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Essays on auctions and Bayesian games with endogenous expectations

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ESSAYS ON AUCTIONS AND BAYESIAN GAMES WITH ENDOGENOUS EXPECTATIONS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. thesis of

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ABSTRACT

The dissertation consists of three self contained, though interrelated chapters. In the first two chapters, I apply the behavioural model of reference dependence to two different games of incomplete information; auctions and bargaining. The main contribution of the models is that they pin down the reference point by endogenising it as the expected price of a good in equilibrium. Modelling games where the utility of a player depends on her beliefs over endogenous variables, introduces mathematical complexity. Indeed, such games lack an existence result, and in the final chapter, I extend methods available in the literature to provide sufficient conditions for the existence of monotone equilibrium in games with endogenous beliefs.

In the first chapter, I model auctions where bidders have reference dependent preferences and may be loss averse. The reference point is defined either as the ex-ante or interim expected price of the good, depending on whether bidders are naive or sophisticated. Equilibrium with consistent reference points are shown to exist and are fully characterised. The model predicts that in equilibrium, bidders both overbid and underbid in comparison to the standard risk neutral Nash equilibrium.

The second chapter extends the model a two player k-double auction, where players are assumed to have preferences that exhibit reference dependence. The expected price of the good is modelled as a player's reference point. In equilibrium the endogenous reference point is said to be consistent if behaviour given the reference point, yields an expected price equal to the reference point itself. Bias is introduced

as an exogenous deviation of the reference point from the expected price of the good. We study the effects of reference dependence and bias on the ex-post efficiency of the auction.In the absence of bias, reference dependence does not alter the efficiency of the model, however we find that efficiency is decreasing in the level of bias.

In the final chapter, I consider a class of Bayesian games where an individual's payoff depends on her beliefs over the actions of her opponents. I model this feedback effect as endogenous (rational) beliefs over equilibrium outcomes, which are influenced by the actions of all players. A *consistent Bayesian equilibrium* is defined, where behaviour given equilibrium beliefs, yields the same beliefs on the equilibrium outcome. Sufficient conditions for equilibrium existence are listed; I use lattice theoretic techniques and exploit monotonicity of equilibrium strategies to show existence. Also provided are comparative static results with respect to the primitive type distributions. The model is then applied to two distinct sub-fields in economics; reference dependent preferences with endogenous reference points and Cournot markets with network effects.

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CHAPTER 1
ENDOGENOUS PRICE EXPECTATIONS AS REFERENCE POINTS IN AUCTIONS

1.1 Introduction

In the standard model of individual behaviour, utility is a function of only the final consumption bundle. Kahneman and Tversky (1974; 1979) propose an alternate specification of individual utility where total utility is a function of both the final consumption bundle and its relation to some reference point or anchor. This behavioural model has received much attention in both economics and psychology under different names: reference dependence and anchor adjustment, respectively. This paper applies reference dependence to first and second price auctions with endogenous reference points and studies its effects on bidder behaviour.

As a motivating example, consider Anne who wishes to purchase a new television. Under the standard model of preferences, Anne would gain some utility from buying (or consuming) the television. Upon purchasing the television, Anne's overall utility would then be the positive utility gained from the purchase of the television, less what it cost to buy it. A transaction would then take place, only if the price of the television is less than its value to Anne.

While intuitive, the above story is incomplete. It can be argued that Anne additionally has expectations over the price of the television. This price may act as a reference point, and Anne would experience additional utility if she pays less than she expected, as she experiences the joy of "saving money." Similarly, "over paying," i.e paying more than the reference point would result in disutility in addition to her standard utility. This additional gain-loss element in Anne's utility, would lead to different predictions from the standard model. Indeed, Anne may refuse to buy the television, even when the price is less than her value, if the price is higher than what she expected to pay.

The choice of reference point is not trivial, as the predictions of the model rely on the particular reference point chosen.[1] It is therefore important to identify an appropriate reference point. In this paper, we exploit the game theoretic structure of auctions to argue that if Anne were to instead buy the television at an auction, she could form rational expectations over the expected price, which in turn would act as her reference point. The reference point would be determined by the behaviour of bidders in the auction and would need to be *consistent* with equilibrium play. We show that such a consistent reference point exists in first and second price auctions, and characterise equilibrium behaviour when bidders have reference dependent preferences and endogenous reference points.

We model first and second price auctions in the independent private values setting, where bidder values are drawn independently from a continuous distribution. Unlike the standard risk neutral model, bidders are assumed to have reference dependent preferences. Reference dependent preferences capture two important inter-related features of human behaviour: gain-loss utility with respect to a reference point, and

[1] If for example Anne's reference point is set to infinity, we get the outrageous prediction that Anne would buy a TV at any price, regardless of her value.

loss aversion. Gain-loss utility is captured by our example above, where Anne gains additional utility (or disutility) from paying less (more) than her reference point. Loss aversion in addition, implies that losses relative to the reference point yield higher disutility compared to the utility derived from gains of equal magnitude. We model reference dependent preferences both with and without loss aversion.

The reference point is endogenised as the expected price of the good in equilibrium. Endogenising the reference point removes a major degree of freedom from the model of reference dependent preferences. Defining the reference point as the expected price of the good is consistent with the motivation of past literature. Such a definition is motivated by bidders experiencing gains in utility when they pay less than what they expected (saving money) and disutility when they pay more than the reference point (over paying).

Bidder expectations can be incorporated into the model in two ways and we define two types of reference points. Naive reference points are modelled as the ex-ante expected price of the good. Naive bidders do not incorporate the effect of their own bid on the expected price of the good and therefore set their reference points equal to the ex-ante expected price. In contrast, the interim or sophisticated reference point, incorporates the effect of a player's own private information on her reference point. Sophisticated bidders understand the effects their own bids have on the expected price of the good and so take expectations once they observe their private information.

In Section 1.3, we consider the case of naive bidders who have reference dependent preferences but are not loss averse. For naive bidders with endogenous reference points, we define a *naive consistent equilibrium*. We characterise the unique symmetric consistent equilibrium, where bidders bid according to increasing bid functions and the reference point is consistent with equilibrium strategies. As the reference point affects bidder behaviour, it determines the expected price in equilibrium. A reference point is said to be consistent when given the reference point, bidder behaviour yields an expected price equal to the reference point.

We find that the introduction of reference dependence alters bidder behaviour in both first and second price auctions. In equilibrium, depending on their values, bidders either overbid or underbid relative to the predictions of the standard model.[2] There exists in both first and second price auctions a cut-off value, where bidders whose values lie below the cut-off overbid, while those above it underbid. The cut-off value is a function of the endogenous reference point and this allows us to predict whether, in expectation, more bidders overbid or underbid in equilibrium. In particular when values are distributed uniformly, we find that in expectation more bidders overbid than underbid in first price auctions and in second price auctions with three or more bidders. We also find that this cut-off is increasing in the number of bidders, so as the auction gets larger more people are predicted to overbid. Given how the size of an auction can be seen as a sign of the objects popularity and desirability, the prediction is intuitive, but it does not conform with laboratory data.[3]

[2]The terms, overbid and underbid, are common in the literature, especially in the case of second price auctions where standard theory predicts that rational bidders should bid their value, which is a weakly dominant strategy. In first price auctions we set our base as the symmetric Nash equilibrium strategy.

[3]For example Kagel and Levin (1993) find that fewer bidders overbid when the size of

In Section 1.4 we analyse the case of sophisticated bidders. Sophisticated bidders incorporate the effect of their own bid on the expected price of the auction and so the reference point is defined as the interim expected price of the good. Unlike the naive case, where the ex-ante formulation implied that the reference point was a real number, for sophisticated bidders the reference point is a function of their value. We modify the definition of a consistent equilibrium to account for this change and define a *sophisticated consistent equilibrium*. We then characterise a symmetric consistent equilibrium in increasing strategies and reference point. We find that like naive bidders, sophisticated bidders also underbid and overbid in comparison to the standard model. However, a lack of tractability prevents us from finding a neat cut-off value.

In Section 1.5 we augment the naive model with loss aversion. The introduction of loss aversion increases the penalty for paying more than the reference point. We find that in equilibrium high value bidders decrease their bids, i.e underbidding is more pronounced. This change in equilibrium strategies, lowers the expected price of the good in the auction, leading to two effects. First, the consistent reference point is lower than the case without loss aversion. Second, because the consistent reference point is lower more bidders now underbid relative to the case without loss aversion. Both these effects combine to lower the revenue generated by the auction, that is loss aversion has a negative effect on revenue. In addition to this, we find that loss aversion breaks down revenue equivalence. While a complete revenue ranking is not

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the auction changes from five to ten bidders.

presented, we find that first price auctions generate higher revenue than second price auctions when the auction has two bidders. For larger auctions however, the revenue comparison is ambiguous. Unlike risk aversion, which only effects bidding behaviour in first price auctions, loss aversion alters bidder behaviour in both first and second price auctions, leading to distinct reference points and therefore different revenues.

Finally, we present a short discussion on the approach used to endogenise the reference point. In particular we focus on the model's use of a nested fixed point argument to establish existence of a consistent equilibrium. We compare our definition to those used by other models of reference dependence with endogenous reference points, and discuss the issue of the robustness of our results.

1.1.1 Related literature

Some models of reference dependence take the reference point as given, that is they model it as an exogenous variable in the utility function. Rosenkranz and Schmitz (2007) apply such a model to first and second price auctions. They set the reference point equal to the weighted average of an exogenous variable and the reserve price. They study the effects of such reference points on bidder behaviour and calculate optimal reserve prices. Shunda (2009) extends their model by including Buy-it now prices in his analysis, explicitly assuming the reference point to be a weighted average of the auction's reserve price and buy-it now price.

Endogenising the reference point has been a recent phenomenon. The objective is to remove a major degree of freedom from the model by using rational expectations to pin down the reference point. There are two methodologies available. Shalev (2000) presents a model where the reference point is set equal to the expected utility of the player in equilibrium, and proves existence of equilibria in extensive form games. Alternatively, Kőszegi and Rabin (2006, 2007) argue that the reference point should in fact be a belief distribution and a player takes expectations over the reference point when evaluating her final consumption bundle. While Kőszegi and Rabin present a decision theoretic model, the two models share some characteristics. Indeed, as pointed out by Kőszegi and Rabin themselves, the uni-dimensional version of their model, with degenerate beliefs is a special case of Shalev's model.

The current paper applies the game theoretic model of Shalev (2000) to auctions and endogenises the reference point in first and second price auctions. In doing so, we present the first application of his model to games of incomplete information. However in line with previous models of reference dependence in auctions, the reference point is modelled as an expected price, instead of expected utility. Furthermore the current paper's model of naive bidders with no loss aversion can be seen as a direct extension of Rosenkranz and Schmitz (2007), where the reference point is endogenised by setting it equal to the ex-ante price of the good in equilibrium.

Lange and Ratan (2010) also study the effects of endogenous reference points in first and second price auctions. However differences in modelling both the reference point and bidder utility, yield remarkably different predictions.[4] For first price

[4]Lange and Ratan (2010) seek to explain why bidders may use different strategies in auctions conducted in the field and the laboratory. They argue that field (or commodity) auctions are different from laboratory (or induced value) auctions, because in field experi-

auctions, Lange and Ratan (2010) predict that bidders both overbid and underbid in commodity auctions. They predict that bidders with higher values overbid while those with lower values underbid. The (naive) model of this paper, like Rosenkranz and Schmitz (2007) predicts the opposite relationship. This is because of the modelling criterion used for the reference point. When the reference point is an expected price, bidders with high values lower their bids to reduce the disutility from paying "too much." Contrast this with Lange and Ratan's model, where overbidding and underbidding are driven by losses in the goods dimension. High value bidders who expect to win the auction, overbid to increase their chance of winning and reduce the chances of loss in the goods dimension. While both explanations are plausible, indeed multiple behavioural factors are most likely at play in the real world, experimental data suggests that most bidders overbid,[5] something that is consistent with the current model.

The second major difference, is the prediction of the current paper that all bidders regardless of type (naive, sophisticated or loss averse) deviate from value bidding in second price auctions. Lange and Ratan (2010) find that in their model, bidders should continue to value bid in a laboratory, where bidder payoffs only have

ments the good for sale is also important. In their model bidders in field auctions also form expectations over winning the object (in the "goods" dimension). They therefore use a two dimensional model of reference dependent preferences, where bidders have reference points in both the money and goods dimension, arguing that the second dimension is not present in the lab. They endogenise the reference point using a modified version of Kőszegi and Rabin (2006), where the reference point is itself stochastic and bidders take expectations over the reference point distribution at each state. Finally, their utility framework only captures losses and does not take into account gains relative to the reference point.

[5]See for example Kagel and Levin (1993) and Kagel, Harstad, and Levin (1987).

the money dimension (as no good is actually won). The current paper can then be seen to explain the phenomenon of over and underbidding in the lab.

Models other than reference dependence have also been applied to auctions, primarily as attempts to explain overbidding in first price auctions. The closest to the current study is Filiz-Ozbay and Ozbay's (2007) application of regret theory to explain over and underbidding in first price auctions. Other models that have been applied include those of ambiguity aversion (Salo and Weber, 1995), level-k thinking (Crawford and Iriberri, 2007) and misperception of probabilities (Dorsey and Razzolini, 2003).

With the exception of Crawford and Iriberri (2007), the above models do not explain deviations from standard value bidding in second price auctions. The model of this paper is to our knowledge, the first model that predicts underbidding in second price auctions conducted in the laboratory.

1.2 Basic model

There are $n > 2$ identical bidders who bid for a single item and their values are independently and identically distributed according to c.d.f $F,$ with support $[0, \omega]$, where F is differentiable over $(0, \omega)$. Bidder i observes her own value, denoted by v_i, and places a sealed bid for the item. The bidder with the highest bid wins and pays a price p, which is determined by the rules of the auction. In a first price auction, the winner pays her own bid, while in a second price auction she pays the highest bid of her opponents. Bidder utility differs depending on whether they are loss averse or not, so we define the utility functions in their respective sections.

For ease of exposition, we define $F_{(1)}(v) = F^n(v)$, as the distribution of the highest value among all bidders in the auction, i.e the first order statistic for n bidders. Let $F_{(2)}(v)$ denote the distribution of the second order statistic, that is the second highest value among all bidders in the auction. Furthermore let $G_{(1)}(v) = F^{n-1}(v)$ be the distribution of the first order statistic for an auction with $(n-1)$ bidders and let $G_{(2)}(v)$ denote the corresponding second order statistic. For any distribution $H(v)$, the density function is represented with the corresponding lower case letter, i.e $h(v)$.

1.3 Naive bidders

We first model the case where bidders are naive and their utility functions display only reference dependence, i.e there is no loss aversion. A naive bidder constructs her reference point at the outset of the auction, before she observes her own value. That is to say her reference point is equal to the ex-ante price of the good. She is naive in that she does not take into account the fact that her value and in turn her bid provides her with private information that affects the price of the good.

While the term “naive” suggests that this formulation is closer to the case of a laboratory setting with inexperienced players, it may also apply to experienced bidders. It is possible that bidders in a lab setting are also naive, however for experienced bidders, the reference point is arguably closer to an ex-ante price of the good. Having participated in a large number of auctions over time, their expectation of the price of a good is perhaps closer to its ex-ante valuation. In the case of auctions with a large number of bidders, this would be especially true, as the individual bidder would not consider her bid to affect the final price of the good.

A bidder's utility if she wins is given by $v - p + \lambda(\alpha - p)$, where v is her value, p the price she pays, α her reference point and λ is a measure of the degree of her reference dependence. We can then reinterpret her total utility as the sum of her standard risk neutral utility and her *gain-loss* utility. Her gain-loss utility, given by $\lambda(\alpha - p)$ simply states that she receives some additional utility or disutility from paying less or more than her reference point, respectively. Furthermore, it is assumed that gain-loss utility can not dominate standard utility, that is $\lambda \in [0, 1]$. If a bidder loses the auction, her utility is simply zero.

The second important component to the model is the criterion used for endogenising the reference point. For naive bidders the reference point is modelled as the ex-ante expected price of the good and when made endogenous, it is a function of equilibrium behaviour. Given how the reference point affects a bidder's bid, it would also affect the expected price of the good in equilibrium. In equilibrium, the reference point then needs to be consistent, i.e behaviour given the reference point should yield an expected price equal to the reference point itself.

Definition 1.3.1. A *naive consistent equilibrium* for an auction, is a consistent reference point and bidding strategies such that given the reference point, bidders play their Bayes-Nash equilibrium strategies, and the ex-ante expected price in equilibrium is equal to the reference point. Formally, for a symmetric equilibrium in increasing strategies, a naive consistent equilibrium is a pair $(\alpha, b(v; \alpha))$, of a consistent reference point α and a strategy $b(v;\alpha)$ such that

1. Given α, the symmetric Bayes-Nash equilibrium strategy is $b(v;\alpha)$.
2. Given b, $\alpha = \int b(z;\alpha)dH(z)$, where $H(z)$ is the distribution of the value that determines the price of the good in equilibrium.

The definition is an extension of Shalev's (2000) definition of a consistent reference points to auctions, where the reference point is defined over the price of the good, instead of the bidder's expected utility. For notational simplicity, the formal definition is expressed for symmetric bid functions that are increasing in value, and therefore express the expected price as an expectation over the distribution of the value that determines the price in the auction (the highest and second highest value in first and second price auctions respectively).[6]

To find the naive consistent equilibrium, we use a nested fixed point argument. We first derive bidder behaviour for exogenous reference points, then given that an equilibrium exists for some exogenous reference point α, we calculate the expected price in equilibrium, which we take as our new reference point α'. The process is repeated and the fixed point of this sequence corresponds to the consistent reference point.

