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Contests: uncertainty and budgets

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CONTESTS: UNCERTAINTY AND BUDGETS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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ABSTRACT

This dissertation adds to the current understanding of contests. Contests are a class of games in which players compete for a prize be expending resources. Some portion of the resources expended cannot be recuperated, even in the event of a loss. Each chapter extends standard models to incorporate realistic features such as nonprobabilistic uncertainty, budgets, dynamics, or intermediate outcomes.

Chapter 1 introduces ambiguity aversion to the all-pay auction and war of attrition. Increasing ambiguity causes weak types to bid lower and strong types to bid higher, in the all-pay auction. In the war of attrition, ambiguity can uniformly decrease the bids. A revenue ranking for the all-pay auction, war of attrition, and standard sealed bid auctions is provided. These results are consistent with much of the experimental literature.

Chapter 2 continues the discussion of ambiguity aversion. The main result is a characterization of the set of increasing equilibria in games like the all-pay auction and war of attrition. Unlike with subjective expected utility, even when beliefs are independent of type, an increasing equilibrium may not exist. Sufficient conditions are provided for such an equilibrium to exist.

Chapter 3 models endogenous budgets in sequential elimination contests. Contestants depend on a strategic group of players to provide resources that will be spent in the contest. We analyze the effect of timing and spending rules on aggregate spending. When budgets are not replenished between stages, spending is higher. When

unspent resources are refunded, total spending is higher than when all spending is a sunk costs.

Chapter 4 introduces an all-pay auction game with an intermediate outcome between winning and losing. When bids are sufficiently different, the player with the highest bid wins a prize, and the other player receives nothing . When bids are close, the outcome is called a tie, and each player receives an intermediate prize. Ties are common in sports, political competition, and war. Equilibrium is characterized for a set of parameters where the tying region is relatively large.

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CHAPTER 1
AMBIGUITY AVERSION AND ALL-PAY AUCTIONS

1.1 Introduction

I introduce ambiguity aversion to the all-pay auction and war of attrition with incomplete information. This paper fits into a large literature, beginning with Ellsberg[32], that demonstrates that ambiguity aversion can explain observations that cannot be explained by standard models with subjective expected utility(SEU). In the finance and macroeconomics literature, ambiguity aversion, has been shown to solve many puzzles regarding asset prices, notably the equity premium puzzle.[1] Mukerji[87] has shown that ambiguity aversion can explain incompleteness of contracts in situations where costly contracting alone gives counter-factual predictions. Kagel and Levin[56] proposed ambiguity aversion as one explanation for overbidding in first-price auctions. Here I show that ambiguity aversion can explain several results in the experimental literature on all-pay auctions and the war of attrition.

The all-pay auction[90] and war of attrition[79] have been used to model a wide variety of strategic environments in which players compete for a prize by expending resources or effort. Environments modeled by a war of attrition include firms competing to determine industry standards[25], firms exiting a crowded market[38], labor strikes[58], and provision of a public good[20]. Some online auctions also share similar qualities with the war of attrition[95]. Some applications of the all-pay auc-

[1] See Guidolin and Rinaldi[44] for a recent survey. See also Hansen and Sargent[45].

tion are political competition[11], litigation[14], and students competing for college admissions[49].

Ambiguity is likely to be present in many environments modeled by the all-pay auction and war of attrition. In these games, players are uncertain about how much another player values a prize. In the SEU model, a player's uncertainty is modeled as a distribution over the other players' values called a belief distribution. The player's interim utility is the expected utility calculated using that distribution. However, in many applications the distribution of the other players' values is hard to learn either because the environment is changing or because of a lack of experience. In the above mentioned applications, it seems unlikely that players can be certain of the distribution of the other players' values.

I find that ambiguity aversion has a significant impact on the outcome of these games. I provide revenue rankings between the all-pay auction, war of attrition, first-price, and second-price auction. With MEU, players expend more resources in the first-price auction than in the all-pay auction. Under conditions that insure the existence of an increasing equilibrium, players expend more in the all-pay auction than in the war of attrition. Players expend less in the war of attrition than in the second-price auction under stronger conditions on the form of ambiguity. I also provide comparative statics regarding how the degree of ambiguity aversion affects strategies. I show that ambiguity decreases bids for low values in the war of attrition. I also provide conditions under which all types bid less. In the all-pay auction, players with low values bid less and players with high values bid more as ambiguity increases.

These observations are also consistent with experimental observations.

The pattern of over and underbidding in the all-pay auction is consistent with a number of experimental papers including Barut, Kovenock, and Noussair[10] and Noussair and Silver[91]. In addition, Hörisch and Kirchkamp[54] find prevalent underbidding relative to the benchmark in the war of attrition which is also consistent with MEU. Hörisch and Kirchkamp[54] also find that the war of attrition generates less expenditure than the all-pay auction. The revenue ranking between the first price and all-pay auction is not confirmed by the experiments. Barut, Kovenock, and Noussair[10] did not find any difference in average revenues between the two types of games. On the other hand, in early rounds of the experiments Noussair and Silver[91] find that high types overbid to such a degree that average expenditure in the all-pay auction is far above that of the first price auction.

1.1.1 Related Literature

In his seminal work, Lo[74] applied the maxmin expected utility(MEU) model to the first-price auction where he derived the unique increasing equilibrium. Following Lo[74], I model ambiguity using the maxmin expected utility model of Gilboa and Schmeidler[41]. With MEU, beliefs are modeled by a set of distributions, Δ, of which any one may generate the other players' values. The player's interim utility is the lowest expected utility generated by any distribution in Δ. Given the other players' strategies, each player chooses an action which maximizes the minimum expected utility. Thus an ambiguity averse player chooses an action which is robust to the worst case distribution.

The revenue equivalence theorem[88][98] implies that when players have SEU and independent beliefs the sum of expenditures in the games studied here is the same as in the first-price and second-price auctions. With a particular form of Δ, Lo[74] showed that the first-price auction generates more revenue than the second-price auction with MEU. Bodoh-Creed[21] showed that more generally no revenue ranking exists between the two auctions when players have MEU. The ranking that I provide for the all-pay auction and first-price auction holds quite generally; the ranking of the all-pay auction and war of attrition holds under a set of conditions that insure that an increasing equilibrium exists. Thus, these revenue rankings are the first general rankings for commonly used mechanisms.

It should be noted that these rankings are the reverse of what Krishna and Morgan[69] find with SEU. Their environment differs from this paper in two ways. They assume that players do not face ambiguity, and beliefs may depend on the player's type. Here players may be ambiguity averse but beliefs are assumed to be independent of type. Thus affiliation of beliefs and ambiguity aversion tend to push the revenue ranking in opposite directions.

The ranking between the first-price and all-pay auction complements the results of Fibich, Gavious, and Sela[35] who study the independent values environment with risk averse players who have SEU preferences. The over and underbidding found here also occurs with risk averse bidders. In the conclusion I discuss some differences between ambiguity aversion and risk aversion in first-price and all-pay auctions.

This paper is also related to the literature on mechanism design with ambiguity aversion. Bodoh-Creed[21] and Bose, Ozdenoren, and Pape[24] analyze abstract mechanism design problems in which agents may be ambiguity averse. In this paper, rather than characterizing the optimal mechanism, I compare some commonly observed mechanisms for allocating objects.

1.2 Model

Consider the environment of a mechanism that allocates m indivisible units of a good to $n > m$ players who can consume at most one unit. Player i values a unit at $v_i \in [\underline{v}, \overline{v}]$. Players compete for a prize by submitting a signal called a bid; player i's bid is $b_i \geq 0$. The allocation depends on the player's bids. The m players with the highest bid receive an object. In the event that there is a tie for the m-th highest bid, the players with the m-th highest bid may receive one of the remaining units with equal probability. The other players do not receive a unit. In addition, players must make a payment that depends on the bids. Player i's payment is given by $t_i : [0, \infty)^n \to [0, \infty)$.

The mechanisms can be split into two main categories, all-pay and winner-pay auctions. In a winner-pay auction, player i's utility $u_i(b_i, b_{-i}, v_i)$ is given by $v_i - t_i(b_i, b_{-i})$ if the player receives an object and zero, otherwise. Examples of winner-pay auctions include the first-price and second-price auctions. In an all-pay auction player i's utility is given by $v_i - t_i(b_i, b_{-i})$ if the player receives an object and $-t_i(b_i, b_{-i})$, otherwise. The all-pay auctions that are analogous to the first-price and second-price auctions are the standard all-pay auction and the war of attrition, respectively. A more detailed description of the mechanisms follows in Section 1.3.[2]

Behavior will be modeled as the equilibrium of a game of incomplete information. It is assumed that players know their own value for the prize when they submit their bid; however, this information is private and not known to the other players. This incomplete information is modeled as ambiguity using the MEU model of Gilboa and Schmeidler[41].

Let D be a set of distributions over $[\underline{v}, \overline{v}]$. Each player believes that the values of the other players are random draws from a distribution in D. For this reason, D is often called the player's ambiguous belief or a multiple prior belief. If G has a density, then g is the density of g, and in general lower case letters refer to the density of the corresponding distribution. Define a strategy for player i to be a measurable function, $s_i : [\underline{v}, \overline{v}] \to [0, \infty)$.[3] Player i's expected utility given that the other player's values are drawn from G is

$$U_i(b_i; v_i, G) = \int_{\underline{v}}^{\overline{v}} \cdots \int_{\underline{v}}^{\overline{v}} [u_i(b_i, s_{-i}(v_{-i}); v_i)] \prod_{j \neq i} dG(v_j) dG(v_1) \dots dG(v_{n-1}) \tag{1.1}$$

where $s_{-i}(v_{-i})$ is taken to be the vector of bids that the other players submit given that the vector of other players' types is v_{-i}.

Since players do not know which distribution generates the other players' values, players choose a bid that maximizes the expected utility for the worst case

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[2]These ex post utility functions is referred to as the private values risk neutral utility.

[3]Measures throughout this paper can be taken to be the Borel measure.

distribution. That is players maximize the minimum expected utility given by

$$V_i(b_i; v_i) = \inf_{G \in D} U(b_i; v_i, G). \tag{1.2}$$

Players are assumed to play Nash equilibrium strategies with respect to the MEU preferences.

Some additional notation will be useful throughout. For the mechanisms studied, a player's expected utility only depends on the distribution of the m-th highest value of the other n-1 players. Because of this it is often convenient to reformulate the problem in terms of the distribution of this order statistic.

$$\Delta = \{H : H \text{ is the distr. of the } m\text{-th highest draw of } n\text{-1 iid draws from } G \in D\} \tag{1.3}$$

The following assumption will insure that the infimum can be replaced by the minimum in the definition of MEU.

A. 1. The set Δ is a compact set of absolutely continuous distributions on $[\underline{v}, \overline{v}]$, with respect to the weak* topology.

I have assumed that the players' preferences are identical conditional on the value for the prize. In addition, I will focus on symmetric equilibrium in which all players use the same strategy. Because of this I will frequently omit the player subscript when referring to a generic player.

1.3 Equilibrium Analysis

In this section, I will derive the symmetric increasing equilibrium strategies for some commonly studied mechanisms. Increasing strategies are strategies which are a strictly increasing function of the player's type. First, I will analyze the all-pay auction(APA) and its winner-pay counterpart, the pay-as-bid auction(PBA). The analysis of these two auctions uses the method first employed by Lo[74] to study the first-price auction which is a special case of the PBA. Then I turn attention to the war of attrition(WOA) and Vickrey's auction[114]. The WOA can be studied in a way analogous to the PBA and APA provided that Δ satisfies an assumption regarding the worst case distribution.

1.3.1 All-pay Auction

The payment function of the APA is that each player pays its own bid whether the player receives a prize or not, $t_i(b_i, b_{-i}) = b_i$. When there is only one unit of a prize to allocate, this is sometimes called a first-price all-pay auction. The only difference from winner-pay or standard first-price auction is that the losers also pay their bids in the APA. In a symmetric increasing equilibrium a player wins a prize if at least n-m players have lower values than that player. The probability of winning a prize given a bid b is $H(s^{-1}(b))$ where H is the distribution of the m-th highest value of the n-1 other players, and s is the increasing strategy the other players are using. The player's utility given a bid is

$$V(b; v) = \min_{H \in \Delta} vH(s^{-1}(b)) - b. \tag{1.4}$$

It is easy to identify a distribution that minimizes the probability of winning given a bid. Let H_M be the lower envelope of the distributions in Δ. If $H \in \Delta$ minimizes the expected utility at a bid of b, then $H(s^{-1}(b))$ minimizes the probability of winning a prize. This implies that $H(s^{-1}(b)) = H_M(s^{-1}(b))$ where $H_M(v) \equiv \min_{H\in\Delta} H(v)$ for all $v \in [\underline{v}, \overline{v}]$. Thus the utility can be rewritten as

$$V(b; v) = vH_M(s^{-1}(b)) - b. \tag{1.5}$$

With this observation the equilibrium can be derived using standard methods. Taking the derivative of equation 1.5 yields

$$v_i \frac{1}{s'(s^{-1}(b))} h_M(s^{-1}(b)) - 1. \tag{1.6}$$

If s is a symmetric equilibrium strategy, then the derivative must be zero at the equilibrium bid $s(v)$ for almost all v. Thus s must solve

$$s'(v) = vh_M(v). \tag{1.7}$$

The initial condition is that the type with the lowest value bids zero. This follows because in a symmetric increasing equilibrium the lowest type wins with probability zero and must pay the bid. The solution to equation 1.7 is an equilibrium as is stated in the following lemma.

Lemma 1.3.1. *The increasing, symmetric equilibrium of the APA is given by*

$$s_a(v) = \int_{\underline{v}}^{v} th_M(t)dt. \tag{1.8}$$

The proof of the lemma follows standard argument once it is observed that the lowest probability of having one of the m highest values is given by H_M so that the utility can be rewritten as 1.5.

1.3.2 Pay-as-bid Auction

The PBA is similar to the APA in that the winner pays its bid, but players that do not win an object make no payment. In the single unit case, this auction is known as the first price auction. The first price auction has been studied in detail under ambiguity aversion[4]. The analysis of the PBA mirrors this previous work and the discussion above.

When other players use a symmetric, strictly increasing strategy $s(v)$ the player's utility is

$$V(b;v) = \min_{H \in \Delta} H(s^{-1}(b))(v-b). \tag{1.9}$$

Just as in the APA the distribution that minimizes the utility is the one which minimizes the probability of winning an object. As before, the minimizing probability is given by $H_M(.)$. So players choose a bid to maximize $H_M(s^{-1}(b))(v-b)$.

Using the same method as in the APA, the first order condition defines an equilibrium. The resulting differential equation is

$$s'(v) = \frac{h_M(v)}{H_M(v)}(v - s(v)). \tag{1.10}$$

This differential equation can be solved with the equilibrium initial condition $s(\underline{v}) = \underline{v}$.

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Lemma 1.3.2. *The increasing, symmetric equilibrium of the PBA is*

$$s_p(v) = v - \frac{\int_{\underline{v}}^{v} H_M(t)dt}{H_M(v)}. \tag{1.11}$$

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[4]See Lo[74]; Levin and Ozdenoren[71]; and Bodoh-Creed[21]

[5]See Milgrom and Weber[85] and Lo[74] for details.

1.3.3 War of Attrition

There are multiple ways to define the payment function for a multi-player war of attrition[20]. The war of attrition is often meant to model a dynamic environment in which players expend effort until they concede to their opponents. Once n-m players concede the remaining players receive a prize without expending any more effort. The payment function of the static analog is that the losers pay there own bid and the m players who receive a prize pay the highest losing bid or the $m+1$-th highest bid. In the single unit case this could be called second-price all-pay auctions since the winner would pay the second highest bid and the losers pay their own bids.

The bidder takes as given that the other bidders are playing the same increasing equilibrium strategy s. For the WOA the MEU of bidding b is given by

$$V(b;v) = \min_{H\in\Delta} vH(s^{-1}(b)) - \int_{\underline{v}}^{s^{-1}(b)} (s(t))\, h(t)dt - \big(1 - H(s^{-1}(b))\big)\, b. \tag{1.12}$$

The first part of the expression is the expected utility of the allocation. The second part is the expected value of winning payments, and the last part is the expected value of losing payments. Unlike in the APA and PBA, it is not so easy to identify the expected utility minimizing distribution. This is because distribution affects both the player's probability of winning and the payment in the event of winning.

In addition, the minimizing distribution will generally depend on the player's value for the prize. The player's value for the prize determines the trade-off between higher payments and a higher probability of winning. Thus a player with a high value is more concerned about the probability of winning than a low value player who is relatively more concerned about the expected payment. Stong[110] shows that in games like the WOA where the minimizing distribution depends on the player's type, an increasing equilibrium may not exist. To insure existence of an equilibrium, one must restrict the form of Δ. The following assumption insures the existence of an equilibrium.

A. 2. $H_M \in \Delta$.[6]

Assumption 8 both makes it easy to identify the minimizing distribution, and the minimizing distribution is independent of the type. To understand why this is the case, observe that H_M first order stochastic dominates(FOSD) every distribution in Δ. Clearly, this means that H_M minimizes the probability of winning. In addition, H_M maximizes the expected payment. This follows because the payment is a weakly increasing function of the m-th highest bid of the $n-1$ bids, and bids are increasing in the players' values. By FOSD the expected payment is minimized by H_M. This argument does not depend on the player's value.

With Assumption 8 the player's MEU can be rewritten as

$$V(b;v) = vH_M(s^{-1}(b)) - \int_{\underline{v}}^{s^{-1}(b)} (s(t))\, h_M(t)dt - \left(1 - H_M(s^{-1}(b))\right) b. \tag{1.13}$$

The first order condition implies that the equilibrium strategy must solve

$$s'(v) = \frac{vh_M(v)}{1 - H_M(v)} \tag{1.14}$$

[6]Let H_M and H be the distributions of the m-th highest value of n-1 iid draws from distributions G_M and G, respectively.[106] By a standard result, H_M FOSD H if and only if G_M FOSD G. Thus, Assumption 8 could be equivalently stated that the lower envelope of D is an element of D.

Lemma 1.3.3. *Assumption 8 implies that the symmetric increasing equilibrium of the WOA is*

$$s_w(v) = \int_{\underline{v}}^{v} \frac{th_M(t)}{1 - H_M(t)} dt. \tag{1.15}$$

1.3.4 Vickrey's Auction

Vickrey's auction is similar to the WOA because the m players with the highest bid pay the highest losing bid. However, unlike the WOA in Vickrey's auction the losers do not make a payment. In the case of one unit, Vickrey's auction is often referred to as the second price auction. It is well known that in Vickrey's auction there is an equilibrium in weakly dominant strategy where each player bids its own value. This equilibrium continues to be an equilibrium with uncertainty about values. I will focus on this equilibrium.

1.4 Comparative Statics

Now that I have described equilibrium behavior, I will discuss how ambiguity affects equilibrium strategies. In particular, I am interested in how increasing ambiguity changes the strategies that players use. Of particular interest is the effect of moving from the case when Δ is singleton, the SEU case, to ambiguous beliefs. To do so I first define what it means to increase ambiguity.

Definition 1.4.1. If $\Delta \subset \tilde{\Delta}$, then $\tilde{\Delta}$ is more ambiguous than Δ.

Let $\tilde{H}_M$ be the lower envelope of $\tilde{\Delta}$. $\tilde{s}_a(v)$ is the equilibrium when the players' beliefs are $\tilde{\Delta}$, and $s_a(v)$ is the equilibrium for Δ. The following results state how ambiguity affects the equilibrium strategies.

Proposition 1.4.2. *Let $\Delta \subset \tilde{\Delta}$, and $H_M(v) > \tilde{H}_M(v)$ for some $v \in (\underline{v}, \overline{v})$. In the APA*

1. *there is a $v^* \in (\underline{v}, \overline{v})$ such that $s_a(v) \geq \tilde{s}_a(v)$ for all $v \in [\underline{v}, v^*]$ with the inequality being strict for some $v \in [\underline{v}, v^*]$, and*
2. *there is a $v^{**} \in (\underline{v}, \overline{v})$ such that $s_a(v) < \tilde{s}(v)$ for $v \in (v^{**}, \overline{v}]$.*

Proof: Part 1: Since $\Delta \subset \tilde{\Delta}$, $H_M(v) \geq \tilde{H}_M(v)$ for all v. $H(v) > \tilde{H}(v)$ for some $v \in (\underline{v}, \overline{v})$ implies that there is an interval of values $[\underline{v}, v')$ such that $h_M(v) \geq \tilde{h}_M(v)$ for almost all values in that interval with the inequality being strict almost everywhere on some open set in $[\underline{v}, v')$. This implies that

$$s_a(v) = \int_{\underline{v}}^{v} t h_M(t) dt \geq \int_{\underline{v}}^{v} t \tilde{h}_M(t) dt = \tilde{s}_a(v) \tag{1.16}$$

for all $v \in [\underline{v}, v')$ with the inequality being strict at v'.

