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Anti-Foundational Categorical Structuralism

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A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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Abstract

The aim of this dissertation is to outline and defend the view here dubbed "anti-foundational categorical structuralism" (henceforth AFCS). The program put forth is intended to provide an answer the question "what is mathematics?". The answer here on offer adopts the *structuralist* view of mathematics, in that mathematics is taken to be "the science of structure" expressed in the language of category theory, a language argued to accurately capture the notion of a *structural property*. In characterizing mathematical theorems as both conditional and schematic in form, the program is forced to give up claims to securing the *truth* of its theorems, as well as give up a semantics which involves reference to special, distinguished "mathematical objects", or which involves quantification over a fixed domain of such objects. One who wishes—contrary to the AFCS view—to inject mathematics with a "standard" semantics, and to provide a secure epistemic foundation for the theorems of mathematics, in short, one who wishes for a *foundation* for mathematics, will surely find this view lacking. However, I argue that a satisfactory development of the structuralist view, couched in the language of category theory, accurately represents our best understanding of the *content* of mathematical theorems and thereby obviates the need for any foundational program.

Keywords: category theory, philosophy, foundations of mathematics, structuralism, properties, schema, conditional, mathematical truth

For Chiara

Acknowledgements

This section began as had many others: I read examples written by friends and mentors, and was left wondering how I could possibly hope to write anything comparable. Here, though, the challenge was not to develop and express some deep insight into a difficult philosophical issue, but rather to write something more personal, yet no less eloquent and sincere.

Indeed, that this task seemed so daunting is a testament to the calibre of my friends and mentors, particularly those mentors I am privileged to count among my friends. Dave DeVidi's encouragement was instrumental in my choice to pursue my interests in philosophy,[1] and I fondly remember wandering past his office hoping to find him available for yet another discussion of lattice-valued semantics, favourite novels, or whatever else came to mind.

After studying with Dave I found myself writing my Ph.D. under the supervision of John Bell, who had also been Dave's Ph.D. supervisor. Dinner conversations about philosophy, music, art and politics at John's would last well into the night, and I am grateful not only for his encouragement and guidance in philosophy, but also for his warmth and hospitality, characteristics that sometimes seem all too rare in the world of academia.

Thanks are due to my readers, all of whom have made suggestions that lead in a number of interesting directions. Insightful comments are of considerable value, and were I to begin my Ph.D. study over again, the comments provided by my readers would no doubt provide the basis for at least a couple

[1] And I hope he hasn't come to regret this, even after reading my thesis!

of Ph.D. theses!

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Finally, the support of my wife Chiara has been absolutely crucial throughout my study, and the thanks she is owed could not possibly be described in this limited space.

Contents

Chapter 1

Foundations

> *Let us now try, guided by the axiomatic concept, to look over the whole of the mathematical universe.* [15, p. 228]

As the name suggests, anti-foundational categorical structuralism (henceforth AFCS[1]), combines a number of the principles and perspectives that have risen to prominence in contemporary philosophy of mathematics, in particular

1. the idea that mathematics concerns *structure*,
2. the move away from *foundational* approaches in the philosophy of mathematics, and
3. the notion of a (mathematical) *category*.

The aim of this chapter is to elucidate the sense in which the program to be proposed in this work is *anti-foundational*. Indeed, it is rather presumptuous to label the program anti-foundational at this stage, as the program may be seen to potentially satisfy some notions of a *foundation for mathematics*, while failing to satisfy others. As the notions of *foundation for mathematics* which the program fails to satisfy are those of principal importance in

[1] This program has also been called *top-down* categorical structuralism in [2, 48].

philosophical enquiries into the foundations of mathematics, I expect that the label will be recognized as appropriate.

1.1 Axiomatic Systems

Crucial to a contemporary discussion of the foundations of mathematics is the notion of an *axiomatic system*. Here the term *axiomatic system* will be used in a deliberately loose sense, including both formal and informal systems, with or without an explicit specification of the admissible rules of inference.[2] This work will focus primarily on formal axiomatic systems, and informal systems will be assumed formalizable. When a formal system is under consideration, I take it that the logical operators are given their usual interpretation.[3] The rise to prominence of the methods of formal representation and symbolic manipulation in mathematics—in some sense a recent development—has brought about a dramatic change in the way that mathematics is typically characterized. The formal methods that emerged from the late 19^{th} and early 20^{th} century developments in logic and set theory are now taken to be an essential, and perhaps principal, component of any philosophical account of contemporary mathematics.

With the increased expressive powers of the various languages of first– and higher-order logic, a number of grand philosophies of mathematics came to the fore. In Frege's *logicist program*, it was hoped that these new languages would be able to bridge the seemingly close-set gap between logic and mathematics, and, in so doing, ultimately free arithmetic from its supposed intuitive foundation. A quite different approach, also dependent on recently developed formal methods, can be found in Hilbert's *formalist program*, according to which it was hoped that appeal to pure intuition could

[2]When no specification is given, the background logic will be assumed to be first-order (classical) logic, unless the axioms require formulation in a second– or higher-order system, or require a restricted set of inference rules (intuitionistic, etc.).

[3]Such a system can be called *semiformal*, as in [18].

be removed, this time to be replaced, at the metamathematical level, by perceptual intuitions concerning the manipulations of concrete objects.

Unfortunately, as the familiar story goes, both of these programs met with failure; indeed, the result which is usually take to signal the defeat of Hilbert's program was itself a significant development in the application of formal techniques.[4] However, despite the failure of these two ambitious programs, formal languages and the associated formal methods developed in the study of these programs have undoubted been instrumental in leading us to the current state of contemporary mathematics. Nowhere has this shift been more apparent than in the contemporary emphasis on *axiomatics*, particularly formal axiomatics in a mathematical context.

1.1.1 Assertory Axioms

The axiomatic method, of course, dates back to antiquity, with one of the better known instances found in the *Common Notions* and *Postulates* of Euclid's *Elements*. These Common Notions and Postulates are usually viewed as having been intended as *self evident* or *obviously true*. As such, the axioms were truth-apt, contentful expressions, what we might now call statements or propositions.[5] The formal axioms of Frege's 1893 *Grundgesetze* [30] were such contentful expressions; the basic laws of thought were taken to be "the most general laws, which prescribe universally the way in which one ought to think if one is to think at all" [30, p. 12]. Indeed, Frege himself observes that his methods might properly be considered Euclidean, insofar as he clearly identifies those propositions (i.e., axioms) which are not to be proved [30,

[4] Incidentally, Gödel's result concerned the system of Russell's *Principia Mathematica*, a system born out of the failure of that system proposed by Frege in his ill-fated *Grundgesetze*. Of course, Gödel's result applies to a more general type of formal system of which Russell's is but one example.

[5] The distinction between *statements*, i.e., declarative sentences, and *propositions*, which may correspond to a number of distinct declarative sentences, will not concern us here, and one may choose either to be the bearers of truth values for the purposes of this work.

p. 2].[6] Following the terminology adopted in some recent writings in the foundations of mathematics [37, 38, 79], systems of this sort—in which the axioms are taken to be propositions—will be called *assertory*.

1.1.2 Algebraic Axioms

A second important class of axioms are *algebraic* axioms[7], which, unlike assertory axioms, are not propositions, and so not meant to be taken as true *simpliciter*. Instead, they are typically taken as *definitions of a type of structure*; the classic example of such axioms are those for rings, groups, fields, topological spaces, etc., of the sort typically found in modern algebra textbooks. Axioms used in this way define a *type of object* at the meta level. The axioms are *propositional schemata*, and any *model* in which the axioms are true is thereby an object of the sort defined. Thus, in a mathematical context, algebraic axioms require some sort of (typically informal) model theory. The following example is typical:

> A *group* is an ordered pair $(G, \star)$, where G is a set and $\star$ is a binary operation satisfying the following axioms:
>
> (i) $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$, i.e., $\star$ is *associative*,
>
> (ii) there exists an element e in G, called an *identity* of G, such that for all $a \in G$ we have $a \star e = e \star a = a$,
>
> (iii) for each $a \in G$ there is an element a^{-1} of G, called an *inverse* of a, such that $a \star a^{-1} = a^{-1} \star a = e$. [27, p. 16–17]

Such a definition immediately raises meta-theoretic questions of *satisfiability*: are there any groups, rings, fields, etc.? If so, how many, and what are the relationships that obtain between such entities? Further, once such

[6] Frege notes, however, that his approach extends that of Euclid's in that he also provides a specification of the admissible *rules of inference*.

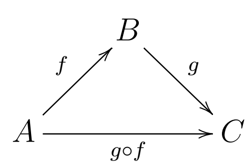
[7] These axioms are sometimes also called *formal* or *schematic* [38].

a move to meta-theoretic questions is made, what is the framework in which we conduct our meta-theoretic investigations? Finally, does this type of axiomatic presentation function correctly as a *definition*? In answering these questions, it will be useful to compare the preceding definition of a group with the definition of the type of structure with which this work will be primarily concerned: *categories*.

1.2 Category Theory

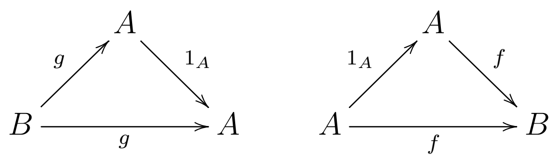
Following roughly the treatment found in [60], a category is a collection of *objects* $A, B, C, \ldots$ and *arrows* $f, g, h, \ldots$, where each arrow f has an associated *domain* and *codomain* (sometimes called the *source* and *target*, respectively), represented as $f : A \to B$. Arrows f, g are composable provided $dom(g) = cod(f)$, and for any such composable pair $f : A \to B$ and $g : B \to C$ there is an arrow $g \circ f : A \to C$ such that the diagram shown below *commutes*, i.e., the arrows obtained by composition on any connected path depend only on the endpoints of that path.

Composition

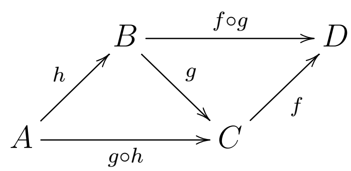


Identity
For each object A there is a (unique) arrow $1_A : A \to A$ such that the diagrams below commute for any $f : A \to B$ and $g : B \to A$. (For a given

object A, 1_A will sometimes be represented in diagrams by A.)



Associativity
For any f, g, h as shown, the diagram below commutes.



When giving a first-order axiomatization of category theory it is customary to use a typed first-order language (with object and arrow types), but this can be dispensed with either by the familiar technique of introducing predicates and rendering the axioms as conditionals, or by giving an "arrows-only" presentation.[8] It is initially useful to think of the arrows as functions and the objects as sets, where a category is then a collection of sets and functions defined on those sets. With this sort of picture in mind, category theory can be roughly described as "the mathematical study of (abstract) *algebras of functions*" [3, p. 1][9]. However, the connection between category theory and set theory will require closer examination, and for the moment it is perhaps more useful to view the category axioms as the axioms of an uninterpreted first-order theory. Consequently, the objects and arrows need not be viewed as elements of, for example, the cumulative hierarchy.

Some familiar types of mathematical objects are very easily described in the language of category theory. For example, a monoid is standardly

[8] For details on the arrows-only definitions, see [52, p. 9].

[9] "Abstract" because "the objects do not have to be sets and the arrows need not be functions" [3, p. 5].

presented as a set of objects with an associative binary operation and a distinguished identity element. So, $\langle \mathbb{N}, +, 0 \rangle$ is a monoid, as addition on the natural numbers is associative, with 0 functioning as the (additive) identity. Similarly, $\langle \mathbb{N}, \times, 1 \rangle$ is also a monoid with the binary operation of multiplication and 1 as the multiplicative identity. In the language of category theory, a monoid is simply a category with one object. Elements of the monoid corresponds to arrows from that single object to itself, and the associativity of arrow composition corresponds to the associativity of the monoid operation, with the identity arrow serving to represent the identity element of the monoid. Thus, categories can be viewed as generalized monoids.

Other important types of mathematical structures[10] also have natural definitions in categorical terms. A *preorder* on a collection of elements is a reflexive, transitive relation. In categorical terms, a preorder is a category in which every pair of objects A and B have at most one arrow from A to B[11], where $A \leq B$ iff there is an arrow $f : A \to B$. Associativity of arrows yields transitivity, and identity arrows yield reflexivity. Similarly, a *partial order* on a collection of elements is a reflexive, transitive, antisymmetric relation. In categorical terms, a partial ordering on a collection of elements corresponds to a category in which every pair of objects A and B have at most one arrow between them, where $A \leq B$ iff there is an arrow $f : A \to B$ as before. Transitivity and reflexivity follow as before, and any arrows $f : A \to B$ and $g : B \to A$ must be identical by the condition on arrows, and so $A = B$. Thus, the arrows f and g must both be 1_A, the unique arrow from A to A. The closed well-formed formulae of a system of first-order logic can be taken as objects in a category, where the arrows $f : A \to B$ are derivations of B from A. Taking as objects the positive integers we can form a category by taking $n \times m$ real-valued matrices as arrows $f : n \to m$, where composition

[10] The term *structure* will come to be used in a more precise sense later in this chapter, but the usage here will agree with that later definition.

[11] Note that the ordering of the domain and codomain of the arrow is important here, as we are allowing the case where there are distinct arrows $f : A \to B$ and $g : B \to A$.

corresponds to matrix multiplication. More obvious examples of categories include the category of sets and functions, groups and group homomorphisms, and the category of continuous functions $f : \mathbb{R} \to \mathbb{R}$. A wealth of similar examples can be found in [3, 52].

1.3 Foundations for Mathematics

There are a number of senses in which one can speak of a *foundation for mathematics*, and, unfortunately, debates concerning the merits of one or another proposed foundation often suffer from this plurality. Thus, Putnam declares that

> I don't think mathematics is unclear; I don't think mathematics has a crisis in its foundations; indeed, I do not believe mathematics either has or needs "foundations." [67, p. 5]

while Mayberry declares that

> ... mathematics not only needs, but in fact has, foundations. Mathematics can no more lack foundations than a building can: wherever a building touches the ground, there, for good or ill, its foundations are to be discovered. Those foundations may have been carelessly laid, they may be shaky or unsound, the whole edifice may threaten to collapse about our ears; but it cannot *lack* foundations. [57, p. 17–18]

In an attempt to map out the various notions of *foundations for mathematics*, Marquis in [53] identifies no less than six separate—but interrelated—senses of the term, and provides examples of single authors shifting their emphasis from one sense to another.

Before exploring the various senses of *foundation for mathematics* it will be convenient to introduce the notion of a *framework*, where a framework is an axiomatic system, algebraic or assertory, and either formalized or presented informally.[12] Zermelo-Fraenkel set theory (with or without the axiom of choice) constitutes a framework, both as an informal system and, for example, as a formal first-order system. Clearly the interpreted system and the formal system are suited to different tasks, and so we may expect to observe a difference in their suitability to various of the possible foundational roles.

1.3.1 Organizational Frameworks

As a tentative account of a *foundation for mathematics*, consider the view that Lawvere offers in "The Category of Categories as a Foundation for Mathematics", wherein "... by "foundation" we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved" [50, p. 1]. The type of foundation Lawvere considers would allow one to characterize mathematics as the investigation of the consequences of those axioms. Let us call a framework that allows for the formulation of definitions (and, given a suitable logic, the production of proofs) that arguably captures *all* of mathematics an *organizational framework*.[13] Is an axiomatic system that constitutes an organizational framework sufficient as a *philosophically adequate* foundation for mathematics?

One important observation about such an approach to foundations is that it yields an account of *mathematics*. As a very rough initial approximation, we might identify mathematics as the subject matter mathematicians are concerned to investigate: groups, rings, fields, functions, geometry, statistics, probability, analysis, computability, graph theory, and a myriad of other areas of mathematical enquiry. However, the inadequacy of lists of this sort highlights one of the important roles that a *framework* for mathematics might play: provide the criteria according to which an area of research counts as

[12] The term *framework* is approximately equivalent to the notion of *linguistic framework* as found in [20] and discussed in [47].

[13] This formulation of the notion of an *organizational framework* does not involve the restriction to first-order systems.

mathematics. Given a set of axioms of the sort envisioned by Lawvere, a research project counts as *mathematics* to the extent that it involves exploring the deductive consequences of definitions (either new or preexisting) in the language of the theory. A first-order axiomatization of ZFC may be argued to provide a framework of this sort, and similarities between the sort of foundation characterized by Lawvere and that offered by a framework like first-order ZFC serve to highlight some important philosophically-motivated concerns about the merits of such an approach.

1.3.2 Epistemology, Semantics, and Ontology

Benacerraf's seminal paper entitled "Mathematical Truth" serves to illustrate two desiderata that I take to be characteristic of the *philosophical* notion of a *foundation for mathematics*. In that paper Benacerraf presents his well-known argument that

> . . . two quite distinct kind of concerns have separately motivated accounts of the nature of mathematical truth: (1) the concern for having a homogeneous semantical theory in which semantics for the propositions of mathematics parallel the semantics for the rest of the language, and (2) the concern that the account of mathematical truth mesh with a reasonable epistemology. . . . almost all accounts of the concept of mathematical truth can be identified with serving one or another of these masters at the expense of the other. [11, p. 661]

For those accounts that emphasize the semantical aspect, there is the problem of explaining how we can come to know anything of the seemingly unusual (typically atemporal, acausal) mathematical objects. Those accounts that emphasize the epistemological aspect—for example, a framework couched in some familiar (formal) logical structure—are also faced with a serious difficulty. Views of this sort (which Benacerraf calls *combinatorial*) are subject to the objection that, while having established a way to come to (apparent) mathematical knowledge, it is not clear that a claim's derivability within a particular formal system is sufficient to warrant the judgment that the given statement is true. Within a particular deductive system $\mathcal{L}$, a separate argument is required to establish the connection between derivability in $\mathcal{L}$ (or $\mathcal{L}$-*truth*) and the concept of truth as it is otherwise understood. Thus, if we accept Benacerraf's request for what he calls a "Tarskian" semantics, a view like that suggested in Lawvere's remark must be supplemented (if indeed the theorems deduced are meant to be taken as true) by an account of the objects—categories—the axioms purportedly describe. What justifies our taking the axioms of Lawvere's system as true of all such categories? The difficulty involved in providing such an account seems particularly imposing given Lawvere's own description of the means by which he arrived at the axioms for his framework:

> The author believes, in fact, that the most reasonable way to arrive at a foundation meeting these requirements [defining the usual objects of mathematics and proving their usual properties] is simply to write down axioms descriptive of properties which the intuitively-conceived category of all categories has until an intuitively-adequate list is attained; that is essentially how the theory described below was arrived at. [50, p. 1]

Of course, an organizational framework of the sort Lawvere aims to provide does serve to clearly define the semantic and epistemological *target* for any philosophical account of the sort Benacerraf identifies; the organizational framework clearly identifies the sentences whose epistemology is to be accounted for, and the sentences whose semantics is to be unpacked. In this way, the construction of an organizational framework is necessary for the development of a characterization of mathematics that accounts for the semantic content of mathematical statements, together with an account of the truth of mathematical statements wherein we appeal to "...the theoretical apparatus employed by Tarski in providing truth definitions, i.e., the analysis of truth in terms of the "referential" concepts of naming, predication, satisfaction, and quantification" [11, p. 677].

Thus, for the purpose of this work, the following three criteria will be taken to be necessary (and sufficient)[14] conditions on any framework that constitutes a philosophically acceptable foundation for mathematics.

1. The framework is an *organizational framework*, and thereby identifies what counts as mathematics.

2. The framework can be linked to an account of the *epistemology* of mathematical statements, providing a characterization of the features in virtue of which mathematical statements are, and can be, known to be true.

3. The framework accounts for the *semantics* of mathematical statements, and proceeds to do so via the theoretical apparatus "of naming, predication, satisfaction, and quantification".

1.4 Classical Approaches to Foundations

The most influential programs of the past century in the philosophical foundations of mathematics can each be seen to present an account aimed at satisfying the three criteria identified above. A brief account of two of these programs, Hilbert's *finitist* program, as well as Frege and Russell's *logicist* program, will be useful when later we come to explore features of the AFCS program.

[14] While I take these conditions to be sufficient to identify a foundation for mathematics, *necessity* is all that is required for the purpose of establishing that the program I aim to defend does indeed fail to count as a *foundation*.

1.4.1 Finitism

Consider Hilbert's *finitist* program, in which an (informally presented) framework was provided by the (assertory) axioms of *finitary proof theory*. Hilbert took this system to play an important justificatory role with respect to mathematical statements. The *meta-theoretic* methods of finitary proof theory were taken by Hilbert to justify the use of axiomatic systems at the *object level*, and this justification proceeded in two steps. First, axiomatic systems of the various branches of mathematics were shown to have an interpretation in arithmetic. Consequently, the consistency of any such system in question was guaranteed, provided the system of arithmetic could itself be shown consistent. The second step, then, involved using the methods of finitary proof theory to establish the consistency of the formalized theory of arithmetic.[15] Mathematics could then be characterized as the investigation of formal systems which could be shown to be consistent by those methods, satisfying the first criterion for foundational programs identified above. As Hilbert notes after presenting a sketch of the system of formal arithmetic, "we are now in a position to carry out our theory of proof and to construct the system of provable formulae, i.e., mathematics" [43, p. 199].

Hilbert's account of the semantics and epistemology of mathematics (by which he satisfies the second and third criteria for foundational programs) proceeds via appeal to the direct, perceptual intuition of concrete symbols:

> . . . something must be given in conception, viz., certain extralogical concrete objects which are intuited as directly experienced prior to all thinking. For logical deduction to be certain, we must be able to see every aspect of these objects, and their properties, differences, sequences, and contiguities must be given, together with the objects themselves, as something which cannot be reduced to something else and which requires no reduction.

[15] It is generally agreed that Gödel's incompleteness result [33] show this second step to be impossible, see [84] for a discussion.

> This is the basic philosophy which I find necessary, not just for mathematics, but for all scientific thinking, understanding, and communicating. The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable. [43, p. 192]

A formal deductive system for arithmetic, shown to be consistent via finitary proof theory, was taken to be epistemically sound insofar as the theorems are true when interpreted as claims about the realm of finite, immediately presented concrete symbols.[16] A proof of the consistency of the system of arithmetic would thereby justify reasoning involving other formal systems, as, for such systems, "proof of consistency is effected by reducing their consistency to that of the axioms of arithmetic" [43, p. 200]. The investigation of other axiomatic systems would be warranted insofar as their theorems could be interpreted as true statements concerning the intuitively given concrete symbols. Finally, in addition to the realm of concrete symbols, *ideal* elements of systems (such as infinite cardinals and the *points at infinity* of projective geometry), could also taken to exist, as Hilbert describes in correspondence with Frege:

> ...if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This is for me the criterion of truth and existence. [79, quoted on p. 69][17]

Thus, via the framework of finitary proof theory, Hilbert's program aimed to characterize the objects of mathematics (and thereby account for the semantic properties of mathematical statements), account for the means by

[16]Hilbert had been particularly interested to justify the use of *ideal* objects in number theory, as most famously advocated in [43].

[17]Note that this quotation is from early in Hilbert's career, but that he maintained such a view (against instrumentalist readings) in his later writings has been argued in [34].

which we can acquire mathematical knowledge (through proofs in consistent axiomatic systems, grounded in the immediately-intuited structure of the concrete objects of mathematics), and identify mathematics as the discipline concerned with the construction of proofs in those consistent axiomatic systems.

1.4.2 Logicism

Frege claims that one of his aims in *Die Grundlagen der Arithmetik* is to show that—contra Kant—synthetic a priori judgements are not required in order to secure the truths of cardinal arithmetic.[18] After having defined numbers as *extensions of concepts* and deriving several important theorems[19], Frege writes

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one. [32, p. 99]

The logical formalism Frege pioneered in the *Begriffsschrift* [31] finally allowed for the formulation of the view that logic and mathematics were not simply closely related (as had long been held), but that the truths of mathematics were in fact logical truths. The claim that mathematics is, in some sense, reducible to logic taken along with suitable definitions of the objects of mathematics is the characteristic tenet of the logicist view, and both Frege and Russell's developments in pursuing that view have shaped much of the debate in the philosophy of mathematics for more than a century.

[18] While Frege may have emphasized this aim in the *Grundlagen*, it has been argued that focus on this aspect of his program does not do justice to the full scope of the logicist program, see [21].

[19] One, the derivation of the Peano axioms from a tentative early definition of number that has come to be called Hume's Principle, forms the core of the neo-Fregean view advocated by Hale and Wright, see [83].

While a great deal of attention is given to their work in arithmetic, the reduction of mathematics to logic was meant to encompass *all* of mathematics. As Russell boldly claims in the Introduction to *The Principles of Mathematics*,

> By the help of ten general principles of deduction and ten other premises of a general logical nature (e.g., "implication is a relation"), all mathematics can be strictly and formally deduced; and all the entities that occur in mathematics can be defined in terms of those that occur in the above twenty premises. In this statement, Mathematics includes not only Arithmetic and Analysis, but also Geometry, Euclidean and non-Euclidean, rational Dynamics, and an indefinite number of other studies still unborn or in their infancy. The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age... [73, p. 4–5]

While the "discovery" Russell claims in the quote above never saw the light of day[20] and the aims of the program were never realized, the logicist program is perhaps the program which most clearly set out to meet the foundational criteria identified here. The logical systems Frege and Russell developed were explicitly taken to provide "organizational frameworks" for the logical truths, insofar as these systems were taken to permit all definitions involving only logical constants, and yield all proofs that proceed only by the most general principles of inference. As Frege explains, "Everything necessary for a correct inference is expressed in full, but what is not necessary is generally not indicated; *nothing is left to guesswork*" [31, p. 12]. While Russell outlines a syntactic distinction that could be drawn between mathematics and logic (casting mathematical truths as a subset of the logical truths with a given form), "But for the desire to adhere to usage, we might identify mathemat-

[20] Gödel's undecidability result in [33], can be taken to have shown the program untenable, as some mathematical truths "escape capture" in systems of the sort Russell had proposed.

ics and logic" [73, p. 9]. Thus, a framework which permitted derivations of all and only the truths of pure logic was thereby a framework that identified *mathematics*—that same set of truths, or a syntactically distinguishable subset of them—as well.

In providing for the semantics of mathematical statements, both Russell and Frege identified a distinguished collection of *logical objects*. Frege invoked the notion of the *extension of a concept*, while Russell made use of the notion of a *class*.[21] In either case, mathematical objects, such as cardinal numbers, were defined as classes (of classes) or extensions of concepts, and the true mathematical propositions expressed truths about those distinctly logical (and, hence, mathematical) objects, in accordance with the third foundational criterion. Similarly, the logical systems on offer were taken to represent the most general rules of inference, and so they provide a secure route to mathematical knowledge, yielding inferences from the definitions of the purely logical (mathematical) objects to truths about those objects. Thus, the successful reduction of mathematics to logic in accordance with the logicist program, taken along with the seemingly secure epistemic and ontological features of the logical systems in question, yields each of the three criteria offered as characteristic of a foundation for mathematics.

We turn now to outline the principal features of a philosophical position, AFCS, which does not satisfy those criteria. What, then, can an advocate of such a position hope to accomplish?

1.5 The AFCS Program

To illustrate one aspect of the AFCS view, consider again the definition of a *group* presented in Section 1.1.2. One can prove from these axioms that, for

[21] Russell would later abandon the primacy of the concept of a *class* in favour of that of a *propositional function*.

any elements a, b, c of a given group,

$$(a \star b \star c)^{-1} = c^{-1} \star b^{-1} \star a^{-1}$$

This result could be described as a theorem of first-order logic (with function symbols) that is conditional in form, taking as antecedent the conjunction of (the universally quantified versions of) the axioms defining a group, and the formula above as consequent. However, as Shapiro observes,

> Say that a theory is 'Fregean' if it is intended to be about a specific subject matter, and that a theory is 'Hilbertian' if it consists of taking the logical consequences of an axiomatization regarded as an implicit definition of a type of structure. Contemporary group theory and ring theory are not pursued, for more than a few minutes, in this Hilbertian manner. Rather, the group theorist studies all groups, developing relationships between them and with other structures. [79, p. 67 ff]

While not strictly required by the group-theoretic theorem above, one might render that theorem as one whose subject matter is *any group*, and so, a theorem of the form "In any group G..." . However, if our starting point is the first-order group axioms, some standard mathematical results about groups will *require* what might be called the "model-theoretic" perspective. For example, the result that the kernel of a group homomorphism $f : G \to H$ is a subgroup of G cannot be rendered simply as a first-order consequence of the universally-quantified axioms, as this theorem involves explicit reference to the group(s) involved in the group homomorphism. Similarly, there are existence theorems, for example, the theorem that, given any two groups G and H, there is at least one group homomorphism $f : G \to H$. One who aims to account for mathematical truths in a manner that appeals to notions of naming, satisfaction, and quantification, has then the task of saying something about those objects, be they groups, sets, or categories. Are there product groups for every pair of groups, powersets for every set, enough arrows for our categories? The axioms which define a category, and similarly the axioms which define a group, do not assert anything about the existence of the categories or groups, but some standard results in mathematics, as with the kernel theorem, seem to involve quantification over groups and other purportedly *mathematical* objects. Taking mathematical propositions of this sort as *true*, then, seems to lead us to the familiar problem Benacerraf highlights: how can we provide a "Tarskian" semantics for our propositions (which would then involve some account of the mathematical objects that figure in those propositions) while at the same time ensuring that our proofs have the epistemic features necessary to secure their truth?

