not metrizable and containing a comeager orbit.*

The example follows from recent work of Kwiatkowska on generalized Ważewski dendrites [18]. What we will need is the following.

Proposition 3.6.2 ([18], Theorem 13)**.** *There is a Fraïssé-HP class $\mathcal{K}$ and an expansion $\mathcal{K}^*$ so that $(\mathcal{K}, \mathcal{K}^*)$ satisfies every property of being an excellent pair except precompactness, which fails.*

It will be helpful to formalize these properties using the notion of a *weak diagram.*

Definition 3.6.3. A *weak diagram* based on $\{J_n : n < \omega\}$ satisfies all the properties of an ω-diagram, except we now allow the J_n to be countably infinite.

The other properties that an ω-diagram might have, i.e. the JEP, AP, Ramsey Property, Minimal Property, and being excellent, all have the same definitions.

Given a weak diagram D, we set $X_D := \{x \in \prod_n J_n : x(n) \cdot i_m = x(m) \text{ for } m \leq n < \omega\}$. Notice that this is the exact same definition as before, and we let G act on X_D in the exact same manner. The key difference is that X_D will only be compact if each J_n is finite, i.e. if D is a diagram. So the action $X_D \times G \to X_D$ turns X_D into a G-space rather than a G-flow. We will prove Theorem 3.6.1 using Proposition 3.6.2 by proving the following.

Theorem 3.6.4. *Suppose D is an excellent weak diagram. Then $M(G) \cong S_G(X_D)$. Furthermore, $X := X_D$ is a rich G-space which has a comeager orbit, and the embedding $i_X^G : X \to S_G(X)$ has comeager image.*

First note that the proof of Proposition 3.5.12 works just the same for weak diagrams, so X_D will have a comeager orbit. Also notice that the forward direction of Proposition 3.5.9 works in the weak setting, so if D has the Minimal Property, then

Lemma 3.6.5. *Suppose D is a weak diagram based on $\{J_n : n < \omega\}$. Then X_D is finitely minimal iff D has the Minimal Property.*

Proof. Fix $y \in J_m$, and let $X_y = \{x \in X_D : x|_m = y\}$. We want to find $g_0, ..., g_{k-1} \in G$ with $X_D = \bigcup_{i<k} X_y g_i$. The Minimal Property ensures that there is some $n \geq m$ so that every $z \in J_n$ embeds y. Find $g_0, ..., g_{k-1}$ so that $\{g_i^{-1}|_m : i < k\} = H_m^n$. Then we have $X_D = \bigcup_{i<k} X_y g_i$ as desired.

Conversely, suppose X_D is not finitely minimal as witnessed by $y \in J_m$. Consider the open set X_y, and fix $g_0, ..., g_{k-1} \in G$. Find some $n \geq m$ with $\{g_i^{-1}|_m : i < k\} \subseteq H_m^n$, and find some $z \in J_n$ which doesn't embed y. It follows that $X_z \cap \bigcup_{i<k} X_y g_i = \emptyset$. $\square$

Lemma 3.6.6. *Suppose D is a weak diagram with the AP. Then $X := X_D$ is a rich G-space. Furthermore, we have $S_G(X) \cong \varprojlim \beta J_n$, and $X \subseteq \varprojlim \beta J_n$ is comeager.*

Proof. Given $y \in J_n$, write $X_y = \{x \in X : x|_n = y\}$. Let $A \subseteq X$ be open, and fix $m < \omega$ in order to consider the subgroup $G_m \in \mathcal{N}_G$. We can write $A = \bigcup_{y \in S} X_y$ for some $S \subseteq \bigcup_{n \geq m} J_n$. Then also $AG_m = \bigcup_{y \in S} X_y G_m$. Notice first that $X_y G_m \subseteq X_{y|_m}$. By AP, $X_y G_m$ is dense in $X_{y|_m}$. It follows that $\overline{AG_m} = \bigcup_{y \in S} X_{y|_m}$, which is open. So given $x \in X$ with $x \in \overline{A}$, we have $x \in \text{int}(\overline{AG_m})$, showing that X is rich.

For the second assertion, notice that if $S \subseteq J_m$, then any $p \in S_G(X)$ contains exactly one of $\bigcup_{y \in S} X_y$ or $\bigcup_{y \in J_m \setminus S} X_y$. This provides a surjection $\psi : S_G(X) \to \varprojlim \beta J_n$. To see that this is injective, suppose $p \neq q \in S_G(X)$. Find $A \in p$, $B \in q$, and $m < \omega$ with $AG_m \cap BG_m = \emptyset$. There are sets $S, T \subseteq J_m$ with AG_m dense in $\bigcup y \in SX_y$ and BG_m dense in $\bigcup_{y \in T} X_y$. It follows that $S \cap T = \emptyset$, $S \in \psi(p)|_m$, and $T \in \psi(q)|_m$, so $\psi(p) \neq \psi(q)$.

For the final assertion, notice that the maps $\rho^n_{i_m} : \beta J_n \to \beta J_m$ occurring in the inverse limit are open. It follows that for each $n < \omega$, the set $X_n := \{p \in \varprojlim \beta H_n : p|_n \in J_n\}$ is open dense, and we have $X = \bigcap_n X_n$. $\square$

We extend the partial right actions to βJ_n for each $n < \omega$ by continuity, and can therefore view $\varprojlim \beta J_n$ as a G-flow. The continuous bijection $\psi : S_G(X) \to \varprojlim \beta J_n$ produced in the proof of the lemma is a G-map.

We now fix $Y = \varprojlim Y_n \subseteq S(G)$ a minimal subflow. Fix $x \in X_D \subseteq \varprojlim \beta J_n$, and consider the G-map $\lambda_x : Y \to \beta J_n$. This gives rise to continuous maps $\lambda^m_x : Y_m \to \beta J_m$ for each $m < \omega$. We want to show that λ_x is an isomorphism.