For a given (exogenous) reference point, the current naive formulation without loss aversion is identical to the model of Rosenkranz and Schmitz (2007), and we state

[6]We restrict attention to symmetric equilibria in this paper. Other equilibrium do exist, for example in a two player second price auction, one player always bidding ω and the other bidding 0, would be a naive consistent equilibrium with the consistent reference point equal to zero.

here their relevant results without proof.

Proposition 1.3.2 (Rosenkranz and Schmitz)**.** *When bidders are not loss averse and have an exogenous reference point α the following results hold.*

1. *In a first price auction the unique symmetric equilibrium bidding function is given by*
$$b_1^n(v;\alpha) = \frac{1}{1+\lambda}\Big[v + \lambda\alpha - \int_0^v \frac{G_{(1)}(w)}{G_{(1)}(v)}dw\Big].$$
2. *In second price auction it is a weakly dominant strategy for a bidder with value v to bid*
$$b_2^n(v;\alpha) = \frac{v+\lambda\alpha}{1+\lambda}.$$
3. *First and second price auctions are revenue equivalent.*

The model predicts overbidding and underbidding in equilibrium with respect to standard theory. Intuitively this is because the additional gain-loss utility encourages (discourages) bidders to bid above (below) what they would in a standard setting. This is clear to see in the case of a second price auction where the bidding function can be re-interpreted as bidding according to a new pseudo-value that is a weighted average of a bidder's value and her reference point. Then she overbids (underbids) compared to the standard model when the reference point is higher (lower) than her true value.

Now consider an endogenous reference point and our requirement of consistency. Given that the bid functions are increasing in player value, consistency requires that the reference point be the fixed point of the mapping

$$Q_t^n(\alpha) = \int b_t^n(z;\alpha) dH_t(z), \tag{1.3.1}$$

where $t \in \{1, 2\}$ is the type of auction, b_t^n is the equilibrium bidding function and $H_t(z)$ is the distribution of the value that determines the price in equilibrium. Application of the Intermediate Value Theorem guarantees the existence of a consistent reference point, as long as the bid function is continuous in the reference point. This is obviously true for the bid functions defined in Proposition 1.3.2.

We can then calculate the consistent reference points explicitly. It is important to identify the correct distribution $H_t(z)$. In the naive case, the price is determined by the highest or second highest value in a first or second price auction, and so $H_t(z) = F_{(t)}(z)$. The following proposition characterises the naive consistent equilibrium for both auctions.

Proposition 1.3.3. *For naive bidders who are not loss averse, the following results hold.*

1. *In a second price auction the unique naive consistent equilibrium in increasing strategies is given by*

$$\begin{aligned} b_2^n(v;\alpha_2) &= \frac{v + \lambda\alpha_2}{1+\lambda}, \\ \alpha_2 &= \int_0^\omega z\, dF_{(2)}(z). \end{aligned}$$

2. *In a first price auction the unique naive consistent equilibrium in increasing*

strategies is given by

$$\begin{aligned} b_1^n(v;\alpha_1) &= \frac{1}{1+\lambda}\Big[v + \lambda\alpha_1 - \int_0^v \frac{G_{(1)}(w)}{G_{(1)}(v)}dw\Big], \\ \alpha_1 &= \int_0^\omega \Big[z - \int_0^z \frac{G_{(1)}(w)}{G_{(1)}(z)}dw\Big]dF_{(1)}(z). \end{aligned}$$

All proofs requiring derivation are relegated to the appendix. For those familiar with auction literature, note that the naive reference points in the two auctions are equal to the expected revenue of a seller in auctions when bidders have standard utility. This implies that the naive consistent reference points are equal in first and second price auctions.[7] Furthermore, given that for exogenous reference points, first and second price auctions are revenue equivalent (Proposition 1.3.2.3), this leads to the corollary that in the current setting the two auctions are revenue equivalent.[8]

Corollary 1.3.4. *In auctions with naive bidders who are not loss averse, first and second price auctions are revenue equivalent.*

An intuitive explanation of the corollary is that the naive specification of the reference point as the ex-ante expected price of the good, sets the reference point equal to the expected revenue of the auction. Since both the reference points are equal, this result is not surprising.

A second corollary to Proposition 1.3.3 is that bidders bid more aggressively, i.e increase their bids (for all values), as the number of bidders increases.

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[7] Derivation of the revenue equivalence result for standard auction can be found in any textbook on auctions, for example Krishna (2010)

[8] In the case of naive bidders, revenue equivalence can be easily generalised to the case with reserve prices. As we do not model the effect of reserve prices in the paper, the proof is not presented here but is reproduced in Appendix A.8 for interested readers.

Corollary 1.3.5. *In auctions with naive bidders who are not loss averse, bids are increasing in the number of bidders.*

The result is plain to see in the case of second price auctions. The distribution of the second order statistic is increasing in n and so is the reference point. Then by the chain rule, bids also increase, as the bid function is increasing in the reference point. To see this intuitively, assume that as another bidder is added to the auction, bidders continue to bid according to their bidding function for n bidders. As there is now another bidder, the expected price, i.e the reference point increases as the probability of drawing a high value opponent increases. As the reference point increases, by Proposition 1.3.2, bidders bid more aggressively, further increasing the reference point. Both these effects combine to raise the final consistent reference point. This is a novel result for second price auctions. While bidder aggression is increasing for first price auctions in the standard model as well, for second price auctions the standard model predicts value bidding regardless of the size of the auction.

The intuition of the result goes further to explain a simple phenomenon of human behaviour: popularity. Consider once again the case of our shopper Anne, and assume she wishes to purchase a new 3D-television. She goes to the auction and observes that there are only a few other connoisseur of bleeding edge technology. In such a case one would argue that her expectation of the price of the television is most definitely lower compared to a case where the auction house is full to the brim. By extension it is also natural then, to assume that our shopper would bid more for the television if the auction house was full rather than empty.

The fact that in second price auctions bidders with values above the reference point underbid and those with values below it overbid, allows us to outline sufficient conditions under which in expectation, more bidders overbid than underbid relative to the standard model.

Corollary 1.3.6. *In a second price auction with naive bidders who are not loss averse, if the distribution of values satisfies either of the following conditions:*

1. *The distribution of values F is symmetric or positively skewed and uni-modal, and*

$$E[v_{(2)}] \geq E[v],$$

where v is a single value drawn from the distribution F and $v_{(2)}$ is the second highest value among all bidders in the auction;

2. *the distribution of values F is negatively skewed, uni-modal and*

$$E[v_{(2)}] \geq E[v] + \sigma(v),$$

where $\sigma(v)$ is the standard deviation of the distribution F;

then the ex-ante expected number of bidders who overbid is greater than the number bidders who underbid.

The proof relies on the relationship between the median and mean of a distribution (Mallows, 1991). The result in its simplest form states that the number of overbids is greater than the number of underbids in an auction if the reference point is higher than the median value. This is because by Proposition 1.3.2.2, bidders with values below (above) their reference point overbid (underbid).

For first price auctions, the cut-off value that determines over and underbidding is a function of the distribution of values. For the common case of uniformly distributed values, explicit computation is possible. For the uniform case, bidders with values above $\left(\frac{n}{n-1}\right)\alpha_1$, underbid and those below it overbid relative to the standard model. We summarise the results for the special case of uniformly distributed values below.

Corollary 1.3.7. *When bidder are naive, are not loss averse and values are distributed uniformly, in expectation more bidders overbid in first price auctions than in second price auctions. Furthermore:*

1. *In a first price auction the ex-ante expected number of bidders who overbid is higher than those who underbid.*
2. *In a second price auction with more than two bidders, the ex-ante expected number of bidders who overbid is higher than those who underbid.*

Recall that over and under bidding are defined in relation to the predictions of the standard model. Bidders with values above the cutoff value underbid while those below it overbid. Since the cutoff value in first price auctions is always above the cutoff for a corresponding second price auctions, the model predicts that in expectation, we should observe more overbidding in first price auctions.

1.4 Sophisticated bidders

Consider now the case of sophisticated bidders, who update their reference points based on their values. For simplicity, we abuse notation slightly and denote the sophisticated or interim reference point by α as well. Note however that for sophisticated bidders, the reference point is a function of their private information, i.e $\alpha : [0, \omega] \rightarrow [0, \omega]$. The particular value of the reference point, for a bidder with value v, is denoted by $\alpha(v) \in [0, \omega]$.

As in the previous section, we require that in equilibrium the reference point be consistent. We modify the definition of a naive consistent equilibrium (Definition 1.3.1) to account for the interim nature of the sophisticated reference point.

Definition 1.4.1. A *sophisticated consistent equilibrium* for an auction, is a consistent reference point and bidding strategies such that given the reference point, bidders play their Bayes-Nash equilibrium strategies, and for each value the interim expected price is equal to the reference point. Formally, for a symmetric equilibrium in increasing strategies and reference point, a sophisticated consistent equilibrium is a pair $(\alpha, b(v; \alpha))$, of a consistent reference point α and a strategy $b(v; \alpha)$ such that

1. Given α, the symmetric Bayes-Nash equilibrium strategy is $b(v; \alpha)$.
2. Given b, for all bidders i and value v, $\alpha(v) = \int b(z; \alpha) dL(z|v)$. Where $L(z|v)$ is the distribution of the value that determines the price of the good in equilibrium, conditional on bidder i with value v placing a bid equal to $b(v; \alpha)$.

As in the case of naive bidders, we once again utilise a nested fixed point argument to characterise a consistent equilibrium. To do so, we first express the bidding strategies for a given (exogenous) reference point, α that is increasing in bidder value.

Proposition 1.4.2. *When bidders have reference dependent preferences and an exogenous reference point α that is increasing, the following results hold.*

1. *In a first price auction the unique symmetric equilibrium bidding strategy is*

$$b_1^s(v;\alpha) = \frac{1}{1+\lambda}\Big[v - \int_0^v \frac{G_{(1)}(w)}{G_{(1)}(v)}dw + \lambda \int_0^v \frac{\alpha(w)}{G_{(1)}(v)}dG_{(1)}(w)\Big]. \tag{1.4.1}$$

2. *In second price auction it is a weakly dominant strategy for a bidder with value v to bid*

$$b_2^s(v;\alpha) = \frac{v+\lambda\alpha(v)}{1+\lambda}. \tag{1.4.2}$$

3. *First and second price auctions are revenue equivalent.*

As in the case for naive bidders, sophisticated bidders also alter their bidding behaviour compared to the standard model. Once again, for second price auctions the bid function can be interpreted as bidders value bidding with respect to an adjusted value, which is a weighted average of their value and reference point. Unlike the case for naive bidders, there is no closed-form cut-off value at which bidders switch from over bidding to underbidding. Finally, note that if the reference point is a constant function, i.e. $\alpha(v) = \bar{\alpha}$ for all $v \in [0, \omega]$, the bid functions defined by Proposition 1.4.2 simplify to the bid functions for naive bidders (Proposition 1.3.2).

Now consider the consistency requirement when the reference point is increasing and a bidder has value v.

$$\alpha_t(v) = \int b_t^s(z;\alpha_t)dL_t(z|v), \tag{1.4.3}$$

where b_t^s is the equilibrium bidding function in a t-th price auction and $L_t(z|v)$ is the distribution of the value that determines the price. Careful attention must be given to two issues.

First, the correct distribution $L_t(z|v)$ needs to be identified. Note that in a first price auction, for a bidder with value v, if she wins, the price is determined by her bid own bid. Otherwise, it is the highest bid of her opponents. We can then re-write equation (1.4.3) for a first price auction as

$$\alpha_1(v) = b_1^s(v;\alpha_1)G_{(1)}(v) + \int_v^{\omega} b_1^s(z;\alpha_1)dG_{(1)}(z). \tag{1.4.4}$$

The first term on the right hand side of the equation is the case when the bidder with value v wins. This happens when she has the highest bid, which given increasing bidding functions, is when she has the highest value. The second term calculates the expected price when she loses the auction.

In a second price auction, whether a player with value v wins or losses is determined once again by the first order statistic of $(n-1)$ bidders. However in case bidder i loses, the price of the good may be determined by either her bid or by the second highest bid of her opponents, which is not independent of the first order statistic. Calculating the conditional distributions and simplifying yields the consistency requirement for sophisticated bidders in a second price auction.

$$\begin{aligned}\alpha_2(v) = &\int_0^{v} b_2^s(z;\alpha_2)dG_{(1)}(z) + (n-1)[1-F(v)]F^{n-2}(v)b_2^s(v;\alpha_2)\\ &+ \int_v^{\omega} b_2^s(z;\alpha_2)dG_{(2)}(z).\end{aligned} \tag{1.4.5}$$

In the above formulation, the right hand side expresses the three disjoint cases that determine the price conditional on a bidder's value v. The first case is the price when the bidder wins the auction and pays the highest bid of her opponents. The second and third terms express the expected price of the good in the case where she losses, considering both the case when her bid determines the price and when the second highest bid is above her bid.[9]

The second issue with the current formulation, is that when deriving the consistency conditions we implicitly assumed that the bidder with the highest value wins the auction, i.e. we assumed that the consistent reference points were increasing in value. So to prove existence, we also need to show that equations (1.4.4) and (1.4.5) define increasing functions. This is indeed the case, as is formalised in the following proposition.

Proposition 1.4.3. *For sophisticated bidders with reference dependent utility, the following results hold.*

1. *In a first price auction there exists a sophisticated consistent equilibrium with bidding strategies given by equation* (1.4.1) *and an increasing consistent reference point, that is implicitly defined by equation* (1.4.4)*.*
2. *In a second price auction there exists a sophisticated consistent equilibrium with bidding strategies given by equation* (1.4.2) *and an increasing consistent reference point, that is implicitly defined by equation* (1.4.5)*.*

[9]Explicit derivation is provided in Appendix A.7.

Note that the proposition does not rule out the existence of other equilibria.[10] However, it is intuitive that the reference point is increasing in bidder value. Bidders with higher values would expect the price of the object to be higher than those with a lower value. The intuition is clear to see in a first price auction, where a bidder's own bid can be seen to act like a reserve price that raises the minimum price at which the good will be sold.

1.5 Loss averse bidders

In this section we augment the naive model by introducing loss aversion. Loss aversion does not change any of the existence results of the naive model.[11] The qualitative results however do change and of interest is the effect of loss aversion on the revenue generated by the auction.

For loss averse bidders, we define utility from winning the auction case by case. If a bidder wins the auction and pays more than her reference point α, her utility is given by $v - p + \lambda_l(p - \alpha)$. When she wins and pays less than she expected, her utility is $v - p + \lambda_g(p - \alpha)$. Loss aversion is captured by assuming $\lambda_g \leq \lambda_l$. We continue to maintain the no dominance of gain-loss utility assumption, so $\lambda_j \in [0, 1]$ for $j \in \{g, l\}$.

[10] Once again, consider a two person second price auction where one player always bids ω while the other bids 0. This is a sophisticated consistent equilibrium when the consistent reference point is $\alpha(v) = 0$, for all $v \in [0, \omega]$.

[11] Which recall relied on the continuity of the equilibrium bidding strategies, which as will be shown later continue to hold. Even in the absence of continuity, monotonicity of the equilibrium strategies in the reference point is sufficient to guarantee existence by Tarski's fixed point theorem.

Once again to characterise the naive consistent equilibrium, as a first step in our nested fixed point argument we prove existence of an equilibrium for exogenous reference points. The following proposition characterises the equilibrium bid functions in the presence of loss aversion. Recall that as the bidders are naive, the reference point is a real number and not a function, i.e $\alpha \in [0, \omega]$.

Proposition 1.5.1. *In an auction with naive loss averse bidders, and exogenous reference point* α*, the following results hold.*

1. *In a first price auction, the unique continuous symmetric equilibrium bidding function is given by*

$$b_1^l(v;\alpha) = \begin{cases} \dfrac{1}{1+\lambda_g}\left[v + \lambda_g \alpha - \displaystyle\int_0^v \frac{G_{(1)}(w)}{G_{(1)}(v)} dw\right] & \text{if } v < \hat{v}, \\ \dfrac{1}{1+\lambda_l}\left[v + \lambda_l \alpha - \displaystyle\int_0^v \frac{G_{(1)}(w)}{G_{(2)}(v)} dw\right] & \text{otherwise.} \end{cases} \tag{1.5.1}$$

   Where $\hat{v}$ *is unique and implicitly defined by the equation*

$$\hat{v} = \alpha + \int_0^{\hat{v}} \frac{G_{(1)}(w)}{G_{(1)}(\hat{v})} dw.$$

2. *In a second price auction, it is a weakly dominant strategy for a bidder with value* v *to bid*

$$b_2^l(v;\alpha) = \begin{cases} \dfrac{v + \lambda_g \alpha}{1+\lambda_g} & \text{if } v < \alpha, \\ \dfrac{v + \lambda_l \alpha}{1+\lambda_l} & \text{otherwise.} \end{cases} \tag{1.5.2}$$

The bid functions are similar to those found by Rosenkranz and Schmitz (2007) (Proposition 1.3.2). However, due the introduction of loss aversion, the coefficient of reference dependence depends on whether a bidder pays more or less than her reference point. Once again, bidders both underbid and overbid in equilibrium depending on the relationship between their value and the reference point. Indeed, for exogenous reference points, the cut-off values are identical to the case without loss aversion. In second price auctions bidders with values above the reference point underbid, while those below it overbid. In first price auctions, $\hat{v}$ defines the cutoff value as a function of the reference point, and is identical to the cutoff value for the case without loss aversion.