In Awodey's "An Answer to Hellman's Question: Does Category Theory Provide a Framework for Mathematical Stucturalism?", we find a presentation of a view Awodey dubs "top-down" structuralism, and his characterization of mathematical propositions as both *schematic* and *conditional in form* is adopted in the AFCS proposal. Unlike those views that identify a privileged class of objects (for example, pure sets, classes, or extensions of concepts), and then "build up" the entities of the various branches of mathematics from those objects, the top-down perspective

> . . . is based instead on the idea of specifying, for a given theorem or theory only the required relevant degree of information or structure, the essential features of a given situation, for the purpose at hand, without assuming some ultimate knowledge, specification, or determination of the 'objects' involved.
>
> Thus according to our view, there is neither a once-and-for-all universe of all mathematical objects, nor a once-and-for-all system of all mathematical inferences. [2, p. 56]

In virtue of this perspective, all the theorems of mathematics are all taken to be conditional in form, with an antecedent condition which functions as a partial *specification of context*, providing that "relevant degree of information".

> Every mathematical theorem is of the form 'if such-and-such is the case, then so-and-so holds'. That is, the 'things' referred to are assumed to have certain properties, and then it is shown, using the tacitly assumed methods of reasoning, that they also have some other properties. . . . Of course, many theorems do not literally have this form, but every theorem has some conditions under which it obtains. [2, p. 58].

It might then be thought that mathematical claims on this view should be taken as universally quantified expressions. Considering the group-theoretic results discussed earlier, we might take it that the theorem concerning the existence of a (group) homomorphism between any two groups is of the form "For all objects G and H, if G is a group and H is a group then there exists a (group) homomorphism from G to H". However, if we ask whether the theorem is *true*, and adopt the usual semantic treatment of expressions involving the universal quantifier, we are left wondering whether there are any groups, and whether the theorem might simply be *vacuously* true. A description of the features of a group via the axioms appearing in the antecedent of a theorem in elementary group theory does not suffice to determine whether or not there are any groups, nor does it give any indication as to how we might try to make this determination. As Awodey explains,

> This lack of specificity or determination is not an accidental feature of mathematics, to be described as universal quantification over all particular instances in a specific foundational system as the foundationalist would have it. . . rather it is characteristic of mathematical statements that the particular nature of the entities involved plays no role, but rather their relations, operations, *etc.*—the structures that they bear—are related, connected, and described in the statements and proofs of theorems.

> The 'schematic' element in mathematical theorems, definitions, and even proofs is not captured by treating the indeterminate objects involved as universally quantified variables, as quantification requires a fixed domain over which the range of the variable is restricted. [2, p. 59]

In adopting this distinction, it seems problematic to account for the truth of a mathematical theorem in terms of the "Tarskian" apparatus required by our third criterion for a foundation. Note that the view of mathematical theorems as conditional in form is *not* to be understood as identifying those theorems with trivial claims of the form "Assuming that we are given an object (or objects) of the following description a*nd that all the steps in the given proof are legitimate*, then such and such follows". Instead, the content of a mathematical theorem is taken *solely to concern properties of the object(s) described*: groups, continuous real-valued functions, objects in a category, and the vast array of other objects treated as singular terms in the various branches of mathematics. Briefly, the theorems of mathematics are not taken to embed the conditions of their proof in the antecedent.[22] The proof, of course, does involve steps which lead from the antecedent to the conclusion, and how are we to know that such rules which permit such steps are *sound*, given objects of the sort that might satisfy the antecedent? Is it legitimate to appeal to the law of excluded middle, or the axiom of choice, when reasoning about the basis of a vector space? In short, the deliberate silence on the status of the objects that might satisfy the description contained in the antecedent precludes any response to concerns about the legitimacy of particular rules of inference employed in a proof. We are led, then, to another view that will feature in the development of the AFCS program: a proof provides grounds for the *assertibility* of a theorem, but does not suffice for the claim that the theorem is *true*.

[22] This allows there to be more than one proof of the same theorem.

Thus, when considering whether the AFCS program constitutes a *foundation* for mathematics, we see that it fails to satisfy the second criterion, which required an account of the way in which mathematical statements can be known to be true. If, contrary to the AFCS proposal, we pack the rules of inference employed in the proof of that theorem into the antecedent of the conditional corresponding to that theorem, then take the theorem as a universally quantified proposition, we obtain a trivial truth, but one which is certainly of little interest, and, on the AFCS view, one which does not correctly represent the content of the theorem. As the AFCS view takes mathematical statements to be *assertible* in virtue of a proof, but *not demonstrated to be true* by that same ground, this second criterion is not satisfied. Further, the third criterion also fails to be satisfied within the AFCS program, insofar as mathematical theorems are taken to be schematic in form, and so the semantics of such statements does not proceed solely via the semantic apparatus identified in the third criterion. Given that the second and third criteria for a foundation fail to be satisfied on the AFCS view, does the view offer an organizational framework? Here the answer is *yes*, but the manner in which the view satisfies this criterion differs from the manner in which the logicist or formalist views satisfy the criterion. It is in describing how the view aims to offer an organizational framework that we have the (long overdue!) appearance of the notion of a *category*, the use of which will also show the program to be a type of mathematical *structuralism*.

As Shapiro remarks, "The slogan of structuralism is that mathematics is the science of structure" [79, p. 61]. While there are a number of ways in which one might proceed in developing this viewpoint (several of which will feature in Chapter 2), it will be argued that the correct rendering of the insights that motivate the structuralist views of mathematics is not to focus our attention on special, intrinsically featureless objects that bear only relational properties, but instead to focus on *structural properties*, those properties which are common to all instances of the *same structure*. The *language* of category theory has the property that (in a manner to be made precise in the coming chapters), objects with the same structure have the same properties expressible in that language. The language of category, then, is capable of expressing properties with the "right level of detail", in contrast to, for example, the language of set theory. Consequently, if mathematics is a "science of structure", and this involves shifting attention away from special sorts of objects to structural properties, the language of category theory will be shown to be particularly well-suited to the task.

So, on this initial sketch of the AFCS view, a mathematical theorem will be argued to be best represented as a schematic conditional, and the correct account of the content of that conditional will be argued to involve a claim about objects with a particular sort of structure: that of a group, field, topological space, etc., and the *structural properties* of any such objects will be argued to be best captured using the language of category theory. Unlike either the logicist or finitist programs, the AFCS program does not identify a single privileged framework of assertory axioms with a special justificatory role. In the case of the logicist program, a system of ramified type theory, for example, was taken to be a framework in which all of the mathematical (logical) objects could be defined, and all of the theorems about those objects could be proved. In the case of the finitist program, finitary proof theory was taken to confer legitimacy on the axiomatic systems of the various branches of mathematics via consistency proofs. The axioms defining a category are not meant to offer a "bedrock" framework of this sort. Thus, while the language of category theory provides a linguistic framework arguably well suited to reflect the structural properties of interest, it constitutes an organizational framework in a manner quite different from that of programs which adopt assertory axioms. Mathematics is identified as the science of structure, and more specifically, the science of structural properties. Structural properties, as will be argued, are best rendered in the language of category theory.

Having outlined the AFCS program, one aspect of the program bears emphasis. The view that mathematical theorems are conditional in form, and involve no commitment to a special class of mathematical objects is not adopted in virtue of a prior commitment to nominalism. Rather, the view is very closely aligned with that expressed by Russell when he claims

> What pure mathematics asserts is merely that the Euclidean propositions follow from the Euclidean axioms—i.e. it asserts an implication: any space which has such and such properties has also such and such other properties. Thus, as dealt with in pure mathematics, the Euclidean and non-Euclidean Geometries are equally true: in each nothing is affirmed except implications. All propositions as to what actually exists, like the space we live in, belong to experimental or empirical science, not to mathematics; when they belong to applied mathematics, they arise from giving to one or more of the variables in a proposition of pure mathematics some constant value satisfying the hypothesis, and thus enabling us, for that value of the variable, actually to assert both the hypothesis and consequent instead of asserting merely the implication. [73, p. 5]

Whether abstract entities are admitted into one's ontology or not, mathematical theorems apply in either case—mathematics is not the arbiter of existence. If one considers a cube and is willing to speak of the symmetries of the cube as objects, those symmetries constitute a group of 24 elements. If one is willing to speak of the squares of a chessboard as objects, and the possible moves of a knight on the chessboard as arrows, the collection of such objects and arrows yields a category. Whether or not such objects are admitted is not a question that one's mathematical program should settle.

This view immediately allows the proponent of AFCS to reply to one line of criticism. Consider Hellman's claim that category theory is inadequate given its failure to address the "problem of mathematical existence",

> This problem as it confronts category theory can be put very simply: the question really just does not seem to be addressed! (We might dub this the *problem of the 'home address'*: *where do categories come from and where do they live?*) [38, p. 136][23]

If one aims at a foundational program of the sort envisioned by the logicists or finitists, this is a reasonable request: if a framework is to account for the truth of mathematical statements (and retains the "face-value" semantics which involves reference to mathematical objects), then there must be enough such mathematical objects to ensure theorems about the existence of exactly two four-element groups (up to isomorphism), the infinity of the natural numbers, and the uncountability of the real numbers. However, if one aims not at a *foundation* by instead at what might be called a *purely* organizational framework, this issue is avoided. If, further, it is correct to hold that mathematics should remain ontologically neutral, it would in fact be a mark of deficiency for the view to imply, or require, the existence of any objects, distinctly mathematical or otherwise.

To summarize, the AFCS program involves the following claims.

1. The structuralist view that "mathematics is the science of structure", is best expanded as the claim that "mathematics is the science of structural properties".

2. Mathematical theorems are both *conditional* and *schematic* in form.

3. Mathematics is taken not to concern any particular, determinate collection of objects, distinctly mathematical or otherwise; mathematical theorems involve no commitment to objects of any sort, structures included.

[23] Hellman's own program of modal structuralism, as presented in [35], addresses this concern via axioms that stipulates the *possibility* of, for example, a model of the (second-order) Peano axioms. If the ontological neutrality of mathematics is to be taken seriously, though, it might naturally be taken that mathematics should no more make claims about *possible* existence than it makes claims about *actual* existence.

4. The language of category theory is better suited to express structural properties than available alternatives.

The AFCS view embodies a number of ideas that have been variously expressed in recent work in the philosophy of mathematics, particularly Awodey [2], Bell [5], and Landry and Marquis [48]. The details of this view will be explored in the chapters to follow, and concerns of the sort raised in Shapiro [79], Hellman [38], and Feferman [29] will be addressed. The development of the AFCS will now proceed via an analysis of the notion of *structure*—and in particular, the notion of a *structural property*—in Chapter 2.

Chapter 2

Structures and Structural Properties

> *The central dogma of the axiomatic method is this:* isomorphic structures are mathematically indistinguishable in their essential properties. [57, p. 19–20]

> *Category theory is the most elaborate and successful instance of an axiomatized theory allowing for a systematic characterization and analysis of the different structures, and the recurring mathematical phenomena that come forward in the latter.* [23, p. 12]

While the claim that mathematics is correctly viewed as the "science of structure" has received considerable attention in recent work in the philosophy of mathematics[1], the proponents of the view each offer distinct—and sometimes incompatible—accounts of their various structuralist projects. In this chapter I will be concerned to distinguish the structuralist element of the AFCS program, and to consider the way in which the category-theoretic account of structure, via *structural properties*, captures what can be identified as the key insight of the structuralist perspective in mathematics. This chapter

[1] See, for example, Shapiro [78], Hellman [35], and Resnik [69].

concludes with a comparison of the category-theoretic treatment of structure to that of two alternative structuralist programs.

2.1 Early Structuralism

An important early work that can be taken to illustrate several of the key *structuralist* insights is Dedekind's 1888 essay on "The Nature and Meaning of Numbers" [24]. In that work Dedekind introduces his construction of the natural numbers, achieved through a characterization of what would now be called an ω-sequence. This definition of an ω-sequence will be reviewed here and then used to illustrate and motivate a number of the insights that reemerge in contemporary structuralist views in the philosophy of mathematics.

Working against a background of informal set theory, Dedekind takes the range of objects in his framework to consist of "every object of our thought", and these objects can be collected up into "systems", where "a system S (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing" [24, p. 21]. Against this informal set-theoretic background, Dedekind appeals to the notion of a *transformation*, where "By a transformation ϕ of a system S we understand a law according to which to every determinate element s of S there belongs a determinate thing which is called the *transform* of s and denoted by $\phi(s)$" [24, p. 24]. Thus, in modern terminology Dedekind provides an informal set-theoretic framework along with the notion of a *function* defined on a set S. A transformation ϕ, which maps S to $\phi(S)$ is *similar* provided it is injective, in which case it has an inverse, $\overline{\phi}$, mapping $\phi(S)$ to S. Two systems S and R are said to be "similar" when there exists a similar transformation ϕ such that $\phi(S) = R$, that is, when there exists a function ϕ mapping S to R that is a bijection. With these notions Dedekind is able to introduce his now familiar definition of an infinite set: a set is *infinite* (now commonly referred to as "Dedekind-infinite") provided there is a similar (i.e., injective) function ϕ which maps S to a proper part of itself.

The final ingredient of Dedekind's definition of a *simply infinite system*—in modern terms, an ω-sequence—involves the definition of a *chain*. A set S is a *chain* (relative to a function ϕ) provided $\phi(S) \subseteq S$. Give a subset A of S and a function ϕ (where S is a chain relative to ϕ), *the chain of* A (relative to ϕ) is defined to be the intersection of all chains containing A. In essence, *the* chain of A is the minimal closure of A under ϕ.[2] With this definition at hand, Dedekind defines a *simply infinite system*:

> A system N is said to be *simply infinite* when there exists a similar transformation ϕ of N in itself such that N appears as chain of an element not contained in $\phi(N)$. We call this element, which we shall denote in what follows by the symbol 1, the *base-element* of N and say the simply infinite system N is *set in order* by this transformation ϕ. [24, p. 33]

Using modern terminology, a set N is *simply infinite* provided there exists an injective function ϕ mapping N to a proper subset of itself, for which one of the elements $a \in N \setminus \phi(N)$ is such that the minimal closure of $\{a\}$ under ϕ is N itself.[3] Consequently, any such N is an ω-sequence, and can be shown to satisfy the Peano axioms.[4]

2.1.1 Structuralist Perspectives

There are two key features of Dedekind's treatment of *simply infinite systems* that can be described as *structural*. First, the elements of any simply infinite system N are described in solely in terms of their *relational properties*; that

[2]That is, if A_0 is the chain of A under ϕ, then $A \subseteq A_0$ and for all $a \in A_0$ we have $\phi(a) \in A_0$, where A_0 is the smallest set with this property.

[3]Here we might observe that Dedekind is not distinguishing between an element and its singleton, as chains have only been defined for sets.

[4]Indeed, Dedekind effectively shows that the Peano axioms hold of any such N, one year prior to the paper in which Peano presents these axioms.

is, the elements of such a simply infinite system N are not required to have any particular *intrinsic* properties. Indeed, given a simply infinite system N (relative to a function ϕ) with base-element b and an element a not in N, we can simply define a function

$$\psi(x) = \begin{cases} \phi(b) & \text{if } x = a, \\ \phi(x) & \text{otherwise.} \end{cases}$$

to get a simply infinite system N' (with a replacing b, and which is now relative to the function ψ). Compare this situation to one in which, instead of simply infinite systems N, we consider systems C of objects of the same colour. We cannot, for a given such system C—say, a system of red objects—replace one such red object with an arbitrarily selected object not already in the system and produce a new system of objects all of the same colour. Any object can *play the role* of a base-element (or similarly, any other element) in a simply infinite system, but this is not true of objects in a system of similarly coloured objects. An element of a system of red objects *must be red*, and replacing it with an object of another colour yields a system which is no longer a system of similarly coloured objects.

A second structural feature of Dedekind's approach concerns not the arbitrary interchangeability of the *elements* of a system, but instead concerns the interchangeability of the *systems* themselves. In an investigation of the properties of simply infinite systems, we might say, with Benacerraf, that "any old ω-sequence would do" [12, p. 189]. The particular features of a given simply infinite system N, such as the particular elements of which it is composed, are irrelevant to the role of N *qua* simply infinite system; the elements of any such N can be used in counting, arithmetic, and so forth. Any simply infinite system could be *taken as* the natural numbers (or taken as a convenient surrogate), and the theorems of arithmetic, purportedly about *the* natural numbers, would apply equally to the elements of that simply infinite system. As Dedekind puts it

> ...it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N set in order by the transformation ϕ, and indeed every theorem in which we leave entirely out of consideration the special character of the elements n and discuss only such notions as arise from the arrangement ϕ, possesses perfectly general validity for every other simply infinite system Ω set in order by a transformation θ and its elements ν [24, p. 48]

Both of these perspectives reflect the *structuralist* focus, which involves not the intrinsic properties of elements or systems themselves, but involves instead the relational properties, i.e., the *relational structure* of the entities in question. In effect, these two different perspectives amount to a difference in emphasis, and are taken here to characterize the *structuralist* view of mathematics. These perspectives are roughly

1. The particular *elements* of a system don't matter, only their standing in relations of the right sort is of mathematical concern—*any suitably-related elements* would do.

2. The particular *system* of elements doesn't matter, only that the system's elements stand in certain relations—*any suitably-structured system* would do.

To illustrate these differences in perspective, consider the case of a jeweller who has grouped emeralds according to colour. If asked to explain to a potential customer the variability permitted *within* colour groupings, we may suppose that any group of similarly coloured stones would suffice. However, it is clearly not the case that a gemstone in one collection could be arbitrarily replaced with a gemstone from another—the colour would be wrong. In this circumstance, then, we have variability of the second sort, but not of the first: any *group* of gemstones would serve to illustrate the variability permitted within a group, but we cannot arbitrarily replace *individual* gemstones in one group with those of another. Now imagine that the jeweller has created several different piles of emeralds on a table, with different numbers of emeralds in each pile, and a customer has requested that a necklace be made with exactly five gemstones. If our jeweller identifies a group of five gemstones, then we have variability of the first sort, but not the second. Any of the *individual* gemstones in the selected group of five could be switched with a gemstone taken from another group (as there would then remain five gemstones in the selected group), but other *groups* may not suffice, as some contain more than or less than five gemstones. The structuralist approach in mathematics can be viewed as highlighting the insensitivity—for mathematical purposes—of both sorts: insensitivity to the *particular elements* in a system of a given type, and insensitivity to the *particular system* of that given type.

While Dedekind's remarks cited earlier may suggest that he favoured the second of these perspectives, other remarks suggest that instead he favoured this first perspective, and indeed it is this first perspective that leads him to a view that

> If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers* [24, p. 33]

This passage from Dedekind can be read in two ways. First, it may be read as suggesting that, attending only the the relational properties of a system's elements, any simply infinite system could *serve as* the natural numbers. Alternatively, it can be read as suggesting that one proceed by a process of abstraction[5] to arrive at *the* natural numbers, which have only those properties common to all ω-sequences, i.e., they have *only* the structural properties possessed in virtue of being an ω-sequence. In support of this latter reading of Dedekind, Dedekind writes in a letter to Heinrich Weber that

> I should still advise that by number... there be understood not the *class* (the system of all mutually similar finite systems), but rather something *new* (corresponding to this class), which the mind *creates*. We are of divine species and without doubt possess creative power not merely in material things (railroads, telegraphs), but quite specially in intellectual things. [80, p. 248, quoting from an 1888 letter found in Dedekind's *Gesammelte Mathematische Werke*]

Of course, placing emphasis on this first perspective need not lead to a commitment to a special sort of "purely relational" (mathematical) object, possessing no intrinsic properties, and we will consider modern structuralist programs that do not involve commitment to objects of this sort.[6]

These structuralist perspectives account for the ease with which structuralist programs may account for the *application* of mathematics in, for example, the sciences. Given that, as Dedekind notes, the theorems of arithmetic will apply to *any* simply infinite system, there is a clear link between the *pure* mathematics of arithmetic—effectively the study of *any* simply infinite system—and the *applied* mathematics of arithmetic, which arises from the (extra-mathematical) determination that a given system of objects *counts as* a simply infinite system, and that the objects in view can *serve as* elements in a simply infinite system. This naturally fits with the schematic, conditional view of mathematics adopted on the AFCS view. Given the determination that a system can be taken to satisfy the axioms defining a

[5]This abstraction process is dubbed "Dedekind abstraction" in [81].

[6]Although, as well shall see, Shapiro's *ante rem* program does invoke such a commitment.

type of structure (thereby satisfying the antecedent of the conditional of a theorem), the consequent of that theorem, having been proven to hold for structures of the relevant sort, is also true of the given system. As the theorem is schematic, there is no *mathematical* constraint on the sorts of entities that can be viewed as systems, nor the sort of entities that can be viewed as elements of systems. Therein lies the key to the universal applicability of mathematics; one willing to treat *forces* as objects will discover that they can then be fruitfully studied as vectors, one willing to treat *symmetries of regular polygons* as objects can appeal to the results of group theory. There are no mathematical prohibitions which concern what can count as a system, or what can serve as an object in a system.

Dedekind's well-known attempt to *prove* the existence of infinite systems is generally taken to have been inadequate. Recall Dedekind's claim that

> My own realm of thoughts, i.e., the totality S of all things, which can be objects of my thought, is infinite. For if S signifies an element of S, then is the thought s', that S can be object of my thought, itself an element of S. If we regard this as transform $\phi(s)$ of the element S then has the transformation ϕ of S, thus determined, the property that the transform S' is part of S; and S' is certainly proper part of S, because there are elements in S (e. g., my own ego) which are different from such thought S' and therefore are not contained in S'. Finally it is clear that if a, b are different elements of S, their transforms a', b' are also different, that therefore the transformation ϕ is a distinct (similar) transformation. Hence S is infinite, which was to be proved. [24, p. 31]

As Hellman rightly notes,

> ...there were at least two flaws in Dedekind's "proof": first, there was the problem of meaningfully iterating a "the thought that..."

> operator an arbitrary finite number of times, obtaining a new object at each stage. And, second, there was the need to collect all such objects via some comprehension principle, which Dedekind did not explicitly articulate. [35, p. 29]

However, the failure of Dedekind's argument as a *proof* of the existence of infinite systems does not prevent him from *using* the results of his mathematical framework to investigate the simply infinite system generated by taking a particular thought a, and considering the minimal closure of $\{a\}$ under ϕ. If *he* is willing to speak in this manner (though we may not be inclined to follow), he can then produce the prime factorization of a particular such element, (secure in the thought that such a factorization is unique up to ordering), or use those elements to create and solve equations.

Before moving to consider recent structuralist programs in the philosophy of mathematics, one further feature of Dedekind's approach bears mention, and it will be a feature that plays a prominent role in *category-theoretic* approaches to structuralism. There is another sense in which a system N counting as a simply infinite system depends on an *external* feature, one that is not intrinsic to the system N (nor is this feature intrinsic to the elements of such a system): whether or not a system N counts as simply infinite depends crucially on the existence of a function ϕ, a function which "sets N in order". Thus, one can conceive of a system of objects that *would* count as a simply infinite system, but for a poverty of functions. While Dedekind says little about the ontological assumptions governing functions, he notes (in article 21) that there are identity functions for any system S, that function composition is defined for all composable pairs (article 25), and that function composition is associative (article 25). That is, Dedekind establishes that taking *functions* as arrows and *systems* as objects yields a category. Whatever the elements of the structures are taken to be, the systems of such elements are described throughout Dedekind's essay entirely in terms of the properties of functions acting on those systems; whether a system is finite, infinite, simply infinite, and so forth depends not on the specific nature of the elements of the system, but instead depends on the "external" characterization of those elements via the functions defined on those systems. Dedekind's treatment of simply infinite systems is quite naturally expressed in the language of the category theory, and the distinctions Dedekind makes concerning the cardinality of his systems—the objects of this category—depends entirely on the features of this ambient category. We will have an opportunity to pursue this account of Dedekind when we explore category-theoretic, or *categorical*, structuralism. Now we will briefly consider how some of Dedekind's structuralist approach *within* mathematics emerged in the modern structuralist approach within the *philosophy of* mathematics.

2.2 Modern Structuralism

The structuralist perspective came to the fore in philosophical circles with Benacerraf's seminal "What Numbers Could Not Be" [10]. Benacerraf in this familiar paper presents the situation of Johnny and Ernie, one of whom is taught, within a set-theoretic framework, the Zermelo definition of the finite ordinals, while the other is taught the von Neumann definition.[7] Discussing their rival accounts of the numbers, Johnny and Ernie notice that their accounts are incompatible; on the Zermelo version, for instance, it can be proven that all finite ordinals are singletons, a result that is (obviously and provably) false on the von Neumann account. Benacerraf's assessment of the problem is that "*the accounts differ at places where there is no connection between features of the accounts and our uses of the words in question*" [10, p. 62]. Benacerraf's response to this observation is to claim that any ω-

[7] On the von Neumann definition (now the standard choice) the finite ordinals begin with $\varnothing$ and each element in the sequence is the set of all its predecessors, so the sequence is $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots$, while the Zermelo definition also begins with $\varnothing$, but then takes each successive element to be the singleton of its predecessor, giving the sequence $\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots$.

sequence can be taken to play the *role* of the natural numbers,[8] but that the natural numbers are not to be identified with any *particular* such collection of objects. Numbers cannot be sets because when presented with incompatible set-theoretic accounts of the numbers we have no criteria to which we can appeal in breaking the tie. Further, numbers can't be *objects* on Benacerraf's view for essentially the same reason: the numbers are characterized only in relational terms, and so an object might *play the role of the number 3*, but to claim that an object *is the number 3* is to predicate a non-relational property of that object, and so any attempt to identify an object as this or that particular number can do so only by smuggling in some inappropriately intrinsic features. Arithmetic, for example, is then "the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions" [10, p. 70].

Benacerraf's discussion of the problem of identifying *the* natural numbers informs many of the debates concerning contemporary structuralist views. Benacerraf's remark that "any ω-sequence will do" invites one to consider the sense in which this claim is intended: any ω-sequence will do *for what purpose*? Benacerraf notes that "For arithmetical purposes the properties of numbers which do not stem from the relations they bear to one another in virtue of being arranged in a progression are of no consequence whatsoever" [10, p. 69–70]. Recalling Dedekind's observation that any theorem we arrive at having ignored "the special character of the elements" of a particular simply infinite system N will also be true of any other simply infinite system M, it is clear that "any ω-sequence will do" is intended in the sense that *any properties of mathematical interest* are common to all ω-sequences. Provided we appeal only to properties possessed by a system *in virtue of being a system of that type* our theorems will be true of all such systems. Enlarging the target from arithmetic to all branches of mathematics and agreeing to call

[8]Benacerraf originally required that the sequence in question have a recursive order relation, and supplied a confused argument to that effect, but has since retracted that additional requirement in [12].

such properties of mathematical interest *structural* properties (as they are common to all systems of a particular type, i.e., all systems of a *particular structure*), we can express the key structuralist insight, reflected in the two structuralist perspectives, as follows: mathematics is concerned only with the study of structural properties—*mathematics is the science of structural properties*. The *particular objects* in a given simply infinite system don't matter (any objects "will do"), because their intrinsic properties are irrelevant to the *mathematical* study of that system; the properties of interest, those the elements exhibit simply in virtue of featuring in a system of that sort, do not involve any intrinsic properties of the elements. The *particular system* does not matter precisely because all of its properties of mathematical interest—the structural properties it possesses simply in virtue of being a system of the relevant sort—are common to *any* system of that sort. Thus we arrive at a positive proposal that captures the central insight of the two (negative) structuralist perspectives identified earlier. Clearly some care is required to articulate precisely what is meant by a *structural property*, and the next section will be concerned to articulate and assess the merits of the category-theoretic account of structural properties.

