

Algebra at the start of Upper Secondary School

A case study of a Norwegian mathematics classroom
with emphasis on the relationship between the mathematics
offered and students' responses

Hildegunn Espeland

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Preface

The work with this thesis has been a long journey, with many stops along the way. I was employed as a doctoral student in the research project Teaching Better Mathematics at the University of Agder from autumn 2006 until 2010. This project was funded by The Research Council of Norway and the Competence Development Fund of Southern Norway.

While working on this project, I had the pleasure of associating with teachers and researchers; all of whom were enthusiastic and very interested in developing mathematics teaching in order for students to improve their learning of the subject, which is ultimately also my goal.

The summer schools and seminars organised by the Nordic Graduate School of Mathematics Education (NoGSME) provided opportunities to meet researchers and doctoral students from other countries, and to discuss professional issues. The work within the project and the doctoral studies have been both interesting and rewarding.

This study was carried out in an upper secondary classroom in collaboration with one teacher. I want to thank her, in particular, for her generosity in opening up her classroom and allowing me to observe and participate in her classroom practice. I am also thankful to the participating students. During the years working with the data, I feel that I have got to know them.

I am very grateful to my two supervisors Professor Barbro Grevholm and Professor Simon Goodchild. They have both been supportive, and have patiently guided me on this long journey. In the final phase Barbro has encouraged me and given generously of her time, to comment and to give input on my study.

Associative Professor Morten Blomhøj was my critical friend on the 90 % seminar, and I am thankful for important advice regarding my work. I am also thankful to Mary Billington, for giving me feedback through the years, for Ingvald Erfjord who helped me with the template, and Antoinette Bergsland for valuable proof-reading and comments on the work.

I have a large family that I love, and I am grateful to them, and to my husband for their patience.

Hildegunn Espeland
Kristiansand, Norway
November, 2016

To Jenny

To my grandchildren:

Hege, Andreas, Sunniva, Oscar, Hanna, Marcus, Ella and Elias.

Abstract

Norwegian students at the end of lower secondary school are according to international comparative studies weak in algebra. This thesis reports a study of a mathematics classroom at the start of upper secondary school, when basic algebra, number, and number operations are recapitulated.

One aim has been to study what kind of mathematics is emphasised. The analysis is done on the basis of the textbook, the tasks, and the presentation in the classroom. Another aim has been to inquire into what experiences students have of the mathematics offered in the classroom. Experiences are not directly accessible, but the analyses are based on students' oral and written responses in the classroom and in interviews, and on all their written task solutions. A third aim appeared during the work with the data, and was to explore how the setting of the mathematics classroom influences students' experiences.

The study, combines the social and the individual perspective, and is based on a conceptual framework with concepts from different theories and studies. Important concepts reflecting the individual perspective are, 'concept image', 'proceptual thinking', 'structure sense', and 'mathematical proficiency'. The social perspective is reflected by some concepts from social interactionism, the 'didactical contract' from the theory of didactical situation, and 'task discourse'.

The data were of different kinds and from different sources: classroom observations, interviews with students and the teacher, written material from students' task solutions; both tests and assigned tasks, teacher's explanation, and the textbook with its resources.

The analyses revealed that the mathematics offered was focused on rules and algorithms, with little or no connection to the underlying concepts. The textbook tasks were of a low level of cognitive demand and did not challenge students to develop their concept images. Important concepts and principles were not explicitly addressed; a request for developing conceptual understanding. The teacher addressed some issues.

Students' responses revealed that problems reported in former studies still exist. The problems are well-known, but the textbook seems not to have been based on research within the area of mathematics education, and thus gave no adequate attention to help those students who struggled to overcome these problems.

The didactical contract, the task discourse, the pressure on curriculum time, and the inadequate teaching material seemed to constrain students in developing conceptual understanding and thus procedural fluency.

Sammendrag

Internasjonale komparative studier viser at norske elever i slutten av ungdomsskolen er svake i algebra. Avhandlingen er en studie av en matematikkklasse ved starten av videregående skole mens elementær algebra, tall og regneoperasjoner repeteres.

Et mål har vært å undersøke hva slags matematikk som vektlegges. Læreboka, oppgavene og presentasjonene i klasserommet danner grunnlag for analysen. Et annet mål har vært å finne ut hva slags erfaringer elevene har av matematikken som blir presentert. Erfaringer er ikke direkte tilgjengelige, men analysen her er basert på elevenes skriftlige og muntlige utsagn i klasserommet og i intervjuer, og på skriftlige løsninger av lærebokoppgaver og testoppgaver. Gjennom arbeidet med dataene dukket et tredje spørsmål opp: Hvordan influerer klasseromsetningen på elevenes erfaringer av matematikken som presenteres?

Det individuelle perspektivet er kombinert med et sosialt perspektiv, og teoretiske begreper benyttet i analysen er hentet fra ulike teorier og studier. Det individuelle perspektivet er reflektert gjennom begrepene 'concept image', 'proceptual thinking', 'structure sence', og 'mathematical proficiency'. Det sosiale perspektivet gjenspeiles i begreper fra sosial interaksjonisme, 'den didaktiske kontrakten' fra 'Theory of didactical situations' (TDS) og 'oppgavediskursen'.

Datamaterialet er data av ulikt slag og fra ulike kilder: klasseroms observasjoner, intervjuer med elever og lærer og skriftlige løsninger både på testoppgaver og lærebokoppgaver. Lærerens forklaringer og læreboka med ressurser er også del av datamaterialet.

Analysene viste at matematikken som ble presentert, var fokusert på regler og algoritmer med få eller ingen koblinger til begrepene de bygger på. Lærebokoppgavene var kognitivt lite krevende og var ikke egnet til å utfordre elevene til å utvikle sine 'concept images', og viktige begreper og prinsipper var ikke eksplisitt nevnt i læreboka, noe som er viktig for å utvikle begrepsforståelse. Læreren brakte inn noen utelatte begreper.

Studien viste at problemer rapportert i tidligere studier fremdeles eksisterer. Problemene burde være velkjente, men det ser ikke ut til at læreboka var bygd på forskning i feltet, og det var derfor ingen fokus på å avdekke problemer og hjelpe de elevene som strevde.

Den didaktiske kontrakten, oppgavediskursen, tidspresset for å dekke læreplanmålene og utilstrekkelig undervisningsmateriell så ut til begrense elevene i å utvikle god begrepsforståelse og i å bruke prosedyrer effektivt og fleksibelt.

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1 Introduction

My quest is to inquire into the relationship between the mathematics offered in a mathematics classroom to students entering upper secondary school, and what these students experience of that mathematics. Observations made in more than 20 years of teaching mathematics, both at lower and upper secondary level, induced the interest in this quest.

As a teacher I experienced students making errors, but to find the basis for the problems students revealed was difficult under the pressure of time and the many demands in the daily work.

In my master study done within the frame of a research project at University of Agder, the KUL-LCM project, test tasks from 230 students in upper secondary school were analysed. Many of the tasks in the test were used in a former large study in Norway – the KIM-study (Quality in the teaching of mathematics) (Brekke, 1995a, 1995b, 2005; Brekke, Rosén, & Grønmo, 2000). Related literature and analyses of these tasks, gave me new insight into students' learning difficulties also experienced in my classes.

In comparing the results from the two studies (KUL-LCM and the KIM study), it appeared that the students in the KIM study performed better 10 years earlier than the students in my study, although the latter students had one more year of schooling. In addition, the students in the KIM study were from the whole age population, while the students in my study were students preparing for university studies. The mathematical problems I found, seemed to be the same as those identified in the KIM-study; even though a lot of material about the findings in this former study had been available for teachers for more than a decade. For me, as a teacher in school, the material had not been made known.

In Norway there is a break between lower and upper secondary school. From grade one to grade ten, the pupils have followed a common curriculum and remained part of the same class. Preschools, primary schools, and lower secondary schools are under the same district administration although each school has its own administration at school level. Upper secondary school has another administration which covers a wider geographical area, and students can choose among different schools within this area.

This means that traditionally there has been little or no contact between upper secondary schools and the lower grades. Courses for in-service teachers are held specifically for teachers from upper secondary schools. It is therefore unusual for teachers from the different levels to have met to discuss matters related to subjects and teaching.

When the opportunity arose to work in a project where teachers from kindergarten or preschool up to upper secondary school were included and would work together, I was eager to participate.¹ It stimulated my interest to find out more about students' understanding of basic mathematical concepts, especially in the area of algebra in the transition from lower to upper secondary school.

There is a huge amount of research carried out on students' conceptions of algebra, and the transition from arithmetic to algebra. Most of this was conducted in the 1980s and 1990s, and often the research design did not include observation of normal classes. It is a moot point whether more research is needed in the area, however, the results in my master study and the findings from PISA and TIMSS show that problems still exist.

Norway has a problem in that many students drop out of upper secondary school, and there is an ongoing discussion about what to do to keep them in school and help them finish their schooling. Studies have shown that low achievement in mathematics from lower secondary school is one important factor (Falch, Nyhus, & Strøm, 2014) causing this drop-out. Algebra is recognised as the foundation for mathematical analysis in many studies both in economics and in science, and TIMSS advanced study 2008 showed that a lack of competence in algebra led to low scores in physics (Lie, Angell, & Rohatgi, 2010).

When students come to upper secondary school they are expected according to the syllabus to be fluent in transforming algebraic expressions. Most upper secondary teachers are fully aware that not all students master these transformations. The issue is then how to deal with problems many students have, and at the same time challenge the students who perform well, when the topics have been covered earlier.

In mathematics textbooks at this level the first part is normally devoted to a recapitulation of basic algebra and the arithmetical operations. The recapitulation of the topic is often treated as self-regulated work with textbook tasks, similar to those the students have met before, while both the teacher and the students are looking forward to starting on something 'new'.

The students in this study are in this transition from lower secondary to upper secondary school. They have left their former schools, and have just met a new school system, which may be challenging for them. According to a study from England about the transition from compulsory

¹ The name of the project was the TBM- project (Teaching Better Mathematics). Information about this project is to be found in Appendix 2.

school to college² (Hernandez-Martinez et al., 2011) it was found that most students in the study were worried about the social dimension of the transition, but for most of them it was later regarded as having given them a feeling of growing up. Another issue for them was that they expected mathematics to be different from what they had experienced earlier; to be more difficult. However, when looking back, they mostly regarded it positively, as challenges to overcome. For weaker students everything seemed to be new in mathematics, and thus it was harder for them. The authors opinion was that the colleges had information about the students entering the college, and thus it is a “curriculum pathways problem, and concomitantly a funding problem” (ibid) to help the students. Also in Norway the upper secondary schools have information about students coming to their schools.

In this study one class is followed during these first weeks in upper secondary school, when algebra is recapitulated. There is generally little classroom research carried out in upper secondary school, and especially during the first weeks in the new school, when basic algebra is recapitulated. The aim in this study is to learn about the relationship between the mathematics offered, and students’ responses. By using the word relationship, I do not intend to find a cause and effect relationship, but to examine how the two phenomena are related. In this way I want to add to the knowledge in the research community about students’ conceptions of topics within basic algebra, and about the teaching/learning going on in the classroom, in the transition from one school system to another.

In order to make the study as transparent as possible I have made an extensive presentation of the data and the results. The organisation of the thesis makes it possible to read it topic by topic. This thesis is aiming at giving the reader a holistic glimpse of an ordinary mathematics classroom in a limited period of time.

1.1 The structure of the thesis

In chapter 2, I have outlined the Norwegian situation with the aim to base the study and the mathematics classroom in the Norwegian context.

In chapter 3, it is discussed what algebra is, and what it means as a subject in school. In addition, the Chapter 4 is a review of former studies focusing on students’ learning and work with number and algebra. That chapter is important as a basis for the analysis of students’ responses.

Chapter 5 has the title: Interpretative framework. Both the social and the individual perspectives are taken into consideration. Important con-

² The students in the study going from compulsory school to college are at about the same age as the students in this study.

cepts within the social perspective are: ‘patterns of interaction’, ‘task discourse’ and ‘didactical contract’. Within the individual perspective ‘concept image’, ‘proceptual thinking’ and ‘structure sense’ are important concepts. These concepts are presented before different kinds of mathematics are discussed. At the end of the chapter, the strands of ‘mathematical proficiency’ are presented.

The heading in chapter 6 is: Opportunities to learn. This chapter includes a review of studies done on what promotes different kinds of learning, and studies on mathematics textbooks and tasks.

The data collection and methods are described in chapter 7, with reflections on the methodology.

In the chapters 8 to 12, students’ observed responses, both written and oral, are categorised and analysed on the basis of former research on algebra and number (chapter 4), and discussed on the basis of the interpretative framework in chapter 5.

In chapter 13, the focus is on the learning situation described through the patterns of interactions going on in the classroom.

Chapter 14 includes short summaries of the students’ work with mathematics presented thoroughly in the chapters 8 to 12. The rest of this chapter is devoted to the research questions and a discussion of the findings.

The last chapter, chapter 15, includes the conclusion and implications for teaching and for further research.

2 The Norwegian context

In the foregoing section, I have given a rationale for my personal interest in this study. In this chapter I will outline the Norwegian context.

The mathematics and the students' mathematical activity is at the core of my investigation of the classroom activity. However, students and their activity do not exist in isolation. Internal and external factors influence both what students do, and how they perform in mathematics. In this chapter I will examine factors which official documents and research reveal as influencing Norwegian mathematics classrooms.

In the first section, reference will be made to Norwegian students' performances in the international comparative studies. The reason for this is that the results from these studies have made a strong impact on policy makers in Norway (Baird et al., 2011), and they form a part of the rationale for this study.

Then the two mathematics syllabi in the L97 and in the K06 curricula are presented. The reason for this is that the students in this study have followed both; the L97 the first 10 years, and during the time of observation they met K06.

The following sections are devoted to factors influencing the daily work in the classroom: Digital tools, the teachers, work plans, prior knowledge and mathematics textbooks.

The last section is a summary of how I see the Norwegian context and a presentation of the research questions.

2.1 Students' mathematics – PISA and TIMSS

The Trends in International Mathematics and Science Study, the TIMSS study, was developed by the International Association for the Evaluation of Educational Achievement (IEA) in order to allow participating nations to compare students' educational achievement across borders. The first TIMSS study was carried out in Norway in 1995, and every 4 years thereafter. Students are assessed in mathematics and science based on test items. Students, teachers and school leaders are in addition asked to answer questionnaires. The study intends to measure students' achievements related to mathematics syllabi. The tasks are thus chosen to correspond to what is supposed to be learned in school.

The Programme for International Student Assessment (PISA) is an internationally standardised assessment jointly developed by the OECD member countries through the OECD's Directorate for Education. The aim is to assess to what extent students near the end of compulsory education have acquired the knowledge and skills that are essential for full par-

participation in society. As in TIMSS, students are expected to solve test items and to answer questionnaires. The first study was carried out in 2000, and every third year after that.

The subjects to be focused on are mathematics, science and reading. The questionnaires have a wider focus than students' performance on test items. The focus is on students' motivation to learn, their beliefs about themselves and the learning situation. Their learning strategies are also examined in addition to differences in performance between groups of students, e.g. the difference between genders and between socioeconomic groups.

In 1995, the TIMSS results in grade 8 showed that Norwegian students' achievements were approximately at the international average with algebra as the weakest topic (Lie, Brekke, & Kjærnsli, 1997). In 2003 Norwegian students performed less well (Grønmo, 2004). It was claimed that Norwegian students in 2003 were estimated to be one year behind the students in 1995. Also this time the weakest topics were algebra and number. The researchers assert there is a correlation between a deliberate low priority of these topics in the syllabus from 1997, and students' achievement.

The PISA study from the same year, 2003, testing students in grade 10 confirmed the results from the TIMSS study (Kjærnsli, Lie, Olsen, Roe, & Turmo, 2004). One of the problematic issues reported was that students lack skills and elementary knowledge in the topic of number. The PISA study also compared the results from the study in 2003 with the results from 2000, and reported a decrease in performance, although not significant.

In 2006 there was still a decrease in the results in mathematics for students in grade 10, and now the decrease was statistically significant (Kjærnsli, 2007). The TIMSS study 2007 (Onstad & Grønmo, 2009) reveals some change. The results were slightly better than the results from 2003, however, students' performance was still far below the results from 1995. The researchers observed that students in grade 8 were allowed to use calculators in 2007, which was a change from 2003, however it was argued that this was taken into account when comparing the two results.

TIMSS advanced, a study of students' performance in mathematics at the end of upper secondary school, was carried out in Norway in 1998 and in 2008. Only students taking the advanced courses in mathematics and physics participated. This study was carried out for the first time in 1995, however, Norway did not participate that year. Instead the same test instruments were used in a study in Norway in 1998, and the results in 2008 were compared to the data from the 1995 study. The results from 1998 showed that Norwegian students at the end of upper secondary school performed above the international mean.

The trend in TIMSS advanced for mathematics (Grønmo, Onstad, & Pedersen, 2010) is in accordance with the trend in the lower secondary schools. Students were tested in the areas of algebra, calculus and geometry. When compared task for task, overall there was a decrease in the proportion of students with correct solutions. When the poor results from 2003 in lower grades were publicly discussed, it was often said that they were nothing to worry about, since by the end of upper secondary Norwegian students perform well above the international mean, referring to the result in 1998. The results, however, from TIMSS advanced 2008, contradict this assertion.

In summary, longitudinal studies such as TIMSS and PISA indicate a setback in mathematical performance among students in both lower and upper secondary school, and especially in the topics of number and algebra. It has been questioned if the setback in physics, which was also documented in TIMSS advanced 2008, partly might be due to the weak performance in manipulating simple algebraic expressions (Angell, et al., 2010). Also the TIMSS study from 2011 (Grønmo et al., 2012) indicates that Norway still lies significantly below the international average with algebra as the weakest topic.

The Norwegian results from the comparative studies are generally considered unsatisfactory and have stimulated much debate in Norway.

2.2 Policy and directives – curricula

The studies in the section above and especially the results from 2003 resulted in bold headlines in the media and discussion both at school level, system level, and in the political arena. Actions were undertaken to deal with the problems. One consequence was a new curriculum introduced in 2006.

The students in my study had 10 years of school within the framework of the L97-curriculum.³ When they entered grade 11 or the upper secondary school, they were introduced to the new curriculum K06⁴ – “The Knowledge Promotion”. Because of this, I will present a short outline of the two curricula and the mathematics syllabi within them. In Norway there is a long tradition of a centralised curriculum. New reforms have come and gone, often felt by the teachers and school leaders as coming too frequently (Kleve, 2007).

³ L97 was the curriculum for the 10-year compulsory school in Norway from 1997. In Norwegian: Læreplanverket for den 10-årige grunnskolen (L97).

⁴ K06 is the curriculum for grade 1 until the end of upper secondary school. This means 13 years of schooling. It was introduced in 2006. In Norwegian: Kunnskapsløftet (K06), official English translation: The Knowledge Promotion.

2.2.1 The curriculum L97

L97 was developed over a relatively short period and came into effect from the school year 1997-1998. One point in the curriculum was that students should be educated to become independent participants in society, and that they should be encouraged to take responsibility for their own learning. It was a detailed goal-oriented curriculum, and teachers and parents could consult the syllabi for each subject and check what had to be learned or taught.

It was built on a constructivist view of learning. In the mathematics syllabus it is asserted:

Learners construct their own mathematical concepts. In that connection it is important to emphasise discussion and reflection. The starting point should be a meaningful situation, and tasks and problems should be realistic in order to motivate pupils. At times pupils may work with incomplete concepts; they make occasional mistakes and show misunderstandings. In a confident and constructive atmosphere such matters are grounds for further learning and deeper insight (Hagness & Veiteberg, 1999, p. 167).

The emphasis is placed on the student as an active learner. Also activities in the classroom such as problem solving and investigation should be promoted to stimulate students' learning. "As they experiment, experience, wonder and reflect, the subject will help to develop the pupils' curiosity and urge to explore. It is important for pupils to experience the learning of mathematics as a process" (ibid p. 165).

The syllabus included specific guidelines for each mathematical domain in each school year. The main areas in the syllabus are outlined in the table below:

Table 2-1: Mathematics syllabus L97 – main areas (ibid p. 168)

Main stages	Main areas				
Lower secondary stage	Mathematics in everyday life	Number and algebra	Geometry	Handling data	Intermediate graphs and function
Intermediate Stage	Mathematics in everyday life	Number	Geometry	Handling data	
Primary Stage	Mathematics in everyday life	Number	Space and shape		

One of the main areas 'Mathematics in everyday life', might appear to be a distinct topic in its own right, however, it was supposed "to be understood more as an attempt to emphasise this aspect in school mathematics,

rather than adding yet another topic to the mathematics syllabus” (Mosvold, 2005, p. 88).

Algebra as a domain comes into the syllabus in grade 8, but is mentioned also in the general part of the curriculum. There, it is said that the work with patterns and regularities in the elementary school should serve as a basis for the further work with the concept of variable. Also it is emphasised that algebra has to be presented in meaningful contexts, and that algebra should be a tool in problem solving, and a language to make thinking and reasoning easier.

In grade 8 there is only one paragraph about algebra: “pupils should have the opportunity to work to develop their understanding of the use of letters and brackets in simple mathematical expressions and formulae, also on the basis of quantities from other subjects or from everyday life” (Hagness & Veiteberg, 1999, p. 179). In grade 9, it is expanded to include work with, and interpretation of formulas, equations, and inequalities with one unknown. The aim is for students to be able to find methods for solving simple equations and inequalities. They should also experience how letters can be used to formulate and prove general arithmetic rules.

In grade 10 students are supposed to work further with formulas, linear equations, and inequalities. It is expressed that they shall describe, judge, and solve problems also through formulas and equations. As for grade 9, they should have further experiences with generalisations in order to make simple proofs. Simple algebraic expressions including fractions with one term in the denominator are to be handled. How to solve systems of linear equations is also expected to be learned.

In mathematics at all levels students were encouraged to produce a “student book”. This was not mentioned in the syllabus but came as an appendix to the information sent out before the exam in 1999 (Eksamenssekretariatet, 1999). This book should be produced by students themselves. There was neither any restriction on what was written therein, nor on the number of books. Normally students wrote rules, formulas and examples. From 1999 this book(s) was allowed in exams with no restriction. Therefore, the use was normally allowed in regular tests. This was the case for the students in my study during their years in lower secondary school, and they often referred to this book(s) during the observations.

What is outlined above is the intended curriculum. A large research program financed by the Norwegian Research Council (Norges Forskningsraad, NFR) included 26 different projects where the assignment was to evaluate the reform from 1997. The aim was to map out differences and development from the former curriculum, to point to weak and strong effects from both what was planned, and what happened unintentionally.

The results should serve as a basis for further plans and actions (Haug, 2004). It was reported (*ibid*) that teachers in the lower secondary school mostly continued their practice from before the reform. Many expressed negative attitudes because the detailed and numerous goals in the subjects gave little space for local initiative, and also many teachers felt there was too much to cover. Teachers thus had to make a choice between making haste and only addressing the surface in the different domains in the subjects, or to choose to leave out some topics and not present all the goals for the students. Some teachers reported that emphasis on students' activity and the intention in the curriculum that students should be promoted to be responsible for their own learning, led in some cases to a teacher's role as a facilitator, and that students were left alone with their work. The same was found in the TIMSS studies. Some even let the students take the role of leaders in the classroom (Klette, 2003).

L97 emphasises what is stated by law, that the school has room for all children whatever background they have. They are divided into groups only on the basis of their age, not on any other criteria. The curriculum states that teachers must have an eye for each individual learner and:

The mode of teaching must not only be adapted to subject and content, but also to age and maturity, the individual learner and the mixed abilities of the entire class (Hagness & Veiteberg, 1999, p. 35).

In order to meet the demand of serving each individual student and his or her needs, most teachers have made work plans, in which tasks for school and home work are listed (Klette, 2007). These plans may be for all the subjects in a period of time, or for one specific subject. In those plans, teachers often make differences between students or groups of students, in order to try to let students at different levels in the class meet tasks appropriate for them (see also section 2.6).

Alseth, Breiteig, and Brekke (2003) conducted an investigation of the implementation of L97 in mathematics classrooms. They tested students in grade 4 and grade 7 and compared the results with results from the Kassel-Exeter study in Norway (Hinna, 1996). Alseth et al. (2003) pointed to the fact that conceptual knowledge and comprehension are to be emphasised in all subjects according to L97, including mathematics. In the mathematics syllabus there is also a warning against learning rules and formulas by rote without understanding. The framework used in their (*ibid*) comparison study, was the theory about conceptual and procedural learning (Hiebert & Lefevre, 1986). Procedural knowledge was defined to be knowledge about how to perform operations in accordance to a step by step algorithm, and conceptual knowledge to be knowledge about the concepts of number and arithmetic operations. The tasks they used were divided into two groups on the basis of these definitions. The conclusion

was that students in grade seven generally performed less successfully than the students tested before the implementation of the L97 reform. Especially it seemed that students' performance on tasks demanding procedural knowledge had decreased, and explicitly they said that the computational skills were better before the reform in 1997. They also did not find evidence that students' conceptual knowledge had improved. On the tests the students had not been allowed to use their "student books" nor their calculators, and the authors therefore asked the question if students relied too much on these tools and thus did not perform so well. The same conclusion was made on the basis of the results from the Norwegian TIMSS study and PISA study in 2003 (Grønmo, 2004; Kjærnsli, et al., 2004).

Klette and her colleagues (Klette, 2003) chose 30 classrooms with the aim of describing Norwegian classrooms after the reforms were introduced during the 1990s. The classes were in elementary and lower secondary school. Each of these classes were followed for one week.

They found that the interaction between teacher and students in all classes, showed signs of acceptance, and the teachers showed that they wanted to tolerate, negotiate, and respect the individual student and groups of students. The interaction seemed to be led by a wish that all students should be seen and be taken care of as far as the teaching situation made it possible. A side effect of this was that the individual student was given space to act, which in some cases disturbed the learning situation. Another possible side effect was that it seemed as if the teachers did not criticise students' work despite the fact that it was poor. Instead they tried to be encouraging and positive, making no demands.

In common for all classes, the teacher was the leader giving turns to speak. On average the teacher talked 60 % of the time during classroom conversations. Mostly the teacher let as many as possible of the students have a word, and the result was that the structure in the conversations according to the authors, was "flat" (ibid). The classroom conversation went on thematically coherent, but there was hardly ever observed utterances indicating a need to expand on something, although some students clearly did not understand.

In all classrooms through the grades and subjects the most dominant activity was individual work. This is also documented in the questionnaires from both teachers and students in the TIMSS study from 2007⁵ (Grønmo & Onstad, 2009). The TIMSS study also documented that according to students, the most frequent work in class was to solve tasks in

⁵ In 2007, the students in the TIMSS study had 7 years with L97 curriculum and almost one year with K06.

the textbook after being introduced to examples similar to the tasks in the textbook. This was also the case internationally.

Also other studies concerning teachers' implementation of L97 indicated that the intentions in the curriculum were not easily carried out in classrooms. It seems as if the traditional teaching practice with the teacher presenting theory and students working individually on tasks was preserved (Alseth, et al., 2003; Kleve, 2007; Mosvold, 2005).

Mosvold (2005) asserts that even in textbooks the curriculum intentions are not easily communicated. He investigated how the intentions in the L97-curriculum to relate mathematics to real life situations, were implemented in textbooks and in classrooms. He found that since textbooks are one of the most important sources for teachers in their practice, the relationship between teaching practice, textbooks and intentions in the curriculum is of great importance especially in curricular reforms.

In this section I have outlined the intended L97-curriculum, as a basis for setting the scene for the students in the study who had been following this curriculum for ten years. Research on the implementation of the L97 curriculum in Norwegian schools, however, indicates that the actual teaching practice changed very little from before the introduction of the L97. For students, the introduction of work plans may have resulted in more isolated learners within the mathematical classroom. Although the emphasis in the mathematics syllabus is on conceptual understanding, students' autonomy, and teachers' connecting mathematics to students' everyday life, the implementation of the ideas takes time and also the impact on students' learning seems not to have occurred as intended.

2.2.2 The curriculum K06

In 2006 a new reform was introduced at all levels in Norwegian schools. This curriculum replaced L97, a curriculum for the first 10 grades, and the Reform 94⁶, revised in 2000, for the upper secondary school. The new reform and curriculum is called "The Knowledge Promotion" (K06). It encompasses the 10-year compulsory school, and upper secondary education and training as a whole. The goal of this curriculum is to help all students to develop fundamental skills for active participation in the so called 'knowledge society' (Ministry of Education and Research, 2006).

Within this new curriculum, K06, the mathematics syllabus contains specific competence aims to be reached at grade 2, grade 4, grade 7, grade 10, and for each year for the different courses in the upper secondary school. The curriculum for upper secondary school is called: 'the Pro-

⁶ Reform 94 was the curriculum for upper secondary education. It was introduced in 1994 and then revised in 2000.

gramme for Specialization in General Studies'. Within this curriculum there are syllabi for two mathematics courses in Vg1 (the first year in upper secondary school), and three courses for Vg2 (the second year in upper secondary school). All students have to follow mathematics courses the two first years in this programme.

For the school in this study, the options are the T course in the first year (T for theoretical). The T programme is more theoretically oriented and serves as a basis for those who need to study mathematics in their further education. After the first year there was no external exam.⁷ External exams were held after the second year and also after the third year for those who continued studying mathematics all three years in upper secondary school.

For the students in this⁸ study the options for the second year is either to follow the R programme which is seen as a more advanced mathematics course; providing a basis for further studies in mathematics or science, or they can follow the S-programme, which is more practically oriented preparing students for further studies in economics and social studies. The figure below illustrates the options offered by the school in this study.

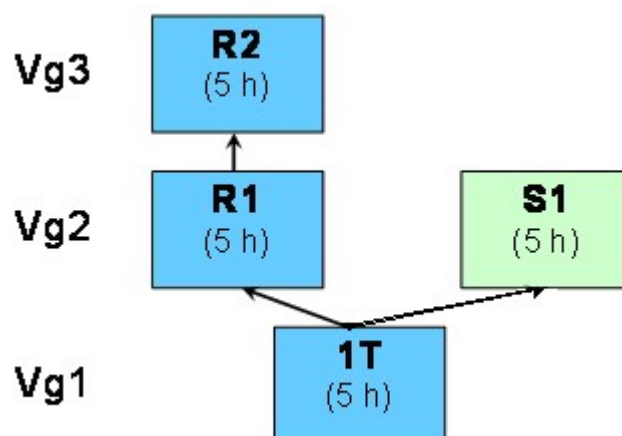


Figure 2-1: Mathematics courses (lessons per week) – options in the observed school

Since the students in this study are meeting the new curriculum only in upper secondary school, I will only go into the curriculum and the mathematics syllabus for what influences students and the teachers in this context.

The mathematics syllabus R94 for upper secondary school from 1994 was goal oriented with specific, detailed goals for each mathematical domain in each year of school, as was the case in L97 for the compulsory

⁷ It must be noted that from 2009, there is also an exam after Vg1, however, it did not influence the students in this study.

⁸ In 2016 the courses are the same.

school. It was easy for students to check if they had reached the specific goals in the mathematics syllabus. This was changed within the new one. The basic educational visions from the old curriculum are still there, however, the goals in the subjects are now listed as statements of competencies that students are expected to achieve. There is now less emphasis on mathematics in an everyday context. It is no longer a main area in the curriculum, but the perspective to take students' everyday life into account is still integrated into most areas in the syllabus.

A new main concept is that of basic skills which are to be applied and developed in all subjects and within all grades. These skills are: oral skills, writing skills, reading skills, arithmetic skills and digital skills.

When it comes to the aims for the education in the area of number and algebra in the first grade in upper secondary school for the 1T course, it is asserted that students shall be able to:

- interpret, process and assess the mathematical content in various texts
- use mathematical methods and aids to solve problems from various subjects and societal fields
- calculate with powers with rational exponents and numbers in scientific notation, algebraic expressions, formulas, expressions with brackets and alphanumerical rational and square expressions, and use quadratic equations to factor algebraic expressions
- solve equations, inequalities and systems of equations of the first and second order and simple equations with exponential and logarithmic functions, using algebra and digital aids
- convert a practical problem into an equation, an inequality or an equation system, solve it and assess the validity of the solution (all curricular text is taken from the official, English version of the mathematics syllabus) (Ministry of Education and Research, 2006).

In the weeks when I was present, there was no attention given to quadratic equations or quadratic inequalities, all the functions and systems of equations were linear. The powers had only exponents within the set of integers (Z). However, what was handled and worked upon in those first weeks serves as basis for further work in algebra and with functions, and is therefore of great importance.

In the above list of what students should be able to do; digital tools are mentioned once.

During the observation, the class followed the progression in the textbook and the expressed competencies to be reached during the first weeks were according to the textbook, to be able to:

- calculate with powers and numbers in scientific notation
- solve equations and inequalities of the first order
- convert a practical problem into an equation
- interpret, process and evaluate the mathematical content in various texts

These competencies are part of the competencies listed above.

2.3 Digital tools

Technological development in the society and the use of technological tools in all areas of private and official life has motivated the politicians to make decisions about how to implement digital tools in schools.

For more than 20 years already, some use of technological tools has been mentioned in the mathematics syllabi. Examples of digital tools are different kinds of handheld calculators, personal computers, software such as Computer algebra systems (CAS), and Learning management systems (LMS).

In the national curriculum L97 the use of digital tools, or what at that time was termed ICT or IT (Information communications technology), was not seen as a main area or a specific topic. It was supposed to be used as integrated suitable tools in other subjects (*ibid*). Calculators and computers were especially mentioned as tools in presenting assignments and projects, not only as tools to make calculations simpler. Word processing was seen as a standard tool, and spreadsheets as useful tools in mathematics.

In L97 it is already stated from grade 2 that calculators and computers are useful in investigating numbers, symbols, and arithmetic operations. From grade 5, students should be able to investigate the limitations and the possibilities of calculators and computers. In other words, they should be encouraged to be critical. Information technology is proposed to be used in lower secondary school in problem solving, and in exploring situations. In grade 8, one goal for students was to be acquainted with the main principles of spreadsheets and their use. Spreadsheets are also mentioned as a tool in grade 10. This means that the students in this study should be familiar with the use of technological or digital tools in mathematics.

In the syllabus, K06, one of the 5 basic skills is digital skills (see section 2.2.2). For mathematics to have digital skills means to:

Use these tools for games, exploration, visualisation and publication, and also involves learning how to use and assess digital aids for problem solving, simulation and modelling. It is also important to find information, analyse, process and present data with appropriate aids, and to be critical of sources, analyses and results.

In the competencies to be attained in the whole syllabus for Vg1T, digital tools are mentioned three times. There is a goal to use algebra and digital tools to solve different kinds of equations and inequalities. When it comes to functions, it is an expectation that students should be able to use digital aids to discuss and elaborate on the different kinds of functions. In probability it is asserted that students should be able to “make binomial proba-

bility models based on practical examples, and calculate binomial probability using formulas and digital aids” (ibid pp. 7-8).

This is what is intended, however, what is the situation? In 1997 ‘The National Network for IT Research and competence in Education’ – ITU was founded, and in 2004 this Network was established as a unit at the University of Oslo. This unit should follow the development in schools when it comes to the implementation of ICT. Every second year a report is published with assessment of the frequency and type of ICT in Norwegian schools, and the degree to which the use of ICT is integrated with pedagogy. A longitudinal survey is conducted and is based on web-based questionnaires. 322 teachers, 3060 students from grade 7, 9, and 11, 307 school leaders from 307 schools answered the questionnaires in 2007. The data was collected in the spring of 2007, only half a year after the introduction of K06.

It was reported that there was an increase in the use of ICT in schools from 2005, especially in the lower grades (Arnseth, Kløvstad, Ottestad, Hatlevik, & Kristiansen, 2007). However, the majority of students in lower grades used computers in schools only occasionally. Especially in the subjects mathematics and science the use of ICT was low. The use of internet and office programmes is most frequent at all grades. There was, however, a big difference between schools in the lower grades.

In grade 11 the situation differed from the lower grades in that all schools used learning management systems. Although the use of ICT in upper secondary school was more part of students’ daily life, only 12 % of the students in the survey reported that they used ICT in mathematics on a daily or weekly basis. The report stated that directives in the form of curricula, and an increase in the number of computers in schools are not enough to enhance integration of digital tools in the pedagogical work in the classrooms. This is in line with international studies (Cuban, Kirkpatrick, & Peck, 2001; Thomas, 2006). The ITU report (Arnseth, et al., 2007) stated that teachers’ lack of competence and school leaders lack of support and systematic work in order to implement the use of ICT were factors limiting the use of ICT in classrooms. The same was concluded in a study about teachers’ implementation of Cabri in lower secondary school (Erfjord, 2008).

From international studies it is claimed that digital tools are often just another tool in the teaching, and the teaching approach is not altered. The classroom organisation is the same, and the tasks are the same as without these new tools (Cuban, et al., 2001). Billington (2009) carried out a study within the same TBM project (see appendix 1) as this study. Her focus was the teaching practice of two teachers, who had implemented the use of computers and pedagogical software in all topics in mathematics. All

students had their own laptop; and textbook, paper, and pencils were absent in the classrooms. Her findings show that the organisation of the classrooms was not changed in the first place, and also the tasks were the same. However, as they developed their practice and used new software, it happened that students solved the tasks in unexpected ways related to the new possibilities within the tools. These ways were not always in accordance with the learning goals within the subject, and the teachers had to change tasks. Billington (ibid) emphasised that the K06 syllabus even though emphasising the use of new tools, does not discuss or make recommendations for new practice. The way to implement the intended curriculum in relation to digital tools seems to depend mainly on the individual teacher.

The Norwegian students are among the students in the TIMSS studies who on a regular basis use calculators in all topics in mathematics. The calculators are advanced calculators. Actually in TIMSS advanced (Grønmo, et al., 2010), students in Norway and Sweden are the students in that study who are reported to use digital tools most frequently, and at the same time they are reported to perform less well.

Research has thus revealed that the quantity of computers and other digital tools are high in schools, however, to implement these tools in the teaching practice as a pedagogical tool, is a long process and depends on the school leaders' and teachers' interest and efforts to reach the goals in the curricula. Although the intention is that digital tools should be integrated in all subjects, the curriculum, K06, gives no clear recommendations about how to use them.

2.4 The teachers

In Norway teachers in lower secondary school are normally educated to teach all subjects, and few teachers have special competence in mathematics, which in TIMSS is defined to be one year with specialisation (Grønmo, 2004). According to the TIMSS study 2003 (ibid) less than 30% of the teachers in the study teaching grade 8, reported that they had participated in courses related to mathematical content during the preceding years. Compared with the international mean Norwegian teachers do not participate in courses related to mathematics as often as teachers in other countries. Grønmo and Onstad (2009) reported in TIMSS 2007 that there are more teachers in lower secondary school, who report that they have participated in courses relevant for mathematics than in the former TIMSS studies. The researchers interpret this to be a changing trend, however, many of the teachers reported that they only joined courses related to the introduction of the new curriculum.

Only a small proportion of the teachers in upper secondary school reported participation in courses to enhance their competencies (Grønmo, et al., 2010). The main difference between teachers in compulsory school and teachers in upper secondary school, is that nearly all teachers in the TIMSS advanced study; teachers in upper secondary school, had specialisation in mathematics and/or in mathematics education.

In a study interviewing a small group of teachers in upper secondary school, Hundeland (2010) found that the teachers knew the curriculum and the directives very well and were loyal to it. This means that they followed the directives coming from the authorities. This was evident in their talk about the obligation to go through the syllabus, which seemed to be the goal in their instruction. The teachers felt the pressure from the system to present all the topics in the syllabus, and at the same time tackle the problem of supporting students at all levels in the same class.

They regarded students' learning to be a gradual process. Therefore, the understanding could come later on, and it was not necessary that all students gained conceptual knowledge in the first place since the students would be given new possibilities later on in the mathematics classes. What was seen as important, was to do mathematics; to solve mathematical tasks. They asserted that the exam was steering much of the activity in the classroom. Also they meant there were too many topics to cover in the syllabus in the time available.

In sum, the teachers in Hundeland's study referred to institutional demands and constraints steering their choices in their daily work. In addition, they told that their main source for both examples and students' work with tasks, was the mathematics textbook.

2.5 Mathematics textbooks

While the mathematics syllabi represent the intended curriculum, textbooks in mathematics are potential links between intended curriculum and classroom activities. In textbooks, the topics are identified in accordance with the syllabus, and organised for students and teachers to follow. For many teachers the textbook is the interpreted syllabus, especially when a new curriculum is being introduced.

In Norway the tradition was that textbooks had to be authorised in order to be used in school. In 2000 this changed. One reason for this was that the curriculum L97 was very detailed, and another that the technological development demanded new thought. As long as the textbooks were controlled by the authorities, teachers were sure they had covered the topics in the syllabus. After 2000, the mathematics teachers in a school have to make their choices from among several mathematics textbooks from

different publishers without being aided by suggestions from the Education Department.

Also without being authorised, the textbooks have a strong position in Norwegian classrooms (Alseth, et al., 2003; Grønmo, et al., 2010; Hundeland, 2010; Mosvold, 2005; Grønmo & Onstad, 2009). Mullis and colleagues (2012), reported that 97 % of grade 8 Norwegian teachers in TIMSS 2011, said they used mathematics textbooks as a basis for instruction. In a recent study based on teachers' responses on a survey, teachers' use of mathematics textbooks in Estonia, Norway, and Finland, was investigated. The findings indicate that Norwegian teachers in lower secondary schools when introducing new concepts, tend to use other resources than the textbook more often than teachers in Estonia and Finland. The vast majority though, relied on the textbook when assigning tasks (Lepik, Grevholm, & Viholainen, 2015).

There is no similar large study from upper secondary school involving Norwegian teachers, but the TIMSS advanced study (Mullis, Martin, Robitaille, & Foy, 2009) found that the only activity taking more than half the time of the lessons in Norwegian mathematics classrooms, was individual work with tasks similar to the examples in the textbook. This implies that textbooks are in extensive use in mathematics classroom also at upper secondary level. On this basis, the mathematics textbook and its resources is one important factor influencing students' experiences of mathematics.

In a study in the USA, the authors Reys and colleagues (2003) found that topics included in textbooks were most likely to be presented in class, and that the pedagogical strategies used by teachers, were influenced by the instructional approaches in the textbooks. Johansson (2006) studying three Swedish teachers with different teaching experiences and at different ages, found the same, and also that the textbook was the main source for assigned tasks. In Hundeland's study (2010) one Norwegian teacher argued that there is no reason for not using the textbook. Not using publisher's resources would be to do the work once again, since professionals writing textbooks had already chosen appropriate tasks for the different topics.

In her study, Johansson (2006) directs attention to the issue of individual teaching in mixed groups of students. Many textbooks in Sweden as also in Norway are graded by level of difficulty. The aim is to afford students, at all levels, tasks which are appropriately challenging for them.

The students in this study were used to such books from lower secondary school, and the textbook they used during observation was graded in this way.

2.6 Work plans

Since work plans are a common way to organise students' work in the school as well as at home (Bergem & Dalland, 2010; Klette et al., 2008), the phenomenon deserves further comment. Legally, every student has the right to be taught according to his/her needs. In mixed ability groups this seems almost impossible, and one assumes that the practice with the work plans started in order to meet those needs (section 2.2.1).

In lower grades the teacher, responsible for a class, normally asks all the other class teachers to deliver the learning goals planned to be reached within the different subjects for the next week(s). Included are assigned tasks in each subject. Then plans for all subjects are collected and presented in one overarching plan. Parents and students welcome such plans, in that they then have an overview of what is to be learned and done in the planned period. In upper secondary schools those plans are normally given for each subject.

In the L97 curriculum it was emphasised that students themselves should take responsibility for own learning. In praxis, this responsibility often turned out to be a responsibility to do the assigned work within the period, no matter when and where. In the PISA+ study (Bergem & Dalland, 2010), the interviewed students told about their decisions: some chose to do all tasks as soon as possible in order to relax or to do other things, other students that they do it the last day in the period covered by the plan. It illustrates that the students often worked with tasks not related to the theoretical focus for the lesson. The students told that they continued to work with the plan where they had left it. In most classes students sat together in pairs or in groups, however, there was little or no cooperation. The reason was according to the students that they were not working with the same tasks.

In addition, the researchers (ibid) found that teachers gave little or no response to assignments. When students had worked with different problems during a lesson, it was not easy for teachers to draw conclusions or to lead discussions about the mathematical topic in focus for the lesson. This is in accordance with the findings from surveys in the main PISA study; that students were often left alone with their work, and that the focus was the doing, not so much the metacognitive activity of reflection and thereby learning.

In sum, it is reported that students did not cooperate so much during seat work, since they worked at different rates on their plans, and that individual work on textbook tasks dominated the work in the classroom. Research reported that the atmosphere in the classrooms was characterised by mutual respect and acceptance. However, through the work plans the

teachers abstained from or lost the possibilities to steer the work in the classroom and to have classroom discussions and to reflect on mutual tasks (Bergem & Dalland, 2010; Klette, 2003; Klette, et al., 2008).

2.7 The task discourse and the didactical contract

In mathematics classrooms the individual work on textbook tasks led Mellin-Olsen (1996) to come up with the notion ‘task discourse’. In interviews with teachers in primary and lower secondary school he found that in schools there could be said to be an institutionalised discourse going on. In mathematics this discourse is characterised or steered by the mathematical tasks, and therefore he introduced the notion ‘task discourse’ as a metaphor for the process of teaching and learning. A mathematical task has a start and an end. And this end is often to be checked by solutions in answer books. He described how mathematical tasks are coming in a sequence, one after another, and it continues until the last task is solved, it may be within a lesson, in home work, on the work plan, or in the book. Mellin-Olsen (*ibid*) found that teachers used words connected to a journey as metaphor for the teaching and learning processes. There is a goal for this journey; the exam, and the teachers have the role as drivers or coachmen. Their responsibility is to bring the passengers, the students, safe to the goal. Students’ responsibility is to keep on and not fall off. Students range themselves according to how far they have come in the textbook. Hundeland (2010) found that the same metaphors were frequently used among teachers in his study. Mellin-Olsen (1996) emphasised that this institutionalised task discourse might constrain teachers’ way of acting.

Mellin-Olsen also showed how stops on the journey implied possibilities for students and teachers to reflect, discuss and negotiate before the journey through the mathematical tasks continued.

The teachers in both studies (Hundeland, 2010; Mellin-Olsen, 1996) refer to the dilemma of going through the syllabus and at the same time supporting each student, who is some place in this sequence of mathematical tasks. As mentioned in the foregoing section, one way to deal with the problem has been the introduction of work plans which have made the important stops on ‘the journey’ more problematic since the students are not working with the same tasks.

The above mentioned studies (*ibid*) are only case studies with few teachers involved, however, as a mathematics teacher, this task discourse is not unfamiliar to me. The task discourse is closely related to the didactical contract described by Brousseau (1997). This contract comprises the tacit and implicit rules regulating the relations between the teacher and the students in a class.

Blomhøj (1994) claims that traditional mathematics classrooms are characterised by some common features that are part of the didactical contract. These features are: The teacher has to carefully demonstrate methods and algorithms presented in the textbook. Students are only offered tasks that can be solved using learned methods and algorithms. A task is solved and finished when its questions are answered. The questions can be answered shortly by a number or by few words. Students' learning is assessed on the basis of their ability to solve such tasks only. Students' roles are to do their best to solve assigned tasks.

Further, Blomhøj (ibid) claims that students' learning is influenced by the type of contract that prevails in the classroom. The contract described above might lead students' motivation to be to fulfill their part of the contract, not to learn mathematics in its full sense. The mathematics has a tendency to be closely bound to the context of the mathematics classroom. Students, however, are mostly satisfied with the situation. Blomhøj, as also Brousseau (1997), claims that such a contract must be broken if students are going to learn mathematics properly.

2.8 Prior knowledge

Students in upper secondary school already have a long experience of school. This means that they have gained knowledge in mathematics and in tackling school life generally. It is asserted that one of the key factors influencing learning outcomes, is the knowledge students have about a topic or subject before instruction (Ausubel, Hanesian, & Novak, 1978). Prior knowledge about a topic includes knowledge about concepts and procedures that a student brings to the learning event. The knowledge might be in accordance with accepted knowledge within the domain, however, it might also be incomplete or incorrect. Thus prior learning can have a positive or negative impact on learning.

'Concept image' was the notion Tall and Vinner (1981) proposed when they elaborated the problem of students taking only some few aspects into account when working with certain mathematical concepts. In the development of constructing mathematical meaning, students' conceptions of a mathematical object may not be in accordance with the formal definition of that object. The notion 'concept image' "consists of all the cognitive structures in the individual's mind that is associated with a given concept. This may not be globally coherent and may have aspects which are quite different from the formal concept definition" (ibid, p. 151).

A concept image is personal and is influenced by all the experiences the person associates with a specific concept. The concept image is thus

limited by the personal experiences an individual student has. Niss (1999) claims:

For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student (Niss, 1999, p. 15).

Although many students can recite formal concept definitions, that is not to say that their concept images are in line with these formal definitions.

De Lima and Tall (2008) discuss how prior experiences may influence current learning. They use the notion 'met before', asserted to be part of the individual's concept image, and defined to be "a mental construct that an individual uses at a given time based on experiences they have met before" (ibid p. 6). Such 'met before-s' may cause problems because they can be used out of the domain in which they are valid. These prior experiences may affect new learning, however, new learning may also affect what is already learned.

The new learning of writing repeated multiplication as powers, for example, may affect the way of writing general multiplication. An over-generalising of rules from old domains to newly learned domains; cognitive obstacles (Brousseau, 1997), is another type of problem caused by incorrect transfer of prior knowledge or experiences.

One example of prior learning causing problems for students is the difference in interpretation of the invisible operation sign between the two mathematical domains; arithmetic and algebra. In arithmetic the invisible operation is addition, however, in algebra it is multiplication. Lee and Messner (2000) reported that college students after working with algebra were confused when asked about the fraction $8\frac{1}{3}$. They suggested that the

operation connecting 8 and $\frac{1}{3}$ was multiplication. This is an indication that the interference also can go in the other direction; conventions in algebra interfere with conventions in arithmetic.

Ausubel (1978) claims that for meaningful learning to take place it is important to relate new material to what the students already know, and that rote learning is more likely to be the outcome if students lack relevant prior knowledge. For teachers it is thus important to find out what knowledge students bring to a course, and use this to plan teaching. The problem is however, that there is a variety of conceptions within a class of students.

2.9 Outline of the situation in Norway

In this chapter I have tried to outline how I see the situation in Norway, to take into account components influencing and making an impact on students' activity in the classroom and their learning of mathematics.

The Norwegian results reported in the comparative studies, and studies within Mathematics Education around the world have reported and are reporting about learning difficulties related to algebra and arithmetic operations. Clements (2013) asks the question: "Why has there not been a marked improvement, given the large amount of mathematics education research conducted around the world, and over a very long period of time, with respect to such fundamentally important curriculum matters?"

In Norway there have been few research reports about algebra in upper secondary school. One is connected to the latest TIMSS advanced study and concludes that many Norwegian students lack skills in manipulating symbolic expressions, and that this lack caused low performances in physics (Pedersen, 2015). Another study in lower grades, also building on TIMSS data and additional task-based interviews of grade 8 students about algebra, concluded that the Norwegian students needed more formal mathematics (Naalsund, 2012). None of these studies are classroom studies.

In this research, students' work with algebra within the classroom during the first weeks in upper secondary school is to be studied in relation to the mathematics offered within that classroom setting.

As far as I know there is no similar study within the Norwegian context, building on earlier studies about algebra and students' problems within the context of institutionalised teaching. The goal is to add to and extend the existing body of research in mathematics education and especially in the Norwegian context.

The figure below illustrates what components are influencing the work in the classroom, as I see it on the basis of what is described in the foregoing sections. All the letters *i* in the figure, stand for the word influence(s).

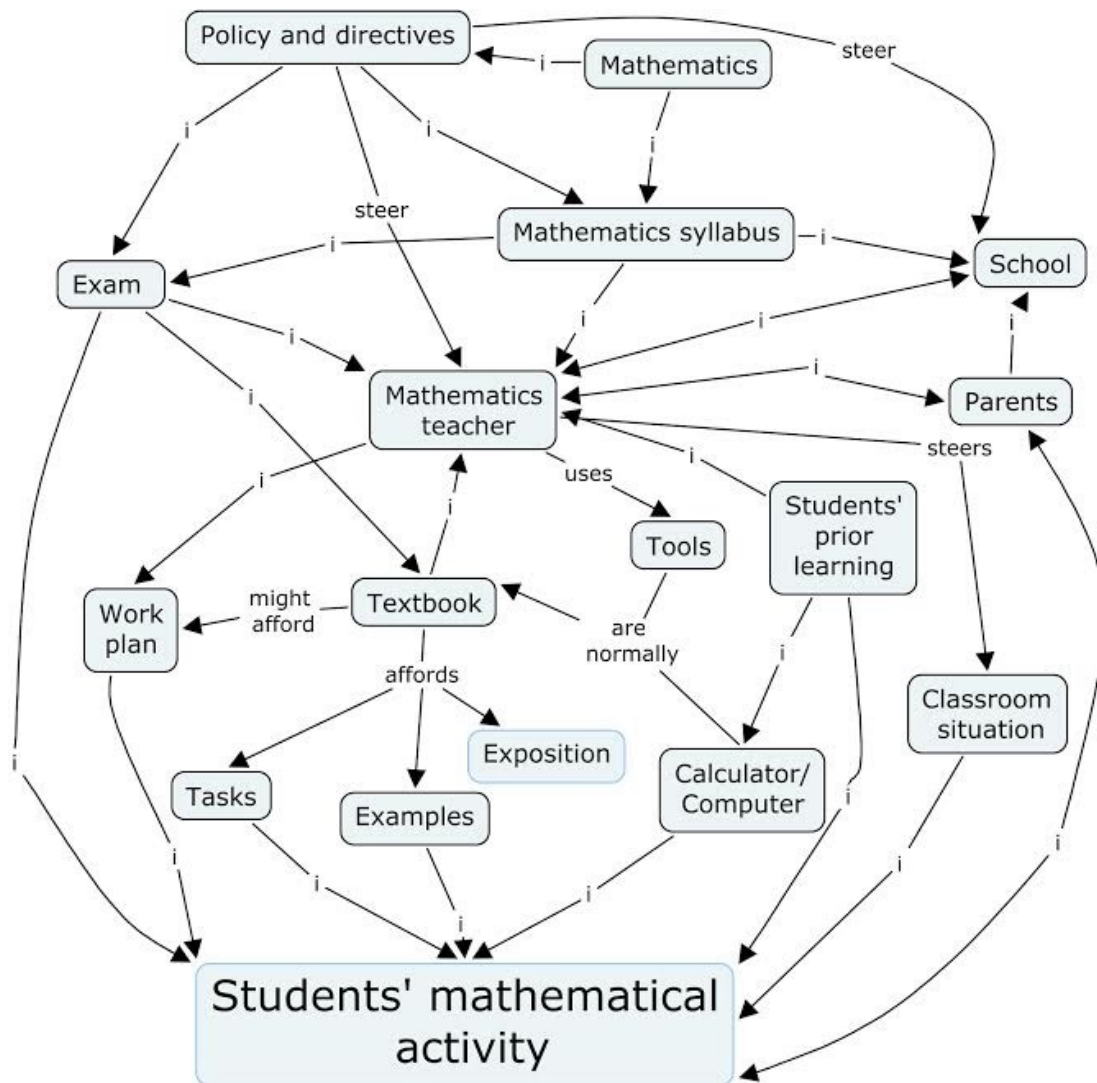


Figure 2-2: Model of the situation in Norway

The curriculum in directives and documents includes the mathematics syllabus, exam, and the whole organisation of the Norwegian school system. However, in my model I have split this into policy and directives, mathematics syllabus, and exam/evaluation. Through policy and directives, the authorities guide and control what is going on in schools. There are special laws for students and parents to refer to if they are not satisfied with the instruction, the resources and/or the way the students are treated in school.

The teacher has to follow up directives coming during the school year. For example in the school year 2007-2008, a new directive about exams was sent out to schools. What was written in the official documents from 2006, should be changed from having no centralised exam after the first year of mathematics in upper secondary school, to centralised exam from

2009 on. This means that new directives have to be taken into account, and as in this example with the exam; these directives have a direct impact on the teaching practice, and students' behaviour. This is evident from Hundeland's (2010) study.

The mathematics, as a science, influences how the mathematics syllabus is formed and also the content of the syllabus. Although the mathematics for mathematicians is different from school mathematics, it is the basis for the transposition into school mathematics. This together with policy and the trends in the society steers what is emphasised in the mathematics syllabus.

What is given in the written documents is the intended curriculum. Then publishers interpret it and produce textbooks with tasks and other resources for teachers and students. Some of these are resources on web-pages such as work plans and time tables. The textbooks include expositions, examples and tasks. As outlined above (section 2.5), researchers (Alseth, et al., 2003; Grønmo, et al., 2010; Hundeland, 2010; Mosvold, 2005; Grønmo & Onstad, 2009) give evidence that textbooks and publishers resources are the most important sources for Norwegian teachers in their work.

Teachers implement the curriculum in the classroom. Then the syllabus has already passed through the interpretation made by authors of teaching and learning resources. Then these resources have to be interpreted by the teachers, who also have their own interpretation of the mathematics syllabus, based on their mathematical competence, their experience as teachers and on the culture in the specific school.

This means that students' experience of mathematics, the mathematics syllabus and the whole curriculum, is influenced by the mathematics teacher, the textbook and publisher's resources. In addition, students have their experiences from earlier work with mathematics in school and out of school, which strongly influence how they experience new input. In addition, their previous knowledge is influenced by new experiences in mathematics. Inferences of new knowledge on previous knowledge are seen for example when students learn the notation of powers. Evident in my data is for example that some students tend to write multiplication as powers.

When it comes to the computer and other digital tools, previous experiences with these tools, influence students' behaviour and their way of using these tools. The use of these tools also makes an impact on the mathematics experienced by the students. There is a big difference between producing graphs by hand and doing the same aided by computer software. When using computers, students have the opportunity to experience a large variety of different graphs in a short time, compared to line drawings by pencil on paper.

The mathematics teacher is the one orchestrating the activity. As mentioned in section 2.4, research gives evidence that many teachers feel constrained and pressed by the curriculum and all the demands on what is to be achieved within a limited period of time. This pressure comes from students, parents, and school leaders. Part of the reason is that for students it is of great importance to achieve well and get good marks for further studies, and for schools, the results are visible and it is important to gain a good reputation.

Although publishers provide learning resources, examples for teaching, and tasks, the teacher has to decide what to use, what to emphasise, and how to present the mathematics. Therefore, publishers' work influences teachers' decisions, and teachers' choices influence what is presented for students. This is all the implemented curriculum. The experienced curriculum, though, might differ from both what is intended and what is implemented, and my emphasis is on students' experiences of a small but important part of the curriculum; elementary algebra.

As shown in section 2.1, the TIMSS studies give evidence that algebra is a weak topic for students in grade 8 (Grønmo, 2004), and also a reason for problems in mathematics and physics in upper secondary school (Angell, et al., 2010; Grønmo, et al., 2010).

The Norwegian Mathematics Council is testing first year science, engineering and economics students' pre-knowledge in mathematics. The test tasks are tasks from the grade 10 mathematics curriculum. The first test was carried out in 1984, and the results have shown a decline since 1984 with the lowest results in 2007. Although the last results, in 2013 (Nortvedt, 2014) showed a tiny improvement (average mean 51%), the results have been on a stable low level the last ten years. A lack of conceptual understanding of fractions and a lack of competence in applying the arithmetic operations are mentioned as factors causing problems in the work with algebra and formulas. Results from these tests show that even when starting higher studies, Norwegian students struggle with basic mathematics.

On this background within the Norwegian context outlined in the foregoing sections, my main research questions are:

What mathematics are students offered in the classroom for learning, or to consolidate already learned mathematics as they enter the upper secondary school?

What experiences of the mathematics offered do students reveal through their responses?

Students' responses include all that is observed of their responses; what they write and what they say.

The learning does not take place in isolation; the student alone with the mathematics. My study of the mathematics offered and students' responses on that mathematics, caused me to reflect on the learning situation, and how the mathematics offered and students' responses on that mathematics were related. It led me to ask a third question:

How does the setting of the mathematics classroom influence students' experiences of the mathematics offered?

The setting of the mathematics classroom, includes the textbook used in the class, the tasks offered, the teacher's explanations, and the computer with the provided software. The students and the teacher are also part of the setting and thus the interaction in the classroom is included. This means that the mathematics offered and students' responses are part of the setting. I think though that to focus particularly on factors constituting the setting of the mathematics classroom, will be fruitful in understanding students' responses to the mathematics offered. I do not intend to search for a cause – effect relationship, but to examine the classroom setting, looking for factors which on the background of former research and theories may influence students' conceptions and learning of mathematics.

The focus is on the mathematics offered in the first weeks of upper secondary school. In this part of the course, the students worked with the first textbook chapter. On the first page of this chapter was written:

Goals:

When you (singular) have read this chapter, you should be able to

- calculate with powers and numbers in scientific notation
- solve linear equations and inequalities
- convert a practical problem into an equation
- interpret, process and evaluate the mathematical content in various texts

These goals are part of the goals stated in the K06 curriculum for the area number and algebra in the K06 curriculum (see section 2.2.2).

These goals, with some minor exceptions, were expected to have been reached during the first 10 years in school. During the first weeks in upper secondary school, the students were being offered a new possibility to learn and to consolidate their previous knowledge.

3 Algebra

Although my focus is on the mathematics offered, students' experiences of this mathematics, and on the classroom setting, a central part of the thesis is devoted to algebra and number. The topic taught during the observation was mainly algebra as a revision of what had been taught in lower secondary school. Therefore, this chapter will be a review of studies done in the area of algebra learning, but first comes a section with a discussion about what algebra is.

3.1 What is algebra?

If going to an encyclopaedia looking for the word algebra, a long list of topics within the area of algebra would appear. However, if one were to ask students in secondary school: What is algebra? my guess is that most of them would answer: calculation with letters instead of numbers. For the students in this study the aim when introduced to algebra in grade 8, was that they should have “opportunity to work to develop their understanding of the use of letters and brackets in simple mathematical expressions and formulae ...” (Hagness & Veiteberg, 1999, p. 179). This makes it likely that they were introduced to algebra as extended arithmetic; calculating with letters as substitute for numbers.

Charbonneau (1996) claims that history of mathematics provides a warning against this view, algebra is much more, and he proposes further that algebraic symbols behave like many other mathematical objects not only numbers, and he points especially to geometry where the mathematical objects, are closer to algebraic symbols than numbers. He exemplifies this by addition of two numbers, then their sum is represented by a quite new number, and the original numbers have vanished. In contrast, when two segments of a line are put together one after another, the new segment includes both the two original ones. This is similar to algebraic symbols. However, when using letters, “then the letter representing the new segment does not have a meaning by itself and therefore cannot be interpreted without specific references to the original segments” (ibid p. 34).

Sfard (1995) presents different views about what algebra is. She proposes that writers mostly have agreed about algebra as the ‘science of generalized computations’, but she asserts that some authors suggest that algebra as such started with the introduction of the modern algebraic symbol system, and thus a definition of algebra must imply this algebraic symbol system. For others this symbol system is not required. Sfard herself states that she uses “the term algebra with respect to any kind of mathematical endeavor concerned with generalized computational pro-

cesses, whatever the tools used to convey this generality” (ibid, p. 18). This definition, emphasises the operational origin of algebraic thinking. By computational processes she refers to the theory of reification (Sfard, 1991). Mathematics is hierarchical and reification occurs on any level, therefore it is possible to talk about computational processes also in abstract algebra, for example as group operations related to plane symmetry. In this way Sfard, as also Charbonneau above, relates algebra to a wider scope than only to numbers.

Although traditional school curricula emphasise the letter-symbolic aspect of algebra, Stacey and MacGregor (2001) say it has been important for educators and researchers also to let other characteristics of algebra come to the forefront. This view has been promoted by recent research done with children in preschool and in lower grades, which shows that small children are able to generalise and to recognise patterns, and even to work with variables (Carraher, Schliemann, & Brizuela, 2001; Schliemann, Carraher, Goodrow, Caddle, & Porter, 2013). Algebraic thinking or algebraic reasoning are the notions used to include these aspects of algebra rather than the letter-symbolic one.

To define algebra is “fraught with difficulty” according to Lins and Kaput (2004, p.48), however, they say that a group of researchers in mathematics education had come to an agreement on two characteristics of algebraic thinking. The first characteristic is based on generalisation, both the acts of generalisations and the expressions of generality. The second involves “reasoning based on the *forms* of syntactically-structured generalisations, including syntactically and semantically guided actions” (ibid p. 48). As I interpret these characteristics, the first can be referred to as generalisation in the way children explore patterns, and the argumentation made to justify their generalisation. This argumentation may be expressed by normal language or by use of more formal mathematical notation. The latter characteristic is based on formal algebraic notation with emphasis on transformational processes. It can be exemplified by the algebraic expression $3(a + 2)$. A syntactically guided action would be to remove the brackets and get $3a + 6$. In a semantically guided action the whole expression $a + 2$ has to be multiplied by 3, that means that a has to be multiplied by 3 and also 2 has to be multiplied by 3. The result is the same in both occasions; what guides the actions is different. The semantics here is semantics within the syntactics. Semantically the expression can also be interpreted based not on the syntactical form, but for example it may be interpreted as the area of a rectangle with one side of length 3 and the other of length $a + 2$.

Mason (1996) says that what he calls algebraic thinking, is an awareness of “detecting sameness and difference, making distinctions, repeating

and ordering, classifying and labelling” (ibid p. 83). He also emphasises the importance of the processes in mathematics. According to him what he includes in algebraic thinking, is the core of algebra. And further he claims that generalisation is the “heartbeat of mathematics” (ibid p.65).

That generalisation is at the core of mathematics is not denied by Wheeler (1996). He, however, pointing to Masons’ definition of algebraic thinking, proposes that to break algebra down to its basic components leads to a list of actions, “in which all the mathematical content appears to have been stripped away” (ibid p. 319). Wheeler asserts that the list proposed by Mason could be a list which could also fit into other subjects, and he mentions biology as an example. Further he suggests that the problem to define algebra may be solved in two different ways. One is to follow Mason in his use of the notion of algebraic thinking; without a demand for mathematical symbols. The other way is to acknowledge these mental operations of very general character, but not to define them to be algebraic before they are linked to an algebraic symbol system.

Kieran (2004a) offers a definition of algebraic thinking in the early grades of school which is aimed to make a bridge from these early grades to the algebraic activity in later grades. She had already suggested a model of algebra built on the different activities students are engaged in in school algebra (Kieran, 2004b). This model involves three categories of activities. Students are involved in the first category, the generational activity, when forming algebraic expressions and equations. Kieran comments that “much of the meaning-building for algebraic objects occurs within the generational activity of algebra” (2004a, p.142). The next category she calls the transformational activity. This involves the rule-based activity of transforming expressions and equations, always on the basis of maintaining equivalence. The last category is the global, meta-level mathematical activities. Within these activities algebra is used as a tool. Kieran mentions “problem solving, modelling, noticing structure, studying change, generalizing, analyzing relationships, proving and predicting” (ibid p. 142). These last activities she claims students can be engaged in without using any algebra. This claim seems to be in accordance with Wheeler’s (1996) critique of Mason’s (1996) definition of algebraic thinking.

Kieran brought in a definition of algebraic thinking, which is integrating algebraic thinking from lower to higher grades:

Algebraic thinking in the early grades involves the development of ways of thinking within activities for which letter-symbolic algebra can be used as a tool but which are not exclusive to algebra and which could be engaged in without using any letter-symbolic algebra at all, such as, analyzing relationships between quanti-

ties, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting (Kieran, 2004a, p.149).

This definition of algebraic thinking is closely connected to what she has called the ‘global meta-level mathematical activities’ (see the paragraph above) in which algebra might be used as a tool.

As the above reveals, algebraic thinking, or reasoning is discussed within the research community of mathematics education. Research done on early algebra’ shows that “young students benefit from opportunities to start from their own intuitive representations and gradually adopt conventional representations, including the use of letters to represent variables as tools for representing and for understanding mathematical relations” (Carraher, Martinez, & Schliemann, 2008, p. 4). In the latest Norwegian curriculum for lower grades one competence to be attained is to find and recognise patterns. Implicitly this is part of algebraic reasoning.

I think it is appropriate to talk about algebraic reasoning in the way Blanton and Kaput (2005) have formulated a definition: “We take algebraic reasoning to be a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways” (ibid p. 413).

Traditionally school algebra has been, as Pimm (1995) expresses it, about “form and transformation” (ibid p. 88), what Kieran calls the transformational activity (Kieran, 2004a, 2004b, 2007a). This Pimm claims to be at the core of algebra. The result has been, that many students seem to manipulate algebraic expressions based only on rules without giving meaning to the symbols they act upon, and without any consideration about why the rules work.