Proposition 3.6.7. *Fix $y \in J_m$. If $\gamma : \text{Emb}(y, x) \to 2$ is a coloring, then at least one of $\gamma^{-1}(\{0\})$ or $\gamma^{-1}(\{1\})$ is not $\mathbf{K}$-syndetic.*

Proof. First notice that Lemma 3.5.14 goes through in the weak setting. By the Ramsey Property, some color class is x-thick. Using Lemma 3.5.14, it follows that the other color class cannot be $\mathbf{K}$-syndetic. $\square$

Proposition 3.6.8. *Suppose $p, q \in Y$ with $x \cdot p|_m = x \cdot q|_m = y \in J_m \subseteq \beta J_m$. Then $p|_m = q|_m$.*

Proof. First note that $(\lambda^m_x)^{-1}(\{y\}) := K \subseteq Y_m$ is clopen. Towards a contradiction, suppose K is not a singleton. Find disjoint clopen sets $U, V \subseteq Y_m$ with $U \cup V = K$. This partition of K gives rise to a coloring $\gamma : \text{Emb}(y, x) \to 2$ depending on whether $x \cdot f \in U$ or $x \cdot f \in V$. As X_D is minimal, both color classes are $\mathbf{K}$-syndetic, contradicting Lemma 3.6.7. $\square$

We conclude the proof of Theorem 3.6.4 by showing that λ_x is injective. It suffices to show that each λ^m_x is injective. Let $D_m = \{p \in Y_m : x \cdot p \in J_m\}$. We have seen that λ^m_x

is injective on D_m. The map λ_x, being a G-map between minimal flows, is pseudo-open. It follows that each λ_x^m is also pseudo-open. Since J_m is dense in βJ_m, it follows that D_m is dense in Y_m. But this implies that $Y_m \cong \beta D_m$ and that λ_x^m is the continuous extension of the injection from D_m to J_m.

3.7 Big Ramsey degrees

This section can be viewed as an extension of section 3.2. We consider here more infinite Ramsey properties that a class $\mathcal{D} = \mathrm{Age}(\mathbf{D})$ of finite structures might possess. Unlike the previous notions we have encountered, these notions will depend on the choice of $\mathbf{D}$. Of course, if $\mathcal{K}$ is a Fraïssé-HP class, then there is a canonical choice of structure to consider.

Definition 3.7.1.

1. Let $\mathbf{D}$ be a countably infinite L-structure, and set $\mathcal{D} = \mathrm{Age}(\mathbf{D})$. Fix $\mathbf{A} \in \mathcal{D}$ and $k < \omega$. Then we say that $\mathbf{A}$ has *Ramsey degree k in* $\mathbf{D}$ if k is least so that for any r with $k \leq r < \omega$, we have $\mathbf{D} \to (\mathbf{D})^{\mathbf{A}}_{r,k}$.

2. If $\mathcal{K}$ is a Fraïssé-HP class and $\mathbf{A} \in \mathcal{K}$, we say that $\mathbf{A}$ has *big Ramsey degree k in $\mathcal{K}$* if $\mathbf{A}$ has Ramsey degree k in $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$.

Note that if $\mathbf{A}$ has Ramsey degree k in $\mathbf{D}$, then $\mathbf{A}$ has Ramsey degree $t \leq k$ in $\mathcal{D}$. A good example to keep in mind is provided by looking at the Fraïssé class $\mathcal{K}$ of finite linear orders. If $\mathbf{A}_2$ is the two element linear order, we can consider the following "Sierpinski-style" coloring of $\mathrm{Emb}(\mathbf{A}_2, \mathbb{Q})$. Enumerate $\mathbb{Q} = \{q_n : n < \omega\}$. Let $\mathbf{A}_2 = \{a, b\}$ with $a < b$, and consider $f \in \mathrm{Emb}(\mathbf{A}_2, \mathbb{Q})$. If $f(a) = q_k$ and $f(b) = q_\ell$, we color f depending on whether $k < \ell$ or $\ell < k$. This coloring shows that $\mathbb{Q} \not\to (\mathbb{Q})^{\mathbf{A}_2}_2$. By a theorem of Galvin [11], this is the worst possible, i.e. the big Ramsey degree of $\mathbf{A}_2$ in $\mathcal{K}$ is 2. D. Devlin in his thesis [7] showed that every finite linear order has finite big Ramsey degree. More precisely, if $\mathbf{A}_k$ is the k-element linear order, then the big Ramsey degree of $\mathbf{A}_k$ in $\mathcal{K}$ is the k-th odd tangent number. The sequence of these numbers starts off $1, 2, 16, 272$, etc. In particular, the "Sierpinski-style" 6-coloring of triples of rationals is not the worst possible.

Just as *thick* and *syndetic* were the appropriate notions of largeness for Ramsey degree in a class, we need analogous notions of largeness for Ramsey degree in a structure. Recall that we write $\mathrm{Emb}(\mathbf{D})$ for $\mathrm{Emb}(\mathbf{D}, \mathbf{D})$.

Definition 3.7.2. Let $\mathbf{D}$ be a countably infinite L-structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and fix $\mathbf{A} \in \mathcal{D}$.

1. $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *large* if for some $\eta \in \mathrm{Emb}(\mathbf{D})$, we have $\eta \circ \mathrm{Emb}(\mathbf{A}, \mathbf{D}) \subseteq S$.

2. $S \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is *unavoidable* if $\mathrm{Emb}(\mathbf{A}, \mathbf{D}) \setminus S$ is not large.

3. If $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is a coloring, then γ is an *unavoidable r-coloring* if $|\mathrm{Im}(\gamma)| = r$, and for each $y \in Y$, we have $\gamma^{-1}(\{y\})$ either empty or unavoidable. We call γ an *unavoidable coloring* if γ is an unavoidable $|\mathrm{Im}(\gamma)|$-coloring.

Notice that every large set is thick, hence every syndetic set is unavoidable. We can now state propositions very similar in spirit to Lemma 3.2.8, Proposition 3.2.9, and Corollary 3.2.10. In many ways the proofs of these are easier, as the largest structure involved (namely $\mathbf{D}$) stays the same throughout inductive proofs, and appeals to compactness are not needed (indeed, appeals to compactness are doomed to fail when working with big Ramsey degree).

Lemma 3.7.3. *Let $\mathbf{D}$ be a countably infinite L-structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and fix $\mathbf{A} \in \mathcal{D}$ and $k < \omega$. Then the following are equivalent.*

1. *$\mathbf{A}$ has Ramsey degree $t \leq k$ in $\mathbf{D}$.*
2. *$\mathbf{D} \rightarrow (\mathbf{D})^{\mathbf{A}}_{k+1,k}$.*

Proposition 3.7.4. *Let $\mathbf{D}$ be a countably infinite L-structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and fix $\mathbf{A} \in \mathcal{D}$ and $k < \omega$. Then the following are equivalent.*

1. *$\mathbf{A}$ has Ramsey degree $t \leq k$ in $\mathbf{D}$.*
2. *For any r with $k \leq r < \omega$ and coloring $\gamma : \mathrm{Emb}(\mathbf{A}, \mathbf{D}) \rightarrow r$, some k colors form a large subset.*
3. *For any large $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$, any r with $k \leq r \leq \omega$, and any coloring $\gamma : T \rightarrow r$, some k colors form a large subset.*

Corollary 3.7.5. *Let $\mathbf{D}$ be a countably infinite L-structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and fix $\mathbf{A} \in \mathcal{D}$ and $k < \omega$. Then if $\mathbf{A}$ does not have Ramsey degree $t < k$ in $\mathbf{D}$, then there is an unavoidable k-coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{D})$.*

As in Propositions 3.2.11 and 3.2.12, we can investigate how large and unavoidable sets behave when considering different finite structures. As before, large sets will "push up," but we will need extra assumptions to push up unavoidable sets.

