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Multi-stage contests : theory and experiments

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MULTI-STAGE CONTESTS:
THEORY AND EXPERIMENTS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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Know ye not that they which run in a race run all, but one receiveth the prize? So run, that ye may obtain.

1 Corinthians 9:24

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ABSTRACT

In a multi-stage contest known as a two-player race, players display two fundamental behaviors: (1) The laggard will make a last stand in order to avoid the cost of losing; and (2) the player who is ahead will defend his lead if it is threatened. Last stand behavior, in particular, contrasts with previous research where the underdog simply gives up. The distinctive results are achieved by introducing losing penalties and discounting into the racing environment. This framework permits the momentum effect, typically ascribed to the winner of early stages, to be more thoroughly examined. I study the likelihood that the underdog will catch up. I find that neck-and-neck races are common when the losing penalty is large relative to the winning prize, while landslide victories occur when the prize is relatively large. Closed-form solutions are given for the case where players have a common winning prize and losing penalty.

Chapter 2 then experimentally examines the prediction of last stand behavior in a multi-battle contest with a winning prize and losing penalty, as well as the contrasting prediction of surrendering in the corresponding contest with no penalty. We find varied evidence in support of these hypotheses in the aggregated data, but more conclusive evidence when scrutinizing individual player behavior. Players tend to adopt one of several strategies. We develop a taxonomy to classify player types and study how the different strategies interact. The last stand and surrendering behaviors have implications for winning margins and the likelihood of an upset, which

we investigate. Behaviorally, players are typically more aggressive when they reach a state in the contest by winning rather than by losing.

The third and final chapter is a distinct departure from the study of multi-battle contests. Using comprehensive census data for Cornwall County, England, I create a panel dataset that spans six censuses (1841–1891)—possibly the largest panel dataset for Victorian England at present. I present the methodology for linking individuals and families across these censuses. This methodology incorporates recent advances in census linking (including the use of machine learning) and introduces new methods for tracking migration and changes in household composition. I achieve a forward matching rate of 43%. The additional inclusion of marriage and death records could allow for well over 60% of the population to be accounted for from one census to the next. Using this new panel, I investigate the frequency with which sons pursue the same occupations that they observed their fathers doing while growing up. For sons that did not follow in their father's footsteps, I identify some correlates that may have contributed to the change.

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1.1 Introduction

In American lore, the name of General George A. Custer is inseparably linked with the term *last stand*. In 1876, in what was then the Territory of Montana, General Custer and his men took up a defensive position in a defiant hope to maintain their ground and their lives as a host of Northern Cheyenne and Lakota warriors descended on them. Another tale of a last stand stretches back to 480 B.C. in ancient Greece. The vast Persian military of Xerxes, which numbered in the hundreds of thousands, made a focused attack on a narrow passage at Thermopylae. When defeat seemed imminent, a rearguard composed of some 300 Spartans, as well as men from Thespis and Thebes, fought to the death in an effort to maintain the pass. The determined struggle to avoid a critical loss in the face of bleak odds poignantly captures the idea of a last stand.

Last stand behavior stands at the junction of two incentives: the incentive to avoid losing and the incentive to win. In order to capture the interplay between these two incentives, I study a multi-stage contest known as a race. In each stage, two opposing players simultaneously commit to a non-recoverable bid (or effort level), and the winner moves one stage closer to victory. The first player to win a pre-specified number of contests becomes the overall winner and receives a prize. To distinguish the incentive to avoid losing from the incentive to win, I introduce two key assumptions.

The first is a negative penalty for the loser of the race—a natural assumption in the context of last stands. The second is discounting, which plays a crucial (albeit subtle) role in allowing the two incentives to be distinguished. The importance of discounting can be seen by first considering a static one-shot contest. Even if players are offered a prize for winning and a penalty for losing, the *difference* between the prize and the penalty is really what players compete over (i.e. the difference becomes the re-normalized prize). The net result is that the distinct effect of each incentive cannot be determined. Even in a multi-stage contest, the two incentives cannot be fully separated in the absence of discounting. Including discounting allows players to appropriately weight the prize and the penalty, based on their relative proximity to winning or losing. In this way, the incentive to avoid losing becomes the dominant motivation for a player nearing defeat, while the incentive to win is the principal drive for a player close to victory.

I identify when it is optimal for a laggard to make a last stand. The last stand may be interpreted as the location in the race when the laggard's incentive to avoid losing is greater than the frontrunner's incentive to win. As such, the location of the last stand will depend upon the relative size of the winning prize to the losing penalty. For instance, if the winning prize is small in comparison to the penalty, the laggard's incentive to avoid losing can trump the frontrunner's incentive to win, even if the frontrunner is only one or two stages from victory. The frontrunner will, however, put up a strong fight if his overall lead in the race is challenged.

While a last stand is an anomaly by itself, its presence changes the fundamental dynamics of how a multi-stage contest unfolds. Instead of the conventional finding of momentum building in favor of an early stage winner, I find that momentum is stymied until after a failed last stand.[1] Even before a last stand actually occurs, the threat of a last stand keeps the leader's progress in check and gives the underdog a realistic opportunity to regain the lead. The last stand itself represents a jump in the underdog's probability of returning the race to a tied position; a probability that rapidly diminishes after an unsuccessful last stand.

Another novel finding is that the underdog will continue to compete at all stages of the contest, even after an unsuccessful last stand. This is significant since related studies find that the underdog typically gets discouraged as momentum builds for his opponent. Since effort is costly and non-recoverable, the underdog loses his incentive to invest in a competition that he has little probability of winning. In a closely related model, Konrad and Kovenock (2009) find that the underdog completely gives up unless there is a separate "intermediate" prize that is awarded to the winner of each stage contest. However, with constant intermediate prizes, there is almost no change throughout the race in expected aggregate effort (it is only when the race is nearly tied and the overall prize becomes contestable that there is a change). My result holds without the use of intermediate prizes. Therefore, the equilibrium effort

[1] An early example of momentum building for the leader is the evolutionary study of duopolies by Budd et al. (1993). Momentum has particular relevance to the US Presidential primaries (and other sequential elections), where it has been studied theoretically (e.g. Callander, 2007; Klumpp and Polborn, 2006) and econometrically (Knight and Schiff, 2010). Alternatives to the momentum effect have also been suggested to try to explain upsets (e.g Strumpf, 2002). In some contest frameworks, the momentum effect is so stark that a player can coast to victory once he has gained only a marginal lead (see Konrad and Kovenock, 2005 and 2009).

levels in this paper are not rigidly tied to an exogenous intermediate prize, but are allowed to dynamically adjust throughout the entire race according to each player's proximity to their finish line.

Work on multi-stage contests has largely been limited to the incentive to win. A simple winning prize was used by Harris and Vickers (1987) in the first formal model of a race.[2] The same was done by Klumpp and Polborn (2006) in their application of the race to the sequential US presidential primary elections. Konrad and Kovenock (2009) generalized the winning prize to allow players to have potentially different valuations of the prize. While each of these papers incorporates the incentive to win, the players in these models lack an incentive to avoid losing. One exception is Sela (2011), who uses losing penalties in a best-of-three contest. While Sela does not use discounting, his central result arises when he does something very similar: allowing the penalties to differ for losing after the second contest versus losing after the third.[3] The model in this paper is not limited to a best-of-three contest, but allows for races of arbitrary size.

Although the combination of winning prizes and losing penalties has not previously been studied in a fully generalized race, it has been studied by Agastya and McAfee (2006) in the one-dimensional analogue of a race: the tug-of-war. Unlike a race, which measures how far each player is from the finish line, a tug-of-war mea-

[2]While an earlier paper, Harris and Vickers (1985), used the term *race*, players moved alternately and could advance any portion of the distance to their goal on their turn.

[3]Sela's result is that if the penalty is substantially larger after the third contest than after the second, effort in a best-of-three all-pay auction may be larger than in a single all-pay auction.

sures the size of the lead that a player has over his opponent. Winning a tug-of-war means achieving a pre-specified winning margin (such as a win-by-two rule in tennis). Although Agastya and McAfee (2006) do identify last stand behavior in a tug-of-war, they place little emphasis on that result. A notable benefit of studying last stand behavior in a race, rather than in a tug-of-war, is that the winning margin is not exogenously set. This makes it possible to analyze how a last stand affects the size of the expected winning margin. I find that the expected winning margin is increasing in the ratio of the winning prize to the losing penalty. Neck-and-neck races are to be expected when the losing penalty outweighs the winning prize; but landslide victories are commonplace when the winning prize dominates.

Two methods have repeatedly surfaced for modeling the individual stage contests in dynamic settings: the all-pay auction and the Tullock contest.[4] Each requires players to commit to a non-refundable bid in each round—a representation of their effort or expenditure in that phase of the competition. The difference between them arises in how a winner is determined. In an all-pay auction, the winner is simply the highest bidder, which leads players to randomize in equilibrium (see Baye et al., 1996). The Tullock contest, on the other hand, operates like a raffle and includes a parameter for governing the degree of exogenous noise.[5] I will use the all-pay auction since it is

[4]For instance, Konrad and Kovenock (2005, 2009), Sela (2011), and Agastya and McAfee (2006) use the all-pay auction, while Tullock contests appear in Klumpp and Polborn (2006) and Harris and Vickers (1987).

[5]For work on the properties of the Tullock contest, see Szidarovszky and Okuguchi (1997), Nti (1999), and Cornes and Hartley (2005).

more conducive to generating closed-form solutions in general environments.[6]

This paper contains a complete closed-form characterization of equilibrium behavior for a race with an arbitrary number of stage contests when players have a common winning prize $Z \geq 0$, losing penalty -1, and discount factor $\delta \in [0.5, 1)$. In addition to these closed-form results, I numerically explore the key features of the model for other parameter configurations (such as asymmetries in the winning prizes and losing penalties; and $\delta < 1/2$ when players have a common prize and penalty).

1.2 The Model

1.2.1 The All-Pay Auction

At each non-terminal stage of the race, players compete in an all-pay auction. I will briefly describe the pertinent features of the all-pay auction before introducing the formal structure of the race.

Players in an all-pay auction must pay their bid, whether they win or lose. Hence, players must maximize their expected payoff from winning, less the cost of their bid. Beyond the monetary interpretation, a player's bid can be thought of more generally as effort. Formally, player $i \in \{1, 2\}$ chooses an effort level $e_i \in \mathbb{R}_+$ to maximize the following:

$$\max_{e_i} \; p_i(e_i,\, e_{-i}) \times \zeta_i - e_i$$

Here, e_{-i} is the effort level of player i's opponent, ζ_i is player i's valuation of the

[6] In using the Tullock contest, Klumpp and Polborn (2006) and Harris and Vickers (1987) both rely on numerical results for describing races with a large number of states. By using the all-pay auction, and also allowing for discounting and a losing penalty, this paper can be viewed as a generalization of the racing model in Konrad and Kovenock (2009).

winning prize, and p_i is the probability that player i wins. Specifically, $p_i = 1$ if $e_i > e_{-i}$, $p_i = 0$ if $e_i < e_{-i}$, and an arbitrary tie-breaking rule can be employed if $e_i = e_{-i}$.

Let $\zeta_H \geq \zeta_L > 0$ denote the high and low prize valuations for a two-player all-pay auction, and let h and λ be the corresponding effort levels. In the unique equilibrium (see Hillman and Riley, 1989, and Baye et al., 1996), the players randomize their effort levels according to the following distributions:

$$F(h) = \begin{cases} h/\zeta_L & \text{if} \quad h \in [0, \zeta_L] \\ 1 & \text{if} \quad h > \zeta_L \end{cases}$$

$$G(\lambda) = \begin{cases} (\zeta_H - \zeta_L + \lambda)/\zeta_H & \text{if} \quad \lambda \in [0, \zeta_L] \\ 1 & \text{if} \quad \lambda > \zeta_L \end{cases} \tag{1.1}$$

Using these strategies, the player with the lower valuation submits a bid that, in expectation, yields a payoff of zero; the player with the higher valuation has an expected payoff of $\zeta_H - \zeta_L$. Equilibrium winning probabilities for the players with the high and low valuations of winning the contest are as follows:

$$p_H = 1 - \frac{\zeta_L}{2\zeta_H} \qquad\qquad p_L = \frac{\zeta_L}{2\zeta_H} \tag{1.2}$$

It is worth noting that the player with the higher prize valuation has at least a one-half probability of winning. Finally, in equilibrium, the expected effort levels for the high and low valuation players are given by:

$$\mathbb{E}[e_H] = \frac{\zeta_L}{2} \qquad\qquad \mathbb{E}[e_L] = \frac{\zeta_L^2}{2\zeta_H} \tag{1.3}$$

1.2.2 Primitives of the (m,n) Race

A two-player race is a dynamic contest structure consisting of a set of successive battles or competitions, which are modeled here as all-pay auctions. In order to be victorious in a race, Player A must win m battles before Player B wins n battles ($m, n \in \mathbb{N}$). Thus every state of the race is marked by a pair (i, j) where i denotes the number of contests that Player A still needs to win in order to claim the race victory, while j is the number of contests that Player B still needs to win. Each player is always aware of the present state of the race. The terminal states where Player A is the victor are $(0, j)$ for all $j \in \{1, ..., n\}$. If Player A succeeds in reaching one of these states he is awarded $Z_A \geq 0$, and Player B incurs a cost of $L_B < 0$. Similarly, Player B is declared the winner of the race at terminal states $(i, 0)$ where $i \in \{1, ..., m\}$. At these states, Player B receives a winning prize of $Z_B \geq 0$, and Player A faces a losing penalty of $L_A < 0$. Both players have a common discount factor $\delta \in (0, 1)$.

Since winning prizes and losing penalties are only awarded at terminal states, it will be necessary to identify contest prizes at each state (i, j) in order to use the all-pay auction at non-terminal states. The contest prize at a non-terminal state can intuitively be thought of as the discounted difference between a player's value of being one state closer to winning the race himself and his value of having his opponent be one state closer to an overall win. In order to formally define the contest prizes at state (i, j) it will be necessary to first designate a valuation that each player has of being at each state. This implicit value of being at (i, j) will be referred to as a player's *continuation value* and will be denoted $v_A(i, j)$ for Player A and $v_B(i, j)$ for

Player B. With this notation, the contest prizes at state (i, j) for Players A and B, respectively, are as follows:

$$\zeta_A(i,j) = \delta\left[v_A(i-1,j) - v_A(i,j-1)\right]$$

$$\zeta_B(i,j) = \delta\left[v_B(i,j-1) - v_B(i-1,j)\right] \tag{1.4}$$

These contest prizes are not fully defined until the continuation values are themselves defined. At each terminal state the continuation values will be defined as the respective winning prize or losing penalty. That is, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, the terminal continuation values are given as:

$$v_A(i,0) = L_A \qquad v_B(i,0) = Z_B$$

$$v_A(0,j) = Z_A \qquad v_B(0,j) = L_B$$

Following Agastya and McAfee (2006) and Konrad and Kovenock (2005), continuation values at non-terminal states are defined as the sum of the expected values of winning and losing a contest, less the cost of a player's bid.[7] Let p_A be the probability that Player A wins the contest at (i, j). At his optimal expected level of effort e_A^*, Player A's continuation value at (i, j) is:

$$\begin{aligned} v_A(i,j) &= p_A \times \delta v_A(i-1,j) + (1-p_A) \times \delta v_A(i,j-1) - e_A^* \\ &= p_A \times \zeta_A(i,j) - e_A^* + \delta v_A(i,j-1) \end{aligned}$$

Note that $p_A \times \zeta_A(i,j) - e_A^*$ is the expected equilibrium payoff of the all-pay auction, which, as was stated previously, is equal to the difference in the contest prizes for

[7]Agastya and McAfee also account for the expected value of a tie.

the high valuation player and equals zero for the low valuation player. Therefore, the continuation values at non-terminal states are the following:

$$v_A(i,j) = \max\{\zeta_A(i,j) - \zeta_B(i,j),\ 0\} + \delta v_A(i,j-1)$$

$$v_B(i,j) = \max\{\zeta_B(i,j) - \zeta_A(i,j),\ 0\} + \delta v_B(i-1,j)$$

Using the definition of the contest prizes, these continuation values can be rewritten as follows:

$$v_A(i,j) = \max\{\delta[v_A(i-1,j) - v_B(i,j-1) + v_B(i-1,j)],\ \delta v_A(i,j-1)\}$$

$$v_B(i,j) = \max\{\delta[v_B(i,j-1) - v_A(i-1,j) + v_A(i,j-1)],\ \delta v_B(i-1,j)\} \quad (1.5)$$

Equation 1.5 completes the set-up of the model. Before solving the model, it will be helpful to define some properties of states within a race. As can be seen from Equations 1.2 and 1.3, the contest prize valuations are critical in determining winning probabilities, expected effort levels, as well as payoffs throughout the race. The following definition will be useful in referring to the player with the higher valuation of winning a contest at state (i,j).

Definition. (Advantaged and Disadvantaged) *Player A is strictly advantaged at* (i,j) *if* $\zeta_A(i,j) > \zeta_B(i,j)$*, weakly advantaged if* $\zeta_A(i,j) = \zeta_B(i,j)$*, and disadvantaged whenever* $\zeta_A(i,j) < \zeta_B(i,j)$*. In the three cases above, Player B is disadvantaged, weakly advantaged, and strictly advantaged, respectively. A player is advantaged if he is either strictly or weakly advantaged.*

Equivalently, a player is advantaged at state (i,j) if his continuation value is equal to the left-hand-side of the maximization argument in Equation 1.5 and weakly advantaged if the two sides of the maximization argument are equal. At terminal states the winner of the race is strictly advantaged and the loser is disadvantaged.

It is important not only to clarify what it means to be advantaged at an isolated state but also what it means in comparison to the neighboring states. A last stand will formally be defined in terms of a sequence of states where a player is advantaged and disadvantaged.

Definition. (Last Stand) *In a two-player race, Player A makes a last stand at state* (i, j) *if he is advantaged at* (i, j) *but disadvantaged at state* (k, j) *for some* $k \in \{1, \ldots, i-1\}$ *and at* (i, ℓ) *for all* $\ell \in \{0, \ldots, j-1\}$*. If Player A is also strictly advantaged at* $(i, j+r)$ *for all* $r \in \{1, \ldots, q\}$ *where* $q \in \mathbb{N}$*, then these states are also part of his last stand. A last stand for Player B is defined symmetrically by reversing the roles of* i *and* j*.*

This definition of a last stand captures the idea of a player making a strong final push before his opponent becomes strictly advantaged *en route* to victory. A key element of the definition is that even if the player making a last stand is able to win a successive number of contests, he will come to a state where he is disadvantaged. While a last stand allows a player to make a critical advance, his opponent will not yield the lead entirely without a fight of his own. Thus, a player making a last stand is an underdog: if he loses, he will be disadvantaged; if he continues to win, then eventually he will still be disadvantaged.

It may seem paradoxical that a player would ever be advantaged at a state where he cannot avoid becoming disadvantaged. The underdog knows that win or lose, his opponent will be very aggressive in future state contests; yet, the underdog has the upper hand right now. The key to this paradox is a careful examination of each player's value of winning such a state contest. Provided that the underdog is substantially behind, the race's leader can afford to let down his guard a little. His incentive to expedite a race victory is not nearly as strong as the underdog's incentive to delay a race loss. Even if the underdog only has a small chance of winning the race, he obtains some benefit from postponing its end. A last stand, therefore, is perhaps more focused on delaying loss than it is on securing victory. Indeed, the odds of ultimate success for the underdog are remote, and when the front-runner sees that his lead has become too small, his incentive to win overpowers the underdog's incentive to avoid losing.

Finally, in order to solve the model, a player's strategy and an equilibrium concept will need to be specified. A *behavioral strategy* for Player A (resp. Player B) consists of a distribution of effort levels $\mathcal{A}_{(i,j)}(e_A)$ (resp. $\mathcal{B}_{(i,j)}(e_B)$) at each state (i,j) where $i \in \{1,...,m\}$ and $j \in \{1,...,n\}$.[8] This paper will examine the unique subgame perfect Nash equilibrium of this racing environment.[9] Equilibrium strategies will follow the format of Equation 1.1 using the contest prizes $\zeta_A(i,j)$ and $\zeta_B(i,j)$.

[8]In a behavioral strategy, each player makes an independent randomization for every information set in the game. For a race where players have complete information about the payoff structure and the current state of the race, a behavioral strategy consists of a randomization of effort levels at each state that is not affected by the path of the race up to that point (i.e. history independent).

[9]Furthermore, the equilibrium is Markov perfect in the sense that, given the winning prize, losing penalty, and discount factor, bidding strategies only depend on the current state (i,j).

1.3 Equilibrium Properties

The starting point for analyzing the dynamic interaction of the two incentives —the incentive to win and the incentive to avoid losing—is the symmetric case where players have a common winning prize $Z \geq 0$ and losing penalty $L < 0$. Without loss of generality, the losing penalty will be normalized to -1 so that Z can represent the relative size of the winning prize in comparison to the losing penalty. I also focus on the case where $\delta \in [0.5, 1)$. (For myopic players with $\delta \in (0, 0.5)$, the biggest differences occur at early stages of the race. These differences are described in A.3, while A.4 studies asymmetries in the winning prize and losing penalty.) The major results will be presented in two parts. In this section, I characterize general solutions for the contest prizes and identify the states where each player is advantaged—including the location of each player's last stand. Section 1.4 then addresses some dynamic implications of last stand behavior, such as the probability that the underdog catches up, and the size of the expected winning margin.

Equilibrium strategies, effort levels, and winning probabilities are all simple constructs of the contest prizes; and calculating contest prizes from continuation values is trivial. The entire problem rests on identifying continuation values for each state of the race. With the continuation values at terminal states set equal to Z for the winner and -1 for the loser, the values at the remaining states can be solved for by backward induction. Lemma 1 in A.1 provides general solutions for all continuation values in an (m, n) race when $\delta \in [0.5, 1)$. (This result is also proved in A.1, while all other proofs are in A.2.) General forms for the contest prizes can then be obtained

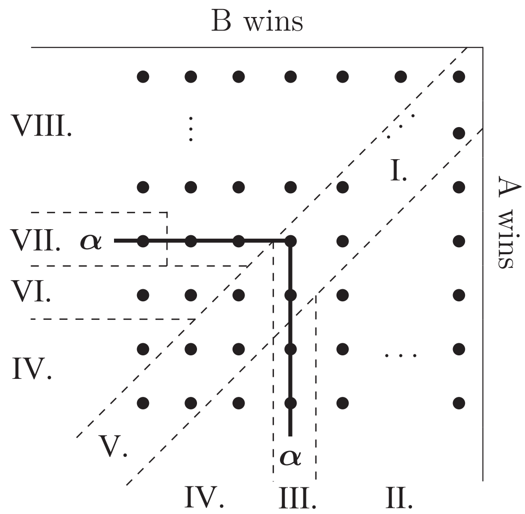

Figure 1.1: Contest prize regions for Player A.

by using these continuation values in Equation 1.4. Since the contest prizes are the basis for equilibrium behavior, they are presented below as a proposition. Figure 1.1 depicts the eight contest prize regions described in this proposition. The race in the figure begins at the bottom-left; wins for Player A advances the race to the right, while wins for Player B moves the race up.

Proposition 1. *Let* $\alpha \in \mathbb{N} \bigcup \{0\}$ *satisfy* $Z \in ((\alpha-1)(1-\delta),\ \alpha(1-\delta)]$*. In a two-player* (m, n) *race with winning prize* $Z \geq 0$*, losing penalty* $L = -1$*, and discount factor* $\delta \in [0.5, 1)$*, the contest prizes for Player A at state* (i, j)*, where* $i \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, n\}$*, are given below; contest prizes for Player B are obtained by reversing the roles of* i *and* j*.*

I. *For* $j \in \{i, i+1\}$ *such that* $i \leq \alpha$*:* $\quad \zeta_A(i, j) = \delta^i[Z - (i-1)(1-\delta)] + \delta^j$

II. *For* $j \geq i+2$ *such that* $i < \alpha$*:* $\quad \zeta_A(i, j) = \delta^i(1-\delta)[Z + 1 - i(1-\delta)]$

III. *For* (α, j) *such that* $j \geq \alpha + 2$*:*

$$\zeta_A(\alpha, j) = (\delta^\alpha - \delta^{j-1})\,[Z - \alpha(1-\delta)] + \delta^\alpha(1-\delta)$$

IV. *For* $j \geq i + 2$ *such that* $i > \alpha$*; or* $i \geq j + 1$ *such that* $j \geq \alpha + 2$*:*

$$\zeta_A(i, j) = \delta^{i+j-\alpha-2}(1-\delta)[Z + 1 - \alpha(1-\delta)]$$

V. *For* $j \in \{i, i+1\}$ *such that* $i > \alpha$*:* $\quad \zeta_A(i, j) = \delta^{i+j-\alpha-1}[Z + 1 - \alpha(1-\delta)]$

VI. *For* $(i, \alpha + 1)$ *such that* $i \geq \alpha + 2$*:*

$$\zeta_A(i, \alpha+1) = \sum_{h=\alpha}^{i-2}(\delta^{h+1} - \delta^i)Z + \alpha\delta^i[(i-\alpha)(1-\delta) + \delta] - (\alpha - 1)\delta^{\alpha+1}$$
$$-\delta^{i-1}[(i - \alpha - 2)(1-\delta)(Z + 1 - \alpha[1-\delta]) + \delta]$$

VII. *For* (i, α) *such that* $i \geq \alpha + 3$*:*

$$\zeta_A(i, \alpha) = \sum_{h=\alpha}^{i-3}(\delta^{i-1} - \delta^{h+1})Z - \alpha\delta^{i-1}[(i-\alpha)(1-\delta) + 2\delta - 1] + (\alpha - 1)\delta^{\alpha+1} + \delta^\alpha$$

VIII. *For* $i > j$ *such that* $j < \alpha$*; or* $i \in \{\alpha+1, \alpha+2\}$ *and* $j = \alpha$*:* $\quad \zeta_A(i, j) = \delta^j(1-\delta)$

To reiterate, these contest prizes represent the implicit reward a player receives from winning a contest at a particular stage of the race—and the implicit reward is simply being one state closer to victory as opposed to one state closer to defeat. However, from the presence of eight different contest prize regions, the value of being one state closer to victory can vary considerably over the course of a race.

As can be seen by its frequent use, α is an important parameter in the contest prizes, and its definition in Proposition 1 will be used throughout this paper. The primary significance of α, as will be shown, is that the underdog makes a last stand when he is α and $\alpha + 1$ states away from his opponent's goal. It is important to recognize that α is a weakly increasing function of the winning prize Z.[10]

The contest prizes in Proposition 1 can be compared at any given state to determine which player is advantaged. Three important behavioral features arise in the pattern of advantages and disadvantages that players have across states: (1) the last stand; (2) the defense of the lead; and (3) a neutral region. Each of these properties is described in the following proposition.

Proposition 2. (The Last Stand and Associated Properties)

1. The Last Stand*: When $m \geq \alpha + 2$ and $n \geq \alpha$ (resp. $n \geq \alpha + 2$ and $m \geq \alpha$) Player A (resp. Player B) makes a last stand before reaching a terminal loss. For Player A, this last stand occurs at (i, j) for $j \in \{\alpha, \alpha + 1\}$ and $i \in \{j + 2, \ldots, m\}$; Player B makes his last stand at (i, j) where $i \in \{\alpha, \alpha + 1\}$ and $j \in \{i+2, \ldots, n\}$. (When $Z = \alpha = 0$, the last stand only occurs at $j = 1$ for Player A and $i = 1$ for Player B.) Additionally, players make their last stand farther and farther away from the end of the race as the size of the winning prize increases relative to the size of the losing penalty (i.e. as α increases).*

2. Defending the Lead*: Immediately off of the main diagonal, players put forth extra effort in order to maintain their lead in the race. That is, Player A is advantaged at $(i, i + 1)$ for $i \in \{1, \ldots, \min\{m, n - 1\}\}$, and Player B is advantaged at $(j + 1, j)$ for $j \in \{1, \ldots, \min\{m - 1, n\}\}$ when $\delta > 1/2$.*

3. Neutral Region*: When a player has a lead of at least two state contests, then,*

[10] An alternative definition of α is $\alpha(Z, \delta) = \lceil Z/(1 - \delta)\rceil$ where $\lceil\cdot\rceil$ denotes the ceiling function for rounding up to the nearest integer. Note that α is also weakly increasing in δ.

before the opposing player makes a last stand, both players have equal contest prizes, and therefore equal winning probabilities and expected effort levels.

The three parts of Proposition 2 are illustrated in Figure 1.2. In the first panel, the winning prize is quite small in comparison to the losing penalty of -1 (specifically, $Z \in (0, 1-\delta]$), and the last stand takes place immediately before a terminal loss. In the second and third panels of Figure 1.2, the winning prize increases by a factor of $1-\delta$, and the last stand occurs successively farther away from a finish line. Additionally, there are neutral regions in each of the panels which are bounded on one side by a last stand and bounded on the other by the frontrunner making a distinct attempt to defend his overall lead. These neutral regions are a reflection of the opposition each player anticipates from advancing toward his own goal.

B wins (top), **A** wins (right)

A	**A**	**A**	**A**	**A**	**A**	**A**	B	○
A	**A**	**A**	**A**	**A**	**A**	B	○	**A**
○	○	○	○	○	B	○	**A**	B
○	○	○	○	B	○	**A**	B	B
○	○	○	B	○	**A**	○	B	B
○	○	B	○	**A**	○	○	B	B
○	B	○	**A**	○	○	○	B	B
B	○	**A**	○	○	○	○	B	B
○	**A**	○	○	○	○	○	B	B

(i) $Z \in (0, 1-\delta]$

B wins (top), **A** wins (right)

B	B	B	B	B	B	B	B	○
A	**A**	**A**	**A**	**A**	**A**	B	○	**A**
A	**A**	**A**	**A**	**A**	B	○	**A**	**A**
○	○	○	○	B	○	**A**	B	**A**
○	○	○	B	○	**A**	B	B	**A**
○	○	B	○	**A**	○	B	B	**A**
○	B	○	**A**	○	○	B	B	**A**
B	○	**A**	○	○	○	B	B	**A**
○	**A**	○	○	○	○	B	B	**A**

(ii) $Z \in (1-\delta, 2-2\delta]$

B wins (top), **A** wins (right)

B	B	B	B	B	B	B	B	○
B	B	B	B	B	B	B	○	**A**
A	**A**	**A**	**A**	**A**	B	○	**A**	**A**
A	**A**	**A**	**A**	B	○	**A**	**A**	**A**
○	○	○	B	○	**A**	B	**A**	**A**
○	○	B	○	**A**	B	B	**A**	**A**
○	B	○	**A**	○	B	B	**A**	**A**
B	○	**A**	○	○	B	B	**A**	**A**
○	**A**	○	○	○	B	B	**A**	**A**

(iii) $Z \in (2-2\delta, 3-3\delta]$

Figure 1.2: The symmetric race with $\delta > 1/2$ for different values of the winning prize, Z. The advantaged player at each state is labeled; a circle indicates that players are weakly advantaged. The last stand is the collection of states near the opponent's goal where the underdog is advantaged.

When $\delta > 1/2$, the defense of the lead extends throughout the entire race. That is not the case for $\delta \leq 1/2$. While the defense of the lead is still present from the last stand to the end of the race, players discount the future too much to defend their lead early on in the race (precise details can be found in A.3).

Finally, it is noteworthy that both players engage in active competition at every non-terminal stage of the race.

Proposition 3. *In the equilibrium of a symmetric race with $\delta \geq 1/2$, both players put forth effort with positive probability in every state contest.*

Proposition 3 starkly contrasts the equilibrium result in Konrad and Kovenock (2009) where players only compete on a limited set of states. However, with the introduction of losing penalties and discounting, this result is fairly intuitive. Both players will actively compete when their contest prizes are strictly positive—and the combination of the incentive to win and the incentive to avoid losing creates an environment where this is always the case.[11] Players not only care about the ultimate outcome of receiving a prize or a penalty, they also care about *when* they receive it. The underdog sees a positive benefit in attempting to stay in the game one more round rather than face a penalty immediately. For the frontrunner, there is an implicit cost to delaying victory, so it is also in his best interest to actively compete.

Before examining the dynamic implications of last stand behavior, it is worth summarizing the principle differences that arise when players are asymmetric in terms

[11]Based on numeric results, it appears that Proposition 3 holds for any combination of positive winning prizes, negative losing penalties, and $\delta \in (0, 1)$.

of their winning prizes and losing penalties (see A.4 for details). The first thing to note is that such an asymmetry changes the location in the race where players are tied in terms of incentives. This in turn changes the locations where players defend their lead and make their last stand. The player who has more at stake in the outcome of the race—based on a larger winning prize, losing penalty, or both—will wait longer before he defends his lead than in the symmetric case. Due to his opponent's weaker incentives, he needs to be farther behind before the race is a close one. If the asymmetries are large enough, the player with more at stake will wait to defend his lead until he is also making his last stand. That is, the finite racing grid blurs together the forces of the last stand and the defense of the lead. Different types of asymmetries (such as having different winning prizes for the two players but a common losing penalty, or vice versa) have similar features when parameter differences are small. The major distinctions arise when there is a large degree of asymmetry. A large difference in the winning prize when players have a common losing penalty leads the player with less at stake to make his last stand farther away from the finish line. The opposite is true when there is a common winning prize but a large difference in the penalty. Both of these features are found when one of the players has both an extra large prize and an extra large penalty.

1.4 Dynamic Implications of Last Stand Behavior

The fundamental goal of this section is to discover how last stand behavior impacts movement within a race. I will specifically examine how a last stand affects

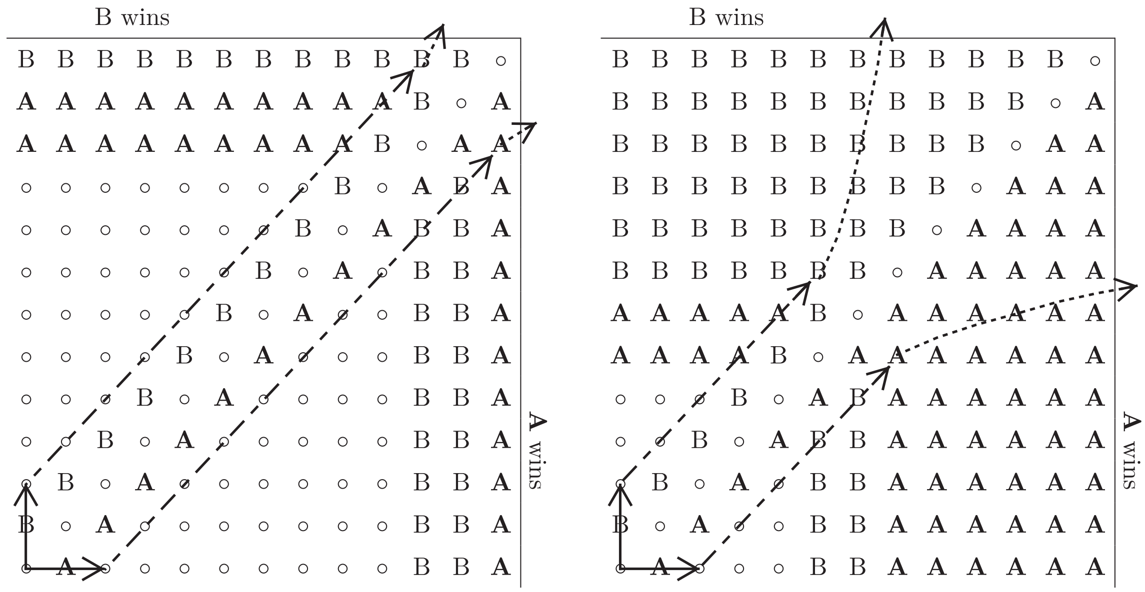

Figure 1.3: The expected path of a race. Left: $\alpha = 2$. Right: $\alpha = 7$. Arrows pass through states that are reached in expectation (players, of course, cannot move diagonally).

(1) the expected winning margin, (2) the probability that the underdog can tie the race, and (3) cumulative expenditures throughout the race. To do this, it is first necessary to ask: What is the expected path of a race? Understanding the expected path that a race follows will serve as a basis for assessing the likelihood that the underdog catches up. Additionally, the expected path will also be useful in identifying expected winning margins, estimating expenditures throughout the race, and in studying how and when momentum builds for the frontrunner. Throughout this section, I will assume an (n, n) race—that is, the race begins when each player is n states away from his goal.

The expected path that a race follows is shown in Figure 1.3. At the initial state (n, n), the race is tied and each player has an equal probability of winning. The winner of the first state will defend his lead at the next state and, hence, has a greater probability of winning there (the probability that the defense of the lead is successful will be covered later on). This pushes the race into the neutral region where each player is equally likely to win any stage contest. When a last stand is reached, the race will move in the direction of the underdog's goal until it is counteracted by the defense of the lead. The net result is that, in expectation, the race will reach one of two states: $(\alpha+1, \alpha-1)$ or $(\alpha-1, \alpha+1)$. From here things change. The winning probability for the frontrunner progressively builds until he wins the entire race.