2.3 Structural Properties

In sharpening the notion of a *structural property*, focus on the natural numbers (via ω-sequences) has the unfortunate consequence that two distinct approaches to identifying structural properties run together. First there is the view that structural properties are those that a system has *in virtue of being a system of a given type*. So, in the case of the natural numbers, characterized by, say, the second-order Peano axioms[9], the structural properties could

[9] Informally these axioms state: (1) every number has a successor; (2) the successor of a number is unique; (3) for any numbers x and y, if the successor of x is equal to the successor of y, then x equals y; (4) there exists a number which is not a successor of any number; and (5) the principle of induction holds. The second-order formulation differs

be taken to be exactly those which are consequences of the axioms. Writing $\mathcal{N}$ and $\mathcal{M}$ to represent models[10] $\langle N, s, 0\rangle$ and $\langle M, s', 0'\rangle$ of the second-order Peano axioms, for any closed sentence A in the (second-order) language of arithmetic, we have

$$\mathcal{N} \vDash A \Leftrightarrow \mathcal{M} \vDash A.$$[11]

As any such A holds (or fails to hold) in both $\mathcal{N}$ and $\mathcal{M}$, *structural properties* might then be identified with those which correspond to the logical consequences of the Peano axioms. Restricting attention to the first-order case[12], which admits of non-standard models, we may still make this identification, as the (now first-order) consequences of the axioms will still hold of any model of the axioms; any properties that serve to distinguish the models may then be deemed non-structural. Thus, for example, any model of the Peano axioms is such that (the interpretation of) the sentences "$2 + 3 = 5$" and "$1 + 4 \neq 7$" both hold, and so those formulae corresponds to structural properties, while the property of "having cardinality $\aleph_0$" will not count as a structural property.[13]

Second, we have the view that the structural properties of a given system are those which are true of *all models of the same sort*. Slightly more precisely, a property may be deemed *structural* provided it holds in all *isomorphic* models. Given that all models of the second-order Peano axioms are isomorphic,[14] when the theory in question is that of natural number arith-

from the first-order formulation in that (5) can, in the second-order formulation, be taken as a single axiom, while it must be taken as an axiom schema in the first-order formulation. PA^2 will be used to indicate the (conjunction of the) second-order Peano axioms.

[10]It is important to note that the notion of *model* invoked here is not meant to involve any restriction to, for example, models constructed in a privileged set-theoretic framework. The notion of *model* invoked will be discussed in detail in Chapter 3.

[11]This is a consequence of the (second-order) categoricity result. For details, see [75, p. 82–83].

[12]First-order presentations of the Peano axioms typically incorporate additional axioms that recursively define addition and multiplication.

[13]The existence of models of the first-order Peano axioms having cardinality $> \aleph_0$ is an immediate consequence of the (upward) Löwenheim-Skolem theorem; see [9, p. 82].

[14]The details of this proof are essentially due to Dedekind, and can be found in [75,

metic this account of structural properties collapses into the first account. If a property is a logical consequence of the Peano axioms then it holds in all models, and so *a fortiori* in all isomorphic models. Conversely, if a property holds in all isomorphic models of the Peano axioms then it holds in *all* models (as they are all isomorphic), and so it is a logical consequence of the Peano axioms.[15]

Consider, however, the notion of a *group* (see Section 1.1.2). The axioms of a group are not categorical, and so, unlike the case with ω-sequences, the two preliminary accounts of *structural properties* come apart. On the second account, where *structural properties* are those *true* (*or false*) *in all isomorphic models*, "having a commutative group operation"[16], "having order 5", and "being generated by a single element" all count as structural properties. However, all of these properties are such that there are both groups that exhibit those properties and groups that do not. Hence, on the first account of *structural properties*, all of these properties would fail to count as structural.

There are three reasons to favour pursuit of this second account of *structural properties*. The first is a purely pragmatic consideration: when mathematicians study groups, they are typically interested in studying exactly the sorts of properties listed above: commutativity, generators, order, subgroups, etc., and all of these are preserved under (group) isomorphism. Second, those properties that would count as *structural* on the first proposal *also* count as *structural* on the second proposal. So, no properties are "lost" in pursuing the second account. Finally, and most importantly, by restricting attention to groups in a particular isomorphism class (rather than taking into account all groups) the two *structuralist perspectives* are preserved. The *particular* elements don't matter, as it is possible to redefine the group operation in

p. 82–83].

[15]It is worth noting that such a property may not be derivable from the Peano axioms in a chosen deductive system, as "full" second-order logic is not recursively axiomatizable. Details are covered in [75].

[16]A commutative group operation is one for which $a \star b = b \star a$ for all $a, b \in G$. A group with a commutative group operation is called an *Abelian* group.

such a way as to permit any object to replace a given object in a particular group,[17] and it is possible to do so in a manner that yields a group isomorphic to the original. Similarly, by definition these properties are exhibited by all groups in the isomorphism class, and so the *particular* group selected for study doesn't matter—any group from the isomorphism class will do. Thus, the second approach to structural properties is consistent with mathematical practice, subsumes the first candidate approach, and preserves both structuralist perspectives. We now turn to examine an account of the notion of a *structural property* cast in the language of category theory.

2.4 Structural Properties in the Language of Category Theory

2.4.1 The Language of Category Theory

Consider a typed first-order language $\mathcal{L}$, with lowercase letters $x, y, z, \ldots$ used for terms of the first type (*arrow type* terms) and uppercase letters $A, B, C, \ldots$ used for terms of the second type (*object type* terms).[18] As is usually the case, letters from the first part of the alphabet are typically used for constants of their respective types, though context will usually suffice to distinguish constants from variables of either type. There are two primitive unary function symbols *Dom* and *Cod* which take arguments of arrow type and yield objects, one primitive unary function symbol 1_{-} which takes an object as argument and yields an arrow, one binary relation symbol $=$ which accommodates arguments of either type, and one binary function symbol $\circ$ which takes two arguments of arrow type and yields an object of arrow type.

[17] This is accomplished simply by introducing a new group operation defined in terms of the previous one, as was done with ω-sequences in Section 2.1.1.

[18] This presentation of the language of *elementary* (first-order) category theory is a modified version of that found in [50]. In this case, composition of arrows, $\circ$, is taken as a (partially-defined) function on arrows, Lawvere makes use of a primitive ternary relation Γxyz, corresponding to $y \circ x = z$.

Note that $f \circ g$ is only defined when $Cod(g) = Dom(f)$, and so $\circ$ is a partial function on the class of arrows. The following clauses then determine the well-formed formulae (wffs) of the language $\mathcal{L}$.

1. For any term of arrow type x and term of object type A, $Dom(x) = A$ and $Cod(x) = A$ are wffs.

2. For any terms of arrow type x, y, and z, $x \circ y$ is a term of arrow type, and $x \circ y = z$ is a wff.

3. For any terms x, y of arrow type and A, B of object type, $x = y$ and $A = B$ are wffs.

4. For any wffs ϕ and ψ, $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \rightarrow \psi$ are wffs.

5. Where a wff ϕ has a free variable x of arrow type, $\forall x\phi$ and $\exists x\phi$ are wffs; similarly for ϕ with a free variable F of object type.

Given such a language $\mathcal{L}$ and using the usual abbreviations, one can form expressions such as

$$\forall A \exists ! x(Dom(x) = A \wedge Cod(x) = B),$$

which expresses the claim that B is a *terminal object*,[19] and

$$\forall x, y((Cod(x) = Cod(y) \wedge Dom(x) = Dom(y)) \rightarrow (z \circ x = z \circ y \rightarrow x = y)),$$

which expresses that z is a *monic*.[20]

2.4.2 Axioms and Definitions in Category Theory

The axioms that define a category were presented in Section 1.2 are here repeated, now presented as formulae in the language $\mathcal{L}$.

[19] *Terminal* objects are the category-theoretic analogue of a singleton set.
[20] *Monic* arrows are the category-theoretic analogue of an injective function.

1. $\forall x \exists A, B(Dom(x) = A \wedge Cod(x) = B)$

2. $\forall x, y((Dom(x) = Cod(y)) \rightarrow (Dom(x \circ y) = Dom(y) \wedge Cod(x \circ y) = Cod(x)))$.

3. $\forall A((Dom(1_A) = A = Cod(1_A)) \wedge (\forall z((Cod(z) = A \rightarrow 1_A \circ z = z) \wedge (Dom(z) = A \rightarrow z \circ 1_A = z))))$

4. $\forall x, y, z((x \circ y) \circ z = x \circ (y \circ z))$

As the notion of *isomorphism* will play a critical role here, we will need to introduce the category-theoretic rendering of this notion. Objects A and B in a category are said to be isomorphic provided there is an arrow between them (in fact, a pair of arrows) with a particular property. Arrows of three types play an important role in category theory. For the sake of brevity, these definitions are presented informally, rather than as formulae in $\mathcal{L}$. Note that two arrows with the same domain and codomain are said to be *parallel.*

- An arrow $f : A \rightarrow B$ is *monic* (represented by $f : A \rightarrowtail B$) provided, given any T and parallel arrows $g : T \rightarrow A$ and $h : T \rightarrow A$, $f \circ g = f \circ h$ implies $g = h$.

- An arrow $f : A \rightarrow B$ is *epic* (represented by $f : A \twoheadrightarrow B$) provided, given any T and parallel arrows $g : B \rightarrow T$ and $h : B \rightarrow T$, $g \circ f = h \circ f$ implies $g = h$.

- An arrow $f : A \rightarrow B$ is *iso* (represented by $f : A \overset{\sim}{\rightarrow} B$) provided there is an arrow $g : B \rightarrow A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. If $f : A \rightarrow B$ is iso, we will sometimes write $A \overset{f}{\sim} B$, or simply $A \sim B$. Such an arrow g is unique, and so typically represented as f^{-1}.

Monics are the category-theoretic analogue of injective functions; if two parallel functions $h, j : T \rightarrow A$ are distinct then they disagree on some value $t \in T$, then $h(t) \neq g(t)$, and so (for an injective function f) $f(h(t)) \neq f(g(t))$, i.e., $f \circ h \neq g \circ h$. The converse of this result is reflected in the definition of a monic. *Epic* arrows are akin to *surjective* functions, although the analogy is less fitting in this case; as McLarty explains "It is better to think of an epic as 'covering enough of B' that any two different arrows out of B must disagree somewhere within the part covered by f" [60, p. 15].[21] Every iso is both epic and monic, but the converse holds only in *balanced* categories.[22] As arrows in a category *need not be functions*[23], it is important to note that these examples are merely suggestive. Monics, epics, and isos are defined in terms of properties of arrows in a category, and arrows are primitive.

2.4.3 The Structural Properties Theorem

The question of principal interest, then, is *given an object A in some category, which predicates expressible in $\mathcal{L}$ correspond to structural properties of A?* Recalling the two structuralist perspectives (see Section 2.1.1), those properties which depend on *particular* elements and *particular* systems are ruled out. In the case of ω-sequences $\langle N, s, 0\rangle$ and $\langle N', s', 0'\rangle$, we would not want to consider, for example, "having successor function equal to s" as a structural property; ω-sequences are required to have a successor relation, but not to have *the same* successor relation. Similarly, having a *particular* base-element a, or for the system to contain a *particular* element b, should not count as structural properties, those particular elements are not essential, we merely require elements that *play the same role*. This leads to the first restriction on predicates corresponding to structural properties in $\mathcal{L}$: structural properties should not involve names of particular elements, i.e., *constants*, in their formulation.

[21] In the category of rings and ring homomorphisms, the inclusion $f : \mathbb{Z} \to \mathbb{Q}$ is monic and epic, but not iso.

[22] In an arbitrary category, an arrow is iso iff it is both monic and *split epic*, where an arrow $f : A \to B$ is split epic provided there is some arrow $g : B \to A$ such that $f \circ g = 1_B$. The details will not concern us here.

[23] Recall the examples given in Section 1.2. For example, taking positive integers as objects and $n \times m$ real-valued matrices as arrows $f : n \to m$.

The first restriction leads directly to a second restriction that shares the same motivation. If we allow parameters—unquantified variables—in the formulae meant to correspond to structural properties, then the problem we hoped to avoid by prohibiting constants would reemerge. Again considering ω-sequences $A = \langle N, s, 0\rangle$ and $B = \langle N', s', 0'\rangle$, take a formula with variables Z, x, and y such as "x is the base-element of the ω-sequence Z and has immediate successor y". Representing this formula as $F(Zxy)$, consider the formula obtained by existentially quantifying over x and leaving y free, $\exists x F(Zxy)$. Structural properties are meant to hold of all models of the same sort—in this case, all ω-sequences—but while all ω-sequences have a base-element (which then has a particular successor element), the successor element of the base-element in one sequence may not be that of the base-element in another. Thus, if a is the successor of the base-element 0 in the ω-sequence A and b is the successor of the base-element $0'$ in the sequence B, then, provided $a \neq b$, we have one sequence that satisfies $\exists x F(Zxa)$ and another that does not. In short, allowing parameters permits the substitution of *particular* elements into formulae, which may then yield predicates corresponding to properties that depend on those particular elements, and so would not count as structural. In summary, the structuralist perspectives yield an account of (the predicates corresponding to) structural properties that prohibits constants and parameters. We now proceed to establish a result that shows the language of category theory to be particularly well suited to express structural properties: given any object A in a category, all formulae in $\mathcal{L}$ which do not involve constants or parameters correspond to structural properties of A. That is,

Structural Properties Theorem. *If Φ is a formula in $\mathcal{L}$ with one free variable of object type and no constants or parameters, then if A and B are isomorphic objects in some category* $\mathbf{C}$*,*

$$\Phi A \Leftrightarrow \Phi B.$$

Proof. (sketch)[24] Fix a category **C** and objects A and B such that $u : A \xrightarrow{\sim} B$. The proof proceeds by structural induction on Φ, and makes use of a mapping F, a function taking objects to objects and arrows to arrows defined as follows.

Let $F(A) = B$, $F(B) = A$, and $F(C) = C$ for any object $C \neq A, B$. Thus, F exchanges objects A and B, and leaves all other objects unaltered. Note that F is a bijection on objects, as F is clearly surjective, and for any object X of **C**, $FF(X) = X$.

The action of F on arrows depends on whether the arrow has either of A or B as domain or codomain. Analogous to the case with objects, the action of F on arrows will be to swap to roles of A and B, leaving other objects fixed. For any $f : X \to Y$ such that $X, Y \neq A, B$, let $F(f) = f$. When either A or B feature in an arrow, composition with either u or u^{-1} will be used to yield an arrow which exchanges A and B. The clauses that determine the action of F on such arrows are given in the table below.
For any $X, Y \neq A, B$,

$$\begin{array}{lcr}
f : X \to A & \mapsto & u \circ f : X \to B \\
f : X \to B & \mapsto & u^{-1} \circ f : X \to A \\
f : A \to X & \mapsto & f \circ u^{-1} : B \to X \\
f : B \to X & \mapsto & f \circ u : A \to X \\
f : A \to B & \mapsto & u^{-1} \circ f \circ u^{-1} : B \to A \\
f : B \to A & \mapsto & u \circ f \circ u : A \to B \\
f : A \to A & \mapsto & u \circ f \circ u^{-1} : B \to A \\
f : B \to B & \mapsto & u^{-1} \circ f \circ u : A \to B
\end{array}$$

A long, but straightforward case analysis suffices to show that, for any arrow f we have $FF(f) = f$, the latter following by the definition of F taken along with the associativity of composition and properties of u and u^{-1}. For

[24]This proof follows the technique presented in [61], in which McLarty establishes a result specific to *natural number* objects in a category of sets (a category with additional axioms appropriate to the specific case investigated in that article). In particular, the core of this proof, the definition of the functor F, carries over from McLarty's proof.

example, consider an arrow $f : A \to X$ as above. We get

$$\begin{aligned} FF(f) &= F(f \circ u^{-1}) && \text{by the definition of } F, \\ &= (f \circ u^{-1}) \circ u && \text{as } f \circ u^{-1} : B \to X, \\ &= f \circ (u^{-1} \circ u) && \text{by associativity,} \\ &= f && \text{as desired.} \end{aligned}$$

The other cases follow similarly. Again, routine case analysis shows F to be surjective (the pre-image of any f can be built up using appropriately chosen composites with u and u^{-1}), and so F is a bijection on arrows.

The strategy is now to use structural induction to establish that

$$\Phi(A) \Leftrightarrow \Phi(A)^F, \tag{2.1}$$

where, for a given formula Ψ, the formula Ψ^F is obtained by replacing every object term X with $F(X)$ and every arrow term f with $F(f)$. As F is a permutation on the objects and arrows of $\mathbf{C}$, there existing an arrow f (or object X) satisfying some condition is equivalent to there existing an arrow $F(f)$ (or object $F(X)$) satisfying that same condition, similarly for universally quantified expressions. As the only constant term appearing in the formula $\Phi(A)$ in (2.1) is, by assumption, A, (2.1) would then simplify to

$$\Phi(A) \Leftrightarrow \Phi(B). \tag{2.2}$$

For the base case of the inductive proof, the atomic formulae are of the form $x = y$ or $X = Y$ for arrow terms or object terms, respectively. Here again, treating cases is lengthy but straightforward. It can be shown that F respects domains, codomains, identities, and composites.[25] That is, for any

[25] These are precisely the conditions that establish that F is a *functor* $F : \mathbf{C} \to \mathbf{C}$, which is an arrow in the category of categories.

arrows f and g and any object X,

$$Dom(F(f)) = F(Dom(f)), \tag{2.3}$$

$$Cod(F(f)) = F(Cod(f)), \tag{2.4}$$

$$F(1_X) = 1_{F(X)}, \text{ and} \tag{2.5}$$

$$F(f \circ g) = F(f) \circ F(g). \tag{2.6}$$

For example, consider the following case of (2.6), with $g : X \to A$ and $f : A \to B$, where $X \neq A, B$. Then we have

$$\begin{aligned} F(f \circ g) &= u^{-1} \circ (f \circ g) && \text{as } f \circ g : X \to B, \\ &= u^{-1} \circ f \circ u^{-1} \circ u \circ g && \text{as } u^{-1} \circ u = 1_A, \\ &= (u^{-1} \circ f \circ u^{-1}) \circ (u \circ g) && \text{by associativity,} \\ &= F(f) \circ F(g). \end{aligned}$$

The other 26 cases of (2.6) follow similarly.

The inductive step of the proof then follows from the earlier observation that quantification is unaffected by "exponentiation by F", and that exponentiation by F leaves the logical operators of a formula unchanged. □

Structural properties are, on one approach, those that do not depend on the features of particular elements or particular systems. This view is reflected in the (negative) structural perspectives identified in Section 2.1.1, and leads to a prohibition against constants or parameters appearing in any formula corresponding to a structural property. Second, in attempting to sharpen the notion of a *structural property*, those properties were tentatively identified as those which are shared by any system of a given type, where two systems N and M being *of a given type* was identified with their being *isomorphic*. The theorem above establishes that the language of category theory unifies these views: all predicates without names or parameters correspond to structural properties, in the sense that any predicate of that sort corresponds to a property common to all isomorphic objects in a category.

Certainly some important questions need to be addressed in order to properly assess the merits of this feature of the language of category theory. If, for example, one is interested in establishing a theorem concerning cyclic groups, one might take cyclic groups as *objects* in a category, where the *arrows* in that category are group homomorphisms. But *which groups are there*? And how are these groups related; i.e., *which homomorphisms are there*? A homomorphism between groups can be represented as an arrow between those groups in a category, but *is there* a homomorphism between two particular groups G and F? Also, should our category $\mathbf{C}$ perhaps contain *all* groups, or *all finitely generated* groups? Recall that textbooks in group theory typically identify groups as *sets*[26], and so it may be more natural to take the language for expressing the structural properties of groups to be the language of some *set theory*, say, ZFC. The language of ZFC has the further advantage of offering a uniform treatment of all its subject matter: groups are sets, functions between groups are sets, elements of groups are sets.[27] One way to describe these concerns is via a comparison to the language of set theory: does category theory afford one the same expressive resources as the language of set theory?

Exploring the AFCS account of *models* of an axiomatic system will be postponed until Chapter 3, but recall that one aspect of the AFCS view is a deliberate silence on Hellman's problem of the "home address", save for advocating the language of category theory as particularly well-suited to capture the notion of *structural property*.[28] Any specification of the particular objects taken to compose models, and, indeed, the nature of the models themselves, is, so to speak, in the hands of the applied mathematician, the physicist, and those who make use of the schematic, conditional theorems of

[26] See the definition cited in Section 1.1.2.

[27] This is a minor complaint against the category-theoretic approach, as an "arrows-only" definition of a category is also available, see [52].

[28] See the discussion in Section 1.5.

mathematics. This view will be further explored in the next chapter.

There is, however, a separate, related issue that merits investigation. It is well known that considerable work has gone into establishing the adequacy of the language of set theory for the purpose of *describing* the objects of all branches of contemporary mathematics. Set-theoretic models of axiomatic theories—for example, the theory of rings—can be described directly *within* the language of set theory. That is, one can translate the axioms for a ring into the language of set theory, then construct a formula in the language of set theory that says "x is a ring", or "z is a (ring) homomorphism between the rings x and y". For all that has been said about the expressive resources of the language of category theory thus far, one might worry whether the language of category theory has this ability, or whether the language of category theory must necessarily reside "one level up" from languages like that of set theory. Must one first *define* the objects of mathematics (and their associated morphisms) in the language of set theory, and then "ascend" to *describe* them as objects (and arrows) in the language of category theory? Note that this concern is distinct from the problem of the "home address"; the concern here is not that the objects must be *shown to exist* in some other theory, but that they must first be *described* in some other theory. While the observations contained in Section 2.5 might be taken to show that sort of translation to be not entirely without benefit, such a "set theory first" approach will be seen to be unnecessary.[29]

A proponent of the AFCS program may, of course, observe that the language of set theory is intended to describe *sets*[30], and this restriction runs counter to the tenets of AFCS program: one may hold the the rigid mo-

[29] Should further conditions need to be imposed for the purposes of defining the objects of interest (existence of products, function spaces, etc.), they can be added directly within the language of category theory, as will come to light in the discussion of the elementary theory of the category of sets.

[30] Here the less common set theories which admit *urelements* are not being considered, but one might observe that, while such elements may be present in the theory, there is little sense to the claim that a set theory is equipped to *describe* such elements.

tions of a tetrahedron form a group, but *rigid motions* are not sets (although the group of rigid motions may be expected to be isomorphic to some set-theoretic group). The particular nature of the models of a given axiomatic system is deliberately unspecified *within* mathematics, a feature that may help to explain the applicability of mathematics in novel contexts such as may be required by a newly developed physical theory.

However, concerns about the expressive resources of the language of category theory as compared to the language of set theory can be addressed directly, and here a particularly strong result obtains. In [49] Lawvere introduces his *elementary theory of the category of sets* (henceforth ETCS). The language of ETCS can be taken to be the language $\mathcal{L}$ described earlier, and eight axioms are added to the axioms that determine a category. It can then be shown that the theory ETCS is inter-interpretable with the theory of BZC, Zermelo set theory with the axiom of choice and the axiom of (bounded) separation.[31] Further, McLarty shows in [62] that an axiom scheme of replacement can be added to yield a theory ETCS+R, which he shows to be inter-interpretable with full ZFC. One who wants to pursue set-theoretic reconstructions of mathematical objects thus finds that the language of category theory has much to offer, but it offers a "function-based" set theory as opposed to the "membership-based" theory like that of ZFC. The technical details will not be pursued here, but the results of the Structural Properties Theorem carry over to ETCS+R; as McLarty notes

> The theory ETCS is structural in the sense that each ETCS set provably has all the same properties as any set isomorphic to it. An ETCS formula can only specify a set up to isomorphism. [62, p. 48]

Some delicacy is required when assessing the inter-interpretation results. One concern is whether the interpretations involved are *homophonic*: are the

[31] See [62] for details.

sets and functions of one theory interpreted as sets and functions of the other? Interpreting ETCS+R in ZFC proves relatively straightforward, with the objects of ETCS+R interpreted as sets in ZFC, and the arrows interpreted as (set) functions. However, the interpretation of ZFC in ETCS+R proves more challenging. The homophonic interpretation is partial: many formulae which involve membership in an essential way are not directly interpretable in ETCS+R. As McLarty explains

> For example there is no homophonic interpretation of the ZF empty-set axiom: There is a set $\varnothing$ such that no set A has $A \in \varnothing$. This relies directly on membership of sets. But consider this ZF theorem: There is a set $\varnothing$ such that for every set A there is exactly one function $\varnothing \to A$. This is homophonically interpretable, and is also a theorem of ETCS. Indeed all isomorphism-invariant theorems of ZF have homophonic interpretation. [62, p. 46]

There is also a *total* interpretation that is not homophonic, which involves the notion of a set's *membership tree*. A set is represented as the bottom node of a tree diagram, with its members corresponding to the base of each branch directly above that bottom node node, and similarly for those members.[32] The details of this interpretation will not be explored here, but the reader is again referred to [62] for additional sources.

Of particular importance for the purpose of this chapter is McLarty's remark that, on the homophonic interpretation of ZF in ETCS, all *isomorphism-invariant* theorems of ZF have homophonic interpretation in ETCS. What of the theorems of ZF that are not isomorphism invariant? On the face of it such theorems would seem to be of little interest to the structuralist—structural properties *are* isomorphism invariant—and this remark highlights an important distinction between the language of set theory and the language of category theory that warrants investigation.

[32] "Moving down" one level in the tree corresponds to "adding brackets", collecting the elements directly above that node, with each branch ending at the top with $\varnothing$.

2.5 Structural Properties in a Set-Theoretic Setting

Despite the similarity between the category-theoretic ("function-based") and set-theoretic ("membership-based") theories of collections discussed above, for the purpose of the structuralist these theories are crucially different. Consider now the language of ZF, which consists of variables of only one type, two primitive binary relation symbols $\in$ and $=$, and no constant symbols. How are the structural properties of its objects to be characterized? As structural properties were viewed as those which did not depend on particular objects (and so in this context, those which do not depend on particular *sets*), one might expect that again those predicates with no parameters and no constant terms would be those which correspond to *structural* properties. This view of structural properties was seen to be motivated by the two structuralist perspectives (See Section 2.1.1). Unfortunately, a problem with this approach emerges when one considers the *other* characterization of structural properties: those which are common to all isomorphic objects (where the objects in question are sets in the present case). On the category-theoretic approach these views were shown to coincide. Unfortunately, these two conceptions of *structural properties* do not coincide in the language of set theory.

Given that set functions can be described directly in the language of set theory[33], one may consider a simple case, where two sets are taken to be isomorphic exactly when they stand in bijective correspondence. Already in this case we see the two conceptions of *structural property* come apart. Consider (using the usual abbreviations) the predicate

$$\Phi(x) =_{df} \forall y(y \in x \rightarrow \exists! z(z \in y)),$$

which corresponds to the property of "having only singleton members", and

[33] Where an n-ary function is a set of ordered $(n+1)$-tuples satisfying the standard uniqueness condition.

take the sets $A = \{1\}$ and $B = \{2\}$.[34] Note that Φ involves no constant terms and no parameters, so exemplifies the first conception of a structural property. Both A and B are themselves singletons, and so isomorphic in the simple sense of being in bijective correspondence. However, the single element of B has two members, and so $\Phi(B)$ fails while $\Phi(A)$ holds; 2 is not a singleton, and so Φ is not preserved under isomorphism.

This example illustrates a difficulty inherent to the set-theoretic framework that is similar to the problem of identifying *the* natural numbers that Benacerraf describes in [10]. In the case considered above, the language of set theory allows us to distinguish—using (predicates corresponding to) properties that do not involve particular elements—two isomorphic sets, but this distinction is irrelevant to their role[35] as singletons. *Qua* singletons, set A serves us just as well as set B, but the language of set theory is, in a sense, too fine grained: it allows for the formulation of predicates that are *structural* in the sense of not depending upon particular objects (and so not involving constants or parameters), but which are *not* structural in the sense of being common to all isomorphic sets. Further, this result can be taken to show that the characterization of *structural properties* as those which do not involve particular objects is inappropriate with respect to the language of set theory—the properties identified by that criterion are *not* structural.

The situation is no better when we take into account more complicated structures along with their associated isomorphisms, as in the case of the ω-sequences that feature in Benacerraf's discussion [10]. In that case, the von Neumann and Zermelo finite ordinals are equivalent *as ω-sequences*, but differ with respect to their set-theoretic properties. Benacerraf's complaint is that, while the Zermelo and the von Neumann finite ordinals both stand as candidates for the title *the* natural numbers, "*the accounts differ at places where there is no connection between features of the accounts and our uses*

[34] Here taking the von Neumann definitions: $1 =_{df} \{\varnothing\}$ and $2 =_{df} \{\varnothing, \{\varnothing\}\}$.