Part 2: The following is an implication of FOSD.

$$\int_{\underline{v}}^{\overline{v}} t \tilde{h}_M(tdt > \int_{\underline{v}}^{\overline{v}} t h_M(t) dt \tag{1.17}$$

So $s_a(\overline{v}) < \tilde{s}_a(\overline{v})$, and the result follows by the continuity of s_a and $\tilde{s}_a$. $\square$

Hopkins and Kornienko[53] show a similar comparative static which implies that if $\tilde{G}$ likelihood ratio dominates G the strategy has the same comparative static even for concave transformation of the ex post utility function. The results of Fibich, Gavious, and Sela[35] together with Proposition 1.4.2 imply that if risk aversion is added to the ambiguity aversion model bids will further decrease for low types and increase for high types. Thus risk aversion and ambiguity aversion are complementary with respect to this comparative static.

Example 1.4.3. $n = 6$ and $m = 2$. Let $\tilde{D} = \{G : |f(v) - g(v)| \leq \epsilon \ (\forall v \in V)\}$. Also let $f(v) = 1$ on $[0, 1]$. It is easy to verify that in this case if $\epsilon \in (0, 1)$,

$$F_m(v) = \begin{cases} (1-\epsilon)v \text{ for } 0 \leq v \leq .5 \\ \\ (1+\epsilon)v - \epsilon \text{ for } .5 < v \leq 1 \end{cases} . \tag{1.18}$$

If we suppose that the true distribution is $F(v) = v$ then an ambiguity neutral bidder's strategy is $s(v) = 4v^5 - \frac{10}{3}v^6$ as shown by Barut, Kovenock, and Noussair[10]. In Figure 1.1, I have graphed the strategy, s, for the no ambiguity case ($\epsilon = 0$) and strategy, s_m, for the case that $\epsilon = .3$. $\square$

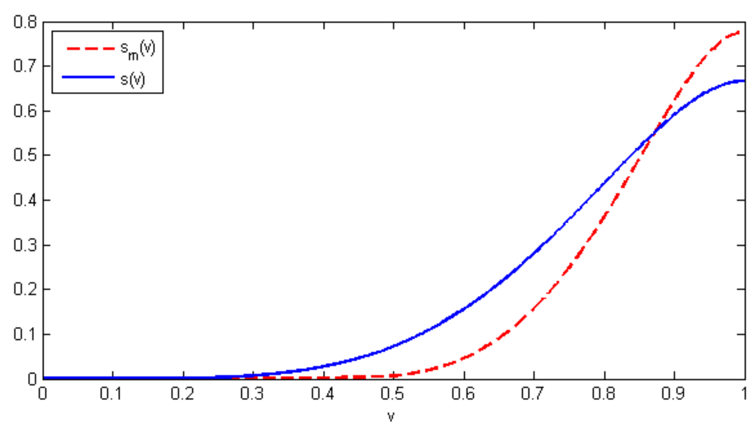

Figure 1.1: All-pay Auction Example: Uniform $n = 6$, $m = 2$

This example has a nice property in that the two strategies cross at only one point. In general this may not be so.

The first part of the Proposition 1.4.2 has an analog for the WOA. Under stronger conditions, increasing ambiguity decreases the players bids for all types.

Definition 1.4.4. If G is greater than $\tilde{G}$ in the monotone hazard rate order,

$$\frac{g(z)}{1 - G(z)} \leq \frac{\tilde{g}(z)}{1 - \tilde{G}(z)}$$

for all $z \in [\underline{z}, \overline{z}]$ the common support of G and $\tilde{G}$.

Proposition 1.4.5. *Δ and $\tilde{\Delta}$ satisfy Assumption 8, and $\Delta \subset \tilde{\Delta}$. $H_M(v) > \tilde{H}_M(v)$ for some $v \in (\underline{v}, \overline{v})$.*

1. *In the WOA, there is a $v^* \in [\underline{v}, \overline{v}]$ such that $s_w(v) \geq \tilde{s}_w(v)$ for all $v \in [\underline{v}, v^*]$, and the inequality is strict for some $v \in [\underline{v}, v^*]$.*
2. *Moreover, if $\tilde{H}_M$ is greater than H_M in the hazard rate order, then $s_w(v) \geq \tilde{s}_w(v)$ for all $v \in [\underline{v}, \overline{v}]$.*

The proof of part one is similar to that of Proposition 1.4.2 and is omitted. The proof of part 2 follows directly from the definition of the monotone hazard rate order.

In contrast it has been observed that ambiguity creates overbidding in first-price auctions[56]. This can also be demonstrated in the PBA. Behavior in Vickrey's auction is not affected by ambiguity since bidding one's own value continues to be an equilibrium.

1.5 Revenue and Welfare

Given the effect of ambiguity aversion on behavior it is natural to ask how ambiguity affects aggregate behavior such as the sum of expenditures. Also, it is natural to ask how the expected revenue varies across the different mechanisms. Here the WOA and APA are compared to each other and to the winner-pay auctions.

Let the distribution $F : [\underline{v}, \overline{v}], \rightarrow [0,1]$ be the distribution that is used to calculate expected utility. One interpretation of F is that a seller believes that the players' values are independent draws from F.[7] H_F is the distribution of the m-th highest value of n-1 independent draws from F. The following assumption guarantees a minimal degree of agreement between the distribution F and the players' beliefs.

A. 3. $F \in D$

Without this assumption it would be possible to create arbitrary revenue rankings by picking F appropriately.

Denote the expected payment of a player with valuation v by $e_l(v)$ for $l \in \{w, a, p, v\}$. w, a, p, and v index the expected payment in the WOA, APA, PBA, and Vickrey's auction, respectively. For instance, in Vickrey's auction,

$$e_v(v) = \int_{\underline{v}}^{v} t h_F(t) dt. \tag{1.19}$$

Since in the all-pay auction the bidders pay their bids,

$$e_a(v) = s_a(v). \tag{1.20}$$

The sum of expected expenditures which is called the revenue is given by

$$R_l \equiv n \int_{\underline{v}}^{\overline{v}} e_l(t) f(t) dt \tag{1.21}$$

for mechanism l.

[7] At the end of this section I discuss the plausible situation in which the sell is also ambiguity averse.

It is particularly useful to compare the revenue from another mechanism to the revenue in Vickrey's auction. The reason for this is that players always bid their own values in Vickrey's auction regardless of the form of ambiguity. Thus the expected revenue only depends on F and does not depend on ambiguity. In addition, the revenue equivalence theorem implies that when there is no ambiguity, the revenue is the same for all of the mechanisms studied here. These facts together imply that comparing the revenue in a mechanism to the revenue in Vickrey's auction is the same as comparing the revenue with ambiguity to the SEU case.

For this reason, I begin by comparing the revenue in the APA to the revenue in Vickrey's auction. It has already been shown by Bodoh-Creed[21] that the first-price and second-price auctions cannot generally be ranked. Given that, in the all-pay auction, ambiguity increases the bid for some types and decreases the bid for others, it is no surprise that the effect of ambiguity on revenue is ambiguous. The following example demonstrates this.

Example 1.5.1. Let the true distribution and belief of the seller be $F(v) = v$ on the interval $[0, 1]$. $n = 2$ and $m = 1$. Let $\Delta = \{G : G(v) = v^\alpha \text{ for } \alpha \in [1, c]\}$. If $c > 1$, then the minimizing distribution is $F_M(v) = v^c$ by FOSD.

$$s(v_i) = \int_0^{v_i} ct^c dt = \frac{c}{c+1} v_i^{c+1} \tag{1.22}$$

$$R_a = 2\int_0^1 s(v)f(v)dv = 2\int_0^1 \frac{c}{c+1} v^{c+1} dv = \frac{2c}{(c+1)(c+2)} \tag{1.23}$$

$$R_v = 2\int_0^1 \int_0^v t dt dv = \frac{1}{3} \tag{1.24}$$

When $c = \frac{3}{2}$, $R_a = \frac{12}{35} > \frac{1}{3}$. When $c = \frac{5}{2}$, $R_a = \frac{20}{63} < \frac{1}{3}$. When $c = 2$, $R_a = \frac{1}{3}$. It can be shown that revenue is increasing in c for $1 \leq c < \sqrt{2}$ and decreasing in c for $c > \sqrt{2}$. This shows that ambiguity can increase or decrease the expected expenditures in the APA.

In contrast to these negative results, the following revenue rankings can be established. Assumption 3 is implicitly assumed throughout this section.

Proposition 1.5.2. *The PBA produces higher expected revenue than the all-pay auction.*

Proof:

$$e_p(v) = \left(v - \frac{\int_{\underline{v}}^{v} H_M(t)dt}{H_M(v)} \right) H_F(v) \tag{1.25}$$

Using integration by parts

$$e_a(v) = s_a(v) = vH_M(v) - \int_{\underline{v}}^{v} H_M(t)dt \tag{1.26}$$

$$e_p(v) - e_a(v) = v\left(H_F(v) - H_M(v)\right) + \left(\frac{H_M(v) - H_F(v)}{H_M(v)} \right) \int_{\underline{v}}^{v} H_M(t)dt \tag{1.27}$$

$$= \left(H_F(v) - H_M(v)\right) \left(v - \frac{\int_{\underline{v}}^{v} H_M(t)dt}{H_M(v)} \right) \tag{1.28}$$

$$= (H_F(v) - H_M(v))s_p(v) \geq 0 \tag{1.29}$$

The inequality follows from the fact that H_M FOSD H and that the strategy $s_p(v)$ is non-negative. Thus each type is expected to bid more in the PBA. □

Proposition 1.5.3. *If $H_M \in \Delta$, the APA gives higher expected revenue than the WOA.*

Proof:

$$e_w(v) = \int_{\underline{v}}^{v} s_w(t)h_F(t)dt + (1 - H_F(v))s_w(v) \tag{1.30}$$

$$= s_w(v)H(v) - \int_{\underline{v}}^{v} s_w'(t)H_F(t)dt + (1 - H_F(v))s_w(v) \tag{1.31}$$

$$= \int_{\underline{v}}^{v} t\frac{h_M(t)}{1 - H_M(t)}dt - \int_{\underline{v}}^{v_i} t\frac{h_M()}{1 - H_M(t)}H_F(t)dt \tag{1.32}$$

$$= \int_{\underline{v}}^{v} t\frac{h_M(t)}{1 - H_M(t)}(1 - H_F(t))dt \tag{1.33}$$

$$\leq \int_{\underline{v}}^{v} th_M(t)dt = e_a(v) \tag{1.34}$$

The inequality follows from FOSD of H_M.□

Proposition 1.5.4. *If H_M is greater than H_F in the monotone hazard rate order, then Vickrey's auction generates more revenue than the WOA.*

Proof: From the proof of Proposition 1.5.3, $e_w(v)$ can be written

$$s_w(v) = \int_{\underline{v}}^{v} t\frac{h_M(t)}{1 - H_M(t)}(1 - H_F(t))dt. \tag{1.35}$$

The revenue in Vickrey's auction can be written

$$s_v(v) = \int_{\underline{v}}^{v} t\frac{h_F(t)}{1 - H_F(t)}(1 - H_F(t))dt. \tag{1.36}$$

By the monotone hazard rate order $s_v(v) \geq s_w(v)$ for all $v \in [\underline{v}, \overline{v}]$. □

Thus the PBA generates higher revenue than the all-pay auction and the WOA generates less than both of them if $H_M \in \Delta$. Since the monotone hazard rate order implies FOSD, under slightly more restrictive conditions the WOA generates the least revenue. This last comparison is not surprising given that bids are uniformly decreasing in the WOA. This suggests that under circumstances in which expenditure is wasteful to society the WOA may be preferable.

In terms of interim utilities the bidders are indifferent between auctions which have the same equilibrium minimizing distributions. This is a direct extension of the payoff equivalence theorem for auctions with SEU.[8] Thus bidders are indifferent between the PBA and APA. If $F_m \in \Delta$ bidders are indifferent between all of the auctions. Lo[74] demonstrates that bidders prefer the second price auction to the first price auction when $F_m \notin \Delta$. The same applies to multiple unit auctions.

1.5.1 Ambiguity Averse Seller

The revenue ranking above can be easily extended to payoff rankings for a seller who is ambiguity averse. Let D_s be the seller's analog of D.

Definition 1.5.5. $F^M(t) \equiv \max_{G \in D_s} G(t)$

I make the following assumptions on D_S.

A. 4. $F^M \in D_s$

A. 5. $D_s \bigcap D \neq \emptyset$

Let $R_l(G)$ represent the expected revenue with respect to distribution G for the auction format indexed by l. The sellers utility is given by

$$V_{s,l} = \min_{G \in \Delta} R_l(G). \tag{1.37}$$

[8]For a generalized version of the payoff equivalence theorem for MEU see Bodoh-Creed[21].

In all of the mechanisms studied, the player's equilibrium expected payment is increasing in the player's value, so the worst case distribution is F^M, when assumption 4 is satisfied. This follows because every distribution in Δ_s dominates F^M in the sense of FOSD.

Let $G \in D \bigcap D_s$. $F_M \geq G \geq F^M$ with the order of FOSD. Thus, all of the revenue rankings above can be extended to payoff ranking for an ambiguity averse seller. Under assumptions 4 and 5, the proofs of the statements are valid with F being replaced by F^M.

1.6 Conclusion

Ambiguity has a significant impact on strategies and expected expenditure in all-pay auctions. Several of the regularities highlighted here are also consistent with experimental data. One of the most robust predictions is that in the all-pay auction low types bid less and high types bid more than in the ambiguity neutral equilibrium. In the studies by Barut, Kovenock, and Noussair[10]; Hörisch and Kirchkamp[54]; and Noussair and Silver[91], this is also observed. In the war of attrition, Hörisch and Kirchkamp[54] find prevalent underbidding relative to the ambiguity neutral case; this is also consistent with ambiguity aversion. Furthermore, Horisch and Kirchkamp[54] observe that expected expenditure is lower in the WOA than the APA.

It should also be noted that Fibicci, Gavious, and Sela[35] show similar results in the APA with risk aversion. One difference between ambiguity and risk aversion is that the highest bid in the APA and first-price auction are the same when there is only ambiguity aversion. However, Fibicci et al.[35] show that with risk aversion a player with the highest type bids higher in the APA than the FPA. Distinguishing between the role of risk aversion and ambiguity aversion is important because risk is to some extent an integral part of the game which cannot be ameliorated without changing the rules; whereas, it seems possible that ambiguity may be reduced as players better understand the information structure. More experimental and theoretical work is needed in this area.

One prediction that is not born out in the experimental literature is the revenue dominance of the PBA. Barut et al.[10] did not find any pattern in ranking the revenue of the PBA and APA. In a single prize setting, Noussair and Silver[91] have higher average revenue in an APA than could ever be conceived of in a first-price auction. This result was partially driven by overly aggressive bidding in early rounds of the experiment. In contrast, Horisch and Kirchkamp[54] find that expected expenditure is lower in WOA experiments than in the APA. This result is consistent with the findings here.

CHAPTER 2
MONOTONE EQUILIBRIUM IN GAMES WITH MAXMIN EXPECTED UTILITY

2.1 Introduction

Since the work of Ellsberg[32], there has been a large body of experimental work that demonstrates that some behavior under uncertainty cannot be explained by maximization of subjective expected utility (SEU). Behavior that contradicts SEU is particularly common when there is ambiguity about the probability of events. In the finance and macroeconomics literature, ambiguity aversion, which generalizes SEU, has been shown to solve many puzzles regarding asset prices, notably the equity premium puzzle.[1] Mukerji[87] has shown that ambiguity aversion can explain incompleteness of contracts in situations where costly contracting alone gives counterfactual predictions. Kagel and Levin[56] proposed ambiguity aversion as an explanation for overbidding in first-price auctions.

I apply ambiguity aversion to a class of games of incomplete information where the all-pay auction[90] and war of attrition[79] are limiting cases[118][89]. These games have been used to model a wide variety of strategic environments in which players compete for a prize by expending resources or effort. Examples include firms competing to determine industry standards[25], firms exiting a crowded market[38], students competing for college admissions[49], and online auctions[95].

Ambiguity is likely to be present in many environments modeled by games of

[1] See Guidolin and Rinaldi[44] for a recent survey. See also Hansen and Sargent[45].

incomplete information. In the games studied here, players are uncertain about how much another player values a prize. With SEU, this uncertainty is modeled using a distribution over the other player's value called a belief distribution. The player's interim utility is the expected utility calculated using that distribution. However, in many applications the distribution of the other player's value is hard to learn either because the environment is changing or because of a lack of experience. Following Lo[74], I model ambiguity using the maxmin expected utility model of Gilboa and Schmeidler[41]. With MEU, beliefs are modeled by a set of distributions, Δ, of which any one may generate the other player's value. The player calculates the expected utility for each distribution in Δ, and the utility is the lowest expected utility for any distribution in Δ. By choosing an action which maximizes the minimum expected utility an ambiguity averse player chooses an action which is robust to the worst case distribution.

I find that ambiguity can have a significant impact on the efficiency of the games studied here. These games may fail to have an efficient equilibrium in the sense that the prize is not always awarded to the player who values it the most. As I explain in the literature review, these results contrast the results in an analogous SEU environment.

I provide a characterization for the increasing, symmetric equilibria of games in this class. Previously, a general characterization of equilibrium was only available for games like the first-price auction where the minimizing distribution is the one which minimizes the probability of winning. Because the games studied here may not have that property, a different method is required to derive the equilibrium. The technique for characterizing equilibria can be applied to other games with MEU. I also provide conditions for an increasing equilibrium to exist.

2.1.1 Related Literature

In his seminal work, Lo[74] applied the MEU model to the first-price auction where he derived the unique increasing equilibrium. His analysis depends on the following observation about the first-price auction. Any bid submitted by a player results in one of two outcomes; either the player wins the object and pays the bid, or the player does not win the object and makes no payment. Because of this, assuming that players bid below their own value, an expected utility minimizing distribution is a distribution in Δ which minimizes the probability of having the highest bid. Thus, the set of expected utility minimizing distributions does not depend on the player's value. As Lo[74] noted, in many auctions the expected utility depends on more than just the probability of having the highest bid. More generally, the minimizing distribution may also depend on the player's own value if the ex post utility depends on more than whether the player wins and the player's own bid.

Since the work of Lo[74], there have been a number of papers that apply MEU to a variety of games and mechanism design environments. These games and mechanisms have the property that the ex post utility, given a bid, only depends on winning or losing. Levin and Ozdenoren[71] study the first-price auction with particular emphasis on uncertainty about the number of bidders. Again, the minimizing distribution is the one which minimizes the probability of winning. Bose, Ozdenoren, and Pape[24] and Bodoh-Creed[21] consider the problem of designing revenue maximizing mechanisms. Both of the mechanism design papers find that, in the optimal mechanism, the set of minimizing distributions is the same as in the first-price auction and does not depend on the type of the player.

In the war of attrition, the worst case distribution may depend on the player's type. In this game the player with the highest bid wins the object and pays the losing bid. The loser pays his own bid and does not receive a prize. Thus the ex post utility in the war of attrition depends on whether the player wins or not, and for the winner, it depends on the losing bid. This makes the war of attrition much different from the first-price auction when there is MEU. The expected utility will depend on both the probability of winning and the expected payment. The relative importance of these two parts will depend on how much the player values the prize. Players with high values will tend to prefer distributions with a higher probability of winning, whereas players with low values will be more concerned with the expected payment. Thus the distribution in Δ that minimizes a player's expected utility may be different for types with different values. As a result, a different method of analysis is needed from the one used for first-price auctions. One consequence is that, whereas the first-price auction has an increasing equilibrium in the symmetric MEU model, this may not be the case with the war of attrition.