The juxtaposition of the two panels in Figure 1.3 highlights a significant point: the expected winning margin is based on the location of the last stand, which in turn is based on the relative size of the winning prize and losing penalty. When the losing penalty is dominant, as in the left panel, races will typically be neck-and-neck. Landslide victories, on the other hand, most likely occur when the winning prize is considerably larger than the losing penalty. The size of the expected winning margin is analytically formalized below.

Proposition 4. *In a symmetric (n, n) race with $\delta \in [0.5, 1)$, the expected winning margin is increasing in the relative size of the winning prize Z to the losing penalty $L = -1$. Specifically, the expected winning margin is:*

$$\alpha + 1 - \sum_{k=1}^{\alpha-1} p_k \quad \text{where} \quad p_k = \frac{1}{2\left[Z + 1 - k(1-\delta)\right]}$$

This result is based on reaching state $(\alpha+1, \alpha-1)$ or $(\alpha-1, \alpha+1)$ in expectation—in which case, the underdog is $\alpha+1$ states away from winning the race.

While the underdog is disadvantaged from here on out, it is still probable that he will win some stage contests. To account for this, the probability p_k that the underdog wins a contest when he is k states away from his opponent's victory is subtracted from $\alpha + 1$ for each $k \in \{1, \ldots, \alpha - 1\}$.[12]

Another point that is demonstrated in Figure 1.3 is that momentum does not build for the frontrunner until after a failed last stand. This is an important clarification in the dynamic contest literature. The typical finding is that momentum begins to build immediately in favor of the player that wins the first stage contest. Thus the initial stage contest carries a disproportionate amount of weight. Here, however, the last stand prevents momentum from building too soon. While in expectation the winner of the initial contest will still win the entire race, the underdog has a realistic chance of catching up before making a last stand. The effect of the last stand, therefore, is felt long before it actually occurs. This result is characterized in the following proposition.

Proposition 5. *Momentum builds for the frontrunner following a failed last stand. More precisely, when* $i \geq j + 2$*, so that Player B is no longer defending the lead, his winning probability at state* $(i, j - 1)$ *is greater than at* (i, j) *if* $j \in \{2, \ldots, \alpha\}$*. Prior to reaching a last stand, however, for* $j \in \{\alpha + 3, \ldots, i - 2\}$*, the winning probabilities*

[12]The probability p_k is obtained by applying Equation 1.2 to parts *II* and *VIII* of Proposition 1, keeping in mind that the roles of i and j need to be reversed in *II* for Player B. Note that p_k is only a first order effect as it does not take into account the probability of successive wins when the underdog is k states away from his opponent's goal. A more formal proof of Proposition 4 is omitted.

at $(i, j-1)$ and (i, j) remain constant at one-half.[13] *A symmetric result holds for Player A.*

Although both players continue to compete following a failed last stand, the incentive to win is stronger than the incentive to avoid losing, and the underdog's probability of catching up quickly diminishes. This decreasing probability is illustrated in Figure 1.4, which plots the likelihood that the underdog can return the race to a tied position when the frontrunner is k states from his goal. As can be seen, the easiest place for the underdog to catch up is immediately after he has lost the initial stage of the race; and with probability $(1-\delta)/2\delta$, he can tie the race again.[14] To appropriately scale the figure, all probabilities have all been normalized by dividing by $(1-\delta)/2\delta$. Analytical expressions for the probabilities pictured in Figure 1.4 are given in A.2.5.

Figure 1.4 illustrates that the last stand is a jump in the probability of catching up—one last hurrah before the probability of tying the race swiftly approaches zero. The bars in Figure 1.4, which begin at the last stand, denote a surprising feature of the model. Previous to a last stand, the probability of catching up is independent of the winning prize Z. While Z determines the proximity of a last stand to the end of the race, its role is limited beyond that: δ is the only parameter that affects the winning probability prior to a last stand. It is only when the underdog is within

[13] Although Player B's winning probability at $(i, \alpha+1)$ is less than at $(i, \alpha+2)$, the probabilities cannot be unambiguously ranked between (i, α) and $(i, \alpha+1)$.

[14] $(1-\delta)/2\delta$ is the probability that the frontrunner loses his defense of the lead before a last stand. It becomes smaller following a last stand.

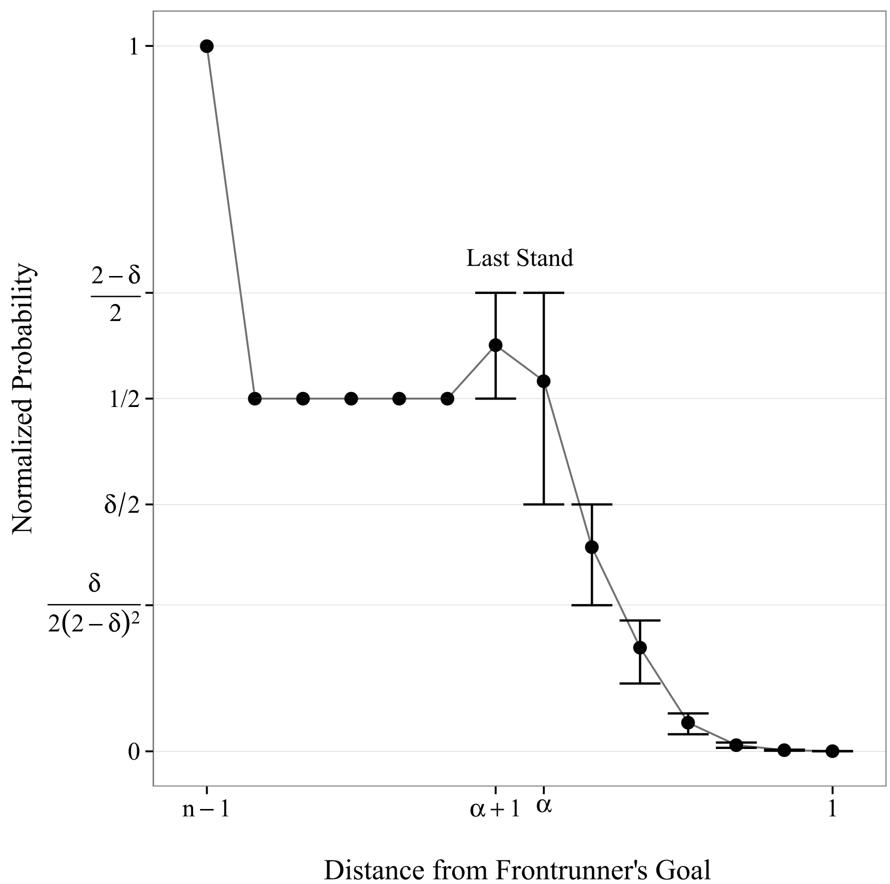

Figure 1.4: Likelihood that the underdog catches up to tied position from the expected race path. Probabilities have been normalized by dividing by $(1-\delta)/2\delta$. Intervals mark the range of probabilities for $Z \in (\,(\alpha-1)(1-\delta),\ \alpha(1-\delta)]$.

$\alpha + 1$ states of the frontrunner's goal that the probability of catching up becomes a function of Z. Initially, at $\alpha + 1$, it is an increasing function of Z. However, for states that are any closer to the frontrunner's goal, the catch-up probability becomes a decreasing function of Z. Figure 1.4 shows the upper and lower bounds of the catch-up probability for $Z \in (\,(\alpha - 1)(1 - \delta),\ \alpha(1 - \delta)]$.

There are three components that seem to impact the rapidity with which the catch-up probability falls in the wake of a last stand. One is that momentum is building for the frontrunner so that he wins contests with increasing frequency (see Proposition 5). The second is closely related. As the frontrunner pulls ahead, catching up requires the underdog to cover more ground. The third factor is that the frontrunner defends his lead more fiercely as the distance to the finish line decreases. In other words, if Player B is the frontrunner, then Player B's winning probability at $(k + 1, k)$ is greater than at $(k + 2, k + 1)$ for $k \in \{1, \ldots, \alpha\}$ (it is constant for $k \in \{\alpha + 1, \ldots, n - 2\}$).[15]

Finally, it is worth investigating how much players can potentially expend throughout the entire race. I will show that while it is possible for players to spend much more in a race than they would in a one-stage contest, that is not the case if players follow the expected path of the race. The following proposition addresses

[15] Player B's winning probability at $(k + 1, k)$ for $k \in \{1, \ldots, \alpha\}$ is

$$1 - \frac{1 - \delta}{2[Z + 1 - (k - 1)(1 - \delta)]},$$

which is decreasing in k. Moreover, Player B's winning probability at $(k+1, k)$ for $k \geq \alpha+1$, $(3\delta - 1)/2\delta$, is less than his winning probability for $k \leq \alpha$.

upper bounds on expenditures. The first part indicates that closer races have the potential for higher expenditures, and the second part specifies the supremum of equilibrium expenditures in a race where neither player ever has more than a one-state lead.

Proposition 6. *Let $Z > 0$, $L = -1$, and $\delta \in [0.5, 1)$.*

1. *In equilibrium, the upper bound on individual expenditures at (i, j) is weakly greater than at $(i+1, j)$ for all i and j such that $i \geq j$. (The same is true when the roles of i and j are reversed.)*

2. *Following the path that includes (k, k) for $k \in \{1, \dots, n\}$ and either $(k+1, k)$ or $(k, k+1)$ for $k \in \{1, \dots, n-1\}$ in an (n, n) race, the highest possible equilibrium expenditure by one player is:*

$$\left(\sum_{k=1}^{\alpha} \delta^k [Z + 1 - (k-2)(1-\delta)] \right) + \left(Z + 1 - \alpha(1-\delta) \right) \left(\delta^{2n-\alpha-1} + (2-\delta) \sum_{k=\alpha+1}^{n-1} \delta^{2k-\alpha-1} \right)$$

Fixing α and letting $n \to \infty$, the supremum is:

$$(\alpha + 2)\delta + \delta^{\alpha+2} \left(\frac{2\delta - 1}{1 - \delta^2} \right)$$

From Equation 1.1, the upper bound of each player's bidding distribution is the smaller of the two contest prizes at a state: ζ_L. Hence, the first part of the proposition could be restated to say that ζ_L weakly increases as the frontrunner's lead decreases. Having the lead is pivotal enough in determining the eventual outcome of a race that the upper bound of the bidding distribution climbs until it reaches a zenith when players are tied. With one exception, the same thing happens to expected effort. For the advantaged player, expected effort is equal to $\zeta_L/2$, and so it also increases as the race becomes closer. On the other hand, expected effort for the disadvantaged player is a function of both the high and low contest prizes ($\zeta_L^2/2\zeta_H$). When the frontrunner defends his lead, ζ_H is high enough to offset the increase in ζ_L so that the disadvantaged player's expected effort is actually smaller than when he is either tied or two states behind.

The second part of the proposition traces the upper bound of the bidding distribution along the highly contested path where the race is always either tied or only one state away from a tie. Although there is a negligible probability that the race would both follow this path and that players would consistently bid at the upper bound of the bidding distribution, Proposition 6 provides a useful benchmark for comparing other measurements of effort. From a contest design standpoint, it is relevant to know both how much *could* be expended in a race versus a simple one-stage contest, as well as how much would *likely* be expended. I will therefore contextualize the upper bound in Proposition 6 by considering (1) expenditures in a one-stage contest, and (2) expenditures along the expected path of a race.

A one-stage contest is equivalent to a race beginning at $(1, 1)$. The upper bound of the bidding distribution at this final stage is the discounted difference between the prize and the penalty: $\delta(Z + 1)$. This value accounts for approximately $1 - \delta$ times the supremum in Proposition 6 as $n \to \infty$.[16] So, in a large race, it is

[16] $1 - \delta$ is the ratio of $\delta(Z + 1)$ to the supremum in Proposition 6 as $Z \to \infty$. As $Z \to 0$,

possible for a player to expend $1/(1-\delta)$ times the maximum he would be willing to pay in a one-stage contest.[17] This result does, however, require the frontrunner to habitually lose the defense of the lead.

The upper bound on cumulative expenditures drops considerably if the frontrunner never loses his defense of the lead. If the only time that a race is tied is at the initial state (n, n), and if it follows a path where one player is always either one or two states ahead of his rival, then the upper bound on a single player's expenditure is only:[18]

$$2\delta - \delta^{\alpha+1} \tag{1.6}$$

Simply by avoiding the states where the race is tied, a player will not spend more than in a one-stage contest. The maximum expenditure in a one-stage contest, $\delta(Z+1)$, can increase without limit as the winning prize increases; Equation 1.6, on the other hand, limits a player's expenditures to no more than 2δ.[19] Since the path that Equation 1.6 is based on constrains the frontrunner to a one or two state lead, it

this ratio goes to $(1-\delta^2)/(3-4\delta^2+2\delta^3)$. These two limits are equal when $\delta = (\sqrt{2})/2$ and remain relatively close for all $\delta \in [0.5, 1)$.

[17] Although high levels of effort may be desirable in athletics since it can increase suspense and excitement, in many contest settings (such as lobbying and legal disputes) expenditures above and beyond the value of the prize are typically viewed as wasteful.

[18] For states $(k+2, k)$ and $(k+1, k)$ where $k < \alpha$, the upper bound is $\sum_{k=1}^{\alpha-2} 2\delta^k(1-\delta) = 2(\delta - \delta^\alpha)$. When $k = \alpha$, the upper bound is $\delta^\alpha(1-\delta)[Z+2-\alpha(1-\delta)]$. For $k \in \{\alpha+1, \ldots, n-2\}$, the upper bound is $(1-\delta)[Z+1-\alpha(1-\delta)]\sum_{k=\alpha+1}^{n-2}(\delta^{2k-\alpha-1} + \delta^{2k-\alpha}) = (\delta^{\alpha+1} - \delta^{2n-\alpha-3})[Z+1-\alpha(1-\delta)]$. The upper bound for $(n, n-1)$ and (n, n) is $\delta^{2n-\alpha-3}(1-\delta+\delta^2)[Z+1-\alpha(1-\delta)]$. Equation 1.6 is the sum of the upper bounds as $n \to \infty$ when $Z = \alpha(1-\delta)$; Equation 1.6 is a strict upper bound for finite n and $Z < \alpha(1-\delta)$.

[19] The upper bound in the one-stage contest is strictly greater than Equation 1.6 for $\alpha \geq 2$ and is equal otherwise. Note that $\delta(Z+1) \leq (\alpha+1)\delta - \alpha\delta^2$.

overestimates expenditures along the expected path of the race by failing to account for the momentum that builds following a failed last stand. If the frontrunner instead won all contests after a failed last stand, the upper bound on expenditures would decrease to:

$$\delta + \delta^{\alpha} - \delta^{\alpha+1} \tag{1.7}$$

Along the expected race path, the upper bound on a player's expenditures must therefore be between Equations 1.6 and 1.7. Since *aggregate* expected expenditures at a state are bounded above by ζ_L, and since players are only tied at the initial state along the expected race path, aggregate expenditures throughout a race will also be bounded above by Equation 1.6 in expectation.

1.5 Simulated Dynamic Results

While the above results demonstrate the dynamic effects of last stand behavior in expectation, Monte Carlo simulations are a pertinent tool for identifying *distributional* outcomes. Three simulated results will serve to complement the findings thus far. The simulations will address the following three questions:

1. How is the distribution of winning margins affected by the relative size of the winning prize to the losing penalty?
2. What is the likelihood that the player who wins the first stage contest will ultimately win the entire race?
3. What is the relationship between the winning margin and cumulative effort?

The first question is addressed in Figure 1.5, which shows the distribution of terminal race locations for three values of the winning prize: $Z = 1/3$, 1, and 3 (the losing penalty remains fixed at -1). Each panel is based on 100,000 simulations in which Player B wins the first contest in a $(12, 12)$ race. Theoretical expected winning margins, as predicted by Proposition 4, are also marked on the panels. Additionally, the key insight of Proposition 4 is evident in Figure 1.5—that is, neck-and-neck races are common when the losing penalty is dominant, but landslide victories are the norm when the winning prize is relatively large. In the top panel, the losing penalty is three times the size of the winning prize and the expected winning margin is 2.52 states. The bottom panel shows the reverse case where the winning prize is now three times the size of the penalty. The result is an increase in the expected winning margin to 8.99 states. Not only do the simulations in Figure 1.5 match the expected winning margins generated by Proposition 4, they also highlight that regardless of whether the winner of the initial stage contest wins the entire race, the winning margin remains the same. Both sides of the bimodal distribution peak at the appropriate point.

The second question complements and extends the discussion surrounding Figure 1.4 in the previous section. Instead of just looking at the probability of moving directly from the expected race path to a tie, the second question asks how much of an indicator the first stage is in predicting the overall winner. Simulations allow the myriad of possible race paths to be considered in addressing this question. Figure 1.5 provides a hint for answering it. Comparing the top panel where $Z = 1/3$ to the bottom panel where $Z = 3$, there does seem to be an increase in the percent of

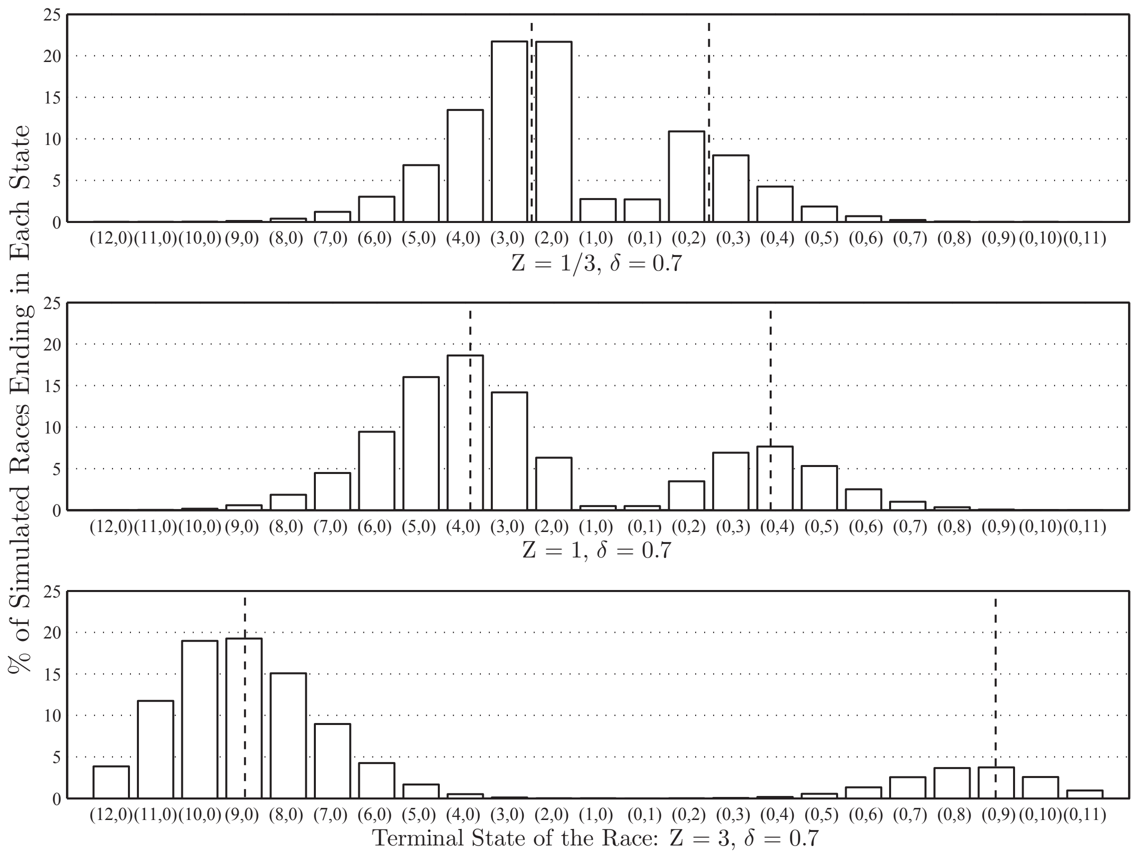

Figure 1.5: Percent of simulated races ending in each terminal state, conditional on Player B winning the initial contest at (12,12). Each panel is based on 100,000 simulations. Theoretical expected winning margins based on Proposition 4 are marked (from top to bottom: 2.52, 3.89, and 8.99).

races that are won by the initial contest winner. (Player B, the initial contest winner, wins the entire race at the left-hand-side states of each panel.) Figure 1.6, however, more thoroughly answers the second question for changes in both δ and Z in a race beginning at $(12, 12)$. Each (Z, δ) pair is based on 50,000 simulations. Notably, the connection between winning the first contest and winning the entire race is increasing in both the discount factor and the size of the winning prize relative to the losing penalty. There is an appreciable change in the winning probability, however, depending on whether the race is long enough for the underdog to make a last stand. The race is too short to make a last stand when $\alpha \geq 11$.[20] Hence, in a race that is too short for a last stand to occur, it is not surprising that the winner of the first contest has a high likelihood of becoming the overall winner. This is especially true considering that momentum builds for the frontrunner beyond a last stand (see Proposition 5), and also since the probability of catching up quickly diminishes in this region (see Figure 1.4). The loser of the first contest has a much greater chance of winning the race when a last stand is possible.

The third question parallels the first part of Proposition 6. As the winning margin decreases, Figure 1.7 shows that average cumulative effort increases. There are two likely factors that contribute to this result: first, based on Proposition 6, the upper bound of the bidding distribution is increasing; second, it takes more stage contests to complete the race. Figure 1.7 is based on 100,000 simulations of a race

[20] A last stand is narrowly missed when $\alpha = 11$, and only a one-state last stand is possible when $\alpha = 10$. For combinations of Z and δ such that $\alpha < 10$, the race is long enough for the underdog to make a full two-row last stand.

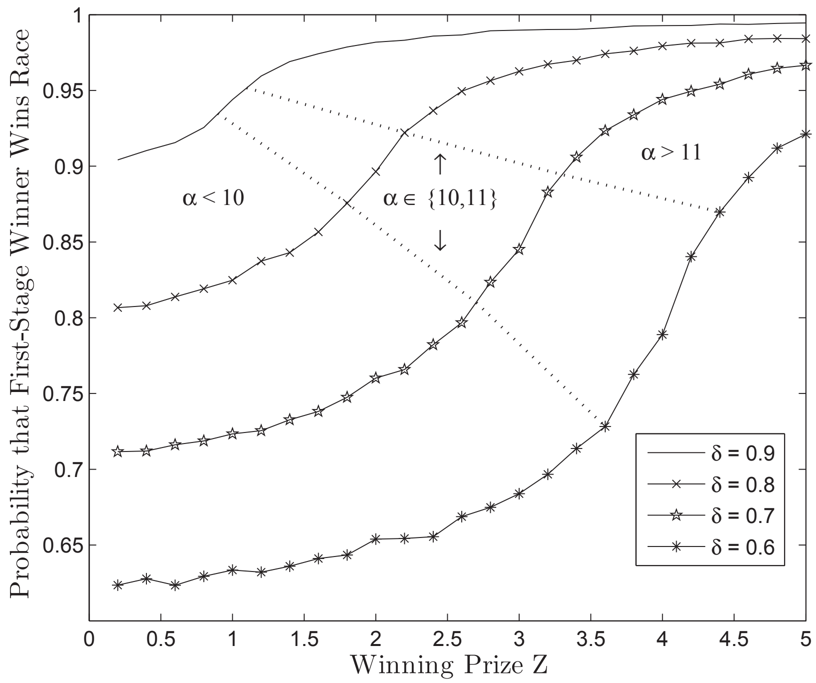

Figure 1.6: Probability of winning a race conditional on winning the first stage contest at (12,12). The race is sufficiently long for the underdog to make a last stand when $\alpha \leq 10$. The value for each $(Z,\, \delta)$ pair is based on 50,000 simulations.

starting at $(12, 11)$ with $\delta = 0.7$ and $Z = 1$.[21] With these parameters, the upper bounds on individual effort in Equations 1.6 and 1.7 are 1.23 and 0.77. These values roughly equal aggregate expenditure at $(2, 0)$ and $(4, 0)$, respectively. It is only when the race reaches $(1, 1)$ that expenditures really peak. However, from an efficiency standpoint, it may be comforting to realize that $(1, 1)$ is rarely ever reached (as can be seen in Figure 1.5).

Another result which is illustrated by Figure 1.7 is that regardless of the terminal location, the winner of the race expends more effort on average than the

[21] The figure can also be thought of as a race beginning at $(12, 12)$ where Player B wins the initial contest (due to the discounted distance to the finish line, both players have an expected effort at $(12, 12)$ of only $4.6{\times}10^{-4}$).

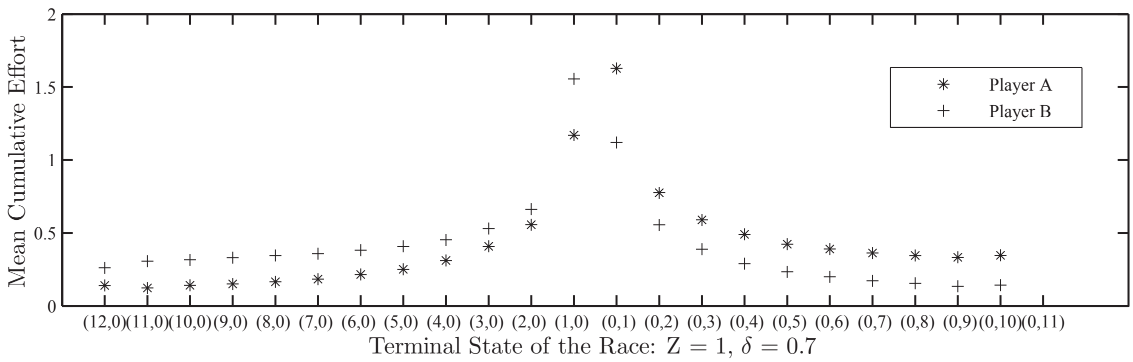

Figure 1.7: Mean cumulative effort for a race starting at (12,11). Based on 100,000 simulations.

loser. This is fairly intuitive since winning a contest requires having a higher bid. Slightly more interesting is that, keeping the winning margin constant, Player A typically spends more when he wins the race than when Player B wins. In order to overtake Player B's initial lead, Player A must spend a greater amount.

1.6 Conclusion

This paper has examined a two-player racing model that includes losing penalties and discounting, in addition to winning prizes. Movement within the race is determined by players competing in successive all-pay auctions. In this environment, players are motivated by both the incentive to win and the incentive to avoid losing. The latter influences the underdog to make a last stand, while the former incites the front-runner to defend his lead. Last stand behavior, in particular, critically alters the standard dynamics of racing models. Perhaps the most important insight is that a last stand prohibits momentum from building for the frontrunner too soon. Keeping the frontrunner in check gives the underdog a realistic chance to overtake the lead.

Moreover, the underdog can deter momentum more effectively as the losing penalty increases relative to the winning prize. The result is that neck-and-neck races are more likely to be observed when there is a substantially large penalty.

It is important to remember that last stands arise in an environment where players behave optimally and can be generated without relying on assumptions of irrational behavior. For the reasons described above, even the threat of an eventual last stand is a valuable asset to the underdog. There is also no time inconsistency in the threat since a last stand provides a jump in the probability of catching up at a critical time in the race. If the underdog successfully overtakes the lead at that point he will likely carry it to victory.

Naturally, there are caveats to the model presented in this paper. Using the all-pay auction, for instance, assumes that the outcome of a contest can hinge on the slightest increase in effort over that of an opponent's. Another caveat is that the race is limited to two players. Yet, three or more firms may be competing in a patent race, and presidential primaries often include at least three major contenders. A race with more than two players may be studied with a single prize or multiple prizes (first place, second place, etc.), as well as with losing penalties. Another interesting direction for further research is to incorporate asymmetric information about the winning prizes and losing penalties, or even about each player's distance from their finish line. In short, the dynamic racing model can be extended in the various ways that the static contest has been examined.

CHAPTER 2
DYNAMIC BEHAVIOR AND PLAYER TYPES IN MAJORITARIAN MULTI-BATTLE CONTESTS

2.1 Introduction

[1]When competition is costly, there are a variety of strategies which may be used in contests that are fought over several sequential battles. Players may opt to forgo the cost of competing by folding when the competition gets tough or even from the get-go, essentially surrendering. Aggressive players may fight at each battle or selectively at key battles. A last stand, for instance, is classically characterized by fierce competition along the brink of an overall loss. In comparing strategies, there is clearly a question of optimality. With a single player type, there are distinct theoretical predictions. If players adopt heterogeneous strategies however, the incidence and resulting interaction of the different player types is an open question. This latter case provides a richer backdrop for analyzing the known theoretical results.

Theoretically, Gelder (2014) provides sufficient conditions under which last stand behavior is optimal. If there is a cost to losing (beyond merely forgoing the winning prize) and if players have a time preference for when they win or lose, then making a last stand becomes a best response for a player on a losing trajectory. The optimality of last stand behavior contrasts with earlier work on dynamic contests where players slacken their effort or even give up entirely if they fall behind. An early example is Fudenberg et al. (1983) who examine preemption in patent races

[1]This chapter is joint work with Dan Kovenock.

by firms who have a marginal lead over their competitors. A more recent example is Konrad and Kovenock (2009), who, like Gelder (2014), examine a contest where players compete in several successive battles until one of the players achieves a critical number of victories. Konrad and Kovenock find that a player who is behind will completely give up unless there is a separate intermediate prize for winning each of the individual battles. This paper experimentally investigates the theoretical predictions of Gelder (2014) and Konrad and Kovenock (2009).

Our laboratory experiment examines competitive behavior in a best-of-seven tournament, a special case of the contest structure in Gelder (2014) and Konrad and Kovenock (2009). This type of a contest is known more generally as a two-player race and stems back to the patent race model of Harris and Vickers (1987). Although previous experimental work has addressed best-of-three tournaments (see Mago et al. 2013; Mago and Sheremeta 2012; Irfanoglu et al. 2010; and Sheremeta 2010), the full dynamics of last stand behavior can only be realized with a larger number of battles. The best-of-seven tournament provides a structure that is large enough to capture the desired dynamics, but small enough to keep the experiment simple. It is also a natural choice as it is used in many championship settings, such as the World Series.

The last stand environment of Gelder (2014) differs from Konrad and Kovenock (2009) through the introduction of a losing penalty and discounting. Since the behavioral predictions of Konrad and Kovenock are innocuous to the introduction of discounting, we can compare the two models solely on the basis of the losing penalty. We examine three prize-penalty combinations: one in which there is no penalty, one with a prize and penalty of equal magnitude, and a third in which the penalty is dramatically larger than the prize. For each of these three cases, the net difference between the prize and penalty is identical; so in a one-shot contest, equilibrium behavior would be the same in each case. The diverging predictions of surrendering versus making a last stand only come into play in a dynamic setting.

Unlike these and other conventional models of dynamic contests which assume a single player type, we find that multiple player types arise endogenously in the laboratory. Different players gravitate to different strategies and repeatedly use those strategies throughout the experiment. We find that the same core set of strategies independently appear time and time again across experimental sessions. The most common is to engage in a bidding war, a strategy which we refer to as escalate when challenged. Our hypothesized surrendering behavior is reflected two distinct strategies, maximin and passive when challenged. We also observe a remarkably pronounced last stand strategy. Defining a formal taxonomy, we classify the occurrence of each strategy and find that the surrendering strategies are particularly prominent in the treatment with no losing penalty. These same surrendering strategies are conspicuously scarce in the other two treatments. Conversely, the last stand strategy is a salient feature of the treatments with losing penalties but less common when a losing penalty is absent.

Winning margins in the overall tournament are naturally affected by last stand behavior. Theory predicts that neck-and-neck outcomes are more likely to be observed when the losing penalty is relatively large, while landslide victories are more probable when the winning prize is dominant. We find evidence to support this hypothesis.

We model each battle of the best-of-seven tournament as an all-pay auction—the highest bidder wins, but all players incur the cost of their own bid (see Hillman and Riley 1989; and Baye et al. 1996). Although the winning prize and losing penalty are fixed, allowing players to expend or conserve resources through the size of their bids makes the tournament a non-constant-sum game. Given the strategic complexity of this environment, we examine two separate subject pools: one in which all subjects had previously participated in some separate contest related experiment, and another comprised of a mix of subjects with and without such prior experience. Although there are a few noted exceptions, the two subject pools behave quite similarly.

Our paper fits within a small but emerging literature on dynamic contest experiments, as well as within a broader literature on contests and tournaments (see Dechenaux et al. 2012 for an extensive survey on experiments involving contests). In terms of "best-of" experiments, we bridge the gap between the work on best-of-three tournaments mentioned previously and the best-of-19 tournament in Zizzo (2002), which was explicitly patterned after Harris and Vickers (1987). Our paper is also closely related to the game of siege experiment by Deck and Sheremeta (2012). In their experiment, players are positioned asymmetrically so that one player (the defender) needs to win two successive battles to be victorious, while the attacker only needs to win one (this is the dynamic counterpart of a weakest link contest). This asymmetric starting point can be reached as an intermediate stage in a best-of-three and a best-of-seven tournament.

This paper also fits within the behavioral literature on heterogeneous player types in experiments. It can be expected that subjects will approach experimental settings from varying degrees of strategic sophistication, with the spectrum ranging from mere guesses to deliberate best-responses. Level-k and other theories have been developed to classify and better understand this range of behavior.[2] Other experimental work seeks to induce player types by allowing subjects to potentially be matched against a computer which is known to be programmed a certain way (see Embrey et al. 2014). Here, instead of inducing player types or focusing on the degree to which players are best-responding, we analyze the frequency and interaction of a handful of player types that arise endogenously. We also provide a rough ranking of these types in terms of average payoffs. To our knowledge, this is the first dynamic contest experiment to examine multiple player types.

We begin by giving a brief description of the theoretical framework (Section 2.2) and then describe how we set up the experiment (Section 2.3). Our analysis is in two parts. First, Section 2.4 provides a baseline analysis of our hypotheses within the aggregate data. Second, we then investigate systematic departures from the theoretical predictions (Section 2.5). This is where we address multiple player types. We also analyze a phenomenon of players bidding more aggressively—all else equal—immediately after winning a battle than after losing one.

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[2] Fragiadakis et al. (2013) assert that while strategic play can largely be captured by these theories, more needs to be done in modeling play that is non-strategic but which follows replicable rules-of-thumb.

2.2 Theory and Hypotheses

The winner of a best-of-seven tournament is the first player to win four battles. To keep track of each player's progress, we can model the state space as a pair (i, j) where i is the number of battles that Player A still needs to win and j is the number of battles that Player B still needs to win.[3] Hence, the tournament begins at state $(4, 4)$ and proceeds until it reaches $(0, j)$ for $(i, 0)$ for $i, j \in \{1, 2, 3, 4\}$. This is depicted in Figure 2.1. Once a player has won four battles, he receives a prize $Z \geq 0$ and his opponent incurs a penalty $L \leq 0$. Each battle consists of players competing in an all-pay auction with the winner of the auction advancing one state closer to victory.[4] The unique equilibrium of the two-player all-pay auction is in mixed strategies with players randomizing their bids between 0 and the smaller of the two players' valuation of the prize (Baye et al. 1996). While both players randomize over this interval, the player with the lower valuation will bid 0 with positive probability. That is, if ζ_H and ζ_L are the high and low valuations of the prize ($\zeta_H \geq \zeta_L > 0$), then the equilibrium bidding distributions are as follows:

$$F_H(h) = \begin{cases} h/\zeta_L & \text{if} \quad h \in [0, \zeta_L] \\ 1 & \text{if} \quad h > \zeta_L \end{cases}$$

---

[3] Tracking the absolute number of wins for each player requires a two dimensional state space—an alternative model, known as the tug-of-war, tracks the relative number of wins with a unidimensional state space. Within the tug-of-war setting, Konrad and Kovenock (2005) predict that laggards surrender when there is no losing penalty, while Agastya and McAfee (2006) find that last stand behavior is possible when there is a penalty.

[4] Although an arbitrary tie-breaking rule typically suffices, the equilibrium in Konrad and Kovenock (2009) requires that ties be awarded to the player who is ahead in the tournament. This assumption allows the frontrunner to coast to victory with a bid of zero when the laggard surrenders. Since this is a rather technical requirement, we use a fifty-fifty tie breaking rule in the experiment.

