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Root Locus Techniques With Nonlinear Gain Parameterization

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ROOT LOCUS TECHNIQUES WITH NONLINEAR GAIN PARAMETERIZATION

THESIS

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Mechanical Engineering in the College of Engineering at the University of Kentucky

by

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ABSTRACT OF THESIS

ROOT LOCUS TECHNIQUES WITH NONLINEAR GAIN PARAMETERIZATION

This thesis presents rules that characterize the root locus for polynomials that are nonlinear in the root-locus parameter k. Classical root locus applies to polynomials that are affine in k. In contrast, this thesis considers polynomials that are quadratic or cubic in k. In particular, we focus on constructing the root locus for linear feedback control systems, where the closed-loop denominator polynomial is quadratic or cubic in k. First, we present quadratic root-locus rules for a controller class that yields a closed-loop denominator polynomial that is quadratic in k. Next, we develop cubic root-locus rules for a controller class that yields a closed-loop denominator polynomial that is cubic in k. Finally, we extend the quadratic root-locus rules to accommodate a larger class of controllers.

We also provide controller design examples to demonstrate the quadratic and cubic root locus. For example, we show that the triple integrator can be high-gain stabilized using a controller that yields a closed-loop denominator polynomial that is quadratic in k. Similarly, we show that the quadruple integrator can be high-gain stabilized using a controller that yields a closed-loop denominator polynomial that is cubic in k.

KEYWORDS: Linear Systems, Root Locus, Feedback Control,
Single-Input Single-Output

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Chapter 1 Introduction and Review of Classical Root Locus

1.1 Introduction

Classical root locus is an established technique for controller design as well as control system analysis [1–7]. The root locus can be interpreted as a graphical representation of the achievable closed-loop poles as a function of a root-locus parameter k. For example, consider the loop transfer function $kL(s) = kz(s)/p(s)$, where $z(s)$ and $p(s)$ are coprime monic polynomials. In this case, the root locus is the set of all achievable poles of the closed-loop transfer function $1/(1 + kL(s))$, for all real values of k. The root locus for positive k is referred to as the positive (or 180°) root locus, while the root locus for negative k is referred to as the negative (or 0°) root locus. Extensions to classical root locus include: multivariable root locus [8–12], root locus for time-varying systems [13], variable gain plots [14], root locus for fractional order systems [15], and logarithmic root locus [16].

In classical root locus, the loop transfer function $kL(s) = kz(s)/p(s)$ is linear in k, and the closed-loop denominator polynomial $kz(s) + p(s)$ is affine in k. In this case, the root locus can be characterized without explicit dependence on k. More specifically, the positive root locus is the set of points λ in the complex plane such that $\angle L(\lambda) = (2n + 1)\pi$, where n is a nonnegative integer; and the negative root locus is the set of points λ in the complex plane such that $\angle L(\lambda) = 2n\pi$, where n is a nonnegative integer. The angle characterizations of the root locus lead to the classical (or affine) root-locus rules, which can be used, along with knowledge of the poles and zeros of $L(s)$, to draw the root locus. However, the affine root-locus rules apply if and only if the loop transfer function and thus the controller are linear in k.

In this thesis, we develop rules that characterize the root locus for classes of loop transfer functions that are nonlinear in k. More specifically, Chapters 2 and 3 consider controllers that are rational functions of k and result in closed-loop denominator polynomials that are quadratic and cubic in k, respectively. As an example, consider the plant $z(s)/p(s)$ and the controller $k^2/(s+k)$. In this case, the closed-loop denominator polynomial is $k^2z(s) + kp(s) + sp(s)$, which is quadratic in k. Controller constructions, which depend nonlinearly on a single parameter k, are considered in [17–20]. For certain minimum-phase systems, the controllers of [17–20] are high-gain stabilizing, meaning that these controllers stabilize the closed-loop system for sufficiently large k. While the results of [17–20] characterize the asymptotic behavior of the closed-loop poles (i.e., the behavior for sufficiently large k), these results do not characterize the locations of the closed-loop poles for either large or small k. In contrast, Chapters 2 and 3 characterize the closed-loop pole locations for all positive k.

Controller structures other than those considered in Chapters 2 and 3 lead to closed-loop denominator polynomials that are quadratic or cubic in k. Chapter 4 presents root-locus rules for a general polynomial that is quadratic in k. These quadratic root-locus rules apply to a linear controller that is a rational function of k and yields a closed-loop denominator polynomial that is quadratic in k. For example, the quadratic root-locus rules in Chapter 4 specialize to the root-locus rules in Chapter 2 if the appropriate controller class is considered.

Some of the rules developed in quadratic and cubic root locus are analogous to affine root locus. For example, portions of the real axis of the quadratic and cubic root loci behave similarly to portions of the real axis of the affine root locus. In contrast, the quadratic and cubic root loci possess other features that differ from affine root locus. For example, quadratic and cubic root loci feature portions of the real axis that may be "double covered" or "triple covered" meaning that there exists two or three distinct $k > 0$ such that a point on the root locus is a closed-loop pole. This cannot occur in affine root locus.

To demonstrate a potential benefit of quadratic and cubic root locus, recall that affine root locus is not high-gain stabilizing for systems, where the relative degree exceeds two. In this case, high gain causes at least one closed-loop pole to diverge to infinity through the open-right-half complex plane. In contrast, the quadratic root locus can be high-gain stabilizing for minimum-phase systems that are relative degree one, two or three. Moreover, the cubic root locus can be high-gain stabilizing for minimum-phase systems that are relative degree one, two, three, or four.

Before discussing quadratic and cubic root locus techniques, we first review the classical or affine root locus.

1.2 Review of Classical Root Locus

Consider the single-input single-output linear time-invariant system

$$y(s) = \beta G(s)u(s), \tag{1.1}$$

where

$$G(s) \triangleq \frac{z_{\mathrm{p}}(s)}{p_{\mathrm{p}}(s)}, \tag{1.2}$$

where $u(s)$ is the input; $y(s)$ is the output; $\beta \in \mathbb{R}$; and $z_{\mathrm{p}}(s)$ and $p_{\mathrm{p}}(s)$ are coprime monic polynomials, where $\deg z_{\mathrm{p}}(s) < \deg p_{\mathrm{p}}(s)$. Next, consider the control

$$u(s) = \frac{1}{\beta}\hat{G}_k(s)(v(s) - y(s)), \tag{1.3}$$

where

$$\hat{G}_k(s) \triangleq k\frac{z_{\mathrm{c}}(s)}{p_{\mathrm{c}}(s)}, \tag{1.4}$$

where $v(s)$ is an external signal; $z_{\rm c}(s)$ and $p_{\rm c}(s)$ are monic polynomials, where deg $z_{\rm c}(s) \le$ deg $p_{\rm c}(s)$; and $z(s) \triangleq z_{\rm p}(s)z_{\rm c}(s)$ and $p(s) \triangleq p_{\rm p}(s)p_{\rm c}(s)$ are coprime.

The closed-loop system (1.1)–(1.4) is shown in Figure 1.1, and the closed-loop transfer function from v to y is given by

$$\tilde{G}_k(s) \triangleq \frac{G(s)\hat{G}_k(s)}{1 + G(s)\hat{G}_k(s)} = \frac{kz(s)}{\tilde{p}_k(s)},$$

where

$$\tilde{p}_k(s) \triangleq kz(s) + p(s). \tag{1.5}$$

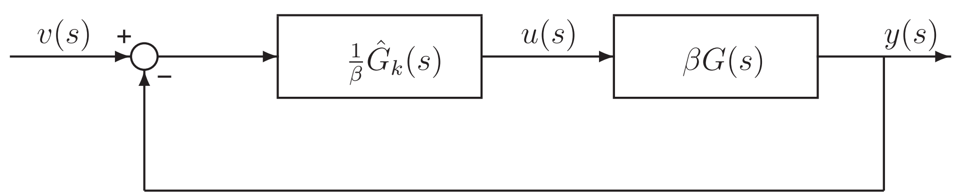

Figure 1.1: The closed-loop system depends on the root-locus parameter k.

We use the following classical definition of the positive root locus.

Definition 1.1. *The root locus is* $\{\lambda \in \mathbb{C} : 1 + G(\lambda)\hat{G}_k(\lambda) = 0,$ *where* $k > 0\}$.

We now present rules that can be used to draw the classical root locus. For more information on the classical root locus, see [21, pp. 381–389]. Facts 1.1 and 1.2 define the root locus starting points for $k = 0$ and describe the root locus symmetry, respectively.

Fact 1.1. *As* $k \to 0$, *the roots of* $\tilde{p}_k(s)$ *approach the roots of* $p(s)$.

Fact 1.2. *The root locus is symmetric about the real axis.*

Next, we present a rule to determine the points on the real axis that are on the root locus. The behavior of the root locus on the real axis depends on the roots of $p(s)$ and $z(s)$ only.

Fact 1.3. *Let $\sigma \in \mathbb{R}$. Then σ is on the root locus if and only if σ lies to the left of an odd number of real roots of $p(s)z(s)$. Furthermore, if σ is on the root locus, then there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.*

Note that the last sentence of Fact 1.3 is usually not included in the statement of the classical root-locus real axis fact. However, the last sentence of Fact 1.3 is a consequence of the closed-loop denominator polynomial (1.5) being affine in k, which implies that there exists at most one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. In contrast, the quadratic and cubic root loci can yield multiple $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

We now describe the asymptotic properties of affine root locus, that is, the properties for sufficiently large $k > 0$. We define

$$n \triangleq \deg p(s), \qquad m \triangleq \deg z(s), \qquad d \triangleq n - m.$$

Furthermore, let $z_1, z_2, \ldots, z_m$ be the roots of $z(s)$, and let $p_1, p_2, \ldots, p_n$ be the roots of $p(s)$. Fact 1.4 characterizes the asymptotic properties of the root locus.

Fact 1.4. *As k tends to infinity, m roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)$, and the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d$,*

$$\lambda_i \triangleq k^{1/d} e^{\jmath\theta_i} + \alpha,$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d},$$
$$\alpha \triangleq \frac{\sum_{j=1}^{n} p_j - \sum_{j=1}^{m} z_j}{d}.$$

Fact 1.4 implies that as k tends to infinity, d roots of $\tilde{p}_k(s)$ tend to infinity along asymptotes centered at α with angles $\theta_1, \theta_2, \ldots, \theta_d$, where $\theta_1, \theta_2, \ldots, \theta_d$ are equally spaced between 0 and 2π radians.

We now examine a point $\tau \in \mathbb{R}$, where there exists $k_\tau > 0$ such that τ is a root of $\tilde{p}_{k_\tau}(s)$ and where an infinitesimal perturbation to k_τ causes the the root of $\tilde{p}_{k_\tau}(s)$ at τ to become complex. These points are break-in and breakaway points. Specifically, roots that become complex under positive perturbation are breakaway points, while roots that become complex under negative perturbation are break-in points.

Definition 1.2. *Let $\tau \in \mathbb{R}$ be on the root locus. Then τ is a break-in or breakaway point if, for all $\epsilon > 0$, there exists $\sigma \in \mathbb{C}$ and $k_\sigma > 0$ such that $|\sigma - \tau| < \epsilon$, $\tilde{p}_{k_\sigma}(\sigma) = 0$, and the imaginary part of σ is nonzero.*

Fact 1.5 characterizes the break-in and breakaway points along the real axis.

Fact 1.5. *Let $\tau \in \mathbb{R}$ be on the root locus. Then τ is a break-in or breakaway point if and only if*

$$\sum_{i=1}^{m} \frac{1}{\tau - z_i} = \sum_{i=1}^{n} \frac{1}{\tau - p_i}.$$

Facts 1.1–1.5 provide rules for constructing the classical root locus. In subsequent chapters, we develop analogous rules for the root locus, where the closed-loop denominator polynomial is quadratic and cubic in the root-locus parameter k. We now present a summary of the subsequent chapters.

1.3 Summary of Chapters

Summary of Chapter 2

Chapter 2 presents the root locus for a controller class that yields a closed-loop denominator polynomial that is quadratic in k. Specifically, Chapter 2 considers controllers where the numerator polynomial is proportional to k^2, and the denominator polynomial includes a pole, whose location is proportional to k. The controller class in Chapter 2 results in a closed-loop denominator polynomial that is quadratic in k. Thus, the root locus rules for this controller require techniques for a polynomial that is quadratic in k. Chapter 2 presents quadratic root-locus rules that are analogous to the classical root-locus rules given by Facts 1.1–1.5.

Summary of Chapter 3

Chapter 3 presents the root locus for a controller class that yields a closed-loop denominator polynomial that is cubic in k. Specifically, Chapter 3 considers controllers where the numerator polynomial is proportional to k^3, and the denominator polynomial includes two poles, whose locations are proportional to k. The controller class in Chapter 3 results in a closed-loop denominator polynomial that is cubic in k. Thus, the root locus rules for this controller require techniques for a polynomial that is cubic in k. Chapter 3 presents cubic root-locus rules that are analogous to the classical root-locus rules given by Facts 1.1–1.4.

Summary of Chapter 4

Chapter 4 presents root-locus rules for a polynomial that is quadratic in k. In contrast to the quadratic root-locus rules in Chapter 2, Chapter 4 does not use a specific controller class to construct the quadratic root-locus rules. If the appropriate controller class is considered, then the quadratic root-locus rules in Chapter 4 specialize to the rules in Chapter 2.

All notation is introduced in the chapter where the notation is used. Furthermore, some notation may change between chapters. Thus, notation is specific to the chapter where it appears.

Chapter 2 Root Locus for a Controller Class that Yields Quadratic Gain Parameterization

This chapter presents rules for constructing the root locus for a class of linear feedback systems, where the closed-loop denominator polynomial is quadratic in the root-locus parameter k. These quadratic root-locus rules apply to a class of controllers that are rational functions of k. In contrast, classical root locus applies to controllers that are linear in k, and thus result in closed-loop denominator polynomials that are affine in k. We provide controller design examples to demonstrate the quadratic root locus. For example, we use quadratic root locus to high-gain stabilize the triple integrator; this is not possible with classical root locus. The results from this chapter have been submitted for publication in [22].

2.1 Introduction

In this chapter, we consider a class of controllers that yield a closed-loop denominator polynomial that is quadratic in the root-locus parameter k. More specifically, we consider a controller class, where the numerator is proportional to k^2, and the denominator includes a pole, whose location is proportional to k. For this controller class, we develop rules that characterize the starting points of the root locus, the segments of the real axis that are on the root locus, the asymptotic behavior of the root locus, and the break-in and breakaway points.

We provide two controller design examples to demonstrate the quadratic root locus. Specifically, we use quadratic root locus to high-gain stabilize the triple integrator.

We also use the quadratic root locus on a minimum-phase system, where the relative degree is two, and show that the settling time can be reduced by increasing k.

2.2 Problem Formulation

Consider the single-input single-output linear time-invariant system

$$y(s) = \beta G(s)u(s), \tag{2.1}$$

where

$$G(s) \triangleq \frac{z_{\rm p}(s)}{p_{\rm p}(s)}, \tag{2.2}$$

where $u(s)$ is the input; $y(s)$ is the output; $\beta \in \mathbb{R}$; and $z_{\rm p}(s)$ and $p_{\rm p}(s)$ are coprime monic polynomials, where deg $z_{\rm p}(s) <$ deg $p_{\rm p}(s)$. Next, consider the control

$$u(s) = \frac{1}{\beta}\hat{G}_k(s)(v(s) - y(s)), \tag{2.3}$$

where

$$\hat{G}_k(s) \triangleq \frac{k^2 z_{\rm c}(s)}{(s - \rho + \gamma k)p_{\rm c}(s)}, \tag{2.4}$$

where $v(s)$ is an external signal; $\rho \in \mathbb{R}$; $\gamma > 0$; $z_{\rm c}(s)$ and $p_{\rm c}(s)$ are monic polynomials, where deg $z_{\rm c}(s) \leq$ deg $p_{\rm c}(s) + 1$; and $z(s) \triangleq z_{\rm p}(s)z_{\rm c}(s)$ and $p(s) \triangleq p_{\rm p}(s)p_{\rm c}(s)$ are coprime. The controller $\hat{G}_k(s)$ is a nonlinear but rational function of k.

The closed-loop system (2.1)–(2.4) is shown in Figure 1.1, and the closed-loop transfer function from v to y is given by

$$\tilde{G}_k(s) \triangleq \frac{G(s)\hat{G}_k(s)}{1 + G(s)\hat{G}_k(s)} = \frac{k^2 z(s)}{\tilde{p}_k(s)}, \tag{2.5}$$

where

$$\tilde{p}_k(s) \triangleq k^2 z(s) + \gamma k p(s) + (s - \rho)p(s). \tag{2.6}$$

In classical root locus, the denominator polynomial of the closed-loop transfer function is an affine function of k. In contrast, $\tilde{p}_k(s)$ is a quadratic function of k. Nevertheless, we use the classical definition of the positive root locus, which is given by Definition 1.1.

This chapter considers the quadratic root locus where $k > 0$. The techniques in this chapter can also be used to develop root locus rules for $k < 0$.

In the next three sections, we present nine facts that characterize the quadratic root locus. Proofs of these facts are provided in Section 2.8.

2.3 Quadratic Root-Locus Rules

In this section, we present four rules for the quadratic root locus. These rules are analogous to Facts 1.1–1.4 of the classical root locus. Facts 2.1 and 2.2 define the root locus starting points for $k = 0$ and describe the root locus symmetry, respectively. These two facts are consistent with classical root locus.

Fact 2.1. *As $k \to 0$, the roots of $\tilde{p}_k(s)$ approach the roots of $(s - \rho)p(s)$.*

Fact 2.2. *The root locus is symmetric about the real axis.*

Next, we present a rule to determine the points on the real axis that are on the root locus. We define

$$t(s) \triangleq \gamma^2 p(s) - 4(s - \rho)z(s), \tag{2.7}$$

and note that $t(s)p(s)$ is the discriminant of $\tilde{p}_k(s)$ with respect to k. The polynomial $t(s)$ is not necessarily monic. Furthermore, if $\deg p(s) \leq \deg z(s)+1$, then the leading coefficient of $t(s)$ can be negative. The behavior of classical root locus on the real axis depends on the roots of $p(s)$ and $z(s)$ only. In contrast, the real axis rule for the quadratic root locus depends on the roots of $p(s)$, $z(s)$ and $t(s)$; the leading coefficient of $t(s)$; and ρ.

Fact 2.3. *Let $\sigma \in \mathbb{R}$. Then σ is on the root locus if and only if either of the following statements hold:*

(a) $\sigma \geq \rho$ and $p(\sigma)z(\sigma) < 0$.

(b) $\sigma < \rho$, $p(\sigma) \neq 0$ and $t(\sigma)p(\sigma) \geq 0$.

Furthermore, if σ is on the root locus, then the following statements hold:

(i) If $\sigma \geq \rho$, then there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

(ii) If $p(\sigma)z(\sigma) \geq 0$ or $t(\sigma) = 0$, then there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

(iii) If $\sigma < \rho$, $p(\sigma)z(\sigma) < 0$ and $t(\sigma) \neq 0$, then there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

Part *(a)* of Fact 2.3 implies that the real axis to the right of ρ is consistent with classical root locus. Specifically, $\sigma \geq \rho$ is on the root locus if and only if σ lies to the left of an odd number of real roots of $p(s)z(s)$. In contrast, part *(b)* of Fact 2.3 shows that the real axis to the left of ρ differs from classical root locus.

It follows from part *(b)* of Fact 2.3 that real roots of $t(s)$ that have odd multiplicity and lie to the left of ρ are boundary points that separate segments of the real axis that are on the root locus from segments of the real axis that are not on the root locus. Furthermore, part *(b)* of Fact 2.3 implies that real zeros (i.e., real roots of $z(s)$), which lie to the left of ρ, can be closed-loop poles (i.e., roots of $\tilde{p}_k(s)$) for finite k. These features are not present in classical root locus.

We now describe the asymptotic properties of the quadratic root locus, that is, the properties for sufficiently large $k > 0$. We define

$$n \triangleq \deg p(s) + 1, \qquad m \triangleq \deg z(s), \qquad d \triangleq n - m. \tag{2.8}$$

Furthermore, let $z_1, z_2, \ldots, z_m$ be the roots of $z(s)$, and let $p_1, p_2, \ldots, p_{n-1}$ be the roots of $p(s)$. Fact 2.4 characterizes the asymptotic properties of $\tilde{p}_k(s)$, that is, the properties for sufficiently large $k > 0$.

Fact 2.4. *As $k \to \infty$, m roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)$, and the d remaining roots satisfy the following statements:*

(a) If $d = 1$, then the remaining root of $\tilde{p}_k(s)$ is approximated by

$$\lambda_1 \triangleq -k^2 - \gamma k + \rho. \tag{2.9}$$

(b) If $d = 2$, then the two remaining roots of $\tilde{p}_k(s)$ are approximated by λ_1 and λ_2, where, for $i = 1, 2$,

$$\lambda_i \triangleq k e^{\jmath \theta_i} + \alpha, \tag{2.10}$$

where

$$\theta_i \triangleq \arg\left(-\frac{\gamma}{2} + \jmath \frac{(-1)^i \sqrt{4 - \gamma^2}}{2} \right), \tag{2.11}$$

$$\alpha \triangleq \sum_{j=1}^{n-1} p_j - \sum_{j=1}^{m} z_j. \tag{2.12}$$

(c) If $d = 3$, then the three remaining roots of $\tilde{p}_k(s)$ are approximated by

$$\lambda_1 \triangleq \alpha + \jmath \sqrt{k/\gamma + \rho/\gamma^2 - \alpha^2}, \tag{2.13}$$

$$\lambda_2 \triangleq \alpha - \jmath \sqrt{k/\gamma + \rho/\gamma^2 - \alpha^2}, \tag{2.14}$$

$$\lambda_3 \triangleq -\gamma k - \gamma^{-2} + \rho, \tag{2.15}$$

where

$$\alpha \triangleq \frac{\sum_{j=1}^{n-1} p_j - \sum_{j=1}^{m} z_j + \gamma^{-2}}{2}. \tag{2.16}$$

(d) *If $d \geq 4$, then the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d-1$,*

$$\lambda_i \triangleq \left(\frac{k}{\gamma}\right)^{\frac{1}{d-1}} e^{\jmath\theta_i} + \alpha, \tag{2.17}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d-1}, \tag{2.18}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n-1} p_j - \sum_{j=1}^{m} z_j}{d-1}; \tag{2.19}$$

and

$$\lambda_d \triangleq -\gamma k - (-\gamma)^{1-d}/k^{d-3} + \rho. \tag{2.20}$$

2.4 Break-in and Breakaway Points

In this section, we examine the break-in and breakaway points of the quadratic root locus. We use the classical break-in and breakaway point definition, which is given by Definition 1.2.

For $i = 1, 2$, define $\kappa_i : \{\sigma \in \mathbb{R} : z(\sigma) \neq 0\} \to \mathbb{C}$ by

$$\kappa_i(\sigma) \triangleq \frac{-\gamma p(\sigma) + (-1)^{i-1}\sqrt{t(\sigma)p(\sigma)}}{2z(\sigma)}, \tag{2.21}$$

which maps the real numbers excluding the real roots of $q(s)$ to the complex numbers. Note that if $\kappa_1(\sigma) > 0$ or $\kappa_2(\sigma) > 0$, then there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, and σ is on the root locus.

We now present facts that characterize break-in and breakaway points. Fact 2.5 characterizes the break-in and breakaway points along the real axis, excluding the real roots of $t(s)z(s)$. Furthermore, Facts 2.6 and 2.7 demonstrate when roots of $z(s)$ and $t(s)$, respectively, are break-in and breakaway points.

Fact 2.5. *Let $\tau \in \mathbb{R}$ be a point on the root locus, and assume τ is not a root of $t(s)z(s)$. Then τ is a break-in or breakaway point if and only if either of the following statements hold:*

(a) $\kappa_1(\tau) > 0$ *and* $\mathrm{d}\kappa_1(\sigma)/\mathrm{d}\sigma|_{\sigma=\tau} = 0$.

(b) $\kappa_2(\tau) > 0$ *and* $\mathrm{d}\kappa_2(\sigma)/\mathrm{d}\sigma|_{\sigma=\tau} = 0$.

Fact 2.6. *Let τ be a real root of $z(s)$, and define $k_\tau \triangleq (\rho - \tau)/\gamma$. Then τ is a break-in or breakaway point if and only if $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ.*

Fact 2.7. *Let τ be a real root of $t(s)$, and define $k_\tau \triangleq 2(\rho - \tau)/\gamma$. Then the following statements are equivalent:*

(a) τ is a break-in or breakaway point.

(b) $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ.

(c) $\tau < \rho$ and $t(s)$ has multiple roots at τ.

2.5 Additional Real Axis Rules

In classical root locus, it is not possible for roots of $z(s)$ to be closed-loop poles. In contrast, Fact 2.3 of quadratic root locus implies that the real roots of $z(s)$ can be closed-loop poles for finite $k > 0$. The following two facts characterize finite values of $k > 0$ for which real roots of $z(s)$ and real roots of $t(s)$ are closed-loop poles.

Fact 2.8. *Let σ be a real root of $z(s)$. Then $\tilde{p}_k(\sigma) = 0$ if and only if $k = (\rho - \sigma)/\gamma$. Furthermore, σ is on the root locus if and only if $\sigma < \rho$.*

Fact 2.9. *Let σ be a real root of $t(s)$. Then $\tilde{p}_k(\sigma) = 0$ if and only if $k = 2(\rho - \sigma)/\gamma$. Furthermore, σ is on the root locus if and only if $\sigma < \rho$.*

2.6 Numerical Examples

In classical root locus, systems where the relative degree exceeds two are not high-gain stabilizable, that is, stable for sufficiently large $k > 0$. In contrast, quadratic root locus can high-gain stabilize minimum-phase systems that are relative degree one, two or three. We consider a relative-degree-three example.

Example 2.1. Consider the system $y(s) = G(s)u(s)$, where $G(s) = 1/s^3$, which is minimum phase and relative degree 3. Part *(c)* of Fact 2.4 states that the asymptote angles for $d = 3$ are $-\pi/2$, $\pi/2$ and π. Thus, if $d = 3$, $z_c(s)$ is asymptotically stable and $\alpha < 0$, then the closed-loop transfer function $\tilde{G}_k(s)$, given by (2.5), is high-gain stable. We let $p_c(s) = s + 30$, $z_c(s) = (s + 10)(s + 15)$ and $\gamma = 1$, and it follows that $\alpha = -2 < 0$ and $d = 3$. Next, we let $\rho = -5$. Note that ρ does not impact the asymptote center α or asymptote angles for $d = 3$. Thus, the controller (2.3), where $\hat{G}_k(s) = k^2(s + 10)(s + 15)/((s + 5 + k)(s + 30))$, high-gain stabilizes the closed-loop transfer function $\tilde{G}_k(s)$, where $G(s)$ is the triple integrator.

To draw the quadratic root locus, we apply the rules from Section 2.3. First, Fact 2.1 implies that the root locus begins at the roots of $(s + 5)p(s)$. Next, Fact 2.3 specifies which points of the real axis are on the root locus. Since $p(s)z(s)$ has no roots in the open-right-half complex plane, part *(a)* of Fact 2.3 implies $(0, \infty)$ is not on the root locus. Since $[-5, 0)$, which includes ρ as a boundary point, is to the left of an odd number of real roots of $p(s)z(s)$, part *(a)* of Fact 2.3 implies that $[-5, 0)$ is on the root locus. Now, part *(b)* of Fact 2.3 is used to determine the points on the real axis to the left of ρ that are on the root locus. Note that $t(s) = (s + 28.96)(s - 8.47)(s + 2.76 + \jmath 2.16)(s + 2.76 - \jmath 2.16)$. Thus, $(-28.96, -5)$ and $(-\infty, -30)$ are on the root locus, while $(-30, -28.96)$ is not on the root locus. Furthermore, since $(-28.96, -15)$ and $(-10, -5)$ are also to the left of an even number of real roots of $p(s)z(s)$, it follows from *(iii)* of Fact 2.3 that these intervals are "double

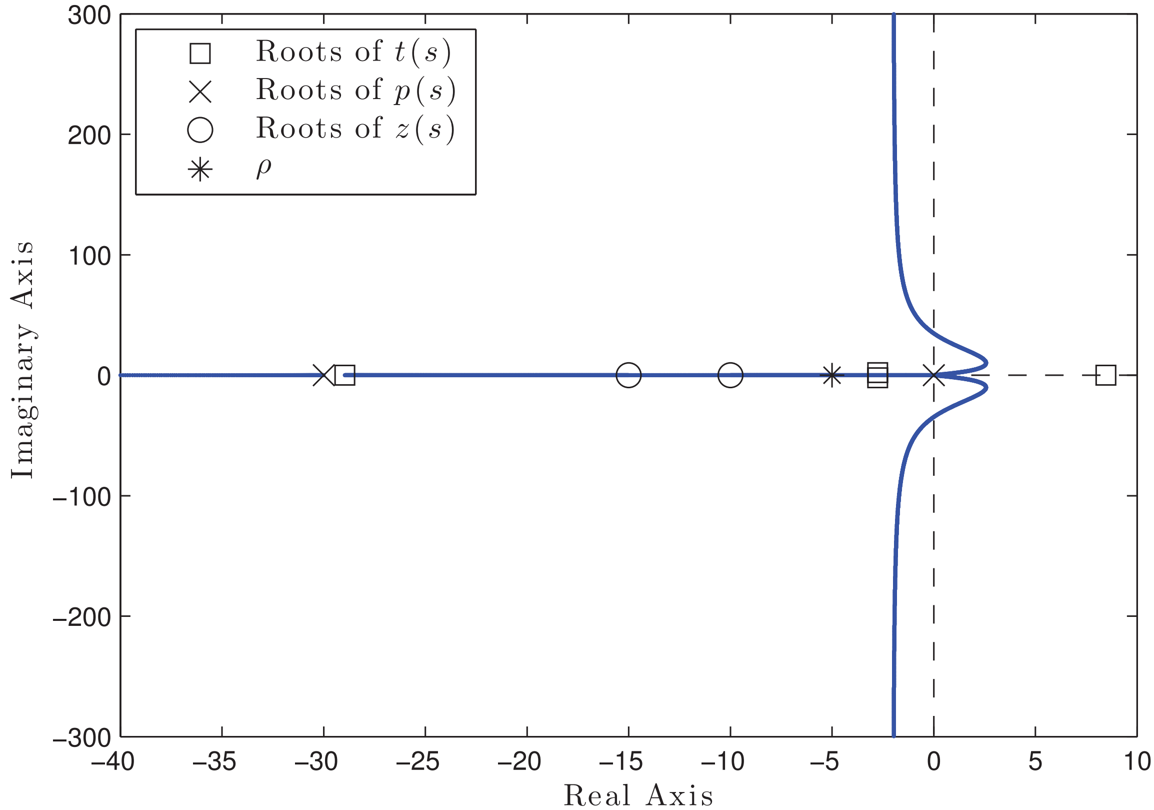

Figure 2.1: The quadratic root locus shows that the triple integrator $G(s) = 1/s^3$ is high-gain stabilized by the controller $\hat{G}_k(s) = k^2(s+10)(s+15)/((s+5+k)(s+30))$. In fact, the closed-loop system is asymptotically stable for all $k > 1414$.

covered", that is, there exist two distinct $k > 0$ such that each point in the interval is a closed-loop pole of $\tilde{G}_k(s)$. The quadratic root locus is plotted in Figure 2.1, and the plot confirms that the closed-loop transfer function $\tilde{G}_k(s)$ is high-gain stable. In fact, $\tilde{G}_k(s)$ is asymptotically stable for all $k > 1414$. △

In classical root locus, a relative-degree-two system has two closed-loop complex-conjugate poles that diverge to infinity along asymptotes with angles $\pm\pi/2$ rad. The real part of these poles converge to the asymptote center, which is finite and determines the settling time associated with the poles. In contrast, the quadratic root locus for a relative-degree-two system has two closed-loop complex-conjugate poles that diverge to infinity along asymptotes with angles that can be selected by choice of γ. We demonstrate this feature in the following example.

Example 2.2. Consider the inverted-pendulum model $\ddot{\theta} - (g/l)\sin\theta = u$, where θ is the angle from the inverted position, g is the acceleration due to gravity, l is the length of the pendulum, and u is the control torque. Let $g/l = 100$ s^{-2}, and assume that the output is the angular velocity, that is, $y = \dot{\theta}$. The linearized transfer function from u to y is $G(s) = s/((s-10)(s+10))$.