The existence of an increasing equilibrium in the war of attrition and all-pay auction has been studied with SEU. An equilibrium exists when the players' values are distributed independently so that each player's belief does not depend on the player's own type[3][5][118]. In the affiliated private values model with players who have SEU, Milgrom and Weber[85] show that to have an increasing equilibrium, in the first-price auction, it is sufficient that a player's belief is increasing in the player's value, in the sense of affiliation. Krishna and Morgan[69] show that, when players have SEU, an increasing equilibrium exists in the war of attrition and all-pay auction if the affiliation is moderate.

In the MEU model, even when the player's belief, Δ, is independent of the player's value, an increasing equilibrium may not exist. This is because the expected utility minimizing distribution may depend on the player's value. The dependence arises because in games like the war of attrition the payment, in the event of winning, depends on the other player's bid. Since the expected payment depends on the other player's strategy, the set of minimizing distributions depends on the equilibrium. This contrasts Krishna and Morgan[69] where the distribution used to calculate expected utility depends on the value exogenously through Bayesian updating.

The remainder of the paper is organized as follows. Section 2.2 formally presents the model. Section 2.3 contains a characterization of the set of symmetric, increasing equilibria in the general model. Sufficient conditions for existence of such an equilibrium are provided. Section 2.4 provides an example that illustrates why an ex post efficient equilibrium may not exist with MEU. Section 2.5 discusses some extensions including the smooth ambiguity aversion model and type dependent ambiguity. Section 2.6 concludes. The appendix contains some of the proofs.

2.2 Model

I begin by describing the class of games studied here. Although all of the results apply to games with any finite number of players, to simplify notation, I consider games with two players, player 1 and player 2.[2] Player i's value for a prize, v_i, comes from the interval $[\underline{v}, \overline{v}]$, with $\underline{v} \geq 0$. Upon learning their own values, both players simultaneously submit a bid. Let $b_i \geq 0$ be player i's bid. The allocation function, $x_i(b_1, b_2)$, determines the probability that player i receives the prize. In addition, each player i has a transfer function, $\tau_i(b_1, b_2)$, which is player i's expenditure as a function of the bids.

I will restrict attention to a specific, well studied class of allocation and transfer functions which includes the all-pay auction and war of attrition[3].

$$x_i(b_1, b_2) = \begin{cases} 1, \text{ if } b_i > b_j \\ 1/2, \text{ if } b_i = b_j \\ 0, \text{ otherwise} \end{cases} \tag{2.1}$$

$$\tau_i(b_1, b_2) = \begin{cases} (1-p)b_i + pb_j, \text{ if } b_i > b_j \\ b_i, \text{ otherwise} \end{cases} \tag{2.2}$$

Where $p \in [0, 1)$. When $p = 0$, the transfer function is that of the all-pay auction. When $p = 1$, the game is a static version of the war of attrition. The usual interpre-

[2]The main savings occurs when I describe beliefs. With multiple players the relevant distribution for calculating expected utility is the distribution of the highest bid of the $n-1$ other players.See the companion paper for generalization of this environment to multiple units and multiple players with single unit demands.

tation of the war of attrition is that players expend resources over time until one side concedes at which point the game terminates instantly. In a dynamic setting, $p < 1$ captures a situation in which a player does not learn about an opponent's concession immediately[3]. p could also be thought of as the probability that the winner pays the losing bid.[3]

The ex post utility of player i is given by the following utility function.

$$u_i(b_1, b_2, v_i) = x_i(b_1, b_2)v_i - \tau_i(b_1, b_2). \tag{2.3}$$

This utility is known as the risk-neutral, private values model. To better understand the role of ambiguity aversion I restrict attention to this benchmark model of ex post utility.[4]

Now I define the interim utility, which is the utility at the point where each player knows his own value but not the other player's value. This is modeled by the maxmin expected utility of Gilboa and Schmeidler[41]. A player's ambiguous belief is a set of distributions over the other player's value. The player's utility is the lowest expected utility generated by any distribution in the set.

Let $\Delta_i(v_i)$ be the belief set for player i with a value of v_i. The expected utility,

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[3]See Bulow and Klemper[25] for some natural ways to extend this description to more than two players.

[4]This model can handle asymmetric information regarding both costs and values. That is, the model also includes the seemingly more general case that $\tilde{u}_i(b_1, b_2, v_i, c_i) = x_i(b_1, b_2)v_i - c_i\tau_i(b_1, b_2)$ where c_i is interpreted as the marginal cost of expenditure. Using an affine transformation $\tilde{u}$ becomes $u_i(b_1, b_2, v_i/c_i) = x_i(b_1, b_2)v_i/c_i - \tau_i(b_1, b_2)$, which is equivalent to the original model as long as $c_i > 0$.

with respect to distribution G, for player i with value v_i and bid b_i is defined as

$$\tilde{U}_i(b_i; v_i, s, G) \equiv \int_{\underline{v}}^{\overline{v}} u_i(b_i, s(v_j), v_i) g(v_j) dv_j \tag{2.4}$$

where $s : [\underline{v}, \overline{v}] \to R^+$ is a measurable strategy and G is a distribution over $[\underline{v}, \overline{v}]$, with density g. The maxmin expected utility is given by

$$\tilde{V}_i(b_i; v_i, s) \equiv \inf_{G \in \Delta_i(v_i)} \tilde{U}(b_i; v_i, s, G). \tag{2.5}$$

If $\Delta_i(v_i)$ is singleton, then the utility corresponds to SEU; otherwise, the player is said to be ambiguity averse.

An equilibrium of the game is analogous to an equilibrium in a game with subjective expected utility; each player's strategy maximizes his utility given the other player's strategy.

Definition 2.2.1. A pure strategy equilibrium is a pair of measurable strategies (s_1, s_2) such that for each player $i \in \{1, 2\}$ and $j \neq i$ and for every $v_i \in [\underline{v}, \overline{v}]$,

$$\tilde{V}_i(s_i(v_i), v_i; s_j) \geq \tilde{V}_i(b_i, v_i; s_j)$$

for all $b_i \geq 0$.

I will further restrict attention to equilibria which are increasing. For the rest of the paper, when I refer to an increasing strategy I mean a strategy which is strictly increasing in the player's value. That is if $v > v'$, then $s(v) > s(v')$ for all $v, v' \in [\underline{v}, \overline{v}]$. There are several reasons for focusing on increasing strategies. First, when the environment is symmetric, the symmetric, increasing equilibria are the ex post efficient equilibria. Also, if Δ is singleton the unique equilibrium is in increasing strategies[3][93].

For an increasing strategy s and for each bid $b \in [s(\underline{v}), s(\overline{v})]$, there is at most one value $z \in [\underline{v}, \overline{v}]$ such that $b = s(z)$. Thus, it will be notationally convenient to think of the players as choosing the value that corresponds to a bid rather than choosing a bid. If a player chooses bid b this is equivalent to choosing $z = s^{-1}(b)$ when the other player is using the continuous, increasing strategy s. Now, with continuous, increasing strategies and this notation, write the expected utility of a player with value v_i who bids $s_j(z)$ as

$$U_i(z; v_i, s, G) \equiv \tilde{U}_i(s_j(z); v_i, s_j, G) = v_i G(z) - p \int_{\underline{v}}^{z} s_j(t) g(t) dt - (1 - pG(z)) s_j(z). \tag{2.6}$$

This follows since with an increasing strategy the probability that a player pays his own bid is given by $1 - pG(z)$. With the complementary probability the player pays the other player's bid. From the point of view of a player, the other player's bid is a random variable determined by the strategy and distribution of the other player's value. The maxmin expected utility is similarly defined as a function of z and is denoted by $V_i(z; v_i, s)$.

A. 6. $\Delta_i(v_i) = \Delta$ for all $v_i \in [\underline{v}, \overline{v}]$, for $i = 1, 2$.

This assumption makes the model as close as possible to the benchmark SEU model in which values are independent and identically distributed. A.6 says that the beliefs for both players are the same and that those beliefs do not depend on the player's value. By focusing on this simple model the role of ambiguity aversion is most transparent. I discuss making the beliefs of the players dependent on the player's value in Section 2.5. Because I focus on a symmetric environment the player subscript is often omitted.[5]

A. 7. Δ is a set of continuously differentiable, strictly increasing distributions on $[\underline{v}, \overline{v}]$. Δ is convex and compact in the sense that the densities form a compact set with respect to the uniform topology.[6]

At this stage, note that A.7 insures that there is a distribution in Δ that minimizes the expected utility. This follows because the expected utility will be continuous in the distribution of the other player's value. Throughout this paper, A.6 and A.7 are implicitly assumed unless otherwise stated.

2.3 Type Dependent Minimizing Distributions

In many instances the analysis of games is facilitated by the assumption that there is a distribution in Δ which first order stochastic dominates(FOSD) the others. Define $F_M(v) \equiv \min_{G \in \Delta} G(v)$. The FOSD assumption is the following.

A. 8. $F_M \in \Delta$.

Note that with this assumption F_M minimizes the probability of winning at any bid. Furthermore, because the payment is a weakly increasing function of the

[5]For a discussion of asymmetry with SEU see Amann and Leininger[3]

[6]For continuously differentiable distributions this is the same as the topology induced by the norm $\max\left\{\sup_{v \in [\underline{v}, \overline{v}]} |F(v)|, \sup_{v \in [\underline{v}, \overline{v}]} |f(v)|\right\}$(Abbott p.164 2001[1], Rudin p.152 1976[99]).

other player's type, when the other player uses an increasing strategy, the expected payment is maximized by F_M. Thus F_M minimizes the expected utility for any increasing strategy, and this fact does not depend on the player's type or bid.

However, for some applications, A.8 may be too restrictive. If Δ is interpreted as a belief, there are many natural ways for the belief to be formed that would not satisfy this assumption.

For instance, A.8 may not be appropriate when players are uncertain about the dispersion in the distribution of types.[7] One application in which the level of variance may be ambiguous is labor strikes. Kennan and Wilson[58] model labor strikes as war of attrition in which each side is uncertain of the other side's cost of conceding in a labor dispute. The variance of the firm's cost can be affected by market conditions or by decisions made by the managers.[8] If the union faces ambiguity about the variance of the firms costs, the first order stochastic dominance assumption may not adequately model the union's information.

2.3.1 Characterization

I show that, with a general form of Δ, the set of equilibria can be analyzed in a way analogous to games with SEU. With SEU, an increasing equilibrium must be a solution to a differential equation. This equation is defined by observing that the derivative of the expected utility must be zero at the equilibrium bid. There

[7] Konrad and Kovenock[64] study the effects of variability in the distribution of types on behavior for some contest environments.

[8] There is a growing literature that studies the strategic importance of risk taking. See Kräkel[67] and Suzuki[111] and the references therein for recent references.

are a couple of difficulties to overcome to apply this method to MEU. The first is to show that the MEU is sufficiently differentiable. This requires an envelope theorem which insures differentiability and gives a formula for the derivative. I show that in equilibrium MEU is right-hand and left-hand differentiable almost everywhere, and that in some sense, which will be made precise, the derivative can be set to zero at the equilibrium bid.[9] Since the derivative of the utility is not uniquely defined at some points, it is useful to think of an equilibrium strategy as a solution to a differential inclusion.

For this section, assumption A.7 is used to insure that the set of minimizing distributions has convenient properties. Since the expected utility is continuous in the distribution and Δ is compact, the set of minimizers $\mathcal{G}_{v,z,s} \equiv \{G \in \Delta : U(z; v, s, G) = V(z; v, s)\}$ is nonempty. The convexity of Δ insures that $\mathcal{G}_{v,z,s}$ is convex valued. Additionally, it is convenient to note that the compactness of Δ implies that all of the densities of the distributions in Δ have a common upper bound $\overline{g}$.

As a matter of notation, the subscript on the player's value is often suppressed since I will be focused on a symmetric environment. A selection of the correspondence $\mathcal{G}_{v,z,s}$ is a function $G_s : [\underline{v}, \overline{v}] \times [\underline{v}, \overline{v}] \to \Delta$ such that $G_{v,z,s} \in \mathcal{G}_{v,z,s}$ for all $v, z \in [\underline{v}, \overline{v}]$.

The first step to characterizing an equilibrium is to establish the differentiability of the equilibrium utility. One can only hope to establish differentiability of the utility if players are using sufficiently smooth strategies. The following lemma says that any increasing, symmetric equilibrium strategy is a Lipschitz continuous

[9] All references to a measure refer to the Lebesgue measure.

function.

Lemma 2.3.1. *If β is an increasing, symmetric equilibrium strategy, then β is Lipschitz continuous with constant $M = \overline{v}\overline{g}/(1-p)^2$.*

In the proof, it is shown that if the strategy is not Lipschitz continuous, there is a profitable deviation for a positive measure of values. Since the equilibrium strategies are Lipschitz continuous, I restrict attention to such strategies without any loss of generality.

The following envelope condition establishes the differentiability of the minimum expected utility, V, and provides a formula for the derivative.

Proposition 2.3.2. *[Envelope Theorem] Let $s(v)$ be a strictly increasing strategy which is Lipschitz continuous with constant $M = \overline{v}\overline{g}/(1-p)^2$.*

1. *$V(z; v, s))$ is absolutely continuous in z.*
2. *V is right-hand and left-hand differentiable in z at v for almost all $v \in [\underline{v}, \overline{v}]$.*
3. *Let $G_{v,z,s} \in \mathcal{G}_{v,z,s}$ be given. If $z > \underline{v}$ and $V(.; v, s)$ is left-hand differentiable at z, then $V'_{-}(z; v, s) \geq U'(z; v, s, G_{v,z,s})$. If $z < \overline{v}$ and $V(.; v, s)$ is right-hand differentiable at z, then $V'_{+}(z; v, s) \leq U'(z; v, s, G_{v,z,s})$. If $z \in (\underline{v}, \overline{v})$ and $V(.; v, s)$ is differentiable at z, then $V'(z; v, s) = U'(z; v, s, G_{v,z,s})$.*

The proof applies the envelope theorems in Milgrom and Segal [83]. If $V(.; v, s)$ is differentiable at z, the derivative of the expected utility is given by

$$V'(z; v, s) = v g_{v,z,s}(z) - (1 - pG_{v,z,s}(z))s'(z). \tag{2.7}$$

Setting this derivative to zero at the equilibrium bid motivates the expression in the statement of Theorem 2.3.3. This is only a heuristic motivation because the envelope theorem *does not* say that the MEU is differentiable at the equilibrium bid.

Theorem 2.3.3. *If β is an increasing, symmetric equilibrium, then there exists a selection $G_{z,v,\beta} \in \mathcal{G}_{z,v,\beta}$ for all $z, v \in [\underline{v}, \overline{v}]$ such that*

$$\beta(v) = \int_{\underline{v}}^{v} \frac{t g_{t,t,\beta}(t)}{1 - p G_{t,t,\beta}(t)}. \tag{2.8}$$

The main idea of the proof is that, for a strategy to be a symmetric equilibrium, the equilibrium bid must be a local maximum. The envelope theorem implies that the left-hand derivative of V is non negative and the right-hand derivative is non positive in equilibrium for almost all values of v. The convexity of the set of minimizing distributions implies that, at almost all equilibrium bids, there is a minimizing distribution, $\check{G}$, such that the derivative of the expected utility, $U'(v; v, \beta, \check{G})$, is zero. It is in this sense that, at the equilibrium bid, the derivative is equal to zero almost everywhere.[10]

The condition in the theorem is only a necessary condition. *A priori*, there seem to be two possible ways for an efficient equilibrium to fail to exist. One is that all strategies satisfying the necessary condition do not identify a global maximum. The other is that there may fail to be a strategy that satisfies (2.8). Since the set of minimizing distributions depends on the strategy played, and the strategy is defined

[10] Ghirardato and Siniscalchi[40] discusses differentiation of ambiguous preferences using set valued derivatives. The additional structure here allows for explicit expressions for the derivatives.

using a selection from the set of minimizing distributions it is not obvious that such a strategy exists.

The existence of a solution to the necessary condition can be understood as the existence of a solution to the following differential inclusion.

$$\beta'(v) \in \left\{ \lambda \in \mathbf{R} : \lambda = \frac{v g_{v,v,\beta}(v)}{1 - p G_{v,v,\beta}(v)} \text{ for } G_{v,v,\beta} \in \mathcal{G}_{v,v,\beta} \right\} \qquad (2.8')$$

A restatement of Theorem 2.3.3 is that an equilibrium strategy must be a solution to this differential inclusion.[11] As stated in Proposition 2.3.4, there is always a solution to (2.8′). That is, there is always a strategy which satisfies the necessary condition. Thus, if there is no efficient equilibrium it is because every solution to the necessary condition fails to identify a global maximum.

Proposition 2.3.4. *There exists a strategy β and a selection $G_{z,v,\beta} \in \mathcal{G}_{z,v,\beta}$ for all $z, v \in [\underline{v}, \overline{v}]$ such that*

$$\beta(v) = \int_{\underline{v}}^{v} \frac{t g_{t,t,\beta}(t)}{1 - p G_{t,t,\beta}(t)}.$$

2.3.2 Existence

Existence of an equilibrium can be established by checking each strategy in the set of strategies that satisfy the necessary condition. For each strategy that satisfies the characterization in Theorem 2.3.3, the utility of submitting a bid can be rewritten under the assumption that the other player uses the candidate strategy. Let β and

[11] The usual form of a differential inclusion problem is to find an absolutely continuous function $x : I \to \mathbf{R}$ such that $x'(t) \in F(x(t), t)$ for almost all t in an interval I where $F : \mathbf{R} \times I \to 2^{\mathbf{R}}$ is potentially multivalued. In this case, I seek a solution to an inclusion of the form $x'(t) \in \hat{F}(x(t), t, x(.))$ where $\hat{F} : \mathbf{R} \times I \times C(I) \to 2^{\mathbf{R}}$.

$G_{v,z,\beta}$ be as in Theorem 2.3.3.

$$
\begin{aligned}
V(z; v, \beta) &= vG_{v,z,\beta}(z) - p \int_{\underline{v}}^{z} \beta(t) g_{v,z,\beta}(t) dt - (1 - pG_{v,z,\beta}(z))\beta(z) \\
&= vG_{v,z,\beta}(z) + p \int_{\underline{v}}^{z} \beta'(t) G_{v,z,\beta}(t) dt - \beta(z) \\
&= \int_{\underline{v}}^{z} \left\{ \frac{v g_{v,z,\beta}(t)}{1 - pG_{v,z,\beta}(t)} - \frac{t g_{t,t,\beta}(t)}{1 - pG_{t,t,\beta}(t)} \right\} (1 - pG_{v,z,\beta}(t)) dt \qquad (2.9)
\end{aligned}
$$

The second line follows from integration by parts and the third follows by applying (2.8).

This is similar to an expression that arises in Krishna and Morgan[69] with subjective expected utility and affiliated distributions. By making assumptions about the unique prior distribution of values they insure that (2.9) is increasing in z for $z < v$ and decreasing for $z > v$. Thus it seems natural to look for conditions under which the same is true with MEU. However, is it is difficult to establish quasi-concavity of (2.9) because the minimizing distribution may depend on v and this dependence is endogenous.

Proposition 2.3.5 below gives a condition under which, in equilibrium, the minimizing distribution is independent of the player's value. A.8 implies that the minimizing distribution is independent of the player's value regardless of the increasing strategy that the other player uses. Proposition 2.3.5 gives a weaker condition than FOSD that insures that, in equilibrium, the minimizing distribution is the same for any bid and value.

Proposition 2.3.5. *If there is a $G^* \in \Delta$ such that for all $G \in \Delta$ and $z \in [\underline{v}, \overline{v}]$*

$$
\overline{v}\left(G^*(z) - G(z)\right) \leq p \int_{\underline{v}}^{z} \frac{t g^*(t)}{1 - pG^*(t)} (G(t) - G^*(t)) dt, \qquad (2.10)
$$

then there is a symmetric equilibrium with the strategy given by

$$\beta(v) = \int_{\underline{v}}^{v} \frac{tg^*(t)}{1 - pG^*(t)} dt. \tag{2.11}$$

Condition (2.10) can be derived directly from (2.9) by starting with strategy (2.11) and imposing that, for any bid and value, G^* is the minimizing distribution. The condition implies a stochastic order which is weaker than FOSD when $p > 0$ and they are equivalent when $p = 0$. This stochastic order allows distributions in Δ to cross G^*. However, G^* must be sufficiently below the others on the lower segments of the support.