B wins

0	$(4,0)$	$(3,0)$	$(2,0)$	$(1,0)$	
1	$(4,1)$	$(3,1)$	$(2,1)$	$(1,1)$	$(0,1)$
2	$(4,2)$	$(3,2)$	$(2,2)$	$(1,2)$	$(0,2)$
3	$(4,3)$	$(3,3)$	$(2,3)$	$(1,3)$	$(0,3)$
4	$(4,4)$	$(3,4)$	$(2,4)$	$(1,4)$	$(0,4)$
	4	*3*	*2*	*1*	*0*

A wins

Figure 2.1: Best-of-seven tournament.

$$G_L(\ell) = \begin{cases} (\zeta_H - \zeta_L + \ell)/\zeta_H & \text{if} \quad \ell \in [0, \zeta_L] \\ 1 & \text{if} \quad \ell > \zeta_L \end{cases} \tag{2.1}$$

Given these distributions, then the expected payoffs are $u_H = \zeta_H - \zeta_L$ and $u_L = 0$; the winning probabilities are $p_H = 1 - \frac{\zeta_L}{2\zeta_H}$ and $p_L = \frac{\zeta_L}{2\zeta_H}$; and the expected bids are $\mathbb{E}[e_H] = \frac{\zeta_L}{2}$ and $\mathbb{E}[e_L] = \frac{\zeta_L^2}{2\zeta_H}$.

The bulk of the analysis in Gelder (2014) and in Konrad and Kovenock (2009) is in extending the one-shot all-pay auction to a dynamic structure where an actual prize is awarded only after a player has achieved a critical number of wins. Hence, it becomes necessary to identify the prize valuations at each interior state (i, j) where $i, j > 0$. These prize valuations are implicitly defined based on the marginal benefit of winning at (i, j) and being one state closer to overall victory versus losing and being one state closer to defeat. When losing is costless—as in Konrad and Kovenock—a player who is behind has a prize valuation of zero, so there is no incentive to compete.[5]

[5]Since the player who is behind receives zero from continuing to lose, and since the expected payoff from winning a single state is also zero, then the prize valuation is zero as well. Konrad and Kovenock also examine the case where there is an intermediate prize for

In Gelder's framework on the other hand, when there is a cost to losing and when players would prefer to win early and lose late, the prize valuations are always strictly positive so that players actively compete at every interior state.[6] The magnitudes of the prize valuations do, however, vary from state to state and across players. Gelder finds that there is a collection of states where the player who is behind in the tournament actually has the higher prize valuation and therefore tends to compete more aggressively. This heightened degree of competition from the underdog is what Gelder terms the last stand.

In terms of incentives, the last stand represents the position in the tournament where the underdog's incentive to avoid losing is stronger than the frontrunner's incentive to win. A player who must avoid losing today, or else incur a sufficiently large penalty, has a stronger motive to compete than the opposing player who may secure the victory tomorrow if not today. The precise collection of states where a last stand occurs depends on the ratio of the winning prize to the losing penalty, as well as the discount factor. The larger the penalty, the closer to the end of the tournament the last stand occurs. The likelihood of the underdog catching up after an unsuccessful last stand is minimal at best. In addition to the last stand, Gelder also finds that the frontrunner will defend his overall lead in the tournament if it

winning each battle. In that setting, the prize valuation for a player who is behind is solely based on the intermediate prize.

[6] An example of when these assumptions may be satisfied is the US presidential primaries. Candidates would typically prefer to secure their party's nomination early in the election cycle to have more time to prepare for the general election. On the losing side, the potential loss of political capital is likely higher for candidates who unmistakably lose at an early stage and are not able to demonstrate their viability for future campaigns.

is threatened. The "defense of the lead" occurs when the frontrunner only has a one-state lead in the tournament, and it entails a much higher expenditure from the frontrunner than from the underdog in expectation. Thus the last stand acts as a defensive push, while the defense of the lead acts as an offensive one.

Based on the theoretical predictions, there are five main hypotheses that we will examine in this experiment:

Hypothesis 1. *Players on a losing trajectory will make a last stand if the penalty for losing is large relative to the winning prize.*

Hypothesis 2. *If there is no losing penalty, then a player who is behind will surrender (or cease to compete).*

Hypothesis 3. *Players with a one battle lead in the tournament will compete more aggressively than their opponent in order to maintain their lead.*

Hypothesis 4. *The expected winning margin is increasing in the size of the winning prize relative to the losing penalty.*

Hypothesis 5. *The winner of the initial battle will win the tournament the majority of the time.*

2.3 Experimental Design

We conducted 18 experimental sessions, each composed of 12 subjects. These sessions were conducted at the Economic Science Institute, Chapman University, in computer labs where the computers were separated by partitions for privacy. The experiment began with subjects reading the instructions on their computer (a copy of the instructions is provided in the appendix). After reading the instructions, subjects were given a short quiz comprised of three possible scenarios for how a best-of-seven tournament could unfold. Subjects were then asked to compute the payoff for each scenario. The purpose of this short quiz was to ensure that subjects had a basic level of comprehension about the structure of the game. The quiz was immediately followed by a short risk preference lottery *à la* Holt and Laury (2002). During the main portion of the experiment, subjects participated in 20 best-of-seven tournaments. Subjects were randomly and blindly paired and re-paired for each of these tournaments via the computer network. At the conclusion of the experiment, subjects completed a demographics survey and were paid in cash based on their performance in two randomly selected tournaments.

Each battle of a best-of-seven tournament was treated as an all-pay auction: subjects placed bids simultaneously and the highest bidder won (ties were broken randomly). In the all-pay fashion, the sum of a player's bids throughout a tournament was deducted from his or her payoff for that tournament. Additionally, the winner of the tournament received a prize and the loser incurred a penalty. Since time preferences for winning or losing in Gelder (2014) were implemented through a discount factor, and since discounting is difficult to replicate in a short laboratory experiment, we followed a common practice from macroeconomic experiments by implementing discounting via a continuation probability (see, for instance, Duffy 2008, and Noussair and Matheny 2000). Until a player had succeeded in winning four battles, there was a 90% probability that the tournament would actually continue from one battle to the next. If a tournament ended prematurely, neither player would receive a prize or a penalty, but players still had to pay their bids. Our justification for this approach is that a continuation probability is equivalent to discounting in terms of the expected payoffs.[7]

We conducted three separate payoff scenarios: the first with a substantial losing penalty and meager winning prize (Win 15 Lose 285), the second with an equal prize and penalty (Win 150 Lose 150), and the third with a sizable prize but no penalty (Win 300 Lose 0). Prizes, penalties, as well as all bids, were denominated in an experimental currency called rupees, where 50 rupees = $1 US dollar. In order to make the stakes comparable across treatments, we fixed the difference between the positive prize and the negative penalty at 300 rupees. The two treatments with non-zero penalties coincide with the Gelder (2014) model, while the treatment with no losing penalty fits the Konrad and Kovenock (2009) model.

For each treatment, we ran a total of six experimental sessions—three of which were open to all individuals in our subject pool, and three were specifically limited to subjects who had previously participated in an experiment involving contests or

[7]The random ending rule may also be thought of as the potential that some exogenous factor suddenly disrupts the conflict (such as the cavalry coming to save the day). An alternative method for implementing discounting is to adjust the size of the prize and the penalty according to the winning margin. Since the winning margin is based on the number of rounds that players compete, this is a present value interpretation of discounting. A benefit of using the random ending rule in an experimental setting is that the order of magnitude of expenditures early in the tournament remains comparable to that of the prize and penalty at later stages of the tournament. Noussair and Matheny (2000) compared both the random ending rule and the present value interpretation of discounting in an experiment involving a single agent dynamic optimization problem.

Table 2.1: Sessions by Treatment.

Sessions	Subject Pool	Prize	Penalty	Bid Observations
3	Experienced	15	285	1650
3	Experienced	150	150	1500
3	Experienced	300	0	1530
3	Mixed	15	285	1590
3	Mixed	150	150	1614
3	Mixed	300	0	1420

contest theory. We will refer to the first subject pool as mixed and to the second as experienced.[8]

A summary of the experimental sessions by treatment is shown in Table 2.1. Bid observations in this table are limited to those during the last ten tournaments since that will be the focus of our analysis.

During a best-of-seven tournament, subjects could see both their own and their opponent's previous bids.[9] They also could see how many rounds they had won or lost, as well as the sum of their bids up to that point in the tournament. An example of the bidding screen is shown in Figure 2.2. The bidding screen would also alert subjects when a tournament had finished, either by a player winning four rounds or by the computer ending the tournament early. After displaying the final outcome and

[8]The Economic Science Institute at Chapman University conducts a large volume of economic experiments and keeps records of the different experiments subjects participate in. We made no attempt to regulate the number of subjects in the mixed sessions who had previously enrolled in a contest related experiment (the average was 6.4 with a minimum of three and a maximum of eleven).

[9]Since equilibrium bidding strategies are Markov perfect, bids at the current state should not be affected by the knowledge of bid realizations at previous states.

payoffs for the tournament, subjects would then be randomly re-matched to begin a new tournament.

In order to cover bids and potential losses, each subject received an initial endowment of rupees at the start of the experiment. At the end of the experiment, two of the best-of-seven tournaments were randomly selected for payment. Subjects were then paid, with winning prizes in the two tournaments being added to their endowment, but bids and losing penalties being subtracted from it. Since losing penalties varied across treatments, and since bids and losing penalties were both deducted from the same account, we wanted to make the treatments comparable in terms of the bidding budget. We accomplished this by varying the initial endowment across treatments so that it was composed of an effective bidding budget (700 rupees) plus the size of the losing penalty. Thus, for penalties of 285, 150, and 0, the endowment was 985, 850, and 700. In paying for two tournaments, we decided to set the bidding budget at more than twice the prize-penalty spread of 300 rupees. We did not want subjects to be budget constrained—especially in tournaments that continued onto the seventh round. In most cases, the bidding budget was more than sufficient.[10] For each round of a best-of-seven tournament, we allowed players to bid between 0 and

[10]Theoretical expected expenditures throughout the entire tournament are as follows: 131.3 (Win 15 Lose 285), 114.4 (Win 150 Lose 150), and 109.4 (Win 300 Lose 0). The bidding budget is large enough to amply cover these amounts. However, if a player happened to consistently bid at the top of the equilibrium bidding distribution along the most expensive path of the tournament, then cumulative expenditures could be as high as 975.3 (Win 15 Lose 285), 1001.7 (Win 150 Lose 150), or 1031.7 (Win 300 Lose 0). Thus there are contingencies of the tournament for which the budget constraint is binding. We faced a dilemma in making the budget large enough that it would not bind in most cases, but also small enough that the losing penalties would still have some bite.

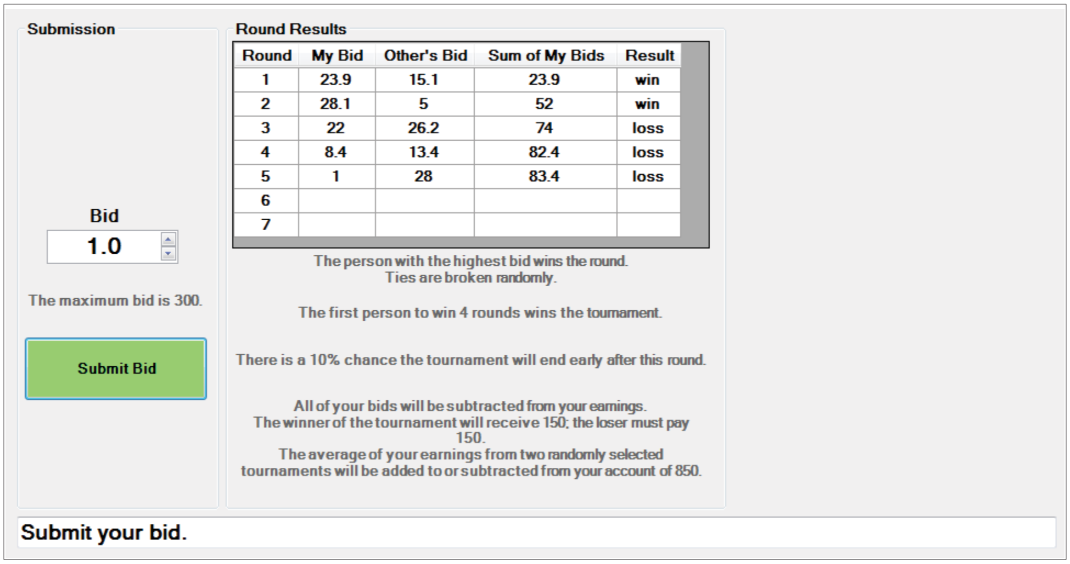

Figure 2.2: Bidding screen during a best-of-seven tournament.

300 inclusive (with up to one decimal place).

2.4 Initial Results

2.4.1 Summary Statistics

Before analyzing the main hypotheses, we will briefly highlight the major summary statistics. The fundamental level of observation is a player's bid at a particular state (i, j) within tournament t. As a whole, the data form a panel with 20 tournament observations per subject and up to seven bid observations per tournament. Variance in the bidding observations is considerably higher during the initial tournaments of the experiment since subjects are learning the structure of the game. The analysis in this paper omits all observations from the first ten (of 20) tournaments. Here we will summarize observations, winning probabilities, and the distribution of

Table 2.2: Bidding Observations by State (i, j) and by Treatment.

		Win 15 Lose 285				**Win 150 Lose 150**				**Win 300 Lose 0**			
Experienced	*0*	39	31	32	22	34	31	30	15	54	27	22	15
	1	54	56	58	44	55	63	45	30	66	47	40	30
	2	95	91	78		95	72	44		102	67	60	
	3	169	122			157	92			170	96		
	4	360				360				360			
Mixed	*0*	40	32	25	22	47	37	20	27	51	32	22	9
	1	66	54	51	44	77	61	51	54	59	41	33	18
	2	101	71	70		118	67	56		91	68	56	
	3	166	98			163	70			156	90		
	4	360				360				360			
Pooled	*0*	79	63	57	44	81	68	50	42	105	59	44	24
	1	120	110	109	88	132	124	96	84	125	88	73	48
	2	196	162	148		213	139	100		193	135	116	
	3	335	220			320	162			326	186		
	4	720				720				720			
		4	*3*	*2*	*1*	*4*	*3*	*2*	*1*	*4*	*3*	*2*	*1*

bids by state and treatment.

Table 2.2 shows the number of observations by treatment at state (i, j) where $i \geq j$. Due to the symmetry of the tournament, whenever one player is at (i, j), their opponent is at (j, i), so the table only shows states where a player is behind or the tournament is tied. The random ending rule causes the total number of observations

Table 2.3: Winning Percentages in State (i, j): Theoretical and Observed.

		Win 15 Lose 285				**Win 150 Lose 150**				**Win 300 Lose 0**			
		4	*3*	*2*	*1*	*4*	*3*	*2*	*1*	*4*	*3*	*2*	*1*
Theory	*1*	56.4	52.4	5.2	50	26.3	26.3	2.6	50	0	0	0	50
	2	52.9	5.6	50		27.8	2.8	50		0	0	50	
	3	5.6	50			2.9	50			0	50		
	4	50				50				50			
Experienced	*1*	27.8	44.6	44.8	50	38.2	50.8	33.3	50	18.2	42.6	45.0	50
	2	36.8	45.1	50		35.8	33.3	50		22.5	46.3	50	
	3	40.2	50			34.4	50			31.8	50		
	4	50				50				50			
Mixed	*1*	39.4	40.7	51.0	50	39.0	39.3	60.8	50	13.6	22.0	33.3	50
	2	29.7	52.1	50		30.5	44.8	50		29.7	42.6	50	
	3	34.9	50			22.1	50			32.7	50		
	4	50				50				50			
Pooled	*1*	34.2	42.7	47.7	50	38.6	45.2	47.9	50	16.0	33.0	39.7	50
	2	33.2	48.1	50		32.9	38.8	50		25.9	44.4	50	
	3	37.6	50			28.1	50			32.2	50		
	4	50				50				50			

to decrease by roughly 10% after each of the first four battles.[11] In successive states, the number of observations continues to decrease through the random ending rule, but also decreases through players winning or losing tournaments.

Theoretical and observed probabilities of winning a battle at each state are

[11]For instance, in the pooled data of the Win 300 Lose 0 treatment, there are a total of 720 observations in the first round at $(4, 4)$. Of these, 90.6% persist to the second round—326 at $(4, 3)$ and an additional 326 at $(3, 4)$.

shown in Table 2.3. Symmetry allows us to again focus on the states where a player is behind or the tournament is tied. The major patterns of competition can be seen by examining the theoretic winning probabilities. For instance, the defense of the lead is reflected by the remote winning probabilities at states $(4,3)$, $(3,2)$, and $(2,1)$ in the Win 15 Lose 285 and the Win 150 Lose 150 treatments. The last stand is evidenced by the underdog having the higher winning probability at states $(4,2)$, $(4,1)$, and $(3,1)$ of the Win 15 Lose 285 treatment. Although not as strong, the underdog is still expected to win roughly a quarter of the time at these three states in the Win 150 Lose 150 treatment.[12] Finally, the tendency to surrender is depicted in the Win 300 Lose 0 treatment by the zero probability of winning a battle when a player is behind in the tournament.

Although the observed probabilities from the laboratory fail to capture the defense of the lead, the basic contrast between making a last stand and surrendering can be seen. In the pooled data for instance, the winning probability at $(4,1)$ falls to 16% in the Win 300 Lose 0 treatment—less than half the corresponding values of 34.2% and 38.6% in the two treatments with a losing penalty. In the mixed group, the Win 15 Lose 285 and Win 150 Lose 150 treatments actually have sizable jumps in the winning probability from $(4,2)$ to $(4,1)$ of 8 to 10 percentage points. While such a jump is absent in the experienced Win 15 Lose 285 treatment, it does boast the highest winning probabilities at $(4,3)$ and $(4,2)$, indicating that players were more

[12]For the Win 150 Lose 150 treatment, a best-of-seven tournament is not large enough to include the states where the player who is behind wins battles with more than one-half probability.

likely to regain lost ground earlier in the tournament.

While the winning probabilities address the relative size of bids between (i, j) and (j, i), it is also informative to have an absolute measure of bids at the different states, both theoretically and in the experiment. The equilibrium bidding distributions can be fully characterized with two sets of numbers: the size of the mass point at zero and the upper bound of the bidding distribution (see Equation 2.1; above the mass points, players uniformly randomize between zero and the upper bound). These are both presented in Table 2.4 by treatment and state. The mass points largely mirror the major features of the theoretical winning probabilities.[13] A prominent feature of the upper bounds is that bids at states where the tournament is tied far and away exceed those at any other state—even when a losing penalty is present. We can also see that bids along the main diagonal of the tournament are increasing in the relative size of the winning prize, while competition off of the main diagonal is increasing in the size of the losing penalty.

As a rule, bids within the experimental data are much lower than predicted when the tournament is tied, but can often be considerably higher than predicted at the other states. Figure 2.3 shows the empirical bidding distributions at each state in the pooled data; and for further detail, the 25th, 50th, and 75th percentile bids for each state and treatment are shown in Table 2.5. A pattern that is present in every experimental treatment, as well as in the theoretical distributions, is that bids

[13] In the Win 300 Lose 0 treatment, the fact that both players always bid zero when one player is ahead is an artifact of the tie-breaking rule in Konrad and Kovenock (2009) that was mentioned previously.

Table 2.4: Theoretical Distributions: Mass Point at Zero (top row); Upper Bound (bottom row).

	Win 15 Lose 285				Win 150 Lose 150				Win 300 Lose 0			
1	0	0	0.90	0	0.47	0.47	0.95	0	1.0	1.0	1.0	0
2	0	0.89	0	0	0.44	0.94	0	0	1.0	1.0	0	1.0
3	0.89	0	0	0.05	0.94	0	0	0	1.0	0	1.0	1.0
4	0	0	0.06	0.13	0	0	0	0	0	1.0	1.0	1.0
1	25.9	27.2	28.5	300.0	15.0	15.0	15.0	300.0	0	0	0	300.0
2	22.0	24.4	244.4	28.5	13.5	13.5	256.5	15.0	0	0	270.0	0
3	19.8	197.9	24.4	27.2	12.2	218.7	13.5	15.0	0	243.0	0	0
4	160.3	19.8	22.0	25.9	185.9	12.2	13.5	15.0	218.7	0	0	0
	4	*3*	*2*	*1*	*4*	*3*	*2*	*1*	*4*	*3*	*2*	*1*

progressively increase as the tournament proceeds to the top-right. It is uncommon for the median bid to be above ten in any treatment until both players have won at least one battle. Thereafter the median bids quickly rise as the tournament becomes more closely contended. By $(1, 1)$, bids of 50 to 100 are commonplace. At states along the left edge of the tournament, at least a quarter of the bids are zero. These mass points are particularly conspicuous in the Win 300 Lose 0 treatment where the median bid at $(4, 2)$ is zero in the pooled data, and even 75th percentile bid is only 0.1 at $(4, 1)$. Not only are these players surrendering at these states, but their rivals are responding with progressively lower bids. By $(1, 4)$ in the experienced group, a mere bid of 1.0 marks the median. There is a different behavior in the Win 15 Lose 285 treatment—the primary difference being what happens at the top of the distribution. Now instead of 0.1, the 75th percentile bid at $(4, 1)$ is 19 or 20. The mixed treatment is particularly suggestive of last stand behavior since the 75th percentile jumps from a bid of 10 at $(4, 2)$ to a bid of 20 at $(4, 1)$.

The shape of the bidding distributions is rather interesting in light of past experimental results. In one-shot contest settings, the experimental bidding distribution is frequently bifurcated with subjects submitting either high or low bids, but largely avoiding bids in the middle range. Ernst and Thöni (2013), for instance, have documented this behavior with the one-shot all-pay auction and have demonstrated that prospect theory provides a possible explanation.[14] Here, however, instead of bifurcated distributions that are concave over the lower bidding range and convex over the upper range, bidding distributions at most states are concave throughout. Even at states $(2, 1)$, $(1, 1)$, and $(1, 2)$ where the shape of the distribution appears to change a little, the curvature goes in the opposite direction with more mass being placed on intermediate bids—slightly convex in the lower bidding range and concave in the upper range.

2.4.2 Fight or Surrender

In analyzing bidding behavior, it is pertinent to identify whether the frontrunner or the underdog is bidding more aggressively at each stage of the tournament—or even if there is a difference. If the underdog is engaging in last stand behavior, then we would expect his bids to increase relative to his opponent's as he nears an overall loss. Therefore, we want to compare the underdog's bid at state (i, j) with the frontrunner's bid at the symmetric state (j, i). To do this, we use a fixed effects regression

[14] The prospect theory explanation has also been applied to Tullock contest experiments (see Sheremeta, 2013).

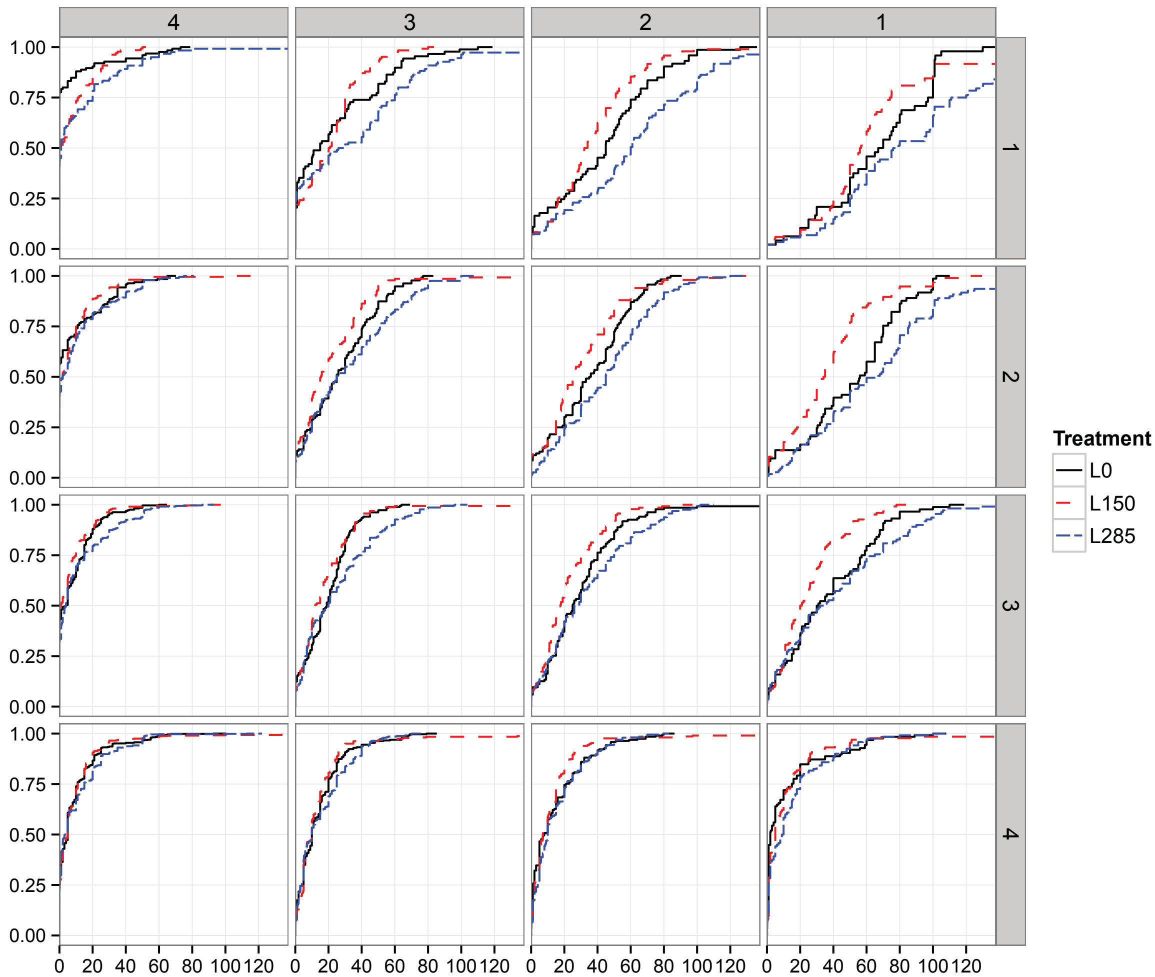

Figure 2.3: CDF of bids at each state by treatment in the pooled data.

Table 2.5: 25th, 50th, and 75th Percentiles of Bids at State (i, j).

		Win 15 Lose 285				**Win 150 Lose 150**				**Win 300 Lose 0**			
		4	*3*	*2*	*1*	*4*	*3*	*2*	*1*	*4*	*3*	*2*	*1*
Experienced		0.0	0.0	40.3	50.0	0.0	3.8	15.0	39.2	0.0	1.0	25.2	49.2
	1	1.0	28.0	67.5	98.0	3.0	20.0	26.0	50.0	0.0	15.0	46.0	66.0
		19.2	60.0	100.0	137.8	12.5	30.0	35.0	60.8	0.1	47.5	65.2	95.0
		0.0	15.0	30.0	38.2	0.0	7.0	7.8	15.0	0.0	5.0	14.0	29.0
	2	5.0	38.1	51.5	77.4	5.0	14.1	20.0	30.0	0.0	20.0	30.0	52.8
		20.0	60.0	70.0	89.0	10.0	25.0	30.4	40.0	5.0	40.0	54.8	70.8
		0.0	8.3	18.0	14.2	0.0	7.0	10.0	11.0	0.0	6.0	10.0	11.5
	3	6.0	25.5	35.0	40.0	5.0	11.0	16.0	16.0	1.0	16.0	20.0	26.0
		20.0	42.0	58.5	70.0	10.0	21.0	25.5	30.9	11.8	27.2	35.5	53.5
		0.1	5.0	5.0	1.0	0.1	5.0	2.0	1.8	0.0	2.0	1.0	1.0
	4	5.5	13.0	15.0	8.0	5.0	7.0	5.3	7.0	3.0	7.0	5.0	1.0
		15.0	25.0	30.0	20.8	10.0	13.0	15.0	15.0	9.0	17.5	16.0	10.0
Mixed		0.0	1.9	27.1	52.8	0.0	7.0	28.8	50.0	0.0	1.0	15.0	49.2
	1	1.1	30.0	50.5	71.5	0.1	25.0	45.0	61.0	0.0	15.0	45.0	75.5
		20.0	60.2	77.5	100.0	10.0	30.0	61.1	87.3	0.1	33.0	60.0	99.0
		0.0	6.9	20.0	35.5	0.0	7.9	18.0	24.5	0.0	14.5	24.5	33.0
	2	1.0	18.7	38.5	50.0	0.2	20.0	34.0	40.2	0.1	30.0	40.5	60.0
		10.0	38.0	54.5	77.5	10.0	39.4	50.0	53.5	19.5	42.2	50.2	70.0
		0.0	5.0	8.0	8.2	0.0	7.6	10.8	10.0	0.0	9.4	18.5	20.0
	3	2.0	10.0	18.9	25.0	0.5	17.0	20.0	25.6	5.0	21.0	30.5	40.0
		10.0	35.0	35.0	54.0	9.8	31.4	34.5	35.1	16.0	29.5	44.0	60.0
		0.1	2.0	2.0	1.0	0.2	5.0	4.0	1.0	0.1	5.0	5.0	1.0
	4	2.0	5.2	8.0	10.0	5.0	10.0	7.5	5.0	5.0	11.6	10.0	5.0
		15.0	25.0	15.0	17.8	15.0	23.0	20.0	11.0	15.0	21.6	27.4	20.0
Pooled		0.0	1.0	31.0	50.8	0.0	5.0	16.8	40.0	0.0	1.0	20.0	49.0
	1	1.0	28.0	60.0	77.6	1.1	20.6	33.0	56.4	0.0	15.0	45.0	68.5
		20.0	60.0	90.0	113.2	11.2	30.0	50.2	74.2	0.1	45.3	63.0	97.8
		0.0	10.0	20.8	36.0	0.0	7.0	15.0	19.5	0.0	10.0	18.0	30.5
	2	2.0	25.0	45.0	67.0	2.0	15.5	25.2	34.5	0.0	25.0	35.5	56.0
		15.0	51.0	64.2	85.5	10.0	34.5	45.1	50.0	11.0	40.8	52.0	70.0
		0.0	6.7	13.0	12.0	0.0	7.0	10.2	10.0	0.0	8.0	14.0	15.0
	3	4.0	20.0	27.5	33.5	1.0	12.1	17.6	20.5	4.0	19.0	25.0	30.5
		15.0	38.5	50.0	63.8	10.0	25.8	31.4	35.0	15.0	28.4	40.0	57.5
		0.1	4.0	4.0	1.0	0.2	5.0	2.0	1.0	0.0	4.2	1.0	1.0
	4	3.8	10.0	10.0	9.5	5.0	10.0	7.1	5.0	5.0	10.0	8.0	2.0
		15.0	25.0	21.0	20.0	11.0	17.5	16.0	15.0	11.0	20.0	21.0	15.0
		4	*3*	*2*	*1*	*4*	*3*	*2*	*1*	*4*	*3*	*2*	*1*

model with cluster robust standard errors where the dependent variable is a player's bid.

To account for the different states within the tournament, we include a set of dichotomous variables, equal to one if a bid is made from that particular state and zero otherwise. Since we are interested in comparing bidding behavior at state (i, j) with state (j, i), we take advantage of the fact that when a categorical variable with N distinct values is represented by a set of $N - 1$ dichotomous variables, the coefficients of the $N-1$ dichotomous variables can be interpreted directly in reference to the omitted N^{th} value. Thus, we are interested in the coefficient for state (i, j) in a regression where (j, i) is the omitted state.[15] Letting $\mathbf{s(j,i)}$ be the vector of dichotomous state variables which omits state (j, i), we use the following fixed effects model to predict player k's bid at time t within the experiment:[16]

Model 1. $\widehat{bid}_{k,t} = \beta_0 + \mathbf{s(j,i)}'_{k,t}\boldsymbol{\beta_s} + f_k + \varepsilon_{k,t}$

Table 2.6 shows the Model 1 coefficients for bids made at (i, j) relative to (j, i) where $i > j$. A common result that holds throughout most of the treatments is that players tend to bid more aggressively when they are behind, and they bid increasingly more aggressively as they fall farther and farther behind. For instance, in the mixed group of the Win 15 Lose 285 treatment, players tend to bid 5.61 more rupees at

[15]Alternatively, for two states that are not omitted, we could obtain the relative difference by subtracting one of the coefficients from the other, taking care to compute the appropriate standard error for the difference.

[16]There are two time components: the tournament number and the bids within each tournament. Since the number of bids per tournament may vary between one and seven, we interpolate the timing of each bid to be at one-seventh intervals between tournaments.

$(4,3)$ than they would at $(3,4)$; this difference then increases until by $(4,1)$, players are submitting bids that average 15.43 rupees higher than at $(1,4)$. Similar increases can be seen in each treatment of the experienced group, although the magnitudes are not quite as large. Given our last stand hypothesis, we would expect to see this type of behavior in the Win 15 Lose 285 treatment and to a lesser degree in the Win 150 Lose 150 treatment. It is somewhat surprising, however, that the regression for the Win 300 Lose 0 treatment produces similar results. In both the mixed and experienced groups of the Win 300 Lose 0 treatment, players submit bids at $(4,3)$ and $(4,2)$ that are significantly greater than the bids they would place at $(3,4)$ and $(2,4)$. In the experienced group, the coefficient for bids at $(4,1)$ relative to $(1,4)$ is a shockingly large and significantly positive 10.31. Given that the overwhelming majority of bids at $(4,1)$ are at or near zero, and that the median bid at $(1,4)$ is one, this coefficient is largely driven by outliers. The mixed treatment, however, does appear to exhibit some of the surrendering behavior we would expect to see when losing is costless. At $(3,1)$ and $(2,1)$ players bid 10 to 14 rupees less than they would at $(1,3)$ and $(1,2)$.

There is little if any evidence from these regressions of players defending their lead in the tournament. Theoretically, this should happen when players are ahead by one state. This may explain the significantly negative coefficient at $(4,3)$ in the mixed Win 150 Lose 150 treatment, and it is also possible that the surrendering behavior at $(3,1)$ and $(2,1)$ in the mixed Win 300 Lose 0 treatment is in fact a defense of the lead. Finally, it is worth noting that the regression coefficients for the pooled data are roughly an average of when the regressions are run separately and that statistical significance carries over in most cases.

2.4.3 Winning Margins and Initial Leads

Our final two hypotheses address the implications of making a last stand versus surrendering—specifically in terms of the size of the winning margin, and also in terms of the importance of winning the initial battle of the tournament. A natural consequence of last stand behavior is that the size of the winning margin tends to decrease. Landslide victories, on the other hand, frequently occur when players are prone to surrendering. Table 2.7 reports the distribution of winning margins for completed tournaments within each treatment.[17] The most pronounced differences in these distributions occur at the endpoints where the winning margin is either one or four battles. In support of our hypothesis, landslide victories are clearly more pronounced in the Win 300 Lose 0 treatment. There, landslide victories account for over 40% of completed tournaments, while the number is between 27% and 31% in the treatments with losing penalties. Neck-and-neck victories likewise increase substantially from the Win 300 Lose 0 treatment to the Win 15 Lose 285 treatment—increasing by 5.5 percentage points in the experienced group and 12.3 percentage points in the mixed group.

The initial battle is often viewed as pivotal in deciding the ultimate outcome of a dynamic contest.[18] It is advantageous in Gelder (2014) but decisive in Konrad and

[17]The counts underlying the percentages in Table 2.7 have been adjusted to account for attrition with the random ending rule (e.g. a winning margin of 4 is equal to $1/\delta$ winning margins of 3).

[18]For example, Klumpp and Polborn (2006) examine the disproportionately large amount

Table 2.6: Bidding Behavior at State (i, j) Compared to (j, i): Model 1.

		Win 15 Lose 285			**Win 150 Lose 150**			**Win 300 Lose 0**		
		4	*3*	*2*	*4*	*3*	*2*	*4*	*3*	*2*
Experienced	*1*	6.37 (2.70) $> 0^{***}$	−4.27 (5.83)	−0.95 (4.43)	5.63 (1.89) $> 0^{***}$	4.01 (2.54) $> 0^{*}$	−1.26 (3.08)	10.31 (2.60) $> 0^{***}$	−0.65 (5.22)	−0.53 (4.95)
	2	4.56 (1.95) $> 0^{***}$	4.47 (2.10) $> 0^{**}$		4.92 (0.81) $> 0^{***}$	0.90 (0.98)		6.28 (1.24) $> 0^{***}$	5.12 (2.61) $> 0^{**}$	
	3	1.32 (1.56)			2.02 (0.73) $> 0^{***}$			2.67 (0.90) $> 0^{***}$		
Mixed	*1*	15.43 (2.17) $> 0^{***}$	14.23 (5.28) $> 0^{***}$	4.10 (3.70)	4.31 (3.96)	3.47 (1.65) $> 0^{**}$	5.38 (3.81) $> 0^{*}$	1.73 (2.04)	−10.14 (2.51) $< 0^{***}$	−13.99 (5.32) $< 0^{***}$
	2	9.62 (1.80) $> 0^{***}$	5.58 (1.50) $> 0^{***}$		0.66 (3.34)	0.55 (2.45)		5.01 (1.45) $> 0^{***}$	−1.19 (4.29)	
	3	5.61 (0.96) $> 0^{***}$			−3.80 (2.66) $< 0^{*}$			2.55 (1.45) $> 0^{**}$		
Pooled	*1*	10.91 (2.02) $> 0^{***}$	4.32 (3.85)	1.29 (3.45)	5.06 (2.30) $> 0^{**}$	3.82 (1.56) $> 0^{***}$	2.24 (2.27)	6.33 (1.69) $> 0^{***}$	−5.08 (2.97) $< 0^{**}$	−6.69 (3.77) $< 0^{**}$
	2	6.72 (1.42) $> 0^{***}$	4.78 (1.21) $> 0^{***}$		2.69 (1.87) $> 0^{*}$	0.74 (1.34)		5.73 (0.85) $> 0^{***}$	1.94 (2.92)	
	3	3.19 (0.79) $> 0^{***}$			−0.82 (1.43)			2.67 (0.91) $> 0^{***}$		
		4	*3*	*2*	*4*	*3*	*2*	*4*	*3*	*2*

Standard errors in parentheses. Significance levels: * 10%, ** 5%, *** 1%.