Proposition 3.7.6. *Let $\mathbf{D}$ be a countably infinite L-structure with $\mathcal{D} = \mathrm{Age}(\mathbf{D})$, and let $\mathbf{A} \leq \mathbf{B} \in \mathcal{D}$. Fix $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{B})$. If $T \subseteq \mathrm{Emb}(\mathbf{A}, \mathbf{D})$ is large, then so is $T_f := \{x \in \mathrm{Emb}(\mathbf{B}, \mathbf{D}) : x \circ f \in T\}$.*

Proof. Fix $\eta \in \mathrm{Emb}(\mathbf{D})$ with $\eta \circ \mathrm{Emb}(\mathbf{A}, \mathbf{D}) \subseteq T$. Now if $x \in \mathrm{Emb}(\mathbf{B}, \mathbf{D})$, we have $\eta \circ (x \circ f) \in T$, so $\eta \circ x \in T_f$, and T_f is large as desired. $\square$

Proposition 3.7.7. *Suppose $\mathbf{K} = \bigcup_n \mathbf{A}_n$ is a Fraïssé structure, and let $m \leq n < \omega$. Fix $f \in H_m^n$. If $S \subseteq H_m$ is unavoidable, then so is $(\rho_f^n)^{-1}(S)$.*

Proof. Fix $\eta \in \mathrm{Emb}(\mathbf{K})$. Find $j \in H_m$ with $\eta \circ j \in S$. As $\mathbf{K}$ is a Fraïssé structure, find $x \in H_n$ with $x \circ f = j$. Then $\eta \circ x \in (\rho_f^n)^{-1}(S)$, so $(\rho_f^n)^{-1}(S)$ is unavoidable as desired. $\square$

We spend the remainder of the section focusing on the case of $\mathbf{K} = \bigcup_n \mathbf{A}_n$ a Fraïssé structure, using Proposition 3.7.7 as a key tool. Two immediate corollaries are the following.

Corollary 3.7.8. *Suppose $m \leq n < \omega$, and let γ be a coloring of H_m. Then γ is unavoidable iff the coloring $\gamma \cdot \rho_{i_m}^n$ of H_n is unavoidable.*

Corollary 3.7.9. *Suppose $\mathbf{A}_m$ has big Ramsey degree R_m. Then if $n \geq m$, $\mathbf{A}_n$ does not have big Ramsey degree $t < R_m$.*

Proof. If $\gamma : H_m \to R_m$ is an unavoidable R_m-coloring, then so is $\gamma \cdot \rho_{i_m}^n : H_n \to R_m$. $\square$

We want to allow the semigroup $\mathrm{Emb}(\mathbf{K})$ to act on colorings in the obvious way. If $\gamma : H_m \to r$ is a coloring and $\eta \in \mathrm{Emb}(\mathbf{K})$, we define $\gamma \cdot \eta : H_m \to r$ via $\gamma \cdot \eta(f) = \gamma(\eta \circ f)$. It turns out that this semigroup action coincides with one that we have already seen.

Proposition 3.7.10. *When $G = \mathrm{Aut}(\mathbf{K})$ as above, $\widehat{G}$ and $\mathrm{Emb}(\mathbf{K})$ are homeomorphic.*

Proof. Suppose $(g_n)_{n<\omega}$ is a Cauchy sequence, and assume that for all $m < \omega$ and all $n \geq m$, that $g_n|_m = g_m|_m$. Then $\eta : \mathbf{K} \to \mathbf{K}$ defined by $\eta|_m = g_m|_m$ is an embedding. Conversely, if $\eta : \mathbf{K} \to \mathbf{K}$ is an embedding, then find $g_n \in G$ with $g_n|_n = \eta|_n$ to create a Cauchy sequence $(g_n)_{n<\omega}$. One should check that both directions of this proof respect equivalence of Cauchy sequences, but this is routine. $\square$

We now check that the semigroup structures of $\widehat{G}$ and $\mathrm{Emb}(\mathbf{K}, \mathbf{K})$ coincide. We can do this by considering the level representation of $S(G)$.

Proposition 3.7.11. *Let $p \in S(G)$. Then $p \in \widehat{G}$ iff each $p|_m \in H_m \subseteq \beta H_m$.*

Proof. Given $\eta \in \widehat{G} \cong \mathrm{Emb}(\mathbf{K})$, we view η as a member of $S(G)$ by considering $\eta|_m$ for $m < \omega$. Conversely, if $p \in S(G)$ with $p|_m \in H_m$ for every $m < \omega$, then $\bigcup_n p|_n \in \mathrm{Emb}(\mathbf{K})$. $\square$

Corollary 3.7.12. *The semigroup structures on $\widehat{G}$ and $\mathrm{Emb}(\mathbf{K})$ coincide.*

Proof. Suppose $\eta, \zeta \in \widehat{G}$, and let $f \in H_m$. Then $\{f\} \in (\eta \cdot \zeta)|_m = \eta \cdot \zeta|_m$ iff $\eta^{-1}(\{f\}) := \{j \in H_m : \eta \cdot j = f\} \in \zeta|_m$. This happens iff $\eta \cdot \zeta|_m = f$, and this is precisely the semigroup structure of $\mathrm{Emb}(\mathbf{K})$. $\square$

From here on out, we will identify $\widehat{G}$ and $\mathrm{Emb}(\mathbf{K})$.

We can now prove several useful facts about unavoidable colorings.

Proposition 3.7.13.

