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Multi-unit auctions with budget-constrained bidders

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MULTI-UNIT AUCTIONS WITH BUDGET-CONSTRAINED BIDDERS

by

Gagan Pratap Ghosh

An Abstract

Of a thesis submitted in partial fulfillment of the
requirements for the Doctor of Philosophy
degree in Economics
in the Graduate College of
The University of Iowa

July 2012

Thesis Supervisor: Professor Srihari Govindan

ABSTRACT

In my dissertation, I investigate the effects of budget-constraints in multi-unit auctions. This is done in three parts. First, I analyze a case where all bidders have a common budget constraint. Precisely, I analyze an auction where two units of an object are sold at two simultaneous, sealed bid, first-price auctions, to bidders who have demand for both units. Bidders differ with respect to their valuations for the units. All bidders have an identical budget constraint which binds their ability to spend in the auction. I show that if valuation distribution is atom-less, then their does not exist any symmetric equilibrium in this auction game.

In the second and third parts of my thesis, I analyze the sale of licenses for the right to drill for oil and natural gas in the Outer Continental Shelf (OCS) of the United States. These sales are conducted using simultaneous sealed-bid first-price auctions for multiple licenses, each representing a specific area (called a tract). Using aspects of observed bidding-behavior, I first make a *prima facie* case that bidders are budget-constrained in these auctions. In order to formalize this argument, I develop a simple extension of the standard model (where bidders differ in their valuations for the objects) by incorporating (random) budgets for the bidders. The auction-game then has a two-dimensional set of types for each player. I study the theoretical properties of this auction, assuming for simplicity that two units are being sold. I show that this game has an equilibrium in pure strategies that is symmetric with respect to the players and with respect to the units. The strategies are essentially pure in the sense that each bidder-type has a unique split (up to a permutation) of his budget between the two auctions. I then characterize the equilibrium in terms of the bid-distribution and iso-bid curves in the value-budget space. I derive various qualitative features of this equilibrium, among which are: (1) under mild assumptions, there always exist bidder-types who submit unequal bids in equilibrium, (2) the equilibrium is monotonic in the sense that bidders with higher valuations prefer more unequal splits of their budgets than bidders with lower valuations and the same budget-level.

With a formal theory in place, I carry out a quantitative exercise, using data from the 1970 OCS auction. I show that the model is able to match many aspects of the data. (1) In the data, the number of tracts bidders submit bids on is positively correlated with budgets (an R^2 of 0.84), even though this relationship is non-monotonic; my model is able to capture this non-monotonicity, while producing an R^2 of 0.89 (2) In the data, the average number of bids per tract is 8.21; for the model, this number is 10.09. (3) Auction revenue in the data was \$1.927 billion; the model produced a mean revenue of \$1.944 billion

Abstract Approved: ______________________________
Thesis Supervisor

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Title and Department

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Graduate College
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Iowa City, Iowa

CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

Gagan Pratap Ghosh

has been approved by the Examining Committee for the thesis requirement for the Doctor of Philosophy degree in Economics at the July 2012 graduation.

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To my mother, Rita Ghosh (1953-2009) who still inspires me

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In my dissertation, I investigate the effects of budget-constraints in multi-unit auctions. This is done in three parts. First, I analyze a case where all bidders have a common budget constraint. Precisely, I analyze an auction where two units of an object are sold at two simultaneous, sealed bid, first-price auctions, to bidders who have demand for both units. Bidders differ with respect to their valuations for the units. All bidders have an identical budget constraint which binds their ability to spend in the auction. I show that if valuation distribution is atom-less, then their does not exist any symmetric equilibrium in this auction game.

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that each bidder-type has a unique split (up to a permutation) of his budget between the two auctions. I then characterize the equilibrium in terms of the bid-distribution and iso-bid curves in the value-budget space. I derive various qualitative features of this equilibrium, among which are: (1) under mild assumptions, there always exist bidder-types who submit unequal bids in equilibrium, (2) the equilibrium is monotonic in the sense that bidders with higher valuations prefer more unequal splits of their budgets than bidders with lower valuations and the same budget-level.

With a formal theory in place, I carry out a quantitative exercise, using data from the 1970 OCS auction. I show that the model is able to match many aspects of the data. (1) In the data, the number of tracts bidders submit bids on is positively correlated with budgets (an R^2 of 0.84), even though this relationship is non-monotonic; my model is able to capture this non-monotonicity, while producing an R^2 of 0.89 (2) In the data, the average number of bids per tract is 8.21; for the model, this number is 10.09. (3) Auction revenue in the data was \$1.927 billion; the model produced a mean revenue of \$1.944 billion

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CHAPTER 1
INTRODUCTION

Traditional auction theory assumes bidders have 'deep pockets'. Here the only consideration in strategically deciding the bids, is the valuation of the object(s) to the bidders. However, in real-world situations this may not be an innocuous assumption. Especially in auctions where the value of the goods on sale might be extremely high or there is exogenous cap on bidding amounts. In these cases, it is natural to think of budget-constraints which bind the ability of bidders to spend.

In this dissertation, I investigate the effects of budget-constraints in multi-unit auctions. This is done in three parts. First, I analyze a case where all bidders have a common budget constraint. Precisely, I analyze an auction where two units of an object are sold at two simultaneous, sealed bid, first-price auctions, to bidders who have demand for both units. Bidders differ with respect to their valuations for the units. All bidders have an identical budget constraint which binds their ability to spend in the auction. I show that if valuation distribution is atom-less, then their does not exist any symmetric equilibrium in this auction game.

In the second and third parts, I study the sale of licenses for the right to drill for oil and natural gas in the Outer Continental Shelf (OCS) of the United States.[1] Multiple licenses, each representing a specific area known as a tract, are sold at a time using simultaneous sealed-bid first-price auctions (FPA). Firms that

[1]These are regions beyond 3 miles from the coast

participate at these auctions typically have a demand for more than one tract. Given this frameowork, an obvious question to ask concerns the characteristics of the firms and the environment which determine the bids that are placed in the auctions. The valuation or the worth of the tracts, is one of these characteristics. However, given a set of valuations for the tracts, are all firms identical in their ability to spend in auctions?

Budget constraints, which limit the set of bids a bidder can place, seems to be a natural characteristic all bidders would posses. These constraints take on a greater significance in multi-unit contexts, since in these environments, bidders not only have to decide the level of their bids but also the number of objects they want to bid on.

1.1 The Case of OCS Wildcat Auctions, 1954-1970

The Department of Interior (DOI) has conducted the OCS auctions since 1954. In this paper I focus on wildcat sales. These are so called as these are in areas where no previous oil exploration has taken place. These auctions present no ex ante informational asymmetries among bidders.

The sale process is as follows. First, DOI announces that certain off-shore area is available for exploration. The announced area would be divided into smaller areas of 5000 or 5760 acres which are called 'tracts'. Firms are then invited to offer nominations as to which tracts they are interested in. On the sale date a large number of tracts (typically, the ones firms showed interest in) would simultaneously go up for sale in sealed-bid, first price-auctions.

There were nine wildcat sales from 1954-1970. Table 1.1 presents summary information about these auctions. What can be seen is that average number of bids per tract is far less than the number of bidders participating in the auction. [2] Clearly this implies some of the bidders (if not all, as is the case in most auctions) are only bidding on subsets of tracts available.

Table 1.1: Wildcat Auctions 1954-70

Auction	No. of tracts receiving one or more bids	No. of bidders	Average no. of bids/tracts
1954-Oct	90	23	3.63
1954-Nov	19	13	4.74
1955-Jul	121	25	3.17
1960-Feb	173	29	2.57
1962-Mar	211	25	2.55
1962-Mar	210	31	3.17
1967-Jul	172	58	4.31
1968-Jun	141	60	3.94
1970-Dec	127	58	8.21

In order to understand precise bidding patterns consider a specific auction. I have picked the 1970 December auction in which fifty-eight bidders participated and submitted a total of 1043 bids.[3]

Figure 1.1 , is a concentration curve. It displays, for each percentage of bidders

[2] A bidder can be an individual firm or a group of firms who submitted joint bids on some tracts.

[3] The number of firms participating was thirty-seven. However, counting each unique group of firms which submitted a bid as a bidder, the number of bidders was fifty-eight

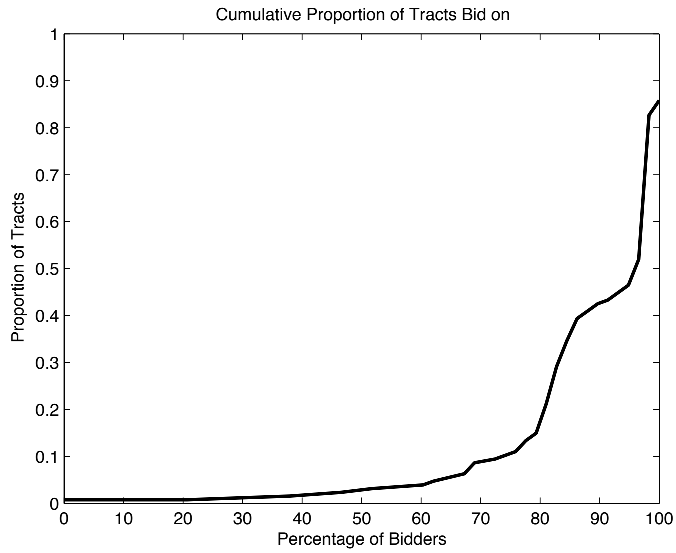

Figure 1.1: 1970 Wildcat Sale

the cumulative proportion of tracts they bid on. For example, the figure indicates that about 90% of bidders submitted bids on less than 45% of tracts. This roughly translates to saying, that out of the fifty-eight bidders, fifty-three of them bid on only 58 tracts or less out of the 127 that were actually bid on. No bidder submitted bids on all the tracts.

In standard auction models where bidders differ only by their valuations, such sparse bidding behavior would be difficult to explain, unless the valuation distributions for each tract had considerable masses in the tails. If this were the case, then it would be possible for bidders to obtain very different valuation-signals with high probability and hence, theoretically one could replicate the sparse bidding pattern seen in the data. However, this does not seem realistic. It is hard to imagine a bidding pattern so sparse in the absence of some other aspect.

Collusion among bidders can be one explanation for sparse bidding. However, Figure 1.2 gives evidence for lack of coordination.[4] It shows the number of bids which were placed on the tracts. Except for four tracts, all tracts received two or more bids. In fact 59 tracts received three or more bids. From table 1.1, one can see that the average number of bids per tract in this auction was 8.21[5]. While the number of bids per tract may not be synonymous with competition, it does indicate a lack of coordination amongst bidders in determining where to place their bids. This evidence suggests collusion may not play a major role in these auctions.

Another possible explanation of sparse bidding in my view is the following. Suppose the bidders participating in the auctions had budget constraints. Then, a budget-constrained bidder may chose to focus on a few and bid on those, rather than bidding on all. The number of tracts he bids on as well as the actual amount bid further depends on the valuation he places on the tracts.

As further evidence of the presence of budget constraints, consider figure 1.3 which depicts the number of tracts each bidder placed a bid on. Bidders are placed on the x-axis in increasing order of total bid-amounts submitted, which is the sum of all the bids that were submitted by a bidder in the auction. Each bar represents an individual bidder.

Table 1.2 gives a brief description of the bidding patterns of of two specific

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[4]In OCS auctions before 1979 firms were allowed to submit joint bids. So legal cartels were allowed to be formed. Despite this aspect Hendricks et al [16] report joint bidding to be low and a lack of general coordiantion

[5]This was considerable higher than previous years

bidders, A and B.[6] While these bidders had similar figures in column 2, they bid on vastly different number of tracts and also submitted very different bids. These bidders were clearly not coordinating since they competed on many tracts. Also, if bidder B was not budget-constrained then it is difficult to rationalize him not bidding on more tracts when A was bidding on many more. Hence, budget constraints along with differences in valuation for the tracts can offer a reasonable explanation of bidding patterns seen in the data.

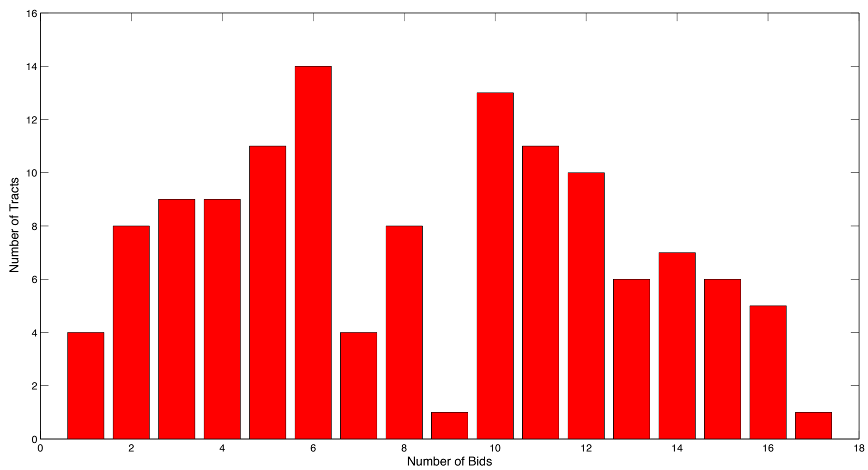

Figure 1.2: Bids Per Tract

In order to qualitatively explain such aspects of bidder behavior in these auctions, I study a theoretical model where, for simplicity, two units of an object are sold using simultaneous first-price auctions to bidders who are budget constrained

[6]All monetary amounts are in terms of 1982 Dollars

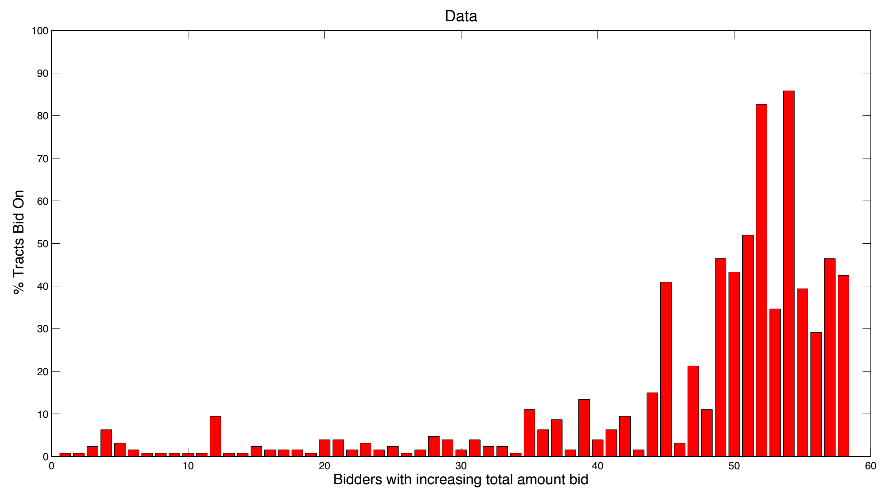

Figure 1.3: No. of Tracts Bid on by Individual Bidders

Table 1.2: Bidding Patterns of Bidders A & B

Bidder Identity	Sum of Total Bid Amounts	No. of Tracts Bids on	High Bid
A	355,871,038	109	8,674,145
B	356,217,670	44	28,623,954

and have a positive valuation for both the units. Bidders are differentiated with respect to their budgets as well as their valuations for the units. Each bidder gets a draw of a budget level from a distribution with a continuous density, which has the interpretation of being the maximum amount he can spend across the two auctions; and nature also draws valuations for each bidder from another distribution but informs each bidder of only the sum of his draws, with the interpretation that when he receives this draw it represents the total value of the two units together, and thus that the expected value of each object is exactly half this signal. The main theoretical result of this paper is identifying and characterizing a symmetric equilibrium. I show that this equilibrium is in pure strategies and is symmetric with respect to the players and with respect to the units. The strategies are pure in the sense of each bidder-type having a unique split of his budget. In equilibrium, when this split is unequal for a bidder-type, he randomizes between the auctions as to which one gets the higher bid. The reason for a bidder to split his budget unequally is that a bidder knows that there might be other bidders who are budget constrained as well. Hence budgets introduce another strategic dimension in the auction game.

With a formal theory in place, the second part of the paper looks at the data from OCS auctions. Since these auctions typically involve dozens of tracts, I use a generalized version of the model, which can be described as follows. As in the simpler model, each bidder receives a draw of a budget. But now Nature's draw of a valuation-signal for each a bidder represents the sum of the values of all tracts being sold. Thus after receiving this draw, each bidder views each tract as having the same value, which is the total worth divided by the number of tracts.

I make an identifying assumption that a bidder's budget is the sum of his observed bids in data. I carry out a quantitative exercise using data from the 1970 OCS auction. I calibrate bidder-specific valuations by targeting the bid distribution in the data. Using these calibrated valuations I evaluate each bidder's best-response bid-vector. The model is able to match many aspects of the data. Table 1.3 provides a brief overview of these results. The coefficient of correlation refers to the correlation between bidder budgets and the number of tracts they submitted bids on.