Table 2.7: Winning Margins by Treatment (in %).

		Winning Margin			
		4	3	2	1
Experienced	L285	27.2	24.1	27.6	21.1
	L150	27.0	27.3	29.4	16.3
	L0	41.0	22.8	20.6	15.6
Mixed	L285	29.3	26.0	22.6	22.1
	L150	31.4	27.4	16.5	24.7
	L0	40.5	28.2	21.5	9.8
Pooled	L285	28.2	25.0	25.2	21.6
	L150	29.4	27.4	22.4	20.9
	L0	40.7	25.4	21.1	12.8

Table 2.8: Percent of Initial Battle Winners to Win the Tournament.

		Win Tournament	End in Lead
Experienced	L285	68.5	70.6
	L150	78.2	78.3
	L0	76.3	77.2
Mixed	L285	79.0	75.0
	L150	79.4	79.4
	L0	78.9	79.4
Pooled	L285	73.7	72.8
	L150	78.8	78.9
	L0	77.6	78.3

Kovenock (2009). In Gelder's framework, the probability of an upset is increasing in the relative size of the losing penalty (as is the strength of the last stand). To a degree this is apparent in the data as well. The first thing that Table 2.8 illustrates is that winning the initial contest is a strong correlate of winning the ultimate tournament—or at least being in the lead at the time the tournament ends (whether by winning or through the random ending rule). Across the different treatments, roughly 70% to 80% of all winners at state $(4, 4)$ went on to win the tournament. Second, the Win 15 Lose 285 treatment appears to have the highest degree of upsets. This is clear in the experienced group where the Win 15 Lose 285 treatment is several percentage points below the other two treatments. The mixed group is less clear since the Win 15 Lose 285 treatment is only lower than the other two treatments when tournaments that ended early are factored in.

2.5 Departures from Theoretical Predictions

Having examined each of the hypotheses predicted by theory, we now investigate two conspicuous behavioral patterns. The first is whether bidding behavior is affected by winning or losing the previous contest. A Markovian property of the equilibrium is that bidding distributions at each state should not be affected by the past history of the tournament. In practice, however, there are several instances where this Markovian property fails to hold. The second issue we address is how individuals behave from tournament to tournament—an analysis which has resulted in a distinct

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of attention that New Hampshire and Iowa receive as the first states to vote in the US presidential primary elections.

taxonomy of player types based on the strategies players use. Although such a set of heterogeneous strategies is not predicted by theory, we find that the distribution of player types across treatments does support our theoretical predictions.

2.5.1 Non-Markovian Bidding

We are interested in seeing whether there are any clear effects (psychological or otherwise) from having won or lost the previous round. We can investigate this issue by constructing a second fixed effects model that factors in the win-loss outcome of the previous state. Since the effect of a win or a loss may potentially differ from one state to another, we expand the representation of a state from (i, j) to (i, j, h) where h denotes whether Player A arrived at (i, j) by winning or losing the previous round. This expanded state space allows us to compare how players behave when they arrive at a state by winning with how they behave when they arrive at the same state by losing. We specifically want to compare the coefficient for $(i, j,\ won)$ in a regression where $(j, i,\ lost)$ is the omitted state with the coefficient for $(i, j,\ lost)$ when $(j, i,\ won)$ is the omitted state. Formally, let $\mathbf{q}(\mathbf{j}, \mathbf{i}, \mathbf{h})$ be the vector of dichotomous variables representing current and previous states, but omitting (j, i, h).[19] Our new fixed effects model is then:

Model 2. $\widehat{bid}_{k,t} = \beta_0 + \mathbf{q}(\mathbf{j}, \mathbf{i}, \mathbf{h})'_{k,t}\boldsymbol{\beta_q} + f_k + \varepsilon_{k,t}$

Results for Model 2 are given in Tables 2.9 and 2.10—the former comparing $(i, j,\ lost)$ to $(j, i,\ won)$, while the latter compares $(i, j,\ won)$ with $(j, i,\ lost)$. Some

[19] State (4,4) is not included because it has no previous state. Also, for $j \in \{1, 2, 3\}$, Player A can only reach state $(4, j)$ by losing, while $(j, 4)$ can only be reached by winning.

Table 2.9: Losing to get to (i, j) vs. Winning to get to (j, i).

		Win 15 Lose 285			**Win 150 Lose 150**			**Win 300 Lose 0**		
		4	*3*	*2*	*4*	*3*	*2*	*4*	*3*	*2*
Experienced	*1*	5.90 (2.84) $> 0^{**}$	−4.83 (7.37)	−3.15 (5.72)	5.86 (1.99) $> 0^{***}$	1.48 (3.49)	−2.03 (5.30)	12.83 (2.74) $> 0^{***}$	−5.57 (5.54)	3.45 (8.06)
	2	3.64 (2.06) $> 0^{**}$	2.78 (2.26)		4.79 (0.93) $> 0^{***}$	0.41 (1.31)		8.43 (1.48) $> 0^{***}$	2.52 (1.97)	
	3	0.16 (1.88)			1.49 (0.72) $> 0^{**}$			4.02 (1.05) $> 0^{***}$		
Mixed	*1*	19.11 (2.43) $> 0^{***}$	11.02 (6.74) $> 0^{*}$	−0.59 (5.30)	3.43 (3.96)	0.65 (3.09)	5.66 (4.78)	1.34 (2.16)	−10.84 (2.93) $< 0^{***}$	−15.09 (6.05) $< 0^{***}$
	2	12.55 (2.18) $> 0^{***}$	6.07 (2.14) $> 0^{***}$		−0.25 (3.33)	−1.47 (4.24)		4.47 (1.77) $> 0^{***}$	−4.88 (7.05)	
	3	7.10 (1.25) $> 0^{***}$			−4.70 (2.66) $< 0^{**}$			1.87 (1.73)		
		4	*3*	*2*	*4*	*3*	*2*	*4*	*3*	*2*

Standard errors in parentheses. Significance levels: * 10%, ** 5%, *** 1%.

states cannot be reached by winning, so Table 2.10 does not include the states where Player A has yet to win a round ($i = 4$). Yet, unlike Tables 2.6 and 2.9, it does include the states where players are tied ($i = j$).

There are three states that can be compared across Tables 2.9 and 2.10 based on whether a player reaches it by winning or losing: $(3, 1)$, $(2, 1)$, and $(3, 2)$. In most cases, the coefficients are larger when a player arrives at one of these states by winning. For instance, in the Win 15 Lose 285 treatment of the mixed group, if a player arrives at $(3, 1)$ by losing at $(3, 2)$, he will bid an average of 11.02 rupees more than he would at $(1, 3)$ having won at $(2, 3)$. However, this difference increases to 26.65 rupees when the underdog reaches $(3, 1)$ by winning at $(4, 1)$. The bidding difference in Model 1 sits between at 14.23 rupees. In addition to being larger, the Table 2.10 coefficients also have a higher likelihood of being significantly positive. Again using the Win 15 Lose 285 treatment of the mixed group as an example, the difference between bids at $(2, 1)$ and $(1, 2)$ is not significant in Model 1. Neither is the difference significant when a player arrives at $(2, 1)$ by losing at $(2, 2)$. However, a player reaching $(2, 1)$ by winning at $(3, 1)$ will bid significantly more than at $(1, 2)$, after losing at $(1, 3)$, by an average of 15.35 rupees. A related example is state $(2, 1)$ in the mixed Win 300 Lose 0 treatment where the coefficient is significantly negative in Tables 2.6 and 2.9 but looses its statistical significance when a player wins the previous state.

A final aspect of Table 2.10 that is worth mentioning is that it allows for bidding behavior to be analyzed at states $(1, 1)$, $(2, 2)$, and $(3, 3)$ based on whether a player arrived at one of these tied positions by winning or losing. As with all of the comparisons involving Model 2, winning or losing the previous round should not alter bidding behavior from a theoretical standpoint. Here, however, there are a handful of instances where it does indeed matter. In both the mixed and experienced groups of the Win 300 Lose 0 treatment, players that lost at $(4, 4)$ but then won at $(4, 3)$ bid more aggressively at $(3, 3)$ than had they won at $(4, 4)$ but lost at $(3, 4)$. Bids at $(2, 2)$ are also strongly connected to the previous state's outcome in the Win 150

Table 2.10: Winning to get to (i, j) vs. Losing to get to (j, i).

		Win 15 Lose 285			**Win 150 Lose 150**			**Win 300 Lose 0**		
		3	*2*	*1*	*3*	*2*	*1*	*3*	*2*	*1*
Experienced	*1*	−1.00 (5.28)	2.81 (3.15)	−4.81 (12.46)	7.89 (4.37) $> 0^{**}$	−1.20 (3.61)	−8.45 (8.73)	19.11 (6.62) $> 0^{***}$	−5.19 (4.12)	0.58 (4.04)
	2	7.73 (3.95) $> 0^{**}$	1.93 (4.28)		0.41 (1.54)	5.86 (1.90) $> 0^{***}$		12.36 (6.08) $> 0^{**}$	−0.09 (3.06)	
	3	2.35 (3.01)			−0.40 (1.65)			3.39 (1.82) $> 0^{**}$		
Mixed	*1*	26.65 (8.54) $> 0^{***}$	15.35 (4.58) $> 0^{***}$	2.26 (6.04)	4.74 (4.17)	3.97 (6.49)	0.68 (15.35)	−7.98 (5.20) $< 0^{*}$	−8.62 (9.05)	−13.06 (14.68)
	2	8.37 (2.79) $> 0^{***}$	0.48 (2.21)		2.46 (3.06)	−18.32 (6.64) $< 0^{***}$		4.55 (3.28) $> 0^{*}$	5.89 (6.03)	
	3	3.14 (2.49)			2.33 (5.20)			6.36 (2.47) $> 0^{***}$		
		3	*2*	*1*	*3*	*2*	*1*	*3*	*2*	*1*

Standard errors in parentheses. Significance levels: * 10%, ** 5%, *** 1%.

Lose 150 treatment. A player in the mixed group bids more aggressively if he reaches $(2, 2)$ by losing (perhaps an element of the defense of the lead), whereas the reverse is true in the experienced group.

2.5.2 Player Types

The analysis thus far has taken an aggregate view of the experimental data. We find, however, that many of the anomalies, as well as expected results at the aggregate level are illuminated by a careful study of individual player behavior. Players tend to adopt one of several distinctive strategies and play it repeatedly. Furthermore, the same set of strategies are independently discovered and used in the different experimental sessions. The heterogeneity of strategies and the response of one strategy to another are key to understanding behavior in this experiment.

As an initial example, consider Figure 2.4 which illustrates bidding behavior along the exterior states where one player has either consistently won or lost.[20] Many of the preeminent strategies and bidding patterns can be identified in this figure. The top row plots the losing trajectory with bids at $(4, 3)$ on the x-axis and the subsequent bids at $(4, 1)$ on the y-axis. The points are further coded by whether the player's bid at $(4, 2)$ increased, decreased, or remained equal to the bid at $(4, 3)$. The winning trajectory is featured on the bottom row, similarly plotting bids at $(3, 4)$ against those at $(1, 4)$.

A striking feature of the bids along the losing trajectory is the concentration of points along the axes and at the origin, with relatively few bids in the interior of the graph. There are four distinct strategies represented here. First, points on or near the y-axis represent a last stand strategy entailing a token bid at $(4, 3)$, frequently another at $(4, 2)$, and then a sharp increase at $(4, 1)$ in the face of a tournament loss. Note that the last stand strategy is much more prevalent in the treatments with a losing penalty than in the Win 300 Lose 0 treatment. The second strategy is one of

[20] A few outlying points have been omitted to focus these graphs on the area of greatest concentration. The points have been slightly perturbed to reduce overlap.

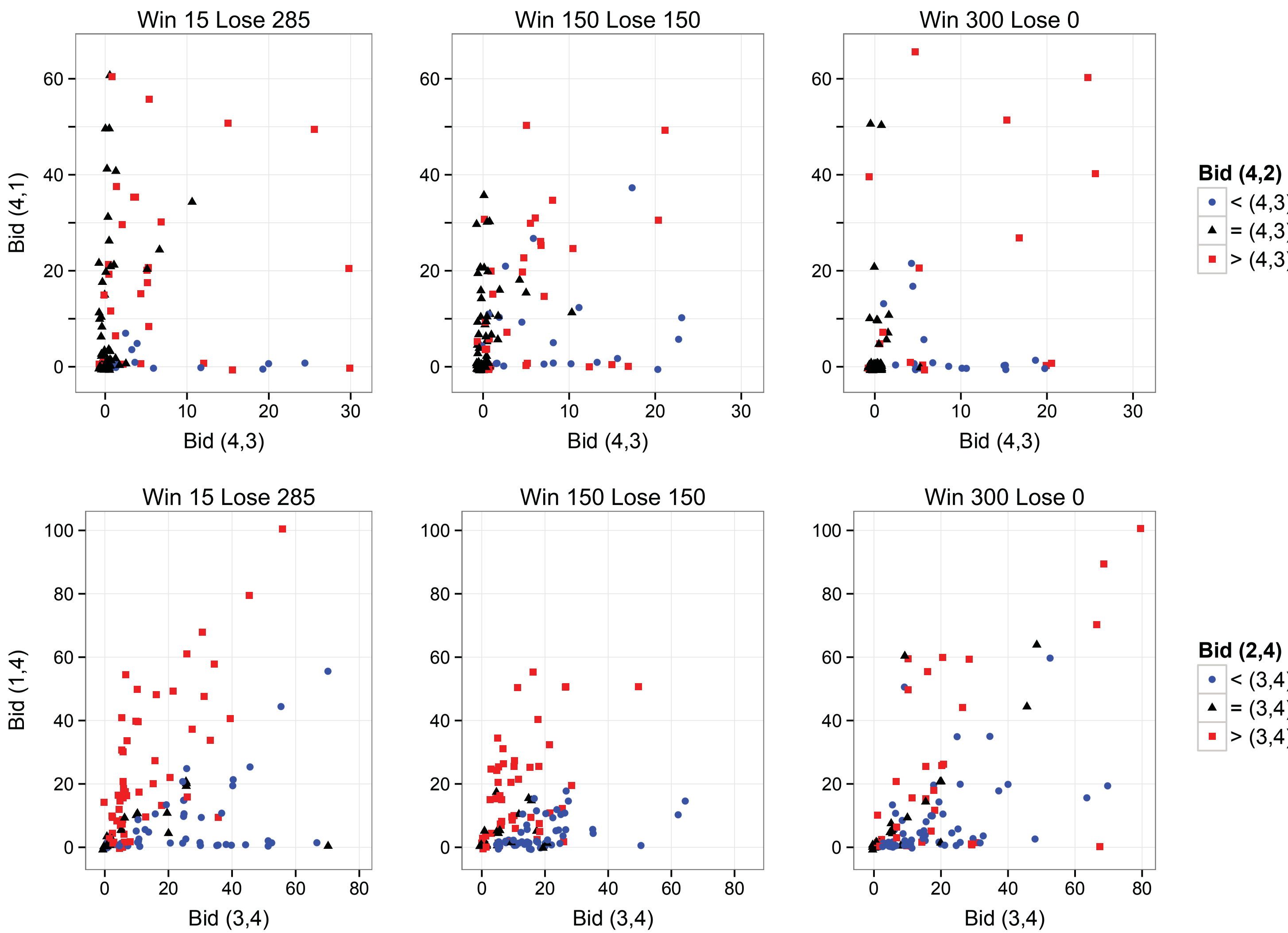

Figure 2.4: Bids along the trajectories leading to (4,1) (top row) and (1,4) (bottom row).

becoming passive and ultimately giving up as resistance mounts. It is represented by the points along the x-axis and away from the origin, indicating that these players were actively competing at $(4, 3)$. A few of these players rallied at $(4, 2)$ while several other began to give way. By $(4, 1)$ however, they had all given up. A third strategy, represented by the cluster of points at the origin, is to give up from the beginning. This is a maximin strategy in the sense that, while anticipating a tournament loss, players maximize their payoff by bidding zero. The fourth strategy—escalation—is represented by the red squares in the interior of the graphs. These are players who were actively competing and increasing their bids, yet continued to be outbid.

Bidding patterns along the winning trajectory clearly interact with those along the losing trajectory. Players either escalate, increasing their bids at (2,4) and (1,4) relative to (3,4); or they curtail their bids as they move toward victory. Since players in the second case continue to win even as they reduce their bids, they are clearly adapting to the strategy of a less aggressive rival. Although some rivals remain passive to the end, the last stand strategy frequently takes advantage of players who have been lulled into thinking they can win at $(1, 4)$ with a minimal bid. Escalation, the other major behavior along the winning trajectory, likely occurs either as a preemptive move or when faced with an aggressive competitor.

In order to further explore the strategies used throughout the experiment, and not just those along the two specific trajectories in Figure 2.4, we individually looked at each of the last ten tournaments for all 216 subjects in the experiment.[21] Similar to

[21] Examples of complete tournament data for six subjects are shown in Tables 2.11 through

the results in Figure 2.4, we identified four major dynamic strategies that were used frequently and repeatedly across the different experimental sessions: maximin, last stand, escalate when challenged, and passive when challenged. There were, however, nuances to these strategies, the most prominent of which was the level of a player's bid at the initial state $(4, 4)$. The initial bid served as a signal to some degree, and most players varied it only slightly from tournament to tournament.[22] A high initial bid typically signaled a fairly aggressive player; low initial bids, on the other hand, carried less information as they were commonly used by both aggressive and non-aggressive players. To capture this feature, we divided players not only by the dynamic strategy they predominantly used, but also by whether their initial bids tended to be low, moderate, or high.[23]

Developing a taxonomy of the different strategies was an iterative process. We began by identifying the four recurring strategies in a small sample of the data and associating loose definitions with each. Working independently, we then classified each player by their most commonly used strategy (if it was at all clear). Differences in our independent classifications lead to more rigorous definitions of the strategies.

2.16. These tables will be described in detail later.

[22]For the last ten tournaments, the 25th, 50th, and 75th percentiles of the standard deviation of initial bids by player are 0.04, 2.10, and 4.89. The consistency of initial bids is likely due, at least in part, to the fact that players were matched at random for each tournament.

[23]We defined high bids as greater than 12 and low bids as less than 2. These cutoffs correspond to the 75th and 39th percentiles of initial bids across all sessions. Some initial bids are fairly common. For instance, 0, 1, 2, 5, and 10 account for 54% of all initial bids, with 0 alone counting for 23%.

The final taxonomy, presented here, is specific enough that a computer can apply these rules to classify the strategy a player uses in each tournament.

Taxonomy. *A player's strategy in an individual tournament is classified as follows:*

1. Escalate when Challenged: *On at least two occasions, the player submits a bid that is both strictly greater than his own last bid and weakly greater than his rival's last bid. The first occurrence may not be after the player has two losses, and the second occurrence may not be at* $(4, 1)$.

2. Maximin: *In a tournament that reaches either* $(i, 1)$ *or* $(1, j)$, *the player may not on any occasion place a bid that is both strictly greater than his own last bid and weakly greater than his rival's last bid. Furthermore, all of the player's bids must be strictly below* 2.

3. Last Stand: *In the absence of escalation, when the player is behind in the tournament with more than one loss, he must bid strictly higher than his own last bid and either* (*i*) *weakly greater than his opponent's last bid; or* (*ii*) *place a bid of at least* 2 *that is strictly greater than the midpoint between his own and his rival's last bid.*

4. Passive when Challenged: *In the absence of escalation, last stand, or maximin, and in a tournament that reaches either* $(i, 1)$ *or* $(1, j)$, *after the player's first loss each bid thereafter must be below the rival's previous bid.*

A player is classified by one of these four strategies if the player uses it in a strict majority of tournaments for which one of the above classifications hold, provided *it is used at least three times. A player is further classified as a low, middle, or high initial bidder if a strict majority of bids at* $(4, 4)$ *are within the range of* 0–1.9*,* 2–12*, or greater than* 12*, respectively.*

Roughly 66% of the 2160 person-level tournament observations can be classified by one of these four strategies. The remaining 34% is largely composed of tournaments that ended before a strategy could be distinguished.[24] Of the 216 players, 201 can be classified by their initial bid, 180 by a strategy, and 169 by both. Examples of players who are classified as using each of the four strategies are shown in Tables 2.11 through 2.16. These tables show bids at each state and the resulting tournament paths for the last ten tournaments of play. To the left of each tournament grid is the number (11–20) and outcome of the tournament (L: loss; W: win; E: end early). Within the cells of the tournament grid, a player's own bid is written on the left and the rival's on the right. Each of the players represented in these tables has been selected from a different session.

Table 2.11 is a classic maximin player who always bids zero, regardless of how low his opponent might bid. Table 2.12 shows a player who is passive when challenged, a behavior evident in tournaments 12, 16, 18, and 19. This table also illustrates the significance of the term *when challenged*. There is no opposition in tournaments 11, 13, and 17, so the player resorts to placing minimal bids—a widespread behavior that does not delineate a unique enough aspect of the player's strategy to be classified.

[24]Another large group of unclassified tournaments are those in which players faced non-aggressive rivals and so did not clearly demonstrate one of the above named strategies.

Table 2.11: Maximin Player from Session L150 MC.

11	0	8							12	0	10						
	0	7								0	10						
	0	6								0	25						
L	0	6							L	0	25						
13	0	1							14	0	1						
	0	1								0	5						
	0	5								0	10						
L	0	20							L	0	1						
15	0	0.1							16	0	1						
	0	5								0	1						
	0	15								0	10						
L	0	30							L	0	1						
17	0	0.1							18			0	0	0	0		
	0	0.1								0	0	0	1				
	0	0.1								0	1						
L	0	0.8							L	0	0						
19	0	0.3							20	0	0.9						
	0	11								0	0.6						
	0	0.2								0	0.1						
L	0	9							L	0	10						

Tournament 20 likewise cannot be classified because of its early termination. The remaining two tournaments, 14 and 15, meet the qualification of escalating when challenged. Since the four passive when challenged tournaments form a strict majority of the six classifiable tournaments, this player is designated as passive when challenged.

In Tables 2.13 and 2.14, players repeatedly use the last stand strategy. Although not required by definition, these two players nearly always begin with a bid of zero. The player in Table 2.13 continues to bid zero, mimicking maximin play, but springs suddenly into action at $(4, 1)$. Evidently expecting this maximin play to continue, the rivals in tournaments 13, 15, 19, and 20 dramatically reduced their

Table 2.12: Passive when Challenged Player from Session L0 EA.

11								
W	3	0	4	0	1	0	1	0
12			0	10				
			0	15				
			0	19				
L	3	2	5	9				
13								
W	4	0	5	0	1	0	1	0
14								
							30	11
W	4	1	10	1	2	1	2	5
15								
W	4	1	6	1	2	1	10	10
16	0	1						
	0	5						
	0	9						
L	4	5						
17								
W	4	0	5	0	2	0	1	0
18	0	1						
	0	3						
	0	11						
L	4	5						
19			0	5				
			0	25				
			0	31				
L	4	2	10	12				
20								
E	4	0	5	0				

bids. The opponent in tournament 14 correctly anticipated the possibility of a last stand and submitted a slightly higher bid. Having bid 15 the previous three times, this opponent may have wanted to avoid being outdone by a bid of 15.1 and so bid 15.2. Even if a last stand is successful at $(4, 1)$ it tends to be dramatically harder to secure a victory at $(3, 1)$ since the rivals have been stirred to action. The last stands in Table 2.14 are quite similar, although it is worth noting that tournaments 13 and 19 do not meet the definition of a last stand but are rather maximin. Last stand bids must be bids that have a credible chance of winning given the previous bids in the tournament.[25]

[25] For tournament 13 to meet the definition of a last stand, the player would have needed

Table 2.13: Last Stand Player from Session L150 MB.

11					12				
						0 11			
E	0 8				E	0 11			
13	20 2	20 31			14	20 15.2	27 75		
	0 15					0 15			
	0 10					0 15			
L	0 14				L	0 15			
15	20 0.2	30 33			16	30 15	30 75		
	0 11					0 15			
	0 22					0 15			
L	0 16				L	0 15			
17	15 5	30 30			18				
	0 5								
	0 5								
L	0 0				E	10 100			
19	30 5	40 38	65 52	90 78	20	20 2	25 66		
	0 10					0 2			
	0 24					0 15			
W	0 16				L	0 15			

The escalate when challenged strategy is demonstrated in Tables 2.15 and 2.16. In eight of the tournaments, the player in Table 2.15 opens with a bid of 12—a bid that is strong enough to consistently win at $(4,4)$ each time it is used. Gaining the lead from the start, the player submits successively lower bids until a rival poses a threat. Typically this occurs when the player loses a round, but in tournaments 11, 18, and 19, the rival's bid at $(4,4)$ is close enough that it registers as a threat. Once challenged, the player escalates. However, in escalating, this player is more judicious than the player in Table 2.16. The difference comes in the rate of escalation. The

to bid at least 12.6 at $(4,2)$ or at least 2.1 at $(4,1)$. There is no clear strategy in tournament 15 since both last stand and maximin are possibilities.

Table 2.14: Last Stand Player from Session L285 MC.

11	41	2	61	51	100	86		
	1	4						
	1	2						
E	0	5						
12	41	2	43	50				
	1	11						
	1	25						
L	0	19						
13	1	9						
	1	3						
	1	24						
L	0	25						
14	21	2	61	45	101	100	41	126
	1	15						
	1	2						
L	0	25						
15								
	0	2						
	0	15						
E	0	25						
16	6	5						
	0	5						
	0	5						
E	0	5						
17	3	1	45	5	0	7		
	0	1						
	0	1						
L	0	44						
18			61	101				
			21	26				
			0	9				
L	0	0	0	2				
19	1	41						
	1	11						
	0	6						
L	0	0						
20	21	11	100	51				
	0	20						
	0	36						
E	0	26						

Table 2.15 player may have two or three rounds of escalation before hitting a bid of twenty, while the Table 2.16 player can easily exceed a bid of twenty in the first of a five or six round escalation. Ultimately, this player expends more in bids than the value of the prize in three of the tournaments (11, 12, and 20) and comes close in another two (15 and 17). Part of the issue is that high initial bids leaves little room to backtrack in a bidding war. Another part of the issue is knowing when to quit. Even if this player would have won the three tournaments that went to $(1, 1)$, the maximin strategy would still have dominated their strategy in terms of expected payoffs.

The distribution of player types within each session is summarized in Table 2.17. Here we classify players both by initial bid and strategy. Several notable

Table 2.15: Escalate when Challenged Player from Session L150 EC.

11								
			50	25	50	50	64	51
W	20	20	21	30				

12								
							22	20.1
					12	4.9	14	15
W	20	5.1	10	0	2	10		

13								
E	12	6.1	10	0.1	2	6.1		

14								
					10	5.1	13	14.8
E	12	5.1	3	0	2	5		

15								
					12	10	16	15
					6	8		
W	12	0	2	0	1	5		

16								
							22	20
					6	5	11	15
W	12	5	10	0	2	10		

17							48	39
							24	28
			8	7	12	9.9	14	18
W	12	0	4	5				

18								
			42	0	16	0	24	0
			19	37				
W	12	11	13	26				

19								
							14	15
E	12	11	18	0	5	0	2	10

20					34	0	40	40
					16	20		
					8	10		
L	12	1	6	0	2	5		

patterns are present in this table. One that is immediately apparent is the widespread distribution of players types across each session. Every session has at least two distinct strategies represented, with escalate when challenged being the most common. Across the low, middle, or high initial bid ranges, escalate when challenged accounts for at least six of the twelve players in all but two sessions. The remaining three strategies represent roughly a quarter of the classifiable players and a fifth of all players. The incidence of these remaining players across treatments is particularly telling, however, in terms of the last stand and surrendering hypotheses.

There is a relative abundance of maximin and passive when challenged players in the Win 300 Lose 0 treatment, a fact which helps to substantiate our hypothesis of

Table 2.16: Escalate when Challenged Player from Session L0 MA.

11								
					70	31	100	49
	40	9	41	15	25	59		
W	25	33						

12			90	80	95	80		
			60	75				
	41	35	45	50				
E	20	40						

13								
W	25	0	10	0	1	0	1	0

14								
							50	21
					15	11	15	21
W	20	5	6	5	6	10		

15					60	55	75	80
					40	45		
			25	19	26	36		
L	20	2	10	19				

16					53	65		
					45	49		
			23	23	31	35		
L	20	1	10	16				

17					70	69.8	80	95.9
			40	36.4	45	52		
	22	21.7	24	32.8				
L	10	15						

18								
			25	18.6	30	27.7	35	10.5
W	10	5.1	10	17.8				

19					66	70		
					45	53		
			30	23	35	39		
L	10	2	10	23				

20							100	101.4
					70	56.6	75	85
	25	24.8	36	35	46	48.9		
L	15	18						

players surrendering when losing is costless. While the Win 15 Lose 285 and Win 150 Lose 150 treatments have one passive when challenged and three maximin players apiece, the Win 300 Lose 0 treatment has three passive when challenged and ten maximin players. The opposite holds true for last stand players. There are only four last stand players in the Win 300 Lose 0 treatment but nine and eleven respectively in the Win 150 Lose 150 and Win 15 Lose 285 treatments.[26]

While players are classified in Table 2.17 by a majoritarian use of a strategy, we can also look at the actual number of times that they play a particular strategy.

[26]Including three last stand players that cannot be classified by initial bid, there are five last stand players in the Win 300 Lose 0 treatment and eleven in each treatment with a losing penalty.

Table 2.17: Distribution of Player Types by Session.

		Low			Mid			High			
		MM	LS	Esc	LS	Esc	Pass	LS	Esc	Pass	Other
L285	Mix A	1	2	6		2			1		
	Mix B		2	4		3			1		2
	Mix C		2		2				7	1	
	Exp A	1		2		6			1		2
	Exp B			1	1				5		5
	Exp C	1	1	3		1		1	2		3
L150	Mix A		2	1		6			2		1
	Mix B		1			2			5		4
	Mix C	2		1		1			2		6
	Exp A	1	2	1		5			1		2
	Exp B		2	4		2	1		2		1
	Exp C		2	1		5			1		3
L0	Mix A	1				4			3	1	3
	Mix B	3				6					3
	Mix C	1	1	3		1			3		3
	Exp A	2	1			5	2				2
	Exp B	2	2	1		4			1		2
	Exp C	1		1		1			4		5
Total	L285	3	7	16	3	12		1	17	1	12
	L150	3	9	8		21	1		13		17
	L0	10	4	5		21	2		11	1	18

11 players can be classified by strategy but not by initial bid. Escalate: L285MB, L285EA, L285EB, L150MB (×2), L150MC, L0EC (×2); Last Stand: L150MB, L150MC, L0MC.

Players who stick exclusively to one strategy throughout the last ten tournaments of the experiment are in fact a minority with nearly 70% using two or more of the classifiable strategies at least once.[27] This practice of shopping around for different strategies can be seen in Figure 2.5, which contains the cumulative distribution of the number of times players use a given strategy during the last ten tournaments. Regardless of treatment, more than one-half of all players have experience using the last stand strategy. Usage rates for maximin and passive when challenged sit at around a quarter, and nearly everyone has tried the classic escalate when challenged strategy. Notwithstanding this shopping around, the usage rates of the different strategies are fairly delineated across treatments, and the distributions can frequently be ranked in terms of stochastic dominance.

For the maximin and passive when challenged strategies, the Win 300 Lose 0 treatment first-order stochastically dominates the other treatments. While the ranking is less clear near the top of the distributions (and the Win 150 Lose 150 treatment actually pulls ahead of the Win 300 Lose 0 treatment for playing maximin eight times), the Win 300 Lose 0 treatment has a pronounced lead earlier on. In the two treatments with losing penalties, only a few players use maximin more than two times, and hardly any use passive when challenged more than once. This seems to indicate that players in the Win 15 Lose 285 and Win 150 Lose 150 treatments are simply trying out these strategies rather than committing to them. Even for the

[27]Of the 216 players, 65 use only one strategy in the tournaments that can be classified; 101 used two, 45 used three, and 5 subjects managed to use each of the four strategies.

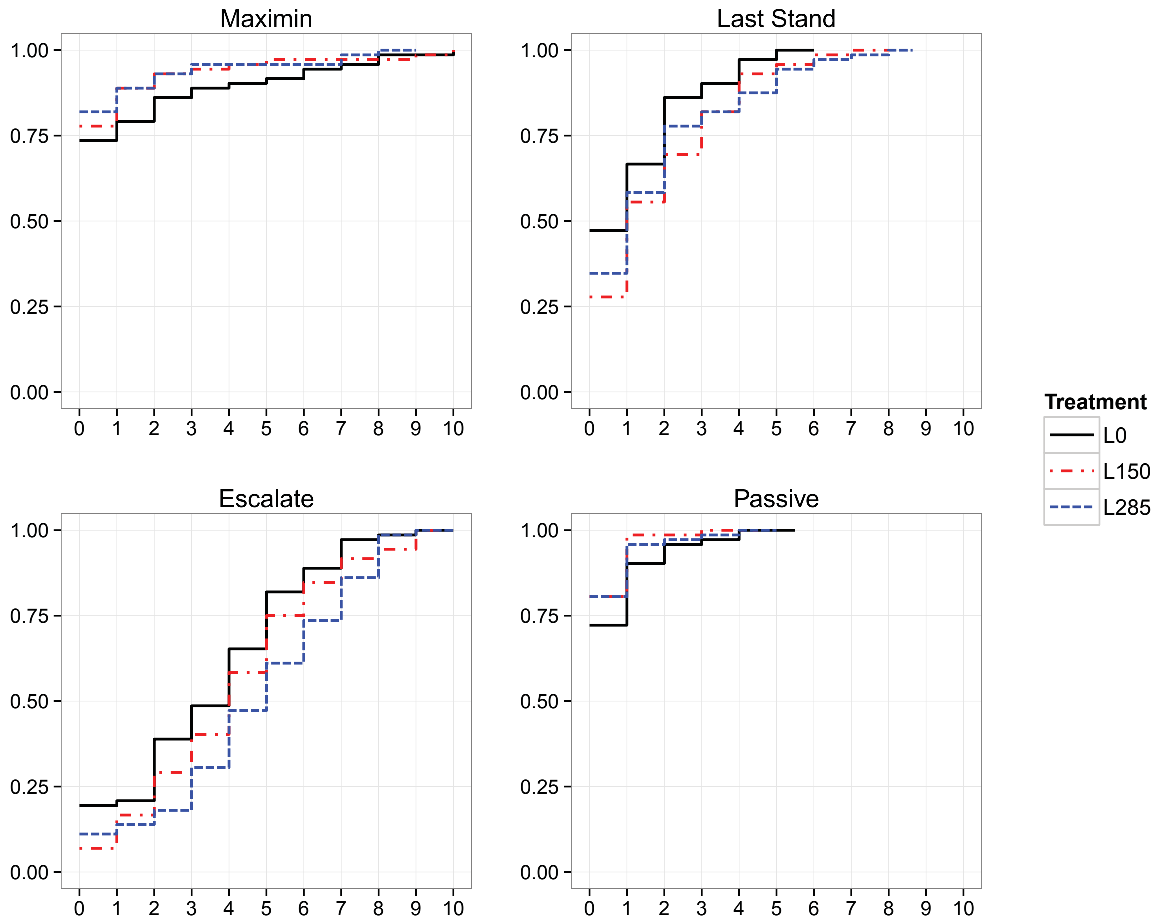

Figure 2.5: CDF of number of times players use a strategy during last ten tournaments.

Win 300 Lose 0 treatment, there are never more than four observations of the passive when challenged strategy. This lack of observations is likely due to two definitional factors which are out of the player's control: first, that his rival is aggressive; and second, that the tournament continues long enough for him to demonstrate that he is adequately passive.

