



2013

# DECENTRALIZED ADAPTIVE CONTROL FOR UNCERTAIN LINEAR SYSTEMS: TECHNIQUES WITH LOCAL FULL-STATE FEEDBACK OR LOCAL RELATIVE-DEGREE- ONE OUTPUT FEEDBACK

James D. Polston

University of Kentucky, [daniel.polston@gmail.com](mailto:daniel.polston@gmail.com)

[Click here to let us know how access to this document benefits you.](#)

---

## Recommended Citation

Polston, James D., "DECENTRALIZED ADAPTIVE CONTROL FOR UNCERTAIN LINEAR SYSTEMS: TECHNIQUES WITH LOCAL FULL-STATE FEEDBACK OR LOCAL RELATIVE-DEGREE-ONE OUTPUT FEEDBACK" (2013). *Theses and Dissertations--Mechanical Engineering*. 24.  
[https://uknowledge.uky.edu/me\\_etds/24](https://uknowledge.uky.edu/me_etds/24)

This Master's Thesis is brought to you for free and open access by the Mechanical Engineering at UKnowledge. It has been accepted for inclusion in Theses and Dissertations--Mechanical Engineering by an authorized administrator of UKnowledge. For more information, please contact [UKnowledge@lsv.uky.edu](mailto:UKnowledge@lsv.uky.edu).

**STUDENT AGREEMENT:**

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained and attached hereto needed written permission statements(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine).

I hereby grant to The University of Kentucky and its agents the non-exclusive license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless a preapproved embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

**REVIEW, APPROVAL AND ACCEPTANCE**

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's dissertation including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

James D. Polston, Student

Dr. Jesse B. Hoagg, Major Professor

Dr. James M. McDonough, Director of Graduate Studies

---

DECENTRALIZED ADAPTIVE CONTROL FOR  
UNCERTAIN LINEAR SYSTEMS:  
TECHNIQUES WITH LOCAL FULL-STATE FEEDBACK OR  
LOCAL RELATIVE-DEGREE-ONE OUTPUT FEEDBACK

---

THESIS

---

A thesis submitted in partial fulfillment of the  
requirements for the degree of Master of Science in  
Mechanical Engineering in the College of Engineering  
at the University of Kentucky

by

James Daniel Polston

Lexington, Kentucky

Director: Dr. Jesse B. Hoagg, Professor of Mechanical Engineering

Lexington, Kentucky

2013

Copyright © James Daniel Polston 2013

## ABSTRACT OF THESIS

### DECENTRALIZED ADAPTIVE CONTROL FOR UNCERTAIN LINEAR SYSTEMS: TECHNIQUES WITH LOCAL FULL-STATE FEEDBACK OR LOCAL RELATIVE-DEGREE-ONE OUTPUT FEEDBACK

This thesis presents decentralized model reference adaptive control techniques for systems with full-state feedback and systems with output feedback. The controllers are strictly decentralized, that is, each local controller uses feedback from only local subsystems and no information is shared between local controllers.

The full-state feedback decentralized controller is effective for multi-input systems, where the dynamics matrix and control-input matrix are unknown. The decentralized controller achieves asymptotic stabilization and command following in the presence of sinusoidal disturbances with known spectrum. We present a construction technique of the reference-model dynamics such that the decentralized controller is effective for systems with arbitrarily large subsystem interconnections.

The output-feedback decentralized controller is effective for single-input single-output subsystems that are minimum phase and relative degree one. The decentralized controller achieves asymptotic stabilization and disturbance rejection in the presence of an unknown disturbance, which is generated by an unknown Lyapunov-stable linear system.

**KEYWORDS:** Adaptive control, Decentralized control, Large-scale systems, Disturbance rejection, Command following

James Daniel Polston  
June 25, 2013

DECENTRALIZED ADAPTIVE CONTROL FOR  
UNCERTAIN LINEAR SYSTEMS:  
TECHNIQUES WITH LOCAL FULL-STATE FEEDBACK OR  
LOCAL RELATIVE-DEGREE-ONE OUTPUT FEEDBACK

by

James Daniel Polston

Dr. Jesse B. Hoagg

---

Director of Thesis

Dr. James M. McDonough,

---

Director of Graduate Studies

June 25, 2013

---

## **Acknowledgments**

I would like to express my sincere gratitude to my advisor Dr. Jesse Hoagg for the continuous support and guidance he has given me over the past two years of graduate work. He deserves a tremendous amount of credit for this work and my development as a researcher and technical writer. Furthermore, I would like to thank him for the amount of time and effort he spends motivating and encouraging all of his students through his invaluable teaching and mentoring.

I would also like to thank the mentors and colleagues I had during my internships with the NASA Academy and NASA Undergraduate Student Research Program. They provided the tools and direction in my undergraduate years that motivated me to pursue a graduate degree.

I thank my mother and father for instilling in me the desire to learn, design, and create at a young age. They have been incredible examples to me in my personal growth, and I thank them wholeheartedly. Additionally, I would like to thank my sisters for everything they have done to make me the person I am today.

I am also grateful for all of the love, support, and encouragement of my friends. In particular, I would like to thank the guys of the Bonnie Brae house for the continued sanity-checks, sarcastic humor, and entertainment.

Finally, I would like to thank Dr. Johné Parker and Dr. Michael Seigler for serving on my committee.

## Table of Contents

<b>Acknowledgments</b>	<b>iii</b>
<b>List of Figures</b>	<b>vi</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Overview of Model Reference Adaptive Control . . . . .	1
1.2 Background and Motivation for Decentralized Adaptive Control . . . . .	2
1.3 Summary of Chapters . . . . .	5
<b>2 Full-State-Feedback Model Reference Adaptive Control</b>	<b>7</b>
2.1 Introduction . . . . .	7
2.2 Problem Formulation . . . . .	8
2.3 Adaptive Stabilization . . . . .	9
2.4 Adaptive Command Following . . . . .	11
2.5 Numerical Examples . . . . .	14
2.6 Conclusions . . . . .	19
<b>3 Decentralized Adaptive Control with Local Full-State Feedback</b>	<b>20</b>
3.1 Introduction . . . . .	20
3.2 Problem Formulation . . . . .	21
3.3 Decentralized Adaptive Stabilization . . . . .	25
3.4 Decentralized Adaptive Command Following and Disturbance Rejection	33
3.5 Numerical Examples . . . . .	41

3.6	Conclusions . . . . .	51
<b>4</b>	<b>Relative-Degree-One Output-Feedback Model Reference Adaptive Control with Exogenous Disturbance</b>	<b>53</b>
4.1	Introduction . . . . .	53
4.2	Problem Formulation . . . . .	54
4.3	Ideal Controller . . . . .	56
4.4	Relative-Degree-One Model Reference Adaptive Control with Disturbance Rejection . . . . .	64
4.5	Numerical Examples . . . . .	68
4.6	Conclusions . . . . .	74
<b>5</b>	<b>Decentralized Relative-Degree-One Output-Feedback Adaptive Control with Exogenous Disturbance</b>	<b>75</b>
5.1	Introduction . . . . .	75
5.2	Problem Formulation . . . . .	76
5.3	Ideal Decentralized Controller . . . . .	78
5.4	Relative-Degree-One Decentralized Adaptive Stabilization and Disturbance Rejection . . . . .	91
5.5	Numerical Examples . . . . .	102
5.6	Conclusion . . . . .	113
<b>6</b>	<b>Conclusions and Future Work</b>	<b>115</b>
	<b>Appendices</b>	<b>117</b>
A	Proofs of Propositions 3.1 , 3.2, and 3.3 . . . . .	117
	<b>Bibliography</b>	<b>120</b>
	<b>Vita</b>	<b>124</b>



## List of Figures

1.1	Schematic diagram of a decentralized control architecture for a large-scale complex system. . . . .	3
2.1	Schematic diagram of adaptive stabilization architecture. . . . .	10
2.2	Schematic diagram of adaptive command following architecture. . . . .	12
2.3	Adaptive stabilization. . . . .	16
2.4	A serially connected, two-mass structure used in Example 2.2. . . . .	17
2.5	Adaptive command following for a mass-spring-dashpot system. . . . .	18
3.1	Schematic diagram of decentralized adaptive stabilization architecture. . . . .	26
3.2	Decentralized adaptive stabilization with local scalar dynamics. . . . .	30
3.3	Decentralized adaptive stabilization with local vector dynamics. . . . .	32
3.4	Schematic diagram of decentralized adaptive command following and disturbance rejection control architecture. . . . .	36
3.5	Decentralized adaptive command following and disturbance rejection with local scalar dynamics. . . . .	43
3.6	The serially connected, $\ell$ -mass structure used in Examples 3.4–3.6. . . . .	45
3.7	Decentralized adaptive command following for a mass-spring-dashpot system. . . . .	46
3.8	Decentralized adaptive command following and disturbance rejection for a mass-spring-dashpot system. . . . .	48

3.9	Decentralized adaptive command following and disturbance rejection for a mass-spring-dashpot system with ten masses. . . . .	50
3.10	Diagram of the planar double pendulum. . . . .	51
3.11	Decentralized adaptive stabilization and disturbance rejection for a planar double pendulum. . . . .	52
4.1	Schematic diagram of MRAC architecture. . . . .	65
4.2	Adaptive command following for an asymptotically stable SISO relative-degree-one system. . . . .	70
4.3	Adaptive command following and disturbance rejection for an asymptotically stable SISO relative-degree-one system. . . . .	72
4.4	Adaptive command following for an unstable SISO relative-degree-one system. . . . .	73
5.1	Schematic diagram of relative-degree-one decentralized adaptive architecture. . . . .	93
5.2	Decentralized adaptive stabilization for an unstable system with $\ell = 2$ . . . . .	104
5.3	Decentralized adaptive disturbance rejection for an unstable system with $\ell = 2$ . . . . .	106
5.4	Decentralized adaptive stabilization for an unstable system with $\ell = 4$ . . . . .	109
5.5	Decentralized adaptive disturbance rejection for an asymptotically stable system with $\ell = 4$ . . . . .	112
5.6	Decentralized adaptive command following for an asymptotically stable system with $\ell = 2$ . . . . .	114

## Chapter 1 Introduction

### 1.1 Overview of Model Reference Adaptive Control

The objective of model reference adaptive control (MRAC) is to force an uncertain system to asymptotically follow the trajectory of a known reference model [1–16]. Classical MRAC techniques are divided into two categories: (i) systems with full-state feedback and (ii) systems with output feedback.

Classical full-state-feedback MRAC applies to multi-input linear time-invariant systems, where the dynamics and input matrices are unknown [1–8]. The goal of full-state-feedback MRAC is to design a control such that all closed-loop signals are bounded and the state of the plant asymptotically follows the state of a reference model. Full-state-feedback MRAC operates under the assumption of matched uncertainty, that is, the plant and reference-model matrices satisfy matching conditions. Full-state-feedback MRAC has been extended to address systems with nonlinearities [9, 10].

Classical output-feedback MRAC applies to single-input single-output (SISO) linear time-invariant systems that are minimum phase [1–8, 11–16]. The goal of output-feedback MRAC is to design a control such that all closed-loop signals are bounded and the output of the plant asymptotically follows the output of a reference model. Output-feedback MRAC operates under the assumptions that the plant is minimum phase, the sign of the high-frequency gain is known, an upper bound on the order of the plant is known, and the relative degree is known. While output-feedback MRAC techniques apply to systems with arbitrary-but-known relative degree, this

thesis focuses on output-feedback MRAC for relative-degree-one systems.

## 1.2 Background and Motivation for Decentralized Adaptive Control

Decentralized control systems are composed of interconnected subsystems, where each local controller has access to information from only the local subsystem. The goal of decentralized control is to design local controllers such that each local subsystem behaves in a desired manner, while no information is exchanged between the local controllers. The performance of each local subsystem is affected by the local control as well as the nonlocal dynamics and nonlocal controls.

The need for decentralized control arises in large-scale complex systems such as interconnected power networks, large flexible structures, and water systems. Decentralized control techniques divide the complex control problem into subproblems, and generally reduce the computational power required for control. Figure 1.1 shows a decentralized control architecture, where each subsystem contains a local sensor, local controller, and local actuator. Each local controller has access to local sensors but does not have access to nonlocal sensors and does not have knowledge of the nonlocal control objectives. See [17–21] for more details on decentralized control.

Classical full-state-feedback MRAC has been extended to address decentralized control with local full-state feedback [22–27]. The controllers in [22], [23] are strictly decentralized, that is, each local controller requires only local full-state measurement and no information is shared between the local controllers. However, the controllers in [22], [23] do not yield asymptotically perfect command following. Furthermore, the errors in [22], [23] converge to residual sets that depend on the interconnection matrices and the controller design parameters. In contrast, asymptotically perfect command following is achieved in [24–27], but these controllers are not strictly decentralized. More specifically, the controllers in [24–27] rely on centralized reference models, meaning that each local controller has access to all reference-model states.

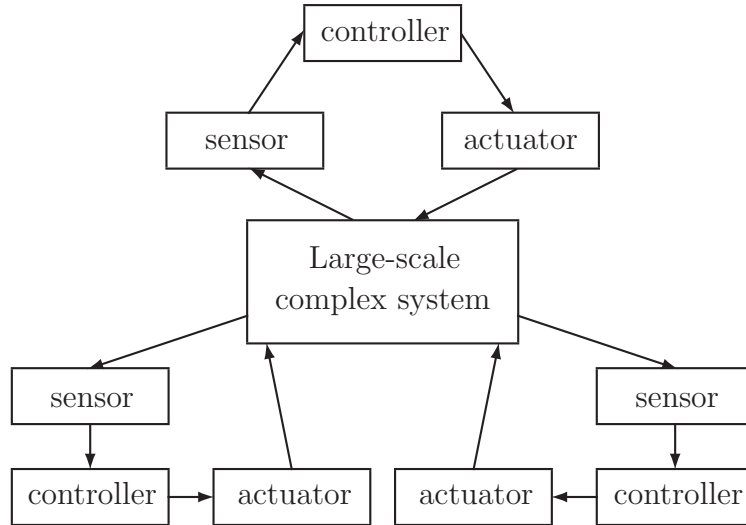


Figure 1.1: Schematic diagram of a decentralized control architecture for a large-scale complex system.

Thus, each local subsystem has knowledge of the control objectives of all nonlocal subsystems.

The controllers in [24–27] require some knowledge of the subsystem-interconnection matrices. For example, [24–26] assumes that an upper bound on the maximum singular value of each subsystem-interconnection matrix is known. In [27], the maximum singular value of each subsystem-interconnection matrix must be less than a fixed bound, which is no larger than 1. Thus, the controller in [27] requires weak subsystem interconnection.

While the adaptive controllers in [24–27] address command following, none of these techniques address disturbance rejection. Furthermore, the approaches of [24–27] are restricted to local subsystems that are single-input, and require that the local control-input matrices are known.

Classical output-feedback MRAC has been extended to address decentralized control for SISO subsystems with local output feedback [28–30]. The approaches of [28–30] address stabilization and command following provided that each local subsystem is minimum phase. The controllers in [28–30] guarantee bounded tracking

errors, but do not drive the tracking errors to zero. In particular, each local tracking error converges to a residual set that depends on the interconnection matrices and the local controller design parameters. The results in [28] are limited to local subsystems that are exactly proper, that is, subsystems with nonzero direct feedthrough. The results in [29] address local subsystems that are relative degree one or two, and the results in [30] address local subsystems that are relative degree greater than two. Decentralized adaptive control using neural networks is addressed in [31–33].

In this thesis, we present decentralized adaptive control techniques for local subsystems with full-state feedback and local subsystems with relative-degree-one output feedback. In Chapter 3, we present a strictly decentralized adaptive controller that uses local full-state feedback and does not require a centralized reference model or sharing of nonlocal reference-model signals. This decentralized adaptive controller allows for multi-input local subsystems, where the local control-input matrices are uncertain. The controller yields asymptotic stabilization and command following in the presence of sinusoidal disturbances with known spectrum. The technique is effective for arbitrarily large subsystem interconnections, provided that a bounding matrix, related to the subsystem-interconnection matrices, is known and that the reference-model dynamics matrix is designed to admit a positive-definite solution to a bounded-real Riccati equation. We provide a construction of the reference-model dynamics matrix, which does admit a positive-definite solution to the Riccati equation.

In Chapter 5, we present an output-feedback decentralized adaptive controller for subsystems that are minimum phase and relative degree one. This controller is strictly decentralized and yields asymptotic stabilization and disturbance rejection, where the disturbance is unknown but generated from a Lyapunov-stable linear system. The technique relies on the assumption that the magnitudes of the subsystem interconnections satisfy a bounding condition.

## 1.3 Summary of Chapters

### Summary of Chapter 2

Chapter 2 presents the classical full-state feedback MRAC technique for linear time-invariant systems. Full-state-feedback MRAC allows for multi-input systems, where the dynamics and control-input matrices are unknown. Full-state feedback MRAC operates under the assumption of matched uncertainty, where three matching assumptions are invoked. The goal of full-state-feedback MRAC is to design a control such that all closed-loop signals are bounded and the state of the plant asymptotically follows the state of a reference model.

### Summary of Chapter 3

Chapter 3 presents a decentralized MRAC technique for linear time-invariant systems, where each local controller uses full-state feedback from the local subsystem. The controller is strictly decentralized, meaning that no information (including reference-model dynamics) is shared between local controllers. This decentralized adaptive controller achieves asymptotically perfect stabilization and command following in the presence of sinusoidal disturbances with known spectrum. Furthermore, the controller is effective for systems with arbitrarily large subsystem interconnections.

### Summary of Chapter 4

Chapter 4 presents classical output-feedback MRAC for SISO linear time-invariant systems that are minimum phase and relative degree one. Classical MRAC is effective for stabilization and command following. In this thesis, we extended classical MRAC to address disturbance rejection, where the disturbance is unknown but generated from a Lyapunov-stable linear system.

### Summary of Chapter 5

Chapter 5 presents a decentralized MRAC method for SISO linear time-invariant subsystems that are minimum phase and relative degree one. The decentralized adap-

tive controller is strictly decentralized, that is, no information is shared between local controllers. This decentralized adaptive controller is effective for stabilization and disturbance rejection, where the disturbance is unknown but generated from a Lyapunov-stable linear system.

All notation is introduced in the chapter where the notation is used. Furthermore, notation may change between chapters. Thus, notation is specific to the chapter in which it appears.



## Chapter 2 Full-State-Feedback Model Reference Adaptive Control

This chapter presents classical model reference adaptive control (MRAC), where all states of the system are available for feedback. The controller is effective for stabilization and command following.

### 2.1 Introduction

In this chapter, we present the classical full-state-feedback MRAC technique for linear time-invariant systems. Full-state-feedback MRAC allows for multi-input systems, where the dynamics matrix and control-input matrix are unknown. Full-state-feedback MRAC operates under the assumption of matched uncertainty, where three matching assumptions are invoked. The goal of classical MRAC is to design a control such that all closed-loop signals are bounded and the state asymptotically follows the state of a reference model. The classical full-state-feedback adaptive controller can be used for stabilization and asymptotic command following. Full-state-feedback MRAC techniques are described in [1–8].

In Section 2.2, we introduce the full-state-feedback MRAC problem. We present a controller for adaptive stabilization in Section 2.3, and extend the controller to address command following in Section 2.4. Examples are given in Section 2.5, and conclusions are given in Section 2.6.

## 2.2 Problem Formulation

For  $t \geq 0$ , consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (2.1)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x(0) \in \mathbb{R}^n$  is the initial condition, and  $u(t) \in \mathbb{R}^m$  is the control input.

Next, consider the reference model

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad (2.2)$$

where  $x_m(t) \in \mathbb{R}^n$  is the reference-model state,  $x_m(0) \in \mathbb{R}^n$  is the initial condition,  $r(t) \in \mathbb{R}^q$  is the bounded reference-model command,  $A_m \in \mathbb{R}^{n \times n}$  is the reference-model dynamics matrix, and  $B_m \in \mathbb{R}^{n \times q}$  is the reference-model input matrix. We assume that  $A_m$  is asymptotically stable, that is, the eigenvalues of  $A_m$  are contained in the open-left-half complex plane. Our goal is to develop an adaptive controller that generates  $u(t)$  such that  $x(t)$  asymptotically follows  $x_m(t)$ . Thus, our goal is to drive the performance

$$e(t) \triangleq x(t) - x_m(t)$$

to zero.

We make the following assumptions regarding the system (2.1) and the reference model (2.2):

(A2.1) There exists a positive-definite matrix  $F \in \mathbb{R}^{m \times m}$ , which need not be known, such that  $\hat{B} \triangleq BF$  is known.

(A2.2) There exists  $K_* \in \mathbb{R}^{m \times n}$ , such that  $A_m = A + BK_*$ .

(A2.3) There exists  $L_* \in \mathbb{R}^{m \times q}$  such that  $B_m = BL_*$ .

The system (2.1) is otherwise unknown. Specifically,  $A$ ,  $B$ , and  $x(0)$  are otherwise unknown. Assumptions (A2.1)–(A2.3) are the standard full-state-feedback MRAC matching conditions. See [1–8] for more details. Note that (A2.2) does not require that  $K_*$  be known.

### 2.3 Adaptive Stabilization

In this section, we address adaptive stabilization, where the reference-model command is zero (i.e.,  $r(t) \equiv 0$ ). Consider the controller

$$u(t) = K(t)x(t), \quad (2.3)$$

where  $K : [0, \infty) \rightarrow \mathbb{R}^{m \times n}$  is given by

$$\dot{K}(t) = -\hat{B}^T P x(t) x^T(t) \Gamma, \quad (2.4)$$

where  $\Gamma \in \mathbb{R}^{n \times n}$  is positive definite, and  $P \in \mathbb{R}^{n \times n}$  is the positive-definite solution to

$$A_m^T P + P A_m + Q = 0, \quad (2.5)$$

where  $Q \in \mathbb{R}^{n \times n}$  is positive definite. The adaptive stabilization architecture is shown in Figure 2.1.

Next, define

$$\tilde{K}(t) \triangleq K(t) - K_*, \quad (2.6)$$

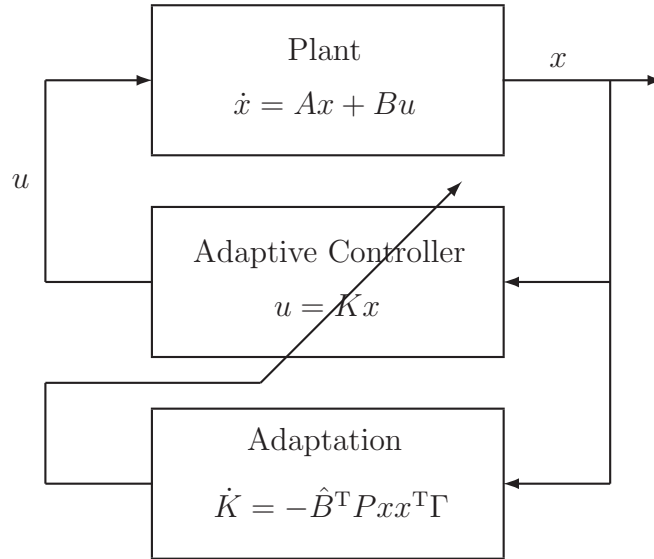


Figure 2.1: Schematic diagram of adaptive stabilization architecture given by (2.1), (2.3), and (2.4).

and it follows from (2.1) and (2.3) that

$$\dot{x}(t) = A_m x(t) + B \tilde{K}(t) x(t). \quad (2.7)$$

The following theorem is the main result on full-state-feedback adaptive stabilization.

**Theorem 2.1.** *Consider the closed-loop system (2.4) and (2.7), where the open-loop system (2.1) satisfies (A2.1)–(A2.2), and  $r(t) \equiv 0$ . Then, the equilibrium  $(x, \tilde{K}) \equiv 0$  is Lyapunov stable. Furthermore, for all initial conditions  $x(0) \in \mathbb{R}^n$  and  $K(0) \in \mathbb{R}^{m \times n}$ , the following statements hold:*

- (i)  $x(t)$ ,  $u(t)$ , and  $K(t)$  are bounded.
- (ii)  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Proof.* Define the Lyapunov function

$$V(x, \tilde{K}) \triangleq x^T P x + \text{tr} F^{-1} \tilde{K} \Gamma^{-1} \tilde{K}^T,$$

where  $P \in \mathbb{R}^{n \times n}$  is the positive-definite solution to (2.5). Evaluating the derivative of  $V$  along the trajectory of (2.4) and (2.7), and using (A2.1) yields

$$\begin{aligned} \dot{V}(x, \tilde{K}) &= \dot{x}^T P x + x^T P \dot{x} + 2 \text{tr} F^{-1} \dot{\tilde{K}} \Gamma^{-1} \tilde{K}^T \\ &= x^T (A_m^T P + P A_m) x + 2 x^T \tilde{K}^T B^T P x + 2 \text{tr} F^{-1} \dot{\tilde{K}} \Gamma^{-1} \tilde{K}^T \\ &= -x^T Q x + 2 \text{tr} (B^T P x x^T \tilde{K}^T + F^{-1} \dot{\tilde{K}} \Gamma^{-1} \tilde{K}^T) \\ &= -x^T Q x, \end{aligned}$$

where  $Q \in \mathbb{R}^{n \times n}$  is positive definite. Therefore, the equilibrium  $(x, \tilde{K}) \equiv 0$  is Lyapunov stable, and for all initial conditions,  $x$  and  $\tilde{K}$  are bounded. Since  $x$  and  $\tilde{K}$  are bounded, it follows from (2.3) and (2.6) that  $K$  and  $u$  are bounded, which confirms (i).

Next, since  $V$  is positive definite and radially unbounded, and  $\dot{V}(x, \tilde{K}) = -x^T Q x$ , it follows from LaSalle's invariance principle [34, Theorem 4.4] that for all initial conditions,  $\lim_{t \rightarrow \infty} x(t) \rightarrow 0$ , which confirms (ii).  $\square$

## 2.4 Adaptive Command Following

In this section, we address adaptive command following. Consider the controller

$$u(t) = K(t)x(t) + L(t)r(t), \tag{2.8}$$

where  $K : [0, \infty) \rightarrow \mathbb{R}^{m \times n}$  and  $L : [0, \infty) \rightarrow \mathbb{R}^{m \times q}$  are given by

$$\dot{K}(t) = -\hat{B}^T P e(t) x^T(t) \Gamma, \tag{2.9}$$

$$\dot{L}(t) = -\hat{B}^T P e(t) r^T(t) \Lambda, \quad (2.10)$$

where  $\Gamma \in \mathbb{R}^{n \times n}$  and  $\Lambda \in \mathbb{R}^{q \times q}$  are positive definite, and  $P \in \mathbb{R}^{n \times n}$  is the positive-definite solution to (2.5). The MRAC architecture is shown in Figure 2.2.

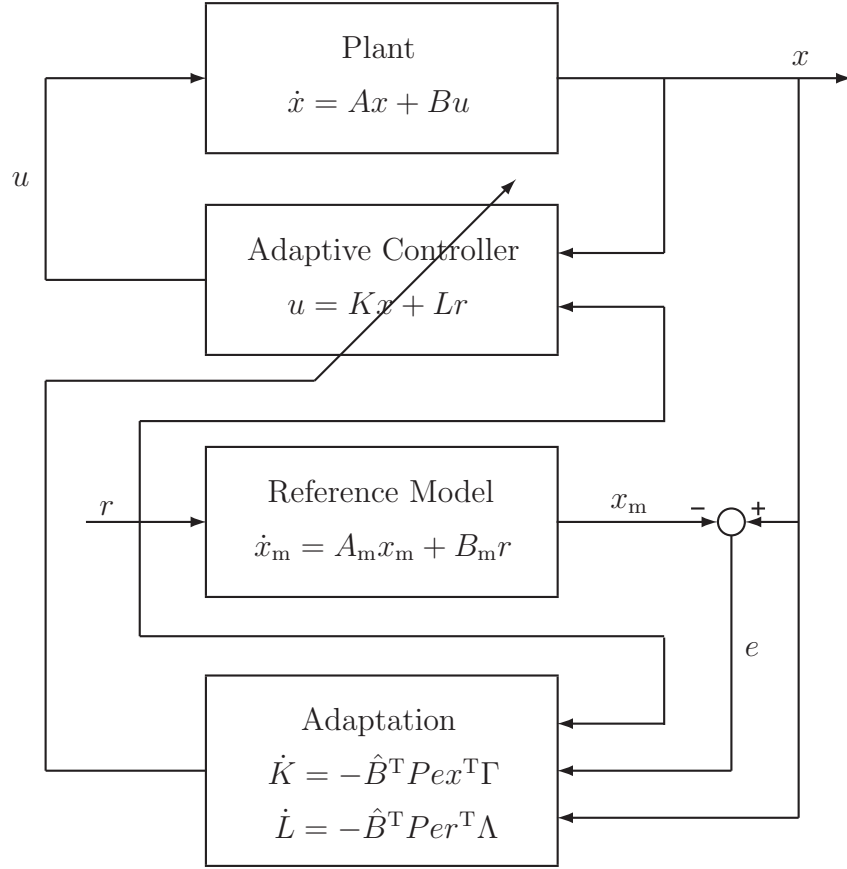


Figure 2.2: Schematic diagram of adaptive command following architecture given by (2.1) and (2.8)–(2.10).

The following theorem is the main result on adaptive command following.

**Theorem 2.2.** *Consider the closed-loop system (2.1) and (2.8)–(2.10), where the open-loop system (2.1) satisfies (A2.1)–(A2.3). Then, for all initial conditions  $x(0) \in \mathbb{R}^n$ ,  $K(0) \in \mathbb{R}^{m \times n}$ , and  $L(0) \in \mathbb{R}^{m \times q}$ , the following statements hold:*

- (i)  $x(t)$ ,  $u(t)$ ,  $K(t)$ , and  $L(t)$  are bounded.

(ii)  $\lim_{t \rightarrow \infty} e(t) = 0$ .

*Proof.* Define

$$\begin{aligned}\tilde{K}(t) &\triangleq K(t) - K_*, \\ \tilde{L}(t) &\triangleq L(t) - L_*,\end{aligned}$$

and it follows from (2.1) and (2.8) that

$$\dot{x}(t) = A_m x(t) + B\tilde{K}(t)x(t) + BL(t)r(t). \quad (2.11)$$

Next, subtracting (2.2) from (2.11), and using (A2.3) yields

$$\dot{e}(t) = A_m e(t) + B\tilde{K}(t)x(t) + B\tilde{L}(t)r(t). \quad (2.12)$$

Define the Lyapunov-like function

$$V(e, \tilde{K}, \tilde{L}) \triangleq e^T P e + \text{tr} F^{-1} \tilde{K} \Gamma^{-1} \tilde{K}^T + \text{tr} F^{-1} \tilde{L} \Lambda^{-1} \tilde{L}^T,$$

where  $P \in \mathbb{R}^{n \times n}$  is the positive-definite solution to (2.5). Evaluating the derivative of  $V$  along the trajectory of (2.9), (2.10), and (2.12), and using (A2.1) yields

$$\begin{aligned}\dot{V}(e, \tilde{K}, \tilde{L}) &= \dot{e}^T P e + e^T P \dot{e} + 2\text{tr} F^{-1} \dot{\tilde{K}} \Gamma^{-1} \tilde{K}^T + 2\text{tr} F^{-1} \dot{\tilde{L}} \Lambda^{-1} \tilde{L}^T \\ &= e^T (A_m^T P + P A_m) e + 2x^T \tilde{K}^T B^T P e + 2r^T \tilde{L}^T B^T P e \\ &\quad + 2\text{tr} F^{-1} \dot{\tilde{K}} \Gamma^{-1} \tilde{K}^T + 2\text{tr} F^{-1} \dot{\tilde{L}} \Lambda^{-1} \tilde{L}^T \\ &= -e^T Q e + 2\text{tr} (B^T P e x^T \tilde{K}^T + B^T P e r^T \tilde{L}^T \\ &\quad + F^{-1} \dot{\tilde{K}} \Gamma^{-1} \tilde{K}^T + F^{-1} \dot{\tilde{L}} \Lambda^{-1} \tilde{L}^T) \\ &= -e^T Q e,\end{aligned} \quad (2.13)$$

where  $Q \in \mathbb{R}^{n \times n}$  is positive definite. Thus,  $0 \leq e^T Q e = -\dot{V}(e, \tilde{K}, \tilde{L})$ . Moreover, integrating from 0 to  $\infty$  yields

$$\begin{aligned} 0 &\leq \int_0^\infty e^T(t) Q e(t) dt \\ &= V(e(0), \tilde{K}(0), \tilde{L}(0)) - \lim_{t \rightarrow \infty} V(e(t), \tilde{K}(t), \tilde{L}(t)) \\ &\leq V(e(0), \tilde{K}(0), \tilde{L}(0)), \end{aligned} \tag{2.14}$$

where the upper and lower bounds imply that  $\int_0^\infty e^T(t) Q e(t) dt$  exists. Thus, it follows from (2.14) that  $V$  is bounded, which implies that  $e$ ,  $\tilde{K}$ , and  $\tilde{L}$  are bounded. Since  $r$  is bounded and  $A_m$  is asymptotically stable, (2.2) implies that  $x_m$  is bounded. Moreover, since  $e$ ,  $x_m$ ,  $\tilde{K}$ , and  $\tilde{L}$  are bounded, it follows that  $x$ ,  $u$ ,  $K$ , and  $L$  are bounded, which confirms (i).

To show (ii), it follows from (2.14) that  $\int_0^\infty e^T(t) Q e(t) dt$  exists. Next, since  $e$ ,  $x$ ,  $r$ ,  $\tilde{K}$ , and  $\tilde{L}$  are bounded, (2.12) implies that  $\dot{e}$  is bounded. Next, since  $e$  and  $\dot{e}$  are bounded, it follows that

$$\frac{d}{dt} \left[ e^T(t) Q e(t) \right] = 2\dot{e}^T(t) Q e(t)$$

is bounded. Thus,  $f(t) \triangleq e^T(t) Q e(t)$  is uniformly continuous. Since  $\int_0^\infty f(t) dt$  exists and  $f(t)$  is uniformly continuous, Barbalat's Lemma implies that  $\lim_{t \rightarrow \infty} f(t) = 0$ . Thus,  $\lim_{t \rightarrow \infty} e(t) = 0$ , which confirms (ii).  $\square$

## 2.5 Numerical Examples

We now present examples to demonstrate adaptive stabilization and command following with full-state-feedback MRAC.



**Example 2.1.** *Adaptive stabilization.* Consider the system (2.1), where

$$A = \begin{bmatrix} 2 & 0 & -5 \\ -1 & -5 & -1 \\ 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad (2.15)$$

Note that  $A$  in (2.15) is unstable with eigenvalues at  $-4.55$  and  $1.28 \pm j3.74$ . We let  $\hat{B} = I_3$ , which satisfies (A2.1). Next, let

$$A_m = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -8 \end{bmatrix}, \quad (2.16)$$

where  $A_m$  is asymptotically stable, and it follows that (A2.2) is satisfied. Next, let  $Q = I_3$ , and let  $P$  be the positive-definite solution to (2.5).

The adaptive controller (2.3) and (2.4) is implemented in feedback with the system (2.1) and (2.15), where  $\Gamma = 10^3 I_3$ . Figure 2.3 shows a time history of  $x(t)$  and  $u(t)$ , where the initial condition is  $x(t) = [-2 \ 2 \ -1]^T$ . The state  $x(t)$  converges asymptotically to zero.  $\triangle$

**Example 2.2.** *Adaptive command following for a mass-spring-dashpot system.* Consider the serially connected structure shown in Figure 2.4, where  $u_1$  and  $u_2$  are control forces, and  $q_1$  and  $q_2$  are the positions of the first and second masses, respectively. The equations of motion for the system are given by (2.1), where

$$A = \begin{bmatrix} -\frac{c_1+c_2}{m_1} & -\frac{k_1+k_2}{m_1} & \frac{c_2}{m_1} & \frac{k_2}{m_1} \\ 1 & 0 & 0 & 0 \\ \frac{c_2}{m_2} & \frac{k_2}{m_2} & -\frac{c_2+c_3}{m_2} & -\frac{k_2+k_3}{m_2} \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \\ 0 & 0 \end{bmatrix}, \quad (2.17)$$

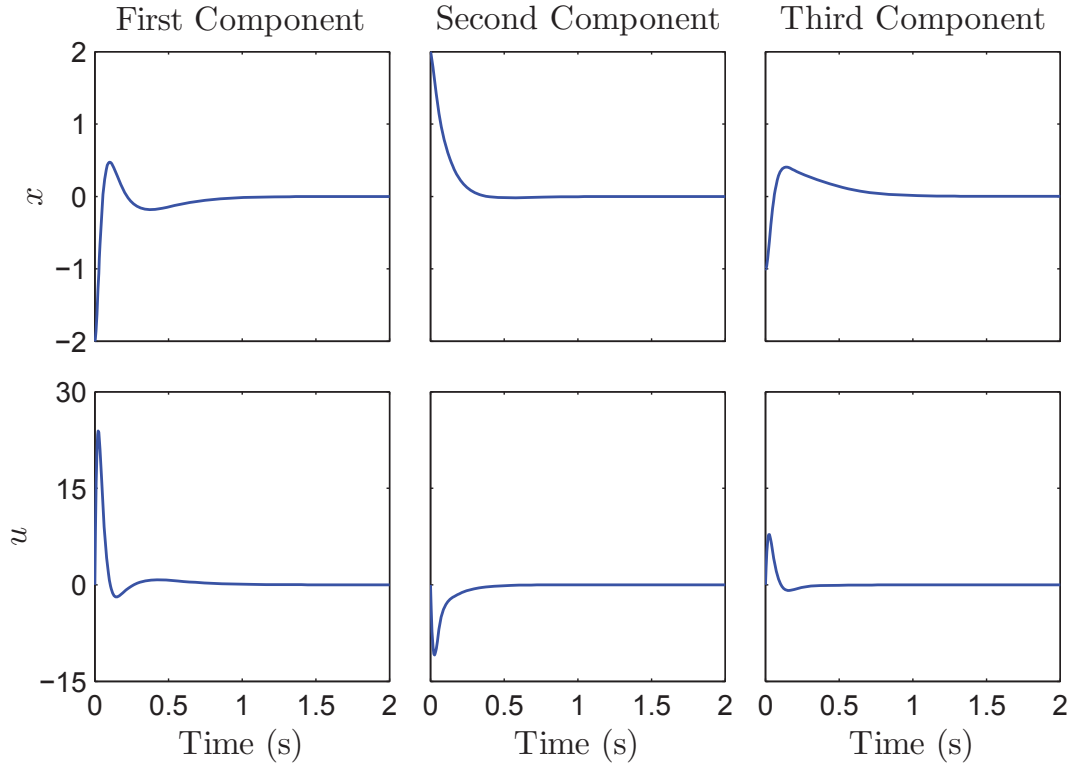


Figure 2.3: *Adaptive stabilization.* The adaptive controller (2.3) and (2.4) is implemented in feedback with the system (2.1) and (2.15). The state  $x(t)$  converges asymptotically to zero.

$$x = \begin{bmatrix} \dot{q}_1 & q_1 & \dot{q}_2 & q_2 \end{bmatrix}^T. \quad (2.18)$$

The masses are  $m_1 = 0.2$  kg and  $m_2 = 0.4$  kg; the damping coefficients are  $c_1 = 5$  kg/s,  $c_2 = 2$  kg/s, and  $c_3 = 3$  kg/s; and the spring constants are  $k_1 = 8$  kg/s<sup>2</sup>,  $k_2 = 9$  kg/s<sup>2</sup>, and  $k_3 = 14$  kg/s<sup>2</sup>.