Lee (1997b) interviewed mathematicians, researchers, teachers, and students about algebraic understanding. She exposed seven metaphors for what algebra is: “algebra is a tool, an activity, a way of thinking, a culture, a generalised arithmetic, a language, and a school subject” (ibid p. 92). Algebra as a language and algebra as an activity were, according to Lee, the richest metaphors she found. Syntactic and semantic were words connected to the view of algebra as a language. These words were also used by interviewees not holding algebra as language in the foreground. The syntax in algebra is related to the rules for manipulation of algebraic expressions and fluency in performing these rules. It was discussed if the semantic had to be associated with the meanings of its referents outside algebra, however, it turned out that several interviewees meant that students had to concentrate on the “semantic of the syntax” (ibid p. 96). And therefore the traditional view of students’ mindless work with symbols

and rules perhaps has to be reconsidered. Meaning may be gained from the algebra itself.

What is then algebra and what is to be taught? As seen in the literature above, algebra is not easy to define. Anderson (1978) also posed this question. He writes that some have called algebra the “study of structure” others “the language of mathematics” in the meaning of syntax and grammar, others again “the codification of mathematical laws”. None of these alone define algebra, however all of them are part of what algebra is.

For this study algebra is taken as synonymous with school algebra, meaning algebra taught in school. Caspi and Sfard (2012) define algebra in school to be formalised meta-arithmetic. Building on the view that mathematics is a discourse creating its own objects, algebra is seen as a sub-discourse. This view is building on Sfard’s (2008) theory of commognition (Thinking as communicating), which will not be the basis for my research, however, I find the definition to be useful. There are according to Caspi and Sfard (2012) two basic types of meta-arithmetical tasks giving rise to algebra. These tasks are questions of numerical patterns, normally described by equalities, and tasks about unknown quantities described as equations.

3.1.1 Approaches to algebra

The students in my study in grade 11, are not expected to have been explicitly exposed to teaching that emphasises algebraic reasoning from the early grades. According to the syllabi they have been following, algebra was introduced in grade 8.

It seems as if it is easier to distinguish between different ways of approaching algebra, and to describe the content that is typical for the different approaches, than to define it. Although the approaches are different, and the content which is emphasised in the different approaches is not quite the same, it is not a talk about disjoint sets of content, rather different aspects and emphases.

As early as 1978, Anderson came up with four different ways of approaching algebra:

- Algebra as generalised arithmetic or classical algebra,
- Algebra as a study of patterns and structure,
- Algebra as an axiomatic study, and
- Algebra through concrete situations (Anderson, 1978, p. 9).

In 1996 different perspectives on algebra were proposed to reflect upon, and the result was the book: “Approaches to algebra; perspectives for research and teaching” (Lee, Bednarz, & Kieran, 1996). The approaches for introduction of algebra were:

- A generalisation perspective
- A problem-solving perspective

- A modelling perspective
- A functional perspective.

Both lists complement each other and include the generalisation perspective. Anderson (1978) has generalised arithmetic, however, algebra as a study of patterns and structure also involves a generalising perspective. He mentions that an awareness of structures can be reached from the source of children's search for patterns. The problem-solving perspective is implicit in classical algebra, although Anderson when referring to classical algebra sees the problem of too much emphasis on manipulative skills.

These examples only illustrate that whatever approach is chosen, it is one out of several possibilities. What is to be noticed, is that the way students have been introduced to algebra will make impact on their further work with the subject.

The functional approach is a fairly new way of approaching algebra. Historically the concept of function came rather late in the development of mathematics. Because of new technology with opportunities for students to easily produce functions and make tables, this approach has been introduced to give meaning to the concept of variable. Some researchers have argued against this approach from different points of view; Pimm (1995) while he meant there was a redefinition of algebra going on in the USA “triggered I feel more by the potentialities of these new systems and the drawbacks of an over-fragmented mathematics curriculum than by any novel epistemological insight” (ibid p.104). Others like Lee (1997a) criticise the functional approach while functions can be expressed without algebra and also algebraic expressions do not need to describe functions.

Kieran (2007a) describes how the influence of new technology has made impact on the curricula in many countries. Approaches to school algebra and content of school algebra have changed from being a topic where form and transformations were the main characteristics to a topic with a new emphasis on the study of families of functions, for example in Israel (Sfard & Linchevski, 1994b).

In Norway it seems as if the functional approach to algebra has not been in focus for curriculum makers, although one of the competencies to promote, is digital competence as a competence on all levels and related to all subjects in the last curriculum (Ministry of Education and Research, 2006).

The students in this study met algebra (letter symbols) in grade 8. This might indicate that the students have met algebra as generalised arithmetic with emphasis on the transformational activity.

In the next section the focus will be on this shift from arithmetic to algebra.

3.2 Algebraic symbol system

Often the distinction between algebraic reasoning with informal representations and algebra expressed by the formal algebraic symbol system has been denoted as a shift from pre-algebra to formal algebra. Students in the study being reported in this dissertation have reached the stage in the curriculum of formal algebra.

Although it is said in the general part of the L97 curriculum that “patterns and regularities in the elementary school should serve as a basis for the further work with the concept of variable”, the mathematics syllabus introduces algebra in relation to literal symbols.

In the literature this shift from arithmetic to algebra has been in focus for many research projects, and there has been said to exist a cognitive gap (Herscovics & Linchevski, 1994) between the two. When dealing with mathematics in lower grades, one has already dealt with a mathematical symbol system. There are symbols for arithmetical operations; and the numerals are symbols. However, after becoming familiar with these symbols, the numerals are often not seen as symbols anymore; they are more likely to be seen as part of the daily life language. It seems as if it has been forgotten for example how the numerals once had to be connected to the concept of numbers. The linkage between concept and its referent has become automatic (von Glasersfeld, 1994) or the name of the concept has come to be seen as the concept itself (Skemp, 1987, p. 12). Steinbring concludes that: “The true mathematical object, that is the mathematical concept, may not be identified with its representations (Steinbring, 2006, p. 137).

The symbolisation in algebra is important and also central for mathematics as a whole. Algebraic symbolism leads to simplified problems in the sense that when confronting a problem, it is possible to translate this into algebraic symbols. Then it is possible to forget the problem in case and operate on the symbols until a solution is reached. This solution then has to be interpreted in terms of the initial problem. This minimises the cognitive workload. It is not necessary to keep in mind the referents for the symbols while working on them.

According to Wheeler (1996) the commonality between words and symbols in arithmetic and in algebra is a general obstacle in algebra learning. The positive side of these commonalities is that students can get access to some of the simpler expressions without much effort. The problematic side is that the symbols and words that are recognised from arithmetic do not always mean the same in an algebraic context. The meaning and the use of the equals sign is an example of a sign whose meaning for most students has to be changed or expanded. Another result of such an

obstacle is the tendency to conjoin numbers and letters in algebra. Also invisible signs cause problems in this shift from arithmetic to algebra. In arithmetic we have the number 43 which means add 3 to 40, while in algebra $3n$ means multiply 3 by n .

Instead of distinguishing between arithmetic and algebra on the basis of the literal symbols, Kieran (1992) distinguishes between the structural and procedural character of mathematics. In arithmetic the result of a computation is a specific number, while in algebra the result might be an expression with several terms. This led Kieran and other researchers to introduce the process-object duality in algebra. This will be the focus for the next section.

3.3 The process-object duality in algebra

In textbooks in school, the notion of the variable is normally introduced by substituting numerical values by letters to build algebraic expressions. This introduction is smooth; however, students are expected to go directly on to transform these expressions and to solve simple equations. According to Kieran (1992) it is often not recognised how demanding this shift is for students; to go from the smooth introduction to work upon these expressions as if they were objects. This requires a structural view on algebra in opposition to a procedural view. By procedural Kieran (*ibid*) means arithmetic operations carried out on numbers and where the outcome is numbers. She uses the equation $2x + 5 = 11$ as an example. Here students can substitute x with different numbers until the solution is found. Then students act seemingly on an algebraic equation, however, what they in reality do, is to work arithmetically. When tasks are getting more complicated, this way of handling algebraic expressions and equations is not sufficient in order to be proficient. Instead algebraic expressions have to be viewed as objects in their own right. To solve equations of the form $ax \pm b = cx \pm d$ one has to view the left side and the right side of the equal sign as two different algebraic expressions which are equivalent. Then the operations to be carried out have to be carried out upon the algebraic expressions, not on numbers. Another example is the algebraic expression $3x + y + 8x$, which can be manipulated to be $11x + y$, which has to be accepted as the solution or answer. Kieran interprets this change in perspective from an arithmetic view on operations to an algebraic one “as a movement from a procedural conception to a structural conception” (*ibid*, p. 393).

By the term procedural, Kieran (*ibid*) refers back to Sfard’s notion operational. Sfard (1991) analysed different mathematical definitions and representations from an ontological – psychological perspective and introduced the duality of structural and operational interpretations of math-

ematical notions. According to her, mathematical concepts can be conceived in two different ways; as objects or as processes. When a mathematical concept is conceived of as an object, Sfard talks about structural conceptions, and when a concept is conceived of as a process, she talks about operational conceptions. The relationship between the two conceptions is according to Sfard complementary and hierarchical. In the case of algebra Sfard and Linchevski (1994b) use the expression $3(x+5) + 1$ as an example. This expression may be interpreted in four different ways: As a “computational process” with a series of steps to be carried out; as representing a certain number although not specified because of the unknown x ; or as a function mapping one number onto another with the whole expression representing change. If one of the coefficients is substituted by a new letter this expression may be interpreted to represent a whole family of functions or as a function in two variables.

In the first case, if the expression was interpreted as a series of steps to be carried out, “it was the computational process rather than any abstract object (except the processed numbers) which gave meaning to the symbols” (ibid, p. 89). That means that in this case the expression is conceived of as an operation, not as a static object to be operated on, as in the fourth interpretation when the expression was seen as a family of functions. The example used, also demonstrates that the same mathematical notion can be interpreted both structurally and operationally.

Gray and Tall (1994) introduced the notion proceptual thinking for this capacity to see the algebraic symbol $3x + 2$ as the process add three x plus two, and at the same time see it as the outcome of the process, the algebraic expression ‘ $3x + 2$ ’. Proceptual thinking is characterised by the ability to “compress stages in manipulation to the point where symbols are viewed as objects that can be decomposed and recomposed in flexible ways” (ibid, p. 132). This process is by Sfard (1991) called reification. Gray and Tall (1994) use the term encapsulation, a term used by Dubinsky (1991). The ability to use the same mathematical notion to “represent both a process and the product of that process” (Gray & Tall, 1994, p. 119) is crucial in the learning of mathematics.

When referring to this area of cognitive psychology, Cobb sums up that “proficiency in a particular domain is therefore seen to involve the conceptual manipulation of mathematical objects whose reality is taken for granted” (Cobb, 2007, p. 19).

4 Reported findings – a review

This chapter is devoted to research on algebra and number. One of the approaches used in this study, is to establish what is known about learning and learning difficulties in the area of school algebra, and to use this literature in categorising students' responses found in this study. The search for literature was guided by the sequencing of the different topics in the textbook used in class. In addition, some issues were searched for, which were more general, such as the equal sign and the different conceptions of the letter symbol.

Much of the reviewed research is done in lower grades, it is, however, relevant for this study for more than one reason. The students in this study have their first weeks in the upper secondary school, and are in the transition from lower to upper secondary school. The study might thus be said to be located within the research field of this transition. Another reason is that knowledge of basic problems lays the foundation for understanding more about what problems students have on higher levels. Some of the reports are rather old, however, the same problems still exist.

There is not much classroom research done in Norway when it comes to teaching and learning algebra in upper secondary schools. In Sweden, however some research similar to mine has been done (Olteanu, 2007; Persson, 2005). Both of them found that also in upper secondary school students struggle with algebra. Some problems are related to students' conceptions of the variable and interpretation of letter-symbols, others are related to problems in arithmetic. Especially, they both report that negative numbers and the minus sign cause problems. In addition, fractions are a stumbling block for students.

In lower grades some studies on algebra have been done in Norway. The KIM project- Kvalitet i Matematikkundervisningen (Quality in mathematics teaching) in the 1990s was a large scale study carried out in lower secondary to investigate students' performance and understanding of algebra and number. Tasks from the CSMS study (Hart, 1981) were used, and booklets were produced to inform teachers about problems students meet in learning mathematics, and also to help teachers to understand why these problems occur so that they might, if possible, be avoided (Brekke, 1995a; Brekke, et al., 2000). Other studies also examining aspects of algebra, are the PISA and TIMSS studies (Grønmo, 2004; Grønmo, et al., 2010; Kjærnsli, 2007; Kjærnsli, et al., 2004; Lie, et al., 1997) see section 2.1 . In my master study (Espeland, 2006) I analysed tests given to 230 students in grade 11, and I also interviewed some students with tasks from the tests as a basis for the interviews. Those tasks were used in the CSMS

study, and in the KIM study in Norway. It was noticeable that the students in grade 11, whom I was considering, performed weaker than the students in the KIM study 10 years earlier. In addition, the students in my study were more than one year older. Furthermore, whereas the students included in the KIM study, represented all students, my study did not include students who were weak in mathematics. It was assumed that weak students struggling with mathematics had not opted for theoretical courses at upper secondary school.

Internationally many studies have been carried out in order to find out what causes problems for students and hinders their learning in algebra. Some studies have focused on students' interpretations and use of symbols and the conception of the variable (Booth, 1984; Brekke, 2005; Küchemann, 1981; Quinlan, 1992; Ursini & Trigueros, 1997; Usiskin, 1988; Wagner, 1981).

Others have been more focused on students' conception of equivalence and the equal sign (Behr, Erlwanger, & Nichols, 1976; 1980; Darr, 2003; Demby, 1997; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Vlassis, 2002). The role of the equal sign is tightly connected to the awareness of algebraic structure and the notion of reification (Sfard & Linchevski, 1994b), encapsulation (Dubinsky, 1991) and procepts (Gray & Tall, 1994). Structure sense is a notion developed by Linchevski and Livneh (1999) and used by others (Hoch & Dreyfus, 2004; Linchevski & Livneh, 2007; Novotná & Hoch, 2008). Another source of problems is the concept of negative numbers and the minus sign (Gallardo, 1995, 2002; Kaur & Sharon, 1994; Linchevski & Williams, 1999). This problem has been investigated further with the focus on the notion of negativity (Vlassis, 2004, 2008).

4.1 Algebraic notation and the variable

Perhaps the most important concept in algebra is the concept of variable. In school algebra a letter is normally used as a place holder for a number. This number can be a specific unknown like the unknown in an equation, it can be a general number as in the general rules for arithmetic, or it can be a number which varies in relation to other numbers as in the case of functions.

In mathematics, this notion of variable is used for all the above-mentioned situations. Some researchers in mathematics education also used the notion variable for all these cases (Drijvers, 2002; Usiskin, 1988). Küchemann (1981) who investigated students' different interpretations of letters in algebra warned against the "blanket use of the term "variable" in generalised arithmetic" (ibid p. 110). He asserted that this use had blurred the meaning of the term itself and also the different meanings

that students assign to the letters. For him it was important to emphasise that the concept of variable includes an interpretation of an unknown, not a specific unknown, but an unknown with a changing value.

Küchemann (ibid) focused on different ways in which students interpreted letters in algebra. His work was part of a large-scale survey of Concepts in Secondary Mathematics and Science in England, the CSMS-project. One of the aims of the study was to categorise mathematics understanding according to Piagetian developmental stages. This aspect of the research has since been questioned; however, the study is still relevant in that it gives a valuable insight into students' thinking about mathematical concepts.

In the study the underlying assumption was that algebra was viewed as generalised arithmetic. Küchemann (ibid) identified six categories of how students interpreted literal symbols. In the first two categories letters are ignored or given a numerical value from the outset.

In the third category letters are used as objects. In this category letters are regarded as shorthand for an object or as an object in its own right. When students are introduced to collecting like terms in algebra, as in the example $2a + 5b + a$, teachers often use the "fruit salad" method of saying for example 2 apples + 5 bananas + one more apple, which can totally mislead students. Another possibility is to see the letter as an object in its own right collecting a's and b's and so on. It is fundamental in algebra that students understand that letter symbols are substitutes for numbers. A lack of such understanding might initially lead to success in solving some types of tasks, it will, however, mislead students and cause problems in the further work with algebra. Another problem occurs when students have to distinguish between the objects themselves and the numbers of objects. If students are given the problem: If cakes cost c pence each and buns b pence each, and 4 cakes and 3 buns are bought, what does $4c + 3b$ stand for? Many students also in higher levels tend to interpret this to be 4 cakes and 3 buns, instead of seeing the expression as an expression for the total price for the 4 cakes and the 3 buns.

The last three categories are as follows: letter used as a specific unknown. Here the letter is regarded as a specific, but unknown number, that can be operated upon directly as in equation solving, or for example in tasks where it is asked for the answer when 4 is added to $3n$. The next category is letter used as a generalised number. This means that the letter is seen as representing, or at least as being able to take, several values rather than just one, for example letters in formulas.

The highest level of understanding, according to Küchemann, is reached when letters are used as variables. By this he asserts that the letter is seen as representing a range of unspecified values, and a systematic re-

relationship is seen to exist between two such sets of values (Küchemann, 1981a, p.104). One example he uses is the statement $5b + 6r = 90$. If the letters here are interpreted as specific unknown numbers, the statement is seen to be true for an unknown, particular pair of numbers. It is also possible to interpret these letters as generalised numbers without having in mind how they change, only that they are different isolated pairs of numbers making the statement true. For these letters to be interpreted as variables, according to Küchemann's meaning of a variable, it is important that there is an established relationship between the two letters b and r in the example above, so that it is seen by the students that for example a change in the value of b causes a change in the value of r .

In a follow-up study, *Strategies and Errors in Secondary Mathematics (SESM)* (Booth, 1984), the tasks and the findings from the algebra part of the CSMS study were used in order to investigate the reasons for the most frequent errors. The researchers interviewed 50 students from the same population as in the CSMS study. In addition, teaching experiments were carried out both in small groups and in whole classes. Booth (ibid) found that some problems students had in interpreting letters as generalised numbers, were due to their cognitive development. Other problems reported seemed to have been resolved during the teaching experiment. An emphasis on open expressions led for example students to accept expressions as $x + 3$ as a final result. This was not obvious in the start, but they developed an acceptance of the 'lack of closure' (see the next section 4.1).

In a recent intervention study in England, the *Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS)*, the first phase was a large scale survey using tasks from the CSMS study. It was concluded that the tasks were still relevant, 30 years on. The algebra test was administered to ca 5000 children from 12 to 14 years old. It was reported (Brown, Hodgen, & Küchemann, 2012) that the results were the same or slightly worse than the results from the CSMS project, and that less than 10 % interpreted the letter symbol as a variable.

Based on the test results and interviews with students in phase 1, on pedagogical principles, and research literature about algebra, rich tasks were designed, and collections of interlinked lessons were outlined. The research team worked with teachers, and after implementing the lessons and the material, many teachers told that the intervention project made them change their ways of teaching. Students' responses on tests indicated that the rate of learning had been double over a year for those who had joined the project compared to those who had not (Brown, Hodgen, & Küchemann, 2014)

When investigating students' conception of variable, Wagner (1981) inquired into students' ability to conserve equations and functions when

changing the literal symbols. In interviews with 29 students between 10 and 18 years from schools in the USA, she found that fewer than half the students, having studied algebra, could state that the functions and equations presented, were conserved when the letters denoting the variables were changed. They seemed to believe that the value of the variable was changed by the changing of the letter symbol, even though the structure of the functions and equations was kept the same. Some gave values according to the ordering of the letters in the alphabet.

Usiskin (1988) describes many uses of the letters (he uses the term variable to be synonymous with the word letter) to which secondary students are exposed. He asserts that these different uses are related to the purposes to which algebra is used, and this again is determined by which approach to algebra is chosen (see section 3.1.1).

MacGregor and Stacey (1997) reported from a longitudinal study in Australia including 22 schools from grade 7 to grade 10. The students were tested at several occasions during three years, and some students were interviewed. They found that students were making the same errors as reported in earlier studies (Booth, 1984; Küchemann, 1981; Wagner, 1981). Also they (Stacey & MacGregor, 1997) found that the difficulties students have in learning to use algebraic notations, appear to have several origins. The authors found evidence that “intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system” caused problems in interpreting and using letter symbols correctly.

‘Interference from new learning in mathematics’, was one such obstacle for some students, especially the older ones. The researchers found evidence that students wrote the right expression or equation, however, misused newly learned rules for solving equations. Another example was students’ use of exponential notation to express multiplication, e.g. x^3 instead of $3x$.

‘Poorly-designed and misleading teaching materials’ was another source for problems with letter symbols. In some schools, students’ performance was relatively poor compared to the others. In these schools the authors discussed with the teachers and found for example that textbooks used in the first algebra course, explicitly stated that letters could be used as abbreviated words and labels.

In one school, they found that students interpreted letters according to their order in the alphabet. The teachers told that they had been working with puzzles and codes in the mathematics lessons. The authors characterised this source of mis-interpretations as: “analogies with symbol systems used in everyday life, in other parts of mathematics or in other school subjects”.

In this section, it is shown that many students have problems in their interpretation and use of letters (the variable) in algebra, and Küchemann (1981) in the CSMS study categorised these different interpretations in six categories. Other studies reported the same problems as was found in the CSMS study.

It was also shown that many students considered that functions and equations changed by changing only the letter variable (Wagner, (1981).

Booth's teaching experiment was promising in that a focus on specific problems helped some students to resolve some of their problems. Also the more recent intervention project, ICCAMS, reported a positive development in students' conception of algebra (Brown, et al., 2014).

Later in this thesis, when referring to students' work, the notion 'variable' will be used only in accordance with Küchemann's definition of the variable. The reason for this is that the notion 'variable' was only used in class in relation to linear function, and that observation done when students were working with functions created a question if all students had the conception of the letter symbols as variables even in that context.

4.2 Lack of closure

One important difference from arithmetic to algebra is that expressions can be the final result in the transformational activity. Collis denoted this as the 'acceptance of lack of closure' (Collis, 1972). In the literature it is reported that many students show a reluctance to accept this lack of closure.

Tall and Thomas (2001) use the example of $7 + 4$ in arithmetic and the expression $7 + x$ in algebra to describe this difference between arithmetic and algebra. Although $7 + 4$ is an open operation or a procept in the vocabulary of Gray and Tall (1994) (see section 3.3), it has a built-in process of computing the numbers to reach to a number result. When it comes to the algebraic expression this is not possible without knowing the value of x . The last expression therefore has to be left open. For students it is not easy to accept this lack of closure. For many this tends to lead students to conjoin the terms, which means that they for example regard $3 + 4n$ to be equal to $7n$, or that they alternatively might assign values to the letter symbol (Küchemann, 1981). Several researchers have reported these phenomena (Booth, 1984; Brekke, 2005; Chalouh & Herscovics, 1988; Küchemann, 1981).

The reluctance among students to accept open expressions is more likely in the beginning of the algebra learning.

4.3 The minus sign and negative numbers

Negative numbers and the meanings of the minus sign are introduced in the lower grades. However, “it is in the transitional process from arithmetic to algebra that the analysis of students’ construction of negative numbers becomes meaningful”(Gallardo, 2002, p. 189). In algebra students are faced with equations and problems where negative numbers are coefficients, constants or solutions.

Studies carried out in Mexico among students in secondary school (age 12-14 years) have shown that students have difficulties in accepting negative solutions when solving linear equations (Gallardo, 1995, 2002; Gallardo & Rojano, 1994). Also students were seen to engage in a lot of acrobatics to avoid negative solutions (Gallardo & Rojano, 1994).

In a study in Sweden, Kilhamn (2011) concluded that students who had a well-developed conception of natural numbers, incorporated negative numbers more easily than students with a poorly developed number sense. She recommended that students were offered more opportunities to develop general number sense and that more time should be spent on subtraction and zero on lower levels.

Gallardo and Rojano (*ibid*) found that the students invent different rules to combine the operation signs for addition and subtraction, some are correct, others lead to errors. Double signs such as $-(-a)$ or $+(-a)$ cause difficulties, and the authors report that the students expressed surprise when they were confronted with the fact that subtraction does not always make smaller; that adding does not always make bigger, and that multiplication is not always repeated addition.

Gallardo (2002, 2003) used historical tasks for students to solve. In their solutions many students avoided using negative numbers where it was possible to figure out the solution without it. This should not be a surprise. This Sfard (2007) took into consideration when designing tasks for her study in which negative numbers were the focus for teaching. She noted:

The choice of the mediators was made carefully, so as to ensure they would not be treatable in terms of the “old” (unsigned) numbers more easily than in terms of the new ones (as is the case with the majority of real-life situations supposedly supporting the use of negative numbers; e.g., questions about changes in temperatures do not, in fact, necessitate manipulating negative numbers) (p. 586).

In Sfard’s study (*ibid*) investigating the introduction of negative numbers in a grade 7 class it was evident that it was hard for students to come to grips with operations on negative numbers.

However, for students the notation and the use of negative numbers is one obstacle, another is the minus sign. When studying the influence of using the balance model in equation solving Vlassis (2002) found the

same difficulties as reported by Gallardo and Rojano (1994). She then went on to consider the problems not so much to be related to the concept of negative numbers as to students' conception of the minus sign. She used the term negativity (Vlassis, 2004, 2008) to represent the multidimensionality of the minus sign.

She categorises the different uses of the minus sign according to the classification given by Gallardo (2002): the unary, the binary and the symmetric functions, and in addition the categorisation from Sfard (2000): operational and structural signifiers. She proposes that the overview of the different uses and functions of the minus sign, illustrates how limited the focus is, when focusing only on the negative number as such.

The unary function is related to the minus sign signalling that the number is a negative number, also called the isolated number. Vlassis (2008) follows Gallardo (2002) in the division of this category into two subcategories, the solution number and the result number. The solution number occurs as a solution to an equation or a problem, when the result number is a result of an operation.

The binary function of the minus sign is the sign for subtraction; an operation sign. This again is divided into two main categories: the arithmetic subtraction which involves completing, taking away and finding the difference between two numbers (Gallardo & Rojano, 1994). The other subcategory is algebraic subtraction. The reason for this category is the change in the meaning of the operation when negative numbers are involved, more generally she (Vlassis, 2008) defines it to be subtraction with rational numbers.

The last function of the minus sign is the symmetric function. Here the operation is about taking the opposite of a number or of a sum. She uses the example of $2a - (4a - 5b + 2c)$. Here the first minus sign is an operation to take the opposite. When talking about relative number, the minus sign is a structural signifier used in examples like $(-x) + x = 0$. In the relative number she explains the minus sign to be part of the number, not representing an action as is the case for taking the opposite of a number. In the table below, the different functions and examples of the minus sign are presented:

Table 4-1: Interpretations of the minus sign

Unary function (structural)		
Notion	Example	Reference
Relative number	$(-x) + x = 0$	(Gallardo, 2002; Gallardo & Rojano, 1994; Vlassis, 2004, 2008)
Solution number	$x + 7 = 4 \Rightarrow x = (-3)$	
Result-number	$15 - 17 = (-2)$	
Formal negative number	$(-5), (-4), \text{etc.}$	
Binary function (operational)		
<i>Arithmetical subtraction</i>		
Taking away	John has 3 marbles, he loses 2. How many does he have left? $3 - 2$	(Gallardo, 2002; Gallardo & Rojano, 1994; Vlassis, 2004, 2008)
Completing	John wants 6 marbles, he has 2. How many does he need? $2 + ? = 6$	
Difference between two numbers	$16 - 14 = 2$	
<i>Algebraic subtracting</i>		
Subtracting negative numbers	$7 - (-3) = 7 + 3$	(Vlassis, 2008)
Symmetric function (operational)		
Taking the opposite of a number	$-(-3) = 3$ $2a - (4a - 5b + 2c)$ (the first minus in the expression)	(Gallardo, 2002; Vlassis, 2004, 2008)

In her study Vlassis (2004) tested students in grade 8 from six classes in three schools in the French speaking part of Belgium. On the basis of their results on a test where they had to simplify 28 polynomials, 12 students were chosen to be interviewed. No polynomials consisted of more than 4 terms, with only one letter symbol in each. In the interviews students were asked to solve tasks from the tests and were asked which models they used.

Vlassis reports that a widespread type of reasoning was to put imaginary brackets around like terms if those were preceded by a minus sign. An example of this was the polynomial $20 + 8 - 7n - 5n$. Some of the students then operated on this expression as if there was a bracket after the first minus sign. The result was then $20 + 8 - (7n - 5n) = 28 - 2n$. Also high achieving students made this mistake, This was also reported in a study by Herscovics and Linchevski (1994) when investigating equation solving, and was also found when students should simplify number strings (Linchevski & Livneh, 1999). The phenomenon was then classified as one type of ‘detachment of a term from its indicated operation’. Vlassis (2004)

found that this reasoning occurred also in tasks where like terms were not grouped initially, but then after students had grouped them. She labelled this phenomenon ‘bracket reasoning’.

Another tendency was to use ‘the signs rule’ especially in the version minus by minus gives plus in not appropriate situations. Vlassis (2004) found that when several minus signs were present in a polynomial, this rule was used. An example is $6 - 5a - 3 - 4a$ where one student said that it would be $5a + 4a$ because of the signs rule, and then she wrote $9a$ minus and then she said that it would be the same $6 + 3$, and the result should be $9a - 9$. It was also found that one student could use ‘the signs rule’ in one occasion and ‘bracket reasoning’ in another, it depended on the numbers and the structure of the task.

There was also evidence of confusion in selecting correct operation sign (ibid). Vlassis found two forms of this incidence. One was students’ tendency to ‘operate from right to left’. One example was the expression $7 - 6n + 13$. Some students subtracted 7 from 13 and then subtracted $6n$ getting the result $6 - 6n$. Vlassis reports that this also was found in the written tests. This tendency she assumed to have its source in students’ avoidance of subtracting a larger number from a smaller one. Another tendency caused by confusion in selecting the correct sign, was that students ‘chose the sign following the number or term’. This tendency was labelled ‘jumping off with the posterior operation’ by Herscovics and Linchevski (1994) and Linchevski and Livneh (1999). One example of this tendency is $6y - 20 + 3y - 12$. Here some of the students subtracted $3y$ from $6y$ in that they use the minus sign following directly after $6y$. Vlassis (2004) hypothesised that the reason for this may be due to students early learning of subtraction with natural numbers, and she uses the example of $6 - 4$ and says if children are told to take away four apples from 6 apples then it is possible that the minus sign is seen to be associated with the term placed before it.

When asked which meaning the minus sign had, all students agreed that the minus sign generally is used for subtraction. Three out of twelve interviewees thought of the minus sign as ‘a sign to split’ the polynomial. And one thought it was there in order to signal the use of the signs rule. None of the students said explicitly that the minus sign could have more than one status. 10 students out of twelve considered the minus sign in the start of a polynomial to be the sign of a negative number.

A striking result was that the high achieving students although performing the manipulations in a correct way, when explaining their strategies, their explanation were poorly described, or they had no explanation. Vlassis (ibid) comments that students were not used to talk mathematics,

and she asserts that this is a sign of a lack of meta-conceptual awareness, which has to be developed in order to use the minus sign flexibly.

Another study carried out by Vlassis' (2008) involved eight grade 8 classes from two schools. All students in these classes were given a test, and on the basis of the results 17 students were chosen for interviews. In the interviews students should solve equations and then prove the solutions to be correct. The students had four possible methods to solve the tasks; to substitute the unknown directly by a number, or to do the inverse operations. Vlassis (ibid) calls these methods arithmetical in line with Filloy and Rojano (1989). The last two methods involve using formal methods, one is to perform the same operation on both sides of the equal sign, and the last method is a shortcut of the latter; to move terms from one side to the other side, changing sign.

The tests and the interviews showed that students had difficulties in distinguishing between the different meanings and uses of the minus sign. The students were given the tasks shown in the table below. These tasks were basis for the interviews.

Table 4-2: Interview tasks and results (Vlassis 2008 p. 563)

Equation	Solution	Structure	Solution ($n = 17$)		
			Correct	Incorrect	Omitted
1 $4 - x = 5$	Negative	Additive	8	7	2
2 $-6x = 24$	Negative	Multiplicative	3	14	-
3 $-x = 7$	Negative	Multiplicative	10	2	5
4 $12 - x = 7$	Positive	Additive	12	5	-
5 $-32 = -8y$	Positive	Multiplicative	6	11	-
6 $5x = 40$	Positive	Multiplicative	14	3	-

It is worth noticing that no student used brackets when presenting negative solutions.

Vlassis reports several difficulties. One is related to the situations where two successive signs emerge, when proving the solution of the equation. It was especially visible in the tasks including multiplication such as in the equations $-6x = 24$ and $-x = 7$.

The first problem she claims, is when there are two consecutive signs as for the multiplication sign and the minus sign in $-6 \cdot -4$. Those students who did not succeed, did not identify the unary function of the minus sign in the solution number - 4. The two signs were seen to have the same status. Some students were helped when Vlassis suggested to write brackets.

Some students corrected their solution number during the interview, while one student asserted he had problems with negative solutions. He could see "how many times 6 goes into 24, and it's 4. But with the minus

I don't feel sure. I often don't feel sure about minuses". Another student did not accept the negative solution suggested by the interviewer. He as well as the others found that 6 goes into 24, 4 times, but "since there is the minus, it can't be done". They failed to see the unary function of the sign attached to the solution (- 4).

Another problem with this task was that some students omitted the multiplication sign. The task turned out to be $-6+x = 24$. The invisible multiplication sign led students to turn the multiplication into a sum. This was actually done by several students and was reported to be the main difficulty in this task.

In the task $-x = 7$, students' problems were that there was a minus sign in front of the x , and that the solution was a negative number. Some succeeded by what Vlassis called the meaningless technique of bringing the minus from the left side over to the right side; from the x to the 7. For those students there was only one minus sign which was moved from one side to another.

Thus they were stuck when they tried to substitute x with (-7). When prompted they came to the result $-7 = 7$. Vlassis (ibid) comments that it helped when she suggested to put brackets around negative 7. One student interviewed meant that $-x$ should be replaced by -7. When prompted he at the end wrote $-7 = 7$, arguing that this was not possible. He had said that minus minus was plus but hesitated to write it. At the end he admitted that he should have used a bracket, because in algebra they had learned to use brackets.

In this task both minus signs were interpreted to be unary, but when substituting the unknown with the solution number, the opposite value⁹ is to be found which means that the first sign has a symmetric function. Vlassis (ibid) relates the problem again to the fact that most students can't see the different functions of the two minus signs.

This fact has been basis for other studies as well, and some researchers have studied students' conception of the 'opposite number'. Gallardo and Hernandez (2005) suggested to focus on the number zero and the number line to enhance students' conception of the symmetric function of the minus sign.

In a rather small study of high school students' conceptions of the minus sign (Lamb et al., 2012b), nine students were interviewed on the basis of several tasks including the minus sign. All nine students seemed to have grasped two of the three main meanings of the minus sign; the minus sign as an operator for subtraction, and the minus sign as a symbol for the

⁹ The notion opposite number will be used for the mathematical notion additive invers in this thesis. This notion, in Norwegian 'motsatt tall', is used synonymously with 'additive invers'.

negative number. Some of the students also moved flexibly between these two meanings. The tasks for students to solve are listed in the figure below:

- | | |
|---------------------------|--|
| 1) $5 + \square = 4$ | |
| 2) $5 - \square = 8$ | |
| 3) $\square + 5 = -2$ | |
| 4) $\square - 5 = -1$ | |
| 5) $\square - 5 = -8$ | |
| 6) Compare $--4$ and -4 | |
| 7) Compare $-x$ and x | |
| | 8) In one textbook we found this definition of 'absolute value'. |
| | $ x = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ |
| | Can you explain this |

Figure 4-1: The tasks given to the students (Lamb, et al., 2012b, p. 40)

The five first tasks were solved without problems, however, the next three happened to cause more struggles than expected. Two of the tasks given to the students will be reported here. In task number 6: “Compare $--4$ and -4 ?” (ibid, p. 40). Six of the nine students compared the two expressions correctly, but five of those referred to the rule that two minus signs make a positive sign and were unsure why the rule works. Only one of the nine students described explicitly that $--4$ was opposite to -4 .

In the other task, “Compare $-x$ and x .” (ibid, p. 40), five of the nine compared the expressions correctly. Four of those five referred to the rule of two negatives make a positive also here. When doing so they tested different values of x . The four students comparing incorrectly asserted that x is always larger than $-x$, because negative x always represents a negative number.

The authors conclude that students mostly move flexibly between the meaning of the minus sign as an operator for subtraction and as a sign symbolising a negative number. This flexibility helped students to efficiently solve the first tasks. However, in the study it was found that those shifts were taken implicitly and that students struggled to explain their executions.

The authors assert that a conception of the minus sign also as a unary operator, taking the ‘opposite of’ might promote a deeper understanding of the variable, algebraic expressions, and symbolically represented definitions. They also suggest that students must be given opportunities to discuss and develop a sense of the different meanings of the minus sign in order to be flexible and to develop a robust concept image of the sign. In another report they suggest tasks and actions for teachers to promote this symbol sense (Lamb et al., 2012a).

Another type of error reported by Vlassis (2008), was related to tasks in which there was a number from which the x should be subtracted as in the equation: $4 - x = 5$. The 4 was subtracted from both sides, or moved over and changed sign. Then the equation should be $-x = 5 - 4$. The minus preceding the x , however, was omitted. It turned out to be the case that the binary sign for subtraction was no longer needed so now it could be removed. The students were not aware of the possibility that the same sign could change status. Here it changed from being a binary sign to a unary sign for the negative x .

To sum up, negative numbers as well as the minus sign cause students a lot of struggles. The different meanings of the minus sign are not easy for students to be aware of, especially since there is no difference in the symbolisation. There are three main functions of the sign: the minus sign as an operator signalling subtraction, as a sign for negative numbers and as a sign for taking the opposite number. In addition, there are occasions in which the same sign changes function, as in the equation $4 - x = 5$ when it is transformed into $-x = 5 - 4$ as in one of the examples above.

Students are normally taught to write negative numbers in brackets, however, when a minus sign is placed in the start of an equality or a polynomial, the negative number is mostly not written within a bracket. Therefore, it should perhaps be no wonder that students are not consistent in placing negative numbers in brackets when substituting letter symbols with negative numbers. As seen above this omission of brackets often causes students to change operation. In cases where two successive minus signs appear as in example $5 - (-4)$, it is often just rote learned that two minuses give plus. This rule is often incorrectly generalised to be used also in examples as $6 - 5a - 3 - 4a = 9a - 6$ as shown above.

In a longitudinal study in Sweden in upper secondary school (Persson, 2005) it was found that one important obstacle when transforming algebraic expressions and solving equations was students' lack of competence in arithmetic operations and especially a lack of understanding the different functions of the minus sign. In that study it was found that it was crucial that the teacher was searching for the reason for the errors made. It was not obvious to see that the problem could be the minus sign or negative numbers. The researchers observed that when the obstacles were addressed, it was possible to overcome the problems.

The type of problems reported in the studies of students' manipulations with polynomials will be applied as categories in the analysis of similar tasks.

Table: 4-3: Minus sign - Problem categories

Categories		Examples
Detachment of a term from the indicated operation ¹⁰	From right to left	$7 - 6n + 13 = 6 - 6n$
	Jumping off with the posterior operation	$6y - 20 + 3y - 12 = 3y - 32$
	Bracket reasoning: The minus sign as a splitting factor	$3 - 5 + 4 = (-6)$ $20 + 8 - 7n - 5n = 28 - 2n$
Sign rules - overgeneralised		$6 - 5a - 3 - 4a = 9a - 6$
Taking the opposite of		$-(-3)$ $2a - (4a - 5b + 2c)$
Omitting brackets – change operation		$-6 \cdot -4 = -6 - 4$
Not acceptance of division by negative numbers		$\frac{24}{-6}$ is impossible

In this section research revealing problems students have when working with negative numbers and the minus sign, have been presented. These problems are closely related to the phenomenon of operating according to the correct ‘order of operations’. In this thesis it is chosen to group all problems related to the the minus sign together, although many of those problems cause students to operate in an inappropriate order.

4.4 Order of operations

From early school years students are familiar with carrying out the computations from left to right, however, addition and multiplication are commutative operations, and in strings of numbers within those fields it is no problem to compute also from right to left. When introduced to the operations of subtraction and division this switch is no longer a possibility. And in problems including multiple operations, the students have to reflect on which operations have to be carried out first.

Herscovics and Linchevski (1994) had experienced as Vlassis (2004) that students calculated from right to left. In Herscovics and Linchevski’s study 22 grade 7 students were interviewed about equation solving. All were from the same class in a school in Montreal and none of them had followed algebra courses. During the interviews they experienced that students when solving the equation $364 = 796 - n$, first transformed the equation to be $n - 796 = 364$. They assumed the reason was that the students had not developed a relational conception of the equal sign, and commented that if the signs had been positive, it would not have mattered if the direction was changed.

¹⁰ The notion ‘detachment of a term from the indicated operation’ is taken from Linchevski and Livneh (1999).

They also found ‘bracket reasoning’ (Vlassis, 2004) (see the section above). In Herscovics and Linchevski’s (1994) terminology this was called ‘detachment of a term from the indicated operation’.

Another problem occurred when students performed addition before multiplication. A large proportion, nearly 30 %, of the students failed in the equation $6 + 9 \cdot n = 60$ because they performed the addition before multiplication. However, in similar tasks in which there was a minus sign instead of the plus sign, the students performed the operations in the correct order. This was explained to be caused by the students’ view of the minus sign as a splitting factor.

In a follow up study Linchevski and Livneh (1999) interviewed 53 students at the end of grade 6. In this study the tasks were designed with the same structure as in the equations, but now as pure numerical strings; no letters included. These students had been taught about order of operations.

The authors found the same problems as in the work with equations. In the expression $5 + 6 \times 10 = ?$, the task was performed by adding 5 and 6 before the multiplication. When there appeared to be a minus sign such as in the task: $17 - 3 \times 5 =$, this happened for only three students, the others correctly carried out the multiplication before the subtraction. This may be due to an interpretation of the minus sign as a splitting factor resulting in bracket reasoning (Vlassis, 2004) as seen above. However, they (Linchevski & Livneh, 1999) also found that when there were both division and multiplication in the expression $(24 : 3 \cdot 2)$, the multiplication was performed first, and in tasks with both addition and subtraction $(27 - 5 + 3)$ the addition was performed first.

Linchevski and Livneh (ibid) interpret these errors to be caused by students generalising the rules for order of operations incorrectly. They had learned that the order of operations is given as: brackets first, then multiplication, division, and then addition and subtraction. The researchers suggest that one reason could be that students interpret the rule to mean that addition and multiplication were on the same level, and therefore it did not matter which was performed first, and that the same was the case with division and subtraction. One could choose what was most convenient. Another suggestion was that the students had interpreted the rule to mean that addition took precedence over subtraction and multiplication over division.

However, the task with both subtraction and addition $(27 - 5 + 3)$, is similar to those referred to earlier in which the phenomenon was called the ‘detachment of intended operation’. For Vlassis (2004) when she asked students about it, they explained that the minus sign was seen as a

splitting sign; as a bracket, and what comes after, has to be operated on first. It might be that students have the same view of the division sign.

Linchevski and Livneh (1999) do not consider such an interpretation here, but later on when reporting other problems in the same article not involving order of operations, they use the examples $24 : 3 \cdot 2$ and $27 - 5 + 3$ to make visible what they call detachment of the term from its indicated operation. Some students had then even put brackets around $5+3$ and $3 \cdot 2$.

It might be that the authors do not see this bracket reasoning as a reason for the problem, but as a description of the phenomenon only.

Linchevski and Livneh (1999) asked the students in the interviews if there were other ways of solving the problems, showing them alternative student solutions. Many students changed their minds, also from correct to incorrect, some kept to their solution. Others accepted both solutions, and meant that one task could give other correct solutions; it depended on the rules.

Linchevski and Livneh (ibid) used the findings to create an intervention program “teaching arithmetic for algebraic purposes” (Linchevski & Livneh, 2007). The aim was to help students to develop structure sense. The results indicated that students who were assumed to have low scores in algebra, progressed in that they avoided making the mistakes reported in the other studies. It was also observed that the other students improved from the instruction program.

Another problem related to order of operations occurs when powers are included in strings of numbers or in algebraic expressions. In the example $2 \cdot 3^2$ some students tend to multiply the base by two before executing the exponentiation.

In higher grades there have been studies about order of operations (Lee & Messner, 2000). College students were reported to discuss the meaning of -3^2 ; the negative number with an exponent without grouping. Some argued that $-3^2 = 9$ rather than $-3^2 = -9$. The argument they gave was that the minus sign should be considered before the exponent.

4.5 Powers – base and exponent

Powers are mentioned in relation to order of operation in the foregoing section. However, the concept of power and what it means for the students, has been in focus for some studies (Pitta-Pantazi, Christou, & Zachariades, 2007; Weber, 2002).

Traditionally students have been introduced to the definition of powers as repeated multiplication of the base. This definition is meaningful as long as the exponents are natural numbers, but can cause troubles when

the powers are more complicated with negative exponents or with fractions as exponents (Weber, 2002).

For this thesis, the basic definition holds for most occasions, however, the students are introduced to scientific notation in which they have to expand their conceptions of exponents from natural numbers to include all numbers within the set of integers, \mathbb{Z} .

In a study (Cangelosi, Madrid, Cooper, Olson, & Hartter, 2013) identifying persistent errors students make when working with powers, and teasing out the reason for those errors, the researchers tested 904 students at a university and then interviewed 18 of them.

Students were asked to decide if some powers would be positive or negative and also to simplify them. Two persistent errors were found. One was what they called the ‘sticky sign’. The expression was -9^2 . When calculating the power, 13 out of 18 treated the expression as if it was in an imaginary bracket, their conception was that the minus sign was ‘stuck’ to the number 9, whether there was a bracket or not. The authors claim this is caused by a lack of a conception of the ‘additive inverse’ or as Vlassis (2008) labelled it; the opposite number.

According to the authors, the other error was due to a lack of the concept of the multiplicative inverse. The power was 2^{-3} and 10 out of 18 failed. Alternative solutions were $2^{\frac{1}{3}}$, -2^3 , $\frac{2}{1/3}$, $\frac{2}{3/1}$, and $\frac{3}{2}$. Some students’ conception of powers with negative exponents was that something should be flipped. However, what to flip and why, was not clear to them. They talked about flipping; not to find the multiplicative inverse of 2^3 .

The authors recommend that textbook authors and teachers should incorporate the additive and multiplicative inverses, when introducing students to the general laws for addition and multiplication.

One error, somewhat related to the ‘sticky sign’, was that some students were reported to equal ab^2 and $(ab)^2$ (Seng, 2010).

Errors occurring when calculating with powers are often caused by interferences from other symbol systems according to MacGregor and Stacey (1997). One example is the tendency shown by some students to mix exponentiation and multiplication. MacGregor and Stacey (ibid) refer to some students’ conceptions of exponents as an instruction to multiply, which causes them to write the multiplication as a power with the multiplier as the exponent. This confusion between what was ‘met before’ and new learning is also reported by de Lima and Tall (2006a), to cause problems when working with powers. This might cause students to multiply the base by the exponent.

4.6 Algebra and fractions

In this section I will present some research on the concept of fraction as a rational number. In the next sections research on students work with fractions will be presented. Then research on simplification of algebraic rational expressions will be reported.

Fractions are often experienced by students as one of the stumbling blocks in mathematics. The concept of fraction is complex, and according to Kieren it includes five different sub-constructs which represent five different “rational number thinking patterns” (Kieren, 1980, p. 134). These sub-constructs are: part-whole, quotients, measures, ratio and operators. To form the full rational number concept, the learner must control these different constructs. However, too often students have learned algorithms and have not been offered opportunities to go in depth into the concept of rational numbers. A well-known example of this is Benny (Erlwanger, 1973) who is enjoying his work with fractions, but, is using different rules for different calculations with fractions.

Kieren (1980) emphasises that the different thinking patterns are not mathematically independent, nor are they psychologically independent. They are closely related, but represent different ways of thinking. According to Kieren (*ibid*), the part-whole sub-construct is historically the main basis for students’ development of the fraction concept. This sub-construct is according to Marshall (1993) related to situations in which continuous or discrete quantities are divided into equal parts. The fraction $\frac{a}{b}$ is then seen as a comparison between the part and the whole. The numerator represents the actual number of units in the part, and the denominator represents the number of units in which the whole is partitioned. In part-whole situations $a \leq b$. Although, the units are made smaller or bigger, the part-whole relation is conserved. This unitising gives basis for the understanding of equivalent fractions (Lamon, 1999).

The quotient sub-construct is closely related to the part-whole sub-construct in that it depends on partitioning. According to Marshall (1993) the difference is that in part-whole situations a and b in the fraction $\frac{a}{b}$ yield the same object or thing. In quotient situations a and b represent different things or objects. The situation represents fair sharing/division of a elements into b groups. One example is “How would you share three pizzas among four friends? How much would each person receive? (*ibid*, p. 275).” Lamon (1999) comments that in such a context the fraction $\frac{3}{4}$ has multiple meanings. It stands for the division 3 divided by 4, and is at the

same time the result of that division, the quotient. However, it might also be interpreted as a ratio, pizza per person. In a quotient situation there are no constraints regarding the size of the fraction.

According to Marshall (1993) the ratio sub-construct is representing a situation in which two quantities are related to each other, not representing a partition of one object. Some number of one object is compared to a number of a second object. One example of such a situation is: “In her recipe, Susie adds 1 cup of sugar for every 3 cups of water. How much sugar should she add for 6 cups of water?” Lamon (1999) asserts that the situation describes ordered pairs and that it implies a proportion. In the fraction $\frac{a}{b}$, a change in a will cause a predictable change in b .

For students this means that they have to grasp that the relationship between the two quantities is constant, and that a change in one of them implies a change in the second. Lamon emphasises that although students seem to have grasped the concept of equivalent fractions, it does not mean that they have a valid conception of the invariance property of ratios.

Marshall (1993) describes the measure situation to be a situation in which a fraction $\frac{1}{b}$ “is used repeatedly to determine a distance”. According to her, the most frequent representation is the number line, alternatively the ruler. Lamon (1999) claims that: “It is unlikely that any other fraction interpretation can come close to the power of the number line for building number sense” (p. 228). In addition, Lamon asserts that students are scaffolded by fractions as measurement to develop understanding for operations on fractions. She refers to her studies which show that students who started learning fractions through the measurement sub-construct of rational numbers, had a good conception of the relative sizes of fractions and were more likely to discover errors when operating on fractions, than students who had started out from other sub-constructs.

The last subconstruct, fraction as operator, is linked to the function concept by Marshall (1993). The operator serves as a function machine, mapping some set or region onto another set or region. For students it is understood as shrinking and enlarging, contracting and expanding or multiplying and dividing (Lamon, 1999). Lamon (ibid) uses the example of the operator $\frac{2}{3}$. This operator instructs one to take $\frac{2}{3}$ of something, which means to divide by 3 and multiply by 2, or it might be regarded as one single operation. The result will be the same no matter in which order the calculations are made.

When discussing the different sub-constructs and approaches in teaching fractions, Lamon concludes that the different interpretations do not give the same access to a deep conception of fractions or rational numbers. She emphasises as does Kieren (1980) that the sub-constructs are intertwined and one construct can enhance another if the teacher is aware of the different interpretations.

Recent research has shown that there is a strong predictive relation between fraction knowledge and later mathematical achievement (Bailey, Hoard, Nugent, & Geary, 2012; Siegler, Fazio, Bailey, & Zhou, 2013). Siegler et al. (ibid) applied two national longitudinal data sets; the ‘British Cohort Study’ from 1980 and 1986 and the ‘Panel Study of Income Dynamics - Child development Supplement’ (PSID-CDS) from 1997 and 2002. In the first study 3677 children born in the United Kingdom during one week in 1970 formed the dataset. The latter dataset included 599 students from USA. The students in both datasets had been tested twice; first as ten years old and then as 15-17 years old. Analysis of the data showed that knowledge of fractions among students in high school correlated strongly with overall mathematics achievement, also with algebra. Knowledge of fraction and their overall mathematics achievement correlated stronger than algebra with overall mathematics achievement.

Across the dataset from both countries, elementary school children’s fraction knowledge and whole number division predict their mathematics performances in high school. This was the result even after controlling for IQ, reading abilities, working memory, family income and education, and knowledge about whole numbers. Bailey et al.’s (2012) study confirmed this result.

Wu (2001) suggests that fractions could be the gateway to algebra and the concept of variable, since the rules for operating on fractions can be generalised. He also claims that for students when meeting fractions, they are introduced for the first time to the world of abstractions (Wu, 2009). Both Wu (ibid) and Siegler et al. (Siegler, Thompson, & Schneider, 2011) propose that fractions should be dealt with as numbers located on the number line. This location on the number line is the property fractions have common with whole numbers. One advantage of applying the number line according to Wu (2009), is that it is possible to present both proper and improper fractions in conceptually similar ways, and that equivalent fractions have the same location on the line. He proposes that defining “a fraction as a point on the number line is a refinement of, not a radical deviation from, the usual concept of a fraction as a ‘part of a whole’” (ibid, p. 12). Siegler et al. (2011) proposed an integrated theory of numerical development which seems to be in accordance with the ideas in the work of Wu (2009). The main point is that “numerical development in-

volves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines” (Siegler et al., 2011, p. 274). The researchers propose that knowledge of the magnitude of fractions would help students in algebra to consider if solution numbers are reasonable.

Although there has been a large amount of research reports on algebra and students’ problems in the topic, only a few have focused on the connection between students’ algebraic reasoning and their knowledge about fractions. However, researchers have claimed that proficiency in dealing with fractions is an important prerequisite in algebra learning e.g. (Brown & Quinn, 2007a, 2007b; Kilpatrick & Izsak, 2008; NMAP, 2008; Wu, 2001). Also researchers have reported that one problem in algebra learning is that students have problems with fractions (Hoffer, Venkataraman, Hedberg, & Shagle, 2007).

4.6.1 Students’ problems with ordinary fractions

There is a long history of research on students’ work with fractions and on their conceptions of rational numbers (Eichelmann, Narciss, Schnaubert, & Melis, 2012). In upper secondary school it is expected that students should have control over and master operations on fractions. However, studies show that also students at higher levels struggle (Brown & Quinn, 2006, 2007a, 2007b). In upper secondary school when working with algebra, students are not asked to transform mixed numbers into improper fractions or opposite. Fractions larger than 1 mostly appear as improper fractions, however, the students have to solve tasks in which there are both whole numbers and fractions. It is also expected that they can transform fractions into whole numbers. This is not obvious for all students.

Eichelmann et al. (ibid) refer to (Shaw, Standiford, Klein, & Tatsuoka, 1982) who found that some junior high school students did not simplify fractions with equal numerator and denominator; they were not transformed into the number 1. This was seen at the end of adding fractions, and according to the author the reason might be that the students simply forgot to do it. It was also found that some regard fractions of type $\frac{1}{a}$ to be equal to a . No reason is given for this. The students were not asked to explain their reasoning.

Padberg (1986) found in his study of 861 students in grade 7 that one of the main problems for students was to work out tasks in which there were both fractions and whole numbers. The types of such tasks are:

$n + \frac{a}{b}$, $\frac{a}{b} + n$, $n - \frac{a}{b}$, $n \cdot \frac{a}{b}$, $n : \frac{a}{b}$, $\frac{a}{b} : n$. There were two main categories of ways to solve the problem. One was to substitute the whole number n by

$\frac{n}{n}$, or to treat the whole number as if it were a numerator with the same denominator as the closest fraction. The same was found in a study among 9 and 10 graders (Brown & Quinn, 2006). Padberg (1986) claimed that the students just followed rules; formal or invented. In many of these tasks it should have been simple for students to check if their results were appropriate, but they did not.

When adding and subtracting fractions with both equal and unequal denominators, most students succeeded in Padberg's study (1986). In Brown and Quinn's study (2006) of 9 and 10 graders, however, 48% did not find the correct sum of $\frac{5}{12} + \frac{3}{8}$. In both studies the most common error was to treat the numerators and the denominators as distinct whole numbers and follow the 'rule' $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$. This 'rule' was explicitly expressed by some students in Padberg's study (1986). It is seen as a logical extension of the natural number system. Padberg found that this phenomenon increased if the addition task followed after a multiplication task, which means that some students generalised the structure in the algorithm of fraction multiplication to fraction addition.

Quinn and Brown asked the students in grade 9 and 10 to find the product of $\frac{1}{2}$ and $\frac{1}{4}$. Only 42 % found the correct product. 26 % misapplied the algorithm for multiplying fractions. Some of them (nearly 6% of all in the study) added the numerators and multiplied the denominators. Others (the same amount) had found the least common denominator before executing the multiplication. The main error found by Wittmann (2013) when testing 428 German students in grade 6 and 7, was of this type: $\frac{a}{b} \cdot \frac{c}{b} = \frac{a \cdot c}{b}$. It was more likely that the tasks were correctly solved when the denominators were unequal. The same was found in the earlier study worked out by Padberg (1986). He reported that this error was more likely to occur if addition/subtraction tasks were solved before the multiplication tasks. Padberg also found that some students solved multiplication tasks in this way: $\frac{a}{b} \cdot \frac{c}{b} = \frac{a \cdot c}{b+b}$. They multiplied the numerator and added the denominators. One reason, he suggested, might be that the students interpreted b multiplied by b to be equal to $2b$. When the denominators were unequal, some students tended to expand one or both to reach to

equal denominators. After this expansion some followed the erroneous path as with the equal denominators (Padberg, 1986; Wittmann, 2013).

In Padberg's study students were asked to respond to the question: "Why do we multiply fractions in exactly this way?" None of the students in the study provided any answer. It seems to be that the rule was just memorised.

In division tasks Padberg (ibid) found that the same structure was used as for multiplication: $\frac{a}{b} : \frac{c}{b} = \frac{a:c}{b}$. In his study 20 % did this at least once, while 4 % did it in all division tasks with equal denominators. One of the tasks was $\frac{4}{5} : \frac{2}{5}$. The researcher had expected that the students could solve this problem intuitively, but that was not the case. They tried to rely on rules. But it was obvious from the way the rule for fraction division was applied, that the rule gave no meaning for many of the students. Some students inverted both fractions, and some inverted the first one. Also for division; none of 681 students could give any reason for the rule and only few could formulate the rule in words. From the article, however, it has not been stated if the students have been taught the reason for the rules.

Brown and Quinn (2006) asked students to reduce the fraction $\frac{24}{36}$ to the lowest factors. The result was that 9 % could not demonstrate any method for reducing it, and 9 % did not reduce it to the lowest factors. 73% of the students reduced it correctly. These were 9th and 10th graders. Padberg (1986) reports that it was more likely that the students simplified fractions if simplification was the aim of the task. More often it happened that result or solution numbers were not reduced. The reason was suggested to be that the students forgot to do it, because simplification was not in focus, and their feeling was that the task was done.

In their study, Brown and Quinn (2006) found that some (4 %) solved the addition task in this way: $\frac{5}{12} + \frac{3}{8} = \frac{5}{12} + \frac{3+4}{8+4} = \frac{5}{12} + \frac{7}{12}$. These students expanded the fraction by addition, not seeing the fraction as a ratio; an indication that those students lack the conception of equivalent fractions.

When fraction tasks were formulated as word problems many students had problems. As with decimal numbers (Fischbein, Deri, Nello, & Marino, 1985) students had problems choosing the correct operation (Brown & Quinn, 2006; Wartha, 2009).

In a longitudinal study in Germany Wartha (ibid) focused upon student's understanding of fractions, and the development of their concep-

tions from grade 5 until the end of grade 7. A pencil and paper test was administered among 2000 students from these three school years, then 36 students were interviewed on the basis of their results on the tests. Wartha (2009) gave students word problems to solve. The students understood that the amounts should be smaller and many students chose division or subtraction, this happened both with unit fractions and other fractions. The researcher claims that the reason for this is the over generalisation of the properties of division and subtraction within the set of natural numbers, where those operations ‘make smaller’. The same was found in word problems in Brown and Quinn’ study (2006).

Another frequent error found, was that some students avoided ordinary fractions and converted the fraction into decimal numbers, which made the solution process even more complicated.

Wartha (2009) also found that many students do not regard fractions as numbers, and this causes problems when trying to compare fractions. The study confirmed that although many students are able to apply the algorithmic rules for operations on fractions, it does not mean that they have a well-developed concept image of fractions as rational numbers.

Wittmann (2013) when studying the consistency of error patterns in operations on fractions, found four distinct groups of students.

The first group consisted of students who solve all fractions tasks with one operation resulting in a correct result. Out of 428 students in seventh grade between 25 % and 33 % belonged to this group. Within this group there are two sub-groups. One contains the students who solve the problems in a flexible way which means that they take the number into consideration, and execute mental computations when that is convenient. The other sub-group represents students who apply one approach, whether it is the most appropriate method or not.

The second main group represents students who consistently show the main error patterns. Wittmann (ibid) asserts that those students might have “internalized an incorrect procedure while performing automation exercises”. The students in these two first main groups represent students who are consistent in their work.

The third group consists of students who mainly proceed well, but occasionally make mistakes. The researchers assume that the reason for the inappropriate approaches might be due to careless mistakes caused by intuitive forms of error patterns, by numbers fitting into an “error pattern”, or caused by the sequencing of the tasks.

In group four are students who are not consistent at all. Wittmann suggests that the reason might be that the students’ choices of approaches are dominated by the given numbers.

The studies reported above indicate that although students meet fractions in early school years, some will still have problems and have a rather limited conception of rational numbers even when entering upper secondary school. In upper secondary school, the algorithmic rules are just recapitulated, and students are expected to move on to more advanced mathematics. Wittmann's study about consistency in carrying out fraction tasks, shows that at least in German seventh grade there is a group of students that is consistent in applying error patterns, and one group in which the students have no clear approach at all. Both groups need help and guidance in applying algorithms, but even more so to gain a concept image of rational numbers that can help them to see the reason behind the rules.

4.6.2 Students' problems with algebraic fractions

There are many studies showing that students have problems simplifying algebraic fractions or strings of fractions. The older studies are focused on typical errors found in testing, not so much on students reasoning when solving the tasks. In more recent studies students have been interviewed on the basis of tasks, or they have been asked to reflect on the solution process. Some studies in this area will be referred to in this section.

The simplest algebraic fractions consist of monomials in both numerator and denominator. For some students it is a challenge to find common factors, for others it is a problem to interpret powers in the correct way. In a study investigating which problems students at grade seven in Malaysia had when simplifying algebraic fractions, it was found that some students interpreted expressions of the type ab^2 to be equal to $(ab)^2$ or to a^2b (Seng, 2010). This will of course lead to problems with fraction simplification. Another misinterpretation of the exponents in powers, is to view them as ordinary numbers. Storer (1956) found that some students when simplifying the task $\frac{a^2}{ab}$ seemed to equal this with $\frac{2}{b}$. He asserted that the error might be caused by "striking without leaving a trace", which means that they had cancelled the literal symbol. The index was left, and then the students had written the index as an ordinary number.

Storer and his colleagues at the university of Birmingham tested more than 1000 students from the age of 14 in different English schools. The test included 52 algebraic fractions which should be simplified. Students who had less than 20 correct answers were left out and this means that tests from 539 students were analysed. The errors were organised into 15 categories. He found that the most frequent category of errors was that the students did not find the common denominator or numerator, one example was $\frac{1}{b} + \frac{1}{d} = \frac{1}{b+d}$. Another frequent error was categorised as partly can-

celling. There were different forms of this. Examples are: $\frac{3}{6a+b} = \frac{1}{2a+b}$, $\frac{3a-b}{b} = 3a-1$ and $\frac{a^2}{ab} = 2b$. Other categories are those where the generalisation of addition of whole numbers into the area of rational numbers, results in the addition of numerators and/or denominators. One example from the test is: $\frac{1}{b} + \frac{1}{d} = \frac{1}{b+d}$. Another problem revealed in half the responses, was to deal with whole numbers and fractions in the same task, which was also a problem in arithmetic (see the section above). One example is: $1 + \frac{a}{b}$. Here 62 % of the students made mistakes. Another mistake made in 8 out of the 52 tasks was to involve subtraction in simplifying fractions. One example is: $\frac{5a}{a} = 4a$. This error was made by only few students.

Lee and Wheeler (1989) found that a large number of the students in their study executed 'partly cancelling'. They tested 354 students in grade 10 and then interviewed 25 of them. One task in the test used in the interviews was the statement: $\frac{2x+1}{2x+1+7} = \frac{1}{8}$ in which students should decide if the statement was definitely true, never true or possibly true. In addition, they should justify their answers. Generally, students concluded that the statement was definitely true, and justified it by telling they cancelled the terms $2x$. The students who cancelled were asked to put in a numerical value for x , but none of them were able to recognise the solution of $x=0$, and that the statement was true for this value.

Hall (2004) conducted a study in a Bermuda school. He interviewed 5 students in order to investigate the reasoning behind the strategies students chose when simplifying a complex algebraic expression. The author claims that he chose this area for two reasons. He had experienced that students have problems and make errors when simplifying algebraic expressions, and he had found that little research is reported on this topic compared to the topic of equation solving.

180 students were asked to simplify the expression: $\frac{x^2 + 3x - 10}{x^2 + 2x - 8}$. The students were in grade 9, 10 and 11. Only 41 of those students (23 %) gave a correct answer, 44 (24 %) simply cancelled the x^2 in both numera-

tor and the denominator and stopped there. 20 (11 %) factorised correctly but stopped without simplifying the expression.

The focus for the study was five students who factorised $\frac{(x+5)(x-2)}{(x+4)(x-2)}$ correctly, and correctly cancelled the factors $(x-2)$ before they cancelled the x -es and reached the solution: $\frac{x+5}{x+4} = \frac{5}{4}$.

These five students were interviewed the next day based on their solutions. The questions posed were:

- 1) Why did you factorise initially?
- 2) Why did you cancel the $(x-2)$'s ?
- 3) Why did you cancel the x -es ?
- 4) If you felt it was sensible to cancel the x -es, why not cancel the x^2 at the beginning?
- 5) Do you now want to cancel the x^2 at the beginning?
- 6) Free discussion based on responses. (ibid p. 6)

Hall (ibid) found that the students justified their factorising of the quadratic expressions by saying that they knew or remembered that those should be factorised. The x squared seemed to be a trigger for this, although one student said he tried to cancel, but then he did not know how to go further on and then factorised. On the question why they cancelled the brackets, four claimed them to be the same, and that they therefore had cancelled them. One student told that she cancelled each term in the bracket. It was not possible to be aware of this from the written solution.

The reason for cancelling the x -es was also that they were the same, or the same on the top as on the bottom. On the questions why they did not cancel the x^2 , since they had cancelled the x -es, there were different answers. This was the question the author had expected to be the most revealing for his focus of investigation, however, "the consensus was that one simply did not cancel x squared at the beginning albeit for a variety of reasons which had more to do with closure and ease of simplifying than with the meaning of the algebraic terms" (ibid p. 12). On question 5, three of the students claimed it would be the same if they cancelled the x^2 in the first line, while one said it then would be another solution, and she knew that $\frac{4}{5}$ was the right one. The last one told that if she cancels x squared, she cannot get any further.

It seems as if Hall (ibid) is searching for evidence of students' problems of accepting the lack of closure, and also evidence to place the students in levels according to Blooms taxonomy (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956), which he at the end finds difficult.

He concludes that there is a variety of rationalisation behind the process of simplifying rational expressions, and that not all of these are visible in the written material.

Olteanu (2012) worked with teachers in a developmental study in Sweden. The teachers were asked to express what they expected to be difficult for students when learning how to simplify algebraic rational expressions. They supposed that students would have problems in perceiving the expression as a whole, and in understanding the relation between parts and the whole, and that for the students there could be different wholes. The difference between terms and factors they presumed students knew, and that only factors would be cancelled. However, test results showed that students struggled to see the difference between factors and terms. The result was that the teachers focused upon what in their view were the problematic issues for students, and ignored what was the really critical aspect for students. The students, however, could not perceive the whole as long as they did not understand the components in the expressions and the relationship between them.

The next phase in the study was a follow-up on one of the teachers. In the next class he taught, he focused on the critical issues, such as the components, not the whole in the expressions. He focused on the difference between factors and terms, on identifying common factors in numerator and denominator, and on the difference between expressions and equations. In addition, it was shown how brackets around the numerator and the denominator could help highlight the whole, and that polynomials had to be factorised. In addition, during classroom discussion, he kept on asking:

What does factorising look like for a polynomial expression?; How do we know when we are finished factorising?; What is the process we use to cancel?; What does cancelling look like?; When do we know we are finished cancelling?(*ibid*).

It is reported in the article that the result is an essential improvement of students' learning and a successful communication in the classroom. It might though be questioned if this learning was on the procedural level of learning only.