The major change in behaviour relative to the naive case without loss aversion is that underbidding for high value bidders is more pronounced. To see why this is the case intuitively, assume we start with the case of no loss aversion and a single measure of reference dependence λ. Then introducing loss aversion such that $\lambda_l > \lambda_g = \lambda$, would only alter the bids of high value bidders, who increase the amount by which they shade their bids.

While loss aversion changes equilibrium bid functions, the naive consistency requirement is the same as in the case of without loss aversion (equation (1.3.1)), as are the relevant distributions. Bid functions continue to be continuous and once again application of the Intermediate Value Theorem yields existence.

Proposition 1.5.2. *For naive bidders who are loss averse, the following results hold.*

1. *In a second price auction when bidders are loss averse, the symmetric naive consistent equilibrium in increasing strategies is the pair of bidding strategies given by equation* (1.5.2) *and the consistent reference point implicitly defined by*

$$(1+\lambda_g)\alpha_2 = (1+\lambda_g)\int_0^{\omega} z dG_{(2)}(z) - \Delta \int_0^{\alpha_2} F_{(2)}(z)dz, \tag{1.5.3}$$

where $\Delta = \lambda_l - \lambda_g$*. Furthermore, the consistent reference point is decreasing in* Δ*.*

2. *In a first price auction when bidders are loss averse, a symmetric naive consistent equilibrium in increasing strategies is the pair of bidding strategies given by equation* (1.5.1) *and the consistent reference point implicitly defined by*

$$(1+\lambda_g)\alpha_1 = (1+\lambda_g)\int_0^{\omega}\left[z - \frac{1}{G_{(1)}(z)}\int_0^z G_{(1)}(w)dw\right]dF_1(z)$$
$$- (n-2)\Delta\Big[\int_0^{\hat{v}} zdF_{(1)}(z) - F(\hat{v})\int_0^{\hat{v}} zdG_{(1)}(z)\Big]. \qquad (1.5.4)$$

Note that when $\Delta = 0$, the equations defining the consistent reference points are equivalent to the case without loss aversion. Further note that when $\Delta > 0$, the additional terms (in relation to the case without loss aversion) in both equations (1.5.3) and (1.5.4) are negative.[12] This implies that introducing loss aversion ($\Delta > 0$) lowers the reference point in both auctions. Recall that in an auction with naive bidders and no reserve price, the reference point is equal to the seller's ex-ante expected revenue and we have our first corollary.

Corollary 1.5.3. *Auctions where bidders exhibit loss aversion (*$\Delta > 0$*) generate lower revenue than auctions where bidders are not loss averse (*$\Delta = 0$*).*

The corollary is intuitive, as the introduction of loss aversion causes high value bidders to lower their bids. Assume that we start from the case where there is no

[12] $\int_0^{\alpha_2} F_2(z)dz \geq 0$ for all $\alpha_2 \geq 0$. Similarly, $\left[\int_o^{\hat{v}} zdF_1(z) - F(\hat{v})\int_0^{\hat{v}} zdG_1(z)\right] \geq 0$ as the first order statistic in increasing in n with respect to first order stochastic dominance and $F(\hat{v}) \in [0,1]$.

loss aversion, and introduce loss aversion such that $\lambda_l > \lambda_g = \lambda$. Consider the case if the reference point remains unchanged, then the same bidders continue to over and underbid, except that while low value bidders do not change their bids, high value bidders lower their bids. This lowers the expected price of the good (the reference point). As the bids and the cut-off value are both increasing in the reference point, a lower reference lowers all bids and implies that more bidders now underbid than before. These effects work together to lower the revenue generated by the auction.

Finally, Proposition 1.5.2 allows for one more observation. Note that in auctions with two bidders, loss aversion has no effect on the revenue generated in a first price auction, as the second term on the right hand side in equation (1.5.4) falls to zero. This results in the first price auction generating higher revenue than the second price auction.

Corollary 1.5.4. *In auctions with only two loss averse bidders, first price auctions generate higher revenue than second price auctions.*

1.6 Note on methodology

As aforementioned the consistent reference points in the paper are found using a nested fixed point argument, where we first find equilibrium in continuous strategies for exogenous reference points and then use these strategies to find a consistent reference point. This process introduces two important issues worthy of discussion.

First, while the methodology is inspired by Shalev's (2000) definition of consistent loss averse equilibria, the proof has one fundamental difference. Shalev models games of complete information with continuous utility functions and allows for mixed strategies. This allows him to prove existence of equilibrium by extending Nash's proof, using a single mapping from the cross product of strategies and reference points into itself. Here, however, due to technical issues and our focus on pure strategies, we are unable to utilise this approach. Our methodology is then closer to the literature on psychological games (see for example, Battigalli and Dufwenberg (2009)).[13]

Second is the issue of existence of equilibria and while a complete discussion is beyond the scope of the paper, we highlight the crucial role of the linear specification of utility in the current model. While we use the continuity of bid functions to establish existence, the methodology itself is generalisable to monotone best responses. Therefore while in standard auctions, sufficient conditions due to Athey (2001) allow for results to be generalised up to log-concave utility functions for first price auctions, here there is no such luxury. Introducing reference dependence as per the specification of Kőszegi and Rabin (2006) requires that both the standard and gain-loss utility be linear. While equilibrium may indeed exist, there is no guarantee that they would be in monotone strategies and hence no guarantee that a consistent reference point would exist. This is due to the fact that non-linear specifications violate the assumptions of Athey (2001) Theorem 7.[14]

Similarly, for the case of second price auctions, in the standard setting value

[13]Though it is worth noting that the fixed point found by both mechanisms (single vs. nested) is the same.

[14]This is easiest to see when we consider the utility of a bidder when she wins, which is no longer log-supermodular in her bid, due to the gain-loss function being convex in losses.

bidding is always the weakly dominant strategy, while here it is not. It can also be shown that linearity in standard utility is a sufficient condition for the weakly dominant strategy to be increasing.[15] Linearity then, is crucial for the study of auctions when bidders have reference dependent utility.[16]

However, given the nature of the applied literature on auctions, the linear specification is indeed general enough to allow the model to be extended to the study of other forms of auctions and contests.

1.7 Conclusion

At its core the paper presents a model for applying endogenous reference points to auctions. The naive model can be seen as an extension of the established model of Rosenkranz and Schmitz (2007), where the reference point is endogenised as the ex-ante expected price of the good in equilibrium. In doing so, it develops a novel methodology for applying reference dependence with endogenous reference points to auctions. The motivation for this approach is closer to the psychological foundations of reference dependence, which argue gain and loss sensations relative to personal

[15]For example consider the case of naive bidders. The generalised utility of a bidder when she wins following the specification of Kőszegi and Rabin (2006) is given by $u(v_i - p) + \mu(u(v_i - p) - u(v_i - \alpha))$, where u is the players standard utility, μ her gain-loss utility and p the price she pays. Then the weakly dominant strategy b^* is implicitly defined by $u(v_i - b^*) = -\mu(u(v_i - b^*) - u(v_i - \alpha))$. Application of the implicit function theorem yields the desired sufficient conditions.

[16]Additionally, note that even when existence can be shown directly, the fixed point requirement needs either that the bid functions be continuous or increasing in the reference point. The latter cannot be guaranteed in the presence of diminishing sensitivity in losses; convexity of the gain-loss function in losses leads to decreasing difference in player action and the reference point.

expectations. Methodologically, the approach is simple and robust to a broad range of model specifications. However, it also highlights the costs of requiring *consistency* on tractability; closed form solutions are difficult to come by.

The application of this model to auctions with linear utility functions predicts overbidding and underbidding, a result consistent with experimental data. The result for naive bidders, that bidder aggression is increasing in the number of bidders, captures the essence of changes in utility linked to the popularity of a good. This is driven by the definition of the reference point; when a good is popular, a bidder should expect to pay more for it.

The paper also serves to highlight the flexibility of reference dependence as a behavioural model. Comparisons with Lange and Ratan (2010), show that while both papers use reference dependent preferences, differences in motivation and methodology yield different results. In the current paper, we capture the behavioural phenomenon of gains and losses in utility due to under and overpaying for an object. Lange and Ratan (2010) model only losses, while allowing for multiple dimensions. Therefore while both papers apply reference dependence to auctions, they yield different predictions.

In Lange and Ratan's model, reference dependence does not alter bidder behaviour in a second price auction conducted in the lab. When the comparison is extended to their specification of field auctions, they predict that high value bidders underbid and low value bidders overbid, while the current paper predicts the opposite relationship. This is driven by the significant differences in utility specification between the two papers. The prediction of overbidding in the model of the current paper is driven by gains in the money dimension. In contrast, for Lange and Ratan, overbidding is driven by expected losses in the goods dimension. Second, Lange and Ratan condition their reference point in the money dimension on the bidder winning the auction, therefore the relevant distributions for the reference point in the two papers are not only different, but Lange and Ratan's monetary reference point is by construction lower than player value (and therefore causes systematic underbidding). Overall, these differences call for future experimental research, to identify the exact behavioural influences at play in the real world.

Finally, the model presented is general and allows room for future extensions. Natural extensions to the model include ascending clock auctions and auctions with reserve and buy-it now prices. The definition of the reference point is also similar to that of an entry threshold (it is the expected price of the good), therefore application to endogenous entry models may prove productive.

CHAPTER 2
REFERENCE DEPENDENCE AND BIAS IN DOUBLE AUCTIONS

2.1 Introduction

The framework of standard expected utility assumes that individual utility is only a function of the final outcome (or bundle). While this allows economists to model various situations, behavioural studies from both economics and psychology strongly suggest that such an assumption may be too restrictive. In this paper, we incorporate two phenomenon that are ignored by standard theory; reference dependence and self-serving bias, and study their effects on equilibrium behaviour in k-double auctions.

Reference dependence, first proposed by Kahneman and Tversky (1974; 1979), models total utility as a function of both the final consumption bundle and its relation to some reference point or anchor. A bidder in a double auction then, may not only receive utility from purchasing the item, but also gain additional utility if she pays less than what she initially expected. This additional gain-loss element may then encourage her to bid higher or lower than what is predicted by standard theory.

Similarly, the phenomenon of bias alters how players attach utility to an outcome. We argue that reference dependence allows for a natural way to incorporate bias. Buyers and sellers in an auction may have different expectations, that are directly effected by their bias. For example, a seller may expect his car to be worth more than its actual market value, while a buyer may be overly harsh in her valuation of the car, influenced by her pursuit of a "good deal." In this paper, we argue that bias alters the reference point of both the seller and buyer, in a self serving manner.

To model bias, we assume that the reference point is of the form of an expected price. Using an ex-ante formulation, we pin down the reference point by requiring it to be equal to the expected price of the good, given equilibrium strategies. This requirement of consistency removes a major degree of freedom from the model, as the predictions of the model rely crucially on the choice of reference point. Bias is then introduced as an exogenous deviation of the players' reference points from the "true" expected price.

A major theme in the study of bargaining models, such as the k-double auction, is that of efficiency. We find in Section 2 that the introduction of reference dependence alone (that is without bias), does not affect the efficiency of the auction compared to the standard model of Chatterjee and Samuelson (1983). While equilibrium bids change to accommodate the gain-loss element introduced by reference dependence, it can be seen, overall, as a simple affine transformation of the standard game.

In Section 3, we introduce bias by modelling it as a systematic misestimation of the expected price of the good. We find that bias alters the game by changing the distribution of bids in a non-affine manner. When compared to the case without bias, we find that bias leads to higher ex-post inefficiency, and that inefficiency is strictly increasing in the level of bias.

Finally, the model of this paper brings together the two disjoint literature's on reference dependent preferences and self serving bias. In doing so, it provides a novel approach of modelling bias with endogenous reference points.

2.1.1 Related literature

Reference dependence has been applied to various economics settings and closest in methodology to the current paper is Rosenkranz and Schmitz's (2007) model of first and second price auctions, where bidders are assumed to have reference dependent preferences. We utilise their utility function and like them, model the reference point as a "reference price." However, unlike Rosenkranz and Schmitz the reference point is defined as the expected price of the good in auction and made endogenous. Other applications of reference dependence to auctions include the study of the effects of regret (Filiz-Ozbay and Ozbay, 2007), buy-it now prices as reference points (Shunda, 2009) and loss aversion with multidimensional reference points on bidder behaviour (Lange and Ratan, 2010).

Endogenising the reference point has been a recent phenomenon in the literature. There are two major methodologies available. Shalev (2000) presents a model where the reference point is set equal to the expected utility of the player in equilibrium, and proves existence of equilibria in extensive form games. Alternatively, Kőszegi and Rabin (2006, 2007) argue that the reference point should in fact be a belief distribution and a player takes expectations over the reference point when evaluating her final consumption bundle. Of the these two methodologies, the consistency requirement of the current paper can be seen to be closest to the methodology suggested by Shalev (2000).

Finally, Hart and Moore (2008), Shalev (2002), Li (2007) and Bram et al. (2012) study games where the reference point is determined by an endogenous variable. Hart and Moore (2008) model a two stage trading game where a contract in the first stage serves as a reference point for the second stage. Li (2007) and Bram et al. (2012) model bargaining games where the reference points are history dependent. Shalev (2002) also studies bargaining games, however focuses on effects of loss aversion on the axiomatic solutions to the Nash bargaining problem.

Experimental studies have highlighted the influence of bias on an individual's perception, and show that individuals systematically misestimate and misinterpret uncertain variables in an egotistical manner. Multiple surveys show that a majority of respondents rate themselves to be more skillful than 50 percent of the population.[1] In the economics literature, this paper joins a small list of recent theoretical papers which seek to construct a formal model for self-serving bias. Most applications fall in the *behavioural law and economics* literature, where bias has been used to explain the phenomenon of bargaining impasse (Babcock and Loewenstein, 1997) and pre-trial negotiations (Farmer and Pecorino, 2002; Langlais, 2008).

Closest in motivation and methodology to the current study are Gallice (2009) and Mori (2013), who also use reference dependent preferences to model self serving bias. Gallice (2009) defines self-serving biased reference points, as reference points that allocate to each player a division greater than the available surplus in bargaining

[1] E.g, driving (Svenson, 1981; McCormick et al., 1986) and teaching ability (Cross, 1977). For a more detailed survey, consult (Babcock and Loewenstein, 1997).

games.[2] Mori (2013), borrowing from Kőszegi and Rabin (2006), defines his reference point as the expected utility of a game. However, the expectation is determined only by the type of game the players play, and is not effected by equilibrium strategies. Unlike the aforementioned papers, the model of this paper, through its requirement of consistency, fully endogenises the reference point, where the strategies of players in equilibrium determine the relevant reference point.

2.2 Reference dependence

We model a two player k-double auction, where a buyer and seller wish to trade a single indivisible object. Both the buyer and seller have some private information regarding the object. The buyer observes her value v, while the seller observes her cost, c. We consider a simple model where both value and cost come from an identical distribution,[3] making the simplifying assumption that both value and cost are distributed independently according to the uniform distribution over the unit interval.

Players observe their private information and simultaneously place bids for the good. The buyer's bid, b, reflects the maximum she is willing to pay to acquire the good. The seller's bid s represents the minimum price she is willing to accept to sell

[2] Similar to a survey in the psychology literature, where the shares in household chores of most married couples added up to more than 100% (Ross and Sicoly, 1979), Gallice (2009) defines self serving bias to exist in a game if the (exogenously given) reference points of players in a game add up to more than the available surplus. He implicitly assumes that players never underestimate their expected share of the surplus (reference point).

[3] This is a simplifying assumption as the analysis can easily be extended to different independent distributions.

the good (it is common to refer to the sellers bid as her "ask"). Trade takes place if the bidder's bid is higher than the seller's ask, i.e $b \geq s$. If the good is traded, the buyer pays the seller p, which is the k-weighted average of the bid and ask. That is, the price is determined by $p = kb + (1-k)s$, where $k \in [0, 1]$ is an exogenous design parameter.

Both players' preferences exhibit reference dependence, that is to say that in addition to standard risk neutral utility, player utility has a gain-loss component with respect to some reference price α. We model this by modifying the utility specification presented by Rosenkranz and Schmitz (2007) for their study of reference dependence in first and second price auctions.[4]

When trade takes place, the buyer's utility is given by $v - p + \lambda(\alpha - p)$, where v is her value, p the price she pays, α her reference point and $\lambda \in [0, 1]$ the measure of reference dependence. Similarly, when trade takes place the utility of the seller is given by $p - c + \lambda(p - \alpha)$, where c is her cost. Note that the above specification simply states that a player's utility is the sum of her standard risk neutral utility, which is the difference of her value and price, and her gain-loss utility. We assume that buyers and sellers are identical in their preferences, and so are both risk neutral in standard utility and share a common measure of reference dependence.[5] While both players have common measures of reference dependence, it is important to note

[4]This specification was also used by Shunda (2009), who extended the model of Rosenkranz and Schmitz (2007) to buy-it now auctions.

[5]The assumption of a common gain-loss function is consistent with the formulation of reference dependence presented by Kőszegi and Rabin (2006), who argue the gain-loss function is *universal* (identical).

that they perceive gains and losses asymmetrically. For example, a realised price lower than the reference point is perceived as a gain by the buyer (she has "saved" money relative to the expected price), but perceived as a loss by the seller (she has received less than what she expected). If there is no trade, player utility is simply zero.