2.4 Example

In this section I provide an example where a symmetric, increasing equilibrium does not exist. Using Theorem 2.3.3, I construct a strategy which is the unique strategy that satisfies the necessary condition. For this strategy, the minimizing distribution for a given bid depends on the value of the player. The dependence is such that although the bid prescribed by the candidate strategy is a local maximum it is not a global maximum for some types.

Let the values come from the interval $[0, 1]$. Let $\Delta = \{G, H\}$ contain two distributions $G(v) = .5\sin(\pi(v - .5)) + .5$ and $H(v) = v$.[12] Furthermore, let $p = .9$.

To construct the unique candidate equilibrium strategy, suppose that β satisfies the conditions of Theorem 2.3.3. Define the strategy $\beta_F(v) \equiv \int_0^v \frac{tl(t)}{1-L(t)} dt$ for any

[12] I could also let Δ be the convex hull of these two distributions; by the linearity of the expectations operator nothing would change.

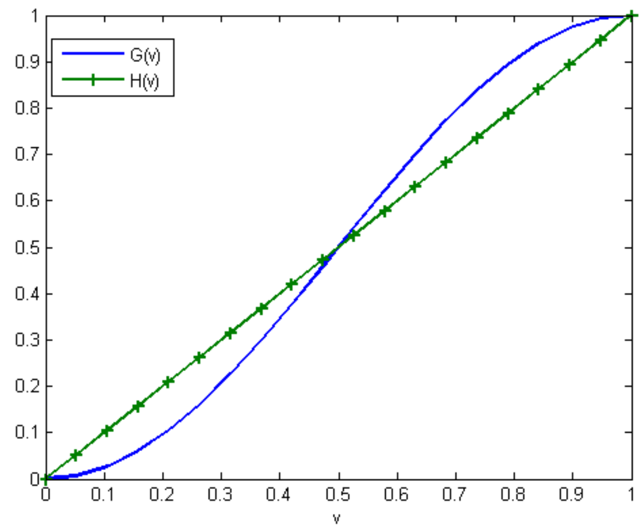

Figure 2.1: Set of Distributions:$\Delta = \{H(v) = v, G(v) = .5\sin(\pi(v - .5)) + .5\}$

distribution $L \in \Delta$. Since G is below H on the interval $(0, .5)$, $U(z; v, \beta, G) < U(z; v, \beta, H)$ for all $z \in (0, .5)$. This follows since the probability of winning is minimized and the expected payment is maximized by G when the player bids $z \in [0, .5]$. Since G minimizes the expected utility on $[0, .5]$, it follows from Theorem 2.3.3 that $\beta(v) = \beta_G(v)$ for all $v \in [0, .5]$. Define v^* as the lowest value such that $U(v^*; v^*, \beta_G, G) \leq U(v^*; v^*, \beta_G, H)$. This crossing is shown in Figure 2.2. Since G continues to minimize the expected utility on $[0, v^*]$, it follows from Theorem 2.3.3 that $\beta(v) = \beta_G(v)$ for all $v \in [0, v^*]$. In the proof of Proposition 2.4.1, I show that if β is an equilibrium, H is the minimizing distribution for a player with value above v^*. From Theorem 2.3.3, this implies that the only candidate for an equilibrium is the strategy defined in the following proposition.

Proposition 2.4.1. *The unique strategy that satisfies the condition of Theorem 2.3.3* *is given by*

$$\beta(v) = \begin{cases} \beta_G(v) \text{ for } v \in [0, v^*] \\ \\ \beta_G(v^*) - \beta_H(v^*) + \beta_H(v) \text{ for } v \in (v^*, 1]. \end{cases} \tag{2.12}$$

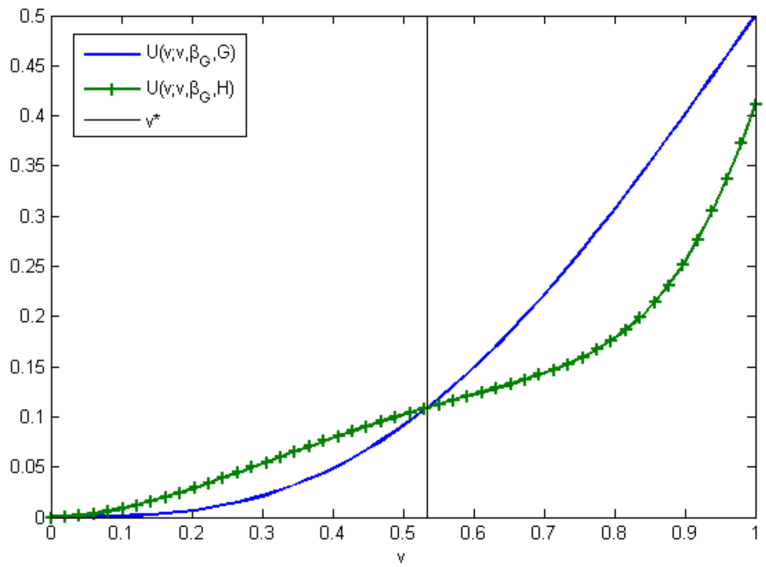

Figure 2.2: Definition of v^*

To see that there is no equilibrium, by straightforward calculation one can show that a player with value just above v^* strictly prefers to bid lower than $\beta(v^*)$, when the other player follows strategy β. The reason is that although a player with value v^* gets the same expected utility from G and H at v^*, a player with a value just above v^* gets a strictly lower expected utility from H at v^*. This is the case because a player with a higher value cares more about the probability of winning. This means that for a player with value above v^* the MEU can be decreasing for transformed bids less than v^*. In this case, this provides a profitable deviation for some values. This situation is depicted in Figure 2.3. The local maximum of $U(z; v^* + .02, \beta, H)$ on the right is the MEU at the candidate equilibrium bid. However, the point of intersection of the two curves provides a higher utility. The source of the profitable deviation is that the minimizing distribution changes with the player's value such that the minimum expected utility is not quasi-concave.

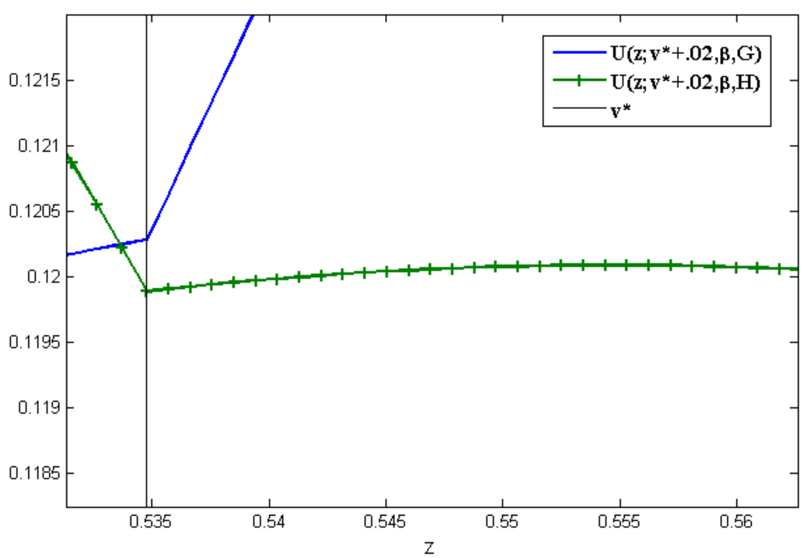

Figure 2.3: The Minimum Expected Utility: $V(z; v^* + .02, \beta)$ is the minimum of the two curves depicted.

This example contrasts the results of a closely related paper by Bodoh-Creed[21]. He analyzes mechanism design problems with general ambiguous beliefs. If $\Delta \subset \tilde{\Delta}$ the reserve price for the revenue maximizing mechanism is lower for $\tilde{\Delta}$. Also, if an ex ante balanced budget bilateral trade mechanism maximizes the seller's revenue, efficient trade increases the more ambiguity the buyer and seller face. If a bilateral trade mechanism is efficient, increasing ambiguity will decrease the ex ante budget deficit. The observation is that increasing ambiguity improves efficiency in a mechanism design environment. In contrast, I show that a fixed mechanism may cease to have an efficient equilibrium if the players' ambiguity is increased.[13]

2.5 Extensions

2.5.1 Smooth Ambiguity Aversion

In the MEU model, I find that without making restrictive assumptions, there may not be an increasing, symmetric equilibrium. The reason for this result is that the minimizing distribution can depend on the value of the player in an endogenous way. This section argues that this result is robust to other specifications of ambiguity aversion. I extend the analysis to the smooth ambiguity aversion model formalized by Klibanoff, Marinacci, and Mukerji[59]. A comprehensive study of games with smooth ambiguity aversion is beyond the scope of this paper; however, I will show how the intuition gained from games with MEU applies to the smooth ambiguity aversion model.

To illustrate the potential issues involved, I describe the smooth ambiguity aversion model as it applies to the class of games studied above. Let $\Delta = \{F(.;\theta)\}_{\theta\in[0,1]}$ be a parametrized set of distributions on $[\underline{v}, \overline{v}]$ where F is measurable

[13] Also in the context of general equilibrium theory, ambiguity aversion tends to improve the efficiency of outcomes as shown by Castro, Pesce, and Yannelis[27].

in θ. Define the expected utility

$$U_\theta(z;v) \equiv vF(z;\theta) - p\int_{\underline{v}}^{z} \beta(t)f(t;\theta)dt - (1 - pF(z;\theta))\beta(z) \tag{2.13}$$

for an increasing strategy β. The player's interim utility from bidding like a player with value z is

$$W(z;v) = \int_0^1 \psi[U_\theta(z;v)]d\theta. \tag{2.14}$$

If $\psi : \mathbf{R} \to \mathbf{R}$ is concave, the player is said to be ambiguity averse. The concavity of ψ means that the lower expected utilities are weighted more heavily in the utility function. It is helpful to write the derivative of the utility function.

$$W'(z;v) = \int_0^1 \psi'[U_\theta(z;v)]\{vf(z;\theta) - (1 - pF(z;\theta))\beta'(z)\}d\theta \tag{2.15}$$

$\psi'[U_\theta(z;v)]$ is a weighting function that weights distributions which yield low expected utility more heavily. Let $\phi(z,v) \equiv \psi'[U_\theta(z;v)]$ denote this weighting function and note that since the expected utility depends on the other player's strategy the weight depends on the strategy.

If a symmetric, increasing equilibrium, β, exists and solves the FOC,

$$\beta'(v) = \frac{\int_0^1 \phi(v;v)vf(v;\theta)d\theta}{\int_0^1 \phi(v;v)(1 - pF(v;\theta))d\theta}. \tag{2.16}$$

Substituting (2.16) into (2.15) yields that

$$\begin{aligned} W'(z;v) = (&\frac{\int_0^1 \phi(z;v)vf(z;\theta)d\theta}{\int_0^1 \phi(z;v)(1 - pF(z;\theta))d\theta} - \frac{\int_0^1 \phi(z;z)zf(z;\theta)d\theta}{\int_0^1 \phi(z;z)(1 - pF(z;\theta))d\theta})\times \\ &\int_0^1 \phi(z;v)(1 - pF(z;\theta))d\theta \end{aligned} \tag{2.17}$$

To insure the quasi-concavity of W, one must make some assumptions regarding the parametrized family of distributions and ψ. In the special case that $p = 0$ and

$\psi(x) = 1 - e^{-ax}$, (2.17) simplifies to the following expression which does not depend on the bid strategy.

$$W'(z;v) = \left(\frac{\int_0^1 e^{-avF(z;\theta)} v f(z;\theta) d\theta}{\int_0^1 e^{-avF(z;\theta)} d\theta} - \frac{\int_0^1 e^{-azF(z;\theta)} z f(z;\theta) d\theta}{\int_0^1 e^{-azF(z;\theta)} d\theta} \right) \int_0^1 \psi'[U_\theta(z;v)] d\theta. \tag{2.18}$$

This motivates the following proposition.

Proposition 2.5.1. *In the all-pay auction with* $\psi(x) = 1 - e^{-ax}$, *if*

$$\gamma(v,z) = \frac{\int_0^1 e^{-avF(z;\theta)} v f(z;\theta) d\theta}{\int_0^1 e^{-avF(z;\theta)} d\theta} \tag{2.19}$$

is increasing in v *for all* z, *then*

$$\beta(v) = \int_{\underline{v}}^{v} \frac{\int_0^1 e^{-atF(t;\theta)} t f(t;\theta) d\theta}{\int_0^1 e^{-atF(t;\theta)} d\theta} dt \tag{2.20}$$

is an equilibrium strategy.

The condition is analogous to the condition developed in Krishna and Morgan[69] in the context of affiliated values. The reason for assuming that $\gamma(v,z)$ is increasing, is to establish the quasi-concavity of equilibrium utility. Furthermore, in this particular case the sufficient condition depends only on the parameters, which include a and the parametrized set of distributions.

In general one might want

$$\hat{\gamma}(v,z) = \frac{\int_0^1 \phi(z;v) v f(z;\theta) d\theta}{\int_0^1 \phi(z;v)(1 - pF(z;\theta)) d\theta} \tag{2.21}$$

to be increasing in v for all z. However, it is not obvious what condition would insure this since the function ϕ depends on the strategy being played and thus is determined in equilibrium. Further work is needed to discover if there are more general conditions that can establish existence of an increasing equilibrium with the smooth ambiguity model.

The reasons for non existence of an increasing equilibrium in both the smooth ambiguity aversion model and MEU are similar. With smooth ambiguity aversion the weighting function ϕ depends on the player's value. MEU is simply the extreme case where weight is only given to the distributions which minimize the expected utility. In either case, restrictions on these weights are needed to insure the existence of an equilibrium. The main difficulty is that the weighting function is usually endogenous.

2.5.2 Type Dependent Ambiguity

The analysis can be extended to allow for ambiguity which depends on the value of the player. This can be used to model an environment where a player may believe that if his own value is relatively high the other player's value will tend to be high as well. To be formal suppose that $\Delta(v)$ is not constant in v. The necessary condition continues to hold as long as the following additional assumptions hold.

A. 9. $\Delta(v)$ is continuous in v.

A. 10. The conditions of A.7 hold for $\Delta(v)$ for all $v \in [\underline{v}, \overline{v}]$.

For the purposes of Theorem 2.3.3 and Proposition 2.3.4 it is sufficient that the set of minimizing distributions is upper semicontinuous and this additional assumption is sufficient for that conclusion. However, to write sufficient conditions for the existence of an increasing equilibrium one must take special care since the minimizing distribution will usually depend on the player's value.

To be more concrete, consider the following model of type dependent ambiguity aversion. Let Π be a set of symmetric joint distributions over the values of the two players. Π can be thought of as a set of priors that players have ex ante. Assume that the distributions in Π are twice differentiable in both arguments. For $F \in \Pi$, let $f(.|v)$ be the distribution of a player's value conditional on the other player's value being v. Let $\Delta(v) = \{G \in C_1([\underline{v}, \overline{v}]) | g = f(.|v)$ for some $F \in \Pi\}$. This means that $\Delta(v)$ arises from prior by prior updating of the priors in Π.[14] From the discussion in Section 2.3, it seems natural to look for conditions under which there is a distribution $G_M \in \Pi$ such that $G_M(.|v)$ is the minimizing distribution in $\Delta(v)$ for any bid. In addition, as in Krishna and Morgan[69] one must be sure that $G_M(.|v)$ depends on v in such a way that an increasing equilibrium exists. The following proposition provides such conditions.

Proposition 2.5.2. *Suppose that there is a $G_M \in \Pi$ such that*

1. *$G_M(.|v)$ FOSD $G(.|v)$ for all $G \in \Pi$ and for all $v \in [\underline{v}, \overline{v}]$*
2. *and*

$$\frac{v g_M(z|v)}{1 - pG_M(z|v)}$$

is increasing in v for all $z \in [\underline{v}, \overline{v}]$,

[14] There are many other ways to model how beliefs depend on v. The appropriate choice of updating rules is beyond the scope of this paper. For an axiomatization of the rule of updating each prior by Bayes rule see Pires[92]

then there exists a symmetric, increasing equilibrium given by

$$\beta(v) = \int_{\underline{v}}^{v} \frac{t g_M(t|t)}{1 - pG_M(t|t)} \tag{2.22}$$

I give a brief outline of the proof as the argument is similar to the discussion in Section 2.3. Condition 1 of Proposition 2.5.2 guarantees that for any z or v, $G_M(.|v)$ will be an expected utility minimizing distribution. Using a straightforward generalization of Theorem 2.3.3, the only candidate equilibrium is given by (2.22). Condition 2 is sufficient to insure that the equilibrium MEU, which has the same form as equation (2.9), is maximized by the candidate equilibrium bid.

2.6 Conclusion

This paper discusses efficiency in a class of games with MEU. In sharp contrast to games with SEU, games with MEU may not have an increasing, symmetric equilibrium. This is because even though Δ, which is interpreted as the ambiguous belief, is independent of the value, the minimizing distribution may depend on the value. This can be resolved if Δ contains a worst distribution according to the relevant stochastic order. In that case there is an equilibrium in which the worst case distribution is the same for all values.

Some may see non existence of efficient equilibrium as a deficiency of the MEU model since increasing equilibria have many convenient properties: they are easy to understand for players, they can be easily characterized, and they are efficient. However, the non existence of an increasing equilibrium illustrates the complexity of games in which the expected utility of a bid does not depend solely on the probability of having the highest bid. When players with limited probabilistic knowledge need to weigh other aspects, such as the expected value of the other player's bid, it is not sufficient to only consider increasing strategies. The potential complexity and the inefficiency which results may explain why, when the mechanism can be chosen, mechanisms such as the first-price auction are prevalent.

It should be noted that this paper illustrates a method that can be used in other environments. For instance, the ambiguity could be made type dependent so that $\Delta(v)$ depends continuously on the type of the player. Here I study only monotone strategies. Thus, games with monotone equilibria in the SEU environment are candidates for this method. A straightforward generalization is the class of contests with spillovers described in Baye, Kovenock, and de Vries[12]. Also, games with a common value component as in Milgrom and Weber[85] can be studied in this way.

An interesting question for future research is the existence of an efficient equilibrium in other models of ambiguity aversion. Using the model of smooth ambiguity aversion by Klibanoff, Marinacci, and Mukerji[59], I provide an example in which assumptions based on the parameters of the model can be used to establish existence. It would also be interesting, to study other models such as the Choquet expected utility[102] model used by Salo and Weber[101] to study the first-price auction.

CHAPTER 3
ELIMINATION CONTESTS WITH ENDOGENOUS BUDGETS

3.1 Introduction

We introduce a sequential model of competition with budgets that are endogenously determined. In classical consumer and producer theory, resource constraints have played a central role. Despite their importance, in many models of strategic interaction, budgets have received little attention. When budgets are considered, they are frequently modeled as exogenous parameters. However, we are interested in applications where budgets are determined by the choices of economic agents.

We have in mind strategic situations in which competitors expend resources to influence an outcome, and the resources they spend are provided by other strategic players. Examples include research departments that compete using funds provided by their institutions, military commanders that depend on their civilian governments for supplies, and politicians that depend on campaign contributions. Two aspects of these examples are that both the competitors and those providing the resources are interested in the outcome of the competition, and competitors and those providing the resources may value the resources or the outcome differently.

We focus on a sequential form of competition called an elimination contest. In an elimination contest, two pairs of contestants compete in two preliminary contests. The winners of the preliminary contests compete for a prize in a final contest. This contest structure is common in sports, political competitions, and labor promotion tournaments.

Contestants in the elimination contest depend on a set of strategic players called backers to provide the resources that will be expended in the contest. Each contestant's objective is to spend resources to maximize the probability of winning the final contest, while taking into account the other players' strategies. Each contestant's spending is constrained by the resources they receive. Backers would like to increase the chance of their contestant winning, but the increased probability of winning is weighed against the cost of providing resources.

A leading application for this model is political campaigns. We believe that, in order to understand the effect of rules governing the timing of elections and campaign finance laws, it is necessary to model the strategic behavior of donors. In our model, donors take into account the spending behavior of their preferred candidate's campaign when making contributions. They are also competing against other campaigns and donors. By modeling these interactions, we are able to describe the equilibrium effect of changes in campaign rules on spending.

We analyze two models; one in which backers provide resources to contestants only at the beginning of the game and another in which backers are permitted to provide resources at each stage of the contest. In a symmetric model, we find that expenditure is lowest when backers can provide resources throughout the contest. This result suggests that spending will be lower if a sufficient amount of time is provided between stages for contestants to raise more donations.