Table 1.3: Summary Results

	Data	Model
No. of Bids	1043	1282
Median Bid	3,253,314	3,810,462
Revenue	1,927,511,599	1,944,565,200
Mean High Bid	15,177,256	15,311,537
R^2	0.84	0.89

The preliminary quantitative exercise I carry out in this paper suggests that a model of multi-unit auctions with budget constraints is an important one to study and can be used to explain bidding patterns in OCS auctions. As a part of future research I aim to generalize the theoretical model developed in this paper to one where many units are sold and bidders can possibly have different valuations for the objects. I also aim to structurally estimate such a model so that I can carry out policy experiments.

The rest of the paper is organized as follows: The next section gives a brief survey of the literature concerning auctions with budget constraints as well the empirical literature on OCS auctions. Section 2, presents the model environment as well as assumptions. I start my analysis in Section 3, where I characterize bidder behavior in terms of best responses to a specific equilibrium, satisfying certain properties. In Section 4, I show, that any symmetric equilibrium would indeed satisfy the properties

I assumed in section 3 and then go on to characterize the said equilibrium. In section 5, I take the more general form of the model (with many units for sale) to the data. I look at a specific auction for the OCS tracts off the coast of Louisiana in December of 1970. Section 6 concludes. Most of the proofs are contained in the appendix unless indicated otherwise.

1.2 Literature Review

1.2.1 Theoretical Literature

Auctions with budget-constrained bidders have received considerable attention in the literature. Some papers have focussed on auctions with single units, where bidders are budget constrained. Che and Gale [5], [6], in a series of papers compared various across auction formats when bidders have budget constraints under varying information structures. Zeng [33] studies an auction for a single unit where budget constrained bidders have the option to declare bankruptcy. Fang and Parreiras [7] characterized equilibrium bidding in second price auctions with budget constraints and interdependent valuations. Kotowski [20] in a recent paper showed the existence of equillbria in first-price auction when bidders are budget constrained and have private valuations for a single object. This paper analyzes the strategic aspect of budget constraints which my paper does well, though in a multi-unit setting.

Palfrey [27], was one of the first to study impact of budget constraints in a multi-unit setting. He studied a discriminatory auction with multiple units and bidders having budget constraints in a complete information setting. He showed that the presence of budget constraints can drastically alter equilibrium predictions of an auction-game. Benoit and Krishna [1] analyzed a multi-object auction environment where bidders have budget constraints in a complete information setting. They provide revenue rankings between sequential and simultaneous auctions. Using a similar complete information environment Szentes [30] identifies mixed equilibria in simultaneous first-price auctions when the objects are either compliments or substitutes. His analysis is without budget constraints. Some applications of auctions with budgets can be found in the contest literature, such as Kvasov [21] and Gavious, Moldovanu and Sela [8]. Brusco, and Lopomo [3] study sequential auctions for multiple objects to bidders who have known budget constraints.

To the best of my knowledge the current paper is the first to consider simultaneous sealed-bid auctions in a multi-dimensional incomplete-information environment. Existing studies of multi-dimensional incomplete-information in auction settings have so far been restricted to single unit sales or optimal auction design. My paper studies the case where there multiple units for sale and bidders demand multiple items. The theoretical results of the paper are related to some results obtained by Palfrey [27], although his analysis was in a complete information setting.

1.2.2 Empirical Studies of OCS auctions

The case of OCS auctions has been studied extensively. Among these the case of wildcat sales was studied by Hendricks, Pinske and Porter [13], Hendricks, Porter and Tan [16] and Campo, Perrigne and Vuong [4]. Using techniques developed by Guerre, Perrigne and Vuong [10] these papers investigate among others certain implications of rational bidding, cognizance of 'winner's curse' , bidding rings and joint bidding. All these papers carry out structural estimations.

My paper is different from preceding research in two dimensions. First, I examine these auctions in a multi-unit environment. Previous research was exclusively concerned with an individual bidder's decision to bid on single tracts. Second, in single-unit environment budget constraints may not appear to be as important a dimension. However, when considering these auctions in a multi-unit framework, budget constraints become that much more important. To the best of my knowledge, my research is the first to consider OCS auctions in a multi-unit environment and explain certain aspects of the data. However, my paper does not carry out a structural estimation.

Other studies such as Hortaçsu and Puller [18] and Hortaçsu and McAdams [19] carry out structural estimations of auctions where multiple items are sold and bidders have demand vectors. These papers study efficiency implications of certain auction formats and carry out policy experiments. The auction formats studied are ones where all units are sold in the *same* auction. The empirical exercise in my paper, while not a structural estimation, considers a different auction format than the ones considered in these studies. In OCS auctions there a large number of *sub-auctions* that take place simultaneously.

CHAPTER 2
COMMON BUDGET CONSTRAINT

2.1 Environment

Two units of a good are sold via two simultaneous FPA, to two bidders. Each bidder i's valuation for each unit is given by a single number, $v^i \in [0, 1]$. The bidders independently draw these valuation from a common, atom-less distribution F with density f, which is positive everywhere.

All bidders also have a budget w which is identical across bidders. I assume that no bidder is able to spend more than his budget.

Each bidder places two bids, $\{b_1, b_2\}$ simultaneously in the two auctions. The highest bid in each auction wins the unit. If a bidder wins both units then his ex-post payoff is $(v - b_1) + (v - b_2)$. Similarly winning one unit only gives a payoff of $(v - b_k)$, where k is the auction he wins. Ties are broken with equi-probability.

Assumption (1)

(i) For all i, (v_i) is drawn independently.

(ii) $F(\cdot)$ has continuous density.

In the absence of the budget constraint, the unique pure strategy equilibrium will be

$$C(v) = \frac{1}{F(v)} \int_0^v x df(x) \tag{2.1}$$

For the budget constraint to be binding for some bidder types, it must be the case that $\frac{w}{2} < C(1)$.

Assumption (2) $w < 2\int_0^1 x df(x)$.

2.2 Equilibrium Characterization

Number the auctions 1 and 2 so b_1 will represent a bidder's bid in the 'first' auction and similarly, b_2. The feasible set of bid pairs (strategy space) for any bidder type (v, w) is given by $B(w) = [\{b_1, b_2\} | b_1 + b_2 \leq w]$, and $B = \Pi_{w \in [\underline{w}, \bar{w}]} B(w)$ is the space of all possible bids. Any element in $b \in B(\cdot)$ is a bid pair. A mixed strategy for any player i is a probability distribution over $B(w_i)$, and is given by a function $\sigma_i(\cdot \mid \cdot) : B \times [0, 1] \rightarrow [0, 1]$

If this auction has a mixed strategy equilibrium then such an equilibrium will generate a two dimensional bid distribution $\bar{H}$, with corresponding marginal distributions, H_1 and H_2. The corresponding supports are $[\underline{b_1}, \bar{b}_1]$ and $[\underline{b_2}, \bar{b}_2]$. I will assume there exists a symmetric equilibrium σ which generates identical bid distributions in both the auctions . That is $H_1 = H_2 = H$ [1]

Every bidder (v, w) has to decide on a bid pair $\{b_1, b_2\} \in B(w)$ to maximize, $(v - b_1)H(b_1) + (v - b_2)H(b_2)$.

Let $[0, \bar{b}]$ be the closure of the support of bids in both auctions which are played in equilibrium with positive probability by some bidder. The lower end is equal to 0 since bidders with v equal to zero will bid zero in both auctions.

[1]Since, for each bidder the value of the two objects is identical it is natural to look for such equilibrium.

Note that

$$\{b_1', b_2'\} \in \underset{\{b_1,b_2\}\in B(w)}{\arg\max}\ (v-b_1)H(b_1)+(v-b_2)H(b_2)$$

$$\Longrightarrow \{b_2', b_1'\} \in \underset{\{b_1,b_2\}\in B(w)}{\arg\max}\ (v-b_1)H(b_1)+(v-b_2)H(b_2)$$

Let the best responses to the bid distribution H is given by $\beta(v)$.

$$\beta(v) = \underset{\{b_1,b_2\}}{\arg\max}\left[\Pi(v,b_1,b_2)|b_1+b_2 \leq w\right] \tag{2.2}$$

Lemma 2.1. *H is continuos at all $b \neq \frac{w}{2}$. That is, the equilibrium bid distribution can have at most one mass point*

Proof. Suppose there is a mass point at $b \neq \frac{w}{2}$. Let the probability mass at b be given by $h(b)$. This implies, there exists an interval of bidder types, $[v', v'']$ who submit b in one of the auctions in equilibrium. Any of these bidders could increase their payoff in the auction they submit b in, by bidding slightly higher. However, as this mass point is part of the equilibrium, this implies that by doing so, they must be decreasing their payoff by the same or larger amount in the other auction. Which in turn implies, that these bidders must be 'constrained'. That is they must be playing $w-b$ in the other auction and there must be a mass point at $w-b$ also. Let this probability mass be $h(w-b)$. For all $v \in [v', v'']$ the payoff is

$$\begin{aligned} \Pi = \quad & (v-b)(H(b)-h(b)) + \frac{1}{2}(v-b)h(b) + \\ & (v-w+b)(H(w-b)-h(w-b)) + \frac{1}{2}(v-w+b)h(w-b) \end{aligned} \tag{2.3}$$

where the second and fourth terms are payoffs from tieing. Since playing the bid pair, $\{b, w-b\}$ is an equilibrium strategy for all $v \in [v', v'']$, any deviations must give weakly lower payoff. Specifically, for v', playing $\{b+\epsilon, w-b-\epsilon\}$ or $\{b-\epsilon, w-b+\epsilon\}$ where $\epsilon > 0$ and small must give lower payoffs. As $\epsilon \to 0$, and since H can only have a countable set of discontinuities, the following inequalities must hold.

$$\frac{1}{2}(v'-b)h(b) \leq \frac{1}{2}(v'-w+b)h(w-b) \tag{2.4}$$

$$\frac{1}{2}(v'-b)h(b) \geq \frac{1}{2}(v'-w+b)h(w-b) \tag{2.5}$$

$$\implies (v'-b)h(b) = (v'-w+b)h(w-b) \tag{2.6}$$

If $b > \frac{w}{2}$, the above implies $h(b) > h(w-b)$. Therefore for all $v > v', (v-b)h(b) > (v-w+b)h(w-b)$ and they will prefer to bid a little higher than b. Similarly, if $b < \frac{w}{2}$, all $v < v'$ will prefer to bid slightly lower than b and bid more in the other auction. Hence there can be no interval of bidder types playing b in equilibrium. $\square$

If there is a mass point at $\frac{w}{2}$, then all bidders who play $\frac{w}{2}$ in one auction in equilibrium must also be playing $\frac{w}{2}$ in the other auction. This is true due to the argument in the previous lemma.

Lemma 2.2. $\bar{b} > \frac{w}{2}$

Proof. Suppose, $\bar{b} < \frac{w}{2}$. Then, H is continuous everywhere. One can then find a symmetric pure strategy equilibrium using standard arguments using differential equations. This would imply that this is also a solution of the unconstrained problem, without a budget constraint. However the unconstrained problem has a unique solution for both bid functions which is given by equation 2.1, which is not possible in the auction with a budget constraint.

Now, suppose, $\bar{b} = \frac{w}{2}$.There can be two cases depending on whether there is a mass point at $\frac{w}{2}$ or not. If there is no mass point then an argument identical to the one in the preceding paragraph will apply.

Suppose there is a mass point. Let the probability mass at $\frac{w}{2}$ be $h(\frac{w}{2})$. First, I can show that H has a connected support to the left of $\frac{w}{2}$. Suppose there exists $b' < \frac{w}{2}$ such that there does not exist (or only a countable set) bidder who bids in $[b', \frac{w}{2}]$. Consider a bidder v bidding $(\frac{w}{2}, \frac{w}{2})$ in the two auctions. This bidder's payoff is given by

$$\begin{aligned}
\Pi &= 2\big[(v - \frac{w}{2})(1 - h(\frac{w}{2})) + \frac{1}{2}(v - \frac{w}{2})h(\frac{w}{2})\big] \\
&= 2\big[(v - \frac{w}{2}) - \frac{1}{2}(v - \frac{w}{2})h(\frac{w}{2})\big] \\
&= 2(v - \frac{w}{2}) - (v - \frac{w}{2})h(\frac{w}{2})
\end{aligned}$$

Suppose this bidder submits the bid pair $\{\frac{w}{2} + \epsilon, \frac{w}{2} - \delta\}$, where $\epsilon < \delta = \frac{w}{2} - b'$, and $\epsilon > 0$ but small. With this strategy, the bidder will win one auction for sure and the other with probability $(1 - h(\frac{w}{2}))$. As $\epsilon \to 0$,

$$\begin{aligned}
\Pi' &= (v - \frac{w}{2}) + (v - \frac{w}{2} + \delta)(1 - h(\frac{w}{2})) \\
&= 2(v - \frac{w}{2}) + \delta(1 - h(\frac{w}{2})) - (v - \frac{w}{2})h(\frac{w}{2}) > \Pi
\end{aligned}$$

which is a contradiction. Hence H has a connected support to the left of $\frac{w}{2}$.

Next, it can be shown that if a bidder submits some bid pair $\{b_1, b_2\}$ such that

$b_1 \geq b_2$ and $b_1 < \frac{w}{2}$ but arbitrarily close, he will benefit from deviating to $(\frac{w}{2} + \epsilon, b_2)$.

$$\begin{aligned}
\Pi(v, b_1, b_2) &\leq (v - b_1)(1 - h(\frac{w}{2}) + (v - b_2)H(b_2) \\
&= (v - b_1) - (v - b_1)h(\frac{w}{2}) + (v - b_2)H(b_2) \\
&= (v - \frac{w}{2} - \epsilon) + (\frac{w}{2} + \epsilon - b_1) - (v - b_1)h(\frac{w}{2}) + (v - b_2)H(b_2) \\
&< (v - \frac{w}{2} - \epsilon) + (v - b_2)H(b_2)
\end{aligned}$$

where the last inequality is true since $(\frac{w}{2} + \epsilon - b_1) \approx 0$ and $(v - b_1)h(\frac{w}{2}) > 0$ $\square$

Lemma 2.3. *For any interval $(b', b'') \subset [0, \bar{b}]$, there must exist a positive mass of bidders who submit some bid $b \in (b', b'')$ in one of the auctions with positive probability.*

Proof. Suppose this was not the case for some b', b''.

Case (1): $b', b'' \neq \frac{w}{2}$

If this were not the case then any bidder submitting a bid b'' in one of the auctions could strictly increase his payoff by bidding b' and not changing the probability of winning in that auction[2]. He can always do this, since this will not conflict with having a budget constraint. Note that this proof would also work if the interval was closed. Then, since H is continuos at b' and b'', one can find some bidder who would want to deviate.

Case(2): $b'' = \frac{w}{2}$

If there was no mass point at $\frac{w}{2}$ then this case is similar to the previous one. Suppose there is a mass point. Then part of the proof in Lemma 2.2 would apply.

[2]Since there can be no mass points in at b' or b''

Case(3): $b' = \frac{w}{2}$

Again, if there was no mass point then this case would be similar. If there was a mass point then, it is not obvious that that a bidder bidding b'' would prefer bidding b', since in this case he would tie with a positive probability and hence lower his payoff. However, bidders who are bidding arbitrarily close to $\frac{w}{2}$ from the left would indeed bid in the interval (b', b''). And since there is a positive mass in all neighborhoods to the left of $\frac{w}{2}$ there would be a positive mass to the right.

Using case (1) and (2), there exists some bidder v such that he bids $b_1 \in (\frac{w}{2} - \epsilon, \frac{w}{2})$ for all $\frac{w}{2} > \epsilon > 0$ in some auction with positive probability. Therefore his bid in the other auction $b_2 \leq w - b_1$. Consider $\epsilon = b'' - \frac{w}{2}$. If this bidder's second bid b_2 is less than $\frac{w}{2}$, then this bidder is not spending his entire budget. Which implies, that by bidding $b_1' > \frac{w}{2}$, he would have higher payoff since there is mass point at $\frac{w}{2}$. Therefore, $b'' > b_2 > \frac{w}{2}$. But this would be true for all bidders who are bidding in the interval $(\frac{w}{2} - \epsilon, \frac{w}{2})$. Hence there would be a positive mass of bidders in $(\frac{w}{2}, b'')$. □

Corallory 2.4. *H is strictly increasing.*

I will now proceed to prove a 'monotonicity' result. Let us define the maximum bid a bidder with valuation v submits in any auction in equilibrium as,

$$\beta_{\max}(v) = \max\{\max\{b_1, b_2\} | \{b_1, b_2\} \in \beta(v)\} \tag{2.7}$$

Lemma 2.5. *Suppose $v < 1$ then there exists a $v' > v$ such that $\beta_{max}(v') \geq \beta_{max}(v)$*

Proof. This proof will be divided up into several cases depending on the value of $\beta_{\max}(v)$.

Case(1) $\beta_{\max}(v) \leq \frac{w}{2}$

Consider a bid pair, $\{\beta_{\max}(v), b_2\} \in \beta(v)$. It must be true that

$$(v - \beta_{\max}(v))H(\beta_{\max}(v)) = (v - b_2)H(b_2)$$

since $\beta_{\max}(v) \leq \frac{w}{2}$. Therefore $\{\beta_{\max}(v), \beta_{\max}(v)\} \in \beta(v)$, since this bid pair is affordable. This implies,

$$(v - \beta_{\max}(v))H(\beta_{\max}(v)) \geq (v - b)H(b), \text{ for all } b \leq \beta_{\max}(v)$$

$$\Longrightarrow (v' - \beta_{\max}(v))H(\beta_{\max}(v)) \geq (v' - b)H(b), \text{ for all } v' > v \text{ and } b \leq \beta_{\max}(v)$$

$$\Longrightarrow (v' - \beta_{\max}(v))H(\beta_{\max}(v)) + (v' - \beta_{\max}(v))H(\beta_{\max}(v)) \geq$$

$$(v' - b_1)H(b_1) + (v' - b_2)H(b_2)$$

where $b_1 \leq \beta_{\max}(v)$ and $b_2 \leq \beta_{\max}(v)$.