We let

$$\hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad (2.19)$$

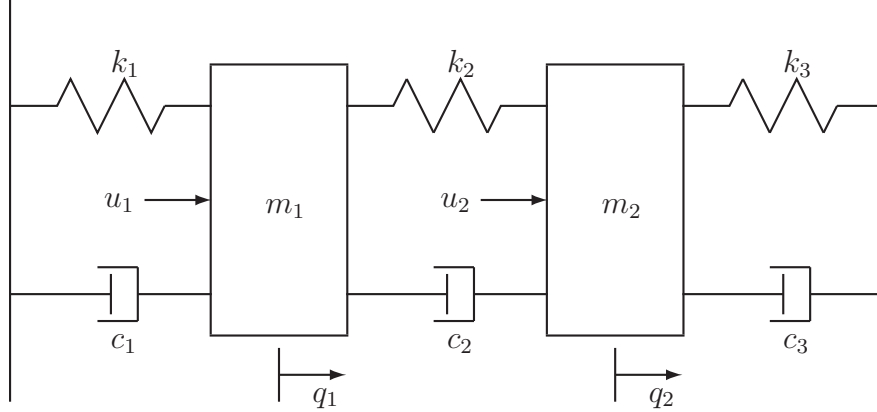


Figure 2.4: A serially connected, two-mass structure used in Example 2.2.

which satisfies (A2.1). Next, let

$$A_m = \begin{bmatrix} -20 & -30 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -20 & -30 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_m = \begin{bmatrix} 30 & 0 \\ 0 & 0 \\ 0 & 30 \\ 0 & 0 \end{bmatrix}, \quad (2.20)$$

which satisfy (A2.2) and (A2.3), respectively. Next, let  $Q = I_4$ , and let  $P$  be the positive-definite solution to (2.5).

The reference-model command is  $r(t) = [ r_1(t) \ r_2(t) ]^T$ , where  $r_1(t) = 0.1 \sin 0.25\pi t$  and  $r_2(t) = 0.2 \cos 0.125\pi t$ . The adaptive controller (2.8)–(2.10) is implemented in feedback with the two-mass system (2.1), (2.17), and (2.18), where  $\Gamma = 10^4 I_4$  and  $\Lambda = 10^4 I_2$ . Figure 2.5 provides a time history of  $x(t)$ ,  $x_m(t)$ ,  $e(t)$ , and  $u(t)$ , where the initial conditions are  $q_1(0) = q_2(0) = 0$  m and  $\dot{q}_1(0) = \dot{q}_2(0) = 0$  m/s. The two-mass system is allowed to run open-loop for 10 seconds, then the adaptive controller is turned on. Figure 2.5 shows  $\lim_{t \rightarrow \infty} e(t) = 0$ .  $\triangle$

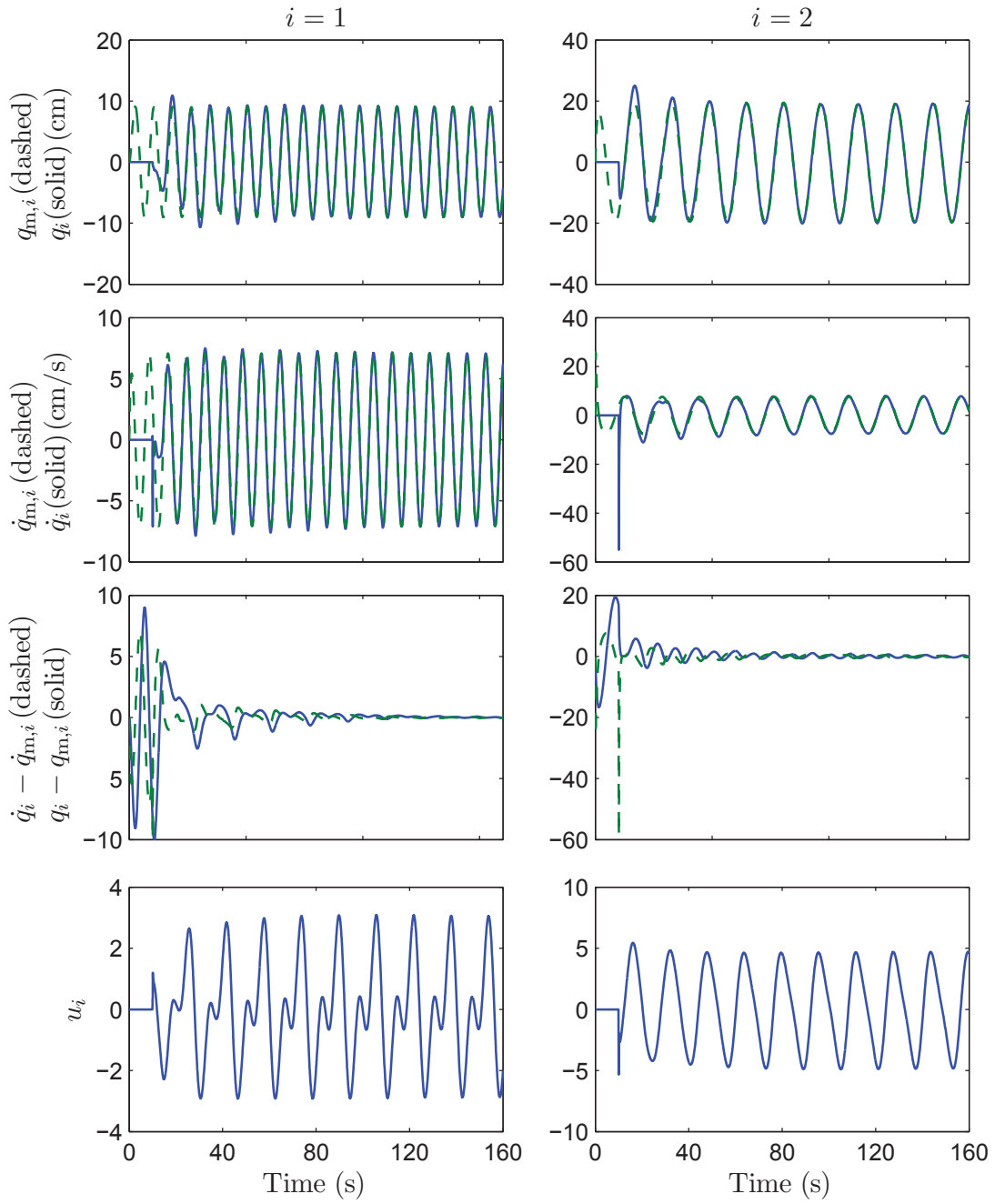


Figure 2.5: *Adaptive command following for a mass-spring-dashpot system.* The adaptive controller (2.8)–(2.10) is implemented in feedback with the two-mass system (2.1), (2.17), and (2.18). The error  $e(t)$  converges asymptotically to zero.

## 2.6 Conclusions

This chapter reviewed the classical full-state feedback MRAC technique for multi-input linear time-invariant systems. The adaptive controller operates under the assumption of matched uncertainty. The controller yields stabilization and asymptotic command following.

## Chapter 3 Decentralized Adaptive Control with Local Full-State Feedback

This chapter presents a decentralized model reference adaptive control method, where each local controller uses full-state feedback from the local subsystem. The controller is strictly decentralized, meaning that no information (including reference-model dynamics) is shared between local controllers. This decentralized controller achieves asymptotically perfect stabilization and command following in the presence of sinusoidal disturbances with known spectrum. Furthermore, the controller is effective for systems with arbitrarily large subsystem interconnections. We provide controller and reference-model design examples to demonstrate the decentralized adaptive controller. The results from this chapter have been submitted for publication in [35].

### 3.1 Introduction

In this chapter, we present a strictly decentralized adaptive controller that uses local full-state feedback and does not require a centralized reference model or sharing of nonlocal reference-model signals. This decentralized adaptive controller allows for multi-input local subsystems, where the local control-input matrices are uncertain. The controller yields asymptotic stabilization and command following in the presence of sinusoidal disturbances with known spectrum. The technique is effective for arbitrarily large subsystem interconnections, provided that a bounding matrix on the subsystem-interconnection matrices is known and that the reference-model dynam-

ics matrix is designed to admit a positive-definite solution to a bounded-real Riccati equation. We provide a construction of the reference-model dynamics matrix, which does admit a positive-definite solution to the Riccati equation.

In Section 3.2, we introduce the decentralized adaptive control problem. We present a controller for decentralized adaptive stabilization in Section 3.3, and extend the controller to address command following and disturbance rejection in Section 3.4. Examples are given in Section 3.5, and conclusions are given in Section 3.6.

### 3.2 Problem Formulation

For  $t \geq 0$ , consider the system

$$\dot{x}_1(t) = \sum_{j=1}^{\ell} A_{1,j}x_j(t) + B_1u_1(t) + D_1w_1(t), \quad (3.1)$$

$\vdots$

$$\dot{x}_\ell(t) = \sum_{j=1}^{\ell} A_{\ell,j}x_j(t) + B_\ell u_\ell(t) + D_\ell w_\ell(t), \quad (3.2)$$

where  $\mathcal{J} \triangleq \{1, 2, \dots, \ell\}$ , for all  $i \in \mathcal{J}$ ,  $x_i(t) \in \mathbb{R}^{n_i}$  is the state,  $x_i(0) \in \mathbb{R}^{n_i}$  is the initial condition,  $u_i(t) \in \mathbb{R}^{m_i}$  is the control input, and  $w_i(t) \in \mathbb{R}^{d_i}$  is the exogenous disturbance.

For each  $i \in \mathcal{J}$ ,  $x_i$  is the *local state*, and  $u_i$  is the *local control*. Moreover, for each  $i \in \mathcal{J}$ , the local control  $u_i(t)$  uses feedback of the local state  $x_i(t)$ , but does not use feedback of the nonlocal states  $\{x_j(t)\}_{j \in \mathcal{J} \setminus \{i\}}$ . Unless otherwise stated, all statements in this chapter that involve the subscript  $i$  are for all  $i \in \mathcal{J}$ .

Next, consider the reference model

$$\dot{x}_{m,i}(t) = A_{m,i}x_{m,i}(t) + B_{m,i}r_i(t), \quad (3.3)$$

where  $x_{m,i}(t) \in \mathbb{R}^{n_i}$  is the state,  $x_{m,i}(0) \in \mathbb{R}^{n_i}$  is the initial condition,  $r_i(t) \in \mathbb{R}^{q_i}$  is the reference-model command,  $A_{m,i} \in \mathbb{R}^{n_i \times n_i}$  is the reference-model dynamics matrix, and  $B_{m,i} \in \mathbb{R}^{n_i \times q_i}$  is the reference-model input matrix. We assume that  $A_{m,i}$  is asymptotically stable, that is, the eigenvalues of  $A_{m,i}$  are contained in the open-left-half complex plane. Our goal is to develop a series of local adaptive controllers that generate  $u_i(t)$  such that  $x_i(t)$  asymptotically follows  $x_{m,i}(t)$  in the presence of the disturbance  $w_i(t)$ . Thus, our goal is to drive the performance

$$e_i(t) \triangleq x_i(t) - x_{m,i}(t)$$

to zero.

In Section 3.3, we develop a controller for decentralized adaptive stabilization. Specifically, we focus on the case where  $w_i(t) \equiv 0$ ,  $r_i(t) \equiv 0$ , and the goal is to stabilize the origin of (3.1)–(3.2). In Section 3.4, we address command following and disturbance rejection.

We make the following assumptions regarding the system (3.1)–(3.2) and the reference model (3.3):

(A3.1) There exists a positive-definite matrix  $F_i \in \mathbb{R}^{m_i \times m_i}$ , which need not be known, such that  $\hat{B}_i \triangleq B_i F_i$  is known.

(A3.2) There exists  $K_{*,i} \in \mathbb{R}^{m_i \times n_i}$  such that

$$A_{m,i} = A_{i,i} + B_i K_{*,i}. \quad (3.4)$$

(A3.3) There exists a known positive-semidefinite matrix  $\Omega_i \in \mathbb{R}^{n_i \times n_i}$  such that

$$\Omega_i \geq \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} A_{i,j}^T. \quad (3.5)$$



(A3.4) There exists a positive-definite matrix  $P_i \in \mathbb{R}^{n_i \times n_i}$  such that

$$A_{m,i}^T P_i + P_i A_{m,i} + Q_i + P_i \Omega_i P_i \leq 0, \quad (3.6)$$

where  $Q_i \in \mathbb{R}^{n_i \times n_i}$  is positive definite and satisfies  $Q_i > \ell I_{n_i}$ .

The system (3.1)–(3.2) is otherwise unknown. Specifically,  $A_{1,1}, \dots, A_{1,\ell}, \dots, A_{\ell,\ell}$ ,  $B_1, \dots, B_\ell$ , and  $x_1(0), \dots, x_\ell(0)$  are otherwise unknown.

Assumptions (A3.1) and (A3.2) are standard full-state-feedback MRAC matching conditions [1–8]. For example, if  $(A_{i,i}, B_i)$  is in controllable canonical form, then (A3.2) is satisfied by a reference-model dynamics matrix  $A_{m,i}$  that is also in controllable canonical form. Note that (A3.2) does not require that  $K_{*,i}$  be known.

Assumption (A3.3) is satisfied if upper bounds on the maximum singular values of  $\{A_{i,j}\}_{j \in \mathcal{J} \setminus \{i\}}$  are known. Specifically,  $\Omega_i \geq \sum_{j \in \mathcal{J} \setminus \{i\}} \sigma_{\max}^2(A_{i,j}) I_{n_i}$ , where  $\sigma_{\max}(\cdot)$  is the maximum singular value, satisfies (A3.3). However,  $\Omega_i$  appears in the Riccati expression (3.6), which may not have a positive-definite solution for all  $\Omega_i$ . Furthermore, the existence of a positive-definite solution  $P_i$  to (3.6) depends on the reference-model dynamics matrix  $A_{m,i}$ . Thus, assumptions (A3.2)–(A3.4) are coupled. In order to satisfy (A3.2)–(A3.4), the known reference-model dynamics matrix  $A_{m,i}$  and the known uncertainty bound  $\Omega_i$  must satisfy (3.4) and (3.5), respectively, and admit a positive-definite solution  $P_i$  to (3.6). Note that the solutions  $P_1, \dots, P_\ell$  are used to construct the decentralized adaptive controller.

Define  $\mathcal{K} \triangleq \{(i, j) \mid i \in \mathcal{J}, j \in \mathcal{J}, i \neq j\}$ . The following result considers the system (3.1)–(3.2), where  $m_i = 1$ ,  $(A_{i,i}, B_i)$  is in controllable canonical form, and for all  $(i, j) \in \mathcal{K}$ ,  $A_{i,j}$  has matched uncertainty. This result provides constructions of  $A_{m,i}$ ,  $\Omega_i$ , and  $\hat{B}_i$  such that (A3.1)–(A3.4) are satisfied. The proof is in Appendix A.

**Proposition 3.1.** Consider the system (3.1)–(3.2), where  $m_i = 1$ . Assume that

$$A_{i,i} = \begin{bmatrix} -a_{i,n-1} & \cdots & -a_{i,1} & -a_{i,0} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} b_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (3.7)$$

and for all  $(i, j) \in \mathcal{K}$ ,  $A_{i,j} = B_i \Delta_{i,j}^T$ , where  $\Delta_{i,j} \in \mathbb{R}^{n_j \times 1}$ ;  $a_{i,0}, \dots, a_{i,n-1} \in \mathbb{R}$ ; and  $b_i \in \mathbb{R}$ . Let  $\alpha_i(s) = \alpha_{i,n-1}s^{n-1} + \cdots + \alpha_{i,1}s + \alpha_{i,0}$  be asymptotically stable, where  $\alpha_{i,0}, \dots, \alpha_{i,n-1}$  are positive. Let

$$A_{m,i} = \begin{bmatrix} -\eta_i \alpha_{i,n-1} & \cdots & -\eta_i \alpha_{i,1} & -\eta_i \alpha_{i,0} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad \hat{B}_i = \begin{bmatrix} \beta_i \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (3.8)$$

where  $\eta_i > 0$  and  $b_i \beta_i > 0$ . Furthermore, let  $Q_i > \ell I_{n_i}$ , and let  $\gamma_i > 0$  satisfy

$$\gamma_i \geq \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{b_i^2}{\beta_i^2} \Delta_{i,j}^T \Delta_{i,j}. \quad (3.9)$$

Then, the following statements hold:

- (i) There exists  $F_i > 0$  such that  $\hat{B}_i = B_i F_i$ .
- (ii) For all  $\eta_i > 0$ , there exists  $K_{*,i} \in \mathbb{R}^{1 \times n_i}$  that satisfies (3.4).
- (iii)  $\Omega_i \triangleq \gamma_i \hat{B}_i \hat{B}_i^T$  satisfies (3.5).
- (iv) For sufficiently large  $\eta_i > 0$ ,  $A_{m,i}$  is asymptotically stable, and there exists a positive-definite matrix  $P_i \in \mathbb{R}^{n_i \times n_i}$  that satisfies (3.6).

Proposition 3.1 provides sufficient conditions under which (A3.1)–(A3.4) are satis-

fied. Specifically, if the reference-model dynamics matrix  $A_{m,i}$  is given by (3.8) and  $\eta_i > 0$  is sufficiently large, then (i)–(iv) of Proposition 3.1 imply that (A3.1)–(A3.4) are satisfied. Note that there is no restriction on the magnitude of the subsystem-interconnection matrices  $\{A_{i,j}\}_{j \in \mathcal{J} \setminus \{i\}}$ . The parameter  $\eta_i > 0$  can be designed using the known bound  $\Omega_i$ . Specifically,  $\eta_i > 0$  can be increased until (3.6) admits a positive-definite solution. Note that the conditions of Proposition 3.1 are not necessary to satisfy (A3.1)–(A3.4).

### 3.3 Decentralized Adaptive Stabilization

In this section, we address decentralized adaptive stabilization, where the disturbances and reference-model commands are zero (i.e.,  $r_i(t) \equiv 0$  and  $w_i(t) \equiv 0$ ). Consider the controller

$$u_i(t) = K_i(t)x_i(t), \quad (3.10)$$

where  $K_i : [0, \infty) \rightarrow \mathbb{R}^{m_i \times n_i}$  is given by

$$\dot{K}_i(t) = -\hat{B}_i^T P_i x_i(t) x_i^T(t) \Gamma_i, \quad (3.11)$$

where  $\Gamma_i \in \mathbb{R}^{n_i \times n_i}$  is positive definite, and  $P_i \in \mathbb{R}^{n_i \times n_i}$  is the positive-definite solution to (3.6). The decentralized adaptive stabilization architecture is shown in Figure 3.1.

Next, define

$$\tilde{K}_i(t) \triangleq K_i(t) - K_{*,i}, \quad (3.12)$$

and it follows from (3.1)–(3.2) and (3.10) that

$$\dot{x}_i(t) = A_{m,i}x_i(t) + B_i \tilde{K}_i(t)x_i(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j}x_j(t). \quad (3.13)$$

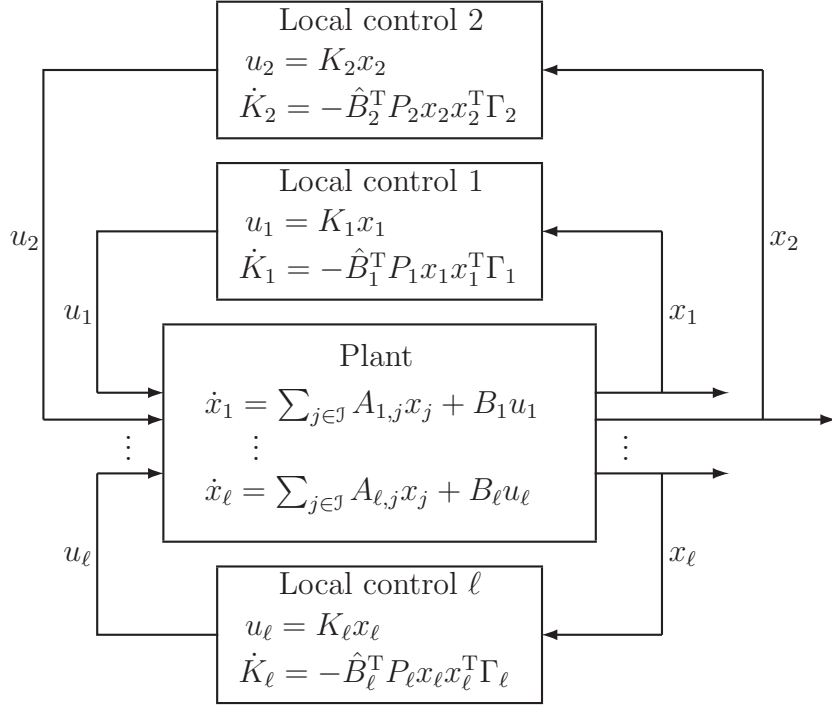


Figure 3.1: Schematic diagram of decentralized adaptive stabilization architecture given by (3.1)–(3.2), (3.10), and (3.11).

The following theorem is the main result on decentralized adaptive stabilization.

**Theorem 3.1.** *Consider the closed-loop system (3.11) and (3.13), where the open-loop system (3.1)–(3.2) satisfies assumptions (A3.1)–(A3.4),  $w_i(t) \equiv 0$ , and  $r_i(t) \equiv 0$ . Then, the equilibrium  $(x_1, \dots, x_\ell, \tilde{K}_1, \dots, \tilde{K}_\ell) \equiv 0$  is Lyapunov stable. Furthermore, for all initial conditions  $x_i(0) \in \mathbb{R}^{n_i}$  and  $K_i(0) \in \mathbb{R}^{m_i \times n_i}$ , the following statements hold:*

(i)  $x_i(t)$ ,  $u_i(t)$ , and  $K_i(t)$  are bounded.

(ii)  $\lim_{t \rightarrow \infty} x_i(t) = 0$ .

*Proof.* For all  $i \in \mathcal{J}$ , define the partial Lyapunov function

$$V_i(x_i, \tilde{K}_i) \triangleq x_i^T P_i x_i + \text{tr } F_i^{-1} \tilde{K}_i \Gamma_i^{-1} \tilde{K}_i^T, \quad (3.14)$$

where  $P_i \in \mathbb{R}^{n_i \times n_i}$  is the positive-definite solution to (3.6). Evaluating the derivative of  $V_i$  along the trajectory of (3.11) and (3.13), and using (A3.1) yields

$$\begin{aligned}
\dot{V}_i(x_i, \tilde{K}_i) &= x_i^\top (A_{m,i}^\top P_i + P_i A_{m,i}) x_i + 2x_i^\top \tilde{K}_i^\top B_i^\top P_i x_i + 2\text{tr} F_i^{-1} \dot{K}_i \Gamma_i^{-1} \tilde{K}_i^\top \\
&\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} x_i^\top P_i A_{i,j} x_j \\
&= x_i^\top (A_{m,i}^\top P_i + P_i A_{m,i}) x_i + 2\text{tr} (B_i^\top P_i x_i x_i^\top \tilde{K}_i^\top + F_i^{-1} \dot{K}_i \Gamma_i^{-1} \tilde{K}_i^\top) \\
&\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} x_i^\top P_i A_{i,j} x_j \\
&= x_i^\top (A_{m,i}^\top P_i + P_i A_{m,i}) x_i + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} x_i^\top P_i A_{i,j} x_j. \tag{3.15}
\end{aligned}$$

Next, note that

$$\begin{aligned}
0 &\leq \sum_{j \in \mathcal{J} \setminus \{i\}} (A_{i,j}^\top P_i x_i - x_j)^\top (A_{i,j}^\top P_i x_i - x_j) \\
&= \sum_{j \in \mathcal{J} \setminus \{i\}} x_i^\top P_i A_{i,j} A_{i,j}^\top P_i x_i + x_j^\top x_j - 2x_i^\top P_i A_{i,j} x_j,
\end{aligned}$$

which combined with (A3.3), implies that

$$\begin{aligned}
2 \sum_{j \in \mathcal{J} \setminus \{i\}} x_i^\top P_i A_{i,j} x_j &\leq \sum_{j \in \mathcal{J} \setminus \{i\}} x_i^\top P_i A_{i,j} A_{i,j}^\top P_i x_i + x_j^\top x_j \\
&\leq x_i^\top P_i \Omega_i P_i x_i + \sum_{j \in \mathcal{J} \setminus \{i\}} x_j^\top x_j. \tag{3.16}
\end{aligned}$$

Using (3.16) and (A3.4), it follows from (3.15) that

$$\begin{aligned}
\dot{V}_i(x_i, \tilde{K}_i) &\leq x_i^\top (A_{m,i}^\top P_i + P_i A_{m,i} + P_i \Omega_i P_i) x_i + \sum_{j \in \mathcal{J} \setminus \{i\}} x_j^\top x_j \\
&\leq -x_i^\top Q_i x_i + \sum_{j \in \mathcal{J} \setminus \{i\}} x_j^\top x_j. \tag{3.17}
\end{aligned}$$

Next, define the Lyapunov function

$$V(x_1, \dots, x_\ell, \tilde{K}_1, \dots, \tilde{K}_\ell) \triangleq \sum_{i \in \mathcal{J}} V_i(x_i, \tilde{K}_i),$$

and it follows from (3.17) that the derivative of  $V$  along the trajectory of (3.11) and (3.13) is

$$\begin{aligned} \dot{V}(x_1, \dots, x_\ell, \tilde{K}_1, \dots, \tilde{K}_\ell) &= \sum_{i \in \mathcal{J}} \dot{V}_i(x_i, \tilde{K}_i) \\ &\leq \sum_{i \in \mathcal{J}} \left( -x_i^\top Q_i x_i + \sum_{j \in \mathcal{J} \setminus \{i\}} x_j^\top x_j \right) \\ &= \sum_{i \in \mathcal{J}} -x_i^\top Q_i x_i + (\ell - 1) \sum_{i \in \mathcal{J}} x_i^\top x_i \\ &= \sum_{i \in \mathcal{J}} -x_i^\top R_i x_i, \end{aligned}$$

where  $R_i \triangleq Q_i - (\ell - 1)I_{n_i}$  is positive definite because  $Q_i > \ell I_{n_i}$ . Therefore, the equilibrium  $(x_1, \dots, x_\ell, \tilde{K}_1, \dots, \tilde{K}_\ell) \equiv 0$  is Lyapunov stable, and for all initial conditions,  $x_i$  and  $\tilde{K}_i$  are bounded. Since  $x_i$  and  $\tilde{K}_i$  are bounded, it follows from (3.10) and (3.12) that  $K_i$  and  $u_i$  are bounded, which confirms (i).

Finally, since  $V$  is positive definite and radially unbounded, and  $\dot{V} \leq \sum_{i \in \mathcal{J}} -x_i^\top R_i x_i$ , it follows from LaSalle's invariance principle [34, Theorem 4.4] that for all initial conditions,  $\lim_{t \rightarrow \infty} x_i(t) = 0$ , which confirms (ii).  $\square$

**Example 3.1.** *Decentralized adaptive stabilization with local scalar dynamics.* Consider the system (3.1)–(3.2), where  $\ell = 3$ ,

$$\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{bmatrix} = \begin{bmatrix} 2.5 & 0.5 & -0.5 \\ -1.5 & -0.5 & -1 \\ 3 & 2 & 1.5 \end{bmatrix}, \quad (3.18)$$

and  $B_1 = 2$ ,  $B_2 = 1.5$ , and  $B_3 = 2.5$ . Note that (3.18) is unstable with eigenvalues at 1 and  $1.25 \pm j1.39$ . We assume that for all  $i, j \in \mathcal{J}$ ,  $A_{i,j}$  is unknown. However, we assume that for all  $(i, j) \in \mathcal{K}$ , an upper bound on the absolute value of  $A_{i,j}$  is known. Specifically, for all  $(i, j) \in \mathcal{K}$ ,  $|A_{i,j}| < 10$ , and the upper bound 10 is known. We also assume that  $\text{sgn}(B_i)$  is known. Our goal is to stabilize the origin of (3.1)–(3.2) using the decentralized adaptive control (3.10) and (3.11). In this example, the disturbances are zero. We consider nonzero disturbances in the next section.

We let  $\hat{B}_i = \text{sgn}(B_i) = 1$ , which satisfies (A3.1). Since  $B_i \neq 0$ , it follows that (A3.2) is satisfied. Since for all  $(i, j) \in \mathcal{K}$ ,  $|A_{i,j}| < 10$ , and the bound 10 is known, we let  $\Omega_i = 200$ , which satisfies (A3.3). Next, let  $Q_i = 4$ , which satisfies  $Q_i > \ell$ . If  $A_{m,i} \leq -\sqrt{\Omega_i Q_i} = -20\sqrt{2}$ , then it follows from the quadratic equation that there exists  $P_i > 0$  that satisfies (3.6), which implies that (A3.4) is satisfied. In this example, we let  $A_{m,i} = -30$ .

The adaptive controller (3.10) and (3.11) is implemented in feedback with the system (3.1)–(3.2) and (3.18), where  $\Gamma_i = 10^5$ . Figure 3.2 shows a time history of  $x_i(t)$  and  $u_i(t)$ , where the initial conditions are  $x_1(0) = 0.5$ ,  $x_2(0) = 0.25$ , and  $x_3(0) = -0.5$ . Moreover, Figure 3.2 shows  $\lim_{t \rightarrow \infty} x_i(t) = 0$ , which agrees with Theorem 3.1.  $\triangle$

**Example 3.2.** *Decentralized adaptive stabilization with local vector dynamics.* Consider the system (3.1)–(3.2), where  $\ell = 3$ , and

$$A_{1,1} = \begin{bmatrix} 2 & -2 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{2,2} = \begin{bmatrix} -3 & 4 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_{3,3} = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (3.19)$$

$$B_1 = \begin{bmatrix} 5 & 0 & 0 \end{bmatrix}^T, \quad B_2 = \begin{bmatrix} -3 & 0 & 0 \end{bmatrix}^T, \quad B_3 = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T. \quad (3.20)$$

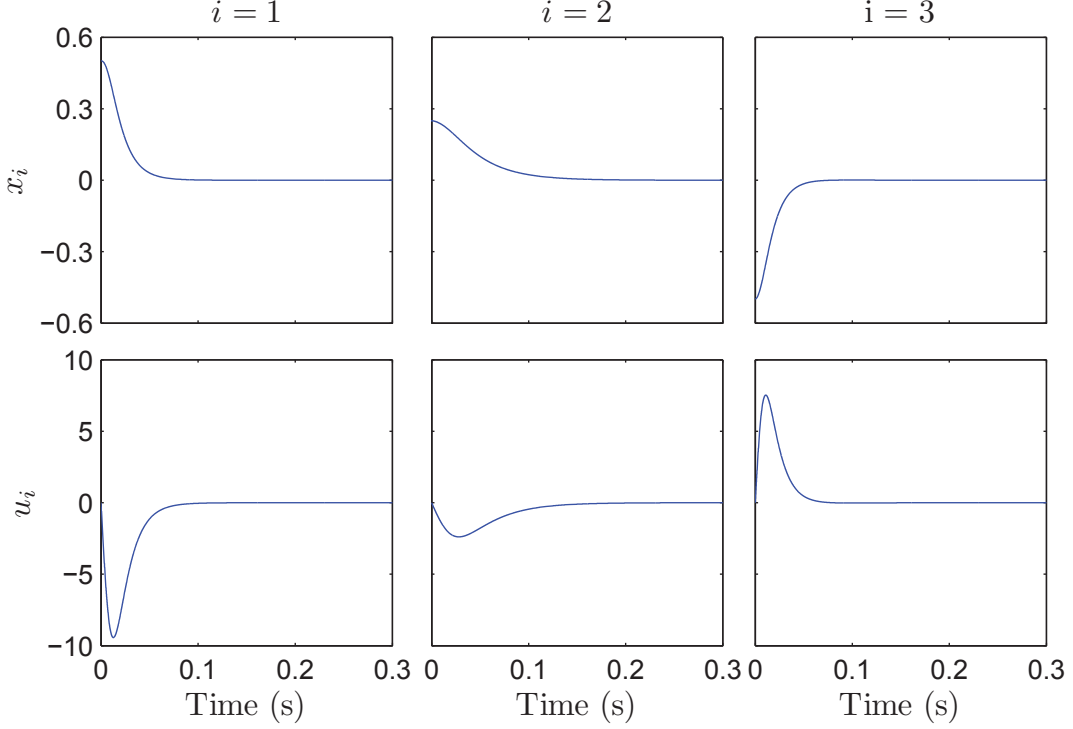


Figure 3.2: *Decentralized adaptive stabilization with local scalar dynamics.* The adaptive controller (3.10) and (3.11) is implemented in feedback with the system (3.1)–(3.2) and (3.18). The state  $x_i(t)$  converges asymptotically to zero.

Furthermore, for all  $(i, j) \in \mathcal{K}$ ,

$$A_{i,j} = B_i \Delta_{i,j}^T, \quad (3.21)$$

where

$$\Delta_{1,2} = \begin{bmatrix} 5 & -2 & -2 \end{bmatrix}^T, \quad \Delta_{1,3} = \begin{bmatrix} -6 & 4 & -2 \end{bmatrix}^T, \quad (3.22)$$

$$\Delta_{2,1} = \begin{bmatrix} -1 & 6 & -3 \end{bmatrix}^T, \quad \Delta_{2,3} = \begin{bmatrix} -5 & 4 & 2 \end{bmatrix}^T, \quad (3.23)$$

$$\Delta_{3,1} = \begin{bmatrix} 5 & -3 & 1 \end{bmatrix}^T, \quad \Delta_{3,2} = \begin{bmatrix} 3 & -1 & -8 \end{bmatrix}^T. \quad (3.24)$$

Note that the dynamics matrix associated with (3.1)–(3.2) and (3.19)–(3.24) is un-



stable with eigenvalues at  $-3.75$ ,  $-0.26 \pm j0.36$ ,  $0.20 \pm j1.57$ ,  $0.65$ ,  $1.38 \pm j6.57$ , and  $1.47$ . For all  $i, j \in \mathcal{J}$ ,  $A_{i,j}$  is unknown. However, we assume that for all  $(i, j) \in \mathcal{K}$ , an upper bound on  $\Delta_{i,j}^T \Delta_{i,j}$  is known. Specifically, for all  $(i, j) \in \mathcal{K}$ ,  $\Delta_{i,j}^T \Delta_{i,j} < 300$ , and the upper bound 300 is known. Furthermore, we assume the sign and an upper bound on the magnitude of  $b_i$  is known, where  $b_i$  denotes the first entry in  $B_i$ . Specifically,  $|b_i| < 10$ , and the upper bound 10 is known.

We let  $\hat{B}_1 = [1 \ 0 \ 0]^T$ ,  $\hat{B}_2 = [-1 \ 0 \ 0]^T$ , and  $\hat{B}_3 = \hat{B}_1$ , which satisfy (A3.1). We let  $\Omega_i = \gamma_i \hat{B}_i \hat{B}_i^T$ , where  $\gamma_i = 6 \times 10^4$ . Since  $\gamma_i \geq \sum_{j \in \mathcal{J} \setminus \{i\}} 100 \Delta_{i,j}^T \Delta_{i,j}$ , it follows from Proposition 3.1 that  $\Omega_i$  satisfies (3.5), which implies that (A3.3) is satisfied. We let  $Q_i = 4I_3$ , which satisfies  $Q_i > \ell I_{n_i}$ . Next, let

$$A_{m,i} = \begin{bmatrix} -\eta_i & -5\eta_i & -6\eta_i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (3.25)$$

where  $\eta_i > 0$ . It follows from Proposition 3.1 that for sufficiently large  $\eta_i > 0$ ,  $A_{m,i}$  is asymptotically stable and there exists a positive-definite matrix  $P_i$  that satisfies (3.6), which implies that (A3.4) is satisfied. In this example, for all  $\eta_i > 492$ , there exists a positive-definite matrix  $P_i$  that satisfies (3.6). We let  $\eta_i = 600$ .