As a first step in our analysis, we assume that the reference point is an exogenous parameter and that both the buyer and the seller have identical reference points. This is a simplifying assumption, made for tractability. Later, when the reference point is made endogenous, the ex-ante nature of the expected price would lead to a single price in equilibrium (and therefore a common reference point).

For the exogenous set-up, note that the new specification of utility is a simple affine transformation of the standard risk neutral utility and so an equilibrium exists. The following proposition formalises the result and fully characterises the equilibrium strategies.

Proposition 2.2.1. *In a k-double auction when players have reference dependent utility and reference point α, there exists an equilibrium in increasing linear strategies given by*

$$b(v,\alpha) = \frac{1}{1+\lambda}\Big[\frac{v}{1+k} + \frac{(1-k)k}{2(1+k)} + \lambda\alpha\Big],$$
$$s(c,\alpha) = \frac{1}{1+\lambda}\Big[\frac{c}{2-k} + \frac{1-k}{2} + \lambda\alpha\Big].$$

The proof follows the standard techniques of solving a k-double auctions and the derivation can be found in the appendix, along with all other mathematical proofs in the paper. Intuitively, the bidding functions in the presence of reference dependence can be interpreted as bidders bidding a weighted average of their standard risk neutral bids and their reference point. This is because the reference dependent utility function can be rewritten as a function where bidders bid according to their "adjusted values" ($\tilde{v} = v + \lambda\alpha$ and $\tilde{c} = c + \lambda\alpha$), in the presence of an additional tax of λ on every dollar they pay.

In the study of double auctions, and bargaining models in general, a major question is with regards to the efficiency of the mechanism. It is known that under standard preferences, double auctions are ex-post inefficient. Uncertainty yields equilibrium strategies such that in some cases trade may not occur even when it may be socially optimal, that is cases when ex-post the buyer values the good more than it costs the seller.

The introduction of reference dependence to the game does not alter this result from standard theory. Given that the transformation discussed above is affine, it is not surprising that an exogenous reference point does not alter the probabilities of trade, nor does it effect the efficiency of the model. We formalise the result in the following corollary.

Corollary 2.2.2. *Efficiency is invariant to the measure of reference dependence, λ. Compared to when players are risk neutral, the addition of reference dependence $(\lambda \neq 0)$ does not alter the efficiency of the mechanism.*

An intuitive explanation of the result is that as aforementioned the game with reference dependence can be seen as players playing the standard game, where the value and cost are $\tilde{v}$ and $\tilde{c}$, respectively (and there is an additional λ tax on price). While bidder value and seller cost were previously distributed uniformly over the interval $[0, 1]$, they are now distributed over $[\lambda\alpha,\, 1 + \lambda\alpha]$. The game is then a simple normalisation away from the standard game, and so there is no change in the probability of trade and ex-post efficiency of the mechanism.

Given the structure of the game, we can calculate the expected price of a good in the auction. By modelling the reference point as the ex-ante expected price of the good, we can make it endogenous. The reference point is then the expected price of the good calculated by the buyer and seller before their own private information is revealed to them. The ex-ante formulation of the reference point is chosen as it simplifies the analysis and yields tractability.

Given the bidding strategies from Proposition 2.2.1, trade occurs only when $v \geq \frac{1+k}{2-k}c + \frac{1-k}{2}$ and in such a case, the price is given by $p(v, c, \alpha) = \frac{1}{1+\lambda}\left[\frac{k}{1+k}v + \frac{1-k}{2-k}c + \frac{1-k}{2(1+k)} + \lambda\alpha\right]$. To calculate the price when trade occurs, we first calculate the appropriate (conditional) distributions. For a seller whose cost is c, trade takes place when buyer value lies in the set $[\frac{1+k}{2-k}c + \frac{1-k}{2},\, 1]$. Additionally, the probability of trade is positive only when $c \in [0,\, \frac{2-k}{2}]$. Since the distributions of v and c are uniform and independent, the expected price for a given α is

$$P(\alpha) = \int_0^{\frac{2-k}{2}} \int_{\frac{1+k}{2-k}c + \frac{1-k}{2}}^{1} p(v, c, \alpha)\, \frac{(1+k)}{2(2-k)}(2 - k + 2c)dv\, \frac{2}{(2-k)}dc.$$

Since the reference point is equal to the expected price of the good, in equilibrium we require it to be consistent. That is, given a reference point, bidder behaviour in equilibrium should yield an expected price equal to the reference point itself. Formally, a consistent reference point α^* is a fixed point of the mapping $P(\alpha)$, that is

$$\alpha^* = P(\alpha^*).$$

Given that the P is continuous and has a constant slope (for given k and λ) such a consistent reference point exists and is unique. The following remark formalises this result.

Remark 2.2.3. In a k-double auction when players have reference dependent utility, the unique consistent reference point is given by

$$\alpha^* = \frac{3(2-k)}{8}.$$

Endogenising the reference point removes a major degree of freedom from the model. This is a major concern with such behavioural models, as under an exogenous specification, one can support different predictions by simply choosing an appropriate exogenous reference point.

We now proceed to incorporate bias into our base model. We find that bias affects the efficiency of the mechanism and that the endogenous reference point is a function of player bias.

2.3 Self serving bias

We now extend our model and incorporate the effects of self serving bias. Maintaining our assumption that players are identical, we introduce a universal measure of bias $\beta \in [0, 1]$.[6] It is assumed that players systematically misestimate the the

[6]Farmer and Pecorino (2002) model bias using a similar deviation. They assume that bias is an exogenous deviation from the "true" (exogenous) probability a trial would succeed.

reference price. We model bias of the self serving variety, that is to say that player misestimations are in their own favour. So, buyers underestimate the reference point, as a lower price is advantageous to them, while sellers overestimate the reference point. Unlike the case without bias where both players have the same reference point α, buyers and sellers now have different reference points; $\alpha_b = (1-\beta)\alpha$ and $\alpha_s = (1+\beta)\alpha$, respectively. Note that while we assume that the level of bias β is identical, it effects the estimations of the two players asymmetrically.

As a first step in our analysis, we consider the case of an exogenous reference point. Player utilities are unchanged from the previous section, and we assume that there is an exogenous true reference point α, that both players systematically miscalculate. The proposition below formalises the equilibrium behaviour in the presence of bias.

Proposition 2.3.1. *In a k-double auction with biased players and exogenous reference point α there exists an equilibrium in increasing linear strategies given by*

$$B(v,\alpha) = \frac{1}{1+\lambda}\Big[\frac{v}{1+k} + \frac{(1-k)k}{2(1+k)} + \frac{(2-k)\underline{v} + k\underline{c}}{2}\Big],$$

$$S(c,\alpha) = \frac{1}{1+\lambda}\Big[\frac{c}{2-k} + \frac{1-k}{2} + \frac{(1-k)\underline{v} + (1+k)\underline{c}}{2}\Big],$$

where $\underline{v} = (1-\beta)\lambda\alpha$ and $\underline{c} = (1+\beta)\lambda\alpha$.

The above formulation assumes that players know each others reference points. This may seem odd, as given rational expectations, this can be seen as arguing that a player knows that she is herself biased. However, this is not the case. An alternative behavioural model of bias is that a player believes that she is not biased but that her opponent is biased. This alternative behavioural formulation where both players assume they are unbiased while the other is biased, is mathematically equivalent to the current formulation.[7]

Recall that the main focus of our study is to model the effects of bias on the efficiency of the auction. Intuitively, bias turns what was an identical bargain into a nonidentical one. Previously in the case without bias, we saw that the introduction of reference dependence did not alter the efficiency of the model. Intuitively this was because while reference dependence changed some of the parameters of the game, it affected the buyer and seller in an identical manner. The game could be seen as a standard game with new adjusted values $\tilde{v}, \tilde{c}$ that were independently and identically distributed uniformly over the interval $[\lambda\alpha, 1 + \lambda\alpha]$.

In the presence of self-serving bias however, the bargain is no longer identical. The analogous adjusted value and cost in the presence of bias are given by $\hat{v} = v + (1 - \beta)\lambda\alpha$ and $\hat{c} = c + (1 + \beta)\lambda\alpha$, and are drawn from different distributions. In fact the new distributions systematically lower the bounds of the buyer's adjusted values, while increasing the bounds of possible seller (adjusted) costs, thereby ruling out many cases of mutually beneficial trade, leading to lower efficiency. The following corollary confirms our intuition.

Corollary 2.3.2. *Bias increases the inefficiency of the mechanism. Furthermore, for*

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[7] A seller may assume that the true reference point is α_s and that the "biased" buyer underestimates it, such that $\alpha_b = \frac{1-\beta}{1+\beta}\alpha_s$. Similarly, the buyer may assume that α_b is true and the biased seller overestimates such that $\alpha_s = \frac{1+\beta}{1-\beta}\alpha_b$. Bias in this setup is multiplicative with a measure $\phi = \frac{1-\beta}{1+\beta}$.

biased agents ($\beta \neq 0$) we have the following results.

1. *For given k, λ and β, ex-post inefficiency is increasing in the reference point, α.*
2. *For given k, λ and α, ex-post inefficiency is increasing in β.*
3. *For given k, β and α, ex-post inefficiency is increasing in the measure of reference dependence, λ.*

The corollary above confirms that bias leads to lower ex-post efficiency. For our analysis, we use the true values and cost (v and c) as our baseline for efficiency. That is to say we continue to assume that ex-post efficiency requires that trade occur whenever it is mutually beneficial ($v \geq c$). Note that this is equivalent to using the case without bias from the previous section as a baseline. It may be argued that the effect of bias should be incorporated in calculating a new baseline for efficiency, such that the definition of "mutually beneficial" trade be altered to $\hat{v} \geq \hat{c}$. The reason we choose not to do so, is that from a policy perspective it is generally argued that bias is a hindrance in the way of achieving optimal choices. Therefore, discussing the full effect of bias on efficiency requires we use the case without bias as a benchmark; accepting bias as a legitimate component of the social welfare criterion undermines such an exercise.[8]

The corollary further details how efficiency of the exogenous model changes, with respect to its parameters. Of interest is the result that efficiency is decreasing

[8]Bias also effects efficiency when the ideal is set to $\hat{v} > \hat{c}$, which implies that trade take place whenever $v \geq c + 2\lambda\beta\alpha$. Comparing this to equilibrium trade yields that even in this case, bias causes inefficiency, due to the underling effect it has on the distribution of values.

in bias. However, this is an incomplete result, since the endogenous reference point is itself a function of bias. To study the effects of bias on efficiency completely, we need to endogenise the reference point.

With the introduction of bias, trade takes place when $v \geq \frac{1+k}{2-k}c + \frac{1-k}{2} + (1+k)\beta\lambda\alpha$, changing the relevant conditional distributions. The distribution of value and cost conditional on trade are Unif$[A = \frac{1+k}{2-k}c + \frac{1-k}{2} + (1+k)\lambda\alpha,\ 1]$ and Unif$[0,\ B = \frac{2-k}{2} - (2-k)\beta\lambda\alpha]$, respectively.

Simple algebra confirms that price continues to be determined by $p(v, c, \alpha)$, and so the consistent reference point is defined as a fixed point of the mapping

$$Q(\alpha) = \int_0^B \int_A^1 p(v, c, \alpha) \, \frac{dv}{1-A} \frac{dc}{B}.$$

Once again, there exists a consistent reference point, and the result is formalised in the following remark.

Remark 2.3.3. In a k-double auction when players have reference dependent utility, the unique consistent reference point is given by

$$\alpha_b^* = \frac{3(2-k)}{4\lambda\beta - 6k\lambda\beta + 8}.$$

We can now study the full effect of bias on the efficiency of the mechanism. Using the case without bias (or equivalently with standard utility, la Chatterjee and Samuelson (1983)) as a baseline, the change in efficiency is directly proportional to $I = \lambda\beta\alpha_b^*$. Corollary 2.3.2 and remark 2.3.3 combine to give us our main result.

Proposition 2.3.4. *When players are biased ($\beta > 0$), inefficiency is increasing in the level of bias β and in the measure of reference dependence λ.*

The proposition confirms the generally held belief that bias (of the self-serving variety) should decrease efficiency and lead to higher probability of failure to reach an agreement. While at its core, the intuition of the result relies on the fact that bias makes the bargain nonidentical and therefore rules out many possible cases where trade may be mutually beneficial, it is important to note that such an explanation excludes some of the subtle dynamics. Note that the consistent reference point, α_b^* is not increasing for all possible values of k, and indeed an increase in bias may lead to a lower reference point. However, the overall change in the bounds of the distributions of $\hat{v}$ and $\hat{c}$, is monotonically increasing in β, which leads to the result that efficiency decreases as players become more biased.

Furthermore, note that the result holds only for self-serving bias. It may be tempting to extend the result to cases where players may display pessimism or "humility," i.e. $\beta < 0$, however this would be incorrect. Two issues prevent such a generalisation. First the bid functions defined in Proposition 2.3.1 do not apply to nonidentical bargains where there are values of the buyer for which trade takes place with probability 1.[9] Second, the formulation of ex-post inefficiency does not take into account cases where trade may occur even though $c > v$. In cases of extreme humility or pessimism, trade may always take place, if bias distorts the bounds such that $\hat{c} \leq \hat{v}$, for all v and c. It is then obvious that such a case is inefficient, and clearly rules out a clean cut generalisation of the above proposition to bias that is

[9]For detailed discussion see Chatterjee and Samuelson (1983), in particular Example 2 of their paper.

not self-serving. However, given the lack of empirical evidence for the existence of self-harming bias, such a generalisation is of little practical value.

2.4 Conclusion

The model of this paper makes two general contributions. First, it details the effects of reference dependence and self-serving bias on equilibrium behaviour in k-double auctions. It outlines equilibrium strategies and analyses the effects of both behavioural phenomenon on the efficiency of the model. We find that without bias, reference dependence does not have any effect on the efficiency of the model (when compared to the standard model) and that inefficiency is increasing in the level of bias.

Second, by endogenising the reference point, it provides a complete theoretical model for modelling bias by way of reference dependence. While the choice of reference point is left to the researcher, and here we use a reference point over price, the requirement for consistency removes a major degree of freedom from such models. Compared to Gallice (2009), reference points are no longer exogenous. In comparison to Mori (2013), the endogenous reference point takes into account strategic behaviour by the players. In essence, the paper exploits recent developments in the reference dependence literature to provide a framework for incorporating self serving bias using endogenous reference points.

The general flexibility of the model allows for exciting prospects for future extensions. Modelling loss aversion and its effects on bidding behaviour and bias would be of interest. Second, while the paper models the effects and comparative statics of bias, the actual causes of bias are not discussed. While more suited to the realm of psychology, experimental testing of the predictions of the model, where bias may indeed be influenced by experimental manipulations, may prove promising. Finally, while beyond the scope of this paper, recent developments in the reference dependence literature would suggest that the methods of this paper can be generalised to different economic games.[10] Therefore studying bias in other economic settings would be of interest.

[10]For decision theoretic problems, the model can be seen as a special case of Kőszegi and Rabin (2006), where instead of consistency one may look for personal equilibria. For games of complete information, the model can be seen as a special case of Shalev (2000)'s *myopic loss averse equilibrium,* or this can be seen as a special case of psychological games with first order beliefs Battigalli and Dufwenberg (2009).

CHAPTER 3
MONOTONE EQUILIBRIA IN BAYESIAN GAMES WITH ENDOGENOUS BELIEFS

3.1 Introduction

The model of purely selfish preferences allows economists to model many situations of interest. However, in a lot of cases the stringent application of selfish preferences, i.e preferences which model an individual's utility to be a function only of her personal consumption, yield results with limited predictive power. While such assumptions simplify the analysis, researchers may wish to incorporate a richer model of preferences. Incorporating the direct effect of the actions of other players on an agent's utility, is one such extension. Psychological games and social network games are two obvious examples where the assumption of selfish preferences are an oversimplification.

As a motivating example, consider a simple game played between two diners; Ann and Bob. After enjoying their meal, both diners need to decide how much to tip the server. In this game, the actions available to the players are the amount they tip and under the standard model of preferences, where both Ann and Bob only maximise their own personal consumption, the optimal tip would be zero. However, such a game is played by diners all around the world and we know that the zero tip prediction is incorrect.

To explain this discrepancy, the utility function may be enriched to account for other factors that may affect Ann and Bob's payoff. One consideration would be that neither player wishes to be viewed as stingy, and would therefore wish to tip close to some set social norm. A player may gain utility from tipping more than the norm and experience disutility otherwise. Such a formulation, while reasonable, would be inconsistent with the assumptions of the standard model if the norm is itself an equilibrium object. For example, the norm may be taken as the average of Ann and Bob's tip. Now, player utilities in this augmented game are no longer just functions of their own actions (how much they tip), but also depend on their beliefs over the actions of their opponents. To consider this new game, we need to model both Ann and Bob's beliefs over the average tip, and this average must be consistent in equilibrium, i.e equilibrium play given particular beliefs over the average tip should yield the same average in equilibrium. The problem becomes more complicated, when we extend the analysis to Bayesian games. For example if we allow for players to have some form of private information that affects their payoffs (Ann may be "nice" and Bob "stingy").

As such then, real world problems require a more general framework. While this realisation is not new, and the literature is full of examples of models where player payoffs are affected by their beliefs over the actions of their opponents, the literature lacks a general existence result, especially for the case of Bayesian games.