Unspent budgets can occur in the elimination contest when a contestant loses the preliminary contest and has some remaining budget. In political contests, the way unspent campaign funds are used after the campaign is often regulated. Although donations can be returned to donors, more often they are spent in a way that is an imperfect substitute for a refund. For example, the funds may be donated to a charity, another campaign, or a political party. In an elimination contest where resources are only provided at the beginning of the game, spending is increasing in the fraction of unspent resources that are returned to backers.

3.1.1 Related Literature

The elimination contest is an example of a contest in which the outcome is determined by the outcome of a number of component contests. Borel's formulation of the Colonel Blotto game is an example of such a contest[22]. In the Colonel Blotto game two players allocate resources from a fixed budget to a number of battle fields in order to try to win as many as possible. Borel's game is restricted in application to situations in which sequential ordering of the component contests is not present or is unimportant.[1]

An important consideration is how spending affects the probability of winning a component contest. We model the probability of winning using the lottery success function proposed by Tullock[113]. In the model, the probability of a player winning the component contest is the contestants expenditure divided by the sum of all of the contestants' expenditures in that component contest.

---

[1]See Roberson and Kovenock[66] for survey of static contests with multiple component contests.

In the literature on elimination contests Harbaugh and Klumpp[46] study a closely related model. Their model includes budget constrained contestants whose objective is to win the final contest. Their main result is that, in equilibrium, disadvantaged players spend more resources in early rounds than do the advantaged players. This contrasts the case of linear costly effort where the advantaged player spends more resources in both rounds. Stein and Rapoport[108] also study the role of budgets in an elimination contest but with contestants who face a cost of expending resources. In both papers budgets are exogenously provided at the beginning of the game. Because of this they cannot study the effect of policy changes on the budgets as we do in this paper.

Friedman[37] studies a related model with endogenous budgets. The contestants' budgets are endogenously determined by strategic players, as in this paper, and the contestants are constrained by the budget. However, in Friedman's model players allocate the budget over a number of contests simultaneously, in order to maximize the sum of contests won. Thus the sequential nature of competition is not present, and players face a different objective.

3.2 Elimination Contest Model

Four contestants compete for a prize in a two stage elimination contest. Let i =1, 2, 3, 4 index the contestants. Contestant 1 competes with contestant 2 and contestant 3 competes with contestant 4 in preliminary contests. The winners of the preliminary contests compete in a final contest to determine who wins the prize.

Contestants compete by expending resources in each contest. The probability of winning a component contest is increasing in the contestant's expenditure as modeled by Tullock. Denote contestant i's expenditure, also called a bid, in the preliminary and final contest as b_i and B_i, respectively. Expenditures are constrained to be non negative. The probability that contestant i wins a preliminary contest given both contestants' expenditures is

$$P_i(b_i, b_j) = \begin{cases} \frac{b_i}{b_i+b_j} \text{ if } b_i + b_j > 0 \\ \\ 1/2 \text{ otherwise} \end{cases} . \tag{3.1}$$

The contest success function in the final contest, as a function of the final contest expenditures, is the same.

Each contestant's expenditure is constrained by a budget in each stage. Contestant i's budget in the preliminary and final contest is given by w_i and W_i, respectively. In each stage, contestants may choose any expenditure between zero and the budget.

The novel contribution of this paper is to analyze a game in which the budgets arise endogenously in the game. This is done by introducing a set of players called backers. There are four backers each of which is paired up with a contestant. The set of players containing contestant i and backer i is called team i. Contestants can only spend the resources that their backers provide in the game. The resources that backer i provides before the preliminary round and before the final round are given by e_i and E_i, respectively. Backers are assumed to not be budget constrained and thus can provide any non negative amount of resources. The resources that backers provide are called contributions.

The relationship between contributions, bids, and budgets is the following. The budget available in the preliminary contest, w_i, is the backer's initial contribution, $w_i = e_i$. The budget in the final contest, W_i is the budget in the preliminary contest less the contestant's preliminary bid plus any addition contribution after the preliminary contest, $W_i = e_i - b_i + E_i$.

It is assumed that contestants are only concerned with maximizing the probability of winning the prize. Suppose that contestants i and j compete together and contestants k and l compete together in the preliminary contests. Contestant i's probability of winning is

$$Q_i = P_i(b_i, b_j)\left(P_k(b_k, b_l)P_i(B_i, B_k) + P_l(b_l, b_k)P_i(B_i, B_l)\right). \tag{3.2}$$

This is defined analogously for the other players.

Backers face a tradeoff between the probability of winning and the expected cost of contributing. The expected cost is a function of how much backers contribute and it may be a function of how much the competitor spends. An important consideration is how contributions that are not spent in the elimination contest are used. The leftover budget for competitor i is.

$$L_i = (1 - P_i(b_i, b_j))(e_i - b_i) + P_i(b_i, b_j)(W_i - B_i). \tag{3.3}$$

The first part is the leftover budget carried over from the preliminary contest times the probability that the contestant loses at the preliminary stage. The second part is the unspent budget at the end of the final contest times the probability of arriving at the final stage. On one extreme the unspent resources may be entirely a sunk cost to the backer. The other extreme is that backers recuperate all of the resources that are unspent. In practice there may be some intermediate value that is recuperated. In general, the expected cost is given by

$$C_i = e_i + P_i(b_i, b_j)E_i - rL_i. \tag{3.4}$$

The constant $r \in [0, 1]$ can be thought of as the fraction of unspent resources that the backer can recuperate.

Backer i's tradeoff between the expected cost and probability of winning is determined by backer i's value of team i winning. The backer's value of winning is v_i. Backer i's utility is

$$U_i = v_i Q_i - C_i. \tag{3.5}$$

The behavior of the players is modeled by subgame perfect Nash equilibrium.

The elimination contest has the following information structure. In the preliminary round backers make contributions simultaneously. Contestants then observe only their own budgets and choose their bids. The winners of the preliminary contest are determined. Each of the backers in the final contest make contributions after observing all of the bids and contributions in the previous stage. The contestants then submit final bids.

3.3 Analysis

The most simple case to study is when all of the teams are symmetric. That is, each backer has the same value of winning v. In this section, we focus on this case.

3.3.1 Fixed Budget Game

This section restricts attention to a situation in which backers can only contribute at the beginning of the game. That is, for each backer i, E_i is constrained to be zero. This is called the fixed budget game. In such a game, contestants only need to decide how to split the budget between the two stages of the elimination contest.

Because of the form of the function P, it is never an equilibrium for both contestants to bid zero at any stage since there would always be a gain to shifting some resources to that stage to insure a win there. Also, in the final contest, contestants always bid their entire remaining budgets. This follows since contestants maximize the probability of winning, and spending more in the last period will always weakly increase the probability of winning. These facts will be used in the derivation of the equilibrium,

Proposition 3.3.1. *There is a unique symmetric equilibrium of the fixed budget game in which backers provide the same contribution*

$$e = \frac{2v}{8-r}. \tag{3.6}$$

Contestants split the budgets evenly over the two stages, that is $b_i = B_i = e/2$ *for all* i.

Proof: First, I show that if three of the contestants bid the same amount in all stages the other contestant's unique best response is to split its budget evenly between the stages.

Lemma 3.3.2. *WLOG, let* $b_2 = b_3 = b_4 = B_2 = B_3 = B_4 = b$. *In the fixed budget* *elimination contest,* $b_1 = B_1 = w_1/2$ *maximizes* Q_1 *for any* $w_1 \geq 0$.

Proof: Contestant 1 solves

$$\max_{b_1 \in [0, w_1]} \frac{b_1}{b_1 + b} \frac{w_1 - b_1}{w_1 - b_1 + b}. \tag{3.7}$$

Any solution is interior and the first order condition is

$$\frac{b}{(b_1 + b)^2} \frac{w_1 - b_1}{w_1 - b_1 + b} - \frac{b_1}{b_1 + b} \frac{b}{(w_1 - b_1 + b)^2} = 0 \tag{3.8}$$

The unique solution to the first order condition is $w_1/2$. $\square$

Next I derive the unique e such that the equilibrium is as in the proposition. To do so I look for the contribution such that if every backer uses the same contribution no backer has an incentive to deviate. Lemma 3.3.2 implies that if all backers use the same contribution an equilibrium of the subsequent subgame is for the contestants to split the budget evenly between stages. Furthermore, if one backer should unilaterally deviate, the contestants will continue to split their budgets evenly. This follows because only the contestant whose backer deviates observes the deviation when preliminary bids are made.

WLOG suppose that $e_2 = e_3 = e_4$. Using lemma 3.3.2 backer 1 solves

$$\max_{e_1 \geq 0} v \left(\frac{e_1}{e_1 + e_2} \right)^2 - e_1 + r \frac{e_2}{e_1 + e_2} \frac{e_1}{2} \tag{3.9}$$

since contestants divide the budgets evenly. The first part is the probability of winning times the value of winning. The expected cost is the contribution minus the recuperated contribution; contributions are only recuperated in equilibrium if the contestant loses the preliminary contest which happens with probability $e_2/(e_1 + e_2)$.

The first order condition is

$$2\frac{ve_1e_2}{(e_1+e_2)^3} - 1 + \frac{re_2}{2(e_1+e_2)} - \frac{re_2e_1}{2(e_1+e_2)^2} = 0. \tag{3.10}$$

Next solve for the e such that if all backers contribute e it is optimal for backer 1 to contribute e. This is the solution to

$$(1/4)v - e + (1/4)re - (1/8)re = 0 \tag{3.11}$$

Which is solved uniquely by

$$e = \frac{2v}{8-r}. \tag{3.12}$$

□

An important consideration in many applications is the expected cost of the contest measured as the expected expenditure. One relevant factor is the level of reimbursement. For a fixed contribution level the expected cost to the backer will be decreasing in the reimbursement rate. However, as seen above the contribution level and spending is increasing in the reimbursement. (The intuition for this is that the marginal cost of contributing is decreasing in the reimbursement rate.) Thus the overall effect of reimbursement on the cost to backers is not obvious. The corollary shows that the the increased fraction of reimbursement compensates the backers for the increased contributions.

Corollary 3.3.3. *In the symmetric equilibrium of the fixed budget game, expected cost is decreasing in the reimbursement rate, r.*

Proof: Expected cost in equilibrium is given by

$$\frac{2v}{8-r} - .5r\frac{v}{8-r} \tag{3.13}$$

Differentiating with respect to r yields

$$\frac{2v}{(8-r)^2} - \frac{.5v(8-r)+.5rv}{(8-r)^2} \tag{3.14}$$

$$= \frac{-2v}{(8-r)^2}. \tag{3.15}$$

Which is negative for $r \in [0,1]$. □

3.3.2 Varying Budgets Game

We now turn our attention to the game in which backers may provide additional resources after the preliminary contest. This is called the varying budgets game. In many applications, there is nothing to prevent the backers from providing more resources in the final contest.

This problem is related to a game in which backers directly control expenditure so that the contestants are not part of the model. Such a game would be equivalent to the game studied by Stein and Rapoport[108] when contestants do not have binding budgets and the cost of expenditure is linear. The solution is found through backward induction. Starting from the final contest the value of winning is the value of the prize. At this stage the backers solve

$$\max_{b_i \geq 0} v_i \frac{b_i}{b_i + b_j} - b_i \tag{3.16}$$

The first order condition yields the best response $br_i(b_j) = \max\{0, \sqrt{v_i b_j} - b_j\}$. The unique equilibrium is to bid $v/4$. Since both sides use the same strategy, each wins with probability half. Thus, the value of competing in the final contest is $v/4$. This means that the value of winning the preliminary contest is $v/4$. So, using a similar argument, in the preliminary stage all teams bid $v/16$. This is all under the assumption that backers can directly control the bid.

When backers are symmetric in their value of winning they can induce the same bids as above, even if they do not directly control the bids. That is, if backers provide contributions equal to the unconstrained equilibrium bids, contestants will bid their entire budgets in each round. Furthermore, backers have no incentive to deviate from this strategy.

Proposition 3.3.4. *In the unique symmetric equilibrium of the varying budgets game,* $e_i = b_i = v/16$ *and* $E_i = B_i = v/4$ *for all* $i = 1, 2, 3, 4$.

The main section of the proof shows that there is no profitable deviation from the described actions. The contestants will not have any incentive to save in the preliminary round since they can expect to receive, in the final contest, much more than what they could save. The backers have no incentive to deviate since they would need to provide too large of a budget in order to induce contestants to save. To finish the proof I show there is no symmetric equilibrium in which all contestants split their budgets.

Now that the equilibrium actions have been described in the fixed and varying budgets games, the two games can be compared. Of particular interest is the expected cost to backers in the two games. The total expected expenditure in the varying budget game is $3v/4$.[2] In the fixed budget game, with full reimbursement of unspent

[2] This is the same expected expenditure as in a game in which four players compete for the prize in a single contest.[113]

funds, the total expected expenditure is given by $6v/7$; when funds are not fully reimbursed the total expenditure can be as high as v. Thus, the varying budget game has significantly lower equilibrium costs than the fixed budget game.

3.4 Conclusion

We have introduced a model of endogenous budgets for a dynamic contest structure. In the symmetric case, we find sharp predictions about the expected expenditure in the two games. The expenditure is lower when backers can contribute throughout the game than when backers only contribute at the beginning. In the elimination contest, we see that total spending increases as the fraction of unused resources returned increases. However, the expected cost to a backer decreases in the fraction returned.

CHAPTER 4
ALL-PAY AUCTIONS WITH TIES

4.1 Introduction

The all-pay auction[90] has been used to model a wide range of competitive environments. In the standard all-pay auction, the player with the highest bid receives a prize and the others do not receive a prize. In equilibrium, ties happen with zero probability(Amann and Leininger[3], Parreras and Rubinchik[93], Baye, Kovenock, and De Vries[11]). However, in many applications ties are not uncommon.

In battles over territory, capturing territory could be considered a win; losing territory is a loss; if the same boundaries persist, the outcome is a tie. In such conflicts, there are barriers that tend to support the status quo. These barriers may be physical, such as a wall, or they may be politically enforced by outside entities. For instance, the boundary between Russia and Ukraine is partially enforced by international norms. In order for one side to claim any new territory, it must expend significantly more than the opposing side; otherwise, the status quo remains. During the spring of 2014, Russia expended much more resources than Ukraine did in a contest over Crimea. As a result Russia gained the territory. In other parts of Ukraine, Russia faces both civilian and military opposition. The outcome seems likely to resemble a tie that gives Russia more regional influence but not complete control.

In political competition, controls are often put into place that maintain the status quo. For instance, to change the Constitution of the United States, a two thirds majority vote in Congress and ratification by three fourths of the states is required. As a result, amendments are rarely even proposed, despite the diverse opinions of law makers. Less stringent super majority rules exist for changing many laws.

Ties are also common in sports. In the World Cup, during group games, teams collect points in order to qualify for the tournament. A win provides 3 points, a tie 1 point, and a loss 0 points. Many other examples in sports exist where the margin of victory or loss affects rankings.

We model the contest as an incomplete information all-pay auction in which, if a player's bid exceeds the competitor's by a large enough margin, the player receives a winning prize. If the player's bid is near the competitor's bid, both receive a prize for tying. If the bid is too low relative to the competitor's, the player does not get a prize. The bids represent sunk expenditure of resources or efforts which cannot be recuperated, regardless of the outcome of the game.

Behavior in all-pay auctions with ties is qualitatively different from standard all-pay auctions. Bidders may all use the same action with positive probability, and there may be gaps in the bid support. This paper provides preliminary results for the analysis of such games. The analysis is demonstrated by characterizing the equilibria for games where the tying region is sufficiently large.

This work is related to a number of complete information pricing games including Shilony[107], Fisher and Wilson[36], and Szech and Weinschenk[112]. A common result is that there can be gaps in the bid support as a result of intermediate outcomes.

Another closely related strand of literature deals with the Tullock Contests[113] with incomplete information(Fey[34], Ryvkin[100] and Wasser[117]). The usual interpretation of the Tullock Contest is that the probability of winning a prize is given by the player's bid divided by the sum of all of the players' bids. Alternatively, the size of the prize is determined by the ratio. Thus, the Tullock Contest could be thought of as a game with a continuum of intermediate outcomes; whereas, we study a game with a discrete number of outcomes.

4.2 General Model

Players 1 and 2 simultaneously submit bids $b_1 \geq 0$ and $b_2 \geq 0$. Players may differ by parameters $v_i \in [\underline{v}, \overline{v}]$ and $c_i \in [\underline{c}, \overline{c}]$ with $\underline{v}$, $\overline{c} \geq 0$, $i = 1, 2$. The payoff of player i given the bids is

$$u_i(b_1, b_2) = \begin{cases} v_i - c_i b_i \text{ if } b_i \geq \frac{b_j+\delta}{\alpha} \\ \\ \beta v_i - c_i b_i \text{ if } \alpha b_j - \delta < b_i < \frac{b_j+\delta}{\alpha} \\ \\ -c_i b_i \text{ if } b_i \leq \alpha b_j - \delta \end{cases} \tag{4.1}$$

It is assumed that $\delta \geq 0$, $\beta \in (0, 1)$, and $\alpha \in (0, 1]$.

A natural interpretation of the model is that if $b_i \geq \frac{b_j+\delta}{\alpha}$, player i wins and receives a prize valued at v_i. If $\alpha b_j - \delta < b_i < \frac{b_j+\delta}{\alpha}$, then player i ties and receives an intermediate prize of value βv_i. The value of losing is normalized to 0. c_i is the unit cost of expending effort or resources in competing for a prize. Alternatively, one could normalize the value of the intermediate outcome to 0 in which case the value of winning is $(1 - \beta)v_i$ and the value of losing is $-\beta v_i$. This normalization would not change the behavior.

If we divide the utility function u_i by the unit cost c_i, we get the new utility function

$$\tilde{u}_i(b_1, b_2) = \begin{cases} \frac{v_i}{c_i} - b_i \text{ if } b_i \geq \frac{b_j+\delta}{\alpha} \\ \\ \beta\frac{v_i}{c_i} - b_i \text{ if } \alpha b_j - \delta < b_i < \frac{b_j+\delta}{\alpha} \\ \\ -b_i \text{ if } b_i \leq \alpha b_j - \delta \end{cases} \tag{4.2}$$

Since this monotonic transformation preserves the preferences of the players, it is without loss of generality to only consider unit costs of one and allow v_i to capture the heterogeneity. For the remainder of this paper, $c_i = 1$ for $i = 1, 2$.

The players' values are drawn independently from the distribution F with density f. The distribution F is atomless and has a density bounded below by $\underline{f}$ and above by $\overline{f}$ on the support $[\underline{v}, \overline{v}]$. Each player learns his own value for the prize before submitting a bid. Each player's value is private information. Behavior is modeled as Bayesian-Nash equilibrium.

Let $\overline{V}(b; s_j)$ be the supremum of the set of types of player j that bids $b_j \leq b$, according to strategy s_j. $\overline{V}(b; s_j) \equiv \sup\{v \in [\underline{v}, \overline{v}] : s_j(v) \leq b\}$. Let $\underline{V}(b; s_j)$ be the infimum of types of player j that bid $b_j \geq b$, according to strategy s_j. $\underline{V}(b; s_j) \equiv \inf\{v \in [\underline{v}, \overline{v}] : s_j(v) \geq b\}$. Let $\gamma(b) = \frac{b+\delta}{\alpha}$. The probability that player i wins with a bid of b_i is

$$P(b_i \geq \gamma(b_j)|s_j) = \int_{\underline{v}}^{\overline{V}(\gamma^{-1}(b_i);s_j)} f(t)dt. \tag{4.3}$$

The probability that player i does not lose with a bid of b_i is

$$P(b_i > \gamma^{-1}(b_j)|s_j) = \int_{\underline{v}}^{\underline{V}(\gamma(b_i);s_j)} f(t)dt \tag{4.4}$$

The expected utility from bidding b_i, for player i, with value v_i, given that player j uses the nondecreasing strategy s_j, is

$$U_i(b_i; v_i, s_j) = v_i P(b_i \geq \gamma(b_j)|s_j) + \beta v_i \left(P(b_i > \gamma^{-1}(b_j)|s_j) - P(b_i \geq \gamma(b_j)|s_j) \right) - b_i \tag{4.5}$$

$$= (v_i - \beta v_i) P(b_i \geq \gamma(b_j)|s_j) + \beta v_i P(b_i > \gamma^{-1}(b_j)|s_j) - b_i.$$

Definition 4.2.1. $h : \mathbf{R}^2 \to \mathbf{R}$ is *supermodular* if, for all $x, y \in \mathbf{R}^2$,

$$h(\max\{x_1, y_1\}, \max\{x_2, y_2\}) h(\min\{x_1, y_1\}, \min\{x_2, y_2\}) \geq h(x_1, x_2) h(y_1, y_2).$$

Note that when player j uses increasing strategy s_j, $U_i(b_i; v_i, s_j)$ is supermodular in (b_i, v_i). This follows from the fact that $h(x, y) = g_1(x) g_2(y)$ with nondecreasing g_1 and g_2 is supermodular, the sum of supermodular functions is supermodular, and the probabilities of winning or not losing are increasing in the bid, for any nondecreasing strategy. In the class of games studied in Athey[5], supermodularity of the expected utility is sufficient for existence of an equilibrium in nondecreasing strategies. In games with intermediate outcomes as described here, the existence of an increasing equilibrium can be established with minor modifications to the arguments. Athey[5] shows that there is an equilibrium with no mass point in the bid distribution except maybe at zero. In fact, this holds for any equilibrium.