The adaptive controller (3.10) and (3.11) is implemented in feedback with the system (3.1)–(3.2) and (3.19)–(3.24), where  $\Gamma_i = 10^6 I_3$ . Figure 3.3 shows a time history of  $x_i(t)$  and  $u_i(t)$ , where the initial conditions are  $x_1(0) = [0.2 \ -0.5 \ 0.2]^T$ ,  $x_2(0) = [0.4 \ -0.2 \ 0.4]^T$ , and  $x_3(0) = [-0.2 \ -0.5 \ -0.2]^T$ . The state  $x_i(t)$  converges asymptotically to zero.  $\triangle$

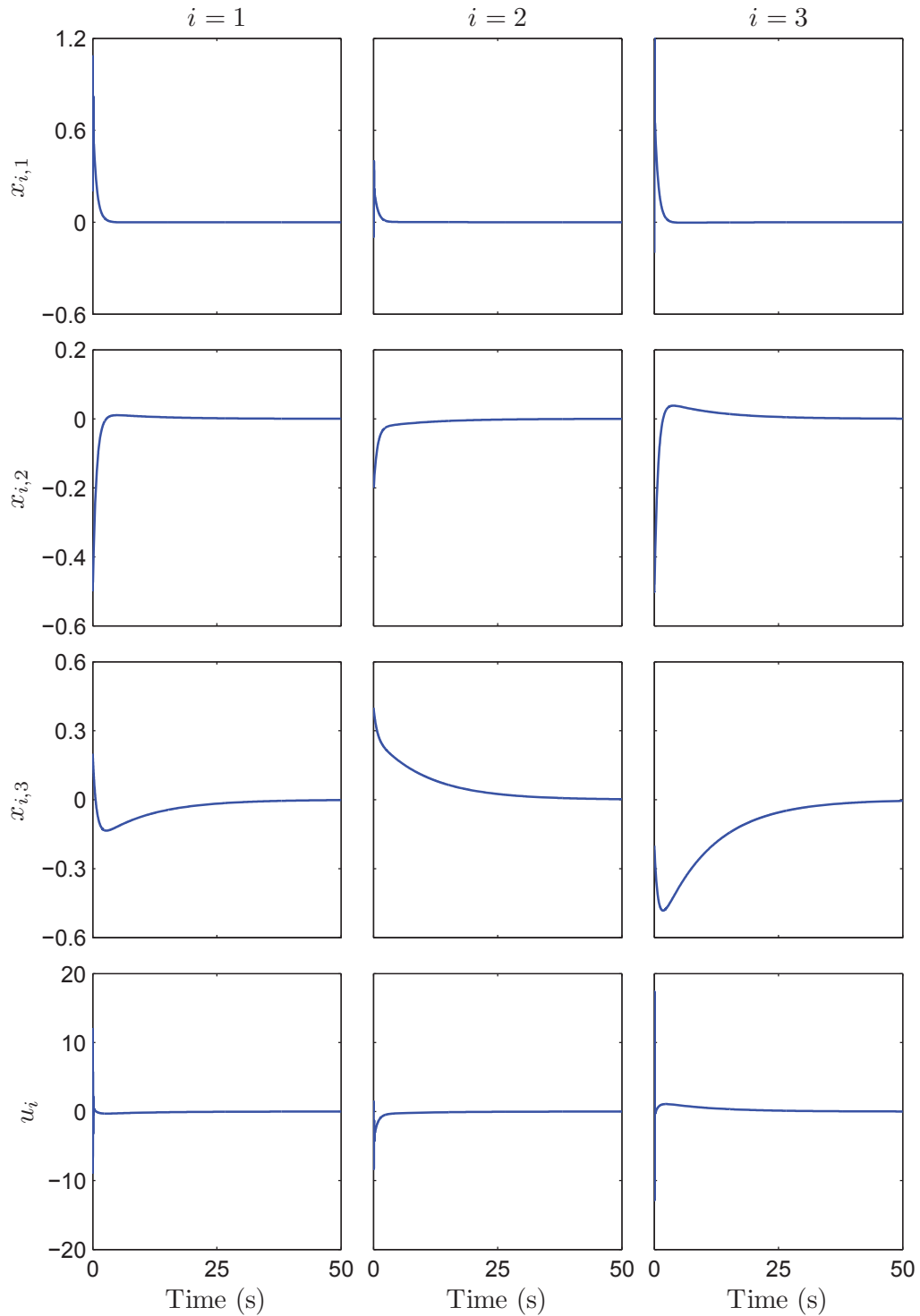


Figure 3.3: *Decentralized adaptive stabilization with local vector dynamics.* The adaptive controller (3.10)–(3.11) is implemented in feedback with the system (3.1)–(3.2) and (3.19)–(3.24). The state  $x_i = [x_{i,1}(t) \ x_{i,2}(t) \ x_{i,3}(t)]^T$  converges asymptotically to zero.

### 3.4 Decentralized Adaptive Command Following and Disturbance Rejection

In this section, we extend the decentralized adaptive stabilization controller presented in Section 3.3 to address command following and disturbance rejection.

We make the following assumptions regarding the reference-model input matrix  $B_{m,i}$  and the disturbance input matrix  $D_i$ :

$$(A3.5) \quad \text{There exists } L_{*,i} \in \mathbb{R}^{m_i \times q_i} \text{ such that } B_{m,i} = B_i L_{*,i}.$$

$$(A3.6) \quad \text{There exists } T_{*,i} \in \mathbb{R}^{m_i \times d_i} \text{ such that } D_i = B_i T_{*,i}.$$

Assumptions (A3.5) and (A3.6) are standard full-state-feedback MRAC matching conditions [1–5]. If the control and disturbance are collocated (i.e.,  $B_i = D_i$ ), then (A3.6) is satisfied by  $T_{*,i} = I_{m_i}$ .

Next, we make the following assumptions regarding the reference-model command  $r_i(t)$  and the disturbance  $w_i(t)$ :

$$(A3.7) \quad \text{There exists } G_i \in \mathbb{R}^{q_i \times 2p} \text{ and } H_i \in \mathbb{R}^{d_i \times 2p} \text{ such that } r_i(t) = G_i \Psi(t) \text{ and } w_i(t) = H_i \Psi(t), \text{ where}$$

$$\Psi(t) \triangleq \begin{bmatrix} \sin \omega_1 t & \dots & \sin \omega_p t & \cos \omega_1 t & \dots & \cos \omega_p t \end{bmatrix}^T \in \mathbb{R}^{2p}, \quad (3.26)$$

and  $\omega_1, \dots, \omega_p$  are nonnegative and known.

$$(A3.8) \quad \text{There exists } N_1 \in \mathbb{R}^{m_1 \times 2p}, \dots, N_\ell \in \mathbb{R}^{m_\ell \times 2p} \text{ such that for all } i \in \mathcal{J},$$

$$\int_0^\infty \left\| B_i N_i \Psi(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} x_{m,j}(t) \right\|^2 dt \quad (3.27)$$

exists, where  $\|\cdot\|$  denotes the Euclidean norm.

Assumption (A3.7) implies that  $r_i(t)$  and  $w_i(t)$  consist of sinusoids with known frequencies. However, the amplitudes and phases are unknown. Note that constant signals are achieved in (3.26) if the frequency is zero.

Assumption (A3.8) is a condition on the trajectories  $\Psi$  and  $x_{m,i}$  as well as the structure of  $B_i$  and  $A_{i,j}$ . Nevertheless, (A3.8) can be verified by matrix matching conditions alone. We now present two results that provide sufficient conditions under which (A3.8) is satisfied. Proofs of these results are in Appendix A.

**Proposition 3.2.** *Assume that  $r_1(t), \dots, r_\ell(t)$  satisfy (A3.7). Furthermore, assume that for all  $W_1 \in \mathbb{R}^{n_1 \times 2p}, \dots, W_\ell \in \mathbb{R}^{n_\ell \times 2p}$ , there exists  $\hat{N}_1 \in \mathbb{R}^{m_1 \times 2p}, \dots, \hat{N}_\ell \in \mathbb{R}^{m_\ell \times 2p}$  such that for all  $i \in \mathcal{J}$ ,*

$$B_i \hat{N}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} W_j = 0. \quad (3.28)$$

*Then, there exists  $N_1 \in \mathbb{R}^{m_1 \times 2p}, \dots, N_\ell \in \mathbb{R}^{m_\ell \times 2p}$  such that for all  $i \in \mathcal{J}$ , (3.27) is satisfied.*

Proposition 3.2 provides matrix matching conditions under which (A3.8) is satisfied. However, the condition (3.28) in Proposition 3.2 cannot be verified without knowledge of  $B_i$  and  $A_{i,j}$ . The next result provides a sufficient condition on the structure of  $A_{i,j}$  under which (A3.8) is satisfied.

**Proposition 3.3.** *Assume that  $r_1(t), \dots, r_\ell(t)$  satisfies (A3.7). Furthermore, assume that for all  $(i, j) \in \mathcal{K}$ ,*

$$A_{i,j} = B_i \Delta_{i,j}^T, \quad (3.29)$$

*where  $\Delta_{i,j} \in \mathbb{R}^{n_j \times m_i}$ . Then, there exists  $N_1 \in \mathbb{R}^{m_1 \times 2p}, \dots, N_\ell \in \mathbb{R}^{m_\ell \times 2p}$  such that for all  $i \in \mathcal{J}$ , (3.27) is satisfied.*

Next, consider the controller

$$u_i(t) = K_i(t)x_i(t) + L_i(t)r_i(t) + M_i(t)\Psi(t), \quad (3.30)$$

where  $K_i : [0, \infty) \rightarrow \mathbb{R}^{m_i \times n_i}$ ,  $L_i : [0, \infty) \rightarrow \mathbb{R}^{m_i \times q_i}$ , and  $M_i : [0, \infty) \rightarrow \mathbb{R}^{m_i \times 2p}$  are given by

$$\dot{K}_i(t) = -\hat{B}_i^T P_i e_i(t) x_i^T(t) \Gamma_i, \quad (3.31)$$

$$\dot{L}_i(t) = -\hat{B}_i^T P_i e_i(t) r_i^T(t) \Lambda_i, \quad (3.32)$$

$$\dot{M}_i(t) = -\hat{B}_i^T P_i e_i(t) \Psi^T(t) \Upsilon_i, \quad (3.33)$$

where  $\Gamma_i \in \mathbb{R}^{n_i \times n_i}$ ,  $\Lambda_i \in \mathbb{R}^{q_i \times q_i}$ , and  $\Upsilon_i \in \mathbb{R}^{2p \times 2p}$  are positive definite, and  $P_i \in \mathbb{R}^{n_i \times n_i}$  is the positive-definite solution to (3.6). The decentralized adaptive command following and disturbance rejection architecture is shown in Figure 3.4.

The following theorem is the main result on decentralized adaptive command following and disturbance rejection.

**Theorem 3.2.** *Consider the closed-loop system (3.1)–(3.2) and (3.30)–(3.33), where the open-loop system (3.1)–(3.2) satisfies assumptions (A3.1)–(A3.8). Then, for all initial conditions  $x_i(0) \in \mathbb{R}^{n_i}$ ,  $K_i(0) \in \mathbb{R}^{m_i \times n_i}$ ,  $L_i(0) \in \mathbb{R}^{m_i \times q_i}$ , and  $M_i(0) \in \mathbb{R}^{m_i \times 2p}$ , the following statements hold:*

(i)  $x_i(t)$ ,  $u_i(t)$ ,  $K_i(t)$ ,  $L_i(t)$ , and  $M_i(t)$  are bounded.

(ii)  $\lim_{t \rightarrow \infty} e_i(t) = 0$ .

*Proof.* Define

$$\tilde{K}_i(t) \triangleq K_i(t) - K_{*,i},$$

$$\tilde{L}_i(t) \triangleq L_i(t) - L_{*,i},$$

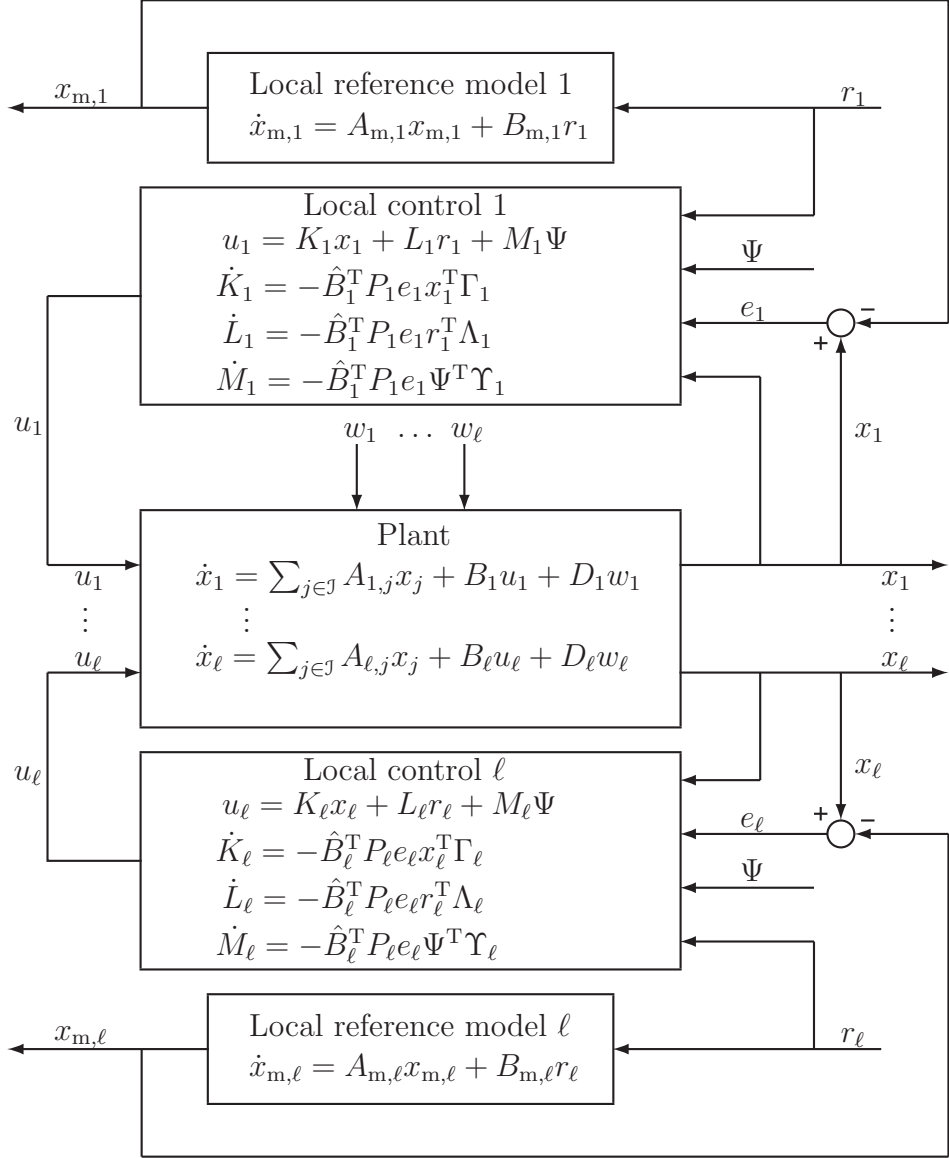


Figure 3.4: Schematic diagram of decentralized adaptive command following and disturbance rejection control architecture given by (3.1)–(3.3) and (3.30)–(3.33).

$$\tilde{M}_i(t) \triangleq M_i(t) - N_i + T_{*,i}H_i,$$

where  $N_i$  is given by (A3.8). Thus, it follows from (3.1)–(3.2) and (3.30) that

$$\begin{aligned} \dot{x}_i(t) &= A_{m,i}x_i(t) + B_i\tilde{K}_i(t)x_i(t) + B_i\tilde{L}_i(t)r_i(t) + B_iM_i(t)\Psi(t) \\ &\quad + D_iw_i(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j}x_j(t). \end{aligned} \quad (3.34)$$

Next, subtracting (3.3) from (3.34), and using (A3.5) yields

$$\begin{aligned} \dot{e}_i(t) &= A_{m,i}e_i(t) + B_i\tilde{K}_i(t)x_i(t) + B_i\tilde{L}_i(t)r_i(t) + B_iM_i(t)\Psi(t) \\ &\quad + D_iw_i(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j}x_j(t), \end{aligned}$$

which implies that

$$\begin{aligned} \dot{e}_i(t) &= A_{m,i}e_i(t) + B_i\tilde{K}_i(t)x_i(t) + B_i\tilde{L}_i(t)r_i(t) + B_i\tilde{M}_i(t)\Psi(t) \\ &\quad + B_iN_i\Psi(t) - B_iT_{*,i}H_i\Psi(t) + D_iw_i(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j}e_j(t) + A_{i,j}x_{m,j}(t). \end{aligned} \quad (3.35)$$

Then, using (A3.6) and (A3.7), it follows from (3.35) that

$$\begin{aligned} \dot{e}_i(t) &= A_{m,i}e_i(t) + B_i\tilde{K}_i(t)x_i(t) + B_i\tilde{L}_i(t)r_i(t) + B_i\tilde{M}_i(t)\Psi(t) \\ &\quad + B_iN_i\Psi(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j}e_j(t) + A_{i,j}x_{m,j}(t). \end{aligned} \quad (3.36)$$

Next, for all  $i \in \mathcal{J}$ , define the partial Lyapunov-like function

$$V_i(e_i, \tilde{K}_i, \tilde{L}_i, \tilde{M}_i) \triangleq e_i^T P_i e_i + \text{tr } F_i^{-1} \tilde{K}_i \Gamma_i^{-1} \tilde{K}_i^T + \text{tr } F_i^{-1} \tilde{L}_i \Lambda_i^{-1} \tilde{L}_i^T + \text{tr } F_i^{-1} \tilde{M}_i \Upsilon_i^{-1} \tilde{M}_i^T,$$

where  $P_i \in \mathbb{R}^{n_i \times n_i}$  is the positive-definite solution to (3.6). Evaluating the derivative

of  $V_i$  along the trajectory of (3.31)–(3.33) and (3.36), and using (A3.1) yields

$$\begin{aligned}
\dot{V}_i(e_i, \tilde{K}_i, \tilde{L}_i, \tilde{M}_i) &= \dot{e}_i^\top P_i e_i + e_i^\top P_i \dot{e}_i + 2\text{tr } F_i^{-1} \dot{K}_i \Gamma_i^{-1} \tilde{K}_i^\top + 2\text{tr } F_i^{-1} \dot{L}_i \Lambda_i^{-1} \tilde{L}_i^\top \\
&\quad + 2\text{tr } F_i^{-1} \dot{M}_i \Upsilon_i^{-1} \tilde{M}_i^\top \\
&= e_i^\top (A_{m,i}^\top P_i + P_i A_{m,i}) e_i + 2e_i^\top P_i B_i N_i \Psi \\
&\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} [e_i^\top P_i A_{i,j} e_j + e_i^\top P_i A_{i,j} x_{m,j}] \\
&\quad + 2x_i^\top \tilde{K}_i^\top B_i^\top P_i e_i + 2r_i^\top \tilde{L}_i^\top B_i^\top P_i e_i + 2\Psi^\top \tilde{M}_i^\top B_i^\top P_i e_i \\
&\quad + 2\text{tr } F_i^{-1} \dot{K}_i \Gamma_i^{-1} \tilde{K}_i^\top + 2\text{tr } F_i^{-1} \dot{L}_i \Lambda_i^{-1} \tilde{L}_i^\top + 2\text{tr } F_i^{-1} \dot{M}_i \Upsilon_i^{-1} \tilde{M}_i^\top \\
&= e_i^\top (A_{m,i}^\top P_i + P_i A_{m,i}) e_i + 2e_i^\top P_i B_i N_i \Psi \\
&\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} [e_i^\top P_i A_{i,j} e_j + e_i^\top P_i A_{i,j} x_{m,j}] \\
&\quad + 2\text{tr} \left[ B_i^\top P_i e_i x_i^\top \tilde{K}_i^\top + B_i^\top P_i e_i r_i^\top \tilde{L}_i^\top + B_i^\top P_i e_i \Psi^\top \tilde{M}_i^\top \right. \\
&\quad \left. - F_i^{-1} (F_i B_i^\top P_i e_i x_i^\top \Gamma_i) \Gamma_i^{-1} \tilde{K}_i^\top - F_i^{-1} (F_i B_i^\top P_i e_i r_i^\top \Lambda_i) \Lambda_i^{-1} \tilde{L}_i^\top \right. \\
&\quad \left. - F_i^{-1} (F_i B_i^\top P_i e_i \Psi^\top \Upsilon_i) \Upsilon_i^{-1} \tilde{M}_i^\top \right] \\
&= e_i^\top (A_{m,i}^\top P_i + P_i A_{m,i}) e_i + 2e_i^\top P_i B_i N_i \Psi \\
&\quad + \sum_{j \in \mathcal{J} \setminus \{i\}} [2e_i^\top P_i A_{i,j} e_j + 2e_i^\top P_i A_{i,j} x_{m,j}]. \tag{3.37}
\end{aligned}$$

Next, note that

$$\begin{aligned}
0 &\leq \sum_{j \in \mathcal{J} \setminus \{i\}} (A_{i,j}^\top P_i e_i - e_j)^\top (A_{i,j}^\top P_i e_i - e_j) \\
&= \sum_{j \in \mathcal{J} \setminus \{i\}} e_i^\top P_i A_{i,j} A_{i,j}^\top P_i e_i + e_j^\top e_j - 2e_i^\top P_i A_{i,j} e_j,
\end{aligned}$$

which combined with (A3.3), implies that

$$\sum_{j \in \mathcal{J} \setminus \{i\}} 2e_i^\top P_i A_{i,j} e_j \leq \sum_{j \in \mathcal{J} \setminus \{i\}} e_i^\top P_i A_{i,j} A_{i,j}^\top P_i e_i + e_j^\top e_j$$



$$\leq e_i^T P_i \Omega_i P_i e_i + \sum_{j \in \mathcal{J} \setminus \{i\}} e_j^T e_j. \quad (3.38)$$

Using (3.6) and (3.38), it follows from (3.37) that

$$\begin{aligned} \dot{V}_i(e_i, \tilde{K}_i, \tilde{L}_i, \tilde{M}_i) &\leq e_i^T (A_{m,i}^T P_i + P_i A_{m,i} + P_i \Omega_i P_i) e_i + 2e_i^T P_i B_i N_i \Psi \\ &\quad + \sum_{j \in \mathcal{J} \setminus \{i\}} [e_j^T e_j + 2e_i^T P_i A_{i,j} x_{m,j}] \\ &\leq -e_i^T Q_i e_i + 2e_i^T P_i B_i N_i \Psi + \sum_{j \in \mathcal{J} \setminus \{i\}} [e_j^T e_j + 2e_i^T P_i A_{i,j} x_{m,j}], \\ &= -e_i^T Q_i e_i + 2e_i^T P_i \xi_i + \sum_{j \in \mathcal{J} \setminus \{i\}} e_j^T e_j, \end{aligned} \quad (3.39)$$

where

$$\xi_i(t) \triangleq B_i N_i \Psi(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} x_{m,j}(t).$$

Then, note that

$$0 \leq \|e_i - P_i \xi_i\|^2 = e_i^T e_i + \|P_i \xi_i\|^2 - 2e_i^T P_i \xi_i,$$

which implies that

$$2e_i^T P_i \xi_i \leq e_i^T e_i + \|P_i \xi_i\|^2 \leq e_i^T e_i + \lambda_{\max}(P_i)^2 \xi_i^T \xi_i, \quad (3.40)$$

where  $\lambda_{\max}(P_i)$  is the maximum eigenvalue of  $P_i$ .

Using (3.40), it follows from (3.39) that

$$\dot{V}_i(e_i, \tilde{K}_i, \tilde{L}_i, \tilde{M}_i) \leq -e_i^T (Q_i - I_{n_i}) e_i + \lambda_{\max}(P_i)^2 \xi_i^T \xi_i + \sum_{j \in \mathcal{J} \setminus \{i\}} e_j^T e_j. \quad (3.41)$$

Next, define the Lyapunov-like function

$$V(e_1, \dots, e_\ell, \tilde{K}_1, \dots, \tilde{K}_\ell, \tilde{L}_1, \dots, \tilde{L}_\ell, \tilde{M}_1, \dots, \tilde{M}_\ell) \triangleq \sum_{i \in \mathcal{J}} V_i(e_i, \tilde{K}_i, \tilde{L}_i, \tilde{M}_i),$$

and it follows from (3.41) that the derivative of  $V$  along the trajectory of (3.31)–(3.33) and (3.36) is given by

$$\begin{aligned} \dot{V} &= \sum_{i \in \mathcal{J}} \dot{V}_i(e_i, \tilde{K}_i, \tilde{L}_i, \tilde{M}_i) \\ &\leq \sum_{i \in \mathcal{J}} \left[ -e_i^\top (Q_i - I_{n_i}) e_i + \lambda_{\max}(P_i)^2 \xi_i^\top \xi_i + \sum_{j \in \mathcal{J} \setminus \{i\}} e_j^\top e_j \right] \\ &= \sum_{i \in \mathcal{J}} -e_i^\top (Q_i - I_{n_i}) e_i + \lambda_{\max}(P_i)^2 \xi_i^\top \xi_i + (\ell - 1) e_i^\top e_i \\ &= \sum_{i \in \mathcal{J}} -e_i^\top R_i e_i + \lambda_{\max}(P_i)^2 \xi_i^\top \xi_i, \end{aligned}$$

where  $R_i \triangleq Q_i - \ell I_{n_i}$ , which is positive definite from (A3.4). Thus,

$$0 \leq \sum_{i \in \mathcal{J}} e_i^\top R_i e_i \leq -\dot{V} + \sum_{i \in \mathcal{J}} \lambda_{\max}(P_i)^2 \xi_i^\top \xi_i. \quad (3.42)$$

Moreover, integrating (3.42) from 0 to  $\infty$  yields

$$\begin{aligned} 0 \leq \int_0^\infty \sum_{i \in \mathcal{J}} e_i^\top(t) R_i e_i(t) dt &\leq V(0) - V(\infty) + \int_0^\infty \sum_{i \in \mathcal{J}} \lambda_{\max}(P_i)^2 \xi_i^\top(t) \xi_i(t) dt \\ &\leq V(0) + \sum_{i \in \mathcal{J}} \lambda_{\max}(P_i)^2 \int_0^\infty \xi_i^\top(t) \xi_i(t) dt, \end{aligned} \quad (3.43)$$

which exists because (A3.8) implies that  $\int_0^\infty \xi_i^\top(t) \xi_i(t) dt$  exists. Thus, it follows from (3.43) that  $V$  is bounded, which implies that  $e_i$ ,  $\tilde{K}_i$ ,  $\tilde{L}_i$ , and  $\tilde{M}_i$  are bounded. Since  $r_i$  is bounded and  $A_{m,i}$  is asymptotically stable, (3.3) implies that  $x_{m,i}$  is bounded. Moreover, since  $e_i$ ,  $x_{m,i}$ ,  $\tilde{K}_i$ ,  $\tilde{L}_i$ , and  $\tilde{M}_i$  are bounded, it follows that  $x_i$ ,  $u_i$ ,  $K_i$ ,  $L_i$ ,

and  $M_i$  are bounded, which confirms (i).

To show (ii), it follows from (3.43) that  $\int_0^\infty \sum_{i \in \mathcal{J}} e_i^T(t) R_i e_i(t) dt$  exists. Next, since  $e_i$ ,  $x_i$ ,  $r_i$ ,  $\Psi$ ,  $x_{m,i}$ ,  $\tilde{K}_i$ ,  $\tilde{L}_i$ , and  $\tilde{M}_i$  are bounded, (3.36) implies that  $\dot{e}_i$  is bounded. Next, since  $e_i$  and  $\dot{e}_i$  are bounded, it follows that

$$\frac{d}{dt} \left[ \sum_{i \in \mathcal{J}} e_i^T(t) R_i e_i(t) \right] = 2 \sum_{i \in \mathcal{J}} \dot{e}_i^T(t) R_i e_i(t)$$

is bounded. Thus,  $f(t) \triangleq \sum_{i \in \mathcal{J}} e_i^T(t) R_i e_i(t)$  is uniformly continuous. Since  $\int_0^\infty f(t) dt$  exists and  $f(t)$  is uniformly continuous, Barbalat's lemma implies that  $\lim_{t \rightarrow \infty} f(t) = 0$ . Thus,  $\lim_{t \rightarrow \infty} e_i(t) = 0$ , which confirms (ii).  $\square$

### 3.5 Numerical Examples

We now present examples that demonstrate the decentralized adaptive controller. Example 3.3 shows perfect command following and disturbance rejection for a system with local scalar dynamics. Example 3.4 demonstrates asymptotically perfect command following for a mass-spring-dashpot system. This result is extended to address disturbance rejection in Example 3.5 and a mass-spring-dashpot system with ten masses in Example 3.6. Finally, Example 3.7 examines the behavior of a nonlinear planar double pendulum.

**Example 3.3.** *Decentralized adaptive command following and disturbance rejection with local scalar dynamics.* Reconsider the unstable system in Example 3.1, but consider nonzero reference-model commands and nonzero disturbances. The plant, reference model, and control parameters satisfying (A3.1)–(A3.4) are the same as in Example 3.1.

The reference-model input constants are  $B_{m,1} = B_{m,2} = B_{m,3} = 30$ , which satisfy (A3.5). The reference-model commands are  $r_1(t) = 2 \sin 1.5\pi t$ ,  $r_2(t) = 1.5 \cos \pi t$  and  $r_3(t) = \sin 0.5\pi t$ . The disturbance input constants are  $D_1 = 0.5$ ,  $D_2 = 1$ , and  $D_3 =$

1.5, which satisfy (A3.6). The disturbances are  $w_1(t) = 2 \sin 1.5\pi t$ ,  $w_2(t) = 3.5 \sin \pi t$  and  $w_3(t) = 10$ . We let

$$\Psi(t) = \begin{bmatrix} \sin 0.5\pi t & \sin \pi t & \sin 1.5\pi t & \sin 2\pi t & \cos 0.5\pi t & \cos \pi t & \cos 1.5\pi t & \cos 2\pi t & 1 \end{bmatrix}^T, \quad (3.44)$$

and it follows that  $r_i(t)$  and  $w_i(t)$  satisfy (A3.7). Since  $A_{i,j}$  has the form given by (3.29), Proposition 3.3 implies that (A3.8) is satisfied.

The adaptive controller (3.30)–(3.33) is implemented in feedback with the system (3.1)–(3.2) and (3.18), where  $\Gamma_i = 10^4$ ,  $\Lambda_i = 10^4$  and  $\Upsilon_i = 10^4 I_9$ . Figure 3.5 shows a time history of  $x_i(t)$ ,  $x_{m,i}(t)$ ,  $e_i(t)$ , and  $u_i(t)$ , where the initial conditions are  $x_1(0) = 0.5$ ,  $x_2(0) = 0.25$ , and  $x_3(0) = -0.5$ . The state  $x_i(t)$  converges asymptotically to  $x_{m,i}(t)$ , and thus,  $\lim_{t \rightarrow \infty} e_i(t) = 0$ , which agrees with Theorem 3.2.  $\triangle$

**Example 3.4.** *Decentralized adaptive command following for a mass-spring-dashpot system.* Consider the serially connected structure shown in Figure 3.6, where  $\ell = 3$ ;  $u_1$ ,  $u_2$  and  $u_3$  are control forces; and  $w_1$ ,  $w_2$  and  $w_3$  are disturbance forces. Furthermore,  $q_1$ ,  $q_2$  and  $q_3$  are the positions of the first, second and third masses, respectively. The equations of motion for the system are given by (3.1)–(3.2), where for all  $i \in \mathcal{J}$ ,

$$A_{i,i} = \begin{bmatrix} -(c_i + c_{i+1})/m_i & -(k_i + k_{i+1})/m_i \\ 1 & 0 \end{bmatrix}, \quad (3.45)$$

$$B_i = \begin{bmatrix} 1/m_i \\ 0 \end{bmatrix}, \quad (3.46)$$

$$D_i = B_i, \quad (3.47)$$

$$x_i = \begin{bmatrix} \dot{q}_i \\ q_i \end{bmatrix}, \quad (3.48)$$

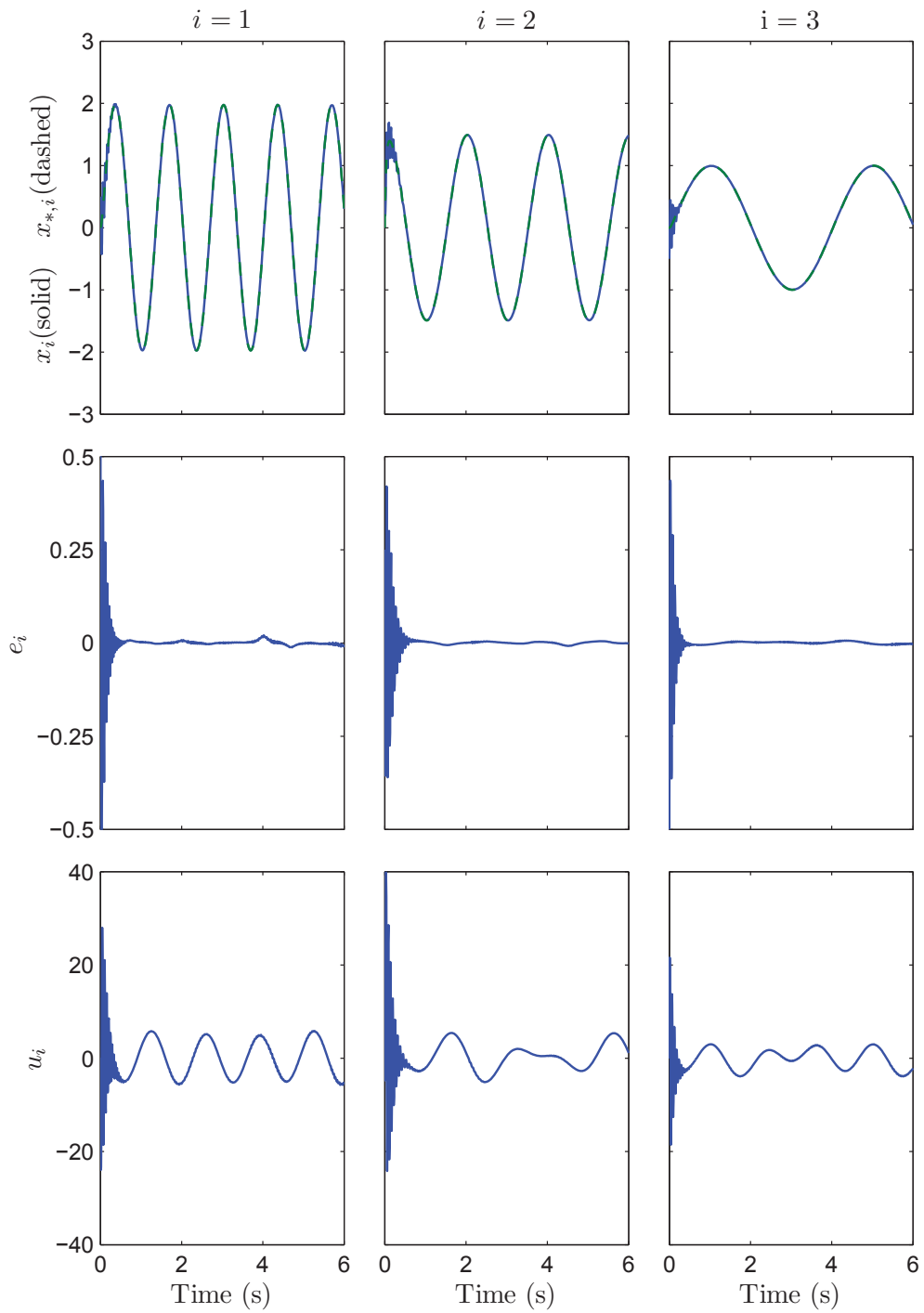


Figure 3.5: *Decentralized adaptive command following and disturbance rejection with local scalar dynamics.* The adaptive controller (3.30)–(3.33) is implemented in feedback with the system (3.1)–(3.2) and (3.18). The error  $e_i(t)$  converges asymptotically to zero.

and for all  $(i, j) \in \mathcal{K}$ ,

$$A_{i,j} = \begin{cases} B_i \begin{bmatrix} c_{\max\{i,j\}} & k_{\max\{i,j\}} \end{bmatrix}, & \text{if } |i - j| = 1, \\ 0_{2 \times 2}, & \text{otherwise.} \end{cases} \quad (3.49)$$

The masses are  $m_1 = 0.5$  kg,  $m_2 = 0.2$  kg and  $m_3 = 0.3$  kg; the damping coefficients are  $c_1 = 3$  kg/s,  $c_2 = 3$  kg/s,  $c_3 = 4$  kg/s, and  $c_4 = 5$  kg/s; and the spring constants are  $k_1 = 10$  kg/s<sup>2</sup>,  $k_2 = 12$  kg/s<sup>2</sup>,  $k_3 = 8$  kg/s<sup>2</sup>, and  $k_4 = 11$  kg/s<sup>2</sup>.

The decentralized adaptive controller in (3.30)–(3.33) is implemented using limited information of the dynamics of the system (3.1)–(3.2). Specifically, the masses  $m_1$ ,  $m_2$  and  $m_3$ ; damping coefficients  $c_1$ ,  $c_2$ ,  $c_3$ , and  $c_4$ ; and spring constants  $k_1$ ,  $k_2$ ,  $k_3$ , and  $k_4$  are unknown. However, we assume bounds on the parameter values are known. For all  $i \in \mathcal{J}$ ,  $m_i > 0.1$  kg; and for all  $i = 1, \dots, 4$ ,  $c_i < 10$  kg/s and  $k_i < 15$  kg/s<sup>2</sup>, and we assume that the bounds 0.1 kg, 10 kg/s, and 15 kg/s<sup>2</sup> are known.

For all  $i \in \mathcal{J}$ , we let  $\hat{B}_i = [1 \ 0]^\text{T}$ , which satisfies (A3.1). We let  $\Omega_i = \gamma_i \hat{B}_i \hat{B}_i^\text{T}$ , where  $\gamma_1 = \gamma_3 = 3.25 \times 10^4$  and  $\gamma_2 = 6.5 \times 10^4$ . Note that  $\gamma_i$  is determined from the bounds on  $m_i$ ,  $c_i$  and  $k_i$ . Since  $\gamma_i$  satisfies (3.9), it follows from Proposition 3.1 that  $\Omega_i$  satisfies (3.5), which implies that (A3.3) is satisfied. We let  $Q_i = 4I_2$ , which satisfies  $Q_i > \ell I_{n_i}$ . Next, let

$$A_{m,i} = \begin{bmatrix} -2\eta_i & -22\eta_i \\ 1 & 0 \end{bmatrix}, \quad B_{m,i} = \begin{bmatrix} 22\eta_i \\ 0 \end{bmatrix}, \quad (3.50)$$

where  $\eta_i > 0$ . It follows from Proposition 3.1 that for sufficiently large  $\eta_i > 0$ ,  $A_{m,i}$  is asymptotically stable and there exists a positive-definite matrix  $P_i$  that satisfies (3.6). In this example, for all  $\eta_i > 255$ , there exists a positive-definite matrix  $P_i$  that satisfies (3.6). We let  $\eta_i = 400$ . Furthermore,  $A_{m,i}$  satisfies (A3.2), and  $B_{m,i}$  satisfies (A3.5).

The reference model commands are  $r_1(t) = 0.1 \sin 0.5\pi t$ ,  $r_2(t) = 0.02$  and  $r_3(t) = 0.1 \sin \pi t$ . In this example, the disturbances are zero. We let  $\Psi(t)$  be the same as in Example 3.3, and it follows that  $r_i(t)$  satisfies (A3.7). Since  $A_{i,j}$  has the form given by (3.29), Proposition 3.3 implies that (A3.8) is satisfied.

The adaptive controller (3.30)–(3.33) is implemented in feedback with the system (3.1)–(3.2) and (3.45)–(3.49), where  $\Gamma_1 = \Gamma_2 = \Gamma_3 = 10^6 I_2$ ,  $\Lambda_1 = \Lambda_2 = \Lambda_3 = 10^6$ ,  $\Upsilon_1 = 10^3 I_9$ ,  $\Upsilon_2 = 10^4 I_9$  and  $\Upsilon_3 = 10^3 I_9$ . Figure 3.7 provides a time history of  $x_i(t)$ ,  $x_{m,i}(t)$ ,  $e_i(t)$ , and  $u_i(t)$ , where the initial conditions are  $q_1(0) = q_2(0) = q_3(0) = 0$  m and  $\dot{q}_1(0) = \dot{q}_2(0) = \dot{q}_3(0) = 0$  m/s. The three-mass system is allowed to run open-loop for 5 seconds, then the decentralized adaptive controller is turned on. Figure 3.7 shows  $\lim_{t \rightarrow \infty} e_i(t) = 0$ . △

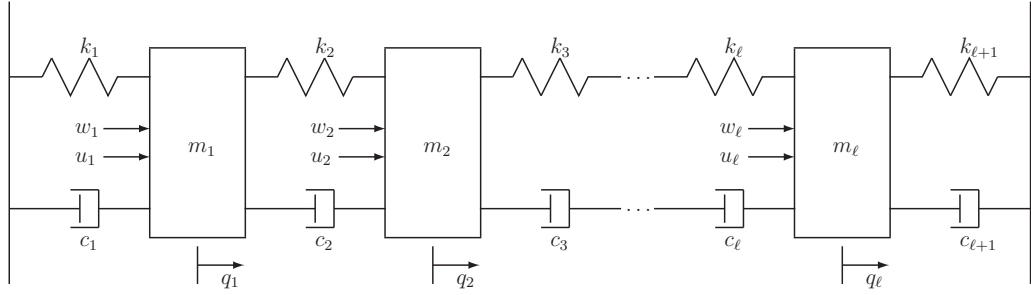


Figure 3.6: The serially connected,  $\ell$ -mass structure used in Examples 3.4–3.6.