There is an interesting wrinkle in the ranking of the distributions for the last stand strategy. While the Win 300 Lose 0 treatment is unmistakably in third place, the ranking of the other two treatments switches midway. The likelihood of ever using the last stand strategy is highest in the Win 150 Lose 150 treatment, however, much of this weight is placed on using this strategy only once or twice. By three times, there is no difference in the usage rate between the Win 150 Lose 150 and the Win 15 Lose 285 treatments. The lead belongs to the Win 15 Lose 285 treatment thereafter. It is fitting that the last stand strategy is employed with greater frequency in the treatment with the larger penalty for losing.

Use of the escalate when challenged strategy is crisply ordered across the three treatments over nearly the entire distribution. The Win 15 Lose 285 treatment has the highest occurrence at nearly every level, followed by the Win 150 Lose 150 treatment, and then the Win 300 Lose 0 treatment. It is not entirely clear why such a clear ordering should exist here. Although the prize-penalty spread remains the same across treatments, it may be that players feel that there is more at stake when there is more to lose (as is the case in prospect theory).

We now turn to the question of how well each of these strategies performs. Figure 2.6 shows the average tournament payoff over the last ten tournaments of play for every player that can be classified by both a strategy and an initial bid. Here, players are coded by strategy, and the escalate when challenged players are also coded by initial bid. Each row denotes a separate session of the experiment, and the sessions are sorted from top to bottom based on the median payoff of the session. The maximin players serve as a baseline for comparison since every player could have guaranteed themselves this payoff. Factoring in the probability that the tournament ends early, consistently bidding zero leads to an average payoff of −207.8 in the Win 15 Lose 285 treatment, −109.4 in the Win 150 Lose 150 treatment, and 0 in the Win 300 Lose 0 treatment. Depending on the actual realization of the number of tournaments that ended early, the maximin payoffs tend to cluster around these points.[28]

More than any other strategy, escalate when challenged with a high initial bid frequently fares worse than maximin. This was the case with the player in Table 2.16 who received the lowest average payoff in the Win 300 Lose 0 treatment. Sessions where multiple players adopted this strategy had an especially hard time since it lead to routine high-stakes bidding wars. These sessions have the lowest median payoffs within each treatment. While escalation can be a harmful strategy if used too aggressively, it can also be a profitable strategy if used with an appropriate amount of moderation. Players who escalate with initial bids in the middle range often surface to the top of the payoff distributions. A mid-range initial bid provides an affordable starting point for bidding wars, and it also frequently results in an early lead in the tournament. Low-range initial bids, on the other hand, frequently forfeit the early lead. This is likely why players who escalate from a low initial bid have lower payoffs than those who escalate from the mid-range. Last stand players, overall, do marginally better than the maximin players. Instead of simply swallowing the losing

[28]The occasional maximin player would win a tournament—frequently by deviating from the maximin strategy when winning looked feasible. There are a few scattered instances where a tournament was actually won with a cumulative bid of zero. Interestingly, there is only one such case where both the winner and the loser were classified as maximin players.

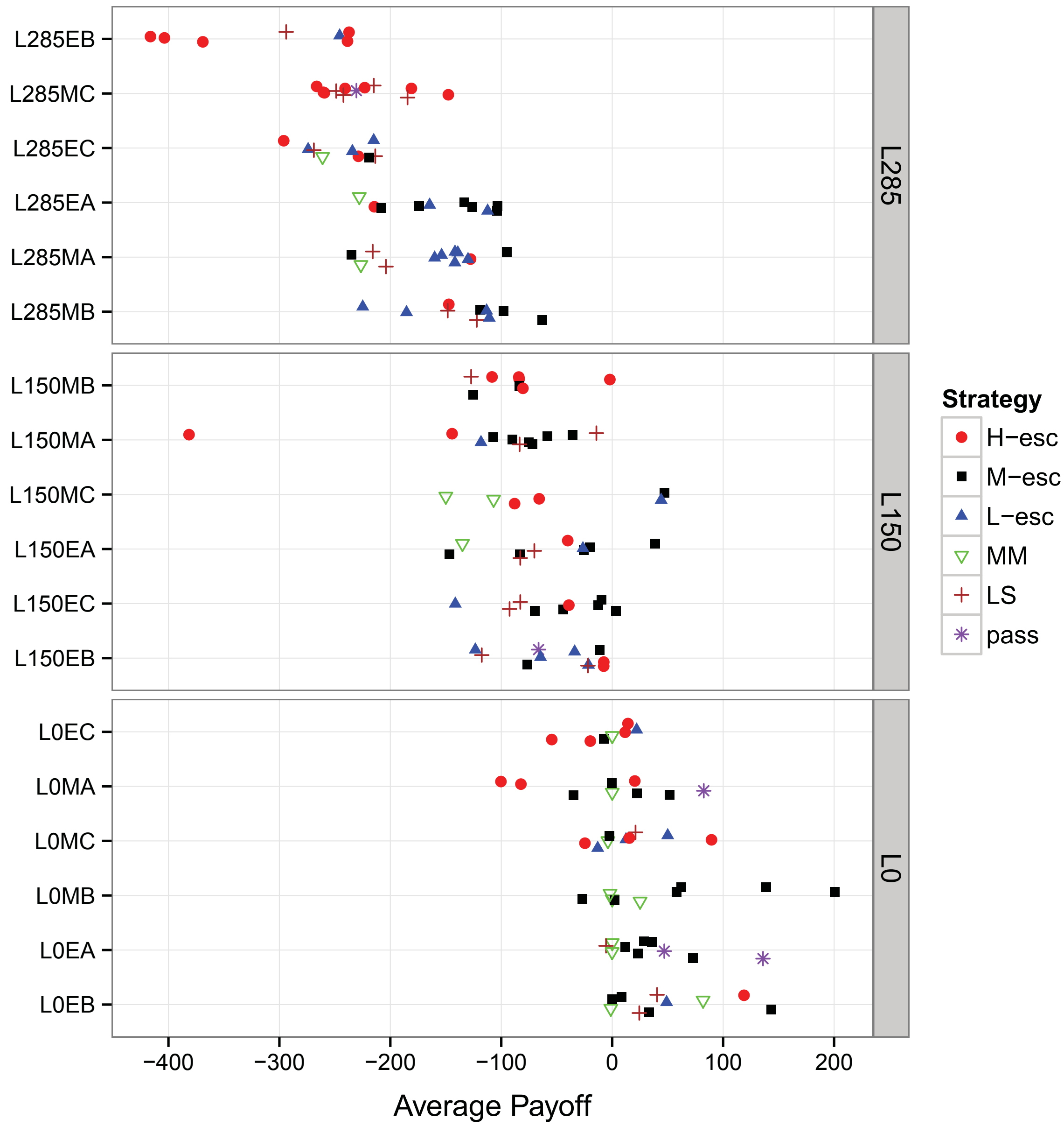

Figure 2.6: Average payoff by player type and session. Sessions are ranked by median payoff.

penalty, the last stand strategy seeks to buy the player time so that the tournament may end early. A last stand is profitable relative to maximin if the bidding cost from successfully delaying the tournament is less than the penalty for losing. Due to the lack of observations, it is difficult to assess the passive when challenged strategy. However, it appears to do very well in the Win 300 Lose 0 treatment where two of the players using it rank first in their sessions. These players are vested enough in the tournament to win against easy competitors, but they are also quick to avoid costly confrontation.

2.6 Conclusion

While there are several anecdotal accounts that can be used to compare last stand and surrendering behavior, this paper provides a controlled laboratory framework to actually assess these behaviors under varying conditions. To do so we examine how aggressively players compete at different stages of a best-of-seven tournament. We find that bidding behavior differs considerably based on whether players face a substantial penalty for losing. When the cost of losing is high, players who have fallen behind frequently submit fairly large bids in an effort to avoid a tournament loss. This is consistent with theory and embodies the notion of a last stand. Within our experiment we specifically identify several players who routinely use a last stand strategy and find that this strategy is much more prevalent in the treatments with losing penalties. Conversely, we find that subjects are more prone to surrendering in tournaments that have no losing penalty. Many subjects even employ a maximin strategy of consistently bidding at or near zero—in essence, surrendering from the start. Although forfeiting the tournament prize, by refusing to compete these players also avoid undue expenditure. Another surrendering player type consists of actively competing until a player faces strong resistance, at which point the player surrenders. The final and most prevalent player type we identify entails engaging in an escalating bidding war. Players who are overly aggressive in these bidding wars perform the worst in terms of average payoffs, while moderately aggressive players often do the best.

Although theory suggests that a player will defend his lead if it is threatened, we find little evidence to support this hypothesis. We do find that the propensity to make a last stand or to surrender leads to reasonable predictions about winning margins. Neck-and-neck tournaments being more likely when the losing penalty is relatively high, and landslide victories occurring more frequently when the prize is large. Tournament upsets are fairly uncommon with the winner of the initial battle going on to win the entire tournament roughly three-quarters of the time.

Given that last stands are fundamentally linked with significant loss—be it life, limb, liberty, or property—there is a natural challenge in designing an experiment which exposes subjects to a loss that is in some sense meaningful, yet minor enough to conform with standard institutional review board guidelines. We suggest, therefore, that any evidence of last stand behavior in our low stakes experiment would be magnified in situations that involve more substantive losses.

CHAPTER 3
METHODOLOGICAL ADVANCES FOR LINKING HISTORICAL CENSUSES: WITH AN APPLICATION TO OCCUPATIONAL FOLLOWING

3.1 Introduction

Beginning in 1841 in the UK and 1850 in the US, census enumerators were charged with individually listing—not just tallying—every man, woman, and child in the country. In addition to recording names, the typical decennial census began to include a person's age, gender, occupation, birthplace, marital status, address, and relationship within his or her household. While census data present a wealth of demographic information for social scientists and genealogists alike, the amount of information conveyed by any census in isolation is limited since it only provides a cross-sectional picture. If, however, censuses were systematically linked so that each individual could be traced throughout their life at ten year increments, the result would be a dynamic and intricately detailed picture of a nation that spans generations.

Given this rich potential, it is surprising how little has been done to systematically link individual and family records across multiple censuses. Three major obstacles have limited large scale census matching projects: (1) the availability of census transcriptions; (2) imperfections in the original census data—a problem which is compounded by imperfections in census transcriptions; and (3) methodological challenges for linking the data given its underlying shortcomings.

Census transcriptions that are freely available to social scientists are primarily limited to 1% and 5% samples—the biggest exceptions being the complete transcriptions of the 1880 US and 1881 UK censuses.[1] In fact, the majority of census linking projects to date have entailed matching one of the 1% or 5% census samples with the corresponding 1880 or 1881 census.[2] The linking process is necessarily difficult since names, ages, and birthplaces frequently have irregular variations from one census to another.[3] Transcribing historic records also has its own set of problems as the quality of original documents may have been compromised by illegible handwriting, ink blots, or improper storage conditions. Transcribers may also be inexperienced, fatiqued, or unfamiliar with names and places.[4] Matching algorithms must be flexible enough to account for substantial variations in the census entries. However, as flexibility increases, so does the problem of duplicate matches and false positives (especially for people with common names). Life cycle changes are yet another consideration:

[1]These transcriptions were coordinated by The Church of Jesus Christ of Latter-day Saints. Complete, albeit proprietary transcriptions for the remaining UK and US censuses in the public domain (UK: 1841–1911; US: 1850–1940) are available through a variety of genealogical websites.

[2]A leader in census matching has been the North Atlantic Population Project of the Minnesota Population Center. It has linked samples from seven US censuses into the 1880 census and has also performed census linking with Norwegian censuses. Joseph Ferrie and Jason Long have also linked small samples into the 1880 US and 1881 UK censuses (see, for instance, Long and Ferrie, 2013, Appendix 2).

[3]Names may be abbreviated or nicknames may be used instead of proper names. Even though ages should systematically increase by ten years from one census to the next, they are often misreported—either accidentally or deliberately. Birthplaces may sometimes be reported as one parish in one census and as an adjoining parish in a subsequent census.

[4]For example, I have repeatedly seen the female biblical name Tamar mistakenly transcribed as James. High quality transcription projects often entail multiple levels of transcribing and checking as an attempt to minimize such errors.

people die, people move, wives adopt their husband's surname. All of these issues should ideally be accounted for.

Here, I present my methodology and the results for linking six consecutive censuses (1841–1891) for Cornwall County, England. Comprehensive transcriptions for these censuses were generously provided by the Cornwall Online Population Project.[5] In total, these transcriptions contain more than 2 million census observations, with roughly one-third of a million observations per census year (see Table 3.1). Methodologically, I implement and build upon the census linking tools introduced in Fu, Christen, and Boot (2011b). A key element of Fu et al. is the utilization of household information in the matching algorithm—information which can greatly aid in selecting true matches from the many false positives that are frequently generated by pairwise comparisons alone.[6] While the algorithm in Fu et al. focuses on tracing the core members of a household through time, I adapt their algorithm to account for household members who move away. The magnitude of observations I am working with is more than a dozen times larger than in Fu et al. The Cornwall data also

[5] This group is affiliated with FreeCen, a non-profit organization whose stated mission is to provide high quality 19th Century census transcriptions free of charge. The transcription methodology and standards for FreeCen can be found at http://www.freecen.org.uk/.

[6] The Minnesota Population Center has deliberately omitted household information from its census linking projects since it is a "source[ ] of potential bias" (Goeken et al., 2011, p. 8). Here the exercise is different. I am not matching a 1% sample into a complete census with the goal of maintaining a representative sample of matches. My ultimate goal is to match (or at least account for) *all* individuals. In doing so, I take more of a jigsaw puzzle approach by putting the easy pieces together first (this paper) and then figuring out where the more difficult pieces fit in (future work). Additionally, by beginning the matching algorithm with pairwise comparisons at the person level, I avoid the pitfalls Goeken et al. allude to for matching on household traits when the composition of a household changes between censuses (p. 8–9).

Table 3.1: Census Observations by Year.

1841	1851	1861	1871	1881	1891	Total
340,901	354,742	372,163	356,364	324,835	318,634	2,067,639

covers a considerably larger geographic area. Given this expanded coverage, tracking migration within Cornwall presents a challenge since comparison sets must be kept small enough to be computationally feasible. I investigate a nested matching algorithm which meets this challenge. For most censuses, I am able to successfully identify about 43% of the population in the subsequent census.

I demonstrate one potential use of the new linked dataset by investigating occupational patterns between fathers and sons. For this exercise, I look across censuses and identify a father's occupation when his son is a boy and then the son's occupation later in life as an adult. Since many of the occupations cannot be clearly ranked monetarily, and since several require unique skill sets, I analyze intergenerational patterns on a sector by sector basis. For most sectors, between one-quarter and one-half of sons pursue jobs in their father's line of work. Using multinomial logistic regression, I find that primogeniture inheritance is clearly apparent in farming but seemingly absent in other sectors.

3.2 Data Formatting

With the underlying goal of maintaining the maximal amount of identifying information, I follow some established procedures and implement others that are new in preparing the data for matching. There are eight variables from the census that I primarily utilize: first name, surname, birth year, birthplace, occupation, gender, address, and household. The gender variable requires little cleaning, and the birth year variable can easily be constructed from the ages reported in the census.[7] I largely follow Fu et al. (2011a) in my standardization of the name variables.[8] I also follow Fu et al. in using the occupational codes that were developed for the 1911 UK Census to numerically code occupational values.[9] The original coding system assigned three-digit codes to each occupation. However, it also grouped occupations into 23 industries, each with between one and eleven sub-industries. To account for these industry divisions and to increase the numeric difference between industries, I extended the three-digit codes into a new seven-digit code where the first two digits are the industry number, the next two digits are the sub-industry number, and the last three digits are comprised of the original three-digit code.

One of the greatest assets of the UK censuses in terms of personal identifying information (and oddly omitted from Fu et al.) is the specificity of birthplaces. In the

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[7]The primary caution is to be mindful of the units in which ages are reported since children's ages are often reported in months, weeks, or even days. Census dates are also important in calculating a child's birth year: June 6 in 1841, and between March 30 and April 7 for 1851–1891.

[8]Names must first be divided into surnames and forenames (this distinction was already made in some transcriptions). I then further divided forenames into first names and middle names (due to the sparsity of middle names and the irregularity of their use from one census to another, I chose to ignore them in the matching process). After removing special characters and capitalizations, I standardized common abbreviations and name variants by assigning uniform values. For example, Chas, Chs, Charley, and Carles were all coded as Charles.

[9]Information on the occupational code system can be found on the official 1911 Census website: http://www.1911census.co.uk/content/default.aspx?127

US, “New York” or “Virginia” is a typical birthplace entry—broad enough to convey no more than trifling information for matching people within those states. The UK censuses, however, commonly specify a person’s birthplace to within a radius of two or three kilometers (the one exception being the 1841 census which only identified a person as having been born either in or out of the county). That said, it is also not uncommon for one parish to be reported as a person’s birthplace in one census and a neighboring parish to be reported in the next. Although geographically close, this information would be lost in the matching process without a reasonable metric for comparing place names. To account for such discrepancies, I used the UK Ordnance Survey’s coordinate system to geocode birthplaces with a northing and an easting.[10] For birthplaces outside the UK, Ireland, and the Channel Islands, I used latitude and longitude degrees.

I use four nested variables to describe a person’s residential location on the night of the census: civil parish, registration district, and two nested groupings of registration districts.[11] This nesting process will be useful in tracking migration. Civil parishes are fairly small, with well over 200 in Cornwall. Registrations districts are substantially larger with portions of 16 within the county, each comprised of as

[10]I used the “1:50 000 Scale Gazetteer” (http://www.ordnancesurvey.co.uk/, downloaded on 13 June 2013). This data, measured in meters and rounded to half a kilometer, provides northing and easting coordinates for thousands of places within the UK. I used the same coordinate system to identify major locations in Ireland and the Channel Islands.

[11]By census regulations, the residence of a visitor or a traveler was recorded as the place where they physically resided on the night of the census. Hence, the residence in the census may not be the person’s permanent domicile.

many as 25 civil parishes apiece.[12] The regional groupings of registration districts first divide Cornwall into six regions, and then into three.[13]

For referencing records within and across censuses, I developed a simple system which nests person, household, regional, and census level information. Individuals are numbered within a household, households are numbered within a census piece (a numbered geographic division of the national census enumeration), and census pieces are numbered within a census. For instance, reading from right to left, record number 9-1856-1128-008 refers to the eighth person in the 1128th household of census piece 1856 in the 18<u>9</u>1 census.[14] Harnessing such a degree of information within the identification numbers themselves has proved to be quite useful.

3.3 Methodology

Before covering details, I will give a brief synopsis of the census matching algorithm. First, pairwise comparisons are done to assess the similarity of records across

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[12]Small portions of Launceston and St. Germans extended beyond Cornwall into Devon County, while bits of Holsworthy and Tavistock crossed into Cornwall County. The remaining 12 registration districts were entirely within Cornwall. Transcription coverage along the Cornwall-Devon border varies from census to census.

[13]The small groupings consist of the following registration districts: 1. Stratton, Launceston, Camelford, and Holsworthy; 2. Liskeard, St. Germans, and Tavistock; 3. Bodmin and St. Austell; 4. St. Columb and Truro; 5. Falmouth and Redruth; and 6. Penzance, Helston, and the Scilly Islands. The larger groupings combine 1–2, 3–4, and 5–6.

[14]Household distinctions are less clear in the 1841 census than they are in subsequent censuses. Household relationship data is also not included in the 1841 census, so I assigned a new household each time the surname changed (this method does, however, separate live-in servants, boarders, and relatives with different surnames into different households). In all censuses, I omit vacant buildings from the numbering process. The transcriptions used by Fu et al. apparently did not include household divisions since they developed a method for automatically detecting breaks between households based on relationship entries (2011a).

censuses. Second, a machine learning algorithm classifies each pairwise comparison as either a match or a non-match. Third, to try to separate false positive matches from true matches, household information is taken into account via a group linking process. So far this is according to the algorithm in Fu et al. (2011b). My contributions are in two additional steps. Fourth, I extend the group linking process to look for individuals who have moved away from their former household members. Fifth, I balance the goal of tracking migration within Cornwall with the computational cost of conducting vast numbers of pairwise comparisons by first identifying the people who did *not* move. These records are then removed, and steps one through four are iterated on the remaining records, looking for people who moved increasingly larger distances.

As I alluded to, it is computationally expensive to systematically compare each record in one census with each record in another. Two censuses with one-third of a million observations apiece would result in over 111 billion pairwise comparisons. In the data linking literature, the concept of blocking has been established to focus pairwise comparisons on sets of data where matches are most likely to occur. For instance, there is little sense in attempting to match a male in one census with a female in another (unless, perhaps, there was an error in the recording process).

I use three variables to define comparison blocks in the census data: gender, surname, and residential location. Thus, pairwise comparisons between censuses will be limited to observations that share the same values for each of these variables. To account for spelling variations and phonetic similarities, the surname variable is coded as a double metaphone. Residential location is based on the four nested variables described earlier, with the county of Cornwall as a fifth.

Since the matching algorithm in Fu et al. (2011b) serves as the basis of the algorithm here, it is worth outlining. Censuses are compared two at a time. After being separated into blocks based on gender, surname, and residential location, observations within each block are then systematically compared across censuses.[15] Pairwise comparisons assess the similarity of records based on five attributes: first name, birth year, the northing and easting of the birthplace geocode, and the occupation code.[16] Each of these five attributes is assigned a score in the unit interval where the score is monotonically increasing in the degree of similarity—1 representing identical values and 0 reflecting no similarity.[17] At this point, a question arises as to how the five similarity scores should be interpreted. How should each score be weighted, and what is the threshold for classifying the record pair as a match? A viable solution is the use of a *support vector machine* (SVM), a technique whereby a computer is trained to identify the characteristics of a true match and to place weights accordingly.

An SVM identifies the separating hyperplane which places the most distance

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[15]Fu et al. (2011a) only uses surnames to define blocks; Fu et al. (2011b) does not specify which variables, if any, are used at the blocking stage.

[16]Fu et al. (2011b) compares surname, first name, gender, age, occupation code, and address.

[17]The specific comparison methods are as follows: first names are compared by the Winkler string comparator (checking for the initial characters, long sets of characters that are similar, and overall similarity); birth years are compared by absolute numeric difference (a score of zero is assigned for differences over 10 years); northings, eastings, and occupational codes are compared by their percentage difference (differences above 15% are given scores of zero for northings and eastings, while 80% is used for occupational codes).

between data from two groups: true matches and false matches. To function, an SVM must first have a training sample of pairwise comparisons which have already been identified as either true or false matches. My training sample is comprised of 478 true matches which I identified by hand and 31,708 false matches. Pairwise comparisons had been done for these false matches because they were in the same block, yet they could not be true matches since I had already identified the unique true match.[18] Based on the characteristics of the elements in the training sample, the SVM is able to identify the desired hyperplane. This hyperplane can then be used to determine the matching status of additional pairwise comparisons.

Formally, let $(\mathbf{x}_k, y_k)$ be a pairwise comparison where $\mathbf{x}_k$ is the vector of similarity scores and $y_k \in \{1, -1\}$ indicates the true or false match status. With the notation for a hyperplane of $\mathbf{w}^T\mathbf{x} + b = 0$, the SVM solves:

$$\min_{\mathbf{w},b,\xi} \quad \frac{1}{2}\mathbf{w}^T\mathbf{w} + C\sum_k \xi_k \qquad \text{s.t.} \quad y_k[\mathbf{w}^T\phi(\mathbf{x}_k) + b] \geq 1 - \xi_k \quad \text{and} \quad \xi_k \geq 0$$

Nonlinear separations between groups are accounted for through the kernel $\phi(\cdot)$, and outliers are factored in through a penalty parameter $C > 0$ and a slack variable ξ (Cortes and Vapnik, 1995).[19] Since birthplace information is limited in the 1841 census, I train two separate SVM models: one that incorporates the birthplace geocodes

[18] For the true matches, I randomly sampled portions of the Penzance and Falmouth registration districts from both the 1861 and 1871 censuses and then looked for the matching record in the corresponding census (either 1861 or 1871). The false matches were derived from pairwise comparisons for these registration districts.

[19] I use the Gaussian Radial Basis Kernel with parameter $\gamma = 0.001$. I also set $C = 200$. My goal in selecting these parameter values was to reduce the degree of both false positives and false negatives.

and one that omits them.

After the SVM determines the matching status of each pairwise comparison, household information is used to address the problem of false positives. The underlying premise being that while an individual (e.g. Mary Smith) may have many potential matches, the probability of detecting the true match increases dramatically if her family members also have matches within the same household. The group linking process assigns a similarity score to each potential household pair. Formally, let $M_{i,j}$ be the set of pairwise comparisons between households i and j that the SVM classified as matches (with its cardinality denoted by $|M_{i,j}|$). Then for each $k \in M_{i,j}$, let S_k be the sum of the similarity scores $\mathbf{x}_k$.[20] Assuming household i is in the earlier of the two censuses and household j is in the latter, denote the number of people in household i as m_i and the number of people in household j who are at least 10 years-old as m_j.[21] The similarity score $\mathbb{S}_{i,j}$ between households i and j is computed as follows:

$$\mathbb{S}_{i,j} = \frac{\sum_{k \in M_{i,j}} S_k}{m_i + m_j - |M_{i,j}|}$$

This formula is increasing in both the number of SVM matches between households and the raw strength of their underlying similarity scores. However, it also takes the number of potential matches between households into account and places downward weight accordingly. The algorithm in Fu et al. (2011b) concludes by selecting the

[20]Fu et al. (2011b) find that this formulation outperforms the alternative where $S_k = 1$.

[21]The purpose of m_i and m_j is to provide weights for the number of potential matches. Therefore, unlike Fu et al. (2011b), I adjust m_j to only include individuals who had been born previous to the last census.

household pairs that have the greatest similarity.[22]

A shortcoming of only selecting the household pair with the maximum similarity score is that it fails to account for evolving household compositions. Family members may move in or out. A household may also be comprised of multiple family units, such as live-in servants, lodgers, or extended family members. Even though core members of a household may be tracked by selecting household pairs with the maximum similarity score, peripheral members are liable to be missed. I introduce a simple multi-level group linking technique to account for such peripheral changes. The first level begins by identifying the household pairs with the highest similarity scores. All SVM matches that are between such household pairs are marked as having cleared this stage of analysis and are separated from the remaining SVM matches. In each successive level, this pattern is repeated: household pairs with the greatest similarity among the remaining SVM matches are flagged, and the SVM matches within those household pairs are separated so that the process can be repeated. This multi-level technique may be iterated any number of times. I find, however, that the number of additional matches decreases sharply with each new level; so I stop after three levels.

The final element of my matching algorithm is to search for matches over an expanding geographic region. Even after separating the data into comparison blocks

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[22]Their notion of similarity, however, is one-sided. For each household in the first census, Fu et al. select the household in the second census with the highest similarity. Yet the converse may not be true (i.e. a different pairing may arise if for each household in the second census, the household in the first census with the highest similarity is selected). To circumvent this problem, I require household pairs to have the maximum similarity score in both directions.

Table 3.2: Matching Stages by Residential Location.

	Type		
Location	A	B	C
Civil Parish	1		
Registration District	2	1	1
RD Small Group	3	2	
RD Large Group	4	3	
Cornwall	5	4	2

by gender and surname, a full pairwise analysis for the entire county of Cornwall is—computationally—a fairly tall order. Furthermore, if such a task is difficult at the county level, consider the difficulty in doing a full pairwise comparison for the entire UK or US. Restricting comparison blocks to smaller geographic regions has clear computational benefits, but it also comes with the potential that erroneous matches may arise at the local level which prohibit true matches from being classified at the global level. The question then is whether or not the same records are consistently classified as matches under different geographic divisions. To investigate this issue, I examine three different matching types which vary in terms of their initial and intermediate geographic divisions (see Table 3.2).

Matching Type A, for instance, begins at the civil parish level. Therefore, all of the matching steps up to this point are conducted among individuals living in the same civil parish. Before repeating the matching steps among individuals within the same registration district, I first remove matches that have a high probability of being correct. Reducing the comparison set in this manner prevents it from becoming too unwieldy when comparisons are done over a larger geographic area. The matches I specifically remove are those where at least two individuals have been matched in a household pair during the multi-level group linking step. For each of the three matching types, I repeat this process, gradually increasing the geographic coverage until the remaining set of records are compared across the entirety of Cornwall.

As a final note on methodology, I used Febrl to calculate the similarity scores (Christen, 2008), and the R package e1071 to conduct the SVM analysis (this package implements the libsvm program by Chang and Lin, 2011).

3.4 Results

Combining the multi-level group linking with the nested geographic matching, there are as many as 15 different matching stages. To highlight the marginal contribution of each, Table 3.3 shows the number of pairwise comparisons to be classified as a match at each stage for the 1871–81 census pair. I report two columns for each of the three matching types: (i) the number of initial matches as given by the algorithm; (ii) a refined set of matches, based on stricter criteria that will be explained shortly. By far, the greatest number of matches occur within the smallest geographic divisions. This should not come as a surprise since a large portion of the population remains settled over time; and if migration does occur, it is presumably easier for individuals to move than for families. The multi-level matching does pick up a fair number of individuals who leave their former household—such as when multiple fam-

Table 3.3: Number of Matches Identified at Each Stage: 1871–1881.

	ML	Type A Init.	Type A Ref.	Type B Init.	Type B Ref.	Type C Init.	Type C Ref.
Civ Par	1	90,785	90,348				
	2	1,266	1,061				
	3	77	61				
Reg Dist	1	11,797	11,731	102,033	101,558	102,033	101,558
	2	228	190	2,088	1,661	2,088	1,661
	3	31	18	232	160	232	160
RD Sm	1	1,776	1,751	1,774	1,754		
	2	52	40	62	52		
	3	8	8	18	16		
RD Lg	1	2,818	2,784	2,812	2,785		
	2	74	47	76	47		
	3	26	18	28	18		
Cornwall	1	2,841	2,821	2,832	2,816	7,391	7,330
	2	117	103	119	103	263	218
	3	50	44	50	46	106	86
Solo Matches		65,487	39,274	63,684	39,537	57,081	37,736
Total		177,433	150,299	175,808	150,553	169,194	148,749

ilies or extended family members are living together in one census but are in different households by the next. The multi-level matching also frequently identifies individuals who are the sole match between two households. However, I group these solo matches separately in Table 3.3, regardless of the stage in which they are found.[23]

Another feature of Table 3.3 is that the number of matches within a geographic

[23]Since records in solo matches remain eligible for the next stage of matching, many of these solo matches are identified multiple times. Table 3.3 gives the number of unique pairwise solo matches.

region is largely unaffected by whether or not the matching is first conducted within a smaller geographic area. This is especially the case among non-solo matches. For instance, Type A has a total of 104,184 non-solo matches across both the civil parish and registration district levels—less than 200 off of the number generated when the matching starts at the registration district level (104,353 matches for Types B and C). Likewise, Type C jumps straight from the registration district level to comparing all of Cornwall, while Types A and B take two additional geographic steps. Yet the number of non-solo matches identified by each type are practically identical (Type A: 7,762; Type B: 7,771; Type C: 7,760). Although the numbers of matches remains similar, there is still a question as to whether they represent the same pairwise matches. Table 3.4 examines the overlap of pairwise matches between the three different matching types. Interestingly, the vast majority of pairwise matches found by any of the matching types were found by all matching types. This holds true both before (93.3%) and after (98.6%) the refinements and suggests that geographic divisions may be chosen to suit computational needs. Looking at the matches that were not identified in all three cases, Types A and C had the most dissimilarities—each conforming well with Type B, but not with each other. Type A has the most unique contributions while Type B has the least. Many of the dissimilarities across matching types center on the solo matches.

The veracity of the solo matches is the most questionable. Yet blanketly removing them would severely compromise the construction of a panel dataset designed to follow individuals over their life cycle: the connection between childhood

Table 3.4: Overlap of Match-Pairs Across Matching Types: 1871–1881.

Match Types	Initial	After Refinements
A, B, C	167,329	148,435
A, B	6,817	1,824
A, C	4	1
B, C	1,550	294
A	3,271	38
B	106	0
C	305	19
Total	179,382	150,611

and adulthood would largely vanish as individuals leave home. I implement a five step refinement process in order to remove solo matches that have a low level of credibility, as well as to hold non-solo matches to a higher standard. The number of matches remaining after each stage of the refinement process is shown in Table 3.5. This refinement process also synthesizes the three different matching types. Thus, before beginning the refinement process, I combine all of the unique pairwise matches from the three different matching types into a single dataset, noting where each pairwise match was identified and the associated number of matches in its household pair.

The first refinement drops all solo matches that were identified by only one of the three matching types. While some of these may be legitimate matches that take advantage of the unique geographic structure of a particular matching types, I reject them for lack of a second witness (either in terms of additional household members or by being acknowledged by a second matching type). In the second refinement, I

Table 3.5: Unique Match-Pairs After Each Refinement.

	1841–51	1851–61	1861–71	1871–81	1881–91
All Match Types	203,165	191,451	193,452	179,382	167,529
Refinement 1	189,812	185,878	188,861	175,839	165,108
2	167,254	175,317	179,289	168,261	158,438
3	115,498	155,408	159,738	151,797	145,285
4	108,904	153,204	157,995	150,688	144,595
5	108,624	153,062	157,870	150,611	144,448

remove all solo matches where the individual is listed as a wife in the latter of the two censuses. Frequently, when a wife is matched, additional members of her household are also matched; so a solo match in this case is likely comparing a woman whose married name happens to coincide with another woman's maiden name.

The final three refinements address duplicates where a record in one census is matched to two or more records in another census. For the third refinement, duplicates are identified, and any duplicate that is a solo match is discarded. The fourth refinement is similar, but is done separately because of the large number of solo duplicates. It first entails re-identifying duplicates among the remaining matches. Then, any match belonging to a household pair comprised of either two or three matches is removed if there is a duplicate anywhere in the household pair. For example, suppose that three members of one household have been matched with three members of another. If two of the household members have also been matched elsewhere, the credibility of the entire household pair as a match comes into question. The fifth and final refinement stage addresses duplicates in household pairs that have

Table 3.6: Number of Matches by the Size of Household Pairs.

Matches/HH	1841–51	1851–61	1861–71	1871–81	1881–91
1	32,148	42,549	41,496	39,538	38,573
2	24,358	28,436	30,510	29,616	28,798
3	18,798	26,475	29,109	27,657	26,652
4	15,219	22,802	24,331	22,959	21,891
5	9,651	16,108	16,315	15,572	14,177
6	5,197	9,759	9,366	8,964	7,754
7	2,273	4,331	4,417	4,056	4,050
8	700	1,769	1,612	1,617	1,498
9	205	583	495	468	522
10+	75	250	219	164	533

at least four matches. By this point, duplicate matches in the remaining data are primarily limited to family members with similar names and attributes. The number of these duplicates is small enough that it would be feasible (although still many hours of work) to manually determine the correct matching status of each. For now, I simply drop the remaining duplicates, leaving the remainder of the household pair intact. I use the refined data from here on out.

Table 3.6 breaks down the refined data by the size of household pairs. For example, between the 1861 and 1871 censuses, 29,109 matches were part of household pairs that consisted of exactly three SVM matches between the households. Across the board, the number of matches steadily declines as the size of household pairs increases. Another feature of Table 3.6 is that the number of matches within each size remains remarkably consistent from one census pair to another between 1851 and 1891. The matching rates are likewise similar.

Forward matching rates, or the percent of the population in one census that can be identified in the next, are given for each census pair in Table 3.7. These rates are presented for all of Cornwall, as well as at the registration district level.[24] It is somewhat intriguing how little variation there is in the forward matching rates at both the aggregate and registration district levels from the 1851 census onward. At the aggregate, the matching rates are all within 2.2 percentage points of each other with an average of 43.1%. The average spread at the registration district level across census pairs is 3.7 percentage points. There is also no clear pattern between the forward matching rates and the populations of the registration districts.[25] The consistency of the matching rates over time suggests a homogeneity in the population structure throughout Victorian Cornwall.

The lower matching rate between the 1841 and 1851 censuses (32%) largely stems from the lack of precision in the 1841 census. As was previously noted, the 1841 census only recorded the place of birth as either in or out of the county. Given that the overwhelming majority were born in the county, I conducted the SVM matching for this census pair without birthplace information. Another known issue with the 1841 census is that ages above 15 were often rounded in five-year increments. In future work, it may be beneficial to specially train an SVM model that is based on

[24] Tavistock, Holsworthy, and the 1861 shipping records are not shown individually, but are included in the totals.

[25] Pearson correlation coefficients between the forward matching rates and the average population of the registration districts range from −0.32 to 0.23 for census pairs between 1851 and 1891. The correlation is stronger for 1841–51 at 0.57. Most of the registration districts in Cornwall had fairly constant populations over this time period.