In a study building on the instrumental approach (Guzmán, Kieran, & Martínez, 2011), the researchers asked the question: Which technical and theoretical aspects are promoted [or emerge] in students' thinking by the use of CAS and an activity designed with a technical-theoretical approach to the simplification of rational algebraic expressions? (*ibid*, p. 482). Eight students in grade 10 in Mexico simplified algebraic rational expressions in pairs. The students worked out some paper-and-pencil tasks before solving the same tasks with CAS calculators. The students were observed during the work session and interviewed. After solving the tasks with calcula-

tors, the students found that they had solved the tasks incorrectly. The tasks and their solutions are shown below:

$$1) \frac{x(3+x)}{x} = \frac{3x+x^2}{x} = \underline{\underline{3+x^2}}$$

$$2) \frac{4x+4y}{x+y} = 4+4 = \underline{\underline{8}}$$

$$3) \frac{3x+4y}{x+y} = 3+4 = \underline{\underline{7}}$$

The students discussed what might be wrong. In the first case they found that they had to cancel two x -es, and adjusted their solution, without factorising the numerator.

The next CAS solution, they found difficult to explain. After some discussion one student found that the task involved division of polynomials. The division was executed and the problem was solved. In the next task the calculator gave the same result as the original task, which showed the students that the fraction could not be simplified.

The researchers concluded that CAS technology could promote theoretical reflections by students, and also help them to find new ways of solving problems as in the example with the division of the polynomial. However, it did not lead the students away from cancelling the variables in the first task, without taking into account that they needed factors. The question was raised: “How to promote in students’ thinking a focus on seeing the expressions in terms of factors?” (ibid, p. 487). Thus, the researchers claimed that the teacher’s role is critical in order for students to take advantage of the CAS technology.

The studies referred above, show that students have had, and still have problems when the task is to simplify algebraic rational expressions. One critical aspect seems to be to make distinction between terms and factors. All the studies above confirm this. Many students still prefer to execute ‘partly cancelling’ in the terminology of Storer (1956). Guzmán, Kieran, and Martínez (2011) reported that all students in their study multiplied the premultiplier x by the bracket before simplifying the first task: $\frac{x(3+x)}{x}$.

This means that they did not have a sense for the structure in the expression (Hoch & Dreyfus, 2004; Kieran, 1992), which is required in order to see what could be done.

Olteanu’s (2012) study shows that when the teachers became aware of what was difficult for their students, they could help by illuminating the critical aspects for them.

Yantz (2013) reported from a study in which the focus was to investigate the relationship between algebraic rational expressions and rational numbers. 107 undergraduate students at a university in the USA were asked to solve three sets of tasks. Each set included two tasks; one numerical and one algebraic. The tasks are shown below (ibid, pp. 37, 38 and 39).

Set A

$\frac{1+6}{2} + \frac{9}{2+3} - 3$	$\frac{x+2}{4} + \frac{6x}{x+1} - 3$
-------------------------------------	--------------------------------------

Set B

$\frac{3}{3-1} \div \frac{9}{9-1}$	$\frac{x}{x-1} \div \frac{x^2}{x^2-1}$
------------------------------------	--

Set C

$\frac{1}{4 \cdot 5} + \frac{3}{5 \cdot 7}$	$\frac{1}{x^2 - x - 2} + \frac{x}{x^2 - 7x + 10}$
---	---

In all the sets of tasks there was a big difference in the frequency of correct solutions between the numerical and the algebraic task, which is shown in the table below:

Table 4-4: Frequency table based on the findings p. 55 and 56 (ibid)

Correct (%)	Numerical	Algebraic	Both correct
Set A	48,6	6,5	3,7
Set B	37,3	6,5	5,6
Set C	41,1	5,6	2,8

In analysing the solutions, the author found the same errors and mistakes that are reported in the studies above. The students in Yantz' study struggled with both the numeric expressions as well as with the algebraic. Less than 14% of the participants answered one or more of the algebraic rational expressions correctly, while 69,2 % answered correctly on one or more of the numerical tasks.

On the basis of statistical analysis of the material the author concludes: A correlation between participants' abilities with algebraic and numeric rational

expressions was found only in the division problem set, and then it could only be categorized as a small effect. The small correlation seen between numeric and algebraic division operations in this study may be related to the consistency with which participants applied procedures in both contexts. The absence of medium or strong correlations between the algebraic and numeric items would suggest that although the participants were more likely to get a numeric item correct and the corresponding algebraic item incorrect, there was no relationship between their abilities in both contexts. Since Hiebert and Carpenter's (1992) framework tells us that algebraic procedural knowledge is connected to conceptual knowledge of number properties, these results suggest that very few participants in this study had made the connection between the two contexts (ibid p. 157).

Eight students were interviewed on the basis of the tasks. Three of them used the same strategies in both tasks in the sets, and expressed that they related algebraic rational expressions to rational numbers. This connection was not found or was found to be weak, by the other students which confirmed the quantitative results from the data.

The author recommends, as does Olteanu (2012), that it is important on higher levels not to assume that the students have a well-developed concept of rational numbers and algebraic rational expressions. Yantz (2013) recommends that in order to create connections between them, it is important not to assume that the students will develop new knowledge and make connections by presenting new material. Instead students should be given the opportunity to develop a conceptual understanding of fractions before entering into new fields of mathematics.

In this chapter the different sub-constructs of fractions have been presented. Also according to research it is claimed that there is a strong correlation between fraction knowledge and high achievement in general mathematics including algebra.

In the two last sub-sections former studies are reported that reveal problems students have working with both numeric and algebraic fractions. These reports show that fraction problems are highly present also in higher grades, and it is recommended that teachers at higher levels take these problems into account and provide new opportunities for students to develop viable conceptions in this area.

4.7 Interpretations of the equal sign

The concept of equality is fundamental to mathematics. From primary school it is represented by the equal sign. After the initial introduction it is handled as a matter of course, and it is rare that it is explicitly the subject of later instruction. However, research has shown that many students at all levels have not developed an adequate conception of the equal sign.

The equal sign is a convention chosen by mathematicians to represent the concept of equality. However, from primary school, children are used

to seeing the equal sign as a sign which demands computation (Behr et al, 1976; Falkner, et al., 1999; Kieran, 1981). All the studies referred to above, state that the meaning of the equal sign is restricted to the context in which it is learned; the context of arithmetic. In arithmetic the interpretation of this symbol as an operator symbol holds for most occasions.

Behr, Erlwanger, and Nichols (1976) were among the first to study students' conception of the equal sign. They carried out clinical interviews of 6 to 12 year-olds and found that: "There is a strong tendency among all of the children to view the = symbol as being acceptable in a sentence only when one (or more) operation signs (+, -, etc.) precede it. Some children, in fact, tell us that the answer must come after the =." (ibid, p. 10)

Arithmetic is about calculation of numbers. Many times this calculation involves several steps on the way to the answer, and it is possible to find examples of students in all grades who write their solutions in this way: " $15 + 20 = 35 + 5 = 40 + 1 = 41$ " (Saenz-Ludlow & Walgamuth, 1998, p. 165). Saenz-Ludlow and Walgamuth claim this to be caused by the arithmetic calculation from left to right, and a view of the equal sign as a 'stop' or 'punctuation' on the way to reaching the goal; that is the answer. Such examples of writing are evidence of not seeing the equal sign as a sign for a relation between the right and the left side of the equal sign. When dealing with algebra, the interpretation of the sign has to be extended to be understood as a relational symbol; including the reflexive, symmetric and transitive properties (Kieran, 1981).

To move from arithmetic to algebra requires an extension in the interpretations of the equal sign described above, which helps the students to also read from right to left (Herscovics & Linchevski, 1994). Kieran (1981) reported from a study of 12 to 14 years old students. These students when asked to explain what they meant by the equal sign, clearly showed evidence of interpreting it in terms of signalling the answer. Also they limited themselves to have one operation on the left side and the answer on the right. Kieran (ibid) proposed that many of students' difficulties when dealing with polynomial expressions, were due to a limited conception of the equal sign. Students found it difficult to give meanings to expressions like $3a$, $a + 3$ and $3a + 5b$, because there was no "equal sign with a number after it" (ibid p. 324). This is clearly connected to the reluctance to accept the lack of closure (see section 4.1).

Other studies show that even some college students, university students, and student teachers have not developed the conception of the equal sign as a sign for equivalence (Clement, 1980; Grevholm, 2003). It might be that some students even at these levels tend to "interpret the equal sign in terms of an operator symbol, albeit at a more sophisticated level, rather than as a symbol for an equivalence relation" (Kieran, 1981, p. 325).

Alibali, Knuth, and Hattikudur (2007) followed 81 students in a three years period from grade 6 to the end of grade 8. The students were all students from an ethnically diverse middle school in the American Midwest. The students were tested at four points in time. In the tests they were asked to tell what meaning they gave to the equal sign. In addition, they should decide if pairs of equations were equal. The number of students who provided a relational understanding of the equal sign increased from grade 6 to the end of grade 8. Also the number of students who used a “recognize - equivalence” strategy in determining whether the equations were equivalent or not, increased, compared to the students who had to solve the equations in order to make a statement about equivalence. The researchers also claimed to have found evidence that students who early on developed a relational understanding of the equal sign, were more successful in solving the equivalent equations problems.

A study (McNeil et al., 2006) about the presentations of the equal sign in four mathematics textbooks used in the USA, and a follow-up study on students in the grades 6, 7, and 8 about their interpretation of the equal sign, indicates that the context in which the equal sign occurs seems to influence students’ interpretation of the equal sign. The results showed that students when presented with equalities in which operations occurred on both sides of the equal sign, were more likely to show a relational interpretation of the equal sign. The authors therefore were concerned about the low percent of occurrence in the textbooks where the equal sign was presented with operations on both sides as in the equality $3 + 4 = 5 + 2$. They concluded that students might hold both interpretations, however, the relational interpretation was for some students only activated in the context of equalities with operations on both sides of the equal signs. The same result was shown in other studies (de Lima & Tall, 2006a; McNeil & Alibali, 2005). This may not mean that all students able to give a relational interpretation on some occasions, have grasped the full concept of equivalence. Studies have, however, shown that relational understanding can be enhanced also in the early school years (Carpenter, Levi, Franke, & Koehler, 2005 ; Falkner, et al., 1999; Whitman & Okazaki, 2003).

4.8 Linear equations

In this section a formal definition for equations will be presented. Then comes a section with review on research revealing students’ conception of equations and equivalence. Thereafter research revealing students’ problems related to equation solving will be presented.

Although equations are presented very early for students in the learning of algebra, one can ask: What is an equation and how is it defined? The Concise Oxford Dictionary of Mathematics offers this definition:

An equation is a statement that asserts that two mathematical expressions are equal in value. If this is true for all values of the variables involved then it is called an identity, for example $3(x-2)=3x-6$, and where it is only true for some values it is called a conditional equation; for example $x^2-2x-3=0$ is only true when $x = -1$ or 3 , which are known as the roots of the equation (Clapham & Nicholson, 2009).

This definition is not clear about expressions with only numbers, but another definition for an equation is: “A mathematical statement of the form “ $A = B$ ”; A is the left member (or side), and B is the right member (or side). When A and B are expressions containing variables, then the equation is an *identity in case it is ...*” (Karush, 1962, p. 94). In this latter definition it is clear that also statements with only numbers as in the example $3 + 4 = 5 + 2$, are in mathematics defined as equations.

Herscovics and Kieran (1980) chose to use the term arithmetic identities when no literal symbol was involved; this in order to avoid confusion for the students, and to make distinction between algebra and arithmetic. The notion equation they used only for algebra.

In the classroom during observation for the research reported in this thesis, the notion equation, was applied only in the algebraic context with specific unknown(s). Thus the notion equation in this thesis will be applied as in Herscovics and Kieran’s (ibid) study.

4.8.1 Conceptions of equations

Some researchers (de Lima & Tall, 2006a, 2006b; Dreyfus & Hoch, 2004; Godfrey & Thomas, 2008) have asked students about their conception of equation. De Lima and Tall (2006b) analysed students’ concept maps about equations. Altogether 117 students from 4 classes in Brazil participated. The students were from 15 to 16 years old and they came from three different schools.

The teachers in each class wrote the word equation on the blackboard, asking each student to give at least one word coming to mind, when seeing this word. In the analysis, the researchers found that many of the words were the same in all classes despite differences of schools and ages. Words like calculation, number, addition, subtraction, multiplication, division, signs, solution, number, unknown, rules, and results led the authors to conclude that “it is likely that students who had suggested these words, see an equation as a calculation in which it is necessary to use rules to find the solution, the value for the unknown” (ibid, p. 2).

There were words in all classes relating equations to the subject of mathematics, and to what is going on in the school. No words related equations to real-world problem solving. There were also no word indicating the balance of two sides in an equation, although the teachers told that this metaphor was applied in the classes.

Only in one class, the word equals was mentioned, but that word was put in categories named symbols, sign and operations. Only two classes mentioned the word unknown. However, the other two classes included letters in their concept maps, and one student said “ x and y ”. One class had the word “variable”, but when grouping the words, the word variable was in 4 out of 8 groups put in categories not related to mathematics at all, indicating that although the word is mentioned, it might be that they do not see the role of the variable, the unknown or letters in equations. Two of the classes had written words representing emotions, like fear, panic, happiness and wish. There was no evidence that the formal meaning of equivalence was familiar to the students.

In another report the investigation was taken further (de Lima & Tall, 2006a). 77 students from three classes participated, and 15 of them were interviewed. These students were also asked about what an equation is. The most frequent response to that question, was that it has to do with calculation, and/or that it is about to find the x . 21 out of 77 students mentioned the unknown, however, none said anything explicitly about the equal sign. The researchers conclude that this might indicate that the students do not see the equal sign as part of the equation, and thus hold a rather operational interpretation of the sign.

In an investigation (Godfrey & Thomas, 2008) students at different levels in the school system were asked to describe what is an equation. In addition, they were asked to describe how to decide from a group of expressions what are equations, and what are not. One task was afforded in which they had to show if they could recognise and use the symmetric, reflexive and transitive properties of equivalence.

The study was carried out in Australia, and three groups of students participated. One group was from lower secondary school, grade 10 (29 students), one group from upper secondary school grade 13 (76 students), and one group from the first year of university studies (30 students). In the latter group, 25 of 30 students studied engineering.

The students were asked to pick out equations in a questionnaire. Those equations, differed according to the grades. Both algebraic expression without an equal sign, conditional equations and identities were included.

The result was that most students did not see equations as equivalence relations, also not the first year university students. Among the other students there was a large group of students whose conception of equation was tied to procedures; an operation was requested in addition to the equals sign. For a minority, but significant group of students this need of an operation overruled the request for an equal sign. Only students at the

university level showed signs of “moving towards a ‘formal world thinking’ about equality” (p.86).

The researchers conclude that a large group of students in lower and secondary school have a ‘strong sense of procedural necessity’ (Godfrey & Thomas, 2008, p. 84), and had not developed a relational understanding of the equal sign. They claim that the development from ‘embodied world’ thinking to ‘symbolic thinking’ and then to the ‘formal mathematical’ thinking might be a slow process, and in this process students’ conception of equation changes gradually. From their study they assert that approximately 25% of the students during transition to university still have an input/output view or a procedural view of equations. Also they found that students’ conception and ability to recognise and use the reflexive, transitive, and symmetric properties develop slowly.

At the end, the authors claim that one reason for students not to develop a relational view on the equal sign, is that the properties for equivalence are used by teachers without mentioning them explicitly. They give the example $x + 6 = 3x + 1$ and say that if teachers just transform it into $2x + 1 = 6$ instead of $6 = 2x + 1$, then they are using the symmetric property implicitly, leaving to the students to abstract this themselves. The same is the case with the other properties. The advice is given that teachers should help students by being explicit about structures and properties of equations.

Dreyfus and Hoch (2004) who were interested in structure sense (see section 3.3) asked some Israeli students about their conceptions of equation. They organised the responses in five groups. The two first were about exercises and doing, which they related to a pure procedural conception of the equation object. The two next were about the form, how to recognise an equation on the basis of the surface structure. Only one type of response they found to be related to an internal structure of equations: “Two sides connected by an equals sign and certain rules for solving” (ibid p. 153). The clue here was the mentioning of some ‘certain rules’. However, there was no student referring to the mathematical properties. The authors conclude that if these responses are typical for high school students, then structure is not in the “realm of awareness of high school students” (ibid, p. 153).

Equivalence of algebraic expressions is at the heart of transformational work in algebra (Kieran & Saldanha, 2005, p. 193), and is the basis for the transformations of one equation to another in the solution process of equations. For students who still have an arithmetic conception not recognising the internal structure and the properties of equations, still holding a conception of the equals sign as an operational sign, it will be problematic to grasp the concept of equation.

Another study (Steinberg, Sleeman, & Ktorza, 1991) reports that many students competent in equation solving and in transforming algebraic expressions, seemed to have no conception of equivalence. In their study Steinberg, Sleeman, and Ktorza (ibid) asked 98 students in the USA in grade 8 and grade 9 to decide if pairs of equations were equivalent. In addition, they were asked to justify their decisions. These students had learned to solve linear equations, by adding or subtracting the same number on both sides of the equation. However, the teachers reported they had not emphasised explicitly the notion of equivalence. One example of the tasks given was the pair of equations:

$$x + 2 = 5 \qquad x + 2 - 99 = 5 - 99$$

The results were grouped in three, due to the justifications. One group included the students who had to solve one or both equations to decide if they were equivalent. The next group based their justifications on transformations of the equations. Some did this on immediate recognition of the second equation as derived from the first, or by transformation of one to fit the other. In the third group the students had given incorrect reasons. Some were based on the surface structure; one justification was that the equations could not be equivalent since one was longer than the other.

Almost a third of the students applied mainly calculations to justify their judgements. The researchers concluded that this result indicates that “many students are not sure that an equation that has been derived by a valid transformation, has the same solution, or they are unable to recognize when an equation has been transformed in a way that does not alter the answer” (ibid, p. 119).

In a study (Ball, Pierce, & Stacey, 2003) the authors had designed a computer test; the ‘Algebraic Expectation Quiz’. Students were shown pairs of algebraic expressions in a limited time, and should decide if the expressions were equivalent or not. 50 students were tested twice; in grade 11 and 12. The authors concluded that to recognise equivalence, even in simple cases, was revealed to be a significant problem (ibid).

Knuth, Stephens, McNeil, and Alibali (2003) found that there is “a strong positive relation between middle school students’ equals sign understanding and their equations-solving performance” (ibid, p. 309). They agreed with Kieran (1992) who argued that “one of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric and transitive character of equality- sometimes referred to as the ‘left-right equivalence’ of the equal sign” (p. 398). This means that the students in order to have a well-developed conception of equation, need to understand what these characters or properties mean.

4.8.2 Equation solving

Some researchers have claimed that there is a “didactic cut” (Filloy & Rojano, 1989) or a “cognitive gap” (Herscovics & Linchevski, 1994) between arithmetic and algebra, and that this demarcation is visible when students go from solving equations of the form $ax + b = c$ to solving equations of the form $ax + b = cx + d$. Filloy and Rojano found that students solved the problems arithmetically when having only one single unknown, and that there was a shift when the unknown was present on both sides of the equal sign.

Herscovics & Linchevski (1994) questioned that the delineation between arithmetic and algebra only was a question of the form in which the equation was presented. They asserted that the mathematical processes involved in the activity of solving the tasks, were of importance. When studying 7 graders before being taught any algebra, they found that the students performed rather well. Their result showed that most of the students were also able to solve equations with the unknown occurring on both sides of the equal sign. The same was found by Tall and de Lima (2006a). On the basis of these findings Herscovics and Linchevski (1994) claimed the cognitive gap to be characterised by students’ inability to operate with or on the unknown. Their result also indicated the need to expand the meaning of the equal sign.

Equation solving is about transformations. Then it is important to understand that these transformations are producing new equations and that the truth-value has to be conserved (Cortés & Pfaff, 2000). According to the relational function of the equal sign it is possible to execute all arithmetic operations, as long as they are performed on both sides of the equal sign. The main point is to choose effective operations to reach the aim; to find the value of the unknown(s).

Most textbooks in lower secondary school instruct students to add or subtract on both sides of the equal sign, before the students are told that this is the same as ‘change side-change sign’. This last rule, the action of transposing, is often said to be a short-cut of the rule ‘do the same operation on both sides’. Kieran, however, asserted: “The procedure of performing the same operation on both sides of an equation emphasises the symmetry of an equation; this emphasis is quite absent in the procedure of transposing” (Kieran, 1989, p. 51).

Cortés and Pfaff (2000) investigated 45 students (17 years old) in grade 10 in a technical high school in France about their processes of equation solving. The data were data from tests, data from recorded group work, and data from individual interviews. The data was collected from two different classes in two subsequent years.

The researchers report that the students used self-justified rules such as “A term passes to the other side by changing sign” and “the coefficient of the unknown passes to the other side by dividing” (ibid, p. 195). The students were grouped into three categories due to their justifications for their solutions:

- a) Students that justified transformations as being identical operations on the two sides of the equation.
- b) Students that provided arithmetical justifications for transformations used in processing numbers.
- c) Students that did not provide any mathematical justification for transformations (ibid, p. 195).

There were only few students in group a) 1 or 2 per class. They were confident in their work also when it came to negative coefficients; they divided by the negative number on both sides of the equal sign.

Some students used different rules; one for multiplication and one for addition. They could say that “a term passes to the other side by changing sign” and “multiplying or dividing on both sides”. Those students came up with the correct properties of an equation when becoming aware of their own justifications. And they correctly answered the question about what is conserved in the solution process.

About half the classes (group b); justified the transposition of numbers by for example saying that addition becomes subtraction and so on, and that the coefficients are transferred to the other side by dividing.

The third group in which students gave no acceptable justification, involved nearly half the students.

There was one interesting student quoted from that group:

Mathieu: $2x - 4.2 = -5.4$; that makes $2x = -5.4 + 4.2$

Teacher: Why "plus" 4.2?

Mathieu: You transpose it and the "minus" sign is transferred as a "plus" sign.

Therefore,... in any case, I learned it like that. I do it automatically. In my head... that, for me, doesn't really mean anything.

Teacher: For you, writing "plus" 4.2 doesn't mean anything?

Mathieu: Here, I do it directly and I don't try to understand. I put the "plus" sign, and that's how I transfer the term.

Teacher: In your opinion, did you learn it like that?

Mathieu: In my opinion, no, I didn't learn it like that. But, for me, I've forgotten it (the justification), I do it automatically... (ibid, p. 198)

This is an example of a student performing operations without any mathematical justification and with no need for one. But he was efficient in his solution process. However, for some students this led to failures. One student solved the equation in this way: $4y + 49 = 93$ became $4y = 93 - 49$, then he made the following error: $y = 93 - 49 - 4$ (ibid, p. 198). He transferred the coefficient in the same way as the number terms.

The students in group b) had a tendency to make errors when the unknown was present on both sides, as in the task: $13x+28 = 5x +49$. This was transformed to $13x+5x=49-28$. And in another task $3x+15= - 50$ the transferring of the coefficient by dividing, led to this error: $x +15= - 50/3$. Some students had problems when the coefficient was negative, then they removed the minus sign from one side to the other side as a plus.

The authors conclude that the self-produced rules: ‘the coefficient is transferred to the other side by dividing’ and ‘you transfer a term to the other side of the equation by changing sign’ are effective for many students to solve equations of the type $ax + b = xc + d$ and some inequalities. However, those students do not necessarily understand the symmetric property of equations, and that the transformations conserve the initial ‘truth value’; the value of the unknown (Steinberg, Sleeman, & Ktorza, 1991).

De Lima and Tall (2006a) gave the students in their study some tasks to solve. The linear equations are listed below:

$$\text{a) } 3x - 1 = 3 + x \quad \text{b) } 5t - 3 = 8 \quad \text{and} \quad \text{c) } 2m=4m$$

When solving them, none of the interviewed students made any claim about doing the same operation on both sides of the equal sign. They talked about ‘change side - change sign’, and the reason they gave, was that it was the way to get to the correct answer.

These students as the students in the study reported by Cortes and Pfaff (2000), talked about shifting symbols and moving them around as mental objects. Some utterances were: “pick this number and put it on the other side of the equals sign”; “I take off the bracket”; “the power two passes to the other side as a square root” (ibid p. 239).

In task c) some students gave the response that the result would be $6m$, and then they found that m was equal to negative 6. The authors conclude that those students probably felt they had to do some operations. This is in line with Godfrey and Thomas (2008) who found that some students did not regard mathematical statements to be equations if no operation symbols were included.

Students were also given some second grade equations in de Lima and Tall’s (2006a) study. One task was:

To solve the equation $(x-3)(x-2)=0$ for real numbers, John answered in one line: “ $x=3$ or $x=2$ ”. Is the answer correct? Analyse and comment on it (ibid, p. 235).

This task was solved and explained in written form. Some students checked the solution by substituting the values for x and calculated. The most common response was to try to solve the equation and compare results. Only three students reached a correct solution.

The way the others solved this task varied, but most of them started directly on multiplying the brackets, only 6 did that correctly. Five students tried to use the quadratic formula. None of the students used the fact that ‘if the product of two numbers is zero, then one of them must be zero’. According to the authors the students relied on one algorithm, or they used ‘guess and check’, concluding that the students responded on a procedural level.

In conversations, the teachers told that they were concerned about the problems students have in algebra, and in order to help them they strongly emphasised the rules for equation solving. However, this emphasis on one rule for solving quadratic equations, seemed to hinder students in being flexible. The authors advice is to base the instruction on experiences that give meaning.

The balance model has been discussed by several researchers (Fillooy & Rojano, 1984, 1989; Pirie & Martin, 1997; Vlassis, 2002) and there is no clear agreement about the usefulness of it. Filloy and Rojano (1989) observed that the balance model, and also the geometric model they used in their study, did not allow the students to deal with the unknown value.

Pirie and Martin (1997) argued that the model makes no sense for modern students, because the scale is rarely in use, and it is therefore problematic to relate the model to real life. Another problem if students should accept the model, would be the action of taking away terms on both sides, the same was argued in de Lima and Tall’s study (2008). In their study, however, no student related their equations to the balance model.

Pirie and Martin (1997) asserted that to use the model would increase the complexity in grasping the concept of equations. They, however, emphasised the effectiveness of the balance model in explaining the meaning of the equal sign, and also the advantage of the model to see the whole equation at one time.

Vlassis (2002) in her study used the balance model. The result shows that where the scales are used, the students created a mental image of the manipulations to be carried out, and also a mental picture of the concept of equality. She expressed the need to emphasise that the model is just a way to demonstrate the principles of equality; the physical object, the scales, are not important. In her study students had comprehended this, and the author reports that the model had in no way created an obstacle for the students. The problem about removing-subtracting mentioned by Pirie and Martin (1997) occurred in Vlassis (2002) study as well, however only in the first lessons. In interviews with 5 students later on, it did not appear at all, and eight months after the experiment students still solved

the equations using the principles correctly by remembering the images of the scales.

Godfrey and Thomas (2003) emphasised in their study the need of being explicit about the properties of equations. In the figure below, they have presented what they see as parts constituting the mathematical equation object. The parts are the variables, the equals sign, the numbers, and the operators. Students' perspective on these parts influence their view of equations.

The figure below illustrates this framework:

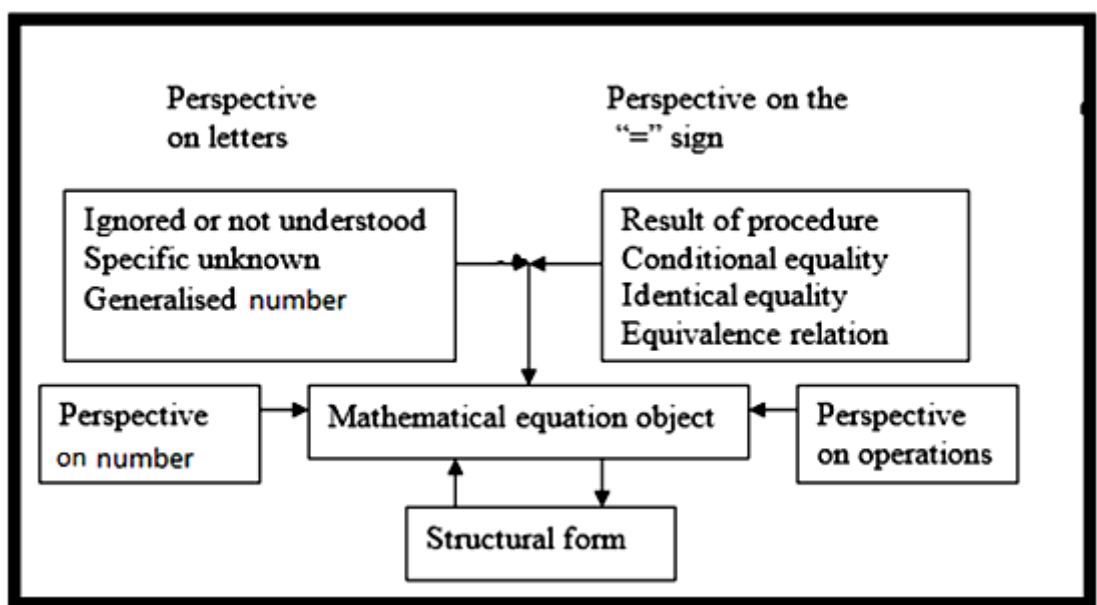


Figure 4-2: Factors influencing the perspectives on equations. Adapted from Godfrey and Thomas (2003, p.402)

The aim is not just to recognise an equation on the basis of its surface structure; that an equals sign and letter symbols are included. In addition, one has to grasp the mathematical object as an entity. This requires a well-developed conception of the letter symbols, both as specific unknowns and as general numbers (section 4.1), and that the perspective on the equal sign is expanded from being an operation sign to the point where the sign is linked with the properties for equivalence; the symmetric, reflexive and transitive properties (section 4.7). Students' perspectives on operations and number are also influencing on their perspective of the concept of equation, although the operations are mostly well-known.

Godfrey and Thomas (ibid) emphasise that the conception of equation is developing, as the perspectives on the different components in the model develop. Each of them influence on the perspective of the equation object as a whole. A consequence for teaching is to explicitly address and

explain the important properties of the equation object and its components.

4.9 Word problems in algebra

What is outlined above is research on students' equation solving. Many word problems are used in the learning of equations in order to lead students into algebraic thinking. The generational activity of generating algebraic expressions and equations from word problems is part of the meaning making activity when students start to learn about algebra (Kieran, 2004b). Another dimension of this activity is the benefit of algebraic symbolism which might unload the cognitive work load in that a problem can be converted into mathematical symbolism, manipulations can be carried out, and then one can go back to the initial problem with the solution and interpret this in the context of the initial problem (see section 3.2).

For students it is often not easy to see the benefits of this way of solving problems in the early phase of algebra. In a brief overview of students' struggles when introduced to algebra, Kieran (2007b) mentions students' problems in converting word problems into algebraic equations. A large body of research shows that "students prefer to solve word problems by arithmetic reasoning rather than first representing the problem by an algebraic equation and then applying algebraic transformations to that equation" (ibid p.1). Also de Lima and Tall (2006b) found that no student seemed to relate equations to real world problems (section 4.8).

Another problem is that although students are confident working on, and manipulating mathematical expressions including letter symbols, it does not mean that they work algebraically. This was discussed by Janvier (1996). Although functions, formulas, and equations may look identical, the different interpretations of them call for different mental processes when solving problems. A formula calls for direct computation as in the formula for finding the area of a circle. If the formula has to be transformed to find the radius, a different reasoning is requested. Then one has to work on the unknown.

Bednarz and Janvier (1996) presented a detailed analysis of typical word problems presented to students in introductory algebra. Generally, they assert that if one can tease out the structure of a problem, then the quantities, known and unknown, and their relationships to each other become clear, open to the problem solver. In order to analyse tasks and present the structure, the authors made schemas. These schemas illustrate the nature of the relationships between the quantities and the links between them and the known and unknown quantities. They realised that the textbooks afforded tasks which could be categorised within three main classes

based on their structure; the quantities, (both known and unknown) involved and the relationships between them. The classes were: Problems of unequal sharing, problems involving a magnitude transformation, and problems involving a non-homogenous magnitude and a rate.

One example of the first class, problems of unequal sharing, they presented in this way:

380 students are registered in three sports activities offered during the season. Basketball has three times as many students as skating and swimming has 114 more students than basketball. How many students are registered in each of the activities? (p.118)

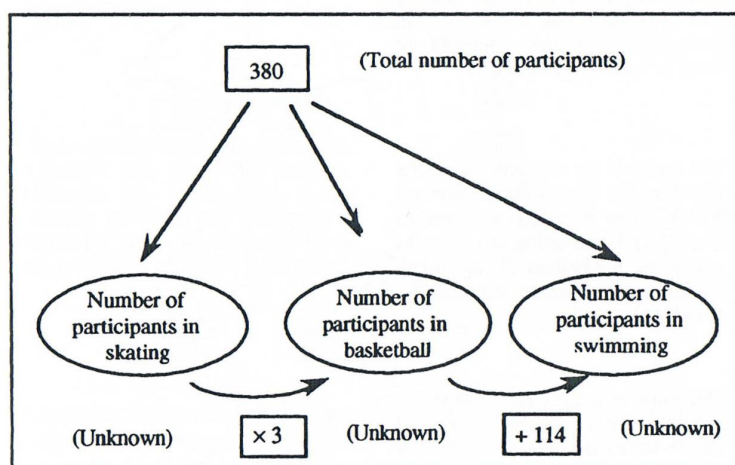


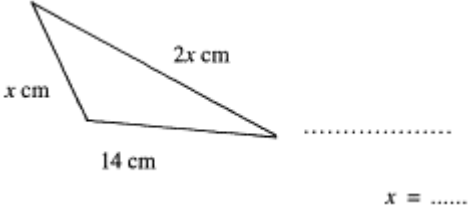
Figure 4-3: General structure of the problem (Bednarz & Janvier, 1996, p.119)

The figure illustrates that the number of students participating in the different types of sport, are linked to each other through the relationships: multiply by 3 and add 114. The known number 380 is the sum of all the unknown quantities. By means of the schema the complexity of the task is teased out. The authors analysed tasks in this way, and through this they could indicate which type of tasks were more problematic for students to solve.

In a study carried out by Stacey and MacGregor (1999) 900 Australian students from 12 schools were tested in solving word problems. In addition, 30 out of them were interviewed individually to illuminate their reasoning behind the written solutions. In the study report, four items and their solutions are presented. In all items students were asked to use algebra.

The problems on the test were simple problems, straight forward applications of linear equations, and the situations described in the problems should be familiar to the students. The students had already learned to solve linear equations.

Table 4-5: Word problems on test (Stacey & MacGregor, 1999, p. 154)

<p>1. Triangle The perimeter of this triangle is 44 cm. Write an algebraic equation and work out x.</p> 	<p>2. Mark Some money is shared between Mark and Jan so that Mark gets \$5 more than Jan gets. Jan gets \$$x$. Use algebra to write Mark's amount. The money to be shared is \$47. Use algebra to work out how much Jan and Mark would get.</p>
<p>3. Bus A bus took students on a 3-day tour. The distance travelled on Day 2 was 85 Km farther than on Day 1. The distance travelled on Day 3 was 125 Km farther than on Day 1. The total distance was 1410 Km. Let x stand for the number of Km travelled on Day 1. Use algebra to work out the distance travelled each day.</p>	<p>4. Number I think of a number multiply it by 8, subtract 3, and then divide by 3. The result is twice the number I first thought of. What was the number? (Hint: write an equation and solve it.)</p>

The first three problems lead to equations with the unknown on only one side of the equal sign. In the terminology of Filloy and Rojano (1989) these are arithmetic equations and should be easy to solve. The last one, the authors expected to be harder to solve, since in this problem the unknown would appear on both sides. However, the last problem was expected to be easier to convert into an equation since it can be translated phrase by phrase.

The authors report that for 10 graders (the majority of the sample) one third of the students wrote correct equations for all 4 problems. Not all students used the equations in their solutions, and some wrote the equations after they had solved the problem arithmetically. Students from classes in which word problems had been taught to be solved algebraically by using equations, were more likely to set up equations and also to solve them correctly. The authors take this as evidence for the claim that algebraic thinking does not develop spontaneously; it has to be taught.

All in all, a great variety of methods were used to solve the problems. Stacey and MacGregor analysed the different routes students went along to reach their answers. Although it was written in the text that students should use algebra, many students ignored it and solved the problems by arithmetical reasoning, or by trial and improvement. Others started by writing an equation, and then switched to arithmetical methods. The authors divided the different non-algebraic solution methods in three catego-

ries: ‘Arithmetic reasoning’, ‘trial and error’, and ‘superficially algebraic route’.

Many students solved all problems by arithmetic reasoning. In the first three problems they mostly succeeded. However, in problem 4 it was harder to reach a correct answer. In problem 2 (Mark), students who initially saw the problem as the division of a whole into parts, divided \$47 by 2 and then adjusted up and down by half of the \$5. Others followed this procedure: first, give Mark his extra \$5. This leaves \$42. Share equally, giving each \$21. Jan gets \$21. Mark gets \$26.

The category ‘trial and error’ was again divided into three sub-categories:

- a) Random: guessing answer in no particular order hoping to guess the correct answer sooner or later by chance
- b) Sequential: trialling numbers in sequence, as in a spreadsheet, hoping that the correct answer will be reached.
- c) Guess-check-improve: Carrying out one or more trials, using the result to choose the next one (ibid p. 157).

In the category superficial algebra, the authors put solutions in which students had written the equations as formulas. For the first three problems this is possible. One example was: $x = (1,410 - (85 + 125)) / 3$

The letter x was set to label the unknown; however, in this case the student did not use an algebraic approach working on the unknown. This is in accordance with how students solve formulas. A formula defines a procedure for calculation, while an equation “specifies the structure of an equality among variables” (ibid, p. 158). One student in the study expressed it like this: “It seems easy that way. You don’t have to rearrange it for working it out” (ibid p. 157).

The authors concluded on the basis of the test and the interviews that there were mainly two reasons for students not choosing an algebraic solution or route to the answer. Students often do not see the logic of formulating the problem into an equation, and for them “using algebra to solve a problem is an extra difficulty imposed by teachers for no obvious purpose” (ibid p. 165). The priority is to calculate the answer. This interferes with what they think an equation is, and also with what they think an unknown is. Students interviewed wanted to start finding the answer and were surprised to be asked to set up an equation. Their comments gave the impression that they did not find it helpful at all.

The other reason was that some students had not learned to transform word problems into equations and had to be helped during the interview. It was evident from the answers students gave, that they saw a separation between finding an answer to the problem and that of doing algebra. The authors concluded that it is crucial for students to grasp the concept of an

equation as a statement of equality, rather than as a formula, in order to be competent in using an algebraic method for solving problems. Another important issue is that many problems in textbooks which students are told to solve algebraically, can easily be solved by arithmetical reasoning or by trial and improvement. The tasks should be designed in order to let students experience the power of algebraic solutions.

For those students who saw the equations as useful on their way towards solving the problems, the researchers found that some students let the letter for the unknown stand for different quantities in one equation. Others let the unknown refer to different quantities at different stages in their solutions.

To sum up, the figure below taken from Stacey and MacGregor (1999) illustrates the different routes the students in the study chose on their way from problem statement to solution. It is evident that although choosing to try algebra, it does not mean that the task is solved algebraically. The different routes in the diagram illustrate this. Even after having written an equation it is not certain that it is solved as an equation; some solutions are also then categorised as ‘trial and error’ solutions.

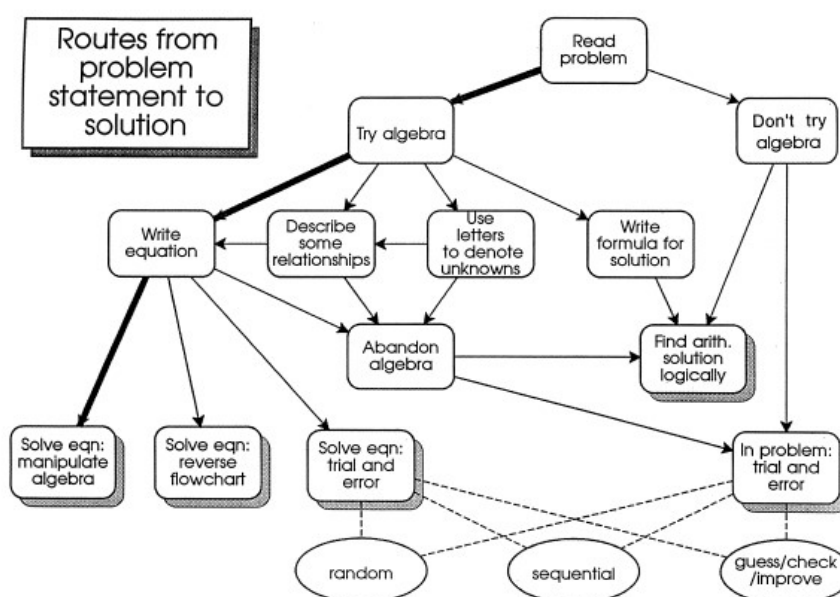


Figure 4-4: Routes from problem statement to solution (Stacey and MacGregor, 1999, p. 155).

In this section word problems are treated as a subject on their own. The figure above illustrates clearly that there is no straightforward way for students to solve a word problem. A word problem is presented as a text in normal language. It cannot be solved directly, it has to be read and interpreted. Then the structure of the problem has to be teased out; the connections between the quantities both known and unknown. Then comes

the process of solution which can follow several routes as is illuminated in the figure above.

The literature referred to above shows that it is more likely that students who have been explicitly taught about how to reason in the transformation of word problems into mathematical language, actually solve the problems as equations.

4.10 Linear inequalities

Wagner and Parker (1993) wrote that the two most investigated algebraic concepts at that time were equations and functions, and added that conceptually there is a big difference between the two concepts. Inequalities they assert to be a 'conceptual intermediary' between equations and functions. In equations the literal symbols represent discrete unknowns, in functions they represent independent and dependent variables, while in inequalities they represent a "whole continuum of numbers" (ibid, p. 126). Their view might hold on most occasions for school algebra, however, the literal symbol in an inequality can be one single number and not a set of numbers as in the inequality $x^2 \leq 0$, or it can represent sets of discrete numbers if related to real life problems. However, what Wagner and Parker describe, is that the unknown in inequalities with few exceptions differ from those in equations and functions. This difference might cause problems for students when interpreting their solutions when solving inequalities.

Sfard and Linchevski (1994a) describe from Israel how equations and inequalities were treated simultaneously as two closely related parts of a single mathematical notion: the propositional formula. Every proposition has its true set, namely the set of all substitutions that turn the proposition into a true proposition. Also the process of solving equations and inequalities was described in set-theoretic terms. This top-down approach is not described in other studies, however, inequalities are reported to be taught in secondary school as a sub-ordinate subject in relationship with equations (Bazzini & Tsamir, 2004).

The concept of inequality is an important concept in mathematics when it comes to trigonometry, linear planning, and investigation of functions (Tsamir, Almog, & Tirosh, 1998). Later on, for those studying mathematics at higher levels, the study of inequalities "include the formal notion of limits, where the epsilon-delta method will certainly benefit from meaningful grounding of inequalities" (Tall, 2004).

In another study from Israel, Tsamir and Almog (Tsamir & Almog, 2001; Tsamir, et al., 1998) comment that little attention is given to the topic. They claim that inequalities are only discussed in the upper grades

of secondary school by mathematics majors, and that this discussion usually is rather poor since the emphasis is on the execution of algorithms instead of focusing on the reason why these algorithms work. Also Boero and Bazzini (2004) claim that inequalities are mostly taught as a sub-topic to equations, not emphasising inequalities as a concept in their own right. This close connection between equations and inequalities is seen in many studies as a frequent reason for students making errors in solving inequalities (Blanco & Garrote, 2007; Sfard & Linchevski, 1994a; Tsamir & Almog, 2001; Tsamir, et al., 1998; Tsamir & Bazzini, 2002; Vaiyavutjamai & Clements, 2006; Verikios & Farmaki, 2006).

Garuti, Bazzini and Boero (2001) propose that teachers often reduce “the difficulties inherent in the ‘variable’ concept and the complexity of the solution process by treating inequalities as a ‘special’ case of equations” (ibid p.10). The inequalities are dealt with in an algorithmic manner which leads to a trivialisation of the topic. The authors assert the result is that students have problems distinguishing between equations and inequalities.

Boero and Bazzini (2004) claim that a functional approach would help students gaining a richer understanding of the concept of inequality, and ensure a “high level of control over the solution processes” (p. 141).

The underlying study (Garuti, Bazzini & Boero, 2001) involved two classes in grade 8 in Italy. The aims of the study were to investigate the feasibility of an early functional approach to inequalities, and to reveal students’ potential and difficulties in learning inequalities through a comparison of functions. The concept of function and variable had been dealt with through work with graphs, tables, and formulas. At first they had used the model of a function machine, transforming values of x into values of y , then the variation of y depending on the variation of x had been emphasised. This was done to enhance a dynamic view of functions and variable as a contrast to the static view of ordered pairs. The result was a “peculiar aspect of variable as a ‘running variable’, i.e. a movement on a set of numbers represented on a straight line (Boero & Bazzini 2004, p. 141). The next stage was to introduce inequalities as a comparison between functions based on the analysis of their formulas. The point by point construction of graphs was discouraged. One individual task is considered in the paper:

Compare the following formulas from the algebraic and graphic points of view. Make hypothesis about their graphs and motivate them carefully, finally draw a sketch of their graphs. A) first function. B) second function.

$$A) y = x^2 - 4x + 4 \quad B) y = -x^2 + 4$$

What does happen with y when $x > 0$ or $x < 0$? Does it increase, decrease?

Are there meeting points between the two graphs? Where? When?

Does the first graph overcome the second one? Where? (ibid p. 12)

The paper shows little evidence of enhanced conceptions of inequalities; however, the authors claim that the approach of studying inequalities within a functional approach seems to be promising for the development of the concept of variable and function.

Sackur (2004) warns against a view that students' use of graphic calculators always will help them in solving inequalities. In a study she conducted, students solved this given inequality problem, $\frac{3}{x} > x + 2$. In the

algebraic transformations made, the expected error of multiplying by x whatever the sign of x could be, occurred. The students who used graphs and compared the two functions $y = \frac{3}{x}$ and $y = x + 2$ made errors as well.

These errors appeared to be of two types. One error was due to the reading of the solution of the inequality. The other came from the writing of the solution although the reading was correct. On the basis of Duval's theory on semiotic registers (Duval, 2000) and Frege's theory of denotation (Frege, 1984), Sackur shows that there are problems arising when an algebraic problem is changed into a problem of graphs. Thus she claims that the use of graphs for solving inequalities needs careful preparation by the teacher.

Kieran (2004c) criticises the studies mentioned above (Garuti, et al., 2001; Tsamir & Almog, 2001; Tsamir, et al., 1998) as being too narrow since there is almost no focus on the manipulative/symbolic aspects of inequalities. While the other researchers set out that the relation between equations and inequalities is a problem, she uses a mathematics lesson from the TIMSS-R 1999 video study of grade 8 classes from seven countries to show how the relation between the two concepts can be used to promote and enhance the learning of the inequalities. She analyses a Japanese algebra lesson to show how it is possible to start with the global meta-level activity of contextualised problem solving, and then end up with the full symbolic form of inequalities. The task given in class was:

It has been one month since Ichiro's mother entered the hospital. He has decided to give a prayer with his small brother at a local temple every morning so that she will soon be well. There are 18 ten-yen coins in Ichiro's wallet and just 22 five-yen coins in the younger brother's wallet. They decided to place one coin from each of them in the offertory box each morning and continue the prayer until either wallet becomes empty. One day after prayer, they looked into their wallets and found the younger brother's amount was bigger than Ichiro's. How many days since they started prayer? (translated version) (ibid, p. 144).

The students worked individually on the task while the teacher circulated and wrote notes about which solutions students had used. Afterwards stu-

dents were asked to come to the blackboard to present their solutions: the teacher had already planned it and made spaces on the blackboard for the different solutions, from the most elementary to the symbolic ones. The students were asked to come to the front in the order of abstractness of their solutions.

The first student actually put coins in a box fixed to the blackboard, and showed his solution by manipulating concrete objects. The next made a table presenting the amount of coins in the wallets each day. On the 14th day the amount was the same and he concluded that the solution had to be day 15. The third student said: “Well, in the beginning, Ichiro had 180 yen, and the smaller brother had 110 yen. And since there is a difference of 70 yen, and since the difference between them becomes smaller by five yen each day, so it’s 70 divided by 10 minus 5. And since by the fourteenth day it becomes exactly the same amount of money, so since on the day after that there will be a difference, so 14 plus one is 15 and it’s the fifteenth day.”

Student four chose x to be the day the brothers have the same amount of money in their wallets and set up a system of simultaneous equations, and found that x is 14 so the fifteenth day is the solution. The fifth student to the blackboard provided a symbolic inequality that was used by the teacher to introduce the inequality symbols and the inequality in its symbolic form. For students to grasp this concept of the inequality the teacher then asked students to find out which values of x would make the inequality true. He asked them to complete a table, and after some minutes they discussed the different values. The teacher said: “ x holds true for 15, 16, 17, and 18; these are the ‘ $<$ ’. The first value of x [13] was a ‘ $>$ ’; the second one [14] was equal = ‘the standard’” (ibid p. 146). He also asked about the value 19, but then one student reacted and said that then the second wallet was empty and the situation was finished.

Kieran makes the comments that in this situation most students used their prior knowledge about equations and equalities to solve the problem. However, this was the teacher’s intention; to let students come up with their own solutions and build on them to make an adaptation into the concept of inequality. Kieran expresses it in this way:

The solution to this inequality, having already been found by the students by means of non-algebraic methods, was regenerated by substituting values, from the vicinity of the solution, into the two algebraic expressions that formed the algebraic inequality. In this way, the relationship between the solution to the linear equality ($180-10x=110-5x$) and those of its two related inequalities ($180-10x>110-5x$ and $180-10x<110-5x$) could be drawn out – implicitly appealing to a number-line interpretation of these solutions (ibid p.147).

She also mentions that none of the students draw graphs in the solution process. Kieran criticises former studies to a certain degree for ignoring

and avoiding approaches to inequalities which links equations and inequalities. She asserts that this study from Japan with grade 8 students might indicate that this interweaving between the two concepts is deeply rooted, and has to be taken into account. She agrees, however, with the findings from other studies that this interweaving is also problematic since the properties underlying valid transformations made in the solutions processes in the two mathematical domains, are different. The question is then how it is possible to help students to be aware of what she calls traps in this connection between the two. She points to the Japanese class and how the contextualised problem solving activity provided meaning for inequalities and their symbolic form. The challenge is, however, how this approach and activity may help students create meaning for the transformational rules which are specific for inequalities. Her last question is: “In which ways, if any, and for which age-ranges of students, can symbol manipulation technology be harnessed so as to provide viable approaches for developing students’ algebraic theorizing with respect to inequalities and their manipulation?” (ibid p. 147).

Sfard and Linchevski (1994a) reported that students in spite of a modern and well-designed curriculum based on set theory, did not “follow the path they were supposed to take while building their concepts” (ibid p. 299). On the basis of three questionnaires given to 280 Israeli students in three different schools, and clinical interviews of some of the same students aged from 14 till 17 years, they concluded that for the majority of students it seems as if equations and inequalities “are meaningless strings of symbols to which certain well-defined procedures are routinely applied” (ibid p. 306). For both equations and inequalities, a standard solving procedure is to simplify until no further simplification can be made. This end of the chain of transformations is then seen to be the solution, and the students were rarely seen interpreting the solution. For inequalities this was even harder than for equations. Probably it is easier to interpret a single number as a solution, than a set of numbers as in $x > a$. The authors express it in this way: “An equation or an inequality seems to be for a student a thing in itself, for which the formal manipulations are the only source of meaning” (ibid p. 306).

Cortes and Pfaff (2000) experienced the same problems with equations, see section 4.8. In order to help students making meaning of equations and show how the manipulative activity was based on the properties of equation, they decided to go the opposite way; to teach inequalities first. Their reasoning was that the students had already proceduralised their own limited rules for equation solving, and that it would be wise to try another approach. The researchers took linear functions as a basis. At

that time the students in the class had already failed on one test with inequalities.

The concept of inequalities was constructed as a relationship between the functions $f(x) = 3,4x$ and $g(x) = -1,2x + 1$ which they were familiar with. The inequality was $-1,2x+1>3,4x$. The students (10 graders) were first asked to substitute the unknown by different values, to see which values kept the inequality true. Then the solution was determined graphically, which was no problem. Afterwards the students were asked to execute algebraic transformations. The idea was to challenge students when coming up with solutions that according to the graphical representation were incorrect. This would afford opportunities to talk about identical operations executed on both sides of the unequal sign, and to discuss why some solutions appeared to be incorrect. The students were then guided to analyse which operations conserved the direction of the sign. Number examples were applied in order to justify that it was allowed and necessary to flip the unequal sign when multiplying or dividing by a negative number. Later on, the teacher wrote the operations executed on both sides and reminded the students that in order to conserve the truth-value (the solutions) of the inequalities, they had to do the same operations on both sides. In addition, the flipping of the unequal sign when dividing/multiplying by a negative number, was made meaningful to them.

A post test showed that the students had made considerable progress both in solving inequalities and also in solving equations, and they now had tools to justify their transformations although still making the shortcut of ‘changing side-changing sign’, and to check if their transformations conserved the ‘truth-value’ given in the initial task.

The problem of interpreting the solution of inequalities is reported by Vaiyavutjai and Clements (2006). They conducted a study in Thailand where 231 grade 9 students in six classes from two schools participated. The students belonged to high-, medium-, and low-stream classes. The students were tested before and after a session of 13 lessons on linear equations and inequalities. In addition, they were given a retention test 6 months later. 18 students were interviewed on the basis of the tests. In grade 7 and grade 8 the students had been working on linear equations, and they were expected to have met inequalities of the form $ax + c > d$ and also they had learned how to solve them before the study was carried out.

The tendency was that students gave single numbers as answers. An example is the inequality: $1 - x < 0$. The percentages of all the students giving ‘single number answers’ at the pre-test, post-test, and retention-test were 68 percent 39 percent and 56 percent respectively. Except from the test shortly after the teaching of the topic, more than half the students gave a ‘single number answer’.

When reading the report, it could be argued that the layout of the test might be one reason for this. The students were asked to show their way of solving the tasks, however at the end they should write the solution in a box. Many students then, wrote only a single number as the answer, also for equations. What strengthens the conclusion, however, is that in interviews students showed clearly that they were not able to interpret the result of their solution processes.

Students in high-stream classes were more likely to think that the answer was a set of numbers, however, interviews revealed also that most of those students had no clear conception of how the solution was related to the initial task. For almost all interviewees; they did not know how to check the solution. The authors conclude that for all but the best students, to solve an inequality meant to perform the same operations as when solving equations. The fact that the solutions were mostly correct, led teachers to think that the students had grasped the concept of inequality. All the teachers in this study thought their students were able to solve all the tasks in the test. The result, however, was that in the end the mean result was less than 10 out of 27.

A similar study in Spain by Blanco and Garrote (2007) was carried out with students in the first year of a two year non-obligatory course for students aiming at studies at the university. 91 students were asked to answer a questionnaire. The students were from 4 different educational centres. They were studying either the option Technology, or the option Nature and Health Sciences. The questionnaire was given after the students had received instruction in the topics that it covered. Most of the participants had studied the concept of the inequality for the first time that school year; only some had prior ideas about the objects being studied.

One of the tasks given was $5 - 3(2 - x) > 4 - 3(1 - x)$. The students according to their answers were divided in three distinct groups. One was a group where students had solved the task correctly coming to the results: $1 > 1$ or $0 > 2$. In addition, they had interpreted the solution to be that there is no value of the unknown satisfying the inequality. The next group had arrived at the correct algebraic solution, but could not interpret the solution correctly. In the third group, the students had not been able to reach the correct symbolic solution.

In this task it seemed to be that many students used the symbols: 'greater than' and 'less than' without attaching meaning to them. They carried the symbols through the steps in the solution process correctly, at the end; however, the sign was replaced by an equal sign when presenting the solution. That the difference between equations and inequalities was blurred for a large proportion of the students, was also evident in

the way they talked about their solutions. Sometimes they were using the term equation to refer to inequalities.

According to the authors this treatment of inequalities as equations caused students to find no meaning in the solution they reached.

Grevholm (2003) found that also among student teachers there was a problem of finding the meaning of inequalities. One fourth of the students in her study, could either not justify their answers correctly or they circled a wrong alternative to the task (proposition E):

Decide if the following propositions A to E are true always, never or sometimes. Draw a circle around the correct answer. If you circle “sometimes” then explain when this proposition is true.

E) $2n > n + 2$ always never sometimes, when

The same student teachers when they should guess how many students in the school who would succeed in solving this task, they estimated it to be 30 % of the students.

That also students in higher education have problems in understanding their solutions to inequalities and also problems relating their solutions to the initial problem, is taken up by Warren (2006). She refers to studies in secondary school (Tsamir & Almog, 2001), and aims in her study with children in elementary school to go beyond the structural similarities between equations and inequalities which creates what Tsamir and Almog (ibid) call a “strong intuitive feeling that similar strategies hold for solving the two types of mathematical entities” (ibid p.522). While Tsamir and Almog (ibid) emphasise the structure, Warren (2006) looks at the development of the conceptions of the notions ‘greater than’, ‘less than’, and ‘equal to’.

These notions are used to describe inequality and equality, and she proposes that a lack in understanding of this terminology may cause students to be confused when working with equations and inequalities. She conducted a longitudinal study in Australia with children from grade 2 and followed them till the end of grade 5. Warren (ibid) found that few children were able to use ‘more’ and ‘less’ in situations where they compared quantities. She analysed textbooks and found that only 5 % of the material was aiming at development of the concepts of the notions ‘more’ and ‘less’. Although her study concerns small children her findings might indicate that students at higher levels have not had the opportunities to develop sufficient conceptions of these notions. She claims that students fail to operate on the level of using the notions in situations where quantities are compared, which is essential in further work with inequalities. Researchers have claimed (Falkner, et al., 1999) that it is important to work with the concept of equivalence in early years. Warren (2006) on the basis

on her findings proposes the need to explore ‘more’ and ‘less’ as non-equivalence.

The study reported by Tsamir & Almog, (2001) included 160 Israeli mathematics majors students in high school. They were given an ‘equation and inequalities’ questionnaire. There were five equations and 10 inequalities to solve in a 90-minute mathematics lesson. Students were asked to explain each step of their solutions. The students had already worked with inequalities for about three months. In addition, they were familiar with drawing graphs of the functions related to all the expressions given. Also they had investigated interrelated behaviour of combined systems of two functions and examined for which values of x , $f(x) < g(x)$. 25 of the students were in individual interviews asked to solve the tasks and to explain their solutions; both the process and the solutions. On the basis of tests and interviews the data was analysed.

Only one of the inequalities in the test was a linear inequality which is the focus for this study. The inequality was: $-8x > 0$. 76 % of the students solved this inequality algebraically correctly. 17 % divided by negative 8 without changing the direction of the sign. Some of the interviewed students referred explicitly to the analogy they drew between equations and inequalities. They assumed that the same solving procedures hold for both types of tasks. One of the students interviewed, understood, when trying to substitute the variable with a value to verify his solution, that there are limitations of this analogy between equations and inequalities.

None of the students mentioned the algebraic expression as a linear function to be presented as a graph, which is assumed to be the best known of the types of functions. In the other more advanced tasks the presentation of the expressions as graphs helped students to reach correct solutions. However, most students in solving all the tasks used algebraic manipulations which often led to incorrect results. The authors discuss whether it should be promoted to teach one way of solving all types of inequalities or if it is useful to teach different procedures for different types of inequalities. The students in their study had been taught the latter; different procedures for different types, and they tried to use those in the test. 5 % of the participants changed the inequality sign to an equal sign in almost all of the nonlinear inequalities, and solved the tasks as equations.

The referenced literature shows that there are problems related to the close connection between equations and inequalities, and that little attention is devoted to inequalities compared to the concept of equation. Some students even regard the solution processes to be identical. In addition, problems are reported in interpreting the solutions found. It is claimed by some researchers that the solution process is a meaningless manipulation of symbols.

However, the example from Japan (Kieran, 2004c) and the study carried out by Cortés and Pfaff (2000) show that it is possible to connect inequalities to equations by careful planning of teaching and tasks.

In the foregoing sections, former research and findings related to the topics treated in the textbook and in the classroom during observations are reviewed. Some of them are old, others are rather new. They cover a long period of mathematics education, and they are based on different theoretical perspectives. In the next section, there will be a short overview of theoretical perspectives in algebra research.

4.11 Theory in algebra research

During the past century the theoretical and philosophical basis for research in algebra and also in all branches of mathematics education has undergone changes. Most research in the first half of the foregoing century was worked out within the positivistic paradigm with behaviouristic ideas on teaching and learning (Thorndike, 1914, 1922; Watson, 1913). Learning was explained without any reference to internal mental processes (Skinner, 1953). Behaviourism presented a simple explanation of learning from an external viewer's perspective and was diminished to be a set of stimulus-response connections.

Two other movements concerned with learning and development appeared during the century; the grand theories: constructivism and socio-cultural theories. Both of them are still the main streams in educational research with their different branches and bridging theories to practice.

Behaviourism had not been concerned with concept development; however, Piagetian genetic epistemology played an important role in the change of focus for research. Genetic epistemology:

deals with both the formation and the meaning of knowledge. We can formulate our problem in the following terms: by what means does the human mind go from a state of less sufficient knowledge to a state of higher knowledge? ... Our problem, from the point of view of psychology and from the point of view of genetic epistemology, is to explain how the transition is made from a lower level of knowledge to a level that is judged to be higher. The nature of these transitions is a factual question. The transitions are historical or psychological or sometimes even biological ... (Piaget, 1970, first lecture)

From focusing on behaviour, the shift implied a focus on students reasoning and conceptions. Clinical interviews were used as a method to inquire into children's and students' beliefs and theories of mathematics.

Behr, Erlwanger, & Nichols' (1976) study of students' conceptions of the equal sign is a good example of how researchers through clinical interviews tried to tease out what students were thinking when solving tasks (see section 4.7). In addition, these authors were interested in studying the development of the concept. They interviewed students from several

grades to see if older students had a more developed conception than children in the lower grades. They found that students in grade 6 had the same problems in interpreting the equal sign as a sign expressing equality, as students in grade 1. They claimed then that the students had an “incomplete conception – indeed a misconception”.

Piaget’s stage theory which followed from the genetic epistemology is another source for many studies. Küchemann’s study (1981) as part of the CSMS study is an example of these studies (see section 4.1). The underlying assumption was that by using carefully chosen tasks, it should be possible by investigating children’s reasoning processes, to find out about their level of understanding (Hart, 1981). Tasks were created in order to check if students’ performances on tests indicated to which cognitive developmental stage they belonged; comparing them to the stages in Piagetian stage theory.

In Piagetian theory development precedes learning (Piaget, 1997), while in the socio-cultural theories learning and development proceeds dialectically. In the tradition of Piagetian stage theory there is the notion of readiness. This follows from the assumption that each stage is characterised with qualitatively different structures of mental schema, and for a child to cognitively move from one stage to another he/she must be cognitively ready for it. Herscovics & Linchevski’s (1994) study is one example in the review above. They stated that “the pupils’ lack of readiness may explain the dismal results achieved in algebra in our secondary schools” (ibid p. 59).

As noted above genetic epistemology is about how knowledge is constructed and is a theory that attempts to explain both how knowledge develops and its origin. Confrey defines it to be “the study of the development (or genesis) of particular concepts over time in children” (Confrey, 1990, p.5). That is why researchers with a constructivist view have partly justified their studies by emphasising the importance of knowing students’ conceptions and difficulties, and the importance of making the results open to teachers (Hart, 1983). To take students’ pre-knowledge into account in teaching was seen to be important, and research investigating students’ conceptions could inform educational practice. Much work in mathematics education is still carried out in this tradition, although researchers do not see the algebraic “curriculum as a given” anymore (Kieran, 2006, p.13). Instead research is done to emphasise the need for a change in school algebra, not just to trace the conceptual development.

In the socio-cultural perspective there is a distinction between spontaneous or everyday concepts and scientific concepts (Daniels, 2001). Spontaneous concepts are concepts a child acquires outside school in in-

formal settings. Scientific concepts are concepts which are introduced by teachers in school.

Vygotsky (1987) writes that concepts are formed from two directions; from the general and from the particular. The scientific concept develops downwards from the abstract to the concrete and to its referent. The spontaneous concept develops in opposite direction from the concrete to the abstract, outside formal systems. In praxis this means that it is possible to use a concept without having grasped the whole meaning of it. When the integration process described above is finished, then the concept is internalised or appropriated. Vygotsky (ibid) warns against rote learning of concepts, as a mindless memorising of words. He uses the concept of flowers as an example of concept formation. He writes that after the concept of rose is assimilated and if the rose is the only type of flower when the concept of flowers is introduced, for a long time the concept of flower is not more general than the concept of rose, however, after having generalised the concept of flowers, the concept of rose is subsumed under the general concept.

The implication for teaching is for teachers to help students to create links between every day and scientific concepts. According to Daniels (2001) the result is that “scientific concepts are developed through different levels of dialogue: in the social space between teacher and taught; and in the conceptual space between the everyday and scientific” (ibid p. 53). In this way, a web of patterns or of conceptual connections is produced.

In an overview of algebra research in the last 30 years Kieran (2006) asserts that the first years were dominated with research on the basis of Piagetian and constructivist theories, focused on students’ errors and misconceptions and teaching experiments to overcome difficulties reported in research. One reason for this cognitive and constructivist basis was mostly that the socio-cultural theories were not that well known among scholars. Another reason may be found within the theory itself. As a reaction to the behaviour oriented focus in the positivistic paradigm, it was of interest to investigate and gain knowledge about the mental learning processes in mathematics. The socio-cultural theories, with their underlying fundamental basis that all higher mental functioning is in its essence cultural and social, seemed not to have had as much to offer in research, as the constructivist theories had at that time. Mathematics as a subject in which one has to act upon abstract objects, is decontextualised in its character, it is also a language with its own symbol system.

According to Confrey (1995) this should fit with how Vygotsky describes the “sign-sign relationships and decontextualized knowledge” (ibid p. 203). In addition, the mathematical symbol system is a cultural-historical achievement which through teaching is to be taken further, or to

be reproduced by new generations. However, working from a socio-cultural perspective will create different research questions than working from the constructivist perspective. Lerman (2006) proposes that the studies reporting weak conceptual knowledge among students, places the responsibility on the students. In a socio-cultural perspective where the social and the cultural are the basis for development, research would have given other answers.

The research on students' conception and problems, with its focus on students errors are by some called the collection of "sad stories" (Lins & Kaput, 2004, p. 53) about what students were not able to do. Although they are sad stories, scholars in mathematics education have gained a considerable amount of information about students' thinking and obstacles in the learning of algebra and other topics in mathematics from these sad stories. As noted above, from another perspective there would have been other stories to tell because the questions would have been different, but as long as the formation of concepts is the focus, the stories would have been sad, however, with supposedly other explanations than those given in the studies above.

Cobb (2007) compares different theoretical perspectives and claims that cognitive perspectives are useful on the classroom level seeing the individual as "reorganiser of activity", while the socio-cultural perspective or social perspective sees the "individual as participant in cultural practices". In this study both perspectives are present.

5 Interpretive framework

This study is trying to answer questions about what students starting in the upper secondary school, experience from the mathematics they are offered during instruction within the classroom setting, and how the setting of the mathematics classroom might influence students' experiences. The focus for instruction and work during the time of observation was number operations and algebra. The students had met these concepts and operations earlier in lower grades, but they are given new opportunities to recapitulate, and to expand their knowledge.

As outlined in section 2.9 the students and the teacher are situated within a larger institutional system; the Norwegian national school system, on the more local level within the institution of one particular upper secondary school, and on the micro level within the particular observed class. Many factors influence the activity in the classroom which are outlined in chapter 2, however, in this thesis, the investigation is limited to focus upon those factors directly influencing the work in the classroom. The background factors will not be investigated, however, during the observation and also during the time of analysis, it is kept in mind that other factors also are playing a role for the students.

The students have 10 years of experiences within mathematics classrooms, and when they enter upper secondary school, they bring these experiences into this new classroom. Each one of the students has his/her own personal experiences, influencing their view on mathematics, how it is to be learned, their view of themselves as individuals, their view of what matters in class, and what they assume is expected of them in relation to their classmates, and to the teacher; the tacit didactical contract (Brousseau, 1997).

When studying the relationship between the mathematical learning goals in this classroom and students' experiences manifested in their responses within the specific mathematics classroom, a theoretical framework is needed. According to Niss (2007), one purpose of a theory is to "provide a structured set of lenses, through which aspects or parts of the world can be approached, observed, studied, analysed or interpreted" (ibid, p. 1309), another is to provide a "safeguard against unscientific approaches to a problem" (ibid, p. 1309). Niss (ibid) further points to the fact that there is no overarching theory or framework not even for specific topics in mathematics education. Instead he suggests to be open about what he denotes an investigational framework. This framework should consist of:

- A perspective on the issues and question to be investigated.