First, we consider the controller

$$u(s) = \frac{k(s+5)}{(s-5)(s+20)}(v(s) - y(s)),$$

where $v(s)$ is an external signal, and we consider the classical root locus, which is shown in Figure 2.2 and demonstrates that two closed-loop poles diverge along asymptotes centered at -5 with angles $\pm\pi/2$ rad. The closed-loop system is asymptotically stable for sufficiently large $k > 0$, but the settling time associated with the diverging poles cannot be changed by choice of k.

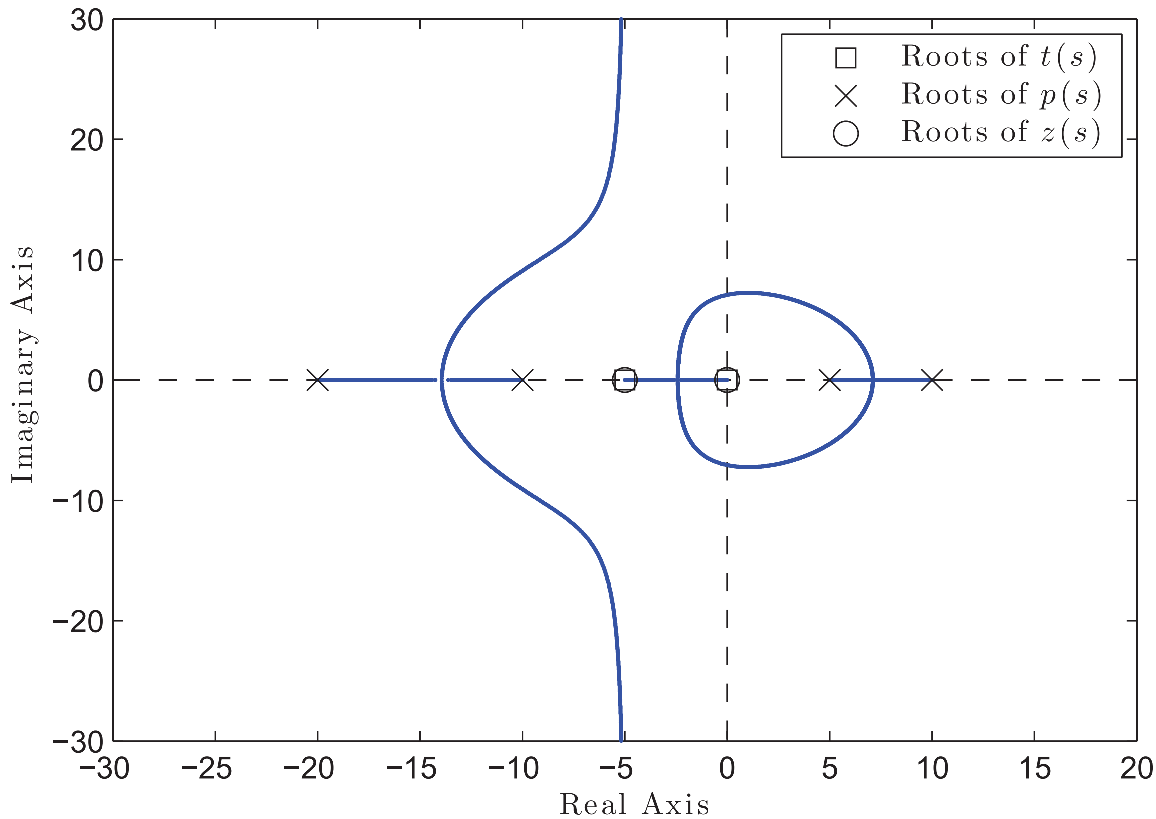

Figure 2.2: The classical root locus shows that, for sufficiently large $k > 0$, the two closed-loop poles diverge along asymptotes centered at -5 with angles $\pm\pi/2$ rad.

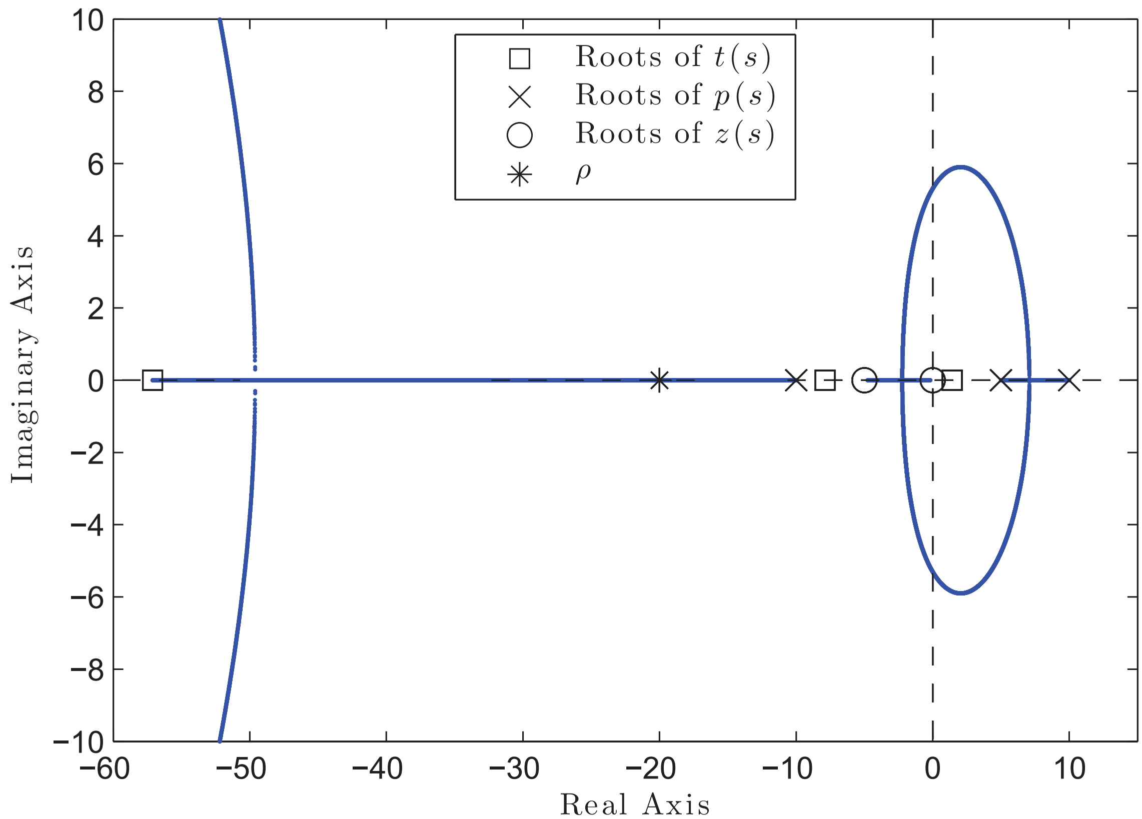

Figure 2.3: The quadratic root locus shows that, for sufficiently large $k > 0$, two closed-loop poles diverge along asymptotes centered at -10 with angles ± 2.42 rad.

Next, we consider the controller

$$u(s) = \frac{k^2(s+5)}{(s-5)(s+20+1.5k)}(v(s)-y(s)),$$

and the associated quadratic root locus, which is shown in Figure 2.3. The quadratic root locus demonstrates that two closed-loop poles diverge along asymptotes, which are determined using part *(b)* of Fact 2.4 and are centered at -10 with angles ± 2.42 rad. The closed-loop system is asymptotically stable for sufficiently large $k > 0$. Furthermore, the settling time associated with the diverging poles can be reduced relative to the settling time of the diverging poles in the classical root locus shown in Figure 2.2. △

2.7 Conclusions

This chapter presented rules for constructing a root locus, where the closed-loop denominator polynomial is quadratic in the root-locus parameter k. These quadratic root-locus rules apply to a controller class that is rational in k. Specifically, this chapter considered the controller class (2.3) and (2.4), where the closed-loop denominator polynomial is quadratic in k. The techniques used in this chapter to develop Facts 2.1, 2.2, 2.3, and 2.5 can be generalized to other controller structures that lead to closed-loop denominator polynomials that are quadratic in k. For example, the real axis rule (i.e., Fact 2.3) determines the real values of σ that are on the root locus by determining the real values of σ such that $\tilde{p}_k(\sigma)$, which is quadratic in k, has positive roots. This technique can be applied to any closed-loop denominator polynomial that is quadratic in k, and can thus be generalized to other controller structures. In contrast, the techniques used to develop the asymptotic properties in Fact 2.4 cannot be directly extended to other controller structures. Fact 2.4 relies on the specific controller class considered in this chapter. Generalizing the quadratic root locus rules to other controller structures is considered in Chapter 4.

2.8 Proofs for Facts 2.1–2.9

Proof of Fact 2.1. If $k = 0$, then $\tilde{p}_k(s) = (s - \rho)p(s)$. □

Proof of Fact 2.2. Since for all $k > 0$, $\tilde{p}_k(s)$ has real coefficients, it follows that the roots of $\tilde{p}_k(s)$ are either on the real axis or occur in complex conjugate pairs. □

Proof of Fact 2.3. First, we show that real roots of $p(s)$ are not on the root locus. Assume $p(\sigma) = 0$, and it follows from (2.6) that $\tilde{p}_k(\sigma) = k^2 z(\sigma)$. Since $p(s)$ and $z(s)$ are coprime, it follows that $z(\sigma) \neq 0$. Thus, there does not exist $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, and σ is not on the root locus.

Next, define

$$a_2 \triangleq z(\sigma), \quad a_1 \triangleq \gamma p(\sigma), \quad a_0 \triangleq (\sigma - \rho)p(\sigma), \tag{2.22}$$

and note from (2.6) that $\tilde{p}_k(\sigma) = a_2 k^2 + a_1 k + a_0$.

We show that *(a)* or *(b)* is necessary for σ to be on the root locus. Assume σ is on the root locus and consider three cases: (A.I) $\sigma > \rho$, (A.II) $\sigma = \rho$ and (A.III) $\sigma < \rho$.

First, assume (A.I) $\sigma > \rho$, and it follows that $a_1 a_0 = \gamma(\sigma - \rho)p(\sigma)^2 > 0$. Since σ is on the root locus, $\tilde{p}_k(\sigma)$ has at least one positive root on the root locus, which implies that a_2 has the opposite sign of a_1 and a_0. Thus, $a_2 a_0 < 0$. Furthermore, since $a_2 a_0 = (\sigma - \rho)p(\sigma)z(\sigma) < 0$ and $\sigma - \rho > 0$, it follows that $p(\sigma)z(\sigma) < 0$. Next, assume (A.II) $\sigma = \rho$, which implies that $\tilde{p}_k(\sigma) = k(a_2 k + a_1)$. Since $\tilde{p}_k(\sigma)$ has at least one positive root, it follows that $-a_1/a_2 > 0$, or equivalently $a_2 a_1 < 0$. Furthermore, since $a_2 a_1 = \gamma p(\sigma)z(\sigma) < 0$ and $\gamma > 0$, it follows that $p(\sigma)z(\sigma) < 0$. Thus, combining cases (A.I) and (A.II), we obtain $\sigma \geq \rho$ and $p(\sigma)z(\sigma) < 0$, which implies *(a)*.

Finally, assume (A.III) $\sigma < \rho$, and consider two cases: (A.III.a) $a_2 = 0$ and (A.III.b) $a_2 \neq 0$. First, assume (A.III.a) $a_2 = 0$, which implies that $a_1^2 - 4a_2 a_0 = a_1^2 \geq 0$. Next, assume (A.III.b) $a_2 \neq 0$. Since $\tilde{p}_k(\sigma)$ has at least one positive root, it follows that the roots of $\tilde{p}_k(\sigma)$ are not complex, which implies that $a_1^2 - 4a_2a_0 \geq 0$. Thus, in both cases, we obtain $a_1^2 - 4a_2a_0 = t(\sigma)p(\sigma) \geq 0$, which implies *(b)*.

Conversely, assume *(a)* or *(b)* hold. Assume *(a)* holds and consider two cases: (B.I) $\sigma > \rho$ and (B.II) $\sigma = \rho$. First, assume (B.I) $\sigma > \rho$. Since $\sigma > \rho$ and $p(\sigma)z(\sigma) < 0$, it follows that $a_2a_0 = (\sigma - \rho)p(\sigma)z(\sigma) < 0$. Thus, part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Next, assume (B.II) $\sigma = \rho$, and it follows that $k_\sigma \triangleq -a_1/a_2$ and 0 are the roots of $\tilde{p}_k(\sigma) = k(a_2k + a_1)$. Since $\gamma > 0$ and $p(\sigma)z(\sigma) < 0$, it follows that $k_\sigma = -\gamma p(\sigma)/z(\sigma)$ is positive. Therefore, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, combining cases (B.I) and (B.II), we obtain that *(a)* implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, which implies that σ is on the root locus, and confirms *(i)*.

Next, assume *(b)* holds, and consider two cases: (C.I) $t(\sigma) = 0$, and (C.II) $t(\sigma) \neq 0$. First, assume (C.I) $t(\sigma) = 0$, which implies from (2.7) that $\gamma^2 p(\sigma) = 4(\sigma - \rho)z(\sigma)$. Furthermore, since $\sigma - \rho < 0$, it follows that $p(\sigma)$ and $z(\sigma)$ have opposite signs, which implies that $p(\sigma)z(\sigma) < 0$. Therefore, $a_1^2 - 4a_2a_0 = t(\sigma)p(\sigma) = 0$ and $a_2a_1 = \gamma p(\sigma)z(\sigma) < 0$. Thus, part *(b)* of Lemma 1 in Appendix A implies that $\tilde{p}_k(\sigma)$ has repeated positive roots. Therefore, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus.

Next, assume (C.II) $t(\sigma) \neq 0$, which implies that $a_1^2 - 4a_2a_0 = t(\sigma)p(\sigma) > 0$. We consider three cases: (C.II.a) $p(\sigma)z(\sigma) > 0$, (C.II.b) $p(\sigma)z(\sigma) = 0$ and (C.II.c) $p(\sigma)z(\sigma) < 0$. First, assume (C.II.a) $p(\sigma)z(\sigma) > 0$, and it follows that $a_2a_0 = (\sigma - \rho)p(\sigma)z(\sigma) < 0$. Therefore, part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Next, assume (C.II.b) $p(\sigma)z(\sigma) = 0$, which implies that $z(\sigma) = 0$ since $p(\sigma) \neq 0$. Therefore, $\tilde{p}_k(\sigma) = p(\sigma)(\gamma k + \sigma - \rho)$ has the sole root $-(\sigma - \rho)/\gamma$, which is positive. Therefore, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Next, assume (C.II.c) $p(\sigma)z(\sigma) < 0$. Since $\gamma > 0$, $\sigma - \rho < 0$ and $p(\sigma)z(\sigma) < 0$, it follows that $a_2a_1 = \gamma p(\sigma)z(\sigma) < 0$, and $a_2a_0 = (\sigma - \rho)p(\sigma)z(\sigma) > 0$.

Furthermore, since $a_1^2 - 4a_2a_0 > 0$, $a_2a_1 < 0$ and $a_2a_0 > 0$, part *(c)* of Lemma 1 in Appendix A implies that there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, combining cases (C.I)–(C.II), we obtain that *(b)* implies that σ is on the root locus. Furthermore, cases (C.II.a) and (C.II.b) confirm *(ii)*, and case (C.II.c) confirms *(iii)*. □

Proof of Fact 2.4. To show that m roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)$, it follows from (2.6) that

$$\frac{\tilde{p}_k(s)}{k^2} = z(s) + \frac{\gamma p(s)}{k} + \frac{(s-\rho)p(s)}{k^2},$$

which implies that, for sufficiently large $k > 0$, $\tilde{p}_k(s)/k^2 \approx z(s)$. Thus, as $k \to \infty$, m roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)$.

Next, define $R \stackrel{\triangle}{=} \max_{i=1,\ldots,m}|z_i|$ and write $p(s) = s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}$ and $z(s) = s^m + b_1 s^{m-1} + \cdots + b_m$, where $a_1, a_2, \ldots, a_{n-1} \in \mathbb{R}$ and $b_1, b_2, \ldots, b_m \in \mathbb{R}$.

First, we show *(a)*. Since $d = 1$, it follows that for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = 1 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{2.23}$$

where the real numbers $c_1, c_2, \ldots$ are coefficients of the Laurent series expansion.

Next, it follows from (2.2), (2.4) and (2.9) that

$$\frac{1}{G(\lambda_1)\hat{G}_k(\lambda_1)} = \frac{-p(\lambda_1)}{z(\lambda_1)}. \tag{2.24}$$

Since $|\lambda_1| \to \infty$ as $k \to \infty$, it follows from (2.23) that, for sufficiently large $k > 0$, $p(\lambda_1)/z(\lambda_1) \approx 1$. Therefore, for sufficiently large $k > 0$, (2.24) yields

$$\frac{1}{G(\lambda_1)\hat{G}_k(\lambda_1)} \approx -1,$$

or equivalently $1 + G(\lambda_1)\hat{G}_k(\lambda_1) \approx 0$. Thus, as $k \to \infty$, one root of $\tilde{p}_k(s)$ is approximated by λ_1, which confirms *(a)*.

Next, we show *(b)*. Since $d = 2$, it follows that, for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = s + c_0 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{2.25}$$

where the real numbers $c_0, c_1, ...$ are coefficients of the Laurent series expansion. Furthermore, note that $c_0 = a_1 - b_1 = -(\sum_{j=1}^{n-1} p_j - \sum_{j=1}^{m} z_j) = -\alpha$.

Next, for $i = 1, 2$, define

$$\bar{\lambda}_i \triangleq k\left(u_k + \jmath(-1)^{i-1} v_k\right) + \alpha. \tag{2.26}$$

where

$$u_k \triangleq -\frac{\gamma}{2} - \frac{\alpha}{2k} + \frac{\rho}{2k}, \tag{2.27}$$

$$v_k \triangleq \frac{\sqrt{4 - (\gamma + \alpha/k - \rho/k)^2}}{2}. \tag{2.28}$$

Since, $|u_k + \jmath v_k| = |u_k - \jmath v_k| = 1$, it follows that (2.26) can be expressed as

$$\bar{\lambda}_i = k e^{\jmath \bar{\theta}_i} + \alpha, \tag{2.29}$$

where, for $i = 1, 2$,

$$\bar{\theta}_i \triangleq \arg\left(u_k + \jmath(-1)^{i-1} v_k\right). \tag{2.30}$$

It follows from (2.26)–(2.28) that, for $i = 1, 2$,

$$\frac{(\bar{\lambda}_i - \rho + \gamma k)(\bar{\lambda}_i - \alpha)}{k^2} = (-k u_k + \jmath(-1)^i k v_k) \times (k u_k + \jmath(-1)^i k v_k)/k^2$$

$$= -1. \tag{2.31}$$

Furthermore, it follows from (2.29) and (2.30) that, as $k \to \infty$, $|\bar{\lambda}_1| \to \infty$ and $|\bar{\lambda}_2| \to \infty$. Thus, for $i = 1, 2$ and sufficiently large $k > 0$, it follows from (2.25) that

$$\frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)} \approx \bar{\lambda}_i + c_0 = \bar{\lambda}_i - \alpha. \tag{2.32}$$

Using (2.32) followed by (2.31) yields, for $i = 1, 2$ and sufficiently large $k > 0$,

$$\begin{aligned} \frac{1}{G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i)} &= \frac{(\bar{\lambda}_i - \rho + \gamma k)p(\bar{\lambda}_i)}{k^2 z(\bar{\lambda}_i)} \\ &\approx \frac{(\bar{\lambda}_i - \rho + \gamma k)(\bar{\lambda}_i - \alpha)}{k^2} = -1, \end{aligned}$$

or equivalently $1 + G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i) \approx 0$. Thus, as $k \to \infty$, two roots of $\tilde{p}_k(s)$ are approximated by $\bar{\lambda}_1$ and $\bar{\lambda}_2$. Finally, it follows from (2.11), (2.27), (2.28), and (2.30) that, as $k \to \infty$, $\bar{\theta}_i$ approaches θ_i . Thus, as $k \to \infty$, $\bar{\lambda}_i$ approaches λ_i, which implies that, as $k \to \infty$, two roots of $\tilde{p}_k(s)$ are approximated by λ_1 and λ_2, which confirms *(b)*.

Next, we show *(c)*. First, we show that two roots of $\tilde{p}_k(s)$ approach λ_1 and λ_2. Since $d = 3$, it follows that, for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = s^2 + f_1 s + c_0 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{2.33}$$

where the real numbers $f_1, c_0, c_1, \ldots$ are coefficients of the Laurent series expansion. Furthermore, note that $f_1 = a_1 - b_1 = -(\sum_{j=1}^{n-1} p_j - \sum_{j=1}^{m} z_j) = -2\alpha + \gamma^{-2}$.

Next, we consider the Taylor series expansion of $k^2/(s - \rho + \gamma k)$ about ρ. For all $s \in \mathbb{C}$ such that $|s - \rho| < \gamma k$, the Taylor series expansion of $k^2/(s - \rho + \gamma k)$ about ρ

is given by

$$\begin{aligned}\frac{k^2}{s-\rho+\gamma k} &= \sum_{j=0}^{\infty} \frac{(-1)^j}{\gamma^{j+1}k^{j-1}}(s-\rho)^j \\ &= \frac{k}{\gamma} - \frac{s-\rho}{\gamma^2} + \sum_{j=2}^{\infty} \frac{(-1)^j}{\gamma^{j+1}k^{j-1}}(s-\rho)^j. \end{aligned} \tag{2.34}$$

For $i = 1, 2$, $|\lambda_i| = \sqrt{k/\gamma + \rho/\gamma^2}$, which implies that as $k \to \infty$, $|\lambda_1| \to \infty$ and $|\lambda_2| \to \infty$. Thus, for $i = 1, 2$ and sufficiently large $k > 0$, it follows from (2.33) that

$$\frac{p(\lambda_i)}{z(\lambda_i)} \approx \lambda_i^2 + (-2\alpha + \gamma^{-2})\lambda_i. \tag{2.35}$$

Next, since for $i = 1, 2$, $|\lambda_i| = \sqrt{k/\gamma + \rho/\gamma^2}$, it follows that as $k \to \infty$, $(\lambda_i - \rho)^2/k$ approaches a constant and for all $j = 3, 4, \ldots$, $(\lambda_i - \rho)^j/k^{j-1}$ approaches zero. In addition, since for $i = 1, 2$ and sufficiently large $k > 0$, $|\lambda_i - \rho| = \sqrt{k/\gamma + \rho/\gamma^2 - 2\alpha\rho + \rho^2} < \gamma k$, it follows from (2.34) that for sufficiently large $k > 0$,

$$\frac{k^2}{\lambda_i - \rho + \gamma k} \approx \frac{k}{\gamma} - \frac{\lambda_i - \rho}{\gamma^2}. \tag{2.36}$$

Using (2.35) followed by (2.36) yields, for $i = 1, 2$ and sufficiently large $k > 0$,

$$\begin{aligned}\frac{p_c(\lambda_i)}{G(\lambda_i)z_c(\lambda_i)} &= \frac{p(\lambda_i)}{z(\lambda_i)} \\ &\approx \lambda_i^2 + (-2\alpha + \gamma^{-2})\lambda_i \\ &= \frac{-k}{\gamma} + \frac{\lambda_i - \rho}{\gamma^2} \\ &\approx -\frac{k^2}{\lambda_i - \rho + \gamma k} \\ &= -\frac{\hat{G}_k(\lambda_i)p_c(\lambda_i)}{z_c(\lambda_i)}, \end{aligned}$$

or equivalently $1 + G(\lambda_i)\hat{G}_k(\lambda_i) \approx 0$. Thus, as $k \to \infty$, two roots of $\tilde{p}_k(s)$ are approximated by λ_1 and λ_2.

Next, to show that the one remaining root of $\tilde{p}_k(s)$ approaches λ_3, it follows from (2.4) and (2.15) that

$$\frac{\hat{G}_k(\lambda_3)p_{\mathrm{c}}(\lambda_3)}{z_{\mathrm{c}}(\lambda_3)} = \frac{k^2}{\lambda_3 - \rho + \gamma k} = -\gamma^2 k^2. \tag{2.37}$$

Next, since, as $k \to \infty$, $|\lambda_3| \to \infty$, it follows from (2.33) that for sufficiently large $k > 0$,

$$\frac{p(\lambda_3)}{z(\lambda_3)} \approx \lambda_3^2 = \gamma^2 k^2 (1 + c_k), \tag{2.38}$$

where

$$c_k \triangleq \frac{\rho^2}{\gamma^2 k^2} - \frac{2\rho}{\gamma^4 k^2} - \frac{2\rho}{\gamma k} + \frac{1}{\gamma^6 k^2} + \frac{2}{\gamma^3 k}. \tag{2.39}$$

As $k \to \infty$, c_k approaches zero. Thus, as $k \to \infty$, (2.38) implies that $p(\lambda_3)/z(\lambda_3) \approx \gamma^2 k^2$, which combined with (2.37) yields

$$\frac{p_{\mathrm{c}}(\lambda_3)}{G(\lambda_3)z_{\mathrm{c}}(\lambda_3)} = \frac{p(\lambda_3)}{z(\lambda_3)} \approx \gamma^2 k^2 = -\frac{\hat{G}_k(\lambda_3)p_{\mathrm{c}}(\lambda_3)}{z_{\mathrm{c}}(\lambda_3)}$$

or equivalently $1 + G(\lambda_3)\hat{G}_k(\lambda_3) \approx 0$. Thus, as $k \to \infty$, one root of $\tilde{p}_k(s)$ approaches λ_3, which confirms *(c)*.

Finally, we show *(d)*. First, we show that $d-1$ roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d-1}$. Since $d \geq 4$, it follows that, for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = s^{d-1} + f_{d-2}s^{d-2} + \cdots + f_1 s + c_0 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{2.40}$$

where the real numbers $f_1, f_2, \ldots, f_{d-2}, c_0, c_1, \ldots$ are real coefficients of the Laurent series expansion. Furthermore, note that $f_{d-2} = a_1 - b_1 = -\left(\sum_{j=1}^{n-1} p_j - \sum_{j=1}^{m} z_j\right) =$

$-(d-1)\alpha$. For all $s \in \mathbb{C}$ such that $|s-\rho| > (k/\gamma)^{\frac{1}{d}}$ and sufficiently large $k>0$, it follows from (2.40) that

$$\frac{p(s)}{z(s)} \approx s^{d-1} - (d-1)\alpha s^{d-2}. \tag{2.41}$$

Next, for all $s \in \mathbb{C}$ such that $(k/\gamma)^{\frac{1}{d}} < |s-\rho| < (k/\gamma)^{\frac{1}{2}}$, it follows that, as $k \to \infty$, $(s-\rho)^2/k$ approaches a constant and for $j = 3, 4, \ldots$, $(s-\rho)^j/k^{j-1}$ approaches zero. Therefore, for sufficiently large $k>0$ and for all $s \in \mathbb{C}$ such that $(k/\gamma)^{\frac{1}{d}} < |s-\rho| < (k/\gamma)^{\frac{1}{2}}$, it follows from the Taylor series expansion (2.34) that

$$\frac{k^2}{s-\rho+\gamma k} \approx \frac{k}{\gamma} - \frac{s-\rho}{\gamma^2}. \tag{2.42}$$

Next, adding (2.41) and (2.42) yields, for sufficiently large $k>0$ and for all $s \in \mathbb{C}$ such that $(k/\gamma)^{\frac{1}{d}} < |s-\rho| < (k/\gamma)^{\frac{1}{2}}$,

$$\frac{p(s)}{z(s)} + \frac{k^2}{s-\rho+\gamma k} \approx d_k(s), \tag{2.43}$$

where

$$d_k(s) \stackrel{\triangle}{=} s^{d-1} - (d-1)\alpha s^{d-2} - (s-\rho)/\gamma^2 + k/\gamma. \tag{2.44}$$

For $k>0$, let $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{d-1}$ be the roots of $d_k(s)$. It follows from classical root locus that as $k \to \infty$, $|\bar{\lambda}_i| \to \infty$. Furthermore, since $d_k(\bar{\lambda}_i) = 0$, it follows that

$$-k/\gamma = \bar{\lambda}_i^{d-1} - (d-1)\alpha\bar{\lambda}_i^{d-2} - (\bar{\lambda}_i - \rho)/\gamma^2. \tag{2.45}$$

Taking the $(d-1)^{\text{th}}$ root of both sides of (2.45) then yields

$$(-k/\gamma)^{\frac{1}{d-1}} = \bar{\lambda}_i \left(1+e_k\right)^{\frac{1}{d-1}}, \tag{2.46}$$

where

$$e_k \triangleq -\frac{(d-1)\alpha}{\bar{\lambda}_i} - \frac{\bar{\lambda}_i - \rho}{\gamma^2 \bar{\lambda}_i^{d-1}}. \tag{2.47}$$

Since as $k \to \infty$, e_k approaches zero, we use the binomial approximation $(1+e_k)^q \approx (1+qe_k)$, where $q = 1/(d-1)$, on (2.46), which yields

$$\begin{aligned}\left(-\frac{k}{\gamma}\right)^{\frac{1}{d-1}} &\approx \bar{\lambda}_i \left(1 - \frac{\alpha}{\bar{\lambda}_i} - \frac{\bar{\lambda}_i - \rho}{(r-1)\gamma^2 \bar{\lambda}_i^{d-1}}\right) \\ &= \bar{\lambda}_i - \alpha - \frac{\bar{\lambda}_i - \rho}{(d-1)\,\gamma^2 \bar{\lambda}_i^{d}}.\end{aligned} \tag{2.48}$$

As $k \to \infty$, $(\bar{\lambda}_i - \rho)/((d-1)\gamma^2\bar{\lambda}_i^d)$ approaches zero, and (2.48) implies that $(-k/\gamma)^{\frac{1}{d-1}} \approx \bar{\lambda}_i - \alpha$. Thus, for $i = 1, 2, \ldots, d-1$ and sufficiently large $k > 0$, solving for $\bar{\lambda}_i$ yields

$$\bar{\lambda}_i \approx (k/\gamma)^{\frac{1}{d-1}} e^{\jmath\phi_i} + \alpha = \lambda_i. \tag{2.49}$$

Next, since for $i = 1, 2, \ldots, d-1$, $d_k(\bar{\lambda}_i) = 0$ and $(k/\gamma)^{\frac{1}{d}} < |\bar{\lambda}_i - \rho| < (k/\gamma)^{\frac{1}{2}}$, it follows from (2.43) that sufficiently large $k > 0$,

$$\begin{aligned}\frac{p_c(\bar{\lambda}_i)}{G(\bar{\lambda}_i) z_c(\bar{\lambda}_i)} = \frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)} &\approx \frac{-k^2}{\bar{\lambda}_i - \rho + \gamma k} \\ &= \frac{-\hat{G}_k(\bar{\lambda}_i) p_c(\bar{\lambda}_i)}{z_c(\bar{\lambda}_i)},\end{aligned}$$

or equivalently $1 + G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i) \approx 0$. Thus, as $k \to \infty$, $d-1$ roots of $\tilde{p}_k(s)$ approach $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{d-1}$. Furthermore, as $k \to \infty$, it follows from (2.49) that, for $i = 1, 2, \ldots, d-1$, $\bar{\lambda}_i$ approaches λ_i. Therefore, as $k \to \infty$, $d-1$ roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d-1}$.

Next, to show that the one remaining root of $\tilde{p}_k(s)$ approaches λ_d, it follows from (2.4) that

$$\frac{\hat{G}_k(\lambda_d) p_c(\lambda_d)}{z_c(\lambda_d)} = \frac{k^2}{\lambda_d - \rho + \gamma k} = -(-\gamma k)^{d-1}. \tag{2.50}$$

Next, since as $k \to \infty$, $|\lambda_d| \to \infty$, it follows from (2.40) that for sufficiently large $k > 0$,

$$\frac{p(\lambda_d)}{z(\lambda_d)} \approx \lambda_d^{d-1} = (-\gamma k)^{d-1} (1 + g_k)^{d-1}, \tag{2.51}$$

where

$$g_k \triangleq -\frac{\rho}{\gamma k} + \frac{k^2}{(-\gamma k)^d}. \tag{2.52}$$

Since $d \geq 4$, it follows that, as $k \to \infty$, g_k approaches zero. Thus, as $k \to \infty$, (2.51) implies that $p(\lambda_d)/z(\lambda_d) \approx (-\gamma k)^{d-1}$, which combined with (2.50) yields

$$\frac{p_{\mathrm{c}}(\lambda_d)}{G(\lambda_d) z_{\mathrm{c}}(\lambda_d)} = \frac{p(\lambda_d)}{z(\lambda_d)} \approx (-\gamma k)^{d-1} = \frac{-\hat{G}_k(\lambda_d) p_{\mathrm{c}}(\lambda_d)}{z_{\mathrm{c}}(\lambda_d)}$$

or equivalently $1 + G(\lambda_d)\hat{G}_k(\lambda_d) \approx 0$. Thus, as $k \to \infty$, one root of $\tilde{p}_k(s)$ is approximated by λ_d, which confirms *(d)*. □

Proof of Fact 2.5. First, we show that real roots of $p(s)$ are not break-in or breakaway points. Let $\mu \in \mathbb{R}$ be a real root of $p(s)$. Since $p(\mu) = 0$ and $p(s)$ and $z(s)$ are coprime, it follows from (2.6) that $\tilde{p}_{k_\mu}(\mu) = 0$ if and only if $k_\mu = 0$. Thus, μ is not on the root locus, and it follows from Definition 1.2 that μ is not a break-in or breakaway point.