One issue with enriching the preference structure in such a manner is of mathematical complexity. In many situations, a researcher does not *a priori* know whether her model would yield an equilibrium. The purpose of this paper is to provide a set of sufficient conditions that guarantee the existence of pure strategy equilibrium in

Bayesian games, when a player's payoff may additionally depend on her beliefs over the actions of her opponents. It is shown that under the given assumptions, equilibrium strategies are increasing in player type and beliefs (where the belief distributions are said to be increasing in the sense of first order stochastic dominance), further simplifying the application of the results.

The basic model of this paper fits best in the wide literature that outlines sufficient conditions for the existence of monotone pure strategy equilibria in games with asymmetric information (see for example, Athey (2001); McAdams (2003); Van Zandt and Vives (2007); Reny (2011)).[1] Indeed at its core, the model relies on the results of the aforementioned papers, namely the existence of monotonic equilibria for a set of given exogenous beliefs. We first show that when belief are exogenously given and fixed, the new model is consistent with its standard analogue. We are therefore able to apply existing techniques to yield existence of a Bayesian Nash Equilibrium. To endogenise the beliefs, we further formalise the structure of belief formation.

We define Consistent Bayesian Equilibrium (CBE), where players take their beliefs as fixed when choosing their equilibrium strategies. We require that beliefs be derived from equilibrium play and that they be consistent. We use a nested fixed point argument to prove existence; existence of BNE is shown for exogenous beliefs and then

[1] Athey (2001) provides sufficient conditions, crucially that of utility functions displaying the single crossing property to yield monotonic equilibria in games with asymmetric information. McAdams (2003) extends this result to allow for multiple dimensions in the action and type space. Reny (2011) further generalises these results to weaker assumptions on the action and types spaces. Van Zandt and Vives (2007) provide alternative conditions, relying instead on supermodularity and increasing differences, while allowing for multidimensional type and actions spaces and a more general class of type distributions.

a fixed point algorithm is run over the set of beliefs, to find a consistent equilibrium. While such an extension allows us to choose any set of sufficient conditions that yield monotonic equilibria for exogenous beliefs, for the purposes of this paper we work with conditions first proposed by Van Zandt and Vives (2007), due to their relative simplicity.

The model can be applied to various fields in economics, from which we present two applications. Applying the model to the theory of reference dependence, yields what is to our knowledge the first general existence result for games where player preferences exihibit gains and losses with respect to a reference point. To highlight the general nature of the model, a second application is chosen from the industrial organisation literature; we analyse Cournot markets with positive network effects and asymmetric costs.

While the applications of the model are methodologically equivalent, they are from disjoint fields. As such then, for the sake of clearer exposition we leave a more detailed literature review to their respective sections. The rest of the paper is organised as follows. We first present the basic model where beliefs are exogenous, and show that it is a special case of standard Bayesian games. We then formalise the process through which endogenous beliefs are formed by defining *feedback functions* and consistent Bayesian equilibria (CBE). We then derive our main result, namely the existence of a consistent Bayesian equilibrium in monotone pure strategies. The result is then applied to the realm of psychological games with particular focus on the literature on reference dependent preferences. As a second application, we show that Katz and Shapiro (1985)'s Fulfilled Expectations Cournot Equilibrium is a CBE, and extend the model of Amir and Lazzati (2011) to allow for asymmetric firms in Cournot markets with positive network effects. The final section presents concluding remarks.

3.2 Model and exogenous beliefs

As a first step in our analysis, we consider the case when beliefs are exogenously given. To do so, we augment the standard model by introducing beliefs over exogenous variables that effect player utility. Later, we endogenise these variables by making them functions of equilibrium outcomes.

The basic components of the model are as follows. We model a static Bayesian game with a finite set of players, $N = \{1, \cdots, n\}$. Each player $i \in N$ has private information that affects her payoff, and we represent it as her type $t_i \in T_i$. Each player i, after observing her own type, chooses an action a_i from the set of possible actions A_i. Define the sets $T = T_1 \times \cdots \times T_n$ and $T_{-i} = \prod_{N\setminus\{i\}} T_k$ (analogously A and A_{-i}), as the sets of the types (actions) of all players in the game and all players except player i, respectively. Types are private information, that is players know their own types but not the types of their opponents. Additionally, each player has a set of beliefs over the actions of her opponents, represented by $s_i \in S_i \subset \mathbb{R}^m$. Then, a player's utility is a function $U_i : A \times T \times S_i \to \mathbb{R}$, expressed as $U_i(a_i, a_{-i}, t_i, t_{-i}, s_i)$, where a_i is her own action, a_{-i} the vector of her opponents actions, t_i her type, t_{-i} the types of her opponents and s_i is the vector of her beliefs.

There are two forms of beliefs incorporated in the model. Given the Bayesian nature of the game, player's have interim beliefs over the types of their opponents. For each player i, with type t_i, we represent her beliefs over the types of her opponents by the c.d.f $P_i(t_{-i}|t_i)$. The type distributions are common knowledge.

Additionally, in the current model player have stochastic beliefs over the actions of their opponents, s_i. Let $\mathcal{M}_i$ be the set of probability measures on S_i, endowed with the partial order of first order stochastic dominance ($\gtrsim_F$),[2] and let $G_i(s_i|t_i) \in \mathcal{M}_i$ represent player i's interim beliefs over s_i. Further note that when the belief distribution is allowed to vary with player type, it itself is a function. Let $G_i \in \mathcal{G}_i$ be such a function such that $G_i : T_i \to \mathcal{M}_i$ and endow $\mathcal{G}_i$ with the point-wise order.[3] Finally for ease of expression, we compress the notation for belief distributions, such that hence forth $G_i(t) = G_i(s|t) \in \mathcal{M}_i$ is the belief distribution given type and $G_i \in \mathcal{G}_i$ are the functions that maps types into their relevant distributions. In the exogenous setting, belief distributions G_i are also common knowledge.

As the beliefs on types are primitives of the model, while the beliefs on the actions of other players will later be endogenised, to avoid confusion we will from now on use the term *beliefs* exclusively for the endogenous beliefs of player i over the actions of her opponents (s_i). Beliefs on types, will simply be referred to as "type distributions."

As beliefs are stochastic, there is a question of how they interact with player

[2]For any two distributions F, F' over a common support X, we say that $F \gtrsim_F F'$ if $F(x) \leq F'(x)$ for all $x \in X$.

[3]$G(s|t_i) \geq G'(s|t_i)$, if for all $t \in T_i$, $G(\cdot|t) \gtrsim_F G'(\cdot|t)$.

utility. In this regard we follow the behavioural literature and model beliefs such that players take expectations over them at every outcome. That is to say, once a game is played and an outcome reached, the player compares the outcome to all the possible s_i in the support of the belief distribution, taking expectations over the belief distribution.

We can then express player i's utility at an outcome where the realised player types are t, players choose actions a and player i's beliefs are given by G_i as

$$u_i(a_i, a_{-i}, t_i, t_{-i}, G_i) = \int_{S_i} U_i(a_i, a_{-i}, t_i, t_{-i}, s_i) dG_i(s_i|t_i).$$

Having defined the components of the model, we now make the following assumptions over the primitives that will allow the use of lattice theoretic techniques.

Assumption 3.2.1. *T_i is endowed with a partial order.*

Assumption 3.2.2. *For each player i, A_i is a compact metric lattice.*

Assumption 3.2.3. *For every player i and each type t_i, $P_i(\cdot|t_i)$ is measurable.*

Assumption 3.2.4. *For all $i \in N$, the payoff function satisfies the following assumptions.*

1. *For all $a \in A$ and $s_i \in S_i$, U_i is measurable in T.*
2. *For all $t \in T$ and $s_i \in S_i$, U_i is continuous in a.*
3. *For all $a \in A$ and $t \in T$, U_i is measurable in s_i.*
4. *U_i is bounded.*

We are interested in determining whether an equilibrium exists under the current exogenous specification, which will later aid our search for equilibrium with endogenous beliefs. To do so, we utilise the interim formulation of Bayes Nash equilibrium and consider a player's maximisation problem given she knows her opponents equilibrium strategies and beliefs.

Let $\sigma_i : T_i \times \mathcal{G} \to A_i$, be player i's measurable strategy. Let Σ_i be the set of player i's strategy, and define Σ, Σ_{-i} as before.

Then given player i knows her type t_i, her interim maximisation problem given a set of strategies σ_{-i} of her opponents and set of beliefs $G \in \mathcal{G} = \prod_N \mathcal{G}_i$ can be expressed as

$$\max_{a_i \in A_i} \left\{ v_i(a_i, t, P_i | G, \sigma_{-i}) = \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}, G), t_i, t_{-i}, G_i(t_i)) dP_i(t_{-i}|t_i) \right\}. \quad (3.2.1)$$

Note that the above formulation assumes that beliefs are not private information; it assumes that player i knows her opponent's beliefs. This assumption will be shown to be consistent with our formulation of endogenous beliefs, where if we assume a player knows the equilibrium strategies (as is typical in the formulation of equilibrium concepts), it will allow her to calculate the equilibrium beliefs of her opponents.

Further note how in this setup, the newly introduced (exogenous) belief distribution is nothing more than a parameter and we can use existing techniques to characterise an equilibrium. Given a set of exogenous belief distributions $G \in \mathcal{G}$, we apply the main theorem of Van Zandt and Vives (2007) which yields the following proposition.

Proposition 3.2.5. *Assume for each player i the following:*

1. *The payoff function U_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), (a_i, t) and (a_i, s_i).*

2. *The type distribution $P_i(t_{-i}|\cdot)$ is increasing in t_i with respect to the partial order of first order stochastic dominance.*

3. *The beliefs G are measurable and increasing in t_i with respect to the partial order of first order stochastic dominance.*

Then there exists a greatest and least Bayesian Nash Equilibrium that are in monotonic strategies, $\sigma_i^*(t_i, G)$.

We present here a short proof of the proposition for completeness, skipping over measurability concerns. Suffice it to say that given exogenous belief distributions, the model is a special case of that of Van Zandt and Vives (2007) and interested readers may consult it for a more detailed treatment of the problem.

For given beliefs, player i's optimisation problem is given by equation (3.2.1). To prove existence through the fixed point algorithm of Van Zandt and Vives (2007) we need to first show that v_i is continuous in a_i and exhibits increasing differences in (a_i, t_i) and (a_i, σ_{-i}). Further, to find an endogenous belief distribution, we need that the equilibrium strategies are also monotonic in beliefs. For this, we need v_i to have increasing differences in player action and her beliefs. The following lemma yields these properties.

Lemma 3.2.6. *If U_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), (a_i, t) and (a_i, s_i), and P_i and G are increasing in t_i, then v_i has the following properties.*

1. *For all $\sigma_{-i} \in \Sigma_{-i}$, v_i is continuous in a_i. It is supermodular in a_i and has increasing differences in (a_i, σ_{-i}).*
2. *For increasing $\sigma_{-i} \in \Sigma_{-i}$ (increasing in type and beliefs), has increasing differences in (a_i, t_i).*
3. *For increasing $\sigma_{-i} \in \Sigma_{-i}$, has increasing differences in (a_i, G).*
4. *For increasing $\sigma_{-i} \in \Sigma_{-i}$, has increasing differences in (a_i, P_i).*

Proof. Consider this case by case. Given that integration preserves supermodularity and continuity, for all t_i, v_i is continuous and supermodular in a_i. It has increasing differences in (a_i, a_{-i}) so it has increasing differences in (a_i, σ_{-i}).

For increasing differences in (a_i, t_i), consider an increase in t_i. When t_i increases, G and P_i both increase. As u_i has increasing differences in (a_i, a_{-i}), (a_i, t_i) and (a_i, s_i), and the condition that σ_{-i} is increasing in t_{-i} and G, we have that v_i has increasing differences in (a_i, t_i).

For the case of increasing differences between actions and distributions we rely on the following lemma.

Lemma 3.2.7. *Assume the function $F(x, y)$ has increasing differences in (x, y), and y is distributed according to the distribution P. Then $f(x, P) = \int_Y F(x, y)dP$ has increasing differences in (x, P).*

Proof. Let $x', x \in X$ such that $x' > x$. Then the difference $H(y) = F(x', y) - F(x, y)$, is increasing due to increasing differences of F in (x, y). Then $f(x', P) - f(x, P) = \int_Y H(y)dP$ is increasing in P. □

So, given that σ_{-i} is increasing in G and v_i has increasing differences in (a_i, a_{-i}) and (a_i, s_i), v_i has increasing differences in (a_i, G).

Finally, by lemma 3.2.7, it is obvious that v_i has increasing differences in (a_i, P_i). □

Consider now player i's best response,

$$a^*(\sigma_{-i}, t_i, G) \quad = \quad \operatorname*{argmax}_{a_i \in A_i} v(a_i, t_i, P_i | G, \sigma_{-i}).$$

Given the properties of v_i from lemma 3.2.6, application of Topkis' theorem yields that player i's best response to increasing strategies of her opponents is a non-empty,[4] complete lattice with a greatest and least element that is monotonic in both her type and her beliefs.

For given beliefs G we can now simply apply Van Zandt and Vives's Greatest Best Response Mechanism (Lemma 6 of their paper) to yield an equilibrium. Let $\bar{\beta}_i(\sigma_{-i}) = \sup a^*(\sigma_{-i})$, and note that since the best responses are monotonic in G, the equilibrium is also in increasing strategies. The result is stated here without proof.

Lemma 3.2.8 (GBR mechanism)**.** *Assume the following;*

1. *The mapping $\bar{\beta}_i$ exists.*

[4]Existence is in fact given by the application of the Extreme Value Theorem. Recall that A_i is a compact metric lattice.

2. *The mapping is increasing in* σ_{-i}.

3. *If* σ_{-i} *is monotone in its arguments* (t_i, G), $\bar{\beta}_i$ *is monotonic.*

Then there exists a Bayesian Nash Equilibrium in monotone strategies.

Application of the above lemma yields the desired result. Note that we skip some steps, in particular the proof of Lemma 3.2.8 requires to check some measurability assumptions. However for the exogenous treatment of beliefs, the beliefs act as an additional parameter that does not affect any of the assumptions of Van Zandt and Vives (2007).[5] Finally, given how the greatest best responses are increasing in beliefs, the fixed point found using the Cournot ttonnement of the GBR mechanism is also increasing in beliefs.[6] □

Having established the results for exogenous beliefs, we now proceed to formally endogenise beliefs, define our equilibrium concept and prove existence.

3.3 Endogenous beliefs

We model beliefs of a player as functions of the actions of all the players in a game. For those familiar with the literature on psychological games, these are sometimes labelled *first order beliefs.*

We introduce an additional design parameter; a *feedback function,* that takes

[5]To see this note that for given beliefs, if we take the function u_i as our starting point, the problem is identical to that of Van Zandt and Vives (2007). For given beliefs then an equilibrium exists in measurable strategies (by the Measurable Maximum Theorem). The only novelty is, that we have increasing differences with respect to the belief distribution, and standard lattice theoretic arguments yield monotonicity of the best response in beliefs.

[6]The GBR mapping converges point wise to the equilibrium strategies. The point-wise limit of monotone functions is monotone.

the actions of all players in the game and generates a player's beliefs. The feedback function is defined over actions in a deterministic setting (i.e when all actions are known), allowing for multiple dimensions (m) of beliefs, let $f_i : A \to \mathbb{R}^m$ be player i's feedback function.

Then given that in equilibrium, outcomes are stochastic (they depend on player types and beliefs), the feedback functions generate corresponding belief distributions. We assume in the current setup that feedback functions are common knowledge. While restrictive, this assumption is equivalent to assuming players know their opponents (type dependent) payoff functions. Indeed in most applications players are assumed to have identical or universal feedback functions. Furthermore we make the following assumption.

Assumption 3.3.1. *For all i, f_i is continuous and bounded.*

There are now two distinct methodologies available to us for endogenising beliefs. These pertain to how the players perceive the effects of their actions on equilibrium beliefs. One way to endogenise beliefs is to assume that players take beliefs as given when deciding on their action, i.e they assume that their actions do not effect their beliefs. We model such a case by defining *Consistent Bayesian Equilibrium.*

When playing a consistent Bayesian equilibrium, player's assume that their actions have no effect on their beliefs. As such then equilibrium strategies $\sigma_i^*(t_i, G)$ are functions of their belief distributions. As equilibrium outcomes are effected by the beliefs of the players, we require that in equilibrium the beliefs be consistent. For a set of strategy profiles and beliefs to be a Consistent Bayesian Equilibrium (CBE), we require two things; given equilibrium beliefs players play their Bayesian Nash equilibrium strategies and that in turn, these strategies yield equilibrium beliefs.

Definition 3.3.2. The set of strategy profile and beliefs, $(\sigma^*, G^*) \in \Sigma \times \mathcal{G}$ is a Consistent Bayesian Equilibrium (CBE) if for all i and and t_i,

1. Given G^*, $\sigma_i^*(t_i, G^*) \in \text{argmax}_{a \in A_i} v_i(a, t, P_i | G^*, \sigma_{-i}^*)$, and;
2. Given σ^* and G^*, for all i the distribution of $s_i = f_i(\sigma^*(t_i, G^*))$ is G_i^*.

Having formally defined our equilibrium concept, we now state our main result, namely the existence of a consistent Bayesian equilibrium in monotone strategies. We apply a nested fixed point argument, where we exploit the result for exogenous beliefs to run a fixed point algorithm on the set of belief distributions.