Case(2) $\beta_{\max}(v) > \frac{w}{2}$, $\{\beta_{\max}(v), b_2\} \in \beta(v)$ such that $\beta_{\max}(v) + b_2 = w$.

Suppose there exists $v' > v$ such that there exists $\{b'_1, b'_2\} \in \beta(v')$, where $b'_1 = \beta_{\max}(v')$, $b'_1 \leq \beta_{\max}(v)$ and $\{b'_1, b'_2\} \neq (\beta_{\max}(v), b_2)$.

$$(v - \beta_{\max}(v))H(\beta_{\max}(v)) + (v - b_2)H(b_2) \geq (v - b'_1)H(b'_1) + (v - b'_2)H(b'_2) \quad (2.8)$$

$$(v' - \beta_{\max}(v))H(\beta_{\max}(v)) + (v' - b_2)H(b_2) < (v' - b'_1)H(b'_1) + (v' - b'_2)H(b'_2) \quad (2.9)$$

Subtracting equation 2.8 from 2.9

$$H(\beta_{\max}(v)) + H(b_2) < H(b'_1) + H(b'_2) \quad (2.10)$$

It is clear that if $b'_1 < \beta_{\max}(v)$ then $b'_2 > b_2$ since H is strictly increasing. Consider the bid pair, $\{w - b_2', b_2'\}$.

$$
\begin{aligned}
& (v - \beta_{\max}(v))H(\beta_{\max}(v)) + (v - b_2)H(b_2) \\
& \geq (v - w + b_2')H(w - b_2') + (v - b_2')H(b_2') \\
\Longrightarrow\ & (v - \beta_{\max}(v))(H(\beta_{\max}(v)) - H(w - b_2')) + (w - b_2' - \beta_{\max}(v))H(w - b_2') \\
& \geq (v - b_2)(H(b_2') - H(b_2)) + (b_2 - b_2')H(b_2') \\
\Longrightarrow\ & (v - \beta_{\max}(v))(H(\beta_{\max}(v)) - H(w - b_2')) \geq (v - b_2)(H(b_2') - H(b_2))
\end{aligned}
$$

The second implication is true since $\beta_{\max}(v)+b_2 = w = w-b_2'+b_2'$ and $w-b_2' \geq b_1' \geq b_2'$. Now, $(v - \beta_{\max}(v)) \leq (v - b_2)$. Hence

$$
\begin{aligned}
& H(\beta_{\max}(v)) - H(w - b_2') \geq H(b_2') - H(b_2) \\
\Longrightarrow \quad & H(\beta_{\max}(v)) + H(b_2) \geq H(b_2') + H(w - b_2') \geq H(b_2') + H(b_1')
\end{aligned}
$$

which is a contradiction.

Case (3) $\beta_{\max}(v) > \frac{w}{2}, \beta_{\max}(v) + b_2 < w$.

In this case, both the bids are local optima for this bidder v. H is continuos at any $b \neq \frac{w}{2}$. Hence, using standard arguments I can prove that H is differentiable at $\beta_{\max}(v)$ and b_2. I will give a sketch of the proof here.

Suppose H was not differentiable at b_2. That is the left slope of H at b_2, $h_-(b_2)$ is not equal to the right slope $h_+(b_2)$. Without loss of generality, $h_-(b_2) < h_+(b_2)$[3].

[3]Proofs of the existence of the one sided slopes, and their finiteness are in the Appendix

Since b_2 is a local optima, for all small $\epsilon > 0$ the following must be true.

$$(v - b_2)H(b_2) \geq (v - b_2 + \epsilon)H(b_2 - \epsilon) \tag{2.11}$$

$$\implies \quad (v - b_2)\frac{H(b_2) - H(b_2 - \epsilon)}{\epsilon} \geq H(b_2 - \epsilon) \tag{2.12}$$

Taking limits as $\epsilon \to 0$

$$\begin{aligned}
& (v - b_2)h_-(b_2) \geq H(b_2) \\
\implies \quad & (v - b_2)h_+(b_2) > H(b_2) \\
\implies \quad & (v - b_2)\frac{H(b_2 + \delta) - H(b_2)}{\delta} > H(b_2 + \delta) \\
\implies \quad & (v - b_2 - \delta)H(b_2 + \delta) > (v - b_2)H(b_2)
\end{aligned}$$

for some $\delta > 0$, which is a contradiction to b_2 being a local optimum. Similarly, H must be differentiable at $\beta_{\max}(v)$ as well. Hence the first order conditions are applicable. In the following h refers to the density of the bid distribution.

$$(v - \beta_{\max}(v))h(\beta_{\max}(v)) - H(\beta_{\max}(v)) = 0 \tag{2.13}$$

$$(v - b_2)h(b_2) - H(b_2) = 0 \tag{2.14}$$

Since $H(\beta_{\max}(v)) > H(b_2)$ the above equations imply $h(\beta_{\max}(v)) > h(b_2)$.

Suppose, $\big\{\{\beta_{\max}(v), b_2\}, \{b_2, \beta_{\max}(v)\}\big\} = \beta(v)$, i.e the bidder has a 'unique' best response. Since the bid distributions are identical, this bidder will randomize equally between $\{\beta_{\max}(v), b_2\}$ and $\{b_2, \beta_{\max}(v)\}$ in equilibrium. If v is the only bidder who plays $\beta_{\max}(v)$ in equilibrium, this implies, $h(\beta_{\max}(v)) = h(b_2) = \frac{1}{2}f(v)$. Therefore there must exist at least one more bidder , $\hat{v} \neq v$ for whom $\{\beta_{\max}(v), b_2\} \in \beta(\hat{v})$. I will show, $\hat{v} > v$ and $b_2 = w - \beta_{\max}(v)$. This $\hat{v}$ is then the required v'.

(i) $\hat{v} < v$

$$(v - \beta_{\max}(v))h(\beta_{\max}(v)) - H(\beta_{\max}(v)) = 0$$

$$\implies (\hat{v} - \beta_{\max}(v))h(\beta_{\max}(v)) - H(\beta_{\max}(v)) < 0$$

Hence this bidder will never bid $\beta_{\max}(v)$ in any auction.

(ii) $\hat{v} > v$, $b_2 < w - \beta_{\max}(v)$.

$$(v - \beta_{\max}(v))h(\beta_{\max}(v)) - H(\beta_{\max}(v)) = 0$$

$$\implies (\hat{v} - \beta_{\max}(v))h(\beta_{\max}(v)) - H(\beta_{\max}(v)) > 0$$

This bidder will prefer to bid more than $\beta_{\max}(v)$ and he can do so, since it is affordable as $\beta_{\max}(v) + b_2 < w$

Now, suppose the bidder v has more than one best response. $(\beta_{\max}(v), b_2') \in \beta(v)$ such that $b_2 \neq b_2'$. Suppose without loss of generality $b_2' < b_2$.

$$(v - \beta_{\max}(v))H(\beta_{\max}(v)) + (v - b_2)H(b_2)$$

$$= (v - \beta_{\max}(v))H(\beta_{\max}(v)) + (v - b_2')H(b_2')$$

$$\implies (v - b_2)H(b_2) = (v - b_2')H(b_2')$$

$$\implies H(b_2) < H(b_2')$$

There must be a positive mass of bidders who bid in the interval (b_2, b_2'). It is easy to show all $v' < v$, prefer b_2 to any bid in this interval. Hence, no $v' < v$ will bid in this interval in any auction. Similarly all $v' > v$ prefer b_2' to any bid in the interval. But since this interval of bids has a positive mass, it must be the case that there exist some $v' > v$ who are playing $b \in (b_2, b_2')$ because they are 'constrained'. That is they must be submitting $w - b$ in the other auction. Clearly, $w - b > \beta_{\max}(v)$. □

Corallory 2.6. $\bar{b} = \beta_{\max}(1)$

Hence, the maximum bid in the equilibrium support of bids belongs to the best response correspondence of the highest valued bidder. This is where the equilibrium will break down.

Lemma 2.7. *If* $\{\bar{b}, b_2\} \in \beta(v)$*, then* $b_2 = w - \bar{b}$*, and* $v = 1$

Proof. First, let us prove, for $v = 1$, if $\{\bar{b}, b_2\} \in \beta(v)$ then $b_2 = w - \bar{b}$. Suppose there exists, $b_2 < w - \bar{b}$ such that $\{\bar{b}, b_2\} \in \beta(1)$. For any $b_2' \in (b_2, w - \bar{b})$, $H(b_2') > H(b_2)$.

$$
\begin{aligned}
& (1 - b_2)H(b_2) \geq (1 - b_2')H(b_2') \\
\implies \quad & (v - b_2)H(b_2) > (v - b_2')H(b_2') \text{ for all } v < 1
\end{aligned}
$$

Hence there will be no bidder other than the highest valued bidder submitting a bid in the interval $(b_2, w - \bar{b})$. Since the valuation distribution is atom-less, this implies that $H(b_2') = H(b_2)$, which intern implies that the highest valued bidder would strictly prefer the bid b_2 to b_2'. So there will be no bidders submitting a bid in the interval $(b_2, w - \bar{b})$, which is a contradiction. There are cases.

Case (1) Let $v < 1$. Again, suppose there exists, $b_2 < w - \bar{b}$ such that $\{\bar{b}, b_2\} \in \underline{(}v)$. As in case(3) of the Lemma 2.5, H must be differentiable at b_2, and the first order conditions will imply,

$$
\begin{aligned}
& (v-\bar{b})h(\bar{b}) - 1 = (v-b_2)h(b_2) - H(b_2) = 0 \\
\implies \quad & h(\bar{b}) > h(b_2) > 0 \\
\implies \quad & (v'-\bar{b})h(\bar{b}) - 1 \lesseqgtr (v'-b_2)h(b_2) - H(b_2) \lesseqgtr 0
\end{aligned}
$$

where $h(\bar{b})$ is the left hand slope of H at $\bar{b}$ and $v' \neq v$. If $v' < v$, then the bidder would prefer bidding less in both auctions and if $v' > v$ then he would prefer bidding more. Therefore there does not exist any $v' \neq v$ such that $\{\bar{b}, b_2'\} \in \underline{(}v')$ where $b_2' \in [0, w-\bar{b})$.

Consider the interval $(b_2, w-\bar{b})$. It must be the case that $H(w-\bar{b}) > H(b_2)$.This implies, there must exist $v' \neq v$ such that there exists $\{b_1', b_2'\} \in \underline{(}v')$ where $b_2' \in (b_2, w-\bar{b})$. By the previous paragraph, this implies, $b_1' < \bar{b}$.

Suppose $v' > v$

$$
\begin{aligned}
& (v-\bar{b})H(\bar{b}) \geq (v-b_1')H(b_1') \\
\implies \quad & (v'-\bar{b})H(\bar{b}) > (v'-b_1')H(b_1') \\
\implies \quad & (v'-\bar{b})H(\bar{b}) + (v'-b_2')H(b_2') > (v'-b_1')H(b_1') + (v'-b_2')H(b_2')
\end{aligned}
$$

Since $\bar{b} + b_2' < w$, the final equation above is a contradiction to $\{b_1', b_2'\} \in \underline{(}v')$.

Using a similar argument, I can show that if $v' < v$, then v' would strictly prefer the bid pair $\{b_1', b_2\}$ to $\{b_1', b_2'\}$.

Case (2) If $v < 1$ and $(\bar{b}, b_2) \in \underline{(}v)$, then $b_2 = w - \bar{b}$.

If such a v exists then using case (2) of the previous lemma and the preceding argument, for all $v \in (\underline{v}, 1)$, $(\bar{b}, w-\bar{b}) \in \underline{(}v')$. This would imply, that there is a mass point at $\bar{b}$ which is not possible since $\bar{b} > \frac{w}{2}$. $\square$

Hence in equilibrium, only the highest valued bidder would submit $\bar{b}$ in the auctions.

Proposition 2.8. *There does not exist any equilibrium where* $H_1 = H_2 = H$.

Proof. Since the bid distributions are identical in equilibrium, then $B(1, \bar{b}, w - \bar{b}) = B(1, w - \bar{b}, \bar{b}) = p < 1$. Consider a deviation for this bidder, where he submits the bid pair $(\bar{b} - \epsilon, w - \bar{b} + \epsilon)$. Then for $\epsilon > 0$ and small,

$$\begin{aligned} & (v - \bar{b})H(\bar{b}) + (v - w + \bar{b})H(w - \bar{b})) \geq \\ & (v - \bar{b} + \epsilon)H(\bar{b} - \epsilon) + (v - w + \bar{b} - \epsilon)H(w - \bar{b} + \epsilon) \\ \implies \quad & (v - \bar{b})(H(\bar{b}) - H(\bar{b} - \epsilon)) - \epsilon H(\bar{b} - \epsilon) \geq \\ & (v - w + \bar{b})(H(w - \bar{b} + \epsilon) - H(w - \bar{b})) - \epsilon H(w - \bar{b} + \epsilon) \end{aligned}$$

Dividing both sides of the equation by ϵ and taking limits as $\epsilon \to 0$ I get,

$$\begin{aligned} & (v - \bar{b})\big(\lim_{\epsilon \to 0} \frac{(H(\bar{b}) - H(\bar{b} - \epsilon)}{\epsilon}\big) - 1 \geq \\ & (v - w + \bar{b})\big(\lim_{\epsilon \to 0} \frac{(H(w - \bar{b} + \epsilon) - H(w - \bar{b}))}{\epsilon}\big) - H(w - \bar{b}) \\ \implies \quad & h(\bar{b}) = \lim_{\epsilon \to 0} \frac{(H(\bar{b}) - H(\bar{b} - \epsilon)}{\epsilon} > \lim_{\epsilon \to 0} \frac{(H(w - \bar{b} + \epsilon) - H(w - \bar{b}))}{\epsilon} = h_+(w - \bar{b}) \end{aligned}$$

What this implies is the the bid distribution has a higher mass in $(\bar{b} - \epsilon, \bar{b})$ than $(w - \bar{b}, w - \bar{b} + \epsilon)$ for ϵ small. However, if there exists a bidder-type who submits $b \in (\bar{b} - \epsilon, \bar{b})$, then, if he spending his entire budget, he will a submit a bid $b' \in (w - \bar{b}, w - \bar{b} + \epsilon)$ in the other auction, which implies that $h_+(w - \bar{b}) \geq h(\bar{b})$. Which is contradiction.

Also, there can not exist a bidder-type v such that this bidder submits $b \in (\bar{b}-\epsilon, \bar{b})$ in one auction but strictly less than $w-b'$ in the other auction as this would imply H is constant, to the left of $w-\bar{b}$ $\square$

CHAPTER 3
PRIVATE BUDGET CONSTRAINTS

3.1 Environment

Consider two objects that are being sold at two simultaneous FPAs to a finite number of bidders. Each bidder i gets a draw of budget-level $w_i \in [\underline{w}, \bar{w}]$ according to an atom-less distribution $G(\cdot)$. The budget level represents the maximum amount a bidder can spend across the auctions.

For each bidder, nature also draws two valuation signals $v_i^1 \in [\underline{v}, \bar{v}]$ and $v_i^2 \in [\underline{v}, \bar{v}]$, according to an atom-less distribution, $\hat{F}(\cdot)$. These are bidder i's valuations for the two objects. The bidders do not observe these signals individually. Instead they observe the sum $\hat{v}_i = v_i^1 + v_i^2$. I assume that bidders do not have the ability to refine this information and hence assume that the value of each unit is $v_i = \frac{\hat{v}_i}{2}$. This information structure is isomorphic to the bidder's drawing a single valuation signal, $v_i \in [\underline{v}, \bar{v}]$ that is distributed according to an atom-less distribution, $F(\cdot)$. v_i is now their valuation for each unit. Seen in this light, this is now a multi-unit auction where bidders value both units identically.

Each bidder i is now described by a type $t_i = (v_i, w_i) \in [\underline{v}, \bar{v}] \times [\underline{w}, \bar{w}]$. The space of all possible types is $T = [\underline{v}, \bar{v}] \times [\underline{w}, \bar{w}]$. Henceforth I drop the subscript i when referring to a bidder, unless mentioned otherwise.

Name the two auctions, auction 1 and auction 2. Each bidder places two bids, (b_1, b_2) simultaneously in the two auctions where b_1 represents a bidder's bid in auction 1 and similarly, b_2. The feasible set of bid pairs (strategy space) for any bidder type (v, w) is given by $B(w) = [\{b_1, b_2\}|b_1 + b_2 \leq w]$, and $B = \Pi_{w \in [\underline{w}, \bar{w}]} B(w)$ is the space of all possible bids. Any element in $b \in B(\cdot)$ is a bid pair.