1. *If $\gamma : H_m \to r$ is an unavoidable r-coloring and $\eta \in \widehat{G}$, then $\gamma \cdot \eta : H_m \to r$ is also an unavoidable r-coloring.*

2. *Suppose $\mathbf{A}_m$ has big Ramsey degree R_m. Suppose γ and δ are both colorings of H_m with γ an unavoidable R_m-coloring. Then there is $\eta \in \widehat{G}$ with $\gamma \cdot \eta \geq \delta \cdot \eta$.*

3. *Suppose $m \leq n < \omega$, let $\gamma \preceq \delta$ be colorings of H_m and H_n, respectively. Suppose δ happens to be unavoidable. Then so is γ.*

Proof. Item (1) is immediate. For item (2), form the product coloring $\gamma \times \delta$, and find $\eta \in \widehat{G}$ so that $|\mathrm{Im}((\gamma \times \delta) \cdot \eta)| \leq R_m$. However, $(\gamma \times \delta) \cdot \eta$ refines both $\gamma \cdot \eta$ and $\delta \cdot \eta$, so we must have $\gamma \cdot \eta \sim (\gamma \times \delta) \cdot \eta \geq \delta \cdot \eta$.

For item (3), notice that δ refines the coloring $\gamma \cdot \pi^n_m$, so $\gamma \cdot \pi^n_m$ is unavoidable. Then by Corollary 3.7.8, so is γ. □

We end this section with a theorem about how big Ramsey degrees on different levels interact, followed by an open question which will gain relevance in the next section.

Lemma 3.7.14. *Suppose $m \leq n < \omega$, and assume $\mathbf{A}_m$ and $\mathbf{A}_n$ have big Ramsey degrees R_m and R_n, respectively. Then there are unavoidable R_m and R_n-colorings γ_m and γ_n of H_m and H_n, respectively, with $\gamma_m \preceq \gamma_n$*

Proof. Write $H^n_m = \{f_0, ..., f_{k-1}\}$. Fix any unavoidable R_m-coloring δ_m of H_m and unavoidable R_n-coloring δ_n of H_n. Form the product coloring $\delta := \delta_n \times \prod_{i<k} \delta_m \cdot \rho^n_{f_i}$. Find $\eta \in \widehat{G}$ with $|\mathrm{Im}(\delta \cdot \eta)| \leq R_n$. Notice that $\delta \cdot \eta$ refines $\delta_n \cdot \eta$, so we must have $\delta \cdot \eta \sim \delta_n \cdot \eta$, meaning $\delta \cdot \eta$ is an unavoidable R_n-coloring. Also, $\delta \cdot \eta$ refines each $\delta_m \cdot \rho^n_{f_i} \cdot \eta$ for $i < k$. Notice that $\delta_m \cdot \rho^n_{f_i} \cdot \eta = \delta_m \cdot \eta \cdot \rho^n_{f_i}$, so we have $\delta_m \cdot \eta \preceq \delta \cdot \eta$. Noting that $\delta_m \cdot \eta$ is an unavoidable R_m-coloring, we set $\gamma_m = \delta_m \cdot \eta$ and $\gamma_n = \delta \cdot \eta$. □

Repeated applications of Lemma 3.7.14 yield the following strengthening.

Proposition 3.7.15. *Suppose $\mathbf{A}_0, ..., \mathbf{A}_{N-1}$ each have big Ramsey degrees $R_0, ..., R_{N-1}$ (recall by convention that $\mathbf{A}_0 = \emptyset$, so $R_0 = 1$). Then there are unavoidable R_m-colorings γ_m of H_m for $0 \leq m < N$ with $(\gamma_m)_{m<N}$ coherent.*

One of the major difficulties in working with big Ramsey degree rather than small Ramsey degree is the lack of compactness; the relevant algebraic object is $\widehat{G}$ rather than $S(G)$. To illustrate this difficulty, we pose the following question.

Question 3.7.16. *Suppose for every $n < \omega$ that $\mathbf{A}_n$ has big Ramsey degree R_n. Are there unavoidable R_n-colorings γ_n of H_n with $(\gamma_n)_{n<\omega}$ coherent?*

3.8 Completion Flows

We have seen in section 2.4 that if X is a G-flow, the action extends to a continuous semigroup action $X \times \widehat{G} \to X$ of the left completion. This is the same action as the one induced by viewing $\widehat{G}$ as a subspace of $S(G)$ in the canonical way.

Definition 3.8.1. Let X be a pre-ambit. A *completion point* of X is any $x \in X$ so that for every $\eta \in \widehat{G}$, we have $x \cdot \eta \in \mathcal{A}(X)$. We call X a *completion flow* if X contains some completion point. A *completion ambit* is an ambit (X, x_0) where x_0 is a completion point.

If X is a completion flow, we call X a *universal completion flow* if there is a surjective G-map from X onto any other completion flow. The following key question remains open.

Question 3.8.2. *Let G be a topological group. Does there exist a universal completion flow for G? If yes, is it unique up to G-flow isomorphism?*

In this section, we will restrict our attention to the case where $G = \mathrm{Aut}(\mathbf{K})$ for a Fraïssé structure $\mathbf{K} = \bigcup_n \mathbf{A}_n$ as usual. Our goal is to connect the question of existence and uniqueness of completion flows to the big Ramsey properties of the class $\mathcal{K}$. For the remainder of the section, we will assume that each $\mathbf{A}_n$ has big Ramsey degree $R_n < \omega$. However, we will need to consider the following strengthening of the property of having finite big Ramsey degrees.

Definition 3.8.3. Let $\mathbf{K}$ be a Fraïssé L-structure with $\mathcal{K} = \mathrm{Age}(\mathbf{K})$. We say that $\mathbf{K}$ *admits a big Ramsey structure* if there is a language $L' \supseteq L$ and an L'-structure $\mathbf{K}'$ so that the following all hold.

1. $\mathbf{K}'|_L = \mathbf{K}$.
2. Every $\mathbf{A} \in \mathcal{K}$ has finitely many L'-expansions to a structure $\mathbf{A}' \in \mathrm{Age}(\mathbf{K}')$; denote the set of expansions by $\mathbf{K}'(\mathbf{A})$.
3. Every $\mathbf{A} \in \mathcal{K}$ has big Ramsey degree $|\mathbf{K}'(\mathbf{A})|$.
4. The function $\gamma_{\mathbf{A}} : \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to \mathbf{K}'(\mathbf{A})$ given by $\gamma(f) = \mathbf{K}' \cdot f$ witnesses the fact that the big Ramsey degree of $\mathbf{A}$ is not less than $|\mathbf{K}'(\mathbf{A})|$.

Call a structure $\mathbf{K}'$ satisfying (1)-(4) a *big Ramsey structure* for $\mathbf{K}$.