Theorem 3.3.3. *Assume for each player i the following:*

1. *The payoff function U_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), (a_i, t) and (a_i, s_i).*
2. *The belief distribution $P_i(t_{-i}|\cdot)$ is increasing in t_i with respect to the partial order of first order stochastic dominance.*
3. *For all $i \in N$, $\mathcal{M}_i$ is endowed with the partial order of first order stochastic dominance.*
4. *The feedback function f_i is increasing in its arguments.*

Then there exists a Consistent Bayesian Equilibrium $(\sigma_i^*(t_i, G), G^*)$ in monotonic strategies and belief distributions that are increasing in type.

Proof. The proof is divided into two steps. We first show that for given beliefs G, such that for each player i, G_i is increasing in player type, a Bayesian Nash Equilibrium exists. We then use this fact and the monotonic best response to calculate updated beliefs, G'. We show that the mapping that generates new beliefs has the fixed point property. To do so, we use a fixed point argument that exploits the fact that the set of beliefs is a chain-complete poset.

First we call on Proposition 3.2.5 to yield the existence of a BNE of the game in increasing strategies $\sigma(t_i, G)$ for a given beliefs $G \in \mathcal{G}$, where for each player i, G_i is increasing in type.

Then, consider the mapping $Q : \mathcal{G} \to \mathcal{G}$, that takes beliefs of players and generates a new distribution of beliefs such that for each player i, $Q_i(G)$ is the distribution of $f_i(\sigma(t, G))$ derived from the primitive type distribution P_i.

Note that the mapping is increasing since higher beliefs yield higher actions (because σ is increasing in beliefs), which in turn yield higher point beliefs (increasing f_i).

Further note that given beliefs that are increasing in type, the mapping Q generates beliefs that are also increasing in type, due to the fact that the underlining type distribution is increasing in player type.

Now consider the set of probability measures on s_i, $(\mathcal{M}_i, \gtrsim_F)$. It is a chain complete poset, as it is the set of probability measures on a compact set S_i. To see why S_i is a compact space, recall that for each i, A_i is a complete lattice and there exist $\bar{a}_i = \sup A_i$ and $\underline{a}_i = \inf A_i$. Then since f is continuous, bounded and increasing,

$S_i = [f(\underline{a}), f(\bar{a})]$.

Then $\mathcal{G}_i$ is also a chain complete poset, since for any chain $C \subset \mathcal{G}_i$, there exists a corresponding chain $C_t \subset \mathcal{M}_i$ for each type, as $\mathcal{G}_i$ is endowed with the point-wise order. Let $\gamma_t \in \mathcal{M}_i$ be the least upper bound of the chain C_t and construct $\gamma(t) \in \mathcal{G}_i$ such that it chooses the corresponding least upper bound for all t. Then $\gamma(t)$ is an upper bound of C by construction. It is also the least upper bound, since by way of contradiction, if a smaller upper bound exists, then for at least some t, $\gamma(t)$ is not the least upper bound of C_t.

We can now complete the proof by applying a fixed point theorem that exploits the fact that the set of beliefs is a chain-complete poset.[7] The fixed point theorem is due to Markowsky (1976) and we state the relevant sections as a lemma without proof. Interested readers may consult Theorem 9 of his paper.

Lemma 3.3.4. *Let X be a chain complete poset and $h : X \to X$ isotone, then the set of fixed points of h, is a non-empty chain complete poset in the induced order.*

□

Extending Van Zandt and Vives (2007) not only yields existence of equilibrium but the current formulation also allows the extension of their comparative statics result. Given the assumption that the feedback functions are increasing, we are able to further extend their comparative static result as it pertains to changes in equilibrium when the interim type distributions increase (according to the order of first order

[7] The use of this theorem was suggested by Paweł Dziewulski.

stochastic dominance). This is trivial once we note that first order stochastic shifts in the type distribution lead to higher actions (Lemma 3.2.6.4), which in turn lead to higher equilibrium beliefs which also increase equilibrium actions. The following proposition formalises this result.

Proposition 3.3.5. *Consider two games consistent with the assumptions of Theorem 3.3.3 that are identical except for their type distributions* $P_i(t_{-i}|t_i)$ *and* $P_i'(t_{-i}|t_i)$*. If* $P_i > P_i'$*, such that for all* t_i*,* $P_i(t_{-i}|t_i) \gtrsim_F P_i'(t_{-i}|t_i)$*; then the greatest equilibrium of the* P *game,* $(\sigma_i(t_i, G), G)$ *is higher than that of the* P' *game,* $(\sigma_i'(t_i, G'), G')$*, i.e.* $(\sigma_i(t_i, G), G) \geq (\sigma_i'(t_i, G), G')$ *in their respective orders.*

3.4 Reference dependent preferences

Kahneman and Tversky (1979) proposed what is perhaps the most well known alternative to standard utility theory. Under what they labelled prospect theory, individual utility was no longer just a function of an individual's final consumption bundle, but there was an additional element of gain-loss utility relative to some reference point or anchor. They outline qualitative properties of gain-loss utility that help explain discrepancies between the predictions of standard theory and experimental data.

The theory has received much attention in the realms of experimental and applied economics, however the definition of the reference point has been characteristically exogenous and ad-hoc. Only recently has the theory been enriched to allow for endogenous reference points.

The models of Kőszegi and Rabin (2006, 2007, 2009) and Shalev (2000) provide a robust theoretical foundation for the application of reference dependence with endogenous reference points. The above papers argue that the an endogenous reference point needs to be consistent with equilibrium outcomes. And while they provide existence results for a limited class of decision problems and games of complete information, the literature lacks a general existence result.

Kőszegi and Rabin model a single player decision problem, where a player's utility for when she consumes x, given reference point r is given by the utility function $U(x|r)$. For lotteries F over x, and beliefs G over the reference point, her expected utility is given by $u(F|G) = \int\int U(x|r)dG(r)dF(x)$. Furthermore, rational expectations imply that if the player chooses lottery F then $G = F$. Existence of a *personal equilibrium* is guaranteed by making a somewhat strong assumption; namely that $u(F|F') > u(F|F') \Rightarrow u(F|F) > u(F'|F)$, which avoids circular choices.[8]

Shalev meanwhile defines a *consistent reference point* for extensive form games, where in equilibrium the reference point of each player is equal to her expected utility. Consistency requires that the behaviour given the reference point yields, for each player, expected utility equal to her reference point. Shalev proves existence for his game by extending Nash's existence result using Kakutani's fixed point theorem. While Shalev models only linear utility, it is trivial to show that his proof can be extended to any continuous utility specification. Indeed the model of Shalev enriched

[8]The assumption can be seen as a weaker version of the "no-regret" condition (Sagi, 2006).

with the utility specification of Kőszegi and Rabin, with degenerate beliefs over the reference points, presents what can be considered the most general theory for games with endogenous reference points.

However, the above methodology does not account for asymmetric information.[9] This is of interest not only due to the fact that most economic settings involve some form of informational asymmetry, but also because reference dependence has more "kick" in such situations. For games of complete information, pure strategy equilibria in standard games continue to be equilibria once the model is enriched with reference dependent preferences.[10] This is no longer the case for asymmetric games as the stochastic nature of the game offers a richer application of the theory, which is not surprising as prospect theory was historically developed to deal with cases with inherent uncertainty.

The application of the model of this paper, then provides a more general result for the existence of equilibria for this class of games. We need to make some modifications to the theory as it stands to allow for the application of our techniques; in particular we can not allow diminishing sensitivity.

We now present the model of reference dependence in greater detail, with special attention to the changes in assumptions needed from the current state of the art to apply the results of the current paper.

[9]One could model incomplete information in Shalev's model by modelling a move by nature in an extensive form game, however then we would need to find a *non-myopic* equilibrium, for which existence is not guaranteed.

[10]This is due to the fact that every player gets what they expect and so gain-loss utility is zero (see Proposition 2 of Shalev (2000)).

3.4.1 Model

We model a game with n players who have reference dependent utility. Players have private information about their type $t_i \in T_i$ and choose an action $a_i \in A_i$. Player i's beliefs over the types of her opponents are given by the type distribution $P_i(t_{-i}|t_i)$. We continue to make the same assumption on these elements as before, however we further refine the utility specification, so that it displays the characteristics of reference dependence.

Following Kőszegi and Rabin (2006) we define reference dependent utility to be the sum of a player's Von Neumann–Morgenstern utility U_i and her gain-loss utility η_i. Explicitly

$$\pi_i(a, t_i|r_i) = U_i(a, t_i) + \eta_i(a|r_i),$$

where $a = (a_1, a_2, ..., a_I)$ is the action vector, t_i is player i's private information about her type and r_i her anchor vector. Given that final consumption may have multiple dimensions, it is further assumed that utility is additive across dimensions, i.e $U_i(a, t_i) = \sum_{k=1}^{m} U_i^k(a, t_i)$ and $\eta_i(a, t_i|r_i) = \sum_{k=1}^{m} \eta_i^k(a, t_i|r_i^k)$.

The reference point is modelled as a design parameter. Like Kőszegi and Rabin (2006) the reference point is modelled as expectations over the outcome. We then define the reference point for each dimension using a feedback function, such that $r_i^k = f_i^k(a)$. Furthermore, it is assumed that gain-loss utility is an increasing function in the difference between the reference point and its realised analogue. That is to say, given an outcome where players choose actions a and the vector of reference points is

r, gain-loss utility for each player i in dimension k is given by

$$\eta_i^k = \mu_i^k(f_i^k(a) - r_i^k).$$

While Kőszegi and Rabin (2006) allow a rich variety of gain-loss functions, here we restrict attention to linear specifications, such that $\mu_i^k \in \mathbb{R}$. The assumption is consistent with those of Kőszegi and Rabin (2006), however it only captures the phenomenon of reference dependence, and does not allow for loss aversion.[11]

3.4.2 Endogenous reference points

We now present one way of endogenising the reference point. The model is a special case of the general model and we gain existence of Consistent Bayesian Equilibrium in *supermodular games with endogenous reference points* as a corollary to Theorems 3.3.3.

Corollary 3.4.1. *If in a game with reference dependent preferences, the following assumptions hold.*

1. *For each player i, the Von Neumann–Morgenstern utility function, U_i is continuous and supermodular in a_i, and has increasing difference in (a_i, a_{-i}) and (a_i, t_i).*
2. *For each player i and every dimension k, $f_i^k(a)$ is increasing in its arguments, is supermodular in a_i and has increasing differences in (a_i, a_{-i}).*

Then there exist a Consistent Bayesian Equilibrium in monotonic pure strategies.

[11] The assumption can be relaxed and the model extended to gain-loss functions that are not "too concave".

Proof. Most of the requirements of Theorem 3.3.3 are trivial to check.[12] We do however need to check that π_i displays increasing differences in (a_i, r_i), which given the differentiability assumptions of the model of reference dependence is equivalent to showing the cross partial is non-negative. Given that the feedback functions are increasing, this is easy to check as, $\frac{\partial^2 \pi_i}{\partial a_i \partial r_i^k} = 0$.

□

The form of consistency yielded by the above corollary is slightly different from those used recently in the applied literature. In particular, here we assume that players are less sophisticated than in certain applied models (see for example, Lange and Ratan (2010) and Gill and Stone (2010)). In the current formulation, we calculate the equilibrium strategies for a given reference point and then update the reference point in the next step. By doing so, we implicitly assume that players do not take into account changes in the reference point due to their own actions. In the aforementioned applications however, player's do take into account how their actions may change their reference point, and so are more sophisticated. However, the current formulation has precedence in the literature. Indeed, it is a direct extension of both the Shalev's (2000)concept of a loss averse equilibrium, and Kőszegi and Rabin's (2006) personal equilibrium.[13]

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[12]Continuity, supermodularity and increasing differences are preserved by addition and increasing transformations.

[13]Shalev's proof of existence of a loss averse equilibrium (Proposition 3 in his paper) also uses a fixed point argument where the mapping finds a player's best response for a given reference point. Kőszegi and Rabin define a lottery F to be an equilibrium iff $U(F|F) \geq U(F'|F)$, for available lotteries F'. They too therefore assume the player does

Also note that in Shalev (2000) the feedback function is assumed to be a players utility function. This is no longer possible, since monotonicity of the feedback function is needed for application of the fixed point theorem (Lemma 3.3.4).[14] So we seek refuge in the more lenient methodology of Kőszegi and Rabin (2006) which allows for flexibility in choosing both the dimensions of the utility framework and the endogenous outcome to be used as a reference point based on sound psychological reasoning.

Finally, another closely related application that has been alluded to before, is the class of psychological games with *first order beliefs.* The class of games first proposed by Geanakoplos, Pearce, and Stacchetti (1989) and extended to account for updated beliefs by Battigalli and Dufwenberg (2009) models games where player utility is directly affected by beliefs over other player's actions.[15] Indeed then, the model of this paper can be seen to also apply to simultaneous psychological games with first order updated beliefs and extends the model in the direction of asymmetric information, while also providing sufficient conditions for the existence of monotone equilibria in pure strategies.

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not update his reference point immediately.

[14]To be more precise, it is no longer interesting. The assumption that the feedback function be increasing in actions would require utility to be increasing, which would yield a trivial equilibrium where all players choose their highest actions.

[15]The full model also allows for higher order beliefs, i.e beliefs over other player's beliefs.

3.5 Network effects in Cournot markets

While the previous application was of a general nature, we now present a more applied and structured application. We extend the model of Amir and Lazzati (2011) by allowing for heterogeneous firms and study the effects of asymmetries in firm costs on Cournot markets with positive network effects.

Amir and Lazzati model a Cournot game, where firms compete for market share in the presence of positive network effects. As in the standard Cournot market, firms compete in the quantity dimension with each other for market share, however additionally, the demand for the good is directly affected by the size of the market; the larger the market (as determined by total output), the higher the demand. Increased competition then has an ambiguous effect on firm profits; the business stealing effect and the network effect work in opposite directions.

Amir and Lazzati model symmetric firms and show the existence of a fulfilled expectations Cournot equilibrium (FECE) developed first by Katz and Shapiro (1985). It is of interest to note that the model of Amir and Lazzati can be seen as a special case of current model and can be extended to account for asymmetric cost without the need of additional assumptions on the basic model. Such an extension allows us to provide insights into the effects of technology shocks that lower costs on the viability of an industry. The observation is both intuitive and comes at a low cost as it is a direct corollary of Proposition 3.3.5. The use of the well known aggregation property to yield increasing differences in Cournot oligopolies, also demonstrates how existing techniques extend to the class of games with endogenous beliefs.

Consider then a Cournot market with n exogenous firms, where each firm i chooses its respective output quantity q_i. Market demand is given by $p(a, s)$, where $a = \sum_{i=1}^{n} q_i$ is total market output and s is the expected size of the market.

We introduce asymmetry to the model by allowing for costs to vary across firms. Continuing with the notation used thus far, let $t_i \in T_i$ be the type of a firm which is its private information and let $P(t_{-i}|t_i)$ be firm i's (interim) distribution over its opponents types.[16] Let $c(x, t_i)$ be the cost of producing x units of the good for a firm of type t_i.[17] We assume that higher types represent higher efficiency, i.e high type firms have lower costs.

We have a choice over how to incorporate the beliefs over the expected market size. While the model of the paper allows the use a distribution of beliefs, we choose to use a point estimate. This both simplifies the notation and unlike the previous application where behavioural considerations may impose the need to consider distributional beliefs, this is not the case for firms.[18]

Then firm i's maximisation problem given its type t_i, exogenous expectations

[16] It is assumed for notational simplicity that firms have identical interim type distributions, which would lead to identical beliefs.

[17] The parametrised cost functions are assumed to be identical for notational simplicity.

[18] In the distributional setting, firms (agents) compare every realised outcome to the distribution of expectations. Such a distributional approach incorporates a richer set of risk attitudes (Kőszegi and Rabin, 2007). Assuming risk neutral firms a degenerate expectation is sufficient. Motivation for a single point may also be drawn from how market forecasts (forecasts of quarterly profits, etc.) are indeed point estimates.

on network size s and its opponents strategies $q_j(t_j, s(t_j))$ is given by

$$\max_q \int_{T_{-i}} \big(p(q + y(t_{-i}, s(t_{-i})), s(t_i)) - c(q, t_i)\big) dP_i(t_{-i}|t_i),$$

where $y(t_{-i}, s(t_{-i})) = \frac{1}{n-1} \sum_{j \neq i} q_j(t_j, s(t_j))$.

Using the well known aggregation property, this problem is equivalent to the case where firm i chooses total industry output while best responding to its rivals total output y. Then compressing notation, firm i's best response can be expressed as

$$a_i(y, t_i, s) = \operatorname*{argmax}_{a \geq y} \int_{T_{-i}} \big(p(a, s(t_i)) - c(a - y, t_i)\big) dP(t_{-i}|t_i).$$

An equilibrium of this game consists of a set of strategies, such that each firm best responds to its opponents strategies, given it knows its own type and takes the expected size of the market as given. Furthermore, consistency requires that for each type the expected size be consistent with the firms type distribution.