Table 3.7: Forward Matching Rates by Registration District.

		Forward Matching Rates (%)				
RD	Ave Pop	1841−51	1851−61	1861−71	1871−81	1881−91
Bodmin	19,600	29.3	40.2	44.1	44.1	45.2
Camelford	8,000	31.4	39.6	45.3	41.6	44.6
Falmouth	23,300	29.8	40.8	40.4	39.7	39.1
Helston	27,600	33.5	47.3	46.4	44.9	47.6
Launceston	16,300	29.0	39.6	42.7	42.0	43.3
Liskeard	29,100	31.1	42.3	43.8	41.0	44.6
Penzance	51,700	33.6	45.8	47.1	44.9	47.0
Redruth	51,100	33.7	44.3	41.2	39.0	43.9
Scilly Islands	2,200	32.1	51.1	48.9	54.4	47.8
St Austell	31,500	32.4	44.2	42.5	43.8	45.6
St Columb	16,500	32.0	43.5	43.3	44.5	44.6
St Germans	16,900	25.8	36.3	39.0	39.7	39.3
Stratton	8,000	28.1	38.6	39.7	40.7	43.3
Truro	39,900	33.2	43.3	43.5	41.9	44.9
TOTAL	344,600	31.9	43.1	42.4	42.3	44.5

a sample of 1841 and 1851 census records (as opposed to the 1861 and 1871 sample used here). Many of the matches that have been identified could be used in such a sample. For that matter, it may also be useful to train an SVM model for each census pair, taking advantage of matches which have been identified here. Larger and more specific training samples may allow the algorithm to better identify the defining traits of matches.

There are additional ways in which the matching rate can be considerably improved. Marriage records provide the vital link for tracing women as their surnames change. Based on the volume of marriage records, approximately 7% of the 1871

Table 3.8: Distribution of Start and End Years in Linked Panel Dataset.

		End Year				
		1851	1861	1871	1881	1891
Start Year	1841	62,489	24,604	9,896	5,515	6,120
	1851	-	60,476	25,253	9,955	11,243
	1861	-	-	53,274	19,367	17,247
	1871	-	-	-	45,776	35,388
	1881	-	-	-	-	74,450

Cornwall population changed their surname before the 1881 census.[26] Not only does the current algorithm miss these brides, but the bias toward selecting matches with multiple members in the household is limiting the number of grooms that are matched as they move away from home. With marriage records in hand, an additional training sample could be developed for tracking brides and grooms at the point of matrimony.

Death records are also pertinent to improving the matching rate. Although these individuals will not be found in the next census, they can at least be accounted for—removing the ambiguity as to whether the person died, emigrated, or cannot be found for some other reason. For monitoring attrition, the value of death records is appreciable. Around 12% of the 1871 Cornwall population died before the 1881 census.[27] Hence, incorporating marriage and death records has the potential to boost

[26]There are 53,444 marriage records in Cornwall (half from brides, half from grooms) between the second quarter of 1871 to the first quarter of 1881 (search performed at http://www.freebmd.org.uk/ on 3 March 2014). A small percentage of these records are likely duplicates.

[27]FreeBMD records 66,967 deaths in Cornwall between the second quarter of 1871 and the

the percentage of records that are accounted for from one census to the next to well over 60%.[28]

In order to construct a panel dataset, I merely link the individual record numbers from the matches in each of the census pairs. There are 714,615 matches across the five census pairs after the refinements (see Table 3.5). Most of these (58.5%) can be strung together in chains which follow individuals over three to six censuses. Put together, the panel dataset contains 461,053 individuals that are matched in two or more censuses; 164,588 that are matched in three or more censuses; 59,976 in four or more censuses; 22,878 in five or more; and 6,120 individuals that can be indentified in all six censuses. While there is still ample room for improvement, as it stands, this is quite possibly the largest panel dataset for Victorian England at present. Table 3.8 shows the distribution of the first census and last census where an individual has been identified.[29] The panel could potentially be improved by matching between non-adjacent censuses. If a person is absent or difficult to identify in one census, it may still be possible to find them in the next.

first quarter of 1881. However, this number includes duplicates, as well as infants who lived and died between censuses. The organization, A Vision of Britian Through Time, reports total deaths and infant deaths during census years. Non-infant deaths at Vision of Britain consistently account for about 65% of the number of death records at FreeBMD for the years 1871, 1881, and 1891. So the 12% death rate between the 1871 and 1881 census is based on 65% of 66,967. See http://www.freebmd.org.uk/ and http://www.visionofbritain.org.uk/ (accessed 3 March 2014).

[28]Ship manifests, census records from neighboring counties, and other emmigration records could boost the matching rate even more; and the demographic picture could be further fleshed out with probate records, obituaries, and local directories.

[29]It is likely that many people are represented as two or more individuals in the panel dataset. For instance, two censuses may match a girl in her childhood. Then two or more subsequent censuses may match the same girl as a married woman.

3.5 Application: Occupational Following

It is not uncommon to learn that a person is in the same occupation as their father or another family member—a pattern which has been named occupational following.[30] Indeed, occupations tend to be perpetuated along family lines, and a vast number of Western surnames (Baker, Cooper, Tanner, etc.) allude to what in many cases was probably a multigenerational family trade. The Victorian era provides an interesting backdrop for examining occupational following. Formal schooling standards were steadily growing during this era, propelled in part by a series of educational legislations that began in 1870.[31] However, basic education in literacy and arithmetic did not of itself prepare a child for a career as a stonecutter or a shoemaker. Those were skills that were often acquired through some form of apprenticeship—frequently with a father teaching his son.

From the new panel data set, I have identified four cohorts of father-son pairs. Each pair gives the father's occupation when the son is between the ages of 7 and 16. In a skills based economy with little formal education, these are formative years for exposing a child to an occupation. The second part of the father-son pair is the son's occupation when he is between the ages of 21 and 30. Given that censuses occur at ten-year intervals, the father's observation is either ten or twenty years before the

[30]The term occupational following at least dates back to Laband and Lentz (1983). Those authors have since used it in a long list of papers, as have other scholars.

[31]Schooling for children ages five to ten became compulsory in Britain in 1880. Information about these educational laws can be found on the UK Parliament's website: http://www.parliament.uk/about/living-heritage/transformingsociety/livinglearning/school/ (accessed 21 June 2014).

son's observation. The cohorts are defined by the census in which the son's occupation is recorded (1861, 1871, 1881, and 1891). For the sons, each of these cohorts represent roughly one-fourth of Cornwall's working male population between the ages of 21 and 30.[32]

For analyzing occupational patterns, I have divided the economy into 18 sectors. The occupational groupings for these sectors, described in Table 3.9, are based in part on the industry divisions employed by the occupational coding scheme for the 1911 UK Census. I have, however, separated and regrouped many occupations based on their rate of occurrence in the data and on the proximity of skills required for different occupations. For instance, due to the overwhelming size of the agricultural industry, I have separated it into six sectors. Other sectors, such as Letters or Fine Crafts, are a miscellany of occupations that are at best loosely related in terms of the type of the product or the requisite training.

The distribution of the workforce across the 18 sectors can be seen in Table 3.10. The table first lists the distribution for all males in Cornwall between the ages of 21 and 30 in each census year from 1861 to 1891. Distributions for the sons and fathers in each cohort make up the remainder of the table. By far, mining is the largest sector, accounting for 41% of the sons in the 1861 cohort (36% among all males 21–30 and 35% of fathers). Over the coming decades, however, the mining sector in Cornwall collapsed so that by the 1891 cohort it employed roughly half as

[32]Many more father-son pairs can be identified. The prominent constraint here is having an occupation listed for both the father and the son during the correct time intervals.

many people. The depletion of the mining sector was a major factor in the decline of Cornwall's population over the later half of the nineteenth century (see Table 3.1).[33] The three primary agricultural sectors—farmers, relatives of farmers, and agricultural labourers—together make up a sizeable block of the economy, and the largest block once mining goes into decline. Sons were more likely to be working on their father's farm than to have their own farm by the time they were 21–30 (and only a smattering of fathers were working on a relative's farm when they had a son aged 7–16). Fishing and boating, woodworking, and the metal and machine sectors each comprises about 5% to 10% of the workforce from decade to decade. Across each of the sectors (mining being the primary exception) there are only minor fluctuations in the distributions from one census to another.[34]

The persistence of occupational following varies considerably across sectors. Combining all cohorts, Table 3.11 provides the transition probabilities from a father's occupation to that of his son's. Percents sum across the rows so that, for instance, of those who had a father in the meat and cereal sector, 46% entered the same sector, 10% went into mining, 5% became woodworkers, etc. Children of miners and of those in the fishing and boating sector have the highest propensity to follow in their father's footsteps (67%). Masonry has a 60% continuance rate, and if farmers and their relatives are combined, then farming comes in at 62%. More commonly,

[33] The Wikipedia article "Cornish Diaspora" estimates that roughly a quarter of a million Cornish people, most of them miners, left the UK between 1861 and 1901.

[34] The Labourer—Other sector does have a sizeable jump between 1861 and 1871 which may well be catching some of the displaced miners.

Table 3.9: Occupational Sectors.

	Industry	Description	Abbr
1	Farmer	Principal farmer on a farm of any size	Farm
2	Farmer Relative	Workers on a relative's farm (e.g. farmer's children)	FRel
3	Cereal & Meat	Millers, butchers, other dealers in meat and grain	Meat
4	Grocer & Baker	Dealers of baked goods, produce, and other groceries	Groc
5	Livestock, etc.	Employed with horses, cows, other livestock, or in sundry agricultural trades	Live
6	Labourer Agr.	Low-skilled, hired agricultural labourers	LabA
7	Labourer Other	Domestic, railway, dockyard, and other low-skilled labourers	LabO
8	Mining	Workers and supervisors in various mines (tin, copper, stone, etc.)	Mine
9	Military & Police	All branches of the military, police, municipal offices	Govt
10	Metal & Machines	Metal manufacturers, blacksmiths, engine operators, etc.	Metl
11	Fine Crafts	Watchmakers, chemists, building workers, and various fine arts	Fine
12	Letters	Medicine, education, law, clergy, printing, clerking, etc.	Ltrs
13	Clothing	Tailors, drapers, milliners, and other cloth, cord, and clothing workers	Clth
14	Fishing & Boating	Fishermen, seamen, pilots, and bargemen	Boat
15	Masonry	Masons and other stonecutters	Masn
16	Shoes & Leather	Shoe and boot makers, saddlers, other workers in leather and skins	Shoe
17	Woodworkers	Carpenters, shipwrights, sawyers, coopers, wheelwrights, etc.	Wood
18	Taverns & Sales	Innkeepers, shopkeepers, brewers, street vendors, etc.	Tvrn

Table 3.10: Percent of Workforce in Each Sector.

		All Males 21 30				Sons				Fathers			
	Ind	1861	1871	1881	1891	1861	1871	1881	1891	1861	1871	1881	1891
1	Farm	3.2	3.4	3.9	5.2	3.7	4.5	4.5	6.7	16.0	16.5	17.4	18.8
2	FRel	3.4	4.8	6.4	4.7	6.6	7.7	10.9	8.3	0.1	0.1	0.1	0.4
3	Meat	1.8	2.0	2.1	2.3	2.0	2.0	2.6	2.7	2.0	2.3	2.4	2.1
4	Groc	0.9	1.1	1.7	2.0	0.8	0.9	1.4	1.8	0.8	1.2	1.7	1.5
5	Live	3.5	1.9	2.4	2.7	1.7	1.6	1.9	2.2	1.2	1.4	1.7	2.1
6	LabA	11.6	12.2	13	15.6	7.7	11.5	12.7	15.1	14.1	15.1	14.6	16.2
7	LabO	5.6	9.1	10.7	10.3	3.6	6.1	7.7	8.2	3.7	3.6	4.8	6.9
8	Mine	35.7	26.7	19.1	18.1	41.4	29.2	20.8	18.4	34.8	29.1	25.5	20.9
9	Govt	1.0	1.0	1.0	1.2	0.5	0.4	0.4	0.6	0.3	0.4	0.3	0.4
10	Metl	5.3	6.1	5.1	6.1	5.8	6.6	5.5	6.3	4.3	5.3	5.2	5.3
11	Fine	1.1	1.5	2.0	2.7	1.1	1.1	1.8	2.3	0.7	0.9	1.1	1.4
12	Ltrs	2.9	3.5	4.7	5.0	2.2	2.5	3.6	3.6	1.3	1.4	1.3	1.1
13	Clth	2.3	2.3	2.0	2.3	2.2	2.3	2.0	2.4	1.7	2.1	1.8	1.6
14	Boat	8.7	8.5	12.1	8.4	4.5	5.4	9.1	7.4	3.2	4.7	7.8	8.4
15	Masn	3.2	4.8	4.7	4.9	4.1	5.5	5.2	5.3	4.5	4.5	4.1	4.2
16	Shoe	3.1	3.1	2.3	1.8	3.9	3.6	2.5	1.8	3.6	4.2	3.4	2.6
17	Wood	6.0	7.3	5.9	5.6	7.6	8.4	6.8	6.1	5.9	6.2	5.8	5.1
18	Tvrn	0.7	0.9	0.8	1.1	0.6	0.7	0.6	0.8	1.8	1.1	1.1	1.2
Employed		22,559	19,624	18,953	19,533	4,358	4,942	4,929	5,323	4,358	4,942	4,929	5,323

however, the rate at which son's enter their father's profession tends to fall between a quarter and a half. Sons of tavern and livestock workers typically scout out more fertile ground in other professions—a mere 12% maintaining their father's trade in each of these sectors.

There is a great deal of transitioning among and into low-skilled occupations. Children of livestock workers and common labourers will commonly switch to the mines or to other labourer positions. Since these sectors are dominated by manual hired labour, cultivating a skill set is likely not as pertinent as simply finding employment. Even for those who had a father in a more skilled or capital intensive sector, large numbers fell back on the security of low-skilled employment. Mining in particular attracted workers from each background in large numbers.

Another feature of Table 3.11 is that, conditional on a son switching away from his father's occupation, there is a wide dispersement of destination occupations. The son of a shoemaker may very well decide to pick up masonry or woodworking, or even be a tailor. The same can also be said for the son of a common labourer. Such transitions necessarily involve the cost of skill and capital acquisition—two principal barriers to entering a new occupation. The full opportunity cost is likely highest for those who have already learned profitable skills at the tutelage of their fathers. While complete reasons for changing occupations may be unclear, the census provides additional information that at least allows for the identification of key correlates.

The linked census dataset makes it possible to tell whether the son is living in a different civil parish as an adult than he did as a child. If he did move, the distance

Table 3.11: Given a Father's Occupation, Percent of Sons Working in Each Sector (All Cohorts).

Father's Occupation	Son's Occupation: Farm	FRel	Meat	Groc	Live	LabA	LabO	Mine	Govt	Metl	Fine	Ltrs	Clth	Boat	Masn	Shoe	Wood	Tavn	**Count**
Farm	20	42	2	1	1	12	3	9		2		2	1	1	1	1	3		3,373
FRel	24	41	3			15		3				3			6		6		34
Meat	3	3	46	2	3	6	6	10	1	4	2	1	2	3	2	1	5		429
Groc	3	2	3	25	1	6	6	8	1	5	4	9	5	4	2	3	10	2	256
Live	4	4	2	2	12	14	11	12		6	3	5	2	4	3	4	11	1	318
LabA	3	3	1		3	37	11	17	1	4	1	1	2	3	4	3	6	1	2,940
LabO	2	2	2	1	3	14	21	15	1	7	2	5	3	7	4	4	8	1	947
Mine	1	1	1	1	1	6	4	67		6	1	2	1	2	2	1	3		5,319
Govt	3			1	1	4	15	10	3	7		10	6	19	3	3	12	1	68
Metl	1	1	1	1	2	6	6	24		37	2	3	2	2	4	2	5	1	986
Fine	2	1		4	2	4	4	9		7	30	8	2	3	11	3	13	2	198
Ltrs	3	2	1	2	2	4	9	15		9	5	37	2	3	3	2	3		249
Clth	1		1	1	2	6	5	9	1	7	3	6	35	4	3	6	7	1	348
Boat	1		1	1	1	4	5	3		2	1	2	2	67	1	2	6		1,203
Masn	1		1	1	2	3	4	8		3	4	2	2	2	60	2	5		844
Shoe	1		3	2	3	5	7	9		5	4	5	4	5	6	30	9	1	674
Wood	2	1	2	2	1	6	6	9		5	3	4	2	4	3	2	46		1,120
Tavn	5	7	6	2	1	8	7	15		5		9	3	5	2	2	12	11	246

can be calculated from the geocodes. Birth order within the son's family can also be approximated since the ages and relations of each household member are included in the census. A caution here is that the oldest son and heir, if that custom is followed, may not be living at home in a given census. Grown children typically leave home, and younger children on occasion can also be found living away from their parents (such as in apprenticeships or as servants). To control for primogeniture, I construct a variable that flags those who are clearly not the firstborn son. Another variable of note is the son's position within the household in the census where he is between 21 and 30. If the son is the head of the household that implies a different set of responsibilities than if he is listed as a son.

Given these variables, I implement a multinomial logistic regression for predicting a son's propensity to switch from one occupation to another. In doing so, I streamline the 18 occupational sectors into just four: farming (Farm and FRel); labour (Live, LabA, LabO, and Govt); mining (Mine); and trade (all other sectors).[35] The question then is, given that a father works in either farming, labour, mining, or trade, what is his son's probability of being employed in each of these industries, conditional on his characteristics in the census. Specifically, whether he moved from one civil parish to another (*Moved*), and if so how far (*Distance* in kilometers); whether he has older brothers (*Not.1st*); if he is the head of his own household (*rel.Head*), living as a son in his father's household, or has some other relationship (*rel.Oth*); and

[35] The inclusion Govt (Military & Police) as a labour sector is based on the assumption that enrollment in the military required little more than being healthy and fit.

also the decade of his cohort.

Table 3.12 shows the regression results for sons who have fathers in the farming or labour categories. For those with fathers in mining or trade, the results are in Table 3.13. In each of the four multinomial regressions, the coefficients are in comparison to the son going into his father's occupation. Since the trade category contains a large number of sectors with diverse specialized skills, I further specified whether the son went into the exact same sector as his father or a different sector in the trade category; so the coefficients in that regression are in reference to the son following his father's footsteps exactly.

The intercepts reveal that—all else equal—sons of labourers are more likely to switch to a trade than to become labourers themselves. This, however, is the exception since, based on the intercepts alone, sons of farmers, miners, and those in trade are highly prone to remain in their father's occupation. Moving from one parish to another is an important indicator of changing occupations. For those growing up on farms, there is a strong correlation between moving and switching to one of the trade sectors. If a son's father is already in one of the trade sectors, then transitioning to any other occupation is typically associated with a move. For instance, both the coefficients for *Moved* and *Distance* are particularly strong for those switching from trade to labour.

In terms of both magnitude and significance, the only time that primogeniture inheritance appears to play a role is in farming. Sons of fathers who are not the firstborn son have a significantly higher occurrence of going into a trade than of

Table 3.12: Multinomial Logistic Regressions for Predicting a Son's Occupation Given His Father's Occupation (Part 1).

	Father: Farm			Father: Labour		
	Labour	Mine	Trade	Farm	Mine	Trade
Intercept	−1.833***	−1.739***	−2.049***	−1.194***	0.064	0.369***
	(0.135)	(0.147)	(0.144)	(0.191)	(0.115)	(0.103)
Moved	0.211	0.334	0.602***	0.193	−0.193	−0.147
	(0.136)	(0.179)	(0.143)	(0.178)	(0.114)	(0.089)
Distance	0.017**	0.001	0.019***	0.000	0.010*	0.012***
	(0.005)	(0.008)	(0.005)	(0.008)	(0.004)	(0.003)
Not.1st	0.156	0.206	0.335**	−0.264	0.076	−0.059
	(0.104)	(0.133)	(0.112)	(0.157)	(0.097)	(0.077)
rel.Head	1.317***	1.205***	0.847***	−0.917***	0.157	−0.315***
	(0.112)	(0.143)	(0.127)	(0.156)	(0.103)	(0.083)
rel.Oth	1.727***	1.104***	1.649***	−2.754***	−0.599***	−0.229*
	(0.176)	(0.257)	(0.181)	(0.465)	(0.160)	(0.108)
1871	−0.210	−0.698***	−0.168	−0.354	−1.198***	−0.638***
	(0.149)	(0.165)	(0.157)	(0.224)	(0.128)	(0.113)
1881	−0.447**	−1.768***	−0.628***	−0.468*	−1.710***	−0.764***
	(0.150)	(0.212)	(0.165)	(0.220)	(0.138)	(0.112)
1891	−0.248	−1.197***	−0.399**	−0.513*	−1.571***	−0.804***
	(0.142)	(0.174)	(0.154)	(0.210)	(0.125)	(0.107)

Standard errors in parentheses. Significance levels: * 5%, ** 1%, *** 0.1%.

Table 3.13: Multinomial Logistic Regressions for Predicting a Son's Occupation Given His Father's Occupation (Part 2).

	Father: Mine			Father: Trade			
	Farm	Labour	Trade	Farm	Labr	Mine	Trade
Intercept	−3.910***	−3.201***	−2.162***	−2.658***	−2.064***	−1.415***	−0.663***
	(0.252)	(0.140)	(0.100)	(0.205)	(0.120)	(0.106)	(0.081)
Moved	0.188	0.489***	0.294**	0.806***	0.746***	0.579***	0.587***
	(0.251)	(0.115)	(0.097)	(0.208)	(0.106)	(0.115)	(0.085)
Distance	0.025***	0.024***	0.022***	0.011	0.013***	0.011**	0.007*
	(0.007)	(0.003)	(0.003)	(0.007)	(0.003)	(0.004)	(0.003)
Not.1st	−0.104	−0.130	−0.052	−0.423*	0.000	0.032	0.009
	(0.198)	(0.096)	(0.077)	(0.178)	(0.086)	(0.090)	(0.063)
rel.Head	−0.440*	0.134	−0.015	−0.434*	0.493***	0.550***	−0.019
	(0.210)	(0.105)	(0.084)	(0.177)	(0.092)	(0.095)	(0.068)
rel.Oth	−0.994*	1.025***	0.773***	−1.776***	0.716***	0.032	0.421***
	(0.479)	(0.140)	(0.119)	(0.475)	(0.129)	(0.161)	(0.101)
1871	0.417	0.680***	0.538***	−0.099	0.122	−0.388**	−0.005
	(0.286)	(0.148)	(0.105)	(0.241)	(0.128)	(0.115)	(0.090)
1881	1.079***	1.422***	0.897***	−0.028	0.156	−0.792***	−0.081
	(0.268)	(0.141)	(0.107)	(0.232)	(0.125)	(0.123)	(0.089)
1891	0.829**	1.652***	1.045***	−0.097	0.410**	−0.803***	−0.030
	(0.293)	(0.142)	(0.110)	(0.234)	(0.122)	(0.123)	(0.089)

Standard errors in parentheses. Significance levels: * 5%, ** 1%, *** 0.1%.

continuing on in farming. Not being the firstborn son also has a positive effect on switching from farming to labour or mining, although the coefficients are not significant. Even for those whose father was not in farming when they were young, having older brothers still reduces the probability of transitioning into farming. This can be seen in each of the other three regressions, although the effect is largest (and significant) for those whose father worked in trade.

The collapse of the mining industry over the decades is evident in each regression. The 1871 cohort is significantly less likely than the 1861 cohort to switch into the mining industry, and sons of miners are more likely to obtain jobs outside of mining. The magnitude of the coefficients only amplify this pattern over the 1881 and 1891 cohorts.

Whether a son still lives in his father's household as a young adult is another telltale sign of the son's occupation. The exact meaning of the sign, however, is specific to the different industries. A son who, like his father, is in farming is likely listed in the census as a son in his father's home. Even if these sons eventually go on to inherit the farm, at the age of 21 to 30 their father is likely still alive and working. Being listed as the head of the household is in many other instances a strong correlate of following a father's occupation. Sons of labourers, for example, are more likely to be labourers themselves than to go into farming or trade if they are the head of the household. Being listed as some other position within the household (perhaps as a brother, son-in-law, hired hand, etc.) produces coefficients that have a similar effect in the different regressions to being listed as the head of the household.

3.6 Conclusion

With over 160,000 individuals linked across three or more censuses, this paper has detailed the construction of what is quite possibly the largest panel dataset for Victorian England at present. The matching algorithm utilizes SVM technology to classify matches; a multi-level group linking technique for monitoring changes in household composition; and a nested matching process for tracking migration. Yet, there are far taller mountains to climb, and the methodologies presented here can certainly be applied on a much larger scale. The course of an entire nation, with the comings and goings of each of its citizens, could be captured over generations in a way never before seen by linking all of its available censuses.

As an example, I have presented a brief analysis of father-son occupational patterns across sectors. This includes assessing the connection between birth order, moving, and household roles on whether a son remains in his father's occupation. The question of primogeniture is one that can certainly be probed in future work.

APPENDIX A
APPENDIX TO CHAPTER 1

A.1 Continuation Values

All the propositions in this paper ultimately stem from Lemma 1, which contains general closed-form expressions for the continuation values at each state in a symmetric race with $\delta \in [0.5, 1)$. This appendix contains the lemma and its proof.

Lemma 1. *In a two-player (m, n) race with winning prize $Z \geq 0$, losing penalty $L = -1$, and discount factor $\delta \in [0.5, 1)$, the continuation values for Player A at state (i, j), $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, are presented below. Substituting the roles of i and j yields the continuation values for Player B.*

1. *For $i \geq j$ such that $j < \alpha$; or $i \in \{\alpha,\ \alpha+1\}$ and $j = \alpha$:*

$$v_A(i, j) = -\delta^j$$

2. *For (i, α) such that $i \geq (\alpha + 2)$:*

$$v_A(i, \alpha) = \left[(i - \alpha - 1)\delta^{i-1} - \sum_{h=\alpha}^{i-2} \delta^h\right] Z + \alpha\Big((i - \alpha - 1)\delta^i - (i - \alpha)\delta^{i-1}\Big) + (\alpha - 1)\delta^\alpha$$

3. *For $i \geq j \geq (\alpha + 1)$:*

$$v_A(i, j) = \delta^{i+j-\alpha-2}\Big[(i - \alpha - 1)(\delta - 1)\Big(Z + 1 - \alpha(1 - \delta)\Big) - \delta\Big]$$

4. *For $j > i$ such that $i \geq \alpha$:*

$$v_A(i, j) = \delta^{i+j-\alpha-2}\Big[(i - \alpha + 1)\delta(Z + \alpha\delta) - (i - \alpha)\Big(Z + (2\alpha - 1)\delta + 1 - \alpha\Big) - \alpha\delta\Big]$$

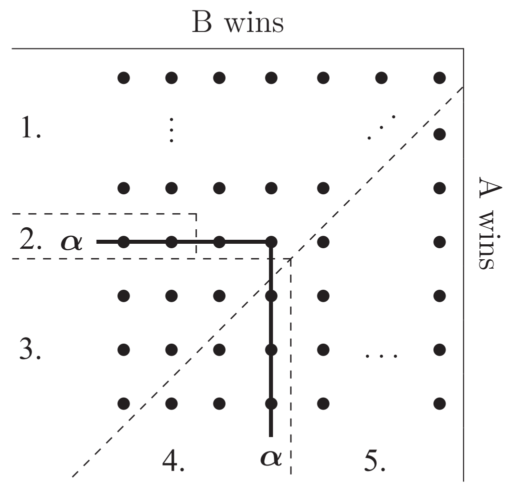

Figure A.1: Continuation value regions for Player A.

5. *For $j > i$ such that $i < \alpha$:*

$$v_A(i, j) = \delta^i \left[Z - i(1 - \delta) \right]$$

Figure A.1 gives a pictorial reference for the five continuation value regions in Lemma 1. To briefly outline the proof, I will begin by solving for the continuation values at the final state $(1, 1)$. From there, I will use an induction argument to establish the continuation values at $(i, 1)$ where $i \in \{1, \ldots, m\}$. A second induction argument will be used to show the continuation values at $(i, 2)$ for $i \in \{2, \ldots, m\}$, conditional on $Z > 2(1 - \delta)$ (i.e. $\alpha > 2$). Each of these first two induction arguments will be presented as a lemma. After using induction on the individual states to establish continuation values for the first two rows, I will use an induction argument on the rows themselves to arrive at the general solutions in parts 1 and 5 of Lemma 1. Parts 3 and 4 follow similarly by beginning with an induction argument on the individual states and ending with an induction argument on the rows. Part 2 only requires induction on the states. Since the transition between continuation value regions is dependent on α, special attention will be taken throughout to account for the size of the winning prize $Z \in (\,(\alpha-1)(1-\delta),\ \alpha(1-\delta)]$.

A.1.1 Continuation Values at (i,1)

Lemma 2. *For $i \in \{3,\ldots,m\}$, the continuation values at state $(i,1)$ are as follows:*

$$v_A(i,1) \quad = \quad \begin{cases} \left[(i-2)\delta^{i-1} - \sum_{h=1}^{i-2}\delta^h\right] Z + (i-2)\delta^i - (i-1)\delta^{i-1} & \textit{if } Z \in [0, 1-\delta] \\ -\delta & \textit{if } Z > 1-\delta \end{cases}$$

$$v_B(i,1) \quad = \quad \begin{cases} \delta^{i-1}Z + \delta^i - \delta^{i-1} & \textit{if } Z \in [0, 1-\delta] \\ \delta[Z-(1-\delta)] & \textit{if } Z > 1-\delta \end{cases}$$

Additionally, $v_A(1,1) = v_B(1,1) = v_A(2,1) = -\delta$ and $v_B(2,1) = \delta[Z-(1-\delta)]$.

Proof. Using Equation 1.5, and remembering that the continuation values at terminal states are equal to the winning prize Z or losing penalty -1, the continuation value for Player A at state $(1,1)$ is: $v_A(1,1) = \max\{\delta[v_A(0,1) - v_B(1,0) + v_B(0,1)],\ \delta v_A(1,0)\} = \max\{\delta[Z-Z-1],\ \delta(-1)\} = -\delta$. By the symmetry of the parameters, $v_B(1,1) = v_A(1,1) = -\delta$. A second appeal to Equation 1.5 produces the following continuation values at $(2,1)$:

$$v_A(2,1) \quad = \quad \max\{-\delta Z - 2\delta^2,\ -\delta\}$$

$$v_B(2,1) \quad = \quad \max\{\delta Z - \delta + \delta^2,\ -\delta^2\}$$

Recall that a player is *advantaged* if the left-hand-side of their maximization argument is (weakly) greater than the right-hand-side. So Player A is advantaged at $(2,1)$ if and only if $-\delta Z - 2\delta^2 \geq -\delta$, or rather if and only if $1 - 2\delta \geq Z$. For $\delta \geq 1/2$ and $Z \geq 0$, this inequality never holds strictly and only holds weakly when $\delta = 1/2$ and $Z = 0$. Therefore, $v_A(2,1) = -\delta$ and $v_B(2,1) = \delta Z - \delta + \delta^2$.

At $(3,1)$, the continuation values are $v_A(3,1) = \max\{(\delta^2 - \delta)Z + \delta^3 - 2\delta^2,\ -\delta\}$ and $v_B(3,1) = \max\{\delta Z - \delta + \delta^2,\ \delta^2 Z - \delta^2 + \delta^3\}$. While the continuation values at $(1,1)$ and $(2,1)$ hold for all values of Z, the continuation values at $(3,1)$ depend on the magnitude of Z. Player A is advantaged at $(3,1)$ if and only if $1 - 2\delta + \delta^2 \geq (1-\delta)Z$, which is satisfied only when $1 - \delta \geq Z$. Noting that either side of the maximization argument can be used when the two sides are equal, the continuation values at $(3,1)$ can be rewritten as follows:

$$v_A(3,1) = \begin{cases} (\delta^2 - \delta)Z + \delta^3 - 2\delta^2 & \text{if } Z \in [0, 1-\delta] \\ -\delta & \text{if } Z > 1 - \delta \end{cases}$$

$$v_B(3,1) = \begin{cases} \delta^2 Z + \delta^3 - \delta^2 & \text{if } Z \in [0, 1-\delta] \\ \delta Z - \delta + \delta^2 & \text{if } Z > 1 - \delta \end{cases}$$

Proceeding to state $(4,1)$, the continuation values are given as

$$v_A(4,1) = \begin{cases} \max\{(2\delta^3 - \delta^2 - \delta)Z + 2\delta^4 - 3\delta^3,\ -\delta\} & \text{if } Z \in [0, 1-\delta] \\ \max\{(\delta^2 - \delta)Z + \delta^3 - 2\delta^2,\ -\delta\} & \text{if } Z > 1 - \delta \end{cases}$$

$$v_B(4,1) = \begin{cases} \max\{(\delta + \delta^2 - \delta^3)Z - \delta^4 + 2\delta^3 - \delta, \\ \qquad \delta^3 Z + \delta^4 - \delta^3\} & \text{if } Z \in [0, 1-\delta] \\ \max\{\delta Z - \delta + \delta^2,\ \delta^2 Z - \delta^2 + \delta^3\} & \text{if } Z > 1 - \delta \end{cases}$$

If $Z \in (0, 1-\delta]$, Player A is advantaged if and only if $1 - 3\delta^2 + 2\delta^3 \geq (1 + \delta - 2\delta^2)Z$, which holds since $1 - 3\delta^2 + 2\delta^3 = (1 + \delta - 2\delta^2)(1 - \delta) \geq (1 + \delta - 2\delta^2)Z$. When $Z > 1 - \delta$, the maximization argument at $(4, 1)$ is identical to the maximization argument at $(3, 1)$, so Player A is disadvantaged. Summarizing the results for $(4, 1)$, we have:

$$v_A(4,1) = \begin{cases} (2\delta^3 - \delta^2 - \delta)Z + 2\delta^4 - 3\delta^3 & \text{if } Z \in [0, 1-\delta] \\ -\delta & \text{if } Z > 1 - \delta \end{cases}$$

$$v_B(4,1) = \begin{cases} \delta^3 Z + \delta^4 - \delta^3 & \text{if } Z \in [0, 1-\delta] \\ \delta Z - \delta + \delta^2 & \text{if } Z > 1 - \delta \end{cases}$$

In order to establish a general pattern, it will be useful to show the continuation values at $(5, 1)$:

$$v_A(5,1) = \begin{cases} \max\{(3\delta^4 - \delta^3 - \delta^2 - \delta)Z + 3\delta^5 - 4\delta^4,\ -\delta\} & \text{if } Z \in [0, 1-\delta] \\ \max\{(\delta^2 - \delta)Z + \delta^3 - 2\delta^2,\ -\delta\} & \text{if } Z > 1 - \delta \end{cases}$$

$$v_B(5,1) = \begin{cases} \max\{(\delta + \delta^2 + \delta^3 - 2\delta^4)Z - 2\delta^5 + 3\delta^4 - \delta, \\ \qquad \delta^4 Z + \delta^5 - \delta^4\} & \text{if } Z \in [0, 1-\delta] \\ \max\{\delta Z - \delta + \delta^2,\ \delta^2 Z - \delta^2 + \delta^3\} & \text{if } Z > 1 - \delta \end{cases}$$

Now fix $m \in \mathbb{N}$ such that $m \geq 4$. Suppose that for $k \in \{3, \ldots, m-1\}$ that the continuation values at $(k, 1)$ have the following form:

$$v_A(k,1) = \begin{cases} \left[(k-2)\delta^{k-1} - \sum_{h=1}^{k-2}\delta^h\right] Z + \\ \qquad (k-2)\delta^k - (k-1)\delta^{k-1} & \text{if } Z \in [0, 1-\delta] \\ -\delta & \text{if } Z > 1-\delta \end{cases}$$

$$v_B(k,1) = \begin{cases} \delta^{k-1}Z + \delta^k - \delta^{k-1} & \text{if } Z \in [0, 1-\delta] \\ \delta Z - \delta + \delta^2 & \text{if } Z > 1-\delta \end{cases}$$

Given this assumption, if $Z \in [0, 1-\delta]$, then Player A's continuation value at $(k+1, 1)$ is given by $v_A(k+1,1) = \max\left\{\left[(k-1)\delta^k - \sum_{h=1}^{k-1}\delta^h\right] Z + (k-1)\delta^{k+1} - k\delta^k,\ -\delta\right\}$. Here, the necessary and sufficient condition for Player A to be advantaged is $(k-1)\delta^{k+1} - k\delta^k + \delta \geq [\sum_{h=1}^{k-1}\delta^h - (k-1)\delta^k]Z$. This condition holds strictly for $Z < 1-\delta$ and weakly when $Z = 1-\delta$ since $(k-1)\delta^{k+1} - k\delta^k + \delta = [\sum_{h=1}^{k-1}\delta^h - (k-1)\delta^k](1-\delta) \geq [\sum_{h=1}^{k-1}\delta^h - (k-1)\delta^k]Z$. Player B, therefore, is disadvantaged (or weakly advantaged when $Z = 1-\delta$), so $v_B(k+1,1) = \delta v_B(k,1) = \delta^k Z + \delta^{k+1} - \delta^k$.