**Example 3.5.** *Decentralized command following and disturbance rejection for a mass-spring-dashpot system.* Reconsider the three-mass structure in Example 3.4, but consider nonzero disturbances. The plant, reference model, and control parameters satisfying (A3.1)–(A3.8) are the same as in Example 3.4. The disturbances are  $w_1(t) = 0.1 \sin 1.5\pi t$ ,  $w_2(t) = 0.005$  and  $w_3(t) = 0.05 \sin 0.5\pi t$ , which satisfy (A3.7), where  $\Psi(t)$  is the same as in Example 3.4.

The adaptive controller (3.30)–(3.33) is implemented in feedback with the system (3.1)–(3.2) and (3.45)–(3.49). Figure 3.8 provides a time history of  $x_i(t)$ ,  $x_{m,i}(t)$ ,  $e_i(t)$ , and  $u_i(t)$ . The three-mass system is allowed to run open-loop for 5 seconds, then the

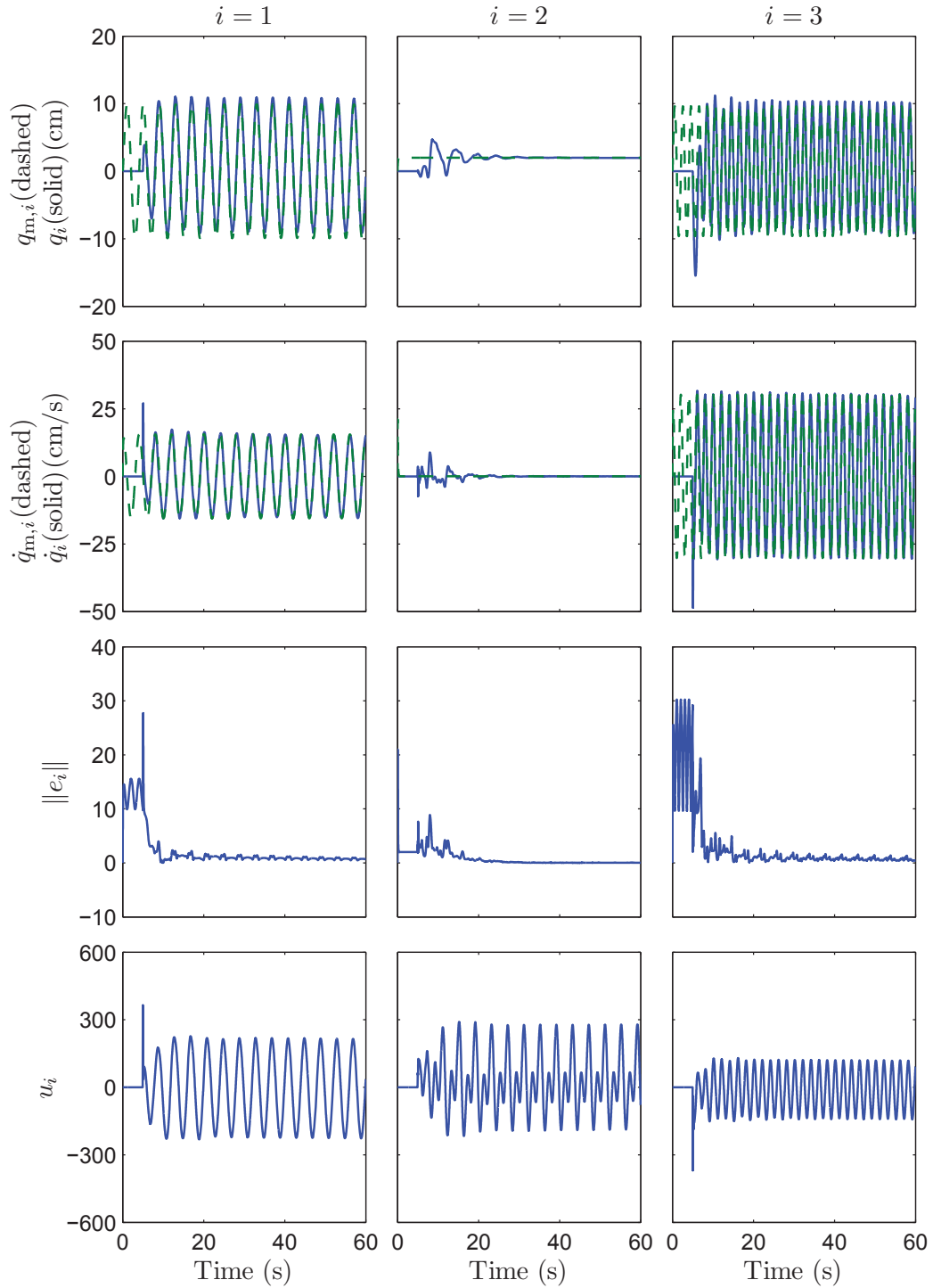


Figure 3.7: *Decentralized adaptive command following for a mass-spring-dashpot system.* The adaptive controller (3.30)–(3.33) is implemented in feedback with the three-mass system (3.1)–(3.2) and (3.45)–(3.49). The error  $e_i(t)$  converges asymptotically to zero.



decentralized adaptive control is turned on. The error  $e_i(t)$  converges asymptotically to zero.  $\triangle$

**Example 3.6.** *Decentralized command following and disturbance rejection for a mass-spring-dashpot system with ten masses.* Consider the serially connected shown in Figure 3.6, where  $\ell = 10$ . The equations of motion for the system are given by (3.1)–(3.2), where  $m_1 = \dots = m_{10} = 0.5$  kg,  $c_1 = \dots = c_{11} = 5$  kg/s, and  $k_1 = \dots = k_{11} = 10$  kg/s<sup>2</sup>; and for all  $i \in \mathcal{J}$ ,  $A_{i,i}$ ,  $B_i$ ,  $D_i$ ,  $x_i$ , and  $A_{i,j}$  are given by (3.45)–(3.49), respectively. We assume bounds on the parameter values are known. For all  $i \in \mathcal{J}$ ,  $m_i > 0.1$  kg; and for all  $i = 1, \dots, 11$ ,  $c_i < 10$  kg/s, and  $k_i < 15$  kg/s<sup>2</sup>, and we assume that the bounds 0.1 kg, 10 kg/s, and 15 kg/s<sup>2</sup> are known.

For all  $i \in \mathcal{J}$ , we let  $\hat{B}_i = [1 \ 0]^T$ , which satisfies (A3.1). We let  $\Omega_i = \gamma_i \hat{B}_i \hat{B}_i^T$ , where  $\gamma_i = 3.25 \times 10^4$ . Note that  $\gamma_i$  is determined from the bounds on  $m_i$ ,  $c_i$ , and  $k_i$ . Since  $\gamma_i$  satisfies (3.9), it follows from Proposition 3.1 that  $\Omega_i$  satisfies (3.5), which implies that (A3.3) is satisfied. We let  $Q_i = 11I_2$ , which satisfies  $Q_i > \ell I_{n_i}$ . Next, let

$$A_{m,i} = \begin{bmatrix} -2\eta_i & -8\eta_i \\ 1 & 0 \end{bmatrix}, \quad B_{m,i} = \begin{bmatrix} 8\eta_i \\ 0 \end{bmatrix}, \quad (3.51)$$

where  $\eta_i > 0$ . It follows from Proposition 3.1 that for sufficiently large  $\eta_i > 0$ ,  $A_{m,i}$  is asymptotically stable and there exists a positive-definite matrix  $P_i$  that satisfies (3.6). In this example, for all  $\eta_i > 423$ , there exists a positive-definite matrix  $P_i$  that satisfies (3.6). We let  $\eta_i = 600$ . Furthermore,  $A_{m,i}$  satisfies (A3.2) and  $B_{m,i}$  satisfies (A3.5).

The reference model commands are  $r_1(t) = r_3(t) = r_5(t) = r_7(t) = r_9(t) = 0.1 \sin 0.5\pi t$  and  $r_2(t) = r_4(t) = r_6(t) = r_8(t) = r_{10}(t) = -0.1 \sin 0.5\pi t$ . The disturbance input matrix  $D_i$  satisfies (A3.6). The disturbances are  $w_1(t) = \dots = w_{10}(t) = 0.05 \sin \pi t$ . We let  $\Psi(t)$  be given by (3.44), and it follows that  $r_i(t)$  and  $w_i(t)$  satisfy (A3.7). Since  $A_{i,j}$  has the form given by (3.29), Proposition 3.3 implies that (A3.8)

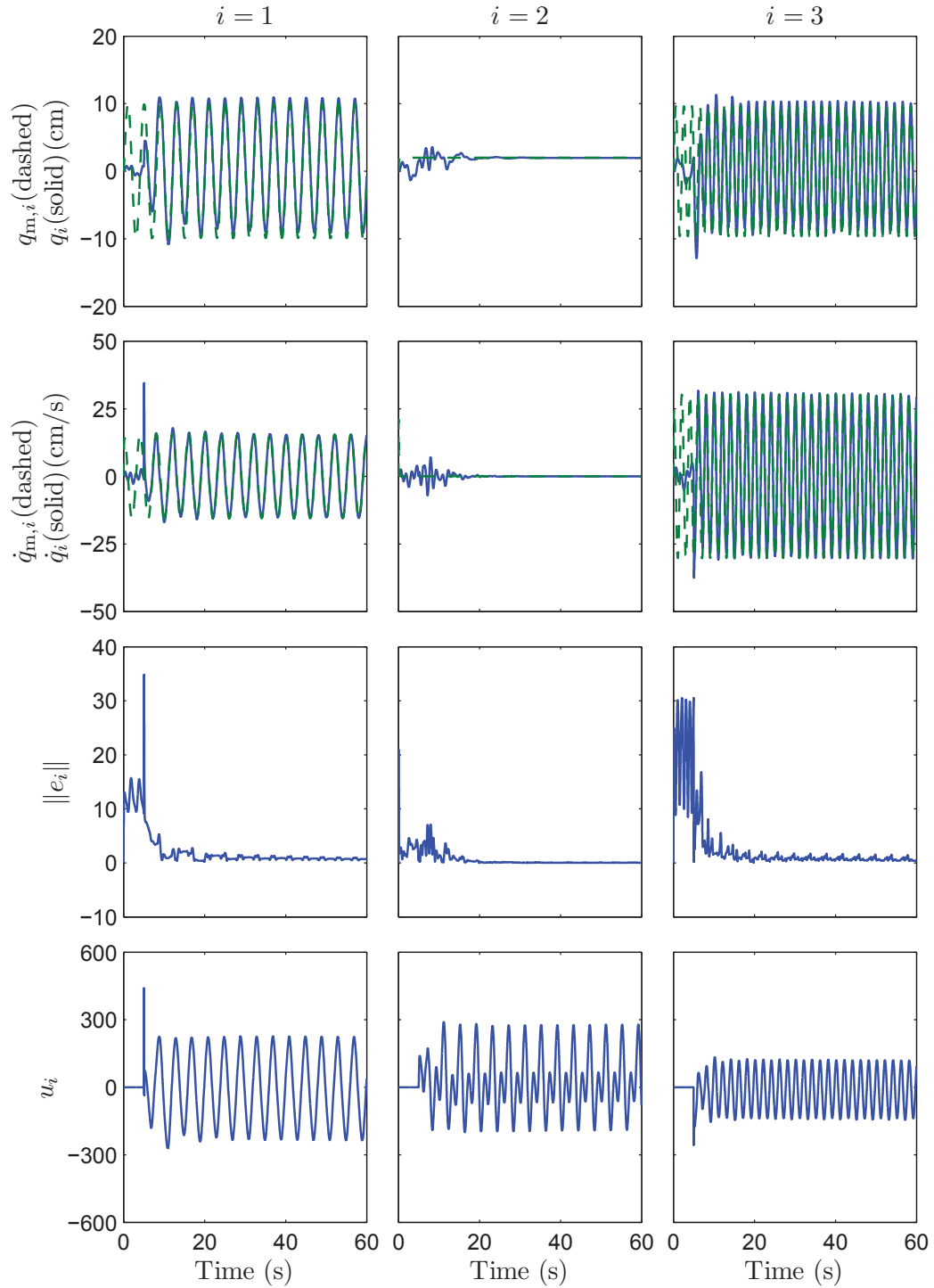


Figure 3.8: *Decentralized adaptive command following and disturbance rejection for a mass-spring-dashpot system.* The adaptive controller (3.30)–(3.33) is implemented in feedback with the three-mass system (3.1)–(3.2) and (3.45)–(3.49). The error  $e_i(t)$  converges asymptotically to zero.

is satisfied.

The adaptive control (3.30)–(3.33) is implemented in feedback with the system (3.1)–(3.2) and (3.45)–(3.49), where for all  $i \in \mathcal{J}$ ,  $\Gamma_i = 10^5 I_2$ ,  $\Lambda_i = 10^4$ , and  $\Upsilon_i = 10^4 I_8$ . Figure 3.9 provides a time history of  $q_i(t)$ ,  $q_{m,i}(t)$ , and  $u_i(t)$ , where the initial conditions are  $q_i(0) = 0$  m and  $\dot{q}_i(0) = 0$  m/s. The ten-mass system is allowed to run open-loop for 5 seconds, then the decentralized adaptive control is turned on. Figure 3.9 shows that  $q_i(t)$  converges asymptotically to  $q_{m,i}(t)$ .  $\triangle$

**Example 3.7.** *Decentralized disturbance rejection for a planar double pendulum.* Consider the planar double pendulum shown in Figure 3.10. The nonlinear equations of motion for the planar double pendulum are

$$\hat{M}(\theta_1, \theta_2) \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \hat{F}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \begin{bmatrix} u_1 + w_1 \\ u_2 + w_2 \end{bmatrix}, \quad (3.52)$$

where

$$\hat{M}(\theta_1, \theta_2) \triangleq \begin{bmatrix} \frac{1}{3}m_1l_1^2 + m_2l_1^2 & \frac{1}{2}m_2l_1l_2 \cos(\theta_1 - \theta_2) \\ \frac{1}{2}m_2l_1l_2 \cos(\theta_1 - \theta_2) & \frac{1}{3}m_2l_2^2 \end{bmatrix}, \quad (3.53)$$

$$\hat{F}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \triangleq \begin{bmatrix} \frac{1}{2}m_2l_1l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (k_1 + k_2)\theta_1 - k_2\theta_2 \\ -\frac{1}{2}m_2l_1l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 - c_2\dot{\theta}_1 + c_2\dot{\theta}_2 - k_2\theta_1 + k_2\theta_2 \end{bmatrix}, \quad (3.54)$$

and for  $i = 1, 2$ ,  $m_i$  is the mass of the  $i^{\text{th}}$  link,  $l_i$  is the length of the  $i^{\text{th}}$  link,  $c_i$  is the damping at the  $i^{\text{th}}$  joint,  $k_i$  is the stiffness at the  $i^{\text{th}}$  joint, and  $\theta_i$  is the angle from the  $i^{\text{th}}$  link to the horizontal plane. Furthermore,  $u_i$  and  $w_i$  are the control and disturbance, respectively, at the  $i^{\text{th}}$  joint. See [36] for more details on the equations of motion.

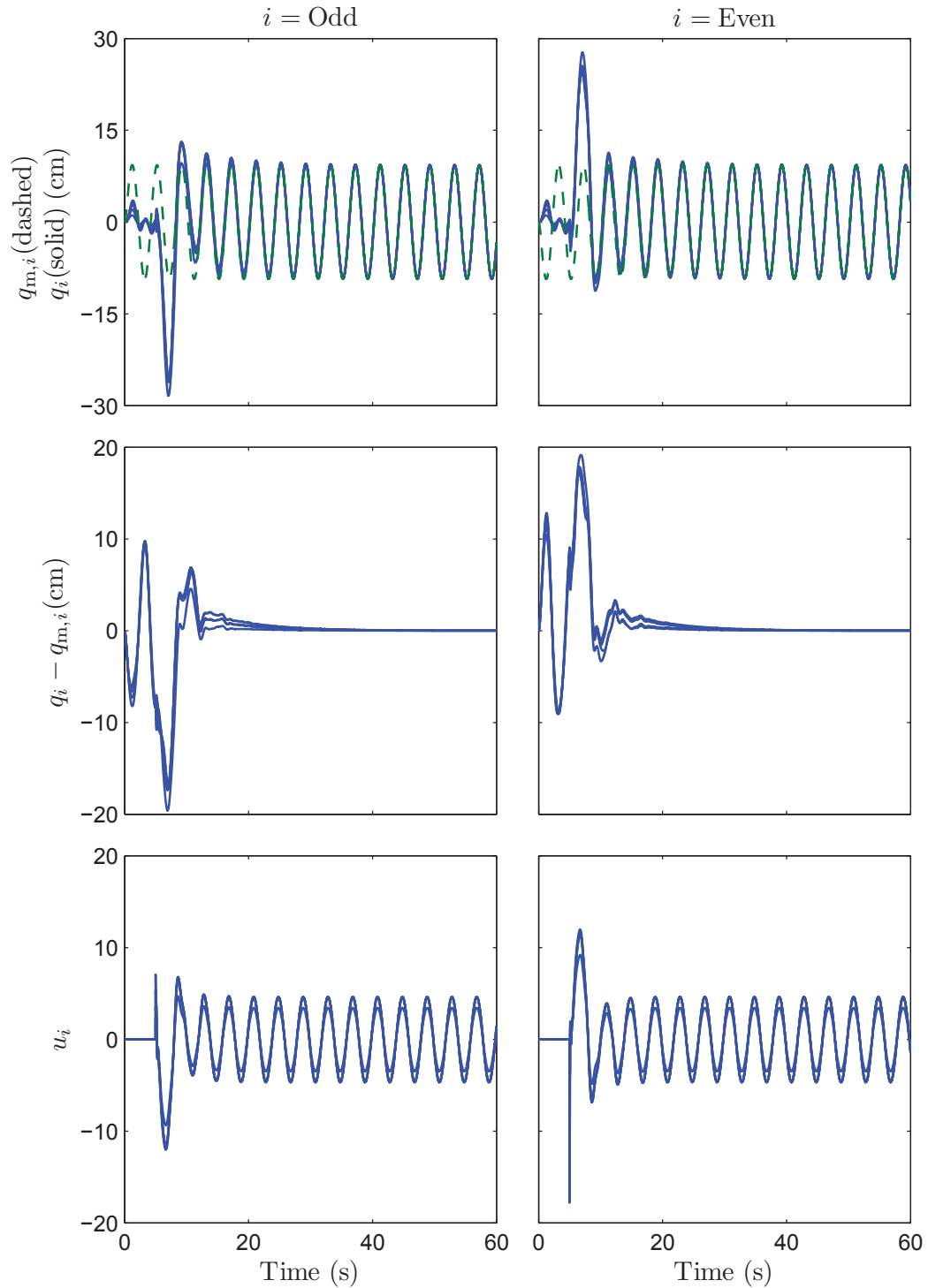


Figure 3.9: *Decentralized adaptive command following and disturbance rejection for a mass-spring-dashpot system with ten masses.* The adaptive controller (3.30)–(3.33) is implemented in feedback with the ten-mass system (3.1)–(3.2) and (3.45)–(3.49). The position  $q_i(t)$  converges asymptotically to the reference-model position  $q_{m,i}(t)$ .

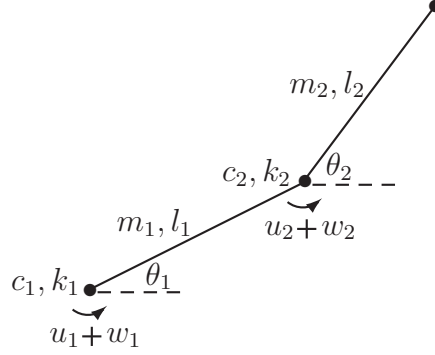


Figure 3.10: All motion of the planar double pendulum is in the horizontal plane.

Let  $x_i \triangleq [ \dot{\theta}_i \ \theta_i ]^T$  and

$$A_{m,i} = \begin{bmatrix} -2\eta_i & -10\eta_i \\ 1 & 0 \end{bmatrix},$$

where  $\eta_i > 0$ . In this example, we let  $\eta_i = 500$ . Note that (3.52)–(3.54) can be expressed in state-space form, with the state  $x_i$  and where all uncertainty is matched.

Let  $m_1 = 2$  kg,  $m_2 = 3$  kg,  $l_1 = 2$  m,  $l_2 = 1$  m,  $c_1 = 10 \frac{\text{kg-m}^2}{\text{rad}}$ ,  $c_2 = 8 \frac{\text{kg-m}^2}{\text{rad}}$ ,  $k_1 = 7 \frac{\text{N-m}}{\text{rad}}$ , and  $k_2 = 5 \frac{\text{N-m}}{\text{rad}}$ .

We let  $Q_i = 3I_2$ ,  $\hat{B}_i = [ 1 \ 0 ]^T$ , and  $\Psi(t) = [ \sin \pi t \ \sin 1.5\pi t ]^T$ . Next, let  $\Omega_i = \gamma_i \hat{B}_i \hat{B}_i^T$ , where  $\gamma_i = 5 \times 10^4$ . The disturbances are  $w_1(t) = \sin \pi t$  and  $w_2(t) = 2 \sin 1.5\pi t$ . The adaptive control is implemented in feedback with the system (3.52)–(3.54), where  $\Gamma_i = 10^7 I_2$  and  $\Upsilon_i = 10^4 I_8$ . Figure 3.11 provides a time history of  $\theta_i(t)$  and  $u_i(t)$ , where the initial conditions are  $\theta_1(0) = \theta_2(0) = 0$  rad and  $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$  rad/s. The nonlinear system is run open-loop and closed-loop, with the decentralized adaptive controller (3.30)–(3.33) implemented in feedback.  $\triangle$

### 3.6 Conclusions

This chapter presented a decentralized adaptive controller for multi-input subsystems with local full-state feedback. This controller is strictly decentralized, that is,

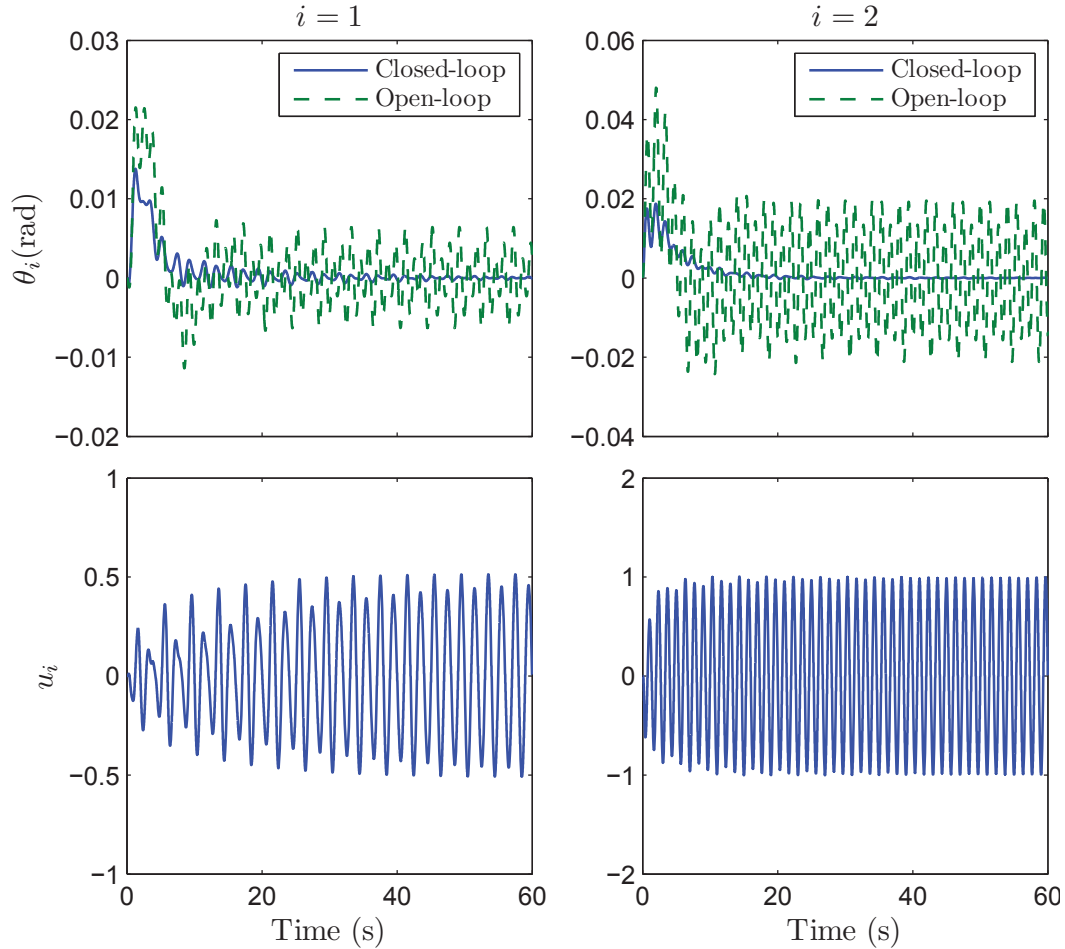


Figure 3.11: *Decentralized adaptive stabilization and disturbance rejection for a planar double pendulum.* The adaptive controller (3.30)–(3.33) is implemented in feedback with the nonlinear system (3.52)–(3.54).

the controller requires only local full-state measurement and no information (including reference-model dynamics) is shared between the local controllers. The controller is effective for stabilization, command following, and disturbance rejection, where the command and disturbance spectrum is known. Furthermore, the controller is effective for systems with arbitrarily large subsystem interconnections provided that the reference-model dynamics matrix  $A_{m,i}$  admit a positive-definite solution to the bounded-real Riccati equation (3.6). We presented sufficient conditions on  $A_{m,i}$  such that (3.6) is satisfied. In this case, the controller yields asymptotically perfect stabilization, command following, and disturbance rejection.

## Chapter 4 Relative-Degree-One Output-Feedback Model Reference Adaptive Control with Exogenous Disturbance

This chapter presents classical model reference adaptive control (MRAC) for single-input single-output (SISO) linear time-invariant systems that are minimum phase and relative degree one. Classical MRAC is effective for stabilization and command following. In this chapter, we extend classical MRAC to address disturbance rejection, where the disturbance is unknown, but generated from a Lyapunov-stable linear system.

### 4.1 Introduction

In this chapter, we present the classical output-feedback MRAC technique for SISO linear time-invariant systems that are minimum phase and relative degree one [1–6]. The goal of output-feedback MRAC is to design a control such that the output of the plant asymptotically follows the output of a reference model. Relative-degree-one output-feedback MRAC operates under the assumptions that the plant is minimum phase, the sign of the high-frequency gain is known, and an upper bound on the order of the plant is known. The classical output-feedback adaptive controller can be used for stabilization and asymptotic command following. In this chapter, classical MRAC is extended to address disturbance rejection. Specifically, the controller presented in this chapter is effective for command following in the presence of an unknown disturbance, provided that the disturbance is generated from a Lyapunov-stable linear system (i.e., the disturbance is a sum of sinusoids).

In Section 4.2, we introduce the output-feedback MRAC problem. We present an ideal fixed-gain controller in Section 4.3, and address command following and disturbance rejection in Section 4.4. Examples are given in Section 4.5, and conclusions are given in Section 4.6.

## 4.2 Problem Formulation

For  $t \geq 0$ , consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t), \quad (4.1)$$

$$y(t) = Cx(t), \quad (4.2)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x(0) \in \mathbb{R}^n$  is the initial condition,  $u(t) \in \mathbb{R}$  is the control input,  $w(t) \in \mathbb{R}^n$  is the exogenous disturbance,  $y(t) \in \mathbb{R}$  is the output, and  $(A, B, C)$  is controllable and observable.

We make the following assumptions regarding the system (4.1) and (4.2):

$$(A4.1) \quad \text{If } \lambda \in \mathbb{C} \text{ and } \det \begin{bmatrix} \lambda I_n - A & B \\ C & 0 \end{bmatrix} = 0, \text{ then } \operatorname{Re} \lambda < 0.$$

$$(A4.2) \quad h \triangleq CB \text{ is nonzero and the sign of } h \text{ is known.}$$

$$(A4.3) \quad \text{There exists a known integer } \bar{n} \text{ such that } n \leq \bar{n}.$$

The system (4.1) and (4.2) is otherwise unknown. Specifically,  $A$ ,  $B$ ,  $C$ , and  $x(0)$  are otherwise unknown. Assumption (A4.1) states that the system (4.1) and (4.2) is minimum phase, that is, the zeros of the transfer function from  $u$  to  $y$  lie in the open-left-half complex plane. Assumption (A4.2) implies that the transfer function from  $u$  to  $y$  is relative degree one.

Let  $\mathbf{p} = \frac{d}{dt}$  denote the differential operator. We make the following assumptions regarding the exogenous disturbance  $w(t)$ :



(A4.4) For all  $t \geq 0$ ,  $w(t)$  is bounded and satisfies

$$\alpha_w(\mathbf{p})w(t) = 0, \quad (4.3)$$

where  $\alpha_w(s)$  is a nonzero monic polynomial with distinct roots that lie on the imaginary axis.

(A4.5) There exists a known integer  $\bar{n}_w$  such that  $n_w \triangleq \deg \alpha_w(s) \leq \bar{n}_w$ .

Assumption (A4.4) implies that  $w(t)$  consists of a sum of sinusoids; however, the disturbance  $w(t)$  and its spectrum are not assumed to be known.

Next, consider the reference model

$$\alpha_m(\mathbf{p})y_m(t) = h_m\beta_m(\mathbf{p})r(t), \quad (4.4)$$

where  $t \geq 0$ ;  $r(t) \in \mathbb{R}$  is the bounded reference-model command;  $y_m(t) \in \mathbb{R}$  is the reference-model output;  $\alpha_m(s)$  is a monic Hurwitz polynomial with degree  $n_m$ ;  $\beta_m(s)$  is a monic Hurwitz polynomial with degree  $n_m - 1$ ;  $\alpha_m(s)$  and  $\beta_m(s)$  are coprime; and  $h_m$  is nonzero. Define

$$G_m(s) \triangleq h_m \frac{\beta_m(s)}{\alpha_m(s)}. \quad (4.5)$$

We now define a strictly positive real transfer function and review the Meyer-Kalman-Yakubovich lemma.

**Definition 4.1.** [3] *A real rational function  $\hat{G}(s)$ , with relative degree one, is strictly positive real if*

- (i)  $\hat{G}(s)$  is asymptotically stable.
- (ii) For all  $\omega \in \mathbb{R}$ ,  $\operatorname{Re}(\hat{G}(j\omega)) > 0$ .

(iii)  $\lim_{\omega \rightarrow \infty} \omega^2 \operatorname{Re}(\hat{G}(j\omega)) > 0$ .

**Lemma 4.1.** [4, 37] *Let  $\hat{A} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ ,  $\hat{B} \in \mathbb{R}^{\hat{n} \times 1}$ , and  $\hat{C} \in \mathbb{R}^{1 \times \hat{n}}$ . If  $\hat{A}$  is asymptotically stable and*

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} \quad (4.6)$$

*is strictly positive real, then there exist positive-definite matrices  $\hat{P} \in \mathbb{R}^{\hat{n} \times \hat{n}}$  and  $\hat{Q} \in \mathbb{R}^{\hat{n} \times \hat{n}}$  such that*

$$\hat{A}^T \hat{P} + \hat{P} \hat{A} + \hat{Q} = 0, \quad (4.7)$$

$$\hat{P} \hat{B} = \hat{C}^T. \quad (4.8)$$

We make the following assumption regarding the reference model (4.4):

(A4.6)  $G_m(s)$  is strictly positive real.

Our goal is to develop an adaptive controller that generates  $u(t)$  such that  $y(t)$  asymptotically follows  $y_m(t)$  in the presence of the disturbance  $w(t)$ . Thus, our goal is to drive the performance

$$z(t) \triangleq y(t) - y_m(t) \quad (4.9)$$

to zero.

### 4.3 Ideal Controller

In this section, we develop the ideal fixed-gain controller. To construct this controller, we assume that the plant (4.1) and (4.2) is known. In the following sections, we relax this condition, but first we consider the ideal fixed-gain controller.

Let  $n_c$  be an integer that satisfies

$$n_c \geq \max \{n_m - 1, \bar{n} + \bar{n}_w\}, \quad (4.10)$$

define

$$\Lambda(s) \triangleq \begin{bmatrix} s^{n_c-1} & s^{n_c-2} & \dots & s & 1 \end{bmatrix}^T, \quad (4.11)$$

and let  $\rho(s)$  be a monic Hurwitz polynomial with degree  $n_c - (n_m - 1)$ , which is non-negative. Next, the matrix transfer function  $\frac{1}{\beta_m(s)\rho(s)}\Lambda(s)$  has the minimal realization  $(A_f, B_f, I_{n_c})$ , where

$$A_f \triangleq \begin{bmatrix} -a_{n_c-1} & \dots & -a_1 & -a_0 \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n_c \times n_c}, \quad B_f \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n_c \times 1}, \quad (4.12)$$

and  $a_0, \dots, a_{n_c-1} \in \mathbb{R}$ . Note that  $\beta_m(s)\rho(s) = s^{n_c} + a_{n_c-1}s^{n_c-1} + \dots + a_1s + a_0$ .

For  $t \geq 0$ , consider the system (4.1) and (4.2) with  $u(t) = u_*(t)$ , where  $u_*(t)$  is the ideal control generated by an ideal fixed-gain controller. Specifically, for  $t \geq 0$ , consider the system

$$\dot{x}_*(t) = Ax_*(t) + Bu_*(t) + w(t), \quad (4.13)$$

$$y_*(t) = Cx_*(t), \quad (4.14)$$

where

$$u_*(t) = \theta_*^T \phi_*(t), \quad (4.15)$$

and

$$\theta_* \triangleq \begin{bmatrix} L_* \\ M_* \\ N_* \end{bmatrix} \in \mathbb{R}^{2n_c+1}, \quad \phi_*(t) \triangleq \begin{bmatrix} U_*(t) \\ Y_*(t) \\ r(t) \end{bmatrix} \in \mathbb{R}^{2n_c+1}, \quad (4.16)$$

where  $L_* \in \mathbb{R}^{n_c}$ ,  $M_* \in \mathbb{R}^{n_c}$ , and  $N_* \in \mathbb{R}$ ; and  $U_*(t) \in \mathbb{R}^{n_c}$  and  $Y_*(t) \in \mathbb{R}^{n_c}$  satisfy

$$\dot{U}_*(t) = A_f U_*(t) + B_f u_*(t), \quad (4.17)$$

$$\dot{Y}_*(t) = A_f Y_*(t) + B_f y_*(t), \quad (4.18)$$

where  $U_*(0) \in \mathbb{R}^{n_c}$  and  $Y_*(0) \in \mathbb{R}^{n_c}$ .

Therefore, the ideal closed-loop system, which consists of (4.13)–(4.18), is given by

$$\dot{\tilde{x}}_*(t) = \tilde{A} \tilde{x}_*(t) + \tilde{B} r(t) + \tilde{D} w(t), \quad (4.19)$$

$$y_*(t) = \tilde{C} \tilde{x}_*(t), \quad (4.20)$$

where

$$\tilde{x}_*(t) \triangleq \begin{bmatrix} x_*(t) \\ U_*(t) \\ Y_*(t) \end{bmatrix} \in \mathbb{R}^{n+2n_c}, \quad (4.21)$$

$$\tilde{A} \triangleq \begin{bmatrix} A & BL_*^T & BM_*^T \\ 0 & A_f + B_f L_*^T & B_f M_*^T \\ B_f C & 0 & A_f \end{bmatrix}, \quad \tilde{B} \triangleq N_* \begin{bmatrix} B \\ B_f \\ 0 \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \quad (4.22)$$

$$\tilde{C} \triangleq \begin{bmatrix} C & 0 & 0 \end{bmatrix}. \quad (4.23)$$

The following lemma guarantees the existence of an ideal fixed-gain controller with certain properties that are used to develop the adaptive controller in the following

section.

**Lemma 4.2.** *Let  $n_c$  satisfy (4.10), and let  $N_* = h_m/h$ . Then, there exists  $L_* \in \mathbb{R}^{n_c}$  and  $M_* \in \mathbb{R}^{n_c}$  such that the following statements hold regarding the ideal closed-loop system (4.19)–(4.23):*

(i)  $\tilde{A}$  is asymptotically stable.

(ii) For all initial conditions  $\tilde{x}_*(0)$  and all  $t \geq 0$ ,

$$\alpha_m(\mathbf{p})\rho(\mathbf{p})y_*(t) = h_m\beta_m(\mathbf{p})\rho(\mathbf{p})r(t). \quad (4.24)$$

(iii)  $\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = G_m(s)$ .

(iv) For all initial conditions  $\tilde{x}_*(0)$ ,  $\lim_{t \rightarrow \infty} [y_*(t) - y_m(t)] = 0$ .

(v) There exists  $\tilde{x}_*(0) \in \mathbb{R}^{n+2n_c}$  such that for all  $t \geq 0$ ,  $y_*(t) = y_m(t)$ .

*Proof.* To show (ii), define  $G_{yu}(s) \triangleq C(sI - A)^{-1}B$ , and it follows from (A4.2) that  $G_{yu}(s)$  can be written as

$$G_{yu}(s) = h \frac{\beta(s)}{\alpha(s)}, \quad (4.25)$$

where  $\alpha(s)$  is a monic polynomial with degree  $n$ , and  $\beta(s)$  is a monic polynomial with degree  $n - 1$ . Moreover, it follows from (A4.1) that  $\beta(s)$  is Hurwitz. Next, define  $G_{yw}(s) \triangleq C(sI - A)^{-1}I$ , and it follows that  $G_{yw}(s)$  can be written as

$$G_{yw}(s) = \frac{1}{\alpha(s)}\sigma(s), \quad (4.26)$$

where  $\sigma(s)$  is a  $1 \times n$  matrix polynomial with degree at most  $n - 1$ . Thus, it follows from (4.13), (4.14), (4.25), and (4.26) that for all  $t \geq 0$ ,

$$\alpha(\mathbf{p})y_*(t) = h\beta(\mathbf{p})u_*(t) + \sigma(\mathbf{p})w(t). \quad (4.27)$$

Next, it follows from (4.15)–(4.18) that  $u_*(t)$  satisfies

$$\beta_m(\mathbf{p})\rho(\mathbf{p})u_*(t) = L_*^T\Lambda(\mathbf{p})u_*(t) + M_*^T\Lambda(\mathbf{p})y_*(t) + N_*\beta_m(\mathbf{p})\rho(\mathbf{p})r(t),$$

which implies that

$$\ell_*(\mathbf{p})u_*(t) = m_*(\mathbf{p})y_*(t) + N_*\beta_m(\mathbf{p})\rho(\mathbf{p})r(t), \quad (4.28)$$

where  $\ell_*(s) \triangleq \beta_m(s)\rho(s) - L_*^T\Lambda(s)$  and  $m_*(s) \triangleq M_*^T\Lambda(s)$ . Since  $\beta_m(s)\rho(s)$  is a monic polynomial with degree  $n_c$ , it follows that the choice of  $L_* \in \mathbb{R}^{n_c}$  uniquely determines  $\ell_*(s)$  and admits all possible monic polynomials with degree  $n_c$ . Therefore, it suffices to show that there exists  $\ell_*(s)$  and  $m_*(s)$  such that (4.24) is satisfied.