Next, let $\tau \in \mathbb{R}$ be on the root locus, and assume $t(\tau)p(\tau)z(\tau) \neq 0$. Since τ is on the root locus, it follows from (2.6) and (2.21) that $\kappa_1(\tau) > 0$ or $\kappa_2(\tau) > 0$. We consider three cases: (A) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) > 0$, (B) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) \leq 0$, and (C) $\kappa_1(\tau) \leq 0$ and $\kappa_2(\tau) > 0$. First, assume (A) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) > 0$. It follows that there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that: $\tau \in (a, b)$; there is at most one break-in or breakaway point on (a, b); for all $\sigma \in (a, b)$, $t(\sigma)p(\sigma)z(\sigma) \neq 0$; and for all $\sigma \in (a, b)$, $\kappa_1(\sigma) > 0$ and $\kappa_2(\sigma) > 0$. It follows from Definition 1.2 that τ is a break-in or breakaway point if and only if τ is the minimizer or maximizer of $\kappa_1(\sigma)$ or $\kappa_2(\sigma)$ on (a, b). Furthermore, τ is the minimizer or maximizer of $\kappa_1(\sigma)$ or $\kappa_2(\sigma)$ on (a, b)

if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$ or $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$, respectively. Thus, τ is a break-in or breakaway point if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$ or $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$.

Next, assume (B) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) \leq 0$. It follows that there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that: $\tau \in (a, b)$; there is at most one break-in or breakaway point on (a, b); for all $\sigma \in (a, b)$, $t(\sigma)p(\sigma)z(\sigma) \neq 0$; and for all $\sigma \in (a, b)$, $\kappa_1(\sigma) > 0$. It follows from Definition 1.2 that τ is a break-in or breakaway point if and only if τ is the minimizer or maximizer of $\kappa_1(\sigma)$ on (a, b). Furthermore, τ is the minimizer or maximizer of $\kappa_1(\sigma)$ on (a, b) if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$. Thus, τ is a break-in or breakaway point if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$.

Finally, assume (C) $\kappa_1(\tau) \leq 0$ and $\kappa_2(\tau) > 0$. Using the same argument as the previous case yields that τ is a break-in or breakaway point if and only if $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$. Combining these three cases yields that τ is a break-in or breakaway point if and only if *(a)* or *(b)* from Fact 2.5 holds. □

Proof of Fact 2.6. Assume τ is a break-in or breakaway point, and it follows from Definition 1.2 that τ is on the root locus. Since τ is a real root of $z(s)$ and on the root locus, it follows from Fact 2.8 that $\tau < \rho$ and $k_\tau = (\rho - \tau)/\gamma$ is the only root of $\tilde{p}_k(\tau)$. Thus, $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ.

Conversely, assume $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ. Since $\tau < \rho$ and $z(\tau) = 0$, Fact 2.8 implies that τ is on the root locus. Thus, Definition 1.2 implies that τ is a break-in or breakaway point. □

Proof of Fact 2.7. It follows from (2.7) that $\gamma^2 p(s) = t(s) + 4(s-\rho)z(s)$. Substituting $p(s) = (t(s) + 4(s-\rho)z(s))/\gamma^2$ into (2.6) and evaluating at $k = k_\tau = 2(\rho-\tau)/\gamma$ yields

$$\tilde{p}_{k_\tau}(s) = \left(4\,(s-\tau)^2\, z(s) + (2(\rho-\tau)+1)\, t(s)\right)/\gamma^2. \tag{2.53}$$

First, we show that *(a)* implies *(b)*. Assume τ is a break-in or breakaway point, and it follows from Definition 1.2 that τ is on the root locus. Since τ is a real root of $t(s)$ and on the root locus, it follows from Fact 2.9 that $\tau < \rho$ and k_τ is the only root of $\tilde{p}_k(\tau)$. Thus, $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ.

Next, we show that *(b)* implies *(c)*. Assume that $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ. It follows from (2.53) that that $t(s)$ has multiple roots at τ.

Finally, we show that *(c)* implies *(a)*. Assume $\tau < \rho$ and $t(s)$ has multiple roots at τ, and it follows from (2.53) that $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ. Since $\tau < \rho$ and $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ, it follows from Definition 1.2 that τ is a break-in or breakaway point. □

Proof of Fact 2.8. Since $z(\sigma) = 0$, it follows from (2.6) that $\tilde{p}_k(\sigma) = p(\sigma)(\sigma - \rho + \gamma k)$. Therefore, $\tilde{p}_k(\sigma) = 0$ if and only if $p(\sigma) = 0$ or $k = (\rho - \sigma)/\gamma$. Since $z(\sigma) = 0$ and $p(s)$ and $z(s)$ are coprime, it follows that $p(\sigma) \neq 0$. Thus, $\tilde{p}_k(\sigma) = 0$ if and only if $k = (\rho - \sigma)/\gamma$. Furthermore, $(\rho - \sigma)/\gamma > 0$ if and only if $\sigma < \rho$, which implies that σ is on the root locus if and only if $\sigma < \rho$. □

Proof of Fact 2.9. Since $t(\sigma) = 0$, it follows from (2.7) that $\gamma^2 p(\sigma) = 4(\sigma - \rho)z(\sigma)$. Substituting $p(\sigma) = 4(\sigma - \rho)z(\sigma)/\gamma^2$ into (2.6) yields

$$\begin{aligned}\tilde{p}_k(\sigma) &= k^2 z(\sigma) + \gamma k \left(\frac{4}{\gamma^2}(\sigma - \rho)z(\sigma)\right) \\ &\quad + (\sigma - \rho)\left(\frac{4}{\gamma^2}(\sigma - \rho)z(\sigma)\right) \\ &= z(\sigma)\left(k + \frac{2}{\gamma}(\sigma - \rho)\right)^2.\end{aligned}$$

Therefore, $\tilde{p}_k(\sigma) = 0$ if and only if $z(\sigma) = 0$ or $k = 2(\rho - \sigma)/\gamma$.

Next, we show by contradiction that $z(\sigma) \neq 0$. Assume that $z(\sigma) = 0$. Thus, $p(\sigma) = 4(\sigma - \rho)z(\sigma)/\gamma^2 = 0$, which is a contradiction, because $p(s)$ and $z(s)$ are coprime. Since $z(\sigma) \neq 0$, it follows that $\tilde{p}_k(\sigma) = 0$ if and only if $k = 2(\rho - \sigma)/\gamma$. Furthermore, $2(\rho - \sigma)/\gamma > 0$ if and only if $\sigma < \rho$, which implies that σ is on the root locus if and only if $\sigma < \rho$. □

Chapter 3 Root Locus for a Controller Class that Yields Cubic Gain Parameterization

This chapter presents rules for constructing the root locus for a class of linear feedback systems, where the closed-loop denominator polynomial is cubic in the root-locus parameter k. These cubic root-locus rules apply to a class of controllers that are rational functions of k and result in closed-loop denominator polynomials that are cubic in k. We provide controller design examples to demonstrate the cubic root locus. For example, we use the cubic root locus to high-gain stabilize the quadruple integrator; this is not possible with classical root locus. The results from this chapter have been submitted for publication in [23].

3.1 Introduction

In Chapter 2, quadratic root-locus rules are developed for a class of controllers that are rational functions of k and yield closed-loop denominator polynomials that are quadratic in k. The controller class considered in Chapter 2 features a numerator polynomial that is proportional to k^2, and a denominator polynomial that includes a pole, whose location is proportional to k. The rules in Chapter 2 can be used to draw the root locus, where the plant is $z(s)/p(s)$ and the controller is $k^2/(s+k)$. In this case, the closed-loop denominator polynomial is $k^2z(s) + kp(s) + sp(s)$, which is quadratic in k.

In this chapter, we extend the techniques of Chapter 2 to develop cubic root-locus rules for a class of controllers that are rational functions of k and that yield closed-loop denominator polynomials that are cubic in k. In particular, the controller class considered in this chapter features a numerator polynomial that is proportional to k^3, and a denominator polynomial that includes two poles, whose locations are proportional to k. For example, consider the plant $z(s)/p(s)$ and the controller $k^3/(s+k)^2$. In this case, the closed-loop denominator polynomial is $k^3z(s)+k^2p(s)+2ksp(s)+s^2p(s)$, which is cubic in k. The cubic root-locus rules developed in this chapter can be used to draw the root locus for $k^3z(s) + k^2p(s) + 2ksp(s) + s^2p(s)$. In fact, this chapter considers the cubic root locus for a more general class of controllers than the one described above. Controller structures that are not considered in this chapter also lead to closed-loop denominator polynomials that are cubic in k. The cubic root locus for other controller structures is discussed in the conclusion.

This chapter develops cubic root-locus rules that describe the behavior of closed-loop poles for small k, the behavior of closed-loop poles for large k, and the segments of the real axis that are on the root locus. The cubic root locus shares certain features with the classical root locus and the quadratic root locus of Chapter 2. For example, some points on the real axis obey the classical root-locus real-axis rule. However, the cubic root locus possesses other features that differ from both the classical root locus and the quadratic root locus. For example, consider a system that is relative degree four. Using classical root locus or quadratic root locus, two of the closed-loop poles diverge to infinity along asymptotes that are directed into the open-right-half complex plane. Thus, neither classical root locus nor quadratic root locus is high-gain stabilizing for systems that are relative degree four. In contrast, the cubic root locus can be high-gain stabilizing for minimum-phase systems that are relative degree one, two, three, or four. The relative-degree-four case is examined in more detail in Section 3.4, where we consider high-gain stabilization of the quadruple integrator.

3.2 Problem Formulation

Consider the single-input single-output linear time-invariant system

$$y(s) = \beta G(s)u(s), \tag{3.1}$$

where

$$G(s) \triangleq \frac{z_{\mathrm{p}}(s)}{p_{\mathrm{p}}(s)}, \tag{3.2}$$

where $u(s)$ is the input; $y(s)$ is the output; $\beta \in \mathbb{R}$; $z_{\mathrm{p}}(s)$ and $p_{\mathrm{p}}(s)$ are coprime monic polynomials; and deg $z_{\mathrm{p}}(s) <$ deg $p_{\mathrm{p}}(s)$. Consider the control

$$u(s) = \frac{1}{\beta}\hat{G}_k(s)(v(s) - y(s)), \tag{3.3}$$

where

$$\hat{G}_k(s) \triangleq \frac{k^3 z_{\mathrm{c}}(s)}{(s-\rho_1+\gamma_1 k)(s-\rho_2+\gamma_2 k)p_{\mathrm{c}}(s)}, \tag{3.4}$$

where $v(s)$ is an external signal; $\rho_1, \rho_2 \in \mathbb{R}$; $\gamma_1, \gamma_2 > 0$; $z_{\mathrm{c}}(s)$ and $p_{\mathrm{c}}(s)$ are monic polynomials, where deg $z_{\mathrm{c}}(s) \leq$ deg $p_{\mathrm{c}}(s) + 2$; and $z(s) \triangleq z_{\mathrm{p}}(s)z_{\mathrm{c}}(s)$ and $p(s) \triangleq p_{\mathrm{p}}(s)p_{\mathrm{c}}(s)$ are coprime. Without loss of generality, we assume $\rho_1 \leq \rho_2$.

The closed-loop system (3.1)–(3.4) is shown in Figure 1.1, and the closed-loop transfer function from v to y is given by

$$\tilde{G}_k(s) \triangleq \frac{G(s)\hat{G}_k(s)}{1+G(s)\hat{G}_k(s)} = \frac{k^3 z(s)}{\tilde{p}_k(s)}, \tag{3.5}$$

where

$$\tilde{p}_k(s) \triangleq k^3 z(s) + k^2\gamma_1\gamma_2 p(s) + k\left(\gamma_1(s-\rho_2)+\gamma_2(s-\rho_1)\right)p(s) + (s-\rho_1)(s-\rho_2)p(s). \tag{3.6}$$

Note that the closed-loop denominator polynomial is a cubic function of k. We use the classical definition of the positive root locus, which is given by Definition 1.1.

This chapter considers the cubic root locus for $k > 0$. The techniques in this chapter can also be used to develop root locus rules for $k < 0$.

3.3 Cubic Root-Locus Rules

In this section, we present four facts that characterize the cubic root locus. Proofs of these facts are provided in Section 3.6. Facts 3.1 and 3.2 define the root locus starting points for $k = 0$ and describe the root locus symmetry, respectively. These two facts are consistent with classical root locus.

Fact 3.1. *As $k \to 0$, the roots of $\tilde{p}_k(s)$ approach the roots of $(s-\rho_1)(s-\rho_2)p(s)$.*

Fact 3.2. *The root locus is symmetric about the real axis.*

Next, we present a rule to determine the points on the real axis that are on the root locus. We define

$$\begin{aligned} t(s) \overset{\triangle}{=} & \gamma_1^2\gamma_2^2(\gamma_1(s-\rho_2)-\gamma_2(s-\rho_1))^2 p(s)^2 \\ & - 27(s-\rho_1)^2(s-\rho_2)^2 z(s)^2 \\ & - 2[\gamma_1(s-\rho_2)+\gamma_2(s-\rho_1)][\gamma_1(s-\rho_2)-2\gamma_2(s-\rho_1)] \\ & \times [2\gamma_1(s-\rho_2)-\gamma_2(s-\rho_1)]p(s)z(s), \end{aligned} \tag{3.7}$$

and note that $t(s)p(s)^2$ is the cubic discriminant of $\tilde{p}_k(s)$ with respect to k. See [24, pp. 153-154] for more information on the discriminant of a cubic polynomial. The behavior of classical root locus on the real axis depends on the roots of $p(s)$ and $z(s)$ only. In contrast, the real axis rule for the cubic root locus depends on the roots of $p(s)$, $z(s)$ and $t(s)$; the leading coefficient of $t(s)$; and ρ_1 and ρ_2.

Fact 3.3. *Let $\sigma \in \mathbb{R}$. Then σ is on the root locus if and only if any of the following statements hold:*

(a) $\sigma \geq \rho_2$ *and* $p(\sigma)z(\sigma) < 0$.

(b) $\sigma < \rho_2$ *and* $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$.

(c) $\sigma < \rho_2$, $p(\sigma) \neq 0$ *and* $t(\sigma) \geq 0$.

Part *(a)* of Fact 3.3 implies that the real axis to the right of ρ_2 is consistent with classical root locus. Specifically, $\sigma \geq \rho_2$ is on the root locus if and only if σ lies to the left of an odd number of real roots of $p(s)z(s)$.

Parts *(b)* and *(c)* of Fact 3.3 imply that the real axis rule to the left of ρ_2 differs from classical root locus. Part *(b)* of Fact 3.3 implies that the real axis behavior differs to the right and left of ρ_1. Specifically, if σ lies between ρ_1 and ρ_2, and σ lies to the left of an even number of real roots of $p(s)z(s)$, then σ is on the root locus. If, on the other hand, $\sigma < \rho_1$ lies to the left of an odd number of real roots of $p(s)z(s)$, then σ is on the root locus.

Part *(c)* of Fact 3.3 implies that additional values of $\sigma < \rho_2$ are on the root locus. Part *(c)* of Fact 3.3 implies that real zeros (i.e., real roots of $z(s)$) which lie to the left of ρ_2 can be closed-loop poles (i.e., roots of $\tilde{p}_k(s)$) for finite $k > 0$. In fact, it follows from (3.7) and part *(c)* of Fact 3.3 that all real roots of $z(s)$ that lie to the left of ρ_2 are on the root locus.

We now describe the asymptotic properties of the root locus, that is, the properties for sufficiently large $k > 0$. We define

$$n \triangleq \deg p(s) + 2, \qquad m \triangleq \deg z(s), \qquad d \triangleq n - m. \tag{3.8}$$

Furthermore, let $z_1, z_2, \ldots, z_m$ be the roots of $z(s)$, and let $p_1, p_2, \ldots, p_{n-2}$ be the roots of $p(s)$. Fact 3.4 characterizes the asymptotic properties of this controller, where γ_1 and γ_2 are equal.

Fact 3.4. *Let* $\gamma_1 = \gamma_2 = \gamma$, *where* $\gamma > 0$. *As* $k \to \infty$, m *roots of* $\tilde{p}_k(s)$ *converge to* *the roots of $z(s)$, and the d remaining roots satisfy the following statements:*

(a) If $d = 1$, then the remaining root of $\tilde{p}_k(s)$ is approximated by

$$\lambda_1 \triangleq -k^3 - \gamma k + \rho_1. \tag{3.9}$$

(b) If $d = 2$, then the two remaining roots of $\tilde{p}_k(s)$ are approximated by

$$\lambda_1 \triangleq -\gamma k + \frac{(\rho_1 + \rho_2)}{2} + \jmath\sqrt{k^3 - \frac{(\rho_1 - \rho_2)^2}{4}}, \tag{3.10}$$

$$\lambda_2 \triangleq -\gamma k + \frac{(\rho_1 + \rho_2)}{2} - \jmath\sqrt{k^3 - \frac{(\rho_1 - \rho_2)^2}{4}}. \tag{3.11}$$

(c) If $d = 3$ and $\gamma \geq \sqrt[3]{27/4}$, then the three remaining roots of $\tilde{p}_k(s)$ are approximated by λ_1, λ_2 and λ_3, where for $i = 1, 2, 3$,

$$\lambda_i \triangleq -\frac{2\gamma k}{3}\left(1 - \cos\left(\frac{\phi + 2\pi i}{3}\right)\right), \tag{3.12}$$

where

$$\phi \triangleq \cos^{-1}\left(1 - \frac{27}{2\gamma^3}\right). \tag{3.13}$$

(d) If $d = 3$ and $\gamma < \sqrt[3]{27/4}$, then the three remaining roots of $\tilde{p}_k(s)$ are approximated by

$$\lambda_1 \triangleq -k\left(2\gamma/3 - x - w\right), \tag{3.14}$$

$$\lambda_2 \triangleq -k(x + w + \sqrt{xw})e^{\jmath\theta}, \tag{3.15}$$

$$\lambda_3 \triangleq -k(x + w + \sqrt{xw})e^{-\jmath\theta}, \tag{3.16}$$

where

$$\theta \triangleq \arg\left(\frac{x + w + 4\sqrt{xw}}{2} - \jmath\frac{\sqrt{3}(x - w)}{2}\right), \tag{3.17}$$

$$x \triangleq \sqrt[3]{(2\gamma^3 - 27)/54 + \sqrt{1/4 - \gamma^3/27}}, \tag{3.18}$$

$$w \triangleq \sqrt[3]{(2\gamma^3 - 27)/54 - \sqrt{1/4 - \gamma^3/27}}. \tag{3.19}$$

(e) If $d = 4$, then the four remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \lambda_3,$ and λ_4, where for $i = 1, 2,$

$$\lambda_i \triangleq \left(\frac{k^2}{\gamma^2 k + \operatorname{sgn}(\rho_1 + \rho_2)\gamma|\rho_1 - \rho_2|} \right)^{\frac{1}{2}} e^{\jmath\theta_i} + \alpha, \tag{3.20}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{2}, \tag{3.21}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n-2} p_j - \sum_{j=1}^{m} z_j + 2\gamma^{-3}}{2}; \tag{3.22}$$

and for $i = 3, 4$,

$$\lambda_i \triangleq -\gamma k + (-1)^i \jmath \sqrt{k}/\gamma. \tag{3.23}$$

(f) If $d \geq 5$, then the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d-2$,

$$\lambda_i \triangleq \left(\frac{k^2}{\gamma^2 k + \operatorname{sgn}(\rho_1 + \rho_2)\gamma|\rho_1 - \rho_2|} \right)^{\frac{1}{d-2}} e^{\jmath\theta_i} + \alpha,$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d - 2}, \tag{3.24}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n-2} p_j - \sum_{j=1}^{m} z_j}{d - 2}; \tag{3.25}$$

and for $i = d-1, d,$

$$\lambda_i \triangleq -\gamma k + \frac{\rho_1 + \rho_2}{2} + (-1)^i \frac{\rho_1 - \rho_2}{2}. \tag{3.26}$$

Part *(b)* of Fact 3.4 implies that if $d = 2$, then the real parts of the two excess roots of $\tilde{p}_k(s)$ approach minus infinity. Part *(c)* of Fact 3.4 implies that that if $d = 3$ and $\gamma \geq (27/4)^{1/3} \approx 1.78$, then the three excess roots of $\tilde{p}_k(s)$ all approach minus infinity along the real axis. In contrast, part *(d)* of Fact 3.4 implies that if $d = 3$ and $\gamma < (27/4)^{1/3}$, then one excess root of $\tilde{p}_k(s)$ approaches minus infinity along the real axis and the two remaining excess roots tend to infinity along asymptotes, whose angles are determined by γ. Furthermore, part *(d)* of Fact 3.4 implies that if $\gamma > 0.794$, then $\theta > \pi/2$ and thus the asymptotes are directed into the open-left-half complex plane. Therefore, parts *(a)*–*(d)* of Fact 3.4 imply that for $d \leq 3$, the d excess roots of $\tilde{p}_k(s)$ all diverge to infinity through the open-left-half complex plane, provided that the controller design parameter $\gamma > 0.794$ for the $d = 3$ case.

Part *(e)* of Fact 3.4 implies that if $d = 4$, then as k tends to infinity, two roots of $\tilde{p}_k(s)$ tend to infinity along the asymptotes centered at α with angles $\pi/2$ and $3\pi/2$, and the two remaining roots of $\tilde{p}_k(s)$ tend to infinity through the open-left-half complex plane at angles that tend to π. Thus, if $\alpha < 0$ and $z(s)$ is asymptotically stable, then the closed-loop transfer function $\tilde{G}_k(s)$, given by (3.5), is high-gain stable. Finally, part *(f)* of Fact 3.4 implies that if $d \geq 5$, then two excess roots of $\tilde{p}_k(s)$ approach minus infinity along the real axis, and the $d-2$ remaining roots of $\tilde{p}_k(s)$ obey the classical root locus rule for relative degree $d-2$. More specifically, these $d-2$ roots of $\tilde{p}_k(s)$ diverge to infinity along asymptotes centered at α with angles $\theta_1, \theta_2, \ldots, \theta_{d-2}$.

3.4 Numerical Examples

In classical root locus, systems where the relative degree exceeds two are not high-gain stabilizable, that is, stable for sufficiently large $k > 0$. In contrast, cubic root locus can high-gain stabilize minimum-phase systems that are relative degree one, two, three, or four. We consider the quadruple integrator, which is relative degree four.

Example 3.1. Consider the system $y(s) = G(s)u(s)$, where $G(s) = 1/s^4$, which is minimum phase and relative degree 4. Part *(e)* of Fact 3.4 states that the asymptote angles for $d = 4$ are $-\pi/2$, $\pi/2$, π, and π. Thus, if $d = 4$, $z_c(s)$ is asymptotically stable and $\alpha < 0$, then the closed-loop transfer function $\tilde{G}_k(s)$, given by (3.5), is high-gain stable. We let $p_c(s) = s+40$, $z_c(s) = (s+8)(s+9)(s+10)$ and $\gamma_1 = \gamma_2 = 1$, and it follows that $\alpha = -5.5$ and $d = 4$. Next, we let $\rho_1 = \rho_2 = -5$. Note that ρ_1 and ρ_2 do not impact the asymptote center α or asymptote angles for $d = 4$. Thus, the controller (3.3), where $\hat{G}_k(s) = k^3(s+8)(s+9)(s+10)/((s+5+k)^2(s+40))$, high-gain stabilizes the closed-loop transfer function $\tilde{G}_k(s)$, where $G(s)$ is the quadruple integrator.

To draw the cubic root locus, we apply the rules from Section 3.3. Fact 3.1 implies that the root locus begins at the roots of $(s+5)(s+5)p(s)$. Next, Fact 3.3 specifies the points on the real axis that are on the root locus. Since $p(s)z(s)$ has no roots in the open-right-half complex plane, part *(a)* of Fact 3.3 implies $(0, \infty)$ is not on the root locus. Since $\rho_2 = -5$ and $[-5, 0)$ is to the left of an even number of real roots of $p(s)z(s)$, part *(a)* of Fact 3.3 implies that $[-5, 0)$ is not on the root locus. Now, parts *(b)* and *(c)* of Fact 3.3 are used to determine the real values to the left of ρ_2 that are on the root locus. Part *(b)* of Fact 3.3 implies that $(-40, -10)$ and $(-9, -8)$ are on the root locus. Next, note that $t(s) = 4(s-11.89)(s+2.98)(s+5)(s+8)(s+9)(s+10)(s+37.44)(s+2.36+\imath 3.57)(s+2.36-\imath 3.57)(s+5+\imath 2.1\times 10^{-4})(s+5-\imath 2.1\times 10^{-4})$.

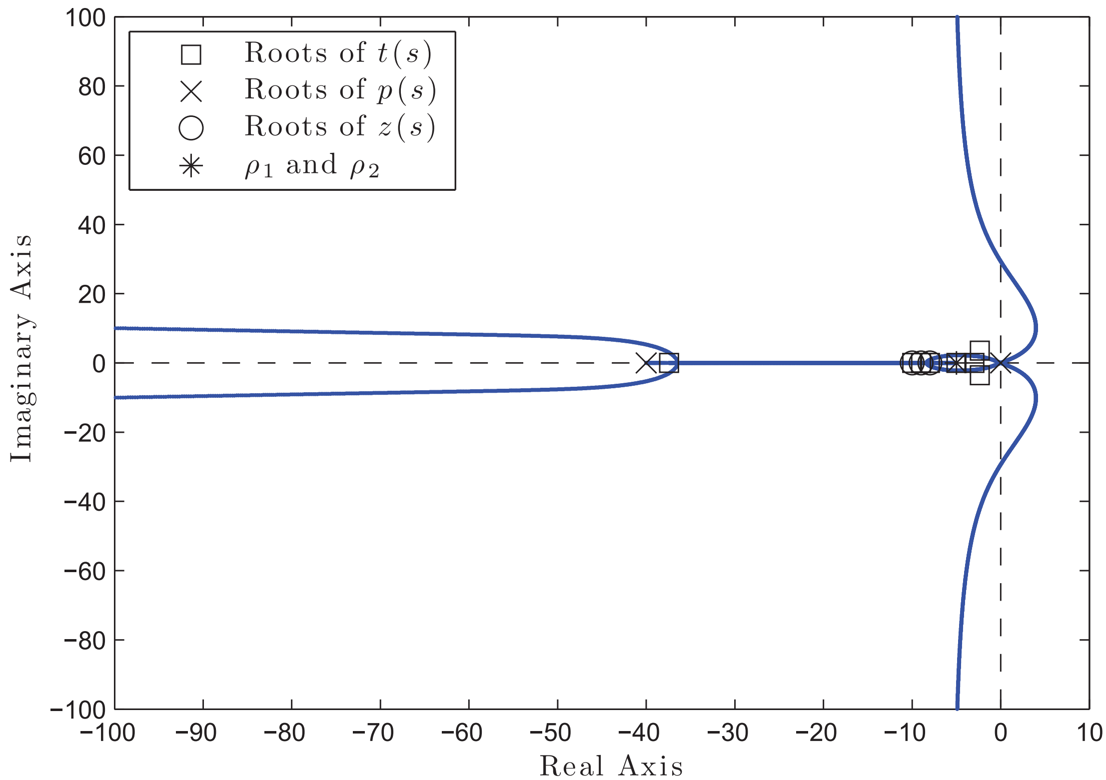

Figure 3.1: The cubic root locus shows that the quadruple integrator $G(s) = 1/s^4$ is high-gain stabilized by the controller $\hat{G}_k(s) = k^3(s+7)(s+8)(s+9)/((s+5+k)^2(s+40))$. In fact, the closed-loop system is asymptotically stable for all $k > 10.6 \times 10^6$.

Thus, part *(c)* of Fact 3.3 implies that $[-37.44, -10]$ and $[-9, -8]$ are on the root locus. Combining parts *(b)* and *(c)* of Fact 2.3 yields that $(-40, -10]$ and $[-9, -8]$ is on the root locus. The cubic root locus is plotted in Figure 3.1, and the plot confirms that the closed-loop transfer function $\tilde{G}_k(s)$ is high-gain stable. In fact, $\tilde{G}_k(s)$ is asymptotically stable for all $k > 10.6 \times 10^6$. △

In classical root locus, it is not possible to high-gain stabilize the two-degree-of-freedom mass-spring-dashpot system shown in Figure 3.2. However, in cubic root locus, it is possible to high-gain stabilize this system, which is demonstrated in the following example.

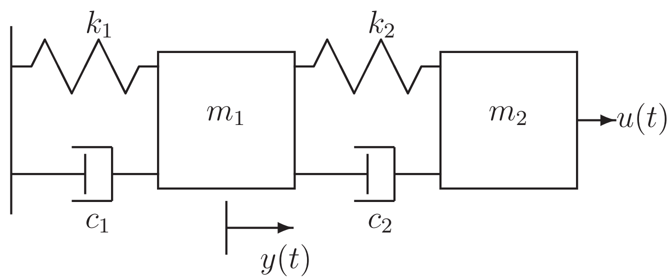

Figure 3.2: The two-degree-of-freedom mass-spring-dashpot system.

Example 3.2. Consider the two-degree-of-freedom mass-spring-dashpot system shown in Figure 3.2, where $u(t)$ is the force input on m_2 and $y(t)$ is the position of m_1. Let $m_1 = 0.5$ kg, $m_2 = 0.2$ kg, $c_1 = 8$ kg/s, $c_2 = 5$ kg/s, $k_1 = 10$ kg/s^2, and $k_2 = 15$ kg/s^2. Thus, the open-loop system from $u(t)$ to $y(t)$ is (3.1), where $\beta = 50$ and $G(s) = (s+3)/((s+38.5)(s+7.4)(s+3.7)(s+1.4))$, which is minimum-phase and relative degree three. Since $G(s)$ is relative degree three, classical root locus cannot high-gain stabilize $G(s)$. Specifically, as k increases, two closed-loop poles tend to infinity through the open-right-half complex plane.

Next, we consider the controller (3.4), where $\hat{G}_k(s) = k^3(s+10)^3/(s(s+7k)^2)$. Note that $\rho_1 = \rho_2 = 0$ and $\gamma_1 = \gamma_2 = 7$, and $\hat{G}_k(s)$ incorporates an integrator. It follows from Fact 3.3 that $(-\infty, -37.4)$, $[-10, -7.4)$ and $(-3.7, -3]$ are on the root locus.