- A set of theoretical constructs (e.g. concepts notions, and assumptions) more or less sharply defined, coined to capture essential entities of significance to the issues and questions in focus of the framework
- A set of preferred methods considered suitable for the investigation of the issues and questions of interest, with particular regard to the ways in which the theoretical constructs come into play. (p. 1297)

The individual student and his or her experiences within the classroom is in focus. At the same time the mathematics offered and the opportunities given to experience the mathematics is central. This means that although the individual student's responses are at the core, the mathematics classroom and what is going on there has to be taken into account. In line with Kilpatrick and colleagues (2001), teaching and learning of mathematics is seen as a product of interactions between the teacher, the students, and the mathematics. In addition, different types of interactions are seen to create different learning opportunities. Therefore, theoretical constructs from different theories will be used.

The epistemological basis for this thesis is to be found in the landscape of constructivism. In accordance with the constructivist assumption about mathematics the underlying assumption for this study is that mathematics is a "human creation, evolving within cultural contexts" and that mathematics is a socio-historical product with its own norms and rules (Heintz, 2000).

Duval explains why mathematical knowledge is different from other knowledge:

... there is an important gap between mathematical knowledge and knowledge in other sciences such as astronomy, physics, biology, or botany. We do not have any perceptive or instrumental access to mathematical objects, even the most elementary We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. This conflicting requirement makes the specific core of mathematical knowledge. And it begins early with numbers which do not have to be identified with digits and the used numeral systems (binary, decimal) (Duval, 2000, p. 61).

Mathematics for the learners is different from the mathematics of the professionals. The latter group has over years learned how to interpret mathematical signs and symbols and to communicate with them and work on them, while the learners are in a developmental process of acquiring mathematical knowledge (Steinbring, 2011).

In order to communicate and to operate with mathematics, certain signs and symbols are needed. These signs and symbols are, like words in ordinary language, not bearers of meanings of their own. The meaning has to be constructed by the individual learner.

The signs have to be interpreted, and this interpretation requires experiences and implicit knowledge – one cannot understand these signs without any presuppositions. Such implicit knowledge, as well as attitudes and ways of using mathematical knowledge, are essential within a culture. Therefore, the learning and understanding of mathematics requires a cultural environment (Steinbring, 2006, p. 136).

Mathematics is thus not to be learned in isolation. Every mathematics classroom might be seen as a sub-culture in which mathematics is to be learned. In order to focus both on the individual learner and on the interaction between the participants in the classroom, Cobb and Bauersfeld (1995) suggestion to coordinate sociological analyses of the micro culture of the classroom with cognitive analyses will be followed.

Some researchers have claimed that this combination of theories is impossible. Lerman (1996) is one of them. He claimed from a socio-cultural perspective that intersubjectivity is not at all in line with the radical constructivist view of learning that each individual constructs its own way of knowing. Steffe and Thompson (2000) objected to this, they claimed that intersubjectivity was part of constructivism from the outset. In a later article in the encyclopaedia of Mathematics Education, Thompson (2014) claims that the core of the misunderstanding was that the notion intersubjectivity was interpreted in different ways by the two ‘fronts’. According to Thompson (*ibid*) for Lerman (1996) intersubjectivity meant: “agreement of meanings” – same or similar meanings. Steffe and Thompson meant “non-conflicting mutual interpretations”, which might actually entail non-agreement of meanings of which the interacting individuals are unaware of” (Thompson, 2014, p. 99).

Thus he concludes in the constructivist tradition of intersubjectivity that they both meant they understood what the other meant; but actually they did not, and “they each presumed they understood what the other meant when in fact each understanding of the other’s position was faulty” (*ibid*, p. 99).

Although there is no consensus within the community of mathematics education, the two different perspectives the individual and the collectivistic or social, will both be present in this thesis. Research done in this tradition has provided much knowledge about the culture of mathematics classrooms (Voigt, 1989, 1994, 1995, 1996), and of sociomathematical norms (Yackel & Cobb, 1996).

Cobb called the combination of the individual and the social perspectives the emergent perspective (Cobb, 1995), another notion is socio-constructivism, which was one of the ‘bridging theories according to Confrey and Kazak (2006). Within this perspective learning is seen both as a “process of active individual construction and a process of mathematical enculturation” (Cobb, 2000, p. 309) and thus the importance of analysing

students' individual mathematical activity as it is situated in the social context of the classroom is emphasised (Cobb, 2000).

To see the mathematical classroom as a specific kind of culture (Voigt, 1998) seems to be useful for the investigation of the classroom situation. Within the culture of the mathematics classroom, the individual learner constructs his/her own conceptions of the concepts to be learned. To describe such processes, constructs from cognitive theories will be applied. In the next sections the constructs from social interactionism and from cognitive theories will be presented and described.

5.1 The social perspective

The main construct from the social perspective is 'pattern of interaction', other related constructs are 'negotiation of meanings', and 'taken as shared'. These constructs will be outlined in this section.

Communication is one important notion accounting for the social or collective perspective. The individual student must actively construct meanings from the experiences in the environment. Language is seen to be the main medium for the individual's available meanings. However, language cannot be transmitted.

A speaker's utterances can function for the listener like 'pointing at something' only, or better as directing the focus of attention, whereas the construction of what might be meant, the construction of references, is with the listener. The speaker's utterances and intentions have no direct access into the listener's system. What the listener's senses receive, undergoes spontaneous interpretation" (Bauersfeld, 1995, p. 273).

For the interactionists this includes more than language; also gestures, body language etc. (Bauersfeld, 2000). The communication in the classroom is seen as a process in which the participants mutually adapt to each other. This is done through negotiation of meanings (Bauersfeld, 1980, 1994).

Negotiation of meanings

Negotiation of meanings (Bauersfeld, 1995; Voigt, 1994, 1995) concerns the notion of the interaction going on in the classroom. With the view that meanings are not transmitted by language, and that each individual constructs its own interpretation from what is experienced, it is not possible to know if two people are assigning the same meaning to one utterance. It is evident from classroom situations that teachers and students often differ in what meaning they assign to an object of classroom discourse. To find a way to resolve this problem of ambiguity, the participants in the classroom have to negotiate meanings in order to arrive at a meaning that is taken-to-be-shared. They are continually modifying their interpretations.

This does not mean that this meaning is held by the individuals in the classroom, but that each individual meaning is compatible with the others.

Meaning-taken-as-shared

Meanings-taken-as-shared is the notion for the meanings negotiated in the classroom, but does not mean that these meanings are actually shared. The discourse is developing as if the participants hold the same conceptions.

Students in the classroom and the teacher come to the classroom with different background experiences and thus interpret the classroom situations in different ways. The tension between these different interpretations Voigt (1989) claims to be the motor for the negotiation of the meaning in the classroom. In spite of these tensions the classroom discourse mostly happens to go on in a smooth way because of a “provisional willingness to cooperate” (ibid, p. 652). This willingness is based on a tacit and implicit agreement, but there is always a risk that the discourse will break down. However, regularities constituted in the classroom minimises the risk. Routine actions are one type of regularities. One example of such a routine in the mathematics classroom is for students to reduce verbal utterances to numbers and key words, which enables the teacher and other students to identify their own expected meanings in what is said.

The term *taken as shared* describes the participants' conviction that meanings are shared, or the participants' willingness to neglect doubts in view of inevitable ambiguities, or the presumption that the meanings will be shared if the others will "read between the lines." (Voigt, 1995, pp. 172-173).

As time goes on in the classroom, the participants in the classroom constitute a culture through the negotiation of meaning, which leaves the students holding an impression that they know what mathematics and mathematics learning is (ibid). Cobb called this the ‘institutionalized knowledge’ which might differ from one classroom to another (Cobb, 1989).

Patterns of interactions

Patterns of interactions are to be found in all social groups. When observing everyday life, it is possible to reconstruct patterns of interactions, so also in the mathematics classroom. The teacher and the students are mostly unaware of them (Bauersfeld, 1995, 2000; Voigt, 1989, 1995):

If the observer looks at the classroom life in the way an ethnographer does who investigates a strange culture, the observer might be astonished by what is taken for granted by the members of this classroom culture. However, in the treadmill of everyday life, the participants would say that they know what mathematics and the classroom practice really are. In everyday classroom situations, the teacher and the students often constitute the context routinely without conflicts and without being aware of the ongoing accomplishments. So, in the participants' experiences the context can seem to be pre-given. In everyday classroom practice the teacher and

the students assume that the context is known which is, in fact, taken-to-be-shared, vague, and implicit (Voigt, 1994, p. 181).

In mathematics classrooms the negotiation of meaning is fragile. One reason is that the mathematical objects in focus are ambiguous to students. Patterns of interaction minimises the risk of collapse, in that the students take their classroom culture to be given and definite, although it is constituted jointly by the participants in the classroom.

Routines reduce the complexity in the classroom and are necessary elements of the classroom. In addition, both teacher and students feel obligations of different kinds in ongoing discourses. When a mathematical task, for example, is intended by the teacher to give students the opportunity to construct conceptual knowledge, and the students do not respond as expected, the teacher may feel obliged to make the problem easier, and to lead students step by step, resulting in students focusing on the procedures. At the same time students might feel obliged to react to teacher's question in a way they interpret teacher's expectation to be (Voigt, 1989). This will result in the kind of pattern which is called the *funnel pattern* or *elicitation pattern*, in which the teacher leads the students in small steps to the solution of the problem.

Obligations and routines are prerequisites for patterns of interactions. These patterns are always jointly constituted and offer one reason for classroom discourses to go on, and to develop without too much effort and without being ever changing (Bauersfeld, 2000; Voigt, 1995). They might, however, be an obstacle for learning mathematics if the patterns turn out to be only joint work in order to arrive at a correct solution narrowed down to small steps (Steinbring, 1989; Voigt, 1989, 1995). It is also a risk for observers to equate successful participation in patterns of interactions with the learning of mathematics. The routines and felt obligations make students think that they know what classroom praxis is and should be (Voigt, 1995).

The result is that it is often difficult to change practice, in that students avoid changes from what is understood to be mathematics and mathematics classroom. Often a culture is constituted in the classroom in which mathematical symbols are mostly connected to particular conventions and methodical rules (Steinbring, 1997, p. 50). Mathematics might then be experienced as rules and conventions rather than as relations and structures.

5.2 The individual perspective

On the individual level, the students construct their own conceptions of mathematics from what they experience. As seen in the foregoing sections mathematics for students is ambiguous, and during the time in mathemat-

ics classes the students have tried to create meanings for abstract mathematical concepts and operations.

Tall and Vinner (1981) proposed the notion ‘concept image’ to denote the conceptions students hold of the formal mathematical concepts they meet. This notion (see section 2.8) “consists of all the cognitive structures in the individual’s mind that is associated with a given concept” (ibid, p. 151). A person’s concept image is developing on the basis of his/her experience and is changing and expanding as the individual gains new experience. The aim is that the concept images will develop to be in line with the formal definitions for the actual concepts.

Gray and Tall (2007) emphasise the importance of language and naming in the formation of mathematical concepts. When a concept is named, it can be referred to, be related to other concepts and situations, and it is possible to focus on its various aspects.

Through the process of mental compression¹¹ of what has been conceived of as extremely challenging in mathematics, it can when suddenly understood, be seen as a whole, and be easily recalled and worked upon (Thurston, 1990). Gray and Tall (2007), see this compression as the underlying mechanism of abstraction that makes mathematical thinking possible “to operate at successively higher levels of sophistication” (ibid, p. 25).

In algebra this flexibility is of great importance. Related to this ability is the ability to see the structure of algebraic expressions. Linchevski and Livneh (1999) used the notion “structure sense”. The authors claimed that if this sense was developed by students, then they are:

able to use equivalent structures of an expression flexibly and creatively. Instruction should promote the search for decomposition and recomposition of expressions and guarantee that the mental gymnastics needed in manipulating expressions makes sense (ibid p.191).

Hoch and Dreyfus (2004) refer to Linchevski and Livneh (1999), however they miss a definition for structure sense. They decided to define the notion of structure sense like this:

Structure sense, as it applies to high school algebra, can be described as a collection of abilities. These abilities include the ability to: see an algebraic expression or sentence as an entity, recognise an algebraic expression or sentence as a previ-

¹¹ Compression is the word Thurston used to describe how previous learned mathematical ideas and processes can be transformed into compact and more precise mathematical objects, seeing them as a whole. He describes: “...once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process” (Thurston, 1990, p. 847).

ously met structure, divide an entity into sub-structures, recognise mutual connections between structures, recognise which manipulations it is possible to perform, and recognise which manipulations it is useful to perform (ibid p. 51).

Structure sense should, according to this definition, be a necessary condition for students to be efficient and precise in their manipulative work with algebra. Implicitly the students then have developed proceptual thinking (Gray & Tall, 1994).

Tall et al. (2001) proposed that the development of mathematical objects followed the routes from perceiving things in the environment, then to act upon those, and finally to reflect on those actions to build theories. The authors express though that there are discontinuities in the learning path taken by different students and that “perception, action and reflection occur in various combinations at a given time and a focus on one more than the others can lead to very different kinds of mathematics” (ibid, p. 82).

5.3 Different kinds of mathematics

When learning mathematics, students might have different impressions and thoughts about what mathematics is. It depends on the classroom culture and what type of mathematics is presented through theory, example and tasks. In this section the discussion in the mathematics education community about mathematical knowledge, especially the divide between conceptual and procedural knowledge will be outlined.

It is stated that “the relationship between computational skills and mathematical understanding is one of the oldest concerns in the psychology of mathematics” (Resnick & Ford, 1981, p. 246). Skemp (1987, p. 156) proposed that there are actually two different subjects being taught in school under the name of mathematics. The difference is recognised in that one of them emphasises instrumental understanding and the other relational understanding. Relational understanding Skemp defined to be “knowing both what to do and why” (Skemp, 1987, p. 153) and instrumental understanding was defined as “rules without reasons” (ibid, p.153). Skemp comments that he was not aware of this latter type of understanding, but realised that many teachers and students mean that to possess rules and to be able to use them is to them understanding, also without a comprehension of why they work.

In his way of distinguishing between different kinds of mathematics, Skemp (ibid) uses the notion understanding, and he describes that “to understand something means to assimilate it into an appropriate schema” (ibid, p. 29). This definition emphasises the individual nature of understanding. It is therefore not always the case that to understand means to have a full understanding.

Other researchers make distinction between skill learning and learning with understanding, between semantics and syntax or between rote learning and meaningful learning, Hiebert and Lefevre (1986) claim that regardless of the labels; the divisions referred to, correspond approximately to their division labelled by procedural and conceptual knowledge. It has to be noticed that it is rather confusing since by this claimed correspondence they do not really distinguish between understanding, knowledge and learning.

Hiebert and Lefevre (*ibid*) acknowledged that to decide what knowledge should be denoted one or the other type is not always easy, and they do not argue that all mathematical knowledge fits within the definitions they gave of the two types.

Conceptual knowledge they defined to be “knowledge that is rich in relationships” (*ibid*, p. 2) forming a web of knowledge, where the connections are as important as the discrete pieces of information. It is also important to note that Hiebert et al. (*ibid*) claim that a unit of conceptual knowledge cannot be an isolated piece of information, and that it is part of conceptual knowledge only if the holder recognises its relationships to other pieces of information.

Conceptual knowledge develops in that pieces of information are linked. This linking process can occur between already existing pieces of information stored in the memory. This happens when students gain new insights and suddenly see unrelated information as related in some way. Another way of developing conceptual knowledge is through connecting new information to existing networks. This latter phenomenon they equate with the labels ‘meaningful learning’ (Ausubel, et al., 1978) and ‘understanding’ (Skemp, 1987).

The connections between pieces of mathematical knowledge can be established on two levels according to Hiebert and Lefevre (1986). At the primary level; knowledge is tied to the context in which it is presented. The degree to which the knowledge can be transferred to other contexts or freed from context, tells about the level of abstractness of the relationships. At the abstract level the knowledge and its relationships are context free or at least less tied to the context. This level is called the reflective level because at this level one might be able to step back and reflect on the pieces of information being connected.

Procedural knowledge is made up of two distinct parts. The formal language or formal symbol system of mathematics forms one part. This means familiarity with mathematical symbols and knowledge of the syntactic rules for operating on them. In addition, the authors (*ibid*) exemplify that at advanced levels also the knowledge about the syntactic configuration of formal proofs is included in procedural knowledge, however, not the logic behind them, but the style in which they are written.

Algorithms, rules, and procedures for solving mathematical tasks make up the other part of procedural knowledge. These are step-by-step instructions to be performed in a linear sequence. The authors also suggest that the procedures they describe are characterised as a production system with an input “firing the system” (ibid, p. 6). They also see procedures as built up hierarchically which means that procedures can be developed to be sub-procedures in super-procedures.

Hiebert and Lefevre although differentiating between the two types of knowledge, emphasised: “Mathematical knowledge in its fullest sense, includes significant, fundamental relationships between conceptual and procedural knowledge” (ibid, p. 9). To be mathematical competent one must possess both types.

Star, however, criticised them for being vague in their definition of conceptual knowledge, and claims “it is defined in terms of the quality of one’s knowledge of concepts, particularly the richness of the connections inherent in such knowledge” (Star, 2005, p. 407). He further claims that concepts do in themselves imply connected knowledge, and asserts that Hiebert and Lefevre’s definition includes only a sub-category of conceptual knowledge which is richly connected to other knowledge or information.

However, my reading of Hiebert and Lefevre, suggests that their notion of levels of abstractness may include levels of richness, although in their definition they refer to conceptual knowledge only as rich in connections.

The primary level of conceptual knowledge is context bound, and lack of rich connections limits the application of the conceptual knowledge from being transferred to non-familiar contexts or to more abstract levels. The possibility to develop conceptual knowledge from being weak in connections and context bound, to knowledge rich in connections on an abstract level freed from context, mirrors an interpretation of conceptual knowledge as lying on a continuum from weak connected knowledge to knowledge rich in connections.

This is also what Star (2005) suggests in his own model, however, claiming that there is also another dimension; deep and surface levels of conceptual knowledge and of procedural knowledge. He defines conceptual knowledge as knowledge about concepts and principles. This means that conceptual knowledge might be on a surface level with weak connections or on a deep level with rich connections. Procedural knowledge, he asserts (ibid), has been equated with superficial and sparsely connected knowledge, often learned by rote. In his view procedures might be learned superficially or deeply with rich connections within the procedural knowledge domain only, without connections to conceptual knowledge.

He defines procedural knowledge to be knowledge of procedures. The characteristics of a deep level of procedural knowledge should be “knowledge of procedures that is associated with comprehension, flexibility, and critical judgement and that is distinct from (but possibly related to) knowledge of concepts” (ibid p. 408). Especially flexibility he introduces as a key indicator of deep procedural knowledge.

The reason for his re-conceptualisation of procedural knowledge is his concern about the status of procedural knowledge in schools and also in research. By detaching knowledge type and quality of knowledge, he proposes that it would be easier to foreground procedural knowledge in research (Star, 2005, 2007). His claim is that most recent research in mathematics education has concentrated on investigating conceptual knowledge with procedural knowledge as a “necessary afterthought” (Star, 2007, p. 134).

Baroody, Feil and Johnson (2007) offer an alternative reconceptualisation of procedural and conceptual knowledge to that outlined by Star (2005). They acknowledge Star’s work and agree that both types of knowledge can lie on a quality continuum between superficial and deep. However, they argue against the characterisation of deep procedural knowledge. Procedural knowledge might exist independently, however deep procedural knowledge includes both knowing how and why procedures work. According to them this is impossible without procedural knowledge being connected to conceptual knowledge.

When introducing these notions of deep and surface levels of knowledge Star (Star, 2000, 2005) refers to de Jong & Ferguson-Hessler (1996) who in turn cite Marton & Säljö (Marton & Säljö, 1976). Marton and Säljö carried out a study in Sweden within a phenomenographic framework. They studied students in higher education and their process of reading academic articles. The interest was mainly to explain why some students understood the text very well, when others did not. They found within this framework, in which the focus was on the experience of the learner, that students’ reading process could be categorised into two main levels. These were described as deep and surface levels of processing, which seemed to correlate with students’ outcomes of learning. Students showing a deep approach to learning, seemed to approach the text with an intention to understand the meaning of the text or to relate it to their own experiences. In contrast, students with a surface approach searched for facts and tried to remember the text. They seemed to see the text as an end in itself. However, when investigating what led to those different levels, it came out that it paralleled students’ initial intentions. Therefore, instead of levels of process of learning or reading one adopted the term surface level and deep level ‘approaches’ to learning’. “This dis-

inction between deep and surface approaches to learning thus appears to be a powerful form of categorisation for differences in learning strategies” (Entwistle, Hanley, & Ratcliffe, 1979. p. 103). Entwistle and Peterson (2004) equate the deep approach to learning with seeking meaning, and the surface approach with an intention to reproduce content.

Related to this surface approach of learning is Mellin-Olsen’s (1981) notion instrumentalism, which he introduced as an educational concept. He defined instrumentalism as a learning strategy derived from a meta-concept of understanding as instrumental understanding. The learner aims for rules, not for relations and structures (ibid, p. 351), and the motivation is not an interest in the mathematics itself, but to demonstrate some knowledge.

From the above it is clear that there is no easy way to define or to make distinctions between different types of knowledge, and it is shown that students’ intentions are important for which type of knowledge is acquired. This is also what Novak (2002) emphasises in his model of meaningful learning which is shown below:

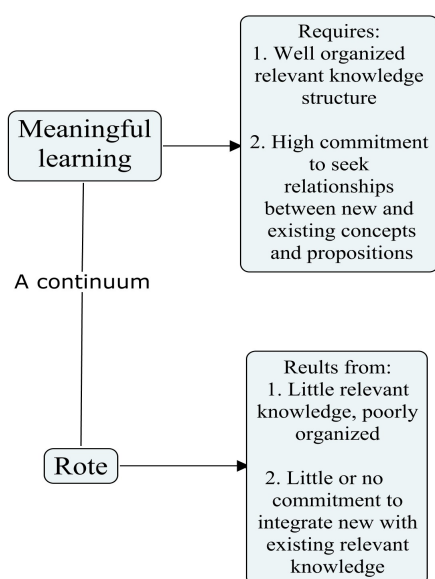


Figure 5-1: Meaningful learning (adapted from Novak (2002. p.552).

In this model, meaningful learning occurs on a continuum from rote learning till meaningful learning.

In Novak’s and also in Ausubel’s theory of learning, students’ own commitment to seek knowledge is a prerequisite for learning. This is in line with what was found to characterise a deep approach to learning (Entwistle & Peterson, 2004) which correlated with students’ outcomes of learning (Marton & Säljö, 1976). Also prior knowledge is emphasised to be highly important as the well-known quote from Ausubel states “If I had

to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel et al., 1978, iv).

In the model above it is also claimed that meaningful learning implies that the learner has a well organised and relevant knowledge structure and that new knowledge has to be connected to that structure. This is what Hiebert and Lefevre (1986) described as one type of development of conceptual knowledge.

Novak (2002) proposes that in early years most concept learning is meaningful and imbedded in context. In school much learning is rote learning of concept definitions or statements of principles, and there is often no careful integration of new concepts and meanings of propositions into already existing knowledge structures, which might result in pure rote learning.

As shown in the figure above, meaningful learning requires both a well-organised relevant knowledge structure, and a high commitment to seek relationships between new and existing concepts and propositions.

Prior knowledge and also students' commitment to seek meaning and connections in their mathematical work is crucial in order to learn, and to avoid instrumentalism. However, to orchestrate optimal classroom situations for meaningful learning is a challenge for teachers. At the level of grade 11 the students have been in schools for years being accustomed to specific cultures of the mathematics classroom and to different patterns of interactions (see section 5.2).

In this section it is shown that there is no agreement in the research community about how to make a distinction between types of knowledge, and in addition, there is a blurred use of notions.

In "Adding it up" Kilpatrick and colleagues (2001) have presented a model of what is required in order to be mathematically proficient. The model includes five different strands, where all the strands are intertwined and none of them can alone describe mathematical proficiency or competence. These strands will be defined and they will also be used in the analysis of students' work.

5.4 Mathematical proficiency

The above section reveals disagreement among mathematics educators about how to distinguish between procedural and conceptual knowledge and learning, however, what is important is to find out what constitutes mathematical competence or proficiency.

Both Niss and Højgaard Jensen (2002) in Denmark and Kilpatrick, Swafford, and Findell (2001) in the USA have worked out criteria for

what mathematical proficiency or competence means. According to the KOM-project mathematical competence:

means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. Necessary, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills, in the same way as vocabulary, orthography, and grammar are necessary but not sufficient prerequisites for literacy (Niss, 2003, p. 7).

In addition, eight distinct competencies were listed:

- Thinking mathematically (mastering mathematical modes of thought)
- Posing and solving mathematical problems
- Modelling mathematically (i.e. analysing and building models)
- Reasoning mathematically
- Representing mathematical entities
- Handling mathematical symbols and formalisms
- Communicating in, with, and about mathematics
- Making use of aids and tools (IT included)

The KOM-project has had an influence on the last curriculum in Norway, and has also made an impact on the design and analysis of the PISA studies. However, in this thesis limiting the focus to algebra and mostly to the transformational activity of the topic, led to the choice of mathematical proficiency as the applied framework. This framework was worked out by Kilpatrick et al. (2001). Five intertwined strands constitute mathematical proficiency. The strands are:

- Conceptual understanding
- Procedural fluency
- Strategic competence
- Adaptive reasoning
- Productive disposition (ibid p. 116-117).

The strands are interwoven and interdependent and to be proficient means that all strands have to be present. A prerequisite developing proficiency is that thinking strategies are emphasised.

In the next section the different strands will be outlined.

5.4.1 Conceptual understanding

Conceptual understanding correlates with what is outlined above about conceptual knowledge and is defined to be “comprehension of mathematical concepts, operations, and relations” (ibid, p. 118). It means knowing more than facts and rules, knowing both how and why. This knowledge may be tacit, in that students can understand before they are able to express their understanding.

One indicator of conceptual understanding is that students are able to use different representations for mathematical situations and choose the most useful representation.

The advantage of conceptual understanding for students is that they need not have to ‘learn’ so much because they can see connections. What is already learned is encapsulated; they have made compressions and knowledge is stored in clusters. What is learned with understanding is less likely to be forgotten, and what might be forgotten can be reconstructed on the basis of the connections made to what is known from before.

What is learned with understanding creates the basis for new learning. It enables students to check themselves, to solve unfamiliar problems, to analyse similarities and differences between different mathematical situations, and to give arguments for why procedures work. Conceptual understanding makes students confident, which again helps them to reach new levels of understanding.

5.4.2 Procedural fluency

Procedural fluency correlates to what is written above in section 5.3 about procedural knowledge, and it is defined to be “skill in carrying out procedures flexibly, accurately, efficiently, and appropriately” (ibid, p.116). Procedural fluency supports conceptual understanding in analysing differences and similarities between methods and procedures for mathematical work. One advantage of procedural fluency such as skills in performing operations and working out procedures efficiently and accurately, is that the focus can shift from a concentration on, and a struggle to follow known procedures step by step. Instead the students can use energy on developing connections and gain an overview of the topics at hand. In that way procedural fluency supports conceptual understanding. Students who learn procedures with understanding can adapt and modify procedures making them easier to use, and transfer them to new contexts.

Procedures learned without understanding are stored as isolated bits of knowledge, and then students often think that there are different procedures for different contexts. They see no connection.

However, students have often learned procedures and methods without understanding. To engage students in activities later, from which an understanding could be developed, is experienced to be a problem. It is thus important to promote teaching for understanding from the beginning.

It is important to practice in order to gain procedural fluency, which can promote knowledge of ways to estimate and to control the results of procedures. This control is also part of procedural fluency.

Kilpatrick et. al. (ibid) claim that procedures can be developed to be general procedures, which can solve classes of problems. Examples are the general rules for arithmetic operations. These rules or procedures can be concepts in their own right. Hiebert and Lefevre (1986) call them super-procedures (see section 5.3).

5.4.3 Strategic competence

Strategic competence is defined to be the ability to formulate, represent, and solve mathematical problems. This strand is connected to problem solving, and problem formulation. For this thesis, this is related to solving word problems. Having strategic competence implies that the students have grasped the problem situation and the relations between the known and the unknown quantities. Then they must represent the problem situation in one way or another and then reformulate it in mathematical symbols and operations before the problem is solved.

It is important that students can represent individual problems, but it is also necessary that they can see that some problems have the same mathematical structure.

In becoming proficient problem solvers, students learn how to form mental representations of problems, detect mathematical relationships, and devise novel solution methods when needed (ibid, p. 126).

When students have not learned procedures to follow, they have to seek for new approaches. This promotes development of flexibility. One indication of a well-developed strategic competence is that a person is able to choose flexibly between different approaches. Related to this study a strategic competent person could choose between reasoning, use of tables, guess and check, or equations depending on what is most suitable in the situation. It is of great importance that students are offered non-routine problems in order to develop strategic competence.

Related to strategic competence is also the competence in choosing the most efficient procedure in computations; which is related to a well-developed procedural fluency.

Strategic competence is intertwined with the other strands. Procedural fluency develops as students use their strategic competence to solve tasks efficiently. In order to solve cognitively challenging problems students are depending on procedural fluency. Also experiences in dealing with complicated problems promote conceptual understanding.

5.4.4 Adaptive reasoning

Adaptive reasoning is defined as thinking logically about the relationships between concepts and situations. The authors claim this strand to be the 'glue' that holds everything together. Students with adaptive reasoning can think logically about what is going on in the mathematical activity. They can explain and justify their work and their arguments. The word justify is used in the sense of 'provide sufficient reason for' (ibid, p. 130). To promote adaptive reasoning, it is said to be important to establish classroom norms which make it an expectation that students shall provide arguments and reasons for their solutions. This is a way to improve conceptual understanding. For students in this thesis it could be to explain

why the rules for fraction operations work. Adaptive reasoning interacts with the other strands especially when working with non-routine problems. When determining if a procedure is appropriate for the actual problem, adaptive reasoning is required and interacts with the strand of strategic competence.

5.4.5 Productive disposition

This strand is defined to be the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics.

This strand has to do with students' motivation and attitude to mathematics and learning (Marton & Säljö, 1976; Novak, 2002) which was a theme in section 5.3. The point in mathematical proficiency is that this productive disposition can be promoted by conceptual understanding and confidence in mathematics. The more concepts a student understands and the more fluent and flexible he or she is in working out procedures, the more sensible mathematics is for him or her.

All the strands are of interest for this study, however, the emphases will be on the first three of them.

5.5 Chapter summary

In this chapter, constructs for both the cognitive and the social perspective have been presented. Although there is disagreement within the mathematics education community about possibilities to combine the two perspectives, in this thesis it is chosen to follow Bauersfeld and Cobb's (1995) suggestion to look at the mathematical activity within the classroom from both perspectives.

Constructs from the social perspective used in this thesis are constructs developed within social interactionism: 'Patterns of interaction', 'negotiation of meanings' and 'meaning taken as shared' (Bauersfeld, 1988; Cobb & Bauersfeld, 1995; Voigt, 1989, 1994, 1996). Another construct is Brousseau's notion 'didactical contract' (Brousseau, 1997).

From the cognitive tradition, the constructs used in the analysis of the data are 'concept map' (Tall & Vinner, 1981), 'procept' and 'proceptual thinking' (Gray & Tall, 1994), and 'structure sense' (Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999). Further an outline was given of different types or perspectives of mathematics (Hiebert & Lefevre, 1986; Skemp, 1987), and different approaches to learning (Entwistle, et al., 1979; Marton & Säljö, 1976; Mellin-Olsen, 1981; Novak, 2002).

The last sections in this chapter were devoted to an outline of the different strands constituting mathematical proficiency (Kilpatrick et al.,

2001). For students to reach the goal of being mathematically proficient, they must be offered opportunities to develop all strands in the proficiency model.

In the next chapter research linking teaching and teaching material to students' learning will be reviewed.

6 Opportunities to learn

In “Adding it up” (Kilpatrick et al., 2001), the ‘opportunity to learn’ is said to be the “single most important predictor of students’ achievement” (p. 334). This opportunity to learn, the authors assert “can be influenced by individual students, their teachers, their schools or school districts, or even the country’s educational system” (ibid, p. 334). Within the classroom the opportunities are dependent on the actual culture constituted interactively among the students and the teacher. In addition, for a mathematical topic to be learned it must at least be covered. However, it also means that if a learning goal is for students to be fluent in using procedures they must have opportunities to develop such fluency. The same is the case for conceptual understanding, problem solving skills and so on.

In this chapter, research done on links between teaching and learning of mathematics, will be presented. In addition, some research done on teaching material such as the mathematics textbook, example tasks, and assigned tasks will be reviewed.

6.1 Teaching and learning - connections

Hiebert and Grouws (2007) reviewed research done on the effects of classroom mathematics on students’ learning. Their aim in the review was to describe patterns found in research that were significant across grade levels. In addition, they looked for studies describing classroom teaching which included well-described measures of students’ learning. Those studies should have been designed to examine the connections between the nature of teaching and what and how students learned.

The investigation of research according to the criteria above, showed that teaching promoting procedural fluency, was characterised by teaching which is:

rapidly paced and includes frequent teacher modelling of procedures, the use of many teacher directed product-type questions, and the making of smooth transitions from demonstration of procedures to substantial amounts of error free practice. The teacher plays a central role in organizing, pacing, and presenting information in this type of teaching in order to meet a well-defined learning goal (ibid, p. 382).

It is noted that none of the studies on which the claim is based, was set out to investigate specifically the relationship between students’ procedural fluency and the teaching, but the tests used in those studies included mostly tasks in which students had to use procedures. Those studies were mostly from the 1970s and 1980s.

Conceptual understanding which was defined to be “the mental connections among mathematical facts, procedures and ideas” (ibid, p.382), the authors found to be promoted by two key features. The two are:

- Teachers and students attend explicitly to concepts
- Students struggle with important mathematics (ibid, p. 383 and 387).

The first point stresses that concepts and connections between mathematical facts, procedures, and ideas must be explicitly presented. Discussions of different solution strategies and how they are related, and attention to relationships among mathematical ideas were reported to enhance conceptual understanding.

The second point emphasises that students should struggle with important mathematics. It is stressed that the word struggle is used “to mean that students expend efforts to make sense of mathematics, to figure something out that is not immediately apparent” (ibid, p. 387). These findings seem to correlate with the goal to encourage students’ ‘commitment to learn’ (Novak, 2002), and a deep level approach to learning (Entwistle & Peterson, 2004; Marton & Säljö, 1976), see section 5.3).

The authors (ibid) claim this is the opposite of presenting facts to be memorised, or asking students to solve only familiar tasks with demonstrated solution methods.

To investigate the links between teaching and learning, or between curriculum materials and students’ learning is not a straightforward task. This is evident also in the review carried out by Stein, Remillard, and Smith (2007). Much of the research they refer to, are comparative studies, and they (ibid) differentiate between what they call ‘conventional’ curriculum materials and ‘standard-based’ curriculum materials.

Conventional programs or materials are characterised by their tendency to explain directly the material and definitions to be learned. The topics are carefully sequenced in that lower-level skills are accumulated before students are given “the opportunity to engage in higher order thinking, reasoning, and problem solving with those skills” (ibid, p. 331). Definitions are explicitly given, and students are expected to master those in addition to standard procedures before they are given opportunities to apply their knowledge.

In contrast, students working with ‘standard-based’ materials are expected to deal with carefully designed tasks in order to engage with mathematical concepts before they are explicitly explained or defined. The purpose is that students themselves should explore features within the concept. Then, the students are exposed to formal definitions and notions and problems in which they are applied. It is expected that the teacher is there to guide in the process to guarantee that students’ conceptions are aligned with formal definitions accepted by the mathematical community.

The studies referred to in the review (Stein, et al., 2007) were carried out in the USA, and they are not immediately transferable to the Norwegian situation. It seems from the studies that at least the standardised (NCTM-standards) materials offer detailed guidelines for each lesson. Also in studies investigating textbooks, what Stein et al. (ibid) call conventional material, it seems as if there is an expectation that each lesson is planned in detail by the author of the textbook.

In Norway the situation has been that the curriculum presented through directives and plans for the mathematics subject has contained detailed prescription of goals to be reached at each level. Out from this the publishers have offered textbooks in which an interpreted version of the official curriculum is presented (see section 2.5). Normally there are few guidelines for teachers on how to use the textbooks except for timetables containing the estimated time allocated to each chapter. In addition, the publishers offer tests for students after each chapter and before the end of each semester. In recent years, the publishers also offer web-sites for students with interactive exercises, links to related web-sites, and sometimes software for computers. Guides for teachers are relatively rare. This means that the normal situation in Norway is as outlined in section 2.5 that most teachers rely on the textbooks, at least in upper secondary school, and most often these textbooks can be said to fit within the category of conventional material in the vocabulary of Stein et al (ibid).

Hiebert and Grouws (2007) as well as Stein et al. (2007), emphasise that in order to look for links or relationships between teaching or the use of curriculum materials and students' learning, one has to investigate mathematics classrooms and how the curriculum materials are implemented. However, much research has been carried out under the assumption that the influence of curriculum materials on students' learning is directly measurable from standardised tests. These studies according to Stein et al. (ibid) are valuable in the sense that they confirm that content matters. This is in accordance with the claim that students must have the opportunity to learn; and what is not included in the content, students will presumably not learn. This is one reason why the textbook and other resources will be investigated in this thesis.

6.2 Mathematics textbooks

The strong position of the textbook in Norwegian mathematics classrooms is reported in several studies (see section 2.5). In addition, the mathematics textbook is one important factor within the setting of the mathematical classroom in this study. Thus some literature on mathematics textbooks will be reviewed, and this will be followed by a review of some studies on mathematical tasks relevant for this study. The intention is not to equate

the content of mathematics textbooks with what is enacted in the classroom.

Also internationally mathematical textbooks or curriculum material are widely used. This has over the years resulted in studies investigating mathematics textbooks (Haggarty & Pepin, 2002; Herbel-Eisenmann & Wagner, 2007; Love & Pimm, 1996; Weinberg, 2010; Weinberg & Wiesner, 2011).

Love and Pimm (1996) examine visible features of texts in mathematics textbooks, and investigate their pedagogical functions. They claim that the most prevalent way to structure mathematics textbooks is the “exposition-examples-exercises model” (p. 385). The exposition part is intended to support students’ learning of specific concepts or topics in mathematics. Sometimes the author(s) may use questions, visual material or tasks for the students to work out (ibid, p. 387). Love and Pimm assert that this part of the model is most influenced by the writer’s own theory of learning. Since the text is written in order to explain or expose a particular part of mathematics, the text is heading to a particular destination which is clearly signalled. Often this destination is signalled by fonts, colours, or in other ways that will catch students’ attention.

The examples in a textbook might serve different purposes. Love and Pimm (ibid) suggest that they are meant to be paradigmatic or generic in some sense. By reading or working out the examples, the students are often offered models of how to solve the following exercises by imitating the worked examples. The assumption is that the students will generalise from the examples. In reviewing research on examples in mathematics education Bills and colleagues (Bills et al., 2006) distinguish between examples of a concept and examples of how to apply a procedure. The latter is further divided into two categories; examples as worked out by the teacher or the textbook author; mostly accompanied with some explanations or comments and exercises for students to complete. How useful an example is, is subjective, depending on the context as well on the learner (ibid). They argued that two attributes make an example pedagogically useful:

- Transparency: making it relatively easy to direct the attention of the target audience to the features that make it exemplary.
- Generalisability: the scope for generalisation afforded by the example or set of examples, in terms of what is necessary to be an example, and what is arbitrary and changeable (p. 135).

In addition, they emphasise that it is important to make underlying generalities and reasoning explicit because students seem to try to generalise from examples; even those students, who are not really engaged in what is being taught. Studies about misconceptions (Clement, 1980; Kaur &

Sharon, 1994; Nesher, 1987) and learning obstacles (Brousseau, 1997) confirm this. Bills and colleagues (2006) emphasise that it is important to give students opportunity to meet a variety of examples, non-examples and counter examples.

However, what teachers or textbooks provide is just one part of the issue. The authors (*ibid*) stress that the most critical influence is what opportunities students are offered in order to work on and with examples.

Pimm and Love (1996) emphasise the transparency but conclude that: ...many texts do not state directly what generalisations the students are assumed to have made. Whether this is to encourage students to form the generalisations for themselves, or whether it indicates the assumed presence of a teacher – or even whether it is simply not thought necessary to include such material – is not clear (*ibid*, p. 387).

Some books do highlight difficulties, and explain explicitly the steps in the solution processes of worked examples. Some present more than one approach and explain why they are applied. This might be part of a meta-cognitive strategy which reflects the author's presence in the text (Pepin & Haggarty, 2001).

The last part of the 'exposition – examples – exercises' model, is the part with exercises made for students to solve. Love and Pimm (1996) claim that this is the principle means by which the students are encouraged to be active readers of the text. They assert though that the exercises often are similar to the immediately previous solved examples and graded from easy to more difficult. This organisation of tasks is according to them (*ibid*) reflecting a belief that learning is best achieved when students are progressing in small well-defined steps. The risk is that the students develop fragmented knowledge.

The exercises may also be presented in parallel sets of differing levels of difficulty in order to offer encouraging exercises for students needing extra practice or for those who need to be stretched. This was the case in the textbooks investigated by Johansson (2006). In some textbooks there is a collection with more varied exercises at the end of each chapter or at the end of the textbook. The parts in the textbook with exercises can combine a variety of those aforementioned structures.

Studies of textbook use have shown that students often skip the exposition part of the text. Instead they go directly to examples, rules or didactical advice emphasised by fonts and colours (Love & Pimm, 1996; Weinberg, Wiesner, Benesh, & Boester, 2011).

Herbel-Eisenmann and Wagner (2007) studied the language used in mathematics textbooks. They claim that there is almost no use of the first personal pronouns, I or we, and that this might distance the reader from the author. The use of the second personal pronoun, you, address the read-

er directly and thus connects the reader to the mathematics, however, it might also be used in a general sense not addressing any particular person. The expressions such as ‘you think’, ‘you know’, ‘you find’, are examples of defining what is common knowledge. In other cases, this pronoun might mask human agency. Examples are: the graphs show you, the equation tells you. Only the reader him/her self decides what the graph shows, or what the equation tells¹².

The imperative mode of verbs is commonly use in mathematical tasks and in rules presented in mathematics textbooks. Morgan (1996) suggests that the imperative mode implies a competent model reader, who can perform and follow the orders given. Rotman (1988) distinguished between inclusive and exclusive commands. Inclusive imperatives in linguistics are verbs of the type, ‘Let’s go’. Verbs such as ‘consider’, ‘define’, ‘prove’ and all their synonyms imply according to him (ibid), that the writer and the reader share a common ‘world’ of mathematics and are thus inclusive imperatives. Herbel Eisenmann and Wagner (2007) call the inclusive imperatives, ‘thinker’ imperatives. The other imperatives are commands demanding that the reader performs specific actions.

The notion ‘closed text’ was applied by Love and Pimm (1996). This notion is used to describe authors’ attempts to lead the reader to follow a clearly defined path. The danger is then that the students might experience mathematics as a subject which is indubitable, and that problems can be solved in only one way. Open texts are on the other hand textbooks offering resources for students to reason about the actual topic, with an expectation that students construct their own conceptions.

Most textbooks lie on a continuum between these two authorial stances; the closed texts and the open texts. The extreme positions might both lead to problems for the reader. If the writer acts solely as a guide, the reader is in danger of missing the important points, while on the other end, a too narrow step by step directing of the reader might lead to a fragmentary conception of mathematics.

Most textbooks relate mathematics to contexts which are expected to be known by the reader. This is the case for both the expository part, the examples, and the exercises. In the report from TIMSS video study 1999 (Hiebert et al., 2003) two tasks are used to describe the distinction between tasks with a real-life context, and one without such a context: “Estimate the surface of the frame in the picture below” (p.84), and “Find the

¹² As distinct from the English language, the Norwegian language has different forms in singular and plural of the second personal pronoun, making the distinction between the two forms clearer. Examples from the actual textbook: ...når **du** har lest - ...when you (sg) have read. Kanskje opplever **dere** ... - perhaps you (pl) are experiencing ...

volume of a cube whose sides measures 3,4 cm” (p.84). The difference between the two examples is that the first is related to a picture, something “real”, while the other problem is about a hypothetical cube. This is a simple illustration distinguishing between real life problems and problems without real life connections.

In this study there are no real ‘real life’ tasks according to this definition. It is well known that within the mathematics education community there has been, and still is, discussion about what is ‘real life’ and what is not, and if it is useful for students when problems, examples and exercises are connected with real life situations. All contextual tasks are set in an imagined or hypothetical context in the actual textbook in this study. The notion ‘real life’ will be used though.

6.3 Mathematical tasks

Mathematical tasks or exercises are important in all mathematics classrooms. In the TIMSS 1999 video study, classrooms in seven countries around the world¹³ were videotaped. It was found that at least 80 % of lesson time was commonly devoted to mathematical tasks (Hiebert, et al., 2003). Although this study concerned mathematics in grade 8, the results from TIMSS advanced (Grønmo, et al., 2010) showed that the situation in grade 13 did not differ that much. Students in upper secondary school participating in mathematics courses, and their teachers answered in questionnaires that the most common activity in the mathematics classes was to solve tasks; tasks similar to the examples provided in the textbook.

The considerable time spent on mathematical tasks in mathematics classrooms is one reason for focusing on them. Another reason is that the tasks students work on, influence their view of the nature of mathematics and what they learn. Stein, Remillard and Smith (2007) claim:

Tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that demand engagement with concepts and that stimulate students to make connections lead to a different set of opportunities for student thinking. (p. 347).

Studies done on tasks implemented in the mathematics classrooms conclude that tasks which are cognitively challenging, and which are not taken over by teachers or higher achieving peers, promote students’ conceptual understanding (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996; Stein, et al., 2007). In the work on mathematical tasks and students’ learning (Stein, et al., 1996; Stein & Smith, 1998) Stein and colleagues developed a framework for analysing mathematical tasks according to the potential levels of cognitive demand. They categorised the

¹³ The countries were Australia, Czech Republic, Hong Kong SAR, the Netherlands, Switzerland, Japan, and the United States.

tasks into two levels ‘the low level of cognitive demand’ and the ‘high level of cognitive demand’, which in turn were divided into two sub-categories each. For the first category, the two subcategories were: ‘memorisation tasks’ and ‘procedures without connections to meaning’. The latter main category was divided into the ‘procedures with connections to meaning tasks’ and ‘doing mathematics tasks’ (for a thorough description of each category see appendix 4).

This categorisation was done for assigned tasks. During classroom observation, however, it was clear that although a task was categorised to be a ‘high level demand task’, it was difficult to keep the high level of challenge (ibid), and often high level tasks were experienced declining into lower cognitive levels. The reason was mostly that students pressed for help, and thus they avoided the mental struggling period, which according to Hiebert and Grows (2007) was one prerequisite for the development of conceptual understanding (section 6.1).

There are several definitions for what is a task. Doyle claims that the concept of task:

calls attention to four aspects of work in class (a) a goal state or end product to be achieved; (b) a problem space or set of conditions and resources available to accomplish the task, (c) the operations involved in assembling and using resources to reach the goal state or generate the product, and (d) the importance of the task in the overall system in the class (Doyle, 1988, p. 169).

A task he said, is defined “by the answers students are required to produce and the routes that can be used to obtain these answers” (Doyle, 1983 p. 161). This means that for him a task (he used the notion ‘academic task’) includes the products students are to produce, the operations they have to execute in order to generate the expected products, and the resources available for students in this production process. Stein et al. (Henningsen & Stein, 1997; Stein, et al., 1996; Stein, et al., 2007) based their definition on Doyle’s work. The difference lies in the duration or length of a task. In their work, a mathematical task is defined to be a classroom activity, whose purpose is to focus students’ attention on a particular mathematical idea. An activity is not classified as a new task unless the underlying mathematical idea of the activity changes.

Niss (1993) defines a task to be “an *oriented activity*, i.e. a set of actions oriented towards undertaking certain *missions* such as orders, proposals, or challenges” (ibid p.17). He further asserts that a task can be formulated orally or in written form, often by using the imperatives: compute, solve, draw, construct, determine, describe and so on. They might also be formulated as questions. In all cases the ‘mission’ is to carry out the task and to come up with a solution or an answer. He listed three categories of tasks: questionnaires, exercises, and problems. The first he de-

defines to be “a collection of questions that usually concern facts” (p.18). There are normally only one or very few options for correct answers to each question. The next category, exercises, are tasks which involve primarily routine considerations or operations to be executed in straightforward combinations, and which are set within a well-defined part of a given mathematical topic. The last category, problems, Niss (ibid) are tasks involving non-routine considerations, or operations, or they are set within a context not related to a specific topic in the syllabus.

Problems and exercises are emphasised as not being absolute concepts. The reason for this is that an exercise might be a problem to one person, while it might be a routine task – an exercise - to another. Types of tasks which have been problems for a student might later in the development be perceived as an exercise for the same person.

In the report from the TIMSS Video Study 1999 (Hiebert, 2003) all mathematical tasks are termed problems, no matter if they are problems to the persons trying to solve them or not. According to the coding manual from the TIMSS Video study a mathematical problem is defined to be an explicit or implicit problem statement that includes an unknown aspect, something that must be determined by applying a mathematical operation. Included in the mathematical problem is the solution or answer, as well as the checking of the answer. This means, in the TIMSS terminology, that as long as there is an expected answer to be found by mathematical operations, there is a mathematical problem.

In this thesis a mathematical task will be defined according to the definition of a mathematical problem in the TIMSS Video Study, and it will include questionnaires, exercises and word problems (Niss, 1993). This means that as long as there is an expected answer to be found by mathematical operations, there is rather a mathematical task than a mathematical problem as in TIMSS.

The reason for the choice of the notion task, instead of problem, is that as Niss (ibid) emphasises, a problem is a problem in relation to the individual student. Another reason is that the assigned tasks were mostly of short duration. This might be due to the mathematical content and to the case that only a short time period was allocated to basic algebra.

However, the investigation has to go beyond the content and the quality of tasks and examples. Stein, Grover, and Henningsen (1996) provided a model in which a task is seen to pass 3 phases. The first phase is the task in the textbook or in other written materials, the second phase is the task set up by the teacher, and the last phase is the task implemented and worked upon by the students in the classroom. This means that the task as it appears in the textbook or in other materials might be altered from phase to phase and even in the classroom situation.

In this thesis the tasks will be categorised in line with former studies about number and algebra, and analysed according to levels of cognitive demand. Then the implementation of the tasks will be focused upon.

6.4 Mathematics syllabus and tasks

According to the perspective outlined above it is important to provide opportunities to learn mathematics. The ultimate aim of mathematics teaching is to enhance students' mathematical proficiency (Kilpatrick, et al., 2001) or mathematical competence (Niss, 2004). According to both frameworks one cannot hold isolated strands of proficiencies or individual mathematical competencies, and at the same time claim to be mathematically competent or proficient. However, in school, different strands or competencies are emphasised at different times, although the aim is to promote connections and an overview of mathematical strands.

The conventional curriculum model as Stein et al. (2007) describe it, is focused on carefully sequenced theory and tasks. This is still the case for most commercial textbooks. Although the goal is that students are to reach a level where mathematics is seen as connected and meaningful, one has also to practice skills in order to be fluent.

One concern has been that the focus on conceptual learning of mathematics would result in students having problems executing procedures and computations. Studies, however, show that students who have followed a reform based curriculum with emphasis on conceptual learning tend to perform as well as students following a conventional program on standardised tests (Swafford, 2003). In addition, those students outperformed students following a conventional program, when it came to tasks testing what had been emphasised in the reform curriculum.

In this chapter, curriculum material, textbooks and tasks have been discussed and related to students' opportunities to learn. It is problematic to make direct links between curriculum material and learning although there are indications that the different features of curriculum material might influence students' learning in different ways (Remillard, Harris, & Agodini, 2014). One result from earlier studies is clear though. Topics covered in the material are more likely to be learned than topics not covered (Stein, et al., 2007). It is also clear that cognitive challenging tasks promote another learning than tasks of a low level of cognitive demand (*ibid*). Grouws and Hiebert (2007) in a meta-study found that different teaching promoted different learning, and came to a conclusion about what promoted conceptual understanding and what promoted procedural fluency (see section 6.1). This will be drawn upon in the further work with the data in this thesis.

7 Methods and methodological considerations

The word methodology has often been used as a synonym for methods. Burton (2002), however, asks researchers to make clear, on which grounds the decisions about methodological issues are made. According to her it is not sufficient only to outline what is done, it is also necessary to explain the rationale for the choices, which are undertaken. She says:

I do not believe that there is ever a case where the researcher's beliefs, attitudes, and values have not influenced a study (p.3).

The purpose of this chapter is to meet Burton's demands and let the reader have a chance to follow the 'journey' of this research process, and to make my assumptions, beliefs and attitudes open to the reader.

The study was carried out within the TBM – LBM projects (see appendix 1) that were based on the philosophy of creating ownership to the projects among all participants, both researchers and teachers. Thus the teachers and their schools worked out local goals to reach within the larger project. This implied an aim of not imposing anything on the teachers, but instead to collaborate with them aiming at what Wagner (1997) calls a co-learning relationship. This made an impact on the way to cooperate with the teachers in the project (Espeland, Goodchild, & Grevholm, 2009). The aim was to join an authentic mathematics classroom, not making impact upon the teaching and the structure of the mathematics lesson.

My main data was collected in the first months of the project, and the study was not aiming at describing a developmental process within the project. Instead, I see the study as a snapshot within the frame of a long developmental project, exploring the mathematical activity within the context of an ordinary classroom.

7.1 The research questions

As a teacher in upper secondary school, I had seen high achieving students struggling with simple fractions, with decimal numbers and with manipulation of simple algebraic expressions, when solving tasks related to more advanced mathematics. This led me to a question about how teachers might respond to students' problems when these basic elements, expected to have been assimilated in lower grades, would occur. Another question was what rationale teachers might have for their responses.

Therefore, the purpose of the study at the outset was to investigate teachers' actions and the rationale for their actions when meeting students' problems related to number and elementary algebra, when working with new topics in upper secondary school.

As time went on within the TBM project the focus changed from being explicitly focused on teachers' action, and rationale for those actions, to focus on the mathematics, and student's mathematical activity when numbers and algebra are recapitulated at the start of upper secondary school.

The purpose for this research is thus to explore the relationship between the mathematics offered in the classroom, and students' experiences of that mathematics within the setting of the mathematics classroom. The research questions are (see section 2.9 at the end):

What mathematics are students offered in the classroom for learning, or to consolidate already learned mathematics as they enter the upper secondary school?

What experiences of the mathematics offered do students reveal through their responses?

How does the setting of the mathematics classroom influence students' experiences of the mathematics offered?

Sub-questions to answer the main questions are:

- What are the learning goals?
- What mathematics is presented in the classroom?
- What can we observe of students' responses to that mathematics?
- What patterns of interactions are observed?

The word response in this context includes all students' task solutions, answers, questions and comments both oral and written during the lessons, interviews and tests.

The setting of the mathematics classroom includes the textbook in use, the work plan, the computer with provided software, teacher's exposition, and the interaction in the classroom.

7.2 The case and the context

Since a classroom has to be investigated in order to explore the mathematics offered, the mathematics students' experience, and how the classroom setting influence students' experiences, it followed that the study would be a case study and that the methods would be qualitative methods. In this section, case study as such is discussed. Then the actual case and its context is presented.

7.2.1 Case study

In a case study the researcher has little control or no control of the factors influencing the events in the classroom. A case study is a "study of singularity conducted in depth in natural settings" (Bassegy, 1999, p. 47).

Mertens (1998) discusses different definitions and views on case studies and concludes that “the commonality in the definitions seems to focus on a particular instance (object or case) and reaching an understanding within a complex context” (ibid, p.166). She also referred to literature discussing whether case study is a method or a research design. Yin (2009) claims that a case study is a method with its own research designs.

What according to Yin (ibid) decides if a case study is an appropriate method for research is the form of the research questions. The most appropriate questions are the ‘how’ and ‘why’ questions. In this thesis, the research questions involve also ‘what’ questions: ‘What mathematics ...and what responses...’ Although ‘what’ questions, the answers searched for are descriptions of the mathematics and of the responses. Thus, the case study as a research method is seen to be appropriate.

In a case study, the theoretical framework and the research questions are the steering guides of how to define the case. Also Miles and Huberman (1994) give the advice to: “attend to several dimensions of the case: its conceptual nature, its social size, its physical location, and its temporal extent”.

In this thesis, the case is the specific class under observation. According to the chosen theoretical framework, the class includes many sub cases, the individual students.

In following Miles and Huberman’s (ibid) advice above, the boundaries for the case related to the social size, the physical location and its temporal extent was set. The specific class is a class located in one of the upper secondary schools participating in the LBM project described in appendix 2.4. The class had 27 students during the time of observation. The contact with the teacher started two weeks before the class started school, then the observation went on intensively in two periods in September and November 2007 with some meetings in between. Then there was a new meeting with the class in May 2008. This means that this case study has boundaries when it comes to size, location and time.

The unit of analysis is the student within the classroom setting. This includes the didactical means: the textbook and its resources, the computer and the software in use, the teaching, the classroom interactions, and student’s mathematical activity working with mathematical tasks.

7.2.2 Participants and the classroom

To talk about a random sample of a population has no significance in my case; also it is not an extreme case. Working in the TBM-project entailed cooperating with someone within the project. I aimed to work with a teacher from upper secondary school. The teacher who accepted to join

this study is an experienced teacher, and she has worked in the same school for many years. When her school decided to participate in the project, she signed on as did four others in her school. As she said, she is used to working in projects since she, together with some colleagues, had recently worked out and set up a plan to use lap-tops in mathematics for all students in the “Programme for Specialization in General Studies” (see section 2.2.2).

The teacher was engaged in teaching many mathematics classes. The class in this study was a class in grade 11. This was the second year of the K06 curriculum (see section 2.2.2) which implied working with a new textbook and a new curriculum. The teacher expressed that the pressure of the lack of time weighed heavily as the new curriculum included more topics than the former curriculum.

The 75 students, attending the programme in this school, came from different schools and mostly they did not know each other. At the start, they made a judgement about their own abilities and goals in mathematics. On the basis of those judgements, they were divided into three different groups. The observed group or class was seen as a high achieving group of students since all of them were aiming towards the highest mathematical education in upper secondary school. In the class, 27 students started; two went to another group and two came as new students during the autumn.

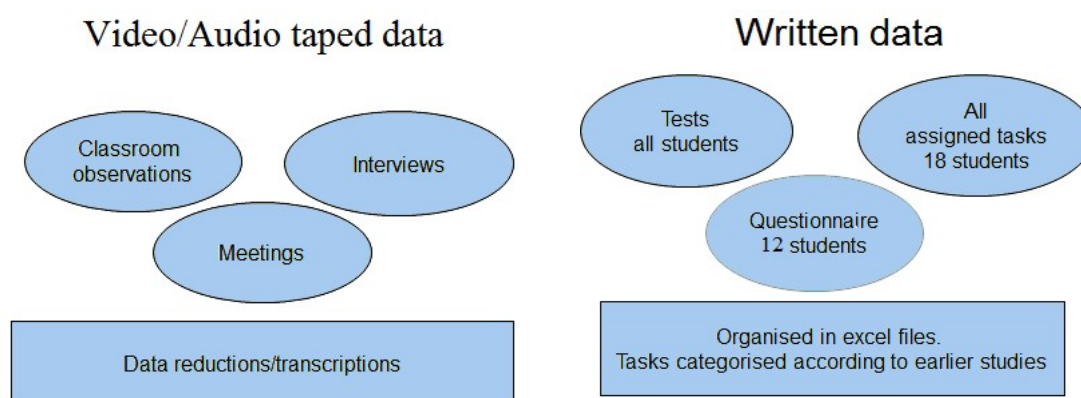
All students had their own computer. The classroom was equipped with all facilities making it convenient for both teacher and the students to use their computers in their daily work. In addition, the technical support was reported by both teacher and students to function well. When I was there, they used their computers to write mathematics and to draw graphs. The programs were MathType and TI-interactive. All written tests including the examination were done on the computer. The classroom can be classified as a paperless classroom. The teacher, however, wrote mostly on the whiteboard during plenary sessions. Exceptions were when the technique of writing the tasks in MathType on the computer was introduced at the end of the plenary sessions.

More background information about the school, the classroom, the teacher, her practice and thoughts, the students and students’ use of, and opinions about the use of computers in mathematics lessons is to be found in appendix 2 and its sub-sections.

7.3 Data and data collection

In the attempt to answer my research questions, I regarded it appropriate to collect different kinds of data. Although classroom observation and interviews play a considerable role in my study, I also draw upon data from

students' written work, from meetings with the teacher about the teaching, and learning, the class and the school, and from some meetings within the TBM/LBM project when teachers' views on topics covered in the observed class came to the fore. In this section I will go into what data I have collected and what methods I have used. An overview of the collected data is presented in the figure below.



Figur 7-1 Types of collected data

In the case study method different kinds of data might be collected, both quantitative and qualitative data (Yin, (2009)). This thesis will rely on qualitative data, and is located within the interpretivist paradigm.

7.3.1 Classroom observations

My main data corpus is data from the classroom. The collection started two weeks after the start of school, after one week with mathematics lessons. I divide the observations into two parts according to the mathematical topic taught and to the period of time. The first period was devoted to number operations and elementary algebra, the second was devoted to linear functions and systems of equations. The teaching followed the textbook, and these periods correspond to chapter 1 and chapter 4 in the textbook. The data from the second period is drawn upon as a comparison to the teaching of the topics in the first chapter. In addition, students' work with tasks related to the learning goals for the first chapter is drawn upon.

To collect data, I videotaped each observed lesson from the start to the end. In addition, the teacher carried with her an audio recorder. The reason for this was to audio tape the conversations between the teacher and the students during seatwork. During the plenary sessions I either administered the video camera, or I sat at the back or in the front of the classroom taking notes. This depended on whether a colleague assisted me or not during the data collection. Mostly we were two.

During the seatwork, I went around with an extra camera, videotaping students' PC-screens and thus students' work. On occasions, I interacted with them during the seatwork asking them questions about what they were doing. It happened that students asked me or my colleague for help. We had discussed this and agreed that ethically we could not refuse to help, when there were others waiting for the teacher. However, mostly they waited for the teacher, or asked their peers.

7.3.2 Interviews

During the collaboration with the teacher, I had many meetings with her, and sometimes with her and some of her colleagues. All the meetings were audio taped. Most of these meetings I will characterise as unstructured interviews, conversational in style. Some of those interviews took place right after a lesson, when we were discussing the lesson, some were planned in the sense that we had made an appointment to meet and to hold an interview. Other interviews were semi-structured in the sense that I had a list of questions to ask, however, the teacher was free to lead the interview in the direction she wanted. For me it was important to listen to her opinions and thoughts. Although this was a strategy, it was as well for me a genuine interest in her practice.

All the students had a short interview twice as to their opinion about using computers instead of paper and pen. Another time they were asked about their experiences of working in groups to find the solutions without the teachers' explanation beforehand. These interviews were structured in the sense that they all were asked the same few specific questions; more in the style of a survey.

Five students were interviewed based on tasks picked out from both the algebra test, and the first ordinary test. Four students were interviewed twice; once in the autumn and once again in the springtime right before finishing the school year. One interviewed student, a girl, changed class during the school year. Another girl, from whom I had a considerable amount of data from the classroom observation, was chosen for interview in the spring.

The reason for choosing task-based interviews was to illuminate the reasoning behind students' solutions. Test results alone give limited information about students' reasoning. Goldin (2000) claims:

In comparison with conventional paper-and-pencil test-based methods, task based interviews make it possible to focus research attention more directly on the subjects' processes of addressing mathematical tasks, rather than just on patterns of correct and incorrect answers in the result they produce (*ibid*, p. 520).

The questions were focused on the solution of the tasks, descriptions of the strategies, and assessment of the solution.

The five students were chosen based on their results on tests. I wanted students representing both top, bottom and middle level. Two girls and three boys were chosen. In the class the proportion between girls and boys was 10:17.

In the autumn the interviews were video and audio taped. Some students felt the filming uncomfortable, and the camera was moved to videotape from a relatively long distance. In the spring the interviews were only audio taped.

From my master study I had positive experiences with focus group interviews. When the students talked together during task solutions, they were more open than when being interviewed alone. Therefore, my intention was to do the same in these interviews, however, this did not function as intended. One of the girls remained rather quiet during the whole interview session. She had to be prompted. This resulted in the boys being interviewed one by one and in the spring all were interviewed individually.

All students had paper and pencil available and their notes were collected. The interviews were of different time duration. They lasted from 6 to 40 minutes. The short interview was with Tord. He had made few errors and was extremely quick in his answers. In addition, he wanted to reach the next lesson, he answered all the questions though.

7.3.3 Tests

Based on my findings from my master study, I conducted an algebra test, this was a paper and pencil test. This test was administered by the teacher after the first week of the school year. At that time algebra had not been the focus for teaching. The tasks were taken from studies such as the CSMS study (see section 4.1) and the KIM-study (see section 1). Most of the tasks were also present in the tests in the LCM-study¹⁴. The reason for not making my own tasks, was to be sure that they had been revealed as purposeful in earlier studies. Another reason was that I could compare the results in this study to the results from the LCM, KIM, and CSMS studies. This test was given twice for students to solve; in August 2007, and 8 months later in May 2008.

In September, 25 out of 27 students were present. In April, 26 students solved the test tasks. At that point in time two students had moved to another class, and two new students had moved in. One student was missing.

¹⁴ The LCM study was part of the KUL Programme (Kunnskap, Utdanning og L ring – Knowledge, Education and Learning) of the Norwegian Research Council (Norges Forskningsraad, NFR). My master study was part of this study. The study relied mainly on tests in grade 4, 7, 9, and 11. The tests were administered by the teachers in the participating schools and were given twice a year in three consecutive years.

That means that only 22 students participated in both test situations. Since only 24 out of the students had followed the same teaching in the autumn, the other students are left out when I refer to the test in the spring. To conclude: 25 students in the autumn and 24 in the spring are in the sample. 22 of them can be compared from autumn to spring. Both tests were administered by the teacher. In May I was there observing. It seemed as if the students worked seriously trying to do their best.

All the ordinary tests were written on the computers and delivered to the teacher as computer files, and I was allowed access to these files. In the first lessons after both periods of observation, students were tested in the topics taught during these observations. Some tasks from the algebra test and some from the ordinary test in September were the basis for the interviews with the students. In addition, one test in December, concerning all the topics taught during the autumn is part of my material. From the latter, only tasks related to the learning goals expressed in the first chapter of the textbook are taken into account. (All tests are presented in appendix 6.)

7.3.4 Questionnaire

At the outset of my study, I planned to give the students a questionnaire with some questions about their views and experiences of mathematics. This questionnaire was mainly a translation of a questionnaire applied by Persson in his study (Persson, 2005). In order not to disturb the teaching, it was decided to send this to the students electronically and ask for answers by e-mail. This was not a success, as less than half the students replied (12 students). In the parts of the thesis describing individual students, the answers in the questionnaire will be drawn upon for the students who answered. (The full questionnaire in Norwegian is to be found in Appendix 7).

7.3.5 The ‘learning book’ and students’ solutions

The teacher and her colleagues had created a web site for all the students attending the mathematics course in Vg1. They had worked out examples, mainly from the textbook, and short versions of rules and didactical advice stored in data files; one file to each sub-chapter in the textbook. The collections of these files were called the ‘learning book’, and were created to help students to be focused on the white board and to participate in the plenary session. For the fast writers the teachers had experienced that students were often distracted by the computers; with a keystroke they could mentally leave the classroom. Thus all worked examples, rules and advice were to be found later on the web-site. In addition, the teachers encour-

aged students later to write their own notes in the same files, and save them on their private computers.

I was interested to see if students had saved their own notes and thus made changes in their 'learning books'. All students answered a short survey, in addition some students down loaded their files in order to show me their work. This was done at the end of the second period in November. Later when looking into their files I realised that some had sent me their complete work done during the autumn term.

At the outset I had rejected the idea of looking into their work with assigned tasks. One reason for this was that students had answers to all the tasks in the back of the textbook, another was the overwhelming amount of data. However, when I looked into the files I had been given access to, I realised that many students had not bothered to check their answers, or they had adjusted them to the right solution in different ways. At that time, I also saw the advantage of comparing the files with the observations made in class. I went back to school, and the students willingly shared the solutions to their assigned work during all the time of observation.

From the date on the files I could see who had changed the files or who had down loaded ready-made solutions from the publisher. The latter files were rejected. In addition, some students had suffered a computer crash which had deleted all their files. Data from 18 out of 27 students constitutes the data from students' saved computer files.

For the students it was helpful to have their computer well organised, because they were allowed to use any resource apart from communication and internet access on tests.

7.3.6 Other data

Other data are official documents from the school, the textbook and its resources, official documents related to the syllabus, and data from meetings and workshops within, or related to the TBM/LBM projects.

In the table below most of the data is presented, with date for the collection, the type of data and who was present.