We can (and will) think about the big Ramsey structure as being given by the collection of unavoidable colorings $\gamma_{\mathbf{A}}$. Given a collection $\{\gamma_{\mathbf{A}} : \mathbf{A} \in \mathcal{K}\}$ of coherent colorings (where the definition of coherent is extended in the obvious way), we can code the colorings into new relational symbols to obtain a structure as in the proof of Proposition 3.4.19. The next proposition shows that having cofinally many coherent colorings suffices.

Proposition 3.8.4. *Let $\mathbf{K} = \bigcup_n \mathbf{A}_n$ be a Fraïssé structure, and suppose each $\mathbf{A}_n$ has finite big Ramsey degree $R_n < \omega$. Assume that for each $n < \omega$, there is an unavoidable R_n-coloring γ_n of H_n so that $(\gamma_n)_{n<\omega}$ is coherent. Then $\mathbf{K}$ admits a big Ramsey structure.*

Lemma 3.8.5. *With γ_n as in the statement of Theorem 3.8.4, then if $\eta \in \widehat{G}$, there is $p \in S(G)$ with $\gamma_n \cdot \eta \cdot p = \gamma_n$.*

Proof. For each $N \geq n$, find $g_N \in G$ so that $\gamma_N \cdot \eta \cdot g_N$ and γ_N agree on i_N. Let $p \in S(G)$ be a cluster point of the g_N. Since $\gamma_n \preceq \gamma_N$ for every $n \leq N < \omega$, we have $\gamma_n \cdot \eta \cdot p = \gamma_n$ as desired. $\square$

Proof of Proposition 3.8.4. First notice that each $\mathbf{B} \in \mathcal{K}$ has finite big Ramsey degree, as for some $n < \omega$, we have $\mathbf{B} \leq \mathbf{A}_n$ (and keeping in mind Proposition 3.7.7). Let $R_{\mathbf{B}} < \omega$ be the big Ramsey degree of $\mathbf{B} \in \mathcal{K}$.

We produce for every $\mathbf{B} \in \mathcal{K}$ with $\mathbf{B} \subseteq \mathbf{K}$ a coloring $\gamma_{\mathbf{B}}$ of $H_{\mathbf{B}} := \mathrm{Emb}(\mathbf{B}, \mathbf{K})$ so that the following items hold.

1. Each $\gamma_{\mathbf{B}}$ is an unavoidable $R_{\mathbf{B}}$-coloring.
2. If $\mathbf{B} \leq \mathbf{C} \in \mathcal{K}$, then $\gamma_{\mathbf{B}} \preceq \gamma_{\mathbf{C}}$, i.e. whenever $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{C})$, then $\gamma_{\mathbf{C}}$ refines $\gamma_{\mathbf{B}} \circ \rho_f^{\mathbf{C}}$. Here $\rho_f^{\mathbf{C}} : H_{\mathbf{C}} \to H_{\mathbf{B}}$ is the map $\rho_f^{\mathbf{C}}(x) = x \circ f$.

Fix $\mathbf{B} \in \mathcal{K}$ with $\mathbf{B} \subseteq \mathbf{K}$, and find $n < \omega$ large enough so that $\mathbf{B} \subseteq \mathbf{A}_n$. Let $i_{\mathbf{B}} : \mathbf{B} \to \mathbf{A}_n$ denote the inclusion embedding. For each $f \in H_{\mathbf{B}}$, let

$$S_f = \{j < R_n : \exists s \in H_n (s \circ i_{\mathbf{B}} = f \text{ and } \gamma_n(s) = j)\}.$$

We define a reflexive graph Γ on $H_{\mathbf{B}}$ by declaring $(f, h) \in \Gamma$ iff $S_f \cap S_h \neq \emptyset$. Define a coloring $\gamma_{\mathbf{B}}$ on $H_{\mathbf{B}}$ by sending $f \in H_{\mathbf{B}}$ to the connected component of f in Γ. Equivalently, $\gamma_{\mathbf{B}}$ is the finest possible coloring with $\gamma_{\mathbf{B}} \preceq \gamma_n$.

First let us argue that $\gamma_{\mathbf{B}}$ is an unavoidable coloring. Fix a connected component $X \subseteq H_{\mathbf{B}}$ of Γ. We can write $X = \{f \in H_{\mathbf{B}} : S_f \subseteq S\}$ for some $S \subseteq R_n$, namely $S = \bigcup_{f \in X} S_f$. But then we also have $X = \{f \in H_{\mathbf{B}} : S_f \cap S \neq \emptyset\}$. Now fix $\eta \in \widehat{G}$ towards showing that $\eta^{-1}(X) \neq \emptyset$. Pick $j \in S$, and find $s \in H_n$ with $\gamma_n(\eta \circ s) = j$. Then $\eta \circ s \circ i_{\mathbf{B}} \in X$, so $\eta^{-1}(X) \neq \emptyset$.

To see that $\gamma_{\mathbf{B}}$ is an unavoidable $R_{\mathbf{B}}$-coloring, let $\delta : H_{\mathbf{B}} \to R_{\mathbf{B}}$ be unavoidable. Find $\eta \in \widehat{G}$ with $\gamma_{\mathbf{B}} \cdot \eta \sim \delta \cdot \eta \preceq \gamma_n \cdot \eta$. Using Lemma 3.8.5, find $p \in S(G)$ with $\gamma_n \cdot \eta \cdot p = \gamma_n$. By Proposition 3.4.5, we have $\delta \cdot \eta \cdot p$ an unavoidable $R_{\mathbf{B}}$-coloring with $\delta \cdot \eta \cdot p \preceq \gamma_n$. We will show that $\gamma_{\mathbf{B}} \sim \delta \cdot \eta \cdot p$. We must have $\gamma_{\mathbf{B}} \geq \delta \cdot \eta \cdot p$; by construction, $\gamma_{\mathbf{B}}$ is the finest possible coloring with $\gamma_{\mathbf{B}} \preceq \gamma_n$. But since γ_B is an unavoidable coloring and $\delta \cdot \eta \cdot p$ is an unavoidable $R_{\mathbf{B}}$-coloring, we must have $\gamma_{\mathbf{B}} \sim \delta \cdot \eta \cdot p$. In particular, $\gamma_{\mathbf{B}}$ is an unavoidable $R_{\mathbf{B}}$-coloring as desired. □

We now see that Question 3.7.16 is equivalent to asking whether having all big Ramsey degrees finite allows us to construct a big Ramsey structure. One of the key theorems of this section gives us a connection between big Ramsey structures and universal completion flows.