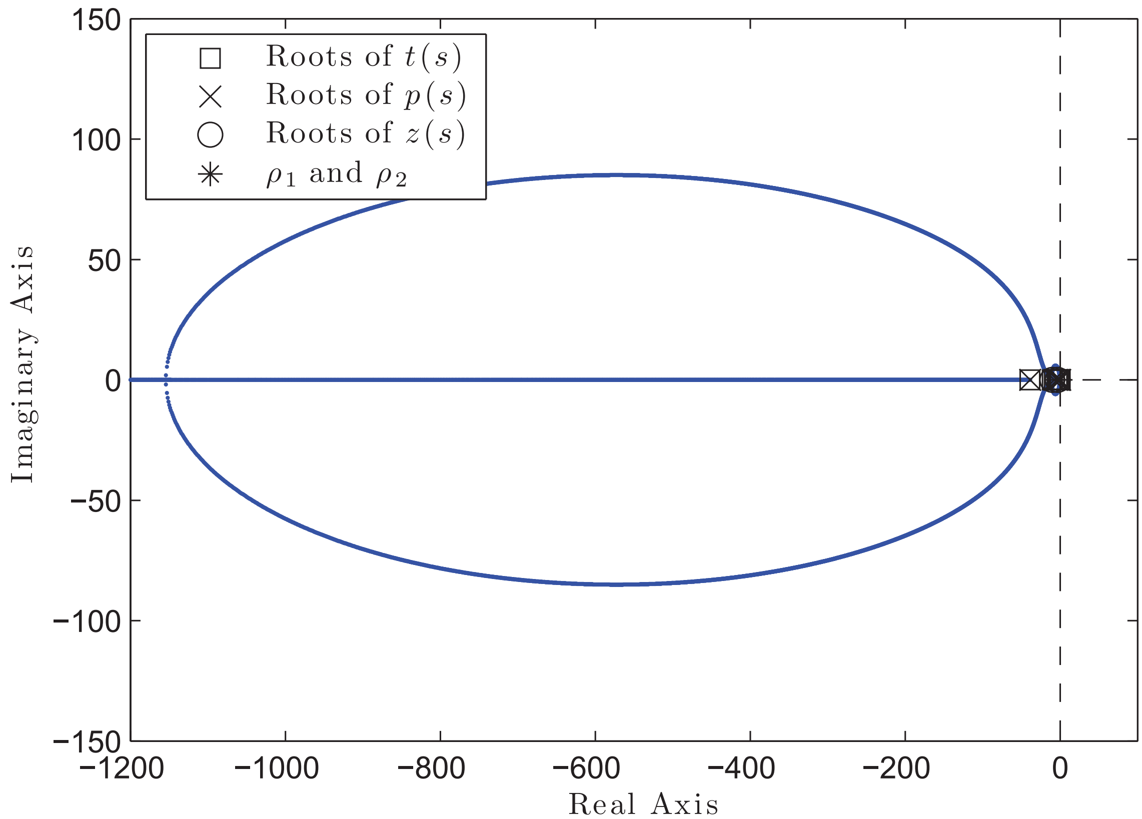

Figure 3.3: The cubic root locus shows that $G(s) = (s+3)/((s+38.5)(s+7.4)(s+3.7)(s+1.4))$ is high-gain stabilized by the controller $\hat{G}_k(s) = k^3(s+10)^3/(s(s+10k)^2)$. As k tends to infinity, three closed-loop poles tend to minus infinity.

Since $d = 3$ and $\gamma = 7 \geq \sqrt[3]{27/4}$, it follows from part *(c)* of Fact 3.4 that the three excess roots of $\tilde{p}_k(s)$ tend to minus infinity along the real axis.

The cubic root locus is shown in Figures 3.3 and 3.4, which demonstrates that, as k tends to infinity, four closed-loop poles tend to the roots of $z(s)$, which are in the open-left-half complex plane, and three remaining closed-loop poles tend to minus infinity as indicated by part *(c)* of Fact 2.4. Thus, $\tilde{G}_k(s)$ is high-gain stable. Furthermore, since $\hat{G}_k(s)$ incorporates an integrator, it follows that the closed-loop system has zero steady-state error to a step input, as shown in Figure 3.5. $\triangle$

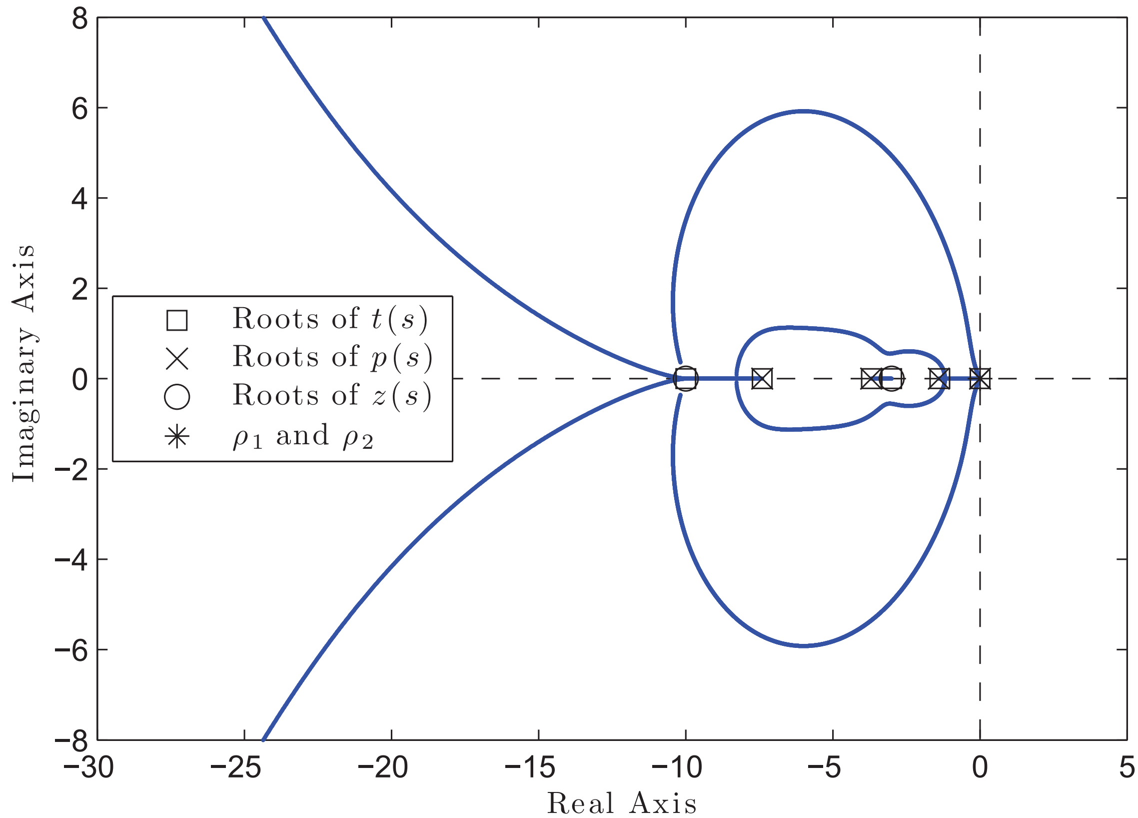

Figure 3.4: A close-up view of the cubic root locus, shown in Figure 3.3, demonstrates that four closed-loop poles converge to the roots of $z(s)$, which are -3, -10, -10, and -10.

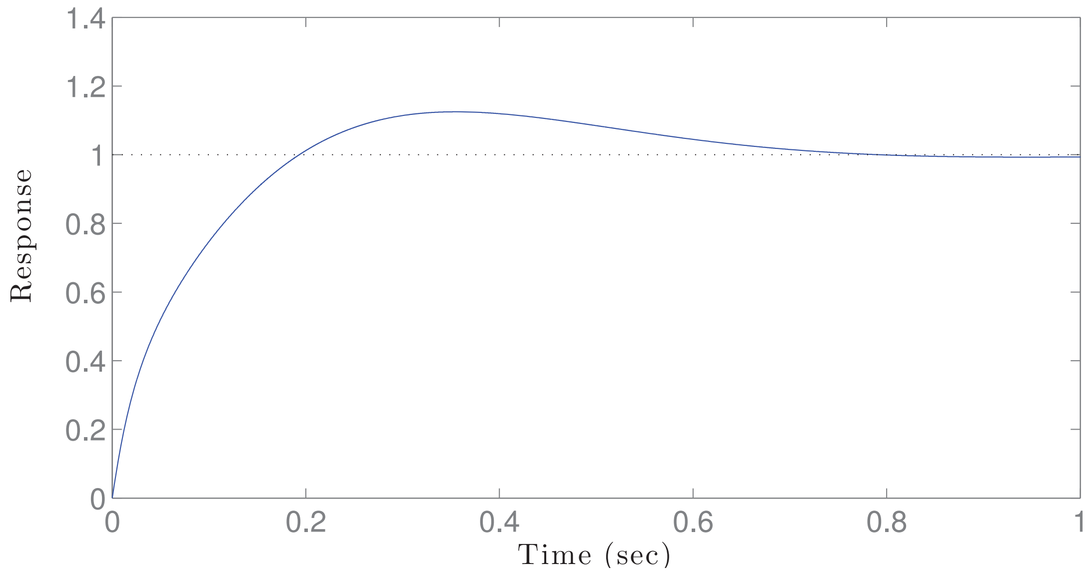

Figure 3.5: The step response of $G(s) = (s+3)/((s+38.5)(s+7.4)(s+3.7)(s+1.4))$ with the controller $\hat{G}_k(s) = k(s+10)^3/(s(s+7k)^2)$, where $k = 1000$ has zero steady-state error to a step command.

3.5 Conclusions

We presented rules for constructing the root locus for a class of controllers that result in closed-loop denominator polynomials that are cubic in the root-locus parameter k. The cubic root-locus rules apply to a controller class that is rational in k.

To develop the cubic root locus, we extended the techniques of Chapter 2. In principle, the techniques of this chapter and Chapter 2 could be extended further to address a class of controllers that result in closed-loop denominator polynomials that are quartic (i.e., 4th order) in k. However, developing root-locus rules for the quartic case may rely on the solution to the quartic equation. The quadratic and cubic root-locus rules do not explicitly use the solutions to quadratic and cubic equations, but the rules are developed using the solutions to quadratic and cubic equations. Moreover, the quadratic and cubic root-locus rules use the quadratic and cubic discriminants, respectively. For example, the real axis rule (i.e., Fact 3.3) for the cubic root locus relies on $t(s)$, which along with $p(s)$ determines the cubic discriminant of $\tilde{p}_k(s)$ with respect to k. Thus, we anticipate that extending the techniques of this chapter to the quartic case may rely on the solution to the quartic equation.

The techniques used in this chapter to develop the real axis rule (i.e., Fact 3.3) can be generalized to other controller structures that result in closed-loop denominator polynomials that are cubic in k. Specifically, Fact 3.3 is developed by using the cubic-polynomial results in Appendix B to determine the real values of σ such that $\tilde{p}_k(\sigma)$, which is cubic in k, has at least one positive root. These values of σ are on the root locus. This technique can be applied to any closed-loop denominator polynomial that is quadratic in k, and can thus be generalized to other controller structures. In contrast, the techniques used to develop the asymptotic properties in Fact 3.4 cannot be directly extended to other controller structures. Generalizing Fact 3.4 to other control structures is an open problem.

3.6 Proofs for Facts 3.1–3.4

Proof of Fact 3.1. If $k = 0$, then $\tilde{p}_k(s) = (s - \rho_1)(s - \rho_2)p(s)$. □

Proof of Fact 3.2. Since for all $k > 0$, $\tilde{p}_k(s)$ has real coefficients, it follows that the roots of $\tilde{p}_k(s)$ are either on the real axis or occur in complex conjugate pairs. □

Proof of Fact 3.3. First, we show that real roots of $p(s)$ are not on the root locus. Assume $p(\sigma) = 0$, and it follows from (3.6) that $\tilde{p}_k(\sigma) = k^3 z(\sigma)$. Since $p(s)$ and $z(s)$ are coprime, it follows that $z(\sigma) \neq 0$. Thus, there does not exist $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, and σ is not on the root locus.

Next, define

$$a_3 \triangleq z(\sigma), \tag{3.27}$$

$$a_2 \triangleq \gamma_1\gamma_2 p(\sigma), \tag{3.28}$$

$$a_1 \triangleq (\gamma_1(\sigma - \rho_2) + \gamma_2(\sigma - \rho_1))p(\sigma), \tag{3.29}$$

$$a_0 \triangleq (\sigma - \rho_1)(\sigma - \rho_2)p(\sigma), \tag{3.30}$$

and note from (3.6) that $\tilde{p}_k(\sigma) = a_3k^3 + a_2k^2 + a_1k + a_0$. Next, it follows that

$$a_3a_2 = \gamma_1\gamma_2 p(\sigma)z(\sigma), \tag{3.31}$$

$$a_3a_1 = (\gamma_1(\sigma - \rho_2) + \gamma_2(\sigma - \rho_1))p(\sigma)z(\sigma), \tag{3.32}$$

$$a_3a_0 = (\sigma - \rho_1)(\sigma - \rho_2)p(\sigma)z(\sigma), \tag{3.33}$$

$$a_2a_1 = \gamma_1\gamma_2(\gamma_1(\sigma - \rho_2) + \gamma_2(\sigma - \rho_1))p(\sigma)^2, \tag{3.34}$$

$$a_2a_0 = \gamma_1\gamma_2(\sigma - \rho_1)(\sigma - \rho_2)p(\sigma)^2, \tag{3.35}$$

$$a_1a_0 = (\gamma_1(\sigma - \rho_2) + \gamma_2(\sigma - \rho_1))(\sigma - \rho_1)(\sigma - \rho_2)p(\sigma)^2, \tag{3.36}$$

and we define

$$D \triangleq t(\sigma)p(\sigma)^2, \tag{3.37}$$

which is the cubic discriminant of $\tilde{p}_k(\sigma)$ with respect to k.

We show that *(a)*, *(b)*, or *(c)* is necessary for σ to be on the root locus. Assume σ is on the root locus, and consider two cases: (A) $z(\sigma) = 0$ and (B) $z(\sigma) \neq 0$. First, assume (A) $z(\sigma) = 0$, which implies that $\tilde{p}_k(\sigma) = a_2k^2 + a_1k + a_0$. Furthermore, since $z(\sigma) = 0$, it follows from (3.7) that $t(\sigma) \geq 0$. We consider three cases: (A.I) $a_2 = 0$, (A.II) $a_0 = 0$ and (A.III) $a_2 \neq 0$ and $a_0 \neq 0$. First, assume (A.I) $a_2 = 0$, which implies that $\tilde{p}_k(\sigma) = a_1k + a_0$. Since there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, it follows that $a_1a_0 < 0$. Furthermore, since $a_1a_0 < 0$ and $\rho_1 \leq \rho_2$, it follows from (3.36) that $\sigma < \rho_2$. Next, assume (A.II) $a_0 = 0$, which implies that $\tilde{p}_k(\sigma) = k(a_2k + a_1)$. Since there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, it follows that $a_2a_1 < 0$. Furthermore, since $a_2a_1 < 0$ and $\rho_1 \leq \rho_2$, it follows from (3.34) that $\sigma < \rho_2$. Finally, assume (A.III) $a_2 \neq 0$ and $a_0 \neq 0$. Since σ is on the root locus, parts *(a)*–*(c)* of Lemma 1 in Appendix A imply that either $a_2a_1 < 0$ or $a_2a_0 < 0$. First, assume $a_2a_1 < 0$, and since $\rho_1 \le \rho_2$, it follows from (3.34) that $\sigma < \rho_2$. Next, assume $a_2a_0 < 0$, and since $\rho_1 \le \rho_2$, it follows from (3.35) that $\sigma < \rho_2$. Thus, (A.I), (A.II) or (A.III) imply that

$$\sigma < \rho_2, \quad t(\sigma) \ge 0. \tag{3.38}$$

Next, assume (B) $z(\sigma) \ne 0$, which implies that $a_3 \ne 0$. We consider five mutually exclusive and collectively exhaustive cases: (B.I) $\sigma > \rho_2$, (B.II) $\sigma = \rho_2$ and $\sigma \ne \rho_1$, (B.III) $\sigma = \rho_2$ and $\sigma = \rho_1$, (B.IV) $\sigma < \rho_2$ and $\sigma \ne \rho_1$, and (B.V) $\sigma < \rho_2$ and $\sigma = \rho_1$. First, assume (B.I) $\sigma > \rho_2$. Since $\gamma_1 > 0$, $\gamma_2 > 0$ and $\sigma > \rho_2 \ge \rho_1$, it follows from (3.28)–(3.30) that a_2, a_1 and a_0 have the same sign. Next, assume for contradiction that $a_3a_0 > 0$. Lemma 2 in Appendix B implies that $\tilde{p}_k(\sigma)$ does not have exactly one positive root, and Lemma 4 in Appendix B implies that $\tilde{p}_k(\sigma)$ does not have three positive roots. Since σ is on the root locus, it follows that $\tilde{p}_k(\sigma)$ has exactly two positive roots. Finally, it follows from *(ii)* in Lemma 3 of Appendix B that a_3, a_2, a_1, and a_0 do not have the same sign, which is a contradiction. Therefore, $a_3a_0 \le 0$, which implies $a_3a_0 < 0$ because $a_3 \ne 0$ and $a_0 \ne 0$. Since $a_3a_0 < 0$ and $\sigma > \rho_2 \ge \rho_1$, it follows from (3.33) that $p(\sigma)z(\sigma) < 0$. Thus, (B.I) implies that

$$\sigma > \rho_2, \quad p(\sigma)z(\sigma) < 0. \tag{3.39}$$

Next, assume (B.II) $\sigma = \rho_2$ and $\sigma \ne \rho_1$, which implies that $\tilde{p}_k(\sigma) = k(a_3k^2 + a_2k + a_1)$. Since $\tilde{p}_k(\sigma)$ has at least one positive root, it follows that a_3, a_2 and a_1 do not all have the same sign. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma > \rho_1$, and $\sigma = \rho_2$, it follows from (3.34) that $a_2a_1 > 0$. Therefore, a_3 has a different sign from a_2 and a_1, which implies that $a_3a_2 < 0$, and thus (3.31) implies that $p(\sigma)z(\sigma) < 0$. Therefore, (B.II) implies that

$$\sigma = \rho_2, \quad p(\sigma)z(\sigma) < 0. \tag{3.40}$$

Next, assume (B.III) $\sigma = \rho_1$ and $\sigma = \rho_2$, which implies that $\tilde{p}_k(\sigma) = k^2(a_3k + a_2)$. Since $\tilde{p}_k(\sigma)$ has at least one positive root, it follows that $a_3a_2 < 0$, and thus (3.31) implies that $p(\sigma)z(\sigma) < 0$. Therefore, (B.III) implies that

$$\sigma = \rho_2, \quad p(\sigma)z(\sigma) < 0. \tag{3.41}$$

Next, assume (B.IV) $\sigma < \rho_2$ and $\sigma \neq \rho_1$, and it follows from (3.30) that $a_0 \neq 0$. We consider three cases: (B.IV.a) $\tilde{p}_k(\sigma)$ has exactly one positive root, (B.IV.b) $\tilde{p}_k(\sigma)$ has three positive roots, and (B.IV.c) $\tilde{p}_k(\sigma)$ has exactly two positive roots. First, assume (B.IV.a) $\tilde{p}_k(\sigma)$ has exactly one positive root. Thus, Lemma 2 in Appendix B implies that $a_3a_0 < 0$. Since $\sigma < \rho_2$ and $a_3a_0 < 0$, it follows from (3.33) that $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$. Thus, (B.IV.a) implies that

$$\sigma < \rho_2, \quad (\sigma - \rho_1)p(\sigma)z(\sigma) > 0. \tag{3.42}$$

Next, assume (B.IV.b) $\tilde{p}_k(\sigma)$ has three positive roots. Thus, Lemma 4 in Appendix B implies that $D \geq 0$, $a_3a_2 < 0$ and $a_3a_0 < 0$. Since $\gamma_1\gamma_2 > 0$ and $a_3a_2 < 0$, it follows from (3.31) that $p(\sigma)z(\sigma) < 0$. Furthermore, since $\sigma < \rho_2$, $p(\sigma)z(\sigma) < 0$ and $a_3a_0 < 0$, it follows from (3.33) that $\sigma < \rho_1$. Thus, (B.IV.b) implies that

$$\sigma < \rho_2, \quad (\sigma - \rho_1)p(\sigma)z(\sigma) > 0. \tag{3.43}$$

Next, assume (B.IV.c) $\tilde{p}_k(\sigma)$ has exactly two positive roots. Therefore, Lemma 3 in Appendix B implies that $D \geq 0$ and $a_3a_0 > 0$. Since $D \geq 0$, it follows from (3.37) that $t(\sigma) \geq 0$. Thus, (B.IV.c) implies that

$$\sigma < \rho_2, \quad t(\sigma) \geq 0. \tag{3.44}$$

Finally, assume (B.V) $\sigma < \rho_2$ and $\sigma = \rho_1$, which implies that $\tilde{p}_k(\sigma) = k(a_3k^2 + a_2k + a_1)$, where it follows from (3.29) that $a_1 \neq 0$. Since $\tilde{p}_k(\sigma)$ has at least one positive root, it follows that the two nonzero roots of $\tilde{p}_k(\sigma)$ are real. Thus, the discriminant of $a_3k^2 + a_2k + a_1$ is nonnegative, which implies that $a_2^2 - 4a_3a_1 \geq 0$. Since $\sigma = \rho_1$, it follows from (3.7) that

$$\begin{aligned} t(\sigma) &= (\gamma_1(\sigma - \rho_2))^2 \left(\gamma_1^2\gamma_2^2 p(\sigma)^2 - 4\gamma_1(\sigma - \rho_2)p(\sigma)z(\sigma)\right) \\ &= (\gamma_1(\sigma - \rho_2))^2 \left(a_2^2 - 4a_3a_1\right), \end{aligned}$$

which is nonnegative. Thus, (B.V) implies that

$$\sigma < \rho_2, \quad t(\sigma) \geq 0. \tag{3.45}$$

We now use cases (A.I)–(A.III) and (B.I)–(B.V) to show that *(a)*, *(b)* or *(c)* holds. First, case (B.I), (B.II) or (B.III) implies (3.39), (3.40) or (3.41), respectively, which implies *(a)*. Case (B.IV.a) or (B.IV.b) implies (3.42) or (3.43), respectively, which implies *(b)*. Case (A), (B.IV.c) or (B.V) implies (3.38), (3.44) or (3.45), respectively, which because $p(\sigma) \neq 0$ implies *(c)*.

Conversely, assume *(a)*, *(b)* or *(c)* holds. First, assume *(a)* holds, and consider two cases: (C.I) $\sigma > \rho_2$ and (C.II) $\sigma = \rho_2$. First, assume (C.I) $\sigma > \rho_2$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma > \rho_2 \geq \rho_1$, and $p(\sigma)z(\sigma) < 0$, it follows from (3.31), (3.33) and (3.34) that $a_3a_0 < 0$, $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 > 0$. Thus, part *(c)* of Lemma 2 in Appendix B implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

Next, assume (C.II) $\sigma = \rho_2$, which implies that $\tilde{p}_k(\sigma) = k(a_3k^2 + a_2k + a_1)$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma \geq \rho_1$, $\sigma = \rho_2$, and $p(\sigma)z(\sigma) < 0$, it follows from (3.32) that $a_3a_1 \leq 0$. We consider two cases: (C.II.a) $a_3a_1 < 0$, and (C.II.b) $a_3a_1 = 0$. First, assume (C.II.a) $a_3a_1 < 0$, and part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus. Next, assume (C.II.b) $a_3a_1 = 0$. Since $a_3 \neq 0$, it follows that $a_1 = 0$, which implies that $\tilde{p}_k(\sigma) = k^2(a_3k + a_2)$. Since $\gamma_1\gamma_2 p(\sigma)z(\sigma) < 0$, it follows from (3.31) that $a_3a_2 < 0$, which implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

Next, assume *(b)* holds, which implies that $\sigma \neq \rho_1$. We consider two cases: (D.I) $\sigma > \rho_1$ and (D.II) $\sigma < \rho_1$. First, assume (D.I) $\sigma > \rho_1$. Since $\sigma > \rho_1$ and $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$, it follows that $p(\sigma)z(\sigma) > 0$. Since $\gamma_1\gamma_2 > 0$, $\sigma > \rho_1$, $\sigma < \rho_2$, and $p(\sigma)z(\sigma) > 0$, it follows from (3.31) and (3.33) that $a_3a_2 > 0$ and $a_3a_0 < 0$. Thus, part *(b)* of Lemma 2 in Appendix B implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

Next, assume (D.II) $\sigma < \rho_1$. Since $\sigma < \rho_1$ and $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$, it follows that $p(\sigma)z(\sigma) < 0$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma < \rho_1$, $\sigma < \rho_2$, and $p(\sigma)z(\sigma) < 0$, it follows from (3.31)–(3.33) that $a_3a_2 < 0$, $a_3a_1 > 0$ and $a_3a_0 < 0$. We consider four cases: (D.II.a) $t(\sigma) < 0$, (D.II.b) $t(\sigma) \geq 0$ and $a_2a_1 - a_3a_0 > 0$, (D.II.c) $t(\sigma) \geq 0$ and $a_2a_1 - a_3a_0 = 0$, and (D.II.d) $t(\sigma) \geq 0$ and $a_2a_1 - a_3a_0 < 0$. First, assume (D.II.a) $t(\sigma) < 0$, and thus (3.37) implies that $D < 0$. Since $a_3a_0 < 0$ and $D < 0$, part *(a)* of Lemma 2 in Appendix B implies that $\tilde{p}_k(\sigma)$ has exactly one positive root. Thus, σ is on the root locus. Next, assume (D.II.b) $t(\sigma) \geq 0$ and $a_2a_1 - a_3a_0 > 0$. Since $a_3a_0 < 0$, $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 > 0$, it follows from part *(c)* of Lemma 2 in Appendix B that there exists exactly one positive $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus. Next, assume (D.II.c) $t(\sigma) \geq 0$ and $a_2a_1 - a_3a_0 = 0$. Since $t(\sigma) \geq 0$, (3.37) implies that $D \geq 0$. Since $a_2a_1 - a_3a_0 = 0$ and $a_3a_1 > 0$, it follows from Lemma 5 in Appendix B that $\tilde{p}_k(\sigma)$ has complex roots on the imaginary axis. Thus, [24, pp. 153-154] implies that $D < 0$. Therefore, case (D.II.c) cannot occur. Next, assume (D.II.d) $t(\sigma) \geq 0$ and $a_2a_1 - a_3a_0 < 0$. Since $t(\sigma) \geq 0$, it follows from (3.37) that

$D \geq 0$. Since $D \geq 0$, $a_3a_0 < 0$, $a_3a_2 < 0$, and $a_2a_1 - a_3a_0 < 0$, Lemma 4 in Appendix B implies that $\tilde{p}_k(\sigma)$ has three positive roots. Thus, σ is on the root locus.

Finally, assume *(c)* holds. We consider three cases: (E.I) $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$, (E.II) $(\sigma - \rho_1)p(\sigma)z(\sigma) < 0$, and (E.III) $(\sigma - \rho_1)p(\sigma)z(\sigma) = 0$. First, assume (E.I) $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$. Since $\sigma < \rho_2$ and $(\sigma - \rho_1)p(\sigma)z(\sigma) > 0$, it follows that this case is identical to *(b)*, which implies from (D.I) and (D.II) that σ is on the root locus.

Next, assume (E.II) $(\sigma - \rho_1)p(\sigma)z(\sigma) < 0$. Since $t(\sigma) \geq 0$, it follows from (3.37) that $D \geq 0$. Since $\sigma < \rho_2$ and $(\sigma - \rho_1)p(\sigma)z(\sigma) < 0$, it follows from (3.33) that $a_3a_0 > 0$. We consider two cases: (E.II.a) $\sigma > \rho_1$ and (E.II.b) $\sigma < \rho_1$. First, assume (E.II.a) $\sigma > \rho_1$. Since $\sigma > \rho_1$ and $(\sigma - \rho_1)p(\sigma)z(\sigma) < 0$, it follows that $p(\sigma)z(\sigma) < 0$. Since $\gamma_1\gamma_2 > 0$ and $p(\sigma)z(\sigma) < 0$, it follows from (3.31) that $a_3a_2 < 0$. Since $D \geq 0$, $a_3a_0 > 0$ and $a_3a_2 < 0$, part *(a)* of Lemma 3 in Appendix B implies that there exists one or two $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus. Next, assume (E.II.b) $\sigma < \rho_1$. Since $\sigma < \rho_1$ and $(\sigma - \rho_1)p(\sigma)z(\sigma) < 0$, it follows that $p(\sigma)z(\sigma) > 0$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma < \rho_1$, $\sigma < \rho_2$, and $p(\sigma)z(\sigma) > 0$, it follows from (3.31), (3.33) and (3.34) that $a_3a_2 > 0$, and $a_2a_1 - a_3a_0 < 0$. Thus, part *(b)* of Lemma 3 in Appendix B implies that exists one or two $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

Finally, assume (E.III) $(\sigma - \rho_1)p(\sigma)z(\sigma) = 0$, which implies that $z(\sigma) = 0$ or $\sigma = \rho_1$. We consider four cases: (E.III.a) $z(\sigma) = 0$ and $\sigma > \rho_1$; (E.III.b) $z(\sigma) = 0$ and $\sigma = \rho_1$; (E.III.c) $z(\sigma) = 0$ and $\sigma < \rho_1$; and (E.III.d) $z(\sigma) \neq 0$ and $\sigma = \rho_1$. First, assume (E.III.a) $z(\sigma) = 0$ and $\sigma > \rho_1$. Since $z(\sigma) = 0$, it follows from (3.27) that $a_3 = 0$, which implies that $\tilde{p}_k(\sigma) = a_2k^2 + a_1k + a_0$. Since $\gamma_1\gamma_2 > 0$, $\sigma > \rho_1$ and $\sigma < \rho_2$, it follows from (3.35) that $a_2a_0 < 0$. Thus, part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus. Next, assume (E.III.b) $z(\sigma) = 0$ and $\sigma = \rho_1$, which implies from (3.27) and (3.30) that $a_3 = 0$ and $a_0 = 0$, respectively. Thus, $\tilde{p}_k(\sigma) = k(a_2k + a_1)$. Since $\gamma_1 > 0$,

$\gamma_2 > 0$, $\sigma = \rho_1$, $\sigma < \rho_2$, and $p(\sigma) \neq 0$, it follows from (3.34) that $a_2 a_1 < 0$. Therefore, there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus. Next, assume (E.III.c) $z(\sigma) = 0$ and $\sigma < \rho_1$. Since $z(\sigma) = 0$, it follows from (3.27) that $a_3 = 0$, which implies that $\tilde{p}_k(\sigma) = a_2 k^2 + a_1 k + a_0$. Since $t(\sigma) \geq 0$, it follows from (3.37) that $D \geq 0$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma = \rho_1$, $\sigma < \rho_2$, and $p(\sigma) \neq 0$, it follows from (3.34) that $a_2 a_1 < 0$. Since $\gamma_1 \gamma_2 > 0$ and $\sigma < \rho_1 \leq \rho_2$, it follows from (3.35) that $a_2 a_0 < 0$. Thus, part *(c)* of Lemma 1 in Appendix A implies that there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

Lastly, assume (E.III.d) $z(\sigma) \neq 0$ and $\sigma = \rho_1$. Since $\sigma = \rho_1$, it follows from (3.30) that $a_0 = 0$, which implies that $\tilde{p}_k(\sigma) = k(a_3 k^2 + a_2 k + a_1$, where it follows from (3.27) that $a_3 \neq 0$ because $z(\sigma) \neq 0$. Furthermore, since $\sigma < \rho_2$ and $p(\sigma) \neq 0$, (3.29) implies that $a_1 \neq 0$. We consider two cases: (E.III.d.i) $p(\sigma)z(\sigma) > 0$, and (E.III.d.ii) $p(\sigma)z(\sigma) < 0$. First, assume (E.III.d.i) $p(\sigma)z(\sigma) > 0$. Since $\gamma_1 > 0$, $\sigma = \rho_1$, $\sigma < \rho_2$, and $p(\sigma)z(\sigma) > 0$, it follows from (3.32) that $a_3 a_1 < 0$. Thus, *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus. Next, assume (E.III.d.ii) $p(\sigma)z(\sigma) < 0$. Since $\sigma = \rho_1 \neq \rho_2$ and $t(\sigma) \geq 0$, it follows from (3.7) that

$$
\begin{aligned}
t(\sigma) &= (\gamma_1(\sigma - \rho_2))^2 \left(\gamma_1^2 \gamma_2^2 p(\sigma)^2 - 4\gamma_1(\sigma - \rho_2)p(\sigma)z(\sigma)\right) \\
&= (\gamma_1(\sigma - \rho_2))^2 \left(a_2^2 - 4a_3 a_1\right) \geq 0,
\end{aligned}
$$

which implies that $a_2^2 - 4a_3 a_1 \geq 0$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $\sigma = \rho_1 < \rho_2$, and $p(\sigma)z(\sigma) < 0$, it follows from (3.31) and (3.32) that $a_3 a_2 < 0$ and $a_3 a_1 > 0$. Since $a_2^2 - 4a_3 a_1 \geq 0$, $a_3 a_2 < 0$ and $a_3 a_1 > 0$, part *(c)* of Lemma 1 in Appendix A implies that there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus. □

Proof of Fact 3.4. To show that m roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)$, it follows from (3.6) that

$$\frac{\tilde{p}_k(s)}{k^3} = z(s) + \frac{\gamma^2 p(s)}{k} + \frac{(2\gamma s - \gamma\rho_1 - \gamma\rho_2)p(s)}{k^2} + \frac{(s-\rho_1)(s-\rho_2)p(s)}{k^3},$$

which implies that, for sufficiently large $k > 0$, $\tilde{p}_k(s)/k^3 \approx z(s)$. Thus, as $k \to \infty$, m roots of $\tilde{p}_k(s)$ converge to the roots of $z(s)$.