Table 7-1: Data collected (R-researcher, T teacher)*

Date	Event/Purpose	Kind of data	Present
070508	Meeting with teachers: Information about the project	Audio	R1, R2, R3 teachers
070619	Meeting in order to start cooperation	Audio + notes	R1, T1, T2
070827	Planning the cooperation	Audio + notes	R1, T1, T2
070904	Classroom observation (2 lessons)	Audio/video/notes	R1, R2
070904	Algebra test	Test on paper	
070910	Classroom observation (3 lessons)	Audio/video/notes	R1, R2
070911	Classroom observation (2 lessons)	Audio/video/notes	R1, R2
070917	Ordinary test algebra + numbers	Students' test on computers	
August- Sept.	Students work on textbook tasks (18 students)	Computer files	
070918	Short meeting with T1	Audio	
070925	Interview with T1, Watch video	Audio	R1 + T1
070925	Interview with five students	Audio/video + students' notes	R1
Oct.-Nov.	Meetings with teachers in the school	Audio	R1, T1, T2, T3, T4
071112	Classroom observation (2 lessons)	Audio/video + students' notes	R1 + T1
071113	Classroom observation (3 lessons)	Audio, 2 videos + students' notes	R1 + T1
071119	Classroom observation (2 lessons)	Audio, 2 videos	R1
071126	Classroom observation (2 lessons)	Audio, 2 videos	R1
071127	Classroom observation (3 lessons)	Audio, 2 videos	R1, R5
071204	Ordinary test - Linear function	Students' tests	
071218	Ordinary test - All topics taught	Students' tests	
080122	Interview teacher + class meeting	Audio	R1, T1
080310	Short questionnaire - digital files		
080519	Interview 4 students	Audio + students' notes	R1
080602	Interview one student	Audio + student's notes	R1
Nov. 2008	Collecting students' computer files	Computer files From 18 students	
Meeting Dec. 2008	Meeting in TBM/LBM	Notes	
Sept. 2009	Conference presentation sept 2009	Notes shortly afterwards	R1 + confer- ence partici- pants
2010	Comments on workshop presentation	Video	TBM/LBM members
2010	Presentation – County's School leader meeting	Notes afterwards	School lead- ers

* T1 is the class teacher, R1 is me.

All data occurring in the TBM project is stored systematically with permission from the NSD - Norwegian Centre for Research Data. In accordance with regulations from the Inspectorate, the data can be stored for an undefined period, however, from 2015 the data is made anonymous and all name lists are deleted. Access is restricted to the participants in the TBM project at the University.

All names of data files have the same format. One example could be: SM-070580-TBM-C-au-ABF-MB-HE.MSV. The first two initials indicate which type of event is recorded. The next is the date, then the initials for the project, and initials for the location. The abbreviation *au* indicates that this recording is an audio recording. The next initials are for the participants in the event.

This particular file is an audio recording from a school meeting on the 8th of May 2007 within the TBM project. The meeting took place in school C and the participating researchers were ABF, MB, and HE. Msv is the type of file, a voice file. In this way the file names give precise information about what, when, where, who, and which type of recording.

7.4 Work with the data

Since the data corpus is of different types, different methods of analysis were applied in the work with the data. In this section these methods will be described.

7.4.1 Video and audio recordings

Many hours of video recordings and audio files were watched and listened to before a data reduction was made.

Data from school meetings in the particular school were partly transcribed after being searched for statements about the particular course and class. Recordings from interviews and meetings with the teacher were fully transcribed, and regular statements were categorised. All references from the teacher are shown to her. She was sent the manuscript for the whole thesis, and in a meeting all text that directly refers to her was shared with her in a process of informant checking.

All the interviews with students were fully transcribed. Afterwards the students' reasoning behind the solutions, and their strategies were categorised according to earlier studies reviewed in chapter 4. This work served as a background for the interpretation of other solutions of the same task and solutions of similar tasks solved by other students on tests and in regular classroom sessions.

The short interviews, more in the style of surveys taken in the classroom about the students' opinions on the paperless classroom and about the use of the 'learning book', were also fully transcribed. Categories were created based on students answers. These were then compared with what was observed in the classroom or said in other interviews.

The data from classroom observation were watched and listened to and a data reduction was made. Then parts of the data material were transcribed, but as time passed, and the video and audio tapes were watched

and listened to time after time, more and more of the data were fully transcribed. As the transcripts were read over and over again, some patterns of interactions appeared to be regularly repeated. These were categorised.

Since the recordings from the classroom were stored in different files, some of the work with the data was to combine the files. This was done in that the transcripts from the different sources were combined in one text file where the time schedule was the common reference.

This file was later combined with the data from students' written work. The computer files with students' written work helped to illuminate what tasks students worked on, and thereby gave a better understanding of the students' responses when solving the tasks in the classroom. The same was the case with the data from the interviews with students.

7.4.2 Students' written task solutions

The computer program excel was used as the main tool when structuring the data. One file was made for each test and one was made for the assigned tasks. First, all students were listed in rows, then the tasks in the columns. All task solutions were investigated for the way they were solved and for errors.

All results were documented with the code 1 for correct solutions, code 0 for not solved, and 11 for errors. Afterwards the worksheets were copied to other worksheets where the errors were categorised according to findings in earlier studies reviewed in chapter 4.

From these excel files it was easy to get an overview of each student and to see if an error occurred only once or more frequently. It also showed how many tasks each students had solved both of the assigned tasks and of test tasks. Changing rows and columns also gave a good overview over which errors occurred in each task.

The next step was to combine the excel file with the solutions of the assigned tasks with the transcripts from the classroom. For each task which was observed to have been solved in the classroom, additional information was written into the excel file. This information from observations was written in a different colour than in the excel file in order to distinguish between the solution process and the saved solution. In addition, data of students from whom I had no saved data files were added into the same excel file. In this way all data from task solving during seatwork was collected in the same file, giving an overview of what was done, and what was observed done, during the time of data collection.

Also the data from the excel file was included in the transcripts as notes about how the tasks were saved in the computer.

Despite all this work done, though, I am fully aware that all the data gives me only a limited overview of what was going on in the classroom. The activity in the classroom is too complex to be grasped by one person although the activity was both video and audio taped. During plenary session a video recorder with its focus on the teacher, the white board and parts of the classroom cannot capture everything, but helped me to get an overview of the interaction between the teacher and the students. The teacher also had the audio recorder in the plenary sessions, which helped me to better capture what was said, because the video recorder was at the back of the classroom and the audio recorder in the front by the teacher. However, sometimes it was still difficult to hear what was said, and it took a long period of listening repeatedly to the two, and sometimes three sources to discern what was said. The positive aspect to this, was that this time-consuming work with the transcriptions and the combination of those with the data from the excel files, helped me to really become familiar with my data.

The findings from the test tasks were compared with the data from the classroom, making it possible to make interpretations of students' mathematics and about some development in time. Contrary to tasks worked upon in the classroom, the test tasks could not be checked and changed. The test situation is different from working in the classroom and this can influence students' task solving since they are in a stressful situation. On the other hand, it might give a better view of their competencies solving tasks, than what is seen from written solutions in the classroom.

It is important to be humble, because test results, as with all written material, does not inform about students' holistic mathematical competencies; only a small area is tested, and it is not known what thoughts lie behind their solutions.

It might be argued that this way of working; focusing on errors shows a negative focus on students' work. It is, however, the case that when working correctly, there is no information about the reasoning. The errors, however, set in the light of earlier studies, give an indication of what reasoning might lie behind the solutions.

Few students (12 out of 27) answered the questionnaire. The responses were delivered in written form, by e-mail. The answers were categorised, and this data is drawn upon in relation to some individual students, and when illuminating students' view of mathematics.

7.4.3 Interviews and conversations with the teacher

The transcriptions of the interviews and the conversations with the teacher both in school and within the setting of project meetings were read, looking for statements about learning, teaching, students' mathematics, the use

of computers in mathematics, and her situation as a mathematics teacher. I searched for patterns within these categories and the results are collected in appendix 2.4 and its sub-sections.

The data are used to ensure that the conclusions drawn about the teacher and her practice in the thesis are in line with her own views and opinions.

7.4.4 The mathematics textbook

Although the classroom was a paperless classroom, the main resource apart from the computer, was the textbook with its resources. In this section the textbook will be presented¹⁵, the reason being the steering role of this book.

The sequencing of the topics in the classroom followed the textbook, and all example tasks were example tasks from the textbook during the first period of observation. The publisher had produced two volumes for the course VG1T, the theoretical mathematics course in grade 1. The first volume is intended to be used in the autumn, and the second in the spring.

The textbook can be said to be a conventional textbook (see section 6.2) in the sense that it follows the exposition-example-exercise model, described by Love and Pimm (*ibid*). The different parts are short, mostly including the whole model. Love and Pimm (*ibid*) assert that textbooks following such a model often have a summary at the end of each main chapter, as does this textbook. The different rules and definitions are assembled on the last pages before a collection of tasks. These tasks are tasks related to more than one sub-chapter and thus more challenging than the other tasks. They were, however, not seen to be assigned during the time of observation.

All example tasks in the first chapter are solved examples; many of them with explanations or recipes for the way they are solved (Bills, et al., 2006). The tasks following the expositions in the sub-chapters are with few exceptions similar to the example tasks, and can be categorised as exercises (Niss, 1993); monitoring common textbooks described in the TIMSS study (see section 2.2.1). The rules and definitions listed up for the reader, might be a sign of a belief that mathematics is best learned in small step presentations of mathematical topics (Love and Pimm, 1996).

At the back of both books are complete lists of answers to each task. An alphabetical register makes it possible to apply the books as reference books. The syllabus for the course, with aims to be reached within each mathematical domain, is presented in the back of the book together with different formulas, and the metric system for length, area and volume.

¹⁵ All quotes from the book and from the resources are my translations from Norwegian to English.

Both books are divided into two parallel tracks, coded with red and blue triangles in the text, signalling the estimated level of difficulty for the current section. The red track is the more challenging. It was followed by all students in this study. The tasks at the end of each main chapter are not differentiated.

In accordance with Love and Pimm's (1996) description of common mathematics textbooks, distinct fonts and coloured boxes are used to inform the reader about important parts in the texts. Rules are placed in coloured boxes, definitions within coloured frames, reminders in coloured elliptic boxes, and example tasks are announced as follows:

Example

This use of coloured boxes and frames makes it easy to distinguish between rules, definitions, examples, reminders and advice.

Although the textbook was a central factor in the classroom, there is no evidence that students read the textbook systematically. One student said that he first tried to solve tasks, if he was stuck he asked one of his classmates, then the teacher. The textbook was used if the teacher was not available, or if he was stuck when working at home. Students, working in the classroom however, were seen looking back to examples, and rules in the textbook. It therefore seemed to be the case that students also in this class tended to skip the exposition, going directly to the rules and examples emphasised by specific fonts or colour boxes (Love & Pimm, 1996; Weinberg, et al., 2011), which might engender the risk that they focus on rules instead of seeking for connections and reasons for the rules.

Every main chapter starts with the learning goals for the actual topic. Then comes an introduction to the chapter; mostly a paragraph or a short section with references to the history of mathematics, or to 'daily life'.

The exposition and the example tasks in the first chapter can be categorised as closed text. The mathematics is divided into parts or sub-chapters, with their exposition and example tasks, which might cause students to develop a fragmentary conception of mathematics without connections between the different parts (Love & Pimm, 1996). Only one approach to task solutions is presented.

Throughout both books, rules, expositions, examples and rules are with some few exceptions presented by the authors using the personal pronoun 'we'. Mostly this 'we' is inclusive not referring to the authors alone. The authors are thus clearly present in the text (Pepin & Haggarty, 2008), which is the opposite of the formal distance often created between author and reader in many mathematics books (Herbel-Eisenmann & Wagner, 2007). An example of this, is the solution of the example task (p. 16):

$$5 - 3(2a - b + 5) - (2b - 5a)$$

← We multiply within the brackets

$$5 - (6a - 3b + 15) - (2b - 5a)$$

← We remove the brackets and change sign
when there is a minus in the front

$$5 - 6a + 3b - 15 - 2b - 5a$$

← We collect like terms

$$-a + b - 10$$

The procedures and rules are in focus, which seems to be the case in most of the two actual chapters (chapter 1 in the first period and chapter 4 in the second period). The examples are transparent, in that it is made clear what it is important to focus upon, and what make them exemplary (Bills, et al., 2006). No activities are suggested for students, other than to solve ordinary textbook tasks; most of them exercises in the way Niss (1993) defined them to be (see section 6.3). An exception to this are some activities in the first chapter (see appendix 8.1). These were, however, not observed to be used in class.

The publisher had in addition to the textbook created a web-site for students and teacher; www.paralleller.no¹⁶. Here one could find work plans¹⁷ for the whole school year, and for shorter periods. Available for the teacher are tests to control if students have achieved the goals for each chapter, and tests for each semester. Both work plans and tests are written in different formats so that teachers can easily make changes.

For students there were interactive tests with each chapter. These tasks were multiple-choice tasks, and feedback was given in the form of results, explanation of the errors made, and reference to information given in the textbook. During the time of observation, neither the teacher nor any student was observed referring to this web-site. The teacher, however, made use of the work plan and the tests.

The work plans were set up with aims and the number of items to be done. Some aims and items were written with bold fonts and some with normal fonts. In the information given, students were told that learning goals written with bold fonts were the most central and important goals. Numbered items written with bold fonts were said to be the easiest or most important items. (A detailed description of the first sub-chapter of the textbook is to be found in appendix 8 with its sub-sections.)

7.4.5 The analysis of the textbook and the assigned tasks

In the two periods of observations the first and the fourth chapter in the textbook were worked through. The analysis of the textbook tasks is thus

¹⁶ It seems that the website has been removed or it has been given another location (in 2015). The textbook is still for sale. The syllabus has been slightly changed.

¹⁷ A description of the work plan is given in appendix 8.6.

limited to the tasks belonging to the parts of those two chapters that were focus for the work in the classroom.

The text, the organisation of the text and the layout in the different sub-chapters were compared to findings in earlier textbook research (see section 6.2), and then compared to the text in the digital 'learning book'.

The assigned textbook tasks, were compared to example tasks in the textbook, before they were analysed according to the mathematical tasks framework (Smith & Stein, 1998) see appendix 4. The categories were tasks of low level of cognitive demands, and tasks of high level of cognitive demand. Tasks belonging to the first category were further divided into two subcategories: 'memorisation tasks' and 'procedures without connections to understanding, meaning or concept tasks'. Tasks belonging to the latter main category were then investigated in order to put them in either the sub-category of 'procedures with connections to understanding, meaning or concept tasks' or the sub-category 'doing mathematics'.

It appeared that most tasks were tasks categorised as 'low level demand tasks' within the sub-category of 'procedures without connections to understanding, meaning or concept tasks' (appendix 4). Since many students though seemed to struggle when solving those tasks, the tasks were further analysed according to the 'procedural complexity' categorisation from TIMSS 1999 Video study (Hiebert, et al., 2003). Procedural complexity is defined as "the number of steps it takes to solve a problem using a common solution method" (ibid p. 70). In line with Hiebert et al. (ibid) the tasks were categorised into three levels of procedural complexity. For all levels the students could use conventional procedures. The levels were:

Table 7-2: Procedural complexity. Adapted from Hiebert et al. (2003, p.70)

Procedural complexity		
	Characteristics	Examples from Hiebert et al. p. 71.
Low level	Solution process requires maximum four decisions by the students and contains no sub-task.	Solve for x $2x + 7 = 2$
Moderate level	Solution process requires at least five decisions by the students and can contain one sub-task.	Solve the set of equations for x and y : $2y = 3x - 4$; $2x + y = 5$
High level	Solution process requires at least five decisions by the students and contains two or more sub-tasks.	Graph the following linear inequalities and find the area of intersection: $y \leq x+4$; $x \leq 2$; $y \geq -1$

7.4.6 Ethical considerations

As a teacher myself I realise how difficult it can be to be a teacher and to handle the complexity of a classroom on the spot. Therefore, I went to the classroom with a sense of humility. To start a study with other people involved, is always a daring deed. Cooperation means using oneself, to make a personal investment. I wished to meet the participants in an “I-you” relation (Buber & Simonsen, 2003), to meet them as whole persons not only as informants.

In a study with only one teacher and one class, the participants will automatically be exposed. Of course all the work done adheres Norwegian laws according to the guidelines and permission given by the NSD. That means that all participants had been given information about the project and thereafter had signed in to the project. It had been emphasised in the information that: anyone who wanted, could withdraw their participation at any time. No one has done this. It was also made clear that what would be written or said, would be done in such a way as to secure the anonymity of participants. In conversations with students, and when I asked some of them for interviews, no one was adverse to participation. Also when I asked for papers they had written, or when I asked them to download what they had done in mathematics on their computers, they willingly did this for me.

Although confidentiality is taken care of, I have been concerned about how to report my study in a way that the participants and especially the teacher recognise as familiar. In order to meet this concern, the teacher has had access to the whole thesis, and as far as possible I have tried to let participants speak for themselves.

7.4.7 Reflections

The data is from different sources: classroom observations, test results from the class' regular tests, and from a test provided by me taken both in the autumn and in the spring, students' stored computer files including the solved assigned tasks. In addition, the textbook is part of the data. I have triangulated (Yin, 2009) the data in order to corroborate the same phenomena.

The analyses of the data, I have mostly made on my own. I have had little opportunity to have my data investigator triangulated (Denzin, 1970) apart from some discussions right after classroom observations and some comments on my analysis in meetings with my supervisors. The lack of this type of triangulation I see as a weakness in my study.

I have tried to compensate by providing a thick description of the classroom, the tasks, and students' responses both in the thesis and in the appendices. In addition, I have tried to give voice to the participants and to bring in their perspectives in the analysis. On the basis of this thick description it ought to be possible for others to make a judgement of the analysis, of the findings, and to what degree it is possible to transfer the findings to other mathematics classes.

After interviews with the teacher, I had a member check; she has received what is written on the basis of interviews and conversations. The students have not had any possibility to check what is written about them.

The thesis is detailed and I have tried to make it readable without losing the aim of being open to the reader about my choices and my analysis.

One of the main methods has been to categorise students' responses according to findings in former algebra research. The tasks used in class and on ordinary tests, are not diagnostic tasks. Only tasks in the algebra test can to a certain degree be categorised as diagnostic.

If I were to do this again on the basis of the findings from this study, I would have made some changes in the sample of the tasks in the algebra test. There is a lack of tasks revealing explicitly students' conceptions of letter symbols, equivalence and of fraction operations. Such tasks would have added valuable information to the study. I have, despite this, categorised the problems found in the tasks worked on in the classroom and on tests.

On one hand this might be criticised as drawing the conclusions too far. What strengthens the conclusions though, are the observed responses from interviews, and from the interactions in the classroom. In this thesis it has been regarded to be a strength to rely on the responses on these tasks because they are mathematical tasks used in the classroom. The situation is authentic.

Another issue is my role in the study. Gold (1958) presents four theoretically possible roles in ethnographic research. These roles range from being a “complete participant”; unknown as researcher among the participants, to the other extreme being a “complete observer”. Between these two he suggests two other roles: “participant as researcher and “observer as participant”. He further emphasises the danger “to go native”; to be too familiar with the group under investigation, when taking the two latter roles.

According to Gold’s definitions of roles, I was never a complete observer. I sometimes interacted with students when they worked with tasks. I did, however, not feel the danger of ‘going native’ but rather the danger of slipping into the role of a teacher. I think the teacher often looked at me rather as a colleague than as a researcher. I have been aware of this, and also to some extent welcomed it. As a teacher myself, I appreciate the generosity the teacher showed me, letting me come into her private practice. I could also offer personal experience to illuminate that I know the complex situation a teacher has to confront every day in school.

Another reflection made is on the interpretative framework. I have chosen a conceptional framework with concepts from different theories and former studies. I might have used a theoretical framework. I considered the French theory: The theory of didactical situations in mathematics (Brousseau, 1997). However, my aim, in addition to inform the mathematics community of mathematics and add to the considerable work done in the area of algebra and classroom research, has been that the thesis should be readable for practitioners in the field, and, as I see it, the concepts and notions used in the theory of didactical situation are too unfamiliar for that purpose.

In this chapter there has been an outline of the data this thesis is built upon and the methods used. In the following chapters the findings will be presented.

8 Students' calculation with numbers and letters

In this chapter there will be an outline of strengths and problems revealed in students' work when it comes to the understanding of literal symbols, invisible operation signs, the commutative law, the distributive law, order of operations, the minus sign, negative numbers, operations on negative numbers, and powers. Except for the interpretation of letters, invisible signs, and the minus sign, the topics were themes in the first chapter of the textbook. (Appendix 8 is an outline of this sub-chapter).

The algebra test will be drawn upon in order to illuminate strengths and problems students had at the outset of the observation. The algebra test focused on an understanding of the literal symbols, which was not addressed in the textbook; supposedly taken for granted by the textbook authors.

When students were solving tasks in the textbook, they had the opportunity to check their solutions. This was not the case when solving test tasks. Test results might thus give a more trustworthy picture of the situation. Some students were interviewed based on some test tasks, which give an illumination of their strategies and reasoning. All tests can be found in appendix 6; the algebra test in 6.1.

In addition, this chapter in the thesis will be based on data from the whole observation period and from interviews. Task solutions and explanations in the classroom related to the aforementioned topics, will be drawn upon to give a picture of the received curriculum. The ordinary tests in September and in December will be used to give a picture of students' performances after being taught and after having practiced in class. Those tests were composed by the authors of the textbook and supposedly adapted by the teacher.

Data from students' seatwork (saved computer files, 18 students) and data from the classroom (27 students) reveals errors known from earlier studies in mathematics education and are categorised according to the reviewed studies (see chapter 4 in this thesis). In appendix 9, there is a more detailed background for the results presented in this chapter.

8.1 Understanding of literal symbols

In this thesis I have used the notion literal symbol instead of the notion 'variable' (see the end of section 4.1). No example task, assigned task, or part of the exposition in the textbook offered opportunities to reason about the meaning or use of literal symbols in algebra. Some few tasks, however, in the algebra test, were given in order to test if students accepted let-

ters as substitutes for general numbers, that different letters could take on the same value, and if students were able to differentiate between the value or number of an object and the object itself (see section 4.1).

The results from the test tasks show that all but one student seemed to accept literal symbols representing number values. In three tasks asking students to pick out expressions representing a certain value of a general number, item 3, 40 % solved all three tasks correctly. One student did not respond, and four students regarded inequivalent expressions to be equivalent. The others picked out only one of two correct expressions, although in the text it was expressed that they should pick out “the or those expressions”. Taking into account that some students might have been satisfied by picking out the first correct expression in the list, it might be inferred that at least 20 % had some problems with equivalent expressions in addition to the 16 % equalling inequivalent expressions. In item 4 in the algebra test; the ‘student-professor problem’ (Clement, Lockhead, & Monk, 1981; MacGregor & Stacey, 1997) was given:

In a school there are 10 students to each teacher.¹⁸ Which of the expression(s) represent the correct relation?

L=number of teachers, E = number of students. Mark the, or those correct expression(s).

10L=E 10E=L L=10E E=10L 10L+E 11E

76 % picked out two expressions (60 %, though, picked out expressions representing the opposite relation). This strengthens the interpretation that some of the students had problems which expressions were equivalent in item 3. In the spring though, the proportion solving all three tasks correctly in item 3, had risen to 67 %; a considerable improvement (see appendix 9.1, and the diagram in appendix 6.1)

Another task similar to a task in the CSMS project (Küchemann, 1981) asked students to consider if a statement is always true, never true or true for a special case. The result is shown in the table underneath.

Table 8-1: Algebra test, item 6 with results (%) and alternative answers

Task 6a)	Correct	Alternatives	No response
$x + y + z = x + p + z$ This: is always true is never true <i>could be true, if</i>	48	Never true 48	4

¹⁸ In Norwegian teacher is 'lærer' (L), and student is elev (E).

48 % of the students regarded correctly the equality to be true if y was equal to p . 48 % of the students regarded the equality to never be true. This means that nearly half the students did not seem to regard the two letter symbols to be able to take on the same value. In the spring, nearly 70 % chose the correct alternative and gave a correct explanation, which is an indication of a positive development. They had then had the experience of working with functions and had seen that two different letter symbols could take on the same value, which might be one reason for the better result.

An important ability according to Küchemann (ibid) is to distinguish between the objects themselves and the numbers/values of objects. One task similar to one in the CSMS study was chosen to check this:

Table 8-2: Algebra test item 7 with results (%) and alternative solutions

Task 7	Correct	Alternatives
One cake costs c kroner. One sandwich costs s kroner. I buy 3 cakes and 4 sandwiches: What does the expression $3c + 4s$ mean?	40	Expression representing numbers of objects (20%) No response or had stopped in the middle of the solution process (36 %) Letters given specific values (4%)

In this item, 40 % or ten students gave an acceptable interpretation of the expression. Five students (20 %) interpreted clearly the expression to be the amount of sandwiches and cakes. They did not differentiate between the price of the cakes/sandwiches and the objects themselves (see section 4.1). The choice of the letters in the task makes it probably more likely to happen since students let c represent the object cake and s the object sandwich¹⁹. One student answered by assigning values to the letters. In the spring, the result was not better; then only nine students (38 %) stated correctly that the expression was the total price.

The interviews confirmed the problems, but when given a similar example in which they had to convert text into mathematical notation, it seemed to be easier. However, when led back to the initial problem, it still was a source of difficulty to differentiate between the price (value) of the object and the object itself. In the lessons, students solved no similar tasks.

In the CSMS-project (Küchemann, 1981; Booth, 1984) and in the KIM-project (Brekke, 2005) it was reported that some students conjoined terms. This was rarely seen during observation, and only by two students on two occasions; an indication that all accepted the lack of closure.

¹⁹ A cake is “kake” in Norwegian, and sandwich is “smørbrød”, the expression was: $3c + 4s$.

The problem to differentiate between the number or value of objects and the objects themselves, might cause problems for some students in transforming word problems into algebraic equations; one student expressed this explicitly when converting text into equations when working with straight lines. The problems of deciding if expressions are equivalent or not, might also create difficulties when solving equations, and when reducing expressions to simpler forms. It was though evident that a larger group of students succeeded in picking out equivalent expressions in the spring than in the autumn.

No example or assigned task was offered for students to discuss or compare expressions, or to interpret the letter symbols. The concept of equivalence was not mentioned in the textbook. This means that there was no new opportunity given (Hiebert & Grouws, 2007; Kilpatrick, et al., 2001) for the students to develop a proper concept image (Tall & Vinner, 1981) of the concept of equivalence, which is the basis for all manipulative work with algebraic expressions (Kieran, 2007).

8.2 The basic properties of addition and multiplication

The first part of the first sub-chapter in the textbook is devoted to the basic properties of addition and multiplication; the commutative, the associative and the distributive properties. These mathematical notions are not applied, but instead explained by daily life language as rules for calculations. Only numbers from the set of natural numbers are used. (A detailed description is to be found in appendix 8.2). There was no task assigned to reason about these properties.

In the algebra test, one task was given that tested the commutative property of both addition and multiplication. In addition, the issue of the invisible multiplication sign was present. The students should decide if two expressions were equal. The result is shown in the table below.

Table 8-3: Algebra test, item 6b with results (%) and alternative answers

Task 6b) $a + b \cdot 2 = 2b + a$	Correct	Alternatives	No response
This: <input type="checkbox"/> <i>is always true</i> <input type="checkbox"/> <i>is never true</i> <input type="checkbox"/> <i>could be true, if.....</i>	72	Never true (12%) Could be true if b=2 (4%)	12

A large part of the group correctly regarded the proposition to be ‘always true’. However, 12 % regarded it to be ‘never true’ and 12 % did not respond. One student responded that it could be true if b was equal to 2.

It might be that the invisible operation sign on the right side of the equal sign, was one reason for some of the students to regard the proposition to be never true. It was evident from the classroom that some students were unsure about the invisible operation sign and asked the teacher about it. The teacher had remarked earlier that she was aware of students' problem with these signs (Herscovics & Chalouh, 1985; Lee & Messner, 2000). During plenary sessions, she used all opportunities given in the example tasks to emphasise that in algebra the invisible operation sign is a multiplication sign. In the spring all students answered correctly; an indication that it was the invisible sign more than the commutative property that had caused a problem for the students in this task.

When working with tasks, the students mostly showed evidence of acting in accordance with the distributive law, although it was a recurring problem that students made mistakes. During seatwork, when checking their results, they mostly realised by themselves, what had gone wrong.

Only four students saved solutions in which they had not followed the distributive law. Three of them did it only in a task with a post-multiplier. Later when working with equations, they mostly put the common denominator as a post-multiplier and performed correct operations.

On the ordinary test in September there were five tasks including eight brackets to be multiplied by a pre-multiplier. Six students out of 27 (22%) failed to follow the distributive law. Five of them failed only once, while one student made this error in three out of eight tasks.

This indicates that the students in general seemed to be competent following the distributive law, and that none consistently made errors.

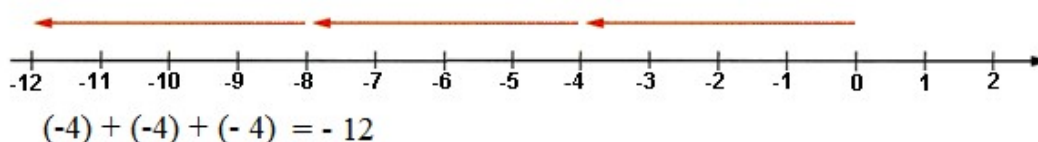
8.3 Negative numbers and the minus sign

The minus sign was the focus for many discussions in the observed lessons. One problem in Norway is that exactly the same sign is applied both as a sign for the operation of subtraction and as a sign for negative numbers. This was not mentioned in class or in the textbook. Some few tasks in the textbook were tasks focusing only on operations on negative numbers. However, students revealed many problems related to the minus sign. During observation, questions caused by what Vlassis (2004) called negativity, were some of the most frequent questions in the classroom.

In this section all the different problems related to negativity will be reported and related to the textbook and the teaching. All available data is drawn upon. (Additional information is to be found in appendix 9.3 with its sub-sections.)

8.3.1 Negative numbers in the teaching material

The textbook (Ekern, Holst, & Guldahl, 2006) had one section devoted explicitly to negative number and the introduction of the set of integers including zero. (A more detailed description is to be found in appendix 8.3). This set was presented in set notation: $\mathbb{Z} = \dots -3, -2, -1, 0, 1, 2, 3 \dots$ and on the number line. The number line was used to visualise the operations of addition and multiplication of a natural number by a negative number. Negative numbers were symbolised by red arrows, the length representing the size of the number, and positive numbers by blue arrows. It was not explicitly said that the size of the number and length of the arrows correlated, but the metaphor was used to exemplify and give meaning to addition and multiplication of negative numbers. One example, $3 \cdot (-4)$ was illustrated as shown below:



Addition of a negative number $5 + (-3)$ was shown in the same way, but said to be normally written as subtraction: $5 - 3 = 2$. There was no comment about this shift in the function of the minus sign from being a sign signalling position to being an operation sign. There was also no presentation of opposite numbers, but the sign rules were highlighted:

The product of a positive and a negative number is negative.
The product of two negative numbers is positive.

In the margin was a short cut: “two like signs give plus, two unlike signs give minus” (ibid, p. 15).

The sign rules for operating on brackets were also presented in a rule box; but there was no explanation of why the rules work (ibid, p. 16).

When we are going to open a bracket with a plus sign or no sign in front, we just remove the bracket.
If we are going to open a bracket with a minus sign in front, we have to change the sign for all terms in the bracket.

There was only one item in the textbook offered for operating on negative numbers. The minus sign however, was included in many tasks.

In the plenary session one example task from the textbook was solved:

$$\begin{aligned} 5 - 3(2a - b + 5) - (2b - 5a) &= \\ 5 - (6a - 3b + 15) - (2b - 5a) &= \\ 5 - 6a + 3b - 15 - 2b + 5a &= \\ \underline{\underline{-a + b - 10}} \end{aligned}$$

This task was an exercise in applying the rule of distributivity and the sign rule shown above.

In the next sections students' work with negative numbers and the minus sign is presented and categorised in accordance with problems revealed in the reviewed literature.

8.3.2 Students' work with negative numbers

Only three textbook tasks were offered explicitly for students to practice operations on negative numbers. One task, 1.10d, is presented under the headline 'double minus sign'. In addition, one task from the algebra test required that a literal symbol was substituted by a negative number and operated upon. (Analyses and more detailed descriptions can be found in appendix 9.3 and its sub-sections).

The table below presents the textbook tasks. The number of students who solved them, and the number of correct answers are listed in the table together with students' alternative answers. (Dataset 18 students.)

Table 8-4: Negative numbers - Textbook item 1.1.10 (Number of students)

Item	Task	Correct solutions	Students solving the task	Alternative solutions
1.1.10a	$(-3)(-2) \cdot 4$	13	14	$(-3)(-2) \cdot 4 = (-6) \cdot 4 = -24$
1.1.10b	$(-1)(-2)(-3)(-4)$	10	13	$(-1)(-2)(-3)(-4) = 2 + 12 = 14$ (A) $(-1)(-2)(-3)(-4) = 3 \cdot 12 = \underline{\underline{36}}$ (B) $(-1)(-2)(-3)(-4) = (-10) = \underline{\underline{-10}}$ (C)

One student came to a negative solution in both tasks. In the second task it seems that she has added the numbers, although the numbers were written in brackets (C). It might be that the invisible multiplication sign caused the error (the responses are copied from the data files). The other two have also applied other operations in parts of the task (A) and (B).

One task in the algebra test included negative numbers; task 5b. This task though, was more complicated than those in the textbook, because literal symbols had to be substituted by given numbers. In addition, there were two terms, one with a power. The task was:

Calculate the expression $3b^2 - abc$ when $a = 3$, $b = -1$, and $c = 5$.

This task was solved correctly by 4 out of 25 students in the autumn and by only 3 students in the spring.

The last term $-abc = -3(-1)5$ is of the same type as those above, but the substitution makes it more complex. Most of the students had substituted the letters by the numbers correctly, but few set negative 1 in a bracket. It was not written within a bracket from the outset, and it might

be that a greater number of the students would have succeeded if that had been done. However, students who initially wrote brackets also showed uncertainty as in the example below:

b $3b^2 - abc$ når $a=3, b=-1$ og $c=5$.

$$3b^2 - abc = 3(-1)^2 - 3(-1) - 5 = 3 - 3 - 3 + 1 - 5 = -13$$

This student has correctly put the negative number within a bracket in the start, has nevertheless changed the multiplication signs into minus signs. The reason behind this cannot be elicited from the data.

If the task had only this last term, the result might have been better. Since the bracket was omitted in the givens, and the students had to focus on the first term before coming back to the last term, it might have caused students to lose sight of the multiplication (they wrote by hand). The results are shown in the table:

Table 8-5: Task 5b algebra test: $3b^2 - abc$ given $a=3, b=-1, c=5$

2nd term $-abc = -3 \cdot (-1) \cdot 5 = 15$	Autumn (August) 25 students	Spring (May) 24 students
Correct	24 %	21 %
-15	24 %	54 %
Mixed operations	51 %	25 %

Only one student solved the last part of the task correctly both in the autumn and in the spring. Although there is no progress from autumn until spring when it comes to correct solutions, there is a development in that the proportion of students who mixes the operations (multiplication and addition/subtraction) decreased. The invisible operation signs seem to cause fewer problems in the spring. The reason might be that more students wrote brackets, but also the emphasis on invisible operation signs during plenary sessions during the year might have helped many of the students to choose correct operations (see section 8.2).

75 % in the spring multiplied correctly but more than 50 % of the students arrived at a negative solution. This might be due to the double minus sign in their solution process which is reported to be a problem (Gallardo & Rojano, 1994; Vlassis, 2004). The minus sign is still a problem in the spring for more than half the students in this task.

8.3.3 The double minus sign

Only three tasks in the textbook were given with two successive minus signs. Only one was on the work plan: $12 - (-7 + 2)$. Four of 13 students solved it incorrectly. For two of them the error might have been caused by a mis-interpretation of the double minus sign.

One task with a double minus sign was given on the ordinary test after the work with the textbook tasks: $-3(a+b)-(-b)\cdot 3$. Three students failed in this task because of the double sign. Thus, it seems to be that nearly 90 % of the students were confident that two consecutive minus signs are equivalent to a positive sign in this task. But more students seemed to have problems when a double minus sign appeared during the solution process (see section 8.3.5).

For those students it might have helped them to be aware of the different meanings and functions of the minus sign (Gallardo, 2002; Lamb, et al., 2012a, 2012b; Vlassis, 2004), but these were not addressed in the textbook nor in the classroom.

8.3.4 Result and solution numbers

According to Gallardo and Rojano (1994) students in their study made efforts to avoid negative solutions.

On the first work plan there was no task with negative numbers as result numbers (Gallardo, 2002; Vlassis, 2004, 2008). On the next plans there were three assigned tasks with negative result numbers, and four tasks with negative solution numbers.

Although a large number of the students made errors in the tasks with negative results numbers (see appendix 9.3.3), it cannot be argued that the students had a tendency to avoid negative solutions. Only one student seems to have ignored the sign in two of the tasks, while having correct solutions in the others.

There were, however, other problems with negative solutions. In one task, three students (dataset 18 students) came to the result $\frac{-3}{3}$. It is correct, but they had not simplified the fraction. In other fraction tasks, when numerator and denominator were equal, but positive, they had written that the result was 1. The lack of simplification, might thus partly be caused by the minus sign.

One episode with Sven working with task $\frac{a}{12} + \frac{a-2}{4} - \frac{a}{3}$ illuminates this suspicion. He had solved the task so far: $\frac{a}{12} + \frac{a-2}{4} - \frac{a}{3} = \frac{a+3a-6-4a}{12} = \frac{-6}{12}$. He checked it and found that it should have been $-\frac{1}{2}$. He then simplified the fraction and came to the result $\frac{-1}{2}$. He points to his solution and asks the teacher:

Sven: In the textbook the solution is said to be $-\frac{1}{2}$. But I have this.

T: That's what you have. You didn't see that?

Sven: Yes, but in the textbook the minus is written in front of the fraction.

T: Minus divided by plus is minus.

(*The teacher goes to the whiteboard and writes: $\frac{-1}{2}$. She then puts in the plus sign: $\frac{-1}{+2}$ and equals this to $-\frac{1}{2}$*). That was what I showed you now.

Sven: Are they equal?

T: They are equal. Because minus divided by plus is minus. If you have a fraction like this (*she writes $\frac{1}{-2}$*), then this will be equal to $-\frac{1}{2}$ as well.

Thus your solution is correct.

Although having reached the correct solution, Sven could not see the equality between the fraction he had written, and the one in the textbook. Even after the teacher had shown him on the white board that the two representations were equal, he was not convinced. The location of the minus sign caused him to think that he was wrong.

Most students ended their solutions without writing the minus in front of the fractions. They placed the sign either in the denominator or in the numerator. It was first interpreted to be so because of convenience, but the episode with Sven shows that there might be a conceptual reason for this.

When helping Sven, the teacher referred to the sign rules, not why they work, which is in line with the exposition part in the textbook (see section 8.3.1). In the case with the solution $\frac{-3}{3}$, it might be that some students avoided simplification because they were not sure where the minus sign 'belonged'

As a conclusion it does not seem that the students tried to avoid negative solutions or negative results as was reported by Gallardo and Rojano (1994). It was at least not a consistent error. From observation in class, it seemed, however, to be a problem to come to this negative solution.

8.3.5 The sign rules

The different sign rules were not easily grasped by all students. By the end of November when working with system of equations, Oscar could not accept the answer in the back of the textbook. He had come to the result $-3x = -6$, and he meant the solution should be negative 2:

Oscar: This here, I can't understand that it will be 2. I thought it was only when multiplying it could be for example minus 6 multiplied by minus 3. (*He points at the screen*). I really thought it was only when we multiply.

R 3: No, it is as if you think in this way: 2 times minus 3. (*She writes $2 \cdot (-3)$ on paper*). How much will that be?

Oscar: 2 times minus 3?

R 3: Yes?

Oscar: Minus 6?

R 3: That's what you have applied here. When you divide minus 6 by minus 3, then it will be 2. That means that 2 times minus 3 is equal to minus 6.

Oscar: Okay.

R 3: Do you understand then?

Oscar: Yes.

R 3: That's good.

Oscar: And now I am going to find the y .

He had learned the sign rules, but had restricted the sign rules to the operation of multiplication and shows surprise in the above episode. It was evident from classroom observations that he was not the only student thinking that it was impossible to divide by a negative number.

Rakel expressed their problem when solving an equation, which caused many students to wonder about the minus sign. At the end of the process she had $-5x = 3$. She expressed her problem in this way: "but if we divide by minus 5, the minus sign will come down there?" It seemed to be that some students thought that was impossible.

In the episode above, Oscar was given a mathematical reason for the result that goes beyond the rule.

For many students the rule about brackets and signs was not an easy matter. This can be illustrated by an example from the plenary session when the topic was equations. The example task was:

$2(3x - 8) = 5 - 3(x + 1)$. The teacher asks the students about the multiplication on the right side of the equal sign:

Tord: It will be $3x$

T: $3x$ and?

Tord: Minus 1, no, no, minus 3.

Tone: -1

Arne: Plus

Some say plus but most students say minus.

Tord: It is minus 3.

Arne: Plus and minus give minus.

T: Do all agree that we will have minus? If I keep the bracket, then I keep that sign (*she points to the plus sign*). Sometimes it is smart, not to execute more than one operation at a time. Here it is pretty simple, then we can do it. But, you have to be conscious about what you are doing. And the reason for us removing the first bracket was?

Eli: There was a plus in the front

T: And when there is a minus in the front, then you have to be aware of it, because it is so easy to make mistakes.

The equation was then solved, but the example shows that not all students would change the sign in the last bracket when removing it. Teacher's advice was that it would be wise to keep the bracket during the operation of

multiplication, and then in the next step remove the bracket and change sign. The same didactical advice she gave several times during seatwork.

In the textbook, examples illustrating the sign rule for brackets and pre-multipliers in the first sub-chapter, the authors followed the same advice; showing two steps in the solution. In the example tasks related to equations, however, the authors executed both operations in one line.

Teacher's advice seemed to work in most cases, but one task:

$2(2a-3)+a(2-b)-5(-b+3)$ on the test in September caused problems. In this task 8 students or 30 % wrote $-5b$ instead of $+5b$. The students had learned that if there is a negative sign in front of a bracket, the signs within the bracket have to be changed when the bracket is removed. Then for the students it might be a problem if they think that the signs are changed, resulting in $+5b$, and there is still a negative sign preceding the bracket causing them to change once again.

The lack of meaning of the sign rules for brackets, might have caused the error. There was only one similar task offered in the work plan, solved by three students, who solved the text task correctly.

It is evident from the saved computer files (18 students) that eight students removed the bracket without changing signs in one or more of the assigned tasks. For five of those students (28 %) it seemed to be a rather stable error in that they also made this error on the ordinary test.

In September, three tasks in the test included brackets preceded by a minus sign, with or without a pre-multiplier. Nine students or 36 % of the students did not change signs correctly when removing the brackets. Two others changed signs when removing a bracket preceded by a positive sign, which might indicate that the rules gave no meaning for them either.

According to Vlassis (2004) this was denoted to be algebraic subtraction and was an algebraic version of the 'opposite of'. The reasoning behind the action of changing sign, might have been more based in mathematics if students had been taught about taking the 'opposite of' (Gallardo & Hernández, 2005; Lamb, et al., 2012a).

In a conversation with the teacher more than one year after the observation period, she referred to Alf, a student who seemingly at the time of classroom observation, had no problems with sign and brackets, saying that he had never really understood why the signs had to be changed.

The teacher said that she knew it could be a problem for some students, but she was surprised that he did not understand the reason for the rule, something she had taken for granted. Her problem, under the severe time pressure, she said, was to know how deeply she should go into the reason for the rules, reasons that she assumed students had become acquainted with earlier.

8.3.6 Detachment of terms from indicated operations

This notion was used by Herscovics and Linchevski (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996), and can be divided into several sub-groups. In the classroom and in students' written material, it is found that terms are just rearranged ignoring the signs. There is evidence in class of what Vlassis (2004) called 'bracket reasoning', and perhaps of 'jumping off with the posterior operation' (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; Linchevski & Livneh, 1999). There are also episodes in which students calculate from right to left (see section 4.3 and section 4.4 and table 4-1). The rest of the section is divided into sub-sections for each of the subgroups.

Rearranging terms – ignoring signs

One of the first assigned tasks was: $5(2b - 3a - 5)$. Four students executed correct multiplications, then they seem to have intended to reorganise the terms in alphabetical order. They just changed the location of two of the terms, ignoring the operation signs: $10b - 15a - 25 = \underline{15a - 10b - 25}$.

The other students solved it correctly, which might be interpreted as that the students were mostly confident with which signs and terms belonged to each other. However, it might also mean that there were no other tasks in which this was tempting to do. The textbook, in all the other tasks in the first chapter, presented the letter symbols in alphabetical order.

One student was observed asking the teacher if she could change the order in an expression while she wanted the first term not to be preceded by a minus sign. That girl rearranged the terms correctly. In other episodes it was evident that a large group of students had problems deciding which operation belonged to which term, and to decide which operation should be applied.

Jumping off with the posterior operation

This was the notion used by Herscovics and Linchevski (1994, 1999) to describe the phenomenon of applying the operation to the right of a term instead of the preceding operation. There is only one possible example of this in the material. Task 1.1.12c was solved by Bill in this way:

$$3(2a - 5) - (6 - 2a) + (3a + 1)2 = 6a - 15 - 6 + 2a + 6a + 2 = \underline{2a - 19}$$

$12a$ disappears. Here the reason might be that the minus behind the first term is applied for the fifth term $6a$. It might also be that he calculated from right to left. This student was observed at one other occasion changing the direction of calculations.

Bracket reasoning and/or calculation from right to left

Bracket reasoning was the notion applied by Vlassis (2004, 2008), and was caused by seeing the minus sign as a splitting factor. Another phenomenon is to calculate from right to left if that seems to be convenient according to the numbers in play (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; Linchevski & Livneh, 1999). In many cases it is not easy to determine what is caused by the minus sign as a splitting factor, and what is caused by changing direction. Both are related to students' uncertainty of what operation sign belongs to which term. This uncertainty was evident in much of the work done both in class and on tests.

Two fraction tasks had a structure that invited bracket reasoning:

Task 1.2.2b

$$\frac{2}{5} - \frac{3}{5} + \frac{4}{5} = \frac{5}{5}$$

Task 1.2.2c

$$\frac{5}{3} - \frac{10}{3} + \frac{2}{3} = \frac{7}{3}$$

The three students solving these tasks, came to the results shown above. There were no other textbook tasks with the same structure. It can be discussed whether the reasoning has been caused by first grouping the two fractions with addition sign in between, before subtracting the first fraction, or if the students have been starting from the right at the outset.

Three of the tasks in item 1 in the algebra test are shown in the table below together with the distribution of correct answers and alternative answers (25 students):

Table 8-6: Algebra test, item 1 with results (%) and alternative answers

Item	Task	Correct	Alternatives	No response
1a)	$2x + 5x$	96	$6x^*$	
1b)	$x + x + 2x$	88	$3x; 5x; x^2 + 2x;$	
1e)	$a - 3a + 2a$	44	$4a$ (48 %)	8

Nearly all students solved the first two tasks (1a and 1b) correctly. However, there is a dramatic change when it comes to the task 1e. The main difference from the other tasks is that 1e includes a minus sign. In addition, there is an invisible coefficient in the first term, as also in the two first terms in task 1b. In task 1b only three students might have been influenced by the case that the number 1 is invisible.

Olav and Selma, who had come to the solution $4a$ in task 1e, were asked about this task in interviews. Olav calculated from right to left, but when the invisible coefficient was made visible and larger than in the second term, he calculated correctly from left to right. He said he knew that

the first coefficient was 1, but assumed that he had felt it easier to start from the right since the second term had a larger coefficient. This tendency to change direction when the number, or the coefficient, in the first term is smaller than in the following term, is reported by (Linchevski & Livneh, 1999; Vlassis, 2008).

Selma, the other student first wrote the answer $4a$, but without any interruption she changed her solution to zero. She explained that her first strategy was to subtract a from $3a$:

Selma: It was the case that I thought you should take away from 3 (a). This means that you should take away the one a . However, there is still a minus so there is a plus there (*she points in the front of the first a*). Therefore, you should just take away from the one so there is still a negative.

From her utterance it seems that she calculated the difference between the two first terms from right to left. She further explained what helped her to find the right solution.

Selma: I forgot there was no plus there (*in front of $3a$*), but a minus. You can sort of say there is a number there, a number there and a number there. (*She points to each term.*) There are several groups. In addition, we have to remember that, this one is a minus, this one before the 3, and that it is not that one, which is minus (*she points to the first term, the a*). That's why I thought it was $2(a)$.

In the last sentence she talks about the difference $a - 3a$. She tells that she has to remember that the minus sign belongs to the term $3a$ and not to the a as it would be if she went from right to left. She tells that she resolved her problem by focusing on terms (calling them groups). She explained how she drew circles around each of them, including the operation signs. From this circling she made clear for herself which operation sign belonged to which term. She had written:


$$(a - 3a) + 2a = \cancel{4a} 0$$

Her notes show that her first solution was $4a$, and then, after realising that she had to be aware of the terms and their signs, she managed to find the right solution. She was also fully aware of the invisible number 1.

None of the two interviewed students seemed to group $3a$ and $2a$ before subtracting it from a . The two interviews show that there is different reasoning behind equal results.

In the autumn 48 % and in the spring 38 % came to the result: $4a$. The proportion of correct responses had increased in the spring, 50 % had then a correct solution. Some with an incorrect solution in the autumn seemed to have progressed, but some with a correct solution in the autumn failed in the spring. What is clear, is that many students have no clear conception of operation signs and terms. For Selma it helped her to consider this

and to make circles around the terms and their operation signs in order to visualise what belonged together.

This could perhaps also have helped Carl and Jone when solving the textbook tasks below:

Carl's task 1.1.12b

$$3x - 2(4x - 3) + 10 =$$

$$3x - 8x + 6 + 10 =$$

$$\underline{\underline{5x + 16}}$$

Jone's task 1.1.12d

$$3(2a + a) - 2(3a - 1) + (8a + 3)(-3) =$$

$$6a + 3a - 6a + 2 + -24a + 9 =$$

$$\underline{\underline{21a + 11}}$$

Carl has seemingly calculated the difference $3x - 8x$ in task 1.12.b from right to left. 3 is smaller than 8 so then it is tempting to change the direction to be $8x - 3x$.

Both students were observed doing the same on other occasions. Seven students seemed according to the written data in two of the textbook tasks to either partly calculate from right to left or to rearrange terms without taking the signs into account. All students, except for one, who detached terms from their intended operation in one or more textbook tasks, had the solution $4a$ when solving the task 1e) in the algebra test.

8.3.7 Summary negativity

The minus sign and negative numbers were the reason for many errors. When looking through the material from students saved computer files, only three students in the data set of 18, had no errors related to negativity although the solutions to the textbook tasks could be checked in the textbook key. Thus this might indicate that the students felt confident that they had arrived at the correct solution.

It was also heard in the classroom that the answer is correct though; only the signs differ. This might indicate that some of the students were not so much focused on the signs, their goal was to reach a "correct" number. Ruth even asked if it mattered if the sign was wrong.

In the test in September, shortly after finishing the first chapter in the textbook, two students made no errors related to negativity, six students made only one error, which means that 70 % had made two or more errors related to negative numbers or the minus sign

The table below presents results from the September test related to negativity. The errors are distributed as shown in the table below:

Table 8-7: September test - Errors related to negativity (27 students)

Types of errors	Number of students %
Double negative sign (section 8.3.3)	11
Do not change signs, or not correctly, when removing brackets preceded by a minus sign or a negative pre-multiplier (section 8.3.5)	44
Do not change sign in binominal numerator if the fraction is negative	30
Ignore the minus sign, addition instead of subtraction	33
Positive solution number; should be negative	19
Detach the term from indicated operation	30

The two first categories are discussed in the sections above. The third category is related to fractions with binomial numerators. The task was²⁰:

$$\frac{3}{2a} + \frac{a}{4b} - \frac{a-2b}{ab}$$

Eight students (30 %) did not change the sign in the binomial when subtracting the fraction. Four students (15%) were not among those who failed in the foregoing category (changing signs when removing brackets). It seemed new to students to meet this, and the teacher reminded the students to treat the numerator as a whole; as an object (Gray & Tall, 1994; Sfard, 1991).

The next category, addition instead of subtraction, refers to two fraction test tasks: 3a) $\frac{1}{3} + \frac{5}{3} - \frac{4}{3}$ and 3b) $\frac{3}{2} + \frac{3}{7} - \frac{7}{5}$. Nine students added all fractions; five in both tasks. There is no explanation for this.

$$\frac{-6}{9} = \frac{4x}{3}$$

$$-6 \cdot 3 = 9 \cdot 4x$$

$$\frac{-18}{36} = \frac{36x}{36}$$

$$\underline{\underline{x = \frac{1}{2}}}$$

Five students in the test ignored one negative solution number, however, they have negative solutions in other tasks, an indication of not avoiding negative solutions (Gallardo & Rojano, 1994). The task was an equation. It might be that the error depends on the switching of sides; keeping the unknown positive, while the solution number changes sign (Cortés & Pfaff, 2000). The symmetric relation symbolised by the equal sign is not taken into account. Both the observation in the classroom and the tests do not reveal a tendency to avoid negative result and solution numbers as such (section 8.3.4).

In two tasks during the solution process, the students could be tempted to detach the terms from their indicated operation. One task was 1c: $-3a - 3b + 3b$. Five students answered $-3a + 6b$ probably calculating from right to left, or by seeing the minus sign as a splitting factor (Vlassis, 2004). Eight students were seen doing this in at least one of the two tasks.

The textbook and the conversation in the classroom was focused on rules. When the teacher asked for reasons for choice of signs, the students

²⁰ This task will be discussed later related to fraction operations.

referred to rules, and these short arguments were accepted. Didactical advice was given rather than reasons for why the rules work. What can be seen is a focus on instrumental understanding (Skemp, 1987) and procedural knowledge (Hiebert & Lefevre, 1986) without connection to conceptual knowledge.

The problem of detachment of indicated operations is not addressed, and seems to be a problem not recognised either by the textbook authors or by the teacher. The different functions of the minus sign is also not addressed.

Kilpatrick et al. (2001) use the example of the general procedures for arithmetic operations, when describing how procedures can be developed to be general procedures. It does not seem that all students have come that far in this area, and that the problem might be that they still lack confidence in the basic matter of what sign belongs to what term. There seemed not to be much progress during the time of observation. In the spring only 50 % succeeded in the algebra test task 1e: $a - 3a + 2a$.

8.4 Powers

Two of the sub-topics in the first sub-chapter in the textbook are powers and scientific notation (see appendix 8.4). There was not much focus on either of these topics. To know about, and to master operations on powers is important in the work with arithmetic and algebraic expressions. The work with scientific notation seems to be the reason for the introduction of powers at the start of the school year. In order to convert decimal numbers into scientific notation the students should have a sense of the size of the numbers written in both forms. Analyses and a more detailed description of students' work is to be found in appendix 9.4.

The textbook had just a short presentation of powers as repeated multiplication. It was defined to be: $a^n = \overbrace{a \cdot a \cdot \dots \cdot a}^{n \text{ factors}}$ and related to scientific notation as a multiple of ten. It was said that the denominator in a fraction could be a power, but then the exponent should be negative as in the example $10^{-5} = \frac{1}{10^5}$. Some example powers were written as repeated multiplication and calculated. Some with positive exponents:

$$a) 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81 \quad b) (-5)^3 = (-5) \cdot (-5) \cdot (-5) = (-5) \cdot 25 = -125$$

and some with negative exponents:

$$a) 10^{-1} = \frac{1}{10^1} = \frac{1}{10}, \quad b) 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \quad c) 10^{-6} = \frac{1}{10^6} = \frac{1}{1000\ 000}.$$

The assigned tasks related to powers and scientific notation were similar to the example tasks (details in appendix 9.4); and were seemingly solved without problems in the classroom. The saved computer files indi-

cate though, that the meaning of negative exponents, and the size of the powers they created were not grasped by all students.

The September test included two tasks presented in the next table:

Table 8-8: September test - Powers – item 2 scientific notation (27 students)

Transform to scientific notation	Correct	Alternatives	No response
a) 0,000065 =	(59 %)	A: $\underline{6,5 \cdot 10^5}$; $\underline{6,5 \cdot 10^6}$ B: $\underline{6,5 \cdot 10^{-6}}$; $\underline{6,5 \cdot 10^{-10}}$ C: $\underline{6,5 \cdot -10^5}$ $\underline{-65 \cdot 10^7}$; $\underline{0,65^{-10.4}}$	7 %
b) 234000000 =	(85 %)	$\underline{2,38 \cdot 10^6}$; $\underline{2,38 \cdot 10^7}$; $\underline{2,38^{10.8}}$	4 %

The table shows that task 2b) is solved correctly by 85 % of the students while only 59 % succeeded in the first, 2a). In category A) the students have divided the decimal number by a power. In this way they avoided the negative exponent which probably carried no meaning for them (Weber, 2002). Two of those had the correct power. In category B, two students multiplied by a power of 10 but with an incorrect negative exponent. Five others (category C) seemed to know that there should be a ‘minus’, and solved the problem in different ways. As the students in Cangelosi et al.’s (2013) study, those students seemed to know about a minus sign but not why and not how.

In one textbook item solved by nine (of 18) students, the same problem with negative exponents was seen, and three students skipped the base 10 when converting decimal numbers to scientific numbers like Else in one of the tasks: $432,1 = 4,321^2$.

These solutions indicate that the students making errors, lack a sense for the numbers they are creating, or they have not clearly grasped the concept of the number system. The textbook did not help much here. The procedure was said to be to move the comma as many steps as the value of the exponent.

Powers as a topic will show up later in the course, then related to the rules for calculating powers with the same base, also with rational exponents. Then the equality $a^{-n} = \frac{1}{a^n}$ is again briefly given in the textbook as a definition of a^{-1} ; also related to bases other than 10. However, this is in the textbook chapter 7.

Results from one task in the algebra test, asking to write $t \cdot t \cdot t$ as a power, showed that all students knew the definition of powers as repeated

multiplication. However, when solving an assigned textbook task, Kari and Rakel were observed discussing powers and the writing.

Kari had expanded the last fraction $\frac{a \cdot 3a}{2 \cdot 3a}$ in a string of fractions and

was going to execute the multiplication:

Kari: Three a squared, do you think that's it?

Rakel: Yes, isn't it, when there are to a-s?

Kari: I can't really see the difference between a squared and just adding another a.

Rakel: Neither do I.

They agreed to check the solution in the textbook, and found that their assumption was right. On the test in September, however, Kari did not use the correct notation in a similar task.

Olav was observed having the same problem with powers. His solution so far was: $\frac{3 \cdot 2 \cdot b}{a \cdot b \cdot 2} + \frac{5 \cdot 2 \cdot a}{a \cdot b \cdot 2} - \frac{a \cdot 2}{2b \cdot a} - \frac{2 \cdot 2}{ab \cdot 2}$. He found that his solution was not in line with the solution in the textbook, and asked for help. The teacher focused on the third term:

T: You have to multiply by a both places because you lack an a in the denominator, and you multiply by a there, and then you have to do the same here as well (*she points to the denominator and then to the numerator*).

Olav: But shouldn't I write $2a$ then?

T: No, you already had 2. (*She points to the number 2 in the denominator*).

Olav: So then it will be a squared.

From the conversation it seems that the teacher thinks that Olav's problem is expansion of fractions, and asks him to multiply by the same factor (a) in the numerator as in the denominator. Olav seems to think that is what he has done. His reply is: "But shouldn't I write $2a$ then?"

He seems to be as unsure as Kari and Rakel in the episode above, when deciding how to write repeated multiplication. He was observed on other occasions revealing the same confusion between multiplication and exponentiation reported by Macgregor and Stacey (1997) (section 4.5).

Several other students were also confused, and at least Olav and Arne continued making this error.

In addition to the interference between the two symbol systems for multiplication and for powers, students' casual talk might promote extra confusion. On several occasions students were observed to talk about $2x$ -es or $2 a$ -s meaning x or a squared. When confronted with this, they always answered correctly that they had to square. This casualness might though have caused trouble when writing.

In the September test five students (19%) made one or more power errors. One example is Ane's solution: $\frac{15ab^2}{5b} = \frac{\cancel{15}^3 a \cancel{b}^2}{\cancel{b} \cancel{b}} = 3a^2$. Two others

cancelled in the same way, but their solution was $3a^2b$. It might be that they have not distinguished between ab^2 and $(ab)^2$, a problem reported earlier (Seng, 2010), or that they have regarded the exponent to be a ‘left over exponent’ to be used. At least it must be a mis-interpretation of the components in powers (Cangelosi, et al., 2013). One student wrote $2a$ instead of a^2 .

Another reported problem is to distinguish between exponentiation of negative numbers and the exponentiation of a number preceded by a minus sign (Lee & Messner, 2000). There were two assigned tasks including the power of a negative number: $(-3)^4$ and $(-5)^3$, but no task challenging students to reason about this distinction.

In December on the semester test, seven (26 %) of 27 students made errors related to powers. Most of them were caused by not following the correct order of operations, and in addition some students multiplied base and exponents in that they equated 2^3 and 6.

To sum up, although knowing the definition of powers as repeated multiplication, some students were unsure about how this should be written when solving tasks. Also some students did the opposite; they wrote ordinary multiplication as powers.

There were few tasks and little talk about numbers written in scientific notation and powers, but the results on the test tasks indicate that a large group of the students lacked a sense for the numbers represented by the powers. This might also be the reason that some students multiplied base and exponent. Another reason might be that there was no clear understanding of the different components constituting a power. In addition, the sloppy talk about powers, might have caused confusion.

Despite these errors, most students seemed to know how to deal with powers in algebraic expressions. The problem was that some students, those who were struggling, seemed not to have progressed at all during the first year in upper secondary school.

Also in the next section, powers will be discussed related to order of operations.

8.5 Order of operations

Some categories under negativity (Vlassis, 2004) (section 8.3.6) could have been categorised under order of operations. However, I have chosen to gather all troubles caused by the minus sign and negative numbers in one section. The reason is that the order of operation would not have been disturbed if the minus sign or negative numbers were not involved in those situations. Another reason is that there were so many questions in

the classroom about tasks with minus sign(s) and troubles in finding the correct solution due to the appearance of that sign.

In the textbook and in the ‘learning book’, the order of operations was presented as plain rules, and there was no task on the work plan offered for students to explicitly practice order of operations (appendix 8.5).

According to Linchevski and Livneh (1999) one frequent error is to add two like ‘terms’ before executing multiplication. One such error was observed in the September test. Carl was the only one, solving the task 1b) in this way: $b + b(5b - 2) - 4b(b + 1) = 2b(5b - 2) - 4b(b + 1)$. The other students succeeded. In the textbook only two tasks of this type were offered and both were not written with bold fonts; only three students had solved them; correctly. One task had a structure that could cause the same mistake $1 + \frac{1}{2}(\frac{5}{6}x - 5)$. The occurrence of the fraction might have prevented students from adding before multiplying.

One item was designed to practice the rules for order of operations, but was not offered on the work plan, four students, however, solved some few tasks in that item. Thus, few students solved tasks including both brackets, powers, multiplication, division, addition, and subtraction.

Else, solving one of those tasks: $4 - 2(-3)^2 - (4 - 8)^2$, suggested to perform multiplication before exponentiation:

- T: No, you are not going to multiply by 2. First, you have to deal with the power. (Else *performs the exponentiation and comes to the second bracket*).
- Else: Yes, look, now am I going to multiply by the power there?
- T: Yes, but since there are numbers only, you can just calculate the bracket before you take it to the power of 2.
- Else: Oh yes.
- T: Yes, when it comes to order of operations you have to calculate the powers first.

The tendency to multiply the base by the preceding factor was observed in the December test. Five students (19%) made this mistake.

Another issue is the exponentiation of a negative number (Lee & Messner, 2000). In the example above, Else suggested that the result would be negative. The teacher first repeats the rule for Else: powers before ordinary multiplication. Secondly, she suggests writing the negative number twice, in brackets, to visualise that she has to multiply two negative numbers, again presenting a rule; the sign rule.

One task in the algebra test had one term with a power. The task was discussed in relation to negative numbers (see section 8.3.2): Calculate $3b^2 - abc$ when $a = 3$, $b = -1$, and $c = 5$. One solution was:

$$3b^2 - abc = \dots 3(-1)^2 - (3 \cdot -1 \cdot 5) = 9 + 20 = 29$$

When solving the task, a large group of students multiplied the base by 3 before exponentiation in the first term. Those students wrote either 9, or making a sign error, writing minus 9. Others multiplied the base by 3, and then multiplied this result by the exponent, writing either 6 or minus 6.

Table: 8-9: Algebra test – task 5b the first term

$3b^2$ given $a = 3$, $b = -1$, $c = 5$	Autumn	Spring
Correct	36 %	50 %
9 or -9	24 %	25 %
6 or -6	4 %	13 %
Other responses	36 %	12 %

The result is better in the spring, but a considerable group of students still have problems, both with the order of operations related to powers, and with the exponent in relation to the base. In addition, many students made sign errors.

This indicates that in the spring, many students still do not follow the correct order of operations. If the students had a well-developed concept image (Tall & Vinner, 1981) of powers and the numbers they represent, they would have executed the operations in correct order.

Another frequent error was to ignore the possibilities to work out the brackets before the other operations (test task December):

$$\frac{2 \cdot (3 \cdot 4 - 3^2) - (4 + 2^3)}{5^2 - 3 \cdot 7}$$

Nearly 80 % of the 21 students who chose this task, carried out the multiplication first. This might be a result of interference from having worked with algebraic expressions. In addition, 39 % of the students did not distinguish between factors and terms and multiplied both 3 and 4 by 2.

The rule for order of operations is one side of the coin. Another is to choose the most efficient order of operations. Some students tended to expand fractions also when the operation was multiplication, or to multiply before checking for possibilities to simplify. This has to do with students' strategic competence (Kilpatrick et. al., 2001), their structure sense (Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999), and the development of proceptual thinking (Gray & Tall, 1994).

8.5.1 Summary order of operations

No task was assigned explicitly for students to practice order of operations. During observation though, problems were observed, and they were

mostly related to bracket reasoning (Vlassis, 2004), and calculations from left to right depending on the size of the numbers (Vlassis, 2004); all related to the minus sign discussed in section 8.3.6.

Other problems were related to the powers in that the base of the powers were multiplied before exponentiation; presumably based on misinterpretation of the components in powers (Cangelosi, et al., 2013) and a lack of number sense (McIntosh, Reys, & Reys, 1992).

In addition, most students did not check the possibility either to calculate brackets before multiplication or to simplify fractions before multiplication. Both phenomena indicate a low level of flexibility and a lack of strategic competence (Kilpatrick et. al., 2001).

8.6 Chapter summary

The topics presented in the textbook, and worked with in collaboration within the classroom, the students have met in earlier grades. The analyses of students' responses indicate that several students have had little experience with some basic mathematical concepts. Basic concepts and principles are neither addressed in class nor in the textbook. Some of these are the meaning of the letter symbols, the concept of equivalence, the different functions of the minus sign and the meaning of powers.

Although most of the assigned tasks are exercises (Niss, 1993), and could be solved by imitating example tasks, the responses give evidence that some students fail or are unsecure because they lack rich experiences with the aforementioned issues.

The properties or rules of the arithmetic operations seem to be known. One issue, the invisible operation sign is addressed and focused upon by the teacher, and the students' performances in that area developed positively.

The main focus is on procedures and rules, without connection to the concepts they are based on. Those students who are challenged and ask for help seem though to have resolved their problems.

9 Fractions – exposition and students’ work

In the textbook fractions are in focus for sub-chapter 1.2: *Calculating with fractions*. As the headline indicates, the emphasis is on the rules for operations on fractions.

Fractions, or rational numbers, are as already mentioned an obstacle in mathematics learning (section 4.6). More sub-concepts of the fraction concept (Lamon, 1999; Marshall, 1993) are presented in the textbook.

Ordinary fractions are introduced as a result of division; as quotients. When introducing simplification and expansion of fractions the two operations are exemplified as a result of partitioning; as part of the whole. Related to the presentation of the set of rational numbers, \mathbb{Q} , fractions are presented on the number line with equivalent fractions located in the same point on the line; which implies that fractions are presented as numbers (Wu, 2009). Two of the fractions with the same location on the number line are equated: $\frac{3}{5} = \frac{6}{10}$. It is commented that this equality is in accordance with expansion of fractions. Equivalence is not explicitly introduced.

Fractions as ratios are not mentioned, but implicitly fractions as measure, and fractions as operators are included by the use of the number line and through multiplication of fractions. The main emphasis, however, is on the rules for fraction operations and on how to apply them, but no explanation is given of the rules.