Lemma 4.2.2. *There are no mass points above zero in an equilibrium bid distribution.*

Given that the strategies are strictly increase except that there may be a set of types $[\underline{v}, v_0]$ that bids zero, the generalized inverse strategy

$$s_i^{-1}(b) \equiv \inf \{v \in [\underline{v}, \overline{v}] : s_i(v) \geq b\}$$

is well defined and continuous for $b \in (\max\{0, s_i(\underline{v})\}, s_i(\overline{v}]$. Define

$$s_i^{-1}(0) \equiv \sup \{v \in [\underline{v}, \overline{v}] : s_i(v) = 0\}$$

when there is a type that bids zero. If $b < s(\underline{v})$ let $F(s^{-1}(b)) = 0$ and $f(s^{-1}(b)) = 0$. The expected utility can be written as

$$U(b_i; v_i) = (v_i - \beta v_i)F(s_j^{-1}(\gamma^{-1}(b_i))) + \beta v_i F(s_j^{-1}(\gamma(b_i))) - b_i. \tag{4.6}$$

Unlike the all-pay auctions without ties, the strategies need not be continuous.[1] However, the inverse strategies are well behave. In fact the inverse strategies are Lipschitz continuous. Thus equilibria can be studied using differential equation methods.

Lemma 4.2.3. *The inverse equilibrium strategies are Lipschitz continuous.*

It follows that if s_i and s_j are equilibrium strategies, for almost all types of player i that do not bid zero

$$\begin{aligned}&\alpha(v_i - \beta v_i)f(s_j^{-1}(\alpha s_i(v_i) - \delta))(s_j^{-1})'(\alpha s_i(v_i) - \delta)\\&+\frac{\beta v_i}{\alpha} f\left(s_j^{-1}(\frac{s_i(v_i) + \delta}{\alpha})\right)(s_j^{-1})'(\frac{s_i(v_i) + \delta}{\alpha}) = 1.\end{aligned} \tag{4.7}$$

This equation is used in the following section to characterize a type of symmetric equilibrium and provide conditions for the existence of such an equilibrium.

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[1]The next section shows an example of such an equilibrium.

4.3 Equilibria Bounded by 2δ

In this section, we will discuss symmetric, increasing equilibria which are bounded above by 2δ. We will further restrict $\alpha = 1$. All such equilibria have a specific form described below. Conditions are provided that insure that δ is sufficiently large for such an equilibrium to exist.

The differential equation argument will use the following technical assumption.

A. 11. The density f is Lipschitz continuous.

We have already seen that there cannot be a mass point above zero in equilibrium. Lemma 4.3.1 describes the gaps in the bid distribution, in other words, the discontinuities in the equilibrium strategies. Define a gap in the bid distribution to be an interval, (c, d), such that for all $\epsilon > 0$, both $(c - \epsilon, c]$ and $[d, d + \epsilon)$ intersect the support of the bid distribution but (c, d) does not intersect the support. Let $\overline{b}$ be the least-upper bound of the bid distribution. This is well defined since any bid above $\overline{v}$ is strictly dominated.

Lemma 4.3.1. *If in an equilibrium bid distribution $\overline{b} < 2\delta$, then there is exactly one gap in the bid distribution given by $(\overline{b} - \delta, \delta)$, and the support contains zero.*

This lemma implies that in any increasing equilibrium with $\overline{b} < 2\delta$, there may be a set of types that bid zero, the rest of the bid strategy is strictly increasing and continuous except for possibly one jump discontinuity. Let v_0 be the infimum of the set of types bidding above 0, and let v_1 be the type that bids δ. The lemma implies that an symmetric equilibrium strategy must be of the following form.

$$s(v) = \begin{cases} 0 \text{ for } v \in [\underline{v}, v_0) \\ s_2(v) \text{ for } v \in [v_0, v_1) \\ s_1(v) \text{ for } v \in [v_1, \overline{v}] \end{cases} \tag{4.8}$$

Where s_1 and s_2 must be continuous and strictly increasing. Furthermore, lemma 4.3.1 implies that $s_1(v_1) = \delta$, and $\lim_{v \to v_1^-} s_2(v) = \overline{b} - \delta$.

To derive such a symmetric equilibrium I rewrite equation 4.7 as

$$(1-\beta)vf(s^{-1}(s(v)-\delta)))(s^{-1})'(s(v)-\delta) = 1 \tag{4.9}$$

for $v \in [v_1, \overline{v}]$ and

$$\beta vf(s^{-1}(s(v)+\delta)))(s^{-1})'(s(v)+\delta) = 1 \tag{4.10}$$

for $v \in [v_0, v_1]$. This follows because types in $[v_0, v_1]$ bid in the interval $[0, \delta)$ and thus never bid high enough to win. When $\overline{b} \leq 2\delta$, types in $[v_1, \overline{v}]$ bid in $[\delta, 2\delta]$ and so never lose.

A solution to equations 4.9 and 4.10 that satisfies the intial conditions $s_1(v_1) = \delta$ and $\lim_{v \to v_1^-} s_2(v) = \overline{b} - \delta$ is said to be a solution to the *equilibrium first order condition.*

For an arbitrary $\overline{b}$, there may not be a strategy that solves the equilibrium first order condition and satisfies $s(\underline{v}) = 0$, as required by equilibrium. This is likely to occur if δ is relatively small and $\overline{b}$ is close to 2δ.

In addition, for a strategy $s(v)$ to be an equilibrium, there are two types of potentially profitable deviations that must be ruled out. One type of deviation that must be ruled out is one in which a player with type v bids another bid within the convex hull of the bid distribution. I will call these local deviations because they can be ruled out using local arguments. Local deviation are ruled out as long as v_1 is indifferent between bidding $\overline{b} - \delta$ and δ. This insures that players with values below v_1 have no incentive to increase their bid to δ and players with values above v_1 do not prefer to bid $\overline{b} - \delta$. The indifference condition is

$$\beta s^{-1}(\delta) - (\overline{b} - \delta) = s^{-1}(\delta)F(s^{-1}(0)) + \beta v_1(\overline{b})(1 - F(s^{-1}(0)) - \delta. \tag{4.11}$$

Condition 4.11 must hold in equilibrium. To the contrary, types near v_1 would have an incentive to deviate either down to $\overline{b} - \delta$ or up to δ.

$x(.;2\delta)$ is defined as the fixed point of the operator in Lemma C.2.2 of the appendix. $x(.,2\delta)$ is constructed such that if the equilibrium first order conditions have a solution for $\overline{b} = 2\delta$ such that the solution $s(.;2\delta)$ satisfies $s(v';2\delta) = 0$ for some $v' \geq 0$, then $x(.;2\delta) = s^{-1}(v;2\delta)$ for all $v > v'$. If for $\overline{b} = 2\delta$ there is no solution such that some type bids zero, $x(.;2\delta)$ is defined to be $\underline{v}$ at the lower bids, and it coincide with $s^{-1}(.;2\delta)$ where the inverse is well defined.

Proposition 4.3.2. *If $\delta < (1 - \beta)\overline{v}$ and $x(0;2\delta) = \underline{v}$, there exists a nondecreasing strategy $s(.)$ that satisfies equation 4.11 and solves the equilibrium first order conditions.*

Intuitively, the condition that $x(0;2\delta) = \underline{v}$ implies that δ is large enough so that the equilibrium first order condition will not allow the strategy to stretch from $s(\overline{v}) = 2\delta$ down to 0 within the bid support. The proof of this proposition starts by constructing an inverse strategy that coincides with the solution to the first order conditions, when such a strategy exists with $s(\underline{v}) = 0$. This inverse strategy is defined for any $\overline{b} \in [\delta, 2\delta]$. The proof shows that $\overline{b}$ can be chosen to satisfy condition 4.11. $\delta < (1-\beta)\overline{v}$ insures that if $\overline{b} = \delta$, bidders prefer to bid δ rather than $\overline{b} - \delta = 0$. If $x(0; 2\delta) = \underline{v}$ there is a $\overline{b}$ such that the opposite is true. By establishing the continuous dependence of $x(., \overline{b})$ on $\overline{b}$ the result is established because $\overline{b}$ can be chosen to make players indifferent between bidding δ and $\overline{b} - \delta$.

Another type of deviation, that must be ruled out, is deviations to bids above $\overline{b}$. In analogous games without intermediate outcomes, this type of deviation is never profitable because by bidding above $\overline{b}$ a player increases the bid without increasing the probability of winning. With intermediate outcomes, increasing the bid in the interval $(2\delta, \overline{b} + \delta)$ does increase the probability of winning. To rule out this type of deviation, when $\beta \leq .5$, it is sufficient to check that type $\overline{v}$ has no incentive to bid $\overline{b} + \delta$. Otherwise, one must check all deviations in the interval $b \in [2\delta, \overline{b} + \delta]$.

Proposition 4.3.3. *A strategy $s(.)$ with upper bound $\overline{b} < 2\delta$ is a symmetric increasing equilibrium if and only if it solves the equilibrium first order conditions and satisfies condition 4.11 and either*

1. *$\beta \leq .5$ and $(1-\beta)\overline{v}F(s^{-1}(\delta)) + \beta\overline{v} - \overline{b} \geq \overline{v} - \overline{b} - \delta$ or*

2. *$\beta > .5$ and $(1-\beta)\overline{v}F(s^{-1}(\delta)) + \beta\overline{v} - \overline{b} \geq (1-\beta)\overline{v}F(s^{-1}(b-\delta)) + \beta\overline{v} - b$ for all $b \in [2\delta, \overline{b} + \delta]$.*

The proof in the appendix shows the following. If $\overline{b}$ is such that condition 4.11 and the equilibrium first order condition can be satisfied, then the strategy which solves these conditions is an equilibrium of a game in which bids are constrained to be in the interval $[0, \bar{b}]$. If either condition 1 or 2 of Proposition 4.3.3 is satisfied, no player has an incentive to unilaterally deviate from this strategy when the bids are not constrained.

The conditions of Proposition 4.3.2 are not too far from the model primitives. But checking the existence of an equilibrium using conditions 1 or 2 of Proposition 4.3.3 may require the construction of the strategies that satisfy the equilibrium first order conditions and condition 4.11. In the following example, we use the conditions of Propositions 4.3.2 and 4.3.3 to find simple bounds on the parameters, which guarantee existence of an equilibrium.

4.4 Example

In this example, let each player's value be distributed uniformly over the interval $[\underline{v}, 1]$. Assume that $\beta \leq .5$ and $\delta \leq (1-\beta)\overline{v}$. Without solving a differential equation we show how Proposition 4.3.3 can be used to identify simple sufficient conditions for the existence of an equilibrium of the type described.

First, we look for conditions under which $x(0, 2\delta) > \underline{v}$. Since, $s(., 2\delta)$ must solve equations 4.9 and 4.10 we can place an upper bound on the slope of the strategy. Using equations 4.9 and 4.10 and that the density is $f = 1/(1-\underline{v})$, it can be seen that

$$x'(b, 2\delta) \geq \frac{1-\underline{v}}{1-\beta} \tag{4.12}$$

for $b \in [0, 2\delta]$. Thus the slope of $s(., 2\delta)$ is less than $(1-\beta)/(1-\underline{v})$. If

$$\frac{1-\beta}{1-\underline{v}}(1-\underline{v}) \leq 2\delta, \tag{4.13}$$

then $s(., 2\delta)$ changes by less that 2δ over the interval $[\underline{v}, 1]$. Therefore, the condition of Proposition 4.3.2 is satisfied.

Lastly, we can guarantee that the first condition of Proposition 4.3.3 is satisfied. This condition can be written in this case as

$$(1-\beta)\frac{s^{-1}(\delta)}{1-\underline{v}} + \beta \geq 1-\delta. \tag{4.14}$$

This is implied by

$$(1-\beta)\frac{\underline{v}}{1-\underline{v}} + \beta \geq 1-\delta. \tag{4.15}$$

Simplifying yields the condition

$$(1-\beta)\frac{1-2\underline{v}}{1-\underline{v}} \leq \delta \tag{4.16}$$

To summarize, using Proposition 4.3.3 we can find a region where an equilibrium of the type described exists. Furthermore, these conditions did not require solving a differential equation. The condition that $\delta \in [(1-\beta)/2, 1-\beta]$ insures that there is a non trivial strategy in which there is no profitable deviation within the bid support. When $\beta \leq .5$, the condition $(1-\beta)\frac{1-2\underline{v}}{1-\underline{v}} \leq \delta$ guarantees that, for such a strategy, there is no benefit from deviating to a bid above the support.

It should be noted that these conditions are sufficient but stronger than necessary. More generally the conditions of the proposition can be checked by solving the relevant differential equation. Below I provide parameters under which an equilibrium strategy can be solved for explicitly.

Restrict the parameters to $\underline{v} = 0$, $\beta = .5$, and $\delta = .25$. δ is chosen so that there is an equilibrium such that the bid support can be contained in the interval $[0, 2\delta]$. Lemmas 4.2.2 and 4.3.1 show that we can restrict attention to equilibria in which the bid strategy is strictly increasing everywhere except that there may be a mass of types at the bottom bidding 0.

I will derive a symmetric increasing strategy s which is piecewise quadratic except where it is zero. Specifically the strategy is of the following form.

$$s(v) = \begin{cases} 0 \text{ for } v \in [0, v_0) \\ s_2(v) \equiv a_2 v^2 + d_2 \text{ for } v \in [v_0, v_1) \\ s_1(v) \equiv a_1 v^2 + d_1 \text{ for } v \in [v_1, 1] \end{cases} \tag{4.17}$$

There are several observations that are used to derive the parameters of this equilibrium.

1. $s_2(v_1) + \delta = s_1(1)$
2. $s_1(v_1) = \delta$
3. $s_2(v_0) = 0$
4. $v_1 v_0 + \beta v_1(1 - v_0) - \delta = \beta v_1 - s(v_1)$

The first three conditions follow from Lemmas 4.2.2 and 4.3.1. They are necessary for the equilibrium bid support to be of the form $[0, \overline{b} - \delta] \bigcup [\delta, \overline{b}]$. The last condition is simply the indifference condition 4.11.

Two other conditions arise from the equilibrium FOC of players with valuations in $[v_0, v_1]$ and $[v_1, 1]$. These conditions relate c_1 to c_2 and d_1 to d_2. Using the

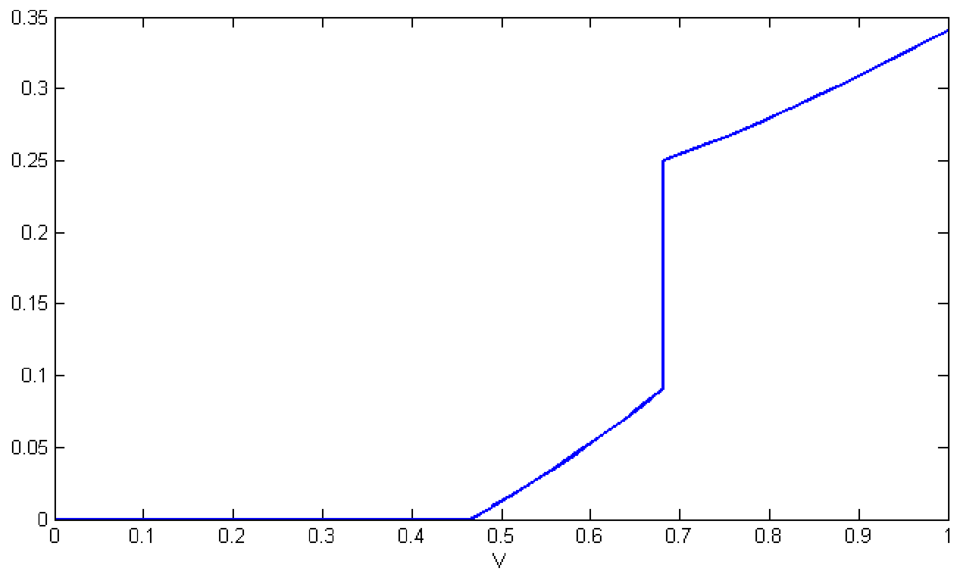

Figure 4.1: Example:Uniform$(0,1)$ distribution, $\beta = .5$, and $\delta = .25$

hypothesized functional form, equations 4.9 and 4.10 are equivalent to

$$\frac{.5v}{2c_2\sqrt{\frac{s_1(v)-\delta-d2}{c_2}}} = 1 \tag{4.18}$$

$$\frac{.5v}{2c_1\sqrt{\frac{s_2(v)+\delta-d1}{c_1}}} = 1. \tag{4.19}$$

If s_1 and s_2 are quadratic, then following two condition must be satisfied.

5. $d_1 = d_2 + \delta$

6. $16c_1c_2 = 1$

The parameters are solved for using these six conditions. It is straightforward to verify that the conditions of Proposition 4.3.3 are satisfied. Figure 4.1 graphs this equilibrium strategy.

4.5 Conclusion

We have introduced a model of competition with intermediate outcomes. General existence and properties of equilibrium are discuss. We characterize the class of equilibria which are increasing and symmetric and bounded by 2δ. The method for calculating an equilibrium is to solve a differential equation. If δ is large enough, shooting methods can be used to find an $\bar{b}$ so that the solution satisfies local constraints.

The main difficulty in studying games with ties is that local constraints are not always sufficient. In standard all-pay auctions it is sufficient to satisfy the initial condition and local constraints. With intermediate outcomes it is not hard to construct examples where local constraints are satisfied, but there is an incentive to deviate to a bid above the support of the strategies.

APPENDIX A
APPENDIX FOR CHAPTER 2

A.1 First Order Conditions

Proof of Lemma 2.3.1: Proceed by contradiction. Let β be an increasing equilibrium strategy and suppose that there exist $v, z \in [\underline{v}, \overline{v}]$ such that $|\beta(v) - \beta(z)| > M|v - z|$ where $M = \frac{\overline{vg}}{(1-p)^2}$. WLOG let $v > z$. The strategy subscript on the minimizing distributions is suppressed to avoid notational clutter.

$$V(v; v, \beta) - V(z; v, \beta) = U(v; v, \beta, G_{v,v}) - U(z; v, \beta, G_{v,z})$$

$$\leq U(v; v, \beta, G_{v,z}) - U(z; v, \beta, G_{v,z})$$

$$= v(G_{v,z}(v) - G_{v,z}(z)) - p\int_z^v \beta(t) g_{v,z}(t)dt - (1 - pG_{v,z}(v))\beta(v) + (1 - p(G_{v,z}(v) + G_{v,z}(z) - G_{v,z}(v)))\beta(z)$$

$$= v(G_{v,z}(v) - G_{v,z}(z)) - p\int_z^v \beta(t) g_{v,z}(t)dt - (1 - pG_{v,z}(v))(\beta(v) - \beta(z)) + p(G_{v,z}(v) - G_{v,z}(z))\beta(z)$$

$$\leq \overline{vg}(v - z) - (1 - p)(\beta(v) - \beta(z)) + p\overline{g}(v - z)\frac{\overline{v}}{1 - p}$$

$$= \frac{\overline{vg}}{1 - p}(v - z) - (1 - p)(\beta(v) - \beta(z)) < 0$$

The first inequality follows from the definition of $G_{v,z}$. The weak inequality on the fourth line follows because the densities are bounded by $\overline{g}$, and $\beta(v) \leq \frac{\overline{v}}{1-p}$ for all v. The bound on β follows since with probability $1 - p$ each player pays his bid so it is a dominated strategy to bid above $\frac{\overline{v}}{1-p}$. The strict inequality follows from the first supposition. Thus any equilibrium is Lipschitz continuous with constant $M = \frac{\overline{vg}}{(1-p)^2}$. Because the last inequality is strict there is a positive measure of types that have a profitable deviation. This contradicts that β is an equilibrium.