Now suppose that $Z > 1-\delta$. The continuation values at $(k+1, 1)$ are identical to those at $(k, 1)$, as well as at $(3, 1)$, so Player B is strictly advantaged at $(k+1, 1)$. Hence, $v_A(k+1,1) = -\delta$ and $v_B(k+1,1) = \delta Z - \delta + \delta^2 = \delta[Z - (1-\delta)]$. □

A.1.2 Continuation Values at (i,2)

Lemma 3. *If $Z > 2-2\delta$, then $v_A(i,2) = -\delta^2$ for $i \in \{2, \ldots, m\}$, and $v_B(i,2) = \delta^2[Z - 2(1-\delta)]$ for $i \in \{3, \ldots, m\}$.*

Proof. By symmetry, the continuation values at $(1, 2)$ can readily be obtained by switching the roles of Player A and Player B at $(2, 1)$. That is, $v_A(1,2) = \delta Z - \delta + \delta^2$

and $v_B(1,2) = -\delta$. Based on the continuation values at $(2,1)$ and $(1,2)$, $v_A(2,2) = v_B(2,2) = -\delta^2$ for all $Z \geq 0$.

At $(3,2)$, the continuation values for all $Z > 1-\delta$ are characterized as follows: $v_A(3,2) = \max\{-\delta^2 Z - 3\delta^3 + \delta^2,\ -\delta^2\}$ and $v_B(3,2) = \max\{\delta^2 Z + 2\delta^3 - 2\delta^2,\ -\delta^3\}$. Player A is advantaged at $(3,2)$ if and only if $2 - 3\delta \geq Z$. However, since $Z > 1 - \delta$, then $2-3\delta > 1-\delta$ implies that $\delta \leq 1/2$. Hence, Player A is disadvantaged if $\delta > 1/2$ and weakly advantaged if $\delta = 1/2$; in either case, $v_A(3,2) = -\delta^2$. Since Player B is advantaged, $v_B(3,2) = \delta^2 Z + 2\delta^3 - 2\delta^2$.

Advancing to $(4,2)$, the continuation values are $v_A(4,2) = \max\{(\delta^3 - \delta^2)Z + 2\delta^4 - 4\delta^3 + \delta^2,\ -\delta^2\}$ and $v_B(4,2) = \max\{\delta^2 Z + 2\delta^3 - 2\delta^2,\ \delta^3 Z + 2\delta^4 - 2\delta^3\}$. Player A is advantaged if and only if $2(1-\delta)^2 \geq (1-\delta)Z$. However, for $Z > 2(1-\delta)$, this inequality fails to hold. Hence, $v_A(4,2) = -\delta^2$ and $v_B(4,2) = \delta^2 Z + 2\delta^3 - 2\delta^2$.

At $(5,2)$, we have the following: $v_A(5,2) = \max\{(\delta^3-\delta^2)Z+2\delta^4-4\delta^3+\delta^2,\ -\delta^2\}$ and $v_B(5,2) = \max\{\delta^2 Z + 2\delta^3 - 2\delta^2,\ \delta^3 Z + 2\delta^4 - 2\delta^3\}$. Note that the continuation values at $(5,2)$ are exactly the same as those at $(4,2)$, so Player B is advantaged.

Fix $m \in \mathbb{N}$ such that $m \geq 5$. For $k \in \{4, \ldots, m-1\}$, assume that the continuation values at $(k,2)$ are as follows: $v_A(k,2) = -\delta^2$ and $v_B(k,2) = \delta^2 Z + 2\delta^3 - 2\delta^2$. The continuation values at $(k+1,1)$ are provided by Lemma 2. These can be combined with the assumption for state $(k,2)$ to calculate the following continuation values at $(k+1,2)$: $v_A(k+1,2) = \max\{(\delta^3 - \delta^2)Z + 2\delta^4 - 4\delta^3 + \delta^2,\ -\delta^2\}$ and $v_B(k+1,2) = \max\{\delta^2 Z + 2\delta^3 - 2\delta^2,\ \delta^3 Z + 2\delta^4 - 2\delta^3\}$. Once again, these are the same continuation values as those at $(4,2)$. Therefore, $v_A(k+1,2) = -\delta^2$ and

$v_B(k+1,2) = \delta^2 Z + 2\delta^3 - 2\delta^2 = \delta^2[Z - 2(1-\delta)].$ □

A.1.3 Proof of Lemma 1

Proof. Fix m, $n \in \mathbb{N}$, and let $Z \geq 0$ and $\delta \in [0.5, 1)$ be given. As has been the case throughout this paper, let $\alpha \in \mathbb{N} \bigcup \{0\}$ satisfy $Z \in ((\alpha-1)(1-\delta),\ \alpha(1-\delta)]$. Since Players A and B have a common winning prize, losing penalty, and discount factor, then the continuation value are also equal for the two players at symmetric states. That is, $v_A(k,\ell) = v_B(\ell,k)$ for all $k \in \{1,\ldots,m\}$ and $\ell \in \{1,\ldots,n\}$. Due to this symmetry, the proof will focus on states where $k \geq \ell$. Hence, parts 1 through 3 of Lemma 1 will be solved for in terms of Player A's continuation values, while parts 4 and 5 will be demonstrated for Player B.

Parts 1 and 5:

First note that by Lemma 2, $v_A(1,1) = v_A(2,1) = -\delta$ and $v_B(2,1) = \delta[Z-(1-\delta)]$ for all α. If $\alpha > 1$ (i.e. if $Z > 1-\delta$), then by Lemma 2, $v_A(k,1) = -\delta$ and $v_B(k,1) = \delta[Z-(1-\delta)]$ when $k \in \{3,\ldots,m\}$. Additionally, $v_A(2,2) = v_A(3,2) = -\delta^2$ and $v_B(3,2) = \delta^2[Z-2(1-\delta)]$ when $\alpha > 1$ (see the proof of Lemma 3). If $\alpha > 2$, Lemma 3 indicates that $v_A(k,2) = -\delta^2$ and $v_B(k,2) = \delta^2[Z-2(1-\delta)]$ for $k \in \{4,\ldots,m\}$. For $\alpha > 2$, let $\ell \in \{1,\ldots,\alpha-2\}$ be given. Suppose that for $k \in \{\ell,\ldots,m\}$ that $v_A(k,\ell) = -\delta^\ell$ and $v_B(k,\ell) = \delta^\ell[Z-\ell(1-\delta)]$ if $k > \ell$ and $v_B(k,\ell) = -\delta^\ell$ if $k = \ell$. By the symmetry of the players, $v_A(\ell+1,\ell) = v_B(\ell,\ell+1)$ and $v_B(\ell+1,\ell) = v_A(\ell,\ell+1)$, so $v_A(\ell+1,\ell+1) = \max\{\delta[v_A(\ell,\ell+1) - v_B(\ell+1,\ell) + v_B(\ell,\ell+1)],\ \delta v_A(\ell+1,\ell)\} = -\delta^{\ell+1}$. At $(\ell+2,\ell+1)$, we have $v_A(\ell+2,\ell+1) = \max\{-\delta^{\ell+1}Z - (\ell+2)\delta^{\ell+2} + \ell\delta^{\ell+1},\ -\delta^{\ell+1}\}$

and $v_B(\ell+2,\ell+1) = \max\{\delta^{\ell+1}[Z-(\ell+1)(1-\delta)],\ -\delta^{\ell+2}\}$. Player A is advantaged at $(\ell+2,\ell+1)$ if and only if $\ell+1-(\ell+2)\delta \geq Z$. Since $Z > (\alpha-1)(1-\delta)$ and $\ell < \alpha-1$, then $Z > \ell(1-\delta)$. However, $\ell+1-(\ell+2)\delta > \ell(1-\delta)$ implies that $\delta < 1/2$, which is a contradiction. Hence, $v_A(\ell+2,\ell+1) = -\delta^{\ell+1}$ and $v_B(\ell+2,\ell+1) = \delta^{\ell+1}[Z-(\ell+1)(1-\delta)]$. Then at $(\ell+3,\ell+1)$ we have $v_B(\ell+3,\ell+1) = \max\{\delta^{\ell+1}[Z-(\ell+1)(1-\delta)],\ \delta^{\ell+2}[Z-(\ell+1)(1-\delta)]\}$, so Player B is advantaged whenever $Z \geq (\ell+1)(1-\delta)$. This condition is satisfied because $\ell+1 \leq \alpha-1$ and $Z > (\alpha-1)(1-\delta)$. Since Player B is advantaged, $v_A(\ell+3,\ell+1) = \delta v_A(\ell+3,\ell) = -\delta^{\ell+1}$. The maximization arguments for the continuation values at $(\ell+4,\ell+1)$ are identical to those at $(\ell+3,\ell+1)$. Assume that $v_A(q,\ell+1) = -\delta^{\ell+1}$ and $v_B(q,\ell+1) = \delta^{\ell+1}[Z-(\ell+1)(1-\delta)]$ for $q \in \{\ell+3,\ldots,m-1\}$. Then $v_B(q+1,\ell+1) = \max\{\delta^{\ell+1}[Z-(\ell+1)(1-\delta)],\ \delta^{\ell+2}[Z-(\ell+1)(1-\delta)]\} = v_B(\ell+3,\ell+1)$, so Player B is advantaged. Hence, $v_A(q+1,\ell+1) = -\delta^{\ell+1}$ and $v_B(q+1,\ell+1) = \delta^{\ell+1}[Z-(\ell+1)(1-\delta)]$.

It remains to demonstrate that $v_A(\alpha+1,\alpha) = v_A(\alpha,\alpha) = -\delta^{\alpha}$. It has already been shown that $v_A(\alpha,\alpha-1) = -\delta^{\alpha-1}$ and $v_B(\alpha,\alpha-1) = \delta^{\alpha-1}[Z-(\alpha-1)(1-\delta)]$. By symmetry, $v_A(\alpha,\alpha-1) = v_B(\alpha-1,\alpha)$ and $v_B(\alpha,\alpha-1) = v_A(\alpha-1,\alpha)$, so $v_A(\alpha,\alpha) = -\delta^{\alpha}$. Then at $(\alpha+1,\alpha)$, we have $v_A(\alpha+1,\alpha) = \max\{-\delta^{\alpha}Z-(\alpha+1)\delta^{\alpha+1}+(\alpha-1)\delta^{\alpha},\ -\delta^{\alpha}\}$. Player A is advantaged at $(\alpha+1,\alpha)$ whenever $\alpha-(\alpha+1)\delta \geq Z$. Note, however, that $Z > (\alpha-1)(1-\delta)$, and $\alpha-(\alpha+1)\delta > (\alpha-1)(1-\delta)$ implies that $\delta < 1/2$. Due to this contradiction, $v_A(\alpha+1,\alpha) = -\delta^{\alpha}$.

Part 2:

The case where $\alpha = 1$ is presented in Lemma 2. There it was shown that $v_A(i,1) =$

$\left[(k-2)\delta^{k-1} - \sum_{h=1}^{k-2} \delta^h\right] Z + (k-2)\delta^k - (k-1)\delta^{k-1}$ for $k \in \{3, \ldots, m\}$. We will now focus on the case where $\alpha > 1$.

As was just shown, when $\alpha > 1$ then $v_A(k, \alpha - 1) = \delta^{\alpha-1}$ and $v_B(k, \alpha - 1) = \delta^{\alpha-1}[Z - (\alpha - 1)(1 - \delta)]$. Additionally, it was shown that Player A is disadvantaged at $(\alpha+1, \alpha)$ and $v_A(\alpha+1, \alpha) = -\delta^\alpha$. Since Player A is disadvantaged at $(\alpha+1, \alpha)$, then Player B is advantaged. Moreover, $v_B(\alpha+1, \alpha) = \delta^\alpha[Z - \alpha(1-\delta)]$. At $(\alpha+2, \alpha)$, Player A's continuation value is given by $v_A(\alpha+2, \alpha) = \max\{(\delta^{\alpha+1} - \delta^\alpha)Z + \alpha\delta^{\alpha+2} - 2\alpha\delta^{\alpha+1} + (\alpha-1)\delta^\alpha,\ -\delta^\alpha\}$. The necessary and sufficient condition for Player A to be advantaged at $(\alpha+2, \alpha)$ is $\alpha\delta^2 - 2\alpha\delta + \alpha = \alpha(1-\delta)^2 \geq (1-\delta)Z$. Since $Z \leq \alpha(1-\delta)$, the inequality is weakly satisfied when $Z = \alpha(1-\delta)$, and it is strictly satisfied otherwise. Hence, Player A is advantaged, and $v_B(\alpha + 2, \alpha) = \delta v_B(\alpha + 1, \alpha) = \delta^{\alpha+1}[Z - \alpha(1 - \delta)]$. Similarly, $v_A(\alpha+3, \alpha) = \max\{(2\delta^{\alpha+2} - \delta^{\alpha+1} - \delta^\alpha)Z + 2\alpha\delta^{\alpha+3} - 3\alpha\delta^{\alpha+2} + (\alpha-1)\delta^\alpha,\ -\delta^\alpha\}$, so Player A is advantaged if and only if $2\alpha\delta^3 - 3\alpha\delta^2 + \alpha = (1+\delta-2\delta^2)\alpha(1-\delta) \geq (1+\delta-2\delta^2)Z$. Therefore, Player A is strictly advantaged when $Z < \alpha(1-\delta)$ and weakly advantaged when $Z = \alpha(1 - \delta)$. The continuation value for Player B is $v_B(\alpha + 3, \alpha) = \delta v_B(\alpha + 2, \alpha) = \delta^{\alpha+2}[Z - \alpha(1 - \delta)]$. Now suppose that for $k \in \{\alpha + 2, \ldots, m - 1\}$ that $v_A(k, \alpha) = [(k - \alpha - 1)\delta^{k-1} - \sum_{h=\alpha}^{k-2} \delta^h]Z + \alpha(k - \alpha - 1)\delta^k - \alpha(k - \alpha)\delta^{k-1} + (\alpha - 1)\delta^\alpha$ and $v_B(k, \alpha) = \delta^{k-1}[Z - \alpha(1 - \delta)]$. Based on this assumption and the continuation values that have already been established, $v_A(k+1, \alpha) = \max\{[(k-\alpha)\delta^k - \sum_{h=\alpha}^{k-1} \delta^h]Z + \alpha(k-\alpha)\delta^{k+1} - \alpha(k+1-\alpha)\delta^k + (\alpha-1)\delta^\alpha,\ -\delta^\alpha\}$. Player A is advantaged at $(k+1, \alpha)$ whenever $\alpha[(k - \alpha)\delta^{k+1} - (k + 1 - \alpha)\delta^k + \delta^\alpha] = [\sum_{h=\alpha}^{k-1} \delta^h - (k - \alpha)\delta^k]\alpha(1 - \delta) \geq [\sum_{h=\alpha}^{k-1} \delta^h - (k - \alpha)\delta^k]Z$. Since $Z \leq \alpha(1 - \delta)$, Player A is advantaged, and therefore

$v_B(k+1,\alpha) = \delta v_B(k,\alpha) = \delta^k[Z - \alpha(1-\delta)]$.

Parts 3 and 4:

The continuation values which have been established for $(\alpha+1,\alpha)$, as well as their symmetric counterparts at $(\alpha,\alpha+1)$ can be used to calculate the continuation values at $(\alpha+1,\alpha+1)$. Namely, $v_A(\alpha+1,\alpha+1) = v_B(\alpha+1,\alpha+1) = -\delta^{\alpha+1}$. The continuation values at $(\alpha+2,\alpha+1)$ are then given as follows:

$$v_A(\alpha+2,\alpha+1) = \max\{\delta^{\alpha+2}[-Z-\alpha\delta+(\alpha-2)],\ \delta^{\alpha+1}[(\delta-1)Z+\alpha\delta^2-2\alpha\delta+(\alpha-1)]\}$$

$$v_B(\alpha+2,\alpha+1) = \max\{(2\delta^{\alpha+2}-\delta^{\alpha+1})Z+2\alpha\delta^{\alpha+3}-(3\alpha-1)\delta^{\alpha+2}+(\alpha-1)\delta^{\alpha+1},\ -\delta^{\alpha+2}\}$$

Here, the necessary and sufficient condition for Player A to be advantaged is $(1-2\delta)Z \geq 2\alpha\delta^2-(3\alpha-2)\delta+(\alpha-1)$. Note that the maximal value of the left-hand-side of the inequality is $(1-2\delta)\alpha(1-\delta) = 2\alpha\delta^2-3\alpha\delta+\alpha$. However, $2\alpha\delta^2-3\alpha\delta+\alpha \geq 2\alpha\delta^2-(3\alpha-2)\delta+(\alpha-1)$ implies that $\delta \leq 1/2$. The condition for Player A to be advantaged, therefore, fails to hold for $\delta > 1/2$ and is weakly satisfied when $\delta = 1/2$. Since the two sides of the maximization arguments in the continuation values are equal when $\delta = 1/2$, then $v_A(\alpha+2,\alpha+1) = \delta^{\alpha+1}[(\delta-1)Z+\alpha\delta^2-2\alpha\delta+(\alpha-1)]$ and $v_B(\alpha+2,\alpha+1) = \delta^{\alpha+1}[(2\delta-1)Z+2\alpha\delta^2-(3\alpha-1)\delta+(\alpha-1)]$ for all $\delta \in [0.5,1)$.

At $(\alpha+3,\alpha+1)$, Player A is strictly advantaged. His continuation value is $v_A(\alpha+3,\alpha+1) = \max\{\delta^{\alpha+2}[2(\delta-1)Z+2\alpha\delta^2-(4\alpha-1)\delta+2(\alpha-1)],\ \delta^{\alpha+1}[(2\delta^2-\delta-1)Z+2\alpha\delta^3-3\alpha\delta^2+(\alpha-1)]\}$, which implies that he is advantaged whenever $(1-\delta)Z \geq (\alpha-1)(1-\delta)^2$. Since $Z > (\alpha-1)(1-\delta)$, this condition is strictly satisfied. Player B's continuation value is then $v_B(\alpha+3,\alpha+1) = \delta v_B(\alpha+2,\alpha+1) = \delta^{\alpha+2}[(2\delta-1)Z+2\alpha\delta^2-(3\alpha-1)\delta+(\alpha-1)]$. Similarly, Player A has a strict advantage at

$(\alpha+4, \alpha+1)$ where his continuation value is given by $v_A(\alpha+4, \alpha+1) = \max\{\delta^{\alpha+3}[3(\delta-1)Z + 3\alpha\delta^2 - (6\alpha-2)\delta + 3(\alpha-1)],\ \delta^{\alpha+1}[(3\delta^3 - \sum_{h=0}^{2}\delta^h)Z + 3\alpha\delta^4 - 4\alpha\delta^3 + (\alpha-1)]\}$. Player A is advantaged if and only if $(1+\delta-2\delta^2)Z \geq (\alpha-1)(2\delta^3-3\delta^2+1) = (1+\delta-2\delta^2)(\alpha-1)(1-\delta)$. Again, this condition is strictly satisfied since $Z > (\alpha-1)(1-\delta)$. Hence, $v_B(\alpha+4, \alpha+1) = \delta v_B(\alpha+3, \alpha+1)$.

For $k \in \{\alpha+2, \ldots, m-1\}$, assume that the continuation values are as follows: $v_A(k, \alpha+1) = \delta^{k-1}(k-\alpha-1)[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + \alpha - 1] - \delta^k$; and $v_B(k, \alpha+1) = \delta^{k-1}[(2\delta-1)Z + 2\alpha\delta^2 - (3\alpha-1)\delta + \alpha - 1]$. As it may not be immediately apparent, it is worth noting that the terms $-2\alpha\delta^{\alpha+2}$, $-(4\alpha-1)\delta^{\alpha+3}$, and $-(6\alpha-2)\delta^{\alpha+4}$ which appear in $v_A(\alpha+2, \alpha+1)$, $v_A(\alpha+3, \alpha+1)$, and $v_A(\alpha+4, \alpha+1)$, respectively, can each be characterized by $-(k-\alpha-1)(2\alpha-1)\delta^k - \delta^k$ for $k \in \{\alpha+2, \alpha+3, \alpha+4\}$. From the continuation values at $(k+1, \alpha)$ and the assumption for those at $(k, \alpha+1)$, Player A's continuation value at $(k+1, \alpha+1)$ is $v_A(k+1, \alpha+1) = \max\{\delta^k(k-\alpha)[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + \alpha - 1] - \delta^{k+1},\ [(k-\alpha)\delta^{k+1} - \sum_{h=\alpha+1}^{k}\delta^h]Z + (k-\alpha)\alpha\delta^{k+2} - (k+1-\alpha)\alpha\delta^{k+1} + (\alpha-1)\delta^{\alpha+1}\}$. Player A is advantaged here whenever $[\sum_{h=\alpha+1}^{k-1}\delta^h - (k-\alpha-1)\delta^k]Z \geq (\alpha-1)[(k-\alpha-1)\delta^{k+1} - (k-\alpha)\delta^k + \delta^{\alpha+1}]$. The left-hand-side of the inequality is equal to the right-hand-side when $Z = (1-\alpha)(1-\delta)$. However, since $Z > (1-\alpha)(1-\delta)$, Player A is strictly advantaged. Player B's continuation value is then given by $v_B(k+1, \alpha+1) = \delta v_B(k, \alpha+1) = \delta^k[(2\delta-1)Z + 2\alpha\delta^2 - (3\alpha-1)\delta + \alpha - 1]$. This generally establishes the continuation values at $(k, \alpha+1)$ for $k \in \{\alpha+2, \ldots, m\}$. It is also noteworthy that the general form for $v_A(k, \alpha+1)$ holds when $k = \alpha+1$ since the expression collapses down to $-\delta^{\alpha+1}$.

The continuation value at $(\alpha+2, \alpha+2)$ can be determined from the continuation values at $(\alpha+2, \alpha+1)$, which were given previously, and those at $(\alpha+1, \alpha+2)$, which are symmetric across players to those at $(\alpha+2, \alpha+1)$. We then have $v_A(\alpha+2, \alpha+2) = v_B(\alpha+2, \alpha+2) = \delta^{\alpha+2}[(\delta-1)Z + \alpha\delta^2 - 2\alpha\delta + \alpha - 1]$. At $(\alpha+3, \alpha+2)$, Player B's continuation value is the following: $v_B(\alpha+3, \alpha+2) = \max\{\delta^{\alpha+3}[(3\delta-2)Z + 3\alpha\delta^2 - (5\alpha-2)\delta + 2(\alpha-1)],\ \delta^{\alpha+3}[(\delta-1)Z + \alpha\delta^2 - 2\alpha\delta + \alpha - 1]\}$. Note that Player B is advantaged if and only if $(2\delta-1)Z \geq -2\alpha\delta^2 + (3\alpha-2)\delta - \alpha + 1$. The lower bound on Z implies that $(2\delta-1)Z > (2\delta-1)(\alpha-1)(1-\delta) = -2\alpha\delta^2 + 2\delta^2 + 3\alpha\delta - 3\delta - \alpha + 1$. Since $-2\alpha\delta^2 + 2\delta^2 + 3\alpha\delta - 3\delta - \alpha + 1 \geq -2\alpha\delta^2 + (3\alpha-2)\delta - \alpha + 1$ holds whenever $\delta \geq 1/2$, then Player B is strictly advantaged at $(\alpha+3, \alpha+2)$. Therefore, $v_A(\alpha+3, \alpha+2) = \delta v_A(\alpha+3, \alpha+1) = \delta^{\alpha+3}[2(\delta-1)Z + 2\alpha\delta^2 - (4\alpha-1)\delta + 2(\alpha-1)]$.

Both players have the same contest prize, and therefore the same winning probabilities and expected effort levels at $(\alpha+4, \alpha+2)$. Identical contest prizes occur when the two sides of the maximization argument in the continuation values are equal (i.e. the players are weakly advantaged). The continuation values at $(\alpha+4, \alpha+2)$ are as follows: $v_A(\alpha+4, \alpha+2) = \delta^{\alpha+4}[3(\delta-1)Z + 3\alpha\delta^2 - (6\alpha-2)\delta + 3(\alpha-1)]$ and $v_B(\alpha+4, \alpha+2) = \delta^{\alpha+4}[(3\delta-2)Z + 3\alpha\delta^2 - (5\alpha-2)\delta + 2(\alpha-1)]$. Similarly, both players are weakly advantaged at $(\alpha+5, \alpha+2)$. Moreover, $v_A(\alpha+5, \alpha+2) = 4\delta^{\alpha+5}[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + (\alpha-1)] - \delta^{\alpha+6}$ and $v_B(\alpha+5, \alpha+2) = \delta^{\alpha+5}[(3\delta-2)Z + 3\alpha\delta^2 - (5\alpha-2)\delta + 2(\alpha-1)]$. Suppose now that $v_A(k, \alpha+2) = \delta^k(k-\alpha-1)[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + \alpha - 1] - \delta^{k+1}$ and $v_B(k, \alpha+2) = \delta^k[(3\delta-2)Z + 3\alpha\delta^2 - (5\alpha-2)\delta + 2(\alpha-1)]$ for $k \in \{\alpha+4, \ldots, m-1\}$. Then at $(k+1, \alpha+2)$, the players are weakly advantaged and we have $v_A(k+1, \alpha+2) = \delta^{k+1}(k-\alpha)[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + \alpha - 1] - \delta^{k+2}$ and $v_B(k+1, \alpha+2) = \delta^{k+1}[(3\delta-2)Z + 3\alpha\delta^2 - (5\alpha-2)\delta + 2(\alpha-1)]$. Thus, the two players have identical contest prizes at $(k, \alpha+2)$ where $k \in \{\alpha+4, \ldots, m\}$. It is also worthwhile to note that $v_A(\alpha+2, \alpha+2)$, $v_A(\alpha+3, \alpha+2)$, and $v_B(\alpha+3, \alpha+2)$ fit the general form for continuation values at $(k, \alpha+2)$ given above.

Before doing the final generalization for parts 3 and 4 of the proposition, it will be helpful to review Player B's continuation values at (k, ℓ) where $\ell \in \{\alpha, \alpha+1, \alpha+2\}$ and $k > \ell$. Player B's continuation value at (k, α) can be written as $v_B(k, \alpha) = \delta^{k-2}[\delta Z + \alpha\delta^2 - \alpha\delta]$. Then at $(k, \alpha+1)$ and $(k, \alpha+2)$, we have the following: $v_B(k, \alpha+1) = \delta^{k-1}[(2\delta-1)Z + 2\alpha\delta^2 - (3\alpha-1)\delta + \alpha - 1]$ and $v_B(k, \alpha+2) = \delta^{k}[(3\delta-2)Z + 3\alpha\delta^2 - (5\alpha-2)\delta + 2(\alpha-1)]$. In the induction argument below, a general form for $v_B(k, \ell)$ is demonstrated for $k \in \{\ell+1, \ldots, m\}$ where $\ell > \alpha$ is given. However, this general form also holds for $k \in \{\ell+1, \ldots, m\}$ when $\ell = \alpha$.

Let $\ell \in \{\alpha+1, \ldots, n-1\}$ be given. Suppose that for $k \in \{\ell, \ldots, m\}$ that

$$v_A(k, \ell) = \delta^{k+\ell-\alpha-2}(k-\alpha-1)[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + \alpha - 1] - \delta^{k+\ell-\alpha-1}$$

Additionally, suppose that for $k \in \{\ell+1, \ldots, m\}$,

$$v_B(k, \ell) = \delta^{k+\ell-\alpha-2}\Big[\Big((\ell-\alpha+1)\delta - (\ell-\alpha)\Big)Z + (\ell-\alpha+1)\alpha\delta^2 - \Big((2\alpha-1)(\ell-\alpha) + \alpha\Big)\delta + (\alpha-1)(\ell-\alpha)\Big]$$

By symmetry, $v_A(\ell+1, \ell+1) = \max\{\delta[v_A(\ell, \ell+1) - v_B(\ell+1, \ell) + v_B(\ell, \ell+1)],\ \delta v_A(\ell+1, \ell)\} = \delta v_A(\ell+1, \ell)$, and $v_A(\ell+1, \ell)$ is given by assumption. Therefore, $v_A(\ell+1, \ell+1) = \delta^{2\ell-\alpha}(\ell-\alpha)[(\delta-1)Z + \alpha\delta^2 - (2\alpha-1)\delta + \alpha - 1] - \delta^{2\ell-\alpha+1}$. At $(\ell+2, \ell+1)$, Player B's continuation value is given by $v_B(\ell+2,\ell+1) = \max\{\delta^{2\ell-\alpha+1}[((\ell-\alpha+2)\delta-(\ell+1-\alpha))Z+(\ell-\alpha+2)\alpha\delta^2-((2\alpha-1)(\ell+1-\alpha)+\alpha)\delta+(\alpha-1)(\ell+1-\alpha)],\ \delta^{2\ell-\alpha+1}(\ell-\alpha)[(\delta-1)Z+\alpha\delta^2-(2\alpha-1)\delta+\alpha-1]-\delta^{2\ell-\alpha+2}\}$. The necessary and sufficient condition for Player B to be advantaged at $(\ell+2,\ell+1)$ is $(2\delta-1)Z \geq (-2\alpha\delta^2+(3\alpha-2)\delta-\alpha+1)$, which is identical to the condition that was shown previously for Player B to be advantaged at $(\alpha+2,\alpha+1)$. Therefore, this condition holds strictly for all $Z > (\alpha-1)(1-\delta)$ and for all $\delta \geq 1/2$, so Player B is strictly advantaged at $(\ell+2,\ell+1)$. Since Player A is disadvantaged, we have $v_A(\ell+2,\ell+1) = \delta v_A(\ell+2,\ell) = \delta^{2\ell-\alpha+1}(\ell+1-\alpha)[(\delta-1)Z+\alpha\delta^2-(2\alpha-1)\delta+\alpha-1]-\delta^{2\ell-\alpha+2}$. Both players are weakly advantaged at $(\ell+3,\ell+1)$. Specifically, $v_A(\ell+3,\ell+1) = \delta^{2\ell-\alpha+2}(\ell+2-\alpha)[(\delta-1)Z+\alpha\delta^2-(2\alpha-1)\delta+\alpha-1]-\delta^{2\ell-\alpha+3}$ and $v_B(\ell+3,\ell+1) = \delta^{2\ell-\alpha+2}[((\ell-\alpha+2)\delta-(\ell+1-\alpha))Z+(\ell-\alpha+2)\alpha\delta^2-((2\alpha-1)(\ell+1-\alpha)+\alpha)\delta+(\alpha-1)(\ell+1-\alpha)]$. Additionally, the contest prizes for the two players are also identical at $(\ell+4,\ell+1)$. Abbreviating the continuation values, we have $v_A(\ell+4,\ell+1) = \delta v_A(\ell+4,\ell)$ and $v_B(\ell+4,\ell+1) = \delta v_B(\ell+3,\ell+1)$. Now for $q \in \{\ell+3,\ldots,m-1\}$ assume that $v_A(q,\ell+1) = \delta^{q+\ell-\alpha-1}(q-\alpha-1)[(\delta-1)Z+\alpha\delta^2-(2\alpha-1)\delta+\alpha-1]-\delta^{q+\ell-\alpha}$ and $v_B(q,\ell+1) = \delta^{q+\ell-\alpha-1}[((\ell-\alpha+2)\delta-(\ell+1-\alpha))Z+(\ell-\alpha+2)\alpha\delta^2-((2\alpha-1)(\ell+1-\alpha)+\alpha)\delta+(\alpha-1)(\ell+1-\alpha)]$. Under these assumptions, both players are again weakly advantaged at $(q+1,\ell+1)$. Equivalently, $v_A(q+1,\ell+1) = \delta v_A(q+1,\ell)$ and $v_B(q+1,\ell+1) = \delta v_B(q,\ell+1)$.

□

A.2 Additional Proofs

A.2.1 Propositions 1 and 2

Propositions 1 and 2 both follow immediately from Lemma 1. As is stated in the main text, the contest prizes in Propositions 1 are derived by inserting the continuation values from Lemma 1 into Equation 1.4. Proposition 2 follows from Lemma 1 because solving for the continuation values at state (i, j) entails solving a maximization problem which identifies a player as strictly advantaged, weakly advantaged, or disadvantaged. Checking which side of the maximization argument holds at each of the myriad of states is a substantial portion of the proof of Lemma 1.

A.2.2 Proof of Proposition 3

In order to demonstrate that players put forth effort with positive probability in every state contest, it suffices to show that the contest prizes are all strictly positive (after all, players will actively compete in a two-player all-pay auction if this condition is met). The positivity of the contest prizes in each of the eight regions in Proposition 1 will be considered in turn. In doing so, it is important to remember that $Z \in (\,(\alpha-1)(1-\delta),\ \alpha(1-\delta)]$. For brevity, I will frequently ignore portions of the contest prizes that are trivially positive (such as multiplicative δ terms).

I: Since $Z > (\alpha-1)(1-\delta)$ and $\alpha \geq i$, then $Z-(i-1)(1-\delta) > (\alpha-1)(1-\delta)-(i-1)(1-\delta) = (\alpha-i)(1-\delta) \geq 0$.

II: $Z+1-i(1-\delta) > (\alpha-1)(1-\delta)+1-i(1-\delta) = (\alpha-1-i)(1-\delta)+1 \geq 1$ since $i \leq \alpha-1$ and $Z > (\alpha-1)(1-\delta)$.

III: $\zeta_A(i,j)$ is increasing in j, so it suffices to check $j=\alpha+2$. $\zeta_A(i,\alpha+2)=\delta^\alpha(1-\delta)[Z+1-\alpha(1-\delta)]$, and $Z+1-\alpha(1-\delta)>1-(1-\delta)=\delta>0$ since $Z>(\alpha-1)(1-\delta)$.

IV and V: Like for *III*, $Z+1-\alpha(1-\delta)>0$ because $Z>(\alpha-1)(1-\delta)$.

VI: Fixing α, there are two terms in *VI* that are based on Z: both are positive, with one being added and the other subtracted.[1] I will show that VI is positive by focusing on its (strict) lower bound where the added term is set to its infimum and the subtracted term is simultaneously set to its maximum. The added term, $\sum_{h=\alpha}^{i-2}(\delta^{h+1}-\delta^i)Z$ has an infimum of $\sum_{h=\alpha}^{i-2}(\delta^{h+1}-\delta^i)(\alpha-1)(1-\delta)$, and the subtracted term reaches its maximum at $\delta^{i-1}[(i-\alpha-2)(1-\delta)+\delta]$. Substituting the infimum and maximum into *VI*, the contest prize at $(\alpha+2,\alpha+1)$ is bounded below by $\delta^{\alpha+2}(1-\delta)>0$. It remains to show that the contest prize is positive at $(i,\alpha+1)$ for all $i>\alpha+2$. With the infimum and maximum in *VI*, the marginal difference between adjoining states, $\zeta_A(i+1,\alpha+1)-\zeta_A(i,\alpha+1)$, is $\delta^{i-1}(1-\delta)^2[(i-\alpha)(1-\delta)-2]$. The sum of the marginal differences, beginning at $i=\alpha+2$ is:

$$(1-\delta)^2\left(\delta^{\alpha+1}[2(1-\delta)-2]+\delta^{\alpha+2}[3(1-\delta)-2]+\delta^{\alpha+3}[4(1-\delta)-2]+\cdots\right)$$

$$=\delta^{\alpha+2}(1-\delta)^2\left(-1-\delta-\delta^2-\delta^3-\cdots\right)=\delta^{\alpha+2}(1-\delta)^2\left(\frac{-1}{1-\delta}\right)=-\delta^{\alpha+2}(1-\delta)$$

This infinite sum of marginal differences can be added to the lower bound at $(\alpha+2,\alpha+1)$ to find a lower bound for $(i,\alpha+1)$, where $i>\alpha+2$: $\delta^{\alpha+2}(1-\delta)-\delta^{\alpha+2}(1-\delta)=0$. Since this lower bound is strictly below the contest prizes, the contest prizes must

[1]The expression $Z+1-\alpha(1-\delta)$ appears in the subtracted term, which was already shown to be positive in *III*.

then be positive.

VII: Given α, Z only affects the term $\sum_{h=\alpha}^{i-3}(\delta^{i-1}-\delta^{h+1})Z$. Since this term is negative, its minimum is achieved when $Z=\alpha(1-\delta)$, which implies that $\zeta_A(\alpha+3,\alpha)\geq \delta^{\alpha}(1-\delta)>0$. For $i\geq\alpha+3$, the marginal difference in the contest prizes is as follows: $\zeta_A(i+1,\alpha)-\zeta_A(i,\alpha)=\delta^{i-1}(1-\delta)(i-\alpha-1)[\alpha(1-\delta)-Z]\geq\delta^{i-1}(1-\delta)(i-\alpha-1)[\alpha(1-\delta)-\alpha(1-\delta)]=0$. The contest prizes are therefore positive since $\zeta_A(i+1,\alpha)-\zeta_A(i,\alpha)\geq 0$ for all $i\geq\alpha+3$ and $\zeta_A(\alpha+3,\alpha)>0$.