Definition 3.5.1. The set of equilibrium strategies and network expectations (a^*, s^*) is a fulfilled expectations equilibrium if for all player i and type t_i

1. Given s^*,

   $$a_i^*(t_i, s^*) \in \text{argmax}_{z \geq y} \int_{T_{-i}} \big(p(z, s^*(t_i)) - c(z - \frac{1}{n-1} \sum_{j \neq i} a_j^*(t_j, s^*), t_i)\big)\ dP(t_{-i}|t_i);$$

   and

2. $s^*(t_i) = \frac{1}{n}\big(a_i^*(t_i, s^*) + \int_{T_{-i}} \{ \sum_{j \neq i} a_j^*(t_j, s^*) \} dP(t_{-i}|t_i)\big)$.

The above definition is an extension of Katz and Shapiro's (1985) concept of a "Fulfilled Expectations Cournot Equilibrium (or FECE)" to the case of asymmetric costs. Unlike the case of complete information, firms choose their output based on their type while taking expectations over the size of the market based on the equilibrium strategies of their opponents and their respective type distributions.

We now list the assumptions needed to apply the main theorem of the paper. All assumptions except for assumption 3.5.7 are identical to those of Amir and Lazzati (2011).

Assumption 3.5.2. *$p(a,s)$is twice continuously differentiable, $\frac{\partial p}{\partial a} < 0$ and $\frac{\partial p}{\partial s} > 0$.*

Assumption 3.5.3. *$c(q,t)$ is twice continuously differentiable, $c(0,\cdot) = 0$ and $\frac{\partial c}{\partial q} > 0$.*

Assumption 3.5.4. *$x_i \leq K$, for each firm i.*

The above assumptions are standard in the literature, stating simply that price is decreasing in total output (the Law of Demand), it is costless to produce zero units and that cost is increasing in a firm's output. Differentiability assumptions are carried over from the previous paper for convenience. The positive network effect is captured by the condition $\frac{\partial p}{\partial s} > 0$, which implies that as the size of the network increases, consumers are willing to pay more for the good. Finally the capacity constraint compactifies the choice set.

Assumption 3.5.5. *For all t,$-\frac{\partial p}{\partial a} + \frac{\partial^2 C}{\partial q^2} > 0$ for $(a,y,s) \geq (y,0,0)$.*

Assumption 3.5.6. *For all t, $p\frac{\partial^2 p}{\partial a \partial s} - \frac{\partial p}{\partial a}\frac{\partial p}{\partial s} > 0$ for $(a,y,s) \geq (y,0,0)$.*

While a detailed discussion on the economic intuition of the above assumptions are left to Amir and Lazzati (2011), the above assumptions guarantee increasing differences in player choice and her opponents' (total) output, as well as increasing differences in her action and the size of the network.

We now impose a restriction on the cost function that yields increasing differences in the player's objective function between her actions and type.

Assumption 3.5.7. *Cost is decreasing in type and* $\frac{\partial^2 c}{\partial t \partial q} \leq 0$.

The above assumption yields increasing differences in player types and her actions. Furthermore it has an intuitive interpretation. It simply states that the marginal cost of production for a firm falls as its type increases. Coupled with assumption 3.5.3, this means that high type firms are more efficient across all feasible output ranges. In other words, types here are a natural measure of firm efficiency, which is what we wish to capture.

Given this set of assumptions, it is obvious that the model is consistent with the requirements of the main theorem, yielding existence of a monotone equilibrium as a corollary.

Corollary 3.5.8. *A Cournot oligopoly with (positive) network effects and asymmetric firms has at least one fulfilled expectations Cournot equilibrium in monotone strategies.*

Furthermore, one can model the effect of cost reducing technologies on equilibrium output. Consider an exogenous shock that increases the distribution of firm efficiency in the sense of first order stochastic dominance. Then as a corollary to Proposition 3.3.5 we have the intuitive result that the extremal equilibrium increases.

Corollary 3.5.9. *For a Cournot oligopoly with (positive) network effects and asymmetric firms, if the distribution of types increases (in the sense of first order stochastic* *dominance), the greatest monotonic equilibrium increases.*

The major contribution of Amir and Lazzati (2011) is its detailed characterisation of *market viability,* i.e. under what primitive assumptions on demand and cost do firms produce positive output in equilibrium. While a detailed study of viability is left to future research, Corollary 3.5.9 provides some intuition regarding the existence of "open standards" in many network industries. An example is the pervasive use of the Peripheral Component Interconnect Bus (PCI) standard for attaching hardware peripherals to desktop computers. Developed by Intel for use with their motherboards, it was adopted as an industry standard and can be found in most desktop computers from the mid-1990s to the present day.[19]

The PCI standard allows for extending the features of a computer by allowing the use of expansion cards. Its prevalence allows different manufacturers to produce computer components for a larger market. Similarly for consumers, having a "PCI slot" allows them to extend the capabilities of their computer from a wide variety of products. PCI as a standard then, presents a network good, and for Intel it is close to a pure network good, high adoption of the standard is crucial to their ability to market their own products. Other firms produce motherboards (and other components) that are compatible with the PCI standard, however it is not reasonable to assume that all firms are homogeneous. Indeed in 1992 it would most likely be true that Intel

[19]The information technology industries are littered with open standards, USB, HTML5 and PDF file format are just some of the standards used by millions everyday. Perhaps the most crucial open standard is the IP networking standard, which is the backbone of the internet.

had the lowest cost, however the industry may not have been *viable* if Intel had not opened the standard to other firms.

Such a case would be close to one of *conditional viability* in Amir and Lazzati (2011). As this is a pure network good $p(x, 0) = 0$,[20] and while Intel may have a cost advantage, it still requires others to enter the market. In such a case, offering the standard royalty-free lowers the costs of its competitors. Such an action would yield a rightward shift in the type distribution for all firms, and so by Corollary 3.5.9 would increase both the equilibrium strategies and expected network size, positively affecting viability. The market would then contain multiple firms with varying levels of efficiency producing different quantities. Intel would benefit from its high efficiency, granting it higher market share in the monotone equilibrium. Open standards can then be explained as a rational choice on part of competitive firms in network industry to increase market viability and in turn their own profits.

3.6 Conclusion

A layman's description of the contribution of this papers are as follows: the paper provides a rule of thumb for the existence of consistent equilibria in games with endogenous beliefs. If for exogenous beliefs, a monotonic equilibrium exists, a consistent equilibrium exists. This methodology allows the researcher to enrich

[20]This assumption may indeed be restrictive, since closed or proprietary standards also exist. However their use is limited to highly specialised markets such as *closed gardens*, where to use company A's product you must buy only from company A. Such a market is designed to not have network effects to begin with. Hardware standards however display network effects and are developed as such.

her model by adding endogenous beliefs, while still using existing techniques she is familiar with. The detailed study of the sufficient conditions and why they are needed hint at simplifications. While not pursued formally, it is easy to see that if beliefs do not depend on player type, the exogenous strategy profile need only be monotonic in beliefs; monotonicity in type is no longer needed as the fixed point algorithm only requires monotonicity in beliefs. Furthermore, while the paper uses Markowsky (1976)'s theorem, the method can be simplified if the exogenous setting yield continuous best responses.[21]

For completeness, the paper presents sufficient conditions under which a pure strategy equilibrium exists in monotone strategies for a general class of games with endogenous beliefs. The fact that strategies are monotone in player type and beliefs is needed for the application of Markowsky's (1976) fixed point theorem that guarantee the existence of a consistent equilibrium.

The additional assumption needed to guarantee existence, namely that feedback functions are monotone is similar in nature to the assumptions used in the literature, though they do indeed limit the number of direct applications available.[22] However, the range of models available is still substantially large; auctions, bargaining

[21] As aforementioned we do not have primitive assumptions that guarantee continuity. Indeed continuity would make a second fixed point argument redundant.

[22] We require not only that the equilibrium is monotonic, but also that the beliefs of interest are monotonic in all player's actions. So for example, one could not in general apply models of disappointment aversion (Gul, 1991), unless standard utility is assumed to be increasing in player actions. The assumption is valid for decision theoretic models where player actions are equivalent to her consumption, but is uninteresting for games as it would lead to trivial equilibria (everyone chooses the highest action).

and contests with beliefs on the final price or effort for example fit the model.[23]

Additionally, the model allows for comparative static results at the same level of generality. When the type distributions increase, in the sense of first order stochastic dominance, the consistent Bayesian equilibrium increases. This too is a direct extension of the results found in the current literature.

Finally, the methodology presented is a simple extension of current techniques and brings with it the advantages of ease of application as well as familiarity. The conditions aid those who wish to apply models with endogenous beliefs to different economic settings. Two such applications were presented, one from the literature on behavioural game theory and the other from the industrial organisation literature.

The applications provided a first step towards a general theory of reference dependent preferences in games and extended Amir and Lazzati (2011) to study the effects of introduction asymmetric costs in Cournot markets with positive network effects. While the latter model was not developed fully to study the issue of viability of markets, we were still able to gain intuition regarding the pervasive practice of developing open standards in the information technology industries.

[23]The basic arguments of this paper extend to auctions. One would however need to replace the assumptions from Van Zandt and Vives (2007) with those of Athey (2001) for the exogenous setting.

APPENDIX A
PROOFS: CHAPTER 1

A.1 Proposition 1.3.3

Substituting the bid functions from Proposition 1.3.2 into the consistency requirement (equation 1.3.1) and simplifying yields the desired result.

A.2 Corollary 1.3.6

From Mallows (1991) the following inequality is known

$$|\mu - m| \leq \sigma,$$

where μ, m, σ are the mean, median and standard deviation of a distribution.[1]

For a continuous, uni-modal distribution $m \leq \mu$ if the distribution is positively skewed and $m \geq \mu$ otherwise (von Hippel, 2005). Then applying the inequality above case by case yields the desired sufficient conditions.

A.3 Proposition 1.4.2

For a second price auction, to calculate the weakly dominant strategy we set player payoff to zero, to calculate the bid at which a player's "worst-case" payoff when she wins is zero.

For a first price auctions, consider bidder i's problem given her opponents bid according to the increasing bid function $b_1^s(v;\alpha)$ and α is increasing. Then her interim

[1]The result is derived through the application of Jensen's inequality.

maximisation problem given her value is v is given by

$$\max_z \Big[v - b_1^s(z;\alpha) + \lambda(\alpha(v) - b_1^s(z;\alpha))\Big] G_{(1)}(z).$$

The first order condition with respect to z yields

$$(v + \lambda\alpha(v))g_{(1)}(v) =(1+\lambda)\big[g_{(1)}(v)b_1^s(v;\alpha) + G_{(1)}(v)\frac{\partial b_1^s}{\partial v}\Big|_z\big].$$

In equilibrium, $z = v$, so

$$\begin{aligned}
(v + \lambda\alpha(v))g_{(1)}(v) =&(1+\lambda)\big[g_{(1)}(v)b_1^s(v;\alpha) + G_{(1)}(v)\frac{\partial b_1^s}{\partial v}\Big|_v\big],\\
(v + \lambda\alpha(v))g_{(1)}(v) =&(1+\lambda)\big[\frac{db_1^s(v;\alpha)G_{(1)}(v)}{dv}\big],\\
(1+\lambda)b_1^s(v;\alpha)G_{(1)}(v) =& \int_0^v (x + \lambda\alpha(x))g_{(1)}(x)dx.
\end{aligned}$$

Integrating by parts and simplifying completes the proof.

Revenue Equivalence

Consider the expected revenue of the seller in a first price auction.

$$\begin{aligned}
R_1 =&\frac{1}{1+\lambda}\int_0^\omega \Big[v - \int_0^v \frac{G_{(1)}(w)}{G_{(1)}(v)}dw + \lambda\int_0^v \frac{\alpha(w)}{G_{(1)}(v)}dG_{(1)}(w)\Big]dF_{(1)}(v),\\
=&\frac{1}{1+\lambda}A_1 + \frac{\lambda}{1+\lambda}\int_0^\omega\int_0^v \frac{\alpha(w)}{G_{(1)}(v)}dG_{(1)}(w)dF_{(1)}(v),\\
=&\frac{1}{1+\lambda}A_1 + \frac{\lambda}{1+\lambda}\int_0^\omega\int_0^v n\alpha(w)dG_{(1)}(w)dF(v),\\
=&\frac{1}{1+\lambda}A_1 + \frac{\lambda}{1+\lambda}\int_0^\omega\int_w^\omega dF(v)n\alpha(w)g_{(1)}(w)dw,\\
=&\frac{1}{1+\lambda}A_1 + \frac{\lambda}{1+\lambda}\int_0^\omega \alpha(w)n(n-1)(1-F(w))F^{n-2}(w)f(w)dw.
\end{aligned}$$

where $A_1 = \int_0^\omega \big[v - \int_0^v \frac{G_1(w)}{G_1(v)}dw\big]dF_{(1)}(v)$ is the seller revenue in a standard risk neutral auction.

Similarly, expected revenue in a second price auction is given by the following equation.

$$\begin{aligned}
R_2 =& \frac{1}{1+\lambda}\int_0^\omega \Big[v + \lambda\alpha(v)\Big]dF_{(2)}(v),\\
=& \frac{1}{1+\lambda}A_2 + \frac{\lambda}{1+\lambda}\int_0^\omega \alpha(v)dF_{(2)}(v),\\
=& \frac{1}{1+\lambda}A_2 + \frac{\lambda}{1+\lambda}\int_0^\omega \alpha(v)n(n-1)(1-F(v))F^{n-2}(v)f(v)dv.
\end{aligned}$$

Where $A_2 = \int_0^\omega vdF_2(v) = A_1$ by standard revenue equivalence arguments. It is then obvious that $R_1 = R_2$.

A.4 Proposition 1.4.3

Existence

While existence can be shown by using the continuity of equilibrium bid functions and exploiting generalisations of Brouwer's fixed point theorem (e.g. Brouwer-Schauder-Tychonoff Theorem). However, we present here a shorter proof using lattice theoretic techniques. This proof also helps motivate the discussion on the robustness of the results (Section 1.6).

Consider the mapping $Q : M \to M$ from the set of measurable functions on the interval $[0,\omega]$. Let $Q(\alpha) = \alpha'$, where for every value v, $\alpha'(v) = \int b_t^s(z,\alpha)dL_t(z|v)$.

Then note that M is a chain-complete poset under the point-wise order.[2] Fur-

[2]For any $a, \hat{a} \in M$, we say that $a > \hat{a}$, if for all $v \in [0,\omega]$, $a(v) > \hat{a}(v)$. For any chain $C \subset M$, C's point wise supremum (upper envelope) is its least upper bound and is measurable.

ther, given that b_i^s is increasing in α for both first and second price auctions,[3] Q is an increasing mapping. Then by a generalisation of Tarski's fixed point theorem to chain-complete posets, due to Markowsky(1976, Theorem 9) a fixed point of Q exists.

First price auctions[4]

Consider the equilibrium bid function as a function of player value for a given reference point α,

$$\beta(v) = b_1^s(v;\alpha) = \frac{1}{1+\lambda}\Big[v - \int_0^v \frac{G_{(1)}(w)}{G_{(1)}(v)}dw\Big] + \frac{\lambda}{1+\lambda}\Big[\int_0^v \frac{\alpha(w)}{G_{(1)}(v)}dG_{(1)}(w)\Big]. \quad \text{(A.4.1)}$$

Note that the function is increasing in value for *any* reference point α. To see this, note that the term in the first square brackets is the bid function in a standard auction with risk neutral players and so is increasing in value. The term in the second set of square brackets, is an integral of the reference points over the first order statistic for $(n-1)$ bidders. Given that the reference point is an expected price, it is by construction non-negative for all values. An increase in value would then (weakly) increase the sum of reference points. $\beta(v)$ is therefore increasing in v.

Consider now the consistency requirement

$$\begin{aligned}\alpha_1(v) &= b_1^s(v,\alpha_1)G_{(1)}(v) + \int_v^{\omega} b_1^s(z;\alpha_1)dG_{(1)}(z),\\ &= \int_o^{\omega} \max\{\beta(v),\beta(x)\}dG_{(1)}(x).\end{aligned}$$

[3]A quick glance at the bid functions defined in Proposition 1.4.2 confirms that bids are increasing in α.

[4]This simpler method of proof was suggested by Steven Stong.

The right hand side is clearly increasing in bidder value, therefore the fixed point of the mapping must also be increasing.

Second price auctions

For second price auctions, application of the implicit function theorem to the consistency requirement for a particular value yields

$$\left[\frac{1}{n-1} - [1-F(v)]F^{n-1}(v)\frac{\lambda}{1+\lambda}\right]\frac{\partial \alpha_2(v)}{\partial v} = (1-F(v))\frac{F^{n-2}(v)}{1+\lambda}.$$

The right hand side is clearly positive. For the reference point to be increasing, we require that the term in the square brackets also be positive. Let $a = F(v) \in [0,1]$ then,

$$\begin{aligned}
[1-F(v)]F^{n-1}(v)\frac{\lambda}{1+\lambda} = (1-a)a^{n-1}\frac{\lambda}{1+\lambda} &\leq (1-a)a^{n-1} \\
&\leq (1-\frac{n-1}{n})(\frac{n-1}{n})^{n-1} \\
&\leq \frac{1}{n}(\frac{n-1}{n})^{n-1} \\
&\leq \frac{1}{n} \\
&\leq \frac{1}{n-1}.
\end{aligned}$$

The second inequality follows from the fact that $\frac{n-1}{n} = \text{argmax}_{a\in[0,1]}(1-a)a^{n-1}$. Therefore the left hand side is always positive.

A.5 Proposition 1.5.1

For the bidding strategy for first price auctions, note that the relevant measure of reference dependence, λ_k $(k \in l, g)$, depends on whether the bid is above or below the reference point. Then the bid function is simply the function from Proposition 1.3.2, with the relevant coefficient of loss aversion.