Definition 3.8.6. Suppose $\mathbf{K}$ admits a big Ramsey structure, which by Proposition 3.8.3 we view as a coherent sequence $\gamma := (\gamma_n)_{n<\omega}$ of unavoidable R_n-colorings of H_n for each $n < \omega$. Form the diagram D_γ as in Example 3.4.15. Then we call the G-flow X_{D_γ} a *big Ramsey flow*.

Theorem 3.8.7. *Suppose* $\mathbf{K}$ *admits a big Ramsey structure. Then any big Ramsey flow for* G *is a universal completion flow, and any two universal completion flows are unique up to* G*-flow isomorphism.*

We will prove Theorem 3.8.7 over the course of the next few propositions. Until otherwise noted, we fix a big Ramsey structure $\gamma = (\gamma_n)_{n<\omega}$ on $\mathbf{K}$; it will help to set $\mathrm{ran}(\gamma_n) = J_n$ for pairwise disjoint sets $\{J_n : n < \omega\}$ each of size R_n. We view γ as an element of X_{D_γ} by setting $\gamma|_n = \gamma_n(i_n)$.

A key ingredient to the proof is the notion of lift introduced in section 2.5. Indeed, the following lemma was the initial motivation for studying lifts of ambits in [36].

Lemma 3.8.8. *Suppose* (X, x_0) *is a completion ambit, and let* $Y \subseteq S(G)$ *be a lift. Then* Y *is a completion flow; any* $y \in Y$ *with* $\lambda_{x_0}(y) = x_0$ *is a completion point of* Y*.*

Proof. Consider any $y \in Y$ with $\lambda_{x_0}(y) = x_0 \cdot y = x_0$. Then for any $\eta \in \widehat{G}$, $\lambda_{x_0}(y \cdot \eta) = x_0 \cdot \eta \in \mathcal{A}(X)$ as x_0 is a completion point. Since $\lambda_{x_0} : Y \to X$ is a strong map, we have $y \cdot \eta \in \mathcal{A}(Y)$, showing that y is a completion point of Y. □

Proposition 3.8.9. *Suppose* (X, x_0) *is a completion ambit, and let* $Y \subseteq S(G)$ *be a lift with level representation* $Y = \varprojlim Y_m$*. Then* $|Y_m| \leq R_m$ *for each* $m < \omega$*.*

Proof. Pick $y \in Y$ with $x_0 \cdot y = x_0$. Then y is a completion point of Y. Let $\psi : Y_m \to k$ be a continuous surjection for some $k < \omega$. Then the coloring $\delta : H_m \to k$ given by $\delta(f) = \psi(y \cdot f)$ is unavoidable. It follows that $k \leq R_m$, so $|Y_m| \leq R_m$. □

Remark. Proposition 3.8.9 applies any time the big Ramsey degrees are all finite, even if there isn't a big Ramsey structure. In particular, if all big Ramsey degrees are finite, then every completion flow of G is metrizable.

Proposition 3.8.10. *(X_{D_γ}, γ) is a completion ambit. If $Y \subseteq S(G)$ is a lift of (X_{D_γ}, γ), then $\lambda_\gamma : Y \to X_{D_\gamma}$ is an isomorphism.*

Proof. Fix $i \in J_m$ and $\eta \in \widehat{G}$. We need to find $g \in G$ with $\gamma \cdot \eta \cdot g|_m = i$. Since γ_m is unavoidable, find $f \in H_m$ with $\gamma_m(\eta \cdot f) = i$. It follows that $\gamma \cdot \eta \cdot f = i$. Pick any $g \in G$ with $g|_m = f$ to complete the proof that γ is a completion point.

Now let $Y \subseteq S(G)$, $Y = \varprojlim Y_m$ be a lift of (X_{D_γ}, γ). Then the map $\lambda_\gamma : Y \to X_{D_\gamma}$ gives rise to maps $\lambda_\gamma^m : Y_m \to J_m$ given by $\lambda_\gamma^m(r) = \gamma \cdot r$. Each map λ_γ^m must be surjective. As $|Y_m| \leq R_m$, λ_γ^m is a bijection, from which it follows that λ_γ is an isomorphism. □

Lemma 3.8.11. *Suppose $Y = \varprojlim Y_m \subseteq S(G)$ is a metrizable subflow, and assume there is $u \in Y$ an idempotent with dense orbit. Then if $\varphi : Y \to Y$ is a surjective G-map, we have that φ is an isomorphism.*

Proof. First note that $u \cdot S(G) = u \cdot Y = Y$. From this it follows that $\varphi = \lambda_{\varphi(u)}$, as $\varphi(y) = \varphi(uy) = \varphi(u)y$. Now each $\lambda_{\varphi(u)}$ gives rise to maps $\lambda_{\varphi(u)}^m : Y_m \to Y_m$ given by $\lambda_{\varphi(u)}^m(r) = \varphi(u) \cdot r$ for each $m < \omega$. Each map $\lambda_{\varphi(u)}^m$ must be surjective; as each Y_m is finite, each $\lambda_{\varphi(u)}^m$ is bijective, hence φ is an isomorphism. □

Proof of Theorem 3.8.7. Instead of working with X_{D_γ}, we instead use Proposition 3.8.10 to work with an isomorphic lift $Y \subseteq S(G)$, and let $y_0 \in Y$ be the unique point with $\gamma \cdot y_0 = \gamma$. Note that y_0 is an idempotent and a completion point. By Lemma 3.8.11, it follows that if Y is a universal completion flow, then it must be unique up to isomorphism.

Now let W be another completion flow, with $w_0 \in W$ a completion point. We need to find a surjective G-map from Y to W. Using Lemma 3.8.8, we may assume that $W \subseteq S(G)$ with W_m finite for each $m < \omega$. For each $m < \omega$, form the colorings $\lambda_{y_0}^m$ and $\lambda_{w_0}^m : H_m \to Y_m$. Each is an unavoidable coloring, and $\lambda_{y_0}^m$ is an unavoidable R_m-coloring. For each $n < \omega$, use Proposition 3.7.13 to find $\eta_n \in \widehat{G}$ so that for every $m \leq n$, we have that $\lambda_{y_0}^m \cdot \eta_n \geq \lambda_{w_0}^m \cdot \eta_n$.

Since y_0 is a completion point, we can find $p_n \in S(G)$ with $y_0 \cdot \eta_n \cdot p_n = y_0$. Let $p \in S(G)$ be a cluster point of the sequence $\eta_n p_n$. Then $y_0 \cdot p = y_0$, so for each $m < \omega$, we have $\lambda_{y_0}^m \cdot p = \lambda_{y_0 \cdot p}^m = \lambda_{y_0}^m$. By Propositions 3.4.5 and 3.7.13, each $\lambda_{w_0}^m \cdot p = \lambda_{w_0 p}^m$ is an unavoidable $|W_m|$-coloring, and $\lambda_{y_0}^m \geq \lambda_{w_0}^m \cdot p$. Let $c_m : Y_m \to W_m$ be the surjective map so that $c_m \circ \lambda_{y_0}^m = \lambda_{w_0 p}^m$.