In this chapter students’ work with fractions will be presented in the light of the presentations given in the textbook, in the ‘learning book’, and in the classroom. The chapter is divided into sub-sections according to the operations involved. Detailed descriptions of the textbook expositions are to be found in appendix 10.1 and 10.4. The work in the classroom is described in the appendices 10.6 and 10.7

9.1 Factorisation and simplification

The technique and formal writing of expansion is exemplified in the textbook. Simplification of fractions is written in two different ways. First, it is explained as dividing the numerator and denominator by the same number:

ber: $\frac{15}{25} = \frac{15:5}{25:5} = \frac{3}{5}$. Then it is said that this operation is mostly written:

$\frac{15}{25} = \frac{3 \cdot \cancel{5}}{5 \cdot \cancel{5}} = \frac{3}{5}$, and the rule is given: “Before simplifying a fraction, one

has to factorise numerator and denominator. Then numerator and denominator must be divided by the same factors” (p. 26). In addition, it is em-

phased that expansion and simplification do not alter the value of the fraction.

In the classroom, expansion was introduced by the example: $\frac{3 \cdot 2}{4 \cdot 2} = \frac{6}{8}$, and simplification was shown to be the opposite operation $\frac{6 : 2}{8 : 2} = \frac{3}{4}$. In addition, this was written in MathType on the computer by cancelling the initial numerator and denominator, with the remaining factors written as superscripts: $\frac{\cancel{6}^3}{\cancel{8}^4} = \frac{3}{4}$. The focus was to find the factor in common for the numbers, 6 and 8, and perform mental factorisation. Most students followed this pattern of writing, shown in MathType.

Solving the first task, only two of 18 students, prime factorised the numbers $\frac{10}{15} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$ before cancelling equal factors. Five students made the division visible; dividing both numerator and denominator by 5. The others followed the pattern the teacher had shown.

One frequent tendency, when the numbers involved were larger numbers, was that the fractions were not reduced to their simplest form. One example is: $\frac{42^{60}}{630^{90}} = \frac{6}{90}$. In this task 25 % of the students observed to solve the task, solved it this way. This tendency is in accordance with Brown and Quinn's report (2006), and is even more frequent, when the results of addition/subtraction tasks should be simplified. Another tendency, reported by Shaw and colleagues (1982), but not so frequent, is to avoid simplification of result fractions when numerator and denominator are equal. There is evidence of this in some students' solutions.

In the classroom it was observed several times that students struggled to find common factors although the numbers were not that large. One such example is the episode with Glenn and Sven. They were observed discussing how it could be possible to go from $\frac{56}{21}$ to $\frac{8}{3}$. (They had found the correct answer in the textbook). They were challenged by the teacher to find the factors in each of the numbers 56 and 21.

- T: 56 and 21 has one factor that goes into both numerator and denominator. If you write 21 as factors, what will it be?
- Sven: Factors?
- Glenn: 7 times 3.
- T: 7 times 3, yes, correct. And then you have to check. Either must 7 go into 56 or 3, this in order to simplify.
- Sven: That... (*he stops there*)
- T: 56. Goes 3 or 7 into 56?

Glenn: 7 goes.

T: Yes 7.

Glenn: 7 times 8.

T: 7 times 8, yes.

Glenn: Yes.

T: Yes, and then you can simplify by seven, you see?

Glenn: Should we just divide by 7?

T: Yes, when you simplify, you must divide by the common factor.

Here Sven seemed to be surprised by the word factors. Although the teacher had stressed the notion of factors and terms earlier, it seemed to mean nothing to Sven. However, when the teacher prompted them, Glenn seemed to grasp what factorisation is about, and he factorised both numbers. However, he asked: “should we just divide by 7?” From this, it seemed that for him the problem was not only the numbers, he seemed to see no connection between simplification and division.

Some of the other students struggling with this task, found that they could use the calculator in the program TI-interactive.

Norwegian students are familiar with the use of hand-held calculators from the early grades (see section 2.3), and it was observed several times that they struggled to perform correct multiplication or division on numbers without that tool. One example of this is from a lesson in November. Ronny is observed when he is going to divide 5 by negative 1:

Ronny: Let me see. How am I dividing 5 by minus 1 on this cell-phone? (*He is struggling with his cell-phone. He has no calculator and the students were asked not to use the computer.*)

Glenn: Oh, 5 divided by minus 1? Five divided by...
(*Ronny waits and calls to Tore*)

Ronny: Tore, what is the result of 5 divided by minus 1?

Tore: That's minus 1.

Ronny seems not to be satisfied by the answer. It is a session of group work and he walks around asking others the same question.

This episode illustrates how simple calculations could be experienced as difficult. It also illustrates how dependent the students are on the technological tools. It might be that the minus sign caused him and the others to wonder (see section 8.3.4). However, at this level this is expected to be known. All students used their cell-phones before they became used to the calculator in the IT-program, and it might be that the habit of relying on these tools prevented them from developing a good number sense (McIntosh, et al., 1992).

When binomials were included in the fractions, all students seemed to be unsure about what to do, and how to simplify. The exposition part in the textbook introducing this, is extremely short. It is stated that the same rules are valid for algebraic fractions as for fractions with only numbers, and when simplifying fractions with binomials, one has to factorise by

setting common factors outside a bracket. Afterwards one can simplify by dividing numerator and denominator by the same expression. It is said to be the opposite of expanding the bracket.

The example task: $\frac{3a+6}{6a-9}$ was applied in the classroom. The immediate suggestion was to cancel the a 's, the literal symbols, reported to be one strategy in Guzmán, Martínez and Kieran's study (2011).

Students also offered other not efficient suggestions, and after a while the teacher applied the number example $\frac{25-5}{5}$ in order to create meaning; to help students realise that the operations they suggested, did not work.

It seemed as if the students accepted her argument, but when asked why their suggestions will not work, the students only pointed to the fact that the answer would be wrong, or that there is a minus sign in the numerator. The teacher clearly waited for the mathematical notions of 'terms' and 'factors'. After many hints and a time of silence Jarl answers 'factors' to the question of what makes a product; an example of a funnel pattern (Voigt, 1995). The teacher had to bring in the notion of 'term' herself, and to explain what it means.

By gesturing, the teacher made circles around each binomial on the whiteboard. She emphasised that both binomials had to be dealt with as entities, and from this she deduced that the terms within the binomials cannot be cancelled; one has to factorise.

Few students had suggestions to what could be done. One was to change the operation sign within the binomial, another to factorise only the first term. The impression from the classroom was that the students really struggled, and the teacher let them take their time.

Her next move was to write the expression $3(a+2)$ asking what could be done. Now many students responded, and they told her to execute the distributed multiplication; $3(a+2) = 3a+6$. The teacher directed the attention to the latter expression and asked them to compare it with the initial task, before emphasising that the operation executed in the factorisation is division.

Neither was to factorise the denominator $6a-9$ easy. To set the factor 3 outside a bracket, was agreed upon, but some responses involved subtraction (Storer, 1956). When the factorisation was completed, the simplification seemed to be a simple matter.

In the example task presenting division of fractions with binomials: $\frac{3}{a+1} : \frac{4}{3a+3}$, it was evident again that the students suggested to cancel the

literal symbols after having flipped the last fraction and transformed it into the operation of multiplication $\frac{3}{a+1} \cdot \frac{3a+3}{4}$. Tore seemed to have grasped the notions, factors and terms emphasised by the teacher earlier, and he applies them in his argumentation.

The teacher wrote both binomials in brackets, and emphasised that the whole brackets constitute factors, and had to be handled as such. Some students suggested to carry out the multiplication. As a response, the teacher asked for possibilities to simplify. Again there were suggestions to subtract. The process of factorising the binomials took a rather long time. The problem was to see brackets as factors, as entities.

There were only few tasks offered on the work plan to practice factorisation and simplification of fractions with binomials; one task in which it was asked for simplification only, one multiplication task, and one division task. In addition, there was one task, where the result had to be factorised in order to be simplified.

It was clear that many students were unsure when solving the first assigned task $\frac{4a+8}{5a+10}$. It was the task with most frequent questions from students. Some just needed the teacher's confirmation on what they were doing. Others were stuck and needed help to factorise and/or to be told once again what factorisation of binomials was about. For many the problem was to find the common factor in the two terms in the numerator. They divided by 2 instead of by 4 in the numerator, and could not cancel the brackets. Others had problems seeing what should be left in the bracket, they did not perform division.

Some students cancelled the literal symbols before they added the remaining numbers, as was also suggested by some students in the plenary session. The structure of the task was such that they then came to the same solution as was written in the textbook. Ronny and some peers were observed solving the task in this way: $\frac{4a+8}{5a+10} = \frac{4\cancel{a}+8}{5\cancel{a}+10} = \frac{12}{15} = \frac{4}{5}$, arguing that they had reached the correct solution. Ronny speaks for them. He argues that there is no need for factorisation:

Ronny: Why should we have to factorise this here, it's just 5 times a plus 10. We can do it like this? (*He points to his solution*)

T: What do you mean by doing it this way?

Ronny: But the answer is correct.

T: You can't do like this.

Ronny: But why not?

(*The teacher goes to the whiteboard and points to the number example*)

she applied in the plenary session: $\frac{25-5}{5}$. She repeats that to cancel the

5's is not what is to be done.)

Ronny: Yes, but that's why there was a minus?

(The teacher then changes the minus in the example to a plus sign).

T: I can do like this.

Ronny: Okay.

T: Okay? (She waits for Ronny's response. He watches the white board).

Ronny: But I got the correct result.

The teacher emphasised that it was just by coincidence that his result and the result in the textbook were identical. She applied the number example once again and told him and his peers to factorise, pointing to the fact that the whole numerator is one factor. Ronny does not seem to agree, but the teacher tells him and the others "to do it the hard way; by factorisation".

From the example above it is clear that these students did not see the necessity of factorisation in order to simplify; they did not see the binomial as an object (Sfard, 1991). It also illustrates the view they had on the 'correct answer'. Since their solution was in line with the answer in the textbook, they did not question their solution method. Ronny argues as if he does not regard the process as important.

Eli was observed when she worked on the same task. She had factorised the fraction, however, not correctly. Her last move is then to cancel term by term within the bracket: $\frac{1(\cancel{a} + \cancel{2})}{5(\cancel{a} + \cancel{2})} = \frac{1}{5}$.

Hall (2004) experienced this phenomenon in his study. He commented that he as a teacher would not have been aware of such a reasoning if he had not interviewed the students. Here it is visible because of the cancelling markers. It might though be that also other students were reasoning in the same way as Eli, although seemingly cancelling the whole bracket.

When asked what she would have done if the expression in the numerator was $3a + 2$, she indicated that a request for simplification is that one of the brackets could be completely cancelled.

In the two other assigned tasks, the main problem was the factorisation or to see the need for factorisation. Not many students, 6 and 11, respectively (dataset 18 students) solved the tasks. Four of them did not succeed in either of them. Also for these two tasks it was possible to come to the same result as was written as correct in the textbook; performing partly cancelling. The tasks were told to be worked out at home.

In the class Ane and Ine solving one of the tasks had written:

$$\frac{2a+6}{5} : \frac{a+3}{15} = \frac{2(a+3)}{5} \cdot \frac{15}{a+3} = . \text{ Ane asked:}$$

Ane: Now we have done this. What are we then going to...

Ine: Eh... Is it correct?

- T: It is perfect.
 Ine: Can we... What are we going to do now?
 T: Yes what should we do?
 Ine: Can we remove this and this? (*She highlights $a+3$ in the numerator and in the denominator*).
 T: Yes, can't we?
 Ane: And the others like that?
 T: Yes, and then you simplify there, yes.

It seems as if they are not sure it is correct, and will not cancel any factor before they have asked the teacher; they want her confirmation. The students tell that they like this type of task.

In students' work plan there were no more opportunities to practice factorisation of binomials. It is clear from the observation that these tasks did not serve their purpose. All assigned tasks were of such a structure that the result would be the same either they were factorised, or if the students started their solutions by cancelling the literal symbols. Actually, students could work out the tasks without having grasped the important difference between factors and terms and carry on with their mathematical work without their erroneous concept images being challenged. Former studies have though showed that it is likely that some students would simplify in the same way as these students did (Guzmán, et al., 2011; Storer, 1956).

In another assigned task with another structure the result had to be simplified in order to reach the solution written in the textbook.

Ronny was observed when recognising that his answer, $\frac{3a-3}{6}$, differed from that in the textbook. He was claiming that he had no clue about how to reach the desired result. He turns to his classmate Jarl:

- Ronny: But, you can't simplify $3a$, there is no a underneath.
 Jarl: What do you want to simplify?
 Ronny: I don't know, but ...yes, look here ... No, I can't simplify.
 Jarl: You have to factorise.
 Ronny: Yes, I have to factorise.
 Jarl: It will be 3 and $a - 1$

Ronny was stuck about how to come to the answer in the textbook. He thought at first that it was impossible to simplify the fraction since the literal symbol was present only in the numerator, this was his only strategy. The result $\frac{a-1}{2}$ in the textbook, made him ask Jarl for help.

Half the students in the data-set of 18 students had solved this task. Three of them did not factorise the numerator, but two of them, Rakel and Tord, correctly divided all terms by 3. It might be that they know why this is functioning. They found at least a way to reach to the expected result.

Ruth in her solution seemed to perform some acrobatics in order to come to the desired result:

$$\frac{2a}{3} - \frac{3+a}{6} = \frac{2a \cdot 4}{3 \cdot 4} - \frac{(3+a) \cdot 2}{6 \cdot 2} = \frac{4a - (6+2a)}{12} = \frac{4a - 6 - 2a}{12} = \frac{\cancel{2}a - \cancel{6}^1}{\cancel{12}^2} = \underline{\underline{\frac{a-1}{2}}}$$

This episode and Ronny's solution are examples of how the answers in the textbook influenced students' actions. Ronny asked for help and his peer instructed him. Ruth just searched for a way to adjust the result she had. The conclusion in Guzman et al.'s study (2011) about rational fractions and the use of CAS, was that when the result using CAS differed from students' solutions, they had the opportunity to try to find another way to a correct result. The answers in the textbook can challenge students in the same way. It is, however, of crucial importance that the students are not left alone (ibid). They need the guidance of a teacher or peers who are more capable. In addition, they must be committed to learn (Entwistle & Peterson, 2004; Mellin-Olsen, 1981; Novak, 2002), not just to find the correct answer.

In September the students should solve a test task, dividing and simplifying rational fractions. The table below shows the test result.

Table 9-1: September test task 4c. Solutions (%). Dataset 27 students

$\frac{3a+15}{6} : \frac{2a+10}{8}$	%	Alternative solutions
Correct	19	
Not finished	3	(A) $\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a+15}{6} \cdot \frac{8}{2a+10} = \frac{(24a+120):12}{(12a+60):12} = \frac{2a+10}{a+5}$
Partly cancelling	60	(B) $\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a+15}{6} \cdot \frac{8}{2a+10} = \frac{\cancel{24}^{12} \cancel{a} + \cancel{120}^{12}}{\cancel{12}^{12} \cancel{a} + \cancel{60}^{12}} = \frac{2+10}{1+5} = \frac{12}{6} = \underline{\underline{2}}$ (C) $\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{8(3a+15)}{6(2a+10)} = \frac{\cancel{24}^2 \cancel{a} + \cancel{120}^2}{\cancel{12} \cancel{a} + \cancel{60}} = \underline{\underline{4}}$
Other solutions	7	
No response	11	

All 24 students who solved the task, had flipped the last fraction, which means they know the algorithm for fraction division. Three students did not solve this task. Only five (19 %), had correctly factorised the binomials and simplified the rational expression. The others, except for two, had

multiplied the factors before trying to cancel/simplify. When simplifying rational expressions with binomials, it is crucial that students have a sense of the structure (Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999), that they see the binomial as an object (Sfard & Linchevski, 1994b), and have strategic competence (Kilpatrick et al., 2001).

One student, (A) after multiplying the factors, he correctly divided all the terms by 12. He was not very successful, however, since he did not factorise the binomials. During seatwork, he was seen doing the same.

The solutions B and C were the most frequent solutions (60 %). In both of them, the students cancelled the literal symbols. Then the solutions differed. In solution B, the students divided all terms by 12. (This student has written the divisors as superscripts). In solution C the first two terms in both numerator and denominator were divided by 12, the two last terms were divided by 60. In B the remaining factor 1, was taken into account. In C there should be a remaining factor 1 in each term in the denominator, these, however, just disappeared. Storer (1956) asserted that this type of error was probably caused by “striking without trace”. This was reported to be a recurring student error when working with numerical fractions (Eichelmann, et al., 2012). It therefore might be that they consider the simplification as an act of pure cancelling (Storer, 1956), forgetting that it is division.

When comparing the solutions on the test with the solutions to the similar textbook tasks, only three students had correct solutions both in the textbook tasks and on the test. Four students solved the tasks correctly in the classroom, however, did not show competence in factorising binomials on the test, Ine and Ane who said they enjoyed this kind of task were in this group. Tore who in the classroom showed that he differentiated between factors and terms, had not factorised any task. The same was the case with Ronny.

Selma solved the test task correctly, but solved just one of the textbook tasks related to factorising binomials. She was observed to struggle and asked the teacher for help. Here it seems to be that she has grasped the concept of factors and terms as she told in the interview (see section 8.3.6).

The main problem for most of the students working with binomials seemed to be to see them as entities and objects, and to distinguish between factors and terms. The assigned tasks confirmed misconceptions rather than challenging students to develop their conceptions.

9.2 Multiplication and division

Multiplication and division of fractions were briefly introduced in the textbook by listing the rules for fractions operations. In addition, it was noted in both the textbook and in the ‘learning book’ that all natural numbers could be written as fractions with 1 as the denominator. One textbook example task was solved: $12 \cdot \frac{4}{3} = \frac{12}{1} \cdot \frac{4}{3} = \frac{12 \cdot 4}{1 \cdot 3} = \frac{2 \cdot 2 \cdot \cancel{3} \cdot 2 \cdot 2}{\cancel{3}} = 16$.

In the classroom the task was solved in this way: $12 \cdot \frac{4}{3} = \frac{\cancel{12}^4 \cdot 4}{\cancel{3}} = \underline{\underline{16}}$, focusing on the operation of division. Multiplication and division of fractions were executed without problems in the classroom. The students knew the algorithms and followed the rules, there was, however, a tendency to forget to simplify the results.

One test task included both multiplication and division as seen in the table below.

Table 9-2: Ordinary fractions – multiplication and division (%)

Task 3c – September test	$\frac{4}{2} \cdot \frac{5}{4} \cdot \frac{7}{5}$
Correct -simplified before executing the operations	7
Correct result, multiplied before simplification	48
Correct process - not simplified	17
Found common denominator - multiplied numerators- kept the common denominator	11
Calculation error	14
Not solved	3

Although 72 % of the students had followed the rules for multiplication and division of fractions, only three (11 %) had looked for the structure and simplified before performing the operations. An example

$$\text{is: } \frac{4}{2} \cdot \frac{5}{4} \cdot \frac{7}{5} = \frac{4}{2} \cdot \frac{5}{4} \cdot \frac{5}{7} = \frac{\cancel{4} \cdot 5 \cdot 5}{2 \cdot \cancel{4} \cdot 7} = \underline{\underline{\frac{25}{14}}}$$

The other students, although finding the correct answer, do not seem to have been searching for the structure in the task (Hoch & Dreyfus, 2004; Kieran, 1992; Linchevski & Livneh, 1999).

One algebraic multiplication task with monomials only, and one division task including binomials were solved in plenary. Both were example tasks from the textbook. In the textbook there were no explanations related to division and multiplication of fractions other than the solved example tasks, and the presentation of the rules offered in relation to ordinary fractions.

In all these tasks most students suggested performing the multiplication before searching for possibilities to simplify or to factorise, although the teacher asked them to search for such a possibility.

One example of students' reluctance to follow the teacher's advice, can be the equation task $2 + \frac{3}{2}x = \frac{1}{3} + x$ solved in plenary. During the solution process, the teacher together with the students agreed to multiply both sides by 6; the least common denominator. On the white board was written: $2 \cdot 6 + \frac{3}{2}x \cdot 6 = \frac{1}{3} \cdot 6 + x \cdot 6$.

The students wanted the teacher to multiply 3 by 6 in the second term, but the teacher simplified by dividing 6 by 2. There were many negative reactions on this, and Tore asked why she did not do, what he had said. To multiply 3 by 6 and then divide 18 by 2, since the result would be the same. The teacher asked them to look for efficient ways to solve tasks, especially when numbers are larger it will pay off to simplify before multiplying the factors.

The same example was used by the teacher asking about the relation between the unknown x and the fraction in the second term in the equation. Many students responded, and some meant that the x belonged both to the numerator and the denominator. The teacher then used a numeric fraction as multiplicand and a natural number as multiplier in order to give meaning to the expression. Not all, however were convinced, expressing that there was a difference between numbers and letter symbols. From this it can be inferred that there is a blurred difference between expansion and multiplication, indicating that some of the students have a limited concept image of equivalent fractions, and fractions as numbers.

The students were as shown, confident with the rules of multiplication and division. However, they did not have strategic competence or flexibility in reaching their solutions. They multiplied before checking for possibilities to simplify. This prevented students from factorising binomials and from simplifying the results, as seen in the foregoing section. For many students the difference between expansion and multiplication was blurred.

9.3 Addition and subtraction

In the textbook it was stated that when adding and subtracting fractions, the denominators had to be equal. If they were not equal, the fractions had to be expanded. This was exemplified by two example tasks. The first had from the outset equal denominators, in the second the two fractions had prime denominators; 3 and 5. "The least number divisible by both 3 and 5

is 15. We thus choose 15 as the common denominator” (p. 27). Then the task was solved.

Addition and subtraction of fractions seemed to be well-known to the students. Nine tasks ranging from simple to more complicated were assigned to practice the rules for addition and subtraction of ordinary fractions. Two of the tasks also included multiplication or division.

Few students had solved these assigned fraction tasks. It might be that they felt competent, and had followed the textbook authors’ advice at the start of the textbook (Ekern, et al., 2006) about not studying the first chapter thoroughly since the topics should be known.

Two of those few students who solved the tasks, presented more of the fraction errors reported in the reviewed literature in section 4.6. One of them, Ruth, was not consistent, as some students in Wittmann’s (2013)

study. One example is: $\left(\frac{3}{5} + \frac{4}{5}\right) \cdot \frac{3}{14} = \left(\frac{3+4}{5+5}\right) \cdot \frac{3}{14} = \frac{7}{10} \cdot \frac{3}{14} = \frac{21}{140}$. Here the stu-

dent has added the numerators and also the denominators before executing the multiplication. This tendency to regard the numerator and the denominator as separate whole numbers is an often reported error in studies about students work with fractions (Eichelmann, et al., 2012; Padberg, 1986).

In another task $\left(\frac{1}{2} + \frac{1}{3}\right) : \frac{10}{9} = \frac{2}{6} \cdot \frac{9}{10} = \frac{18}{60}$ the same student multiplied the

denominators and added the numerators, before executing the division.

These two examples are again revealing the issue common for most of the students. They do not look for the structure, but multiply the factors not taking into account that the numbers invite simplification before multiplication, as is also commented upon in the former sub-chapters.

In the textbook, addition and subtraction of algebraic fractions were introduced through two example tasks. The only comment is: “When adding and subtracting fractions, one has to expand the fractions in order to let them have equal denominators”.

$$\frac{a+3}{4} - \frac{a-1}{6} \qquad \frac{2}{x+2} - \frac{1}{3x+6}$$

During the solution process, brackets are written around the binomials, but there is no comment upon this. However, in the margin the advice is given, that it is wise to put binomials on one fraction line before the

brackets are removed: $\frac{(a+3) \cdot 3}{4 \cdot 3} - \frac{(a-1) \cdot 2}{6 \cdot 2} = \frac{(a+3) \cdot 3 - (a-1) \cdot 2}{12}$

There is no instruction given of how to factorise binomials, except for one note at the start, saying that we can factorise expressions with two terms by setting common factors outside a bracket. During the solution

process of the example tasks the brackets are not mentioned any more. According to the studies referred to earlier (section 4.6.2) this factorisation is an obstacle for students at many levels (Guzmán, et al., 2011; Hall, 2004; Olteanu, 2012) and that many teachers are not aware of what is crucial for students when simplifying algebraic fractions (Olteanu, 2012).

The same example tasks were worked out in the plenary session in cooperation between the teacher and the students. In the first, the denominators were numbers only: $\frac{a+3}{4} - \frac{a-1}{6}$. The students told the teacher that

the least common denominator would be 12, and Carl told her to “times up and down” to get the common denominator. She revoices (Herbel-Eisenmann, et al., 2009) his utterance, but emphasises the word ‘expand’, and writes on the whiteboard $\frac{a+3 \cdot 3}{4 \cdot 3}$ asking if she can let it be like this.

More than half the class tell her to put a bracket around the expression $a+3$. The problem comes when the fraction $\frac{a-1}{6}$ is to be subtracted from $\frac{a+3}{4}$. The two fractions are expanded and are to be placed on a common fraction line. The teacher has written $3a+9$ in the numerator and asks Else to continue:

Else: It will be minus 2a minus 2.

(Some of the other students express that it should be plus. The teacher writes $-2a$ - and listens to the students. She turns towards them.)

T: Now, you seem to be unsure about what to do?

(The students continue their suggestions. Some say minus, some say plus, and some say that negative and negative is plus.)

T: Is it going to be minus 2 or plus 2?

(The students continue the discussion.)

T: Will some of you have minus? *(There are some quiet murmurs).* We can do it like this, *(she writes $-2a-2$ and says:)* but then I do like this. *(She puts the binomial in a bracket).* Sometimes it is smart not to work out too many operations at the same time.

She gives the didactical advice to perform only one operation at a time to keep control. The solution so far is shown to the right:

Else interrupts saying there is something that she wonders about:

Else: The number, 2, there was a plus in front of it? *(second line in the solution).*

T: That is true, but it is the negative sign in front of the bracket that causes the change within the bracket.

$$\begin{array}{r} \frac{a+3}{4} - \frac{a-1}{6} \\ \frac{(a+3) \cdot 3}{4 \cdot 3} - \frac{(a-1) \cdot 2}{6 \cdot 2} \\ \frac{(3a+9) - (2a-2)}{12} \\ \frac{3a+9-2a+2}{12} \end{array}$$

It seems that Else accepts the answer. The teacher

once again emphasises that it is wise to write an extra line, writing the binomials in brackets signalling that they are entities. She argues that it is not much extra work since the copy-function in MathType makes it convenient. In addition, the little extra time might give them opportunity for some extra reasoning.

When solving two tasks on an earlier work plan, in which the brackets had post-multipliers, some few students showed uncertainty (section 8.2). Here Else expresses their problem, and the teacher answers. This was a problem not explicitly caused by the fractions. It might be that Else's understanding of the rule was bound to the context of earlier tasks with pre-multipliers.

The next example, $\frac{2}{x+2} - \frac{1}{3x+6}$ causes much more trouble for the

students. Here one has to factorise the binomials in order to find the common denominator. This was as difficult as it appeared to be in the former section about factorisation and simplification. Also this time the teacher applies a number example. To find the common denominator the teacher points to the fact that in the common denominator one has to find both initial denominators as factors in the new denominator.

Overall, 12 tasks were assigned for the students to practice addition and subtraction of algebraic fractions. Six of them included binomials in the numerator, but none of them had binomials in the denominator; thus regarded as being easier than the last example task.

Most students seemed to find the common denominator without much difficulty. Some students, however, wrote expansion as multiplication; regarding the factor to "belong to the whole fraction" as some students expressed it. Some were corrected by their peers in this matter.

Another issue exemplifying how the use of the computer could have an influence on the formal writing was observed when Glenn was per-

forming expansion: $\frac{x+1}{2} - \frac{x-2}{3} = \frac{(x+1)^3}{2^3} - \frac{(x-2)^2}{3^2}$. Although the mul-

tiplication was written apparently as exponentiation, he correctly expanded the fractions. The teacher noticed his writing and explained that it was only when cancelling factors, that the remaining factors could be written as indexes. Glenn and a couple of other students changed their writing after this.

Some few students struggled to find the least common denominator during seatwork. One of them was Atle who asked for help when solving

the task: $\frac{2}{3} + \frac{5}{a} - \frac{a}{2}$.

Atle: How can I solve this?

- T: What are the denominators then?
 Atle: Those. (*He points to the denominators in the fractions on the screen*)
 T: None of them are equal.
 Atle: No, none are equal but what am I going to do then?
 T: When none are equal, then all should be included, shouldn't they?
 Atle: Yes? (*a bit reluctant*).
 T: Yes, they have to.
 Atle: Yes (*his yes is still not convincing*).
 T: Because if the 2 has to go into the common denominator, the 2 must be included, and if all are different then all must be included. What will it be then?

Atle seems to have no strategy for finding a common denominator in this situation. The teacher tries to explain that if the initial denominators have no common factors, all factors have to be included. She clearly expresses that she means factors, but for Atle this is not obvious. He comes up with the suggestion:

- Atle: 3 plus a minus 2?
 T: No pluses.
 Atle: $3a$ minus 2
 T: No minuses either, but 2 times 3 times a .
 Atle: 2 times 3 times a ?
 J: Yes, that has to be the common denominator.
 Atle: Okay.
 T: Okay. You just accepted it? You did not get it?
 Atle: I did not.
 T: Okay. But if you had a 5 instead of the a , what would the common denominator be then?
 Atle: Then it would be 2 times, 3 times a , no, I mean 5.
 T: Yes!
 Atle: I think I grasped it.

During the conversation, the teacher seemed to recognise that Atle does not follow her reasoning. At the end she confronts him with her doubt, and he confirms it. Then she changes her strategy in that she substitutes the literal symbol a by the number 5 and asks what the common denominator then would be. Although he makes a mistake at first, his voice and also his utterance: "I think I grasped it", seems to confirm that the use of a pure number example helps him back on the right track. This would indicate that the presence of a literal symbol caused the problem. Therefore, it might be that he differentiates between factors and terms as long as he operates in the area of pure numbers, but that he has not transferred this knowledge to the area of algebra.

Fractions with binomials connected by the minus sign seemed to be the main problem for the students. Else came to a wrong solution because she had not changed the sign in the last binomial, when bringing two bi-

nomials together on the same fraction line: $\frac{x+3}{5} - \frac{2x-1}{5} = \frac{x+3-2x-1}{5}$.

However, in the next task, when she had to expand the fractions before subtraction, she wrote brackets, and this seemed to help her to reach the correct solution. This was what the teacher also recommended when there was no need for expansion. This is in line with Vlassis (2004) findings, that brackets helped students to perform correct actions when working with algebraic expressions; not fractions. Also it was reported that the action of applying brackets helped students to see the structure of the tasks (Hoch & Dreyfus, 2004).

There were few tasks on the work plan including both whole numbers and fractions, but it was observed that it was not easy for students to see how whole numbers and fractions were related when it came to the operation of multiplication.

Two tasks written with normal font were solved by two students.

Rakel's end solution was: $\frac{8a}{12} + \frac{4a}{12} + \frac{6a}{12} - \frac{24}{12} + \frac{24}{12} - 10 = \frac{\cancel{18a}^6 - 10}{\cancel{12}^2} = \frac{3a}{2} - 10$. She

kept the number 10 through the whole process until she puts it on the fraction line (Padberg, 1986). Her incorrect partly cancelling causes the solution to be in line with the answer in the textbook.

Paul was observed asking the teacher about the same issue in another task. 1.2.21 c: $-3x + \frac{x}{2}(x-3) - \frac{x}{3}(4-2x)$. It turns out that the problem is the term $-3x$. He says he has multiplied $3x$ by 4. The teacher asks what common denominator he has, and he suggest 9 and then 12. The teacher now concentrates on the number 3:

T: Can you write number 3 as a fraction? What will that be?

Paul: I thought I had done that.

T: If you have the number 3, and it should be written as a fraction, what will you do?

Paul: The number 3 as a fraction? Then it will be written 3 over 9. Isn't it so?

T: Over 9?

Jone: 3 firsts.

T: 3 firsts. Thank you. So if you now want 12 as the common denominator you have to expand by 12.

Paul: Yes

The teacher focused on the number 3. Jone, after a while, who solved the same task, suggested to write $3x$ as a fraction with 1 as the denominator. This helped Paul to resolve his problem. He followed the advice, and saved a correct solution.

The revealed problems related to addition and subtraction, were mostly not related to fraction operations. More than half the students solving

the assigned numerical fraction tasks had one or more errors related to the minus sign; they had the opportunity though to check their solutions.

The table below presents two test tasks. Dataset 27 students.

Table 9-3: Ordinary fractions –addition/subtraction - September test (%)

Task 3a and 3b - September test	$\frac{1}{3} + \frac{5}{3} - \frac{4}{3}$	$\frac{3}{2} + \frac{3}{7} - \frac{7}{5}$ *
Correct solution	70	37
Write expansion as multiplication		11
Just addition, ignoring the minus sign	30	14
Multiplies by the common denominator (treated as equation)		11
Calculation error		22
Not solved		11

*The sum is not 100%; some solutions fit into more than one category

70 % had reached the correct solution in the first task, while 30 % had just added; ignoring the minus sign.

In the next task, one had to find a common denominator. All solving that task, found that it should be 70, but 11 % wrote the expansion as multiplication, but performed the expansion correctly. Three others (11 %) cancelled the denominators. For these students this occurred in only this task. This might be an interference from equation solving. Three students had not solved the problem. It might be that those students did not find a common denominator for three numbers. These students solved the first task correctly, and also one algebra task (the table below) with different denominators. Five students made different calculation errors.

Another task in the same test asked students to solve one algebraic fraction task. The table below shows the results. (Dataset 27 students).

Table 9-4: Algebraic fractions – addition/subtraction - September test (%)

Task 4b	$\frac{3}{2a} + \frac{a}{4b} - \frac{a-2b}{ab}$ *
Correct	33
Do not change sign in binomial numerator	33
Expansion written as ordinary multiplication	22
Treat as equation, cancelling the denominator	11
Partly cancelling	22
Errors related to power	3
Change the sign in the binomial before it is located on a common fraction line.	7

*The sum is not 100%; some solutions fit into more than one category.

Only 33 % solved the task correctly. During observation and from the saved computer files, it was evident that some few students struggled when searching for a common denominator. Here it seems that everyone solving the task, succeeded in that matter, except for Arne who did not give any response. The students asking for help during the lessons seem to have grasped what it is about, and how to find it. They also seemed to be confident in expanding the fractions, although six students still write the operation as ordinary multiplication. Three of those then multiply the fractions by the common denominator as if the task were an equation. These three were not the same as in the task above.

The category with most students making errors is the category of ‘do not change sign in the binomial numerator’. The teacher gave the didactical advice to put the binomials in brackets. However, 14 students did not follow this advice on the test. In the test task the distribution of bracket use and correct results was as follows:

Table 9-5: Sign and use of bracket - item 4b (Numbers are number of students)

Bracket		No bracket	
Correct sign	Incorrect sign	Correct sign	Incorrect sign
11	1	6	8

From this single task, the tendency is that it is more likely that the students come up with a correct sign when they write brackets, only one student did not. Here it was necessary to expand, what might have helped students to write brackets. 14 students did not write brackets though, and less than half of them wrote a correct positive sign. This is consistent with Vlassis’ (2004) observation in her study.

In December a choice of two alternatives were given, one algebraic fraction and one with ordinary fractions. 18 out of 27 students chose the ordinary one. The results are presented in the table below:

Table 9-6: Ordinary fractions – multiplication and addition (%) 18 students

Task 2b alternative 1 - December test	$\frac{2}{3}(\frac{1}{2} + \frac{2}{4})$
Correct result, but multiplied before simplification	55
Multiplies by the common denominator (treated as equation)	17
The result not simplified	6
Add numerator and multiply denominator or opposite	11
Found common denominator for all three fractions.	11

As seen in the table, 55 % solved the task correctly. Some students treated the task as an equation in that they multiplied by the common denomina-

tor and then simplified, as if they were solving an equation. Selma did that in both tests.

Also this time the fractions had such factors in the numerators and in the denominators that they invited simplification before the multiplication should be executed. None of the students had done that. Some even expanded all the fractions in order to have a common denominator. The question is if those students differentiated between terms and factors. At least they did not take the structure of the task into account before starting to solve it.

One task (item 1b alternative 2) in the September test was solved by 9 of 27 students. The others solved the task above. The task was:

$$\frac{1 + \frac{2}{a}}{\frac{a}{2} - \frac{2}{a}} =$$

This task weighted double -twice as many points as the task above with ordinary fractions.

The task was given in a form they had not met during the teaching, and that might be the reason why so few students chose it. It was evident though that the way those students tried to solve the task, was in conflict with the properties of and operations on fractions. None of the students succeeded.

There were two main problems in this task. One was to see the need of finding a common denominator and to multiply with this in both numerator and the denominator as an expansion. The other main problem was to operate on the whole number, in relation to the fractions. Some of them split the fraction vertically and divided those fractions.

9.4 Summary fractions

The students showed competence in executing the fraction operations. Problems revealed in former studies, were though also found in this study. There is evidence that many students did not simplify fractions to their simplest form (Brown & Quinn, 2006; Eichelmann, et al., 2012; Padberg, 1986), and some few students did not simplify result numbers when numerator and denominators were identical (Shaw, 1982).

There is little tradition in upper secondary school to work to develop number sense (McIntosh, et al., 1992), and there is no focus on basic number sense in the textbook. The teacher, challenged those who asked, to find the prime factors, and explained how to find the least common denominator.

Some responses indicated a lack of understanding fractions as numbers. One example is: $\frac{42^6}{630^{90}} = \frac{6^2}{90^{30}} = \frac{2}{30^{15}} = \underline{\underline{15}}$. Three students had cor-

rectly simplified the fraction, but by cancelling 2 in the numerator, the result 1 as an index is not written. It might be that this ‘cancelling without trace’ (Storer, 1956) caused them to regard the solution to be 15. It is, however, clear that if these students had a well-developed concept image (Tall & Vinner, 1981) of the fraction concept, they would not have regarded the denominator 15, to be equal to the number 15. In September one of those students, regarded to be a confident and high mastering student, equated $\frac{-18}{36}$ with -2 ; an indication that this error was not just a coincidence.

When working with unfamiliar tasks as the complex fraction in the foregoing section, and one of the equations in the algebra test, it was evident that the students’ concept images of the fraction concept was limited. The test task was: $\frac{x+1}{x+4} = \frac{4}{5}$. None of the students solved it correctly in the autumn. In the spring one student succeeded. The distribution of solutions is shown in the table below.

Table 9-7: Task with binomial - algebra test (%)

Task 9c $\frac{x+1}{x+4} = \frac{4}{5}$	Autumn % (25 students)	Spring % (24 students)
Correct		4
A) Multiplies by the binomial $(x + 4)$ but do not follow fraction rules/equation rules		13
B) $\frac{x+1}{x+4} = \frac{4}{5} \rightarrow \frac{x}{x} = \frac{4-1}{5-4} \rightarrow \frac{x}{x} = \frac{3}{1}$	16	
C) Divides the fraction vertically $\frac{x}{x} + \frac{1}{4} = \frac{4}{5}$	32	21
Other not understandable solutions	36	33
No response	16	29

It was still a problem for students in the spring to regard binomials as entities. The recurring error of treating fractions as a composite of two natural numbers that can be operated on as single numbers, was the most frequent error in the autumn (Eichelmann, et al., 2012). This is the reason for the solutions in category B. Some students wrote their solution as in the table, others as $x=3$ and $x=1$. In the autumn eight students did this, in the spring it was five. In category C, the fraction on the left side is divided vertically from the outset.

It is though positive that some few students seemed to have viewed the binomial $x+4$ as an entity in the spring, in the autumn there were none.

From the sub-sections above, the problems students reveal, when solving fraction tasks can be summarised as follows:

- Factorisation/simplification
 - To factorise and simplify larger numbers - (Brown & Quinn, 2006; Eichelmann, et al., 2012; McIntosh, et al., 1992; Padberg, 1986)
 - To factorise binomials (Guzmán, et al., 2011; Olteanu, 2012)
 - To factorise powers - to interpret the different components (Seng, 2010)
 - When binomials involved – to avoid partly cancelling (Guzmán, et al., 2011; Hall, 2004; Olteanu, 2012)
 - When denominator and numerator are identical – to not simplify (Shaw, 1982)
 - To distinguish between factors and terms
- Multiplication/Division/Expansion
 - To differentiate between multiplication and expansion
 - To choose an efficient strategy
- Addition/subtraction
 - When larger numbers - to find the LCD (lack of number sense) (McIntosh, et al., 1992)
 - When literal symbols involved – to find the LCD²¹
 - When binomials involved – to find the correct sign and the LCD

It is evident from both the classroom and from test tasks that most students struggled to factorise binomials. The textbook gave little help. The teacher emphasised factorisation of brackets as opposite to multiplying brackets. From the plenary session, it appeared that the notions factors and terms were not well known. In the textbook those notions are mentioned once: “We can factorise a fraction with two terms by putting a common factor outside a bracket” (p. 35). The teacher emphasised the difference between the two notions, and also asked the students to pick out terms and factors in tasks on the whiteboard.

The strongest tendencies observed was on one hand that most students did not search for possibilities for simplification before they performed multiplication. This happened both when monomials and binomials were involved. It might indicate an operational (Sfard, 1994) or procedural (Gray & Tall, 1991) view of mathematics. They just started performing operations without analysing the task or taking the structure into account. It seemed as if they had one strategy only. Their limited way of solving tasks was somewhat reinforced through the textbook authors’ choice of tasks; presenting only exercises (Niss, 1993) of the same structure as the example tasks. In the classroom there was reluctance and sometimes

²¹ LCD – least common denominator

heavy argumentation on the part of the students against the teacher's wish to take the structure into account, and to consider choice of strategy.

On the other hand, the larger group of students (80 %) did not see binomials as entities or objects, although it was emphasised by the teacher. Few tasks including binomials were provided, and they were of the same structure resulting in the same solution; whether the simplification was partly cancelling, or if the binomials were factorised correctly. They seemed to create or consolidate misconceptions rather than offering them opportunities to develop their concept images (Vinner & Tall, 1981), and prevented students from developing proceptual thinking (Gray & Tall, 1994)

The mathematics offered related to fractions was mostly procedural mathematics (Hiebert & Lefevre, 1986) focusing on the rules for fraction operations without a connection to the reason for the rules; the why (Skemp, 1987). Fractions were first introduced for the students in grade 4, and the fraction operations should be learned in grade 8 (L97).

The responses indicated though that most students lacked conceptual understanding (Kilpatrick et al., 2001) of fractions (see table 9-7 above) although they knew the rules and mostly performed the operations when binomials were not included. They did, however, not take the structure of the task into account, what was one reason for the problems working with fractions including binomials. They seemed to have one strategy only.

Although the exposition in the textbook started by introducing more sub-concept of fractions, the tasks were of a low level of cognitive demand (Smith, Stein, Arbaugh, Brown, & Mossgrove, 2004; Stein, et al., 1996) similar to the example tasks. They can be characterised as exercises (Niss, 1993) on a low to moderate level of procedural complexity (Hiebert, et al., 2003). They were ordered from easy to more complicated (Love & Pimm, 1996), and no task asked students to reason about why the rules work.

Students' responses indicate that issues crucial for students' learning were not addressed.

10 Equations – exposition and students’ work

Formal equations had been a focus for students’ work in mathematics also in lower grades. In the textbook, equations were introduced in a sub-chapter under the headline: Equations and inequalities. The first part of the sub-chapter is devoted to equations. One word problem is presented:

Multiply my age by 5 and subtract 23 years. You will get the same answer by multiplying my age by 2 before adding 25 years. How old am I? (p. 40).

The text is transformed into two algebraic expressions by letting x stand for the unknown age. By equating the two expressions, an equation is created. To solve an equation is according to the textbook to find those values of x making the proposition true.

Then the rules are highlighted in a rule box: “We can add or subtract the same number on both sides of the equal sign. We can multiply or divide by the same number except for the number zero, on both sides of the equal sign”. In the margin it is commented: “When we add or subtract the same number on both sides, we say that we move over and change signs”. In all worked out examples it is written that the numbers or expressions are added or subtracted on both sides, but it is not shown in the writing.

Five example tasks are solved. The first is the word problem presented above. The other tasks are presented from simple to more complicated. The first is without brackets, the next with brackets. Then comes a ratio equation, before the last two example tasks are presented under the heading: Equations with fractions.

The exposition part is rather short, and there is no explicit reference to the properties of equalities nor to equivalence relations. In addition, the authors assert that the rules ‘doing the same on both sides’ and ‘change side – change sign’ are equivalent. However, Kieran (1989) argues strongly that these two rules indicate two completely different ways of thinking (see section 4.8).

In the file made by the teacher and her colleagues, the ‘learning book’ it is stated:

If you execute an operation on one side of the equal sign, you must remember to execute the same operation on the other side as well.

As an extra piece of advice is given:

It is always smart to control the value you find, by substituting the unknown in the initial equation with the found value.

In the ‘learning book’, cross multiplication is presented related to ratio equations. In the textbook nothing is said about controlling the result, and cross multiplication is not mentioned.

The work plan included 20 tasks similar to the example tasks. They were ordered according to procedural complexity (Hiebert, et al., 2003). The five ‘last’ tasks on the work plan can be categorised to be of moderate level of procedural complexity (ibid) with five or more decisions to be taken during the solution processes. Those tasks were, however, solved by only nine of the students in the data set of 18 students. Three of those had picked out only one or two of those five tasks.

The introduction of equations in the classroom and students work will be presented in the following sub-chapters. The sub-chapters will be head-lined according to categories revealed in other studies (see section 4.8). Analyses and more a detailed description can be found in appendix 11 and its sub-sections.

10.1 Equations and the equal sign

The teacher started the introduction of equations by saying that she was intrigued about what students know about equations. Some responses were: ‘there is an unknown’ and ‘this unknown is to be found’. ‘The unknown can be labelled by any letter’, and ‘in an equation there are two sides which are equal’. The teacher applies the move of revoicing (Herbel-Eisenmann, Drake, & Cirillo, 2009) the responses and asks follow up questions. In line with former studies (de Lima & Tall, 2006b) none of the students came up with the equal sign as a required component in an equation. At the end, the teacher writes an equal sign on the white-board saying that there is a left and a right side, and that there has to be an equal sign in between them.

This equal sign seemed to trigger Ivar, who responded:

Ivar: Aren’t then all arithmetical tasks equations?

T: What did you say? (*she could not hear his response.*)

Ivar: Aren’t actually almost all arithmetic tasks equations?

T: No, not all are equations. What we have been working on until now for instance, has been to simplify fractions.

Ivar: Strictly speaking, aren’t then those fractions equations?

T: No, there is no unknown to be found. Instead, the task is to operate on the fractions, either to simplify or to add them. What is special about equations is that we have a left and a right side.

Some students sitting close to Ivar argue eagerly, but it is hard to hear them all.

Else: But it is an equation, because in a way we have the answer. You have something on both sides of the equal sign. That’s why it is an equation.

T: Is it this what you mean?

She goes to the white board where it is written: $Vs=Hs$, which means the left side is equal to the right side.

Else: If you have something to add for instance, then you have to find what will be on the other side of the equal sign.

- T: But you are not going to find the value of the unknown. That's what is special about equations.
- Eli: But both sides are equal? (*Eli supports Else's view*)
- T: Yes, both sides are equal?
Eli continue arguing, but cannot be heard. Ivar brings in inequalities.
- Ivar: But the same is the case with inequalities.
- T: But in inequalities the sides are unequal. One side is less or larger than the other. Tomorrow we are going to work on them.

The students have brought in the unknown, the need of finding this unknown, and the equality of two sides. From the teacher's side it is important to make distinction between equations and arithmetical tasks, and thus she uses the word equation for algebraic equations only; as do Herscovics and Kieran (1980).

However, Ivar enters the scene with the question: "are not all arithmetical tasks in some sense equations?" The teacher replies by pointing to the fact that although working with fractions, they cannot be said to be equations because there is no unknown to be found. More students follow up on Ivar's thought, and from Else's utterance it is clear that she thinks that number strings or strings of fractions might be viewed as equations. The reason is that when operating on them, one put in equals signs on the way to the solution: "You have to find what will be on the other side of the equal sign". The teacher once again emphasises that the point in such tasks is not to find the value of an unknown. Eli, however tries to argue that "both sides are equal". The teacher seems to agree, but is interrupted when Ivar suggests that the same is the case with inequalities. It is unclear if he meant that the left and the right sides are equal also in inequalities, or if he meant that also in inequalities, one has to find the value of an unknown. At least it seems as if the teacher interprets his utterance to mean that he regards the left and the right sides to be equal in inequalities as well, and she responds to this.

The students here discuss the definition of an equation. In mathematics some definitions are not clear if they include arithmetic equalities (see section 4.8). Here it could be that some of the students have equalities in mind, however, when talking about fractions and addition and answers to come, it is likely that the students here do not differentiate between the equal sign as a relational sign and a sign to announce a result or an answer to come, and thus have a procedural view of equations rather than a relational as reported in the literature (de Lima & Tall, 2006a, 2006b; Godfrey & Thomas, 2008).

The examples below from some students' work with equations at the start of the semester also indicate such a view.²²

Selma	Olav
$3x - 5 = 10 =$	$d) \frac{2}{3} = \frac{6}{x} =$
$3x \cancel{-5} = 10 + 5 =$	$\frac{2x}{\cancel{2x}} = \frac{18}{2} = 9 =$
$\frac{\cancel{3}x}{\cancel{3}} = \frac{15}{3} =$	$x = \underline{5}$
$x = \underline{5}$	

The equal signs at the end of the lines make it impossible for them to regard equal sign to signal a binary relation between two sides. After some lessons, Selma writes the equations in a formally correct way, while Olav continues to write two equal signs in each line in the first 14 tasks. He had solved nearly all equation tasks in the textbook. In addition, in the example above he cancels the unknown. He was consistent in doing so as long as the unknown's coefficient was different from ± 1 , although in the questionnaire (see section 7.3.4) he explained how to divide by the coefficient in order to find one x . In the algebra test before the teaching of equations his solution on one of the equation tasks was:

$$4x - 15 = 75$$

$$75 + 15 = \frac{90}{4x} = \frac{\cancel{4x}}{\cancel{4x}} = 22,50$$

He is able to find the correct solution but his writing is highly informal, and his view of the equal sign cannot be a relational view. Little by little his writing changes, also he stops cancelling the unknown. However, in the first ordinary test this cancelling is back again, but he in his mind operates correctly finding the unknown. He was the only one in the questionnaire explaining that it was smart to substitute the unknown by the solution number to check the answer (dataset 12 students).

This writing of extra equal signs was also seen in the work of two other students in the test in September. Selma ended her solution writing an expression not a solution in two of her test tasks. In December only one student wrote more than one equal sign in each line in the solution.

The teacher showed in the program MathType that it was possible to apply a function 'align at =', making the equal sign as the centre of each line. This might make the structure more visible to the students, but few students applied the function; ten students in the first test in September and only five in December.

²² All solutions to assigned tasks and test tasks presented in the thesis are students' own solutions.

It is not possible to verify the students' view of the equal sign. The students were not asked about it, but some students' writing might be evidence of an operational view. Others writing the equations in a formally correct way might also have the same view though.

10.2 Equation solving

When presenting how to solve equations, the teacher used the example task: $x + 7 = 11$, and asked students what should be done:

Rakel: We move the number 7 over to the other side and change the sign.

T: Why are we doing that? Did you hear what she said?

Rakel: We have to get the numbers on one side and x on the other.

T: Yes, it's right what you say. I have to move over and change sign.

She writes on the white board

$$x + 7 = 11$$

$$x = 11 - 7$$

T: Why am I writing it, like this?

Tore: x has to stand alone and then you have to...

T: What have I actually done when I moved it over?

Tore: *(after a short silence)* Actually you have just subtracted 7 from both sides.

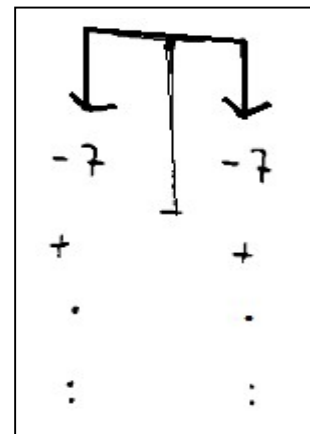
T: Yes, that's what I have done. Really good that you noticed it. Actually, I have subtracted 7 from both sides. When executing the same operation on both sides of the equal sign, the balance is kept. Do you see that?

She draws a balance on the whiteboard and writes minus 7 under the balance on both sides, and with gestures she acts as if it is in balance. Also in the equation she writes negative 7 on both sides.

Further, she emphasised that even though the operation is mostly not written on both sides, mathematically the same operation is executed on both sides of the equal sign. This she made visible by writing the arithmetical operation signs on both sides under the balance as shown to the right, saying that these are the operations to apply in order to find the unknown in linear equations.

The first example task to work out is $2(3x - 8) = 5 - 3(x + 1)$ from the textbook.

To perform distributed multiplication is the first suggested step. The teacher agrees, and then the students and the teacher execute the operations in cooperation, as in all plenary sessions. When solving this equation task, the minus sign causes discussion and uncertainty. In the end, she advises the students to keep the brackets until the multiplication is performed. Then they can remove the bracket and change the sign, if the preceding sign is negative.



For each suggested operation, she revoiced (Herbel-Eisenmann, et al., 2009) students' utterances and emphasised that the activity of moving one term from one side to the other, mathematically is either subtraction or addition of the term on both sides of the equal sign, and that the operation to choose, is the opposite of the 'given' operation.

Ending up with $9x = 18$. Arne is eager to participate:

Arne: We divide by 9.

T: Yes, but why should we?

Arne: In order to find only x .

T: To have only x ?

Arne: We want to know the value of one x .

T: Yes, we will know the value of one x .

Arne: We don't want the value of $9x$

T: What is actually standing between 9 and x ?

More students answer: Times

T: Yes, it is times and then I have to...?

Arne: Divide because that is the opposite function.

Arne tells what to do, however, the teacher wants him also to explain why. In addition, she uses the task to focus on the invisible sign, in order to let the students see the need of executing the opposite operation.

The next example task, $2 + \frac{3}{2}x = \frac{1}{3} + x$, revealed problems related to fractions and the task is referred to in section 9.2. In order to visualise that the same operation is executed on both sides of the equation, the teacher writes the left and the right side in brackets before she multiplies by 6:

$$\left(2 + \frac{3}{2}x\right) \cdot 6 = \left(\frac{1}{3} + x\right) \cdot 6.$$

Arne interrupts by asking why it is multiplied by 6 and not 4 for example. The teacher asks Alf to explain. Alf answers that the reason is that 6 is the least common denominator for the fractions.

The last example task was worked out on the computers, the teacher on her PC, her screen projected on the whiteboard, and the students on their PC-s: $\frac{4x}{3} - \frac{1}{5}(2x + 1) = \frac{2}{3}\left(x - \frac{3}{5}\right)$. The challenge in this task is that the brackets have to be multiplied by fractions. When discussing with the students what to do first, the teacher advises them to multiply the brackets by the fractions first of all. The reason for this, she says, is that since there are fractions, and one has to multiply by a common denominator, there is a risk that one multiplies the common denominator by both the bracket and the factor outside the bracket. She does not, however, go into the difference between factor and terms this time. It shows, however, that, from her experience, she is aware of the risk of regarding each factor as a term when multiplying by a common denominator.

When approaching the solution to this equation, Ronny is a bit worried:

Ronny: Yes, but then we have $4x = -3$.

T: Yes? What about that?

Ronny: I just wonder about what am I to do?

T: Okay, I come to $4x = -3$ and then you think it was a bit strange that it was negative 3 on the right side?

Ronny: No, but, I can't remember what to do!

T: But remember? You have to be focused. We are executing opposite operations all the time. I have multiplied in order to get rid of the fractions. And now I have?

Ronny: Yes, but now it will be a fraction again.

First it seemed as if he was stuck because of the minus sign on the right side, however, the utterance “but now it will be a fraction again” indicates that he is stuck because the solution will be a fraction, and the point in the start was to get rid of the fractions by multiplying by the least common denominator. Other students had similar utterances during seatwork; they did not want fractions as solutions. The teacher, however, explained that fractions always represented an exact number. Some few students seemed to interpret this to mean that the teacher preferred fractions, and they were observed discussing how to transform the solutions into fractions. Ronny, however, was concerned about fractions as solution number.

The first tasks on the work plan were simple. Some of them were of the form with the unknown on only the left side. Filloy and Rojano (1989) proposed such tasks to be solved arithmetically. However, all students seemed to operate on the unknown (Kieran, 1992).

When solving the tasks, many students asked for help when checking their solutions. Others made errors seemingly without noticing it. However, their expressed problems were not related to the equations as such. The problems were mostly related to the minus signs, the fractions, the distributive law and so on. None were seen to control their solution except for the comparison with the textbook answers.

One episode from the classroom illuminates students' limited conception of equations as a symmetric relation. Tone and Ruth were working, and Tone seemed to be stuck and glanced at Ruth' work. They had on their screens:

Tone

$$(x+1) \cdot 2 = 5(1-x) + 2x$$

$$2x + 2 = 5 - 5x + 2x$$

$$2 - 5 = -2x - 5x + 2x$$

Tone: Do we have the same on the screen now? However, look here.

She points to her own screen, and she follows the terms on her right side of the equation, while she calculates.

Ruth

$$(x+1) \cdot 2 = 5(1-x) + 2x$$

$$2x + 2 = 5 - 5x + 2x$$

$$2x - 2x + 5x = 5 - 2$$

Tone: Minus $2x$ minus $5x$ will be minus $7x$

Ruth: Yes, minus $7x$

Tone: And then plus $2x$ is minus $5x$. (*She correctly calculates the x -es on the right side of the equal sign.*)

Ruth: But that $5x$! You have the $5x$ on the other side. When you take the $5x$ over to that side, then it will be plus. Since you have the $5x$ on that side, that is the reason that it has not become positive yet. That's why the two...

Tone: Isn't it actually the same now?

Now Ruth has correct numbers on the right side and the $5x$ -es on the left side of the equal sign, while Tone has them on the right side. Ruth discovers this and denies that they have the same solution:

Ruth: No. You will have a negative solution and I will have a positive one.

Tone, who believes what Ruth claims, deletes her third line and starts over again.

This is an example of how students can mislead each other. Ruth did not regard Tone's solution to lead to a correct result. Neither of them had the competence to understand that in an equation it does not matter from which side you read. These two students had not really discovered the power of equations and the property of symmetry. Their cooperation on this task did not help them to advance their thinking.

Also other students asked if it mattered on which side the x was located. One example is Paul. He is going to solve one of the more complicated tasks and asks: "Can I move the x over to the other side?" He had discovered that he would then avoid a negative coefficient, but he is not sure if it is allowed. This is also an example of students' talk when solving equations. They 'move' numbers and expressions as if they were physical objects (Lima & Tall, 2006b, 2008).

This was also evident in the questionnaires from 11 of the 12 students who answered it (see section 7.3.4). Selma who wrote more than one equation sign above, expressed equation solving in this way:

"I would have moved all x -es over to one side and the numbers to the other. Afterwards I would remove the number in front of the x , and that is done by division. However, when one divides on one side, the same has to be done on the other side". From her explanations it seems that she thought of the terms as physical objects (de Lima & Tall, 2006a), and that they had two rules to follow, one for division and one for addition/subtraction (Cortés & Pfaff, 2000).

This was at the start of equation solving, and it might be that teacher's recurring emphasis on 'doing the same operation on both sides', has influenced their thinking. However, during observation it was heard only once that a student referred to balance and of doing the same on both sides (Alf

explaining for Atle), others overheard were talking about ‘moving and changing sides’.

In her explanation of equation solving, Selma did not mention that the signs should be changed when terms are moved from one side to another, but she changed when solving tasks. There were, however, examples of students ‘changing side, but not sign’; which according to Cortes et al. (2000) and Kieran (1989) is a common error when relying on the rule ‘change side-change sign’. One student still did that in the tests both in September and in December. Others were not consistent.

In addition, there was evidence of performing different operations on the two sides in that some multiplied or divided by different numbers on the two sides. One example was Ruth’s solution. The task was:

$\frac{3x}{2} - 4 = \frac{x}{2} + 1$. In the next line she wrote: $\frac{3x}{\cancel{2}} \cdot \cancel{2} - 4 = \frac{x}{\cancel{2}} \cdot \cancel{2} + 1$. Here she has multiplied only the fractions by 2. She must have had a problem seeing the left and the right sides as entities or objects (Sfard, 1991). More students made such errors when both fractions and whole numbers were included. Fractions and whole numbers is one problem, but it also illustrates not seeing the equation as a binary relation.

10.3 Ratio equations

Some tasks were categorised in the work plan as ratio equations. They were solved in the textbook by multiplying both sides by the common denominator. The teacher demonstrated how they could perform cross multiplication in such tasks. The reason for this method, she told the students, was that it might be a benefit for them later when working with trigonometry. She applied the example task from the textbook: $\frac{3}{10} = \frac{x}{18}$. When she solved it, she said that she multiplied in this way $\frac{3}{10} \times \frac{x}{18}$, but at the same time she swapped the products to the opposite sides, resulting in $10x = 54$.

She applied the symmetric property of equations without making it explicit to the students. When solving equations, some students, who strictly followed the teacher’s example also when solving inequalities, got into trouble later when solving inequalities (section 11.2.3).

Most students who applied this method succeeded, but some students, who performed cross multiplication solved the tasks unconventionally, the example underneath is from Ruth’s files:

In the second line (in all her solutions) she seems to cross only from.

$$\begin{aligned} \frac{2}{3} &= \frac{6}{x} \\ \frac{2 \cdot x}{3 \cdot 6} &= \frac{2x}{18} \\ \frac{2x}{2} &= \frac{18}{2} \\ x &= \underline{\underline{9}} \end{aligned}$$

right to left: This could have been mathematically correct if she had equated $\frac{2 \cdot x}{3 \cdot 6}$ with 1, but the next thing she does, is to leave the equation and just carry out the multiplication $\frac{2 \cdot x}{3 \cdot 6} = \frac{2x}{18}$. This is an identity, however, she has probably in her mind applied the equal sign as a ‘here comes the result’ sign. Then in the third line, she is back to the equation. It is a question if she is aware that she is going in and out of equation solving.

It is also a question what she is thinking mathematically going from line 2 to line 3. She did this in all similar tasks.

During the lessons, some students asked if they had to perform this cross multiplication. The teacher told them to apply the most convenient method.

10.4 Summary equations

All assigned tasks were variants of the solved example tasks, and the progression in the textbook went from easy to more procedurally complicated tasks. This might reflect a belief that learning is best achieved when the topic is presented in small well-defined steps (Love & Pimm, 1996).

In the model made by Godfrey and Thomas (2003) (section 4.8) they found that students’ perspective on the four factors: the equal sign, letter symbols, operations, and the structural form, influence how students will perceive the object of equation.

In the textbook, the perspective on the equal sign as such is not addressed. The perspective: ‘equivalence relation’ is implicitly there through equation solutions presented as operating in the same way on both sides of the equal sign, and through the metaphor of a balance used in the classroom. To perform the same operation on both side, was heavily emphasised by the teacher, but also then without applying the word equivalence. At the same time the latter is blurred in the textbook through the rule ‘move over and change sign’ which is equated with performing the same operations on both sides.

In all tasks offered in the former sub-chapters, the perspective of the equal sign is highly procedural, announcing the result of a procedure. In the plenary, there was no student heard opposing Else’s and Ivar’s perspective about all arithmetical tasks as equations. Also those students writing more than one equal sign in each line in the solution of equations, seemed to have a perspective on the sign as a result of a procedure. It seemed, however to be the case that most of them changed this form of writing during the work with equations. In the December test only Tone

had one occasion of this in her solution. Two students, however, reverted to their former habit in the spring.

It is hard to say which perspective those students writing the equations in a formally correct way hold on the equal sign, Students' talk about 'moving over and changing sign' seems to be evidence against holding a perspective on the equal sign as an equivalence relation with symmetric, transitive and reflexive properties. Another example is Tone and Ruth's conversation about solutions depending on which side the unknown is written on (section 10.2).

One task in the algebra test asked students to transform the formula of the volume for the cylinder, to be a formula for the height in the cylinder. The formula was: $V = \pi \cdot r^2 \cdot h$. In the autumn, 32 % succeeded, in the spring 40 %. In a task in the December test, task 7c, the relation: $v(t) = 90 - 3t$ was given. The students were asked to find a formula for t ; six students (22 %) succeeded.

These results indicate that a large group of students have not grasped the symmetric property of the equal sign, and to transform what is taught about performing the same operation on both sides of the equal sign, from equation solving to transformation of formulas. Godfrey and Thomas (ibid) stressed the need of being explicit about the properties of equations, and the equal sign. It might have helped these students.

In the equations solved during observation, the perspective on letter symbols is implicitly the specific unknown. Letter symbols are applied also in the other sub-sections in the textbook, but the perspective on letter symbols is not explicitly stated; the danger being that the students extrapolate from the specific unknown which they are more familiar with. There is no occurrence leading students to a perspective of letter symbols as generalised numbers or variables in Küchemann's (1981) sense. It might be inferred that the authors take it for granted that students have an adequate conception of letter symbols.

In the classroom it was asserted that in an equation it did not matter which letter was applied for the unknown, no one argued against it and the teacher confirmed it (section 10.1); a meaning taken to be shared (Voigt, 1995). In the algebra test, however, the students were asked to create an equation in which the solution should be $y = 4$. Both in the autumn and in the spring some few students changed the letter symbol for the unknown to be x .

In section 8.1, it was shown that a rather large group of students had problems differentiating between the value of an object and the object itself. The students seemed though in the algebra test to accept letter symbols as generalised numbers, and when solving equations, it was clear that

the students searched for the specific unknown. There was, however, no sign of students controlling their solutions.

Another perspective according to Godfrey and Thomas (2003) was the perspective on number. There is no indication that students had an inadequate conception of numbers as such, but there is evidence that the number sense was rather poor in some situations; especially related to powers (section 8.4) and fractions. Powers were not included in the equation tasks, but fractions were found to be a problem for students when solving equations.

The perspective on operations was the next perspective. The problems were related to the minus sign and to fraction operations. The textbook presented the rules, but gave no explanation about why they work. In addition, many students seemed to struggle when multiplication, division and factorisation should be performed. This is evident in many solutions in all topics; most students though are not making those errors consistently. They seemed to rely on technological tools.

The structural form of equations seemed not to be clear for all students. In December Tore wrote the solution as an expression $x - 3$. It might be that he just made a writing error. In September, Selma ended up with two solutions written as expressions. In the algebra test, when asked to write two equations, given the solutions, Alf wrote expressions in the autumn. In the spring both Olav and Alf presented expressions instead of equations. Also the writing of extra equal signs indicates an improper conception of the form of linear equations.

The table below presents the proportion of correct solutions on equation test tasks:

Table 10-1: Test tasks – equations. Correct results.

Equation test tasks		September test (27 students)	December test
	Item 6 (September)		
a)	$\frac{-6}{9} = \frac{4x}{3}$	44 %	
b)	$2 \cdot \frac{7}{x} = \frac{5}{25}$	44 %	
c)	$\frac{x}{2} - x = 2\left(\frac{x}{2} - \frac{2}{3}\right)$	48 %	
	Item 1c (December)		
Alt 1	$2 + 5x - 1 = x$		83 % (6 students)
Alt 2	$\frac{2}{3}\left(x + \frac{3}{4}\right) = \frac{1}{6}x - 1$		52 % (21 students)

In the test, one week after the work in the classroom, less than half the students solved the equations correctly. Only five students solved all equations correctly. In the December test the proportion of correct solutions was slightly higher (52 % of those solving alternative 2 solved it correctly). Some students might have progressed. However, it shows that the perspective on the different components and the conception of equations as objects are not satisfactory among many of the students.

When solving equations, few students asked questions specifically related to equation solving. Few students solved all tasks on the work plan, and even fewer the tasks of moderate level of procedural complexity. However, most problems are seemingly related to fractions, the minus sign, differentiation between factors and terms, and it was evident that students struggled more to find the correct solution when the denominators were larger numbers, and when the fractions included binomials. The latter types of errors are discussed under negative numbers and signs in section 8.3.5. and in section 9.3.

It can be concluded that the mathematics offered in the textbook, is focused on the rules and routines for equation solving; making no distinction between ‘change side – change sign’ and ‘performing the same operations on both sides of the equal sign’. The teacher although accepting students’ utterances about ‘changing side – changing sign’, encouraged them to think that they performed the same operations on both sides. She even showed it in the writing, implicitly emphasising the relational meaning of the equal sign.

Students’ oral responses indicate though that most students are used to mentally move numbers and expressions as if they were physical objects (de Lima & Tall, 2008), and it seems that they continue this way of thinking from lower grades.

The properties of the concept of equation is not addressed, and it might be that the students would have been better solving equations, if those properties were explicitly offered. In relation to the solving of inequalities, it would have been helpful to know the properties of equations.

11 Inequalities – exposition and students’ work

In this chapter the introduction of the concept of inequality will be presented, both as introduced in the textbook and in the classroom. Then comes a part with a model of students’ conceptions of inequalities interpreted from their written materials, from interviews, and from classroom observations. Analyses and more detailed descriptions can be found in appendix 12 and its sub-sections.