Next, let  $\ell_*(s) = \bar{\ell}_*(s)\alpha_w(s)\beta(s)$ , where  $\bar{\ell}_*(s)$  is a monic polynomial with degree  $n_c - n_w - n + 1$ . Now, it suffices to show that there exists  $\bar{\ell}_*(s)$  and  $m_*(s)$  such that (4.24) is satisfied.

Multiplying (4.27) by  $\bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})$  yields

$$\bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p})y_*(t) = h\ell_*(\mathbf{p})u_*(t) + \bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\sigma(\mathbf{p})w(t).$$

Since (A4.4) implies that  $\bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\sigma(\mathbf{p})w(t) = 0$ , it follows that

$$\bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p})y_*(t) = h\ell_*(\mathbf{p})u_*(t). \quad (4.29)$$

Next, combining (4.28) and (4.29) yields

$$\bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p})y_*(t) = hm_*(\mathbf{p})y_*(t) + hN_*\beta_m(\mathbf{p})\rho(\mathbf{p})r(t),$$

which implies that

$$\left[ \bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - hm_*(\mathbf{p}) \right] y_*(t) = hN_*\beta_m(\mathbf{p})\rho(\mathbf{p})r(t).$$

Since  $N_* = h_m/h$ , it follows that

$$\left[ \bar{\ell}_*(\mathbf{p})\alpha_w(\mathbf{p})\alpha(\mathbf{p}) - hm_*(\mathbf{p}) \right] y_*(t) = h_m\beta_m(\mathbf{p})\rho(\mathbf{p})r(t). \quad (4.30)$$

Next, we show that there exist polynomials  $\bar{\ell}_*(s)$  and  $m_*(s)$  such that  $\bar{\ell}_*(s)\alpha_w(s)\alpha(s) - hm_*(s) = \alpha_m(s)\rho(s)$ . First, note that  $\deg \bar{\ell}_*(s)\alpha_w(s)\alpha(s) = n_c + 1 = \deg \alpha_m(s)\rho(s)$  and  $\deg m_*(s) = n_c - 1$ . Thus, since  $\bar{\ell}_*(s)$  is a monic polynomial with degree  $n_c - n_w - n + 1$  and  $m_*(s)$  is a polynomial with degree  $n_c - 1$ , it follows from [2, Theorem 2.3.1] that the roots of  $\bar{\ell}_*(s)\alpha_w(s)\alpha(s) - hm_*(s)$  can be assigned arbitrarily by choice of  $\bar{\ell}_*(s)$  and  $m_*(s)$ . Therefore, there exists polynomials  $\bar{\ell}_*(s)$  and  $m_*(s)$  such that

$$\bar{\ell}_*(s)\alpha_w(s)\alpha(s) - hm_*(s) = \alpha_m(s)\rho(s). \quad (4.31)$$

For all  $t \geq 0$ , (4.24) follows from (4.30) and (4.31). Thus, we have confirmed (ii).

To show (iii), note that  $G_m(s) = h_m\beta_m(s)/\alpha_m(s)$ . Next, it follows from (4.19), (4.20), and (4.24) that  $\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = G_m(s)$ , which confirms (iii).

To show (iv), it follows from (4.4) that  $y_m(t)$  satisfies

$$\alpha_m(\mathbf{p})\rho(\mathbf{p})y_m(t) = h_m\beta_m(\mathbf{p})\rho(\mathbf{p})r(t). \quad (4.32)$$

Subtracting (4.32) from (4.24) implies that

$$\alpha_m(\mathbf{p})\rho(\mathbf{p})[y_*(t) - y_m(t)] = 0. \quad (4.33)$$

Since  $\alpha_m(s)\rho(s)$  is Hurwitz, (4.33) implies that  $\lim_{t \rightarrow \infty} [y_*(t) - y_m(t)] = 0$ , which confirms (iv).

To show (i), it follows from (4.24) and (iii) that the roots of  $\alpha_m(s)\rho(s)$  are eigenvalues of  $\tilde{A}$ . Furthermore, since  $\ell_*(s) = \bar{\ell}_*(s)\alpha_w(s)\beta(s)$ , it follows from (4.28) and (4.29) that the roots of  $\beta(s)$  are eigenvalues of  $\tilde{A}$ . Thus,  $n+n_c$  eigenvalues of  $\tilde{A}$  coincide with the  $n+n_c$  roots of  $\alpha_m(s)\beta(s)\rho(s)$ . The remaining  $n_c$  eigenvalues of  $\tilde{A}$  coincide with the eigenvalues of  $A_f$ , which are the roots of  $\beta_m(s)\rho(s)$ . It follows from (A4.1) that  $\beta(s)$  is Hurwitz, and it follows from (A4.6) that  $\alpha_m(s)$  and  $\beta_m(s)$  are Hurwitz. Since, in addition, the eigenvalues of  $\tilde{A}$  coincide with the roots of  $\alpha_m(s)\beta_m(s)\beta(s)\rho^2(s)$  and  $\rho(s)$  is Hurwitz, it follows that  $\tilde{A}$  is asymptotically stable, which confirms (i).

To show (v), it follows from (iii) that (4.4) has the realization

$$\dot{\tilde{x}}_m(t) = \tilde{A}\tilde{x}_m(t) + \tilde{B}r(t), \quad (4.34)$$

$$y_m = \tilde{C}\tilde{x}_m(t), \quad (4.35)$$

where  $\tilde{x}_m(0) \in \mathbb{R}^{n+2n_c}$ .

Subtracting (4.34) and (4.35) from (4.19) and (4.20), respectively, yields

$$\dot{e}_*(t) = \tilde{A}e_*(t) + \tilde{D}w(t), \quad (4.36)$$

$$z_*(t) = \tilde{C}e_*(t), \quad (4.37)$$

where  $e_*(t) \triangleq \tilde{x}_*(t) - \tilde{x}_m(t)$ ,  $e_*(0) = \tilde{x}_*(0) - \tilde{x}_m(0)$ , and  $z_*(t) \triangleq y_*(t) - y_m(t)$ .

Next, note that (4.3) has the realization

$$\dot{x}_w(t) = A_w x_w(t), \quad (4.38)$$

$$w(t) = C_w x_w(t), \quad (4.39)$$

where  $A_w \in \mathbb{R}^{n_w \times n_w}$ ,  $C_w \in \mathbb{R}^{n \times n_w}$ , and  $x_w(0) \in \mathbb{R}^{n_w}$ . Thus, it follows from (4.36)–



(4.39) that

$$\dot{e}_s(t) = A_s e_s(t), \quad (4.40)$$

$$z_*(t) = C_s e_s(t), \quad (4.41)$$

where

$$e_s(t) \triangleq \begin{bmatrix} e_*(t) \\ x_w(t) \end{bmatrix}, \quad A_s \triangleq \begin{bmatrix} \tilde{A} & \tilde{D}C_w \\ 0 & A_w \end{bmatrix}, \quad (4.42)$$

$$C_s \triangleq \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}. \quad (4.43)$$

It follows from (iv) that  $\lim_{t \rightarrow \infty} z_*(t) = 0$ . Since  $\lim_{t \rightarrow \infty} z_*(t) = 0$  and  $\tilde{A}$  is asymptotically stable, [38, Lemma 3.1] implies that there exists  $S \in \mathbb{R}^{(n+2n_c) \times n_w}$  such that

$$\tilde{A}S - SA_w = \tilde{D}C_w, \quad (4.44)$$

$$\tilde{C}S = 0. \quad (4.45)$$

Define

$$Q \triangleq \begin{bmatrix} I & -S \\ 0 & I \end{bmatrix},$$

and it follows from (4.42)–(4.45) that

$$\bar{e}_s(t) \triangleq Q^{-1}e_s(t), \quad (4.46)$$

$$\bar{A}_s \triangleq Q^{-1}A_sQ = \begin{bmatrix} \tilde{A} & -\tilde{A}S + \tilde{D}C_w + SA_w \\ 0 & A_w \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_w \end{bmatrix}, \quad (4.47)$$

$$\bar{C}_s \triangleq C_sQ = \begin{bmatrix} \tilde{C} & -\tilde{C}S \end{bmatrix} = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}. \quad (4.48)$$

Thus, using the change of basis (4.46)–(4.48), it follows from (4.40) and (4.41) that

$$\dot{\bar{e}}_s(t) = \bar{A}_s \bar{e}_s(t), \quad (4.49)$$

$$z_*(t) = \bar{C}_s \bar{e}_s(t), \quad (4.50)$$

which implies that

$$z_*(t) = \bar{C}_s e^{\bar{A}_s t} \bar{e}_s(0) = \tilde{C} e^{\tilde{A} t} [e_*(0) + Sx_w(0)]. \quad (4.51)$$

Next, let  $\tilde{x}_*(0) = \tilde{x}_m(0) - Sx_w(0)$ , which implies that  $e_*(0) = -Sx_w(0)$ . Thus, it follows from (4.51) that there exists  $\tilde{x}_*(0)$  such that for all  $t \geq 0$ ,  $z_*(t) = 0$ , which confirms (v).  $\square$

#### 4.4 Relative-Degree-One Model Reference Adaptive Control with Disturbance Rejection

Let  $U(t) \in \mathbb{R}^{n_c}$  and  $Y(t) \in \mathbb{R}^{n_c}$  satisfy

$$\dot{U}(t) = A_f U(t) + B_f u(t), \quad (4.52)$$

$$\dot{Y}(t) = A_f Y(t) + B_f y(t), \quad (4.53)$$

where  $U(0) \in \mathbb{R}^{n_c}$  and  $Y(0) \in \mathbb{R}^{n_c}$ , and  $A_f$  and  $B_f$  are given by (4.12). Define

$$\phi(t) \triangleq \begin{bmatrix} U(t) \\ Y(t) \\ r(t) \end{bmatrix} \in \mathbb{R}^{2n_c+1}, \quad (4.54)$$

and consider the controller

$$u(t) = \theta^T(t) \phi(t), \quad (4.55)$$

where  $\theta : [0, \infty) \rightarrow \mathbb{R}^{2n_c+1}$  is given by

$$\dot{\theta}(t) = -\text{sgn}(h)z(t)\Gamma\phi(t), \quad (4.56)$$

and  $\Gamma \in \mathbb{R}^{(2n_c+1) \times (2n_c+1)}$  is positive definite. The MRAC architecture is shown in Figure 4.1.

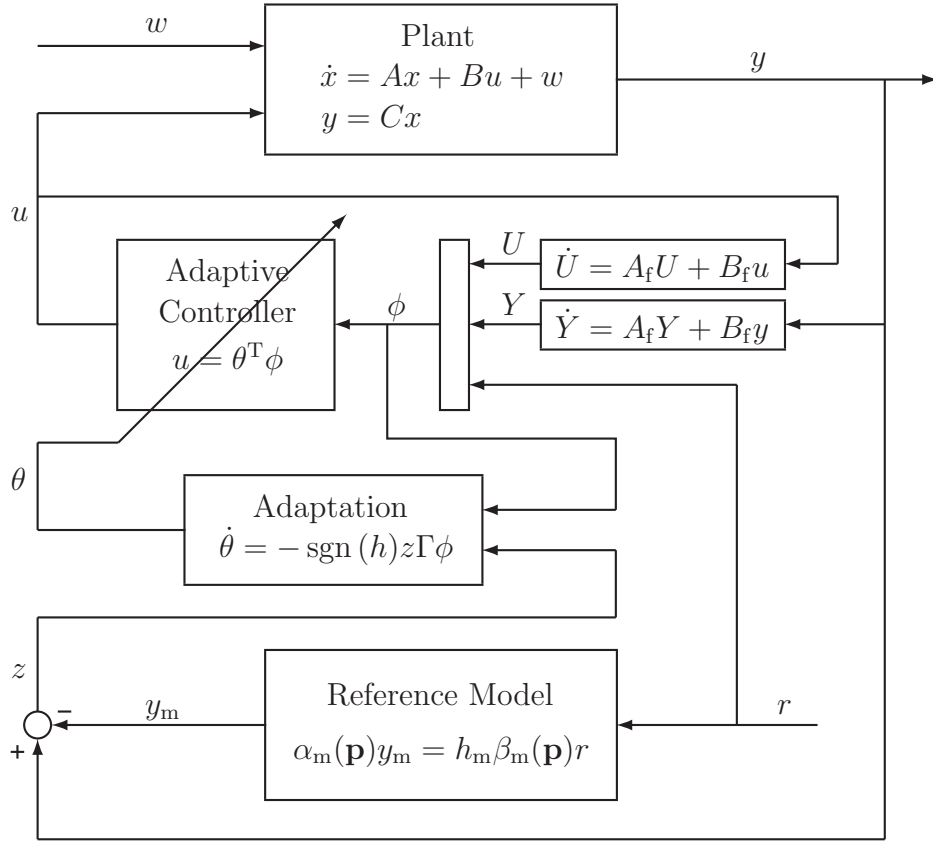


Figure 4.1: Schematic diagram of MRAC architecture given by (4.1), (4.2), and (4.52)–(4.56).

**Theorem 4.1.** Consider the closed-loop system (4.1), (4.2), and (4.52)–(4.56), where  $n_c$  satisfies (4.10) and the open-loop system (4.1) and (4.2) satisfies assumptions (A4.1)–(A4.6). Then, for all initial conditions  $x(0) \in \mathbb{R}^n$ ,  $U(0) \in \mathbb{R}^{n_c}$ ,  $Y(0) \in \mathbb{R}^{n_c}$ , and  $\theta(0) \in \mathbb{R}^{2n_c+1}$ , the following statements hold:

(i)  $x(t)$ ,  $u(t)$ ,  $\theta(t)$ ,  $U(t)$ , and  $Y(t)$  are bounded.

(ii)  $\lim_{t \rightarrow \infty} z(t) = 0$ .

*Proof.* Let  $\theta_* \in \mathbb{R}^{2n_c+1}$  be given by (4.16), where  $N_* \triangleq h_m/h$ , and  $L_* \in \mathbb{R}^{n_c}$  and  $M_* \in \mathbb{R}^{n_c}$  are the ideal controller parameters given by Lemma 4.2. Define  $\tilde{\theta}(t) \triangleq \theta(t) - \theta_*$ , and it follows that the closed-loop system (4.1), (4.2), and (4.52)–(4.55) is given by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \frac{1}{N_*}\tilde{B}\tilde{\theta}^T(t)\phi(t) + \tilde{B}r(t) + \tilde{D}w(t), \quad (4.57)$$

$$y(t) = \tilde{C}\tilde{x}(t), \quad (4.58)$$

where  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$ , and  $\tilde{D}$  are given by (4.22) and (4.23), and

$$\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ U(t) \\ Y(t) \end{bmatrix} \in \mathbb{R}^{n+2n_c}.$$

Next, consider the ideal closed-loop system (4.19)–(4.23), where  $N_* = h_m/h$ ;  $L_*$  and  $M_*$  are given by Lemma 4.2; and  $\tilde{x}_*(0)$  is the initial condition given by part (v) of Lemma 4.2.

Define  $e(t) \triangleq \tilde{x}(t) - \tilde{x}_*(t)$ , and subtracting (4.19) and (4.20) from (4.57) and (4.58), respectively, yields

$$\dot{e}(t) = \tilde{A}e(t) + \frac{1}{N_*}\tilde{B}\tilde{\theta}^T(t)\phi(t), \quad (4.59)$$

$$z(t) = \tilde{C}e(t), \quad (4.60)$$

where part (v) of Lemma 4.2 implies that  $y(t) - y_*(t) = y(t) - y_m(t) = z(t)$ .

Assumption (A4.6) and part (iii) of Lemma 4.2 imply that  $G_m(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$

is strictly positive real. Since, in addition, part (i) of Lemma 4.2 implies that  $\tilde{A}$  is asymptotically stable, it follows from Lemma 4.1 that there exist positive-definite matrices  $P \in \mathbb{R}^{(n+2n_c) \times (n+2n_c)}$  and  $Q \in \mathbb{R}^{(n+2n_c) \times (n+2n_c)}$  such that

$$\tilde{A}^T P + P \tilde{A} + Q = 0, \quad (4.61)$$

$$P \tilde{B} = \tilde{C}^T. \quad (4.62)$$

Next, define the Lyapunov-like function

$$V(e, \tilde{\theta}) \triangleq e^T P e + \frac{1}{|N_*|} \tilde{\theta}^T \Gamma^{-1} \tilde{\theta}, \quad (4.63)$$

where  $P \in \mathbb{R}^{(n+2n_c) \times (n+2n_c)}$  is the positive-definite solution to (4.61).

Evaluating the derivative of  $V$  along the trajectory of (4.56) and (4.59), and using (4.61) and (4.62) yields

$$\begin{aligned} \dot{V}(e, \tilde{\theta}) &= e^T P \left( \tilde{A}e + \frac{1}{N_*} \tilde{B} \tilde{\theta}^T \phi \right) + \left( \tilde{A}e + \frac{1}{N_*} \tilde{B} \tilde{\theta}^T \phi \right)^T P e + 2 \frac{1}{|N_*|} \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \\ &= -e^T Q e + 2 \left( \frac{1}{N_*} \right) e^T P \tilde{B} \tilde{\theta}^T \phi + 2 \frac{1}{|N_*|} \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \\ &= -e^T Q e + 2 \left( \frac{1}{N_*} \right) e^T \tilde{C}^T \tilde{\theta}^T \phi + 2 \frac{1}{|N_*|} \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \end{aligned} \quad (4.64)$$

$$= -e^T Q e + 2z \tilde{\theta}^T \phi \left( \frac{1}{N_*} - \frac{\text{sgn}(h)}{|N_*|} \right). \quad (4.65)$$

Next, it follows from (4.62) that

$$h_m = \tilde{C} \tilde{B} = \tilde{B}^T P \tilde{B}^T > 0. \quad (4.66)$$

Since  $N_* = h_m/h$ , it follows from (4.66) that  $\text{sgn}(h) = \text{sgn}(h_m/h) = \text{sgn}(N_*)$ . Then,

it follows from (4.65) that  $\dot{V}(e, \tilde{\theta}) = -e^T Q e$ , which implies that

$$0 \leq e^T(t) Q e(t) = -\dot{V}(e(t), \tilde{\theta}(t)). \quad (4.67)$$

Integrating (4.67) from 0 to  $\infty$  yields

$$0 \leq \int_0^\infty e^T(t) Q e(t) dt = V(e(0), \tilde{\theta}(0)) - \lim_{t \rightarrow \infty} V(e(t), \tilde{\theta}(t)) \leq V(e(0), \tilde{\theta}(0)), \quad (4.68)$$

where the upper and lower bounds imply that  $\int_0^\infty e^T(t) Q e(t) dt$  exists. Thus, it follows from (4.68) that  $V$  is bounded, which implies that  $e$  and  $\tilde{\theta}$  are bounded. Since  $r$  and  $w$  are bounded and  $\tilde{A}$  is asymptotically stable, (4.19) implies that  $\tilde{x}_*$  is bounded. Since  $e$  and  $\tilde{x}_*$  are bounded, it follows that  $\tilde{x}$  is bounded, which implies that  $x$ ,  $U$ , and  $Y$  are bounded. Then, (4.54) and (4.55) imply that  $u$  is bounded. Thus,  $x$ ,  $u$ ,  $\theta$ ,  $U$ , and  $Y$  are bounded, which confirms (i).

To show (ii), it follows from (4.68) that  $\int_0^\infty e^T(t) Q e(t) dt$  exists. Next, since  $e$ ,  $\tilde{\theta}$ , and  $\phi$  are bounded, (4.59) implies that  $\dot{e}$  is bounded. Next, since  $e$  and  $\dot{e}$  are bounded, it follows that

$$\frac{d}{dt} [e^T(t) Q e(t)] = 2\dot{e}^T(t) Q e(t)$$

is bounded. Thus,  $f(t) \triangleq e^T(t) Q e(t)$  is uniformly continuous. Since  $\int_0^\infty f(t) dt$  exists and  $f(t)$  is uniformly continuous, Barbalat's Lemma implies that  $\lim_{t \rightarrow \infty} f(t) = 0$ . Thus,  $\lim_{t \rightarrow \infty} e(t) = 0$ , and it follows from (4.60) that  $\lim_{t \rightarrow \infty} z(t) = 0$ , which confirms (ii).  $\square$

## 4.5 Numerical Examples

We now present examples to demonstrate adaptive command following for SISO systems that are relative degree one and minimum phase.

**Example 4.1.** *Adaptive command following for an asymptotically stable SISO relative-degree-one system.* Consider the system (4.1) and (4.2), where

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (4.69)$$

$$C = \begin{bmatrix} 1 & 8 \end{bmatrix}, \quad (4.70)$$

which satisfies (A4.1) and (A4.2). Note that  $A$  in (4.69) is asymptotically stable with eigenvalues at  $-2$  and  $-1$ . We let  $\bar{n} = n = 2$ , which satisfies (A4.3). For this example, we let  $w(t) = 0$ , and consider the command following problem without disturbance. Next, consider the reference model (4.5), where

$$G_m(s) = \frac{2(s+9)}{s^2+7s+6}, \quad (4.71)$$

which satisfies (A4.6). The reference-model command is  $r(t) = 0.4 \sin 2\pi t + 0.1 \sin \pi t$ . Next, let  $n_c = 2$ , which satisfies (4.10), and consider (4.52) and (4.53), where

$$A_f = \begin{bmatrix} -14 & -45 \\ 1 & 0 \end{bmatrix}, \quad (4.72)$$

which has eigenvalues at  $-9$  and  $-5$ . Note that  $A_f$  has an eigenvalue equal to the zero of  $G_m(s)$ , which is  $-9$ .

The adaptive controller (4.52)–(4.56) is implemented in feedback with the system (4.1), (4.2), (4.69), (4.70), and (4.72), where  $\Gamma = 10^5 I_5$ . Figure 4.2 provides a time history of  $y(t)$ ,  $y_m(t)$ ,  $z(t)$ , and  $u(t)$ , where the initial conditions are zero. The system is allowed to run open-loop for 5 seconds, then the adaptive controller is turned on. The performance  $z(t)$  converges asymptotically to zero.  $\triangle$

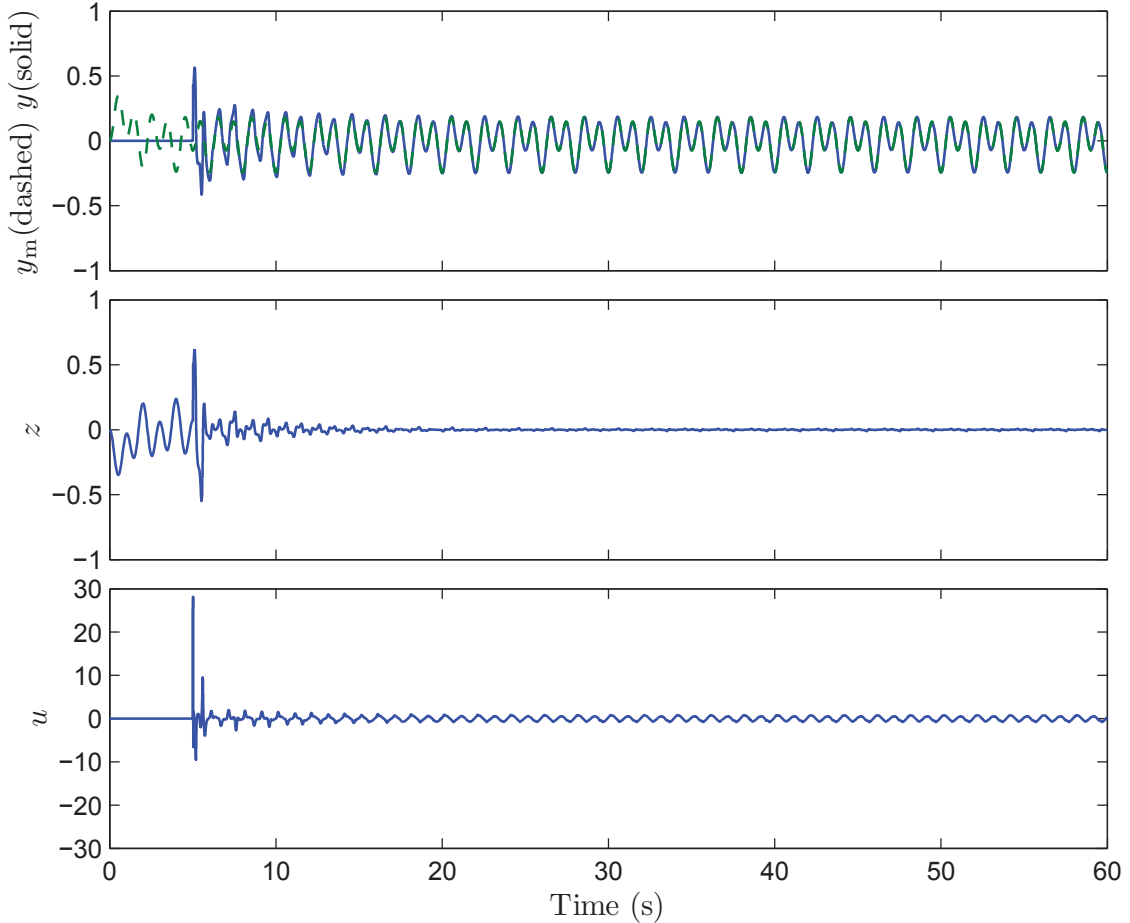


Figure 4.2: *Adaptive command following for an asymptotically stable SISO relative-degree-one system.* The adaptive controller (4.52)–(4.56) is implemented in feedback with the system (4.1), (4.2), (4.69), (4.70), and (4.72). The performance  $z(t)$  converges asymptotically to zero.

**Example 4.2.** *Adaptive command following and disturbance rejection for an asymptotically stable SISO relative-degree-one system.* Reconsider the system in Example 4.1, but consider nonzero disturbance. The plant and reference-model parameters, satisfying (A4.1)–(A4.3) and (A4.6), are the same as in Example 4.1. The disturbance is  $w(t) \triangleq [w_1(t) \ w_2(t)]^T$ , where we let  $w_1(t) = 0.02 \sin 0.5\pi t$  and  $w_2(t) = 0.04 \sin 2\pi t$ , which satisfies (A4.4). Note that the disturbance spectrum is unknown and the disturbance is unmeasured. We let  $\bar{n}_w = n_w = 4$ , which satisfies



(A4.5). Next, let  $n_c = 6$ , which satisfies (4.10), and consider (4.52) and (4.53), where

$$A_f = \begin{bmatrix} -18 & -107 & -268 & -327 & -194 & -45 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad (4.73)$$

which has eigenvalues at  $-9$ ,  $-5$ , and four at  $-1$ .

The adaptive controller (4.52)–(4.56) is implemented in feedback with the system (4.1), (4.2), (4.69), and (4.70), where  $\Gamma = 10^5 I_{13}$ . Figure 4.3 provides a time history of  $y(t)$ ,  $y_m(t)$ ,  $z(t)$ , and  $u(t)$ , where the initial conditions are zero. The system is allowed to run open-loop for 5 seconds, then the adaptive controller is turned on. The performance  $z(t)$  converges asymptotically to zero. Thus,  $y(t)$  follows  $y_m(t)$ , while rejecting the disturbance  $w(t)$ .  $\triangle$

**Example 4.3.** *Adaptive command following for an unstable SISO relative degree-one-system.* Consider the system (4.1) and (4.2), where

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (4.74)$$

$$C = \begin{bmatrix} 1 & 6 \end{bmatrix}, \quad (4.75)$$

which satisfies (A4.1) and (A4.2). Note that  $A$  in (4.74) is unstable with eigenvalues at  $-2$  and  $1$ . We let  $\bar{n} = n = 2$ , which satisfies (A4.3). For this example, we let  $w(t) = 0$ , and consider the command following problem without disturbance. Next,

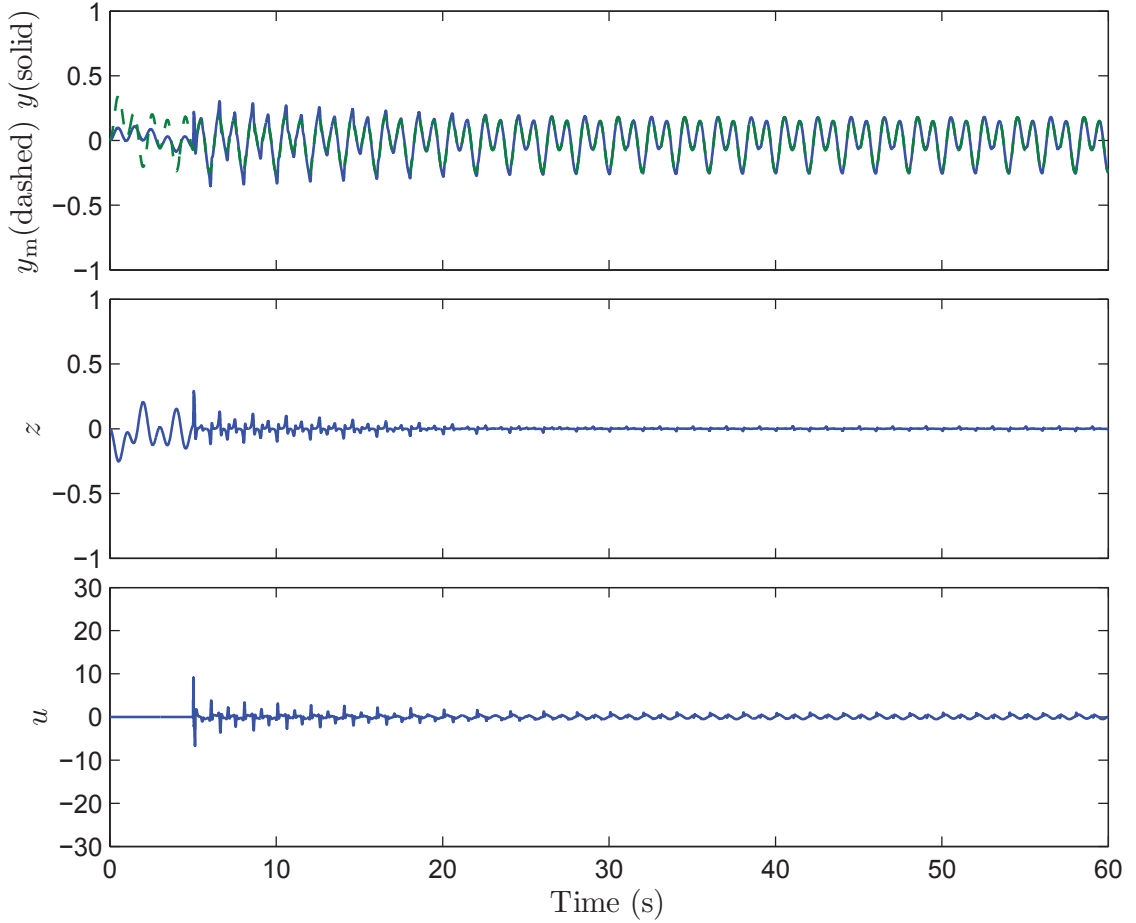


Figure 4.3: *Adaptive command following and disturbance rejection for an asymptotically stable SISO relative-degree-one system.* The adaptive controller (4.52)–(4.56) is implemented in feedback with the system (4.1), (4.2), (4.69), (4.70), and (4.73). The performance  $z(t)$  converges asymptotically to zero.

consider the reference model (4.5), where

$$G_m(s) = \frac{2(s+4)}{s^2 + 7s + 6}, \quad (4.76)$$

which satisfies (A4.6). The reference-model command is  $r(t) = 4 \sin 2\pi t + 3 \sin 1.5\pi t$ .

Next, let  $n_c = 2$ , which satisfies (4.10), and consider (4.52) and (4.53), where

$$A_f = \begin{bmatrix} -9 & -20 \\ 1 & 0 \end{bmatrix}, \quad (4.77)$$

which has eigenvalues at  $-5$  and  $-4$ . Note that  $A_f$  has an eigenvalue equal to the zero of  $G_m(s)$ , which is  $-4$ .

The adaptive controller (4.52)–(4.56) is implemented in feedback with the system (4.1), (4.2), (4.74), (4.75), and (4.77), where  $\Gamma = 10^2 I_5$ . Figure 4.4 provides a time history of  $y(t)$ ,  $y_m(t)$ ,  $z(t)$ , and  $u(t)$ , where the initial conditions are zero. The performance  $z(t)$  converges asymptotically to zero.  $\triangle$

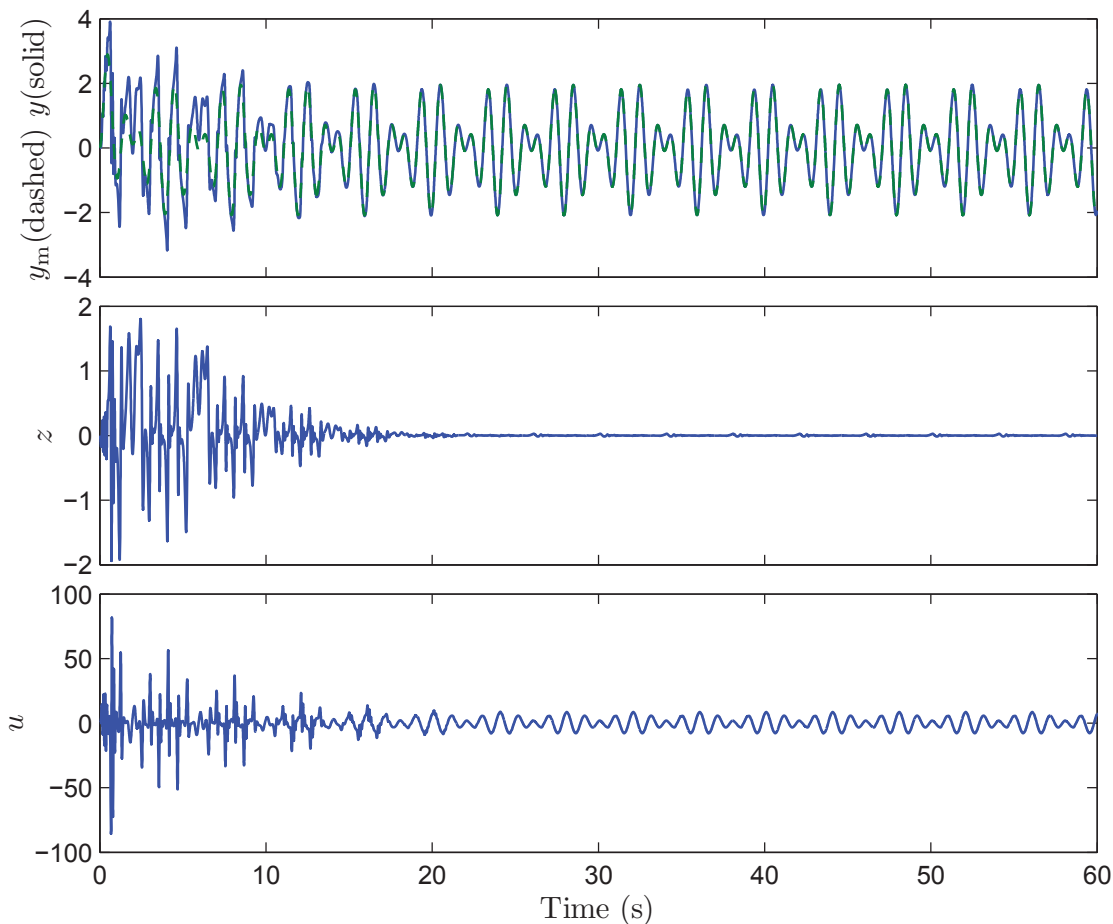


Figure 4.4: *Adaptive command following for an unstable SISO relative-degree-one system.* The adaptive controller (4.52)–(4.56) is implemented in feedback with the system (4.1), (4.2), (4.74), (4.75), and (4.77). The performance  $z(t)$  converges asymptotically to zero.

## 4.6 Conclusions

This chapter presented an adaptive controller for SISO linear time-invariant systems that are minimum phase and relative degree one. This controller is effective for command following and disturbance rejection, where the disturbance spectrum is unknown.

## Chapter 5 Decentralized Relative-Degree-One Output-Feedback Adaptive Control with Exogenous Disturbance

This chapter presents a strictly decentralized model reference adaptive controller for single-input single-output (SISO) linear time-invariant subsystems that are minimum phase and relative degree one. This decentralized adaptive controller requires only local output measurement and no information is shared between the local controllers. The controller is effective for stabilization and disturbance rejection, where the disturbance is unknown, but generated from a Lyapunov-stable linear system.

### 5.1 Introduction

In this chapter, we present an output-feedback decentralized model reference adaptive control (MRAC) technique for SISO linear time-invariant subsystems that are minimum phase and relative degree one. The decentralized adaptive controller is strictly decentralized, meaning that no information is shared between local controllers. Moreover, the decentralized adaptive controller presented in this chapter is effective for stabilization and disturbance rejection in the presence of an unknown disturbance, provided that the disturbance is generated from a Lyapunov-stable linear system (i.e., the disturbance is a sum of sinusoids). The decentralized adaptive controller operates under the assumption that the magnitude of the subsystem interconnections satisfy a bounding condition.

In Section 5.2, we introduce the output-feedback decentralized MRAC problem. We present an ideal decentralized controller in Section 5.3. We address adaptive

stabilization and disturbance rejection in Section 5.4. Examples are given in Section 5.5, and conclusions are given in Section 5.6.