To show *(a)*–*(e)*, define $R \triangleq \max_{i=1,\ldots,m}|z_i|$ and write $p(s) = s^{n-2} + a_1 s^{n-3} + \cdots + a_{n-2}$ and $z(s) = s^m + b_1 s^{m-1} + \cdots + b_m$, where $a_1, a_2, \ldots, a_{n-2} \in \mathbb{R}$ and $b_1, b_2, \ldots, b_m \in \mathbb{R}$.

First, we show *(a)*. Since $d = 1$, it follows that for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = \frac{1}{s} + \sum_{i=2}^{\infty} \frac{c_i}{s^i}, \tag{3.46}$$

where the real numbers $c_2, c_3, \ldots$ are real coefficients of the Laurent series expansion.

Next, it follows from (3.2), (3.4) and (3.9) that

$$\frac{1}{G(\lambda_1)\hat{G}_k(\lambda_1)} = \frac{-(\lambda_1 + \gamma k - \rho_2)p(\lambda_1)}{z(\lambda_1)}. \tag{3.47}$$

Since $|\lambda_1| \to \infty$ as $k \to \infty$ and $(\gamma k - \rho_2)/\lambda_1 \to 0$ as $k \to \infty$, it follows from (3.46) that, for sufficiently large $k > 0$,

$$\frac{(\lambda_1 - \rho_2 + \gamma k)p(\lambda_1)}{z(\lambda_1)} = 1 + \frac{\gamma k - \rho_2}{\lambda_1} + (\lambda_1 - \rho_2 + \gamma k)\sum_{i=2}^{\infty} \frac{c_i}{\lambda_1^i} \approx 1. \tag{3.48}$$

Therefore, for sufficiently large $k > 0$, combining (3.46) and (3.47) yields

$$\frac{1}{G(\lambda_1)\hat{G}_k(\lambda_1)} \approx -1,$$

or equivalently $1 + G(\lambda_1)\hat{G}_k(\lambda_1) \approx 0$. Thus, as $k \to \infty$, one root of $\tilde{p}_k(s)$ is approximated by λ_1, which confirms *(a)*.

Next, we show *(b)*. Since $d = 2$, it follows that, for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = 1 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{3.49}$$

where the real numbers $c_1, c_2, \ldots$ are real coefficients of the Laurent series expansion.

Next, for $i = 1, 2$ define

$$\bar{\lambda}_i \triangleq -\gamma k + \frac{\rho_1 + \rho_2}{2} + \jmath(-1)^{i-1} \frac{\sqrt{4k^3 - (\rho_1 - \rho_2)^2}}{2}. \tag{3.50}$$

For $i = 1, 2$, it follows from (3.4) and (3.50) that

$$\begin{aligned}
\frac{z_{\mathrm{c}}(\bar{\lambda}_i)}{\hat{G}_k(\bar{\lambda}_i)p_{\mathrm{c}}(\bar{\lambda}_i)} &= \frac{(\bar{\lambda}_i - \rho_1 + \gamma k)(\bar{\lambda}_i - \rho_2 + \gamma k)}{k^3} \\
&= \frac{(\bar{\lambda}_i - \rho_1 + \gamma k)(\bar{\lambda}_i - \rho_1 + (\rho_1 - \rho_2) + \gamma k)}{k^3} \\
&= \frac{(\bar{\lambda}_i - \rho_1 + \gamma k)^2 + (\rho_1 - \rho_2)(\bar{\lambda}_i - \rho_1 + \gamma k)}{k^3} \\
&= \Big((\rho_2 - \rho_1)^2/4 + \jmath(\rho_2 - \rho_1)(-1)^i \sqrt{4k^3 - (\rho_1 - \rho_2)^2}/2 \\
&\quad - \left(4k^3 - (\rho_1 - \rho_2)^2\right)/4 - (\rho_2 - \rho_1)^2/2 \\
&\quad - \jmath(\rho_2 - \rho_1)(-1)^i \sqrt{4k^3 - (\rho_1 - \rho_2)^2}/2\Big)/k^3 \\
&= -1.
\end{aligned} \tag{3.51}$$

Furthermore, for $i = 1, 2$, it follows from (3.2), (3.5) and (3.51) that

$$\frac{1}{G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i)} = -\frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)}. \tag{3.52}$$

Since $|\bar{\lambda}_i| \to \infty$ as $k \to \infty$, it follows from (3.49) that, for sufficiently large $k > 0$,

$p(\bar{\lambda}_i)/z(\bar{\lambda}_i) \approx 1$. Therefore, it follows from (3.52) that, for sufficiently large $k > 0$, $1/(G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i)) \approx -1$, or equivalently $1 + G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i) \approx 0$. Thus, as $k \to \infty$, two roots of $\tilde{p}_k(s)$ are approximated by $\bar{\lambda}_1$ and $\bar{\lambda}_2$. Furthermore, for $i = 1, 2$ and sufficiently large $k > 0$, it follows from (3.10), (3.11) and (3.50) that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are approximated by λ_1 and λ_2, respectively, which confirms *(b)*.

Next, we show *(c)* and *(d)*. Since $d = 3$, it follows that, for all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = s + c_0 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{3.53}$$

where the real numbers $c_0, c_1, \ldots$ are coefficients of the Laurent series expansion. For sufficiently large $k > 0$ and for all $s \in \mathbb{C}$ such that $|s| > \sqrt{k}$, it follows from (3.53) that $p(s)/z(s) \approx s$. Thus, for sufficiently large $k > 0$ and for all $s \in \mathbb{C}$ such that $|s| > \sqrt{k}$, it follows from (3.53) that

$$\begin{aligned} &\frac{p(s)}{z(s)}(s - \rho_1 + \gamma k)(s - \rho_2 + \gamma k) + k^3 \\ &\qquad \approx s(s - \rho_1 + \gamma k)(s - \rho_2 + \gamma k) + k^3 \\ &\qquad = s^3 + (2\gamma k - \rho_1 - \rho_2)s^2 + (\gamma k - \rho_1)(\gamma k - \rho_2)s + k^3 \\ &\qquad \approx \mu_k(s), \end{aligned} \tag{3.54}$$

where

$$\mu_k(s) \triangleq s^3 + 2\gamma k s^2 + \gamma^2 k^2 s + k^3. \tag{3.55}$$

Next, we define

$$\Delta_k \triangleq k^6(4\gamma^3 - 27), \tag{3.56}$$

which is the cubic discriminant of $\mu_k(s)$ with respect to s. We consider two cases: (A) $4\gamma^3 - 27 \geq 0$ and (B) $4\gamma^3 - 27 < 0$. First, assume (A) $4\gamma^3 - 27 \geq 0$, which implies that $\Delta_k \geq 0$ because $k > 0$. Since the cubic discriminant $\Delta_k \geq 0$, it follows from [25, p. 97] that $\mu_k(s)$ has three real roots. Furthermore, it follows from the closed-form solution of a cubic polynomial with three real roots that, for $i = 1, 2, 3$, the roots of $\mu_k(s)$ are given by (3.12) and (3.13). Next, assume (B) $4\gamma^3 - 27 < 0$, which implies that $\Delta_k < 0$ because $k > 0$. Since the cubic discriminant $\Delta_k < 0$, it follows from [25, p. 97] that $\mu_k(s)$ has one real root and two complex conjugate roots. Furthermore, it follows from the closed-form solution of a cubic polynomial with one real root and two complex-conjugate roots that the roots of $\mu_k(s)$ are given by (3.14)–(3.19).

Since for sufficiently large $k > 0$ and $i = 1, 2, \ldots, 6$, $|\lambda_i| > \sqrt{k}$ and $\mu_k(\lambda_i) = 0$, it follows from (3.54) that

$$\frac{p(\lambda_i)}{z(\lambda_i)}(\lambda_i - \rho_1 + \gamma k)(\lambda_i - \rho_2 + \gamma k) + k^3 \approx \mu_k(\lambda_i) = 0,$$

which implies that

$$\begin{aligned} -1 &\approx \frac{p(\lambda_i)(\lambda_i - \rho_1 + \gamma k)(\lambda_i - \rho_2 + \gamma k)}{k^3 z(\lambda_i)} \\ &= \frac{1}{G(\lambda_i)\hat{G}_k(\lambda_i)}, \end{aligned}$$

or equivalently $1 + G(\lambda_i)\hat{G}_k(\lambda_i) \approx 0$. Thus, for case (A) $4\gamma^3 - 27 > 0$, as $k \to \infty$, three roots of $\tilde{p}_k(s)$ are approximated by λ_1, λ_2 and λ_3, which confirms *(c)*. Similarly for case (B) $4\gamma^3 - 27 < 0$, as $k \to \infty$, three roots of $\tilde{p}_k(s)$ are approximated by λ_4, λ_5 and λ_6, which confirms *(d)*.

Finally, we show *(e)* and *(f)*. If $|\rho_1| \leq |\rho_2|$, then let $\bar{\rho}_1 \triangleq \rho_1$ and $\bar{\rho}_2 \triangleq \rho_2$. If $|\rho_1| > |\rho_2|$, then let $\bar{\rho}_1 \triangleq \rho_2$ and $\bar{\rho}_2 \triangleq \rho_1$. Since $d \geq 4$, it follows that, for all $s \in \mathbb{C}$

such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is given by

$$\frac{p(s)}{z(s)} = s^{d-2} + f_{d-3}s^{d-3} + \ldots + f_1 s + c_0 + \sum_{i=1}^{\infty} \frac{c_i}{s^i}, \tag{3.57}$$

where the real numbers $f_1, \ldots, f_{d-3}, c_0, c_1, \ldots$ are real coefficients of the Laurent series expansion. Furthermore, note that $f_{d-3} = a_1 - b_1 = -\left(\sum_{j=1}^{n-2} p_j - \sum_{j=1}^{m} z_j\right) = -(d-2)\alpha$. For all $s \in \mathbb{C}$ such that $|s - \bar{\rho}_2| > (k/\gamma^2)^{\frac{1}{d}}$ and for sufficiently large $k > 0$, it follows from (3.57) that

$$\frac{p(s)}{z(s)} \approx s^{d-2} - (d-2)\alpha s^{d-3}. \tag{3.58}$$

Next, we consider the Taylor series expansion of $k^3/((s - \bar{\rho}_1 + \gamma k)(s - \bar{\rho}_2 + \gamma k)$ about $\bar{\rho}_2$. For all $s \in \mathbb{C}$ such that $|s - \bar{\rho}_2| < \gamma k$, the Taylor series expansion of $k^3/((s - \bar{\rho}_1 + \gamma k)(s - \bar{\rho}_2 + \gamma k)$ about $\bar{\rho}_2$ is given by

$$\begin{aligned}\frac{k^3}{(s - \bar{\rho}_1 + \gamma k)(s - \bar{\rho}_2 + \gamma k)} = {} & \frac{k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)} - \frac{2k^2(s - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2} \\ & + \frac{k(\bar{\rho}_1 - \bar{\rho}_2)(s - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2} + \ldots\end{aligned}$$

Next, for all $s \in \mathbb{C}$ such that $(k/\gamma^2)^{\frac{1}{d}} < |s - \bar{\rho}_2| < (k/\gamma^2)^{\frac{1}{2}}$, it follows that as $k \to \infty$, $k(\bar{\rho}_1 - \bar{\rho}_2)(s - \bar{\rho}_2)/(\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2)$ approaches zero and the higher-order terms of the Taylor series approach zero. Therefore, for sufficiently large $k > 0$ and for all $s \in \mathbb{C}$ such that $(k/\gamma^2)^{\frac{1}{d}} < |s - \bar{\rho}_2| < (k/\gamma^2)^{\frac{1}{2}}$, it follows from the Taylor series expansion that

$$\frac{k^3}{(s - \bar{\rho}_1 + \gamma k)(s - \bar{\rho}_2 + \gamma k)} \approx \frac{k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)} - \frac{2k^2(s - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2}. \tag{3.59}$$

Adding (3.58) and (3.59) yields, for sufficiently large $k > 0$ and for all $s \in \mathbb{C}$ such that $(k/\gamma^2)^{\frac{1}{d}} < |s - \bar{\rho}_2| < (k/\gamma^2)^{\frac{1}{2}}$,

$$\frac{p(s)}{z(s)} + \frac{k^3}{(s - \bar{\rho}_1 + \gamma k)(s - \bar{\rho}_2 + \gamma k)} \approx \nu_k(s), \tag{3.60}$$

where

$$\nu_k(s) \triangleq s^{d-2} - (d-2)\alpha s^{d-3} + \frac{k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)} - \frac{2k^2(s - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2}. \tag{3.61}$$

For $k > 0$, let $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{d-2}$ denote the roots of $\nu_k(s)$. Note that as $k \to \infty$, $k^2/(\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)) - 2k^2(s - \bar{\rho}_2)/(\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2)$ approaches infinity. Thus, it follows from classical root locus that, as $k \to \infty$, $|\bar{\lambda}_i| \to \infty$. Furthermore, since $\nu_k(\bar{\lambda}_i) = 0$, it follows that

$$\frac{-k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)} = \bar{\lambda}_i^{d-2} - (d-2)\alpha\bar{\lambda}_i^{d-3} - \frac{2k^2(\bar{\lambda}_i - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2}.$$

Taking the $(d-2)^{\text{th}}$ root of both sides yields

$$\left(\frac{-k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)}\right)^{\frac{1}{d-2}} = \bar{\lambda}_i(1 + o_k)^{\frac{1}{d-2}} \tag{3.62}$$

where

$$o_k \triangleq -\frac{(d-2)\alpha}{\bar{\lambda}_i} - \frac{2k^2(\bar{\lambda}_i - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2\bar{\lambda}_i^{d-2}}. \tag{3.63}$$

Since as $k \to \infty$, o_k approaches zero, we use the binomial approximation $(1 + o_k)^q \approx (1 + qo_k)$, where $q = 1/(d-2)$, on (3.62), which yields

$$\begin{aligned}\left(\frac{-k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)}\right)^{\frac{1}{d-2}} &\approx \bar{\lambda}_i\left(1 - \frac{\alpha}{\bar{\lambda}_i} - \frac{2k^2(\bar{\lambda}_i - \bar{\rho}_2)}{\gamma(d-2)(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2\bar{\lambda}_i^{d-2}}\right)\\ &= \bar{\lambda}_i - \alpha - \frac{2k^2(\bar{\lambda}_i - \bar{\rho}_2)}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2\bar{\lambda}_i^{d-3}}.\end{aligned} \tag{3.64}$$

For $d = 4$, as $k \to \infty$, $-2k^2(\bar{\lambda}_i - \bar{\rho}_2)/(\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2\bar{\lambda}_i)$ approaches $-1/\gamma^3$. For

$d > 4$, as $k \to \infty$, $-2k^2(\bar{\lambda}_i - \bar{\rho}_2)/(\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)^2\bar{\lambda}_i)$ approaches 0. Let $\bar{\alpha}$ denote α given by (3.22) if $d = 4$, and let $\bar{\alpha}$ denote α given by (3.25) if $d \geq 5$. Thus, for sufficiently large $k > 0$, (3.64) implies that

$$\left(\frac{-k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)}\right)^{\frac{1}{d-2}} \approx \bar{\lambda}_i - \bar{\alpha}. \tag{3.65}$$

Thus, for $i = 1, 2, \ldots, d-2$ and sufficiently large $k > 0$, solving for $\bar{\lambda}_i$ yields

$$\bar{\lambda}_i \approx \left(\frac{k^2}{\gamma(\bar{\rho}_2 - \bar{\rho}_1 + \gamma k)}\right)^{\frac{1}{d-2}} e^{\jmath\phi_i} + \bar{\alpha} = \lambda_i, \tag{3.66}$$

Next, since for $i = 1, 2, \ldots, d-2$, $\nu_k(\bar{\lambda}_i) = 0$ and $(k/\gamma^2)^{\frac{1}{d}} < |\bar{\lambda}_i - \bar{\rho}_2| < (k/\gamma^2)^{\frac{1}{2}}$, it follows from (3.60) that, for sufficiently large $k > 0$,

$$\frac{p_{\mathrm{c}}(\bar{\lambda}_i)}{G(\bar{\lambda}_i)z_{\mathrm{c}}(\bar{\lambda}_i)} = \frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)} \approx \frac{-k^3}{(\bar{\lambda}_i - \bar{\rho}_1 + \gamma k)(\bar{\lambda}_i - \bar{\rho}_2 + \gamma k)} = \frac{-\hat{G}_k(\bar{\lambda}_i)p_{\mathrm{c}}(\bar{\lambda}_i)}{z_{\mathrm{c}}(\bar{\lambda}_i)},$$

or equivalently $1 + G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i) \approx 0$. Thus, as $k \to \infty$, $d-2$ roots of $\tilde{p}_k(s)$ are approximated by $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{d-2}$. For $d = 4$, (3.66) implies that as $k \to \infty$, $d-2$ roots of $\tilde{p}_k(s)$ are approximated by $\bar{\lambda}_1$ and $\bar{\lambda}_2$. Furthermore, as $k \to \infty$, it follows from (3.66) that $\bar{\lambda}_1$ and $\bar{\lambda}_2$ are approximated by λ_1 and λ_2. Therefore, as $k \to \infty$, two roots of $\tilde{p}_k(s)$ are approximated by λ_1 and λ_2. For $d \geq 5$, (3.66) implies that as $k \to \infty$, $d-2$ roots of $\tilde{p}_k(s)$ are approximated by $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{d-2}$. Furthermore, as $k \to \infty$, it follows from (3.66) that $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{d-2}$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d-2}$. Therefore, as $k \to \infty$, $d-2$ roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d-2}$.

Next, we show that for $d = 4$, the two remaining roots are approximated by λ_3 and λ_4, and for $d \geq 5$, the two remaining roots approach $-\infty$. For $i = d-1, d$, define

$$\begin{aligned}\bar{\lambda}_i \triangleq\; & -\gamma k + (\rho_1 + \rho_2)/2 \\ & + (-1)^i\sqrt{-k^{5-d}(-\gamma)^{2-d} + (\rho_1 - \rho_2)^2/4}.\end{aligned} \tag{3.67}$$

Next, it follows from (3.4) and (3.67) that, for $i = d-1, d$,

$$\frac{-\hat{G}_k(\bar{\lambda}_i)p_{\rm c}(\bar{\lambda}_i)}{z_{\rm c}(\bar{\lambda}_i)} = \frac{-k^3}{(\bar{\lambda}_i - \rho_1 + \gamma k)(\bar{\lambda}_i - \rho_2 + \gamma k)}$$
$$= (-\gamma k)^{d-2}. \tag{3.68}$$

For $d \geq 5$, sufficiently large $k > 0$ and $i = d-1, d$, (3.57) and (3.67) imply that $p(\bar{\lambda}_i)/z(\bar{\lambda}_i) \approx (-\gamma k)^{d-2}$. Combining with (3.68) implies that, for sufficiently large $k > 0$,

$$\frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)} \approx (-\gamma k)^{d-2} = \frac{-\hat{G}_k(\bar{\lambda}_i)p_{\rm c}(\bar{\lambda}_i)}{z_{\rm c}(\bar{\lambda}_i)},$$

or equivalently $1 + G(\bar{\lambda}_i)\hat{G}_k(\bar{\lambda}_i) \approx 0$. Thus, as $k \to \infty$, two roots of $\tilde{p}_k(s)$ are approximated by $\bar{\lambda}_1$ and $\bar{\lambda}_2$. Furthermore, it follows from (3.26) and (3.67) that as $k \to \infty$, $\bar{\lambda}_{d-1}$ and $\bar{\lambda}_d$ are approximated by λ_{d-1} and λ_d, respectively. Therefore, as $k \to \infty$, the two remaining roots of $\tilde{p}_k(s)$ are approximated by λ_{d-1} and λ_d, which confirms *(f)*.

For $d = 4$ and sufficiently large $k > 0$, (3.57) and (3.67) imply that $|p(\bar{\lambda}_i)/z(\bar{\lambda}_i)| \approx \gamma^2 k^2$ and $\angle p(\bar{\lambda}_i)/z(\bar{\lambda}_i) \approx 0$. Combining with (3.68) implies that, for sufficiently large $k > 0$ and $i = 3, 4$,

$$\left|\frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)}\right| \approx \left|\frac{-\hat{G}_k(\bar{\lambda}_i)p_{\rm c}(\bar{\lambda}_i)}{z_{\rm c}(\bar{\lambda}_i)}\right|.$$

Since, in addition, both angles are approximately zero, it follows that as $k \to \infty$, two roots of $\tilde{p}_k(s)$ approach $\bar{\lambda}_3$ and $\bar{\lambda}_4$. Furthermore, it follows from (3.23) and (3.67) that as $k \to \infty$, $\bar{\lambda}_3$ and $\bar{\lambda}_4$ are approximated by λ_3 and λ_4, respectively. Therefore, as $k \to \infty$, the two remaining roots of $\tilde{p}_k(s)$ are approximated by λ_3 and λ_4, which confirms *(e)*. □

Chapter 4 Root Locus with Quadratic Gain Parameterization

In this chapter, we present rules for constructing the root locus for a polynomial that is quadratic in the root-locus parameter k. These quadratic root-locus rules extend the results of Chapter 2. Chapter 2 focuses on a controller class, where the numerator polynomial is proportional to k^2 and the denominator polynomial includes a pole, whose location is proportional to k. In contrast, this chapter presents quadratic root-locus rules for a more general class of polynomials with quadratic gain parameterization.

4.1 Introduction

In Chapter 2, we consider a controller class that yields a closed-loop denominator polynomial that is quadratic in the root-locus parameter k. Specifically, Chapter 2 considers a controller class, where the numerator is proportional to k^2, and the denominator includes a pole, whose location is proportional to k. The quadratic root-locus rules in Chapter 2 only apply to controllers with the properties described above.

In contrast, this chapter presents quadratic root-locus rules for a general polynomial with quadratic gain parameterization. Thus, the root-locus rules presented in this chapter apply to all linear controllers that yield a closed-loop denominator polynomial that is quadratic in k and satisfy three relative degree requirements. More specifically, the quadratic root-locus rules of this chapter apply to the polynomial $k^2q(s)+kr(s)+t(s)$ provided that $0 \leq \deg\, q(s) < \deg\, r(s) < \deg\, t(s)$. Since the quadratic root-locus rules in Chapter 2 satisfy these three conditions, it follows that the quadratic root-locus rules in Chapter 2 are a special case of the quadratic root-locus rules in this chapter.

4.2 Problem Formulation

Consider the polynomial

$$\tilde{p}_k(s) \triangleq k^2\gamma_1 q(s) + k\gamma_2 r(s) + t(s), \tag{4.1}$$

where $\gamma_1 > 0$; $\gamma_2 > 0$; $q(s)$, $r(s)$ and $t(s)$ are monic polynomials; and $0 \leq \deg q(s) < \deg r(s) < \deg t(s)$. We use the following classical definition of the positive root locus.

Definition 4.1. *The root locus is* $\{\lambda \in \mathbb{C} : \tilde{p}_k(\lambda) = 0,$ *where* $k > 0\}$.

This chapter considers the quadratic root locus where $k > 0$. The techniques in this chapter can also be used to develop root locus rules for $k < 0$. In the next two sections, we present six facts that characterize the quadratic root locus. Proofs of these facts are provided in Section 4.7.

4.3 Quadratic Root-Locus Rules

In this section, we present four facts that describe the quadratic root locus. Facts 4.1 and 4.2 define the root locus starting points for $k = 0$ and describe the root locus symmetry. These two facts are consistent with classical root locus.

Fact 4.1. *As* $k \to 0$*, the roots of* $\tilde{p}_k(s)$ *approach the roots of* $t(s)$*.*

Fact 4.2. *The root locus is symmetric about the real axis.*

Next, we present a rule to determine the points on the real axis that are on the root locus. We define

$$w(s) \triangleq \gamma_2^2 r(s)^2 - 4\gamma_1 q(s)t(s), \tag{4.2}$$

which is the discriminant of $\tilde{p}_k(s)$ with respect to k. The polynomial $w(s)$ is not necessarily monic. Furthermore, if $\deg r(s) \le \deg q(s)t(s)$, then the leading coefficient of $t(s)$ can be negative. The real axis rule for the quadratic root locus depends on the roots of $q(s)$, $r(s)$, $t(s)$, and $w(s)$, and the leading coefficient of $w(s)$.

Fact 4.3. *Let $\sigma \in \mathbb{R}$. Then σ is on the root locus if and only if any of the following statements hold:*

(a) $q(\sigma)t(\sigma) < 0$.

(b) $q(\sigma) = 0$ *and* $r(\sigma)t(\sigma) < 0$.

(c) $w(\sigma) \ge 0$ *and* $q(\sigma)r(\sigma) < 0$.

Furthermore, if σ is on the root locus, then the following statements hold:

(i) If $q(\sigma)t(\sigma) \le 0$ or $w(\sigma) = 0$, then there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

(ii) If $w(\sigma) > 0$, $q(\sigma)r(\sigma) < 0$ and $q(\sigma)t(\sigma) > 0$, then there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

We note that parts *(a)*–*(c)* of Fact 4.3 are not mutually exclusive, but these parts are collectively exhaustive. For example, parts *(a)* and *(c)* of Fact 4.3 can occur simultaneously. Part *(a)* of Fact 4.3 implies that the real axis to the left of an odd number of real roots of $q(s)t(s)$ is on the root locus. Part *(b)* of Fact 4.3 implies that the real roots of $q(s)$ that are to the left of an odd number of real roots of $r(s)t(s)$ are on the root locus. Part *(c)* of Fact 4.3 implies that if the discriminant of $\tilde{p}_k(\sigma)$ (i.e, $w(\sigma)$) is nonnegative, then the real axis to the left of an odd number of real roots of

$q(s)r(s)$ is on the root locus. Moreover, parts *(a)–(c)* of Fact 4.3 demonstrate that real roots of $q(s)$, $r(s)$ and $t(s)$ can be on the root locus for finite $k > 0$.

In classical root locus, if σ is on the root locus, then there exists exactly one $k > 0$ such that the closed-loop denominator polynomial (i.e., $\tilde{p}_k(\sigma)$) is zero. In contrast, parts *(i)* and *(ii)* of Fact 4.3 imply that if σ is on the root locus, then there exists either one or two $k > 0$ such that $\tilde{p}_k(\sigma) = 0$.

We now describe the asymptotic properties of the quadratic root locus, that is, the properties for sufficiently large $k > 0$. We define

$$l \triangleq \deg q(s), \quad m \triangleq \deg r(s), \quad n \triangleq \deg t(s), \tag{4.3}$$

$$d_1 \triangleq m - l, \quad d_2 \triangleq n - m, \quad d \triangleq d_1 + d_2 = n - l. \tag{4.4}$$

Note that d_1, d_2 and d are positive integers. Let $q_1, q_2, \ldots, q_l$ be the roots of $q(s)$; $r_1, r_2, \ldots, r_m$ be the roots of $r(s)$; and $t_1, t_2, \ldots, t_n$ be the roots of $t(s)$. Fact 4.4 characterizes the asymptotic properties of $\tilde{p}_k(s)$, that is, the properties for sufficiently large $k > 0$. In Fact 4.4, we assume that $\gamma_1 = \gamma_2 = 1$. The extension of Fact 4.4 to the case where $\gamma_1 > 0$ and $\gamma_2 > 0$ remains open.

Fact 4.4. *Assume $\gamma_1 = \gamma_2 = 1$. As $k \to \infty$, l roots of $\tilde{p}_k(s)$ converge to the roots of $q(s)$, and the d remaining roots satisfy the following statements:*

(a) If $d_1 < d_2 - 2$, then the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d$,

$$\lambda_i \triangleq k^{2/d} e^{\jmath\theta_i} + \alpha, \tag{4.5}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d}, \tag{4.6}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n} t_j - \sum_{j=1}^{l} q_j}{d}. \tag{4.7}$$

(b) If $d_1 > d_2+1$*, then the* d *remaining roots of* $\tilde{p}_k(s)$ *are approximated by* $\lambda_1, \lambda_2, \ldots, \lambda_d$*, where for* $i = 1, 2, \ldots, d_1$*,*

$$\lambda_i \triangleq k^{1/d_1} e^{\jmath\theta_i} + \alpha_1, \tag{4.8}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d_1}, \tag{4.9}$$

$$\alpha_1 \triangleq \frac{\sum_{j=1}^{m} r_j - \sum_{j=1}^{l} q_j}{d_1}; \tag{4.10}$$

and for $i = 1, 2, \ldots, d_2$*,*

$$\lambda_{d_1+i} \triangleq k^{1/d_2} e^{\jmath\phi_i} + \alpha_2, \tag{4.11}$$

where

$$\phi_i \triangleq \frac{2\pi i - \pi}{d_2}, \tag{4.12}$$

$$\alpha_2 \triangleq \frac{\sum_{j=1}^{n} t_j - \sum_{j=1}^{m} r_j}{d_2}. \tag{4.13}$$

Part *(a)* of Fact 4.4 demonstrates that if $d_1 < d_2 - 2$, then the asymptotic properties of $\tilde{p}_k(s)$ are similar to classical root locus. More specifically, as k tends to infinity, d roots of $\tilde{p}_k(s)$ tend to infinity along asymptotes centered at α with angles $\theta_1, \theta_2, \ldots, \theta_d$. Part *(b)* of Fact 4.4 demonstrates that if $d_1 > d_2 + 1$, then the asymptotic properties of $\tilde{p}_k(s)$ are similar to a "double root locus", that is, the asymptotic properties of $\tilde{p}_k(s)$ are similar to the superposition of two classical root loci. More specifically, as k tends to infinity, d_1 roots of $\tilde{p}_k(s)$ tend to infinity along asymptotes centered at α_1 with angles $\theta_1, \theta_2, \ldots, \theta_{d_1}$. In addition, as k tends to infinity, d_2 roots of $\tilde{p}_k(s)$ tend to infinity along asymptotes centered at α_2 with angles $\phi_1, \phi_2, \ldots, \phi_{d_2}$.

Note that parts *(a)* and *(b)* of Fact 4.4 are not collectively exhaustive. The following conjecture describes the numerically observed asymptotic properties of $\tilde{p}_k(s)$ for all $d_1 > 0$ and $d_2 > 0$ such that $d_2 - 2 \leq d_1 \leq d_2 + 1$. A proof of this conjecture remains open.

Conjecture 4.1. *Assume $\gamma_1 = \gamma_2 = 1$. As $k \to \infty$, l roots of $\tilde{p}_k(s)$ converge to the roots of $q(s)$, and the d remaining roots satisfy the following statements:*

(a) If $d_2 - 2 \leq d_1 < d_2$, then the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d$,

$$\lambda_i \triangleq k^{2/d} e^{\jmath\theta_i} + \alpha, \tag{4.14}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d}, \tag{4.15}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n} t_j - \sum_{j=1}^{l} q_j}{d}. \tag{4.16}$$

(b) If $d_1 = d_2$, then the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d$,

$$\lambda_i \triangleq k^{1/d_1} e^{\jmath\theta_i} + \alpha, \tag{4.17}$$

where

$$\theta_i \triangleq \begin{cases} \frac{2\pi i + 2\pi/3}{d_1}, & i = 1, 2, \ldots, d_1, \\ \frac{2\pi i - 2\pi/3}{d_1}, & i = d_1 + 1, d_1 + 2, \ldots, d, \end{cases} \tag{4.18}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n} t_j - \sum_{j=1}^{l} q_j}{d}. \tag{4.19}$$

(c) If $d_1 = d_2+1$, then the d remaining roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$, where for $i = 1, 2, \ldots, d_1$,

$$\lambda_i \triangleq k^{1/d_1} e^{\jmath\theta_i} + \alpha_1, \tag{4.20}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d_1}, \tag{4.21}$$

$$\alpha_1 \triangleq \frac{\sum_{j=1}^{m} r_j - \sum_{j=1}^{l} q_j + 1}{d_1}; \tag{4.22}$$

and for $i = 1, 2, \ldots, d_2$,

$$\lambda_{d_1+i} \triangleq k^{1/d_2} e^{\jmath\phi_i} + \alpha_2, \tag{4.23}$$

where

$$\phi_i \triangleq \frac{2\pi i - \pi}{d_2}, \tag{4.24}$$

$$\alpha_2 \triangleq \frac{\sum_{j=1}^{n} t_j - \sum_{j=1}^{m} r_j - 1}{d_2}. \tag{4.25}$$

It follows from part *(a)* of Conjecture 4.1 that if $d_2 - 2 \leq d_1 < d_2$, then the asymptotic properties of $\tilde{p}_k(s)$ are expected to be similar to part *(a)* of Fact 4.4. If $d_1 = d_2$, then part *(b)* of Conjecture 4.1 implies that the asymptote angles $\theta_1, \theta_2, \ldots, \theta_d$ given by (4.18) are not expected to be equally spaced between 0 and 2π radians. For example, in Chapter 2, part *(b)* of Fact 2.4 demonstrates that θ_1 and θ_2 given by (2.11) are not equally spaced between 0 and 2π radians.