11.1 Introduction of the topic

In the textbook, the concept of inequality is presented in the same sub-chapter as equations, directly after equations, and before tasks to practice rules for both equations and inequalities. This organisation of the topic of inequality as a sub-topic of equations is reported by researchers (Boero & Bazzini, 2004; Tsamir & Bazzini, 2002).

The authors focus in the textbook on the different inequality signs and their meanings. The number line is used as a tool to show that numbers to the right are larger than those to the left. Then an example of an inequality is presented and solved; $x + 3 > 5$. The solution $x > 2$ is commented upon in this way: “we see that x has to be a number larger than 2” (p. 45).

In the classroom the teacher started by asking students what they know about inequalities from before. One immediate response was:

Tord: It is sort of, as an equation, but the sign is different, it is to be drawn like this. (*He draws an inequality sign in the air*).

The teacher building on this response introduces the inequality signs, by asking for the meanings of the signs. Jarl says that the two signs are ‘less than’ and ‘larger than’. The teacher remarks that in the primary school they had presumably been talking about crocodiles, and writes $x > 2$ on the whiteboard. Arne, tells her that the statement means x larger than 2, since the number on the side with the opening, is the larger number. He recognised the signs because of the remark about the crocodile. The teacher built further on this metaphor, when talking about the different inequality signs, before she turned over to focus on how the inequality statement $x \geq 2$ could be interpreted and presented on the number line.

T: Have you thought about what this expression means on the number line?

Else: It means all the numbers up to 2.

T: Up to 2?

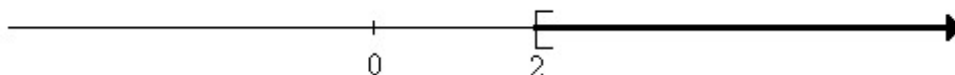
Else: No, it means all the numbers from 2 and upwards.

T: Is the number 2 included?

Else: Yes.

The teacher asks students to join her interpreting the algebraic statement by illustrating it on the number line. She continues asking the students which sign they would use to illustrate that the number 2 is included in the set of numbers. There is silence in the classroom. Then Arne responds by gesturing and telling that he would apply a special sign for brackets.

The teacher draws a number line on the whiteboard and puts in this ‘special bracket’ which shows the set from 2 and upwards, 2 included. She marks the set which satisfies the inequality:



She also introduces other inequality expressions such as number sets and presents them both in set notation and on the number line, emphasising that all numbers in the sets are included. The explanation in the textbook, might be interpreted to mean one specific number within the set; an interpretation held by some students.

Solutions of inequalities as sets of numbers was not mentioned in the textbook. The teacher, however, introduced set notation before she, in line with the textbook, applied an inequality to check if the rules for equation solving would hold also for inequalities. The actual inequality was $4 > 2$.

After adding and subtracting numbers on both sides, the teacher divided and multiplied the inequality by a positive number on both sides. All students agreed that the statement was still true. Finally, she divided both sides by negative 2: $\frac{4}{-2} > \frac{2}{-2}$

T: If I divide 4 by negative 2, what will it be?

Tord: Minus 2

T: I will have negative 2? Plus, divided by minus; you remember? That will be minus, and 4 divided by 2, is 2. The result is minus 2. And the rest? What will that be?

Student chorus: Minus 1

T: Is minus 2 larger than minus 1? (*She writes $-2 > -1$.*) Yes, isn't it? can it be correct?

Else: If you write plus instead of minus in the front, then it will be correct.

Others: We could remove both minus signs.

T: Do you know what we have to do?

Tord: We flip it.

T: Yes, we flip the sign, and when do we have to flip the sign?

Tord: When we multiply or divide by something minus.

Multiplying and dividing the statement by positive numbers on both sides of the sign were found by the students to be satisfactory; the statement was still true. However, when dividing by a negative number, the result turned out to be a contradiction. One suggestion was to remove the minus

sign in front of 2, others wanted to remove both minus signs. Tord then tells her to flip the sign, and he explains that the sign must be flipped, when it is divided or multiplied by a negative number in order to keep the statement true. The number line was then used to illustrate that he was right.

In the class, the conclusion was that one could operate on an inequality as if it were an equation with the exception of division and multiplication by a negative number.

The textbook listed the rules for inequalities:

We can:

- add and subtract the same number on both sides of the inequality sign
- multiply and divide by the same positive number on both sides of the inequality sign
- multiply and divide by the same negative number on both sides of the inequality sign, if we at the same time flip the sign
- not divide by zero (p.46).

Three context free inequality tasks were presented and solved in the textbook. Each step was explained as in the example underneath:

$$\frac{1}{3}x + \frac{1}{2} \leq \frac{7}{6} + x$$

$$2x + 3 \leq 7 + 6x \quad \leftarrow \text{We multiply both sides by the common denominator 6.}$$

$$2x - 6x \leq 7 - 3 \quad \leftarrow \text{We subtract 3 and 6x from both sides.}$$

$$-4x \leq 4 \quad \leftarrow \text{We divide both sides by -4 and have to flip the inequality sign}$$

$$x \geq -1$$

This task was also used as an example task in the plenary session. The students, however, did not seem so eager to participate as in the other lessons. After having no response, the teacher says:

T: We can think that we have an equal sign, because we are going to solve it as if it were an equation.

Arne: What sign is written between one half and 7 sixths?

T: What sign it is? Does anybody know?

Both Arne and the other students seem to be unsure. There is silence until Jone finally tells what is correct, and the teacher revoices it before she adds that the sign shows that the task is an inequality not an equation.

Although the plenary session had started with an introduction and a presentation of the different inequality signs, it did not seem that Arne and the other students were sure about their meanings, especially of the sign ‘larger or equal to’; at least Arne needed a confirmation.

The fraction in the task caused some questions about the relation between the fraction and the unknown as in the equation task in section 9.2.

After a while the interaction went on as follows:

- T: Now we have $-4x \leq 4$. Have you, Ine, an idea about what to do?
- Ine: We can divide by minus 4.
- T: Divide by minus 4, and why?
- Ine: In order to search for one x .
- T: Now I have an inequality and I want to divide by minus 4, what do I need to take into account?
- Ronny: We have to flip the sign.
- T: Yes, we have to flip the sign, and then we have larger than or equal to. What is the result?
- Roy: Minus 1.

The teacher has changed the sign only in the last line $x \geq -1$ and points to the line above saying that the sign has to be changed already when writing the sign for division. She deletes the sign ‘less than or equal to’ and substitutes it with the sign ‘larger than or equal to’. She sums up that to solve an inequality is nearly the same as solving an equation. The exception is that the sign has to be flipped when dividing or multiplying by a negative number as demonstrated by dividing the example $4 > 2$ by a negative.

The teacher followed students’ inputs about the inequality sign. This was in line with the exposition in the textbook. But then she led the focus over to the meaning of expressions of the kind $x \geq 2$, and to sets of numbers. This was not mentioned at all in the textbook related to inequalities. However, when solving the task, she is again in line with the textbook about the close connection to equations.

11.2 Students’ work with inequalities

The first textbook chapter offers only two items. The reason for this might be that in relation to functions and graphs the topic will be handled once again later. All tasks in both items are in pure mathematical language. (A detailed overview of solutions and other responses is to be found in appendix 12.3 and 12.4). In this section the responses will be categorised according to problems mostly reported in earlier studies about inequalities (see section 4.10).

The assigned items included 14 tasks. Four of them would have a negative coefficient at the end of the solution process, if collecting the x -es on the left side of the inequality. That means that in only four out of 14 tasks it was required that students took into account that the properties of inequalities are not the same as for equations. This might mask a not well developed concept image of the concept, as in most of the tasks it is possible to perform exactly the same operations as for equations and reach a correct solution (Vaiyavutjamai & Clements, 2006).

In the fourth chapter, one word problem was given where it would be natural to apply an inequality. All students who solved the task, solved it

as an equation before they checked the result and decided which interval was the correct one.

11.2.1 The signs

The signs symbolising inequality were introduced and discussed, and inequality expressions were visualised on the number line. Some students, however, seemed not to be sure about their meanings, especially the signs \leq and \geq . Ane was one of them; she was observed having this solution:

$3 - 2x \leq 5(3 - x)$	She raises her hand for help, and the teacher appears.
$3 - 2x \leq 15 - 5x$	She watches Ane's screen, and realises that she has stopped having found the solution set for $3x$.
$5x - 2x \leq 15 - 3$	She asks Ane to finish it; to divide by 3, the coefficient. But Ane has another question:
<u><u>$3x \leq 12$</u></u>	

- Ane: Does it matter if there is an extra line?
 T: An extra line?
 Ane: A line underneath there? (*She points on the screen*)
 T: Do you mean that double line there? (*she seems to mean the double line below the end result*). Is it that line you are talking about?
 Ane: I'm going to show you (*she uses her cursor and points to the inequality sign*). That line under there.
 T: Oh, is that what you mean? That means that the values can also be equal.
 Ane: It can be equal?
 T: Yes, but it doesn't matter for the solution process. But you have to finish this and find the x .

Ane does not finish the task while the teacher watches her at work, but it can be seen that she ends up dividing by 3: $\frac{\cancel{\beta}x}{\cancel{\beta}} \leq \frac{12}{3}$ and the solution $x = 4$.

It might be tempting to interpret that she wrote the equal sign because of the teacher's utterance: "the values can also be equal" above. She had however, already solved several inequality tasks writing the solution as equalities.

Another student wrote some solutions as equalities during seatwork, although operating correctly on the inequality in other matters. On the test in September one task was a simple inequality: $-3x + 6 < 2(x + 8)$. In this task 7 students or 26 % ended with an equality. This was also seen in other studies and regarded to be caused by the blurred connection between the two concepts, equations and inequalities (Blanco & Garrote, 2007; Tsamir & Almog, 2001).

In the interviews the five students were asked to solve the test task above and explain their actions and thoughts. On the test they applied the correct inequality sign. During the whole process of solving the task, they all spoke about 'x is' or 'x is equal to' when explaining their actions in the interview situation. This was also heard from other students during seat work. The signs though were correctly written.

When solving an inequality task on the test in December, 4 out of 27 students ended up with an equality statement. None of them did the same in September. Two of the students who did this in September did not participate. Four of them avoided a negative coefficient before x by keeping the unknown on the right side of the inequality sign. One did not complete the solution.

11.2.2 Flipping the sign

Both the textbook and the teacher offered an example inequality in order to check which of the procedures from equation solving would hold for inequalities. The interaction between the teacher and the students seemed to show a negotiation of meaning resulting in all students taking the reason for the flipping of the inequality sign as shared.

This appeared not to be the case. In the break Arne, who participated in the interaction in the plenary session stopped the teacher asking:

Arne: Can you explain why we have to flip the inequality sign?

T: Of course. Yes?

Arne: At the end there. Why is the sign pointing that way?

He points to his solution. He has aligned the end result to be the same as that in the textbook.

T: Oh. Because...but I showed that on the white board.

Arne: Yes, you did, but I did not really grasp it. Why should we?

T: Because otherwise the inequality will not be true.

She goes to the white board and Ivar comes and joins them. She applies the example written on the white board.

T: You agree that $4 > 2$ is true?

Arne: Yes.

T: And then I divide by negative 2. You see, if I do not flip, I will have $-2 > -1$. which is not true.

Arne: Yes.

T: However, if I flip the inequality sign, it will be true. Thus we have to flip the sign every time we divide or multiply by a negative number, if not, the inequality will not be true.

Arne is an active listener, he participates in her explanation by nodding and confirming his acceptance. In the rest of the tasks, he correctly turns the sign when dividing. This is also the case for most of the students observed during seatwork, but several students ask when the sign has to be flipped. One issue is that many of them flip the sign only at their end solutions. They clearly do not see the expressions as objects but as procedures to be performed (Sfard & Linchevski, 1994b) not regarding the division to have been executed before the result of the division is found. In December all who flipped the sign, seemed to follow this thought.

Some few students consequently collected the x -es on the side where the coefficient would be positive. In that way they avoided problems with

the inequality signs. In the December test 7 students (30 %) ²³ did that, three of them, however, failed in other ways. Three students flipped the sign although dividing by a positive number.

11.2.3 The connection inequalities – equations

It has already been shown how more than a quarter of the students wrote the solution as an equality on the first test, and how Tord expressed an inequality to be a kind of equation at the start of the plenary session above. Also there was other evidence that this connection was blurred.

In the lessons about equations, the teacher had solved a ratio task, and introduced the students to cross multiplication. The example was a word problem and it was transformed into mathematical notation as: $\frac{3}{10} = \frac{x}{18}$.

When the teacher solved it, she said she cross multiplied, and in the next line she wrote $10x = 54$ (see section 10.3). She had applied the symmetric property of equations implicitly, leaving to the students to create meaning to the action (Godfrey & Thomas, 2008). Here it made no difference to the result, because the relation was an equality relation. However, when working with inequalities it was found to be a problem for Ronny.

He was working on the task $\frac{3x}{4} \geq \frac{4}{3}$. He had cross multiplied coming to the result $16 \geq 9x$ and from this to $\frac{16}{9} \geq x$. When checking his result with that in the textbook he is really confused. He asks his classmates (the teacher is in the adjoining room):

Ronny: When we cross multiply equalities, is there then some “mumbo-jumbo” about it? Is it okay to cross multiply?

He is sitting in the first row and turns around waiting for answers from his peers. Nobody responds, he gets no response and calls for the teacher:

Ronny: Listen, when I cross multiply, do I have to do something to this thing?
(he means the inequality sign).

T: With that thing *(she laughs)*. You don't have to do anything to that thing.
(She has not seen the task yet).

Ronny: But I don't remember.

T: Okay?

Ronny: When it is larger than or equal to, should I flip it or what should I do?
The teacher asks the students to take a break and sits down beside Ronny. He tells his departing classmates that he can't find what is wrong.

Ronny: I have got x larger than.

T: But you have to read it in opposite direction. You have to start from x .

²³ The test had two alternative tasks, only one task ended in a negative expression with the unknown. 23 students solved this alternative.

He is asked to read it from the side where the x is. For the teacher who has not the right solution in mind, it seems at first that she thinks his problem relates only to how to read the solution, and this leads to a discussion about how to read the inequality.

However, Ronny at the end asserts that his solution is not in line with that in the textbook, and he shows the teacher the initial task.

T: Okay, but you should have placed the x on the other side.

Ronny: I multiplied like this and like this. That's what I did.

He explains by words and by gestures how he has multiplied. He has multiplied in this way: $\frac{a}{b} \geq \frac{c}{d} \Rightarrow bc \geq da$ and his result is $4 \cdot 4 \geq 9x$. He has

followed teacher's demonstration when she cross multiplied, and at the same time implicitly applied the symmetric property of equations.

Here the teacher asks him instead to multiply by the common denominator, since the x then will be kept on the side where it was from the outset. She makes the comment that for him perhaps it is not a good idea to cross-multiply inequalities. Ronny replies:

Ronny: But I can then multiply the other way.

T: Yes, you can do that.

Ronny: If I multiply in the other direction, it will be okay. Then I keep the x on that side.

$\frac{3x}{4} \geq \frac{4}{3}$	line1	He suggests to multiply in the opposite direction.
$16 \geq 9x$	line2	Then the x will be kept on the left side. The teacher agrees. What was not discussed, was the difference between equations and inequalities. There is a question whether the teacher understands his thinking.
$\frac{9x}{9} \geq \frac{16}{9}$	line3	When looking into his saved task, (solution to the left) it turns out that he has just changed places for the two sides, operating as if the symmetric property for equations also holds for inequalities.
$\underline{\underline{x \geq \frac{16}{9}}}$		

Actually it is an example of a problem caused by being not explicit about the properties applied (Godfrey & Thomas, 2008), and also that the distinction between equations and inequalities is blurred.

Later in the lesson, Tore, had the same experience as Ronny and he asks his classmates for help:

Tore: What was your result on that cross multiplication?

Ronny: You must remember always to multiply by the x , so that x ends up on the correct side. You must not do as usual.

He moves his right hand from bottom right, up to left. It seems as if he means that the way he had cross-multiplied, is the 'correct' way to do it.

Ronny: Because if you have $3x$ on one side and 4 on the other, then you have to multiply upwards in such a way that you keep x on the correct side. (*He*

points to the left). This is not what you normally would have done, but it is not the same as for equations.

Bill: There is no difference.

Ronny: I'm sure. It makes a difference. I just did it.

Carl: It doesn't matter if x is on the 'wrong' side.

Ronny: It does.

What they mean by correct and wrong side is not easy to say, but it might be that at least Carl means that the left side is the correct side, and the right is the incorrect, however, his saved computer files show that he sometimes avoids negative unknowns by collecting the x -es on the right side. It might be that he then thinks that he places them on the 'wrong' side.

That day some students started the seatwork working on equations; others started on inequalities and then went back to the equations. It might therefore be the case, that the others were working on equations, when joining the discussion. Tore is focused on finding the correct solution:

Tore: What was the result then?

Ronny ignores Tore's question and he continues:

Ronny: Yes, because, if you multiply 3 the way I did it; $3x$ times 3. Then you get the answer on the wrong side. Then you have to turn the sign.

Tore: Then I have to turn it?

Bill: But, if you multiply towards left, then you get it on the left side.

Carl: If the x is on the left or on the right side doesn't matter.

Ronny: Though. For sure it does.

Carl: No, not really. Actually it doesn't matter. The result is the same. It is just the same.

Ronny: But then you will get it less than, you are going to...

Carl: Yes, ok. If it is like that.

Ronny: That was what I meant.

Carl: I didn't think about that.

Here the discussion goes from focusing on which side to put the x , over to the reason for being concerned about it. Ronny introduces the sign when saying: "then you have to turn the sign". What really makes Carl aware of the importance of Ronny's explanation is the utterance "But then you will get less than". This changes the situation from being disagreement to get insights into the difference in handling an equation in relation to an inequality. This means that there might be other situations where one has to be careful about how to handle the inequality signs apart from the situation of multiplying or dividing by negatives.

Tore who started the discussion seems to be pragmatic. He has not really participated in the discussion. He just accepts that the sign should be turned, and he ends the episode by asking for the solution: "Yes, but you got x something, x is $\frac{16}{9}$, isn't it?". From his responses, it is impossible to

say anything about his interpretation of the inequality sign. He does not mention it at all when asking for the solution. He talks about x is. However, his solution is exactly the same as Ronny's above. He just moves the terms over to the opposite sides of the inequality, as if symmetry was one inequality property.

The discussion is important, but it is possible it just resulted in another rule: If you cross multiply in an inequality you have to take care that the x , after the multiplication, is on the side that x was initially; before transformations. It also represents students' poor command of mathematical language, and how this blurred the discussion.

It is clear in this class that the work with equations (Garuti, et al., 2001) influenced students' conception of inequalities. At least in this episode with Ronny, he is becoming aware that equations and inequalities have to be treated differently, however, it does not appear that he has a mathematical justification for it, it seems rather likely that he produced a self-made rule, which worked in this situation (Cortes & Pfaff, 2000).

Neither in the classroom nor in the textbook are the properties of inequalities and of equations made explicitly visible for students. In order to promote conceptual learning these properties should have been introduced and exemplified (Hiebert & Grouws, 2007). The examples in the textbook and in the classroom demonstrate that division and multiplication by negative numbers make the statements not true. They give meanings to the rules, but do not really give mathematical arguments for why executing the same operations on both sides do not work for all numbers, as it does when solving equations.

11.2.4 Interpretation of solutions

No task was designed to promote discussion about the result. Although the introduction started by interpreting the conventions for writing sets and solutions of inequalities, no one was heard, during the observation, to mention anything about what the produced solutions meant. One student, Alf, was prompted by me, to explain what his result $x > 2$ meant. His answer was that the unknown is a number between 2 and "upwards". This might mean that he sees the solution to be one specific number in the set, not the whole set.

In the interviews when the students were asked about how they interpreted the solution $x > -2$ on the test task, two of five showed evidence of regarding the solution as a set of numbers.

Kristin said she had no clue, while the other two students' responses can be exemplified by one of them:

Olav: It is minus 1, minus 2, zero, 1, 2... They are lying there, and then it is minus 2, but at the same time you don't know the x .

R1: You don't know the x ?

- Olav: No, but you know that it has to be larger than minus 2.
 R1: What does that mean?
 Olav: It means that it is larger.
 R1: Some specific number which is larger?
 Olav: Yes.

Olav clearly felt that he did not know the x . According to him it had to be one specific number within the set of numbers. The way one solution in the textbook was commented on, gives rise to the same interpretation.

The teacher was not so concerned about this because she had experienced that the concept of inequality would develop during the work with functions and graphs related to inequality.

Some students seemed to solve the tasks in a correct way, but did not find the solution set for one x . In several tasks Roy saved his solution before he had found the solution set for one x . The responses on the tasks he was observed to solve, show that he finished the solution finding a number set for x if the result of dividing by the coefficient resulted in a whole number. When not, he ended his solution as in the first solution presented below. Both examples are from his work (video-taped on his computer screen).

$$\begin{array}{rcl}
 & 5x + 2 > 2x + 8 & \\
 1) & 5x - 2x > 8 - 2 & \\
 & \underline{3x} > 7 & \\
 & & 3 - 2x \leq 15 - 5x \\
 2) & \frac{3x}{3} \leq \frac{12}{3} & \\
 & \underline{x} \leq 4 &
 \end{array}$$

If the division by the coefficient resulted in a whole number, the solution set was given for one x (example 2 above). It might be that he avoided fractions in the final solution. He was consequent in this matter. The interesting question is if he reflects over the solutions. It seems as if he instead performs the procedures and has made his own rules for what a solution is. It does not seem that he is aware of what solution set satisfies the inequality. The same was the case with Ane in the example above (section 11.2.1). Their solutions might indicate a lack of reflection, and they are clearly not controlling their solutions.

11.2.5 Summary inequalities

The assigned tasks afforded had different levels of difficulties, however, not according to the concept of inequality. All tasks were variations of example tasks in the textbook, and thus not of a high level of cognitive demand (Smith, et al., 2004; Stein & Smith, 1998). The students could go to the examples and imitate the solutions. There was no contextual task offered in the first chapter under the heading inequalities.

The table below shows the inequality tasks in the December test. The students could choose either one of the two alternatives; both are shown in the table.

Table 11-1: Inequality test tasks results. Data set 27 students (%)

Inequality tasks	September (27 students)	December alt 1 (3students)	December alt 2 (24 studentms)
Task	$-3x + 6 < 2(x + 8)$	$\frac{x}{4} + \frac{1}{3} > 2$	$2\left(1 - \frac{y}{2}\right) + 3y < 4\left(y - \frac{1}{2}\right)$
Correct	22	67	50
Equal sign	22	33	13
Not flipping sign	26		8

It is evident that the students performed better in December than in September. In September a large proportion of the students had sign errors causing the x coefficient to be positive when keeping the unknown on the left side, and it is thus not easy to know if they would flip the sign. However, 26 % did not flip the sign having negative coefficient either, and one student flipped it, although he had a positive coefficient. 22 % ended their solution with an equal sign.

In December the result was better, 50 % had a correct solution, but all who kept the unknown on the left side, 30 %, flipped the sign only after the result of the operation was presented; in the last line. They did not seem to see the expression $\frac{-4y}{-4}$ as both an operation and the result of that operation (Gray & Tall, 1994), which might indicate a lack of proceptual thinking.

In the textbook the inequality signs were presented, and it was told that the larger numbers were located to the right on the number line. In the classroom the focus was on set notation and in interaction with the students, number sets and set notation was exemplified and explained on the number line. The crocodile mouth was used as metaphor for the signs at the start. In the classroom set solutions were said to be interpreted as including all numbers within the set. The exposition in the textbook could be interpreted to mean one specific number within the set.

The distinction between the solution process of equations and inequalities was exemplified with a number inequality both in the textbook and in the classroom. From then on the focus was upon the rules for solving inequalities, and the close connection between solving equations and inequalities was emphasised.

In the table below the main problems related specifically to the process of solving inequalities are presented.

Table 11-2: Inequality problems

Problems	Revealed by	Possible reasons
<i>Blurred relation inequalities / equations</i> }	<i>Use of the equal sign</i> <i>Inequality sign not flipped</i> <i>Sloppy language</i> }	<i>Properties for the two concepts are not addressed</i>
<i>Interpretation of solutions</i> }	<i>Interpreted as a specific number within the solution set</i> }	<i>Interpretation of textbook exposition</i> <i>No tasks requesting interpretation of the solution</i>

There is strong evidence that students regarded inequalities as types of equation which blurred the distinction between the two concepts and led students to make errors (Blanco & Garrote, 2007; Sfard & Linchevski, 1994a; Tsamir & Almog, 2001; Tsamir, et al., 1998; Tsamir & Bazzini, 2002; Vaiyavutjamai & Clements, 2006; Verikios & Farmaki, 2006).

The teacher, and also the textbook authors, applied a simple inequality to give meaning to the operation of flipping the sign. That example might be obvious for mathematicians, but it is not so for many of the students. Arne asked above: “why do we flip?”, and Kristin, Ane and others asked the teacher during seatwork: “when do we have to flip, is it only when there is a minus sign?”

Neither in the classroom nor in the textbook are the properties of inequalities and of equations made explicitly visible for the students. In order to promote conceptual learning, these properties should have been introduced and exemplified (Hiebert & Grouws, 2007).

One student claimed that inequalities were even easier to solve than equations. She was careful and kept the unknowns such that she could divide by a positive number. She failed, however, in both the September and the December test because she treated the tasks in accordance with her utterance, but forgot to keep the unknowns positive as she did in the classroom.

In the second period of observation, one assigned word problem was expected to be solved as an inequality, none of the students wrote inequalities. They solved the problem as an equation and had to reason about what interval could fit the word problem.

It can be concluded that the mathematics offered in the textbook also in the area of inequalities is focused on rules and procedures. Except for showing that to divide or multiply by a negative number leads to contradiction, and that to flip the inequality sign then makes the proposition true, there is little or no explanation of the concept of inequality.

The explanation of the solution set, is written in such a way that it can be interpreted to mean a specific number within the set, not the whole set. The teacher though went beyond the textbook when it came to the set notation and the meaning of inequality solutions. The tasks were, however, context free and did not challenge students to reflect over their solutions.

Students' responses indicate that they regarded the process of solution as a sequence of routine actions, not worth spending much time on. The interviewees told that they had no strategy when they saw the task. One said that he thought of equations, and went on solving the task, while keeping in mind the exception of not dividing/multiplying by a negative number without flipping the sign. Nobody checked their solutions. When students talked during the work with inequalities, they talked about ' x is'. Sfard and Linchevski's (1994a) conclusion that students' conception of inequalities "are meaningless strings of symbols to which certain well-defined procedures are routinely applied" (ibid p. 306), seems also to fit for this class. However, as the teacher emphasised; the topic will be introduced once again in relation to functions.

Neither in class, nor in the textbook the properties for inequalities were presented, and thus not compared with the properties of equations.

12 Word problems

In the textbook and also in the plenary sessions there was no explicit exposition about word problems, and how to transform problem situations into formal mathematical language. One problem, however, was transformed in the introduction to equations (see section 10):

Eli says to her friend: Multiply my age by 5 and subtract 23 years. You get the same value by multiplying my age by 2 before you add 25 years. How old am I? (p. 40)

This problem is not complicated in that it is possible to transform it phrase by phrase, by direct mathematising (Voigt, 1995). In addition, the age was said to be written as x , the unknown.

A ratio word problem was also introduced:

Trude is going to mix juice with 20 litres of water in the ratio 3:10. How much juice does she need?

We call the unknown volume of juice for x . Then we have:

$$\frac{x}{20} = \frac{3}{10}$$

$$\frac{x}{20} \cdot 20 = \frac{3}{10} \cdot 20 \text{ We multiply by 20 on both sides}$$

$$x = \frac{60}{10}$$

$$x = 6 \quad \text{Trude needs 6 liters of juice}$$

In the first chapter, nothing more was said about converting word problems into mathematical notation. In chapter 4 in the textbook, coded the blue track (the easy track), there was a short paragraph about converting word problems with two unknowns into two equations. The blue track was not for the students in this study, and the location of this exposition might be evidence of the authors regarding this transformation to be trivial for students interested in mathematics.

In the next section, the tasks in which students are asked to convert word problems into mathematical language, will be presented together with their responses. Those tasks and their results will be compared with some tasks on ordinary tests. To solve word problems at this level requires that students formulate the problem as an equation and that they solve this equation.

12.1 Converting text into equations

In total there are 18 word problems in the sub-chapters of the first main chapter of the textbook. Then the items at the end of the chapter are not taken into account since none of those tasks were included in the work

plan from the publisher. Five of the 18 tasks were written with bold letters; which means that they were most likely to be solved.

The table below presents those tasks and the number of students who solved them. Dataset 18 students.

Table 12-1: Work plan – word problems (equations)

	Item 1.3.2	Item 1.3.5	Item 1.3.6	Item 1.3.9	Item 1.3.10
Students who solved the task	9	7	14	7	6
Not solved	9	11	4	11	12

The table shows that a large proportion of the students generally seemed to be reluctant to engage in word problems, or they were lacking competence to solve them.

In the plenary sessions in September only one word problem was presented as an example, the ratio word problem about juice and water referred to above. This task represents a class of problems; ratio word problems. The structure of the task is $\frac{a}{b} = \frac{c}{d}$; one of the letter symbols representing the unknown. The structure is straight forward if the students are familiar with the notation of ratios, and what this notation means.

The teacher presented the problem and the students responded immediately saying that Trude needs 6 litres of juice. The teacher, however, stressed the point that they together should make it formal and write an equation. She asks some girls:

Ane: As an equation?

T: Yes, can't we? Can we try?

Ane: But, it is 6, it is 6 litres of juice.

Tore: We have no unknown. It is only a simple calculation.

The teacher realises that the numbers in the example task are not convenient, and thus she changes 20 litres into 18. Bill respond immediately:

Bill: 18 times 3 divided by 10 is equal to x .

T: How did you reason to come to that solution?

Bill: I just multiplied by the ratio.

The example task was initially an example of how to convert the word problem into an equation. The students however, did not see the point in this transformation. This corresponds to the findings in Stacey and MacGregor's (1999) study (section 4.9). After the quantity of water was changed, and when prompted, Bill presents the volume of juice, as a formula, which is not to work algebraically according to Janvier (1996).

The first assigned word problem in the textbook is not more suitable than the example task for converting normal text into an equation. The task is:

Berit buys 15 apples and pays 48 Nkr. Call the price for each apple x and write an equation. Solve it and find the price for one apple (p. 47).

To find the answer to the problem was indeed not the problem, but to set up the equation. Most students had to be reminded by the teacher to do what was requested according to the textbook, although she herself agreed with the students that there really was no point making an equation in that problem. The data showed that most students put in an x and wrote

$$x = \frac{48}{15},$$
 which is arithmetical reasoning despite the presence of the x

(Janvier, 1996).

The next task was about sugar and berries. Here the ratio 1: 4 between sugar and berries in jam was given, and students were asked to find the amount of sugar needed to make jam from 2 kg of berries.

Tore and Ronny discussed if the solution should include an equation. They had written the answer only. Tore wonders how it should be done, while Ronny asserts that it is not written in the task that they have to write an equation. Both agree that on a test they would certainly not solve the problem as an equation. The teacher comes:

Ronny: We can just simply write the answer, can't we?

T: No, you have to explain how you found it.

Ronny: Then we have to try to write it as an equation?

She admits that an equation is not requested in the text, but says they have to explain their method, not just present the solution. Tore expresses that he does not know how to convert the problem into an equation. The teacher explains when the ratio is given, they have one side of the equation already. Tore is still waiting for instruction on how to proceed, then the teacher asks him what represents the berries and what represents the sugar in the given ratio. Tore is not sure, and the teacher tells him what is mentioned first in the text, represents the first number in the ratio. Then Tore claims he understands.

Here it seems that both the textbook and the teacher have taken it for granted that the students knew about ratios, and how to write equations based on equal ratios. Although struggling to convert the problem into an equation, the students must have had practical experience of proportional thinking, since most of them solved the tasks arithmetically without any struggle. Only few students, as shown in table 12-1 above, solved the other three items on the work plan.

Two of the tasks had a similar structure, task 1.3.5 and task 1.3.10. The two tasks are presented below:

Task 1.3.5

Nina has an equal number of each coin value:
 1 NOK coins, 5 NOK, 10 NOK coins and 20 NOK coins. It adds up to 144 NOK. How many coins does Nina have?

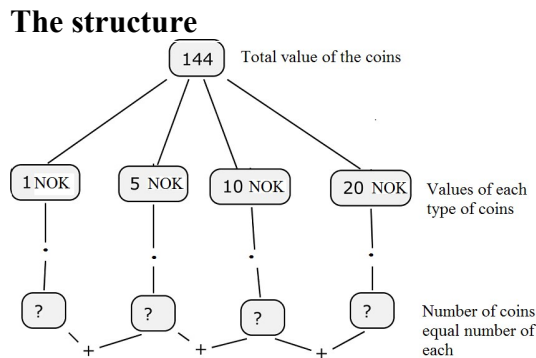


Figure 12-1: The structure of item 1.3.5

In this task each coin value has to be multiplied by the number of coins of each value. What eases the task is that this number of coins are the same for all values. If the students can see that this amount is the unknown, the x , it should be easy to set up an equation. However, what might complicate the task is that the question is not directly about the unknown, the x , but to find the amount of coins Nina had altogether.

The next task is shown below:

Task 1.3.10

Silje returns bottles. For the bigger ones she gets 2,50 NOK each, and for the smaller 1,00 NOK each. Silje has three times as many smaller bottles as bigger ones. Altogether, she gets 110 NOK. Create an equation and find the amount of each kind of bottle.

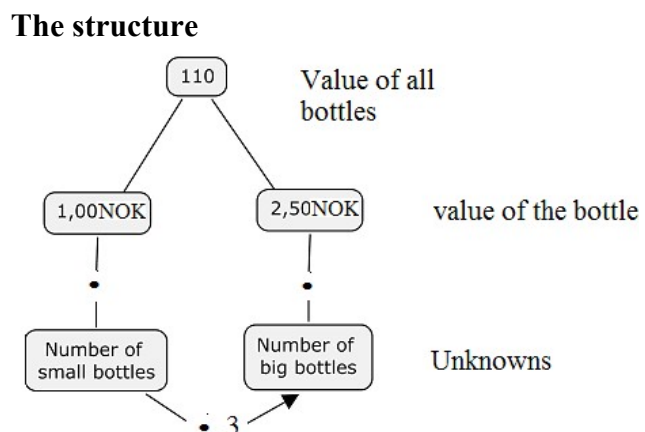


Figure 12-2: The structure of item 1.3.10

This task is a bit more complicated in the structure because in addition to the additive relation between the values of the two types of bottles, the relation between the numbers of the two types of bottles is multiplicative. Here it becomes problematic if the students have problems differentiating between the values of the objects at hand and the amount of those objects (Küchemann, 1981), which seemed to be the case for a large proportion of the students (see section 8.1).

The third task, an age problem, is presented below:

Task 1.3.9

The age of Knut and Karianne is altogether 36 years. Knut is two years older than Karianne. Let x be the age of Karianne. Create an equation and solve it. How old is Knut?

The structure

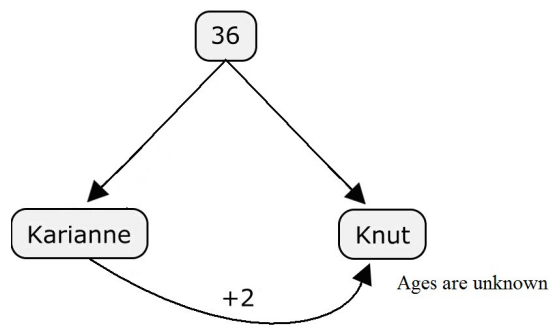


Figure 12-3: The task and the structure of item 1.3.9

This task might be easier although the unknowns here are also related to each other, but it is an additive relation.

Nine students had solved some of the solutions (9 out of 18). One of those had solved only 1.3.5, and two others had solved only 1.3.9. The other 6 had solved all three items. Three of those had solved the ratio items by setting up a formula, while they solved the others by proper equations.

One of the students, Tone, really struggled with the item 1.3.5 and asked for help. She had written $144/4$ on her screen.

T: What does the text say? You have an equal amount of each type of coin. If you call the amount of one type of coin x , that is where you can start.

Tone: What?

The teacher asks her for each type of coin and, Tone comes to the expression $1x + 5x + 10x + 20x$. The teacher follows up and explains that in order to get an expression for the sum, one has to add, since the sum of the value of the coins should be 144. She emphasises that the amount of each value is the same. Tone asks if this will be equal to 144.

Tone: Am I going to do it like this? (she writes $=144$)

T: Yes, fine. Did you understand what you did?

Tone: (whispering) No.

T: Hm?

Tone: I do not know why I should write x .

Her first suggestion was to divide 144 by 4. One reason might be that her strategy was to divide the total value by the number of different coin values or it might be that she guessed and checked to come to the number 4. In this problem it is important to differentiate between the amount of coins and their values (Küchemann, 1981). Tone seems not to have seen this difference, and that might be the reason for her not to understand why she should write an x . In the algebra test she was the one assigning values to the unknown in the expression representing the total price (see section 8.1).

The dialogue between Tone and the teacher is an example of the funnel pattern (Voigt, 1995). Tone is led in small steps through the solution, but at the end she admits that she has not grasped the point of using the x .

The teacher responded to Tone's statement about not knowing why there should be an x , by saying that x is equal to 4 and then she patiently goes through the solution; substituting x by 4. Her strategy seems to be to help Tone to create meaning for the found solution, and to show that it fits into the initial word problem.

In the age problem above, Tone was stuck again. She worked together with Ruth who had written: $\frac{36}{2} = \frac{x}{2} + 2$ and then she asked for help.

The teacher came, but Ruth did not explain her thinking behind what she had written. The teacher asked her what Karianne's age is and Ruth answers it is x .

T: And what is Knut's age then?

Ruth: Must be y or something like that.

T: He is then, read the text once again.

Ruth: He is, oh yes, Knut is $x + 2$.

T: Okay. Then you have x and $x + 2$.

Ruth: Yes then $x + 2$ is equal to 36.

T: That sounds okay. You chose Karianne's age to be x .

Ruth: Yes.

T: Just delete what you had written.

Now Ruth writes: $x + 2 = 36$.

T: Yes, then you have only Knut's age. And he is two years more than Karianne. He is $x + 2$.

Ruth: Then it will be $x + x + 2$

Ruth although struggling, seems to be comfortable with the concept of the unknown, however, she struggles with how to express the relationship between the unknown and the known quantities. Led by the teacher's comments, she managed to find the equation. Tone sitting next to her says at the end that she did not understand.

The teacher then asks Ruth to explain for Tone. She does this eagerly and explains in detail. The teacher listens and puts the two expressions for ages in brackets during Ruth's explanation. After Ruth had explained for her, Tone says she understands.

In this episode, the teacher encouraged Ruth to express her understanding of the problem, and in this way Ruth guided her peer, and at the same time had the experience of expressing orally a mathematical explanation.

In contrast to Tone, Ruth seems to have grasped the representations of the ages, although her start was incorrect. This is confirmed in her way of explaining what the expressions represented. The teacher at the end points

to the text and says that they always should follow it closely and reason about how to represent the information given.

Item 1.3.10 was solved by six students. One of them, Kari, had formulated an incorrect equation. The others had saved their correct solutions and had related their results to the initial question in the task. Arne had to ask for help though, because he had written multiplication as power.

12.2 Word problem - Test tasks

On the first regular test students had in September, they were given two word problems; one of them was a ratio problem:

In a children's party Stine decides to mix some juice. She has 2 litres of juice in a bottle. On the bottle is written that the ratio between juice and water should be 1:7.

How much water does Stine need if she is going to follow the recipe on the bottle?

The other was an age problem:

The brothers Tor, Erling and Jonas are altogether 37 years. Tor is double the age of Erling and 3 years older than Jonas. How old are the brothers?

The table below shows the results:

Table 12-2: September test – word problems (27 students)

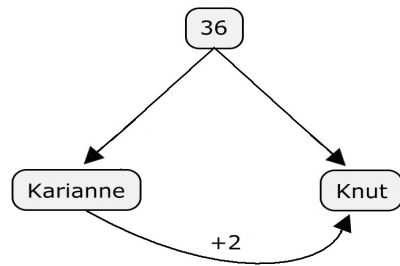
	Ratio problem %	Age problem %
Correct equation	15	37
Correct arithmetic solution	51	19
Correct answer no justification	15	11
Incorrect equation	4	7
Incorrect calculations	11	11
No response	4	15

From the table it is seen that in the ratio problem, few students (4) felt the need or felt competent to apply equations to solve the problem. The main group gave some explanation for their arithmetic calculations and came to the correct solution. Four just gave the answer, giving no justification for their solutions.

In the age problem, 10 (37 %) students solved the task by setting up and solving a correct equation.

The age problem on the test was more complicated than those problems students had solved in the textbook. Here two quantities are related to a third one.

Age problem from textbook



Age problem from September test

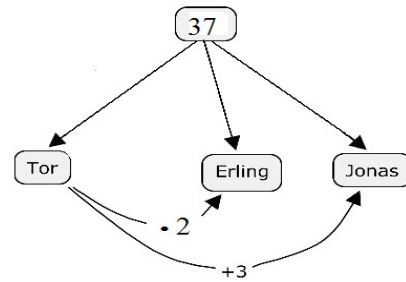


Figure 12-4: Comparison of the two age problems

In the figure above the structure of the problems are illustrated according to the schema developed by Bednarz and Janvier (1996). As shown in the other schemas, none of the problems on the work plan had the same structure as this test problem. However, most students gave the correct answer (67 %). Only seven in the class had solved the textbook problem. Four of those students solved the test task correctly, solving an equation. In this problem, the structure is so complicated that it seems as if nearly half the students felt that equation solving was the best strategy, although two of them did not succeed.

Two students justified their correct solutions by saying that they ‘tried and failed’. One student, Tone, who in the classroom had a problem understanding why there should be an x involved, (see the foregoing section), succeeded in finding the correct solution by applying a table.

12.3 Summary word problems

The data confirms what has been shown in other studies that students are reluctant to set up equations if the structure of the problem makes it easy to solve it arithmetically (Stacey & MacGregor, 1999). It seems as if students prefer to rely on arithmetic reasoning, also when they know that equation solving is the focus for learning, as long as the structure of the problem is easy enough. As Ronny said: “I would never write an equation if I got such a problem on a test” (about the ratio problem). He and Tore were not among those who applied an equation in the test task with a given ratio. However, both wrote equations in the age problem. Tore who asked for help in the lesson succeeded and got a correct solution.

On the work plan for chapter 4 in which students had to work with linear equations and solve system of equations, five items required that students represented relationships in mathematical notations before solving the problems.

In two tasks they should find out the algebraic expressions for linear functions from a text. This did not seem to be a problem for the students, all in the dataset of 18 students (saved computer files) had solved the task. The reason might be that in such problems they knew that the equations should have the form: $y = ax + b$. Very few students solved the other assigned word problems in that chapter.

There is no data explaining students' reluctance to solve word problems, but in addition to the fact that some of the tasks were more easily solved arithmetically, it might be that the students had to use more effort to solve those with a more complicated structure than to start on tasks already written in mathematical notation. Another reason for some students might be the problem of distinguishing between letter symbols standing for the object, and the value/amount of that object (see 8.1).

The result was that few students experienced the generational activity of algebra (Kieran, 2007a) which is regarded as being the meaning making activity of the topic.

From this chapter it can be concluded that the generational activity of algebra is not devoted much space in the textbook although one goal for the first textbook chapter is: "To convert a practical problem into an equation" (see section 2.9). It might be that the textbook authors expect that the students at this level have had rich experiences with such work, and that it is not needed to go into this topic for students in order to learn or consolidate basic algebra.

The examples offered are poor, and especially in the ratio problem the numbers are of a size not inspiring students to create equations. The same is the case with the ratio problems assigned for students to solve. In the plan from the publisher only three tasks were written with bold fonts, and in addition five tasks were written with normal fonts. The teacher had chosen five of them and written them with bold fonts thereby demanding the students to solve them. The students were nevertheless offered few opportunities to experience the meaning-making activity of algebra.

The responses in the classroom and the reluctance to engage with word problems, might indicate that most students have had few experiences with this activity also in lower grades.

13 Patterns of interaction

Patterns of interactions are jointly constituted (section 5.1) by the teacher and students on the basis of routines and obligations (Bauersfeld, 2000; Voigt, 1995). Also from this class, it is possible to reconstruct such patterns. In this chapter, these patterns will be described. The reason for this is that different patterns of interaction might provide different learning opportunities (Anghileri, 2006; Voigt, 1994; Wood, Williams, & McNeal, 2006). Another reason is that the patterns of interactions gives an important insights in the learning situation.

In the plenary sessions the interaction between teacher and students followed much the same pattern from lesson to lesson.

Every time a new²⁴ topic was introduced, the teacher asked for prior knowledge (Ausubel, et al., 1978) (see section 10.1 and 11.1). One example is from the introduction to equations:

T: The topic for today is equations. And so I am very intrigued about what you already know about equations.

Ronny: Unknown!

T: An equation is something that you will meet over and over again especially for those of you who will choose to study mathematics all the three years here in school. The sooner you learn the principles the better. But you said that there is an unknown. That is right. What do we call it then?

Else: That depends on what name you will give it.

The conversation went on in the same style, with the teacher listening to students' input. Her style was to revoice (Herbel-Eisenmann, et al., 2009) what students said, which made it easier for the whole class to capture what was said, and thereby to make it easier for students to join in. In addition, this revoicing was applied to focus on correct terms as in the episode with Carl who told her to "times up and down" to get the common denominator. She revoices his utterance, but applies the word 'expand' (section 9.3). On other occasions revoicing was applied to emphasise important underlying concepts, when students responded, as when students suggested to 'move over and change sign' she revoiced the utterance and added that this 'movement' mathematically was to apply the same operation on both sides of the equal sign (section 10.2). At other times this revoicing was followed by 'why questions' as in the episode with Arne in the same section (10.12).

All inputs were taken seriously, and she built on them, no matter how elementary they seemed to be. In the plenary session when students were asked about inequalities, there was also a discussion about the inequality

²⁴ New in the sense that it was introduced for the first time in this class. It does not mean that it was new to the students.

sign and how to read it. After a remark from teacher's side about the use of the metaphor of a crocodile's mouth in primary school, one student responded correctly using the metaphor. The teacher then used it in her further explanation (section 11.2.1). The fact that prior knowledge was taken seriously, was probably one reason for students to feel confident to engage in such patterns of interaction.

Another pattern was that example tasks were solved on the whiteboard with the teacher asking students to join her in the solution process. Most often a large number of students were eager to participate. As the teacher was writing on the whiteboard, the students suggested what to do. Students' responses were mostly rather short.

When, drawing from her experience, the teacher wanted to emphasise reasons for suggested operations, she asked follow up questions. One example is from the plenary session about simplifying algebraic fractions.

The fraction $\frac{8x^2y}{24xy}$ should be simplified:

- Ine: Can't we cancel some which are similar?
 T: Yes, tell me which of them.
 Ine: We can cancel the y's.
 T: Yes, I can simplify by cancelling the y. What am I doing then?
 Ine: We cancel both.
 T: What operation is it?
 Ine: We divide.

In this example Ine suggests to simply cancel the y's. The teacher is not satisfied and asks what operation that is. To that Ine answers "we divide". This is one of several examples of a routine of reducing verbal utterances to numbers and keywords, which is often seen especially in mathematics classrooms (Voigt, 1989). The example is at the same time an example of a move the teacher makes in order to emphasise what is going on mathematically, although the responses are very short.

When students seemed not to agree about what would be the next step in the solution, the teacher gave them time to discuss without intervening.

One example is when the task $\frac{a+3}{4} - \frac{a-1}{6}$ is going to be solved (see section 9.3). The teacher and the students take turns smoothly until the fractions after being expanded $\frac{3a+9}{12} - \frac{2a-2}{12}$ are going to be subtracted.

Then one student has the opinion that the number 2 in the last fraction is going to be subtracted. Other students join in, some agree and others suggest it should be added. They discuss what the sign is going to be, some give arguments others just agree with their peers. Also then the inputs are short, pointing to rules, without mathematical argumentation.

The teacher asks after a while if they want a minus sign, the students continue their discussion, until the teacher finally gives the advice to put the numerators in brackets: $\frac{(3a + 9) - (2a - 2)}{12}$. After being presented with this, all the students seemed to agree that the number 2 has to be added; the *meaning was taken to be shared*. However, that was not the case before the teacher intervened.

She let them discuss until she herself or some student presented a convincing argument. Several times the teacher used number examples as counter examples. One example is when the students argued for simplifying fractions term by term (see section 9.1).

Sometimes the interaction was not that smooth. One example is to be seen in section 9.2, when the teacher challenged the students to change strategy; to be more efficient and flexible. Then some of the students responded negatively, they did not want to change. Another example can be seen in section 12.1 when the students could see no benefits in applying equations to solve the given ratio problems.

Most of the interaction patterns shown above can be said to be examples of the initiation-response-follow up/evaluate pattern. This pattern is characterised by the teacher posing a question, students respond, and the teacher follows up or evaluates the responses. Former studies have shown that the students had little or no impact on the classroom discourse within such interaction patterns (Cazden, 1988; Mehan, 1979). Studies of Norwegian classrooms, however, have shown that this is not the case (Klette, 2010)

Norwegian students have possibilities to participate and contribute; their roles have changed in terms of their “possibilities for initiation, negotiation and involvement”, so also in this study. The teacher was open to, and developed on students’ ideas, and problems, as shown in some of the episodes above.

Students answers are short in style though, pointing to rules, possibly through a long-time familiarity with the didactical contract for traditional mathematics classroom (Blomhøj, 1994) which has conditioned them to respond in this way. If the teacher expects more than short answers is not known, at least she responds to students’ suggestions in such a way that they do not feel unsuccessful, but are willing to carry on. She offers them, however, opportunities to go beyond the rules and the procedures.

During the seatwork other patterns of interaction were observed. Mostly when students asked for help, they did it because they had checked their answer with that in the textbook key, and had found that it did not

correspond. If they did not find the error, they asked for help. Mostly the teacher went patiently through their solutions together with them.

Another frequent pattern occurred when in their solution process students asked questions such as: what am I going to do now, or I do not know how to carry on. Then the teacher's reaction often was to ask questions. One example is an episode with Ronny solving a fraction task:

Ronny: What am I going to do in order to multiply a fraction by a bracket?

T: Yes? Can you tell me. Do you have any suggestion?

Ronny: We multiply the fraction by those. (*He points to the terms in the bracket*).

T: Yes, you multiply. (*Ronny seems to hesitate*) What will you have then?

You multiply $\frac{1}{4}$ by x ?

Ronny: It will be one fourth of x ?

T: Yes.

Ronny: Just like that? (*he writes*)

T: Yes, that's how it is.

Another pattern which may be called the confirmation pattern, occurred in situations when students had solved a problem, or were in the middle of the solution process. It seemed they did not feel secure and asked if the method or part of the solution was correct. If this was a way to make contact with the teacher, when she was nearby, or if they really felt uncertain, is hard to say. The teacher confirmed what they had done. One example of this is from the lesson working with equations. The teacher is watching Peter's screen from behind:

Peter: Is it just like this. Is it okay what I have done?

T: Yes, it is.

This last pattern was more frequent in the first lessons, then nearly one quarter of the interactions during seatwork were of this kind. In the last part of the observed lessons there were fewer such episodes.

When students were working on tasks, they interacted with each other as well as with the teacher. The interaction between the students was not so well captured, but from both video- and audio-recordings it was possible to capture some of the conversations.

One typical pattern observed, was interactions in which one student asked one peer or group of peers about their solutions or answers. When their solutions did not coincide, they either asked others or they started to search for differences or for errors.

One example of this is the episode when Tone and Ruth compared each others screens in section 10.2. Another is when Tore turns to his peers and asks if they have got the same result as his, in section 11.2.23.

Mostly it seemed that students were more focused on getting a correct answer, than to discuss the mathematics. However, some such episodes were also observed. One is the episode when Ronny realised that it was

not straightforward to cross multiply when solving linear inequalities in section 10.2.3. However, also this time the arguments were held on the surface level, pointing to rules. They did not go into the properties of equations and inequalities.

Other patterns, not interaction pattern between students and teacher or between students, were also observed in the classroom. One pattern discovered during seat work, and confirmed by the saved computer files, was that a large number of students mostly skipped the few assigned word problems.

Another pattern, seen in those files, was that students did the first assigned tasks, then ever fewer students solved tasks further on in the work plan. This might mean that a significant number of students did no home work.

The students worked on the computers, which made it easy for the teacher as well as for the observer(s) to follow the work on the screens. Often when students asked for help, they just pointed to the screen using the indexicals 'this' and 'that'.

It is possible that the use of the computer made 'reduced verbal utterances' even more reduced. In the introduction to linear functions, the students wrote on paper. The teacher had prepared tasks in which students had to give reasons for their answers. The writing on the paper was often not easy to read, which caused students to have to explain what was written. The utterances were also then not rich in words, but richer than when working with tasks on the computers.

The transformational character of the activity during the work with the first textbook chapter, the topics presented which the students had met in earlier grades, and also the limited time, might explain the focus on thematic patterns heading towards a specific solution.

A summary of found patterns is presented in the table below:

Table 13-1: Patterns observed in the mathematics classroom

Patterns of interactions:	
Connecting to prior knowledge	Teacher's action at the start of each topic
Revoicing	Students' utterances were revoiced by the teacher, sometimes just to broadcast what was said, other times to widen it, to emphasise mathematical terms or to ask follow up question
Prior knowledge and suggestions	Teacher treats seriously what students bring in, and builds on those inputs in such a way that they continue coming with responses
Tasks solved in cooperation	Teacher works out tasks on the whiteboard together with the students
Emphasise reasons for suggestions	The teacher asked follow up questions
Use of number examples as counter examples	To create meaning as to why operations work or do not work, number examples were applied
Didactical advice	Experienced difficulties – teacher gave didactical advice about how to avoid errors
Initiated discussions by the teacher	Experienced problems – teacher challenged students, provoking discussions about strategies and old habits. The students were given time to discuss when there was no agreement about an answer to teacher's input.
Confirmation	Many students asked if their solution was correct
Return questions	When asking 'what to do', the teacher responded with a new question
Comparing solutions	What is your solution? Comparing screens to see what went wrong, or the end result. Focus on the 'correct answer'. No student was seen to control their equation/inequality solutions
Other patterns:	
Correct answer	Students were seen to search for the correct answer in the textbook. Often the initial answer was just adjusted. Also there was evidence of students not regarding the process as important; although incorrect.
Skipping word problems	Few students had solved word problems
Doing little or no homework	The saved files showed evidence that few students had solved the last tasks on the work plans
Using the cursor instead of using mathematical language	The pointing on the screen with the cursor minimised the need for explanations
A reluctance to do work 'not usual' in mathematics lessons	Expressed that they preferred the lesson pattern: teacher's exposition and then seat work

Although coming from different schools, no one seemed to question the culture of the mathematics classroom, which might indicate that they are following a familiar didactic contract (Brousseau, 1997). The interaction was already characterised by routines (Voigt, 1989) even though they had only seen each other a few times, when the observation started. The teacher and the students in a class are according to Voigt (1994) interacting and communicating with each other through the rules and expectations they already have with each other, and the style of interaction is hard to change. Voigt (1995) claimed there is a risk that patterns turn out to be just joint work in order to arrive at a correct solution narrowed down to small steps, which in reality is an obstacle for learning.

These students have had 10 years in school and in mathematics classes. It might thus be argued that most patterns observed were also patterns students had experienced from lower grades. This was confirmed in conversations with some of the students.

The teacher was expected to introduce the topics and to lead to the solution process on the white board. This expectation was explicitly expressed by the students in an interview about their preferred way to work. She was also expected to help them find their errors and correct them. These expectations were based on former experiences with mathematics classrooms, and correlate with the features of a traditional mathematics classroom (Blomhøj, 1994).

When the teacher challenged the students to change strategy, the smooth interaction (the didactical contract) was broken. Then some students objected so powerfully that the teacher succumbed, and seemingly no learning occurred (section 9.2 and section 12.1). They wanted to act as they always had.

The interaction patterns, shown above, constituting the classroom culture seemed to confirm that students' experiences of mathematics, at least in this part of the syllabus, are mostly based on rules and conventions and that they have not experienced mathematics as relations and structures (Steinbring, 1997).

There are examples of focusing patterns (Wood, 1994) when the teacher draws students' attention to critical aspects by posing questions, and then letting the students discuss themselves, thus leaving to the students the responsibility of solving tasks. Students' argumentation, however, was held on a superficial level of pointing to rules.

14 Findings: Summaries and Discussion

In chapters 8 to 12, students' work with mathematics in the first weeks of attending upper secondary school is presented. The expositions in the textbook, teacher's explanations, the teacher and the students' joint work with example tasks, and the students' work with assigned tasks have been presented. Students' responses, are compared with, and categorised according to findings from earlier studies, and reveal that problems and common errors found in earlier studies are also found here. That is one level of this study; to present and discuss the mathematics offered and to be informed about what experiences the students' have of the mathematics offered at the start of upper secondary school.

The study has also a further aim; to study how the learning situation; the setting of the mathematics classroom influences students' experiences. The learning situation includes the learning goals, the teaching, the textbook with its expositions and tasks, the computer, classroom interactions and the didactical contract.

In this chapter I will provide short summaries of the findings in chapter 8 to chapter 12, categorised according to findings of students' responses in former studies. In the subsequent sections, those findings will be discussed more explicitly in relation to the interpretative framework in chapter 5, and in relation to the research questions.

14.1 Problems and common errors

In this section the findings are presented in accordance with the ordering of the topics in the textbook.

14.1.1 Literal symbols and calculations

The textbook examples, assigned tasks and the first plenary sessions during the time of observation²⁵ revolved around manipulation of number strings and algebraic expressions. The assigned tasks were exercises (Niss, 1993) similar to the example tasks and thus of low cognitive demand (Smith & Stein, 1998), and also of low procedural complexity (Hiebert, et al., 2003).

The students manipulated and transformed algebraic expressions, without referring to the meaning of the letter symbols, or what they represent. Steinbring (2006) emphasised that signs have no meanings of their own, and have to be interpreted. It seemed that the textbook authors had taken for granted that students have no problems with the different inter-

²⁵ The students had been instructed on how to use their computers and the software programs MathType and Word in plenary sessions before the observation started.

pretations of the letter symbols (Küchemann, 1981). The textbook did not explicitly focus on the mathematical symbol system, and there were no examples or assigned tasks to provoke errors that commonly occur. Thus there were no or few opportunities (Hiebert & Grouws, 2007; Kilpatrick, et al., 2001) given for students to explicitly develop their interpretation of the symbols.

The algebra test indicated that less than half the students, 40 % in the autumn and 38 % in the spring (section 8.1), differentiated clearly between an object and the value of that object (Küchemann, 1981). The difficulty in making this differentiation might be one reason why few students solved the assigned word problems (see chapter 12). To solve those tasks they had to decide what the letter symbols should represent. Tone's interaction with the teacher (section 12.1) is an example of this problem. She did not know why she had to bring in an x .

It may be questioned whether all students were aware of the possibility that different letter symbols could take on the same value, since only half the students in the autumn, showed evidence of accepting this in a task in the algebra test (section 8.1, table 8-1). In the Spring the proportion had risen to 72 %; indicating a positive development. The work with simultaneous equations and different functions might have offered the students opportunities to develop their concept images (Tall & Vinner, 1981).

The concept of variable in the sense of Küchemann (1981) (section 4.1), was not in focus, and there was no sign of such an interpretation among the students in this period of observation. This is the reason for using the notion letter symbol, instead of the notion variable, in this thesis. All students seemed though to accept the 'lack of closure' (Collis, 1972) and letter symbols representing specific unknowns and possibly general numbers.

Students' responses indicate that nearly all students are confident when it comes to elementary calculation with numbers and letters. The introduction of the minus sign, powers, or fractions caused difficulties (ch. 8 and 9). Most students seemed to follow the law of distribution although some students failed occasionally. It seemed not to be a consistent error. The tasks were simple though; no assigned task included two binomial factors.

The concept of equivalence, underlying all transformational work in algebra (Kieran, 2007), and thus tacitly and implicitly present in students' work, was not addressed by the textbook or in the classroom. Results from the algebra test indicate that one third had problems picking out equivalent expressions (section 8.1). The results in the spring were far better, indicating a positive development.

Earlier studies about learning, confirm the importance of being explicit about the concepts students are going to learn (Hiebert & Grouws, 2007), and also to name the concepts (Gray & Tall, 2007) in order to give the students possibilities to refer to the concept and to discuss its properties. Equivalence seemed not to be part of the students' vocabulary. No student was observed to use equivalence as a tool to control their work.

There was no task or discussion in the classroom about letter symbols as such, apart from one episode in which a student suggested that it did not matter which letter represented the unknown in equations (section 10.1). The teacher welcomed that comment.

One cannot say for certain how the work with algebra was introduced in lower secondary school for these students since they came from different schools using different textbooks. The former curriculum (L97, see section 2.2.1) emphasised the need to encourage discussion and reflection in the classrooms. Alseth and colleagues (2003) studying mathematics textbooks, and the introduction of algebra in these books, found that not much was asked about reflection and discussion.

In a recent study of six mathematics textbooks introducing algebra in lower secondary school in Norway, Kongelf (2015) found that there was little or no emphasis on letter symbols representing general numbers. The connection to numbers was weak, and the way algebra was presented laid the ground for misconceptions especially 'letter symbols as objects'. The focus was on manipulation with little explanation of why these manipulations were being performed. He concludes that, "the incorrect formulations, illustrations and mathematical reasoning, create conditions for development of misconceptions" (ibid, p. 104).

14.1.2 Invisible signs

The teacher addressed the invisible operation sign when it was 'present' in example tasks. It was not addressed in the textbook. Those who asked for help, seemed to have been helped, and in the spring all came to the correct solution of the task presented in section 8.2, table 8-3.

In tasks involving negative numbers the invisible operation sign was still a problem for many students. In the spring, 25 % mixed the invisible multiplication sign with the addition/subtraction signs or chose incorrect operations (algebra test). This proportion decreased from 51 % to 25 % from autumn till spring (see table 8-5 section 8.3.2); a positive development.

The invisible number (number 1) was addressed several times by the teacher during both plenary sessions and during seatwork. The students seemed to be aware of it. This was confirmed in the interviews (section 8.3.6). It might however, still be a factor causing students to change direc-

tion in calculation (see section 8.3.6 - Olav's response solving task 1e in the algebra test).

14.1.3 The minus sign and negative numbers

The minus sign gave rise to many problems in students' work. The use of the number line indicates that the textbook authors wanted to give meaning for the work with negative numbers, but the exposition ended in the sign rules without any explanation. The function of the minus sign as a sign for 'taking the opposite of' and the concept of 'opposite numbers' (the additive inverse) were not addressed at all.

The responses on the few tasks offered explicitly to practice operations on negative numbers, indicate that some few students were not confident with these operations. One issue was to handle two successive minus signs (section 8.3.3); to take the opposite number, another was negative numbers as base in powers.

One issue was the sign rules for brackets. On the first ordinary test, 37 % did not change the sign in at least one task when the bracket was removed (section 8.3.5 and 8.3.7), and two changed sign when the bracket was preceded by a plus sign. Repeatedly in the classroom, students were observed to choose an incorrect sign when removing brackets, but then the students could check their results and ask for help.

In tasks with binomials as numerator the same problem occurred. The teacher had given the advice to write the binomials in brackets. This helped many students to write the correct sign. However, if they had not understood the reason for the sign rules for brackets, it was just another rule. Alf's correct solutions and his comment (section 8.3.5) that he had never understood why the signs should be changed, exemplify this.

Students had no tools to control their work, but if the concept of equivalence had been known, they could have substituted the letter symbols by numbers, and compared their end solutions with the initial expressions to see if the value was equal. This might also have helped them to see the brackets and binomials as objects (Sfard, 1991).

Students stating that they did not know that the sign rules also worked when dividing by a negative number (section 8.3.5), is an indication that they had been offered little variation in the examples (Bills, et al., 2006), but most importantly that they should have learned why the rules work.

The tendency to detach the terms from their indicated operations, causing students to either calculate from right to left, operate as if there was a bracket involved, or to rearrange terms without considering the operation signs was also caused by the presence of the minus sign. This happened both with arithmetic and algebraic expressions (see section 8.3.6). No assigned task was designed to provoke such errors, but in one task in

the algebra test 48 % in the autumn, and 38 % in the spring failed because they detached terms from their indicated operations. They had no clear conception of what sign belonged to which term.

The errors revealed related to negativity (Vlassis, 2004) show a lack of conceptual understanding (Kilpatrick et al., 2001) for basic mathematics, and that such a lack creates problems for students in developing procedural fluency (ibid).

It might certainly have helped students to have a conception of the minus sign including all three functions of the sign (Gallardo & Hernández, 2005; Gallardo & Hernández, 2007; Vlassis, 2004). In addition, there is a large group of students who need to be informed about the importance of considering which sign ‘belongs’ to which term. To interpret and to understand the mathematical language, was not addressed in the learning material.

The mathematics offered was highly procedural (Hiebert & Lefevre, 1986) and promoted instrumental learning (Skemp, 1987). The tasks were exercises (Niss, 1993) and did not challenge students’ concept images (Tall & Vinner, 1981).

14.1.4 Powers

The concept of power was presented in the textbook in relation to scientific numbers. In the work with powers, problems revealed in former studies, were also found here. The textbook presented the definition of power

as $a^n = \overbrace{a \cdot a \cdot \dots \cdot a}^{n \text{ factors}}$ and used the example $10^{-5} = \frac{1}{10^5}$ to introduce negative

exponents. The method to transform scientific notation into decimal numbers was told to be to move the comma as many steps as the value of the exponent. There was no explicit connection made to the number system.

In the algebra test all students showed that they knew the definition of powers as repeated multiplication. Errors included writing the multiplier as exponent (Glenn in section 9.3), or the opposite; writing exponentiation as ordinary multiplication. Some students expressed uncertainty about the writing (see section 8.4). This phenomenon was reported in earlier studies (de Lima & Tall, 2006a; Stacey & MacGregor, 1997). Some few students seemed to lack an understanding of the meaning of the different components in a power. This was evident when simplifying frac-

tions as in the example: $\frac{15ab^2}{5b} = \frac{\cancel{15}^3 a \cancel{b}^2}{\cancel{b}} = 3a^2$.

Some had problems in deciding in which order the powers should be calculated; either to multiply the base before exponentiation or to calculate the power first. One example is the episode with Else in section 8.5.

In December 24 % (task 1a) failed because of calculating in an incorrect order. Some of those calculated also the base by the exponent (12 %).

Most students who at the outset had a problem, still seemed to have troubles in the spring. This might indicate that they had not been offered rich experiences of the concept of powers, their concept images had not been developed (Tall & Vinner, 1981).

Later in the school year, the powers were in focus in the textbook once again. Also in that textbook chapter, the underlying assumption was that powers and the operation of powers are well known. The focus is to learn more efficient methods through new rules.

The results on the test (task 2a section 8.4, table 8-8) indicate that nearly 40 % of the students had problems with powers with negative exponents. If those students do not develop an understanding of what this negative exponent means, and a sense for the size of the numbers the powers represent, it does not help learning the new rules for adding and multiplying exponents when multiplying/dividing powers. They will still lack the meaning of those rules and operations.

14.1.5 Order of operations

The responses indicating problems related to order of operations, were mostly related to powers and the minus sign. In the textbook, only rules were presented in relation to ‘order of operations’, and no task²⁶ was explicitly offered on the work plan to practice the rules for order of operations. There were no opportunities offered to solve tasks including both powers, multiplication/division and addition/subtraction.

Two test tasks, however, one in September and one in December, had the potential for challenging students. In the first task, the design was such that it might have been tempting to add before multiplication, but only one student did this (see section 8.5), indicating that the students were confident in this matter. There was, however, no other task to challenge them. In one test task, December, 24 % of those who solved the task, multiplied by the base before the exponentiation (see the foregoing section).

Problems related to the minus sign were grouped together, although the category ‘detachment of indicated operation’ could have been classified as problems with order of operations.

²⁶ This must have been a mistake from the publisher’s side, since the item said to be related to order of operations, was an item about scientific notation. The task designed for practicing the rules for order of operations was listed on the work plan, but in a font signalling it as not being important. Only three students in the sample had solved some few of the tasks in the item.

14.1.6 Fractions

As in the foregoing sections, the assigned tasks offered for students to solve, were all similar to the example tasks; of low cognitive demand (Smith & Stein, 1998) and of low procedural complexity (Hiebert, et al., 2003). The textbook introduced the concept of fraction by some of its sub-concepts, but the focus was on the rules for fraction operations. There was no explanation given for the rules in the textbook.