## 5.2 Problem Formulation

For  $t \geq 0$ , consider the system

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t) + B_1 \sum_{j \in \mathcal{J} \setminus \{1\}} \delta_{1,j} y_j(t) + D_1 w(t), \quad (5.1)$$

$\vdots$

$$\dot{x}_\ell(t) = A_\ell x_\ell(t) + B_\ell u_\ell(t) + B_\ell \sum_{j \in \mathcal{J} \setminus \{\ell\}} \delta_{\ell,j} y_j(t) + D_\ell w(t), \quad (5.2)$$

$$y_1(t) = C_1 x_1(t), \quad (5.3)$$

$\vdots$

$$y_\ell(t) = C_\ell x_\ell(t), \quad (5.4)$$

where  $\mathcal{J} \triangleq \{1, 2, \dots, \ell\}$ , for all  $i \in \mathcal{J}$ ,  $x_i(t) \in \mathbb{R}^{n_i}$  is the state,  $x_i(0) \in \mathbb{R}^{n_i}$  is the initial condition,  $u_i(t) \in \mathbb{R}$  is the control input,  $w(t) \in \mathbb{R}^m$  is the exogenous disturbance,  $y_i(t) \in \mathbb{R}$  is the output, and  $(A_i, B_i, C_i)$  is controllable and observable.

For each  $i \in \mathcal{J}$ ,  $x_i$  is the *local state*,  $u_i$  is the *local control*, and  $y_i$  is the *local output*. Moreover, for each  $i \in \mathcal{J}$ , the local control  $u_i$  uses feedback of the local output  $y_i$ , but does not use feedback of the nonlocal outputs  $\{y_j\}_{j \in \mathcal{J} \setminus \{i\}}$ . Unless otherwise stated, all statements in this chapter that involve the subscript  $i$  are for all  $i \in \mathcal{J}$ .

We make the following assumptions regarding the system (5.1)–(5.4):

$$(A5.1) \quad \text{If } \lambda \in \mathbb{C} \text{ and } \det \begin{bmatrix} \lambda I_{n_i} - A_i & B_i \\ C_i & 0 \end{bmatrix} = 0, \text{ then } \operatorname{Re} \lambda < 0.$$

$$(A5.2) \quad h_i \triangleq C_i B_i \text{ is nonzero and the sign of } h_i \text{ is known.}$$

$$(A5.3) \quad \text{There exists a known integer } \bar{n}_i \text{ such that } n_i \leq \bar{n}_i.$$

The system (5.1)–(5.4) is otherwise unknown. Specifically,  $A_1, \dots, A_\ell, B_1, \dots, B_\ell, C_1, \dots, C_\ell, D_1, \dots, D_\ell, \delta_{1,1}, \dots, \delta_{1,\ell}, \dots, \delta_{\ell,\ell}$ , and  $x_1(0), \dots, x_\ell(0)$  are otherwise unknown. Assumption (A5.1) states that each local subsystem of (5.1)–(5.4) is minimum phase, that is, the zeros of the transfer function from  $u_i$  to  $y_i$  lie in the open-left-half complex plane. Assumption (A5.2) implies that the transfer function from  $u_i$  to  $y_i$  is relative degree one.

Let  $\mathbf{p} = \frac{d}{dt}$  denote the differential operator. We make the following assumptions regarding the exogenous disturbance  $w(t)$ :

(A5.4) For all  $t \geq 0$ ,  $w(t)$  is bounded and satisfies

$$\alpha_w(\mathbf{p})w(t) = 0, \quad (5.5)$$

where  $\alpha_w(s)$  is a nonzero monic polynomial with distinct roots that lie on the imaginary axis.

(A5.5) There exists a known integer  $\bar{n}_w$  such that  $n_w \triangleq \deg \alpha_w(s) \leq \bar{n}_w$ .

Assumption (A5.4) implies that  $w(t)$  consists of a sum of sinusoids; however, the disturbance  $w(t)$  and its spectrum are not assumed to be known.

Next, consider the reference model

$$\alpha_{m,i}(\mathbf{p})y_{m,i}(t) = h_{m,i}\beta_{m,i}(\mathbf{p})r_i(t), \quad (5.6)$$

where  $t \geq 0$ ;  $r_i(t) \in \mathbb{R}$  is the bounded reference-model command;  $y_{m,i}(t) \in \mathbb{R}$  is the reference-model output;  $\alpha_{m,i}(s)$  is a monic Hurwitz polynomial with degree  $n_{m,i}$ ;  $\beta_{m,i}(s)$  is a monic Hurwitz polynomial with degree  $n_{m,i} - 1$ ;  $\alpha_{m,i}(s)$  and  $\beta_{m,i}(s)$  are coprime; and  $h_{m,i}$  is nonzero. Define

$$G_{m,i}(s) \triangleq h_{m,i} \frac{\beta_{m,i}(s)}{\alpha_{m,i}(s)}, \quad (5.7)$$

let  $\gamma_i > 0$ , and define

$$F_i(s) \triangleq \frac{G_{m,i}(s)}{1 - \gamma_i G_{m,i}(s)} = \frac{h_{m,i}\beta_{m,i}(s)}{\alpha_{m,i}(s) - \gamma_i h_{m,i}\beta_{m,i}(s)}. \quad (5.8)$$

We make the following assumption regarding the reference model (5.7):

(A5.6)  $F_i(s)$  is strictly positive real.

Note that  $F_i(s)$  depends on the parameter  $\gamma_i > 0$ , which is discussed in the following section.

Our goal is to develop an adaptive controller that generates the control  $u_i(t)$  such that  $y_i(t)$  asymptotically follows  $y_{m,i}(t)$  in the presence of the disturbance  $w(t)$ . Thus, our goal is to drive the performance

$$z_i(t) \triangleq y_i(t) - y_{m,i}(t)$$

to zero.

### 5.3 Ideal Decentralized Controller

In this section, we develop the ideal decentralized controller. To construct this controller, we assume the plant (5.1)–(5.4) is known. In the following sections, we relax this assumption, but first we consider the ideal decentralized controller.

Let  $n_{c,i}$  be an integer that satisfies

$$n_{c,i} \geq \max\{n_{m,i} - 1, \bar{n}_i + \bar{n}_w\}, \quad (5.9)$$

define

$$\Lambda_i(s) \triangleq \begin{bmatrix} s^{n_{c,i}-1} & s^{n_{c,i}-2} & \dots & s & 1 \end{bmatrix}^T, \quad (5.10)$$



and let  $\rho_i(s)$  be a monic Hurwitz polynomial with degree  $n_{c,i} - (n_{m,i} - 1)$ , which is nonnegative. Next, the matrix transfer function  $\frac{1}{\beta_{m,i}(s)\rho_i(s)}\Lambda_i(s)$  has the minimal realization  $(A_{f,i}, B_{f,i}, I_{n_{c,i}})$ , where

$$A_{f,i} \triangleq \begin{bmatrix} -a_{n_{c,i}-1} & \cdots & -a_{1,i} & -a_{0,i} \\ 1 & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n_{c,i} \times n_{c,i}}, \quad B_{f,i} \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n_{c,i} \times 1}, \quad (5.11)$$

and  $a_{0,i}, \dots, a_{n_{c,i}-1} \in \mathbb{R}$  such that  $\beta_{m,i}(s)\rho_i(s) = s^{n_{c,i}} + a_{n_{c,i}-1}s^{n_{c,i}-1} + \dots + a_{1,i}s + a_{0,i}$ .

For  $t \geq 0$ , consider the system (5.1)–(5.4) with  $u_i(t) = u_{*,i}(t)$ , where  $u_{*,i}(t)$  is the ideal control generated by an ideal decentralized controller. Specifically, for  $t \geq 0$ , consider the system

$$\dot{x}_{*,i}(t) = A_i x_{*,i}(t) + B_i u_{*,i}(t) + B_i \sum_{j \in \mathcal{I} \setminus \{i\}} \delta_{i,j} y_{*,j}(t) + D_i w(t), \quad (5.12)$$

$$y_{*,i}(t) = C_i x_{*,i}(t), \quad (5.13)$$

where

$$u_{*,i}(t) = \theta_{*,i}^T \phi_{*,i}(t), \quad (5.14)$$

and

$$\theta_{*,i} \triangleq \begin{bmatrix} L_{*,i} \\ M_{*,i} \\ N_{*,i} \end{bmatrix} \in \mathbb{R}^{2n_{c,i}+1}, \quad \phi_{*,i}(t) \triangleq \begin{bmatrix} U_{*,i}(t) \\ Y_{*,i}(t) \\ r_i(t) \end{bmatrix} \in \mathbb{R}^{2n_{c,i}+1}, \quad (5.15)$$

where  $L_{*,i} \in \mathbb{R}^{n_{c,i}}$ ,  $M_{*,i} \in \mathbb{R}^{n_{c,i}}$ , and  $N_{*,i} \in \mathbb{R}$ ; and  $U_{*,i}(t) \in \mathbb{R}^{n_{c,i}}$  and  $Y_{*,i}(t) \in \mathbb{R}^{n_{c,i}}$

satisfy

$$\dot{U}_{*,i}(t) = A_{f,i}U_{*,i}(t) + B_{f,i}u_{*,i}(t), \quad (5.16)$$

$$\dot{Y}_{*,i}(t) = A_{f,i}Y_{*,i}(t) + B_{f,i}y_{*,i}(t), \quad (5.17)$$

where  $U_{*,i}(0) \in \mathbb{R}^{n_{c,i}}$  and  $Y_{*,i}(0) \in \mathbb{R}^{n_{c,i}}$ .

Therefore, the ideal closed-loop system, which consists of (5.12)–(5.17), is given by

$$\dot{\tilde{x}}_{*,i}(t) = \tilde{A}_i \tilde{x}_{*,i}(t) + \tilde{B}_i r_i(t) + \tilde{E}_i \sum_{j \in \mathcal{I} \setminus \{i\}} \delta_{i,j} \tilde{C}_j \tilde{x}_{*,j}(t) + \tilde{D}_i w(t), \quad (5.18)$$

$$y_{*,i}(t) = \tilde{C}_i \tilde{x}_{*,i}(t), \quad (5.19)$$

where

$$\tilde{x}_{*,i}(t) \triangleq \begin{bmatrix} x_{*,i}(t) \\ U_{*,i}(t) \\ Y_{*,i}(t) \end{bmatrix} \in \mathbb{R}^{n_i + 2n_{c,i}}, \quad (5.20)$$

$$\tilde{A}_i \triangleq \begin{bmatrix} A_i & B_i L_{*,i}^T & B_i M_{*,i}^T \\ 0 & A_{f,i} + B_{f,i} L_{*,i}^T & B_{f,i} M_{*,i}^T \\ B_{f,i} C_i & 0 & A_{f,i} \end{bmatrix}, \quad \tilde{B}_i \triangleq N_{*,i} \begin{bmatrix} B_i \\ B_{f,i} \\ 0 \end{bmatrix}, \quad (5.21)$$

$$\tilde{C}_i \triangleq \begin{bmatrix} C_i & 0 & 0 \end{bmatrix}, \quad (5.22)$$

$$\tilde{E}_i \triangleq \begin{bmatrix} B_i \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{D}_i \triangleq \begin{bmatrix} D_i \\ 0 \\ 0 \end{bmatrix}. \quad (5.23)$$

The following result provides properties of the ideal closed-loop system (5.18)–(5.23).

**Lemma 5.1.** *Let  $n_{c,i}$  satisfy (5.9), and let  $N_{*,i} = h_{m,i}/h_i$ . Then, there exists*

$L_{*,i} \in \mathbb{R}^{n_{c,i}}$  and  $M_{*,i} \in \mathbb{R}^{n_{c,i}}$  such that the following statements hold regarding the ideal closed-loop system (5.18)–(5.23):

(i)  $\tilde{A}_i$  is asymptotically stable.

(ii) For all initial conditions  $\tilde{x}_{*,1}(0), \dots, \tilde{x}_{*,\ell}(0)$ , all  $t \geq 0$ , and all  $i \in \mathcal{J}$ ,

$$\alpha_{m,i}(\mathbf{p})\rho_i(\mathbf{p})y_{*,i}(t) = h_{m,i}\beta_{m,i}(\mathbf{p})\rho_i(\mathbf{p})r_i(t) + h_i\ell_{*,i}(\mathbf{p}) \left[ \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} y_{*,j}(t) \right], \quad (5.24)$$

where  $\ell_{*,i}(s) \triangleq \beta_{m,i}(s)\rho_i(s) - L_{*,i}^T \Lambda_i(s)$ .

(iii)  $\tilde{C}_i(sI - \tilde{A}_i)^{-1} \tilde{B}_i = G_{m,i}(s)$ .

*Proof.* To show (ii), define  $G_{yu,i}(s) \triangleq C_i(sI - A_i)^{-1} B_i$ , and it follows from (A5.2) that  $G_{yu,i}(s)$  can be written as

$$G_{yu,i}(s) = h_i \frac{\beta_i(s)}{\alpha_i(s)}, \quad (5.25)$$

where  $\alpha_i(s)$  is a monic polynomial with degree  $n_i$ , and  $\beta_i(s)$  is a monic polynomial with degree  $n_i - 1$ . Moreover, it follows from (A5.1) that  $\beta_i(s)$  is Hurwitz. Next, define  $G_{yw,i}(s) \triangleq C_i(sI - A_i)^{-1} D_i$ , and it follows that  $G_{yw,i}(s)$  can be written as

$$G_{yw,i}(s) = \frac{1}{\alpha_i(s)} \sigma_i(s), \quad (5.26)$$

where  $\sigma_i(s)$  is a  $1 \times m$  matrix polynomial with degree at most  $n_i - 1$ . Thus, it follows from (5.12), (5.13), (5.25), and (5.26) that for all  $t \geq 0$ ,

$$\alpha_i(\mathbf{p})y_{*,i}(t) = h_i\beta_i(\mathbf{p})u_{*,i}(t) + h_i\beta_i(\mathbf{p}) \left[ \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} y_{*,j}(t) \right] + \sigma_i(\mathbf{p})w(t). \quad (5.27)$$

Next, it follows from (5.14)–(5.17) that  $u_{*,i}(t)$  satisfies

$$\beta_{m,i}(\mathbf{p})\rho_i(\mathbf{p})u_{*,i}(t) = L_{*,i}^T\Lambda_i(\mathbf{p})u_{*,i}(t) + M_{*,i}^T\Lambda_i(\mathbf{p})y_{*,i}(t) + N_{*,i}\beta_{m,i}(\mathbf{p})\rho_i(\mathbf{p})r_i(t),$$

which implies that

$$\ell_{*,i}(\mathbf{p})u_{*,i}(t) = m_{*,i}(\mathbf{p})y_{*,i}(t) + N_{*,i}\beta_{m,i}(\mathbf{p})\rho_i(\mathbf{p})r_i(t), \quad (5.28)$$

where  $m_{*,i}(s) \triangleq M_{*,i}^T\Lambda_i(s)$ . Since  $\beta_{m,i}(s)\rho_i(s)$  is a monic polynomial with degree  $n_{c,i}$ , it follows that the choice of  $L_{*,i} \in \mathbb{R}^{n_{c,i}}$  uniquely determines  $\ell_{*,i}(s)$  and admits all possible monic polynomials with degree  $n_{c,i}$ . Therefore, it suffices to show that there exists  $\ell_{*,i}(s)$  and  $m_{*,i}(s)$  such that (5.24) is satisfied.

Next, let  $\ell_{*,i}(s) = \bar{\ell}_{*,i}(s)\alpha_w(s)\beta_i(s)$ , where  $\bar{\ell}_{*,i}(s)$  is a monic polynomial with degree  $n_{c,i} - n_w - n_i + 1$ . Now, it suffices to show that there exists  $\bar{\ell}_{*,i}(s)$  and  $m_{*,i}(s)$  such that (5.24) is satisfied.

Multiplying (5.27) by  $\bar{\ell}_{*,i}(\mathbf{p})\alpha_w(\mathbf{p})$  yields

$$\begin{aligned} \bar{\ell}_{*,i}(\mathbf{p})\alpha_w(\mathbf{p})\alpha_i(\mathbf{p})y_{*,i}(t) &= h_i\ell_{*,i}(\mathbf{p})u_{*,i}(t) + h_i\ell_{*,i}(\mathbf{p}) \left[ \sum_{j \in \mathcal{I} \setminus \{i\}} \delta_{i,j}y_{*,j}(t) \right] \\ &\quad + \bar{\ell}_{*,i}(\mathbf{p})\alpha_w(\mathbf{p})\sigma_i(\mathbf{p})w(t). \end{aligned} \quad (5.29)$$

Since (A5.4) implies that  $\bar{\ell}_{*,i}(\mathbf{p})\alpha_w(\mathbf{p})\sigma_i(\mathbf{p})w(t) = 0$ , it follows from (5.29) that

$$\bar{\ell}_{*,i}(\mathbf{p})\alpha_w(\mathbf{p})\alpha_i(\mathbf{p})y_{*,i}(t) = h_i\ell_{*,i}(\mathbf{p})u_{*,i}(t) + h_i\ell_{*,i}(\mathbf{p}) \left[ \sum_{j \in \mathcal{I} \setminus \{i\}} \delta_{i,j}y_{*,j}(t) \right]. \quad (5.30)$$

Next, combining (5.28) and (5.30) yields

$$\bar{\ell}_{*,i}(\mathbf{p})\alpha_w(\mathbf{p})\alpha_i(\mathbf{p})y_{*,i}(t) = h_im_{*,i}(\mathbf{p})y_{*,i}(t) + h_iN_{*,i}\beta_{m,i}(\mathbf{p})\rho_i(\mathbf{p})r_i(t)$$

$$+ h_i \ell_{*,i}(\mathbf{p}) \left[ \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} y_{*,j}(t) \right],$$

which implies that

$$\begin{aligned} \left[ \bar{\ell}_{*,i}(\mathbf{p}) \alpha_w(\mathbf{p}) \alpha_i(\mathbf{p}) - h_i m_{*,i}(\mathbf{p}) \right] y_{*,i}(t) &= h_i N_{*,i} \beta_{m,i}(\mathbf{p}) \rho_i(\mathbf{p}) r_i(t) \\ &+ h_i \ell_{*,i}(\mathbf{p}) \left[ \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} y_{*,j}(t) \right]. \end{aligned}$$

Since  $N_{*,i} = h_{m,i}/h_i$ , it follows that

$$\begin{aligned} \left[ \bar{\ell}_{*,i}(\mathbf{p}) \alpha_w(\mathbf{p}) \alpha_i(\mathbf{p}) - h_i m_{*,i}(\mathbf{p}) \right] y_{*,i}(t) &= h_{m,i} \beta_{m,i}(\mathbf{p}) \rho_i(\mathbf{p}) r_i(t) \\ &+ h_i \ell_{*,i}(\mathbf{p}) \left[ \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} y_{*,j}(t) \right]. \end{aligned} \quad (5.31)$$

Next, we show that there exist polynomials  $\bar{\ell}_{*,i}(s)$  and  $m_{*,i}(s)$  such that  $\bar{\ell}_{*,i}(s) \alpha_w(s) \alpha_i(s) - h_i m_{*,i}(s) = \alpha_{m,i}(s) \rho_i(s)$ . First, note that  $\deg \bar{\ell}_{*,i}(s) \alpha_w(s) \alpha_i(s) = n_{c,i} + 1 = \deg \alpha_{m,i}(s) \rho_i(s)$  and  $\deg m_{*,i}(s) = n_{c,i} - 1$ . Thus, since  $\bar{\ell}_{*,i}(s)$  is a monic polynomial with degree  $n_{c,i} - n_w - n_i + 1$  and  $m_{*,i}(s)$  is a polynomial with degree  $n_{c,i} - 1$ , it follows from [2, Theorem 2.3.1] that the roots of  $\bar{\ell}_{*,i}(s) \alpha_w(s) \alpha_i(s) - h_i m_{*,i}(s)$  can be assigned arbitrarily by choice of  $\bar{\ell}_{*,i}(s)$  and  $m_{*,i}(s)$ . Therefore, there exist polynomials  $\bar{\ell}_{*,i}(s)$  and  $m_{*,i}(s)$  such that

$$\bar{\ell}_{*,i}(s) \alpha_w(s) \alpha_i(s) - h_i m_{*,i}(s) = \alpha_{m,i}(s) \rho_i(s). \quad (5.32)$$

For all  $t \geq 0$ , (5.24) follows from (5.31) and (5.32). Thus, we have confirmed (ii).

To show (iii), note that  $G_{m,i}(s) = h_{m,i} \beta_{m,i}(s) / \alpha_{m,i}(s)$ . Next, it follows from (5.18), (5.19), and (5.24) that  $\tilde{C}_i(sI - \tilde{A}_i)^{-1} \tilde{B}_i = G_{m,i}(s)$ , which confirms (iii).

To show (i), it follows from (5.24) and (iii) that the roots of  $\alpha_{m,i}(s) \rho_i(s)$  are eigenvalues of  $\tilde{A}_i$ . Furthermore, since  $\ell_{*,i}(s) = \bar{\ell}_{*,i}(s) \alpha_w(s) \beta_i(s)$ , it follows from (5.28) and

(5.30) that the roots of  $\beta_i(s)$  are eigenvalues of  $\tilde{A}_i$ . Thus,  $n_i + n_{c,i}$  eigenvalues of  $\tilde{A}_i$  coincide with the  $n_i + n_{c,i}$  roots of  $\alpha_{m,i}(s)\beta_i(s)\rho_i(s)$ . The remaining  $n_{c,i}$  eigenvalues of  $\tilde{A}_i$  coincide with the eigenvalues of  $A_{f,i}$ , which are the roots of  $\beta_{m,i}(s)\rho_i(s)$ . It follows from (A5.1) that  $\beta_i(s)$  is Hurwitz, and it follows from (A5.6) that  $\alpha_{m,i}(s)$  and  $\beta_{m,i}(s)$  are Hurwitz. Since, in addition, the eigenvalues of  $\tilde{A}_i$  coincide with the roots of  $\alpha_{m,i}(s)\beta_{m,i}(s)\beta_i(s)\rho_i^2(s)$  and  $\rho_i(s)$  is Hurwitz, it follows that  $\tilde{A}_i$  is asymptotically stable, which confirms (i).  $\square$

Let  $N_{*,i} = h_{m,i}/h_i$ , and let  $L_{*,i} \in \mathbb{R}^{n_{c,i}}$  and  $M_{*,i} \in \mathbb{R}^{n_{c,i}}$  be the ideal controller parameters given by Lemma 5.1. Part (iii) of Lemma 5.1 implies that  $G_{m,i}(s) = \tilde{C}_i(sI - \tilde{A}_i)^{-1}\tilde{B}_i$ , and thus, it follows from (5.8) that

$$F_i(s) = \tilde{C}_i(sI - \tilde{A}_i - \gamma_i\tilde{B}_i\tilde{C}_i)^{-1}\tilde{B}_i. \quad (5.33)$$

Moreover, since (A5.6) states that  $F_i(s)$  is strictly positive real, and part (i) of Lemma 5.1 states that  $\tilde{A}_i$  is asymptotically stable, it follows from Lemma 4.1 that there exist positive-definite matrices  $P_i \in \mathbb{R}^{(n_i+2n_{c,i}) \times (n_i+2n_{c,i})}$  and  $Q_i \in \mathbb{R}^{(n_i+2n_{c,i}) \times (n_i+2n_{c,i})}$  such that

$$(\tilde{A}_i + \gamma_i\tilde{B}_i\tilde{C}_i)^T P_i + P_i(\tilde{A}_i + \gamma_i\tilde{B}_i\tilde{C}_i) + Q_i = 0, \quad (5.34)$$

$$P_i\tilde{B}_i = \tilde{C}_i^T. \quad (5.35)$$

Next, we invoke an assumption regarding the interconnections  $\delta_{1,i}, \dots, \delta_{\ell,i}$ :

(A5.7) For all  $i \in \mathcal{J}$ ,

$$\sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \leq 2\gamma_i \left( \min_{j \in \mathcal{J}} \frac{\lambda_{\min}(Q_j)}{\ell \lambda_{\max}(P_j \tilde{E}_j \tilde{E}_j^T P_j)} \right). \quad (5.36)$$

Assumption (A5.7) limits the magnitude of the interconnections. Note that the

upper bound given by (5.36) depends on  $\gamma_i$ , which can be arbitrarily large provided that (A5.6) is satisfied. However, (A5.6) also involves the reference model (5.7), which affects  $Q_i$  and  $P_i$ , which also appear in the upper bound given by (5.36).

The next result provides additional properties of the ideal closed-loop system (5.18)–(5.23) with  $r_i(t) \equiv 0$ .

**Lemma 5.2.** *Consider the ideal closed-loop system (5.18)–(5.23), which satisfies assumptions (A5.1)–(A5.7). Let  $N_{*,i} = h_{m,i}/h_i$ , and let  $L_{*,i} \in \mathbb{R}^{n_{c,i}}$  and  $M_{*,i} \in \mathbb{R}^{n_{c,i}}$  be given by Lemma 5.1. Assume that  $r_i(t) \equiv 0$ . Then, the following statements hold:*

(i) *If  $w(t) \equiv 0$ , then the equilibrium  $(\tilde{x}_{*,1}, \dots, \tilde{x}_{*,\ell}) \equiv 0$  of (5.18) is asymptotically stable.*

(ii) *For all initial conditions  $\tilde{x}_{*,1}(0), \dots, \tilde{x}_{*,\ell}(0) \in \mathbb{R}^{n_i+2n_{c,i}}$ ,  $\lim_{t \rightarrow \infty} y_{*,1}(t) = \dots = \lim_{t \rightarrow \infty} y_{*,\ell}(t) = 0$ .*

(iii) *There exists  $\tilde{x}_{*,1}(0), \dots, \tilde{x}_{*,\ell}(0) \in \mathbb{R}^{n_i+2n_{c,i}}$  such that for all  $t \geq 0$ ,  $y_{*,1}(t) = \dots = y_{*,\ell}(t) = 0$ .*

*Proof.* To show (i), define the partial Lyapunov function

$$V_i(\tilde{x}_{*,i}) \triangleq \tilde{x}_{*,i}^T P_i \tilde{x}_{*,i}, \quad (5.37)$$

where  $P_i$  is the positive-definite solution to (5.34). Evaluating the derivative of  $V_i$  along the trajectory of (5.18) with  $r_i(t) \equiv 0$  and  $w(t) \equiv 0$  yields

$$\dot{V}_i(\tilde{x}_{*,i}) = \tilde{x}_{*,i}^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) \tilde{x}_{*,i} + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{x}_{*,i}^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_{*,j}. \quad (5.38)$$

Next, define

$$\varepsilon_i \triangleq \frac{\lambda_{\min}(Q_i)}{\ell \lambda_{\max}(P_i \tilde{E}_i \tilde{E}_i^T P_i)}, \quad (5.39)$$

and note that

$$\begin{aligned}
0 &\leq \sum_{j \in \mathcal{J} \setminus \{i\}} \left[ \sqrt{\varepsilon_i} \tilde{E}_i^\top P_i \tilde{x}_{*,i} - \frac{1}{\sqrt{\varepsilon_i}} \delta_{i,j} \tilde{C}_j \tilde{x}_{*,j} \right]^\top \left[ \sqrt{\varepsilon_i} \tilde{E}_i^\top P_i \tilde{x}_{*,i} - \frac{1}{\sqrt{\varepsilon_i}} \delta_{i,j} \tilde{C}_j \tilde{x}_{*,j} \right] \\
&= (\ell - 1) \varepsilon_i \tilde{x}_{*,i}^\top P_i \tilde{E}_i \tilde{E}_i^\top P_i \tilde{x}_{*,i} + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 \tilde{x}_{*,j}^\top \tilde{C}_j^\top \tilde{C}_j \tilde{x}_{*,j} - 2 \delta_{i,j} \tilde{x}_{*,i}^\top P_i \tilde{E}_i \tilde{C}_j \tilde{x}_{*,j} \\
&\leq \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} \tilde{x}_{*,i}^\top \tilde{x}_{*,i} + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 \tilde{x}_{*,j}^\top \tilde{C}_j^\top \tilde{C}_j \tilde{x}_{*,j} - 2 \delta_{i,j} \tilde{x}_{*,i}^\top P_i \tilde{E}_i \tilde{C}_j \tilde{x}_{*,j},
\end{aligned}$$

which implies that

$$\sum_{j \in \mathcal{J} \setminus \{i\}} 2 \delta_{i,j} \tilde{x}_{*,i}^\top P_i \tilde{E}_i \tilde{C}_j \tilde{x}_{*,j} \leq \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} \tilde{x}_{*,i}^\top \tilde{x}_{*,i} + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 \tilde{x}_{*,j}^\top \tilde{C}_j^\top \tilde{C}_j \tilde{x}_{*,j}. \quad (5.40)$$

Next, using (5.40), it follows from (5.38) that

$$\dot{V}_i(\tilde{x}_{*,i}) \leq \tilde{x}_{*,i}^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_{*,i} + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 \tilde{x}_{*,j}^\top \tilde{C}_j^\top \tilde{C}_j \tilde{x}_{*,j}. \quad (5.41)$$

Next, define the Lyapunov function

$$V(\tilde{x}_{*,1}, \dots, \tilde{x}_{*,\ell}) \triangleq \sum_{i \in \mathcal{J}} V_i(\tilde{x}_{*,i}),$$

and it follows from (5.41) that the derivative of  $V$  along the trajectory of (5.18) is given by

$$\begin{aligned}
\dot{V}(\tilde{x}_{*,1}, \dots, \tilde{x}_{*,\ell}) &= \sum_{i \in \mathcal{J}} \dot{V}_i(\tilde{x}_{*,i}) \\
&\leq \sum_{i \in \mathcal{J}} \left[ \tilde{x}_{*,i}^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_{*,i} \right]
\end{aligned}$$



$$\begin{aligned}
& + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 \tilde{x}_{*,j}^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_{*,j} \Big] \\
& = \sum_{i \in \mathcal{J}} \tilde{x}_{*,i}^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_{*,i} \\
& \quad + \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{J} \setminus \{j\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 \tilde{x}_{*,j}^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_{*,j} \\
& = \sum_{i \in \mathcal{J}} \tilde{x}_{*,i}^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_{*,i} \\
& \quad + \sum_{i \in \mathcal{J}} \tilde{x}_{*,i}^T \tilde{C}_i^T \tilde{C}_i \tilde{x}_{*,i} \left( \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_j} \delta_{j,i}^2 \right) \\
& \leq \sum_{i \in \mathcal{J}} \tilde{x}_{*,i}^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_{*,i} \\
& \quad + \frac{1}{\varepsilon} \tilde{x}_{*,i}^T \tilde{C}_i^T \tilde{C}_i \tilde{x}_{*,i} \left( \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \right), \tag{5.42}
\end{aligned}$$

where  $\varepsilon \triangleq \min_{j \in \mathcal{J}} \varepsilon_j$ . Since (A5.7) implies that  $\sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \leq 2\gamma_i \varepsilon$ , it follows from (5.34), (5.35), and (5.42) that

$$\begin{aligned}
\dot{V}(\tilde{x}_{*,1}, \dots, \tilde{x}_{*,\ell}) & \leq \sum_{i \in \mathcal{J}} \tilde{x}_{*,i}^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I + 2\gamma_i \tilde{C}_i^T \tilde{C}_i \right) \tilde{x}_{*,i} \\
& = \sum_{i \in \mathcal{J}} \tilde{x}_{*,i}^T \left( -Q_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_{*,i} \\
& \leq \sum_{i \in \mathcal{J}} -\frac{1}{\ell} \lambda_{\min}(Q_i) \tilde{x}_{*,i}^T \tilde{x}_{*,i},
\end{aligned}$$

which is negative definite. Therefore, it follows from Lyapunov's direct method that the equilibrium  $(\tilde{x}_{1,*}, \dots, \tilde{x}_{\ell,*}) \equiv 0$  is asymptotically stable, which confirms (i).

To show (ii), it follows from (5.18) and (5.19) that

$$\dot{\tilde{x}}_*(t) = \tilde{A} \tilde{x}_*(t) + \tilde{B} r(t) + \tilde{D} w(t), \tag{5.43}$$

$$y_*(t) = \tilde{C} \tilde{x}_*(t), \tag{5.44}$$

where

$$\tilde{x}_*(t) \triangleq \begin{bmatrix} \tilde{x}_{*,1}(t) \\ \vdots \\ \tilde{x}_{*,\ell}(t) \end{bmatrix} \in \mathbb{R}^{\tilde{n}_*}, \quad (5.45)$$

$$y_*(t) \triangleq \begin{bmatrix} y_{*,1}(t) \\ \vdots \\ y_{*,\ell}(t) \end{bmatrix} \in \mathbb{R}^\ell, \quad r(t) \triangleq \begin{bmatrix} r_1(t) \\ \vdots \\ r_\ell(t) \end{bmatrix} \in \mathbb{R}^\ell, \quad (5.46)$$

$$\tilde{A} \triangleq \begin{bmatrix} \tilde{A}_1 & \delta_{1,2}\tilde{E}_1\tilde{C}_2 & \cdots & \delta_{1,\ell}\tilde{E}_1\tilde{C}_\ell \\ \delta_{2,1}\tilde{E}_2\tilde{C}_1 & \tilde{A}_2 & & \delta_{2,\ell}\tilde{E}_2\tilde{C}_\ell \\ \vdots & & \ddots & \\ \delta_{\ell,1}\tilde{E}_\ell\tilde{C}_1 & \delta_{\ell,2}\tilde{E}_\ell\tilde{C}_2 & & \tilde{A}_\ell \end{bmatrix} \in \mathbb{R}^{\tilde{n}_* \times \tilde{n}_*}, \quad (5.47)$$

$$\tilde{B} \triangleq \begin{bmatrix} \tilde{B}_1 & 0 \\ & \ddots \\ 0 & \tilde{B}_\ell \end{bmatrix} \in \mathbb{R}^{\tilde{n}_* \times \ell}, \quad \tilde{D} \triangleq \begin{bmatrix} \tilde{D}_1 \\ \vdots \\ \tilde{D}_\ell \end{bmatrix} \in \mathbb{R}^{\tilde{n}_* \times m} \quad (5.48)$$

$$\tilde{C} \triangleq \begin{bmatrix} \tilde{C}_1 & 0 \\ & \ddots \\ 0 & \tilde{C}_\ell \end{bmatrix} \in \mathbb{R}^{\ell \times \tilde{n}_*}, \quad (5.49)$$

and  $\tilde{n}_* \triangleq \sum_{i \in \mathcal{J}} n_i + 2n_{c,i}$ . It follows from (i) that  $\tilde{A}$  is asymptotically stable.

Next, it follows from part (ii) of Lemma 5.1 that

$$\tilde{\alpha}(\mathbf{p})y_*(t) = \tilde{\beta}(\mathbf{p})r(t), \quad (5.50)$$

where

$$\tilde{\alpha}(s) \triangleq \begin{bmatrix} \alpha_{m,1}(s)\rho_1(s) & -\delta_{1,2}h_1\ell_{*,1}(s) & \cdots & -\delta_{1,\ell}h_1\ell_{*,1}(s) \\ -\delta_{2,1}h_2\ell_{*,2}(s) & \alpha_{m,2}(s)\rho_2(s) & & -\delta_{2,\ell}h_2\ell_{*,2}(s) \\ \vdots & & \ddots & \\ -\delta_{\ell,1}h_\ell\ell_{*,\ell}(s) & -\delta_{\ell,2}h_\ell\ell_{*,\ell}(s) & & \alpha_{m,\ell}(s)\rho_\ell(s) \end{bmatrix}, \quad (5.51)$$

$$\tilde{\beta}(s) \triangleq \begin{bmatrix} h_{m,1}\beta_{m,1}(s)\rho_1(s) & & & 0 \\ & & \ddots & \\ & & & h_{m,\ell}\beta_{m,\ell}(s)\rho_\ell(s) \\ 0 & & & \end{bmatrix}. \quad (5.52)$$

Note that (5.43)–(5.52) imply that  $\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = \tilde{\alpha}^{-1}(s)\beta(s)$ . Then, since  $\tilde{A}$  is asymptotically stable, it follows that  $\det \tilde{\alpha}(s)$  is Hurwitz. Next, since  $r(t) \equiv 0$ , it follows that  $\tilde{\alpha}(\mathbf{p})y_*(t) = 0$ . Since  $\det \tilde{\alpha}(s)$  is Hurwitz, it follows that for all initial conditions  $\tilde{x}_*(0)$ ,  $\lim_{t \rightarrow \infty} y_*(t) = 0$ , which implies that for all initial conditions  $\tilde{x}_{*,1}(0), \dots, \tilde{x}_{*,\ell}(0)$ ,  $\lim_{t \rightarrow \infty} y_{*,1}(t) = \cdots = \lim_{t \rightarrow \infty} y_{*,\ell}(t) = 0$ , which confirms (ii).

To show (iii), note that (5.5) has the realization

$$\dot{x}_w(t) = A_w x_w(t), \quad (5.53)$$

$$w(t) = C_w x_w(t), \quad (5.54)$$

where  $A_w \in \mathbb{R}^{n_w \times n_w}$ ,  $C_w \in \mathbb{R}^{m \times n_w}$ , and  $x_w(0) \in \mathbb{R}^{n_w}$ .