The highest bid in each auction wins the unit. If a bidder wins both units then his ex post payoff is $(v - b_1) + (v - b_2)$. Similarly winning one unit only gives a payoff of $(v - b_k)$, where k is the auction he wins.

Assumptions:

(i) For all i, (v_i, w_i) is drawn independently. $F(\cdot)$ and $G(\cdot)$ are independent and have positive and continuous densities.

(ii) $\underline{w} > 0$.

(iii) $\underline{v} > \bar{w}$

Assumption (i) is standard. Item (ii) implies that the lowest budget level is positive. The third assumption ensures that it is individually rational for any bidder to expend all his budget. Then, under many parameter values, in equilibrium all bidders would spend all their budget.[1] This assumption is restrictive in the sense that it requires the lowest valuation to be relatively high. However, there are three reason due to which I believe studying an auction-game where bidders exhaust their budgets is beneficial.

First, theoretically, under a mild condition, I can show that there always exist a set of bidders who expend their entire budget in equilibrium. This condition is that there exists a w, $w < 2\int_0^{\bar{v}} x df(x)$. This ensures that there are bidder-types who

[1]Examples can be found in the appendix in $A.7$

can not play the equilibrium strategy as dictated by a single unit first-price auction equilibrium, in each individual auction. Many qualitative features of the auction-game studied under assumption (iii) are extendible to one with this less strict condition.

Second, the aim of the theoretical part of my paper is to build a model which can offer qualitative predictions for the specific auctions I am considering. Since I aim to do an empirical exercise where the identifying assumption is that bidders exhaust their budgets, considering a theoretical model which has the same properties is helpful.

Finally, in an auction-game where many units are up for sale, exhausting one's budget is not an unrealistic assumption. For example in the OCS auctions the largest total bid-amounts spent were lower than the average winning bid times number of tracts.

3.2 Bidder Behavior

If the lowest valuation is high enough, all bidders expend their entire budget in equilibrium. Therefore, from now on I take the feasible bid space to be $B(w) = [\{b_1, b_2\} | b_1 + b_2 = w]$. A mixed strategy for any player i is a probability distribution over $B(w_i)$, and is given by a function $\sigma_i(\cdot \mid \cdot) : B \times T \to [0, 1]$

3.2.1 Best Responses

If this auction has a mixed strategy equilibrium then such an equilibrium generates a two-dimensional bid-distribution $\bar{H}$, with marginal distributions, say H_1 and H_2. Consider a symmetric equilibrium σ which generates identical bid distributions in both the auctions. That is $H_1 = H_2 = H$.[2] Using techniques developed by Govindan and Wilson [9] one can show the existence of an equilibrium in behavioral strategies which generates such a bid distribution.

Every bidder (v, w) has to decide on a bid pair $\{b_1, b_2\} \in B(w)$ to maximize, $(v - b_1)H(b_1) + (v - b_2)H(b_2)$. Let $[\underline{b}, \bar{b}]$ be the closure of the support of equilibrium bids.

Note that if

$$\{b_1', b_2'\} \in \underset{\{b_1,b_2\}\in B(w)}{\arg\max}\ (v - b_1)H(b_1) + (v - b_2)H(b_2)$$
$$\Longrightarrow \{b_2', b_1'\} \in \underset{\{b_1,b_2\}\in B(w)}{\arg\max}\ (v - b_1)H(b_1) + (v - b_2)H(b_2)$$

Also, since $B(w) = \big\{\{b_1, b_2\} | b_1 + b_2 = w\big\}$, each bidder essentially has to decide how to split his budget level into bids he submits in the two auctions. If any bidder with budget level w chooses a split α, then his bids in the auctions are $\{\alpha w, (1 - \alpha)w\}$. Hence a strategy σ, essentially, assigns a probability distribution over all possible splits of budget levels, for all bidders. Given this formulation, define $\alpha(v, w)$ as

$$\alpha(v, w) = \arg\max_{\alpha}(v - \alpha w)H(\alpha w) + (v - (1 - \alpha)w)H((1 - \alpha)w) \tag{3.1}$$

Note, that when $\alpha \in \alpha(v, w)$, then $(1 - \alpha) \in \alpha(v, w)$. Therefore, in searching for α which maximizes a bidder's payoff, I can restrict my attention to $\alpha \geq 1/2$, knowing

[2]Since, for each bidder the value of the two objects is identical it is natural to look for such equilibrium. Furthermore,I show that the if the bidders play best responses to such an equilibrium then the bid distribution generated by their strategies precisely have this structure.

that the mirror image is also a maximizer. Clearly, if σ is an equilibrium, then it must be the case that $\sigma(t, \{\alpha w, (1-\alpha)w\}) > 0$ if and only if $\alpha \in \alpha(v, w)$

Lemma 3.1. *If σ is a mixed strategy equilibrium, that generates identical bid distributions in both auctions, then this equilibrium has the property that the set of bidder types t for whom $\alpha(\cdot, \cdot)$ has more than one element has measure zero.*

This lemma is helpful in isolating unique best responses for each bidder, in terms of how they split their budget level. With the restriction $\alpha \geq 1/2$ one can now think of $\alpha(\cdot, \cdot)$ as a function.

In order to gain further insight into the best response functions of bidders, for the moment assume that the equilibrium bid distribution is continuously differentiable with second derivatives existing almost everywhere. Let $h(\cdot)$ represent the density of the bid cumulative distribution function. In the next section, I prove that this property is in fact true for all symmetric equilibria of the type $H_1 = H_2 = H$.

Since H is continuous, I know $\alpha(\cdot, \cdot)$ is upper semi-continuous. Also, since the equilibrium is differentiable, the first order conditions (FOC) are applicable. Therefore, if $\alpha = \alpha(v, w)$ then α must satisfy

$$(v - \alpha w)h(\alpha w) - H(\alpha w) = (v - (1-\alpha)w)h((1-\alpha)w) - H((1-\alpha)w) \qquad (3.2)$$

One point to note from equation 3.2 is that if any bidder type t has $\alpha(v, w) > 1/2$, then it must be that $h(\alpha w) > h((1-\alpha)w)$.

Lemma 3.2. *The best response function satisfies the following properties* *(i)* $\alpha(\cdot, w)$ *is non-decreasing. Also, if* $\alpha(v, w) > 1/2$*, then for all* $v' > v$*,* $\alpha(v', w) > \alpha(v, w)$

(ii) $\alpha(v, \cdot)$ *is non-increasing.*

The second part of the lemma follows from the first. When there are two bidders with same budget but different values, then the bidder with the higher value is more 'constrained.' Therefore, increasing the v while keeping w the same, is similar to keeping v constant but reducing the w.

The intuition behind the lemma is that as v increases, the values of the objects to the bidder are increasing. If the bidder is constrained, in equilibrium he would rather increase the chance of winning at least one unit rather than place equal bids in both auctions. This lemma gives an indication of the equilibrium having qualitative features that can explain the bidding patterns seen in the OCS auctions. Bidding on a few tracts is akin to unequal bidding in the model.

Lemma 3.3. $\alpha(\cdot, \cdot)$ *is continuous in both arguments.*

Lemmas 3.2 and 3.3 show that there exists a pure strategy equilibrium in terms of budget splits. Therefore, a bid pair in terms of $\beta_{\max}(v, w) = \alpha(v, w)w$ and $\beta_{\min}(v, w) = (1 - \alpha(v, w))w$, provides a unique strategy for each bidder. Of course, bidders may choose to randomize the auctions in which they submit the higher and lower bid.

Lemma 3.4. $\beta_{\max}(\cdot, w)$*,* $\beta_{\max}(\cdot, w)$ *are continuous. Also for any* w*, if,* $v' = \sup\{v \mid \beta_{\max}(v, w) = \beta_{\min}(v, w)\} < \bar{v}$*, then* $\beta_{\max}(\cdot, w)$ *and* $\beta_{\min}(\cdot, w)$ *are strictly increasing* *and strictly decreasing, respectively over* $(v', \bar{v}]$

Proof. The proof follows from 3.2 and 3.3. $\square$

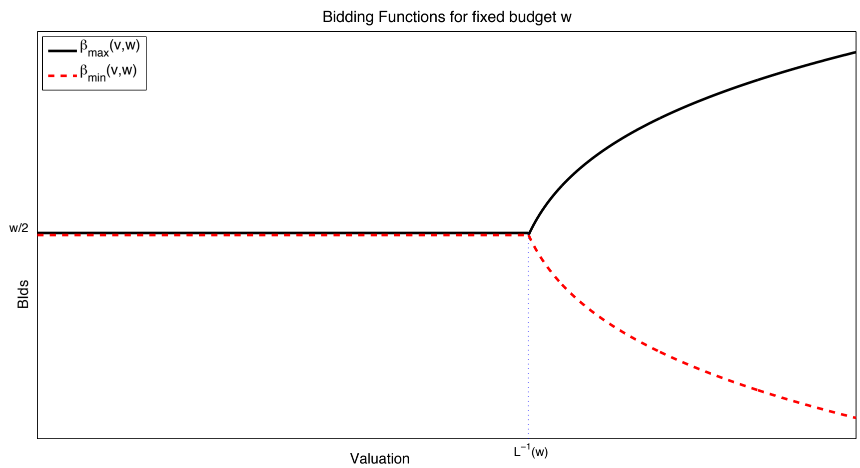

Figure 3.1: Bidding Behavior

Figure 3.1, depicts the sketch of hypothetical best response functions for bidders with a fixed budget. The budget level is such that, some bidder types split their budgets equally while some do an unequal split.

3.3 'Pure Strategy' Equilibria

Proposition 3.5. *There does not exist any equilibrium with* $\alpha(v, w) = 1/2$ *for all* (v, w)

Proof. Suppose such an equilibrium exists. Consider the best response of a bidder $(v, \underline{w})$. Under the strategy $\alpha(\cdot, \cdot) = 1/2$, this bidder has an expected payoff of zero, since $H(\underline{w}/2) = 0$, as the budget distribution, $G(\cdot)$ is atom-less. This bidder can do strictly better by bidding $(w/2 + \epsilon, 0)$ $\square$

Lemma 3.6. *The equilibrium distribution satisfies the following properties*

(i) H *is atom-less. That is* H *is continuous.*

(ii) H *is strictly increasing.*

(iii) The support of bids played in equilibrium is connected. That is for all $b \in [\underline{b}, \bar{b}]$*, there is some bidder* (v, w) *who submits* $\{b, b'\}$ *with positive probability where,* b' *is some other feasible bid.*

(iv) H *is continuously differentiable.*

Part (iv) implies that any symmetric equilibrium is differentiable and hence I can use first order conditions to further describe the structure of the equilibrium.

Until now, I have been silent concerning the support of equilibrium bid distribution. In essence, $[\underline{b}, \bar{b}] \subset [0, \bar{w}]$. In light of the previous lemma I can now make some observations about the ends of the support and which bidders submit them.

Lemma 3.7. $[\underline{b}, \bar{b}]$ *is the support of bids in equilibrium.*

(i) $\underline{b} < \underline{w}/2$ *and* $\bar{b} \geq \bar{w}/2$.

(ii) There exists v *such that* $\alpha(v, \underline{w}) > 1/2$

(iii) $\alpha(\bar{v}, \underline{w})$ *is such that* $(1 - \alpha(\bar{v}, \underline{w}))\underline{w} = \underline{b}$

(iv) Consider the bidder $(\bar{v}, \bar{w})$*. Then* $\alpha(\bar{v}, \bar{w})\bar{w} = \bar{b}$.

Having established some properties about the equilibrium distribution and the support of bids in equilibrium, I can state the equilibrium of the auction. By a pure strategy I mean a unique split of budget levels. An equilibrium is defined by the function $\alpha(\cdot,\cdot)$.

Theorem 3.8. *An equilibrium of the auction game is given by* $\alpha(v,w)$ *which satisfies the FOC in equation* 3.2*, where the bidders play the strategy* σ *with*

$$\sigma(v,w,(\alpha w,(1-\alpha)w)) = \sigma(v,w,((1-\alpha)w,\alpha w)) = 1/2$$

Proof. Playing any element in $\alpha(v,w)$ is a best response to a symmetric equilibrium bid distribution H. I have proved that for all (v,w), $\alpha(v,w)$ contains two bid pairs which are mirrors of each other. Hence, randomizing equally over them is also a best response.

Also, such a strategy generates bid distributions which are identical. Any bidder (v,w) places $\alpha(v,w)w$ and $(1-\alpha(v,w))w$ in auction with probability 1/2 each, and does the same in the other auction. This implies that the bid densities in each auction are the same and hence have the same bid distributions. □

3.3.1 Iso-Bid Curves

Based on the equilibrium, one can divide the valuation-budget space into two regions. In the first region (E), bidders split their budgets equally in equilibrium, and in the second region (U) bidders do an unequal split. These regions are separated by a curve which is the set of bidders for whom doing an equal split of their budget is neither a maxima nor a minima: an equal split of one's budget is always satisfy the FOC. This can be seen by substituting $\alpha = 1/2$ in equation 3.2. However, for some bidders this, may be a maxima and for some this may be a minima. Define this curve as $L(v)$,

$$L(v) = \inf\{w \mid \alpha(v, w) = 1/2\} \tag{3.3}$$

Lemma 3.9. *If $v > v'$ then $L(v) \geq L(v')$. Also if $L(v') > \underline{w}$, then $L(v) > L(v')$. Therefore L is invertible.*

Now, define two regions which are separated by $L(v)$.

$$E = \{(v, w) \mid w \geq L(v)\} \text{ and } U = \{(v, w) \mid w < L(v)\} \tag{3.4}$$

E is the region in the primitive space where bidders, submit equal bids i.e. split their budgets equally and U is the set of bidders who do an unequal split. Note, that there can be parameter values for which E may not exist.

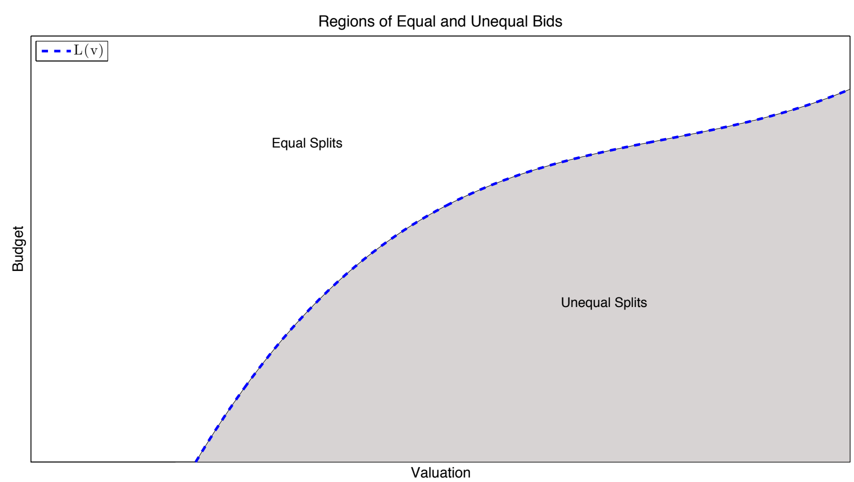

Figure 3.2: Bidder Type Space

Iso-bid lines are the loci of all bidder types who play the same bid with positive probability. In other words, iso-bid line for a bid $b \in [\underline{b}, \bar{b}]$ is given by $I(b)$,

$$I(b) = \{(v, w) \mid \alpha(v, w)w = b \text{ or } (1 - \alpha(v, w)w) = b\} \tag{3.5}$$

In order to understand the iso-bid curves better, I can separate each curve $I(b)$ into two parts. $I(b) = \underline{I}(b) \cup \bar{I}(b)$ where

$$\underline{I}(b) = \{(v, w) \mid (1 - \alpha(v, w))w = b\} \tag{3.6}$$

$$\bar{I}(b) = \{(v, w) \mid \alpha(v, w)w = b\} \tag{3.7}$$

$\underline{I}(b)$ is the set of all those bidder-types who submit b as their lower bid in one of the auctions and similarly, $\bar{I}(b)$ is the set of bidders who submit b as their high bid.

Proposition 3.10. *For all $b \in [\underline{b}, \bar{b}]$,*

(i) $I(b)$ is closed.

(ii) If $(v, w) \neq (v', w')$ and both belong to $\underline{I}(b)$, then either $v \geq v'$ and $w \geq w'$ or $v \leq v'$ and $w \leq w'$. Similarly, If $(v, w) \neq (v', w')$ and both belong to $\bar{I}(b)$, then either $v \geq v'$ and $w \leq w'$ or $v \leq v'$ and $w \geq w'$.

(iii) If $b' > b$, then if $(v, w) \in \underline{I}(b)$ and $(v, w') \in \underline{I}(b')$, then $w' > w$. The same property applies to $\bar{I}(b)$.

(iv) $\underline{I}(b)$ and $\bar{I}(b)$ are closed and connected.

(v) For $b' \neq b$, $\underline{I}(b) \cap \underline{I}(b') = \bar{I}(b) \cap \bar{I}(b') = \phi$.

(vi) For $b' \neq b$, $I(b) \cap I(b')$ is either empty or contains exactly one bidder type (v, w). That is any two Iso-bid curves can only cross once, if they do.