[35] The *role* of these sets can be taken to be either identified implicitly by the isomorphism type, or subject to some prior specification which then determines the isomorphism type.

of the words in question" [10, p. 62]. The formula Φ constructed in the example above serves to illustrate one of the inconsequential differences between ω-sequences that can be expressed in the language of set theory. One aspect of this difficulty is due to the language of set theory itself, as the language *permits the expression of non-structural properties*. In contrast, the category theoretic framework can be augmented by axioms (all expressed in the language $\mathcal{L}$) that allow for the definition of a *natural number object*, the category-theoretic analogue of an ω-sequence. In fact, given one such natural number object, there are provably infinitely many such objects.[36] The category theoretic account of an isomorphism remains as before: two objects A and B are isomorphic provided there exists an iso $f : A \xrightarrow{\sim} B$. Again here, it is provable that (continuing to restrict attention to predicates that do not involve constants or parameters) "All natural number objects are indiscernible in this theory. They provably have all the same properties" [61, p. 494]. Thus, taking the first account of structural property, distinct natural number objects share all structural properties when expressed in the language of category theory. Further, framed in the language of category theory, the first and second accounts of *structural property* coincide; those properties without constants or parameters are common to all isomorphic objects.[37]

To summarize, structural properties on one view may be taken to correspond to those predicates in which particular objects do not feature (so no constants, no parameters), or they can be taken to be those which are common to all isomorphic objects. These two views coincide in the language

[36] One proof of this result proceeds by taking different successor relations defined on the same object, see [61, p. 493].

[37] McLarty considers the more complicated case where systems of objects and one or more arrow are taken into consideration, and so the *structural properties* are not represented by predicates with a single free variable, but also include variables for any number of distinguished arrows as well. In the case of natural number objects, two distinguished arrows are admitted, one corresponding to the successor operation and one to the selection of a base-element. In [61] McLarty establishes a more complicated Structural Properties Theorem involving these natural number objects.

of category theory: all predicates in which there are no constants and no parameters correspond to structural properties—they are common to all isomorphic objects. In the language of set theory, these two views come apart; predicates not involving constants or parameters *do not* all correspond to structural properties. In the language of set theory, we have recourse only to the account that identifies *structural properties* as those properties common to all isomorphic structures. But which are those?

In the language of category theory, there is a *syntactic* criterion to which we can appeal in identifying (predicates corresponding to) structural properties. In the language of set theory, a predicate's avoidance of constant terms and parameters is, as we have seen, not sufficient to guarantee its preservation under isomorphism. The structuralist who appeals to the language of category theory finds that—rather than having to rely on the language of set theory in order to describe the mathematically relevant features of the objects under consideration—it is the language of category theory which serves to separate the wheat from the chaff, yielding only isomorphism-invariant properties. And it is exactly the isomorphism-invariant properties that are of interest to the structuralist.

It is worth remarking here that the problem facing the structuralist who adopts a set-theoretic framework is *not* addressed merely by producing a criterion according to which properties of the language can be identified as being preserved under isomorphism. That structuralist faces a further question: why use a language which so readily *allows for the formulation of* (predicates corresponding to) *properties not preserved under isomorphism*? If such properties are of no interest to the structuralist, why should they be admitted at all into the framework of a structuralist program?

2.6 Alternative Structuralist Programs

Given the suitability of the language of category theory to the structuralist view, it is perhaps surprising to note that some key proponents of structuralism have not adopted, and, in some cases, have actively resisted the use of the language of category theory in developing their programs. In this final section of the chapter, we will consider the features of two current structuralist programs, and explore their alternatives to structuralist programs framed in the language of category theory.

Before considering alternative structuralist programs it will be useful to observe that the programs do not use the term "structure" univocally. One may follow Shapiro and use "structure" as a sortal concept, where a structure is taken to be a sort of object, or one may adopt the view that an object *has* or *exhibits* a particular structure, as when one speaks of the finite von Neumann ordinals exhibiting the structure of an ω-sequence. Denoting these two options *structure*$_1$ and *structure*$_2$, Benacerraf notes that

> ...the Empire State Building, although a paradigmatic concrete object, is an imposing structure$_1$, as is the union of the R_α for all $\alpha < \aleph_\omega$, as is $\mathbb{R}$, the structure$_1$ of the real numbers between 0 and 1; whereas it is also true that the structure$_2$ of the Empire State Building has never been repeated in any other building... [12, p. 184]

On the AFCS program, "structure" is used in the sense of structure$_2$, and proponents of the AFCS program satisfy Benacerraf's description of those who

> ...represent mathematical theories as being about structures$_2$— the *structural* or *relational* features that systems of "objects" might exhibit, without any special concern about whether there are or could be any systems of objects that indeed exhibit them... [12, p. 185]

Thus "structure" will typically be used here in the sense of structure_2. However, both Hellman and Shapiro use the term "structure" in the sense of structure_1[38]; context should suffice to disambiguate. Despite this difference in usage, all parties endorse the slogan that "mathematics is the science of structure", although the terminological differences point to their differing motivations and goals in the development of these alternative structuralist programs.

2.6.1 Shapiro's *Ante Rem* Structuralism

Shapiro's structuralist program is a *foundational* program according to the criteria presented in Section 1.3.2. As Shapiro notes, "My structuralist program is a realism in ontology and a realism in truth-value…" [76, p. 72]. The "realism in ontology" points to Shapiro's aiming to satisfy the third foundational criterion, as

> …the *ante rem* structuralist interprets statements of arithmetic, analysis, set theory, and the like, at face value. What appear to be singular terms are in fact singular terms that denote bona fide objects. [76, p. 11]

Given the view that mathematical theorems are *true*, Shapiro endeavours to describe the manner by which we come to *know* that the theorems of mathematics are true (Chapter 4 of [76]), thereby satisfying the second foundational criterion. Finally, his structuralist perspective, and particularly the *structure theory* he presents (Chapter 3 of [76]), provides a unified view of mathematics, and thereby aims to satisfy the first foundational criterion. In short,

[38] Although their usages do not entirely coincide either, with Hellman's *structure* corresponding roughly to Shapiro's *system*. Hellman, for example, speaks of distinct structures potentially being "pairwise isomorphic" [35, p. 19], while on Shapiro's view it is distinct *systems* that may be pairwise isomorphic. On Shapiro's view, provided all systems of a given sort are pairwise isomorphic (i.e., the axioms defining the system type are *categorical*) they determine a *structure*, and any isomorphic structures are identical [76, p. 93].

> I [Shapiro] try to say what mathematics is about, how we come to know mathematical statements, and how we come to know about mathematical objects. [76, p. 8]

One of the key features of Shapiro's structuralist program is the distinction between "places-are-offices" and "places-are-objects" perspectives. Consider, for example, the white queen's bishop in the game of chess. Different physical objects have played the *role* of the white queen's bishop, so the usage of the term "white queen's bishop" can be associated with the *role* of that chess piece. Taking terms of this sort to indicate a role is to adopt the places-are-offices perspective. Alternatively,

> When we say that the Speaker presides over the House and that a bishop moves on a diagonal, the terms "Speaker" and "bishop" are singular terms, at least grammatically. Prima facie, they denote the offices themselves, independent of any objects or people that may occupy the offices. This is the *places-are-objects* perspective. [76, p. 10]

Considering the natural numbers, number terms may designate places-as-offices: both the Zermelo $\{\{\varnothing\}\}$ and the von Neumann $\{\varnothing, \{\varnothing\}\}$ occupy the "office" of the number 2, insofar as they occupy the corresponding position in their respective ω-sequences.

One central feature of Shapiro's program is that he holds that there is a sort of canonical occupant of some roles; in the case of the natural numbers, for example, there is an object uniquely suited to the title *the* number 2. One way to articulate the view is to consider how Shapiro characterizes the *structure* vs. *system* distinction. Both the Zermelo finite ordinals and the von Neumann are *systems*, where a *system* is taken to be a collection of objects and relations on those objects [76, p. 73]. A system can be thought of as a model of some set of axioms, and—provided the set of axioms is *coherent*—to each isomorphism class of models of a particular collection of axioms there corresponds a *structure*. Structures, however, are *not* to be viewed as equivalence classes of systems, instead they are best viewed as involving a shift from the places-are-offices perspective to the places-are-objects perspective, a shift that is considered legitimate when the axioms are both *categorical* and *coherent*.

Coherence is, by Shapiro's admission, a difficult notion to capture, and cannot be identified with deductive consistency, as there are consistent second-order theories that are not satisfiable.[39] Instead, coherence is taken to be "something more like satisfiability" [76, p. 95]. To complete the picture, Shapiro's adopts a stipulation concerning when structures are to be identified: "we stipulate that two structures are identical if they are isomorphic" [76, p. 93]. Thus, Shapiro's view is that

> A purported implicit definition characterizes *at most* one structure if it is *categorical*—if any two models of it are isomorphic to each other. A purported implicit definition characterizes *at least* one structure if it is *coherent* [76, p. 73]

As the axioms of (second-order) Peano arithmetic are taken to be both *categorical* and *coherent*, they serve to characterize a unique structure answering to the title *the* natural numbers, and at the appropriate position in this structure we find *the* number 2, an *office* as far as other systems are concerned, but an *object* in the *structure* of natural numbers. Shapiro dubs his view *ante rem* structuralism after the ancient view of universals which holds that universals exist independently of any particular instantiation, just as struc-

[39] Shapiro offers the following example

> Let P be the conjunction of the second-order axioms for Peano arithmetic and let G be a standard Gödel sentence that states the consistency of P. By the incompleteness theorem, $P \;\&\; \neg G$ is consistent, but it has no models. Indeed, because every model of P is isomorphic to the natural numbers, G is true in all models of P. Clearly, $P \;\&\; \neg G$ is not a coherent implicit definition of a structure, despite its deductive consistency. [76, p. 135]

tures are taken to exist independently of whether there are any systems that exemplify them (see [76, p. 9]).

Concerns raised about certain features of Shapiro's program will for the most part be ignored here, as our focus here will be solely on those aspects of Shapiro's program that involve explicit rejection of some of the central tenets of the AFCS program. A first such concern that needs to be addressed involves Shapiro claims concerning the view he calls *eliminative* structuralism: a structuralist view which acknowledges only the legitimacy of talk of *systems* of some sort, avoiding the reification of structures and avoiding the "places-are-objects" perspective. The AFCS program is of this sort. Furthermore, Shapiro argues that eliminative structuralists are pushed to adopt ontological assumptions in order to "make sense of a substantial part of mathematics" [76, p. 86]. As the proponent of the AFCS view considers such ontological assumptions to be misplaced if taken to be part of *mathematics*, it will be necessary to reply to Shapiro's arguments in support of this view.

Shapiro's reason for concern about the ontological commitment required by the eliminative structuralist involves the proposed interpretation of mathematical theorems as involving (implicit) quantification over *systems*. If Φ is a sentence in the language of arithmetic, for example, "$2 + 3 = 5$", then

> According to eliminative structuralism, Φ amounts to something in the form:
>
> (Φ') for any system S, if S exemplifies the natural-number structure, then $\Phi[S]$,
>
> where $\Phi[S]$ is obtained from Φ by interpreting the nonlogical terminology and restricting the variables to the objects in S. If the background ontology is finite, then there are no systems that exemplify the natural-number structure, and so Φ' and $(\neg\Phi)'$ are both true. Because mathematics is not vacuous, this is unacceptable. [76, p. 86]

This concern is also presented earlier in [76], where Shapiro claims that

> ...eliminative structuralism requires a background ontology to fill the places of the various structures. Suppose, for example, that there are only finitely many objects in the universe. Then there are no natural-number systems, and every sentence in the language of arithmetic turns out to be true. For example, the above rendering of "$2 + 3 = 5$" [as a conditional in the form of Φ'] is true because there are no natural-number systems, but the renderings of "$2 + 3 = 0$" and "$2 + 3 \neq 5$" are also true. If the background ontology is not big enough, then mathematical theories will collapse into vacuity. [76, p. 9]

This concern is echoed in Parsons [66], who also uses this line of argument to illustrate a seemingly fatal problem for eliminative structuralism. Taking a formula like Φ' to give the "canonical" representation of a mathematical theorem of arithmetic,

> ...on the eliminative reading, if there are no simply infinite systems, then for any [natural number system] $N, 0, S$ the statement...giving the 'canonical form' of an arithmetic statement A is vacuously true. But then both A and $\neg A$ have true canonical forms, which amounts to the inconsistency of arithmetic. [66, p. 310]

The proponent of the AFCS program does not take the program to be in any way dependent on the number of objects in the universe, much less a program that leads to an inconsistent arithmetic!

The obvious response to these concerns involves a discussion of the notion of a *system* or *model* of a collection of axioms, such as the Peano axioms (in either their first– or second-order formulation). Recall that the proponent of the AFCS version aims to remain neutral on questions concerning models. Are they sets? Are there many of them? Do any of them "contain" infinitely many elements? All of these questions are treated on the AFCS program as questions concerning not mathematics proper, but the *application* of mathematics. In keeping with this view, the proponent of the AFCS program treats such claims about *models* of the Peano axioms as not involving *quantification over* models, but instead treats talk of models as *schematic*. This is related to the failure of mathematical theorems, read as conditional and schematic, to be *true*. These key aspects of the AFCS program will be explored in more detail in Chapter 4.

For present purposes, it suffices to note that the proponent of the AFCS program speaks of *systems* or *models* as Benacerraf suggests: "without any special concern about whether there are or could be any" such entities [12, p. 185]. If indeed there were no models of the axiomatic systems that characterize the subject matter of the various branches of mathematics—if there are no groups, no ω-sequences, no topological spaces, it would indeed be remarkable, but that would be a concern for the physicist and the applied mathematician, not for the pure mathematician, nor for the proponent of the AFCS program. The proponent of the AFCS program denies that mathematical theorems typically involve vacuous antecedents, while refraining from affirming that there are models that satisfy the antecedent specification. There is simply no commitment to any ontological claims about models.

Shapiro, however, does take it to be necessary for his (foundational) program to account for an ontology of *structures*, and that is the role of his *structure theory*, which stipulates the existence of certain key structures (via an axiom of Infinity) and stipulates principles that allow for new structures to be built up from others (via axioms like Replacement and "Powerstructure") (see Chapter 3 of [76]). "In effect, structure theory is a reworking of second-order Zermelo-Fraenkel set theory" [76, p. 95]. It is curious that Shapiro's concern about the existence of mathematical systems can be addressed within his own program by laying down an axiom that simply asserts the existence of the structures (each of which also counts as a system) required! However, Shapiro's metaphysical view (and other metaphysical views) can be accommodated on the AFCS program. Shapiro has no concerns about vacuity because his structure theory asserts the existence of the structures he requires, a set theorist may happily assert that both the von Neumann finite ordinals and the Zermelo finite ordinals establish that the Peano axioms have a model, and Dedekind may be content to assert his thoughts suffice to show the existence of ω-sequences. The nominalist may hesitate to commit to an infinite totality, and some branches of mathematics may have no applications on the nominalistic conception. This is not to say that those branches are any less "mathematical" than those accepted by the nominalist. A child may refuse to eat vegetables, but that child cannot legitimately claim that vegetables are not food, or that recipes for ratatouille have no place in a cookbook.

There is, however, another response to the problem identified by Shapiro, and that is to note that the canonical representation Φ' does not quite capture all aspects of the AFCS program (or, presumably, other eliminative structuralist programs). The relationship between the antecedent and the consequent in a conditional taken to correspond to a mathematical theorem is not captured solely by the truth-functional behaviour of the material conditional—the antecedent is meant to be related to the consequent by the availability of a *proof*. The difference is rather like that of the following two sentences of first-order logic

1. $\forall x((Fx \land Gx) \to Fx)$, and
2. $\forall x((Fx \land Gx) \to \neg Fx)$.

An interpretation $\mathcal{I}$ of the first-order language may be such that no elements x in the universe of the interpretation satisfy Gx, and so both statements are "true-in-$\mathcal{I}$". However, formula 1 will easily be recognized as having a feature that distinguishes it from formula 2: formula 1 is a first-order theorem,[40]

[40] Here the logical framework has been left unspecified, but this particular theorem holds in, for example, classical and intuitionistic logic. The distinctions between systems will be

while formula 2 is not. Thus, considering, for example, a system of first-order classical logic, we have

$$\vdash \forall x((Fx \land Gx) \to Fx), \text{ but}$$
$$\nvdash \forall x((Fx \land Gx) \to \neg Fx).$$

A (semantic) proof of $\forall x((Fx \land Gx) \to \neg Fx)$ would require that, for any interpretation and any a in the associated universe of discourse, if $Fa \land Ga$ holds, then so does $\neg Fa$. All—except possibly the dialethic logician—would hold such a circumstance to be impossible. Formula 1 can be asserted (perhaps as a schematic conditional, "For any properties F and G. . . ") not solely because of its truth-conditional behaviour, but because of the availability of a proof. Formula 2 does not share this feature, and its assertibility depends on the particular properties taken to be represented by F and G, along with the features of the objects in the universe of discourse.

While the proponent of the AFCS program aims not to identify a single, privileged framework in which to prove theorems, we may nevertheless abuse the turnstile notation[41] to observe that, again taking Φ to stand for the sentence "$2 + 3 = 5$", the theorem is characterized on the AFCS view as

$$\vdash (\text{if } \mathcal{N} \vDash \text{PA then } \mathcal{N} \vDash \Phi), \tag{2.7}$$

and a rather unremarkable proof suffices to establish this result. However, we *do not* have

$$\vdash (\text{if } \mathcal{N} \vDash \text{PA then } \mathcal{N} \vDash \neg\Phi).$$

In order to obtain *this* result, one would need either a proof that models of the Peano axioms are impossible (for example, if the Peano axioms were inconsistent), or a proof that Φ fails in all models of the Peano axioms. As neither Shapiro nor Parsons provide such a result, even the eliminative structuralist

of no consequence here.

[41] Here the turnstile is used to indicate provability *simpliciter*, without the specification of a proof system. Provability will be discussed in more detail in Chapter 4.

seems to have available a consistent arithmetic. Note that using the standard model-theoretic notation (though continuing to abuse the turnstile), we can express (2.7) as

$$\vdash (\text{PA} \vDash \Phi), \tag{2.8}$$

and so we might more succinctly express our reply as noting that the AFCS program takes the conditional expressions corresponding to mathematical theorems as meta-theoretic, involving the notion of *entailment* rather than the notion of *implication*.

A further aspect of Shapiro's system that bears on the AFCS program concerns his treatment of the structuralist perspectives identified in Section 2.1.1. Shapiro makes a number of remarks that clearly indicate his endorsement of these perspectives. The "particular elements of a system don't matter" perspective is reflected in his remark that ". . . anything at all can "be" 2—anything can occupy that place in a system exemplifying the natural-number structure. The Zermelo 2 ($\{\{\varnothing\}\}$), the von Neumann 2 ($\{\varnothing, \{\varnothing\}\}$), and even Julius Caesar can each play that role" [76, p. 80]. That "the particular system doesn't matter" perspective (and his recognition of the importance of the notion of *isomorphism*) is reflected in his remark that

> No matter how it is to be articulated, structuralism depends on a notion of two systems that exemplify the "same" structure. That is its point. Even if one eschews structures [treated as objects] themselves, we still need to articulate a relation among systems that amounts to "have the same structure." [76, p. 90]

As remarked above, in Shapiro's structure theory he takes isomorphic structures (structures$_1$) to be identical (although *systems* that exhibit that same structure$_2$ are not identified). Do structures (in the sense of structures$_1$ here) have all the same *structural properties*? Certainly they do given Shapiro's identification of isomorphic structures—isomorphic *structures* share the same properties because they are, in virtue of the isomorphism, identical. The same is not to be said of isomorphic *systems*, though, and Shapiro's structure theory treats both *systems* and *structures*. Consider the "Subclass" axiom, for example, which states that

> If S is a structure and c is a subclass of the places of S, then there is a structure isomorphic to the system that consists of c but with no relations and functions. [76, p. 94]

While Shapiro does not give a completely formalized presentation of the axioms of his structure theory, axioms such as "Powerstructure" involve set-theoretic notions. The "Powerstructure" axiom asserts

> Let S be a structure and s its collection of places. Then there is a structure T and a binary relation R such that for each subset $s' \subseteq s$ there is a place x of T such that $\forall z(z \in s' \equiv Rxz)$. [76, p. 94]

Thus, structure theory treats both structures and systems, where a system may be isomorphic to, but distinct from, the structure it exemplifies. Of note, within structure theory the systems are "constructed" out of the places of structures. As Shapiro explains,

> Because structures, places, relations, and functions are the only items in the ontology [of structure theory], everything else must be constructed from those items. Thus, a *system* is defined to be a collection of places from one or more structures, together with some relations and functions on those places. For example, the even-number places of the natural-number structure constitute a system, and on this system, a "successor" function could be defined that would make the system exemplify the natural-number structure. The "successor" of n would be $n + 2$. Similarly, the finite von Neumann ordinals are a system that consists of places in the set-theoretic hierarchy structure, and this system also exemplifies the natural-number structure [76, p. 93–94]

Now we find exactly the ingredients of the problem faced in developing versions of the structuralist program in membership-based set theory—there is a difficulty in articulating the notion of a *structural property*. Structure theory permits structures (structures$_1$) and *distinct isomorphic systems*. Their being isomorphic is important because the isomorphism ensures that they have all the same properties of interest—the *structural* (in the sense of structure$_2$) properties. But which are those?

Given the finite von Neumann ordinals, the Subclass axiom suggests that there are systems $A = \{1\}$ and $B = \{2\}$ (and, as a consequence of that axiom, a single *structure* isomorphic to both). The language of set theory (in particular, the membership relation appearing in the "Powerstructure" axiom) permits the construction of the formula

$$\Phi(x) =_{df} \forall y(y \in x \to \exists! z(z \in y)),$$

shown in Section 2.5, which does not involve constants or parameters but which serves to distinguish isomorphic systems A and B, and which therefore counts as non-structural. The problems of the set-theoretic approach, then, seem to have been inherited by the structure theory proposed by Shapiro. If isomorphic systems (structures$_1$ among them) are interchangeable in virtue of their exhibiting the same properties of interest, why use a language that serves to carve out their irrelevant properties?

2.6.2 Hellman's Modal Structuralism

In [76] Shapiro identifies a third alternative to the eliminative and *ante rem* varieties of structuralism: modal structuralism. Like eliminative structuralism, modal structuralism avoids commitment to a realm of structures (structures$_1$) or, indeed, to any special class of objects particular to mathematics. However, unlike the eliminative programs, the modal program is committed to asserting the *possibility* of there being systems that exemplify certain of the key mathematical structures. Shapiro holds that the modal structural option fares better than eliminative programs like AFCS, in at least the respect that "there is an attenuated threat of vacuity" [76, p. 10], given that the modal translations of sentences of, for example, arithmetic come out to something of the form "In every *possible* natural-number system...", and so are vacuous not merely in the case that natural-number systems *fail to exist*, but are vacuous only in the case that such systems are *impossible*.

The most detailed account of modal structuralism is given in Hellman's *Mathematics Without Numbers* [35]. Interestingly, Hellman's program is not obviously a *foundational* program according to the criteria of Section 1.3.2. On the "realist" view that Hellman proposes, mathematical theorems are taken to be true, and objectively so: "mathematical discourse is understood as consisting of statements or propositions that have determinate truth value, independent of our minds" [35, p. 2]. In accordance with this view, Hellman does explicitly aim to satisfy the second criterion for a foundational program, in that "a philosophical interpretation of mathematics ought to admit of an extension that reasonably accounts for how we come to know or justify that mathematics which we can reasonably be claimed to know or be capable of knowing" [35, p. 3].

With respect to the first criterion, Hellman aims to tackle a project whose origins he attribute to Putnam[42], which Hellman describes as aiming

> ...to develop explicit translation patterns of mathematical theories into suitable modal theories—capable of standing independently of set theory—and then to justify these as "equivalent for mathematical purposes." Like structuralism, the idea of "mathematics as modal logic" has remained at the level of some seemingly promising suggestions, but it has not been developed even to the point at which a serious philosophical assessment would

[42]Hellman cites Putnam's view as presented in [67].

> become possible. One of our goals in what follows has been to remedy this situation. [35, p. 8]

Thus Hellman's program is, in some respects, like that proposed in Russell's *The Principles of Mathematics* ([73], see Section 1.4.2) insofar as it locates mathematics within a type of logic: modal logic in Hellman's program, the theory of classes in Russell's. Hellman may thus be considered to aim at satisfying the first criterion for a foundational program: mathematics is identified, identified as a part of a system of (second-order) *modal logic*. Of course, some axioms adopted in Hellman's framework explicitly assert principles (for example, one asserts the "logico-mathematical" possibility of an ω-sequence), that are not offered as *purely* logical truths. Like Russell, Hellman does not aim at a clear separation between mathematics and logic. As Hellman clarifies,

> ...in employing the phrase "second-order logic", we are referring to a well-known notation and its metatheory; we are not committed to the view that it is "genuine logic". Nor are we committed to any particular way of drawing a line between logic and mathematics. As we see it, structuralism does not need to draw such a line. [35, p. 21]

Thus, while reluctant to draw a line between mathematics and logic, *unlike* Russell, Hellman does not explicitly adopt the view that mathematical theorems can be characterized as *logical truths*. Indeed, it is the modal notion that features in Hellman's program that most clearly prevents such an identification. In the presentation of [35] the (primitive) modal notion is itself a partly *mathematical* notion—a "logico-mathematical modality" [35, p. 15]—and so any specification of the modal framework (prior to the introduction of axioms explicitly postulating the possibility of ω-sequences, complete ordered fields, etc.) may be thought to already embed mathematical content. Mathematics, then, is not *reduced* to modal logic; rather, mathematics is already a part of the modal framework employed. Interestingly, in later writings Hellman drops the "mathematico-" qualification, describing the strategy as involving "taking a logical modality as primitive" [36, p. 103], and similarly, involving the "use of a primitive modal operator, for (second-order) logical possibility" [37, p. 198]. Thus, Hellman's more recent approach leaves open the possibility of defending his program as a sort of contemporary "modal logicism", although Hellman does note that in some cases—like that of modal set theory—"the assumption of logical possibility is a reasonable working hypothesis" [37, p. 204], and so not likely to be defended as a logical truth.

More difficult to assess, though, is whether Hellman's program accounts for the truth of a mathematical theorem via the mechanisms of "naming, predication, satisfaction, and quantification". Hellman aims at developing a theory that is nominalistically acceptable, and the second-order quantifiers and the modal language he requires both need to be given nominalistically acceptable readings. Hellman notes that he wants

> ... to avoid literal quantification over abstract structures, possible worlds, or intensions, in order to provide a genuine alternative to objects-platonism [which involves commitment to abstract (mathematical) objects], in which literal reference to such objects is eliminated [35, p. 16]

It is useful to outline the key methods of Hellman's program in order to identify the features he hopes will avoid "literal reference" to abstract objects.[43] As Hellman explains (here discussing the special case of natural number arithmetic),

> Beginning with the standard Peano-Dedekind axioms for the natural numbers, PA^2, involving just successor, $'$, and the second-order statement of mathematical induction, we treat an arbitrary

[43] Abstract objects of the sort that feature so prominently in Shapiro's program.

> sentence S of first– or second-order arithmetic (in which any function constants have been eliminated by means of definitions in terms of $'$) as elliptical for the modal conditional
>
> $$\Box\forall X\forall f[\wedge PA^2 \rightarrow S]^X('/f),$$
>
> in which a unary function variable f replaces $'$ throughout and the superscript X indicates relativization of all quantifiers to the domain X. This is a direct, modal, second-order statement to the effect that 'S holds in any model of PA^2 there might be'. [36, p. 105]

To this translation scheme for sentences of natural number arithmetic, Hellman adds a modal existence postulate,[44]

$$\Diamond\exists X\exists f[PA^2]^X('/f),$$

asserting the possibility of an ω-sequence. Real analysis and Zermelo-Fraenkel set theory are given an analogous treatment in [35]: a translation scheme is provided, along with a modal existence postulate of the relevant sort.