Next, it follows from part *(c)* of Conjecture 4.1 that if $d_1 = d_2 + 1$, then the centers α_1 and α_2 include terms in the numerator that do not depend on the roots of $q(s)$,

$r(s)$ or $t(s)$. For example, in Chapter 2, part *(c)* of Fact 2.4 demonstrates that the center α given by (2.16) includes a term in the numerator that depends on γ, which is equivalent to γ_2 in this chapter.

Conjecture 4.1 assumes all polynomials are monic (i.e., $\gamma_1 = \gamma_2 = 1$). If $\gamma_1 \neq 1$ or $\gamma_2 \neq 1$, then part *(b)* of Fact 2.4 suggests that the asymptote angles $\theta_1, \theta_2, \ldots, \theta_d$ in part *(b)* of Conjecture 4.1 depend on γ_1 and γ_2. Moreover, part *(c)* of Fact 2.4 suggests that the centers α_1 and α_2 in part *(c)* of Conjecture 4.1 depend on γ_1 and γ_2.

4.4 Break-in and Breakaway Points

In this section, we examine the break-in and breakaway points of the quadratic root locus. We use the classical break-in and breakaway point definition, which is given by Definition 1.2.

For $i = 1, 2$, define $\kappa_i : \{\sigma \in \mathbb{R} : q(\sigma) \neq 0\} \to \mathbb{C}$ by

$$\kappa_i(\sigma) \triangleq \frac{-\gamma_2 r(\sigma) + (-1)^{i-1}\sqrt{w(\sigma)}}{2\gamma_1 q(\sigma)}, \tag{4.26}$$

which maps the real numbers excluding the real roots of $q(s)$ to the complex numbers. Note that if $\kappa_1(\sigma) > 0$ or $\kappa_2(\sigma) > 0$, then there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, and σ is on the root locus.

We now present two facts that characterize break-in and breakaway points. Fact 4.5 characterizes the break-in and breakaway points along the real axis that are not real roots of $q(s)$; and Fact 4.6 characterizes the break-in and breakaway points along the real axis that are real roots of $q(s)$.

Fact 4.5. *Let $\tau \in \mathbb{R}$ be a point on the root locus, and assume τ is not a root of $q(s)$. Then τ is a break-in or breakaway point if and only if either of the following statements hold:*

(a) $\kappa_1(\tau) > 0$ *and* $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$.

(b) $\kappa_2(\tau) > 0$ *and* $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$.

Fact 4.6. *Let* $\tau \in \mathbb{R}$ *be a root of* $q(s)$ *that is on the root locus, and define* $k_\tau \triangleq -t(\tau)/(\gamma_2 r(\tau))$. *Then* τ *is a break-in or breakaway point if and only if* $\tilde{p}_{k_\tau}(s)$ *has multiple roots at* τ.

4.5 Numerical Examples

We now present examples that demonstrate the quadratic root locus. The first three examples show how the degree of $r(s)$ affects the asymptotic properties of $\tilde{p}_k(s)$. The last example demonstrates the quadratic root locus for a controller that yields quadratic gain parameterization.

Example 4.1. Consider the polynomial (4.1), where $q(s) = s+10$, $r(s) = (s+40)^2$, $t(s) = (s+20)^7$, and $\gamma_1 = \gamma_2 = 1$, which implies that $l = 1$, $m = 2$ and $n = 7$. Thus, $d_1 = m - l = 1$, $d_2 = n - m = 5$ and $d = d_1 + d_2 = 6$.

In order to determine the points of $\tilde{p}_k(s)$ that are on the real axis, note that $w(s) = -4(s+22.9)(s+10)(s+22.1+\jmath 2.2)(s-22.1-\jmath 2.2)(s+19.7+\jmath 3.3)(s+19.7-\jmath 3.3)(s+16.8+\jmath 1.9)(s+16.8-\jmath 1.9)$, which has real roots at -22.9 and -10. We apply Fact 4.3 to determine the points of $\tilde{p}_k(s)$ that are on the root locus. Part *(a)* of Fact 4.3 implies that $(-20, -10)$ is on the root locus. Part *(b)* of Fact 4.3 implies that -10 is not on the root locus. Part *(c)* of Fact 4.3 implies that $[-22.9, -10)$ is on the root locus. Combining parts *(a)*–*(c)* of Fact 4.3 implies that $[-22.9, -10)$ is on the root locus. Furthermore, part *(i)* of Fact 4.3 implies that for $\sigma \in -22.9 \cup [-20, -10)$, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Part *(ii)* of Fact 4.3 implies that for $\sigma \in (-22.9, -20)$, there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. A close-up view of the real axis of the quadratic root locus is shown in Figure 4.1.

Next, we examine the asymptotic properties of $\tilde{p}_k(s)$. Since $d_1 < d_2 - 2$, part

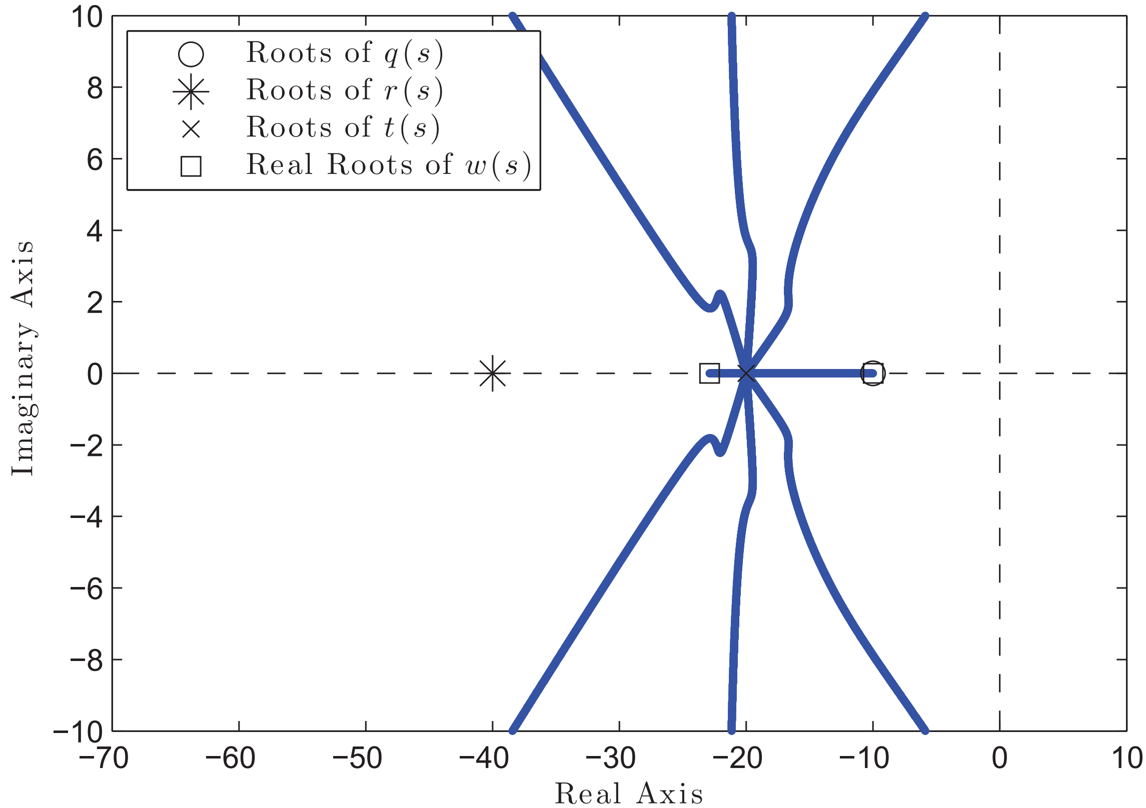

Figure 4.1: The quadratic root locus for $q(s) = s + 10$, $r(s) = (s + 40)^2$ and $t(s) = (s + 20)^7$ shows that $[-22.9, -10)$ is on the root locus.

(a) of Fact 4.4 implies that the six remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha = -21.7$ with angles $\pi/6$, $\pi/2$, $5\pi/6$, $7\pi/6$, $3\pi/2$, and $11\pi/6$. The quadratic root locus is shown in Figure 4.2. △

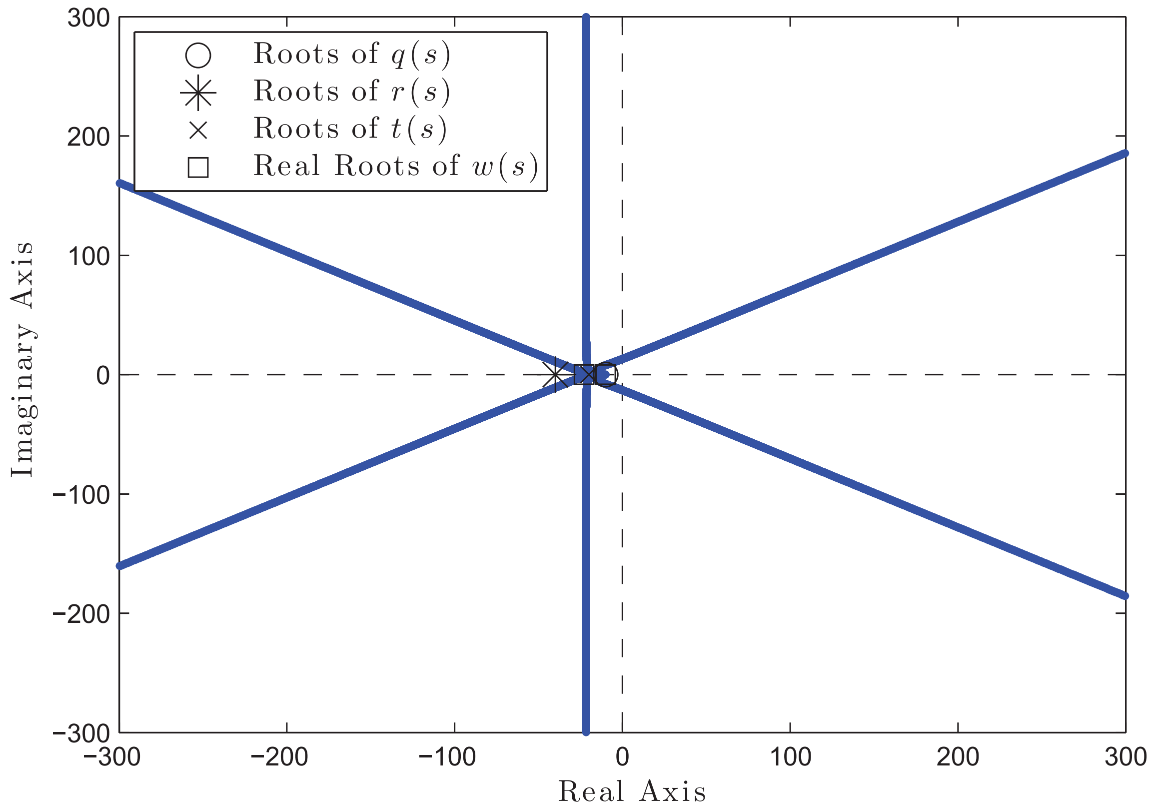

Figure 4.2: The quadratic root locus for $q(s) = s + 10$, $r(s) = (s + 40)^2$ and $t(s) = (s + 20)^7$ shows that six roots of $\tilde{p}_k(s)$ tend to infinity along the asymptotes centered at $\alpha = -21.7$ with angles $\pi/6$, $\pi/2$, $5\pi/6$, $7\pi/6$, $3\pi/2$, and $11\pi/6$.

Example 4.2. Reconsider Example 4.1, where $r(s) = (s + 40)^4$ instead of $r(s) = (s + 40)^2$. Thus, $d_1 = m - l = 3$, $d_2 = n - m = 3$ and $d = d_1 + d_2 = 6$.

In order to determine the points of $\tilde{p}_k(s)$ that are on the real axis, note that $w(s) = -3(s + 28.7)(s - 93.9)(s + 23.2 + \jmath 24.2)(s - 23.2 - \jmath 23.2)(s + 27.6 + \jmath 10.2)(s + 27.6 - \jmath 23.2)(s + 28.5 + \jmath 4.2)(s + 28.5 - \jmath 4.2)$, which has real roots at -28.7 and 93.9. We apply Fact 4.3 to determine the points of $\tilde{p}_k(s)$ that are on the root locus. Part *(a)* of Fact 4.3 implies that $(-20, -10)$ is on the root locus. Part *(b)* of Fact 4.3 implies that -10 is not on the root locus. Part *(c)* of Fact 4.3 implies that $[-28.7, -10)$ is on the root locus. Combining parts *(a)*–*(c)* of Fact 4.3 implies that $[-28.7, -10)$ is on the root locus. Furthermore, part *(i)* of Fact 4.3 implies that for $\sigma \in -28.7 \cup [-20, -10)$, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Part *(ii)* of Fact 4.3 implies that for $\sigma \in (-28.7, -20)$, there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. A close-up view of the real axis of the quadratic root locus is shown in Figure 4.3.

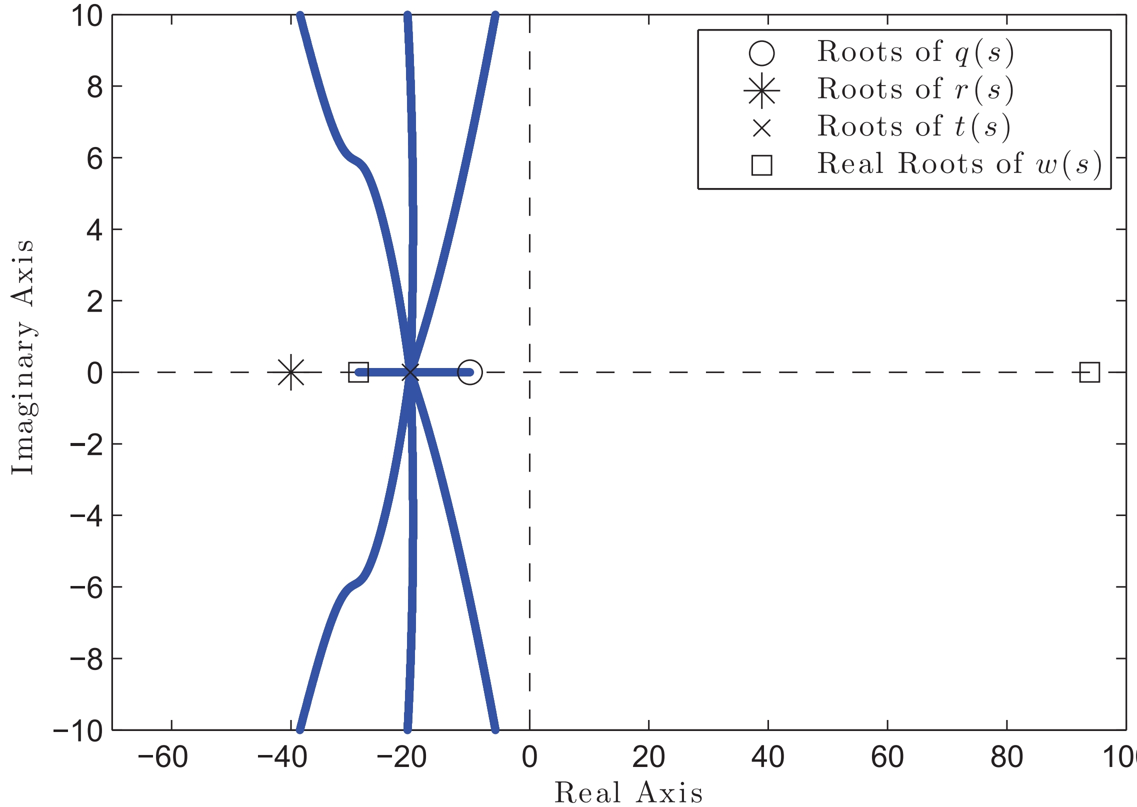

Figure 4.3: The quadratic root locus for $q(s) = s + 10$, $r(s) = (s + 40)^4$ and $t(s) = (s + 20)^7$ shows that $[-28.7, -10)$ is on the root locus.

Next, we examine the asymptotic properties of $\tilde{p}_k(s)$. Since $d_1 = d_2$, part *(b)* of Conjecture 4.1 implies that the six remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha = -21.7$ with angles $2\pi/9$, $4\pi/9$, $8\pi/9$, $10\pi/9$, $14\pi/9$, and $16\pi/9$. Note that the asymptote angles are not equally spaced between 0 and 2π radians. The quadratic root locus is shown in Figure 4.4. △

Example 4.3. Reconsider Example 4.1, where $r(s) = (s + 40)^5$ instead of $r(s) = (s + 40)^2$. Thus, $d_1 = m - l = 4$, $d_2 = n - m = 2$ and $d = d_1 + d_2 = 6$.

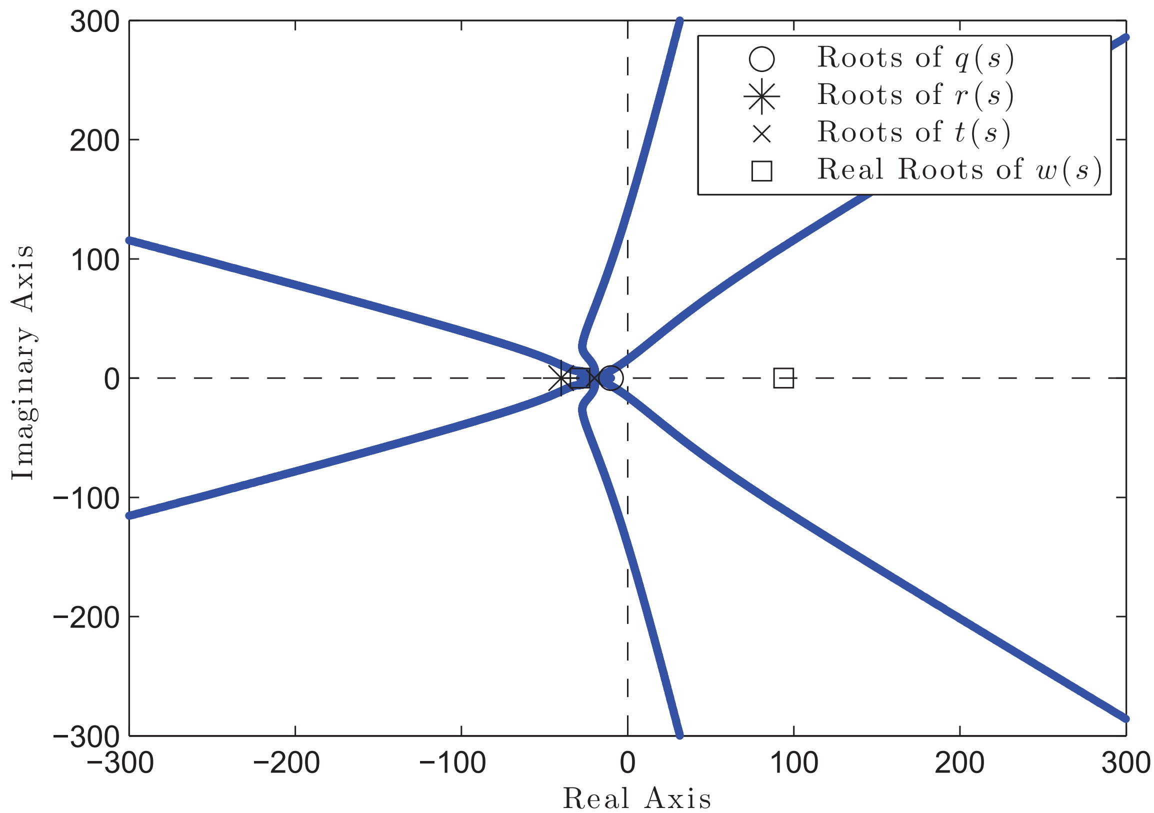

Figure 4.4: The quadratic root locus for $q(s) = s+10$, $r(s) = (s+40)^4$ and $t(s) = (s+20)^7$ shows that the six remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha = -21.7$ with angles $2\pi/9$, $4\pi/9$, $8\pi/9$, $10\pi/9$, $14\pi/9$, and $16\pi/9$.

In order to determine the points of $\tilde{p}_k(s)$ that are on the real axis, note that $w(s) = (s+64.3)(s+31.4)(s+50.4+\jmath 17.3)(s+50.4-\jmath 17.3)(s+37.1+\jmath 13.8)(s+37.1-\jmath 13.8)(s+32.9+\jmath 7.9)(s+32.9-\jmath 7.9)(s+31.7+\jmath 3.6)(s+31.7-\jmath 3.6)$, which has real roots at -64.3 and -31.4. We apply Fact 4.3 to determine the points of $\tilde{p}_k(s)$ that are on the root locus. Part *(a)* of Fact 4.3 implies that $(-20,-10)$ is on the root locus. Part *(b)* of Fact 4.3 implies that -10 is not on the root locus. Part *(c)* of Fact 4.3 implies that $[-31.4,-10)$ is on the root locus. Combining parts *(a)*–*(c)* of Fact 4.3 implies that $[-31.4,-10)$ is on the root locus. Furthermore, part *(i)* of Fact 4.3 implies that for $\sigma \in -31.4 \cup [-20,-10)$, there exists exactly one $k>0$ such that $\tilde{p}_k(\sigma)=0$. Part *(ii)* of Fact 4.3 implies that for $\sigma \in (-31.4,-20)$, there exists two distinct $k>0$ such that $\tilde{p}_k(\sigma)=0$. A close-up view of the real axis of the quadratic root locus is shown in Figure 4.5.

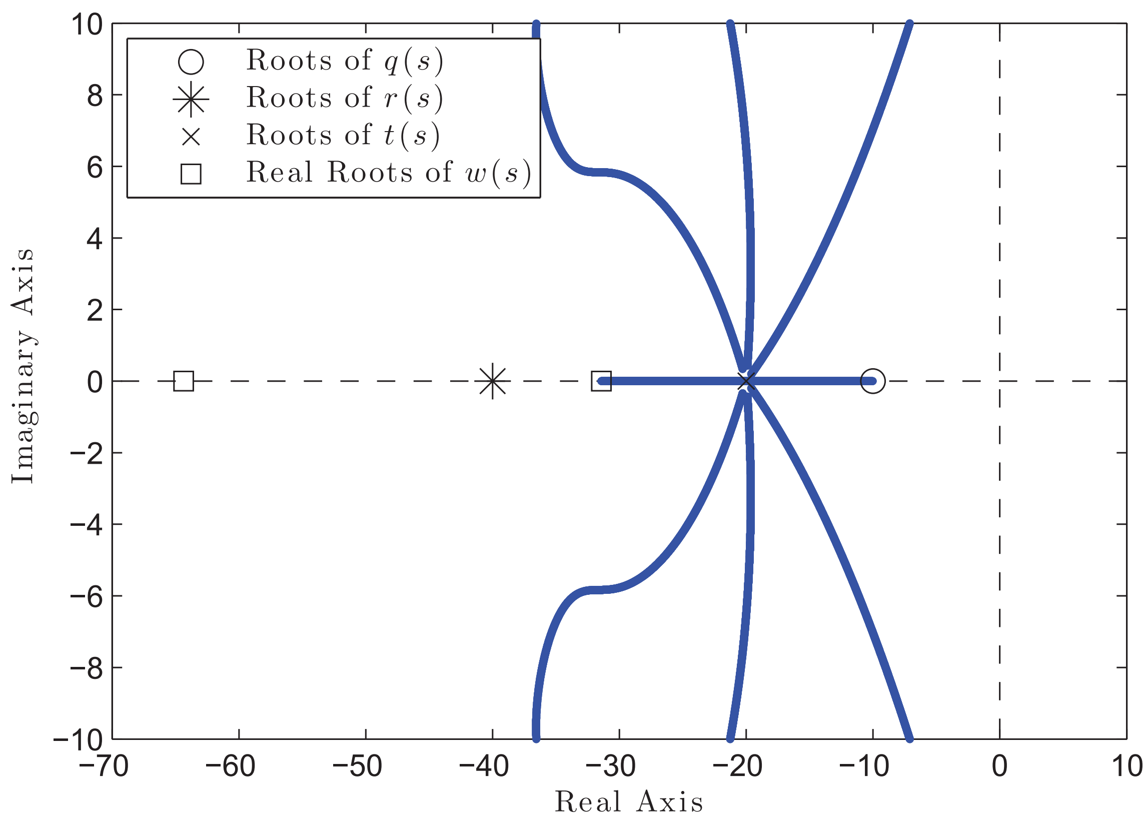

Figure 4.5: The quadratic root locus for $q(s)=s+10$, $r(s)=(s+40)^5$ and $t(s)=(s+20)^7$ shows that $[-31.4,-10)$ is on the root locus.

Next, we examine the asymptotic properties of $\tilde{p}_k(s)$. Since $d_1 > d_2 + 1$, part *(b)* of Fact 4.4 implies that four of the remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha_1 = -47.5$ with angles $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. Furthermore, the two remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha_2 = 30$ with angles $\pi/2$ and $3\pi/2$. The quadratic root locus is shown in Figure 4.6. △

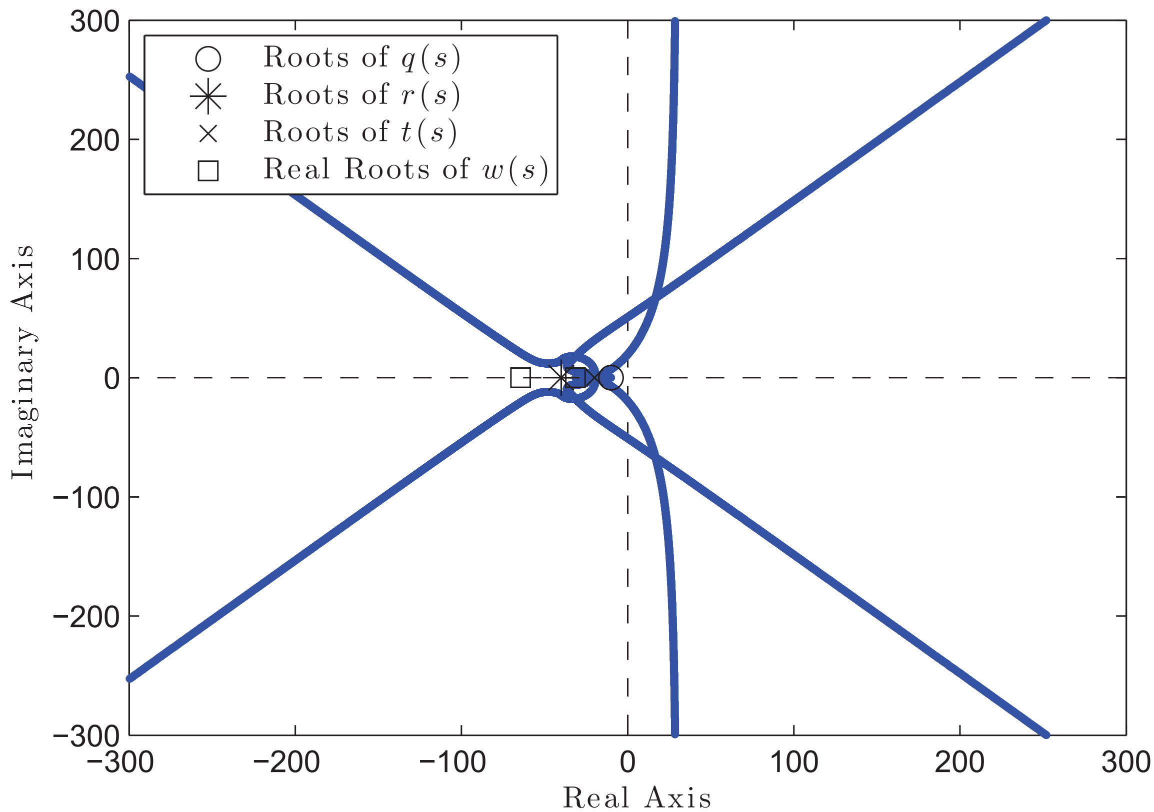

Figure 4.6: The quadratic root locus for $q(s) = s+10$, $r(s) = (s+40)^5$ and $t(s) = (s+20)^7$ shows that four of the remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha_1 = -47.5$ with angles $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. Furthermore, two of the remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at $\alpha_2 = 30$ with angles $\pi/2$ and $3\pi/2$.

Next, we reconsider the triple integrator from Chapter 2. Specifically, we show that the triple integrator can be high-gain stabilized using a simpler controller class than the controller class in Chapter 2.

Example 4.4. Consider the single-input single-output linear time-invariant system

$$y(s) = G(s)u(s), \tag{4.27}$$

where

$$G(s) \triangleq \frac{1}{s^3}. \tag{4.28}$$

Note that $G(s)$ is minimum phase and relative degree 3. In Chapter 2, we high-gain stabilize (4.28) using the quadratic root locus for the special controller class considered in Chapter 2. We now use the generalized quadratic root locus to construct a simpler controller that high-gain stabilizes (4.28). Specifically, consider the control

$$u(s) = \hat{G}_k(s)(v(s) - y(s)), \tag{4.29}$$

where

$$\hat{G}_k(s) \triangleq \frac{k^2(s+b)^2}{s^2 + k(s+a)}, \tag{4.30}$$

where $v(s)$ is an external signal and a and b are real numbers. Note that (4.30) cannot be expressed given by (2.4), and thus, (4.30) is not in the controller class in Chapter 2.

The closed-loop transfer function from v to y is given by

$$\tilde{G}_k(s) \triangleq \frac{G(s)\hat{G}_k(s)}{1 + G(s)\hat{G}_k(s)} = \frac{k^2(s+b)^2}{\tilde{p}_k(s)}, \tag{4.31}$$

where $\tilde{p}_k(s)$ is given by (4.1), where $\gamma_1 = \gamma_2 = 1$, and specifically,

$$q(s) \triangleq (s+b)^2 = s^2 + 2bs + b^2, \tag{4.32}$$

$$r(s) \triangleq (s+a)s^3 = s^4 + as^3, \tag{4.33}$$

$$t(s) \triangleq s^5. \tag{4.34}$$

Thus, $d_1 = 2$, $d_2 = 1$ and $d = 3$. Since $d_1 = d_2 + 1$, part *(c)* of Conjecture 4.1

implies that as k tends to infinity, two roots of $\tilde{p}_k(s)$ approach $-b$, one root of $\tilde{p}_k(s)$ approaches minus infinity, and the two remaining roots of $\tilde{p}_k(s)$ approach infinity along asymptotes centered at α_1 with angles $\pi/2$ and $3\pi/2$. Moreover, note that $\sum_{j=1}^{m} r_j = -a$ and $\sum_{j=1}^{l} q_j = -2b$, which implies that $\alpha_1 = (-a+2b+1)/2$. Therefore, if $a > 2b+1$, then $\alpha_1 < 0$. Thus, if $b > 0$ and $a > 2b+1$, the controller class (4.30) high-gain stabilizes the triple integrator.