We now show that $w_0 \cdot py_0 = w_0 \cdot p$. Since y_0 is an idempotent, we have $w_0 \cdot p \cdot i_m = c_m(y_0 \cdot i_m) = c_m(y_0 \cdot y_0 \cdot i_m) = w_0 \cdot p \cdot y_0 \cdot i_m$. Since this holds for each $m < \omega$, we have $w_0 \cdot py_0 = w_0 \cdot p$ as desired.

Consider the G-map $\lambda_{w_0p} : Y \to W$. To check that λ_{w_0p} is surjective, it suffices to check that $w_0py_0 = w_0p$ has dense orbit. So let $m < \omega$, and consider $r \in W_m$. We have seen that $\lambda^m_{w_0p} : H_m \to W_m$ is an unavoidable $|W_m|$-coloring, so in particular it is surjective. Find some $g \in G$ with $w_0pg|_m = r$. Hence λ_{w_0p} is surjective. $\square$

We now collect some examples of automorphism groups whose universal completion flows can be described using Theorem 3.8.7.

Example 3.8.12 (A countable set). The simplest example of an automorphism group with a metrizable universal completion flow is the group $G = S_\infty = \mathrm{Aut}(\mathbf{K})$, where $\mathbf{K} = \mathbb{N}$ is just a countable set with no additional structure. If we let $\mathbf{A}_n \subseteq \mathbf{K}$ be a subset of size n, we see by Ramsey's theorem that the small Ramsey degree and the big Ramsey degree of $\mathbf{A}_n$ are both $n!$ (recall that we are considering embedding versions of Ramsey degree, so the $n!$ comes from the automorphisms of $\mathbf{A}_n$). It follows that the universal minimal flow $M(S_\infty)$ is the universal completion flow of S_∞. This is just the space of linear orders on a countable set. In particular, if $<$ is *any* linear order of $\mathbb{N}$, then $\langle \mathbb{N}, < \rangle$ is a big Ramsey structure.

More generally, whenever $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ and $\mathcal{K}$ is a Fraïssé class where the big and small Ramsey degrees are finite and equal, then $M(\mathrm{Aut}(\mathbf{K}))$ is the universal completion flow of $\mathrm{Aut}(\mathbf{K})$. It would be interesting to find other examples of Fraïssé classes where the big and small Ramsey degrees coincide.

Example 3.8.13 (Finite distance ultrametric spaces). Another family of examples are the classes of finite distance ultrametric spaces. Fix $S \subseteq (0, \infty)$ with $|S| = r < \omega$, and let $\mathcal{K}_0$ be the class of finite ultrametric spaces with distances from S. The big Ramsey behavior of these classes was described by Nguyen Van Thé in [24]. To describe the big Ramsey structure, it is useful to instead work with the class $\mathcal{K}$ of rooted finite trees of height at most r. Structures in $\mathcal{K}$ are of the form $\langle T, \preceq, L_0, ..., L_r \rangle$, where $\preceq$ is the partial order and L_i is a unary predicate saying that a node is on level i of the tree (it should be remarked that this class is not hereditary, but we will discuss Fraïssé classes without HP in the next section). Then $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$ is the rooted, countably-branching tree of height r. If $\mathbf{K}_0 = \mathrm{Flim}(\mathcal{K}_0)$, then $\mathbf{K}_0$ can be identified with the set of leaves of $\mathbf{K}$, and $\mathrm{Aut}(\mathbf{K}) \cong \mathrm{Aut}(\mathbf{K}_0)$. Then we have the following.

Proposition. Let $\mathbf{K}' = \langle \mathbf{K}, \leq \rangle$, where $\leq$ is a linear order in order type ω which extends the tree order. Then $\mathbf{K}'$ is a big Ramsey structure for $\mathbf{K}$.

It follows that $X_{\mathbf{K}'}$, the space of linear orderings on $\mathbf{K}$ which extend the tree order, is the universal completion flow for $G = \mathrm{Aut}(\mathbf{K})$. It should be noted that this is not the same space as $M(G)$. Nguyen Van Thé describes $M(G)$ in [25]; this is the space of all *convex* linear orderings on the leaves of $\mathbf{K}$. Here, a linear order of the leaves is convex if whenever $s, t, u \in \mathbf{K}$ are leaves with $s \leq t \leq u$, then the meet of s and u is an initial segment of t.

Example 3.8.14 (The rational linear order). We next consider the example from the introduction, the rational linear order $\langle \mathbb{Q}, \leq \rangle$. The group $G = \mathrm{Aut}(\mathbb{Q})$ is extremely amenable, but it is not hard to see that G admits a non-trivial completion flow; the space of linear orders on $\mathbb{Q}$ is a good example, as for instance any linear order of order type ω is a completion point. As was mentioned in the introduction, this is not the universal completion flow. A good account of the big Ramsey behavior of $\mathbb{Q}$ can be found in Todorčević's book [30].

To construct the universal completion flow, first consider the binary tree $2^{<\omega}$. If $x, y \in 2^{<\omega}$, we set $x \wedge y$ to be the longest common initial segment of both x and y. We set $|x|$ to be the unique $n < \omega$ so that $x \in 2^n$. If $x \in 2^n$ and $m < n$, write $x|_m$ for the restriction of x to domain m. We say that x and y are *comparable* if either $x \wedge y = x$ or $x \wedge y = y$; otherwise we say x and y are *incomparable*. Define $x \prec y$ if x and y are incomparable and $x(|x \wedge y|) < y(|x \wedge y|)$, which in the case of the binary tree means $x(|x \wedge y|) = 0$ and $y(|x \wedge y|) = 1$. A subset $A \subseteq 2^{<\omega}$ is an *antichain* if no two distinct elements of A are comparable. Notice that if A is an antichain, then $\prec$ is a linear order on A.

It is possible to build an antichain $Q \subseteq 2^{<\omega}$ so that $\langle Q, \prec \rangle \cong \langle \mathbb{Q}, < \rangle$, and we freely identify Q with $\mathbb{Q}$. We now define the 4-ary relation R as follows. If $p \leq q \leq r \leq s \in \mathbb{Q}$, we set $R(p, q, r, s)$ iff $|p \wedge q| \leq |r \wedge s|$. We then have the following.