The key, then, to Hellman's avoiding commitment to abstract objects it taken to lie in the modal element of his program. When Hellman provides the translation of the modal conditional for a mathematical theorem, say S of natural number arithmetic, the necessity operator distinguishes between the intended 'S holds in any model of PA^2 there *might be*' as distinct from the (non-modal) 'S holds in any model of PA^2 there (actually) *is*'. While this manoeuvre may avoid explicit commitment to ω-sequences and other mathematical entities, there is the additional concern about the second-order machinery: does second-order logic commit one to an ontology of sets, or set-like entities over which the second-order quantifiers range? Here Hellman

[44] Exponentiation with respect to X and the substitution of f for $'$ are as in the translation scheme.

goes on to develop a sophisticated program that involves an appeal to the notions of *plural quantification*[45] and mereology, the details of which will not concern us here.[46]

Granting Hellman's claim to have provided a nominalistically acceptable way of reading the second-order quantifiers, what of the modal notion employed? As Burgess and Rosen observe, a nominalist who appeals to modalities as a way of avoiding reference to abstract objects must defend "primitivism", the "acceptance of modal logical distinctions [involving *possibility* and *necessity*] as undefined" [16, p. 124]. The modal structuralist who is also a nominalist cannot give the necessity and possibility operators an interpretation involving non-actual *possible worlds*, and the *possibilia* (possible, but non-actual objects) that inhabit such non-actual worlds. Causally isolated from us and *abstract* by any account, one can hardly think of a better example of exactly the sort of entity to which the nominalist wishes to avoid commitment! Similarly, Hellman wishes to avoid any dependence on a prior theory of sets, given that the modal machinery he develops is going to be used to show, among other things, how a nominalistically acceptable set theory can be developed. Hellman, aware of these concerns, notes in his discussion of modal set theory that

> ...possibilia are not recognized as objects...we do not quantify over possible worlds or intensions; we simply use modal operators...
>
> We are accustomed to giving set-theoretical semantics for modalities, and for a variety of logical purposes this is perfectly in order. But the msi [modal structural interpretation] of set theory, while aiming to respect such semantics as part of set theory, nevertheless, requires that its notion of logical possibility stand on its own. It functions as a primitive notion, and must not be thought

[45] The notion of plural quantification is developed in Boolos [13].
[46] See, e.g., [35, 36].

> of as requiring a set-theoretical semantics in order for it to be intelligible. Instead, of course, we may give modal axioms. [35, p. 59–60]

Despite the prospects of having outlined a nominalistically acceptable modality for his purposes, in so doing Hellman has blocked himself off from an account of the truth of mathematical theorems using the "referential" semantic apparatus called for in the third foundational criterion identified in Section 1.3.2. Hellman's nominististically-motivated need to treat the modal notion as primitive requires, for example, that the modal existence postulate

$$\Diamond \exists X \exists f [PA^2]^X ('/f)$$

is *not* simply understood as a claim about objects in an accessible possible world, true because entities among the possibilia satisfy the matrix of existentially quantified expression. The possible world account of the modal terms is unavailable to the nominalist, and so the reading of the modal existence postulate as true in virtue of such (possibly non-actual) entities is unavailable. Given its set-theoretic characterization, a Kripke-style semantics is also unavailable. In virtue of the necessarily primitive modality, Hellman's modal structuralist program cannot account for the truth of mathematical theorems appealing only to the resources Benacerraf identifies in [11].

Here the proponent of the AFCS program can simply agree with Hellman's remark that "modal primitives for mathematics are problematic" [37, p. 205]. Of course, the AFCS program itself fails to count as a foundational program according to the criteria of Section 1.3.2, and so it provides little grounds for criticism! As was the case when considering Shapiro's *ante rem* structuralist program, our concern here will be to consider any of Hellman's claims that conflict with the principles of the AFCS program, and to consider the extent to which Hellman's program adequately captures the notion of a *structural property*.

Among the most obvious differences between the AFCS and modal structuralist programs is the appeal to a (primitive) modality. Hellman does not argue explicitly for the view that modal notions should be invoked in accounting for our understanding of mathematical theorems, instead he takes up the view that "*mathematics is the free exploration of structural possibilities, pursued by* (*more or less*) *rigorous deductive means*" [35, p. 6], and then proceeds to argue that such a view satisfies certain of his desiderata concerning philosophical accounts of mathematics. Some of the desiderata clearly point toward what is here considered a foundational program: accounting for the *a priori*, objective truth of mathematical theorems, etc. As such, these features of the program are of little interest to the proponent of AFCS, who aims to offer a similarly structured argument: given the AFCS account of mathematics, treating theorems as schematic and conditional in form, assertible rather than true, etc., is anything further required?[47]

However, the modal structuralist[48] and the proponent of the AFCS program do both aim at an eliminative structuralism; one that does not involve commitment to actual structures (structures$_1$). Save for the modal operator, the "hypothetical component"[49] bears some resemblance to that suggested on the AFCS proposal, but the AFCS proposal does not endorse any principle resembling what Hellman terms the "categorical component" of his translation scheme: the claim that a system of the relevant sort is *possible*.[50] What, then, is the role of this categorical component?

As Hellman explains, the categorical component corresponds to "an indispensable "working hypothesis" of underlying mathematical practice" [35, p. 27]: given such a categorical assumption in the modal reconstruction of

[47]And of those features which may be *prima facie thought* required, can they in fact *be* provided?

[48]Hellman's modal structuralism is the only modal view considered here, and so the references to modal structuralism can be assumed to refer to Hellman's program.

[49]The hypotheticals in question are the conditionals of which $\Box\forall X\forall f[\wedge PA^2 \rightarrow S]^X('/f)$ is one example.

[50]Here, $\Diamond\exists X\exists f[PA^2]^X('/f)$ is one example.

mathematics, one can view the categorical component as corresponding to a step in, for example, a proof in arithmetic: informally put, "Assume an ω-sequence" [35, p. 27]. Of course, such an informal claim might also be taken to correspond to an assumption that begins a conditional proof, discharged when the conditional itself is introduced. However, Hellman offers another reason for adopting categorical claims of this sort, and it has a now familiar form:

> ...a categorical assumption to the effect that "ω-sequences are possible" is indispensable and of fundamental importance. Without it, we would have a species of "if-thenism", i.e. a modal if-thenism, and this would be open to quite decisive objections, analogous to those which can be brought against a naïve, non-modal if-then interpretation...the very same situation would obtain in the case of modal conditionals if ω-sequences were *not possible*, i.e. if there could (logically) be no standard realization of the PA^2 axioms...In that case, the translation scheme would not respect negation: all the original sentences A would be translated as true. Thus, it is absolutely essential to affirm, categorically, an appropriate version of [the categorical component for ω-sequences] [35, p. 26–27]

To the familiar problem, then, the familiar solution. Again we note that the proponent of the AFCS program takes the analogous conditionals to involve reference to *models*, and so the conditional is at the level of the metatheory, involving the notions of *model*, *interpretation*, and *satisfaction*. The necessitated, universally quantified conditional in the hypothetical component of the modal structuralist program corresponds to a non-modal, schematic, metatheoretic conditional on the AFCS program. Again, *entailment* rather than *implication* is involved, and the schematic element—reflecting a deliberate neutrality on the status of models—prevents[51] treatment of the conditional

[51] As will be discussed in the coming chapters.

as *true*.

In developing the modal approach Hellman comes to advocate, he does consider (quite early on in his discussion, which may hint at the "naturalness" of such an approach), an attempt to render the hypothetical component of his translation procedure using notions invoked in the AFCS translation. Hellman offers this move as one way to address the concern to "respect the full, classical truth-determinateness of the mathematical theory [arithmetic at this stage] in question" [35, p. 16]. Then,

> ...one way of accomplishing this would be simply to use the language of set theory, since we know how to express both "ω-sequence" and "satisfies" in terms of set membership; (1.1) [the natural language expression of the hypothetical component] could then be made precise by
>
> $$\Box\forall X(X \vDash \wedge \mathrm{PA}^2 \supset X \vDash S),$$
>
> ...One disadvantage of this choice is that the translates all become metalinguistic, and this is surely an awkwardness, if not a fatal misrepresentation of arithmetic discourse. But even more serious is the problem that the structuralist programme, so articulated, becomes just a piece of modal set theory... [35, p. 18]

Hellman holds that branches of mathematics like natural number arithmetic should be capable of a development independent of set theory, and the proponent of the AFCS program is sympathetic to this view. However, one might note that talk of *models* or *satisfaction* need not be given a set-theoretic treatment. This line of thought will be pursued in Chapter 3. Further, it seems that a structuralist might naturally adopt this metalinguistic translation as exactly in keeping with the modal structural program: Hellman endorses the characteristic structuralist slogan—"Any ω-sequence will do" [35, p. 18]—and what is an ω-sequence if not a *model* of the (full, second-order) Peano axioms? In avoiding model-theoretic notions, Hellman instead represents the Peano axioms and their "models" directly in the modal, second-order language of his chosen framework, exploiting the second-order machinery that permits quantified variables ranging over functions and those ranging over predicates, along with the possibility of using relativized quantifiers.

As established in the preceding section, the proponent of the AFCS program can avoid commitment to actual or possible ω-sequences, and still avoid the threat of vacuity. Appealing to a schematic, meta-theoretic rendering of a mathematical theorem, neither assumptions concerning the actual existence or the possible existence of mathematical objects are required to avoid the threat of vacuity. Of course, this will be seen to come at the cost of treating mathematical theorems as *true*, but subsequent chapters may suffice to allay any concerns on that point. In disallowing meta-theoretic notions in the modally bound, quantified conditionals of Hellman's hypothetical components, Hellman does indeed require the categorical assumptions to guard against vacuity. But *are* ω-sequences possible? Are any infinite collections possible? Can an adequate account of mathematics avoid even these seemingly modest ontological claims?[52]

Turning now to the role of category theory in articulating those insights of the structuralist view, Hellman has been a vocal opponent of the suitability of category-theoretic approaches to structuralism [38, 40]. Many of Hellman's objections are raised against category theory as part of a foundational program, and as such those objections will not be considered here. Others (for example, dealing with the notions required in order to *understand* category theory) will be taken up in later chapters. For the moment, then, we turn to consider the treatment of *structural properties* available on the

[52] In Hellman's appeal to mereology, from which he is able to derive claims like the categorical component for arithmetic, Hellman is driven to assert a mereological axiom of infinity, an assertion concerning the possibility of infinitely many individuals. We are then left to wonder with Hellman: "What sort of evidence can we have for the various modal-existence postulates arising in mathematics. . . ?" [39, p. 556].

modal structuralist program.

That Hellman acknowledges the importance of the relation between isomorphic systems and the preservation of structural properties is clear: the system Hellman proposes suffices for the derivation of a categoricity theorem[53] for arithmetic—all "possible" ω-sequences are isomorphic—but Hellman goes on to note that

> . . . one may be tempted to suppose that, in recovering the categoricity of PA^2, the structuralist has accomplished whatever could reasonably be demanded by way of an internal justification for the translation schemes. For (1.11) [the categoricity theorem] is a direct way of saying, within the second-order framework, that our axioms characterize a unique type of mathematical structure; obviously, then, it does not matter "which one" we are "talking about" when we are doing the mathematics of such structures. Isn't our justification complete?
>
> It would be pleasant to conclude this, but overly sanguine. For, while the inference just drawn from (1.11) may indeed be intuitively obvious, really it demands a proof. For the inference pertains to language used to describe the structures, viz. the sentences of $\mathcal{L}(\mathrm{PA})$; yet (1.11) itself says nothing about these sentences. And, remember, it is a translation scheme—a representation of sentences of a given mathematical language—that is to be justified. There is thus a further step, from categoricity to a claim involving language, that needs to be taken. [35, p. 40]

Thus, Hellman is led to produce his Elementary Equivalence Theorem (see [35, p. 41]) a modal analogue of the Structural Properties Theorem for ω-

[53] We return now to using the term "categorical" to describe axiomatic systems for which all models are isomorphic.

sequences: roughly, all ω-sequences exhibit the same structural properties.[54] However, any optimism this result may encourage is short-lived, as axioms for categorical systems (natural number systems and the system of real numbers, being two prominent, important examples) are the exception rather than the rule. Groups, fields, topological spaces, and so on, are decidedly *non*-categorical in their axiomatic presentations, and constitute an important part of contemporary mathematics. What of their structural properties?

One might expect here that, given the reconstruction of ZF set theory in the modal framework, the mathematical objects of the various branches of mathematics might be recovered in this modal set theory, and so the argument that the language of set theory is ill-suited to the structuralist program would carry over to the modal structural program. Such a view is encouraged by Hellman's remark that

> . . . set theory represents both a great opportunity and a challenge to the [modal structural] approach; an opportunity since, as is well known, so much mathematics can be represented within set theory. In so far as set theory yields to a ms [modal structural] treatment, so does all set-theoretically representable mathematics. (Thus, model theory—of special interest to logicians, but not directly representable in the second-order framework of the msi [modal structural interpretation]—would become available, at least indirectly.) [35, p. 53–54]

However, Hellman seems to deny this view in later work, noting that in his *Mathematics Without Numbers*

> . . . it was left open how to treat generally some of the most important structures or spaces in mathematics, e.g., metric spaces, topological spaces, differentiable manifolds, and so forth. This

[54] The categorical version of this theorem presented by McLarty in [61], explicitly concerned with natural number structures, is a particularly good category-theoretic match for Hellman's result.

> may have left the impression that such structures would have to be conceived as embedded in models of set theory, whose modal-structural interpretation depends on a rather bold conjecture, *e.g.*, the logical possibility of full models of the second-order ZF axioms. [36, p. 100]

Perhaps, then, such structures are to be conceived as independent of any set-theoretic reconstruction[55], instead treated in the manner of ω-sequences: translate the axioms and theorems into necessitated, quantified second-order conditionals, and adopt a *possible existence* postulate. For non-categorical theories like group theory, such an approach is clearly inadequate: asserting the possibility of a group doesn't satisfy the modal structuralist's concern about vacuity. Is the assumed group Abelian? If not, theorems about Abelian groups collapse into vacuity. Is the assumed group finite? If not, theorems concerning finite groups collapse into vacuity. It seems then, we must assume a group for every (group) isomorphism type in order to secure the standard group theoretic results, and, moreover, since such results often concern claims about *all* groups (for example, their each being isomorphic to some group of permutations), we are pushed to assume them "simultaneously", as it were. On a standard model-theoretic account in a set theory like ZF, even assuming a single group for each isomorphism type would involve a proper class of (possible) groups—quite a number of objects to assume!

It seems, then, that a recovery of branches of mathematics involving non-categorical axiomatic presentations within Hellman's program does require a retreat to model theory, a model theory recovered via the modal structural treatment of set theory.[56] As these are typically "membership-based" theories, we arrive once more at the uncomfortable position in which isomorphic objects (groups, etc.) can be distinguished in the set-theoretic language,

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[55] A point of agreement with the AFCS program.

[56] The theories which are treated in Hellman's [36] are treated in higher-order analysis, and so built up from, e.g., sets of sets of reals. Algebraic theories such as those at issue here are not discussed.

while such distinctions are deemed, on the structuralist view, to be irrelevant. Why take on this extra baggage? One might expect, here, that a modal structural recovery of a "function-based" set theory, like that of ETCS, may offer a way out, and this is indeed an interesting possibility. However, granting that the threat of vacuity is *not* threatening on the AFCS view, one might want to continue exploring an option that does not enter into the difficulties faced by an appeal to a primitive modality.

Chapter 3

Definitions and Primitive Notions

> *. . . it will be found, in what follows, that the definitions are what is most important, and what most deserves the reader's prolonged attention.* [74, p. 12]

> *. . . nothing in the axioms says functions are not ducks* [61, p. 491]

There are at least two senses in which one might speak of *fundamental mathematical notions*:[1]

1. notions that play a central role *within* mathematics (such as the notions of a *limit* or a *field*), and

2. notions required in order to *understand* those notions used within mathematics (possibly including, for example, the notions of a *rule* or a *property*).

[1] Here the term "notion" is used in approximately the sense of *concept*. The term "notion" is used here to highlight the fact that the arguments of this chapter are taken to be independent of any particular account of the nature of concepts.

Determining whether a given notion falls into one or the other category may not be a straightforward task, as some of the notions that play a central role in mathematics are used (*prima facie* without change in meaning) outside of mathematics proper. For example, of which sort is the notion of a *set*? Is the informal, extra-mathematical notion required in order to understand that notion codified in ZFC? Similarly, at least some notions of the second sort will have to be taken as primitive, and it may be argued that certain notions cannot properly be taken as such.

In identifying a framework for mathematics, one will either implicitly or explicitly classify notions as being of one or the other sort, and some debates relevant to the defence of category theory as a framework for mathematical structuralism turn on issues related to exactly this project of classification. Feferman, for example, has argued that on the view of the categorical structuralist one must inappropriately

> *presume as understood* the ideas of *operation* and *collection*. . . at each step we must make use of the unstructured notions of operation and collection to explain the structural notions to be studied.
>
> It follows that a theory whose objects are supposed to be highly structured and which does not explicitly reveal assumptions about operations and collections cannot claim to constitute a foundation for mathematics, simply because those assumptions are unexamined. [29, p. 150]

While Feferman is explicitly concerned with foundational programs, the scope of his concern can be take to present a difficulty for frameworks more generally. One interpretation of Feferman's critique, in light of the distinction between the two types of fundamental notions, is to view the categorical structuralist as having (perhaps unknowingly) wrongly classified the notions of *operation* and *collection*: if they are notions of the first sort—distinctly mathematical notions to be *explained* by a philosophical account of mathematics—it is a mistake to treat them as notions of the second sort. Alternatively, Feferman may be content to class *operation* and *collection* as notions of the second sort, but view it as inappropriate to take them as *primitive*—given their seemingly essential role in developing the categorical structuralist program, failing to provide a detailed account of those notions may be considered a serious omission.

This chapter offers a response to Feferman's concern, a response which provides a background against which to later develop the *schematic* treatment of mathematical theorems offered on the AFCS program.

3.1 Models

Expanding on the concern about *operations* and *collections*, Feferman notes that, for example,

> ... we say that a group consists of a collection of objects together with a binary operation satisfying such and such conditions. Next, when explaining the notion of *homomorphism* for groups or *functor* for categories, etc., we must again understand the concept of operation... The *logical* and *psychological priority* if not primacy of the notions of operation and collection is thus evident. [29, p. 150]

Thus, one can read Feferman's concern as applying generally to *axiomatic definitions*: given a collection of axioms, be they axioms taken to characterize a group, ring, ordered field, etc., simply understanding the manner in which such axioms are meant to *serve as definitions* requires a prior understanding of *operation* and *collection*.

In developing a reply to Ferferman's concern it will be necessary to first clarify the intended sense of "logical" and "psychological" *priority*. In speaking of "logical priority" Feferman notes that "My use of 'logical priority' refers...to order of definition of concepts, in the cases where certain of these *must* be defined before others" [29, p. 152], whereas "psychological priority" "has to do with the "natural order of understanding" [29, p. 152]. As Feferman grants that the notion of "psychological priority" is somewhat unclear,[2] here we will only be concerned to deal with the claim involving "logical priority".

The strategy of the AFCS program is to concede Feferman's claim that that definition of a category presupposes the notions of *operation* and *collection*, but to deny the claim that this presupposition is illegitimate. The proponent of the AFCS program holds that the notions of *operation* and *collection* are so ubiquitous that any attempt to present a view that avoids the presupposition of those notions is doomed to failure. For example, the mechanisms involved in understanding the *truth of a proposition* in some natural language may plausibly be taken to presuppose both of these notions. Understanding that the statement "Everyone here likes the smell of freshly-brewed coffee" is true can be taken to involve a number of such presuppositions. Understanding that the statement comes out true is a given situation may plausibly be taken to involve an *operation*, mapping the term "here" to some location, as well as a *collection*, the people[3] over which the quantifier is taken to range. There may be an additional function invoked in accounting for the role of the quantifier in this expression: the notion of a *valuation*, mapping the variable ranging over people that figures in this expression to an element of the collection of people that constitute the *universe of discourse* for this expression. Indeed, the expression bound by the quantifier can be viewed as a *propositional function*, yielding a proposition whenever the name of an object in the universe of discourse is substituted into the open sentence cor-

[2] In response to an objection from Mac Lane, Feferman concedes that psychological priority "...is admittedly 'fuzzy' but not always 'exceedingly' so" [29, p. 152].

[3] The interpretation could instead be taken to involve a collection of, for example, objects in a room, in which case the proposition would then be relativised to a class of people in the usual way.

responding to the matrix of the quantified expression. One's understanding of the role of the quantifier in this expression could then be taken to proceed via an appeal to the notion of a further operation, that corresponding to substitution into a propositional function.

Of course, one might object that here we have merely *described* language-related understanding in terms of *operation* and *collection*, but we have not argued against there being a necessary order of dependence. This much may be conceded, but the example above should serve to indicate that the extremely general notions of *operation* and *collection* easily transcend their treatment in any theory specifically concerned with their use in mathematics, and appealing to a prior understanding of these notions in developing a mathematical framework is as legitimate as appealing to a prior understanding of some of the most general features of language.

The advocate of the AFCS program can be somewhat more specific about those notions the program takes as primitive. In particular, the program takes the notion of a *model* as primitive, and so presupposes any of those notions necessary to understand the notion of a *model*. A *model* can be characterized in terms of the notion of an *interpretation* of some collection S of statements in given formal language.[4] An *interpretation* of a given formal language is understood as involving a *collection* (the *universe of discourse*) and some number of *properties* and *relations* defined on the universe of discourse, corresponding to predicate and relation symbols, respectively, of the language. The interpretation of constant symbols or function symbols may also be required, and each of which is taken to correspond to an *element of the universe of discourse* or a *function defined on the universe of discourse*, respectively. For a given collection S of statements in the formal language, a *model* $\mathcal{M}$ of S consists of a universe of discourse, properties, relations, and designated functions and elements of the universe of discourse, corresponding

[4]While the first-order case is treated here, the type of formal language is left unspecified, and additional clauses required to interpret higher-order vocabulary are to be added as needed.

to an interpretation in which each statement in S is true. Thus, the notion of *truth* is also presupposed. On the AFCS view, a model is taken to be an *object*, which may then legitimately serve as the referent of a singular term.

Of course, any reader familiar with the "standard" set-theoretic semantics for a first-order language will be familiar with these notions, and may worry that they presuppose a considerable amount of set theory—itself a prominent branch of *mathematics*, and so there may be a concern about circularity. Here, unfortunately, the categorical structuralist can offer little in the way of a direct reply to this concern. The program takes as primitive those notions—whatever they may be—required in order to understand that "Everyone thinks highly of Palin" is a substitution instance of an expression of the form "$\forall x Hxp$", a statement which is *true* in a circumstance where the quantifier is taken to range over all the people in a particular room, each of whom happens to think highly of the individual named "Palin". The "set-theoretic" presuppositions fall short of requiring anything akin to the Axiom of Infinity or the Axiom of Choice, and whatever fragment of the set-theoretic machinery is, in fact, required in order to understand this notion of a *model* is accepted as necessary.[5] *Prima facie*, the "thinks highly of" relation does not require any "extensional" account—involving a *set of ordered pairs*, for example—in order to be correctly understood by a speaker of the English language, and similarly it may be possible to dispense with some of the other informal set-theoretic notions typically invoked when defining components involved in the notion of a model. An interpretation of "Fa" in which the universe of discourse is taken to be the collection of all films, "F" is taken to be the property *is a modern classic*, and "a" represents the film *Låt den råtte komma in* yields a model of "Fa", and any notions or principles involved in recognizing this situation to obtain are taken as primitive on the AFCS program. Insofar as these notions are plausibly taken to be involved in the more general understanding of any natural language, this element of

[5]The set-theoretic notions required can be found in, e.g., §2.4 of [8].

the AFCS program should be deemed unproblematic.

3.2 Category-Theoretic Definitions

The notion of a *model* of some set of axioms—like those that form the familiar definition of a *group*—is taken to be primitive. In accordance with the view developed in Chapter 2, such models are to be treated as *objects in a category*, as the language of category theory has been shown to be particularly well-suited to the structuralist's focus on structural properties.

On this picture, then, the following two questions arise:

1. Which other models feature in the *ambient* (background) category?
2. Given a category of models, how can these models be manipulated to yield other models of the same (or of a different) sort?

The deliberately lack of specificity inherent in taking the notion of a model as primitive affords a great deal of flexibility in addressing this first question. In essence, the ambient category provides the *context* of the study of the mathematical objects under consideration, and there are no restrictions placed on contexts admitted as legitimate on the AFCS view. Consider again the study of the mathematical notion of a group. The ambient category for models of the group axioms may be that consisting of monoids as objects and monoid homomorphisms as arrows, in which case the ambient category contains objects which are not themselves models of the group axioms. Alternatively, the ambient category may be that containing (only) groups as objects and group homomorphisms as arrows, where groups are, for example, identified as sets in ZFC. As a third option, groups may instead be identified themselves as categories,[6] where the ambient category is taken to be that of,

[6] A group can be defined as a category with one object for which every arrow is iso.

for example, all *small* categories,[7] with functors[8] as arrows. In this latter case again we have objects in the ambient category which are not models of the axioms in question.

3.2.1 Products, Equalizers, and Coequalizers

In addressing the second question concerning the manipulation of objects in a category, the suitability of the language of category theory to the expression of some of the principal techniques of mathematical definition becomes apparent. Take, for example, the notion of forming a *product* of two mathematical objects. Considering first the Cartesian product of two sets in a set-theoretic framework, one may proceed by defining, for example, the Kuratowski ordered pair, where the ordered pair $\langle x, y\rangle =_{def} \{\{x\}, \{x, y\}\}$. Given two sets A and B, their product, symbolized as $A \times B$, can be defined as $\{\langle x, y\rangle | x \in A \text{ and } y \in B\}$. Of course, other definitions of the ordered pair are possible, for example, $\langle x, y\rangle =_{def} \{x, \{x, y\}\}$. For mathematical purposes, the choice between these alternative definitions of ordered pair is of no consequence, as both exhibit the essential feature that

$$\langle x_1, y_1\rangle = \langle x_2, y_2\rangle \Leftrightarrow x_1 = x_2 \text{ and } y_1 = y_2.$$

The categorical treatment of the notion of a product is most naturally expressed by means of a diagram. Given objects A and B, a *product* diagram for A and B is an object P and arrows p_1 and p_2 (called *projections*)

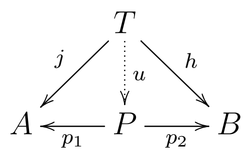
$$A \xleftarrow[p_1]{} P \xrightarrow[p_2]{} B$$

such that, for any object T and arrows j and h with $A \stackrel{j}{\leftarrow} T \stackrel{h}{\rightarrow} B$, there is a

[7]A category is *small* provided its collection of arrows is a set.

[8]Briefly, a *functor* F is a function between categories that maps objects to objects and arrows to arrows while respecting domains, codomains, identities and composites. For example, the F which appears in the Structural Properties Theorem is a functor.

unique arrow u such that the diagram below commutes.



For two objects A and B in a category, there may be no product for A and B, or any number of products. Given any two distinct products

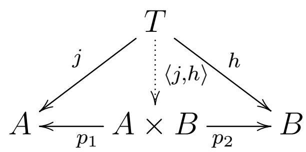
$$A \overset{p_1}{\leftarrow} P \overset{p_2}{\rightarrow} B \text{ and } A \overset{q_1}{\leftarrow} Q \overset{q_2}{\rightarrow} B$$

for objects A and B, it is a routine exercise to show that P and Q are necessarily isomorphic. As Awodey observes, "The categorical definition of a product…is one of the first examples of category theory being used to give a purely structural characterization of an important basic mathematical notion" [1, p. 220 ff]. In the present context, one might interpret Awodey's remark as highlighting the definition's embodiment of the two structuralist perspectives identified in Section 2.1.1. The *particular elements* of objects A and B of a category are not explicitly involved in the definition of the product, and, indeed, there may be no such elements. For example, the category-theoretic definition of a product applies in the case where the category is a partially-ordered collection,[9] in which case a product of two objects A and B is the greatest lower bound of the two elements. Similarly, the *particular system*—in this context, the *particular product diagram*—is of no consequence, as, given a pair of elements A and B in a category, all products for A and B are isomorphic.