VIII: $\delta^j(1-\delta)$ is trivially positive since $\delta\in[0.5,1)$.

A.2.3 Proof of Proposition 5

If $i\geq j+2$ and $j\in\{\alpha+3,\ldots,i-2\}$, then states $(i,j-1)$ and (i,j) are within the neutral region where each player has a one-half probability of winning each contest (see Proposition 2, Part 3). For $j\in\{2,\ldots,\alpha\}$, this proposition can be restated in terms of the underdog: the underdog's winning probability is *lower* at $(i,j-1)$ than at (i,j). I will demonstrate this result in terms of the underdog's winning probability.

Without loss of generality, suppose that Player A is the underdog. Based on the winning probabilities in Equation 1.2, it suffices to check the following inequality:

$$\frac{\zeta_A(i,j-1)/2\zeta_B(i,j-1)}{\zeta_A(i,j)/2\zeta_B(i,j)}<1$$

This inequality does indeed hold when $i\geq j+2$ if $j\in\{2,\ldots,\alpha-1\}$:

$$\frac{\zeta_A(i,j-1)/2\zeta_B(i,j-1)}{\zeta_A(i,j)/2\zeta_B(i,j)}=\frac{Z+1-j(1-\delta)}{Z+1-(j-1)(1-\delta)}<1$$

For comparing (i, α) and $(i, \alpha - 1)$, it is easier to merely note that Player A is disadvantaged at $(i, \alpha - 1)$ and hence has a lower winning probability than at (i, α) where he is advantaged.

A.2.4 Proof of Proposition 6

Part 1:

At any state, each player can potentially bid up to the smaller of the two contest prizes, ζ_L (see Equation 1.1). It therefore suffices to show that $\zeta_L(i, j) \geq \zeta_L(i + 1, j)$ for i and j such that $i \geq j$. I will check that this is the case for each of the relevant contest prize regions in Proposition 1.

This result clearly holds within contest prize region VIII since $\zeta_L(i, j) = \zeta_L(i+1, j)$. Additionally, comparing regions I and VIII, $\zeta_L(j, j) = \delta^j[Z+1-(j-1)(1-\delta)] > \zeta_L(j + 1, j) = \delta^j(1 - \delta)$ since $Z > (\alpha - 1)(1 - \delta)$ and $\alpha \geq j$. At (i, α) for $i \geq 2$, Player B is disadvantaged so ζ_L is in region III. Within region III, $\zeta_L(i, \alpha) - \zeta_L(i + 1, \alpha) = (\delta^i - \delta^{i-1})[Z - \alpha(1 - \delta)] \geq 0$ since $Z \leq \alpha(1 - \delta)$. Going from region III to VIII, $\zeta_L(\alpha + 2, \alpha) = \delta^\alpha(1 - \delta)[Z + 1 - \alpha(1 - \delta)] \leq \delta^\alpha(1 - \delta) = \zeta_L(\alpha + 1, \alpha)$. In region VI, the only relevant state is $(\alpha + 2, \alpha + 1)$ where the contest prize has the same form as in region IV (i.e. $\zeta_L(\alpha+2, \alpha+1) = \delta^{\alpha+1}(1-\delta)[Z+1-\alpha(1-\delta)]$). Comparing region IV (and the one state from region VI) with region V, $\zeta_L(j + 1, j) = (1 - \delta)\zeta_L(j, j)$. Finally, within region IV, $\zeta_L(i + 1, j) = \delta\zeta_L(i, j)$.

Part 2:

The highest possible equilibrium expenditure along the specified race path is simply the sum of the ζ_L at those states. The summation of ζ_L at (k,k) and $(k+1,k)$ for $k \in \{1,\ldots,\alpha\}$ is:

$$\begin{aligned}
&\sum_{k=1}^{\alpha} \delta^k [Z+1-(k+2)(1-\delta)] \\
&\qquad \leq \sum_{k=1}^{\alpha} \delta^k [(\alpha+2-k)(1-\delta)+1] \qquad \text{since } Z \leq \alpha(1-\delta) \\
&\qquad =(1-\delta)\left[(\alpha+2)\sum_{k=1}^{\alpha}\delta^k - \sum_{k=1}^{\alpha} k\delta^k\right] + \sum_{k=1}^{\alpha}\delta^k \\
&\qquad =(1-\delta)\left[\frac{(\alpha+2)\delta(1-\delta^\alpha)}{1-\delta} - \frac{\delta(1-\delta^\alpha)}{(1-\delta)^2} + \frac{\alpha\delta^{\alpha+1}}{1-\delta}\right] + \frac{\delta(1-\delta^\alpha)}{1-\delta} \\
&\qquad =(\alpha+2)\delta - 2\delta^{\alpha+1}
\end{aligned}$$

Summing ζ_L at (k,k) for $k \in \{\alpha+1,\ldots,n\}$ and $(k+1,k)$ for $k \in \{\alpha+1,\ldots,n-1\}$ yields:

$$\bigg(Z+1-\alpha(1-\delta)\bigg)\left(\delta^{2n-\alpha-1} + (2-\delta)\sum_{k=\alpha+1}^{n-1}\delta^{2k-\alpha-1}\right)$$

Again, since $Z \leq \alpha(1-\delta)$:

$$\begin{aligned}
&\leq \delta^{2n-\alpha-1} + (2-\delta)\sum_{k=\alpha+1}^{n-1}\delta^{2k-\alpha-1} \\
&= \delta^{2n-\alpha-1} + (2-\delta)\left(\frac{\delta^{\alpha+1}-\delta^{2n-\alpha-1}}{1-\delta^2}\right) \\
&< \delta^{\alpha+1}\left(\frac{2-\delta}{1-\delta^2}\right) \qquad \text{by letting } n \to \infty.
\end{aligned}$$

Finally,

$$(\alpha+2)\delta - 2\delta^{\alpha+1} + \delta^{\alpha+1}\left(\frac{2-\delta}{1-\delta^2}\right) = (\alpha+2)\delta + \delta^{\alpha+2}\left(\frac{2\delta-1}{1-\delta^2}\right)$$

A.2.5 Catch-up probabilities

Figure 1.4 plots the probability that the underdog will catch up directly from the expected race path to a tied position. Here, I explicitly present the non-normalized probabilities for catching up. These probabilities can be calculated using Equation 1.2 and the contest prizes in Proposition 1.

Without loss of generality, I will assume that Player A is the underdog. Then at $(k+1, k)$, where $k \in \{\alpha+1, \ldots, n-1\}$, Player A's probability of catching up is $(1-\delta)/2\delta$. The catch-up probability is one-half that amount at $(k+2, k)$ for $k \in \{\alpha+2, \ldots, n-2\}$. At $(\alpha+3, \alpha+1)$, the probability of catching up is an increasing function of $Z \in (\,(\alpha-1)(1-\delta),\ \alpha(1-\delta)]$:

$$\left(\frac{1-\delta}{2\delta}\right)\left(1-\frac{\delta(1-\delta)[Z+1-\alpha(1-\delta)]}{2\left([1-\delta^2]Z+\alpha\delta[1+\delta-\delta^2]+1-\delta-\alpha\right)}\right) \in \left[\frac{1-\delta}{4\delta},\ \frac{(2-\delta)(1-\delta)}{4\delta}\right)$$

Hereafter, though, the catch-up probabilities are decreasing functions of Z. The probabilities at $(\alpha+2, \alpha)$ and $(\alpha+1, \alpha-1)$ are as follows:

$$\left(\frac{1-\delta}{2}\right)\left[\frac{1}{Z+1-\alpha(1-\delta)}-\frac{1}{2}\right] \in \left[\frac{1-\delta}{4},\ \frac{(2-\delta)(1-\delta)}{4\delta}\right)$$

$$\frac{1-\delta}{\left(2[Z+1-(\alpha-1)(1-\delta)]\right)^2} \in \left[\frac{1-\delta}{4(2-\delta)^2},\ \frac{1-\delta}{4}\right)$$

When the underdog is j states from the frontrunner's goal, $j \in \{1, \ldots, \alpha-2\}$, the probability of catching up to a tied position from the expected race path is:

$$\frac{1-\delta}{\left(2[Z+1-j(1-\delta)]\right)^{\rho(j)}}$$

where

$$\rho(j) = \alpha+1-j-\sum_{k=j}^{\alpha-1}\left(\frac{1}{2[Z+1-k(1-\delta)]}\right)$$

Upper and lower bounds can be found by substituting Z with $(\alpha-1)(1-\delta)$ and $\alpha(1-\delta)$, respectively.

A.3 Discounting Less Than 1/2

We have seen that for $\delta > 1/2$, the front-runner always defends his lead—regardless of the distance to the finish line. However, when $\delta = 1/2$, the notion of defending the lead largely vanishes. Player A only defends his lead at $(i, i+1)$ where $i \in \{1, \ldots, \alpha\}$, and Player B only defends his lead at $(j+1, j)$ for $j \in \{1, \ldots, \alpha\}$. Farther out in the race, both players are weakly advantaged immediately off of the main diagonal. The story when $\delta < 1/2$ is quite similar to when $\delta = 1/2$. There is, however, a slight wrinkle.

Designating the states where each player is advantaged requires a finer partition of Z when $\delta < 1/2$.[2] Specifically, $\alpha \in \mathbb{N}$ must be chosen such that $Z \in ((\alpha-1)(1-\delta),\ \alpha-(\alpha+1)\delta]$ or $Z \in (\alpha-(\alpha+1)\delta,\ \alpha(1-\delta)]$. These two possible sets that Z may be in have lengths $1-2\delta$ and δ, respectively. Figure $A.2$ represents the symmetric race when $\delta < 1/2$ for the four divisions of Z when $\alpha \in \{1, 2\}$.

As can be seen, the underdog player makes a last stand when he is α and $\alpha+1$ state contests away from losing the race. This is identical to when $\delta \geq 1/2$. The distinguishing feature for the case when $\delta < 1/2$ is what happens to defending the lead. For $Z \in ((\alpha-1)(1-\delta),\ \alpha-(\alpha+1)\delta]$, the player making a last stand is advantaged at the state prior to the main diagonal.[3] That is, Player A is advantaged at $(\alpha+1, \alpha)$

[2]The results for the symmetric race with $\delta < 1/2$ are based on numerical calculations for various parameter values. A.5.1 outlines the algorithm for numerical computations.

[3]The definition of a last stand needs to be amended for $Z \in ((\alpha-1)(1-\delta),\ \alpha-(\alpha+1)\delta]$ as follows: Player A makes a last stand if he is advantaged at state (i, j) but disadvantaged *or weakly advantaged* at state (k, j) for some $k \in \{1, \ldots, i-1\}$ and disadvantaged at (i, ℓ) for all $\ell \in \{0, \ldots, j-1\}$.

B wins

A	**A**	**A**	**A**	**A**	**A**	**A**	∘	
A	**A**	**A**	**A**	**A**	**A**	∘	B	
∘	∘	∘	∘	∘	∘	B	B	**A** wins
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	

(i) $Z \in (0, 1-2\delta]$

B wins

A	**A**	**A**	**A**	**A**	**A**	B	∘	
A	**A**	**A**	**A**	**A**	**A**	∘	**A**	
∘	∘	∘	∘	∘	∘	B	B	**A** wins
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	
∘	∘	∘	∘	∘	∘	B	B	

(ii) $Z \in (1-2\delta, 1-\delta]$

B wins

B	B	B	B	B	B	B	∘	
A	**A**	**A**	**A**	**A**	**A**	∘	**A**	
A	**A**	**A**	**A**	**A**	∘	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	**A** wins
∘	∘	∘	∘	∘	B	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	

(iii) $Z \in (1-\delta, 2-3\delta]$

B wins

B	B	B	B	B	B	B	∘	
A	**A**	**A**	**A**	**A**	B	∘	**A**	
A	**A**	**A**	**A**	**A**	∘	**A**	**A**	
∘	∘	∘	∘	∘	B	B	**A**	**A** wins
∘	∘	∘	∘	∘	B	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	
∘	∘	∘	∘	∘	B	B	**A**	

(iv) $Z \in (2-3\delta, 2-2\delta]$

Figure A.2: Symmetric race for $\delta < 1/2$.

and at $(\alpha+2, \alpha+1)$, while Player B is advantaged at $(\alpha, \alpha+1)$ and at $(\alpha+1, \alpha+2)$.[4] Hence, players only defend their lead when they need fewer that α state contest wins to claim the victory in the race. On the other hand, for $Z \in (\alpha - (\alpha+1)\delta,\ \alpha(1-\delta)]$, Player A defends his lead at $(\alpha, \alpha+1)$, and Player B does so as well at $(\alpha+1, \alpha)$. This resembles the case where $\delta = 1/2$ except that Players A and B are strictly advantaged (as opposed to weakly so) at $(\alpha+2, \alpha+1)$ and $(\alpha+1, \alpha+2)$, respectively.

One further note is that if $Z = \alpha(1-\delta)$, then Player A is weakly advantaged at (i, α) for $i \in \{\alpha+2, \ldots, m\}$. This value of Z is the borderline between making a last stand at α and $\alpha+1$ states away from losing the race or making a last stand at $\alpha+1$ and $\alpha+2$ states away from a terminal loss. Hence, the effect of the last stand at (i, α) for Player A (or (α, j) for Player B) is somewhat weakened. This also holds true when $\delta \geq 1/2$.

A.4 The Asymmetric Race

The presence of a last stand is robust to various asymmetries in the winning prizes and losing penalties. However, asymmetries in the parameters of the race distort what it means for a player to be in the lead. The added incentive of a large winning prize, compared to that of an opponent's, alters equilibrium behavior in such a way that the player with the smaller prize has to get much closer to his end goal before the player with the larger prize feels that his lead is being threatened. If the difference in the winning prizes is sufficiently large, holding all else equal, the player

[4]When $Z = \alpha - (\alpha+1)\delta$, the advantages at $(\alpha, \alpha+1)$ and at $(\alpha+1, \alpha)$ are weak; otherwise, they are strict.

with the higher winning prize will defend his lead at the same time that he makes his own last stand. That is, the two forces will merge together.

Due to the analytical difficulty of expanding the parameter space to accommodate for player specific winning prizes and losing penalties, this section will rely on numerical calculations (the algorithm is presented in A.5.1). Three types of asymmetries will be examined in this paper. A *winning prize asymmetry* denotes the case where the winning prizes differ, but the losing penalties remain the same (i.e. $Z_A \neq Z_B$, but $L_A = L_B$). Conversely, in a *losing penalty asymmetry*, players have common winning prizes but not losing penalties (i.e. $Z_A = Z_B$, but $L_A \neq L_B$). Finally, a *mean preserving spread asymmetry* refers to the case where $Z_A = \gamma Z_B$ and $L_A = \gamma L_B$ for $\gamma > 0$. Different underlying incentives are captured by each of these asymmetries. The incentive to win is isolated in winning prize asymmetries, while losing penalty asymmetries focus on the incentive to avoid losing. Both of these incentives are featured in mean preserving spread asymmetries.

When differences in the parameters are small in magnitude, these three types of asymmetries are closely related to each other. Figure $A.3$ contains four panels which illustrate increasing levels of asymmetry in the parameters. Panel (i) contains the baseline symmetric case where $Z_A = Z_B = 1$, $L_A = L_B = -1$, and $\delta = 0.7$. Interestingly, all three types of asymmetry are simultaneously represented in each of the other three panels. While the numeric values which fit each panel vary for each type of asymmetry, there is a commonality across asymmetry types in the evolution of the race as parameters become increasingly different.

B wins

B	B	B	B	B	B	B	B	B	B	B	B	○
B	B	B	B	B	B	B	B	B	B	B	○	**A**
B	B	B	B	B	B	B	B	B	B	○	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	○	**A**	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	B	○	**A**	**A**	**A**	**A**
○	○	○	○	○	○	B	○	**A**	B	**A**	**A**	**A**
○	○	○	○	○	B	○	**A**	B	B	**A**	**A**	**A**
○	○	○	○	B	○	**A**	○	B	B	**A**	**A**	**A**
○	○	○	B	○	**A**	○	○	B	B	**A**	**A**	**A**
○	○	B	○	**A**	○	○	○	B	B	**A**	**A**	**A**
○	B	○	**A**	○	○	○	○	B	B	**A**	**A**	**A**
B	○	**A**	○	○	○	○	○	B	B	**A**	**A**	**A**
○	**A**	○	○	○	○	○	○	B	B	**A**	**A**	**A**

A wins

(i) Baseline Symmetric Case: $Z_A = Z_B = 1$, $L_A = L_B = -1$, $\delta = 0.7$

B wins

B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**	**A**
○	○	○	○	○	○	B	B	**A**	B	**A**	**A**	**A**
○	○	○	○	○	B	B	**A**	B	B	**A**	**A**	**A**
○	○	○	○	B	B	**A**	**A**	B	B	**A**	**A**	**A**
○	○	○	B	B	**A**	**A**	○	B	B	**A**	**A**	**A**
○	○	B	B	**A**	**A**	○	○	B	B	**A**	**A**	**A**
○	B	B	**A**	**A**	○	○	○	B	B	**A**	**A**	**A**
B	B	**A**	**A**	○	○	○	○	B	B	**A**	**A**	**A**
B	**A**	**A**	○	○	○	○	○	B	B	**A**	**A**	**A**

A wins

(ii) Asymmetries: $Z_B = 1.0001$; or $L_B = -1.0001$; or $Z_B = -L_B = 1.0001$

B wins

B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**	**A**
○	○	○	○	○	○	B	B	**A**	B	**A**	**A**	**A**
○	○	○	○	○	B	B	**A**	B	B	**A**	**A**	**A**
○	○	○	○	B	B	**A**	**A**	B	B	**A**	**A**	**A**
○	○	○	○	B	B	**A**	○	B	B	**A**	**A**	**A**
○	○	○	B	B	**A**	○	○	B	B	**A**	**A**	**A**
○	○	○	B	B	**A**	○	○	B	B	**A**	**A**	**A**
○	○	○	B	B	○	○	○	B	B	**A**	**A**	**A**
○	○	B	B	**A**	○	○	○	B	B	**A**	**A**	**A**

A wins

(iii) Asymmetries: $Z_B = 1.01$; or $L_B = -1.005$; or $Z_B = -L_B = 1.0035$

B wins

B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**	**A**
○	○	○	○	○	○	B	B	**A**	$\underline{\text{B}}$	**A**	**A**	**A**
○	○	○	○	○	○	B	B	**A**	B	**A**	**A**	**A**
○	○	○	○	○	○	B	B	B	B	**A**	**A**	**A**
○	○	○	○	○	○	B	B	B	B	**A**	**A**	**A**
○	○	○	○	○	○	B	B	B	B	**A**	**A**	**A**
○	○	○	○	○	B	B	**A**	B	B	**A**	**A**	**A**
○	○	○	○	○	B	B	**A**	B	B	**A**	**A**	**A**
○	○	○	○	○	B	B	○	B	B	**A**	**A**	**A**

A wins

(iv) Asymmetries: $Z_B = 1.1^*$; or $L_B = -1.028$; or $Z_B = -L_B = 1.0175$

Figure A.3: Races with increasing levels of winning prize, losing penalty, and mean preserving spread asymmetries. *In (iv) when $Z_B = 1.1$, Player A is advantaged at (4,6). This is marked by $\underline{\text{B}}$.

In the symmetric case, both players are weakly advantaged along the main diagonal and leads in the race are defended immediately off of the main diagonal. When there is a hairline separation in the parameters of the two players, then the player who has the higher winning prize, lower losing penalty, or larger spread will be advantaged on the main diagonal. This is shown in Panel (ii) of Figure $A.3$. Here, the winning prize asymmetry is $Z_B = 1.0001$, the losing penalty asymmetry is $L_B = -1.0001$, and the mean preserving spread asymmetry is $Z_B = 1.0001$ and $L_B = -1.0001$; all other parameters are the same as in the symmetric case.

The collection of states where players defend their lead diagrammatically forms an arm along the line where players are tied in the race, after adjusting for differences in the winning prizes and losing penalties. As if fixed on a hinge at the crux of the last stands of both players, this arm swings toward the finish line of the player who has the smaller incentive to win or avoid losing (or both). The higher the degree of asymmetry in the parameters, the farther this arm swings. This concept is illustrated in Panels (iii) and (iv) of Figure $A.3$. Since the arm represents the states where the players are nearly tied in the race, it is therefore significant that an increase in the winning prize of one player does not have the same effect on the arm as an equal decrease in the same player's losing penalty. In Panel (iii), the winning prize asymmetry is $Z_B = 1.01$, while the losing penalty asymmetry is only $L_B = -1.005$. Additionally, the winning prize asymmetry in Panel (iv) is almost four-times the magnitude of the corresponding losing penalty asymmetry ($Z_B = 1.1$, $L_B = -1.028$). The incentive to avoid losing is in some sense stronger than the incentive to win. Both of these incentives are incorporated in the mean preserving spread asymmetry, and in order to fit Panels (iii) and (iv) to this type of asymmetry, even smaller losing penalties and winning prizes are needed. Specifically, $Z_B = -L_B = 1.0035$ in Panel (iii), and $Z_B = -L_B = 1.0175$ in Panel (iv).

The arm of states where players defend their lead can entirely merge with the last stand of the player who has the higher incentive to win, avoid losing, or both. However, in a sufficiently large race, the likelihood that this arm will be distinctly separated from the last stand at some point in the race increases as δ increases. An example of this can even be seen in the symmetric case where players do not defend their lead prior to a last stand occurring when $\delta \leq 1/2$, but they do for $\delta > 1/2$. For every set of winning prizes and losing penalties, it appears that there exists a value of δ above which players distinctly defend their lead.

It is now time to analyze larger degrees of asymmetry. Figure $A.4$ depicts a winning prize asymmetry in Panel (i) ($Z_B = 2.5$), a losing penalty asymmetry in Panel (ii) ($L_B = -2.5$), and a mean preserving spread asymmetry in Panel (iii) ($Z_B = -L_B = 2.5$). Unless otherwise specified, the parameter values in each of these panels are as follows: $Z_A = Z_B = 1$, $L_A = L_B = -1$, and $\delta = 0.7$.

An immediate distinction between Figures $A.3$ and $A.4$ is that all three asymmetry types can be represented in a single panel when the magnitude of the asymmetries is small. However, for larger differences in the parameters, each type of asymmetry reveals its own nuanced behavior. The winning prize asymmetry in Panel (i) of Figure $A.4$ shows that Player B's strong incentive to win pushes back Player

B wins

B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	**A**	**A**
B	B	B	B	B	B	B	B	B	B	B	**A**	**A**
B	B	B	B	B	B	B	B	B	B	**A**	**A**	**A**
B	B	B	B	B	B	B	B	B	B	**A**	**A**	**A**
B	B	B	B	B	B	B	B	B	B	**A**	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**	**A**	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**	**A**	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**	**A**	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**	**A**	**A**

A wins

B wins

B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**

A wins

B wins

B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	B
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	B	**A**
B	B	B	B	B	B	B	B	B	B	B	B	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**
A	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	**A**	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**
$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	B	B	**A**

A wins

Figure A.4: Races with large degrees of asymmetry. The baseline parameters for these diagrams are $Z_A = Z_B = 1$, $L_A = L_B = -1$, and $\delta = 0.7$. **(i)** *Winning Prize Asymmetry* (Top Left): $Z_B = 2.5$; **(ii)** *Losing Penalty Asymmetry* (Top Right): $L_B = -2.5$; **(iii)** *Mean Preserving Spread Asymmetry* (Bottom): $Z_B = 2.5$ and $L_B = -2.5$.

A's last stand. Player A's incentive to avoid losing is only able to outweigh Player B's incentive to win when Player A is 9 and 10 states away from a loss. In Panel (ii), Player B's incentive to avoid losing overpowers Player A's incentive to win, even though Player A is only 2 and 3 states away from winning. The losing penalty asymmetry causes Player B to bring his last stand closer to Player A's finish line. On the other hand, Player A makes his last stand in Panel (ii) at the same distance from losing as Player B does in Panel (i); in both cases, the player making the last stand has a losing penalty of -1 and the opposing player has a winning prize of 1. The mean preserving spread asymmetry in Panel (iii) represents a combination of the effects of the winning prize and losing penalty asymmetries. Player B's incentive to win and Player A's incentive to avoid losing mimic the incentives of the winning prize asymmetry. Likewise, the incentives from the losing penalty asymmetry also carry over. In either case, the location of a last stand is jointly determined by the defending player's incentive to avoid losing and the leading player's incentive to win.

A.5 Numerical Calculations

Numeric results in this paper are based on computations performed in MATLAB. This appendix will first outline the algorithm for solving the static features of races (i.e. continuation values, contest prizes, winning probabilities, etc.). The Monte Carlo simulations used for examining the dynamic features of a race will then be described.

A.5.1 Static Race Computations

Parameters:

Define the tolerance level for calculations (`e`), the size of the race (`m`, `n`), the discount factor (`d`), the winning prizes (`Za`, `Zb`), and the losing penalties (`La`, `Lb`).

Race Grids:

Define grids for the continuation values (`Va=zeros(m+1,n+1); Vb=zeros(m+1,n+1)`), contest prizes (`zetaA = zeros(m,n); zetaB = zeros(m,n)`), winning probabilities (`probA = zeros(m,n); probB = zeros(m,n)`), and a grid to designate which player is advantaged (`Adv = 2.*ones(m,n)`). Terminal states only need to be included in the grid for the continuation values, so its dimensions are $(m+1, n+1)$ for a race beginning at (m, n).

Continuation Values:

Set the terminal continuation values equal to the respective winning prize or losing penalty: `Va(1,:)=Za; Va(:,1)=La; Vb(:,1)=Zb; Vb(1,:)=Lb`. Note that the terminal states are indexed by a one since the grid does not contain states indexed by zero. Now solve for the continuation values at non-terminal states:

```
for j = 2:(n+1)
   for i = 2:(m+1)
    Va(i,j) = max(d*(Va(i-1,j) - Vb(i,j-1) + Vb(i-1,j)), d*Va(i,j-1));
    Vb(i,j) = max(d*(Vb(i,j-1) - Va(i-1,j) + Va(i,j-1)), d*Vb(i-1,j));
   end
end
```

Contest Prizes and Advantaged States: The transition from the $(m+1, n+1)$ to the (m, n) index used throughout this paper can be done when solving for the contest prizes. A single grid can be used to denote which player is advantaged at each state ($1 =$ Player A is strictly advantaged; $0 =$ both players are weakly advantaged; $-1 =$ Player B is strictly advantaged).

```
for j = 1:n
   for i = 1:m
     zetaA(i,j) = d*(Va(i,j+1) - Va(i+1,j));
     zetaB(i,j) = d*(Vb(i+1,j) - Vb(i,j+1));
   end
end
for j = 1:n
    for i = 1:m
        if zetaA(i,j) - zetaB(i,j) > e
            Adv(i,j) = 1;
        elseif abs(zetaA(i,j) - zetaB(i,j)) < e
            Adv(i,j) = 0;
        else
            Adv(i,j) = -1;
        end
    end
end
```

Winning Probabilities

The winning probabilities follow Equation 1.2 and are based on knowing which player is advantaged at each state.

```
for j=1:n

    for i = 1:m

        if Adv(i,j) == 1        %Player A is adv. at (i,j)

               probA(i,j) = 1 - (zetaB(i,j))/(2*zetaA(i,j));

               probB(i,j) = (zetaB(i,j))/(2*zetaA(i,j));

        elseif Adv(i,j) == -1   %Player B is adv. at (i,j)

               probA(i,j) = (zetaA(i,j))/(2*zetaB(i,j));

               probB(i,j) = 1 - (zetaA(i,j))/(2*zetaB(i,j));

        elseif Adv(i,j) == 0    %Both players are adv. at (i,j)

            probA(i,j) =  1/2;

            probB(i,j) =  1/2;

        end

    end

end
```

Expected effort levels for each player can be found through a similar algorithm.

A.5.2 Monte Carlo Simulations

All simulations use the Mersenne Twister pseudorandom number generator *mt19937ar* (Seed 1). The primary role of the pseudorandom numbers is to generate effort levels according to the equilibrium distributions in Equation 1.1. For the advantaged player, this is a one step operation of drawing a pseudorandom number from the appropriate interval. However, since the disadvantaged player has a mass point at zero, a pseudorandom number must first be drawn to see whether the player will exert any effort; if so, a second pseudorandom number determines the effort level.

Once drawn, the effort levels are compared, and the player with the higher effort level wins the contest and moves one state closer to winning the race (ties are broken with a coin flip). This process is repeated until one of the players reaches a terminal state, at which time the terminal location of the race, the identity of the winner, and each player's aggregate effort are recorded. This is one simulation of the race. A loop is used to simulate the race a pre-specified number of times for each parameter combination (either 50,000 or 100,000 times), and additional loops are used to repeat the simulation process for different parameter values. The full MATLAB code is available upon request.

APPENDIX B
APPENDIX TO CHAPTER 2

B.1 Experiment Instructions

Thank you for your willingness to participate in this experiment. You will have the opportunity to earn some money as part of this experiment—the exact amount you earn will be based on both your choices and the choices of the other participants. Funding has been provided by the Economic Science Institute. You will be paid privately at the conclusion of the experiment.

In order to preserve the experimental setting, we ask that you DO NOT talk with the other participants, make loud noises, or otherwise disturb those around you. You will be asked to leave and will not be paid if you violate this rule. Please raise your hand if you have any questions.

There are two parts to this experiment.

Part 1

In the first part of the experiment, you will be given a set of 15 choices. You will be asked to choose between receiving \$1 for sure (Option A) and receiving \$3 with some probability and nothing otherwise (Option B). The probability of winning \$3 in Option B varies across the 15 choices. You will receive payment for one of your choices. The computer will draw a number between 1 and 15 at random, and you will be paid for your choice corresponding to that number. If you chose Option A, you will receive \$1. If you selected Option B, the computer will randomly draw another number between 1 and 20, and the result of that draw will determine whether you are paid \$3 or \$0.

Are there any questions?

Part 2

The second part of the experiment consists of 20 best-of-7 tournaments. In each tournament, you will be paired at random on the computer with another participant. The winner of each tournament will receive a prize and the loser will incur a penalty.

The currency for this part of the experiment is rupees, and the exchange rate is 50 rupees = 1 US dollar. As part of this experiment you have received an account with 850 rupees (equivalent to \$17.00). This account is in addition to the \$7.00 show up fee. The prize for winning a tournament is 150 rupees, and the penalty for losing is 150 rupees.

In order to win a tournament you must be the first player to win 4 contests. A contest consists of entering a bid on the computer screen. The computer will allow bids that are either whole numbers or have up to one decimal point that are between 0 and 300 inclusive. You win a contest if your bid is higher than your opponent's (in case a tie occurs, the computer will decide the winner randomly, giving each player a 50% chance of winning). Once both players have entered their bids, the computer will display the two bids and indicate which player is the winner. The computer will also display past bids and the total number of contests that each player has won so far in the tournament.

After each contest, there is a 10% chance that the tournament will suddenly end.

The computer will randomly determine whether or not to end the tournament by selecting an integer between 1 and 10 (each number is equally likely to be drawn). If a 1, 2, 3,. . ., 9 is drawn, then the tournament will continue, and you will return to the bidding screen to bid in another contest. However, if the computer draws a 10, then the tournament will end early. Numbers that the computer has drawn previously may be drawn again. Given that no player has won 4 contests, there is always a 90% chance of continuing to the next round of the tournament. The following table shows the percent of all tournaments that are expected to reach a given round provided that no player has won 4 contests by that round.

Round	1	2	3	4	5	6	7
% of Tournaments	100%	90%	81%	73%	66%	59%	53%

Earnings

Your earnings for each tournament are based on your bids and whether you win or lose the tournament. All of your bids throughout the tournament will be subtracted from your earnings. Please note that each of your bids will be subtracted regardless of whether you win or lose each contest.

The prize of 150 rupees will be added to your earnings if you win the tournament, and the penalty of 150 will be subtracted from your earnings if you lose.

Here are some examples to illustrate how your earnings for a tournament are calculated. If you win a tournament in six rounds, then your earnings are as follows:

$$150 - (\text{Round 1 bid}) - (\text{Round 2 bid}) - (\text{Round 3 bid}) - (\text{Round 4 bid}) - (\text{Round 5 bid}) - (\text{Round 6 bid})$$

Similarly, your earnings for losing a tournament in five rounds are given below:

$$-150 - (\text{Round 1 bid}) - (\text{Round 2 bid}) - (\text{Round 3 bid}) - (\text{Round 4 bid}) - (\text{Round 5 bid})$$

If the computer does end the tournament before one of the players has won 4 contests, then neither player receives a prize or incurs a penalty. However, your bids will still be subtracted from your earnings. For example, if the computer stops the tournament after three rounds, you earn the following:

$$-(\text{Round 1 bid}) - (\text{Round 2 bid}) - (\text{Round 3 bid})$$

When a tournament ends, either by a player winning 4 contests or by the computer ending it early, the computer will display your earnings for that tournament. You will then be paired at random with another participant for the next tournament.

We ask for your patience as there may be a short pause between tournaments. This may happen, for example, if your tournament ended early, but your next randomly selected partner is still competing in a tournament.

Payment

At the end of the experiment, 2 of the 20 best-of-seven tournaments will be selected at random. Your payment will be based on the average of your earnings in those 2 tournaments. The average will be added to your 850 rupee account and then converted from rupees to dollars (50 rupees = 1 US dollar). Positive earnings will increase the balance in your account, while negative earnings will decrease it. You will be paid the balance of your account.

Quiz #1

Your account initially has 850 rupees. The winning prize is 150, and the losing penalty is −150.

Contest 1: Your bid: 45 Your opponent's bid: 73

Contest 2: Your bid: 92 Your opponent's bid: 100

Contest 3: Your bid: 21 Your opponent's bid: 21

Tournament randomly terminated after 3rd contest.

How many rupees would you receive from this portion of the experiment if this tournament was selected for payment?

Quiz #2

Your account initially has 850 rupees. The winning prize is 150, and the losing penalty is −150.

Contest 1: Your bid: 295 Your opponent's bid: 23

Contest 2: Your bid: 70 Your opponent's bid: 150

Contest 3: Your bid: 51 Your opponent's bid: 40

Contest 4: Your bid: 80 Your opponent's bid: 20

Contest 5: Your bid: 72 Your opponent's bid: 80

Contest 6: Your bid: 51 Your opponent's bid: 70

Contest 7: Your bid: 200 Your opponent's bid: 175

How many rupees would you receive from this portion of the experiment if this tournament was selected for payment?

Quiz #3

Your account initially has 850 rupees. The winning prize is 150, and the losing penalty is −150.

Contest 1: Your bid: 27 Your opponent's bid: 295

Contest 2: Your bid: 41 Your opponent's bid: 150

Contest 3: Your bid: 200 Your opponent's bid: 40

Contest 4: Your bid: 20 Your opponent's bid: 78

Contest 5: Your bid: 31 Your opponent's bid: 83

How many rupees would you receive from this portion of the experiment if this tournament was selected for payment?

This is the end of the instructions. If you have any questions, please raise your hand and a monitor will come by to answer them. If you are finished with the instructions, please click the Start button. The instructions will remain on your screen until the experiment begins. We need everyone to click the Start button before we can begin the experiment.

APPENDIX C
APPENDIX TO CHAPTER 3

C.1 Data Sources

1. GB Historical GIS / University of Portsmouth, Cornwall RegC through time — Life and Death Statistics, *A Vision of Britain through Time*. This work is based on data provided through www.VisionofBritain.org.uk and uses historical material which is copyright of the Great Britain Historical GIS Project and the University of Portsmouth.

2. McCormick, M., *Cornwall Online Census Project*, 1841 [computer file]. Colchester, Essex: UK Data Archive [distributor], November 2005. SN: 5221, http://dx.doi.org/10.5255/UKDA-SN-5221-1.

3. McCormick, M., *Cornwall Census Returns*, 1851 [computer file]. Colchester, Essex: UK Data Archive [distributor], March 2011. SN: 6738, http://dx.doi.org/10.5255/UKDA-SN-6738-1.

4. McCormick, M., *Cornwall Online Census Project*, 1891 [computer file]. Colchester, Essex: UK Data Archive [distributor], July 2004. SN: 4978, http://dx.doi.org/10.5255/UKDA-SN-4978-1.

5. UK Ordnance Survey, *1:50 000 Scale Gazetteer*. Contains Ordnance Survey data © Crown copyright and database right 2013.

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