Proof. (i) Closedness of $I(\cdot)$ follows from the upper semicontinuity of $\alpha(\cdot, \cdot)$. Consider a sequence of bidder types (v_n, w_n), such that for each n, $\alpha(v_n, w_n)w_n = b$ or $(1 -$

$\alpha(v_n, w_n))w_n = b$. That is, $\{(v_n, w_n)\} \subset I(b)$. Separate this sequence into those bidders for whom $\alpha(v_n, w_n)w = b$ and those for whom $(1-\alpha(v_n, w_n))w_n = b$. Let the former be $\{v_k, w_k\}$ and the latter $\{v_j, w_j\}$.

$$(v_k, w_k) \in \{(v_n, w_n) \mid \alpha(v_n, w_n)w = b\} \tag{3.8}$$

$$(v_j, w_j) \in \{(v_n, w_n) \mid (1-\alpha(v_n, w_n))w = b\} \tag{3.9}$$

Clearly, $\{(v_k, w_k)\} \cup \{(v_j, w_j)\} = \{(v_n, w_n)\}$ .Therefore, at least one or both must converge to (v, w). Without loss of generality, $(v_k, w_k) \to (v, w)$. $\alpha(\cdot, \cdot)$ is upper hemi-continuous and $\alpha(v_k, w_k)w_k = b$. Therefore, $\alpha(v, w)w = b$. Hence, $(v, w) \in I(b)$

(ii)The proof follows from lemma 3.2. For example, consider any $(v, w) \in \underline{I}(b)$. For all $v' < v$ and $w' > w$, $\alpha(v', w') \leq \alpha(v, w)$, which implies $(1 - \alpha(v'w'))w' > (1 - \alpha(v, w))w$, since $w' > w$. Hence $(v'w')$ can not be in $\underline{I}(b)$. The other cases can be done similarly.

(iii) Suppose to the contrary $w \geq w'$. Then $b = (1 - \alpha(v, w))w < (1 - \alpha(v, w'))w' = b'$ implies, $\alpha(v, w) > \alpha(v, w')$ which is a contradiction since $\alpha(\cdot, w)$ is non-increasing.

(iv)Proving that $\underline{I}(b)$ and $\bar{I}(b)$ are closed is done in an identical manner as $I(b)$. Connectedness of $\underline{I}(b)$, follows from continuity of $\alpha(v, w)$.

(v) Suppose to the contrary, there exists $(v, w) \in \underline{I}(b) \cap \underline{I}(b')$. This implies that this bidder has two equilibrium splits of his budget that he can play. This would be a violation of the split being unique.

(vi) Consider $b < b'$, without loss of generality. Suppose to the contrary, there exist (v, w) and (v', w'), $(v, w) \neq (v'w')$, such that $(v, w), (v', w') \in I(b) \cap I(b')$.

$$
\begin{aligned}
I(b) \cap I(b') &= \underline{I}(b) \cup \bar{I}(b) \cap \underline{I}(b') \cup \bar{I}(b') \\
&= \underline{I}(b') \cap \bar{I}(b) \cup \underline{I}(b) \cap \bar{I}(b')
\end{aligned}
$$

where the second equality follows by part (iv).

Suppose $(v, w) \in \underline{I}(b) \cap \bar{I}(b')$. If there exists $(v', w') \neq (v, w)$ which also belongs to $\underline{I}(b) \cap \bar{I}(b')$, then using part (ii), one can easily show that $v' > v$ and $w' = w$. This implies, for all $(\hat{v}, w)$, $\hat{v} \in (v, v')$, $\alpha((\hat{v}), w) = \alpha(v, w) = \alpha(v', w)$, which a contradiction to lemma 3.2 (i). Therefore $\underline{I}(b) \cap \bar{I}(b')$ can have at most one element.

Clearly, $\underline{I}(b') \cap \bar{I}(b)$ is empty, since $\underline{I}(b')$ lies weakly above $\bar{I}(b')$ and $\underline{I}(b)$ lies weakly above $\bar{I}(b)$. And due to part (iii), $\underline{I}(b')$ lies above $\underline{I}(b)$ $\square$

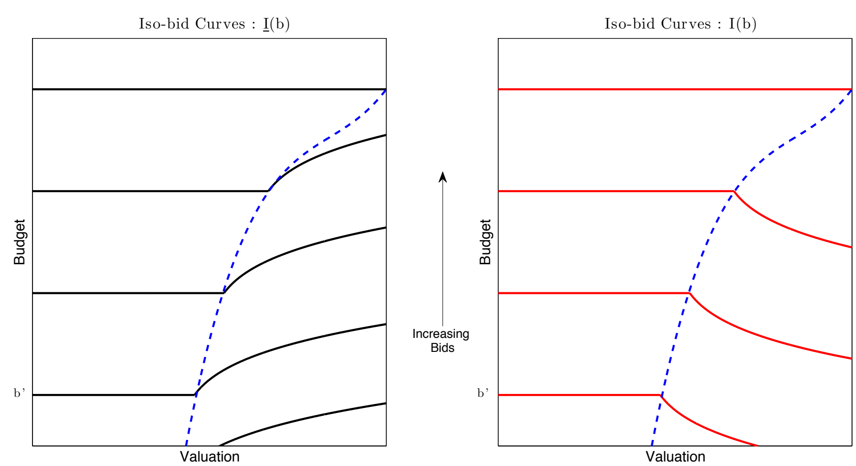

Figure 3.3: Iso-Bid Curves

Given these properties of $I(\cdot)$, the following figures depict some hypothetical iso-bid curves. For the purpose of this figure, I assume that the curve $L(v)$, exists. In the figure it is shown to be strictly increasing. The figure on the left displays $\underline{I}(\cdot)$ and the one on the left displays $\bar{I}(\cdot)$.

From the figure, one can locate each bidder type's bid pair. In order to see that, take (v, w). Simply read off the iso-bid curve this bidder lies on, in each figure. This gives the bid pair submitted by the bidder.

The most interesting feature of the equilibrium is the set of bidders who split their budget unequally. These bidders, despite valuing the objects identically, prefer 'loading up' one bid rather than equally splitting their wealth. The intuition for this feature is that bidders are aware of other bidders also being budget constrained. The increase in one bid is the effect of this realization.

Another point to note is that while the set U always exists the set of bidders who equally split their budgets, may not. If the bidder $\{\bar{v}, \underline{w}\}$ does not equally split his budget then lemma 3.2, implies that no bidder would do so. In this case, the iso-bid curves take the shape of the curves in the region U.

CHAPTER 4
OFFSHORE DRILLING RIGHTS AUCTIONS

In this section I present the results of an empirical exercise to determine whether certain features of the data such as figures 1.3 and 1.2 are, indeed, outcomes of bidders playing a Bayes-Nash equilibrium in the model analyzed in the previous sections. Specifically I examine the auction that took place in December 1970.

In the previous sections, the model analyzed was one where two units were being sold. However, in the OCS auction I examine, multiple tracts are sold simultaneously. I do not explicitly extend the results of the model to one with many units. While doing the empirical exercise, the only assumption I maintain is that there exists an equilibrium in terms of a bid distribution, in a more general model, and bidders 'best respond' to the equilibrium distribution.

The empirical exercise in this paper is *not* a structural estimation, but a calibration exercise. The data supplies many of the variables that are required for the exercise. Then, I pick some parameters (which show up in a bidder's maximization problem) to match certain aspects of the auction data. Using this method I evaluate the model's performance by comparing variables generated by the model against their counterparts in the data.

4.1 Data

In the 1970 auction, 127 tracts were offered and each tract received at least one bid. The auction data available includes all bids submitted by all bidders along with bidder identities and tract locations. Some details of the this auction can be found in table $B.1$.

Along with auction data, I also have access to ex post 'valuations' of each tract, which is defined as discounted revenues less discounted drilling costs and royalty payments. These are taken from a study done by Hendricks, Pinske and Porter in [13]. The interested reader is referred to [13] for further details. For my purposes it suffices to know that for each tract I have some measure of expost worth.

4.2 Recovery of Variables of Interest from Data

In solving any bidder's maximization problem there are three objects of interest. The equilibrium bid distribution, a bidder's budget and his valuation for each tract. In this section I demonstrate how each was recovered from the data.

4.2.1 Bid Distribution, H

The theoretical bid distribution H described in the theory section is the bid distribution bidders *expect* to face. This expected distribution is a result of the primitives of the model, namely the value and the budget distribution. H is not the same as the bid distribution which was obtained in the 1970 auction, H_{1970}. The latter is the result of a specific realization of bidder-types. In other words, suppose there are fifty-eight bidders who independently draw their types from T. Expecting to play against a bid distribution H, they submit their best-response bid-vectors. When aggregated, these in turn would form an auction specific bid distribution.

In order to find H, consider a larger data set of many auctions. There were nine wildcat auctions held between 1954 and 1970. I claim that aggregating bids from these auctions is a reasonable way of obtaining H. First note that even though number of tracts vary from sale to sale, the ratio of productive tracts to tracts drilled does not show much variation. This ratio is around 50%. Second, while the mean net discounted revenues for all tracts, differ across these auctions, the mean and median net revenues of productive tracts do not show a great deal of variation across the larger sales (except 1968) as can be seen in table $B.2$. Coupled with the fact that all these sales were for tracts in the Gulf of Mexico, mostly off the coast of Texas and Louisiana, this evidence seems to the suggest that ex ante bidders did not have any reason to believe that the net revenue which would accrue from the productive tracts would be very different across auction. This does not seem to be an unrealistic assumption. Also, the set of firms that participate in this auctions does not change a lot from one auction to the other.

Hence one can assume, that the budget and valuation distribution from which the bidders draw their types is fixed across these nine auctions. In essence, these are nine realizations of the hypothetical bid distribution H. Therefore the overall empirical bid distribution, which is a mixture of the nine auction-specific distributions can be thought of as H. In figure 4.1 I depict the empirical H as well as H_{1970}. In the appendix I complete the exercise for all the auctions in figure $B.1$. [1]

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[1]Note that figure also shows that the number of tracts or the number of bidders does not seem to have a great influence on the realization of bid distributions

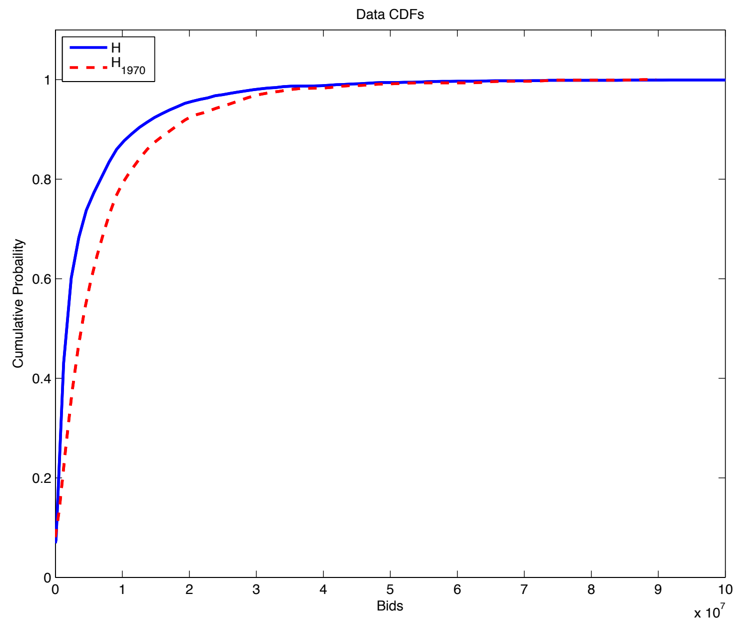

Figure 4.1: Data Bid Distributions

4.2.2 Budget Levels : w

In the theoretical model, the feasible bid space was $B(w) = [\{b_1, b_2\}|b_1 + b_2 = w]$. Extending this empirically, the budget level faced by any specific bidder is simply the sum of the bids he submitted. This is the crucial assumption for identifying bidder budget levels. This is reasonable for two reasons.

First, no bidder submitted bids in all the sub-auctions. In fact, fifty-six out of the fifty-eight bidders submitted bids on 60 tracts or less. If they had bidding budgets higher than the sum of their total bids, they might have been bidding on more tracts.[2] This indicates that the sum of the budgets should be upper bound on

[2] Considering most tracts got more than two bids indicates that most tracts were deemed valuable ex-ante. In the absence of budgets, it can not be Nash behavior to submit zero

bidding budgets.

Of course, bidders may not have valued the other tracts as much. However, the model built in this paper is spare, and the aim is to see how much traction this simple mode has in explaining the data.

Second, there has never been a default on the part of the bidders: no bidder was ever 'exposed' by submitting bids which were higher than his budget. Thus the sum of submitted bids appear to be an lower bound on bidder budget levels.

There are fifty-eight distinct bidders in the data. For each, the budget level is given by

$$w_i = \sum_{k=1}^{127} b_k^i, \text{ for } i = 1, 2 \ldots 58$$

where b_k^i represents bidder $i's$ bid on the k^{th} tract.[3] Now I have 58 distinct budget levels.

4.2.3 Tract Valuations : v

Two assumptions in the theoretical model regarding valuations need to be revisited here.

Private Values: In the empirical literature concerning OCS auctions, both private as well as common value paradigms have been considered. In reality, the true specification probably lies somewhere between. In order to use the theoretical

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bids in the other tracts unless they thought that they were worth less than the reserve price, which is a nominal amount of \$15 per acre.

[3] Obviously for all bidders, there is a number of tracts where their bids are zero.

model developed and to test the equilibrium predictions made, I shall consider the valuations to be of private nature.

Single-Dimensional Value-Signal: In order to simplify the analysis in the theoretical part of this paper, although I allowed for inter-bidder heterogeneity of valuations, once a bidder was fixed, there was no heterogeneity with respect to the valuation of each tract. The reason for this was that bidders only observe the sum of the values and interpret the value of each tract to be half that sum. They were not able to refine their information any further.

In the case of OCS auctions, this is indeed a strong assumption as tracts in a sale are located over a large area and hence may have different valuations. Typically, bidders engage in refining their initial information by engaging in tract specific surveys, using seismic and geological studies. [4] However, as a first pass at the problem, it may be instructive to see how far this model can go in explaining the bidding patterns, even with a strong assumption of bidders only receiving information about the sum of the valuations of the tracts.

As in the theoretical section, assume bidders observe a valuation signal $\hat{v}$

$$\hat{v} = \sum_{k=1}^{127} v^k$$

Bidders assume that the value of each tract is, $v = \hat{v}/127$. For the purposes of this study, I require a grid of v's, since I carry out a maximization routine for many (v, w) pairs.

[4] Fault lines might be in different areas. The reports regarding presence of hydro carbons may indicate a higher probability of oil pool being located over specific tracts rather than all

As indicated earlier, due to previous work done by Hendricks et. al [13], I have access to information regarding revenues, drilling costs and royalty payments. However these have a wide range with the minimum reported net revenue being \$-15,619,000 and maximum being \$516,760,000.

Instead of using these values directly, I create a grid of possible valuations using average revenue and average high bid (winning bid) data from table $B.1$. The mean high bid in this auction was \$15,177,000 and the average revenue was \$19,597,270. So, on an average, bidders expect to get back a third of their bid as profits. I take this ratio to create the grid of the valuations, $[\underline{v}, \bar{v}]$.

$$\underline{v} = \min\big\{\text{winning bids}\big\} + \frac{1}{3}\big(\min\big\{\text{winning bids }\big\}\big)$$
$$\bar{v} = \max\big\{\text{winning bids}\big\} + \frac{1}{3}\big(\max\big\{\text{winning bids }\big\}\big)$$

This method for creating a grid of valuations is not sacrosanct. The view I take is that if I observe a bid in the data, then it must be the case that the bidder thought the value of the tract to be higher than the bid, in the very least.

4.3 Bidder's Problem (Revisited)

Translating the bidder's problem in 3.1 to one in which he has chose his bids over 127 tracts, a bidder type $t_i = (v_i, w_i)$ solves

$$\max_{\{\alpha_k\}} \sum_{k=1}^{127} (v_i - \alpha_k^i w_i) H(\alpha_k w) \tag{4.1}$$

$$\text{subject to } 0 \leq \alpha_k^i \leq 1 \text{ for all } k$$
$$\sum_{k=1}^{127} \alpha_k^i \leq 1$$

Here, I allow for the complete action space in terms of feasible bids. The idea behind the empirical exercise is to supply the bidder's maximization problem with all the parameters and functions which I get from data and then solve for the best-reply for each bidder.

4.4 Calibration

The bid distribution in equation 4.1 is the overall bid distribution H in the data. Using this H, I solve the bidder's maximization problem for all the bidder budget levels, $\{w_1, \ldots, w_{58}\}$ which I observe in the data. The last parameter in a bidder's problem is v. I create a grid of 1000 evenly spaced v, $\{v^1, \ldots, v^{1000}\} \in [\underline{v}, \bar{v}]$, with $v^1 = \underline{v}$ and $v^{1000} = \bar{v}$. In essence the type space is $T = \{w_1, \ldots, w_{58}\} \times \{v^1, \ldots, v^{1000}\}$.

For each bidder-type, $(v, w) \in T$, I solve the optimization problem in 4.1. Consequently, for each pair (v, w) $\alpha(v, w) = (\hat{\alpha}_1(v, w) \ldots \hat{\alpha}_{127}(v, w))$, is the best-response vector. Using these I can find out the bids.