We calculate the value at which the bidder switches from the gain domain to the loss domain, i.e we calculate the value $\hat{v}$ at which a bidder experiencing gains (i.e her bid is below the reference point), bids an amount equal to the reference point. That is

$$\begin{aligned}\alpha =& \frac{1}{1+\lambda_g}\left[\hat{v} + \lambda_g \alpha - \int_0^{\hat{v}} \frac{G_{(1)}(w)}{G_{(1)}(v)} dw\right],\\ \hat{v} =& \alpha + \int_0^{\hat{v}} \frac{G_{(1)}(x)dx}{G_{(1)}(\hat{v})}.\end{aligned}$$

Alternatively, we could have used the bid function for bidders with value above the reference point. It yields an identical condition, therefore showing continuity.

Note that $\hat{v}$, is the fixed point of the mapping $\mathcal{V}(z) = \alpha + \int_0^z \frac{G_{(1)}(x)dx}{G_{(1)}(z)}$. The mapping is continuous and and has a slope of less than one, therefore $\hat{v}$ is unique.

A.6 Proposition 1.5.2

Writing out the consistency requirements and simplifying yields the implicit functions that define both α_1 and α_2.

To show that α_2 is decreasing in Δ, application of the implicit function theorem yields

$$\begin{aligned}(1+\lambda_g)\frac{\partial \alpha_2}{\partial \Delta} =& -\int_0^{\alpha_2} F_{(2)}(z)dz - \Delta \frac{\partial \alpha_2}{\partial \Delta} F_{(2)}(\alpha_2),\\ \frac{\partial \alpha_2}{\partial \Delta} =& \frac{-\int_0^{\alpha_2} F_{(2)}(z)dz}{(1+\lambda_g - \Delta F_{(2)}(\alpha_2))}.\end{aligned}$$

Recall that by definition $\Delta \in [0, \lambda_l]$ which implies that $\frac{\partial \alpha_2}{\partial \Delta} \leq 0$.

A.7 Derivation of $\alpha_2(v)$

There are two mutually exclusive cases to consider. The case when bidder i wins after bidding according to $b_2^n(v;\alpha)$ and the case where she loses. Let y_1 be the highest of $(n-1)$ values, then when she wins the price is determined by

$$\begin{aligned}\theta(z) =& P(y_1 = z|y_1 < v),\\ =& \frac{P(y_1 < v|y_1 = z)P(y_1 = z)}{P(y_1 < v)},\\ =& \frac{g_{(1)}(z)}{G_{(1)}(v)}.\end{aligned}$$

When the bidder loses, the price is determined by the second highest price. Define y_2 as the second highest value of bidder i's opponents, the distribution of the value that determines price when the bidder losses the auction is given by

$$\begin{aligned}\gamma(z) =& P(y_2 = z|y_1 > v),\\ =& \frac{P(y_1 > v|y_2 = z)P(y_2 = z)}{P(y_1 > v)}.\end{aligned}$$

Consider $P(y_1 > v|y_2 = z)$. It is unity when $z > v$, and for the case of $z < v$ we derive it explicitly. Let $g_{(12)}(y_1, y_2)$ be the joint distribution of the highest and second highest of $(n-1)$ values.

$$\begin{aligned}P(y_1 = v|y_2 = z) =& \frac{g_{(12)}(v, z)}{g_{(2)}(z)},\\ =& \frac{(n-1)(n-2)f(v)f(z)F^{n-3}(z)}{(n-1)(n-2)[1-F(z)]F^{n-3}(z)f(z)},\\ =& \frac{f(v)}{1-F(z)}.\\ P(y_1 > v|y_2 = z) =& \frac{1-F(v)}{1-F(z)}.\end{aligned}$$

Substituting yields

$$\gamma(z) = \begin{cases} \frac{[1-F(v)]g_{(2)}(z)}{[1-F(z)](1-G_{(1)}(v))} = \gamma_1(z) & \text{if } z < v \\ \\ \frac{g_{(2)}(z)}{1-G_{(1)}(v)} = \gamma_2(z) & \text{if } z \geq v \end{cases}.$$

The reference point for a second price auction is then given by

$$\alpha_2(v) = G_{(1)}(v)\int_0^v b_2^s(z;\alpha)\theta(z)dz$$
$$+ [1 - G_{(1)}(v)]\Big[\int_0^v b_2^s(v;\alpha)\gamma_1(z)dz + \int_v^\omega b_2^s(z;\alpha)\gamma_2(z)dz\Big].$$

Simplifying yields equation (1.4.5).

A.8 Revenue equivalence in the presence of reserve prices.

We present here the proof for revenue equivalence in the presence of reserve prices when bidders are naive and have reference dependent preferences.

For the case with reserve prices, the proof entails calculating the reference point for a given reserve price and then showing that $\alpha_1(r) = \alpha_2(r)$ for all r. In the presence of reserve prices, the reserve price acts as a lower bound on the expected price at which a bidder can buy the object. The naive reference point is then defined by

$$\alpha_t(r) = \int \max\{r, b_t^n(z;\alpha)\}dH_t(z). \tag{A.8.1}$$

Similarly given $\alpha(r)$ bidders bid according to the following strategies (see Rosenkranz and Schmitz (2007) for complete derivation).

$$b_1(v;\alpha(r)) = \frac{1}{1+\lambda}\Big[v + \lambda\alpha - \int_{\max\{0,\tilde{v}(r,\alpha)\}}^{\max\{v,\tilde{v}(r,\alpha)\}} \frac{G_{(1)}(w)}{G_{(1)}(v)}dw\Big],$$
$$b_2(v;\alpha(r)) = \frac{v+\lambda\alpha}{1+\lambda},$$

where $\tilde{v}(r,\alpha) = (1+\lambda)r - \lambda\alpha$ is the screening value given r and $\alpha(r)$. We also know from past literature that the two auctions are revenue equivalent for a given reference point and reserve price.

We now characterise the reference point in the two auctions and then show that they are equal.

Consider a second price auction.

To solve for the reference point as a function of the reserve price explicitly, note that for all r there exists a value $\tilde{v}(r,\alpha_2(r)) = (1+\lambda)r - \lambda\alpha_2(r)$ such that bidders with value $v \leq \tilde{v}(r,\alpha_2(r))$ do not participate in the auction. If $\tilde{v}(r,\alpha_2(r)) \leq 0$ then the reference point is the same as the one given in Proposition 1.3.3. Calculating the maximum such r, we get

$$\alpha_2(r) = \int_0^\omega z dF_{(2)}(z), \qquad \text{if } r \leq \frac{\lambda}{1+\lambda}\int_0^\omega z dF_{(2)}(z) = \tilde{r},$$

For $r > \tilde{r}$, the consistent reference point $\alpha^2(r)$ is implicitly defined by the equation,

$$\alpha^2(r) = F_{(2)}(\tilde{v}(r,\alpha^2(r)))r + \int_{\tilde{v}(r,\alpha^2(r))}^{\omega} \frac{z+\lambda\alpha^2(r)}{1+\lambda} dF_{(2)}(z).$$

In the above equation we consider two cases. First, if the second highest value is below the reserve price then the expected price is r otherwise the price is determined by the second highest bid. The above can be simplified to yield

$$\alpha^2(r) = F_{(2)}(\tilde{v}(r, \alpha^2(r)))\tilde{v}(r, \alpha^2(r)) + \int_{\tilde{v}(r,\alpha^2(r))}^{\omega} z dF_{(2)}(z).$$

So combining the results the consistent reference point is given by,

$$\alpha_2(r) = \begin{cases} \int_0^\omega z dF_{(2)}(z) & \text{if } r \leq \tilde{r} \\ \alpha^2(r) & \text{if } \tilde{r} \leq r \leq \omega \\ \omega & \text{if } r = \omega \end{cases}$$

To show that the function is indeed continuous, note that plugging in $r = \tilde{r}$ and $\alpha^* = \int_0^\omega z dF_{(2)}(z)$ implies $\tilde{v} = 0$ and these values solve the implicit equation defining the reference point.

Consider now a first price auction.

Before analysing the case with reserve prices, it is noted that $\alpha_1 = \alpha_2$ for the case without a reserve price. Then for first price auctions we have that the same $\tilde{r}$ such that for all $r \leq \tilde{r}$, $\alpha_1(r) = \alpha_1$.

For $r > \tilde{r}$ the consistent reference point $\alpha^1(r)$ is implicitly defined by,

$$\alpha^1(r) = F_{(1)}(\tilde{v}(r, \alpha^1(r)))r + \int_{\tilde{v}(r,\alpha^1(r))}^{\omega} \frac{1}{1+\lambda}\Big[z + \lambda\alpha^1(r) - \int_{\tilde{v}(r,\alpha^1(r))}^{z} \frac{G_{(1)}(w)}{G_{(1)}(z)} dw\Big] dF_{(1)}(z),$$

simplifying,

$$\alpha^1(r) = F_{(1)}(\tilde{v}(r, \alpha^1(r)))\tilde{v}(r, \alpha^1(r)) + \int_{\tilde{v}(r,\alpha^1(r))}^{\omega} \Big[z - \int_{\tilde{v}(r,\alpha^1(r))}^{z} \frac{G_{(1)}(w)}{G_{(1)}(z)} dw\Big] dF_{(1)}(z).$$

Rearranging and dropping the functional arguments for simplicity,

$$\alpha^1 = F^n(\tilde{v})\tilde{v} + \int_{\tilde{v}}^{\omega} z\, dF^n(z) - n\int_{\tilde{v}}^{\omega}\int_{\tilde{v}}^{z} G_{(1)}(w)dwf(z)dz.$$

Changing the order of integration yields,

$$\alpha^1 = F^n(\tilde{v})\tilde{v} + \int_{\tilde{v}}^{\omega} z\, dF^n(z) - n \int_{\tilde{v}}^{\omega} \int_{w}^{\omega} f(z)dzG_{(1)}(w)dw,$$
$$= F^n(\tilde{v})\tilde{v} + \int_{\tilde{v}}^{\omega} z\, dF^n(z) - n \int_{\tilde{v}}^{\omega} [1 - F(w)]F^{n-1}(w)dw,$$

Integrating by parts and simplifying,

$$\alpha^1 = F^n(\tilde{v})\tilde{v} + [zF^n(z)]_{\tilde{v}}^{\omega} - \int_{\tilde{v}}^{\omega} F^n(z)dz - n \int_{\tilde{v}}^{\omega} F^{n-1}(z)dz + n \int_{\tilde{v}}^{\omega} F^n(z)dz,$$
$$= \omega + (n-1) \int_{\tilde{v}}^{\omega} F^n(z)dz - n \int_{\tilde{v}}^{\omega} F^{n-1}(z)dz. \qquad \text{(A.8.2)}$$

Revenue equivalence.

Note that the screening value in the two auction is the same if the reserve price and reference point are the same. So the auction are revenue equivalent for all $r \leq \hat{r}$.

For $r > \hat{r}$ consider the reference point in a second price auction. Compress the notation by dropping the functional arguments for simplicity.

$$\alpha^2 = F_{(2)}(\tilde{v})\tilde{v} + \int_{\tilde{v}}^{\omega} zdF_{(2)}(z),$$
$$= (nF^{n-1}(\tilde{v}) - (n-1)F^n(\tilde{v}))\tilde{v} + \int_{\tilde{v}}^{\omega} zd(nF^{n-1}(z) - (n-1)F^n(z)),$$
$$= nF^{n-1}(\tilde{v})\tilde{v} - (n-1)F^n(\tilde{v})\tilde{v} + n \int_{\tilde{v}}^{\omega} zdF^{n-1}(z) - (n-1) \int_{\tilde{v}}^{\omega} zdF^n(z).$$

Integrating by parts and simplifying yields,

$$\alpha^2 = \omega + (n-1) \int_{\tilde{v}}^{\omega} F^n(z)dz - n \int_{\tilde{v}}^{\omega} F^{n-1}(z)dz. \qquad \text{(A.8.3)}$$

Comparing equations (A.8.2) and (A.8.3) implies that $\alpha_1(r) = \alpha_2(r)$.

APPENDIX B
PROOFS: CHAPTER 2

B.1 Proposition 2.2.1

We assume that both the buyer and seller use linear strategies in equilibrium. We express these candidate strategies as

$$b(v, \alpha) = dv + e + f\alpha,$$

$$s(c, \alpha) = gc + h - l\alpha.$$

Then consider the buyer's problem

$$\max_{b} \int_{o}^{\frac{b-h-l\alpha}{g}} \Big(v + \lambda\alpha - (1+\lambda)\big[kb + (1-k)(gx + h + l\alpha)\big]\Big) dx.$$

The first order condition yields

$$b(v, \alpha) = \frac{v + \alpha(\lambda + (1+\lambda)kl) + (1+\lambda)kh}{(1+\lambda)(1+k)}.$$

Similarly, the sellers maximisation problem can be expressed as

$$\max_{s} \int_{\frac{a-e-f\alpha}{d}}^{1} \Big((1+\lambda)\big[k(dx + e + f\alpha) + (1-k)s\big] - \lambda\alpha - c\Big) dx,$$

and its corresponding first order conditions yields

$$s(c, \alpha) = \frac{c + \alpha(\lambda + (1+\lambda)(1-k)f) + (1+\lambda)(1-k)(d+e)}{(1+\lambda)(2-k)}.$$

Comparing coefficients yields the desired result.

B.2 Corollary 2.2.2

For ex-post efficiency, trade should take place whenever $v \geq c$. Consider now the case when trade takes place in the current model.

$$b(v,\alpha) \geq s(c,\alpha),$$

$$\frac{1}{1+\lambda}\Big[\frac{v}{1+k} + \frac{(1-k)k}{2(1+k)} + \lambda\alpha\Big] \geq \frac{1}{1+\lambda}\Big[\frac{c}{2-k} + \frac{1-k}{2} + \lambda\alpha\Big],$$

$$\frac{v}{1+k} \geq \frac{c}{2-k} + \frac{1-k}{2} - \frac{(1-k)k}{2(1+k)},$$

$$v \geq \frac{1+k}{2-k}c + \frac{(1-k)}{2}.$$

The condition is invariant to λ, i.e the measure of reference dependence.

B.3 Proposition 2.3.1

To simplify the analysis, define $\hat{v} = v + (1-\beta)\lambda\alpha$ and $\hat{c} = c + (1+\beta)\lambda\alpha$. Note that the $\hat{v}$ and $\hat{c}$ are also random variables distributed uniformly over $[\underline{v} = (1-\beta)\lambda\alpha, \bar{v} = 1 + (1-\beta)\lambda\alpha]$ and $[\underline{c} = (1+\beta)\lambda\alpha, \bar{c} = 1 + (1+\beta)\lambda\alpha]$, respectively.

Assume now for this transformed game that the equilibrium strategies are linear and take the form

$$\hat{B}(x) = hx + i,$$

$$\hat{S}(y) = jy + l.$$

Then the buyer's problem when her value is x is given by

$$max_B \int_{\underline{c}}^{\frac{B-l}{j}} \Big(x - (1+\lambda)\big[kB + (1-k)(jz+l)\big]\Big)\, dz.$$

The first order condition yields

$$\hat{B}(x) = \frac{x}{(1+\lambda)(1+k)} + \frac{k}{1+k}(l + \underline{c}j).$$

Similarly, the sellers whose value is y maximisation problem can be expressed as

$$\max_{S} \int_{\frac{S-i}{h}}^{\bar{v}} \Big((1+\lambda)\big[k(hz+i) + (1-k)S\big] - y \Big)\, dz,$$

and its corresponding first order conditions yields

$$\hat{S}(y) = \frac{y}{(1+\lambda)(1+k)} + \frac{(1-k)}{(2-k)}(\bar{v}h + i).$$

Comparing coefficients and simplifying yields the desired result.

B.4 Corollary 2.3.2

Consider once again the cases when trade occurs.

$$\begin{aligned}
b(v,\alpha) \geq & s(c,\alpha), \\
\frac{v}{1+k} + \frac{(1-k)k}{2(1+k)} + \frac{(2-k)\underline{v} + k\underline{c}}{2} \geq & \frac{c}{2-k} + \frac{1-k}{2} + \frac{(1-k)\underline{v} + (1+k)\underline{c}}{2}, \\
\frac{v}{1+k} \geq & \frac{c}{2-k} + \frac{1-k}{2} - \frac{(1-k)k}{2(1+k)} \\
& + \frac{(1-k-2+k)\underline{v} + (1+k-k)\underline{c}}{2}, \\
v \geq & \frac{1+k}{2-k}c + \frac{1-k}{2} + \frac{(1+k)(\underline{c} - \underline{v})}{2}, \\
v \geq & \frac{1+k}{2-k}c + \frac{1-k}{2} + (1+k)\beta\lambda\alpha.
\end{aligned}$$

The results of the corollary are obvious from the above equation.

B.5 Proposition 2.3.4

Consider

$$\begin{aligned} I &= \lambda\beta\alpha_b^*, \\ &= \frac{3(2-k)\lambda\beta}{4\lambda\beta - 6k\lambda\beta + 8}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial I}{\partial \beta} &= \frac{6\lambda(2-k)(1+k)}{(4\lambda\beta - 6k\lambda\beta + 8)^2} \geq 0, \\ \frac{\partial I}{\partial \lambda} &= \frac{6\beta(2-k)(1+k)}{(4\lambda\beta - 6k\lambda\beta + 8)^2} \geq 0. \end{aligned}$$

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