$\square$

Proof of Envelope Theorem: The proof uses the envelope theorems of Milgrom and Segal[83] (hereafter MS).

Part 1: $U(z; v, s, G)$ is absolutely continuous in z since each $G \in \Delta$ is absolutely continuous and s is absolutely continuous. By MS Theorem 2 to prove (1) it is sufficient to show that there exists $B > 0$ such that $|U'(z; v, s, G)| \leq B$ for almost all $z \in [\underline{v}, \overline{v}]$ and for all $G \in \Delta$.

Suppose $s(.)$ is differentiable at z. For all $G \in \Delta$,

$$|U'(z; v, s, G)| = |vg(z) - (1 - pG(z))s'(z)| \leq \frac{\overline{vg}}{(1-p)^2}. \tag{A.1}$$

The inequality follows since s is Lipschitz continuous with constant $M = \frac{\overline{vg}}{(1-p)^2}$ and g is bounded by $\overline{g}$. Since s is differentiable almost everywhere the result is proved.

The following definition is used in proving Part 2.

Definition A.1.1. The collection of functions $\{l(.; G)\}_{G\in\Delta}$ is equidifferentiable at z if

$$\frac{l(z'; G) - l(z; G)}{z' - z} \tag{A.2}$$

converges uniformly as $z' \to z$.

This condition is satisfied for instance if $\{l'(.; G)\}_{G\in\Delta}$ is an equicontinuous collection. Suppose $\{l(.; G)\}_{G\in\Delta}$ and $\{h(.; G)\}_{G\in\Delta}$ are equidifferentiable at v and $f(.)$

is differentiable at v. Then $\{l(.;G)+h(.;G)\}_{G\in\Delta}$ and $\{l(.;G)f(.)\}_{G\in\Delta}$ are equidifferentiable at v.

Part 2: From MS Theorem 3, it is sufficient to show that $\{U(.;v,s,G) : G \in \Delta\}$ is equidifferentiable at v wherever s is differentiable. Since Δ is compact in the sense that the densities form a compact set in $C([\underline{v},\overline{v}])$, by the Arzelà-Ascoli theorem the densities of the distributions in Δ are an equicontinuous set. Thus, Δ is equidifferentiable. Since s is bounded and g is bounded above by $\overline{g}$ for all $G \in \Delta$, $\left\{\int_{\underline{v}}^{z} s(t)g(t)dt\right\}_{G\in\Delta}$ is equidifferentiable. $\{(1-pG(.))s(.)\}_{G\in\Delta}$ is equidifferentiable at v whenever s is differentiable at v. Since equidifferentiability respects sums $\{U(.;v,s,G)\}_{G\in\Delta}$ is equidifferentiable at v whenever s is differentiable at v. Since s is differentiable almost everywhere the result follows from MS.

Part 3: is a direct consequence of Theorem 1 in MS.

□

Lemma A.1.2. $\mathcal{G} : [\underline{v},\overline{v}]\times[\underline{v},\overline{v}]\times IBC([\underline{v},\overline{v}]) \to 2^{\Delta}$ *is upper semicontinuous, compact valued, and convex valued. Here* $IBC([\underline{v},\overline{v}])$ *is the space of increasing, continuous functions on* $[\underline{v},\overline{v}]$ *which are bounded by* $\frac{\overline{v}}{1-p}$ *and it is endowed with the uniform topology.*

Proof: First I show that

$$U(z;v,\beta,G) = vG(z) - \int_{\underline{v}}^{z} p\beta(t)g(t)dt - (1-pG(z))\beta(z) \tag{A.3}$$

is continuous in all of its arguments. Focus on the middle summand as continuity of the rest follows easily.

Let $|z-\hat{z}|<\epsilon_1$ and $\|g-\hat{g}\|_\infty<\epsilon_2$ and $\|\beta-\hat{\beta}\|_\infty<\epsilon_3$. WLOG let $z>\hat{z}$

$$\left|\int_{\underline{v}}^{z} p\beta(t)g(t)dt-\int_{\underline{v}}^{\hat{z}} p\hat{\beta}(t)\hat{g}(t)dt\right|$$

$$\leq \left|\int_{\underline{v}}^{\hat{z}} p\beta(t)g(t)-p\hat{\beta}(t)\hat{g}(t)dt\right|+\left|\int_{\hat{z}}^{z} p\beta(t)g(t)dt\right|$$

$$=\left|\int_{\underline{v}}^{\hat{z}} p\beta(t)g(t)-\hat{\beta}(t)g(t)+\hat{\beta}(t)g(t)-p\hat{\beta}(t)\hat{g}(t)dt\right|+\left|\int_{\hat{z}}^{z} p\beta(t)g(t)dt\right|$$

$$\leq \int_{\underline{v}}^{\hat{z}} pg(t)\left|\beta(t)-\hat{\beta}(t)\right|dt+\int_{\underline{v}}^{\hat{z}} p\hat{\beta}(t)\left|g(t)-\hat{g}(t)\right|dt+\left|\int_{\hat{z}}^{z} p\beta(t)g(t)dt\right|$$

$$< p\overline{g}\epsilon_3(\overline{v}-\underline{v})+\frac{p\overline{v}}{(1-p)^2}\epsilon_2(\overline{v}-\underline{v})+\frac{p\overline{vg}}{(1-p)^2}\epsilon_1$$

By making ϵ_1, ϵ_2 and ϵ_3 small enough the result is obtained. Berge's maximum theorem establishes that $\mathcal{G}_{v,z;s}\equiv\arg\min_{G\in\Delta}U(v;z,s,G)$ is u.s.c. and compact valued.

That $\mathcal{G}_{v,b;s}$ is convex valued follows from the convexity of Δ and the linearity of $U(v;z,s,G)$ as a function of G.

□

Proof of Theorem 2.3.3: Let β, an increasing, symmetric equilibrium, be differentiable at v. It follows from the proof of the envelope theorem that $V'_+(.;v,\beta)$ and $V'_-(.;v,\beta)$ both exist at v. Since $\beta(v)$ is an equilibrium $V'_+(v;v,\beta)\geq 0\geq V'_-(v;v,\beta)$. Furthermore, by MS Theorem 3

$$V'_+(v;v,\beta)=\lim_{z\to v+} vg_{v,z}(v)-(1-pG_{v,z}(v))\beta'(v) \tag{A.4}$$

and

$$V'_-(v;v,\beta)=\lim_{z\to v-} vg_{v,z}(v)-(1-pG_{v,z}(v))\beta'(v). \tag{A.5}$$

Since $\mathcal{G}_{v,z;\beta}$ is upper semicontinuous and since $vg(v)-(1-pG(v))\beta'(v)$ is a continuous function of G, there is a $\hat{G}\in\mathcal{G}_{v,z;\beta}$ such that $V'_+(v;v,\beta)=v\hat{g}(v)-(1-p\hat{G}(v))\beta'(v)$.

There is also a $\overline{G} \in \mathcal{G}_{v,z;\beta}$ such that $V'_{-}(v; v, \beta) = v\overline{g}(v) - (1 - p\overline{G}(v))\beta'(v)$. Since $\mathcal{G}_{v,z;\beta}$ is convex valued there exists a $\check{G} \in \mathcal{G}_{v,z;\beta}$ such that $v\check{g}(v) - (1 - p\check{G}(v))\beta'(v) = 0$. Thus there is a selection $G_{v,z,\beta}$ of $\mathcal{G}_{v,z,\beta}$ such that $\beta'(v) = \frac{vg_{v,v;\beta}(v)}{1-pG_{v,v,\beta}(v)}$ a.e. $v \in [\underline{v}, \overline{v}]$. By Lemma2.3.1, β is absolutely continuous and can thus be written as in the theorem.

□

A.2 Main Theorem

Definition A.2.1.

$$\lambda(v, z; s) \equiv \left\{ \lambda \in \mathbf{R} : \lambda = \frac{vg(z)}{1 - pG(z)} \text{ for some } G \in \mathcal{G}_{v,z,s} \right\} \tag{A.6}$$

Lemma A.2.2. *$\lambda(v, z; s)$ is upper semicontinuous and has compact, convex values.*

Proof:

$$\hat{\lambda}(v, G) \equiv \frac{vg(v)}{1 - pG(v)} \tag{A.7}$$

By the continuity of $\hat{\lambda}$ and Lemma A.2, $\lambda(v, b; s)$ is upper semicontinuous and compact valued. The continuity of $\hat{\lambda}(v, .)$ implies that $\lambda(v, b; s)$ is convex valued since $\mathcal{G}_{v,z,s}$ is convex valued. □

The proof of Proposition 2.3.4 uses a convergence result which is useful in the study of differential inclusions. For reference, I state a version of the theorem which is proved in Aubin and Cellina[6].

Proposition A.2.3. *[Convergence Theorem] Let F be a u.s.c. map from $\mathbf{R}^2$ to the closed, convex subsets of $\mathbf{R}$. Let I be an interval of $\mathbf{R}$ and $x_k(.)$ and $y_k(.)$ be* *measurable functions from* I *to* $\mathbf{R}^2$ *and* $\mathbf{R}$*, respectively, satisfying for almost all* $t \in I$*, for every* ϵ*-ball,* $B_\epsilon(0)$*, in* $\mathbf{R}^2 \times \mathbf{R}$ *there is a* $k_0 \equiv k_0(t, \epsilon)$ *such that for all* $k \geq k_0$*,* $(x_k(t), y_k(t)) \in \text{graph}(F) + B_\epsilon(0)$*.*

If

- $x_k(.)$ *converges almost everywhere to a function* $x(.)$ *from* I *to* $\mathbf{R}^2$*,*
- $y_k(.)$ *belongs to* $L_1(I, \mathbf{R})$ *and converges weakly to* $y(.)$ *in* $L_1(I, \mathbf{R})$*,*

then for almost all $t \in I$*,* $(x(t), y(t)) \in \text{graph}(F)$*, i.e.* $y(t) \in F(x(t))$*.*

Proof of Proposition 2.3.4:

The proof follows the technique in Aubin and Cellina (pages 128-129, [6]). For completeness and since the differential inclusion here is slightly different from theirs, I include the details. Let $M = \frac{\overline{v}g}{(1-p)^2}$.

$$\mathcal{K} = \{x \in C([\underline{v}, \overline{v}]) : x \textit{ is Lipschitz with constant } M \textit{ and } x(\underline{v}) = 0\} \tag{A.8}$$

$\mathcal{K}$ is compact by the Arzelà-Ascoli Theorem.

$$\mathcal{J}(s) \equiv \{z \in \mathcal{K} : z'(v) \in \lambda(v, s(v); s)\} \tag{A.9}$$

A fixed point of $\mathcal{J}$ satisfies the conditions for the type of strategy described in the statement of the theorem.

I now show that the Kakutani-Glicksberg-Fan fixed point theorem applies to $\mathcal{J}(.)$. First I argue that $\mathcal{J}(.)$ is non empty. For any continuous $s(.)$, $\lambda(v, s(v); s)$ is u.s.c. as a function of v. Thus $\lambda(v, s(v); s)$ has a measurable selection. If w(.) is such a selection, then $\int_{\underline{v}}^{v} w(t)dt$ is in $\mathcal{J}(x)$. That $\mathcal{J}(.)$ is convex valued is straight forward since $\lambda(v, s(v); s)$ is convex valued.

To establish upper semicontinuity, since $\mathcal{K}$ is compact it is sufficient to show that $\mathcal{J}$ has a closed graph. This is done through the convergence theorem. Let $x_k \in \mathcal{K}$ and $z_k \in \mathcal{J}(x_k)$ be such that $x_k \to x$ and $z_k \to z$. Let $y_k = z_k'$ for all k. Since $||y_k||_\infty \leq M$ for all k, by the Banach-Alaoglu theorem there is a subsequence of $\{y_k\}$ and y such that $||y||_\infty \leq M$ and $\int_{\underline{v}}^{\overline{v}} y_k(t)\phi(t)dt \to \int_{\underline{v}}^{\overline{v}} y(t)\phi(t)dt$ for all $\phi \in L_1[\underline{v}, \overline{v}]$. Since $L_\infty[\underline{v}, \overline{v}]$ is a subset of $L_1[\underline{v}, \overline{v}]$, $\{y_k\}$ converges to y weakly as a sequence in $L_1[\underline{v}, \overline{v}]$.

Since y_k converges weakly it converges pointwise almost everywhere. $y_k(v) \in \lambda(v, x_k(v); x_k)$ so by the u.s.c. of λ in all of its arguments there exists a $k_0(v, \epsilon)$ s.t. for all $k \geq k_0$, $(x_k(v), y_k(v)) \in \text{graph}(\lambda(v, .; x)) + B_\epsilon(0)$ for almost all $v \in [\underline{v}, \overline{v}]$. By the convergence theorem $y(v) \in \lambda(v, x(v); x)$ for almost all $v \in [\underline{v}, \overline{v}]$.

Since $y = z'$, $J(.)$ is upper semicontinuous. By the Kakutani-Glicksberg-Fan fixed point theorem, there exists a strategy $\beta \in J(\beta)$. Such a strategy satisfies the conditions of the proposition.

$\square$

A.3 Existence

Proof of Proposition 2.3.5: I will first prove that if β is the strategy played by the other player, then G^* is the worst case distribution. To that end, suppose that for some v and z in $[\underline{v}, \overline{v}]$ there is another distribution which gives a strictly lower expected utility. Using a similar expression to (2.9), the previous statement is equivalent to

$$\int_{\underline{v}}^{z} \left\{ \frac{vg^*(t)}{1 - pG^*(t)} - \frac{tg^*(t)}{1 - pG^*(t)} \right\} (1 - pG^*(t))dt$$

$$> \int_{\underline{v}}^{z} \left\{ \frac{vg(t)}{1-pG(t)} - \frac{tg^*(t)}{1-pG^*(t)} \right\} (1-pG(t))dt$$

For some $G \in \Delta$.

By canceling and collecting terms this implies

$$\overline{v}(G^*(z) - G(z)) > \int_{\underline{v}}^{z} tg^*(t) \left(1 - \frac{1-pG(t)}{1-pG^*(t)} \right) dt \tag{A.10}$$

However this contradicts the hypothesis about G^*. Since z and v where arbitrary G^* always minimizes the expected utility. So, β satisfies the necessary condition. For β, (2.9) reduces to

$$\int_{\underline{v}}^{z} (v-t)g^*(t)dt \tag{A.11}$$

which is clearly maximized at v. □

Proof of Proposition 2.4.1 I first show that, in a neighborhood above v^*, H is the minimizing distribution for any equilibrium strategy. By calculation, it is can be shown that H continues the be the minimizing distribution thereafter.

Observe that $h(v^*) < g(v^*)$ and $H(v^*) < G(v^*)$ together imply that $\beta'_H(v^*) < \beta'_G(v^*)$. Let β satisfy the conditions of Theorem 2.3.3 and let $\hat{\beta}$ be as in Proposition 2.4.1. By Theorem 2.3.3, $\beta_H(v) \leq \beta(v) \leq \beta_G(v)$ for all $v \in [v^*, v^{**}]$ where v^{**} is such that $h(v) < g(v)$ and $H(v) < G(v)$ both continue to hold on the interval. This implies that $U(v; v, \beta, G) \geq U(v; v, \beta_G, G)$ and $U(v; v, \hat{\beta}, H) \geq U(v; v, \beta, H)$ for all $v \in [v^*, v^{**})$. Define $\tilde{U}(v; s, F) \equiv U(v; v, s, F)$ for all $F \in \Delta$.

$$\tilde{U}'(v; \beta_G, G) = G(v) \geq H(v) = \tilde{U}'(v; \hat{\beta}, H) \tag{A.12}$$

for all $v \in (v^*, v^{**}]$. Since $U(v^*; v^*, \beta_G, G) = U(v^*; v^*, \hat{\beta}, H)$ and the expected utilities are absolutely continuous on $(v^*, v^{**}]$, $U(v^*; v^*, \beta_G, G) \geq U(v^*; v^*, \hat{\beta}, H)$ for all $v \in$

$(v^*, v^{**}]$. Thus, H is the worst case distribution when both players follow β. By direct calculation v^{**} can be taken to be $\overline{v}$.

APPENDIX B
APPENDIX FOR CHAPTER 3

B.1 Proof of Proposition 3.3.4:

We first show that the actions described are part of an equilibrium.

There is no profitable deviation from the prescribed equilibrium behavior in the final stage of the contest. The contestants will always bid the entire budget in the last round. Conditional on the players following the strategies in the preliminary round the unique equilibrium of the final subgame is for both backers to contribute $v/4$.

In order to show that there are not profitable deviations in the preliminary round we must consider the behavior in the final round following a unilateral deviation in the preliminary stage. The final stage subgames are completely described by the amount of resources each contestant who wins carries over to the final. These subgames will be represented by the pair $(w_1 - b_1, w_2 - b_2)$.

We are only interested in unilateral deviations so WLOG suppose that the other backers and contestants used the prescribed bids and thus arrive at the final stage with no resources. Thus we are interested in subgames of the type $(w_1 - b_1, 0)$.

Backer i's best response to W_j $j \neq i$ is

$$e_i(W_j) = \max\left\{0, \sqrt{vW_j} - W_j - (w_i - b_i)\right\} \tag{B.1}$$

Thus the Nash equilibrium of subgame $(w_1 - b_1, 0)$ is $\varepsilon_1 = v/4 - (w_1 - e_1)$ and $\varepsilon_2 = v/4$ if $w_1 - b_1 \leq v/4$; if $w_1 - b_1 > v/4$, $e_1 = 0$ and $e_2 = \max\left\{0, \sqrt{v(w_1 - b_1)} - (w_1 - b_1)\right\}$.

Contestants spend their entire budgets.

The only way that saving resources can benefit contestant 1 is if the amount of saved resources exceeds the resources backer 1 would have provided in the final. If players follow the prescribed strategies, there is no incentive for contestants to save. This follows because $e_i = v/16$ for all i so if a contestant saves it decreases the probability of winning the preliminary contest without increasing the probability of winning the final since $v/16 < v/4$.

Now focus on the backer's preliminary strategy. To do so we will need to describe contestant 1's behavior after backer 1 one deviates. It will be useful to introduce the following transformation: $\varepsilon_1 = w_1 = \alpha v$ for $\alpha \geq 0$. Using this notation, if contestant 1 saves resources in the preliminary stage, the optimal allocation solves

$$\max_{0 \leq b_1 \leq \alpha v} \frac{b_1}{b_1 + v/16} \frac{\sqrt{\alpha v - b_1}}{\sqrt{v}}. \tag{B.2}$$

The FOC is

$$\frac{v/16}{(b_1 + v/16)^2} \frac{\sqrt{\alpha v - b_1}}{\sqrt{v}} - \frac{b_1}{b_1 + v/16} \frac{1}{2\sqrt{v(\alpha v - b_1)}} = 0 \tag{B.3}$$

which is equivalent to

$$\frac{v}{8}(\alpha v - b_1) = b_1(b_1 + v/16). \tag{B.4}$$

This has the unique positive root

$$b_1(\alpha) = v\frac{\sqrt{9 + 128\alpha} - 3}{32}. \tag{B.5}$$

Now let us derive contestant 1's best response to α. First, we identify the value of α such that the contestant is indifferent between splitting the budget and bidding the entire budget. The value of spending the entire budget is $.5 * \alpha v/(\alpha v + v/16)$, since in the final round both contests with spend the same amount $v/4$. Thus the α that solves

$$\frac{\alpha v}{2(\alpha v + v/16)} = \frac{b_1(a)}{b_1(a) + v/16}\sqrt{(av - b_1(a))/v} \tag{B.6}$$

is the indifference point. Equation B.6 has a unique solution at $\alpha' \approx .545$.