To calibrate the model, I identify fifty-eight bidder types who, when best-responding to the over all bid distribution, generate a bid distribution which looks like H_{1970}. Since I observe the budget levels in the data under my identifying assumption, this task boils down to simultaneously choosing a valuation signal for each budget level.

The metric used to measure the distance between the calibrated bid distribution H_c and H_{1970}, is the sup-norm metric. In other words, I use the Kolmogorov-

Smirnov metric. Figure 4.2 shows the three distributions[5]. Figure 4.3 depicts the bid densities. [6]

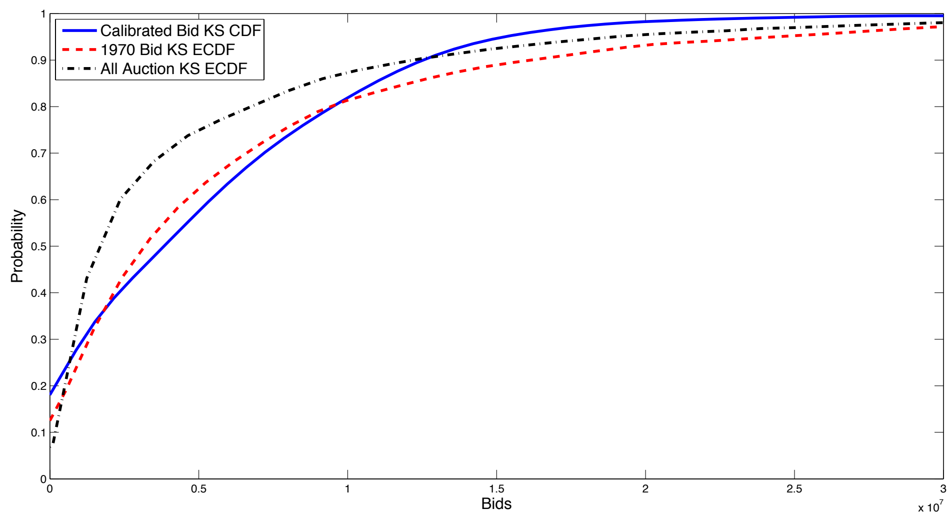

Figure 4.2: Bid Distributions

4.5 Results

In this section I present the results from the empirical exercise.

[5]The distributions are only for bids upto $30,000,000, since most of the mass is in this support. Complete figure of bid cumulative density function are shown in the appendix in figure $B.2$

[6]Carrying out the two sample Kolmogorv-Smirnov test for larger data set and bid data from 1970 as well as calibrated bids and bid data from 1970, the null hypothesis was not rejected at the 5% significance level.

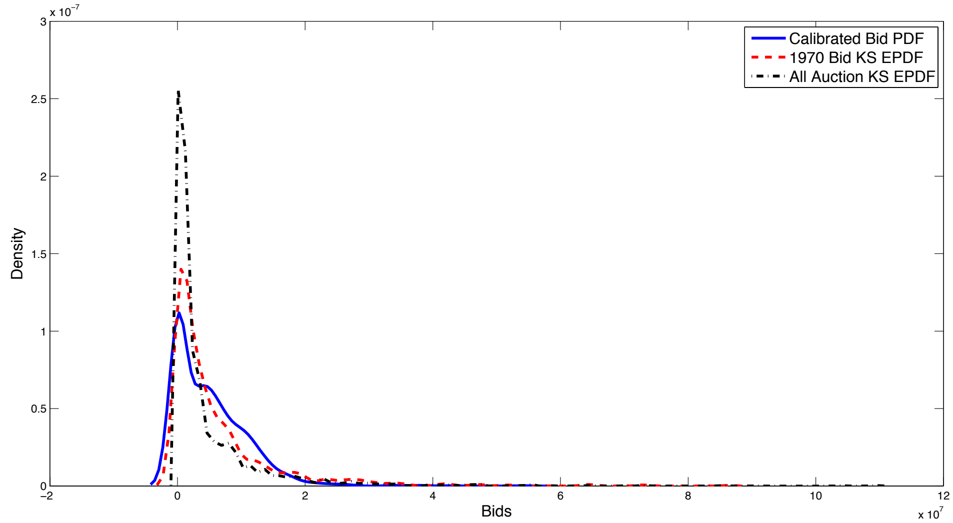

Figure 4.3: Bid Densities

4.5.1 Calibrated Tract Valuations

Recall bidders A and B mentioned in the introduction who had similar budgets but had vastly different bidding behavior. My calibration exercise assigns a higher valuation draw to bidder B as can be seen in table 4.1.

Table 4.1: Calibration for Bidders A and B

Bidder Identity	Sum of Total Bid Amounts	No. of Tracts Bids on	Calibrated v
A	355,871,038	109	65,410,000
B	356,217,670	44	100,490,000

This is indeed an indication of the theoretical model having some merit. Looking back to figure 3.3, it was possible for two bidder-types to bid vastly different amounts in the two auctions even when they had similar budgets. The bidder with the higher valuation could bid less on a tract than a bidder with the lower valuation. A similar idea seems to follow here. Bidder B despite having a similar budget as bidder A bid on fewer tracts as he received a higher value signal.

For each budget level, and hence for each bidder i, I now have a unique calibrated valuation v_i. This in the eyes of the model is the realization of the fifty-eight bidder types $\left\{(v_i, w_i)\right\}_{i=1}^{58}$. As a result I can state results regarding model predicted number of bids per bidder,number of bids per tract and revenue.

4.5.2 Tracts Bid On

For each bidder (v_i, w_i) I have $\{\hat{\alpha_k^i}\}_{k=1}^{127}$ which are generated as best responses. In order to calculate the bids, this vector needs to be multiplied by the budget level which gives us the bids of each bidder on the 127 tracts.

$$b_k^i = \alpha_k^i w_i \text{ for } k = 1, 2, 3 \ldots 127$$

Using the bid vector so generated I can obtain the number of tracts a bidder i submitted bids on.

$$\text{Tracts bid on } (i) = [\# b_k^i | b_k^i > 0 \text{ for } k = 1, 2, 3 \ldots 127]$$

Figure 1.3 depicts how many tracts each bidder i submitted bids on in the data. Figure 4.4 compares the data with the model generated 'tracts bid on' for each bidder.

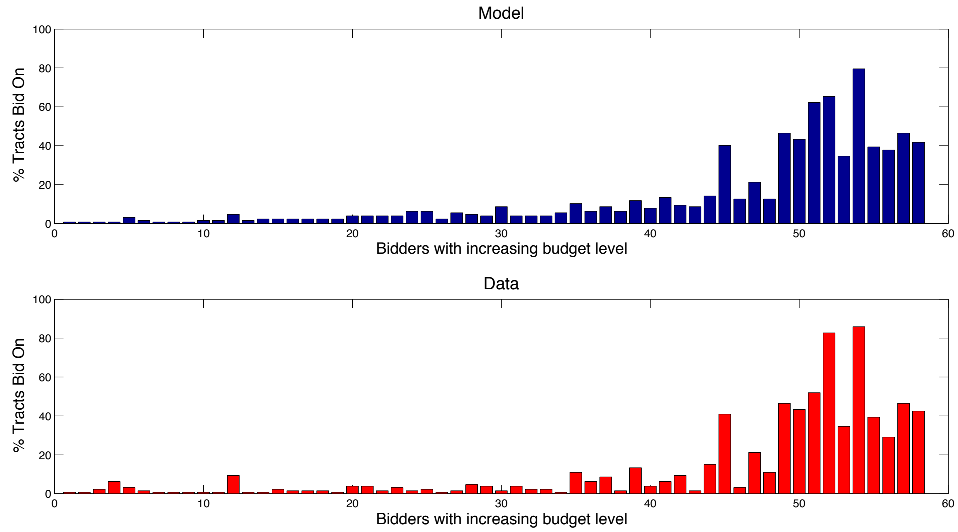

Figure 4.4: Tracts Bid On

The model has reasonable success in this dimension. In the data, coefficient of correlation between 'tracts bid on' and budget levels, is 0.84 and the model produces 0.89. What is far more impressive is that the model captures the non-monotonicty in this relationship. This evidence suggests that budget constraints have a profound influence on bidding behavior in these auctions.

4.5.3 Bids Per Tract

The equilibrium of the auction game analyzed in the theoretical section predicts that when budget-constrained, bidders have a unique split of their budgets. However, they randomize over which tracts they submit bids on. Using a similar idea in the more general model, once bidders decide how many tracts they will submit bids on, they may choose to randomize which tracts they submit bids on. As a result of this randomization the number of bids per tract can vary. Seen in this light, figure 1.2 is one realization of many possible randomizations each bidder can play. Each realization can be vastly different from the other.

For example, consider an auction with two bidders and two units. Suppose in equilibrium each bidder, draws a type for whom submitting one positive bid is optimal. As they are indifferent over which tract they submit this bid on, with probability 1/2 we see each tract receiving one bid and with probability 1/2 we see one tract receiving two bids.

Replicating events such as 1.2 is hard. However, I undertake a thought experiment. In the data I can see how many tracts each bidder submitted bids on. Suppose each bidder was choosing these tracts at random. In such a case one can have many possible outcomes of the random play. What I mean by this is the following. Suppose, in equilibrium a bidder chooses to bid on 20 out of 127 tracts. If he were to randomize where he would submit the bids, in one realization of his random play he could bid on tracts 1, 7, 8, 10, 45 and so forth. In other randomization, he could randomize over 2, 5, 9, 11. For each bidder, I take thousand of such randomizations.

For a given randomization, I count how many tracts got one bid, how many got two bids etc. Thus, for each randomization realization I have a figure such as 1.2 showing the number of tracts which received 1, 2, 3 bids etc. Take a simple average of the number of tracts receiving a given number of bids, across one thousand randomizations. I can do a similar exercise for the model generated 'tracts bid on' for each bidder. The results of this experiment are shown in figure 4.5

The figure on top is the model generated 'average' bids per tract. The one in the middle is the data average randomization and finally the bottom is the actual data.

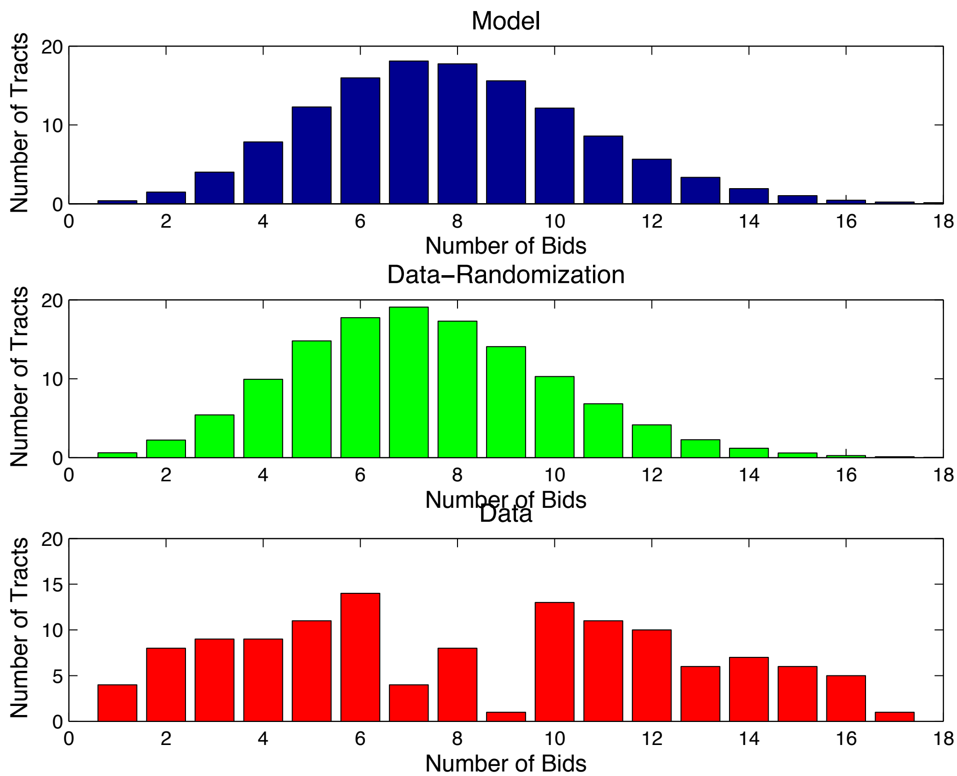

Figure 4.5: Bids Per Tract

The model and data randomizations follow bell shaped curves and are very close to each other. The reason for this is that model did a good job in replicating bidding behavior in terms of tracts bid on by each bidder. However both the model and data randomizations miss the longer tails seen in the data.

Having tracts which receive low number of bids or very high number of bids imply that most bidders either got very good or very bad valuation signals for those tracts. The grid of valuations had an upper bound which was well below the highest valuation see in data. Coupled with the assertion that bidders only drew a single v for all prevents the model from capturing heterogeneity between tracts for any specific bidder. This in turn implies that the model can not replicate the longer tails.

4.5.4 Auction Revenue

Auction revenue as generated by the model is a random variable. As bidders randomize over which tracts they bid on as well as the bids that are placed on those tracts, the winning bids and hence the revenue generated would change accordingly. I compute the revenue for one thousand randomizations. The distribution has a distinct bell shape. In figure 4.6 I also plot the revenue which was observed in the data. As can be seen this is clearly within the 95% confidence interval. A similar exercise can be done for the mean high-bid. In table 4.2 I report the mean revenue and mean high-bid as produced by the model.

4.5.5 Summary of Results

I present table 4.2 as a summary of the results obtained. The coefficient of correlation, R^2 refers to the correlation between bidder budgets and the number of tracts they submitted bids on.

The model is not able to match the highest bid seen in the data as well as the mean bid. The latter is a result of the data having more bids higher than the highest

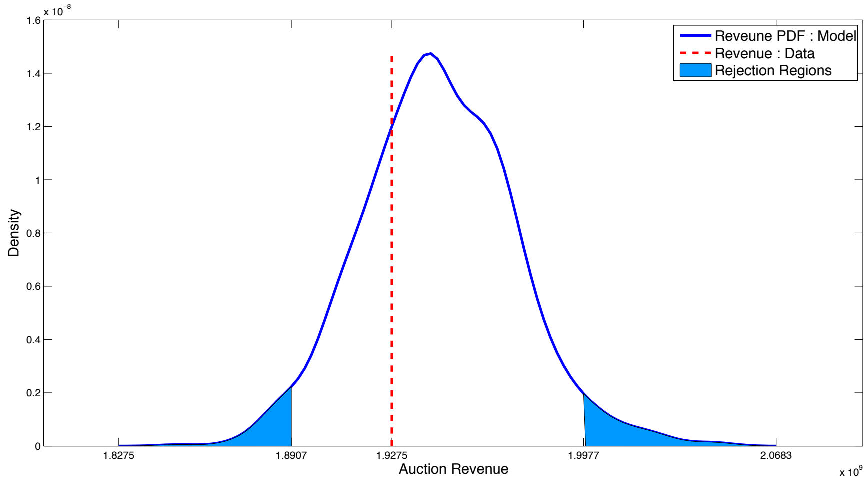

Figure 4.6: Auction Revenue

Table 4.2: Summary of Results for 1970

	Data	Model
No. of Bids	1043	1282
Mean Bid	6,245,821	5,079,766
Median Bid	3,253,314	3,810,462
High Bid	86,447,926	54,460,519
Revenue	1,927,511,599	1,944,565,200
Mean High Bid	15,177,256	15,311,537
R^2	0.84	0.89

bid produced by the model. I conjecture, that the tract valuations being identical and hence the model being conservative about the grid of valuations is the cause for this.

CHAPTER 5
CONCLUSION

Traditionally, the sale of licenses for the right to drill for oil and natural gas in the OCS of continental United States have been analyzed as single unit first-pice auctions. Since multiple licenses are sold simultaneously I take a novel approach by analyzing these auctions as multi-unit auctions. Using empirical evidence, I make a case for the bidders participating in these auctions to be budget-constrained. I develop a simple extension of the standard model by incorporating (random) budgets for the bidders. The auction-game then has a two-dimensional set of types for each player. I study the theoretical properties of this auction, assuming for simplicity that two units are being sold.

I applied the general version of the model developed to the data for a specific auction which took place in 1970 or drilling rights in the OCS off the coast of Louisiana. The model was able to match many aspects of the data, lending credence to the idea that these auctions need to be analyzed in multi-unit framework with budget-constrained bidders.

5.1 Future Work

The model I have developed and analyzed is based on a set of assumptions. Some of these may seem extreme in the case of OCS auctions. Specifically, a single valuation for all tracts. As discussed previously, this is an extreme assumption when there are many tracts on sale. I aim to generalize the theoretical model to one in which many units are sold and the units can have different valuations for the same bidder. While complicating the model considerably, I believe this will be a fruitful exercise in developing techniques to structurally estimate the valuation vectors for such multi-unit models.

Currently the empirical exercise is a calibration exercise to recover the unknown parameters. Traditionally auction data has been analyzed using structural techniques. These techniques are extremely useful in recovering distributions of unknown parameters, which can be used to carry out counter-factual and policy experiments.

I aim to make the empirical exercise in my paper more robust by developing an estimation strategy to recover the bidder valuation parameters. Theoretical results regarding single crossing of iso-bid curves, if applicable to a more general model with many units and valuations can provide identification results which can aid in the structural estimation.