When there is product diagram for objects A and B in a category it is customary to denote a selected product diagram by $A \overset{p_1}{\leftarrow} A \times B \overset{p_2}{\rightarrow} B$, and, given any diagram $A \overset{j}{\leftarrow} T \overset{h}{\rightarrow} B$, the unique $u : T \rightarrow A \times B$ will typically

[9] Partially ordered collections can be treated as categories with an arrow from A to B iff $A \leq B$. See the discussion in Section 1.2.

denoted by $\langle j, h\rangle$. The commuting diagram in the definition of a product then becomes



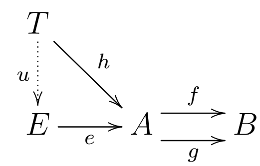
The definition of a product in a category is noteworthy for the following feature:

> . . . one and the same categorical definition describes also products of topological spaces, groups, vector bundles on a smooth manifold, or whatever. The definition. . . provides a uniform, structural characterization of a product of two objects in terms of their relations to other objects and morphisms [arrows] in a category, in contrast to 'material' set-theoretic definitions which depend on specific and often irrelevant features of the objects involved, introducing unwanted additional structure. [1, p. 220]

Notice that the definition of the (Cartesian) product of two *sets* does not extend directly to, for example, the product of two groups on the set-theoretic definition of a group. In a set-theoretic framework, the Cartesian product of the groups $\langle G, \star\rangle$ and $\langle G', \star'\rangle$ is not an ordered pair, and so, *a fortiori*, not a group.

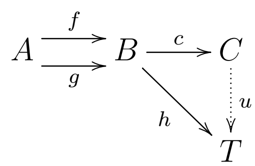
Other standard techniques for providing definitions of mathematical objects can also be characterized in the language of category theory. Consider the notion of an *equationally defined subset*, and the notion of the *kernel of a homomorphism*. These notions are both generalized by the category-theoretic notion of an *equalizer*. Given *parallel* arrows $f, g : A \to B$ (arrows with the same domain and codomain), an arrow $e : E \to A$ *equalizes* f and g provided $f \circ e = g \circ e$. Given such f, g, E, and e, the arrow e is an *equalizer* for f and g provided e equalizes f and g, and for any $h : T \to A$ which equalizes f and g, there is a unique $u : T \to E$ such that $e \circ u = h$. That is, e

is an equalizer for f and g provided, given any other arrow h that equalizes f and g, there is a unique arrow $u : T \to E$ such that the following diagram commutes.



It is useful to consider the definition of an equalizer in the set-theoretic case, in a category where the objects are sets and the arrows set functions. For two functions $f, g : A \to B$, a function h that *equalizes* f and g is one whose range is contained in the set of elements $a \in A$ such that $f(a) = g(a)$. Of course, a function that equalizes f and g may not map onto the entire set $\{a | a \in A \text{ and } f(a) = g(a)\}$; however, the "universal" condition on an *equalizer* $e : E \to A$, ensures that any such equalizer is maximal in this sense. One equalizer is given by taking $E =_{def} \{a | a \in A \text{ and } f(a) = g(a)\}$ and taking e to be the corresponding inclusion map $e : E \hookrightarrow A$. Of course, such equalizers are not unique in this context; any set F in bijective correspondence with E will yield another equalizer; if $d : F \to E$ is a bijection, then $e \circ d : F \to A$ is also an equalizer.

The *dual* of a category-theoretic statement is given by reversing arrows (and so, reversing composites and exchanging domains/codomains). The dual of the definition of an equalizer yields the definition of a *coequalizer*, which can be considered a generalization of the notion of a quotient by an equivalence relation. A *coequalizer* for a pair of parallel arrows $f, g : A \to B$ is an arrow $c : B \to C$ such that $c \circ f = c \circ g$, and for any arrow $h : B \to H$ such that $h \circ f = h \circ g$, there is a unique arrow $u : C \to H$ such that the following diagram commutes.



Again, it is useful to consider this definition in the context of a category whose objects are sets and whose arrows are set functions. For a given equivalence relation R defined over the elements of some set B (and so, $R \subseteq B \times B$), consider the two projections $r_1 : \langle a, b\rangle \mapsto a$ and $r_2 : \langle a, b\rangle \mapsto b$, with $r_i : R \to B$. The coequalizer for r_1 and r_2 is then $c : B \to B/R$, with $c : b \mapsto [b]_R$. In the context of a category of groups and group homomorphisms, consider the coequalizer of any homomorphism $f : G \to H$ and the trivial homomorphism $g : G \to H$ which maps every element of G to the identity element of H. Letting K be the kernel of f, the function $c : H \to H/K$ is a coequalizer of f and g.

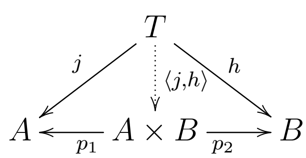
In general, given parallel arrows $f, g : A \to B$ in a category of sets and set functions, a coequalizer $c : B \to C$ for f and g can be constructed by considering the equivalence relation $\sim$ on elements of B generated by $\{\langle f(a), g(a)\rangle | a \in A\}$.[10] Take $C = B/\sim$, and let $c : B \to C$ be defined by $c : b \mapsto [b]_\sim$. For any $a \in A$ we have $f(a) \sim g(a)$, and so $[f(a)]_\sim = [g(a)]_\sim$, i.e., $(c \circ f)(a) = (c \circ g)(a)$. Thus we get $c \circ f = c \circ g$ as desired. Now consider any $h : B \to T$ such that $h \circ f = h \circ g$. Let $u : C \to T$ be defined by $u : [b]_\sim \mapsto h(b)$. It can be shown that u is well defined, and that $u \circ c = h$. As c is onto[11] we also get that u is the *unique* function with the property that $u \circ c = h$. Thus, c is a coequalizer for f and g.

[10]Note that $B \times B$ is an equivalence relation, and so the equivalence relation generated is a subset of $B \times B$, and is also such that elements not in the image of A through f nor in the image of A through g are related only to themselves.

[11]In categorical terms, c is epic.

3.2.2 Elements and Subobjects

In considering examples of the products, equalizers, and coequalizers involving categories of sets and (set) functions, we have frequently appealed to the notion of an *element* of an object, i.e., a member of a set. While the category-theoretic treatment of products, equalizers, and coequalizers abstracts from—and so does not appeal to—the notion of an *element* of an object, it is nevertheless possible to introduce this notion directly within the language of category theory. The notation used in modified diagram of the product,



is motivated by the method for recovering the notion of an element of an object.

A *terminal* object in a category is an object A such that every object in the category has exactly one arrow to A. In a category with sets as objects and set functions as arrows, terminal objects correspond to singleton sets, and it is useful to think of terminal objects as abstract singletons. A category may have any number of terminal objects, and any two terminal objects are isomorphic.[12] Arrows from terminal objects are easily shown to be monic,[13] and so the elements of a set S (viewed as an object in a category of sets) are in one-to-one correspondence with the arrows from any terminal object to that set. Since all terminal objects are isomorphic, it is customary to represent a selected terminal object as 1, in which case the elements $x \in S$ can be identified with the arrows $x : 1 \to S$. As the definition of a terminal object

[12]The composite of the unique arrows between two terminal objects A and B is, without loss of generality, the unique arrow $f : A \to A$ in the category, which must then be the identity arrow for A.

[13]See the definitions in Section 2.4.2.

can be presented entirely in the language of category theory,[14] categories that have a terminal object, 1, allow *elements* of a object S to be identified with arrows $x : 1 \to S$.[15]

Not all categories have terminal objects, and some of those that do are such that other features of the category prevent the arrows from terminal objects being treated as elements in the manner described above. A one-element group is terminal in the category of groups and group homomorphisms, but one-element groups are also *initial* objects. An *initial* object I is the dual of a terminal object: every object A in the category has exactly one arrow $f : I \to A$ from a given initial object. Clearly, then, terminal objects which are also initial (called *zero* objects) do not stand in one-to-one correspondence with the elements of an arbitrary object. Thus the elements of a group, in the category of groups and group homomorphisms, cannot be identified with arrows from a one-element group, as one-element groups are zero objects in the category of groups and group homomorphisms.[16]

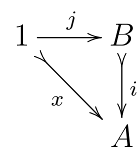
Of course, if one aims to treat elements of groups, it is possible to use the language of category theory to provide an alternative definition of a group as a type of object in an arbitrary category, and this method of defining a group does allow elements of groups to be identified with arrows from a terminal object. To accomplish this, is is useful to first introduce the notion of a *subobject*. If A is an object in a category, then any monic arrow $f : B \to A$ can be viewed as identifying a "part" of A. Monic arrows correspond to injective functions in a category of sets and set functions, and the images of such functions (and so, in this sense, the functions themselves) determine

[14] See the examples at the end of Section 2.4.1.

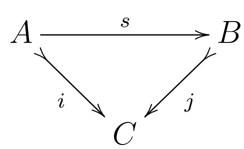
[15] In addition to elements in a category with domain 1, any arrow $x : T \to S$ can be viewed as a *generalized* element of S, sometimes denoted $x \in_T S$. Arrows $x : 1 \to S$ are sometimes called *global* elements. Henceforth, the notation $x \in S$ will be used to abbreviate global elements $x \in_1 S$.

[16] While arrows from terminal objects in the category of groups and group homomorphisms don't suffice for identifying the elements of a group G, the arrows from any group isomorphic to the integers to G do suffice. See [55, p. 103–104].

subsets of A. The treatment of arrows as subobjects may be viewed as a natural generalization of this feature of injective functions. For an element $x : 1 \to A$ and subobject $i : B \to A$, $x \in i$ provided x "factors through" i, that is, when there exists an arrow $j : 1 \to B$ such that $i \circ j = x$, i.e., such that the following diagram commutes.



A relation of *inclusion*, $\subseteq$, can be defined on subobjects i, j of a given object A; if $i : A \rightarrowtail C$[17] and $j : B \rightarrowtail C$, i is *included* in j (represented as $i \subseteq j$) provided there is an $s : A \to B$ such that the diagram below commutes.



Such an s is unique and monic. If both $i \subseteq j$ and $j \subseteq i$, then i and j are said to be equivalent (represented as $i \equiv j$), in which case their domains are isomorphic. Note that the category-theoretic treatment of subobjects in a "function-based" set theory differs in several ways from the treatment of subsets in a "membership-based" theory like ZFC. In the category-theoretic treatment, elements $x : 1 \to A$ are identified with their singletons, and elements are not themselves *objects*, in particular, an object A in a category is not an element of any other object. However, fundamental results, such as the result that, if $i \subseteq j$, then for all x, $x \in i \Rightarrow x \in j$, are available.[18]

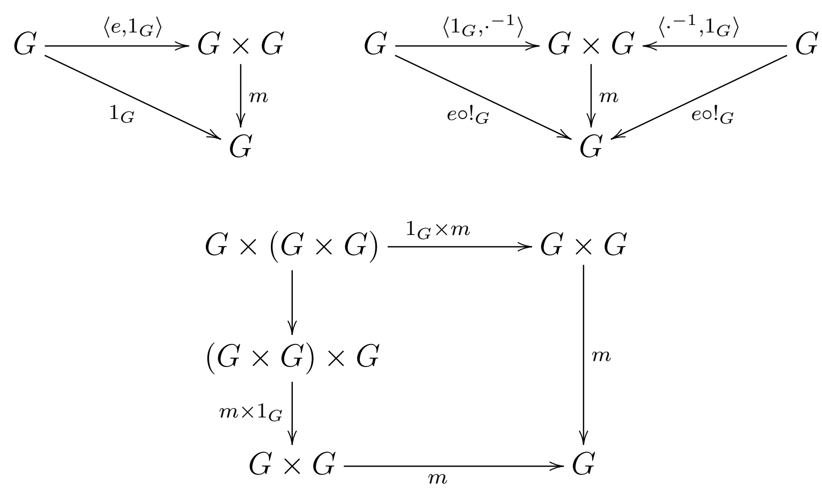
Relations, then, can be viewed as subobjects of the relevant product. A binary relation defined over the elements of an object A is simply a monic

[17] Henceforth, the usual convention of denoting a monic using an arrow with a "tail", $\rightarrowtail$, (both in diagrams and arrow descriptions), will be adopted.

[18] Note that the converse may fail, and must be added as an additional axiom if needed.

arrow $r : R \rightarrowtail A \times A$. Note that, for any elements $x, y : 1 \to A$, the ordered pair $\langle x, y \rangle$ is an element of $A \times A$. Monic arrows *from* a product $A \times A$ to A can be viewed as binary operations defined on (pairs of elements of) A.

With these category-theoretic notions to hand, we now return to explore the characterization of a group as an object with associated arrows in a category. Assuming the category to have a terminal object and products, an object G of the category is a *group* provided there is an element $e : 1 \to G$ (the identity element), an arrow $m : G \times G \to G$ (the multiplication operation), and an arrow $\cdot^{-1} : G \to G$ (the inverse operator), such that e is an identity with respect to m, m is associative, and $\cdot^{-1}$ yields an inverse for any element of the group. All of the conditions on e, m, and $\cdot^{-1}$ can be expressed via commuting diagrams. Following the presentation of Chapter 3 of [60][19], G is a group provided each of the diagrams below commutes.[20]



The commutativity of these diagrams corresponds to (proceeding clockwise from the upper left) e serving as a unit with respect to m, $\cdot^{-1}$ mapping an element to its inverse (again with respect to m), and the associativity

[19]A similar treatment can be found in Chapter 4 of [3].

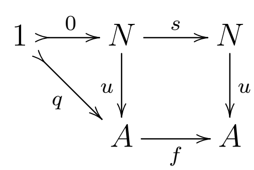
[20]Here, $!_G : G \to 1$ is the (unique) arrow from G to the terminal object 1, and the unlabelled arrow from $G \times (G \times G)$ to $(G \times G) \times G$ is an arrow that witnesses the associativity of the product (such arrows exist for any finite product, and are iso).

of m, respectively. Other algebraic structures (for example, *rings*) can be treated analogously.

Thus, we see that it is possible to define a group as an object with associated arrows in *any* category with a terminal object and products for any pair of objects. While the AFCS view does not require that groups be defined in this (or any other) particular manner, this example serves to further illustrate some of the familiar definitions that can be cast entirely in the language of category theory.

3.2.3 Natural Numbers and Beyond

Following McLarty's treatment in [61] (which in turn draws on Lawvere's treatment in [49]), the collection of natural numbers can be characterized as an *object* N in a category, taken along with *arrows* $s : N \to N$ (the successor arrow) and $0 : 1 \rightarrowtail N$ (the zero arrow) such that N "supports recursive definition". That is, for any $f : A \to A$, and $q : 1 \rightarrowtail A$, there is a unique arrow $u : N \to A$ such that the following diagram commutes.



Intuitively, u is the unique function on N defined by the recursion data q and f, i.e., we set $u(0) = q$, and for (global) elements x of N we have $u(s(x)) = f(u(x))$.

The ambient category in which natural number objects are typically identified has, in addition to the usual axioms of category theory,

1. an axiom positing the existence of a terminal object 1,

2. products, equalizers, and coproducts for each pair of objects,

3. an axiom of non-triviality, which gives that, where $1 \xrightarrow{i_1} 1+1 \xleftarrow{i_2} 1$ is a coproduct diagram, the arrows i_1 and i_2 are distinct,[21]

4. an axiom that establishes the ambient category to be *well-pointed* (this axiom is sometimes called "1 Generates"), which yields that subobjects are determined by (global) membership,[22] and,

5. an axiom which asserts the existence of a *stable* natural number object, which is a natural number object that supports recursive definition with parameters.[23]

All these axioms save the last are satisfied if the ambient category is a type of category called a *well-pointed topos*, and in such a topos any natural number object will be stable (though the *existence* of natural number objects is independent of the axioms determining a well-pointed topos). A topos can be roughly characterized as a category with properties akin to those of a set theory like ZFC; "toposes are categories which allow the constructions used in ordinary mathematics" [60, p. 6]. Toposes have proved a fruitful topic of research, and are studied in detail in, e.g., [6, 60].[24] For present purposes, it suffices to note that all toposes have terminal objects, products (and coproducts) for all pairs of objects, as well equalizers and coequalizers for all parallel arrows.

For an ambient category that satisfies the axioms identified above, it can be proved that a natural number object satisfies (suitably translated

[21] Coproducts are the category-theoretic dual of products, and can be informally thought of as an abstraction of the set-theoretic notion of a disjoint union.

[22] For subobjects i and j of an object A, if $i \subseteq j$, then any $x : 1 \to A$ is such that $x \in i \Rightarrow x \in j$, although the converse may fail. This axiom expresses the converse. Note that subobjects in any category are always determined by *generalized* membership. (See note 15.)

[23] This last axiom of stability is required in order to allow for the standard inductive definitions of addition and multiplication, which involve parameters.

[24] Bell's [6], in particular, highlights the close relationship between toposes and set theories in developing the formal systems of *local set theory*; toposes are shown to be (in a sense that can be made precise) the natural models of local set theories.

versions of) the Peano axioms, that any two natural number objects $N, s, 0$ and $N', s', 0'$ are *isomorphic*, that there are infinitely many natural number objects, and, most importantly, that any two natural number objects have all the same *structural properties*. This last result lies at the heart of McLarty's [61], and his argument for this result has been adapted to yield the Structural Properties Theorem of Section 2.4.3.

The axioms above (and, *a fortiori*, the axioms for a well-pointed topos, taken with the axiom stipulating the existence of a stable natural number object), also permit the construction of other familiar number systems, in the standard manner. As described in [62], one can use the definition of addition for a given natural number object N along with the relevant products, to produce a pair of arrows $f, g : (N \times N) \times (N \times N) \to N$ with $f : \langle m, n, m', n' \rangle \mapsto m+n'$, and $g : \langle m, n, m', n' \rangle \mapsto m'+n$. Taking an equalizer for this pair of arrows yields a subobject of $s : S \to (N \times N) \times (N \times N)$, the elements of which are ordered 4-tuples $\langle m, n, m', n' \rangle$ with $m+n' = m'+n$. We can then consider the two projections $s_1, s_2 : S \to N \times N$, with $s_1 : \langle m, n, m', n' \rangle \mapsto \langle m, n \rangle$ and $s_2 : \langle m, n, m', n' \rangle \mapsto \langle m', n' \rangle$. The coequalizer $z : N \times N \to Z$ for arrows s_1 and s_2 then yields the integers, Z, as $(N \times N)/\sim$, where $(a, b) \sim (c, d)$ provided $a+d = c+b$, i.e., $a-b = c-d$. The existence of the relevant products, equalizers, and coequalizers, is thus seen to permit the usual construction of the integers from the natural numbers, and the rational numbers can be similarly constructed. The real numbers can be constructed by Dedekind cuts or Cauchy sequences, although in some contexts the results are not equivalent.[25]

3.3 The Lens of Category Theory

As was the case with groups, the AFCS program does not require that the natural numbers, rationals, reals, etc., be given the category-theoretic treat-

[25] In [6] Bell establishes the result (attributed to Johnstone, see [44]) that "Dedekind cuts within a local set theory need not be (conditionally) order-complete" [6, p. 226].

ment described above, nor does it require they be given any other particular account. Indeed, the related aspect of the AFCS proposal is simply that the models of the axioms that define some type of system be treated as *objects* within a category-theoretic framework, as the language of category theory has been shown to effectively isolate the structural properties of any such objects. There is no requirement that the models *are* categories, no requirement that the models be defined in—or otherwise recovered in—a fixed, predetermined category of some sort (for example, a well-pointed topos), and no requirement that the models belong to a particular, fixed collection of categories, such as the collection of toposes.[26]

The aim in introducing the various category-theoretic methods of definition and demonstrating their use in constructing familiar mathematical objects is to further illustrate the expressive power of the language of category theory. In Chapter 2, the language of category theory was shown to be well-suited to capturing the notion of (mathematical) *structure*, as the language of category theory preserves the structural properties of isomorphic objects. When those objects are models of some type of system, the language of category theory can be used to capture the mathematically relevant features of those objects, independently of that object's "internal" composition. Models may be conceived as consisting of spacial points, sequences of thoughts, rotations of objects, etc., but the language of category theory sharpens the focus, permitting one to isolate and study only those properties shared by all models isomorphic to a selected model, i.e., only the structural properties. In this chapter, the expressive resources of the language of category theory have been further explored, and it has been observed that, starting with some given collection of models of a given type, framed in the language of category theory, one may appeal to the standard means

[26] Other approaches, aimed at developing *foundational* programs using the resources of category theory, are often led to constrain the realm of mathematical objects in this way, and the results of this chapter are of particular importance to such programs (see, e.g., [6, 50, 49, 62]).

of mathematical definition and construction in producing other models, of either the same type or of a different type. Such techniques permit the familiar construction of product groups, equationally-defined subsets, kernels of morphisms, quotients, and combinations thereof.

Category theory is thus capable of "standing on its own", in at least the sense that it does not dependent on the expressive resources of a membership-based set theory in order to characterize standard mathematical constructions on mathematical objects. This observation serves to address a concern related to that raised by Feferman with which we began the chapter. Hellman, for example, has worried that category theory has yet to be shown

> ... autonomous from set theory in a strong sense: not only is its primitive basis capable of standing on its own and sufficient for *some* recovery of ordinary mathematics, even if via a detour through set-theoretic constructions... but, without any such detour, it can achieve a genuinely distinctive, intelligible conceptual development throughout, not just in its initial stages.... unless and until it [strong autonomy] is achieved, the charge that category theory is 'parasitic' on set theory in its recovery of ordinary mathematics will surely linger. [38, p. 133]

Of course, the call for a "distinctive, intelligible conceptual development" is difficult to clearly address. Are the methods and notions of category theory sufficiently distinct (presumably from a membership-based set theory)? Are the methods and notions sufficiently intelligible? This concern is reminiscent of Feferman's call for "psychological priority", and here Hellman is led to admit that his request is "not a precise distinction", and that "perhaps we would only recognize 'strong autonomy' if we saw it" [38, p. 133].

While Hellman's concerns may be at least partially addressed within this chapter, a group of related concerns merit further examination. Recall Feferman's worry about the notions of *operation* and *collection*, which were seen to be focused on the role of the axiomatic method in understanding the axioms of category theory, or, indeed, axiomatic systems more generally. The response offered here was to observe that, insofar as the mechanisms involved are so ubiquitous as to be plausibly involved in the understanding of any natural language, demanding an account of these features from a philosophy of mathematics—foundational or otherwise—is inappropriate. We see this request again in Hellman's discussion of McLarty's (foundational) proposal to pursue an account of "a (meta) category of categories",[27] about which Hellman remarks that

> when we speak of the "objects" and "arrows" of a metacategory of categories as *categories* and *functors*, respectively, what we really mean is "structures (or at least "interrelated things") satisfying the algebraic axioms of CT [category theory]", i.e. we are using "satisfaction" which is normally understood set-theoretically. . . clearly there is some dependence on a background that explicates *satisfaction* of sentences by structures, and this background is not "category theory" itself. . . [40, p. 157]

While this concern should also have been addressed by the foregoing (and, indeed, it was again a concern raised in discussion of *foundational* programs), it does suggest that other metatheoretic issues merit some attention. In turning his attention to Awodey's anti-foundational proposal,[28] very close in spirit to the AFCS view considered here, Hellman worries that such programs are faced with a dilemma: either the language of category theory arrives at the meanings of its terms (*arrows* and *objects*)[29] via the notion of *satisfaction* (and so "falling back on *prima facie* set-theoretic notions after all" [40, p. 159]), or

> . . . what we are really presented with is a kind of formalism, in

[27]See [62, 63].

[28]See [2].

[29]Since "arrows-only" presentations of the axioms of category theory are possible, Hellman restricts his attention to the notion of an arrow or "morphism." See [52, p. 9].

> which theorems in conditional form, together with definitions, are all there is to mathematics, that is, we just give up on the notion of mathematical truth as anything beyond deductive logical validity. [40, p. 158]

While the first horn of this dilemma should now be sufficiently dulled, it remains to consider the status of *mathematical truth* on the AFCS view, and it is to that discussion that we now turn in Chapter 4.

Chapter 4

Domains, Truth, and Proof

> *. . . nothing capable of proof ought to be accepted without proof.* [24, p. 14]
>
> *Mathematics is "correct" but not "true"* [51, p. 443]

To hope to do justice to an account of roles of *proof*, *truth*, and *domains* in mathematics—over the course of a single chapter—is optimistic at best, foolish at worst. Thus, this chapter is best viewed as a sketch of the accounts of these notions either best suited to, or required by, the central tenets of the AFCS program.

4.1 Theorems as Conditionals

Recalling the discussion of Section 1.5, the AFCS program adopts Awodey's characterization of mathematical theorems as both *schematic* and *conditional*. Of the claim about the conditional form of mathematical theorems, Awodey asserts that mathematical theorems are such that

> . . . the 'things' referred to are assumed to have certain properties, and then it is shown, using the tacitly assumed methods of reasoning, that they also have some other properties. . . . Of course,

> many theorems do not literally have this form, but every theorem has some conditions under which it obtains. [2, p. 58].

A quick glance over results established in a variety of textbooks in mathematics offers support for this view. Examples of theorems presented in this form abound, and the following two examples are typical. A recent textbook on abstract algebra contains the theorem that "In a ring with identity every proper ideal is contained in a maximal ideal" [27, p. 254]. A classic textbook in real analysis states Minkowski's inequality as the theorem

Minkowski's Inequality. *Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^P(E)$, then so does their sum $f+g$, and, moreover,*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$[1]

At least *prima facie*, Awodey's claim seems plausible as a descriptive claim about the form of many theorems in contemporary presentations of mathematical results. However, some well-known worries (aimed at nominalistic philosophies of mathematics) seemingly become relevant when one considers those theorems which "do not literally have this form", and so which require rewriting as conditionals. Is it legitimate to rewrite the theorem that *there are infinitely many primes* as the theorem that *in any ω-sequence, there are infinitely many primes*? Is it legitimate to rewrite the theorem that *there are exactly two groups of order 4* as the theorem that *for any group G, if G has order 4 then G is isomorphic to a cyclic group of order 4 or G is isomorphic to the product of two cyclic groups of order 2*? The initial expressions of these theorems seem to embed ontological commitments that do not appear in their conditional translations; do such translations accurately represent the content of the original theorems?

At least one aspect of this question of representation of content can be safely ignored: what did the mathematician who produced the result believe

[1] See [72, p. 141].

him or herself to have established? Did he or she regard the statement as involving a commitment to abstract objects, be they numbers, groups, or the like? That, when presented with theorems in the original forms above, whose grammatical structure is such that they "appear to assert the existence of mathematical objects, and to be true only if such objects exist", many mathematicians "assent verbally to them [theorems with apparent existential commitment] without conscious silent reservations" seems wholly irrelevant to the question of the correct representation of the *mathematical content* of such theorems [71, p. 516]. If these results are indeed *theorems*, and not, for example, a mathematician's *conjectures*, the concern of both the mathematician and the philosopher of mathematics is to investigate how to best represent the *mathematical content* of those theorems, not to capture any aspect of the mathematician's *attitude towards* those theorems.

In aiming to understand how to best express the content of any such theorem, one is naturally led to consider the *warrant* for the assertion of that theorem. Mathematics is a unique area of research in that the warrant for asserting a mathematical claim is provided by a *proof* of that claim; "Mathematics differs from all other sciences in requiring that its propositions be proved" [58, p. 3]. The *proof* of a mathematical theorem then yields information about the correct description of that proof, i.e., the statement of the theorem that has been proven. A proof of the infinity of the primes may be taken by the mathematician who produced the result to describe a feature of *the* natural numbers, but the *methods* employed in the proof will be readily seen to apply to a whole class of systems, even in the special case of those systems which admit of a categorical[2] description. Such a procedure is typical of contemporary mathematics, and, indeed, is at the core of both the axiomatic method and the structuralist view of mathematics. As Mayberry notes,

[2]Here, *categorical* is used to describe an axiomatic system for which all models are isomorphic.

> ...in the axiomatic approach there is no unique, particular system of natural numbers; there is instead an absolutely infinite species of mutually isomorphic *simply infinite systems*...
>
> ...the axiomatic method allows us to dispense with the "mathematical objects" of tradition. [58, p. 194–195]

Thus, while a mathematician may envision having proved a result that holds for a particular system of objects—*the* natural numbers, for example—the methods employed in the proof typically generalize, and so apply to a whole class of systems.

Of course, it would be disingenuous to represent, for example, Euclid's proof of the infinity of the primes as a general claim about ω-sequences (or, indeed, to present this theorem in its more general, ring-theoretic version). However, proponents of the AFCS program (and those, like Awodey, who would endorse at least some of its central tenets) need not view the procedure of rendering a theorem in conditional form as a process of *translation*—aimed at *preserving the content of the original theorem*—but may instead view the rendering in conditional form as reflecting the contemporary mathematical concern to produce a sharper, more accurate, more general, and ultimately more useful version of the original theorem. The aim of the AFCS conditional rendering is not simply to capture the *original* or *originally intended* content of a mathematical theorem, but instead to properly reflect *our current best understanding of that theorem.* This contemporary understanding of a theorem will typically be informed by a study of the methods employed in proving the theorem, and may be motivated by an aim towards increasing the scope of applicability of the theorem, to better identify the conditions under which it obtains, and so to more precisely identify the types of systems to which it applies. One need not claim to have accurately represented the *original* theorem, but instead to have distilled and sharpened the original result in order to better display the generality permitted in view of the methods employed in establishing that initial result.