Next, let $a = 10$ and $b = 1$, which implies that $q(s)$ is asymptotically stable and $\alpha_1 = -7/2 < 0$. Since $d_1 = 2$, $d_2 = 1$, $q(s)$ is asymptotically stable, and $\alpha_1 < 0$, it follows that the closed-loop transfer function $\tilde{G}_k(s)$ given by (4.31) is high-gain stable. Next, we apply the quadratic root-locus rules in Section 4.3. First, Fact 4.1 implies that the root locus begins at the roots of $t(s)$. In order to determine the points of $\tilde{p}_k(s)$ that are on the real axis, note that $w(s) = s^5(s-0.04)(s+8.02+\jmath 5.32)(s+8.02-\jmath 5.32)$. We apply Fact 4.3 to determine the points of $\tilde{p}_k(s)$ that are on the root locus. Part *(a)* of Fact 4.3 implies that $(-\infty,-1)\cup(-1,0)$ is on the root locus. Part *(b)* of Fact 4.3 implies that -1 is not on the root locus. Part *(c)* of Fact 4.3 implies that $(-\infty,-1)\cup(-1,0)$ is on the root locus. Combining parts *(a)*–*(c)* of Fact 4.3 implies that $(-\infty,-1)\cup(-1,0)$ is on the root locus. Furthermore, part *(i)* of Fact 4.3 implies that for $\sigma \in (-\infty,-1)\cup(-1,0)$, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. The quadratic root locus is shown in Figure 4.7. Note that $\tilde{G}_k(s)$ is asymptotically stable for all $k > 6.2$. △

4.6 Conclusions

We presented rules for constructing the root locus for a class of polynomials that are quadratic in the root-locus parameter k. These quadratic root-locus rules apply to controller classes that are rational functions of k and yield a closed-loop denominator that is quadratic in k. The controller class in Chapter 2 is a special case of the more general structure considered in this chapter.

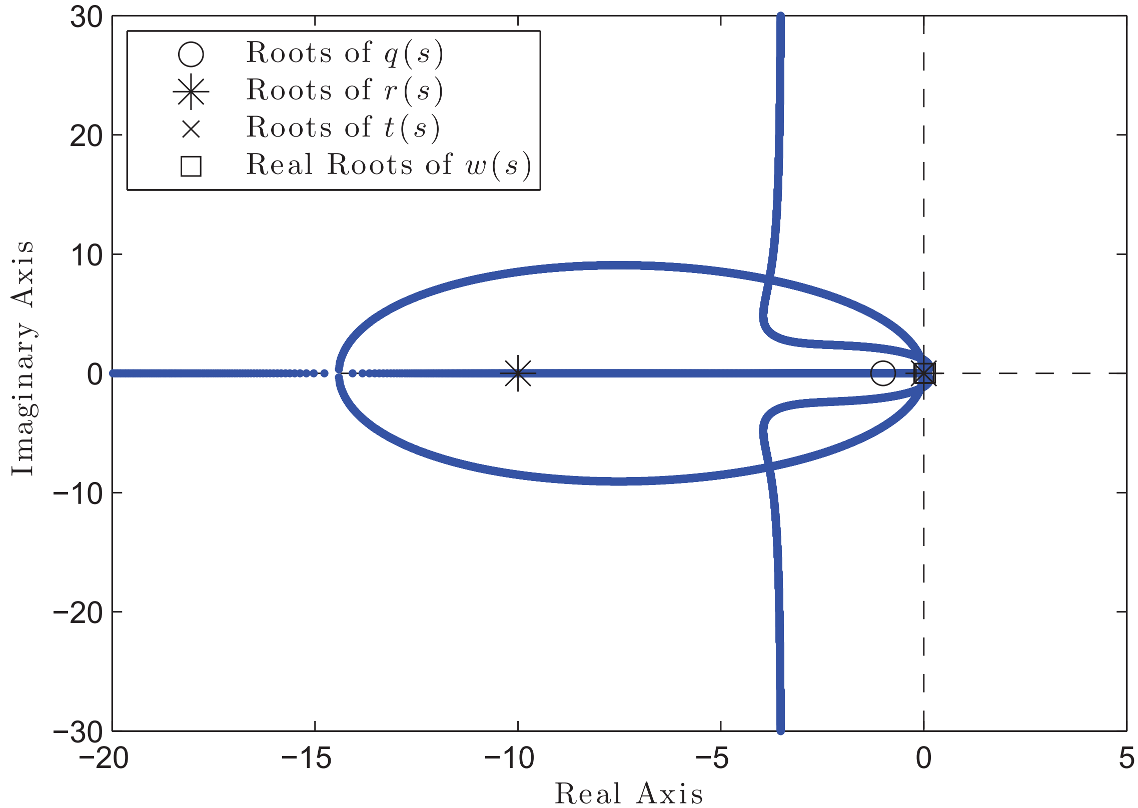

Figure 4.7: The quadratic root locus shows that the triple integrator $G(s) = 1/s^3$ is high-gain stabilized by the controller $\hat{G}_k(s) = k^2(s+1)^2/(s^2 + k(s+10))$. In fact, the closed-loop system is asymptotically stable for all $k > 6.2$. Furthermore, the quadratic root locus shows that $(-\infty, -1) \cup (-1, 0)$ is on the root locus.

To develop the quadratic root locus, we extended the techniques of Chapter 2. In principle, the techniques of this chapter could be extended further to address closed-loop denominator polynomials that are cubic in k, which would extend the techniques developed in Chapter 3. For example, the asymptote rule in this chapter (i.e., Fact 4.4) is constructed in part by finding the four largest terms of $\tilde{p}_k(s)$ in an annulus that depends on k. This technique can be applied to polynomials that are cubic in k.

4.7 Proofs for Facts 4.1–4.6

Proof of Fact 4.1. If $k = 0$, then $\tilde{p}_k(s) = t(s)$. □

Proof of Fact 4.2. Since for all $k > 0$, $\tilde{p}_k(s)$ has real coefficients, it follows that the roots of $\tilde{p}_k(s)$ are either on the real axis or occur in complex conjugate pairs. □

Proof of Fact 4.3. First, we show that *(a)*, *(b)* or *(c)* are necessary for σ to be on the root locus. Assume σ is on the root locus, and consider three cases: (A) $q(\sigma)t(\sigma) < 0$; (B) $q(\sigma) = 0$; (C) $t(\sigma) = 0$; and (D) $q(\sigma)t(\sigma) > 0$. First, assume (A) $q(\sigma)t(\sigma) < 0$, which implies *(a)*. Next, assume (B) $q(\sigma) = 0$, which implies that $\tilde{p}_k(\sigma) = \gamma_2 r(\sigma)k + t(\sigma)$. Since σ is on the root locus, it follows that there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, $-t(\sigma)/(\gamma_2 r(\sigma)) > 0$, which implies that $r(\sigma)t(\sigma) < 0$ because $\gamma_2 > 0$, which implies *(b)*. Next, assume (C) $t(\sigma) = 0$, which implies that $\tilde{p}_k(\sigma) = k(k\gamma_1 q(\sigma) + \gamma_2 r(\sigma))$. Since σ is on the root locus, it follows that there exists $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, $-\gamma_2 r(\sigma)/(\gamma_1 q(\sigma)) > 0$, which implies that $q(\sigma)r(\sigma) < 0$ because $\gamma_1\gamma_2 > 0$. Furthermore, since $t(\sigma) = 0$, it follows from (4.2) that $w(\sigma) = \gamma_2^2 r(\sigma)^2 - 4\gamma_1 q(\sigma)t(\sigma) = \gamma_2^2 r(\sigma)^2 \geq 0$. Finally, assume (D) $q(\sigma)t(\sigma) > 0$. Since σ is on the root locus, it follows that $\tilde{p}_k(\sigma)$ has at least one positive root. Since $\gamma_1 > 0$ and $q(\sigma)t(\sigma) > 0$, it follows that $\gamma_1 q(\sigma)t(\sigma) > 0$. Therefore, part *(a)* of Lemma 1 in Appendix A implies that $\tilde{p}_k(\sigma)$ does not have exactly one positive root. Thus, $\tilde{p}_k(\sigma)$ has two positive roots. Therefore, parts *(b)* and *(c)* of Lemma 1 in Appendix

A imply that $w(\sigma) \geq 0$ and $q(\sigma)r(\sigma) < 0$. Combining (C) and (D) implies *(c)*.

Conversely, assume *(a)*, *(b)* or *(c)* holds. First, assume *(a)* holds. Since $\gamma_1 > 0$ and $q(\sigma)t(\sigma) < 0$, it follows that $\gamma_1 q(\sigma)t(\sigma) < 0$. Thus, part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

Next, assume *(b)* holds. Since $q(\sigma) = 0$, it follows that $\tilde{p}_k(\sigma) = \gamma_2 r(\sigma)k + t(\sigma)$. In addition, since $r(\sigma)t(\sigma) < 0$ and $\gamma_2 > 0$, it follows that $-\gamma_2 t(\sigma)/r(\sigma) > 0$. Thus, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, and σ is on the root locus.

Finally, assume *(c)* holds. We consider four cases: (D) $w(\sigma) = 0$; (E) $w(\sigma) > 0$ and $q(\sigma)t(\sigma) < 0$; (F) $w(\sigma) > 0$ and $q(\sigma)t(\sigma) = 0$; and (G) $w(\sigma) > 0$ and $q(\sigma)t(\sigma) > 0$. First, assume (D) $w(\sigma) = 0$. Since $w(\sigma) = 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, and $q(\sigma)r(\sigma) < 0$, it follows from (4.2) that $t(\sigma) \neq 0$. Therefore, part *(b)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus. Next, assume (E) $w(\sigma) > 0$ and $q(\sigma)t(\sigma) < 0$. Since $\gamma_1 > 0$ and $q(\sigma)t(\sigma) < 0$, it follows that $\gamma_1 q(\sigma)t(\sigma) < 0$. Therefore, part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus. Next, assume (F) $w(\sigma) > 0$ and $q(\sigma)t(\sigma) = 0$. Since $q(\sigma)t(\sigma) = 0$ and $q(\sigma)r(\sigma) < 0$, it follows that $t(\sigma) = 0$. Therefore, $\tilde{p}_k(\sigma) = k(\gamma_1 q(\sigma)k + \gamma_2 r(\sigma))$. Since $\gamma_1 > 0$, $\gamma_2 > 0$ and $q(\sigma)r(\sigma) < 0$, it follows that $-\gamma_2 r(\sigma)/(\gamma_1 q(\sigma)) > 0$, which implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Thus, σ is on the root locus. Finally, assume (G) $w(\sigma) > 0$ and $q(\sigma)t(\sigma) > 0$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $q(\sigma)r(\sigma) < 0$, and $q(\sigma)t(\sigma) > 0$, it follows that $\gamma_1\gamma_2 q(\sigma)r(\sigma) < 0$ and $\gamma_1 q(\sigma)t(\sigma) > 0$. Since $w(\sigma) > 0$, $\gamma_1\gamma_2 q(\sigma)r(\sigma) < 0$ and $\gamma_1 q(\sigma)t(\sigma) > 0$, part *(c)* of Lemma 1 in Appendix A implies that there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, σ is on the root locus.

To show *(i)* and *(ii)*, assume σ is on the root locus. First, we show *(i)*. Assume $q(\sigma)t(\sigma) \leq 0$ or $w(\sigma) = 0$, and consider four cases: (H) $q(\sigma)t(\sigma) < 0$, (I) $q(\sigma) = 0$,

(J) $t(\sigma) = 0$, and (K) $w(\sigma) = 0$. First, assume (H) $q(\sigma)t(\sigma) < 0$. Since $\gamma_1 > 0$ and $q(\sigma)t(\sigma) < 0$, it follows that $\gamma_1 q(\sigma)t(\sigma) < 0$. Thus, part *(a)* of Lemma 1 in Appendix A implies that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Next, assume (I) $q(\sigma) = 0$, which implies that $\tilde{p}_k(\sigma) = \gamma_2 r(\sigma)k + t(\sigma)$ and that there exists at most one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Since σ is on the root locus, it follows that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Next, assume (J) $t(\sigma) = 0$, which implies that $\tilde{p}_k(\sigma) = k(\gamma_1 q(\sigma)k + \gamma_2 r(\sigma))$. Since σ is on the root locus and $\tilde{p}_k(\sigma)$ has one nonzero root, it follows that there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Finally, assume (K) $w(\sigma) = 0$, which implies that $\tilde{p}_k(\sigma)$ has repeated real roots. Since σ is on the root locus and $\tilde{p}_k(\sigma)$ has repeated real roots, it follows that the repeated roots are positive. Thus, there exists exactly one $k > 0$ such that $\tilde{p}_k(\sigma) = 0$. Therefore, cases (H), (I), (J), and (K) confirm *(i)*.

Next, we show *(ii)*. Assume $w(\sigma) > 0$, $q(\sigma)r(\sigma) < 0$ and $q(\sigma)t(\sigma) > 0$. Since $\gamma_1 > 0$, $\gamma_2 > 0$, $q(\sigma)r(\sigma) < 0$, and $q(\sigma)t(\sigma) > 0$, it follows that $\gamma_1\gamma_2 q(\sigma)r(\sigma) < 0$ and $\gamma_1 q(\sigma)t(\sigma) > 0$. Since $w(\sigma) > 0$, $\gamma_1\gamma_2 q(\sigma)r(\sigma) < 0$ and $\gamma_1 q(\sigma)t(\sigma) > 0$, part *(c)* of Lemma 1 in Appendix A implies that there exists two distinct $k > 0$ such that $\tilde{p}_k(\sigma) = 0$, which confirms *(ii)*. □

Proof of Fact 4.4. To show that l roots of $\tilde{p}_k(s)$ converge to the roots of $q(s)$, it follows from (4.1) that

$$\frac{\tilde{p}_k(s)}{k^2} = q(s) + \frac{r(s)}{k} + \frac{t(s)}{k^2},$$

which implies that, for sufficiently large $k > 0$, $\tilde{p}_k(s)/k^2 \approx q(s)$. Thus, as $k \to \infty$, l roots of $\tilde{p}_k(s)$ converge to the roots of $q(s)$.

Next, write

$$q(s) = s^l + a_1 s^{l-1} + \cdots + a_l, \tag{4.35}$$

$$r(s) = s^m + b_1 s^{m-1} + \cdots + b_m, \tag{4.36}$$

and

$$t(s) = s^n + c_1 s^{n-1} + \cdots + c_n, \tag{4.37}$$

where $a_1, a_2, \ldots, a_l, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n \in \mathbb{R}$. Thus,

$$\tilde{p}_k(s) = k^2 \left(s^l + \sum_{i=1}^{l} a_i s^{l-i} \right) + k \left(s^m + \sum_{i=1}^{m} b_i s^{m-i} \right) + s^n + \sum_{i=1}^{n} c_i s^{n-i}. \tag{4.38}$$

Note that $a_1 = -\sum_{j=1}^{l} q_j$, $b_1 = -\sum_{j=1}^{m} r_j$ and $c_1 = -\sum_{j=1}^{n} t_j$.

To show *(a)*, assume $d_1 < d_2 - 2$. Let $\epsilon < 1/(dn^2)$ be positive, and for $k > 1$, define $\mathfrak{T}_k \triangleq \{s \in \mathbb{C} : k^{2/d-\epsilon} < |s| < k^{2/d+\epsilon}\}$. We show that for sufficiently large $k > 0$ and for all $s \in \mathfrak{T}_k$, the four terms of (4.38) with the largest magnitude are $k^2 s^l$, $a_1 k^2 s^{l-1}$, s^n, and $c_1 s^{n-1}$. For all $k > 1$ and all $s \in \mathfrak{T}_k$, it follows that

$$|s|^{n-1} > (k^{2/d-\epsilon})^{n-1} = k^{(2n-2)/d-\epsilon(n-1)}, \tag{4.39}$$

$$k^2 |s|^{l-1} > k^2 (k^{2/d-\epsilon})^{l-1} = k^{(2n-2)/d-\epsilon(l-1)}. \tag{4.40}$$

Since $l < n$ and $\epsilon < 1/(dn^2)$, it follows from (4.39) and (4.40) that for all $k > 1$ and $s \in \mathfrak{T}_k$,

$$|s|^{n-1} > k^{(2n-2)/d-\epsilon(n-1)} > k^{(2n-2)/d-(n-1)/(dn^2)} = k^{(2n^3-2n^2-n+1)/(dn^2)},$$

$$k^2 |s|^{l-1} > k^{(2n-2)/d-\epsilon(l-1)} > k^{(2n-2)/d-\epsilon(n-1)} > k^{(2n^3-2n^2-n+1)/(dn^2)}.$$

Moreover, for all $k > 1$ and $s \in \mathfrak{T}_k$, it follows that

$$|s|^n > |s|^{n-1} > k^{(2n^3-2n^2-n+1)/(dn^2)}, \tag{4.41}$$

$$k^2|s|^l > k^2|s|^{l-1} > k^{(2n^3-2n^2-n+1)/(dn^2)}. \tag{4.42}$$

Next, we show that the magnitude of each of the remaining terms of (4.38) is smaller than $k^{(2n^3-2n^2-n+1)/(dn^2)}$. For all $k > 1$ and $s \in \mathcal{T}_k$, it follows that

$$|s|^{n-2} < (k^{2/d+\epsilon})^{n-2} = k^{(2n-4)/d+\epsilon(n-2)}, \tag{4.43}$$

$$k^2|s|^{l-2} < k^2(k^{2/d+\epsilon})^{l-2} = k^{(2n-4)/d+\epsilon(l-2)}, \tag{4.44}$$

$$k|s|^m < k(k^{2/d+\epsilon})^m = k^{(d+2m)/d+\epsilon m}. \tag{4.45}$$

Since $l < n$ and $\epsilon < 1/(dn^2)$, it follows from (4.43)–(4.45) that for all $s \in \mathcal{T}_k$,

$$|s|^{n-2} < k^{(2n-4)/d+\epsilon(n-2)} < k^{(2n-4)/d+(n-2)/(dn^2)} = k^{(2n^3-4n^2+n-2)/(dn^2)}, \tag{4.46}$$

$$k^2|s|^{l-2} < k^{(2n-4)/d+\epsilon(l-2)} < k^{(2n-4)/d+\epsilon(n-2)} < k^{(2n^3-4n^2+n-2)/(dn^2)}, \tag{4.47}$$

$$k|s|^m < k^{(d+2m)/d+\epsilon m} < k^{(d+2m)/d+m/(dn^2)} = k^{(n^3-ln^2+2mn^2+m)/(dn^2)}. \tag{4.48}$$

Since n is a positive integer, it follows from (4.41), (4.42), (4.46), and (4.47) that for sufficiently large $k > 0$ and all $s \in \mathcal{T}_k$,

$$|c_1||s|^{n-1} > k^2|a_2||s|^{l-2} > k^2|a_3||s|^{l-3} > \cdots > k^2|a_l|, \tag{4.49}$$

$$k^2|a_1||s|^{l-1} > |c_2||s|^{n-2} > |c_3||s|^{n-3} > \cdots > |c_n|. \tag{4.50}$$

Next, since $d_1 < d_2 - 2$, it follows that $n + l - 2m - 2 > 0$. Since $0 \le l < m < n$ and l, m and n are integers, it follows that $n \ge 2$, which implies that $n - 1 - n^2 < -n$. Furthermore, since $m \le n - 1$, $n + l - 2m - 2 > 0$ and $n - 1 - n^2 < -n$, it follows that

$$n^3 - ln^2 + 2mn^2 + m \le n^3 - ln^2 + 2mn^2 + n - 1$$

$$
\begin{aligned}
&= 2n^3 - 2n^2 + n - 1 - n^2(n + l - 2m - 2) \\
&< 2n^3 - 2n^2 + n - 1 - n^2 \\
&< 2n^3 - 2n^2 - n,
\end{aligned}
$$

which combined with (4.48) implies

$$k|s|^m < k^{(2n^3 - 2n^3 - n)/(dn^2)}. \tag{4.51}$$

Therefore, it follows from (4.41), (4.42) and (4.51) that for sufficiently large $k > 0$ and for all $s \in \mathcal{T}_k$,

$$|c_1||s|^{n-1} > k|s|^m > k|b_1||s|^{m-1} > k|b_2||s|^{m-2} > \cdots > k|b_m|, \tag{4.52}$$

$$k^2|a_1||s|^{l-1} > k|s|^m > k|b_1||s|^{m-1} > k|b_2||s|^{m-2} > \cdots > k|b_m|. \tag{4.53}$$

Thus, (4.41), (4.42), (4.49), (4.50), (4.52), and (4.53) imply that for sufficiently large $k > 0$ and for all $s \in \mathcal{T}_k$, the four terms of (4.38) with the largest magnitude are $k^2 s^l$, $a_1 k^2 s^{l-1}$, s^n, and $c_1 s^{n-1}$.

Therefore, for sufficiently large $k > 0$ and for all $s \in \mathcal{T}_k$,

$$\tilde{p}_k(s) \approx \mu_k(s), \tag{4.54}$$

where

$$\mu_k(s) \stackrel{\triangle}{=} k^2 s^l + k^2 a_1 s^{l-1} + s^n + c_1 s^{n-1}. \tag{4.55}$$

Since $a_1 = -\sum_{j=1}^{l} q_j$ and $c_1 = -\sum_{j=1}^{n} t_j$, Lemma 6 in Appendix C implies that as k tends to infinity, $d = n - l$ roots of $\mu_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$ given by (4.5)–(4.7). Thus, it follows from (4.54) that for sufficiently large $k > 0$, d roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$ given by (4.5)–(4.7), which confirms *(a)*.

To show *(b)*, assume $d_1 > d_2 + 1$. First, we show that d_1 roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d_1}$. Let $\epsilon < 1/(d_1 n^2)$ be positive and, for $k > 1$, define $\mathcal{U}_k \triangleq \{s \in \mathbb{C} : k^{1/d_1 - \epsilon} < |s| < k^{1/d_1 + \epsilon}\}$. We show that for sufficiently large $k > 0$ and for all $s \in \mathcal{U}_k$, the four terms of (4.38) with the largest magnitude are $k^2 s^l$, $a_1 k^2 s^{l-1}$, $k s^m$, and $b_1 k s^{m-1}$. For all $k > 1$ and $s \in \mathcal{U}_k$, it follows that

$$k|s|^{m-1} > k(k^{1/d_1-\epsilon})^{m-1} = k^{1+(m-1)/d_1-\epsilon(m-1)} = k^{(2m-l-1)/d_1-\epsilon(m-1)}, \tag{4.56}$$

$$k^2|s|^{l-1} > k^2(k^{1/d_1-\epsilon})^{l-1} = k^{2+(l-1)/d_1-\epsilon(l-1)} = k^{(2m-l-1)/d_1-\epsilon(l-1)}. \tag{4.57}$$

Since $l < m$ and $\epsilon < 1/(d_1 n^2)$, it follows from (4.56) and (4.57) that for all $k > 1$ and all $s \in \mathcal{U}_k$,

$$k|s|^{m-1} > k^{(2m-l-1)/d_1-\epsilon(m-1)} > k^{(2m-l-1)/d_1-(m-1)/(d_1 n^2)} = k^{(2mn^2-ln^2-n^2-m+1)/(d_1 n^2)},$$

$$k^2|s|^{l-1} > k^{(2m-l-1)/d_1-\epsilon(l-1)} > k^{(2m-l-1)/d_1-\epsilon(m-1)} > k^{(2mn^2-ln^2-n^2-m+1)/(d_1 n^2)}.$$

Moreover, for all $k > 1$ and $s \in \mathcal{U}_k$, it follows that

$$k|s|^m > k|s|^{m-1} > k^{(2mn^2-ln^2-n^2-m+1)/(d_1 n^2)}, \tag{4.58}$$

$$k^2|s|^l > k^2|s|^{l-1} > k^{(2mn^2-ln^2-n^2-m+1)/(d_1 n^2)}. \tag{4.59}$$

Next, we show the magnitude of each of the remaining terms of (4.38) is smaller than $k^{(2mn^2-ln^2-n^2-m+1)/(d_1 n^2)}$. For all $k > 1$ and all $s \in \mathcal{U}_k$, it follows that

$$k|s|^{m-2} < k(k^{1/d_1+\epsilon})^{m-2} = k^{(2m-l-2)/d_1+\epsilon(m-2)}, \tag{4.60}$$

$$k^2|s|^{l-2} < k^2(k^{1/d_1+\epsilon})^{l-2} = k^{(2m-l-2)/d_1+\epsilon(l-2)}, \tag{4.61}$$

$$|s|^n < (k^{1/d_1+\epsilon})^n = k^{n/d_1+\epsilon n}. \tag{4.62}$$

Since $l < m$ and $\epsilon < 1/(d_1 n^2)$, it follows from (4.60)–(4.62) that

$$
\begin{aligned}
k|s|^{m-2} &< k^{(2m-l-2)/d_1+\epsilon(m-2)} < k^{(2m-l-2)/d_1+(m-2)/(d_1n^2)} \\
&= k^{(2mn^2-ln^2-2n^2+m-2)/(d_1n^2)}, && (4.63) \\
k^2|s|^{l-2} &< k^{(2m-l-2)/d_1+\epsilon(l-2)} < k^{(2m-l-2)/d_1+\epsilon(m-2)} \\
&< k^{(2mn^2-ln^2-2n^2+m-2)/(d_1n^2)}, && (4.64) \\
|s|^n &< k^{n/d_1+n/(d_1n^2)} = k^{(n^3+n)/(d_1n^2)}. && (4.65)
\end{aligned}
$$

Since m is a positive integer and $n \geq m+1$, it follows that $-m+1 > -n^2+m-2$, which implies from (4.58), (4.59), (4.63), and (4.64) that for sufficiently large $k > 0$ and for all $s \in \mathcal{U}_k$,

$$
k|b_1||s|^{m-1} > k^2|a_2||s|^{l-2} > k^2|a_3||s|^{l-3} > \cdots > k^2|a_l|, \tag{4.66}
$$

$$
k^2|a_1||s|^{l-1} > k|b_2||s|^{m-2} > k|b_3||s|^{m-3} > \cdots > k|b_m|. \tag{4.67}
$$

Next, since $d_1 > d_2 + 1$, it follows that $2m - n - l - 1 > 0$. Furthermore, since m is a positive integer and $n \geq m+1$, it follows that $n - n^2 < -m + 1$. Since $2m - l - n - 1 > 0$ and $n - n^2 \leq 1 - m$, it follows that

$$
\begin{aligned}
n^3 + n &= 2mn^2 - ln^2 - n^2 + n - n^2(2m - l - n - 1) \\
&< 2mn^2 - ln^2 - n^2 + n - n^2 \\
&< 2mn^2 - ln^2 - n^2 - m + 1,
\end{aligned}
$$

which combined with (4.65) yields

$$
|s|^n < k^{(2mn^2-ln^2-n^2-m+1)/(d_1n^2)}. \tag{4.68}
$$

Therefore, it follows from (4.58), (4.59) and (4.68) that for sufficiently large $k > 0$

and for all $s \in \mathcal{U}_k$,

$$|b_1|k|s|^{m-1} > |s|^n > |c_1||s|^{n-1} > \cdots > |c_n|, \tag{4.69}$$

$$|a_1|k^2|s|^{l-1} > |s|^n > |c_1||s|^{n-1} > \cdots > |c_n|. \tag{4.70}$$

Thus, (4.58), (4.59), (4.66), (4.67), (4.69), and (4.70) imply that for sufficiently large $k > 0$ and for all $s \in \mathcal{U}_k$, the four terms of (4.38) with the largest magnitude are $k^2 s^l$, $a_1 k^2 s^{l-1}$, ks^m, and $b_1 k s^{m-1}$.

Therefore, for sufficiently large $k > 0$ and for all $s \in \mathcal{U}_k$,

$$\tilde{p}_k(s) \approx \nu_k(s), \tag{4.71}$$

where

$$\nu_k(s) \triangleq k\left(ks^l + ka_1 s^{l-1} + s^m + b_1 s^{m-1}\right). \tag{4.72}$$

Since $a_1 = -\sum_{j=1}^{l} q_j$ and $b_1 = -\sum_{j=1}^{m} r_j$, Lemma 6 in Appendix C implies that as k tends to infinity, $d_1 = m - l$ roots of $\nu_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d_1}$ given by (4.8)–(4.10). Thus, it follows from (4.71) that for sufficiently large $k > 0$, d_1 roots of $\tilde{p}_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_{d_1}$ given by (4.8)–(4.10).

Next, we show that $d_2 = d - d_1$ roots of $\tilde{p}_k(s)$ are approximated by $\lambda_{d_1+1}, \lambda_{d_1+2}, \ldots, \lambda_d$. Let $\epsilon < 1/(d_2 n^2)$ be positive and, for $k > 1$, define $\mathcal{V}_k \triangleq \{s \in \mathbb{C} : k^{1/d_2 - \epsilon} < |s| < k^{1/d_2 + \epsilon}\}$. We show that for sufficiently large $k > 0$ and for all $s \in \mathcal{V}_k$, the four terms of (4.38) with the largest magnitude are ks^m, $b_1 k s^{m-1}$, s^n, and $c_1 s^{n-1}$. For all $s \in \mathcal{V}_k$, it follows that

$$s^{n-1} > (k^{1/d_2 - \epsilon})^{n-1} = k^{(n-1)/d_2 - \epsilon(n-1)}, \tag{4.73}$$

$$k|s|^{m-1} > k(k^{1/d_2 - \epsilon})^{m-1} = k^{1+(m-1)/d_2 - \epsilon(m-1)} = k^{(n-1)/d_2 - \epsilon(m-1)}. \tag{4.74}$$

Since $m < n$ and $\epsilon < 1/(d_2 n^2)$, it follows from (4.73) and (4.74) that for all $s \in \mathcal{V}_k$,

$$|s|^{n-1} > k^{(n-1)/d_2 - \epsilon(n-1)} > k^{(n-1)/d_2 - (n-1)/(d_2 n^2)} = k^{(n^3 - n^2 - n + 1)/(d_2 n^2)},$$
$$k|s|^{m-1} > k^{(n-1)/d_2 - \epsilon(m-1)} > k^{(n-1)/d_2 - \epsilon(n-1)} > k^{(n^3 - n^2 - n + 1)/(d_2 n^2)}.$$

Moreover, for all $k > 1$ and for all $s \in \mathcal{V}_k$, it follows that

$$|s|^n > |s|^{n-1} > k^{(n^3 - n^2 - n + 1)/(d_2 n^2)}, \tag{4.75}$$
$$k|s|^m > k|s|^{m-1} > k^{(n^3 - n^2 - n + 1)/(d_2 n^2)}. \tag{4.76}$$

Next, we show the magnitude of each of the remaining terms of (4.38) is smaller than $k^{(n^3 - n^2 - n + 1)/(d_2 n^2)}$. For all $s \in \mathcal{V}_k$, it follows that

$$|s|^{n-2} < (k^{1/d_2 + \epsilon})^{n-2} = k^{(n-2)/d_2 + \epsilon(n-2)}, \tag{4.77}$$
$$k|s|^{m-2} < k(k^{1/d_2 + \epsilon})^{m-2} = k^{(n-2))/d_2 + \epsilon(m-2)}, \tag{4.78}$$
$$k^2|s|^l < k^2(k^{1/d_2 + \epsilon})^l = k^{l/d_2 + \epsilon l}. \tag{4.79}$$

Since $m < n$ and $\epsilon < 1/(d_2 n^2)$, it follows from (4.77)–(4.79) that

$$|s|^{n-2} < k^{(n-2)/d_2 + \epsilon(n-2)} < k^{(n-2)/d_2 + (n-2)/(d_2 n^2)} = k^{(n^3 - 2n^2 + n - 2)/(d_2 n^2)}, \tag{4.80}$$
$$k|s|^{m-2} < k^{(n-2)/d_2 + \epsilon(m-2)} < k^{(n-2)/d_2 + \epsilon(n-2)} < k^{(n^3 - 2n^2 + n - 2)/(d_2 n^2)}, \tag{4.81}$$
$$k^2|s|^l < k^{l/d_2 + \epsilon l} < k^{l/d_2 + l/(d_2 n^2)} = k^{(ln^2 + l)/(d_2 n^2)}. \tag{4.82}$$

Since n is a positive integer, it follows from (4.75), (4.76), (4.80), and (4.81) that for sufficiently large $k > 0$ and for all $s \in \mathcal{V}_k$,

$$|c_1||s|^{n-1} > k|b_2||s|^{m-2} > k|b_3||s|^{m-3} > \cdots > k|b_m|, \tag{4.83}$$
$$k|b_1||s|^{m-1} > |c_2||s|^{n-2} > |c_3||s|^{n-3} > \cdots > |c_n|. \tag{4.84}$$

Next, since $l \leq n-2$ and n is a positive integer, it follows that

$$\begin{aligned} ln^2 + l &\leq (n-2)n^2 + n - 2 \\ &= n^3 - 2n^2 + n - 2 \\ &< n^3 - n^2 - n + 1, \end{aligned}$$

which combined with (4.82) yields

$$k^2|s|^l < k^{(n^3-n^2-n+1)/d_2}. \tag{4.85}$$

Therefore, it follows from (4.75), (4.76) and (4.85) that for sufficiently large $k > 0$ and for all $s \in \mathcal{V}_k$,

$$|c_1||s|^{n-1} > k^2|s|^l > |a_1|k^2|s|^{l-1} > \cdots > k^2|a_l|, \tag{4.86}$$

$$|b_1|k|s|^{m-1} > k^2|s|^l > |a_1|k^2|s|^{l-1} > \cdots > k^2|a_l|. \tag{4.87}$$

Thus, (4.75), (4.76), (4.83), (4.84), and (4.86), and (4.87) imply that for sufficiently large $k > 0$ and for all $s \in \mathcal{V}_k$, the four terms of (4.38) with the largest magnitude are k^2s^l, $a_1k^2s^{l-1}$, ks^m, and b_1ks^{m-1}.