APPENDIX A
PROOFS APPENDIX

A.1 Proof of Lemma 3.1

I prove, that for any fixed budget level, there can only be a countable set of bidders who are indifferent between two splits. Since this applies to all budget levels, the set of all such bidders have measure zero.

Consider two bidder types, (v, w) and (v', w), such that $v' > v$. I need to show, if $\alpha \in \alpha(v, w)$ and $\alpha > 1/2$, then there does not exist α', $1/2 \leq \alpha' < \alpha$ such that $\alpha' \in \alpha(v', w)$ [1]. Suppose to the contrary, such an α' does exist.

Since $\alpha \in \alpha(v, w)$,

$$
\begin{aligned}
& (v - \alpha w)H(\alpha w) + (v - (1-\alpha)w)H((1-\alpha)w) \\
& \geq (v - \alpha' w)H(\alpha' w) + (v - (1-\alpha')w)H((1-\alpha')w) \qquad \text{(A.1)} \\
\implies & (v - \alpha w)(H(\alpha w) - H(\alpha' w)) + (\alpha' - \alpha)wH(\alpha' w) \\
& \geq (v - (1-\alpha)w)(H((1-\alpha')w) - H((1-\alpha)w)) + (\alpha' - \alpha)wH((1-\alpha')w) \\
\implies & H(\alpha w) - H(\alpha' w) > H((1-\alpha')w) - H((1-\alpha)w)
\end{aligned}
$$

Where the second inequality is a result of re-arranging the first. The third inequality follows $\alpha > 1/2$ and $\alpha' < \alpha$ and H strictly increasing. Consider bidder

[1] If $\alpha = 1/2$, and $\alpha' < \alpha$, then $1 - \alpha' > \alpha$ and the proof is trivial

(v', w). Since, $\alpha' \in \alpha(v', w)$

$$(v' - \alpha w)H(\alpha w) + (v' - (1-\alpha)w)H((1-\alpha)w) \\ \leq (v' - \alpha' w)H(\alpha' w) + (v' - (1-\alpha')w)H((1-\alpha')w) \tag{A.2}$$

Subtracting equation $A.1$ from equation $A.2$,

$$(v' - v)(H(\alpha w) + H((1-\alpha)w) \leq (v' - v)(H(\alpha' w) + H((1-\alpha')w)$$

$$\implies H(\alpha w) - H(\alpha' w) \leq H((1-\alpha')w) - H((1-\alpha)w)$$

The second inequality is true since $v' > v$. The last inequality gives the necessary contradiction.

A.2 Proof of Lemma 3.2

(i) $\alpha(\cdot, w)$ is non-decreasing, following lemma 3.1. I show that once a bidder-type splits his budget unequally, then for all bidder-types with higher valuations and same budget level, the splits are more unequal.

Suppose there exists $v < v'$ such that $\alpha = \sup\{\alpha(v, w)\} = \inf\{\alpha(v', w)\}$. Since $\alpha(\cdot, w)$ is non-decreasing; for all bidders with $\hat{v} \in (v, v')$ and the same budget w, $\alpha(\hat{v}, w) = \alpha$.

Consider the bidder (v, w). Since α is a best response for this bidder, the FOC in equation 3.2 apply. Now, consider bidders $(\hat{v}, w)$. Replacing v, with $\hat{v}$, in equation 3.2,

$$(\hat{v} - \alpha w)h(\alpha w) - H(\alpha w) > (\hat{v} - (1-\alpha)w)h((1-\alpha)w) - H((1-\alpha)w) \tag{A.3}$$

where the inequality follows from $\hat{v} > v$ and $h(\alpha w) > h((1-\alpha)w)$. This implies, that by increasing α, the bidder $(\hat{v}, w)$ could increase his payoff, a contradiction to $\alpha(\hat{v}, w) = \alpha$.

(ii) Since for almost all bidders there exists unique split of their budget, I need to show this property of $\alpha(v, \cdot)$, locally. [2] Consider a bidder $(v, w) \in (\underline{v}, \bar{v}) \times (\underline{w}, \bar{w})$. This bidder submits the bid pair $(\alpha w, (1-\alpha)w)$, where $\alpha = \alpha(v, w)$ and satisfies.

$$(v - \alpha w)h(\alpha w) - H(\alpha w) = (v - (1-\alpha)w)h((1-\alpha)w) - H((1-\alpha)w) \quad \text{(A.4)}$$

The above equation implies $h(\alpha w) > h((1-\alpha)w)$. Since α is the maximizer it must also satisfy Second Order Conditions (SOC) [3]. The SOC are

$$(v - \alpha w)h'(\alpha w) - 2h(\alpha w) \leq -(v - (1-\alpha)w)h'((1-\alpha)w) + 2h((1-\alpha)w) \quad \text{(A.5)}$$

The left hand side of the SOC is the second order change in the expected payoff of the bidder (v, w) if he were to bid a little more in the auction in which he is placing the higher bid and the right hand side is the change from bidding more in the other auction. Since the FOC imply that first order effects are equal; while determining bids when the budget levels change locally, SOC are used. Clearly, when the budget levels go up, the bidder would first want to increase the bid in the auction in which he placed the lower bid. Similarly, when budget level goes down, using equation $A.5$

[2] Also, given Lemma 3.3, which does not use this property; this is indeed sufficient

[3] Since I have proved FOC are applicable everywhere, one can use them to say that the first order effects are equal. While I do not prove the differentiability of the bid density everywhere (I know it is differentiable almost everywhere) all we need is the one sided second derivatives of H to exist. This is true since H is continuously differentiable

with the reverse signs, (as this gives the change in payoffs from reducing bids), the bidder first reduce the bid in the auction with the higher bid.

This implies that as w increases, the lower bid would rise first (weakly) and the split reduces, weakly. The opposite effect takes place as w decreases.

(iii) The proof of this part is similar to the previous. Essentially, the idea is that for small increases (decreases) in budget, $\alpha(v, \cdot)$ can not be too 'far' away.

A.3 Proof of Lemma 3.3

Continuity of $\alpha(\cdot, w)$: Recall, in Lemma 3.2 , I have already shown $\alpha(\cdot, w)$ is non-decreasing, and strictly increasing if $\alpha(\cdot, w) > 1/2$. Clearly, if $\alpha(\bar{v}, w) = 1/2$, then the result trivially follows from 3.2. This also implies, if there exists $v \in (\underline{v}, \bar{v})$, such that $\alpha(v, w) = 1/2$, then $\alpha(\cdot, w)$ is continuous over $[\underline{v}, v]$, as it is the constant function. What remains to be shown is continuity of $\alpha(\cdot, w)$ over $(v', \bar{v}]$, where $v' = \sup\{v \mid \alpha(v, w) = 1/2\}$. Let v' be so defined.

$\alpha(\cdot, w)$ is upper semi-continuous, due to theorem of maximum. From lemma 3.2, , $\alpha(\cdot, w)$ is strictly increasing for $v \in (v', \bar{v}]$. If I prove, for all $v \in (v', \bar{v}]$, $\alpha(v, w)$ is a singleton, then the proof is complete. Suppose to the contrary, there exists v such that $\alpha', \alpha'' \in \alpha(v, w)$ and $\alpha' < \alpha''$, without loss of generality. From lemma 3.2, for all $v' > v$, $\alpha(v', w) \geq \alpha''$ and for all $v' < v$, $\alpha(v', w) \leq \alpha'$.

Consider the FOC for v. Since $\alpha', \alpha'' \in \alpha(v, w)$, they both must satisfy the following equality.

$$v = \frac{\alpha w h(\alpha w) - (1-\alpha) w h((1-\alpha)w) + H(\alpha w) - H((1-\alpha)w)}{h(\alpha w) - h((1-\alpha)w)} \tag{A.6}$$

Taking derivatives I get the slope of the right hand side as

$$\frac{aw^2h'(\alpha w)+(1-\alpha)w^2h'((1-\alpha)w)+2(h(\alpha w)+h((1-\alpha)w))}{h(\alpha w)-h((1-\alpha)w)}- \tag{A.7}$$

$$\frac{(\alpha wh(\alpha w)-(1-\alpha)wh((1-\alpha)w))(wh'(\alpha w)+wh'((1-\alpha)w))}{(h(\alpha w)-h((1-\alpha)w))^2}+$$

$$\frac{(H(\alpha w)-H((1-\alpha)w))(wh'(\alpha w)+wh'((1-\alpha)w))}{(h(\alpha w)-h((1-\alpha)w))^2} \tag{A.8}$$

Substituting the FOC into the above,

$$\frac{w\big(2(h(\alpha w)+h((1-\alpha)w))-((v-\alpha w)h'(\alpha w)+(v-(1-\alpha)w)h'((1-\alpha)w))\big)}{h(\alpha w)-h((1-\alpha)w)}>0 \tag{A.9}$$

where the inequality follows from $h(\alpha w)-h((1-\alpha)w)>0$, and the second order condition for α, being a maximizer from equation $A.5$. The above equation must be satisfied by both α' and α''. But if, $\alpha''>\alpha'$, it must imply that there exists $\hat{\alpha}\in\alpha(v,w)$, $\hat{\alpha}\neq\alpha',\alpha''$. However it can not be the case that equation $A.9$ is satisfied by $\hat{\alpha}$ as well since the left hand side of equation $A.9$ is a continuous function.

Continuity of $\alpha(v,\cdot)$. Uniqueness of $\alpha(v,w)$ follows from the previous part. From upper-semi continuity of the $\alpha(\cdot,\cdot)$, continuity of $\alpha(v,\cdot)$ follows.

A.4 Proof of Lemma 3.6

(i) I prove the right continuity of $H(\cdot)$. Left continuity can be proved analogously.

First, suppose there exists a $b\in[\underline{b},\bar{b})$ such that there exists a $b_n\to b$ such that $\lim_{n\to+\infty}H(b_n)=b'>H(b)$. So, there, is a distinct 'jump' in the bid distribution at b. Such a jump can only take place if and only if, there exists a positive mass of bidders, who place the bid b in the auctions with positive probability. In other words, there must exist a set $M = [v_l, v_h] \times [w_l, w_h]$[4], such that for all $(v, w) \in M$, $\sigma(v, w, b, b_2) > 0$ with some b_2.

Consider bidders such that $(v, w) \in (\underline{v}, \bar{v}] \times (b, \bar{w}]$, i.e. they have enough budget to place bid $b + \epsilon$ in one of the auctions. This is a profitable deviation within one auction. The only reason a bidder with $w \in (b, \bar{w}]$ would not to want to submit $b + \epsilon$ is because such a deviation would cause a loss in the other auction which would be greater than the gain in the present auction. For such an occurrence there must be another point of discontinuity in the bid distribution. However, there can only be a countable set of discontinuities in the bid distribution[5]. This implies, that the set of budget levels for which the bids of $(b, w - b)$ are both points of discontinuity is countable. Therefore the set of bidders who would be willing play b with positive probability has measure zero.

Suppose, $b = \bar{b}$. If $\bar{w} > \bar{b}$, then the previous argument would apply. We are left with the case, $\bar{w} = \bar{b}$. However the only bidders who can place a bid equal to the highest budget level are precisely the bidders with the highest budget level, a set with mass zero.

(ii) The proof of this part is standard. Suppose there exists $b > b'$, and $H(b) = H(b')$, $\sigma(v, w, b_1, b_2) = 0$, for all (v, w) and $b_1 = \hat{b}$ or $b_2 = \hat{b}$, where $\hat{b} \in (b, b']$. This is so, as any bidder bidding $\hat{b}$ can increase his payoff by bidding b, which does

[4]this set does not have to be closed. It only needs to have the property that $v_l \neq v_h$ and $w_l \neq w_h$

[5]The set points of discontinuity of a real monotone function is countable.

not affect the probability of winning but reduces the payment. Since H does not have mass points, there would exist a neighborhood of b' such that any bidder submitting bids in $b' + \epsilon$ for ϵ small, would prefer to deviate to b causing a mass point at b.

(iii) Essentially, I need to show that the for any $b \in (\underline{b}, \bar{b})$, there exists a bidder (v, w), who submits b with positive probability. Suppose not. Since b is contained strictly within the interval spanned by the highest and the lowest bid, $b < \bar{w}$.

Since no bidder submits b, then no bidder with the budget level $w \geq b$ submits $(w - b)$. Let $w_l = \max\{b, \underline{w}\}$ and $w_h = \min\{\bar{w}, \bar{b}\}$. Clearly, $w_l < w_h$ and $(w_l, w_h) \subset (\underline{b}, \bar{b})$. However, no bidder would place a bid in (w_l, w_h), a contradiction to H strictly increasing.

(iv) I prove differentiability in three steps. First proving that the slope of the bid distribution (left and right hand) are positive. Second, showing that they are finite. And finally, that for all interior points in the support of the bid distribution the right and the left hand slope are equal.

Lemma A.1. *Consider $b \in (\underline{b}, \bar{b})$. For any $b_n \uparrow b$ and for any $b_m \downarrow b$*

$$\lim_{n\to\infty} \frac{H(b) - H(b_n)}{b - b_n} > 0 \text{ and } \lim_{m\to\infty} \frac{H(b_m) - H(b)}{b_m - b} > 0$$

Proof. Suppose, to the contrary, there exists, $\{b_n\}_{n=1}^{\infty}$, such that $b_n \uparrow b$ and

$$\lim_{n\to\infty} \frac{H(b) - H(b_n)}{b - b_n} = 0$$

For simplicity, I assume, $\{b_n\}_{n=1}^{\infty}$ is a strictly increasing sequence. Consider a bidder (v, w), who submits b. That is $\alpha(v, w)w = b$ or $(1 - \alpha(v, w))w = b$. I consider the former case, however this is without loss of generality. Since this bidder must be playing his best responses, for all $b_n \in \{b_n\}_{n=1}^{\infty}$

$$(v-b)H(b)+(v-(1-\alpha(v,w))w)H((1-\alpha(v,w))w) \geq$$
$$(v-b_n)H(b_n)+(v-(1-\alpha(v,w))w)H((1-\alpha(v,w))w) \qquad \text{(A.10)}$$

Rearranging the equation, and taking limits as $b_n \to b$

$$(v-b)\lim_{n\to\infty}\frac{(H(b)-H(b_n))}{b-b_n} \geq \lim_{n\to\infty} H(b_n) \qquad \text{(A.11)}$$

where the left hand side is zero and the right hand side equals $H(b) > 0$. Hence I arrive at the desired contradiction. The intuition for the proof is simple. If the left hand slope of H at b is zero this implies that any bidder, bidding b would strictly prefer to deviate to a slightly smaller bid. This deviation is affordable, reduces his payment in case he wins and does not affect the probability of winning an object with a lower bid, too greatly.

In similar fashion one can prove that the right hand side, slope at any b is positive. The difference in the proof is that instead of considering a bidder who bids b in equilibrium one has to consider a sequence of bidders (v_m, w_m) who play $\{b_m\}$ in equilibrium. One can show that for large M, a bidder (v_M, w_M) would prefer to deviate and bid b. $\square$

Lemma A.2. *Consider $b \in (\underline{b}, \bar{b})$. For any $b_n \uparrow b$ and for any $b_m \downarrow b$*

$$\lim_{n\to\infty}\frac{H(b)-H(b_n)}{b-b_n} < \infty \text{ and } \lim_{m\to\infty}\frac{H(b_m)-H(b)}{b_m-b} < \infty$$

Proof. I consider the case $b_n \uparrow b$. Suppose,

$$\lim_{n\to\infty} \frac{H(b) - H(b_n)}{b - b_n} = \infty$$

That is the left hand slope of H at b is infinite. Consider a bidder (v, w) who has b as one of his bids. Since $\alpha(v, w)$ is unique for almost all (v, w), take $\alpha = \alpha(v, w)$ and $b = \alpha w$. From Lemma 3.2, $\alpha(\cdot, w)$ is non-decreasing. Hence for all $v' < v$, $\alpha(v', w) \leq \alpha$. Consider v' close to v. Their equilibrium split of budget is close to α. However, reducing their bid a little from b, to $b - \epsilon$, reduces their payoff by a large amount. Hence for some interval $(\hat{v}, v)$, $\alpha(v', w) = \alpha$.

By a similar reasoning, there exists a neighborhood of the budget level w, such that for all $(v', w') \in (\hat{v}, v) \times (\hat{w}, w)$, $\alpha(v', w')w' = b$. The reason is that when the budget of bidders with valuations $v' \in (\hat{v}, v)$, is reduced from w to some slightly lower budget level, these bidders would not lower their bid of b as this leads to a large loss in their payoff.

As a result there exists a mass point in the bid distribution at b, which is a contradiction.

In this sketch I assumed $\alpha(v, w)$ to be unique. If this is not the case, then due to Lemma 3.6, there exist a set of bidders who's $\alpha(\cdot, \cdot)$ is close to α, and a similar treatment would apply. □

What remains to be shown is that they are equal.