Here we may observe that the structuralist perspectives identified in Section 2.1.1 motivate the rendering of mathematical theorems in conditional form. The *particular objects* of a system don't matter, only that they stand in the relations specified in the conditions given in the theorem. The *particular system*—the model of the antecedent conditions specified in the theorem—doesn't matter: the proof establishes that the results of the theorem apply to *any* system satisfying those conditions. The ubiquity of the structuralist perspectives among the mathematical community, along with the rise of the axiomatic method which so naturally accommodates those perspectives, may also explain why many of the results of contemporary mathematics are already expressed in conditional form.

4.2 Theorems as Schematic

Recalling the discussion of Section 1.5, a further aspect of the AFCS program is the view that mathematical theorems are best expressed as *schematic* in form, i.e., the variables ranging over models which appear in (conditional) mathematical theorems are *not* taken to fall within the scope of a universal quantifier ranging over a fixed domain of objects. The reasons offered for adopting this view of mathematical theorems as schematic may also serve to clarify some otherwise curious features of mathematical discourse. In this section, we consider why the treatment of mathematical theorems as schematic squares well with the AFCS view, and reflects some of the original structuralist motivations. In the next section we will explore in greater detail some potential alternatives to the schematic approach, and establish why these alternatives are unworkable on the AFCS program.

Consider the above-mentioned ring-theoretic result that *any ring with identity is such that every proper ideal is contained in a maximal ideal.*[3] In

[3] A *ring* is an additive Abelian group with a multiplication operation that is associative and which distributes over addition. A ring *with identity* has an element a such that $ar = ra = r$ for all elements r of R, i.e., a ring with identity has a *multiplicative identity* in

accordance with the view developed in Section 2.6.1,[4] the theorem is taken to involve a claim about *rings*, and as rings are simply identified as *models of the ring axioms*, the conditional theorem is *metatheoretic*, insofar as it concerns the properties of *models* of a collection of axioms.[5] As a schematic conditional, this theorem may be represented as[6]

$$\begin{aligned}&\text{if } (\mathcal{R} \vDash \text{the ring axioms, and has an identity})\\&\text{then } (\text{any ideal } \mathcal{I} \text{ of } \mathcal{R} \text{ is contained in some maximal ideal } \mathcal{J} \text{ of } \mathcal{R})\end{aligned} \quad (4.1)$$

Before moving to consider reasons for regarding this theorem as best understood to involve treating the variables[7] ranging over rings as *schematic*, it is helpful to consider why the apparent reference to rings should be treated as *metatheoretic*: involving the notions of a *model*, *satisfaction*, and *truth*, as discussed in Section 3.1. Contrary to this view, in pursuing his modal structuralist program Hellman deliberately rejects a (modal) translation of the theorems of arithmetic into a form that invokes metalinguistic treatments of, e.g., *ω-sequence* and *satisfaction*,[8] claiming that such a translation "…is surely an awkwardness, if not a fatal misrepresentation of arithmetic discourse" [35, p. 18].[9] Hellman does not expand on his reasons for this assessment, but this metalinguistic translation, which invokes the metatheoretic notions of *model* and *satisfaction*, seems entirely fitting given the structural-

addition to the additive identity coming from the group axioms. Note that some definitions of *ring* require that a ring has a multiplicative identity (in which case the additional qualification in this theorem is unnecessary).

[4]See especially the discussion surrounding formula 2.8.

[5]Here the axioms that determine a ring (with identity) are first order and finite in number, but this need not be the case in general.

[6]Here we again use the standard symbols for metatheoretic notions as outlined in Section 2.6.1. Note that an ideal of a ring $\mathcal{R}$ is itself a ring.

[7]The ideals of a ring are subrings, provided one uses a definition of *ring* that does not require that rings have a multiplicative identity.

[8]Accounts which proceed via the usual set-theoretic treatment.

[9]See Section 2.6.2.

ist perspectives. Recall Dedekind's observations that[10]

> ... it is clear that every theorem regarding numbers, i.e., regarding the elements n of the simply infinite system N set in order by the transformation ϕ,... possesses perfectly general validity for every other simply infinite system Ω set in order by a transformation θ and its elements ν [24, p. 48]

Dedekind, of course, considered only (informally treated) *sets* as candidate simply infinite systems, but he was also content to allow that his infamous sequence of thoughts could be taken to constitute such a system. While a contemporary structuralist may not follow Dedekind in taking Dedekind's thoughts to constitute a *set* in the contemporary sense of the term, his accommodating view *is* carried forward to the AFCS program, and *any model that one is willing to sanction* may count as a system of the relevant sort. For example, the mathematical theorems of arithmetic will apply to any ω-sequence one is willing to admit, no matter what its constitution. As ω-sequences are identified as *models* of the second-order Peano axioms, a structuralist is naturally led to adopt, even in the case of "the" natural numbers, a metatheoretic view.

Hellman, however, does not dispense with the notions of, e.g., *satisfaction*, he instead prefers to avoid treating these notions as metatheoretic in order to avoid their usual, *set-theoretic* construal.[11] Thus, despite Hellman's earlier remark about the "awkwardness" of the translation involving metatheoretic notions, instead he rejects such a translation for reasons related to the typical set-theoretic account of metatheoretic notions like *satisfaction*. First, such a translation makes, for example,

> ... number theory dependent on set theory in a way that, from a mathematical point of view, it would be desirable to avoid.

[10] See Section 2.1.1.

[11] Hellman instead uses the resources of second-order logic to express *satisfaction* directly in the object language.

> There is good motivation for understanding number theory and analysis as capable of standing on their own. Surely we should resist saddling them—as basic mathematical theories—with the problem associated with "Cantor's universe" [35, p. 14]

If—as suggested on the AFCS program—it is legitimate to take the notion of a *model* as primitive,[12] Hellman's concern on this point is acknowledged, and the problem avoided. However, Hellman is also concerned that, on a set-theoretic approach, the objects of mathematics must then be recovered as *sets*, in which case

> What will be missed is the full *generality* of structuralism: arithmetic or analysis investigates relations holding within *arbitrary* structures of the appropriate type—not just within those that happen to be recognized in a weak set theory. [35, p. 14]

In keeping with the general structuralist perspective, *any* system should suffice, not simply those which are sets.

While taking the notion of *model* (and related notions) as primitive goes some way toward addressing Hellman's second concern as well, sensitivity to this concern is also reflected in the AFCS treatment of mathematical theorems as conditional and schematic. First, in treating the theorems as *conditionals* (as Hellman himself does in his preferred modal framework), there is no commitment to the *actual* satisfaction, or, indeed, the *satisfiability*, of the antecedent conditions—involving objects like rings, metric or topological spaces—featuring in those conditional statements.[13] This ontological neutrality goes some way towards accommodating the view that there should

[12] Taking associated notions, such as *satisfaction*, as primitive as well. See the discussion in Section 3.1.

[13] Recall that Hellman's modal structural approach does later involve explicitly asserting that w-sequences, etc., are *possible*, a manoeuvre aimed at tempering the threat of vacuity. See the discussion in Section 2.6.2 concerning Hellman's program and see Section 2.6.1 for a response to concerns about vacuity within the AFCS program.

be no restrictions placed on the sorts of objects that may count as groups, ω-sequences, etc.

Unfortunately, the apparent gain afforded by the treatment of theorems as conditional in form is lost if the conditional is treated as governed by a *universal quantifier* ranging over models. Recall Awodey's remark that

> The 'schematic' element in mathematical theorems, definitions, and even proofs is not captured by treating the indeterminate objects involved as universally quantified variables, as quantification requires a fixed domain over which the range of the variable is restricted. [2, p. 59]

Thus, the "generality of structuralism" is reflected in the AFCS program by treating any variables (like $\mathcal{R}$ in theorem 4.1) which could be viewed as ranging over groups, ω-sequences, metric spaces, and the like—ranging over the "indeterminate objects" that feature in the (conditional) theorems—as schematic, thereby avoiding the treatment of those variables as ranging over a *fixed domain* of given objects. Groups need not be sets, and they need not be recovered as elements of some other, privileged collection serving as the domain of quantification.

In rejecting a single, fixed universe of mathematical objects, the AFCS proposal resembles that program proposed by Bell in [5, 6], which is also framed in the language of category theory, and which also seeks to permit *variability* in the range of acceptable universes of mathematical discourse. Bell notes that

> From the set-theoretical point of view, the term "group" signifies a *set* (equipped with a couple of operations) satisfying certain elementary axioms in terms of the elements of the set. Thus the set-theoretical interpretation of this concept is always referred to the *same* framework, the *universe of sets*. [5, p. 410–411]

Bell's suggested program can be described as accommodating the "generality of structuralism" via a loosening of this restriction on the class of models: instead of being limited to the universe of sets,[14] one may instead appeal to the universe provided by a *topos*.

> Any topos may be regarded as a mathematical domain of discourse or 'world' in which mathematical concepts can be interpreted and mathematical constructions performed. [6, p. 238]

The *topos of sets*, then, is but one possible universe of mathematical objects, and Bell's proposed program allows for groups, rings, and other types of objects to be recovered in *any* topos universe, not just the topos of sets. Groups, for example, can be interpreted on Bell's view as objects with suitable arrows in any topos, where a group may be characterized in an arbitrary category (and so, *a fortiori*, an arbitrary topos) in the manner described in Section 3.2.2. However, recall that the AFCS proposal is to require only that a group be treated as

1. a model of the appropriate sort (in this case, a model of the group axioms), and

2. an *object* in a category.

When the category-theoretic method of characterizing a group was presented in Section 3.2.2, it was noted[15] that such an approach was *permitted*, but not *required* on the AFCS view. Requiring that, for example, models of the group axioms are *objects with associated arrows in a topos* fares no better in respecting the open-ended nature of the structuralist perspectives than requiring that a model of the group axioms be a particular sort of *set*.

[14] Bell also notes that there is also a sense in which "the" cumulative hierarchy of sets admits (or suffers from, depending one one's aims!) a degree of variability—as witnessed by, e.g., the Löwenheim-Skolem theorem, or Cohen's independence results. See [5, 6] for a discussion.

[15] See the discussion in Section 3.3.

In taking models as primitive, the AFCS proposal necessarily treats models of axioms—groups, topological spaces, ordered fields, etc.—as in some sense *opaque*. Whatever their origins, whatever their "home address", an essential component of the AFCS view is that models be framed in the language of category theory in order to best capture their structural properties, the properties of mathematical interest. Taking models as primitive, as described in Chapter 3, is intended to accommodate *any* sort of "internal structure" that may be supposed for the model(s) in question.

4.3 Domains

The AFCS program agrees with Hellman's aim to respect the "full generality of structuralism", and seeks to do so by

1. not restricting the sorts of entities which can be viewed as models of the various axiomatic systems, and so *not committing to a single, definitive universe of such models*, as well as

2. not restricting the universes—the collections of models—which may provide the context for the interpretation of a conditional, schematic theorem of mathematics.

The purpose of this section is to further examine the reasons for, and consequences of, these two aspects of the AFCS view.

4.3.1 The Single Domain Option

Concerning this first aspect we are again reminded of Dedekind's approach, which could be viewed as leading away from the study of *the* natural numbers to the study of *simply infinite systems*. Where before there may have been thought to be a single, definite object satisfying a given axiomatic definition, the mathematician is led to recognize the potential for a multitude of distinct objects satisfying that definition; the definition alone does not suffice to fix a single model. This process of replacing the *constant* with the *variable* has been identified by Bell as leading naturally to the category-theoretic approach, which "may be said to bear the same relation to abstract algebra as the latter does to elementary algebra" [5, p. 409]. For the purpose of the current discussion, it suffices to note that one might consider the replacement of constancy with variability to be a hallmark of the structuralist approach to mathematics, and so a view which purports to be a *structuralist* view, but which characterizes mathematics as the study of a *fixed* collection of objects (models, in the present case) seems *prima facie* at odds with the structuralist position. In the case under consideration, why should variability stop at the level of models?

It is useful here to contrast the AFCS view with the set-theoretic structuralist view (STS) considered by Hellman in [39]. On such a view, *systems* (groups, fields, etc., and so *models*) are identified as types of *sets*, and one may hold that, for example, the cumulative hierarchy is the universe of *all* models. However, adopting this view is then to deny that the structuralist perspectives should be applied to the set-theoretic framework that functions at the meta-level.[16] As Hellman explains,

> Here we encounter a massive exception to the structuralist point of view, in that, on its face-value interpretation, set theory itself is *not* treated structurally: its axioms are not understood as defining conditions on structures of interest but are taken as assertions of truths in an absolute sense. [39, p. 540]

Of course, those who defend set-theoretic structuralist programs are often clear on this point. Mayberry, for example, observes that on his view

> ...the *logical* dependence of axiomatics on the set-theoretical concept of mathematical structure requires that set theory already

[16]See [79] for a discussion of the issues associated with the shift from *algebraic axioms at the object level* to *assertory axioms at the meta-level*.

> be in place before an account of the axiomatic method, understood in the modern sense of axiomatic *definition*, can be given. It follows necessarily, therefore, that *we cannot use the modern axiomatic method to establish the theory of sets.* [58, p. 7]

As a consequence of this view, the theory of sets developed by Mayberry constitutes only "...a partial description of the absolute universe of sets in which *all* conventional structures, including all models ... are to be found" [58, p. 243].

A set-theoretic structuralist view might, at the very least, be hoped to at least partially reflect the "open-ended" aspect of the structuralist perspectives, evident in some of the earliest versions of structuralism. Recall that Dedekind opens [24] with the sentence "In what follows I understand by *thing* every object of our thought", and he goes on to claim that

> It very frequently happens that different things, $a, b, c, \ldots$ for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a *system* S; we call the things $a, b, c, \ldots$ *elements* of the system S, they are *contained* in S; conversely, S *consists* of these elements. Such a system S (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing... [24, p. 21]

The view that *any* entity—any *thing*—can feature as an object in a system, and that *any* suitably-structured system can count as a simply infinite system, a group, metric space, etc., lies at the heart of the structuralist view as characterized in Section 2.1.1. One who accepts the structuralist perspectives, then, should prefer a program that preserves these principles—as the AFCS program seeks to do—to any that do not.

Unfortunately, the familiar logical and set-theoretic paradoxes point to our having to tread carefully when treating a *single, all encompassing universe* that would then be expected to contain *all* models. Such a universe is typically barred from being treated as an element of itself, on pain of contradiction, and so it seems that the universe, which was intended to be *all encompassing*, must instead be *restricted*. Naive comprehension is replaced with *restricted* comprehension; some collections may be deemed "too large" to be considered sets, and terms like *species* or *proper class* may be introduced. Such restrictions, already at odds with the structuralist motivations, introduce a further tension between the *principles encoded* in typical characterizations of the universe of sets—the universe of models—and the universe itself. On a theory such as ZFC, the axiom (schema) of comprehension is restricted in such a way that some problematic totalities are not deemed sets. Thus, Hellman is led to wonder "...what *prevents* the "collectibility" of "all sets".... And why aren't such collections subject to operations analogous to those of set theory itself, including formation of singletons, power collections, and so on?" [39, p. 540]. As Mac Lane remarks,

> Understanding Mathematical operations leads repeatedly to the formation of totalities: The collection of all prime numbers...the manifold of all lines in 3-space...the set of all power series expansions for a function (its Riemannian surface) or the category of all topological spaces. There are no upper limits.... This is the idea of a *totality*, and these are some of its many formulations. After each careful delimitation, bigger totalities appear. No set theory and no category theory can encompass them all—and they are needed to grasp what Mathematics does. [51, p. 390]

Accounts of a single universe of mathematical objects, typically a universe of sets (which in the case at hand, would be treated as a *category* of sets), are generally required to draw boundaries around their proposed universe for reasons of consistency, but it is *the presence of the boundaries themselves* that is at odds with the generality present in the key structuralist perspectives.

4.3.2 The Multiple Domain Option

Perhaps this generality can be obtained in following Bell to admit not a *single* universe of models, but rather a whole class of such universes. On the single domain option, it was observed that there was a tension between the AFCS aim—motivated by the structuralist perspectives—to allow *anything one was willing to sanction* to stand as a *model* of some axiomatic definition, and the restrictions required in order to articulate a single domain view that does not lead to inconsistency. It may be hoped that allowing variability in the domains might permit one to preserve the intended generality of the notion of *model* intended on the AFCS view. Dedekind's thoughts may yield a model of the second-order Peano axioms, and the symmetries of the cube should be able to stand as a model of the group axioms. While no one, single domain may contain both these as well as *all* other models, allowing multiple domains, none of which are held to contain *all* models, may yet permit the intended generality.

On Bell's program, each topos provides a mathematical universe, and the models of various axiomatic systems, the groups, rings, natural numbers systems, etc., can be recovered in these topos universes,[17] with no one, privileged universe. Note, however, that the AFCS program does not use categories in the way that Bell uses toposes in developing his view.[18] In particular, on the AFCS view the language of category theory is not used in *defining* the objects of mathematical interest (models of axioms characterizing groups, vector spaces, and so forth), it is instead taken as a framework in which to *relate* those entities, whatever their origins. A group, for example, on the AFCS view is a model of the group axioms, but is not necessarily a set, nor necessarily an object with associated arrows in a topos. Given a particular collection of models meant to serve as the background for the interpretation

[17]Note that the existence of a natural number object is independent of the other topos axioms. See the discussion in Section 3.2.3.

[18]See [6] for details of the approach via the notion of a *local set theory*.

of a mathematical theorem, the AFCS view requires only that those models be treated as *objects* in a *category.* The ambient category for the interpretation of a theorem in group theory might be a category of groups where each group is a set in ZFC, or a category of monoids, where each monoid is a string rewriting system, etc., but no constraints are placed on the description of the *internal* structure of such entities, save their being (candidate) models of the relevant sort.

Given that the AFCS view requires that models be treated as objects in a category (and so, that variables like $\mathcal{R}$ in theorem 4.1 range over objects in a category), each "universe" on the AFCS would then be a category (of models). A multiple domain view appropriate to the AFCS program would thus involve providing some characterization of the "realm" of *categories.* To this end, the theory of multiple domains appropriate to the AFCS program might be expected to resemble the *category of categories* proposed by Lawvere in [50], and further refined in, e.g., McLarty's [59]. Of course, one need not be committed to the view that *all* categories may serve as categories of objects that are models of some axiomatic definition, but certainly those categories which can be viewed as *categories of models* would be expected to be contained in the category of *all* categories.

A *category of categories* is a category whose *objects* are themselves categories, and whose arrows are *functors*[19] between categories. Assuming the availability of an identity functor for each category and the ability to form functor composites, it is an immediate consequence of the definition of a functor that functors between categories satisfy the conditions on arrows in a category (where the categories themselves are taken as objects), and thus a collection of categories and functors between them together constitute a *category* of categories.

Some of the remarks made in the presentation of Lawvere's [50] suggest

[19]Recall again that a *functor* F is a function between categories that maps objects to objects and arrows to arrows while respecting domains, codomains, identities and composites.

that the intended generality concerning the composition of models may indeed be available in a category of categories, as "By a category we of course understand (intuitively) any structure which is an interpretation of the elementary theory of abstract categories" [50, p. 4].[20] Thus, it would seem we are permitted in granting that a category of (models of the axioms for) groups may contain the group of symmetries of a cube, or whatever else we are inclined to admit as a candidate model of the group axioms. However, such optimism is short-lived, as the move away from the "intuitive" conception to the more definite (axiomatic) presentation of the category of categories requires that one abandon at least some of these intuitions. As in the single domain theory, the danger of paradox requires modifications that are at odds with the desired generality. The category of categories is, after all, a *category*. Is it *an object of itself*? As Hellman observes, "we certainly had better avoid such things as 'the category of exactly the non-self-applicable categories'!" [40, p. 157]. McLarty offers a reply to Hellman's concerns[21] about the axioms Lawvere presents in [50], "The Category of Categories as a Foundation for Mathematics", (sometimes referred to as the *CCAF* axioms)[22] but his reply to these concerns does not bode well for the preservation of the generality sought on the AFCS program.

> When we axiomatize a metacategory of categories by the axioms CCAF, the categories are *not* 'anything satisfying the algebraic axioms of category theory'. . . They are *anything whose existence follows from the* CCAF *axioms.* [63, p. 52]

Such a position may be required for consistency (although it is not immediately clear that, for example, admitting a category of all categories as an object in the category of categories would lead to inconsistency).[23] However,

[20] Lawvere's *elementary theory of abstract categories* consists of the (first-order) axioms that define a category.

[21] These concerns are also raised in [38].

[22] See also McLarty's strengthening of those axioms as presented in [59].

[23] See McLarty's brief remarks in [63, p. 52] and [59, p. 1243], in which he speculates

McLarty goes on to note that, "even if there is such a category [of "all" categories] it will not be *the* category of *absolutely* all categories" [63, p. 52]. Again, then, we are faced with boundaries that run contrary to the generality that motivates the AFCS view.

What, then, of any category containing the group of symmetries of the cube? Such a category should stand as a possible background for the interpretation of a theorem concerning groups. Clearly such a category is not something "whose existence follows from the CCAF axioms", as the axioms say nothing about the nature of the objects involved in any category whose existence can be established on their basis. It seems the best one may hope for is that, for a given category $\mathbf{C}$ of models of the group axioms that contains the group of symmetries of the cube, there is some category $\mathbf{C}'$, whose existence *does* follow from the CCAF axioms, such that $\mathbf{C}$ is isomorphic to $\mathbf{C}'$. That is, we do not have the existence of $\mathbf{C}$ as an element of the category of categories, but we may get the existence of the next best thing, a category $\mathbf{C}'$, isomorphic to $\mathbf{C}$. This situation seems at least partly in keeping with the structuralist perspectives of Section 2.1.1; if we view the given category $\mathbf{C}$ of groups as a *system*, that our system involves an element (i.e., an object) that is the group of symmetries of the cube shouldn't matter: another similarly structured system, with a different element playing the "role" of the group of symmetries of the cube, should suffice. Doesn't the Structural Properties Theorem establish that we can just as easily work with $\mathbf{C}'$ instead of $\mathbf{C}$?

There is, however, a crucial distinction between the group axioms that serve to *define* what counts as a group, and the category axioms which serve to *organize the framework* in which *models* of the group axioms are to be considered. Once the axioms for a type of structure—like the axioms for a group—have been provided, models of those axioms may be taken to have any "internal" structure that one is inclined to admit. For example, one might be

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that a category of categories may be admitted in the manner in which a "set of all sets" can be admitted in Quine's *New Foundations* axioms [68].

interested in groups recovered as *sets* in ZFC, or one might consider groups recovered as *rigid motions of platonic solids*. In either case, one might then take these objects, along with their associated homomorphisms as arrows, as the background framework—the *ambient category*—for the interpretation of some group-theoretic result. *Once this choice of background framework has been fixed, the structural properties of the objects have also been fixed*. The categories in question are categories whose objects are particular sorts of *models*, and the *structural properties* of these models are given by interpreting the language of category theory with respect to the chosen framework. In the language of category theory, an object A having a single non-identity arrow $f : A \to A$ corresponds to a structural property of an object A, as it can be expressed in the language of category theory without names or parameters. In the case where our groups are taken to be *sets in ZFC*, the corresponding property is one concerning non-trivial automorphisms, where an automorphism is a particular sort of function mapping *sets* to *sets*. If our ambient category is instead that in which groups are taken to be *rigid motions of the platonic solids*, the claim again concerns automorphisms, where in this case the automorphisms are functions mapping *rigid motions of a solid* to *rigid motions of a solid*. In the former case, the *structural properties concern sets*, in the latter case, the *structural properties concern rigid motions of platonic solids*. In short, *the language of category theory must be interpreted in the chosen ambient category in order to determine the structural properties*: different ambient categories yield different structural properties. Even though there may be an ambient category of *groups in ZFC* that is isomorphic, as a category, to the ambient category of *groups of rigid motions of the platonic solids*, each ambient category determines distinct structural properties. In moving from $\mathbf{C}$ to $\mathbf{C}'$, then, we *change the subject*.

Of course, while the structural properties of two distinct ambient categories may themselves be distinct, if the categories in question *are* isomorphic, those structural properties lie in a natural correspondence given by the representation of those properties in the language of category theory. In the cases discussed above, we would be able to move from structural properties of groups consisting of sets to structural properties of groups consisting of rigid motions via the representation of those properties in the language of category theory. As outlined above, the structural property corresponding to an expression in the language of category theory concerning the existence of a single non-identity arrow for an object A can be *interpreted* in one ambient category or another. Consequently, the *representation* in the *language of category theory* of the properties in question thereby allow us to identify properties in one category that correspond to property in another. Perhaps, then, it doesn't matter that the move from $\mathbf{C}$ to $\mathbf{C}'$ "changes the subject"; we may be able to translate back to our original category. Again, doesn't the Structural Properties Theorem license exactly this sort of move?

A reply to this question gets to the heart of the problem with the multiple domains approach via a *category of categories*. Taking $\mathbf{C}$ and $\mathbf{C}'$ as above, The application of the Structural Properties Theorem requires that both $\mathbf{C}$ and $\mathbf{C}'$ are objects *in the same (ambient) category*. That theorem establishes a connection between objects that are isomorphic in the (ambient) category, and the structural properties (determined by that ambient category) of those objects. If the ambient category is taken to be the category of categories, then we are *prevented* from applying the theorem: the shift from $\mathbf{C}$ to $\mathbf{C}'$ was suggested precisely because $\mathbf{C}$ is presumed *not* to be an element of the category of categories! This situation is much like the difficulty in working with ZFC to establish a theorem about (pure) sets, and then hoping to appeal to that theorem when discussing the set of books on a desk.[24]

Would it be legitimate simply to *add* this category $\mathbf{C}$ to the category of categories? This new, extended category of categories would then contain all categories in question, and in similar circumstances involving other cate-

[24]Perhaps something akin to the use of *urelements* in set theory could also be used with the category of categories, but it is not immediately clear that this method, if possible, would address the present concern.

gories we might simply add categories as needed. This option, of course, is untenable. The axioms for the category of categories were proposed in order to circumscribe our intuitive understanding of the universe of categories, and were initially proposed as a part of a foundational approach aimed at producing "a single system of first-order axioms in which all the usual mathematical objects can be defined and all their usual properties proved" [50, p. 1]. Despite Lawvere's admitted foundational concerns (shared by McLarty), such a systematization is directly relevant to the multiple domain approach on the AFCS program, which requires an answer to the question "*what are the categories of models?*". In order to maintain the AFCS commitment to strict neutrality with respect to the sorts of entities that can *be* models, this question concerning categories *of models* is necessarily subsumed under the question: "*what are the categories?*". Any answer to this question, of course, is aimed at *sharpening and improving upon* on our intuitive conception. The well-known difficulties encountered when trying to tame naive views in developing early systems of modern logic and set theory show just how carefully these systems must be developed, and just how cautious we must be when working with intuitive views. It runs counter to the principal aim of these related projects—projects aimed at producing a more precise, systematic approach to address the delicate questions of category-theoretic existence—if we are simply to ignore the systems that have been developed and return to the intuitive conception whenever it proves convenient!

It seems, then, the proponent of the AFCS view is unable to accept either the single domain view or the multiple domain view. Granting that quantification must be quantification over a *fixed domain*, the situation may be expressed as follows. A mathematical theorem, like that in theorem 4.1, involving a variable $\mathcal{R}$, can be represented as having the form

$$\text{if } \mathcal{R} \vDash \text{Ring} + \text{Id then } \mathcal{R} \vDash M,$$ [25]

[25] Recall the discussion involving expression 2.7 in Section 2.6.1.

which may be more succinctly represented as

$$F(\mathcal{R}) \to G(\mathcal{R}), \tag{4.2}$$

with the obvious definitions of F and G. Accordingly, the single domain option involves treating the variable $\mathcal{R}$ as ranging over a single, fixed universe of *all* models, in which case we might represent the theorem as having the form

$$\forall \mathcal{R}(F(\mathcal{R}) \to G(\mathcal{R})).$$

On the AFCS program, that single domain would be a category, and so a category of *all models*. Calling this category of all models $\mathbf{M}$, we can then describe the theorem as being of the form

$$\forall \mathcal{R} \text{ in } \mathbf{M}(F(\mathcal{R}) \to G(\mathcal{R})). \tag{4.3}$$

The truth of (4.3) would then involve a claim about *all models*, but the difficulty in respecting the AFCS silence on *what counts as a model*, along with the familiar difficulties associated with providing a consistent account of such totalities (in particular, the difficulty in providing an account that avoids drawing boundaries running contrary to the open-ended character of mathematics highlighted by Mac Lane and Hellman), seems to leave us without a characterization of $\mathbf{M}$ sufficient to justify (4.3). That is, without the ability to make substantial claims about $\mathbf{M}$, it is difficult to see what might justify the assertion that (4.3) is *true*.