Since  $r(t) \equiv 0$ , it follows from (5.43)–(5.49), (5.53), and (5.54) that

$$\dot{x}_s(t) = A_s x_s(t), \quad (5.55)$$

$$y_*(t) = C_s x_s(t), \quad (5.56)$$

where

$$x_s(t) \triangleq \begin{bmatrix} \tilde{x}_*(t) \\ x_w(t) \end{bmatrix} \in \mathbb{R}^{\tilde{n}_*+n_w}, \quad A_s \triangleq \begin{bmatrix} \tilde{A} & \tilde{D}C_w \\ 0 & A_w \end{bmatrix}, \quad (5.57)$$

$$C_s \triangleq \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}. \quad (5.58)$$

Next, it follows from (ii) that  $\lim_{t \rightarrow \infty} y_*(t) = 0$ . Since  $\lim_{t \rightarrow \infty} y_*(t) = 0$  and  $\tilde{A}$  is asymptotically stable, [38, Lemma 3.1] implies that there exists  $S \in \mathbb{R}^{\tilde{n}_* \times n_w}$  such that

$$\tilde{A}S - SA_w = \tilde{D}C_w, \quad (5.59)$$

$$\tilde{C}S = 0. \quad (5.60)$$

Define

$$R \triangleq \begin{bmatrix} I_{\tilde{n}_*} & -S \\ 0 & I_{n_w} \end{bmatrix} \in \mathbb{R}^{(\tilde{n}_*+n_w) \times (\tilde{n}_*+n_w)},$$

and it follows from (5.57)–(5.60) that

$$\bar{x}_s(t) \triangleq R^{-1}x_s(t), \quad (5.61)$$

$$\bar{A}_s \triangleq R^{-1}A_sR = \begin{bmatrix} \tilde{A} & -\tilde{A}S + \tilde{D}C_w + SA_w \\ 0 & A_w \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & A_w \end{bmatrix}, \quad (5.62)$$

$$\bar{C}_s \triangleq C_sQ = \begin{bmatrix} \tilde{C} & -\tilde{C}S \end{bmatrix} = \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}. \quad (5.63)$$

Thus, using the change of basis (5.61)–(5.63), it follows from (5.55) and (5.56) that

$$\dot{\bar{x}}_s(t) = \bar{A}_s \bar{x}_s(t), \quad (5.64)$$

$$y_*(t) = \bar{C}_s \bar{x}_s(t), \quad (5.65)$$

which implies that

$$y_*(t) = \bar{C}_s e^{\bar{A}_s t} \bar{x}_s(0) = \tilde{C} e^{\tilde{A} t} [\tilde{x}_*(0) + Sx_w(0)]. \quad (5.66)$$

Next, let  $\tilde{x}_*(0) = -Sx_w(0)$ . Thus, it follows from (5.66) that there exists  $\tilde{x}_*(0)$  such that for all  $t \geq 0$ ,  $y_*(t) = 0$ , which confirms (iii).  $\square$

#### 5.4 Relative-Degree-One Decentralized Adaptive Stabilization and Disturbance Rejection

In this section, we address decentralized adaptive stabilization and disturbance rejection for relative-degree-one subsystems. Let  $U_i(t) \in \mathbb{R}^{n_{c,i}}$  and  $Y_i(t) \in \mathbb{R}^{n_{c,i}}$  satisfy

$$\dot{U}_i(t) = A_{f,i} U_i(t) + B_{f,i} u_i(t), \quad (5.67)$$

$$\dot{Y}_i(t) = A_{f,i} Y_i(t) + B_{f,i} y_i(t), \quad (5.68)$$

where  $U_i(0) \in \mathbb{R}^{n_{c,i}}$  and  $Y_i(0) \in \mathbb{R}^{n_{c,i}}$ , and  $A_{f,i}$  and  $B_{f,i}$  are given by (5.11). Define

$$\phi_i(t) \triangleq \begin{bmatrix} U_i(t) \\ Y_i(t) \\ r_i(t) \end{bmatrix} \in \mathbb{R}^{2n_{c,i}+1}, \quad (5.69)$$

and consider the controller

$$u_i(t) = \theta_i^T(t) \phi_i(t), \quad (5.70)$$

where  $\theta_i : [0, \infty) \rightarrow \mathbb{R}^{2n_{c,i}+1}$  is given by

$$\dot{\theta}_i(t) = -\text{sgn}(h_i) z_i(t) \Gamma_i \phi_i(t), \quad (5.71)$$

where  $\Gamma_i \in \mathbb{R}^{(2n_{c,i}+1) \times (2n_{c,i}+1)}$  is positive definite. The relative-degree-one decentralized adaptive architecture is shown in Figure 5.1.

Let  $\theta_{*,i} \in \mathbb{R}^{2n_{c,i}+1}$  be given by (5.15), where  $N_{*,i} \triangleq h_{m,i}/h_i$ , and  $L_{*,i} \in \mathbb{R}^{n_{c,i}}$  and  $M_{*,i} \in \mathbb{R}^{n_{c,i}}$  are the ideal controller parameters given by Lemma 5.1. Define

$$\tilde{\theta}_i(t) \triangleq \theta_i(t) - \theta_{*,i}. \quad (5.72)$$

Thus, it follows from (5.1)–(5.4) and (5.67)–(5.70) that the closed-loop system is given by

$$\dot{\tilde{x}}_i(t) = \tilde{A}_i \tilde{x}_i(t) + \frac{1}{N_{*,i}} \tilde{B}_i \tilde{\theta}_i^\Gamma(t) \phi_i(t) + \tilde{B}_i r_i(t) + \tilde{E}_i \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{C}_j \tilde{x}_j(t) + \tilde{D}_i w(t), \quad (5.73)$$

$$y_i(t) = \tilde{C}_i \tilde{x}_i(t), \quad (5.74)$$

where  $\tilde{A}_i$ ,  $\tilde{B}_i$ ,  $\tilde{C}_i$ ,  $\tilde{E}_i$ , and  $\tilde{D}_i$  are given by (5.21)–(5.23), and

$$\tilde{x}_i(t) \triangleq \begin{bmatrix} x_i(t) \\ U_i(t) \\ Y_i(t) \end{bmatrix} \in \mathbb{R}^{n_i + 2n_{c,i}}. \quad (5.75)$$

The following theorem is the main result on decentralized adaptive stabilization, where the reference-model commands and the disturbances are zero (i.e.,  $r_i(t) \equiv 0$  and  $w(t) \equiv 0$ ).

**Theorem 5.1.** *Consider the closed-loop system (5.71) and (5.73), where  $n_{c,i}$  satisfies (5.9), the open-loop system (5.1)–(5.4) satisfies assumptions (A5.1)–(A5.7),  $w(t) \equiv 0$ , and  $r_i(t) \equiv 0$ . Then, the equilibrium  $(\tilde{x}_1, \dots, \tilde{x}_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) \equiv 0$  is Lyapunov stable. Furthermore, for all initial conditions  $x_i(0) \in \mathbb{R}^{n_i}$ ,  $U_i(0) \in \mathbb{R}^{n_{c,i}}$ ,  $Y_i(0) \in \mathbb{R}^{n_{c,i}}$ , and  $\theta_i(0) \in \mathbb{R}^{2n_{c,i}+1}$ , the following statements hold:*

- (i)  $x_i(t)$ ,  $u_i(t)$ ,  $\theta_i(t)$ ,  $U_i(t)$ , and  $Y_i(t)$  are bounded.

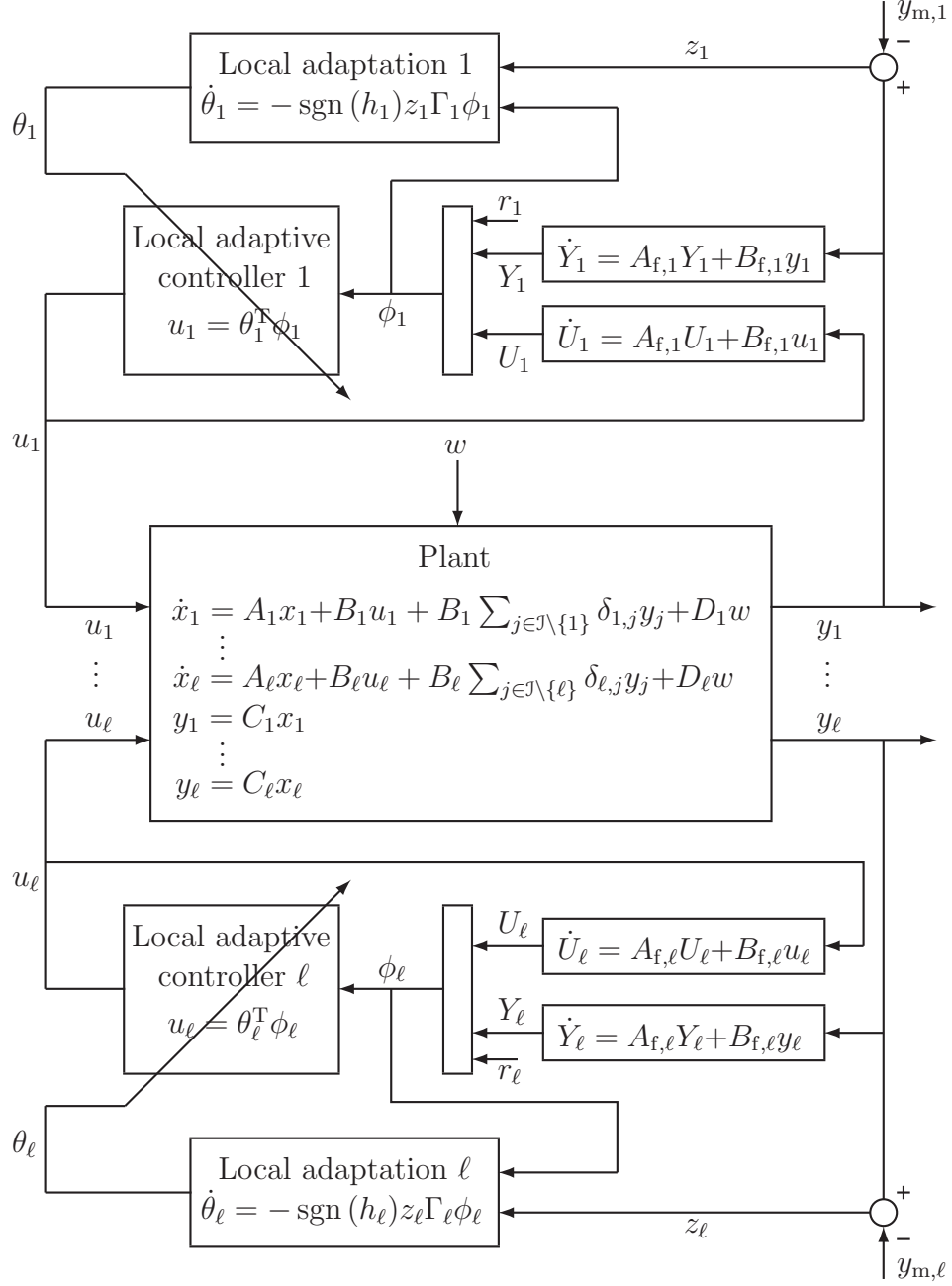


Figure 5.1: Schematic diagram of relative-degree-one decentralized adaptive architecture given by (5.1), (5.4), and (5.67)–(5.71).

(ii)  $\lim_{t \rightarrow \infty} \tilde{x}_i(t) = 0$ .

*Proof.* Define the partial Lyapunov function

$$V_i(\tilde{x}_i, \tilde{\theta}_i) \triangleq \tilde{x}_i^T P_i \tilde{x}_i + \frac{1}{|N_{*,i}|} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i, \quad (5.76)$$

where  $P_i \in \mathbb{R}^{(n_i+2n_{c,i}) \times (n_i+2n_{c,i})}$  is the positive-definite solution to (5.34).

Evaluating the derivative of  $V_i$  along the trajectory of (5.71) and (5.73) with  $w(t) \equiv 0$  and  $r_i(t) \equiv 0$ , and using (5.35) and (5.74) yields

$$\begin{aligned} \dot{V}_i(\tilde{x}_i, \tilde{\theta}_i) &= \tilde{x}_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) \tilde{x}_i + \frac{2}{N_{*,i}} \tilde{x}_i^T P_i \tilde{B}_i \tilde{\theta}_i^T \phi_i + \frac{2}{|N_{*,i}|} \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\theta}_i \\ &\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j \\ &= \tilde{x}_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) \tilde{x}_i + \frac{2}{N_{*,i}} \tilde{x}_i^T \tilde{C}_i^T \tilde{\theta}_i^T \phi_i + \frac{2}{|N_{*,i}|} \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\theta}_i \\ &\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j \\ &= \tilde{x}_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) \tilde{x}_i + 2y_i \tilde{\theta}_i^T \phi_i \left( \frac{1}{N_{*,i}} - \frac{\text{sgn}(h_i)}{|N_{*,i}|} \right) + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j. \end{aligned} \quad (5.77)$$

Next, it follows from (5.35) that  $h_{m,i} = \tilde{C}_i \tilde{B}_i = \tilde{B}_i^T P_i \tilde{B}_i > 0$ . Since  $N_{*,i} = h_{m,i}/h_i$ , it follows that  $\text{sgn}(h_i) = \text{sgn}(h_{m,i}/h_i) = \text{sgn}(N_{*,i})$ . Then, it follows from (5.77) that

$$\dot{V}_i(\tilde{x}_i, \tilde{\theta}_i) = \tilde{x}_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) \tilde{x}_i + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j. \quad (5.78)$$

Next, define

$$\varepsilon_i \triangleq \frac{\lambda_{\min}(Q_i)}{\ell \lambda_{\max}(P_i \tilde{E}_i \tilde{E}_i^T P_i)}, \quad (5.79)$$



and note that

$$\begin{aligned}
0 &\leq \sum_{j \in \mathcal{J} \setminus \{i\}} \left[ \sqrt{\varepsilon_i} \tilde{E}_i^T P_i \tilde{x}_i - \frac{1}{\sqrt{\varepsilon_i}} \delta_{i,j} \tilde{C}_j \tilde{x}_j \right]^T \left[ \sqrt{\varepsilon_i} \tilde{E}_i^T P_i \tilde{x}_i - \frac{1}{\sqrt{\varepsilon_i}} \delta_{i,j} \tilde{C}_j \tilde{x}_j \right] \\
&= (\ell - 1) \varepsilon_i \tilde{x}_i^T P_i \tilde{E}_i \tilde{E}_i^T P_i \tilde{x}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 \tilde{x}_j^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_j - 2 \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j \\
&\leq \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} \tilde{x}_i^T \tilde{x}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 \tilde{x}_j^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_j - 2 \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j,
\end{aligned}$$

which implies that

$$\sum_{j \in \mathcal{J} \setminus \{i\}} 2 \delta_{i,j} \tilde{x}_i^T P_i \tilde{E}_i \tilde{C}_j \tilde{x}_j \leq \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} \tilde{x}_i^T \tilde{x}_i + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 \tilde{x}_j^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_j. \quad (5.80)$$

Next, using (5.80), it follows from (5.78) that

$$\dot{V}_i(\tilde{x}_i, \tilde{\theta}_i) \leq \tilde{x}_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_i + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 \tilde{x}_j^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_j. \quad (5.81)$$

Next, define the Lyapunov function

$$V(\tilde{x}_1, \dots, \tilde{x}_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) \triangleq \sum_{i \in \mathcal{J}} V_i(\tilde{x}_i, \tilde{\theta}_i),$$

and it follows from (5.81) that the derivative of  $V$  along the trajectory of (5.71) and (5.73) is given by

$$\begin{aligned}
\dot{V}(\tilde{x}_1, \dots, \tilde{x}_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) &= \sum_{i \in \mathcal{J}} \dot{V}_i(\tilde{x}_i, \tilde{\theta}_i) \\
&\leq \sum_{i \in \mathcal{J}} \left[ \tilde{x}_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_i \right. \\
&\quad \left. + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 \tilde{x}_j^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_j \right]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in \mathcal{J}} \tilde{x}_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_i \\
&\quad + \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{J} \setminus \{j\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 \tilde{x}_j^T \tilde{C}_j^T \tilde{C}_j \tilde{x}_j \\
&= \sum_{i \in \mathcal{J}} \tilde{x}_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_i \\
&\quad + \sum_{i \in \mathcal{J}} \tilde{x}_i^T \tilde{C}_i^T \tilde{C}_i \tilde{x}_i \left( \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_j} \delta_{j,i}^2 \right) \\
&\leq \sum_{i \in \mathcal{J}} \tilde{x}_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_i \\
&\quad + \frac{1}{\varepsilon} \tilde{x}_i^T \tilde{C}_i^T \tilde{C}_i \tilde{x}_i \left( \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \right), \tag{5.82}
\end{aligned}$$

where  $\varepsilon \triangleq \min_{j \in \mathcal{J}} \varepsilon_j$ . Since (A5.7) implies that  $\sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \leq 2\gamma_i \varepsilon$ , it follows from (5.34), (5.35), and (5.82) that

$$\begin{aligned}
\dot{V}(\tilde{x}_1, \dots, \tilde{x}_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) &\leq \sum_{i \in \mathcal{J}} \tilde{x}_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I + 2\gamma_i \tilde{C}_i^T \tilde{C}_i \right) \tilde{x}_i \\
&= \sum_{i \in \mathcal{J}} \tilde{x}_i^T \left( -Q_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) \tilde{x}_i \\
&\leq \sum_{i \in \mathcal{J}} -\frac{1}{\ell} \lambda_{\min}(Q_i) \tilde{x}_i^T \tilde{x}_i,
\end{aligned}$$

which is nonpositive. Therefore, the equilibrium  $(\tilde{x}_1, \dots, \tilde{x}_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) \equiv 0$  is Lyapunov stable, and for all initial conditions,  $\tilde{x}_i$  and  $\tilde{\theta}_i$  are bounded. Since  $\tilde{x}_i$  is bounded, it follows from (5.75) that  $x_i$ ,  $U_i$ , and  $Y_i$  are bounded. Moreover, since  $\tilde{\theta}_i$ ,  $U_i$ , and  $Y_i$  are bounded, it follows from (5.70) and (5.72) that  $\theta_i$  and  $u_i$  are bounded, which confirms (i).

Finally, since  $V$  is positive definite and radially unbounded, and

$$\dot{V} \leq - \sum_{i \in \mathcal{J}} \frac{1}{\ell} \lambda_{\min}(Q_i) \tilde{x}_i^T \tilde{x}_i,$$

it follows from LaSalle's invariance principle [34, Theorem 4.4] that for all initial conditions,  $\lim_{t \rightarrow \infty} \tilde{x}_i(t) = 0$ . Thus, it follows from (5.74) that  $\lim_{t \rightarrow \infty} y_i(t) = 0$ , which confirms (ii).  $\square$

Next, we extend the analysis for the relative-degree-one decentralized adaptive controller to address disturbance rejection. The following theorem is the main result on decentralized adaptive disturbance rejection for SISO relative-degree-one subsystems that are minimum phase, where the reference-model commands are zero (i.e.,  $r_i(t) \equiv 0$ ).

**Theorem 5.2.** *Consider the closed-loop system (5.1)–(5.4), and (5.67)–(5.71), where  $n_{c,i}$  satisfies (5.9), the open-loop system (5.1)–(5.4) satisfies assumptions (A5.1)–(A5.7), and  $r_i(t) \equiv 0$ . Then, for all initial conditions  $x_i(0) \in \mathbb{R}^{n_i}$ ,  $U_i(0) \in \mathbb{R}^{n_{c,i}}$ ,  $Y_i(0) \in \mathbb{R}^{n_{c,i}}$ , and  $\theta_i(0) \in \mathbb{R}^{2n_{c,i}+1}$ , the following statements hold:*

(i)  $x_i(t)$ ,  $u_i(t)$ ,  $\theta_i(t)$ ,  $U_i(t)$ , and  $Y_i(t)$  are bounded.

(ii)  $\lim_{t \rightarrow \infty} y_i(t) = 0$ .

*Proof.* Consider the closed-loop system (5.73) and (5.74) with  $r_i(t) \equiv 0$ , which is given by

$$\dot{\tilde{x}}_i(t) = \tilde{A}_i \tilde{x}_i(t) + \frac{1}{N_{*,i}} \tilde{B}_i \tilde{\theta}_i^T(t) \phi_i(t) + \tilde{E}_i \sum_{j \in \mathcal{I} \setminus \{i\}} \delta_{i,j} \tilde{C}_j \tilde{x}_j(t) + \tilde{D}_i w(t), \quad (5.83)$$

$$y_i(t) = \tilde{C}_i \tilde{x}_i(t). \quad (5.84)$$

Next, consider the ideal closed-loop system (5.18) and (5.19) with  $r_i(t) \equiv 0$ , which is given by

$$\dot{\tilde{x}}_{*,i}(t) = \tilde{A}_i \tilde{x}_{*,i}(t) + \tilde{E}_i \sum_{j \in \mathcal{I} \setminus \{i\}} \delta_{i,j} \tilde{C}_j \tilde{x}_{*,j}(t) + \tilde{D}_i w(t), \quad (5.85)$$

$$y_{*,i}(t) = \tilde{C}_i \tilde{x}_{*,i}(t), \quad (5.86)$$

where  $N_{*,i} = h_{m,i}/h_i$ ;  $L_{*,i} \in \mathbb{R}^{n_{c,i}}$  and  $M_{*,i} \in \mathbb{R}^{n_{c,i}}$  are the ideal controller parameters given by Lemma 5.1; and  $\tilde{x}_{*,i}(0)$  is the initial condition given by part (iii) of Lemma 5.2.

Define  $e_i(t) \triangleq \tilde{x}_i(t) - \tilde{x}_{*,i}(t)$ , and subtracting (5.85) and (5.86) from (5.83) and (5.84), respectively, yields

$$\dot{e}_i(t) = \tilde{A}_i e_i(t) + \frac{1}{N_{*,i}} \tilde{B}_i \tilde{\theta}_i^T(t) \phi_i(t) + \tilde{E}_i \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} \tilde{C}_j e_j(t), \quad (5.87)$$

$$y_i(t) = \tilde{C}_i e_i(t), \quad (5.88)$$

where part (iii) of Lemma 5.2 implies that  $y_i(t) - y_{*,i}(t) = y_i(t)$ .

Define the partial Lyapunov-like function

$$V_i(e_i, \tilde{\theta}_i) \triangleq e_i^T P_i e_i + \frac{1}{|N_{*,i}|} \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i, \quad (5.89)$$

where  $P_i \in \mathbb{R}^{(n_i+2n_{c,i}) \times (n_i+2n_{c,i})}$  is the positive-definite solution to (5.34).

Evaluating the derivative of  $V_i$  along the trajectory of (5.71) and (5.87) with  $r_i(t) \equiv 0$ , and using (5.35) and (5.88) yields

$$\begin{aligned} \dot{V}_i(e_i, \tilde{\theta}_i) &= e_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) e_i + \frac{2}{N_{*,i}} e_i^T P_i \tilde{B}_i \tilde{\theta}_i^T \phi_i + \frac{2}{|N_{*,i}|} \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\theta}_i \\ &\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j \\ &= e_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) e_i + \frac{2}{N_{*,i}} e_i^T \tilde{C}_i^T \tilde{\theta}_i^T \phi_i + \frac{2}{|N_{*,i}|} \tilde{\theta}_i^T \Gamma_i^{-1} \dot{\theta}_i \\ &\quad + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j \\ &= e_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) e_i + 2y_i \tilde{\theta}_i^T \phi_i \left( \frac{1}{N_{*,i}} - \frac{\text{sgn}(h_i)}{|N_{*,i}|} \right) + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j. \end{aligned} \quad (5.90)$$

Next, it follows from (5.35) that  $h_{m,i} = \tilde{C}_i \tilde{B}_i = \tilde{B}_i^T P_i \tilde{B}_i > 0$ . Since  $N_{*,i} = h_{m,i}/h_i$ , it follows that  $\text{sgn}(h_i) = \text{sgn}(h_{m,i}/h_i) = \text{sgn}(N_{*,i})$ . Then, it follows from (5.90) that

$$\dot{V}_i(e_i, \tilde{\theta}_i) = e_i^T (\tilde{A}_i^T P_i + P_i \tilde{A}_i) e_i + 2 \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j. \quad (5.91)$$

Next, define

$$\varepsilon_i \triangleq \frac{\lambda_{\min}(Q_i)}{\ell \lambda_{\max}(P_i \tilde{E}_i \tilde{E}_i^T P_i)}, \quad (5.92)$$

and note that

$$\begin{aligned} 0 &\leq \sum_{j \in \mathcal{J} \setminus \{i\}} \left[ \sqrt{\varepsilon_i} \tilde{E}_i^T P_i e_i - \frac{1}{\sqrt{\varepsilon_i}} \delta_{i,j} \tilde{C}_j e_j \right]^T \left[ \sqrt{\varepsilon_i} \tilde{E}_i^T P_i e_i - \frac{1}{\sqrt{\varepsilon_i}} \delta_{i,j} \tilde{C}_j e_j \right] \\ &= (\ell - 1) \varepsilon_i e_i^T P_i \tilde{E}_i \tilde{E}_i^T P_i e_i + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 e_j^T \tilde{C}_j^T \tilde{C}_j e_j - 2 \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j \\ &\leq \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} e_i^T e_i + \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 e_j^T \tilde{C}_j^T \tilde{C}_j e_j - 2 \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j, \end{aligned}$$

which implies that

$$\sum_{j \in \mathcal{J} \setminus \{i\}} 2 \delta_{i,j} e_i^T P_i \tilde{E}_i \tilde{C}_j e_j \leq \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} e_i^T e_i + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_j^T \tilde{C}_j^T \tilde{C}_j e_j. \quad (5.93)$$

Next, using (5.93), it follows from (5.91) that

$$\dot{V}_i(e_i, \tilde{\theta}_i) \leq e_i^T \left( \tilde{A}_i^T P_i + P_i \tilde{A}_i + \frac{(\ell - 1) \lambda_{\min}(Q_i)}{\ell} I \right) e_i + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_j^T \tilde{C}_j^T \tilde{C}_j e_j. \quad (5.94)$$

Next, define the Lyapunov-like function

$$V(e_1, \dots, e_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) \triangleq \sum_{i \in \mathcal{J}} V_i(e_i, \tilde{\theta}_i),$$

and it follows from (5.94) that the derivative of  $V$  along the trajectory of (5.71) and (5.87) is given by

$$\begin{aligned}
\dot{V}(e_1, \dots, e_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) &= \sum_{i \in \mathcal{J}} \dot{V}_i(e_i, \tilde{\theta}_i) \\
&\leq \sum_{i \in \mathcal{J}} \left[ e_i^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) e_i \right. \\
&\quad \left. + \frac{1}{\varepsilon_i} \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{i,j}^2 e_j^\top \tilde{C}_j^\top \tilde{C}_j e_j \right] \\
&= \sum_{i \in \mathcal{J}} e_i^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) e_i \\
&\quad + \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{J} \setminus \{j\}} \frac{1}{\varepsilon_i} \delta_{i,j}^2 e_j^\top \tilde{C}_j^\top \tilde{C}_j e_j \\
&= \sum_{i \in \mathcal{J}} e_i^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) e_i \\
&\quad + \sum_{i \in \mathcal{J}} e_i^\top \tilde{C}_i^\top \tilde{C}_i e_i \left( \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{1}{\varepsilon_j} \delta_{j,i}^2 \right) \\
&\leq \sum_{i \in \mathcal{J}} e_i^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) e_i \\
&\quad + \frac{1}{\varepsilon} e_i^\top \tilde{C}_i^\top \tilde{C}_i e_i \left( \sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \right), \tag{5.95}
\end{aligned}$$

where  $\varepsilon \triangleq \min_{j \in \mathcal{J}} \varepsilon_j$ . Since (A5.7) implies that  $\sum_{j \in \mathcal{J} \setminus \{i\}} \delta_{j,i}^2 \leq 2\gamma_i \varepsilon$ , it follows from (5.34), (5.35), and (5.95) that

$$\begin{aligned}
\dot{V}(e_1, \dots, e_\ell, \tilde{\theta}_1, \dots, \tilde{\theta}_\ell) &\leq \sum_{i \in \mathcal{J}} e_i^\top \left( \tilde{A}_i^\top P_i + P_i \tilde{A}_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I + 2\gamma_i \tilde{C}_i^\top \tilde{C}_i \right) e_i \\
&= \sum_{i \in \mathcal{J}} e_i^\top \left( -Q_i + \frac{(\ell-1)\lambda_{\min}(Q_i)}{\ell} I \right) e_i \\
&\leq \sum_{i \in \mathcal{J}} -\frac{1}{\ell} \lambda_{\min}(Q_i) e_i^\top e_i \\
&= \sum_{i \in \mathcal{J}} -\xi_i e_i^\top e_i,
\end{aligned}$$

where  $\xi_i \triangleq \frac{1}{\ell} \lambda_{\min}(Q_i)$ , which is positive. Therefore,  $\dot{V}$  is nonpositive and  $\dot{V} \leq -\sum_{i \in \mathcal{J}} \xi_i e_i^T e_i$  implies that

$$0 \leq \sum_{i \in \mathcal{J}} \xi_i e_i^T e_i \leq -\dot{V}. \quad (5.96)$$

Moreover, integrating (5.96) from 0 to  $\infty$  yields

$$0 \leq \int_0^{\infty} \sum_{i \in \mathcal{J}} \xi_i e_i^T(t) e_i(t) \leq V(0) - \lim_{t \rightarrow \infty} V(t) \leq V(0), \quad (5.97)$$

where the upper and lower bounds imply that  $\int_0^{\infty} \sum_{i \in \mathcal{J}} \xi_i e_i^T(t) e_i(t)$  exists. Thus, it follows from (5.97) that  $V$  is bounded, which implies that  $e_i$  and  $\tilde{\theta}_i$  are bounded. Since  $w$  is bounded, it follows from part (ii) of Lemma 5.2 that  $\tilde{x}_{*,i}$  is bounded. Since  $e_i$  and  $\tilde{x}_{*,i}$  are bounded, it follows that  $\tilde{x}_i$  is bounded. Since  $\tilde{x}_i$  is bounded, it follows from (5.75) that  $x_i$ ,  $U_i$ , and  $Y_i$  are bounded. Moreover, since  $\tilde{\theta}_i$ ,  $U_i$ , and  $Y_i$  are bounded, it follows from (5.70) and (5.72) that  $\theta_i$  and  $u_i$  are bounded, which confirms (i).

To show (i), it follows from (5.97) that  $\int_0^{\infty} \sum_{i \in \mathcal{J}} \xi_i e_i^T(t) e_i(t)$  exists. Next, since  $e_i$ ,  $\tilde{\theta}_i$ , and  $\phi_i$  are bounded, (5.87) implies that  $\dot{e}_i$  is bounded. Next, since  $e_i$  and  $\dot{e}_i$  are bounded, it follows that

$$\frac{d}{dt} \left[ \sum_{i \in \mathcal{J}} \xi_i e_i^T(t) e_i(t) \right] = 2 \sum_{i \in \mathcal{J}} \xi_i \dot{e}_i^T(t) e_i(t) \quad (5.98)$$

is bounded. Thus,  $f(t) \triangleq \sum_{i \in \mathcal{J}} \xi_i e_i^T(t) e_i(t)$  is uniformly continuous. Since  $\int_0^{\infty} f(t) dt$  exists and  $f(t)$  is uniformly continuous, Barbalat's Lemma implies that  $\lim_{t \rightarrow \infty} f(t) = 0$ . Thus,  $\lim_{t \rightarrow \infty} e_i(t) = 0$ , and it follows from (5.88) that  $\lim_{t \rightarrow \infty} y_i(t) = 0$ , which confirms (ii).  $\square$

## 5.5 Numerical Examples

We now present examples that demonstrate the decentralized adaptive controller for SISO subsystems that are relative degree one and minimum phase. Examples 5.1 and 5.3 show stabilization for second-order subsystems, where  $\ell = 2$  and  $\ell = 4$ , respectively. Examples 5.2 and 5.4 show asymptotic disturbance rejection for second-order subsystems, where  $\ell = 2$  and  $\ell = 4$ , respectively. Example 5.5 examines the command following problem for second-order subsystems, where  $\ell = 2$ .

**Example 5.1.** *Decentralized adaptive stabilization for an unstable system with  $\ell = 2$ .* Consider the system (5.1)–(5.4), where  $\ell = 2$ ,

$$A_1 = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}, \quad (5.99)$$

$$B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (5.100)$$

$$C_1 = C_2 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad (5.101)$$

$$D_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (5.102)$$

which satisfies (A5.1) and (A5.2). The interconnections are given by  $\delta_{1,2} = 2$  and  $\delta_{2,1} = 1$ . Moreover, the complete dynamics matrix

$$A \triangleq \begin{bmatrix} A_1 & \delta_{1,2}B_1C_2 \\ \delta_{2,1}B_2C_1 & A_2 \end{bmatrix} = \begin{bmatrix} -1 & 6 & 2 & 4 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is unstable. We let  $\bar{n}_i = n_i = 2$ , which satisfies (A5.3). For this example, we let



$w(t) \equiv 0$  and  $r_i(t) \equiv 0$ , and consider the stabilization problem.

Next, we consider the reference model

$$G_{m,i}(s) = \frac{s + 7}{s^2 + 20s + 79},$$

and let  $\gamma_i = 10$ . Thus,  $F_i(s)$ , given by (5.8), satisfies (A5.6). Next, let  $n_{c,i} = 2$ , which satisfies (5.9), and consider (5.67) and (5.68), where

$$A_{f,i} = \begin{bmatrix} -8 & -7 \\ 1 & 0 \end{bmatrix},$$

which has eigenvalues at  $-7$  and  $-1$ . Note that  $A_{f,i}$  has an eigenvalue equal to the zero of  $G_{m,i}(s)$ , which is  $-7$ .

The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4) and (5.99)–(5.102), where  $\Gamma_i = 10^3 I_5$ ,  $w(t) \equiv 0$ , and  $r_i(t) \equiv 0$ . Figure 5.2 provides a time history of  $x_i(t)$  and  $u_i(t)$ , where the initial conditions are  $x_1(0) = [1 \ 0]^T$ ,  $x_2(0) = [-1 \ 0]^T$ , and  $U_1(0) = U_2(0) = Y_1(0) = Y_2(0) = [0 \ 0]^T$ . The state  $x_i(t)$  converges asymptotically to zero.  $\triangle$

**Example 5.2.** *Decentralized adaptive disturbance rejection for an unstable system with  $\ell = 2$ .* Reconsider the system in Example 5.1, but consider nonzero disturbance. The plant and reference-model parameters, satisfying (A5.1)–(A5.3) and (A5.6), are the same as in Example 5.1. The disturbance is

$$w(t) = \begin{bmatrix} \sin 0.25\pi t & \sin 0.5\pi t & \sin 0.75\pi t & \sin \pi t \end{bmatrix}^T,$$

which satisfies (A5.4). Note that the disturbance spectrum is unknown and the disturbance is unmeasured. We let  $\bar{n}_w = n_w = 8$ , which satisfies (A5.5). Next, let

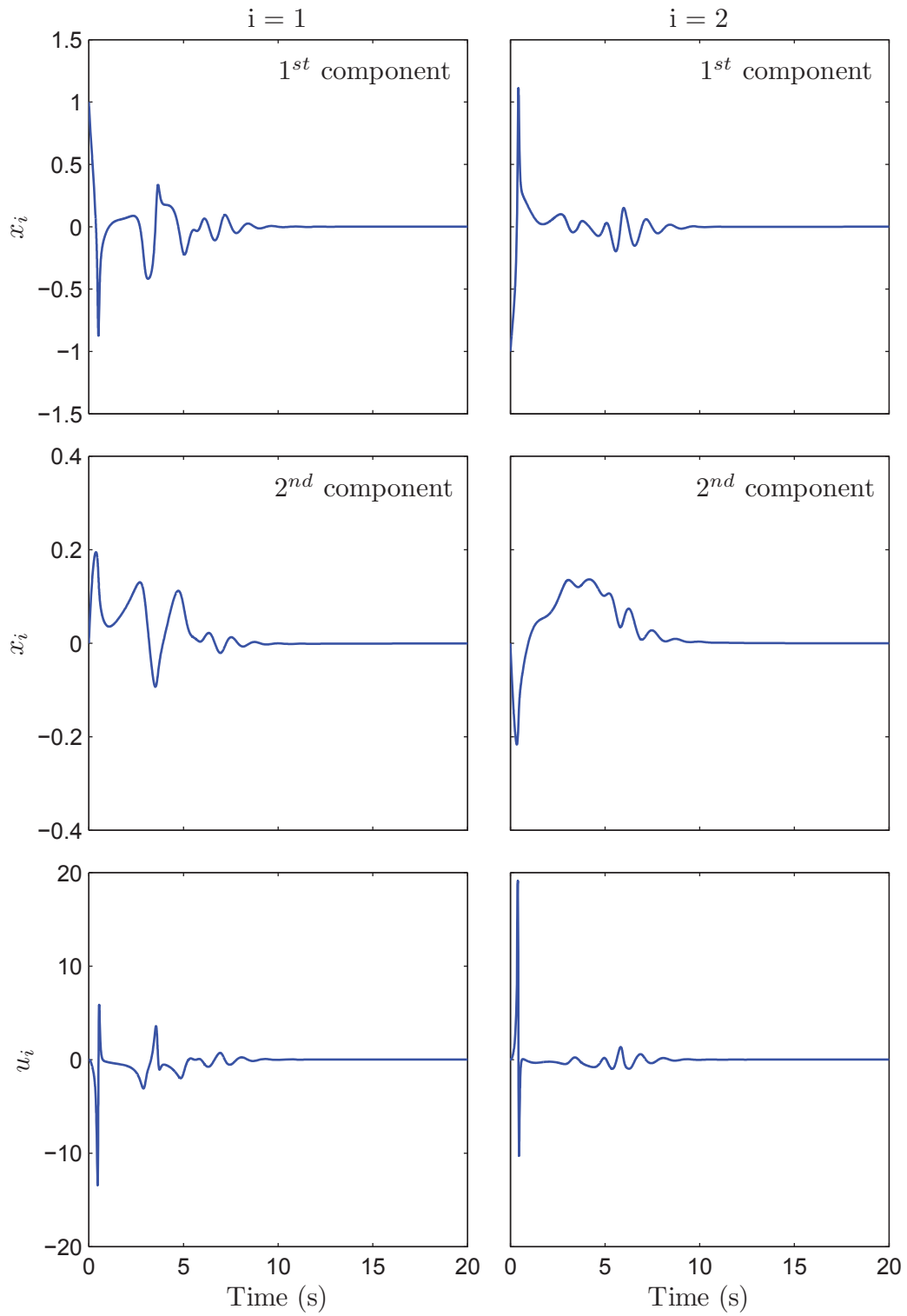


Figure 5.2: *Decentralized adaptive stabilization for an unstable system with  $\ell = 2$ . The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4) and (5.99)–(5.102). The state  $x_i(t)$  converges asymptotically to zero.*

$n_{c,i} = 10$ , which satisfies (5.9), and consider (5.67) and (5.68), where

$$A_{f,i} = \begin{bmatrix} -16 & -99 & -336 & -714 & -1008 & -966 & -624 & -261 & -64 & -7 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

which has one eigenvalue at  $-7$  and nine at  $-1$ .