Proposition. The structure $\mathbf{Q}' := \langle \mathbb{Q}, \leq, R \rangle$ *is a big Ramsey structure for* $\langle \mathbb{Q}, \leq \rangle$.

We can then interpret the space $X_{\mathbf{Q}'}$ as a space of total pre-orders on $W := \mathbb{Q} \cup [\mathbb{Q}]^2$. Let L be a total pre-order of W, and let E_L be the induced equivalence relation on W. Then $L \in X_{\mathbf{Q}'}$ iff $L|_{\mathbb{Q}}$ is a linear order, and given $a < b < c \in \mathbb{Q}$, we have $\neg E_L(\{a, b\}, \{b, c\})$, and $\{a, c\}$ is E_L-equivalent to the L-least of $\{a, b\}$ or $\{b, c\}$.

Example 3.8.15 (The random graph). The *Random graph*, often called the *Rado graph*, is the Fraïssé limit of the class of all finite graphs. A countable graph $\langle Q, E \rangle$ is isomorphic to the Rado graph iff for any disjoint and finite $F_0, F_1 \subseteq Q$, then there is $x \in Q \setminus (F_0 \cup F_1)$ so that $\neg E(x, y)$ for each $y \in F_0$ and $E(x, z)$ for every $z \in F_1$.

It can be shown that the big Ramsey degree of any finite subgraph of the Rado graph is finite by using Milliken's tree theorem. To construct a big Ramsey structure, we follow the presentation of Laflamme, Sauer, and Vuksanovic [20]. Once again, we consider the binary tree $2^{<\omega}$. We call a subset $T \subseteq 2^{<\omega}$ *transversal* if $|x| \neq |y|$ for any distinct $x, y \in T$. If $T \subseteq 2^{<\omega}$ is transversal, we can give T a graph structure E, where if $x, y \in T$ and $|x| < |y|$, we set $E(x, y)$ iff $y(|x|) = 1$. Now let $\langle Q, E \rangle$ be a Rado graph, and fix an enumeration $Q = \{q_n : n < \omega\}$. To each q_n, we associate an element $x_n \in 2^n$, where for $m < n < \omega$, we set $x_n(m) = 1$ iff $E(q_m, q_n)$.

Theorem 7.6 from [20] now gives us an unavoidable coloring for each finite subgraph. To turn this into a Ramsey structure, we need to perform one extra step. Find a subset $T \subseteq Q$ of the Rado graph so that $\langle T, E|_T\rangle$ is isomorphic to the Rado graph, and so that $\{x_n : q_n \in T\} \subseteq 2^{<\omega}$ is an antichain. By doing this, we can ensure that the collection of "non-diagonal" tuples as defined in [20] is empty.

To describe the resulting structure, it will be useful to instead assume that we have mapped Q into a transversal antichain in $2^{<\omega}$ which respects the graph structure. With this identification, we now define $\prec$ and the 4-ary relation R as before.

Proposition. The structure $\mathbf{Q'} := \langle Q, E, \prec, R\rangle$ *is a big Ramsey structure for the Rado graph* $\langle Q, E\rangle$.

Similar to the example of the rationals, the space $X_{\mathbf{Q'}}$ can be described as a space of pairs (L_0, L_1), where L_0 is a linear order of Q and L_1 is a total preorder of $Q \cup [Q]^2$. Describing precisely which pairs are in the closure of the big Ramsey structure seems to be somewhat more difficult.

Example 3.8.16 (The orders $\mathbb{Q}_n$ and the tournament $\mathbf{S}(2)$). The last examples we will consider are the dense local order $\mathbf{S}(2)$ and the orders $\mathbb{Q}_n$. The dense local order is a countable *tournament*, a directed graph $\langle S, E\rangle$ where for distinct $x, y \in S$ exactly one of $E(x, y)$ or $E(y, x)$ holds. One way to construct $\mathbf{S}(2)$ is to consider a countable dense set of points on the unit circle so that no two points are exactly π radians apart. Then set $E(x, y)$ iff y is less than π radians counterclockwise from x. Then $\langle S, E\rangle$ is isomorphic to $\mathbf{S}(2)$.

The big Ramsey behavior of the structure $\mathbf{S}(2)$ is studied by Laflamme, Nguyen Van Thé, and Sauer in [19]. The trick to analyzing $\mathbf{S}(2)$ is to instead analyze the structure $\mathbb{Q}_2 := \langle \mathbb{Q}, <, P_0, P_1\rangle$, where $\langle \mathbb{Q}, <\rangle$ is the rational order, and each P_i is a dense subset of $\mathbb{Q}$ with $\mathbb{Q} = P_0 \sqcup P_1$. The structures $\mathbb{Q}_n$ are defined similarly; they are rational orders with a distinguished partition into n dense pieces. The authors of [19] prove a slight generalization of Milliken's theorem to obtain big Ramsey results for the structures $\mathbb{Q}_n$, the "colored" version alluded to in the title of [19]. However, once this is proven, the big Ramsey structures for $\mathbb{Q}_n$ are easy to describe; namely, if $\langle \mathbb{Q}, <, R\rangle$ is a big Ramsey structure for the rational order, then $\langle \mathbb{Q}, <, P_0, ...P_{n-1}, R\rangle$ is a big Ramsey structure for $\mathbb{Q}_n$.

Using the big Ramsey structure for $\mathbb{Q}_2$, one obtains a big Ramsey structure for $\mathbf{S}(2)$ as follows. Represent $\mathbf{S}(2)$ as $\langle S, E\rangle$, where S is a dense subset of the unit circle as before. Then we can view $\mathbb{Q}_2$ as a structure with underlying set S. We let P_0 be those points below the x-axis, and P_1 be the points above. Let S^1 be the unit circle; define the map $\varphi : S \to S^1$ by setting $\varphi(x) = x$ for $x \in P_0$ and $\varphi(x) = x \cdot e^{i\pi}$ for $x \in P_1$. Note that φ is an injection with $\mathrm{Im}(\varphi)$ contained below the x-axis. Then for $x, y \in S$, we set $x < y$ iff $\varphi(y)$ is to the right of $\varphi(x)$. Then if $\mathbb{Q}_2' := \langle S, <, P_0, P_1, R\rangle$ is a big Ramsey structure for $\mathbb{Q}_2$, then $\mathbf{S'} := \langle S, E, <, P_0, P_1, R\rangle$ is a big Ramsey structure for $\mathbf{S}(2)$.

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