Lemma A.3. *Consider $b \in (\underline{b}, \bar{b})$. For any $b_n \uparrow b$ and for any $b_m \downarrow b$*

$$\lim_{n\to\infty} \frac{H(b) - H(b_n)}{b - b_n} = \lim_{m\to\infty} \frac{H(b_m) - H(b)}{b_m - b}$$

Proof. For ease of notation, let $h_-(b)$ be the left hand slope of H at b. This is given by the left hand side of the equation in the statement of the lemma and similarly $h_+(b)$. Suppose to the contrary, $h_-(b) < h_+(b)$. Let (v, w) be the bidder who plays b with positive probability. That is $\alpha(v, w)w = b$ or $(1 - \alpha(v, w))w = b$. There are two cases to be analyzed.

Case (i) Suppose $\alpha(v, w) = 1/2$. In this case the bidder plays b in both auctions and hence $b = w/2$. It easy to see that this bidder would deviate to bidding $(b + \epsilon, b - \epsilon)$, since by doing so the bidder defeats an additional mass of people in the auction in which he bids more which is greater than the mass of people he loses to in the auction he bids less.

Formally, consider the bid pair $(b+\epsilon, b-\epsilon)$. If the bidder by bidding $\alpha(v, w) = 1/2$ is playing a best response, then

$$(v - b)H(b) + (v - b)H(b) \geq (v - b - \epsilon)H(b + \epsilon) + (v - b + \epsilon)H(b - \epsilon)$$

Rearranging, dividing by ϵ and taking limits as $\epsilon \to 0$

$$\begin{aligned}
&(v - b) \lim_{\epsilon \to 0} \frac{(H(b) - H(b - \epsilon))}{\epsilon} + \lim_{\epsilon \to 0} H(b - \epsilon) \\
&\geq (v - b) \lim_{\epsilon \to 0} \frac{H(b + \epsilon) - H(b)}{\epsilon} + \lim_{\epsilon \to 0} H(b - \epsilon) \qquad \text{(A.12)} \\
\implies &(v - b)h_-(b) + H(b) \geq (v - b)h_+(b) + H(b)
\end{aligned}$$

The last inequality implies $h_-(b) \geq h_+(b)$, a contradiction.

Case (ii) $\alpha(v, w) > 1/2$. Suppose, $\alpha(v, w)w = b$. So, this bidder submits the bid pair $(b, w - b)$ with $b > w - b$. [6]. Since this bidder must be playing a best response

[6]Similarly, the case $(1 - \alpha(v, w))w = b$ can be proved

in equilibrium, for all $\epsilon > 0$

$$(v-b)H(b) + (v-w+b)H(w-b)$$
$$\geq (v-b+\epsilon)H(b-\epsilon) + (v-w+b-\epsilon)H(w-b+\epsilon) \tag{A.13}$$
$$(v-b)H(b) + (v-w+b)H(w-b)$$
$$\geq (v-b-\epsilon)H(b+\epsilon) + (v-w+b+\epsilon)H(w-b-\epsilon) \tag{A.14}$$

That is, submitting a different bid pair such as increasing the bid in one auction and reducing the bid on the other must weakly lower the bidder's payoffs. Rearranging, equation $A.13$ the equations, dividing both sides by ϵ and taking limits as ϵ goes to zero,

$$\begin{aligned}
&(v-b)\lim_{\epsilon\to 0}\frac{(H(b)-H(b-\epsilon))}{\epsilon} - \lim_{\epsilon\to 0} H(b-\epsilon) \\
&\geq (v-w+b)\lim_{\epsilon\to 0}\frac{H(w-b+\epsilon)-H(w-b)}{\epsilon} - \lim_{\epsilon\to 0} H(w-b+\epsilon) \\
\implies &(v-b)h_-(b) - H(b) \geq (v-w+b)h_+(w-b) - H(w-b)
\end{aligned} \tag{A.15}$$

Carrying out a similar procedure with $A.14$,

$$(v-b)h_+(b) - H(b) \leq (v-w+b)h_-(w-b) - H(w-b) \tag{A.16}$$

Using $A.15$ and $A.16$ and $h_-(b) < h_+(b)$, I get a number of inequalities which help describe behavior of bidders in the neighborhood of (v, w).

1. $h_-(b) > h_+(w-b)$. Since $b > w-b$, by assumption and H is strictly increasing.
2. $h_-(w-b) > h_+(w-b)$. Notice, since $h_-(b) < h_+(b)$, $A.15$ and $A.16$ imply

$$(v-w+b)h_-(w-b) - H(w-b) > (v-w+b)h_+(w-b) - H(w-b)$$

which gives the desired strict inequality. This is an important consequence of H not being differentiable at b. It implies, that H must also not be differentiable at $w - b$.

3. The right hand slope of the payoff of bidder (v, w) at b is greater than the right hand slope at $w - b$. This can be seen from $A.15$ and $A.16$. Similarly, the left hand slope at b is less than the left hand slope at $w - b$.

$$(v - b)h_+(b) - H(b) > (v - w + b)h_+(w - b) - H(w - b) > 0 \tag{A.17}$$

$$0 < (v - b)h_-(b) - H(b) < (v - w + b)h_-(w - b) - H(w - b) \tag{A.18}$$

The strict inequalities with respect to zero follow from the earlier equations. Equation $A.17$, implies, that for a bidder with value v, increasing the bid b a little bit gives a strictly higher payoff than increasing the bid $(w - b)$. And similarly equation $A.18$ implies, if this bidder had to lower one of his bids, he would prefer to lower b since, by doing so the loss would be smaller as compared to lowering $(w - b)$.

Formally, consider a bidder (v, w'), $w' = w + \epsilon$. Notice, since the argmax is upper semi continuous, and it is unique for almost all bidder-types, for any neighborhood of $(b, w - b)$, one can find a neighborhood of (v, w), such that their best responses are within the set. And any neighborhood of (v, w) contains bidders of type (v, w'). [7]

Suppose, (v, w') submits the bid pair $(b', w' - b')$, with $b' > w' - b'$, since $(b', w' - b')$ must be in a neighborhood of $(b, w - b)$. Equations $A.16$ and $A.17$ imply,

---

[7]Unless $w = \bar{w}$, but in that case one can consider bidders with lower budget levels and the proof follows

$b' \geq b$, and $w' - b' \geq w - b$. Also, equation $A.17$ would imply, for ϵ small, $b' = b + \epsilon$ and $w' - b' = w - b$. Conversely, if $w' = w - \epsilon$, equation $A.18$ would imply, $b' = b = \epsilon$ and $w' - b' = w - b$. Either way, this shows that for a neighborhood of w, given by $(w - \epsilon, w + \epsilon)$, for all bidder-types, (v, w'), $w' \in (w - \epsilon, w + \epsilon)$, one of their bids is always $w - b$.

Let $b' = w' - w + b$, be the higher bid of these bidders. For all $b' \in (b - \epsilon, b + \epsilon)$, $(b', w - b)$ is played by (v, w') in equilibrium. Since, H is a strictly increasing continuous function, it can only have a countable set of points of non-differentiability. Hence for almost all $b' \in (b - \epsilon, b + \epsilon)$, $h_-(b') = h_+(b')$. Collect all points of differentiability in $(b - \epsilon, b + \epsilon)$, and call it $D(b)$. $D(b)$ must be dense in $(b - \epsilon, b + \epsilon)$. If $b' \in D(b)$, then

$$(v - b')h_-(b') - H(b') = (v - b')h_+(b') - H(b') \tag{A.19}$$

Since (v, w'), must be playing best responses, using a similar technique as used previously,

$$(v - b')h_-(b') - H(b') \geq (v - w + b)h_+(w - b) - H(w - b) \tag{A.20}$$

$$(v - b')h_+(b') - H(b') \leq (v - w + b)h_-(w - b) - H(w - b) \tag{A.21}$$

I now show that there exists a neighborhood of v, such that for all $v' \in (v - \delta, v + \delta)$ the bidder-types (v', w'), also bid $(b', w - b)$. This is easy to see if both the above inequalities were strict. If this were true, by changing v to some v', the inequalities remain strict and hence, the bidder would not increase or decrease either of his bids.

I have shown earlier, that $h_-(w - b) > h_+(w - b)$. And since $h_-(b') = h_+(b')$, one of the above equations, must be a strict inequality. Without loss of generality, suppose $A.20$ is strict and $A.21$ is an equality. This implies $h_+(b') > h_-(w-b)$. Consider $v' \in (v, v-\delta)$ and δ small.

$$(v'-b')h_-(b') - H(b') > (v'-w+b)h_+(w-b) - H(w-b) \tag{A.22}$$

$$(v-b')h_+(b') - H(b') < (v'-w+b)h_-(w-b) - H(w-b) \tag{A.23}$$

Where the first equation follows from $A.20$ being strict and δ small and the second from $h_+(b') > h_-(w-b)$. Therefore (v', w'), $v' \in (v, v-\delta)$ would submit $(b', w-b)$. Since, $D(b)$ is dense, and we can find a neighborhood such as $(v, v-\delta)$ for each $b' \in D(b)$, the set of bidders submitting the bid $w-b$ has positive mass, which gives H a mass point at $w-b$, a contradiction.

The case, $h_-(b) > h_+(b)$, can be done in a similar fashion. All though I have assumed $\alpha(v, w)$ to be singleton, even if it wasn't, one can show there always is a set of bidders, with the same value and lower or higher budget levels who would bid in a neighborhood of α. $\square$

The density of a differentiable cumulative distribution function can not have 'jump discontinuities'. Also, since the bid distribution has a finite slope everywhere, it can not have asymptotic discontinuity. Therefore the only type of discontinuity it can have are point discontinuities. Hence it is continuously differentiable almost everywhere.

Since I use second order conditions for some of the proofs, I need to show that H is twice differentiable almost everywhere. Using a Taylor-series expansion of the payoff function, and applying the same technique as in the proof of differentiability, one can show the differentiability of the bid density function, almost everywhere.

A.5 Proof of Lemma 3.7

(i) The proof follows from the non-existence of equillbria where $\alpha(v,w) = 1/2$ for all $(v,w) \in T$.

(ii)Since $\underline{b} < \underline{w}/2$, there must exist a bidder $(v',w), w \geq \underline{w}$ for whom $(1 - \alpha(v',w))w = \underline{b}$. Clearly, since $w \geq \underline{w}$, $\alpha(v',w) > 1/2$. Since $\alpha(v,\cdot)$ is non-increasing, $\alpha(v,\underline{w}) \geq \alpha(v,w)$. This implies, $(1-\alpha(v,\underline{w})\underline{w} \leq (1-\alpha(v,w))w = \underline{b}$. But $(1 - \alpha(v,\underline{w}))\underline{w} < \underline{b}$ would be a contradiction as $\underline{b}$ is the lowest bid. Therefore $\alpha(v,\underline{w})\underline{w} = \underline{b}$.

(iii) The proof of this part follows from the previous part and the fact that $\alpha(\cdot,w)$ is non-decreasing.

(iv) If $\bar{b} = \bar{w}/2$, the result is trivial. If $\bar{b} > \bar{w}/2$ then the proof follows from part (ii) and part (iii) of 3.2.

A.6 Proof of Lemma 3.9

Suppose to the contrary, $w = L(v) < L(v') = w'$. That is $\alpha(v,w) = 1/2$. Since $\alpha(\cdot,w)$ is non-decreasing, this implies, $\alpha(v',w) = 1/2$. Therefore $w \geq L(v')$, which is contradiction.

Suppose $L(v') > \underline{w}$. I have already shown,$L(v) \geq L(v')$. Suppose $L(v) = L(v') = w'$. Then for all $\hat{v} \in [v',v]$, $\alpha(\hat{v},w') = 1/2$, since $\alpha(\cdot,w')$ is non-decreasing. Also, for all $w < w'$, $\alpha(\hat{v},w) > 1/2$ due to lemma 3.2 and the definition of $L(\cdot)$.

As noted earlier, $\alpha(v,w) = 1/2$ is always an optima. Therefore for any bidder

$(\hat{v}, w')$, the SOC in equation $A.5$, must hold for $\alpha = 1/2$. Substituting in equation $A.5$,

$$(\hat{v} - \frac{w'}{2})h'(\frac{w'}{2}) - 2h(\frac{w'}{2}) \leq 0$$

Notice, that there must exist a subset of $[v', v]$, such that the inequality is strict. Suppose to the contrary for for all $\hat{v} \in [v', v]$, the above equation equaled zero. One can reach a contradiction for any value of $h'(\frac{w'}{2})$, since $(\hat{v} - \frac{w'}{2}) > 0$ and $2h(\frac{w'}{2}) > 0$. Without loss of generality, I can assume that this subset is in fact (v', v) [8]. Re-writing the SOC for such bidders,

$$(\hat{v} - \alpha w)h'(\alpha w) - 2h(\alpha w) < -(\hat{v} - (1-\alpha)w)h'((1-\alpha)w) + 2h((1-\alpha)w)$$

for $\alpha = 1/2$. Like in Lemma 3.2, the left hand is the second order change in the expected payoff of the bidder $(\hat{v}, w)$ if he were to bid a little more in the auction in which he is placing the higher bid and the right hand side is the change from bidding more in the other auction. Since the right hand side is large and the first order effects are equal, for a bidder $(\hat{v}, w)$, where $w = w' - \epsilon$, and ϵ is small, it is better to strictly reduce the higher bid first and bid the same in the other auction, i.e. $w - b$. This is true for all $\hat{v} \in [v', v]$, which implies there is a mass point at $w - b$, a contradiction.

A.7 Examples of $G(\cdot)$

Consider a $G(\cdot)$ which is concave. Then, if $g(\bar{w}) > \bar{w}/2$, all bidders expend their budget. In other words, as long as the mass of bidders near the highest budget level is not 'too low' then every bidder-type prefers expending his budget.

[8] One can work with a smaller set. All one needs is the set to be an interval

If $G(\cdot)$ is uniform and $\bar{w}/2 < \underline{w}$ then all bidders expend their budget.

These examples show cases where bidders expend their entire budgets. However it is my conjecture that as long as $\underline{v} > \bar{w}$, with high probability (in terms of primitive space) bidders would expend their budgets. The intuition is as follows. If the distribution had considerable mass near the lower end of the budget-support then these bidders would expend their budgets as they know that there is large mass of bidders with similar low budgets. Then the bidders who may not expend their budgets are the very 'rich', which constitute a smaller mass as under the concave distribution. If on the other hand most of the mass is concentrated around the upper end of the budget support then bidders know that with a high probability they compete against 'rich' bidders and hence in order to beat them need to expend their budgets.

APPENDIX B
ADDITIONAL FIGURES AND TABLES

Table B.1: December 1970 Auction

No. of Tracts	No. of Bidders	No. of Bids	Median Bid	Mean Bid	Mean High Bid	Mean Expost Value
127	58	1043	\$3,482,766	\$6,245,821	\$15,177,000	\$19,597,270

I present sample statistics about tract valuations of productive tracts. A tract is productive if it was drilled for oil and oil was found. I also provide ratio of tracts hit to tracts drilled for each auction. Tract valuations are given by

$$v = \text{Total Revenue - Costs - Royalty payments} \tag{B.1}$$

Table B.2: Tract Valuations across Auctions.

Auction	Hit Ratio	Mean v (all tracts)	Mean v (productive tracts)	Median v (productive tracts)
1954-Oct	0.69	6,994,883	34,823,805	18,616,467
1954-Nov	0.40	243,551	10,758,156	10,758,156
1955-Jul	0.42	21,55,707	25,791,836	14,221,833
1960-Feb	0.52	18,363,903	66,172,716	36,251,909
1962-Mar	0.48	11,172,410	47,200,788	22,257,408
1962-Mar	0.47	9,693,390	42,815,374	20,214,567
1967-Jul	0.41	10,549,570	58,678,295	30,273,240
1968-Jun	0.23	-869,242	9,130,089	7,544,201
1970-Dec	0.57	19,597,270	60,825,122	30,156,846

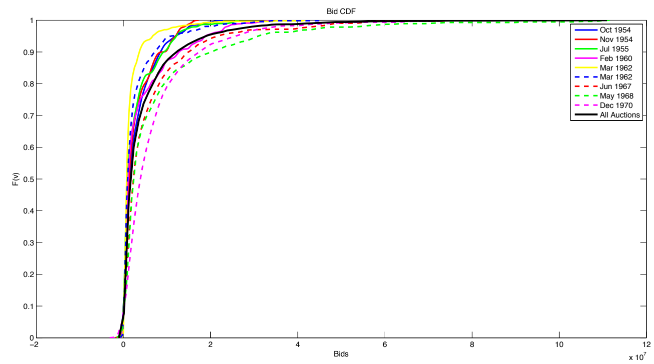

Figure B.1: Bid Distributions for All Auctions

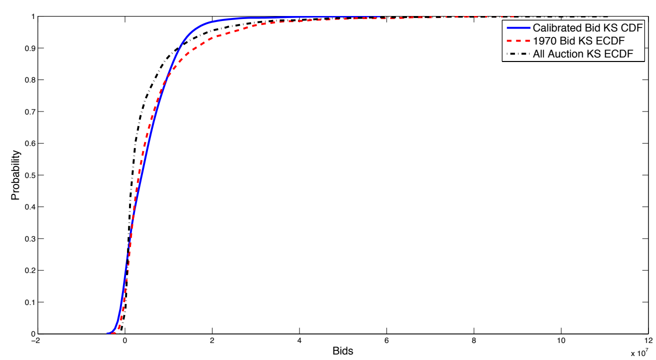

Figure B.2: Bid Distributions

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