The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4), and (5.99)–(5.102), where  $\Gamma_i = 10^7 I_{21}$  and  $r_i(t) \equiv 0$ . Figure 5.3 provides a time history of  $y_i(t)$  and  $u_i(t)$ , where the initial conditions are  $x_1(0) = x_2(0) = [0 \ 0]^T$  and  $U_1(0) = U_2(0) = Y_1(0) = Y_2(0) = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$ . The output  $y_i(t)$  converges asymptotically to zero while rejecting the disturbance  $w(t)$ .  $\triangle$

**Example 5.3.** *Decentralized adaptive stabilization for an unstable system with  $\ell = 4$ .* Consider the system (5.1)–(5.4), where  $\ell = 4$ ,

$$A_1 = A_3 = \begin{bmatrix} -5 & 14 \\ 1 & 0 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -4 & 5 \\ 1 & 0 \end{bmatrix}, \quad (5.103)$$

$$B_1 = \cdots = B_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (5.104)$$

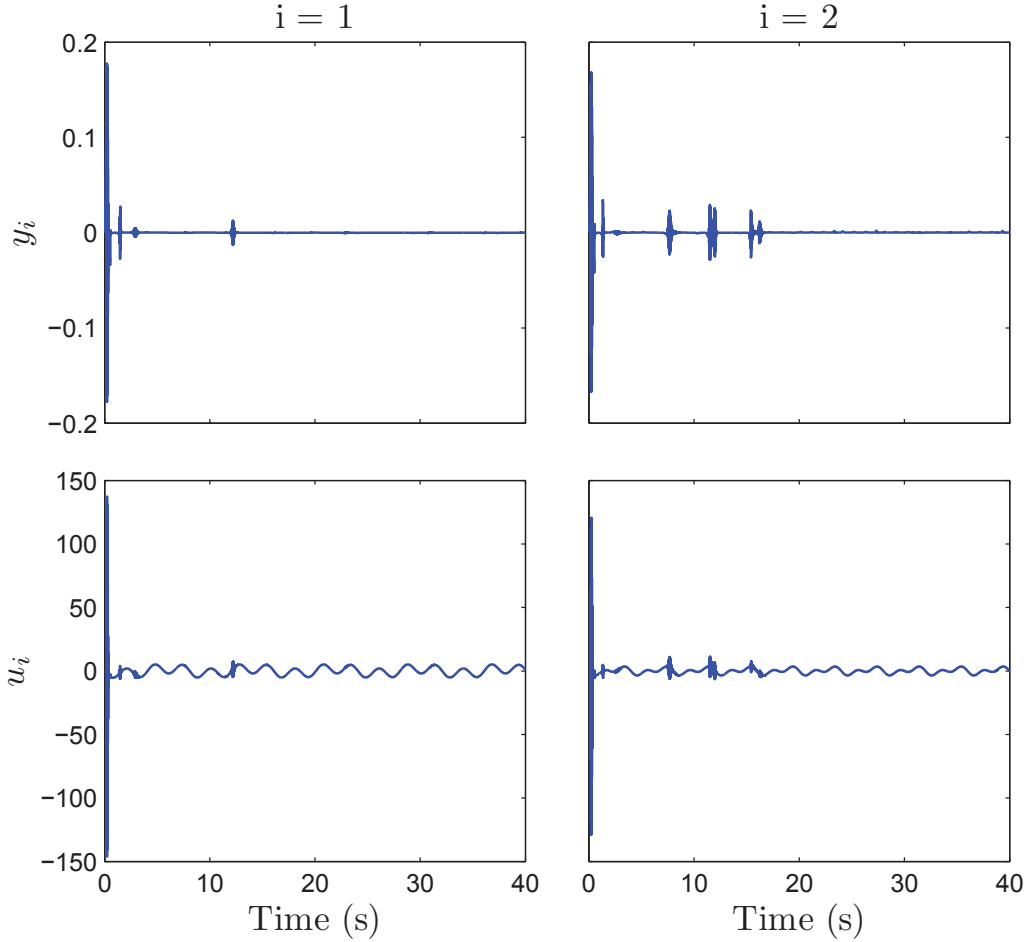


Figure 5.3: *Decentralized adaptive disturbance rejection for an unstable system with  $\ell = 2$ . The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4) and (5.99)–(5.102). The output  $y_i(t)$  converges asymptotically to zero while rejecting the disturbance  $w(t)$ .*

$$C_1 = \cdots = C_4 = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad (5.105)$$

$$D_1 = D_3 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D_2 = D_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (5.106)$$

which satisfies (A5.1) and (A5.2). The interconnections are given by  $\delta_{1,2} = \delta_{2,1} = \delta_{2,3} = \delta_{2,4} = \delta_{3,4} = 2$ ,  $\delta_{3,2} = \delta_{4,2} = \delta_{4,3} = 1$ , and  $\delta_{1,3} = \delta_{1,4} = \delta_{3,1} = \delta_{4,1} = 0$ .

Moreover, the complete dynamics matrix

$$\begin{aligned}
A &\triangleq \begin{bmatrix} A_1 & \delta_{1,2}B_1C_2 & \delta_{1,3}B_1C_3 & \delta_{1,4}B_1C_4 \\ \delta_{2,1}B_2C_1 & A_2 & \delta_{2,3}B_2C_3 & \delta_{2,4}B_2C_4 \\ \delta_{3,1}B_3C_1 & \delta_{3,2}B_3C_2 & A_3 & \delta_{3,4}B_3C_4 \\ \delta_{4,1}B_4C_1 & \delta_{4,2}B_4C_2 & \delta_{4,3}B_4C_3 & A_4 \end{bmatrix} \\
&= \begin{bmatrix} -5 & 14 & 2 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & -4 & 5 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -5 & 14 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

is unstable. We let  $\bar{n}_i = n_i = 2$ , which satisfies (A5.3). For this example, we let  $w(t) \equiv 0$  and  $r_i(t) \equiv 0$ , and consider the stabilization problem.

Next, we consider the reference model

$$G_{m,i}(s) = \frac{s + 7}{s^2 + 21s + 88},$$

and let  $\gamma_i = 10$ . Thus,  $F_i(s)$ , given by (5.8), satisfies (A5.6). Next, let  $n_{c,i} = 2$ , which satisfies (5.9), and consider (5.67) and (5.68), where

$$A_{f,i} = \begin{bmatrix} -8 & -7 \\ 1 & 0 \end{bmatrix},$$

which has eigenvalues at  $-7$  and  $-1$ . Note that  $A_{f,i}$  has an eigenvalue equal to the zero of  $G_{m,i}(s)$ , which is  $-7$ .

The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4) and (5.103)–(5.106), where  $\Gamma_i = 10^2 I_5$ ,  $w(t) \equiv 0$ , and  $r_i(t) \equiv 0$ . Figure 5.4 provides a time history of  $x_i(t)$  and  $u_i(t)$ , where the initial conditions are  $x_1(0) = x_3(0) = [1 \ 0]^T$ ,  $x_2(0) = x_4(0) = [-1 \ 0]^T$ , and  $U_1(0) = \cdots = U_4(0) = Y_1(0) = \cdots = Y_4(0) = [0 \ 0]^T$ . The state  $x_i(t)$  converges asymptotically to zero.  $\triangle$

**Example 5.4.** *Decentralized adaptive disturbance rejection for an asymptotically stable system with  $\ell = 4$ .* Consider the system (5.1)–(5.4), where  $\ell = 4$ ,

$$A_1 = A_3 = \begin{bmatrix} -10 & -21 \\ 1 & 0 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} -12 & -27 \\ 1 & 0 \end{bmatrix}, \quad (5.107)$$

and  $B_1, \dots, B_4$ , and  $C_1, \dots, C_4$  are given by (5.104)–(5.105), respectively, and

$$D_1 = D_3 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad D_2 = D_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad (5.108)$$

which satisfies (A5.1) and (A5.2). The interconnections  $\delta_{1,1}, \dots, \delta_{1,4}, \dots, \delta_{4,4}$  are the same as in Example 5.3. Moreover, the complete dynamics matrix

$$A \triangleq \begin{bmatrix} A_1 & \delta_{1,2}B_1C_2 & \delta_{1,3}B_1C_3 & \delta_{1,4}B_1C_4 \\ \delta_{2,1}B_2C_1 & A_2 & \delta_{2,3}B_2C_3 & \delta_{2,4}B_2C_4 \\ \delta_{3,1}B_3C_1 & \delta_{3,2}B_3C_2 & A_3 & \delta_{3,4}B_3C_4 \\ \delta_{4,1}B_4C_1 & \delta_{4,2}B_4C_2 & \delta_{4,3}B_4C_3 & A_4 \end{bmatrix}$$

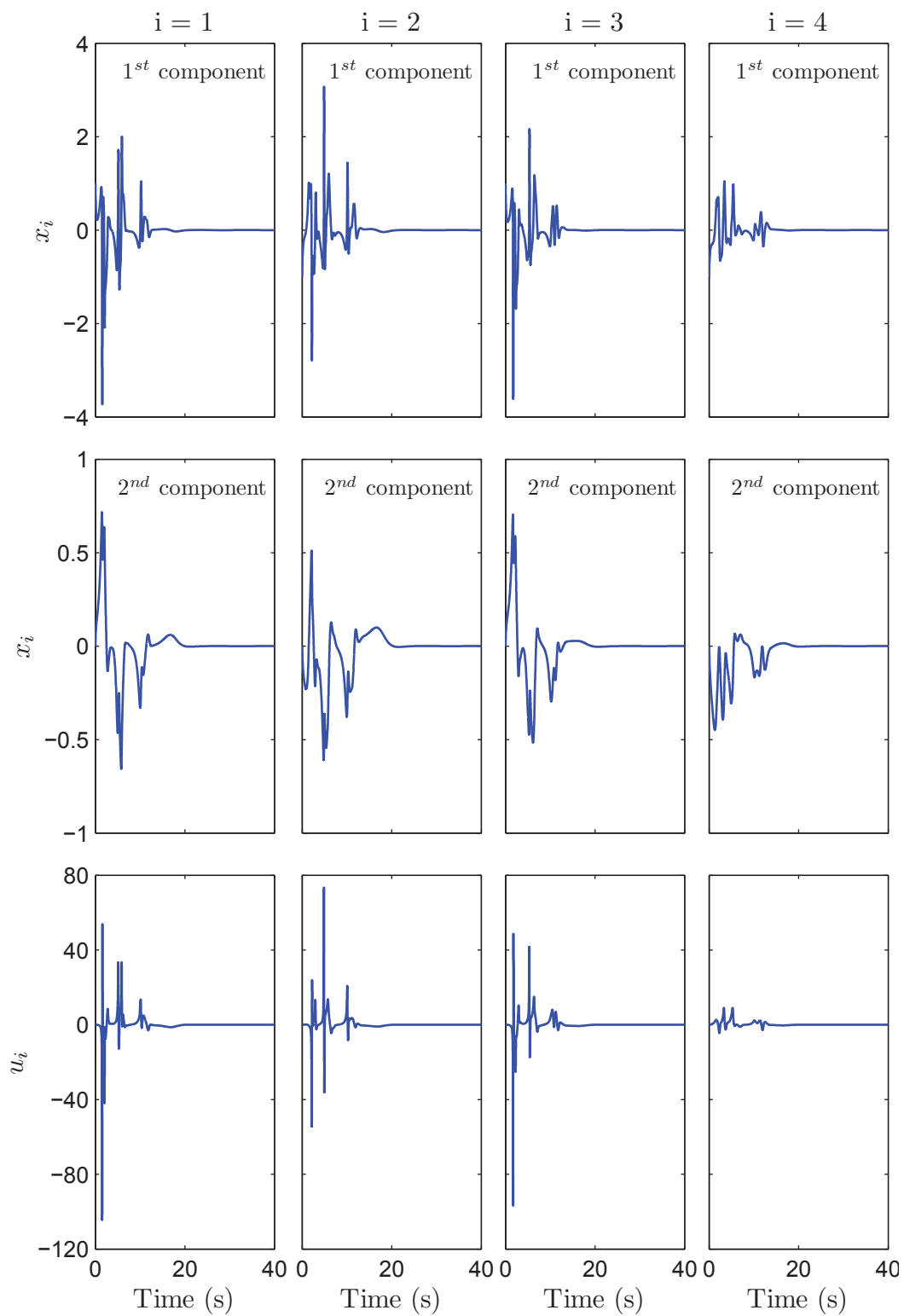


Figure 5.4: *Decentralized adaptive stabilization for an unstable system with  $\ell = 4$ .* The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4) and (5.103)–(5.106). The state  $x_i(t)$  converges asymptotically to zero.

$$= \begin{bmatrix} -10 & -21 & 2 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & -12 & -27 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -10 & -21 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 2 & -12 & -27 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

is asymptotically stable. We let  $\bar{n}_i = n_i = 2$ , which satisfies (A5.3). For this example, we let  $r_i(t) \equiv 0$ , and consider the disturbance rejection problem. The reference-model parameters satisfying (A5.6) are the same as in Example 5.3. The disturbance is given by  $w(t) = [\sin 0.25\pi t \quad \sin 0.5\pi t]^T$ , which satisfies (A5.4). Note that the disturbance spectrum is unknown and the disturbance is unmeasured. We let  $\bar{n}_w = n_w = 4$ , which satisfies (A5.5). Next, let  $n_{c,i} = 6$ , which satisfies (5.9), and consider (5.67) and (5.68), where  $A_{f,i}$  is given by

$$A_{f,i} = \begin{bmatrix} -12 & -45 & -80 & -75 & -36 & -7 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

which has one eigenvalue at  $-7$  and five at  $-1$ .

The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4), (5.104)–(5.106), and (5.107), where  $\Gamma_i = 10^5 I_{13}$  and  $r_i(t) \equiv 0$ . Figure 5.5 provides a time history of  $y_i(t)$  and  $u_i(t)$ , where the initial conditions are  $x_1(0) =$



$x_3(0) = [1 \ 0]^T$ ,  $x_2(0) = x_4(0) = [-1 \ 0]^T$ , and  $U_1(0) = \dots = U_4(0) = Y_1(0) = \dots = Y_4(0) = [0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$ . The system is allowed to run open-loop for 10 seconds, then the decentralized adaptive control is turned on. The output  $y_i(t)$  converges asymptotically to zero while rejecting the disturbance  $w(t)$ .  $\triangle$

**Example 5.5.** *Decentralized adaptive command following for an asymptotically stable system with  $\ell = 2$ .* Consider the system (5.1)–(5.4), where  $\ell = 2$ ,

$$A_1 = \begin{bmatrix} -10 & -16 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -11 & -28 \\ 1 & 0 \end{bmatrix}, \quad (5.109)$$

and  $B_1$  and  $B_2$ ,  $C_1$  and  $C_2$ , and  $D_1$  and  $D_2$  are given by (5.100)–(5.102), respectively, which satisfies (A5.1) and (A5.2). Furthermore, the interconnections are given by  $\delta_{1,2} = 2$  and  $\delta_{2,1} = 1$ . Moreover, the complete dynamics matrix

$$A \triangleq \begin{bmatrix} A_1 & \delta_{1,2}B_1C_2 \\ \delta_{2,1}B_2C_1 & A_2 \end{bmatrix} = \begin{bmatrix} -10 & -16 & 2 & 4 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & -11 & -28 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is asymptotically stable. We let  $\bar{n}_i = n_i = 2$ , which satisfies (A5.3). For this example, we let  $w(t) \equiv 0$ , and consider the command following problem. Although Theorems 5.1 and 5.2 do not address command following, we use this example to explore the command following properties of the decentralized adaptive controller. The reference-model parameters, satisfying (A5.6), are the same as in Example 5.1. The reference-model commands are  $r_1(t) = 0.5 \sin 0.25\pi t$  and  $r_2(t) = 0.4 \sin 0.5\pi t$ . Next, let  $n_{c,i} = 2$ ,

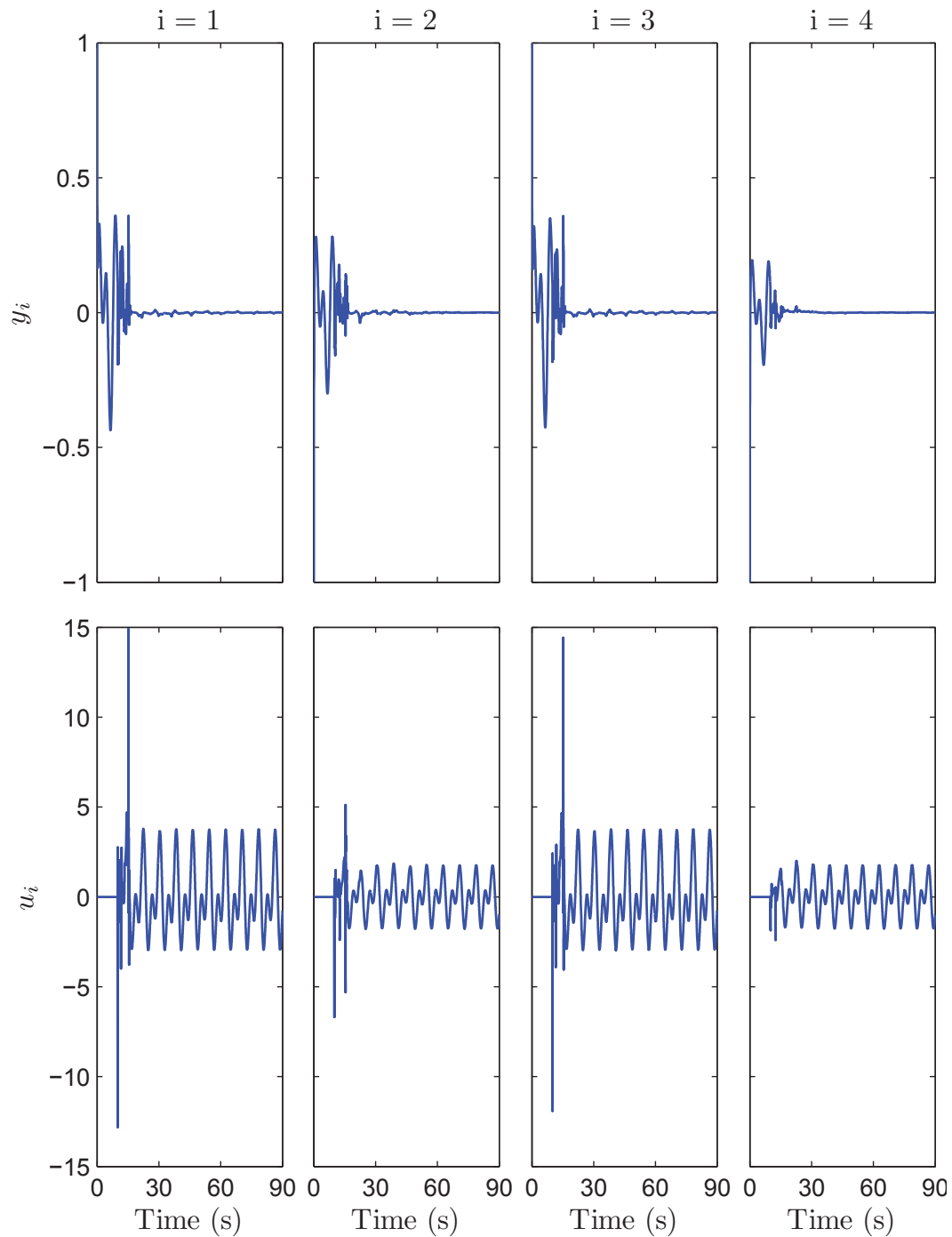


Figure 5.5: *Decentralized adaptive disturbance rejection for an asymptotically stable system with  $\ell = 4$ . The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4), (5.104)–(5.106), and (5.107). The output  $y_i(t)$  converges asymptotically to zero while rejecting the disturbance  $w(t)$ .*

which satisfies (5.9), and consider (5.67) and (5.68), where

$$A_{f,i} = \begin{bmatrix} -8 & -7 \\ 1 & 0 \end{bmatrix},$$

which has eigenvalues at  $-7$  and  $-1$ . Note that  $A_{f,i}$  has an eigenvalue equal to the zero of  $G_{m,i}(s)$ , which is  $-7$ .

The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4), (5.100)–(5.102), and (5.109), where  $\Gamma_i = 10^2 I_5$  and  $w(t) \equiv 0$ . Figure 5.6 provides a time history of  $y_i(t)$ ,  $y_{m,i}(t)$ ,  $z_i(t)$ , and  $u_i(t)$ , where the initial conditions are zero. The system is allowed to run open-loop for 5 seconds, then the decentralized adaptive control is turned on. The performance  $z_i(t)$  does not converge to zero, but does remain bounded. △

## 5.6 Conclusion

This chapter presented a decentralized adaptive controller for SISO subsystems that are minimum phase and relative degree one. This controller is strictly decentralized, that is, the controller requires only local output measurement and no information is shared between the local controllers. The controller is effective for stabilization and disturbance rejection, where the disturbance is unknown but generated from a Lyapunov-stable linear system. The decentralized adaptive controller requires that the magnitude of the subsystem interconnections satisfy a bounding condition. In a command following example, the controller yields bounded-but-nonzero performance.

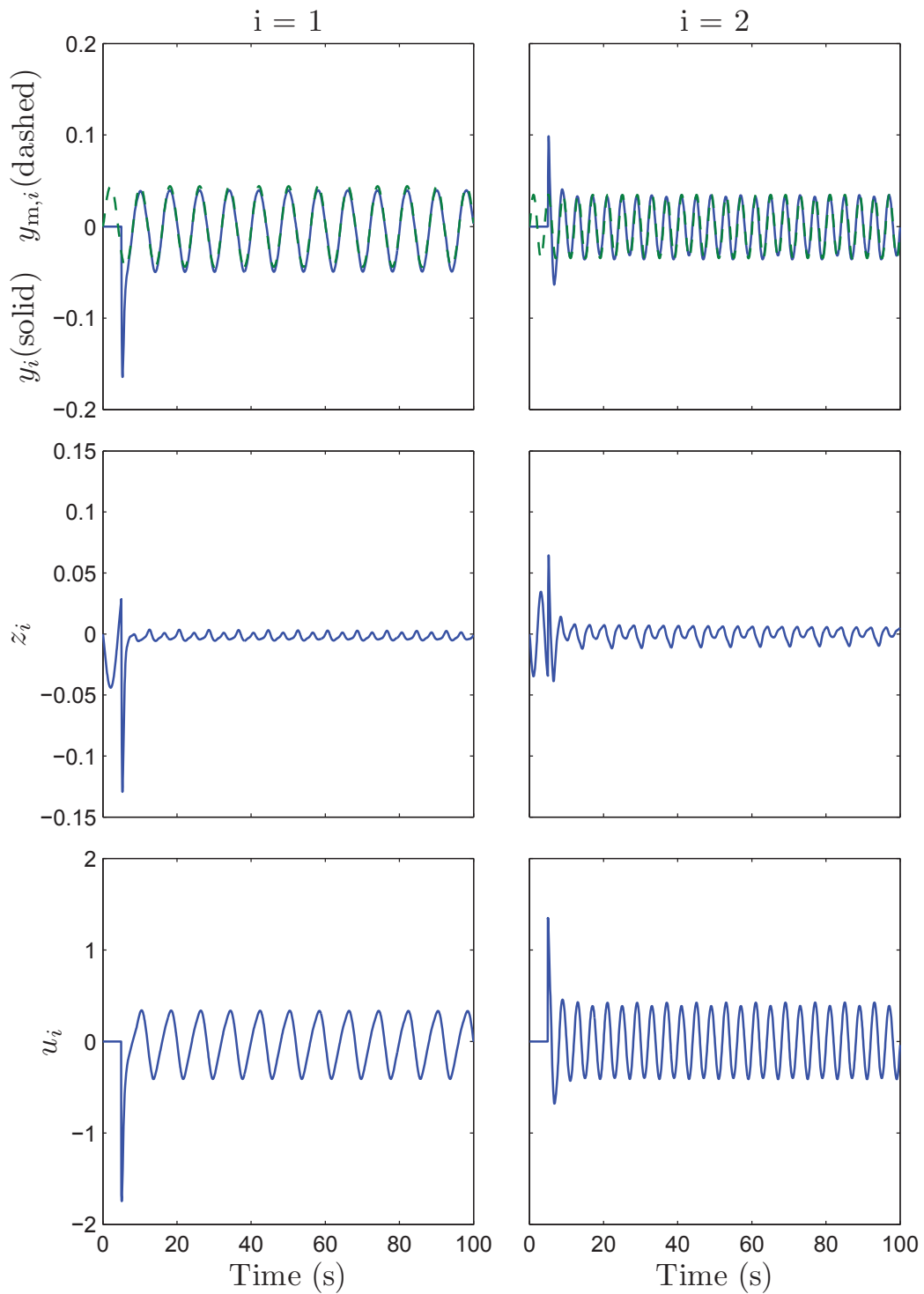


Figure 5.6: *Decentralized adaptive command following for an asymptotically stable system with  $\ell = 2$ . The adaptive controller (5.67)–(5.71) is implemented in feedback with the system (5.1)–(5.4), (5.100)–(5.102), and (5.109). The performance  $z_i(t)$  does not converge to zero, but does remain bounded.*

## Chapter 6 Conclusions and Future Work

This thesis presented decentralized adaptive control techniques for subsystems with local full-state feedback and local output feedback with relative degree one. In Chapter 2, we introduced classical full-state-feedback model reference adaptive control (MRAC) for multi-input linear time-invariant systems. We presented three matching conditions for the dynamics matrix and control-input matrix. Classical MRAC yields asymptotic stabilization and command following.

In Chapter 3, we presented a strictly decentralized adaptive controller for linear time-invariant systems that use local full-state measurements and do not share information between local controllers. The controller does not require a centralized reference model, meaning that nonlocal reference-model signals are unknown to each local controller. The controller yields asymptotic stabilization and command following in the presence of sinusoidal disturbance with known spectrum. The technique is effective for arbitrarily large subsystem interconnection matrices, provided that a bounding matrix on the subsystem interconnection matrices is known and that the reference-model dynamics matrix is designed to admit a positive-definite solution to a bounded-real Riccati equation. We presented a construction for the reference-model dynamics matrix, which guarantees that a positive-definite solution to the Riccati equation exists. Future work for the decentralized controller in Chapter 3 includes extensions to unmatched uncertainties and unknown nonlinearities.

Chapter 4 presented classical output-feedback MRAC for SISO linear time-invariant systems that are relative degree one. The output-feedback adaptive controller oper-

ates under the assumptions that the plant is minimum phase, the sign of the high-frequency gain is known, and an upper bound on the order of the plant is known. In this chapter, classical MRAC was extended to address disturbance rejection, where the disturbance is unknown but generated from a Lyapunov-stable linear system. The adaptive controller yields asymptotic command following in the presence of unknown sinusoidal disturbances.

In Chapter 5, we presented a strictly decentralized adaptive controller for SISO linear time-invariant systems that are relative degree one and minimum phase. This decentralized adaptive controller requires only local output measurements and does not share information between the local controllers. The controller is effective for stabilization in the presence of unknown sinusoidal disturbances. The decentralized adaptive controller operates under the assumption that the magnitudes of the subsystem interconnections satisfy a bounding condition. However, this bounding condition relies on information that is not assumed to be known. Developing a method for verifying this bounding condition with the assumed model information is an open problem. Moreover, an extension to address asymptotically perfect command following is an open problem.

## Appendices

### A Proofs of Propositions 3.1 , 3.2, and 3.3

*Proof of Proposition 3.1.* To show (i), let  $F_i \triangleq \beta_i/b_i$ , which is positive because  $b_i\beta_i > 0$ . It follows from (3.7) and (3.8) that  $\hat{B}_i = B_i F_i$ , which confirms (i).

To show (ii), let

$$K_{*,i} \triangleq \begin{bmatrix} (-\eta_i \alpha_{i,n-1} + a_{i,n-1})/b_i & \cdots & (-\eta_i \alpha_{i,0} + a_{i,0})/b_i \end{bmatrix}.$$

Next, it follows from (3.7) and (3.8) that for all  $\eta_i > 0$ ,  $A_{m,i} = A_{i,i} + B_i K_{*,i}$ , which confirms (ii).

To show (iii), let  $\Omega_i \triangleq \gamma_i \hat{B}_i \hat{B}_i^T$ . Next, it follows from (3.9) that

$$\Omega_i \geq \sum_{j \in \mathcal{J} \setminus \{i\}} \left( \frac{b_i}{\beta_i} \right)^2 \hat{B}_i \Delta_{i,j}^T \Delta_{i,j} \hat{B}_i^T = \sum_{j \in \mathcal{J} \setminus \{i\}} B_i \Delta_{i,j}^T \Delta_{i,j} B_i^T = \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} A_{i,j}^T, \quad (\text{A.1})$$

which confirms (iii).

To show (iv), let  $\varepsilon_i > 0$  and let  $Q_i \in \mathbb{R}^{n_i \times n_i}$  be positive definite and satisfy  $Q_i > \ell I_{n_i}$ . It follows from [39, part (ii) of Lemma A.2] that there exists  $\eta_{s,i} > 0$  such that for all  $\eta_i > \eta_{s,i}$ , there exists positive definite  $\hat{P}_i \in \mathbb{R}^{n_i \times n_i}$  such that

$$A_{m,i}^T \hat{P}_i + \hat{P}_i A_{m,i} + Q_i + \varepsilon_i I_{n_i} = 0. \quad (\text{A.2})$$

Next, [39, part (iii) of Lemma A.2] implies that there exists  $\eta_{*,i} > \eta_{s,i}$  such that for

all  $\eta_i > \eta_{*,i}$ ,

$$\hat{B}_i^T \hat{P}_i^2 \hat{B}_i \leq \frac{\varepsilon_i}{\gamma_i}, \quad (\text{A.3})$$

where  $\gamma_i$  is given by (3.9). Thus, it follows from (A.3) that for all  $\eta_i > \eta_{*,i}$ ,

$$\hat{P}_i \Omega_i \hat{P}_i = \gamma_i \hat{P}_i \hat{B}_i \hat{B}_i^T \hat{P}_i \leq \gamma_i \hat{B}_i^T \hat{P}_i^2 \hat{B}_i I_{n_i} \leq \varepsilon_i I_{n_i}. \quad (\text{A.4})$$

Combining (A.2) and (A.4) yields for all  $\eta_i > \eta_{*,i}$ ,

$$A_{m,i}^T \hat{P}_i + \hat{P}_i A_{m,i} + Q_i + \hat{P}_i \Omega_i \hat{P}_i \leq 0. \quad (\text{A.5})$$

Thus, for all  $\eta_i > \eta_{*,i}$ ,  $A_{m,i}$  is asymptotically stable and  $P_i = \hat{P}_i$  satisfies (3.6), which confirms (iv).  $\square$

*Proof of Proposition 3.2.* Note that  $x_{m,i}(t)$  is the solution to the asymptotically stable linear time-invariant system (3.3), where the input is  $r_i(t)$ . Since, in addition,  $r_i(t)$  satisfies (A3.7), it follows that there exists  $\hat{W}_i \in \mathbb{R}^{n_i \times 2p}$  and  $f_i : [0, \infty) \rightarrow \mathbb{R}^{n_i}$  such that

$$x_{m,i}(t) = \hat{W}_i \Psi(t) + f_i(t), \quad (\text{A.6})$$

where  $\int_0^\infty \|f_i(t)\|^2 dt$  exists. Next, it follows from the assumptions of Proposition 3.2 that there exists  $\hat{N}_1 \in \mathbb{R}^{m_1 \times 2p}, \dots, \hat{N}_\ell \in \mathbb{R}^{m_\ell \times 2p}$  such that for all  $i \in \mathcal{J}$ ,

$$B_i \hat{N}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} \hat{W}_j = 0. \quad (\text{A.7})$$



Therefore, it follows from (A.6) and (A.7) that

$$\begin{aligned}
\int_0^\infty \left\| B_i \hat{N}_i \Psi(t) + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} x_{m,j}(t) \right\|^2 dt &= \int_0^\infty \left\| \left[ B_i \hat{N}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} \hat{W}_j \right] \Psi(t) \right. \\
&\quad \left. + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} f_j(t) \right\|^2 dt \\
&= \int_0^\infty \left\| \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} f_j(t) \right\|^2 dt \\
&\leq 2^{\ell-1} \sum_{j \in \mathcal{J} \setminus \{i\}} \int_0^\infty \|A_{i,j} f_j(t)\|^2 dt \\
&\leq 2^{\ell-1} \sum_{j \in \mathcal{J} \setminus \{i\}} \lambda_{\max}(A_{i,j}^\top A_{i,j}) \int_0^\infty \|f_j(t)\|^2 dt
\end{aligned}$$

exists because  $\int_0^\infty \|f_j(t)\|^2 dt$  exists. Thus,  $N_1 = \hat{N}_1, \dots, N_\ell = \hat{N}_\ell$  satisfies (3.27).  $\square$

*Proof of Proposition 3.3.* Let  $W_1 \in \mathbb{R}^{n_1 \times 2p}, \dots, W_\ell \in \mathbb{R}^{n_\ell \times 2p}$ . Define

$$\hat{N}_i \triangleq - \sum_{j \in \mathcal{J} \setminus \{i\}} \Delta_{i,j}^\top W_j,$$

and it follows that

$$B_i \hat{N}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} B_i \Delta_{i,j}^\top W_j = 0.$$

Since  $A_{i,j} = B_i \Delta_{i,j}^\top$ , it follows that

$$B_i \hat{N}_i + \sum_{j \in \mathcal{J} \setminus \{i\}} A_{i,j} W_j = 0.$$

Thus, it follows from Proposition 3.2 that there exists  $N_1 \in \mathbb{R}^{m_1 \times 2p}, \dots, N_\ell \in \mathbb{R}^{m_\ell \times 2p}$  such that for all  $i \in \mathcal{J}$ , (3.27) is satisfied.  $\square$

## Bibliography

- [1] G. Tao. *Adaptive Control Design and Analysis*. Wiley, 2003.
- [2] P. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice Hall, 1996.
- [3] P. Ioannou and B. Fidan. *Adaptive Control Tutorial*. SIAM, 2006.
- [4] K. S. Narendra and A. M. Annaswamy. *Stable Adaptive Systems*. Prentice Hall, 1989.
- [5] K. J. Åström and B. Wittenmark. *Adaptive Control*. Addison-Wesley, second edition, 1995.
- [6] S. Sastry and M. Bodson. *Adaptive Control: Stability, Convergence, and Robustness*. Prentice Hall, Englewood Cliffs, New Jersey, 1989.
- [7] I. D. Landau. *Adaptive Control: The Model Reference Approach*. Marcel Dekker, New York, 1979.
- [8] H. Kaufman, I. Barkana, and K. Sobel. *Direct Adaptive Control Algorithms: Theory and Applications*. Springer, New York, second edition, 1998.
- [9] J. Hong and D. S. Bernstein. Adaptive stabilization of nonlinear oscillators using direct adaptive control. *Int. J. Contr.*, 74:432–444, 2001.
- [10] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear & Adaptive Control Design*. Wiley, 1995.

- [11] R. V. Monopoli. Model reference adaptive control with an augmented error signal. *IEEE Trans. Autom. Contr.*, 19:474–484, 1974.
- [12] K. S. Narendra and L. S. Valavani. Stable adaptive controller design—direct control. *IEEE Trans. Autom. Contr.*, 23(4):570–583, 1978.
- [13] A. S. Morse. Global stability of parameter adaptive control systems. *IEEE Trans. Autom. Contr.*, 25:433–440, 1980.
- [14] G. Kreisselmeier and B. D. O. Anderson. Robust model reference adaptive control. *IEEE Trans. Autom. Contr.*, 31(2):127–133, 1986.
- [15] J. Sun. A modified model reference adaptive control scheme for improved transient performance. *IEEE Trans. Autom. Contr.*, 38:1255–1259, 1993.
- [16] G. Tao and P. A. Ioannou. Model reference adaptive control for plants with unknown relative degree. In *Proc. Amer. Contr. Conf.*, pages 2297–2302, 1989.
- [17] S. H. Wang and E. J. Davison. On the stabilization of decentralized control systems. *IEEE Trans. Autom. Contr.*, 18(5):473–478, 1973.
- [18] D. D. Siljak. *Decentralized Control of Complex Systems*. Academic Press, 1991.
- [19] N. R. Sandell, P. Varaiya, M. Athans, and M. G. Safonov. Survey of decentralized control methods for large scale systems. *IEEE Trans. Autom. Contr.*, 23(2):108–128, 1978.
- [20] P. R. Pagilla. Robust decentralized control of large-scale interconnected systems: general interconnections. In *Proc. Amer. Contr. Conf.*, pages 4527–4531, June 1999.
- [21] O. Huseyin, M. E. Sezer, and D. D. Siljak. Robust decentralised control using output feedback. *IEE Proc. Contr. Theor. App.*, 129(6):310–314, 1982.

- [22] D. T. Gavel and D. D. Siljak. Decentralized adaptive control: structural conditions for stability. *IEEE Trans. Autom. Contr.*, 34(4):413–426, 1989.
- [23] L. Shi and S. K. Singh. Decentralized adaptive controller design for large-scale systems with higher-order interconnections. *IEEE Trans. Autom. Contr.*, 8:1106–1118, 1992.
- [24] B. M. Mikrin. Decentralized adaptive control with zero residual tracking errors. In *Proc. 7th IEEE Mediterranean Conf. Contr. Autom.*, pages 388–398, Haifa, Israel, June 1999.
- [25] K. S. Narendra and N. O. Oleng. Exact output tracking in decentralized adaptive control systems. *IEEE Trans. Autom. Contr.*, 47(2):390–395, 2002.
- [26] K. S. Narendra, N. O. Oleng, and S. Mukhopadhyay. Decentralised adaptive control with partial communication. *IEE Proc. Contr. Theor. Appl.*, 153(5):546–555, 2006.
- [27] P. R. Pagilla, R. V. Dwivedula, and N. B. Siraskar. A decentralized model reference adaptive controller for large-scale systems. *IEEE/ASME Trans. Mechatron.*, 12(2):154–163, 2007.
- [28] P. Ioannou and P. Kokotovic. Decentralized adaptive control of interconnected systems with reduced-order models. *Automatica*, 21(4):401–412, 1985.
- [29] P. Ioannou. Decentralized adaptive control of interconnected systems. *IEEE Trans. Autom. Contr.*, 31(4):291–298, 1986.
- [30] C. Wen and Y. C. Soh. Decentralized model reference adaptive control without restriction on subsystem relative degrees. *IEEE Trans. Autom. Contr.*, 44(7):1464–1469, 1999.

- [31] J. T. Spooner and K. M. Passino. Decentralized adaptive control of nonlinear systems using radial basis neural networks. *IEEE Trans. Autom. Contr.*, 44(11):2050–2057, 1999.
- [32] N. Hovakimyan, E. Lavretsky, B.-J. Yang, and A. J. Calise. Coordinated decentralized adaptive output feedback control of interconnected systems. *IEEE Trans. Neural Networks*, 16(1):185–194, 2005.
- [33] S. N. Huang, K. K. Tan, and T. H. Lee. Decentralized control design for large-scale systems with strong interconnections using neural networks. *IEEE Trans. Autom. Contr.*, 48(5):805–810, 2003.
- [34] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 1996.
- [35] J. D. Polston and J. B. Hoagg. Decentralized adaptive control for systems with local full-state feedback. In *Proc. Conf. Dec. Contr.*, Firenze, Italy, December 2013. (submitted).
- [36] A. V. Morozov, J. B. Hoagg, and D. S. Bernstein. Retrospective cost adaptive control of a planar multilink arm with nonminimum-phase zeros. In *Proc. Conf. Dec. Contr.*, pages 3706–3711, Atlanta, GA, December 2010.
- [37] K. R. Meyer. On the existence of Lyapunov functions for the problem on Lur’e. *SIAM J. Contr.*, 3:373–383, 1965.
- [38] J. B. Hoagg, M. A. Santillo, and D. S. Bernstein. Internal model control in the shift and delta domains. *IEEE Trans. Autom. Contr.*, 53:1066–1072, 2008.
- [39] J. B. Hoagg and D. S. Bernstein. Lyapunov-stable adaptive stabilization of nonlinear time-varying systems with matched uncertainty. *Int. J. Contr.*, 80(6):872–884, 2007.

## Vita

James “Daniel” Polston was born in Lexington, Kentucky, the son of Larry and Cheryl Polston. After graduating from West Jessamine High School in Nicholasville, Kentucky, he entered the University of Kentucky in Lexington, Kentucky, to study mechanical engineering. He received a bachelor’s of science degree in mechanical engineering in December of 2011. During the following years, he pursued a master’s of science degree in mechanical engineering at the University of Kentucky with a focus in control systems.