Additionally, there is no incentive to save more than v into the final round, since at any bid higher than v, contestant 1's competitor would bid 0 in the last round. Thus the best response is

$$BR_1(\alpha) = \begin{cases} \max\left\{v\frac{\sqrt{9+128\alpha}-3}{32}, \alpha v - v\right\} \text{ for } \alpha > \alpha' \\ \\ \alpha v \text{ otherwise.} \end{cases} \tag{B.7}$$

Let α'' satisfy

$$b_1(\alpha'') = \alpha'' v - v. \tag{B.8}$$

$\alpha'' = (\sqrt{(}129) + 31)/32$ which is approximately 1.327. This is the point where for an $\alpha \geq \alpha''$ the contestant saves exactly v.

Now we check deviations in which backer 1 provides enough resources that contestant 1 does not spend everything in the prelim. We have already shown that if contestant 1 does not save it is better for backer 1 to bid $v/16$. There is never an incentive to contribute such that $\alpha > \alpha''$ since doing so only increases the preliminary round bid and the utility is decreasing for preliminary bids great than 3/16 when the final contribution is 1.

We focus on the following maximization problem.

$$\max_{\alpha' \leq \alpha \leq \alpha''} \frac{vb_1(\alpha)}{b_1(\alpha)+v/16} \frac{\sqrt{\alpha v - b_1(a)}}{\sqrt{v}} - \alpha v + r(\alpha v - b_1(a)) \frac{v/16}{b_1(\alpha)+v/16} \tag{B.9}$$

It is sufficient to focus on the case that $r = 1$ since the backer's utility from inducing the contestant to save is always highest when any remaining resources are returned in the event of a loss.

The objective function has exactly 3 roots given by 0, .0172, and .527. Using these facts it can be easily shown that the objective is negative above .527. In fact α' maximizes the objective on the relevant range. Contributing α' yields a utility of $\approx -.00475v$ which is much less than the equilibrium utility of contributing $v/16$. Since there is no incentive to give more than $v/16$ when $r = 1$, there is no incentive for smaller r.

It is also straightforward to rule out symmetric equilibria in which contestants split their budgets. It follows from Lemma 3.3.2 that if all the backers provide the same budget and contestants split the budget the split is half in each stage. If $e_i < v/2$ and contestants save less than $v/4$ into the next stage, then in the equilibrium of the final subgame all of the backers would increase the budget to $v/4$. However, by subgame perfection arguments, if this is the case, a contestant is better off spending the whole budget in the preliminary round since he will receive $v/4$ in the last round. But in a symmetric equilibrium if $e_i > v/2$ the backer's utility is

$$U_i = v_i/4 - e_i + re_i/2 < 0. \tag{B.10}$$

Which contradicts equilibrium since $e_1 = E_1 = 0$ yields a utility of 0. $\square$

APPENDIX C
APPENDIX FOR CHAPTER 4

C.1 Preliminaries

Proof of lemma 4.2.2: Suppose that there is a mass point at $b > 0$ in the equilibrium bid distribution for some player. It follows that there is a gap below both $b - \delta$ and $b + \delta$. If this were not so, there would be a profitable deviation for those bidding near $\alpha b - \delta$ or $\frac{b+\delta}{\alpha}$ to increase the bid an arbitrarily small amount to increase the probability of winning or tying by a discrete amount. So there cannot be types bidding just below $\alpha b - \delta$ and $\frac{b+\delta}{\alpha}$. But if there are gaps below $\alpha b - \delta$ and $\frac{b+\delta}{\alpha}$, then types bidding at b can decrease their bid without decreasing the probabilities of tying or winning. Therefore, this cannot be an equilibrium bid distribution. $\square$

Proof of Lemma 4.2.3 I will call a trivial equilibrium one in which all types bid 0. By the definition of the inverse given in the text the result is trivial in this case.

Let $\overline{b} > 0$ be the least-upper bound of a players bid support. This is well defined since bidding more than $\overline{v}$ is a dominated strategy. WLOG let player j's equilibrium bid strategy be s_j.

By way of contradiction suppose that $|s_j^{-1}(b) - s_j^{-1}(\hat{b})| > M|b - \hat{b}|$ where $M = \max\left\{\frac{\alpha}{(\underline{v}-\beta\underline{v})\underline{f}}, \frac{1}{\alpha\beta\underline{v}\underline{f}}\right\} + 1$. Without any loss we can let b and $\hat{b}$ be in the bid support. This follows because the generalized inverse is constant outside of the support. WLOG let $b > \hat{b}$.

Let supp represent the equilibrium bid support. I will prove the following statement. Either (case 1) there is a positive measure of types that bid in $(\alpha\hat{b} - \delta - \epsilon, \alpha\hat{b} - \delta] \bigcap \text{supp}$ or (case 2) there is a positive measure of types that bid in $(\frac{\hat{b}+\delta}{a} - \epsilon, \frac{\hat{b}+\delta}{a}] \bigcap \text{supp}$, for all $\epsilon > 0$. If not there would be an $\epsilon > 0$ such that decreasing the bid would not decrease the probability of winning or tying. This would contradict that $\hat{b}$ is in the bid support.

Case 1 Since $b > \hat{b}$, $\alpha b - \delta > 0$. Let $b_n \in (\alpha\hat{b} - \delta - 1/n, \alpha\hat{b} - \delta] \bigcap \text{supp}$. This sequence is well defined since there are no mass points above zero, and if $\alpha\hat{b} - \delta = 0$, $\{b_n\}$ is the zero sequence. It follows that $b_n \to \alpha\hat{b} - \delta$. Since the inverse is continuous by construction $\hat{v}_n = s_j^{-1}(b_n)$ has a limit. Let the limit be $\hat{v}$. Since each b_n is in the bid support, $s_j(\hat{v}_n) = b_n$.

$$\begin{aligned} U(\alpha b - \delta, \hat{v}) - U(\alpha\hat{b} - \delta, \hat{v}) = (v - \beta v)\left[F\left(s_j^{-1}(b)\right) - F\left(s_j^{-1}(\hat{b})\right)\right] + \\ \beta v\left[F\left(s_j^{-1}(\alpha(\alpha b - \delta) - \delta)\right) - F\left(s_j^{-1}(\alpha(\alpha\hat{b} - \delta) - \delta)\right)\right] - \alpha(b - \hat{b}) \\ \geq (\underline{v} - \beta\underline{v})\underline{f}M(b - \hat{b}) - \alpha(b - \hat{b}) > 0 \end{aligned} \tag{C.1}$$

Note that the expected utility is continuous in the bid and value except at the bid δ/α. If $\alpha\hat{b} - \delta = 0$ the above contradicts equilibrium. This is true because $s_j(\hat{v}) = 0$, and the inequality implies that a bid of $\alpha b - \delta$ is strictly preferred to 0 by type $\hat{v}$.

If $\alpha\hat{b} - \delta > 0$ this yields a contradiction because $s_j(v_n) = b_n$ for all $n > 0$. So by definition of equilibrium $U(\alpha b - \delta, \hat{v}_n) - U(b_n, \hat{v}_n) \leq 0$. But by continuity $U(\alpha b - \delta, \hat{v}_n) - U(b_n, \hat{v}_n) \to U(\alpha b - \delta, \hat{v}) - U(\alpha\hat{b} - \delta, \hat{v}) > 0$ which is impossible.

Case 2: Now let $b_n \in (\frac{\hat{b}+\delta}{\alpha} - 1/n, \frac{\hat{b}+\delta}{\alpha}] \bigcap \text{supp}$. Now $b_n \to \frac{\hat{b}+\delta}{\alpha}$. Similarly, redefine

$\hat{v}_n = s_j^{-1}(b_n)$ and define the limit to be $\hat{v}$.

By a similar argument as above.

$$U(\frac{b+\delta}{\alpha}, \hat{v}) - U(\frac{\hat{b}+\delta}{\alpha}, \hat{v}) \geq \beta \underline{v}(f) M(b-\hat{b}) - \frac{(b-\hat{b})}{\alpha} > 0 \tag{C.2}$$

By a similar argument to Case 1 this is a contradiction.

□

Proof of Lemma 4.3.1: Suppose that (c, d) is a gap in the bid distribution. If $c \leq \delta$, then there must also be a gap containing $(c + \delta, d + \delta)$. This follows because decreasing the bid in $(c + \delta, d + \delta)$ does not decrease the probability of winning or tying. Similarly, if $c \geq \delta$, there must also be a gap at $(c - \delta, d - \delta)$. By a similar argument, there must be a gap in the bid distribution at $(\overline{b} - \delta, \delta)$.

By lemma 4.2.2, there is not a mass point at d, $d+\delta$, and $d-\delta$ unless $d-\delta = 0$. By continuity of the bid distribution, if $d - \delta \neq 0$, there is an $\varepsilon > 0$ such that if G is the bid distribution $\overline{v}(G(d + \varepsilon) - G(d)) < d - c$. This implies that a bid of c is strictly preferred to any bid in $(c, d + \varepsilon)$. This follows because the decrease in utility caused by either an increase in the probability of tying and decrease in the probability of winning or a decrease in the probability of tying with no accompanying increase in probability of winning is offset by a significant decrease in the payments when a player decreases the bid from $d + \varepsilon$ to c.

But then if $d-\delta \neq 0$, $[d, d+\varepsilon)$ cannot intersect the equilibrium bid distribution. However, this would contradict that (c, d) is a gap. Thus $(\overline{b} - \delta, \delta)$ is the only gap. □

C.2 Characterization

It is useful to solve for the equilibrium strategies by analyzing the inverse of the strategy wherever the strategy is strictly increasing. We make the change of variables $x(b;\bar{b}) = s^{-1}(b)$, where $\bar{b}$ is the upper bound of s. Using this transformation we rewrite equations 4.9 and 4.10 as

$$x'(b;\bar{b}) = \frac{1}{(1-\beta)x(b+\delta;\bar{b})f(x(b;\bar{b}))} \tag{C.3}$$

for $b \in [0, \bar{b} - \delta]$ and

$$x'(b;\bar{b}) = \frac{1}{\beta x(b-\delta;\bar{b})f(x(b;\bar{b}))} \tag{C.4}$$

for $b \in [\delta, \bar{b})$. In addition, the initial conditions $x(\bar{b};\bar{b}) = \bar{v}$ and $x(\bar{b}-\delta;\bar{b}) = x(\delta;\bar{b})$ must be satisfied. These conditions are necessary for the increasing strategy to prescribe a bid for each type and are analogous to the initial conditions of the equilibrium first order conditions.

It is also convenient to rewrite equation 4.11, which says that a player with type $x(\delta;\bar{b})$ must be indifferent between bidding $\bar{b} - \delta$ and δ. Using the new notation this can be written

$$\beta x(\delta;\bar{b}) - \bar{b} + \delta = x(\delta;\bar{b})F(x(0;\bar{b})) + \beta x(\delta;\bar{b})(1 - F(x(0;\bar{b}))) - \delta \tag{C.5}$$

$$\rightarrow (1-\beta)x(\delta;\bar{b})F(x(0;\bar{b})) = 2\delta - \bar{b}. \tag{C.6}$$

The existence of a solution to equations C.3 and C.4 given the initial conditions can be established for any value of $\bar{b}$ using functional analysis arguments. The first step is to reformulate the problem as an integral equation. This equation will define

$x(., \bar{b})$ over the entire interval $[0, 2\delta]$. This continuous extension will be such that the $x(., \bar{b})$ is flat over the intervals $(\bar{b} - \delta, \delta)$ and $(\bar{b}, 2\delta)$.

Define $x_-(b; \bar{b}) \equiv x(b - \delta; \bar{b})$ and $x_+(b; \bar{b}) \equiv x(b + \delta; \bar{b})$.

Lemma C.2.1. *An increasing, absolutely continuous function $x(.; \bar{b})$ satisfies the first order conditions C.4 and C.3 and the initial conditions $x(\bar{b}; \bar{b}) = \bar{v}$ and $x(\bar{b} - \delta; \bar{b}) = x(\delta; \bar{b})$ if and only if*

$$x(b; \bar{b}) = \bar{v} - \int_b^{\bar{b}} g(t, x(t), x_-(t), x_+(t); \bar{b}) dt \tag{C.7}$$

where

$$g(b, x, x_-, x_+; \bar{b}) = \begin{cases} \frac{1}{(1-\beta) f(x) x_+} \text{ for } b \in [0, \bar{b} - \delta] \\ 0 \text{ for } b \in (\bar{b} - \delta, \delta) \\ \frac{1}{\beta f(x) x_-} \text{ for } b \in [\delta, \bar{b}] \\ 0 \text{ for } b \in (\bar{b}, 2\delta] \end{cases} \tag{C.8}$$

For an arbitrary $\bar{b} \in [\delta, 2\delta]$ there is no guarantee that there is a strategy satisfying the equilibrium first order conditions such that $x(0, \bar{b}) \geq 0$; however, this is a prerequisite for an equilibrium as shown by Lemma 4.3.1. Since it will be useful to define a strategy for every $\bar{b}$, the following lemma proves the existence and uniqueness of a function $x(.; \bar{b})$ which maps $[0, 2\delta]$ into $[\underline{v}, \bar{v}]$. When $x(0; \bar{b}) > \underline{v}$, this function will solve the equation in lemma C.2.1.

Lemma C.2.2. *Assuming A.11, the mapping $T : C([0, 2\delta]^2) \to C([0, 2\delta]^2)$ defined by*

$$T(x)(b; \bar{b}) = \max \left\{ \underline{v}, \bar{v} - \int_b^{\bar{b}} g(t, x(t), x_-(t), x_+(t); \bar{b}) dt \right\} \tag{C.9}$$

has a unique fixed point $x(b, \bar{b})$*. Furthermore, the fixed point depends continuously on* $\bar{b}$*.*

Proof: The proof uses the contraction mapping theorem to show, using standard methods, that $T : CB([0, 2\delta]^2) \to CB([0, 2\delta]^2)$ as defined above is a contraction mapping(Wolfgang Walter, 1998[115]). Let $CB([0, 2\delta]^2)$ be defined as the space of continuous functions that map $[0, 2\delta] \times [0, 2\delta]$ into the interval $[\underline{v}, \overline{v}]$.

The first step in the argument is to observe that $g(.)$ as defined in Lemma C.2.1 satisfies Lipschitz condition that there is an $L > 0$ such that

$$|g(b, x, x_-, x_+; \bar{b}) - g(b, y, y_-, y_+; \bar{b})| \leq L \max\{|x, y|, |x_- - y_-|, |x_+ - y_+|\} \quad \text{(C.10)}$$

for all b and $\bar{b}$ in $[0, 2\delta]$ and for all y, y_-, y_+, x, x_-, and x_+ in $[\underline{v}, \overline{v}]$. This follows from boundedness and Lipschitz continuity(LC) of the density f and from the fact that the product of bounded LC functions is LC.

Let the Euclidean norm $|\mathbf{x} - \mathbf{y}|_3 = \max\{|x, y|, |x_- - y_-|, |x_+ - y_+|\}$ for $\mathbf{x} = (x, x_-, x_+)$ and $\mathbf{y} = (y, y_-, y_+)$.

$CB([0, 2\delta]^2)$ endowed with the norm given by

$$||z|| = \sup_{(u,v) \in [0,2\delta]^2} |z(u, v)| e^{-2L(|v-u|)}$$

is a Banach space. This follows because the norm $||.||$ is equivalent to the sup norm.[1] To shorten notation let $\mathbf{x}(b; \bar{b}) \equiv (x(b; \bar{b}), x_-(b; \bar{b}), x_+(b; \bar{b}))$.

[1]Two norms $||.||_A$ and $||.||_B$ are equivalent if there are $C, D > 0$ such that $C||x||_A \leq ||x||_B \leq D||x||_A$ for all x. $e^{-2(\overline{v}-\underline{v})}||x||_{sup} \leq ||x|| \leq ||x||_{sup}$ for all $x \in CB([\underline{v}, \overline{v}]^2)$.

$$|Tx(b,\overline{b}) - Ty(b,\overline{b})| \leq \int_b^{\overline{b}} |g(t,\mathbf{x}(t;\overline{b});\overline{b}) - g(t,\mathbf{y}(t;\overline{b});\overline{b})|dt$$

By LC of g the right hand side is bounded above by

$$L\int_b^{\overline{b}} |\mathbf{x}(t;\overline{b}) - \mathbf{y}(t;\overline{b}|_3 dt.$$

$$\leq L\int_b^{\overline{b}} e^{2L(\overline{b}-t)}||\mathbf{x}-\mathbf{y}||dt \leq \frac{1}{2}||\mathbf{x}-\mathbf{y}||e^{2L(\overline{b}-b)}.$$

Since b is arbitrary this shows that

$$||T\mathbf{x} - T\mathbf{y}|| \leq \frac{1}{2}||\mathbf{x}-\mathbf{y}||.$$

By the contraction mapping theorem, $T(.)$ has a unique fixed point in $CB([0,2\delta]^2)$.

□

Proof of Proposition 4.3.2: If $\delta < (1-\beta)\overline{v}$, then when $\overline{b} = \delta$, a type with value $x(\delta,\delta)$ strictly prefers to bid δ. That is from C.6

$$(1-\beta)x(\delta,\overline{b})F(x(0),\delta) = (1-\beta)\overline{v} > \delta = 2\delta - \overline{b}.$$

Let $b' = \sup\{b \in [0,2\delta] : x(b;2\delta) = \underline{v}\}$. If $b' = 0$, the condition is satisfied by $\overline{b} = 2\delta$.

Suppose that $b' > 0$. By the continuous dependence of $x(.;\overline{b})$ on $\overline{b}$, there is an $\Delta > 0$ such that if $\overline{b}' = 2\delta - \Delta$, $x(0,\overline{b}') = \underline{v}$. For $\overline{b}'$, a bidder with type $x(\delta,\overline{b}')$ strictly prefers to bid $\overline{b}' - \delta$ rather than δ. That is

$$(1-\beta)x(\delta,\overline{b}')F(x(0,\overline{b}')) = 0 < \Delta = 2\delta - \overline{b}'.$$

By continuous dependence, there must be a $\overline{b}$ such that equation C.6 holds.

Proof of Proposition 4.3.3

Necessity follows from the discussion in the text. I will show sufficiency.

By substituting into the FOC, it is straightforward to verify that at a bid $b \in (0, \bar{b} - \delta) \bigcup (\delta, \bar{b}]$, if $v > x(b; \bar{b})$, the utility is increasing in the bid, and if $v < x(b; \bar{b})$, the utility is decreasing in the bid. Proposition 4.3.2 insures that a player with value $x(\delta, \bar{b})$ is indifferent between bidding δ and $\bar{b} - \delta$. However, a player with a higher(lower) value strictly prefers to bid δ(respectively $\bar{b} - \delta$). Thus no type has an incentive to deviate from the equilibrium to a bid in $[0, \bar{b}]$.

Clearly, no type can benefit by unilaterally deviating to a bid above $\bar{b} + \delta$, since by bidding $\bar{b} + \delta$ a player wins with certainty. Bidding in $(\bar{b}, 2\delta)$ is worse than bidding $\bar{b}$ as argued in lemma 4.3.1.

Furthermore, if $\beta \leq .5$, $\bar{b} + \delta$ is preferred to any bid in $(2\delta, \bar{b} + \delta)$. To prove this last assertion, for almost all $b \in (0, \bar{b} - \delta)$.

$$U'(b; x(b; \bar{b})) = \beta x(b, \bar{b}) f(x(b + \delta, \bar{b})) x'(b + \delta, \bar{b}) - 1 = 0$$

This implies that for $\bar{v}$,

$$U'(b; \bar{v}) = \beta \bar{v} f(x(b + \delta, \bar{b})) x'(b + \delta, \bar{b}) - 1 > 0.$$

Now

$$U'(b + 2\delta; \bar{v}) = (1 - \beta) \bar{v} f(x(b + \delta, \bar{b})) x'(b + \delta, \bar{b}) - 1$$

is sure to be positive if $\beta \leq .5$. Thus the utility is increasing in the bid on the interval $(2\delta, \bar{b} + \delta)$. The first condition of the proposition insures that $\bar{b}$ is preferred to $\bar{b} + \delta$.

If $\beta > .5$, the utility may not be increasing over the interval $(2\delta, \overline{b} + \delta)$. Thus it is necessary to check that there is no bid in $(2\delta, \overline{b} + \delta)$ which is preferred to $\overline{b}$ by a player with type $\overline{v}$. This is insured by condition 2. $\square$

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