On the multiple domain view, we attempt to remedy this situation by allowing $\mathbf{M}$ to vary. As each such $\mathbf{M}$ is taken to be a *category* of models, $\mathbf{M}$ in (4.3) can be treated as a variable, ranging over the collection of *all* categories of models. A mathematical theorem, then, might be represented as having the form

$$\forall \mathbf{M}(\forall \mathcal{R} \text{ in } \mathbf{M}(F(\mathcal{R}) \to G(\mathcal{R}))). \tag{4.4}$$

Again, the AFCS program's neutrality on the status of models requires that we instead investigate the collection of all possible categories (as we lack any prior criteria according to which we may distinguish those categories which can be viewed as categories of *models*), independently of the nature of their objects, which leads us to appeal to an account of the *category of categories*. Here too we find that concessions required to carefully articulate the (presumed) characteristics of the category of categories prevent us from achieving the desired generality in the interpretation of mathematical theorems, and we are led to reject (4.4) as properly reflecting the content of a mathematical theorem on the AFCS view.

Having rejected both (4.3) and (4.4), we are left to treat mathematical theorems as having the *schematic* form of (4.2). If instead we were to treat theorems as instead involving a quantifier ranging over *all models*, and assuming that the interpretation of quantified expressions involves appeal to a fixed domain over which the quantifier ranges,[26] we would arrive at the form shown in (4.3), which was rejected in virtue of the incompatibility of the AFCS program and the single domain view. If instead we pursue the multiple domain view, which again admits quantification over the (now varying) domains, a theorem would have the general form shown in (4.4), which was rejected in virtue of the incompatibility of the AFCS program and the multiple domain view.

Note that, had the single domain view proved acceptable, theorems having the form of (4.3) could have been ascribed a truth value in accordance with the standard semantic account. Similarly, had the multiple domain view proved tenable, theorems of the form (4.4) could also have been ascribed a truth value. The situation is somewhat analogous to treating a first-order expression of the form $Fx \rightarrow Gx$. If we introduce a quantifier to form the expression $\forall x(Fx \rightarrow Gx)$, then with a fixed domain, and on the intended

[26]It remains to be investigated whether or not this claim is legitimate: does the intelligibility of a quantifier always require a fixed domain over which the quantifier is taken to range?

interpretation, that closed formula would express a statement, and so would obtain a truth value. On this analogy, the multiple domain view corresponds to allowing the *interpretations* to vary, in which case we may have been able to identify those sentences which are *true on every interpretation*, i.e., the logical truths, corresponding to the statement "On all interpretations, $\forall x(Fx \to Gx)$ is true".

Given, then, that the AFCS view leads to the treatment of mathematical theorems as schematic, can they be ascribed a *truth value*? If not, what exactly is established by a *mathematical proof*?

4.4 Truth and Proof

Chapter 3 concluded with Hellman's claim that category theory either falls back on a set-theoretic account of *satisfaction*, or "...we just give up on the notion of mathematical truth as anything beyond deductive logical validity" [40, p. 158]. It was argued in Chapter 3 that we may legitimately take the notion of *model* (and so, the notion of *satisfaction* on which it depends) as primitive, but the status of the notion of *mathematical truth* remains to be explored.

Recall that taking the notion of model as primitive on the AFCS program is intended to best capture what could be called the *metaphysical neutrality* of the structuralist perspectives identified in Section 2.1.1. The proponent of the AFCS program takes mathematics to be concerned with the *structural properties* of any model one is willing to admit, and so any principles which either implicitly or explicitly serve to limit the class of models that can be treated on the AFCS view are to be avoided. We arrive, then, at an immediate difficulty in trying to unpack the notion of "deductive logical validity": *deductive validity* in which system of logic?

Note that *deductive validity* is here being applied to the conditional sentence corresponding to a mathematical theorem (and so, being applied to a *sentence* rather than an *argument*). For present purposes, one may assume that deductive validity is framed in terms of the notion of "following from", which Beall and Restall express in their principle (V) (where "V" is used for *validity*).

> (V) A conclusion, A, *follows from* premises, Σ, if and only if any case in which each premise in Σ is true is also a case in which A is true. [4, p. 476]

A *valid argument* is then taken to be a collection of sentences Σ along with a sentence A for which condition (V) obtains, and a *sentence* A is taken to be valid provided the argument from *no premises* to A is valid, i.e, provided A is true in all cases. A *logic*, then, involves a specification of the relevant sorts of *cases* to be considered:

> To use (V) to develop a *logic* you must specify the cases over which (V) quantifies, and you must tell some kind of story about which kinds of claims are true in what sorts of cases. For example, you might give an account in which cases are *possible worlds*... On the other hand, you might spell out such cases as set-theoretic constructions such as *models* of some sort. [4, p. 477]

Of course, in light of the earlier arguments of this chapter, the AFCS program requires that deductive validity *not* be characterized as involving quantification over all models, and so an account of validity appropriate to the program must be given in some other manner.

Already in the course of developing the AFCS program we have had occasion to discuss first- and second-order classical logic, as well as modal logic (the latter surfacing in Hellman's structuralist program). The system of intuitionistic logic, initially motivated by Brouwer's Intuitionist program in the philosophy of mathematics, is another familiar option. Unfortunately, as debates between intuitionistic, classical, and mathematicians inclined to even more exotic systems have made clear, these systems themselves may be held to embed metaphysical principles that the AFCS program seeks to avoid. As Heyting's INT. remarks to the classical mathematician, "Your argument [for excluded middle] is metaphysical in nature…It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not" [42, p. 2].

Nonetheless, given the variety of logical systems on offer, one might hope that some particularly weak logic (perhaps some system of free minimal logic) may be able to accommodate the strict neutrality requirements of the AFCS program. Thus, it may be possible to admit certain mathematical theorems (for example, the theorem that *in any group G, the identity element of G is unique* may be one such theorem) as valid, i.e., *logically true*, but such metaphysically neutral theorems will be expected to be the exception, rather than the rule.

Consider, for example, the ring-theoretic theorem discussed earlier: "In a ring with identity every proper ideal is contained in a maximal ideal" [27, p. 254]. A *proof* of this result will go some way to clarifying the sense—if any—in which this may be considered a *logical truth*, and there are several things to note about the proof of theorem. The proof found in [27] runs essentially as follows.

Proof. Consider a ring R with identity and I a proper ideal of R. Let $\mathcal{S}$ be the set of all proper ideals of R containing I. $\mathcal{S}$ is non-empty (as $I \in \mathcal{S}$) and is partially ordered by inclusion. If $\mathcal{C}$ is a chain in $\mathcal{S}$, then let $J = \bigcup_{A \in \mathcal{C}} A$. Then for any elements $a, b \in J$, there are ideals A and B in $\mathcal{C}$ such that $a \in A$ and $b \in B$, where either $A \subseteq B$ or $B \subseteq A$. Without loss of generality, assume $A \subseteq B$, then as $a, b \in B$ and B is an ideal, we have $a - b \in B$, and so $a - b \in J$. A similar argument establishes that J is closed under (left and right) multiplication by elements of R, and so J itself is an ideal. Further, J is proper, as if $1 \in J$ again we must have some $B \in \mathcal{C}$ such that $1 \in B$, contrary to the definition of $\mathcal{S}$ (as $B \in \mathcal{C} \subseteq \mathcal{S}$). Consequently, $J \in \mathcal{S}$, and is clearly an upper bound for $\mathcal{C}$. Thus, any chain $\mathcal{C}$ in $\mathcal{S}$ has an upper bound, and so by Zorn's Lemma[27] there is a maximal ideal in $\mathcal{S}$ (which is thus a proper ideal containing I), as desired. □

Of note, this proof has the following characteristics.

1. It is informal (and the logical system unspecified).

2. It makes use of *Zorn's Lemma*, a principle (classically) equivalent to the Axiom of Choice.[28]

3. It uses various set-theoretic principles (and notation) without clarification, and so potentially with imprecision. For instance, if a ring is intended to be treated as an ordered triple $\langle R, +, * \rangle$, then the definition of J does not yield a ring (and so, does not yield an ideal), as a union of ordered triples is not itself an ordered triple.

Each of these characteristics is an obstacle to the treatment of this theorem as *valid*. An unfortunate observation for those wishing to treat mathematical proofs as valid, then, is that *proofs with characteristics of this sort are standard fare in mathematics*.

Certainly one course of action is to seek to remove these obstacles. For example, the use of Zorn's Lemma in the proof suggests that Zorn's Lemma could simply be incorporated into the antecedent of the conditional form of this theorem. Such a move may indeed be held to yield a more accurate statement of the theorem (for those keen to track the use of Zorn's Lemma), as the domain of applicability of the theorem is then more clearly stated. Of course, Zorn's Lemma is likely selected for such a move because of its link to the Axiom of Choice, but the thought that *moving all such "controversial"*

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[27] *Zorn's Lemma* states that if P is a non-empty, partially ordered set in which every chain has an upper bound then P has a maximal element.

[28] Interestingly, the principle is not intuitionistically equivalent to the Axiom of Choice (which is objectionable to Intuitionists as it implies the law of excluded middle). Indeed, Zorn's Lemma can be shown to have, in a precise sense, no "non-constructive" logical consequences. See [7, p. 12].

principles into the antecedent of the conditional (thereby serving as a restriction on the range of applicability of the theorem) allows *all mathematical theorems to be characterized as logical truths* again depends on the viability of a *neutral* logical framework in which to construct the proof connecting the antecedent to the consequent. Further, if we are to recover the bulk of modern mathematics as logical truths in this manner, the logical system must also have sufficient expressive power to encode the relevant mathematical notions. While such a system may be *possible*, the considerable difficulties in even being able to clearly state the requirements for such a system render this option unappealing at best.

In some sense, then, the state of contemporary mathematics may be described as particularly hostile to this sort of recovery of mathematical theorems as *logical truths*. As Awodey remarks,

> The laws, rules, and axioms involved in a particular piece of reasoning, or a field of mathematics, may vary from one to the next, or even from one mathematician or epoch to another. The statement of the inferential machinery involved thus becomes a (tacit) part of the mathematics; functional analysis makes heavy use of abstract functions and the axiom of choice, some theorems in algebra rely on the continuum hypothesis; many arguments in homology theory are purely algebraic, once given the non-algebraic objects that they deal with; theorems in constructive analysis avoid impredicative constructions; nineteenth-century analysis employed other methods than modern-day analysis, and so on. The methods of reasoning involved in different parts of mathematics are not 'global' and uniform across fields or even between different theorems, but are themselves 'local' or relative. [2, p. 56]

Given the difficulties faced in attempting to recover mathematical theorems as logical truths, which options remain? One way of framing the difficulty encountered in this chapter is to note that there is a discrepancy between the mathematical methodology—proof—and the anticipated use of the resulting *proven theorems*. A fundamentally structuralist aspect of the AFCS view is that, while the theorems themselves involve no commitments to the existence of groups, sets, topological spaces, etc., those theorems can be used to yield correct results in their *application*, that is, *when one supposes a model*. While the theorem above involves no commitment to the existence of rings, one who is concerned to study the integers constructed, for example, as sets in ZFC, can correctly conclude that every ideal in the ring of integers is contained in some maximal ideal. If points in physical space can be taken to model the axioms for a particular type of metric space, then the area of a region can be correctly calculated using the methods of calculus. *Truth* does enter the picture, then, but only as part of the *application* of a mathematical theorem: one takes it to be true that there exists a model of a certain sort, and concludes that such a model has the property that features in the consequent of the theorem.

The AFCS program takes seriously the possibility that our ends exceed our means, and so theorems once thought proven may need to be rejected. If we were able to produce a ring with identity and an ideal of that ring not contained in a proper ideal, we would have reason to return to study the supposed proof of that theorem in the hopes of understanding how our reasoning failed in the case at hand. The proof of a mathematical theorem is not simply produced, then filed away never to be viewed again, the theorem simply added to the list of those that have been proved. A previously accepted proof may be shown to be flawed, and unanticipated cases may also lead us to scrutinize theorems that had otherwise seemed correct. This picture of mathematical development has been persuasively presented by Lakatos in his *Proofs and Refutations* [46], in which he charts the development of Euler's Formula. Euler's Formula states that, for all polyhedra, $V-E+F=2$, where V is the number of vertices, E the number of edges, and F the number of faces.[29] The picture of the evolution of a mathematical theorem is precisely in keeping with the AFCS view, and the process of "updating" mathematical theorems[30] to best reflect our current, best understanding of that theorem fits well with the view Lakatos presents.

Another way to characterize the AFCS perspective on the status of mathematical truth and mathematical proof is to borrow Dummett's notion of the *harmony*, as presented in [26], which concerns a sort of balance between the introduction and elimination rules for a logical constant. As Dummett explains,

> Any one given logical constant, considered as governed by some set of logical laws, will satisfy the criterion for harmony provided that it is never possible, by appeal to those laws, to derive from premisses not containing that constant a conclusion not containing it and not attainable from those premisses by other laws that we accept. [26, p. 219]

Thus, harmony for a logical constant is a property related to that constant's behaving as a conservative extension of the language. Relevant to the discussion here, Dummett goes on to remark that

> The requirement that this criterion for harmony be satisfied conforms to our fundamental conception of what deductive inference accomplishes. An argument or proof convinces us because we construe it as showing that, given that the premisses hold good according to our ordinary criteria, the conclusion must also hold *according to the criteria we already have for its holding*. [26, p. 219]

[29] It is perhaps worth note that I first encountered this theorem in a course on *graph theory*, the theorem having been recast to concern *planar graphs*, with not a polyhedron in sight!

[30] As described in Section 4.1.

With this account of deductive inference to hand, we see why the proponent of the AFCS program may be required to abandon the claim that mathematical theorems are *logical truths*: the proof that establishes a theorem may not necessarily secure the conclusions warranted by that theorem's usage in *every* context in which it may be used. Despite deliberate silence on the status of models, *there remains the possibility that implicit assumptions figuring in the proof of a theorem may not be legitimate for the models figuring in some application of the theorem.*

Such tacit assumptions may appear in a proof, and identifying those assumptions may be made more difficult by the informal presentation of the proof itself. It is intended that a mathematical theorem be broadly applicable: establishing that, for example, a result holds for *all* rings, whatever their "internal constitution", but the methods employed in producing the proof may have implicitly restricted the domain of applicability. Do the classical rules of inference employed in a proof apply to *all* rings treated in *any* category? When appealing to the Axiom of Choice in the course of a proof, do we implicitly restrict our theorem, or does the theorem remain "universally" applicable? In short, the silence on the status of models adopted on the AFCS view seems to prevent one from being justified in claiming that a proof secures the unrestricted scope of applicability of a mathematical theorem. Silence is maintained in order to admit models of *any sort*, but this silence also appears to prevent one from claiming that *proof techniques are sound* with respect to reasoning about models of *any* sort.

Much of the preceding discussion has not involved the notion of a category: how do categories fit into the discussion of this section? One of the central claims of this work is that the language of category theory is particularly well-suited to express the content of a mathematical theorem from the perspective of the structuralist. It is important to note that this *need not* involve rewriting theorems so that they are expressed in the language of category theory; the view is not "revolutionary" in this sense. What instead is being suggested is that, given a mathematical theorem, *our current best account of the structuralist view of the content of that theorem is expressed in language of category theory.* The language of category theory is thus employed in *expressing the content* of a mathematical theorem, but the theorems themselves need not be translated into that language.

It is likely that the most unpalatable aspect of the view sketched here is the apparent inability to secure the truth of the theorems of mathematics. Happily, this is one aspect of the view that is most readily abandoned. The view that mathematical theorems cannot be taken to be true is a view that seems to be a consequence of the deliberate neutrality on the status of models. As it is common for accounts of validity to proceed via an appeal to models (typically sets in a some naive set theory), it is not surprising that a view which aims to maintain neutrality on the status of models encounters difficulties with the notion of validity. There is, perhaps, some comfort to be found in the observation that some prominent mathematicians (and, certainly, one of the most prominent category theorists) of the past century also held the view that mathematical theorems are not true. On Mac Lane's view, mathematical theorems can be *correct*, but not *true*. Mac Lane holds that

> This view means that the philosophy of Mathematics need not involve questions about epistemology or ontology. If Mathematical theorems do not assert truths about the world, we need not inquire as to how we know or would come to know such truths. (We of course *do* need to inquire how we recognize a correct proof, but getting the recognition is a major part of advanced education in Mathematics, and is usually not considered as part of epistemology.) This observation means that the philosophy of Mathematics cannot be much advanced by many of the books entitled "Mathematical Knowledge", in view of the observation that such a title usually covers a book which appears to involve little knowledge

> of Mathematics and much discussion of how Mathematicians can (or cannot) know the truth. [51, p. 443–444]

The proponent of the AFCS view, then, is at least in good company!

Many, of course, will not be satisfied with a view that cannot recover an account of mathematical truth, and will consider this shortcoming a sort of *reductio ad absurdum* of the AFCS view. Those who do not feel as though they have gone too far down the rabbit hole, though, may wonder: if we refrain from calling our mathematical theorems *true*, are they any less useful? If mathematical theorems are not true, is the *science of mathematics* in any way diminished?

Chapter 5

Conclusion

> *In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of.* [50, p. 1]

> *We have no objection against a mathematician privately admitting any metaphysical theory he likes...* [42, p. 2]

What is *mathematics*? The preceding chapters contain one articulation of a structuralist response to this question: mathematics is the science of structure. The particular sharpening of the structuralist position on offer here is that *mathematics is the science of structural properties*, and the particular program developed here has been dubbed *anti-foundational categorical structuralism*.

The structure of this work can be roughly considered to involve an account of each of the terms in the name of the program. The rise of the use of axiomatic definitions in characterizing mathematical terms leads naturally to the key *structuralist* perspectives identified in Section 2.1.1. Axiomatic definitions allow for multiple instantiations, many instances of the sort of entity defined. However, it not clearly required that we recover those entities, i.e., those *models*, as objects of a particular sort: as sets, as physical objects, as entities of any other restricted collection. Attempts to constrain the admissible collection of models often encounter technical difficulties, but even if this were this not the case any such constraints are in tension with the generality that is, in part, characteristic of the structuralist perspective. Indeed, part of the success of the axiomatic method is a consequence of exactly this lack of specificity: we leave open the possibility of applying theorems in unexpected areas, in contexts that had not been anticipated when those axiomatic definitions were produced. In preserving this open-ended aspect of the structuralist view, the notion of *model* is taken to be primitive. In understanding what mathematics concerns, then, we are led away from a description in terms of a *privileged subject matter*, and instead to talk of mathematical descriptions, or mathematical features, of otherwise non-mathematical aspects of the world. These *features of mathematical interest* are identified here as the *structural properties*, and the language of *category theory* has been shown, via the Structural Properties Theorem, to be particularly well suited to encode those structural properties.

However, in order to best preserve the open-ended aspect of the structuralist view, it has been seen *necessary to reject* (or, at the very least, it has been seen *exceedingly difficult to preserve*) other features that have been variously defended as essential components of any philosophical account of *mathematics*. In taking the notion of *model* as primitive is has been seen to be difficult to recover a suitable notion of (mathematical) *truth*, as the link between the *methods of mathematical proof* and the *structural properties of models* those proofs are taken to concern is difficult to establish, given the program's deliberate silence on the nature of those models. In abandoning mathematical truth, the AFCS program is seen not to count as a *foundation* for mathematics, in accordance with the criteria presented in Section 1.3.

Given these aspects of the AFCS program, is the AFCS program acceptable as a philosophical account of *mathematics*? The program is here offered as an *explication* of *mathematics*, from the structuralist perspective, where an *explication* "consists in transforming a given more or less inexact concept into an exact one or, rather, in replacing the first by the second" [19, p. 3]. Following Carnap, a successful *explicatum* can be taken to "fulfil to a sufficient degree" four criteria: similarity to the *explicandum*, exactness, fruitfulness, and simplicity [19, p. 5]. How, then, does the AFCS proposal fare as an explication of *mathematics*?

Certainly one criterion, that of *fruitfulness*, is clearly satisfied by a component of the AFCS program: the language of category theory. As Awodey remarks, "category theory provides a framework (indeed, *the* currently dominant one) for the practice of modern abstract mathematics" [2, p. 54], and Corry claims (as cited at the beginning of Chapter 2) that

> Category theory is the most elaborate and successful instance of an axiomatized theory allowing for a systematic characterization and analysis of the different structures, and the recurring mathematical phenomena that come forward in the latter. [23, p. 12]

It is less clear how well the program fares with respect to the other criteria. While the characterization of the notion of a *structural property* via the language of category theory may be considered *exact*, Carnap holds that an *explicatum* is *exact* insofar as the necessary definitions are incorporated "into a well-constructed system of scientific either logicomathematical or empirical concepts" [19, p. 3]. Given that the AFCS view does not acknowledge a single, privileged system of proof for mathematical theorems, *is* the proposed program sufficiently exact? In taking the notion of *model* as primitive, can the resulting program be considered particularly *simple*? Of course, candidate explicatums for mathematics are to be assessed relative to one another, and the arguments contained in this work are taken to establish that the AFCS program fares better than those other structuralist programs explicitly considered.

Ultimately, the language of category theory may not be uniquely suited, or even best suited, to precisely express the notion of a *structural property*, and consequently the AFCS program may not best suit the development of the structuralist view. Instead, the proponent of the AFCS program offers a pragmatic line of argument, aimed at establishing, in part, that the crucial *structural properties* are well rendered within the program, but leaving open the possibility that the language of category theory may be surpassed by some other means of description, in the way that, within mathematics, the language of category theory has gone some way to replace the language of set theory that was its precursor. The AFCS program is not expected to be the final stage in the development of the structuralist view in mathematics, but the arrows of category theory may serve to point us in the right direction.

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Curriculum Vitae

Darren McDonald

Areas of Specialization

- Philosophy of Mathematics
- Philosophy of Logic
- Philosophical/Mathematical Logic

Areas of Competence

- Philosophy of Language
- Philosophy of Science
- History of Analytic Philosophy

Education

- Doctor of Philosophy
    - Ph.D. thesis entitled "Anti-Foundational Categorical Structuralism" written under the supervision of Prof. John L. Bell—University of Western Ontario, May 2012
    - Visiting Research Scholar—University of St Andrews,
      July 2006–June 2007
        - Attended biweekly seminar meetings of the Logical and Metaphysical Foundations of Classical Mathematics project and weekly meetings of the Epistemology project at the Arché Research Centre for the Philosophy of Logic, Language, Mathematics and Mind
        - under the supervision of Prof. Peter Clark

      September 2004–June 2005
        - attended and presented work at weekly seminar meetings of the Logical and Metaphysical Foundations of Classical Mathematics project at the Arché Research Centre
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- Master of Arts in Philosophy
    - masters thesis entitled "Quantum Logic: Formal Semantics and the Quantum Conditional" written under the supervision of Prof. David DeVidi—University of Waterloo, October 2003

- Bachelor of Mathematics
    - Honours Pure Mathematics, Philosophy Minor, Co-operative program (With Distinction)—University of Waterloo, June 2002

Refereed Publications

- "Explication and the Foundations of Number Theory". *Eidos*. (2) Volume XVIII (June 2004), 1–18.

Presentations and Other Publications

- "Ontology and Uniformity". *Selected Papers Contributed to the Sections of GAP.6, Sixth International Congress of the Society for Analytical Philosophy, Berlin, 11–14 September 2006.* Paderborn: mentis, 2008.
    - "Ontology and Uniformity" was presented at the Sixth International Conference organized by the German Society for Analytic Philosophy (GAP) —Freien Universität Berlin, September 2006
- "Explication and the Foundations of Number Theory" was presented at the Philosophy Graduate Student Association (PGSA) Conference—University of Waterloo, March 2004

Scholarships, Awards, and Distinctions

- Western Graduate Thesis Research Award—awarded in 2006 by the Faculty of Arts and Humanities at the University of Western Ontario in the amount of $300
- Mary Routledge Fellowship—awarded in 2006 by the Faculty of Arts and Humanities at the University of Western Ontario in the amount of $300
- Robert T. Jones Jr. Chevening Scholarship—awarded in 2004 by British Council, funded by the Canadian Robert T. Jones, Jr. Scholarship Foundation and the Foreign and Commonwealth Office in the amount of $40,000
- Nominated for a Graduate Student Teaching Award—University of Western Ontario (2004)
- SSHRC Doctoral Fellowship—awarded in 2003 by the Social Sciences and Humanities Research Council of Canada in the amount of $76,000 ($19,000 annually over 4 years)
- Ontario Graduate Scholarship (OGS)—awarded in 2003 by the Government of Ontario in the amount of $15,000 (declined)
- President's Scholarship for Graduate Study (PSGS)—awarded in 2003 by the University of Western Ontario in the amount of $15,000 (this scholarship, when combined with a SSHRC Doctoral Fellowship, becomes a Graduate Tuition Scholarship)
- Graduate Tuition Scholarship (GTS)—awarded in 2003, a full tuition scholarship for the duration of the SSHRC Doctoral Fellowship (4 years), approximately equivalent to $24,000 in funding (this has since been renamed the Western Graduate Research Scholarship)
- Certificate for "Outstanding Achievement in Graduate Studies"—awarded in 2003 by the University of Waterloo
- Nominated for an Alumni Gold Medal—University of Waterloo (2003)
- University of Waterloo Graduate Entrance Scholarship—awarded in 2002 by the University of Waterloo in the amount of $5000

- St. Jerome's Founder's Scholarship—awarded in 1996 by the University of St. Jerome's (federated with the University of Waterloo) in the amount of $1600
- Governor General's Bronze Medal—awarded in 1995 at St. Joseph-Scollard Hall Secondary School for the highest academic achievement in the graduating class

Teaching and Other Education Experience

- Teacher in Mathematics and Theory of Knowledge (for the International Baccalaureate Diploma Program and GCSE curricula)—Rome International School, September 2010 to present
  - 22 teaching hours per week
- Teaching Assistant for Philosophy and the Mind (PY1004)—University of St Andrews, January 2007 to May 2007
  - involved running two 1-hour tutorial sessions each week, as well as grading
- Instructor for Introduction to Philosophy (PHIL 02E)—University of Western Ontario, September 2005 to December 2005
  - 2 lecture hours and 1 tutorial hour each week, as well as grading
- Teaching Assistant for Critical Thinking (PHIL 021)—University of Western Ontario, September 2003 to April 2004
  - involved running two 1-hour tutorial sessions each week, as well as grading
- Instructor for Critical Thinking (PHIL 145)—University of Waterloo, May 2002–August 2002
  - co-taught the course, 3 biweekly lecture hours
- Teaching Assistant for Critical Thinking (PHIL 145)—University of Waterloo, January 2002–April 2002
  - graded all student work for the course
- Instructor for Formal Logic (PY204)—Wilfrid Laurier University, November 2001–December 2001
  - after Professor Graham Solomon's untimely death, I was selected to teach the second half of his course, focusing on predicate logic
- Head Math Tutor—The Learning Centre, Seneca College, May 2000–August 2000 and September 1999–December 1999
  - tutored in business mathematics, statistics, and logic at a tutorial drop-in centre
  - developed supplementary documentation for a business mathematics program

- Assignment marker—University of Waterloo
    - Introduction to Combinatorics (MATH 239), May 1999–August 1999
    - Linear Algebra II (MATH 235), September 1998–December 1998
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Professional Activities

- Conference Co-Organizer for the 7th Annual Logic, Math and Physics Graduate Conference—May 2006

- Commented on "You Can't Mean That: Yablo's Figuralist Account of Mathematics" by S. Hoffman in the Canadian Philosophical Association (CPA) 50th Annual Congress—York University, May, 2006

- Commented on "Epistemic Contexts and Cognitive Control: Toward an Analysis of Mathematical Thought" by J. Keränen in the 5th Annual Logic, Math and Physics (LMP) Conference—University of Western Ontario, May, 2004

Extracurricular Activities and Personal Interests

- Mountain biking, photography, tennis, literature, music, computers