Therefore, for sufficiently large $k > 0$ and for all $s \in \mathcal{V}_k$,

$$\tilde{p}_k(s) \approx \eta_k(s), \tag{4.88}$$

where

$$\eta_k(s) \triangleq ks^m + kb_1s^{m-1} + s^n + c_1s^{n-1}. \tag{4.89}$$

Since $b_1 = -\sum_{j=0}^{m} r_j$ and $c_1 = -\sum_{j=0}^{n} t_j$, Lemma 6 in Appendix C implies that as k tends to infinity, $d_2 = d - d_1$ roots of $\eta_k(s)$ are approximated by $\lambda_{d_1+1}, \lambda_{d_1+2}, \ldots, \lambda_d$ given by (4.11)–(4.13). Thus, it follows from (4.88) that for sufficiently large $k > 0$, d_2

roots of $\tilde{p}_k(s)$ are approximated by $\lambda_{d_1+1}, \lambda_{d_1+2}, \ldots, \lambda_d$ given by (4.11)–(4.13), which confirms *(b)*. □

Proof of Fact 4.5. Let $\tau \in \mathbb{R}$ be on the root locus, and assume $q(\tau) \neq 0$. Since τ is on the root locus, it follows from (4.1) and (4.26) that $\kappa_1(\tau) > 0$ or $\kappa_2(\tau) > 0$. We consider three cases: (A) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) > 0$, (B) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) \leq 0$, and (C) $\kappa_1(\tau) \leq 0$ and $\kappa_2(\tau) > 0$. First, assume (A) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) > 0$. It follows that there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that: $\tau \in (a, b)$; there is at most one break-in or breakaway point on (a, b); for all $\sigma \in (a, b)$, $q(\sigma) \neq 0$; and for all $\sigma \in (a, b)$, $\kappa_1(\sigma) > 0$ and $\kappa_2(\sigma) > 0$. It follows from Definition 1.2 that τ is a break-in or breakaway point if and only if τ is the minimizer or maximizer of $\kappa_1(\sigma)$ or $\kappa_2(\sigma)$ on (a, b). Furthermore, τ is the minimizer or maximizer of $\kappa_1(\sigma)$ or $\kappa_2(\sigma)$ on (a, b) if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$ or $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$, respectively. Thus, τ is a break-in or breakaway point if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$ or $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$.

Next, assume (B) $\kappa_1(\tau) > 0$ and $\kappa_2(\tau) \leq 0$. It follows that there exists $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that: $\tau \in (a, b)$; there is at most one break-in or breakaway point on (a, b); for all $\sigma \in (a, b)$, $q(\sigma) \neq 0$; and for all $\sigma \in (a, b)$, $\kappa_1(\sigma) > 0$. It follows from Definition 1.2 that τ is a break-in or breakaway point if and only if τ is the minimizer or maximizer of $\kappa_1(\sigma)$ on (a, b). Furthermore, τ is the minimizer or maximizer of $\kappa_1(\sigma)$ on (a, b) if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$. Thus, τ is a break-in or breakaway point if and only if $d\kappa_1(\sigma)/d\sigma|_{\sigma=\tau} = 0$.

Finally, assume (C) $\kappa_1(\tau) \leq 0$ and $\kappa_2(\tau) > 0$. Using the same argument as the previous case yields that τ is a break-in or breakaway point if and only if $d\kappa_2(\sigma)/d\sigma|_{\sigma=\tau} = 0$. Combining these three cases yields that τ is a break-in or breakaway point if and only if *(a)* or *(b)* from Fact 4.5 holds. □

Proof of Fact 4.6. Assume τ is a break-in or breakaway point, and it follows from Definition 1.2 that τ is on the root locus and that $\tilde{p}_k(s)$ has multiple roots at τ.

Since $q(\tau) = 0$, it follows from (4.1) that $\tilde{p}_k(\tau) = \gamma_2 k r(\tau) + t(\tau)$, which implies that k_τ is the only root of $\tilde{p}_k(\tau)$. Therefore, $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ.

Conversely, assume $\tilde{p}_{k_\tau}(s)$ has multiple roots at τ, and it follows from Definition 1.2 that τ is a break-in or breakaway point. □

Chapter 5 Conclusions and Future Work

This thesis presented rules for constructing root loci, where the closed-loop denominator polynomial is quadratic or cubic in the root-locus parameter k. These root-locus rules apply to a class of controllers that are rational functions of k. In Chapter 2, we characterized the root locus for a class of controllers that yields a closed-loop transfer function, whose denominator polynomial is quadratic in k. More specifically, we considered a controller class, where the numerator polynomial is proportional to k^2, and the denominator polynomial includes a pole, whose location is proportional to k. We developed nine rules that characterize the starting points of the root locus, the segments of the real axis that are on the real axis, the asymptotic behavior of the root locus, and the break-in and breakaway points on the real axis. We showed that the triple integrator can be high-gain stabilized using the quadratic root locus. In Chapter 4, we extended the quadratic root locus rules to accommodate a more general controller class than the controller class in Chapter 2.

In Chapter 3, we characterized the root locus for a class of controllers that yields a closed-loop transfer function, whose denominator polynomial is cubic in k. More specifically, we considered a controller class, where the numerator polynomial is proportional to k^3, and the denominator polynomial includes two poles, whose locations are proportional to k. We developed four rules that characterize the starting points of the root locus, the segments of the real axis that are on the real axis and the asymptotic behavior of the root locus. We showed that the quadruple integrator can be high-gain stabilized using the cubic root locus.

Chapter 3 considered a specific controller class that yielded a closed-loop transfer function, whose denominator polynomial is cubic in k. However, the cubic root-locus rules in Chapter 3 could be extended to accommodate a more general controller class. In principle, an extension of the cubic root-locus rules in Chapter 3 is similar to the Chapter 4 extension of the quadratic root-locus rules in Chapter 2. For example, the real-axis rule in Chapter 2 is a special case of the real-axis rule in Chapter 4. Similarly, the real-axis rule in Chapter 3 is a special case of a real-axis rule for a general polynomial that is cubic in k. This extension is an open problem.

In Chapter 4, we characterized the root locus for a polynomial is quadratic in the root-locus parameter k. Chapter 4 extended the controller class of Chapter 4 by considering a general polynomial that is quadratic in k. The quadratic root-locus rules of Chapter 4 apply to a broad class of controllers that yield closed-loop denominator polynomials that are quadratic in the root-locus parameter k. In Chapter 4, we developed six rules that characterize the starting points of the root locus, the segments of the real axis that are on the real axis, the asymptotic behavior of the root locus, and the break-in and breakaway points on the real axis. The asymptote rule of Chapter 4 (i.e., Fact 4.4) is not collectively exhaustive. In Conjecture 4.1, we provided numerically demonstrated results for the cases that are not included in Fact 4.4. A proof of Conjecture 4.1 is an open problem.

Appendices

A Results for Quadratic Polynomials

Lemma 1. *Consider the polynomial* $a(k) = a_2k^2 + a_1k + a_0$, *where* $a_2, a_1, a_0 \in \mathbb{R}$, $a_2 \neq 0$, *and* $a_0 \neq 0$. *The following statements hold:*

(a) $a(k)$ *has exactly one positive root if and only if* $a_2a_0 < 0$.

(b) $a(k)$ *has repeated positive roots if and only if* $a_1^2 - 4a_2a_0 = 0$ *and* $a_2a_1 < 0$.

(c) $a(k)$ *has two distinct positive roots if and only if* $a_1^2 - 4a_2a_0 > 0$, $a_2a_1 < 0$ *and* $a_2a_0 > 0$.

Proof. Let $\sigma_1 \in \mathbb{C}$ and $\sigma_2 \in \mathbb{C}$ be the roots of $a(k)$. Note that if $\mathrm{Im}(\sigma_1) \neq 0$, then $\sigma_1 = \bar{\sigma}_2$. It follows that

$$a(k) = a_2(k - \sigma_1)(k - \sigma_2) = a_2k^2 + a_1k + a_0,$$

where

$$a_1 \triangleq -a_2(\sigma_1 + \sigma_2), \quad a_0 \triangleq a_2\sigma_1\sigma_2.$$

First, we show *(a)*. Since $a_2a_0 = a_2^2\sigma_1\sigma_2$ and $a_2a_0 \neq 0$, it follows that $a(k)$ has exactly one positive root if and only if $a_2a_0 < 0$.

To show *(b)* and *(c)*, the roots of $a(k)$ are

$$\sigma_1 \triangleq \frac{-a_1 + \sqrt{a_1^2 - 4a_2a_0}}{2a_2}, \quad \sigma_2 \triangleq \frac{-a_1 - \sqrt{a_1^2 - 4a_2a_0}}{2a_2}. \tag{A.1}$$

To show *(b)*, assume σ_1 and σ_2 are repeated positive roots of $a(k)$. Since $\sigma_1 = \sigma_2$, it follows from (A.1) that $a_1^2 - 4a_2a_0 = 0$. Next, since $\sigma_1 = -a_1/(2a_2) > 0$, it follows that $a_2a_1 < 0$. Conversely, assume $a_1^2 - 4a_2a_0 = 0$ and $a_2a_1 < 0$, and it follows from (A.1) that $\sigma_1 = \sigma_2 = -a_1/(2a_2)$. Furthermore, since $a_2a_1 < 0$, it follows that $\sigma_1 > 0$ and $\sigma_2 > 0$. Thus, $a(k)$ has repeated positive roots.

Finally, we show *(c)*. Assume $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_1 \neq \sigma_2$. It follows from (A.1) that $a_1^2 - 4a_2a_0 > 0$. Furthermore, since $-a_1/a_2 = \sigma_1 + \sigma_2 > 0$ and $a_0/a_2 = \sigma_2\sigma_1 > 0$, it follows that $a_2a_1 < 0$ and $a_2a_0 > 0$. Conversely, assume $a_1^2 - 4a_2a_0 > 0$, $a_2a_1 < 0$ and $a_2a_0 > 0$. Since $a_1^2 - 4a_2a_0 > 0$, it follows from (A.1) that σ_1 and σ_2 are real roots and $\sigma_1 \neq \sigma_2$. Furthermore, since $a_2a_1 = -a_2^2(\sigma_1 + \sigma_2) < 0$ and $a_2a_0 = a_2^2\sigma_1\sigma_2 > 0$, it follows that $\sigma_1 + \sigma_2 > 0$ and $\sigma_1\sigma_2 > 0$. Finally, since $\sigma_1 + \sigma_2 > 0$, $\sigma_1\sigma_2 > 0$, $\sigma_1 \neq \sigma_2$, $\sigma_1 \in \mathbb{R}$, and $\sigma_2 \in \mathbb{R}$, it follows that $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_1 \neq \sigma_2$, which confirms *(c)*. □

B Results for Cubic Polynomials

Consider the cubic polynomial $a(k) \stackrel{\triangle}{=} a_3k^3 + a_2k^2 + a_1k + a_0$, where $a_3, a_2, a_1, a_0 \in \mathbb{R}$ and $a_3 \neq 0$. Since $a(k)$ has real coefficients, it follows that $a(k)$ has at least one real root. Let σ_1 be a real root, and let σ_2 and σ_3 be the remaining roots, which are potentially complex. Next, it follows that $a(k) = a_3k^3 + a_2k^2 + a_1k + a_0 = a_3(k - \sigma_1)(k - \sigma_2)(k - \sigma_3)$, which implies that $a_2 = -a_3(\sigma_1 + \sigma_2 + \sigma_3)$, $a_1 = a_3(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$ and $a_0 = -a_3\sigma_1\sigma_2\sigma_3$. Thus, it follows that

$$a_3a_0 = -a_3^2\sigma_1\sigma_2\sigma_3, \tag{B.1}$$

$$a_3a_1 = a_3^2(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3), \tag{B.2}$$

$$a_3a_2 = -a_3^2(\sigma_1 + \sigma_2 + \sigma_3), \tag{B.3}$$

$$a_2a_1 - a_3a_0 = -a_3^2(\sigma_1 + \sigma_2)(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3). \tag{B.4}$$

Next, we define the discriminant of $a(k)$, which is given by

$$D \triangleq 18a_3a_2a_1a_0 - 4a_2^3a_0 + a_2^2a_1^2 - 4a_3a_1^3 - 27a_3^2a_0^2. \tag{B.5}$$

If $D < 0$, then $a(k)$ has one real root and two complex roots with nonzero imaginary parts [24, pp. 153-154]. If $D \geq 0$, then $a(k)$ has three real roots [24, pp. 153-154].

Lemma 2. *Consider the polynomial $a(k) = a_3k^3+a_2k^2+a_1k+a_0$, where $a_3, a_2, a_1, a_0 \in \mathbb{R}$, $a_3 \neq 0$ and $a_0 \neq 0$. Then $a(k)$ has exactly one positive root if and only if any of the following statements hold:*

(a) $a_3a_0 < 0$ and $D < 0$.

(b) $a_3a_0 < 0$ and $a_3a_2 \geq 0$.

(c) $a_3a_0 < 0$, $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 > 0$.

Proof. First, we show that *(a)*, *(b)* or *(c)* is necessary for $a(k)$ to have exactly one positive root. Assume $a(k)$ has exactly one positive root. Without loss of generality, let σ_1 be the positive root. We consider three cases: (1) σ_2 and σ_3 are complex with nonzero imaginary parts; (2) $\sigma_2 \leq 0$, $\sigma_3 \leq 0$ and $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$; and (3) $\sigma_2 \leq 0$, $\sigma_3 \leq 0$ and $\sigma_1 + \sigma_2 + \sigma_3 > 0$.

First, assume σ_2 and σ_3 are complex with nonzero imaginary parts, and it follows from [24, pp. 153-154] that $D < 0$. Since $a(k)$ has real coefficients, it follows that σ_2 and σ_3 are complex conjugates, which implies that $\sigma_2\sigma_3 = |\sigma_2|^2$. Thus, (B.1) implies that $a_3a_0 = -a_3^2\sigma_1\sigma_2\sigma_3 = -a_3^2\sigma_1|\sigma_2|^2 < 0$, which confirms *(a)*.

Next, assume $\sigma_2 \leq 0$, $\sigma_3 \leq 0$ and $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$. Since $a_3 \neq 0$ and $a_0 \neq 0$, it follows from (B.1) that $a_3a_0 = -a_3^2\sigma_1\sigma_2\sigma_3 \neq 0$. Since, in addition, $\sigma_1 > 0$, it follows that $\sigma_2 < 0$ and $\sigma_3 < 0$, which implies that $a_3a_0 < 0$. Since $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$, it follows that from (B.3) that $a_3a_2 \geq 0$, which confirms *(b)*.

Finally, assume $\sigma_2 \leq 0$, $\sigma_3 \leq 0$ and $\sigma_1 + \sigma_2 + \sigma_3 > 0$. Since $a_3a_0 = -a_3^2\sigma_1\sigma_2\sigma_3 \neq 0$ and $\sigma_1 > 0$, it follows that $\sigma_2 < 0$ and $\sigma_3 < 0$, which implies that $a_3a_0 < 0$. Since $\sigma_1 + \sigma_2 + \sigma_3 > 0$, it follows from (B.3) that $a_3a_2 < 0$. Next, since $\sigma_2 < 0$, $\sigma_3 < 0$ and $\sigma_1 + \sigma_2 + \sigma_3 > 0$, it follows that $\sigma_1 + \sigma_2 > 0$ and $\sigma_1 + \sigma_3 > 0$. Therefore, it follows from (B.4) that $a_2a_1 - a_3a_0 = -a_3^2(\sigma_1 + \sigma_2)(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) > 0$. Thus, $a_3a_0 < 0$, $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 > 0$, which confirms *(c)*.

Conversely, assume *(a)*, *(b)* or *(c)* hold. First, assume *(a)* holds. Since $D < 0$, it follows from [24, pp. 153-154] that σ_2 and σ_3 are complex conjugates, which implies that $\sigma_2\sigma_3 = |\sigma_2|^2$. Furthermore, since $a_3a_0 < 0$, it follows from (B.1) that $a_3a_0 = -a_3^2\sigma_1\sigma_2\sigma_3 = -a_3^2\sigma_1|\sigma_2|^2 < 0$, which implies that $\sigma_1 > 0$. Therefore, $a(k)$ has exactly one positive root.

Next, assume *(b)* or *(c)* hold. Since $a_3a_0 = -a_3^2\sigma_1\sigma_2\sigma_3 < 0$, it follows that $a(k)$ has exactly one positive root or three positive roots. Assume for contradiction that $a(k)$ has three positive roots. It follows from (B.3) and (B.4) that $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 < 0$. Since $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 < 0$, it follows that neither *(b)* nor *(c)* hold, which is a contradiction. Thus, $a(k)$ has exactly one positive root. □

Lemma 3. *Consider the polynomial* $a(k) = a_3k^3 + a_2k^2 + a_1k + a_0$, *where* $a_3, a_2, a_1, a_0 \in \mathbb{R}$, $a_3 \neq 0$ *and* $a_0 \neq 0$. *Then* $a(k)$ *has exactly two positive roots if and only if either of the following statements hold:*

(a) $D \geq 0$, $a_3a_0 > 0$ *and* $a_3a_2 < 0$.

(b) $D \geq 0$, $a_3a_0 > 0$, $a_3a_2 \geq 0$, *and* $a_2a_1 - a_3a_0 < 0$.

Furthermore, if $a(k)$ *has exactly two positive roots, then the following statements hold:*

(i) The positive roots of $a(k)$ *are distinct if and only if* $D > 0$.

(ii) a_3, a_2, a_1, *and* a_0 *do not all have the same sign.*

Proof. First, we show that *(a)* or *(b)* is necessary for $a(k)$ to have exactly two positive roots. Assume $a(k)$ has exactly two positive roots. Since $a(k)$ has exactly two positive roots, it follows that that the third root must be nonpositive and real, which implies that $D \geq 0$ [24, pp. 153-154]. Without loss of generality, let $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 \leq 0$. Since $a_3 \neq 0$ and $a_0 \neq 0$, it follows that $a_3 a_0 = -a_3^2 \sigma_1 \sigma_2 \sigma_3 \neq 0$. Since, in addition, $\sigma_1 \sigma_2 > 0$, it follows that $\sigma_3 \neq 0$, which implies that $\sigma_3 < 0$ and $a_3 a_0 > 0$. To show that *(a)* or *(b)* hold, we consider two cases: (1) $\sigma_1 + \sigma_2 + \sigma_3 > 0$ and (2) $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$.

First, assume $\sigma_1 + \sigma_2 + \sigma_3 > 0$, and it follows from (B.3) that $a_3 a_2 = -a_3^2(\sigma_1 + \sigma_2 + \sigma_3) < 0$. Thus, $D \geq 0$, $a_3 a_0 > 0$ and $a_3 a_2 < 0$, which confirms *(a)*.

Next, assume $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$, and it follows from (B.3) that $a_3 a_2 = -a_3^2(\sigma_1 + \sigma_2 + \sigma_3) \geq 0$. Next, since $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$, it follows that $\sigma_1 + \sigma_3 < 0$ and $\sigma_2 + \sigma_3 < 0$. Therefore, it follows from (B.4) that $a_2 a_1 - a_3 a_0 = -a_3^2(\sigma_1 + \sigma_2)(\sigma_1 + \sigma_3)(\sigma_2 + \sigma_3) < 0$. Thus, $D \geq 0$, $a_3 a_0 > 0$, $a_3 a_2 \geq 0$, and $a_2 a_1 - a_3 a_0 < 0$, which confirms *(b)*.

Conversely, assume *(a)* or *(b)* hold. Since $D \geq 0$, it follows that $a(k)$ has three real roots [24, pp. 153-154]. Since, in addition, $a_3 a_0 = -a_3^2 \sigma_1 \sigma_2 \sigma_3 > 0$, it follows that $a(k)$ has exactly two positive roots or three negative roots. Assume for contradiction that $a(k)$ has three negative roots, which implies that $\sigma_1 < 0$, $\sigma_2 < 0$ and $\sigma_3 < 0$. Therefore, it follows from (B.3) and (B.4) that $a_3 a_2 > 0$ and $a_2 a_1 - a_3 a_0 > 0$. Since $a_3 a_2 > 0$ and $a_2 a_1 - a_3 a_0 > 0$, it follows that neither *(a)* nor *(b)* hold, which is a contradiction. Thus, $a(k)$ has exactly two positive roots.

To show *(i)*, assume $a(k)$ has exactly two positive roots and $D > 0$, and it follows from [24, pp. 153-154] that the two positive roots of $a(k)$ are distinct. Conversely, assume $a(k)$ has exactly two positive roots and $D = 0$, and it follows from [24, pp. 153-154] that at least two roots of $a(k)$ are equal. Furthermore, since $a(k)$ has two positive roots and one negative root, it follows that the positive roots are equal.

To show *(ii)*, assume $a(k)$ has exactly two positive roots, which implies that *(a)* or *(b)* hold. First, assume *(a)* holds. Since $a_3a_2 < 0$, it follows that a_3 and a_2 have opposite signs. Next, assume *(b)* holds. Without loss of generality, let $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 \leq 0$. Since $a_3a_2 \geq 0$, it follows from (B.3) that $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$. Since $\sigma_1 > 0$ and $\sigma_1 + \sigma_2 + \sigma_3 \leq 0$, it follows that $\sigma_2 + \sigma_3 < 0$. Thus, $\sigma_1(\sigma_2 + \sigma_3) + \sigma_2\sigma_3 = a_1/a_3 < 0$, which implies that a_3 and a_1 have opposite signs. □

Lemma 4. *Consider the polynomial* $a(k) = a_3k^3 + a_2k^2 + a_1k + a_0$, *where* $a_3, a_2, a_1, a_0 \in \mathbb{R}$, $a_3 \neq 0$ *and* $a_0 \neq 0$. *Then* $a(k)$ *has three positive roots if and only if* $D \geq 0$, $a_3a_0 < 0$, $a_3a_2 < 0$, *and* $a_2a_1 - a_3a_0 < 0$. *Furthermore,* $a(k)$ *has three distinct positive roots if and only if* $D > 0$, $a_3a_0 < 0$, $a_3a_2 < 0$, *and* $a_2a_1 - a_3a_0 < 0$.

Proof. First, we show that $D \geq 0$, $a_3a_0 < 0$, $a_3a_2 < 0$, and $a_2a_1 - a_3a_0 < 0$ are necessary for $a(k)$ to have three positive roots. Assume $a(k)$ three positive roots. Since $\sigma_1 > 0$, $\sigma_2 > 0$ and $\sigma_3 > 0$, it follows from (B.1), (B.3) and (B.4) that $a_3a_0 < 0$, $a_3a_2 < 0$ and $a_2a_1 - a_3a_0 < 0$. Since $a(k)$ has three real roots, it follows from [24, pp. 153-154] that $D \geq 0$. Assume, in addition, that the roots of $a(k)$ are distinct, and [24, pp. 153-154] implies that $D > 0$.

Conversely, assume $D \geq 0$, $a_3a_0 < 0$, $a_3a_2 < 0$, and $a_2a_1 - a_3a_0 < 0$. Since $D \geq 0$, it follows from [24, pp. 153-154] that $a(k)$ has three real roots. Since, in addition, $a_3a_0 < 0$, it follows from (B.1) that $a(k)$ has one positive root and two negative roots, or $a(k)$ has three positive roots. Assume for contradiction that $a(k)$ has one positive root and two negative roots. Without loss of generality, let $\sigma_1 > 0$, $\sigma_2 < 0$ and $\sigma_3 < 0$. Since $a_3a_2 < 0$, it follows from (B.3) that $\sigma_1 + \sigma_2 + \sigma_3 > 0$. Since $\sigma_2 < 0$, $\sigma_3 < 0$ and $\sigma_1 + \sigma_2 + \sigma_3 > 0$, it follows that $\sigma_1 + \sigma_2 > 0$ and $\sigma_1 + \sigma_3 > 0$. Therefore, (B.4) implies that $a_2a_1 - a_3a_0 > 0$, which is a contradiction. Thus, $a(k)$ has three positive roots. Assume, in addition, that $D > 0$, and [24, pp. 153-154] implies that the roots of $a(k)$ are distinct. □

Lemma 5. *Consider the polynomial* $a(k) = a_3k^3+a_2k^2+a_1k+a_0$, *where* $a_3, a_2, a_1, a_0 \in \mathbb{R}$, $a_3 \neq 0$ *and* $a_0 \neq 0$. *Then* $a(k)$ *has nonzero roots on the imaginary axis if and only if* $a_2a_1 - a_3a_0 = 0$ *and* $a_3a_1 > 0$.

Proof. First, we show that $a_2a_1 - a_3a_0 = 0$ and $a_3a_1 > 0$ are necessary for $a(k)$ to have nonzero roots on the imaginary axis. Assume $a(k)$ has nonzero roots on the imaginary axis. Since σ_2 and σ_3 are complex conjugate roots on the imaginary axis, it follows that $\sigma_2 + \sigma_3 = 0$ and $\sigma_2\sigma_3 = |\sigma_2|^2 > 0$. Thus, (B.2) and (B.4) imply that $a_3a_1 = a_3^2|\sigma_2|^2 > 0$ and $a_2a_1 - a_3a_0 = 0$.

Conversely, assume $a_2a_1 - a_3a_0 = 0$ and $a_3a_1 > 0$. Since $a_3 \neq 0$ and $a_2a_1 - a_3a_0 = 0$, it follows from (B.4) that $\sigma_1 + \sigma_2 = 0$, $\sigma_1 + \sigma_3 = 0$ or $\sigma_2 + \sigma_3 = 0$. Now, we show that $\sigma_1 + \sigma_2 \neq 0$ and $\sigma_1 + \sigma_3 \neq 0$. Assume for contradiction that $\sigma_1 + \sigma_2 = 0$. Since σ_1 is real, it follows from (B.2) that $a_3a_1 = a_3^2\sigma_1\sigma_2 = -a_3^2\sigma_1^2 < 0$, which is a contradiction. Thus, $\sigma_1+\sigma_2 \neq 0$. The same argument shows that $\sigma_1+\sigma_3 \neq 0$. Therefore, $\sigma_2+\sigma_3 = 0$, and (B.2) implies that $a_3a_1 = -a_3^2\sigma_2^2$. Since $a_3a_1 = -a_3^2\sigma_2^2 > 0$, it follows that $\sigma_2^2 < 0$, which implies that $\text{Im}\ \sigma_2 \neq 0$. Thus, σ_3 is the complex conjugate of σ_2. Since $\sigma_2 + \sigma_3 = 0$ and $\text{Im}\ \sigma_2 = -\ \text{Im}\ \sigma_3 \neq 0$, it follows that σ_2 and σ_3 are on the imaginary axis. □

C Classical Root Locus Asymptotes

Let $z(s)$ and $p(s)$ be monic polynomials, and define

$$m \triangleq \deg z(s), \qquad n \triangleq \deg p(s), \qquad d \triangleq n - m. \tag{C.1}$$

Furthermore, let $z_1, z_2, \ldots, z_m$ be the roots of $z(s)$ and $p_1, p_2, \ldots, p_n$ be the roots of $p(s)$. We present the following classical root locus asymptote rule.

Lemma 6. *Consider the polynomial* $\mu_k(s) \triangleq k^a z(s) + p(s)$, *where* $k > 0$, a *is a positive integer and* $\deg z(s) < \deg p(s)$. *As* $k \to \infty$, m *roots of* $\mu_k(s)$ *converge to* *the roots of* $z(s)$*, and the* d *remaining roots are approximated by* $\lambda_1, \lambda_2, \ldots, \lambda_d$*, where for* $i = 1, 2, \ldots, d$*,*

$$\lambda_i = k^{a/d} e^{\jmath\theta_i} + \alpha, \tag{C.2}$$

where

$$\theta_i \triangleq \frac{2\pi i - \pi}{d}, \tag{C.3}$$

$$\alpha \triangleq \frac{\sum_{j=1}^{n} p_j - \sum_{j=1}^{m} z_j}{d}. \tag{C.4}$$

Proof. For sufficiently large $k > 0$, it follows that

$$\frac{\mu_k(s)}{k^a} = z(s) + \frac{p(s)}{k^a} \approx z(s),$$

which implies that, as $k \to \infty$, $\mu_k(s)/k^a \approx z(s)$. Thus, as $k \to \infty$, m roots of $\mu_k(s)$ converge to the roots of $z(s)$.

Next, we show that the d remaining roots of $\mu_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$. We write $z(s) = s^m + a_1 s^{m-1} + \cdots + a_m$ and $p(s) = s^n + b_1 s^{n-1} + \cdots + b_n$, where $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n \in \mathbb{R}$. Furthermore, note that $a_1 = -\sum_{j=1}^{m} z_j$ and $b_1 = -\sum_{j=1}^{n} p_j$.

Define $R \triangleq \max_{j=1,2,\ldots,m} |z_j|$. For all $s \in \mathbb{C}$ such that $|s| > R$, the Laurent series expansion of $p(s)/z(s)$ is

$$\frac{p(s)}{z(s)} = s^d + c_1 s^{d-1} + \cdots + c_d + \sum_{j=1}^{\infty} \frac{f_j}{s^j}, \tag{C.5}$$

where $c_1, c_2, \ldots, c_d, f_1, f_2, \cdots \in \mathbb{R}$. Furthermore, note that $c_1 = b_1 - a_1 = -d\alpha$. Thus, for sufficiently large $k > 0$ and all $s \in \mathbb{C}$ such that $|s| > k^{1/2d}$, it follows from (C.5) that

$$k^a + \frac{p(s)}{z(s)} \approx \bar{\mu}_k(s), \tag{C.6}$$

where

$$\bar{\mu}_k(s) \triangleq s^d - d\alpha s^{d-1} + k^a. \tag{C.7}$$

Next, let $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_d$ be the roots of $\bar{\mu}_k(s)$, which implies that for $i = 1, 2, \ldots, d$,

$$-k^a = \bar{\lambda}_i^d - d\alpha\bar{\lambda}_i^{d-1}. \tag{C.8}$$

Taking the d^{th} root of both sides yields

$$(-k^a)^{1/d} = \bar{\lambda}_i(1 - d\alpha)/\bar{\lambda}_i)^{1/d}. \tag{C.9}$$

Since for $i = 1, 2, \ldots, d$, as $k \to \infty$, $|\bar{\lambda}_i| \to \infty$, it follows that as $k \to \infty$, $| - d\alpha/\bar{\lambda}_i|$ tends to zero. Thus, we apply the binomial approximation $(1 - d\alpha/\bar{\lambda}_i)^{1/d} \approx 1 - \alpha/\bar{\lambda}_i$, which implies that for $i = 1, 2, \ldots, d$,

$$(-k^a)^{1/d} = k^{a/d}e^{\jmath\theta_i} \approx \bar{\lambda}_i - \alpha. \tag{C.10}$$

Therefore, for sufficiently large $k > 0$ and $i = 1, 2, \ldots, d$, it follows from (C.10) that

$$\bar{\lambda}_i \approx k^{a/d}e^{\jmath\theta_i} + \alpha = \lambda_i. \tag{C.11}$$

Next, it follows from (C.6) that for sufficiently large $k > 0$ and $i = 1, 2, \ldots, d$,

$$k^a + \frac{p(\bar{\lambda}_i)}{z(\bar{\lambda}_i)} \approx 0,$$

or equivalently $\mu_k(\bar{\lambda}_i) = k^a z(\bar{\lambda}_i) + p(\bar{\lambda}_i) \approx 0$. Thus, for sufficiently large $k > 0$, d roots of $\mu_k(s)$ are approximated by $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_d$. Moreover, it follows from (C.11) that for sufficiently large $k > 0$, $\bar{\lambda}_i \approx \lambda_i$, which confirms that d roots of $\mu_k(s)$ are approximated by $\lambda_1, \lambda_2, \ldots, \lambda_d$. □

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Vita

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