Students' responses related to the work with fractions, reveal that most students were confident using fraction operations related to ordinary fractions. It was observed to be a problem to factorise larger numbers. The teacher's exposition was focused on finding a common factor for reducing the fractions, not so much on prime factorisation, but the students who asked about factorising numbers, were helped (Glenn and Sven in section 9.1). Mostly the fractions were not reduced to their simplest form, when they were result numbers after calculations. These issues were reported in earlier studies (Brown & Quinn, 2006; Eichelmann, et al., 2012; Padberg, 1986).

The work with algebraic fractions was more difficult. Errors reported earlier (Brown & Quinn, 2006; Guzmán, et al., 2011; Hall, 2004; Olteanu, 2012; Storer, 1956) were also found in this study.

Some few students struggled to factorise powers (section 8.4), and the majority of students had problems simplifying fractions with binomials. Few tasks including binomials were offered, and those were of the same structure, resulting in the same answer either the binomials were factorised, or if the students cancelled the letter symbols before they added and simplified what was left (partly cancelling, see the test task in section 9.1, table 9-1).

In the classroom, some students even argued with the teacher, saying that there was no need for factorisation; they had the 'correct' result (the episode with Ronny in section 9.1). The teacher's number example did not seem to convince all of them, 81 % failed on the subsequent test task.

It seemed that the students had no clear conception of simplification of fractions and the difference between factors and terms. Although some could point out the factors and the terms, there seemed to be no strong connection between factors and simplification.

The teacher emphasised the difference between factors and terms and that simplification is division; dividing both numerator and denominator by the same factor. It might be that the poor sample of tasks, hindered them from developing conceptual understanding (Kilpatrick, 2001) for simplification of fractions with binomials. The teacher used the opportunities she had, to emphasise that binomials were entities and as such factors.

Implicitly she tried to help them to see those expressions as objects (Sfard, 1991) or as procepts (Gray & Tall, 1994).

Students' main problem might be that they had only one strategy when multiplying fractions. With few exceptions they performed the multiplication before they searched for common factors. When binomials were included they then had terms not factors; they nevertheless simplified by partly cancelling. They seemed not to be used to reflect or consider the structure of the task. This is a sign of a purely operational interpretation of the expressions; not seeing the expressions as objects (Sfard, 1991; Sfard & Linchevski, 1994b). Students' lack of proceptual thinking (Gray & Tall, 1994) might be one reason for their inefficient strategy.

The teacher tried to convince them to change this strategy and look for the structure of the tasks (Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999) before carrying out the operations. Students were observed disagreeing with her in the episode with Tore in section 9.2. In that episode it might have been the small size of the numbers involved, that led students to see no need for a change in their strategy. This is also an example of how difficult it is to change thinking when procedures without connections are already learned (Kilpatrick et al., 2001). This form of thinking seemed to be the reason for errors both when working with ordinary fractions and with equations.

In general, addition and subtraction went well, although some students were seen to struggle in the first tasks, when letter symbols were included in the denominator (Atle in section 9.3). The test results show that all students succeeded in finding a common denominator. It was evident though both from task solutions and from responses during plenary that the difference between expansion and multiplication was quite blurred for some students (see at the end of section 9.2). The teacher was aware of this and addressed the issue.

Another issue was the problem of subtracting or adding fractions with binomials in the numerator (no assigned addition/subtraction task included binomials in the denominator). The students were reminded by the teacher to handle the binomials as entities, and gave the advice to place the numerators in brackets. In the test in September, most students using brackets changed the sign if the fraction was to be subtracted (see section 9.3 and table 9-5). However, a third of the students did not change sign; bracket or not, an indication of not regarding the binomial as an object (Sfard, 1991) and of having problems with the sign rules (see section 8.3.5).

One can question though, whether the use of brackets, was just another rule for the students, like the rule for changing sign when the bracket had to be subtracted. From the test task about simplification it did not seem to

help them to regard the binomials as entities or objects in relation to multiplication and factorisation.

There was little variation in the assigned textbook tasks, and none of them seemed to challenge students to change inefficient strategy, that is to proceed to 'go on doing' without first reflecting on the structure of the task. The students were not offered fraction tasks explicitly challenging their conception of fractions, or which asked them to reflect over the reason why the fraction rules work.

Unfamiliar fraction tasks revealed that most students struggled when the tasks involved binomials. This was evident when they should simplify a complex fraction (test task section 9.3), and when they should solve the equation $\frac{x+1}{x+4} = \frac{4}{5}$. None in the autumn and only one in the spring solved the latter task correctly. (Algebra test task section 9.4).

These examples and the fact that some students equated the fraction $\frac{1}{15}$ with the natural number 15 (section 8.4) illustrate a lack of conceptual understanding of the concept of fraction. Instead they used self-created procedures violating mathematical laws.

The concept of equivalent fractions was not explicitly focused upon in the textbook, although some fractions with the same value had the same location on the presented number line.

The responses revealed a lack of conceptual understanding of the different components in rational binomial expressions and how these relate to each other (Olteanu, 2012; Seng, 2010), a lack of structure sense (Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999) and proceptual thinking (Gray & Tall, 1994). The students had, however, been offered very few tasks and thus little or no opportunities for such thinking during the observation.

The focus was on learning rules and not on why they work; promoting instrumental rather than relational understanding of mathematics (Skemp, 1987).

14.1.7 Equations

The teacher emphasised the relational aspect by focusing on the left-right balance, when she introduced equations. She accepted, however, students' rule of thumb, *change side-change sign*, but demanded that they perform the same operation on both sides of the equal sign. The textbook equated those two approaches, which are two completely different ways of thinking. To execute the same operation on both sides is based on the symmetric property, which is absent in the transposing activity (Kieran, 1989).

It was evident from the classroom that the symmetric property was not part of all students' concept images of equations (Tall & Vinner, 1981)

although the teacher explicitly emphasised that they always performed the same operation on both sides. One example is the episode with Tone and Ruth (section 10.2),.

When equations were introduced, some students expressed clearly that they had not a relational conception of the equal sign (section 10.1). Some of them wrote more than one equal sign indicating the same (see Selma's example section 10.2) during the first weeks of observation. They changed their writings but whether the students then had changed their way of thinking, may be questioned. They continued saying that they changed side and changed sign. Some students, though, 'forgot' to change sign. Both responses indicate a transposing procedure (Cortés & Pfaff, 2000; Kieran, 1989). A couple of students went back to the same writing later.

When the unknown had visible coefficients at the end of the solution process, it was evident that the students executed the same operation on both sides, which indicates that the students had two rules, one for subtraction/addition, and one for division/multiplication. It might also be that they had one rule; to move numbers and expressions from one side to the other as if they were physical objects (Lima & Tall, 2006b, 2008).

The equations in the textbook tasks were as in the other sub-chapters exercises according to Niss (1993) and thus similar to the example tasks in the textbook. However, some of them might be categorised in the moderate level of procedural complexity (Hiebert, et al., 2003). In general, the problems revealed were mostly related to fractions, the minus sign, negative numbers, or the problem to differentiate between factors and terms (section 9.3).

The teacher introduced cross multiplication as a tool when solving ratio equations. For some few students this seemed to complicate the process of equation solving, as seen in Ruth's solutions in the classroom (section 10.3).

The properties of equations, the symmetric, the reflexive and the transitive properties were not explicitly presented. The concept of equivalence was not addressed in relation to equations either, and none of the students were seen to control their answers; a procedure which could have prevented many errors, and promoted more meaning.

14.1.8 Inequalities

The teacher related the topic to the notation of number sets, which was not presented in the textbook related to inequalities. The main point, however, was to learn the procedures to solve inequalities algebraically. The difference between the procedures for solving equations and inequalities was emphasised. The main point was: solve it as an equation; the difference

turns up when the coefficient before the unknown is negative, then the inequality sign has to be flipped. Both in the textbook and in the classroom, this rule was exemplified by a numerical example.

There were few tasks offered, and all were similar to the example tasks; and thus of low cognitive demand (Smith & Stein, 1998). In November, related to straight lines, one word problem was an inequality. All the students, however, solved it as an equation, and afterwards they reasoned, based on the solution and the context, as to how they should express the solution.

Although the number sets were introduced in the classroom and the meanings of those sets were 'taken as shared' (Voigt, 1994), it seemed that some students thought of the unknown as a specific unknown within the set (Olav in section 11.2.4). This could also be interpreted from the short exposition part in the textbook.

Some students solved the inequalities in exactly the same manner as they solved equations, but with inequality signs. They talked about x is. Others followed the rules for inequalities, but wrote an equal sign at the end. On the test in September 46 % of the students followed one or other of these options.

The observations in class and the responses from the tests, illuminate the problems caused by the juxtaposition between the solution processes of equations and inequalities. To differentiate between the two, is important in order to develop rich concept images (Tall & Vinner, 1981) of the two concepts. The teacher laid a foundation in that she related inequalities to sets of numbers. There seemed to be consensus in class that the solution included all numbers within the solution set. This meaning taken as shared, was not part of each individual's conception though. The teacher, however, was not so concerned since the topic would be presented later, in relation to graphs and functions. Then the concept of inequality would become clearer for the students.

The students, however, might have benefited from working with the properties of inequalities and to compare these with the properties of equations before the work with functions and graphs. Then the students would have had an opportunity to go beyond the work of manipulating expressions in order to come to a result. They would have a possibility to reach a new level of understanding of the concepts. Students' concept images (*ibid*) are always constrained by their experiences, and an emphasis on one part of a concept might constrain students' possibility to develop a concept image in line with the formal definition of the concept.

14.1.9 Word problems

To convert texts into mathematical notation is the generational activity and the meaning giving activity in algebra according to Kieran (2007).

The work plan included only five word problems, and few students, solved these. It seemed that they preferred to solve the tasks already expressed in mathematical notation. Some of the few students solving these tasks, struggled to convert the problems into mathematical notation.

There was no exposition in the textbook related to word problems. Thus it seems that the competence to convert text into equations was taken for granted in the textbook, although it was stated as a learning goal for the first chapter.

Ratio problems were offered both as example tasks and as assigned tasks (section 12.1). Some students responded that they would not solve these as equations. One issue was that the numbers invited arithmetical thinking, another was that the students seemed unfamiliar with writing ratio equations. Thus some wrote formulas when prompted to write equations (Bill in section 12.1). Students' informal proportional thinking seemed to be correct though. In the test in September only a small number of students, 11 %, solved the ratio problem as an equation, most students just wrote the result, presumably after mental calculations.

In the other assigned word problems, it seemed hard to tease out the relation between the different quantities; known and unknowns. Some very few students solved them without any problem, while some, such as Tone and Ruth, had to be led by the teacher through each step in the process (section 12.1).

One test task (September) was an age problem. 14 students (dataset 27) set up an equation, 11 of those came to a correct equation and a correct solution. It might be that it was harder to solve this problem arithmetically, causing students to see the benefit of writing equations. In addition, students had been practicing more equation solving.

Later in the autumn when working with straight lines, the students were more successful. The reason might be that they used the formula for linear functions, and thus the relations were given. At that time too, students were observed to struggle to differentiate between values of an object and the object themselves (section 8.1). The assigned word problems not related to linear functions were solved by only few students at that time (22% solved some of them).

14.2 Discussion

In this section the findings will be discussed. The first part is devoted to a discussion of the mathematics offered in the classroom. The next section

discuss students' experiences, before the learning situation is discussed in the last section.

14.2.1 The mathematics offered

The first research question leading this study has been:

What mathematics are students offered in the classroom for learning, or to consolidate already learned mathematics as they enter the upper secondary school?

In the chapters 8 to 12, the mathematics offered is presented together with students' responses. In those chapters the findings are presented in accordance with the different topics covered during observation. Those topics constitute one level of the mathematics offered. The next level is to discuss what kind of mathematics is offered.

The mathematics offered in this classroom during the first weeks of term, is mostly focused upon rules and procedures; the transformational activity of algebra (Kieran, 2007). The topics were not new to the students. Therefore it might be that the textbook authors regarded the actual concepts to be known, and that the recapitulation of rules is what was seen to be needed to develop fluency in the application of known procedures.

The tasks, except for the word problems, were, of low cognitive demand (Stein & Smith, 1998). In addition, the tasks were sequenced according to the topic at hand. One student expressed explicitly, that his strategy was to check the location of the tasks, in order to find out what to do. Love and Pimm (1996) warned against such a sequencing (see section 6.2), in order to avoid fragmented knowledge.

According to Bills and colleagues (2006), example tasks are of crucial importance for the learning of mathematics, and they emphasise the importance of giving students a wide variety of examples.

There was little or no variation in the sample of textbook tasks and examples. Most of them were non-contextual tasks, written in mathematical notation to be solved by applying known procedures. Both the example tasks and the assigned tasks were sequenced from easy to more complicated. The possible complexity was of procedural character (Hiebert, et al., 2003). The assigned tasks were exercises (Niss, 1993) and could be solved by imitating the example tasks in the textbook. Exceptions were the word problems. There were also no counter examples or non-examples presented, which is important according to Bills and colleagues (2006). Only one way to solve the example tasks was presented, and reflection was not encouraged.

In the classroom, however, the teacher presented counter examples, when students' hypotheses of the next step(s) in the solution process was

against mathematical conventions (one example in section 9.1). These counter examples were pure number examples, and appeared to help some students to gain conceptual understanding (Kilpatrick et al., 2001).

In this study, it is shown how textbook expositions, example tasks and assigned tasks failed to address important issues. The example tasks, and also the assigned tasks related to simplification of fractions including binomials, are examples of how the selection even can lead to, or maintain misconceptions. The structure of those tasks made students feel that they were competent reaching a ‘correct’ answer although acting contrary to the mathematics they were intended to learn (see section 9.1). The structure of the tasks prevented students from being challenged and so to experience a need to change their incorrect concept images (Tall & Vinner, 1981).

The algebra learned prior to the start of upper secondary school was allocated little time for recapitulation. The aim seemed to be to hasten through a series of tasks to freshen up already known rules. The introduction to the first chapter in the textbook is symptomatic of this attitude:

Supposedly, you will not need to study everything in this chapter thoroughly. Get an overview of the content, and look into it later if you need to (p. 8).

Some sub-concepts of fractions are presented in the textbook, some operations of negative numbers are illustrated on the number line, and some rules are exemplified by examples from ‘daily life’; an indication of a willingness to give a basis for some of the rules. However, important basic concepts and properties are not addressed. Examples thereof are equivalence and the properties of equations and inequalities. Most rules are presented without any connection to conceptual understanding (Kilpatrick et al., 2001). According to the textbook, “The aim of this chapter is to give you a basis for the mathematical work in Vg1” (p.8).

International comparative studies, TIMSS and PISA, identify algebra as the weakest topic at the end of lower secondary school in Norway (see section 2.1). The tests administrated by the Norwegian Mathematics Council show that first year Norwegian university students, entering studies requiring mathematical knowledge, are still struggling with basic mathematics after ending upper secondary school (section 2.9).

Thus it should be known that many students entering the upper secondary school need a new opportunity to go into basic mathematics concepts. They need more than to practice rules. Those rules must be connected to conceptual understanding (Kilpatrick et al., 2001).

Educational studies suggest that the conceptual basis for rules and procedures should be explicitly presented (Hiebert & Grouws, 2007; Kilpatrick, et al., 2001) in order to create opportunities for learning.

In their work about concept development, Tall and Vinner (1981) emphasised that the students have no chance to develop a rich concept image without rich experiences with the concepts. Hiebert and Grouws (2007) found from their review of former studies, that conceptual understanding requires that concepts are explicitly presented, and that students be offered tasks which cause them to struggle; tasks with a high level of cognitive demand (Stein, et al., 2007; Stein & Smith, 1998). It seems that the textbook expositions and the tasks design are not based on these research findings.

The teacher, introduced some issues of conceptual character, for example: the presentation of terms and factors (section 9.1), equations as relations (section 11.1) and inequalities related to sets and set notation (section 11.1). In situations when individual students clearly were challenged and asked questions, they seemed to make progress. Examples are Atle in section 9.3 (common denominators) and Else in section 8.4 (powers). Selma, who was observed not to know the difference between factors and terms, and thus had problems in following the correct order of operations, and to factorise binomials asked, for help. In the interview (section 8.3.6) she told that she had learned the difference. She was one of five students who correctly solved the test task simplifying fractions with binomials (section 9.1).

The mathematics textbook is closely followed by the teacher and by the students in their work with tasks. The learning goals for the first chapter was stated in the start of the textbook,

When you (singular) have read this chapter, you should be able to

- calculate with powers and numbers in scientific notation
- solve linear equations and inequalities
- convert a practical problem into an equation
- interpret, process and evaluate the mathematical content in various texts (p. 7)

The textbook and the teacher introduce the rules for arithmetic calculations, the negative numbers and fractions before presenting equations and inequalities. The two first goals are thus addressed. The third goal, to convert word problems into equations, is not given any exposition, but some few tasks were offered.

The last goal in the list might have been thought of as being implicit in the work with the example tasks and assigned tasks. There were no explicit references made to interpretation or evaluation of tasks, results or texts in the textbook. Nor was there any mention of reflection about results or premises. Neither was it set as a learning goal in other chapters of the

textbook, and it was not suggested to read other texts apart from the textbook.

The mathematics offered in the textbook in this part of the mathematics course is thus highly procedural (Hiebert & Lefevre, 1986) focusing on rules and procedures, the how in mathematics (Skemp, 1987) and with little or no connection to conceptual understanding (Kilpatrick et al., 2001). In the classroom, some conceptual issues were though addressed by the teacher.

14.2.2 Students' experiences

Students' experiences cannot be observed directly. Thus, students' responses, oral and written, are the means illuminating students' experiences. Students' responses are presented topic by topic in the chapters 8 to 12. In this section, these responses will be discussed on a more general level.

The second research question leading this study was:

What experiences of the mathematics offered, do students reveal through their responses?

The students in this study have finished ten years in compulsory school, and are in the transition from one school to another school. Thus, the responses observed, are to a large degree, responses also on the mathematics they have been offered in lower grades.

The teacher asked for students' pre-knowledge in the topics presented, and the responses revealed that a large group of the students have partial conceptions and limited experiences with the topics in focus during the first weeks in upper secondary school.

The students mostly knew the rules, but some responses gave evidence that those rules were not connected to conceptual understanding. One example is the sign rules for brackets (section 8.3.5).

When the teacher asked follow up questions, or invited for argumentation, the students' arguments were held on the superficial level of rules, often expressed with keywords (see chapter 13).

The heavy emphasis on rules did not help all students to develop procedural fluency (Kilpatrick et al., 2001) in the work with equations and inequalities. In the regular test in September, students should solve three equations. For each of those tasks, less than half the students solved them correctly, and only 19 % solved all three test tasks correctly. In December, half the students solved the equation test task correctly. In September, only a fifth solved the inequality task correctly. In December, approximately half the students succeeded.

The problems were mostly related to basic mathematics; fractions, the minus sign and the difference between factors and terms. In addition, the

blurred difference between equations and inequalities seemed to cause problems when solving inequalities.

None of these issues were addressed in the textbook. The teacher, though, introduced some of them, but it seemed difficult to engage the students in issues they meant were learned on lower levels (Kilpatrick et al., 2001); an example is teachers' suggestion to search for the structure of the task before multiplying fractions (section 9.2).

The letter symbols and possible interpretations of these symbols (Küchemann, 1981), were not addressed. Students' responses indicate that nearly 40 % interpreted letter symbols as objects (section 8.1). The students in general accepted letter symbols as specific unknown. There is an indication that most students interpreted letters as general numbers, but both from a test task and from interviews, there is a question whether all students accepted that different letters could take on the same value (section 8.1).

In general, students did not use the formal mathematical vocabulary. They had a limited grasp of mathematical language, and talked about 'times up and down' (section 9.3), which might be one reason for some students not to differentiate between multiplication and expansion of fractions. They talked about 'x is' in the meaning of 'x is equal to' when working with inequalities (11.2.4), and we have 'two x-es', meaning x squared, and we 'change side and change signs' when solving equations. This way of talking is perhaps a consequence of blurred conceptions of the actual concepts, or it may lead to blurred conceptions.

The concept of equivalence and the different properties of equations and inequalities were not presented, neither in everyday Norwegian language, nor in formal mathematical notation. Whether they were known from lower grades is difficult to say.

In the process of developing proceptual thinking Gray and Tall (2007) emphasise the importance of abstracting and compressing expressions into entities or thinkable objects. In this process they point to the importance of naming. In order to think of and talk about a concept, it must be named. The students in this study do not seem to have words for some important concepts underlying the work they perform. Examples are factors and terms (section 9.1); although introduced by the teacher, and equivalence. There is evidence of how difficult it is to ask and to explain when the students have no word for what they try to express. The episode in the classroom when Ronny tried to explain that cross multiplication might be a problem when working with inequalities, is an example of this lack of concepts and naming (section 11.2.3).

When working with binomial numerators, it was evident that few students showed evidence of proceptual thinking (Gray & Tall, 1994), or

looked at the binomials as objects (Sfard, 1991). It might also be questioned if the same was the case with brackets.

The responses give evidence that students are aiming for answers and results more than to reason about why they reach these results. The questions they asked, were mostly specific to the individual task, few questions were asked about general mathematical issues. Both their oral responses, and the way they worked, indicate a procedural view of mathematics. This view was confirmed in the interviews. When asked what their strategy or thoughts were when going to solve a task, one student answered: “I just want to finish it, to try to find the correct answer”. The others did not respond to the question, they just started the work telling about their actions as they proceeded. There was no evidence of prior analysis about what to find, or what sort of task was to be solved. When asked if they afterwards thought about what was found, they all said “no”

Most students, though, expressed a positive attitude to the learning situation, and worked eagerly solving tasks in the classroom.

14.2.3 The learning situation

The third and last question has been:

How does the setting of the mathematics classroom influence students' experiences of the mathematics offered?

Included in the expression ‘the setting of the mathematics classroom’ is the mathematics textbook in use, with its examples, tasks and work plan. Teacher’s expositions, and the interactions going on in the classroom are other components of this setting. In addition, the computer with its programmes in use is part of the setting. In this section, these factors will be discussed in relation to students’ responses on the mathematics offered.

The textbook was a steering factor in this part of the course²⁷ and it is evident that concepts and principles crucial for students’ learning were not addressed. The understanding of the minus sign with its different functions and meanings (section 8.3.7), and the conceptual basis for the sign rules are examples of issues not addressed.

The expositions in the textbook and the tasks offered, indicate that the textbook authors mainly have interpreted the goals to be to learn techniques to solve tasks. They have emphasised and highlighted rules to follow. Even the number properties, the commutative, the distributive and the associative properties were presented as pure rules to be followed

²⁷ In the second part of observation when working with functions, the textbook was not followed so tightly. The teacher offered other tasks and other examples than those in the textbook.

when calculating with numbers and letter symbols. The formal notions, though, for these properties were not presented.

The example tasks were mostly context free, solved in one way only, and are classified as exercises (Niss, 1993). There was thus little or no possibilities offered from the authors' side to discuss the basic underlying concepts for the applied procedures; to link procedural fluency to conceptual understanding (Kilpatrick et al., 2001).

Except for the few word problems, the mathematics was related to the transformational activity of algebra (Kieran, 2007a). The activity of converting word problems into equations, represented the generational activity of algebra, but was not devoted any exposition in the textbook in this first chapter.

As stated in section 6.2, it is not possible to predict a direct link between the textbook and students' learning. The implementation of the learning material has to be investigated. However, Stein and colleagues' (2007) review of research in this area, confirm that content matters, and that it is likely that what is not included will not be learned. In this study, the implementation of the textbook examples and tasks have been studied, and students' responses confirm that the missing links in the textbook between the rules presented and a conceptual understanding of those rules, is also a missing link for at least a large group of the students in this class.

In the classroom the students used their computers, and pen and paper were absent during the work with the first chapter. The topics in focus in this part of the course were not made easier or more understandable by the use of the computers, something the teacher was highly aware of (see appendix 2.4.1 about teacher and computers). The computer were used as a tool for writing in this part of the course. What was a problem earlier, remained a problem. Although the students later in the course were introduced to CAS (Computer Algebra System) it is a question whether they had a satisfactory understanding of the reasons for the answer the computer generated. The basis for the transformations of algebraic expressions was not a theme in the textbook further during the course.

There were few questions and problems related to the use of the computers and the software. For each topic, the teacher solved one task in cooperation with the students; she writing on her computer, the students on theirs. During these solution processes, the problems students had related to the computer and the software were resolved. The teacher patiently answered questions and showed on her projected computer screen what to do and how to do it. She also went around if that was needed. The service system in school functioned well.

One benefit of the computer in the work with the first textbook chapter, was that students when working with tasks on the computer, were so

absorbed by what was on the screen that they many times forgot to take breaks. It was also easy to follow their process towards solutions; they used the copy functions and did not hesitate to write many lines, and it was easy to read. The stored computer files gave them easy access to solved tasks.

Later in the school year, they expressed that the internet connection turned the computer into a distraction during the mathematics lessons. The teacher was well aware of this, and she had taken precautions. She had in advance created computer files, the so-called 'learning book' including the task process through to the solution of the example tasks. These files were located on the web site so that students could download them later (see section 7.3.5). By so doing she encouraged them to focus on the white board and the classroom interaction, and most students participated willingly.

In chapter 13, the patterns of interaction in the classroom have been described and categorised. The teacher was sensitive to students' pre-knowledge and responses, and she built on them in line with the IRF pattern typical for Norwegian classrooms, described by Klette (2010). She, also drawing from her extensive experience, brought in issues not addressed in the textbook, those she was aware of as being problematic.

It is evident that both the teacher and the students were mostly bound to the tacit didactical contract of a traditional mathematics classroom (Blomhøj, 1994). Taking into consideration that the class is observed in the first weeks in upper secondary school, it might be concluded that the students have brought their expectations from lower grades into this classroom. Students, who were asked, confirmed this assumption. This contract seemed also in this class to lead to a form of mathematics closely bound to the mathematics classroom with responses short in style, based on rules and conventions rather than on structure and relations and reasons why rules work.

In the episodes when the contract was broken, some students immediately resisted. One example is the teacher's resistance to follow students' suggestions to multiply before simplifying fractions. She wanted them to change strategy and be more flexible (section 9.2). When working with later chapters in the textbook, she also broke with the contract, let them work to find regularities on their own and reflect on them. The students then told that the optimal learning situation was to work with tasks after the teacher had told them what to do, and how to do it.

The students were focused on the correct answer. This was evident when they asked for help. Most students wanted to find the errors in order to correct them, not to find out why they made these errors. This questioning and some students' tendency to align their answer with that in the

textbook, might indicate that at least some students had an instrumental approach to learning (Mellin-Olsen, 1981). The motivation might not be an interest in the mathematics itself, but to demonstrate some knowledge. In this study this means to have a good mark or grade. The answers in the questionnaire (section 7.3.4), to the question: Is mathematics important for you? confirm such a view. None of the twelve, who responded, showed signs of personal interest in the subject as such, it was even seen by some as a hindrance to be overcome, in order to get access to further studies.

The textbook taking for granted that the needed conceptual basis for the work with procedures and rules in the first chapter are known to the students, and the provided tasks of a low level of cognitive demand, did not challenge the students to reflect over the work they carried out. Some tasks even promoted misconceptions, such as the tasks about simplifying rational expressions (section 9.1).

This fact together with the tacit didactical contract, the task discourse, and the patterns of interactions thus limited the teacher's opportunities to provide rich experiences with the concepts. Although the teacher from her teaching experience, knows much about the problems students have, related to elementary algebra, she is hampered by the example tasks in the textbook, the assigned textbook tasks which all are of a low cognitive demand (Smith & Stein, 1998). She is also impeded by the task discourse (Mellin-Olsen, 1996) which is reinforced by the work plans.

Another problem expressed by the teacher, is the short time frame allocated to this recapitulation. She told that the curriculum (K06)²⁸ included more topics than the former (L97), and thus she felt even more stressed timewise. The teacher knows that students have many problems. Her dilemma is how to help students to resolve those problems within the limited time, knowing that there are many more topics to cover which are new to the students²⁹.

The setting of the mathematics classroom is thus not optimal for providing opportunities to learn or to consolidate already learned mathematics in the start of the upper secondary school.

²⁸ The two curricula are outlined in chapter 2 section 2.2.1 and 2.2.2.

²⁹ The school has later changed textbook, one more lesson is allocated during the school year to give students extra time with basic mathematics.

15 Conclusion and implications

In this study, the mathematics offered was highly procedural (Hiebert & Lefevre, 1986) focusing on rules and procedures with few or no connections made to conceptual understanding (Kilpatrick et al., 2001). Students' responses reveal that a large group of students in this class entered the upper secondary school with partial conceptions and errors reported in studies done on lower levels. It is clear that the mathematics offered does not help these students to develop conceptual understanding for the rules and procedures they are set to practice. Thus the students do not reach the goal of procedural fluency (ibid) either.

Concepts and properties are not explicitly expressed; a prerequisite for conceptual learning (Hiebert & Grouws, 2007; Kilpatrick, et al., 2001). All tasks are solved in one way, and the low level of cognitive demand does not challenge students to raise important questions about conceptual matters. The result is that students' experiences with concepts and properties are limited. The prerequisites for development of their concept images (Tall & Vinner, 1981) are missing, and the result is a fragmentary knowledge in the area of basic algebra.

This situation seems to be due to poor resources. It is also due to the tacit didactical contract (Blomhøj, 1994); the students' expectations to what mathematics and a mathematics classroom is. The limited time allocated this recapitulation is another reason.

The students in this class seemed to be on a journey (Mellin-Olsen, 1996) through, at least, this first chapter in the textbook. They followed the work plan and were guided by the teacher and peers, working with topics they had met on lower levels, while the time available was a strong constraining factor.

The responses give evidence that students are aiming for answers and results more than to reason about why they reach these results, and that their experiences of mathematics is that mathematics is about rules and procedures. There are also indications of students' holding an instrumental approach (Mellin-Olsen, 1981) to learning mathematics.

From the observations in class, it seems to be hard to change this view of algebra and mathematics in general, as long as the participants stick to the tacit didactical contract (Blomhøj, 1994).

15.1 Implications - teaching resources

The Norwegian school system with little or no contact between the two school levels, the lower and the upper secondary school, might be one reason for the way the topics are treated both in the classroom, and in the

textbook. Neither the authors, who are often upper secondary teachers, nor the teachers in upper secondary schools have much knowledge about how the topics have been presented and worked with on the lower levels.

The textbook and the way the recapitulation of basic mathematics is presented and tasks offered, is said to be one reason for the problems revealed. Other available Norwegian textbooks for this course do not differ much when it comes to the way they treat the recapitulation of algebra. None of the most frequently used textbooks (Andersen, Jasper, Natvig, & Aadne, 2006; Oldervoll, Orskaug, Vaaje, & Hanisch, 2006; Pedersen, Erstad, Heir, Borgan, & Engeset, 2006; Sandvold et al., 2006) go into why the rules for fraction operations work, or present fractions as numbers. One book presents opposite numbers (the additive inverse), but none present the function of the minus sign as ‘taking the opposite of’, and none give reasons for the sign rules. All the textbooks present the rule for equations ‘change side, change sign’ although they start with an explanation of doing the same on both sides. There is no explicit explanation of equivalence in either of them related to this basic algebra, and there is no encouragement to control the solutions of equations. This shows that the textbook used in this study was not exceptional.

This study has shown the importance of the tasks offered. In Norway there is no central authority controlling the textbooks (see section 2.5), and thus the responsibility lies with the teacher and her colleagues. In the daily work the teacher feels pressure from many instances, with many individual students, a heavily-felt time pressure and demands from school authorities, students and their parents. This additional responsibility for what is presented in the textbook, and what is afforded in each task, is an almost impossible task for a teacher, especially when a new syllabus is being introduced.

In summer, 2015, there were bold headlines in the newspapers about the result on the final exam in lower secondary school. One headline was: This year’s mathematics exam, the weakest ever. Kjersti Væge, the leader of the National Centre for Mathematics Education in Norway, commented³⁰ that: “The mathematics teaching for students is mostly about learning rules – and finding a correct answer”. In addition, she claims that there is a need to emphasise understanding in mathematics, with not so much focus on memorising rules. These were comments on the situation in lower grades, but also from this study it seems to be that memorised rules was steering students’ work.

³⁰ Kjersti Væge in the Norwegian newspaper, VG, 2015.07.02. The translation into English is mine.

In her work studying students' difficulties with learning algebra, Naalsund (2012) pointed at a number of issues that should be given attention in the teaching of students in the lower secondary school. Some of them are highly relevant also in upper secondary school, although it might be harder to engage students in the activities since they have already met and applied what should have been learned with more conceptual understanding from the outset (Kilpatrick et al., 2001). One such important issue according to her, is the equal sign. The equal sign representing the symmetric, reflexive and transitive properties is the basis for students' conception of equation as a left-right balance. Another issue was the lack of understanding of equivalence.

In a meeting in the TBM/LBM projects, after I had reported my findings about equations, teachers in both lower and upper secondary school expressed that they had not been aware of the importance of the meaning of the equal sign. This indicates that also many teachers are not conversant with research in the area of mathematics education. It is very important that mathematics teachers be made aware of basic underlying mathematical concepts, particularly those leading to common students' errors.

Despite the important work done within the KIM project about diagnostic teaching (see the introduction to chapter 4), the results and booklets produced seem not to be well known by Norwegian teachers, and have not influenced the introduction of algebra in Norwegian mathematics textbooks.

In the future there will still be a need for teachers in upper secondary school to meet students' need when it comes to number sense and to basic algebra, and to help them gain an understanding that goes deeper than to understand how the rules are to be used in familiar settings. All the strands in 'Mathematical Proficiency' (Kilpatrick et al., 2001) should be involved. For teachers to reach these goals they need didactical means, textbooks or other resources, affording research based expositions, example tasks and assigned tasks designed to challenge students and to provide opportunities for new learning to occur.

The work done by Swan and colleagues in England (Swan, 2006), give evidence that carefully designed teaching material and tasks, together with encouragement of teachers to use the material, have made impact on students' learning and on the beliefs and practices of experienced teachers. In the intervention study, ICCAM (Increasing Competence and Confidence in Algebra and Multiplicative Structures) teachers were enabled to use formative assessment in their mathematics classrooms (Hodgen, Coe, Brown, & Küchemann, 2014). As in Swan's project above, the teacher were encouraged to assess students' pre-knowledge and to offer appropri-

ate tasks challenging students. Tasks designed on the basis of former research findings were offered. The researchers (Hodgen et al., 2014) reported that the intervention in their study had a considerable positive effect on students' learning compared to control groups.

This could also be a way to enhance students' learning in Norway. Rich tasks offer opportunities for students on all levels of performance, and also well-performing students would have benefitted from tasks promoting discussion, reflection and conceptual understanding.

15.2 Implications for teaching

The study has identified several problems students have when recapitulating basic algebra. Those problems are exposed in the chapters from 8 to 12. One common implication for all topics is that in order to enhance procedural fluency, it is important to give opportunities to develop conceptual understanding for the rules and algorithms. One aim is then to offer challenging tasks promoting reflection.

Some issues, which should be attended to in the recapitulation of basic algebra, are listed below:

General issues

- Interpretation of letter symbols
- Invisible operation signs and coefficients
- The minus sign – and its different functions including ‘taking the opposite of
- Reasons for the sign rules and brackets as procepts
- Factors and terms and the differences between them in connection to simplification of fractions
- Scientific numbers – connected to the position system
- Explicit presentation of the concept of equivalence

Fractions

- Fractions as numbers, fractions as measure
- Explicit presentation of equivalent fractions
- Explicit differentiation between multiplication and expansion
- Reasons for fraction operations
- Focus on the different components in rational expressions
- Binomials as procepts

Equations

- Focus on the equal sign
- Focus on the properties of equations
- Emphasis on the relational structure – (work against ‘change side –change sign’).
- Check the solutions by use of the concept of equivalence.

Inequalities

- Present the properties, and make the differences between equations and inequalities clear.
- Focus on the result and its meaning
- Check the solutions by use of the concept of equivalence.

Word problems

- Explicit focus on the relations between the known and unknowns

Kieran (2007) categorised and described algebra in relation to activity. Norwegian classrooms are characterised by activity, but it is shown in this study that not all algebraic activity promotes learning.

Many of the issues listed above are mostly implicitly present in mathematics textbooks and tasks. The main point is to make them explicitly visible for students, and to offer challenging tasks promoting reflection. Olteanu's study (2012) about simplification of rational expression is a study of how explicit attention to known issues helped students to gain understanding of the different components involved, and thus also about the whole expressions.

The time pressure is heavily felt by both teacher and students, but the question is whether it is possible to implement some of the issues above by giving other more challenging tasks, tasks of a high level of cognitive demand, promoting connections. This requires new textbooks that go deeper and make more connections than the former textbooks³¹. Another suggestion is, to encourage students to ask more 'why' questions. Examples could be to ask students in groups to find arguments for the rules. It could also be to give different solutions – the whole process – of manipulative tasks and let them decide what is correct and what is not and to argue for their decision; with arguments that go deeper than referring to rules or procedures.

The aim of the teaching should be to promote reflection and metacognition. This means to break the didactical contract of a traditional mathematics classroom, in which task solutions are expected to be very short. The socio-mathematical norms must be changed (Yackel & Cobb, 1996), which is a challenge after many years in school. There is though a golden opportunity for change when students enter a new school.

³¹ In a meeting, the teacher told that the school has changed the textbook, and have allocated one more mathematics lesson per week, in order to focus more on basic algebra.

15.3 Contributions to and implications for the community of mathematical education

The aims in this study have been to investigate the mathematics offered, students' experiences of that mathematics, and how the learning situation influences students' experiences.

One of the approaches was to compare students' responses with findings in earlier research in the area of algebra learning. The result of this comparison shows that problems found in former studies, are problems also in this class. It means that the findings are not novel findings, but they add on to the rich area of research done on algebra.

In Norway, there have been no studies of this kind at the upper secondary school level. This study is thus first and foremost a study of Norwegian students in the context of a Norwegian classroom.

Some specific findings should be of interest for the community of mathematics education:

The minus sign

The way students treated the minus sign is revealed in many studies related to negative numbers, and to the phenomenon of detachment of indicated operations on lower levels. This study gives evidence that the latter phenomenon has a high frequency in this class. This came as a surprise on this level, and in 2014 the students entering another upper secondary school were tested (123 students) and more than 40 % failed in the task $a - 3a + 2a$ which was one task in the algebra test (section 8.3.6). Some classes in the same school have later been asked to solve similar tasks. When asked for their solutions, they appeared to fall into two groups; one correct and obviously one incorrect, and half the class supported each one. When discussing the results, some students declared that "both are correct; it depends on how you think".

These findings should lead to more attention to such phenomena in school on higher levels. Further research on this issue would be fruitful.

Another issue for further research is the introduction of the opposite number in the teaching and in the mathematics textbook.

Lack of strategic competence and flexibility in algebra

It was evident that students lack strategic competence, it was especially visible when multiplying/dividing fractions. When asked in interviews, the interviewed students showed clearly that they did not reflect on or analyse tasks before solving them. One reason might be the character of the topics and the kind of tasks offered. Another might be limited experi-

ences with example tasks, or a procedural thinking that one had to follow procedures only.

There has been little focus on strategic competence and flexibility in relation to manipulative work in algebra. In the class some students objected to the teacher's suggestion to search for possibilities to simplify before multiplying fractions, and they seemed to have only one way to solve tasks with no control mechanism at the end.

It seemed that lack of strategic competence and flexibility was connected to a reliance on learned procedures, to a lack of mathematical argumentation and a lack of understanding of equivalence.

This lack of strategy and flexibility is mostly connected to problem solving and to contextual tasks, but it would have been worthwhile to investigate the connection between the aforementioned issues in further research.

Similar research or intervention studies

In this study, the teacher was a competent and experienced teacher devoted to her work, and the students were eager to work and committed to learning in their sense of the word. This study has shown that the teaching material, the learned patterns of interaction and the tacit didactical contract were constraints to students' learning, although the students were keen to reach the learning goals.

The mathematics offered, students have met before, and it would have been worthwhile to study how algebra is introduced in classroom on lower levels.

As long as Norwegian results related to algebra and number are lower than desired, intervention programmes such as those reported from the ICCAM project (Hodgen, et al., 2014) and from the work carried out by Swan and colleagues (Swan, 2006) should be of interest also in Norwegian classrooms.

15.4 Afterword

The study is a case study of a Norwegian mathematics classroom. Based on the outline of the Norwegian context in chapter 2, I argue that this mathematics classroom is not an atypical mathematics classroom. This is the first and only existing study of Norwegian upper secondary students learning algebra, exploring the classroom mathematical activity in such an in-depth analysis.

The basis and validation for this study is Norwegian students' problems reported in TIMSS and PISA, related to number and algebra. In addition, the problems students still encounter with basic mathematics when entering university studies requiring knowledge in mathematics (Nortvedt,

2014), make the study relevant. The initiative for this study was also based on my own experiences seeing students struggling with basic mathematics.

The rich data collected is scrutinised and analysed not alone based on earlier research, but also on the interviews and conversations with the teacher about students' mathematics, about her own practice, and on students' responses through their work, in interviews, and in the classroom.

The thesis provides not only a rich description of what is going on in this Norwegian classroom, but relates the observations to findings in other parts of the world and analyses them based on theoretical constructs and views.

My findings confirm research findings from earlier studies about the learning of algebra. Those studies are mainly studies of students on lower levels, but they are still highly relevant as they deal with the pre-knowledge of upper secondary students. The study of this particular mathematics classroom is thus a contribution to the community of mathematics education. It provides an exploration of what strengths and problems exist, and how the learning situation influences students' experiences in this classroom. It gives evidence that what is not addressed cannot be learned. It also confirms that concepts, their properties and mathematical principles must be explicitly presented also on this level even though the students have met them before.

It reveals the potential for improvement in mathematics textbooks, provided they base their exposition on research results. The deep and detailed image of what is actually going on in the classroom has the potential to enlighten teachers and other researchers in a way which has not been available earlier. My wish is that Norwegian mathematics teachers can be inspired from the results of this study.

My activities with the data from this specific classroom and my research into the literature has inspired me in my own work to eagerly search for students' reasoning and to base my teaching on their thinking. It has also given me a solid basis in understanding how students may experience what is offered, what problems may occur, and why they occur.

16 References

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About the appendices

My study is a qualitative case study within the interpretivist paradigm, and it is not possible to apply the same quality criteria as for studies carried out within the positivist paradigm. In these appendices, I will provide a more thorough description of the context, the tasks, the tests and the interaction in the classroom.

In the first appendix the twin projects LBM and TBM are presented. The school, the class, and the teacher are presented in the second appendix.

A complete list of the data collection, both the first and second period, is included in the third appendix.

The fourth appendix is a presentation of the levels of cognitive demand in the Mathematical Tasks Framework (Smith & Stein, 1998).

In appendix 5, an example of the digital ‘learning book’ (a digital file) is presented; written by the teachers in the observed school.

Appendix 6 includes the tests; the algebra test they had in September 2007 and in May 2008. In addition, the regular tests in September and the test in December³² are included. The results are presented in diagrams.

Appendix 7 is the questionnaire sent to the students per e-mail, and appendix 8 includes an outline of the first textbook chapter.

The appendices 9 to 12 are appendices including students’ work and examples of the interaction going on in the classroom. These appendices are organised by topics as the chapters 8 to 12 in the thesis. The purpose is to provide the reader with a more thorough description of the learning situation, the tasks offered, students’ responses and examples of my analyses.

There is no appendix devoted to word problems. There was no exposition offered on this topic and all students’ work related to this topic is described in the thesis.

I have chosen not to include references.

³² The whole test is included, but only the tasks related to the topics in the first textbook chapter, are written with bold fonts.

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Appendix 1: The LBM and the TBM projects

The research reported in this thesis was carried out within the projects ‘Learning Better Mathematics’³³ (LBM) and ‘Teaching Better Mathematics (TBM), hereafter referred to as TBM and LBM.

TBM was a joint project involving a consortium of five institutions of higher education located in different regions in Norway. University of Agder headed the project. Each institution within the consortium was running its own local project, in which they collaborated with local schools and school owners. The main aim of the TBM project, as stated in the project description was:

To develop and improve mathematics education in Norway with special reference to the improvement of learning experiences for pupils in mathematics classrooms and to the education of new mathematics teachers.³⁴

The TBM project was a developmental research project with the aim to improve students’ learning, and at the same time investigate and report the characteristics and the nature of the developmental process and its outcomes.

At the University of Agder, the TBM project was connected to a local project, the LBM project. The two projects existed as twin projects, although LBM was supported by ‘Sørlandets kompetansefond’, and TBM by ‘The Research Council in Norway’ (Project No. 76442/S20). Through LBM, the TBM-project collaborated with regional pre-schools, schools and teachers within these schools. A council of school authorities in the region ran LBM.

The local council owns pre-schools, primary schools, and lower secondary schools, while upper secondary schools are owned by the regional county. This division of ownership and leadership has traditionally led to little contact and collaboration between teachers from different school levels in Norwegian schools. In that sense the project was unique, covering all levels of lower education.

Each participating school appointed a leader. These leaders, The B-leaders, met regularly together with the TBM-group at the university. Another leader group, the A-leaders, consisted of the TBM - project leader at the University and three leaders representing the owners of the participating pre-schools and schools. This leader group met regularly to discuss financial and management issues.

³³ The title LBM in Norwegian means ‘Lær bedre matematikk’ which literally translated means ‘Learn better Mathematics’.

³⁴ From the web-site: http://prosjekt.hia.no/tbm/ProjectDescr/AUC_PFoU_Proposal.pdf

Each participating school set its own developmental goal for the work within the project. This was important in order to promote ownership and leadership within each school. Although two leader groups existed, the organisation and the pace of the developmental work, was led by the participating teachers within the schools. Nothing should be imposed upon them from the group at the University. However, the members of the research group at the University supported and guided teachers according to the teachers' wishes and needs. Researchers visited schools when invited to school meetings or classrooms.

All participating members met regularly at workshops. In these events, TBM members often presented some theory, that the teachers through the B-leaders had asked for. Consistent with the conceptual basis of the project, however, much time was spent in small group activity. The groups solved tasks, mostly mathematical problems, but it occurred often that the tasks were didactical issues to be discussed. The group work included preparations for innovative work in the classrooms and reflection on experiences from such work. One of the aims was through the work with mathematical problems to find ways to implement them in the classrooms in a way that could make students wonder and ask questions. Another aim was to build long-lasting communities of teachers from different schools.

During the study with this thesis, I was employed as a doctoral student within the TBM-project.

Appendix 2: The specific context of the class

This appendix presents the specific school, the class, the teacher and the students. The information about the school is taken from the official web site and from conversations with teachers and the principle of the school. The presentation of the teacher and the class is based on audiotaped conversations both in school and in meetings within the LBM-project, and on videotaped classroom observations.

2.1 The school

The school is a large school in Norway. About 1200 young people were students. The school had more than 200 teachers and 50 other support staff. Although the school offered the more academic Programme for Specialization in General Studies, most students attended the vocational training programs. In the program for general studies, students had to choose courses within the program for Science and Mathematics.

Since 2004, the school had been running an internal project integrating computers in the teaching. The teacher in this study was one of two leading this project. The aim was that all students should have their own laptops. The students had to pay a fee or a deposit for the machines and the software to be used. When the students finished their course of studies at the school, the computers became their properties.

The school was divided into departments according to grades and study programs. In the spring of 2007, the school joined the LBM project, five teachers started to attend the LBM workshops regularly, and they continued with this until the end of the project in the autumn of 2010.

2.2 The classroom

The classroom was equipped with desks and chairs arranged in rows. Although looking like a traditional classroom, it can be described as a technology-enhanced classroom. All the students had their own laptops. Electric contacts for students to connect their computers, were hanging from the ceiling. In addition, electric cables were on the floor for students not located close to those from the ceiling. A wireless network operated throughout the whole school area. A video projector was installed hanging from the ceiling, and on the front wall, covering most of the wall, was a whiteboard.

Although the classroom was a paperless classroom, the class might be characterised as a traditional mathematics class during the time work-

ing with the first chapter in the textbook. The students at that time used the computers only as a tool for writing. The programs in use were MathType and Microsoft Word.

Together with a parallel class, the observed class had their own hall and a large ‘quiet’ room with desks having partitions for students to work independently and in silence. The students could work in this room after plenary sessions in the classroom. It was mostly the same few students who went there from lesson to lesson. The teacher sometimes went into this room to see if anyone needed of help. Also from time to time, I went in there with the camera and video recorded students’ work on their lap top screens and talked with the students.

2.3 The class

75 students starting in the 11th grade attended the course Vg1T (Theoretical Mathematics). Most of these students were aiming at studies in mathematics and science and then this course is required. Shortly after coming to school, they were asked to estimate their skills in mathematics. Based on those responses, they were divided into three classes. In the observed class there were from the start 17 boys and 10 girls. The course was allocated five 45 minutes lessons each week grouped in two blocks of two and three lessons.

15 of the students in the class were following a special study program following the ‘normal’ curriculum three days a week. The other two days they worked in industry. Those students had to spend one extra year at upper secondary school to prepare for further studies at universities. After 4 years, they had a general university and college admission certification, and in addition a trade certificate. This class was called 1TAF (Technical and general subjects)³⁵. Most of those students explained that they were aiming at education in engineering. All the students in the class, also the students following the TAF program, were hoping to complete the highest education in mathematics possible in upper secondary school.

When the lessons started, the students started the work quickly. They plugged in and turned on the computers without being told to do so. This contrasts to my experience both as a teacher and as an observer in other classes. Normally it takes some minutes for students to find the right notebook, the pencils, the calculator, the ruler and other necessary items. The students in this class had their ordinary textbooks, but apart from this, the environment was paperless.

³⁵ In Norwegian: Teknisk, allmennfaglig utdanning (TAF)

When the teacher in the second period of observation introduced linear functions, she wanted students to draw the graphs by hand. Then she had to bring paper and pencils since the students normally did not use these items in the class.

2.4 The teacher

The teacher had more than 20 years of experience. She had specialisation in mathematics as do most Norwegian teachers at upper secondary school (see section 2.4 in the thesis). Although she had a high achieving class, she was aware that despite good marks from lower secondary school, many students would struggle with algebra and mathematics in general.

Both from observation in class and from conversations with the teacher, it was evident that she cared about the students and tried to help them with their individual mathematical problems, no matter how basic they were. The challenge was, according to her, to balance between the time available and the demands from the syllabus.

Many times during our collaboration, she mentioned that the content in the new syllabus had increased compared to the former one, and that she had experienced this as problematic. There was just too little time available to let students have the possibility to become confident with the mathematical topics. She asserted that the time pressure she felt, many times prevented her from working as she would have liked to.

When watching the films from the classroom, she expressed that she was surprised that she seemed to be calm and patient in her work in the classroom. That was not always what she felt during the lessons. Her aim was to encourage the students to think for themselves, and to create situations promoting mathematical thinking, but sometimes, when many students were waiting for help, she admitted that she told them what to do.

Another experienced challenge, was to ask the right questions and ask them in such a way that she could meet students on their own level. Her aim during seatwork was to listen so carefully that she could find the roots of students' problems.

When talking about her way of teaching, she said that she searched for students' thoughts and their prior knowledge, in order to build on that. From her experience, though, she usually had a prior expectation of what would come. She welcomed comments and answers fitting in, but if they caused a shift in focus, she mostly asked for alternatives. Although being fully aware that some students are quiet during plenary sessions, she assumed they probably had some of the same ideas or questions as those who were orally active.

In order to promote students' learning, she emphasised the importance of teachers' dedication and enthusiasm for what they teach.

2.4.1 The teacher and computers in mathematics

The teacher and her colleague were the first in their school to introduce the use of computers in all lessons in mathematics. When the LBM-project started, they had already worked 2 years with classes where all the students had their own laptops. Both teachers emphasised that the use of computers does not make mathematics easier, and that a recurring problem is that often students do not use the computers as they should in the lessons. Students sometimes communicated with others, played games, or checked Facebook during seatwork. This was especially seen among first graders. Later on, most students realised that they had no time for this³⁶.

In order to keep the students focused on what was going on in plenary, the teachers had made what they called 'The learning book'. This was a collection of digital files containing the examples presented in class during plenary sessions. In addition, a summary of rules related to each sub-chapter was included. The teachers experienced that if students should write notes on their laptops during plenary sessions, it would be too tempting for the fast writers to sneak into other computer programs.

The reason for introducing the computers as students' main tool in their mathematical activity, was that digital competence is one of the basic competencies in the new curriculum. In mathematics, the advantage of working with functions and their graphs was emphasised by the mathematics teacher. A graph, she said, is drawn by just some few keystrokes, and therefore students have the possibility to get experience from many different types of graphs and functions. She clearly had seen benefits from this, in her mathematics classes in grade 13. Her experience was that those students could sketch graphs given the algebraic expressions. They were also seen to check algebraic results by drawing a graph. This did not happen automatically earlier without the computers.

In addition, she regarded it better to use computers than graphical calculators. Graphs are more visible on the computer screen, and it is easy to change units on the axis. Another advantage was that students' writings on the computers are easy to read. The copy function in Math-Type makes it easy for students to copy lines when solving algebraic

³⁶ In a conversation with a group of students in the observed class in the spring, the students uttered the same as was observed by the teacher. They admitted they had wasted a lot of time during the school year, but at the time of the conversation they had realised that there was no time for this. The aim for the following year was to be more focused on the mathematics. During the first period of observation it was rarely seen, but in the second period, students were observed going into other programs as soon as they had finished the assigned tasks.

problems. It is thus likely that students write more lines compared to when hand writing, and thus it is easier to see which operations are executed, and the order of those operations. This is an advantage since all tests and exams are digital.

For students it is easier to find their earlier work, since all material is stored in the computer. In addition, they do not forget calculators, notebooks rulers etc.; all are to be found on their laptops.

The school had a well-functioning service system for the computers, and the Wi-Fi functioned well throughout the whole school. The teacher appreciated the system and the service people. When a student had problems with the computer, it was delivered for service, and the problems were solved smoothly and fast.

2.4.2 The teacher and students' mathematics

The teacher meant that students often do not analyse the tasks, searching for the structure, neither do they form a strategy about how to solve tasks. Many of them just start doing. Mathematics for them is often fragmentised knowledge. The result is that they sometimes succeed and sometimes fail, even when solving similar tasks. Thus, she wanted students to express their reasoning, giving her the opportunity to help them analyse mathematical problems and tasks. She wanted to teach them to be systematic and strategic in their work, and to give them time to practice and work.

Methods can be shown and explained, the students, however, have to make the mathematics their own through mathematical activity. She questioned that students should explore all methods on their own; that would take too much time, and time is a constantly-felt constraint.

She expressed the need for students to understand, but she had also experienced how students can come to understand later. She said it thus is possible to practice rules and methods without understanding immediately; understanding can come later, and for some students it takes a considerably time to learn. The main thing is that the students themselves are committed to learning.

From three years' experience of teaching the same students, she had learned what to emphasise at the start in order to ease further learning in mathematics.

The start in upper secondary school gives students a new opportunity. She was aware that some students might feel unsecure starting in a new school. They have to find their roles in the new class. Some lose confidence in mathematics since their new classmates seem to be more successful. The teacher told about some students also in the observed class

with whom she had had individual conversations about such problems. Some students developed positively after individual feedback.

She expressed that she would like to encourage students to talk more mathematics. Students often do not see the point of communicating mathematics orally since for them mathematics, is written mathematics. She tells that her emphasis at the start of the school year is mostly on the formal writing of mathematics. The oral mathematical argumentation is then often not focused upon. She had also experienced that students when asked to write explanations, found it difficult to know what to write. They were not used to giving reasons for their actions.

Another problem she mentioned, was that even in grade 13 students have problems using correct mathematical notions. They talk about 'minus-ing', multiplying 'up and down' etc. To come to grips with this, she let them write definitions in their own words, before writing them in precise mathematical language.

She expressed the need for summing up and showing connections. This is what she does after each textbook chapter. Again, it is a time constraint because she then has to push students to work faster, to free time for this summary. For students it is often before tests and exams that they see these connections as being worthwhile since they then are motivated to work with the mathematics as a whole not only as fragmented parts as in the textbook.

2.4.3 The teacher and our collaboration

The teacher is positive, and open for new things. This is evident in her involvement in the internal IT-project, and in her participation in the LBM project. She has also had former experiences with observers in her class. Several times, she told, that she just forgot that my colleague, the cameras, the audio recorder and me were present. She also confirmed that the students did not change behaviour when observers were present in the class.

She said that she used a lot of time in planning her teaching, and that our discussion, and the discussions in the workshops at the university, gave her ideas for the work in the classroom.

Appendix 3: Overview over data collection

Table 3-1: Overview of data (R for researchers and T for teachers; T1 is the class teacher, R1 is me)

Date	Event/Purpose	Kind of data	Present
070508	Meeting: Information about the project	Audio	R1, R2, R3 Mathematics teachers
070619	Meeting in order to start cooperation	Audio + notes	R1, T1, T2
070827	Planning the cooperation	Audio + notes	R1, T1, T2
070904	Classroom observation (2 lessons)	Audio + 2 video + notes	R1, R2
070904	Algebra test	Test on paper	
070910	Classroom observation (3 lessons)	Audio + 2 video	R1, R2
070911	Classroom observation (2 lessons)	Audio + 2 video	R1, R2
070917	Ordinary test algebra + numbers	Students' test on computers	
August-Sept.	Students work on textbook tasks (18 students)	Computer files	
070918	Short meeting with T1	audio	
070925	Interview with T1, Watch video	audio	R1 + T1
070925	Interview with five students	Audio + video + students' notes	R1
071030	Meeting	Audio	R1, T1, T2, T3, T4
071030	Meeting to plan linear function	audio	R1, T1
071106	Meeting to plan linear function	audio	R1, T1, T2
071112	Classroom observation (2 lessons)	Audio + 2 videos + students' notes	R1 + T1
071113	Classroom observation (3 lessons)	Audio + 2 videos + students' notes	R1 + T1
071119	Classroom observation (2 lessons)	Audio + 2 videos	R1
071126	Classroom observation (2 lessons)	Audio + 2 videos	R1
071127	Classroom observation (3 lessons)	Audio + 2 videos	R1, R5
071204	Ordinary test Linear functions	Students' tests Computer files	
071218	Ordinary test All topics taught in the autumn	Students' tests Computer files	
080122	Interview teacher + class meeting	Audio	R1, T1
080310	A questionnaire about the 'learning book'	Audio	Students
080519	Interview 4 students	Audio + students' notes	R1
080602	Interview one student	Audio + student's notes	R1
November 2008	Students work on textbook tasks from the first chapters in the textbook	Computer files From 18 students	
Meeting	Meeting in TBM/LBM	Notes shortly af-	All participants

December 2008		terwards	in the project
Presentation	Conference Sept. 2009	Notes shortly afterwards	R1 + conference participants
Workshop 2010	Comments after my presentation	Video	TBM/LBM members
Meeting Spring 2010	Presentation School leader meeting	Notes shortly afterwards	School leaders in Agder

Appendix 4: Mathematical tasks – levels of cognitive demands (Smith & Stein, 1998, p. 348)

Lower-level demands (memorization)

- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory.
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.

Lower-level demands (procedures without connections to meaning)

- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers instead of on developing mathematical understanding.
- Require no explanations or explanations that focus solely on describing the procedure that was used.

Higher-level demands (procedures with connections to meaning)

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Higher-level demands (doing mathematics)

- Require complex and non-algorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.

- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions.
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

Appendix 5: The 'Learning book' – an example

Examples and summary of rules from the first sub-chapter, an example from the 'Learning book' made by teachers at school (digital file):

Tall og regneregler (Kap 1.1)

Regneregler for addisjon og multiplikasjon

$$a + b = b + a$$

$$(a + b) + c = a + (b + c)$$

$$a \cdot b = b \cdot a$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Negative tall

Produktet av to positive tall blir positivt, $a \cdot b = ab$

Produktet av to negative tall blir positivt, $(-a) \cdot (-b) = ab$

Produktet av et positivt og et negativt tall blir negativt, $a \cdot (-b) = -ab$, $(-a) \cdot b = -ab$

Skal vi løse opp en parentes med minustegn foran, må vi endre fortegnet foran alle leddene i parentesen.

Potenser

En potens består av et grunntall og en eksponent. I potensen 2^5 er 2 grunntall og 5 eksponent.

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

Tall på standard form

Vi bruker standard form når vi må regne med svært store eller svært små tall. Et tall er skrevet på standardform når det er skrevet som $\pm a \cdot 10^n$, der $1 \leq a < 10$ og n er et helt tall.

$$\text{Eks. } 245.000.000.000 \text{ km} = 2,45 \cdot 10^{11} \text{ km}$$

I et regnestykke hvor du må gjøre flere operasjoner gjelder disse regnerekkefølgene:

Parenteser

Potenser

Multiplikasjon og divisjon

Addisjon og subtraksjon

Appendix 6: Tests

In this appendix, the different tests are presented: The algebra test, and the regular tests: the September test and the December test.

6.1 Algebra test

Test i algebra Navn.....

Testen skal gjøres uten lommeregner, men du kan bruke kladd. Svar skal skrives på arket. Vis utregning der det er mulig.

1) Forenkling av algebraiske uttrykk

Skriv enklere dersom det er mulig:

a $2x + 5x$

b $x + x + 2x$

c $t \cdot t \cdot t$

d $2y \cdot y^2$

e $a - 3a + 2a$

f $10x + 3(4 - 3x) + 8$

g $5a - 2(7 - a) + 6$

h $8x + 15 + 4x - 5$

2) Lag en likning der:

a Løsningen er $x = 10$
.....

b Løsningen er $y = 4$
.....

3) Skriv opp det/de uttrykket/uttrykkene nedenfor som står for et tall som er:

$$k + 5; \quad k + 2; \quad 2k; \quad k + k; \quad 0,5k; \quad \frac{k}{2}; \quad \frac{k}{5}; \quad 5k$$

a Dobbelt så stort som k

b Halvparten av k

c 2 mer enn k

4) På en skole er det 10 elever for hver lærer. Hvilke av disse uttrykkene forteller dette?

L = antall lærere, E = antall elever. Sett ring rundt det eller de riktige uttrykkene.

$$10L = E \quad 10E = L \quad L = 10E \quad E = 10L \quad 10L + E \quad 11LE$$

5) Finn verdien av uttrykkene (vis utregning):

a $a + b - c$ når $a = 1$, $b = 2$ og $c = 3$

$$a + b - c = \dots\dots\dots$$

b $3b^2 - abc$ når $a = 3$, $b = -1$ og $c = 5$.

$$3b^2 - abc = \dots\dots\dots$$

6) Alltid sant, aldri sant eller kan være sant

a $x + y + z = x + p + z$ Dette:

er alltid sant er aldri sant kan være sant, nemlig når.....

b $a + b \cdot 2 = 2b + a$ Dette:

er alltid sant er aldri sant kan være sant, nemlig når.....

c $\frac{2x+1}{2x+1+5} = \frac{1}{6}$ Dette:

er alltid sant er aldri sant kan være sant, nemlig når.....

7) En kake koster c kroner. Et smørbrød koster s kroner. Jeg kjøper 3 kaker og 4 smørbrød.

Hva står da $3c + 4s$ for?

.....

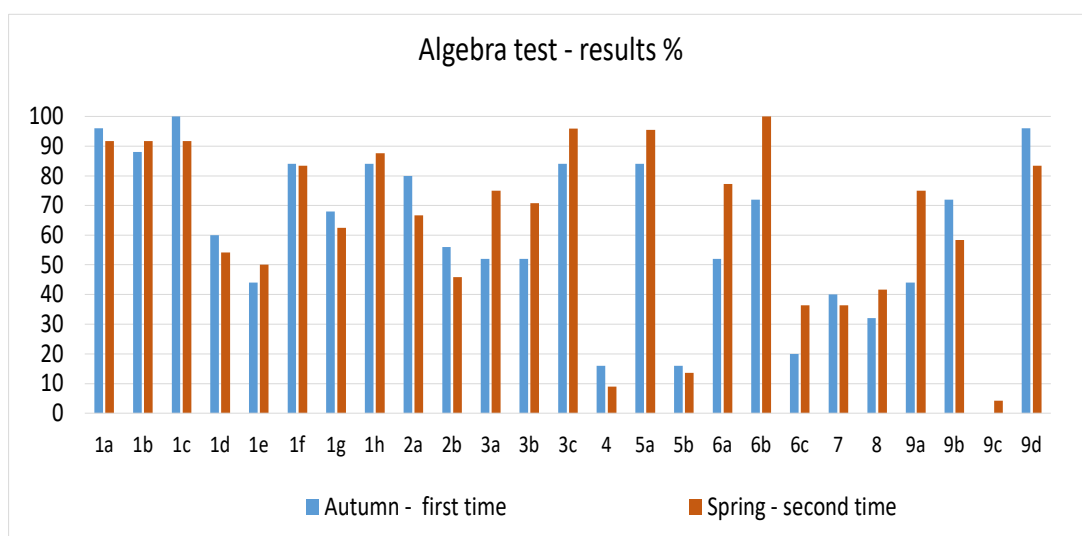
8) Volumet av en sylinder er: $V = \pi \cdot r^2 \cdot h$

Kan du skrive denne om slik at vi får en formel for høyden i en sylinder

$h =$

9) Løs likningene nedenfor:

<p>a $4x - 15 = 75$</p>	<p>b $123 + 2x = 195 - x$</p>
<p>c $\frac{x+1}{x+4} = \frac{4}{5}$</p>	<p>d $5x = 45$</p>



6.2 September test – 90 Minutes (Computers allowed)

Kapittel 1 Regning med tall og bokstaver

Oppgave 1

Løs opp parentesene og trekk sammen:

- a) $(12a - 4b) - 10a$
- b) $2(2a - 3) + a(2 - b) - 5(-b + 3)$
- c) $b + b(5b - 2) - 4b(b + 1)$
- d) $-3(a + b) - (-b) \cdot 3$

Oppgave 2

Skriv svarene på standardform:

- a) 0,000065
- b) 234000000

Oppgave 3

Regn ut:

- a) $\frac{1}{3} + \frac{5}{3} - \frac{4}{3}$
- b) $\frac{3}{2} + \frac{3}{7} - \frac{7}{5}$
- c) $\frac{4}{2} \cdot \frac{5}{4} \div \frac{7}{5}$

Oppgave 4

Regn ut og forkort så mye som du kan i svarene:

- a) $\frac{15ab^2}{5b}$
- b) $\frac{3}{2a} + \frac{a}{4b} - \frac{a-2b}{ab}$
- c) $\frac{3a+15}{6} \div \frac{2a+10}{8}$

Oppgave 5

Stine skal arrangere bursdag for sønnen sin Jarle og vil glede alle åtteåringene med å lage 12 liter av Stines brusspesial som består av Cola og Appelsinbrus.

- a) Stine har bare to liter Appelsinbrus. Hva blir blandingsforholdet i Stines brusspesial?

To timer ut i selskapet begynner Jarle og vennene å bli slitne av all lekningen. For å gi barna litt energi bestemmer Stine seg for å blande saft til dem. Hun har to liter saft, og på flaska står det at blandingsforholdet mellom saft og vann skal være 1:7.

- b) Hvor mye vann må Stine bruke dersom hun skal følge oppskriften på flaska?

Oppgave 6

Løs likningene og ulikhet:

a) $\frac{-6}{9} = \frac{4x}{3}$

b) $2 \cdot \frac{7}{x} = \frac{5}{25}$

c) $\frac{x}{2} - x = 2 \left(\frac{x}{2} - \frac{2}{3} \right)$

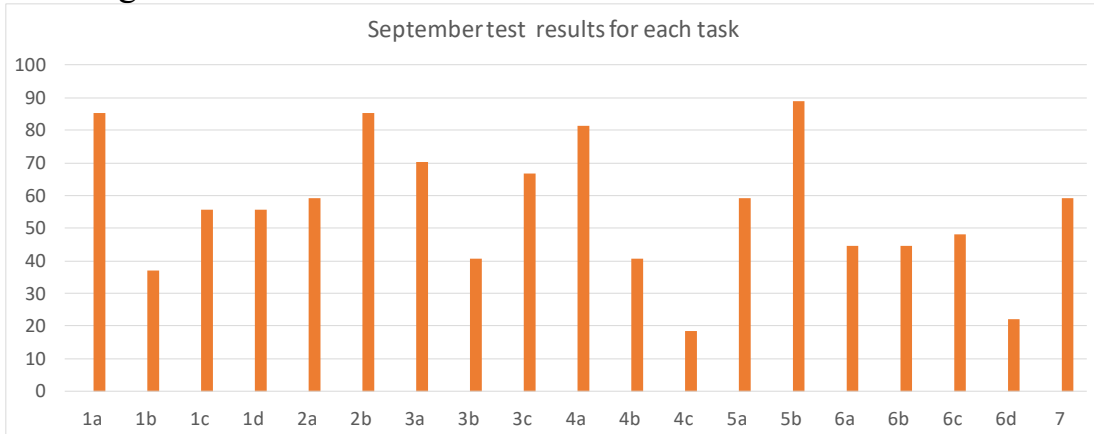
d) $-3x + 6 < 2(x + 8)$

Oppgave 7

Brødrene Tor, Erling og Jonas er til sammen 37 år. Tor er dobbelt så gammel som Erling og tre år eldre enn Jonas.

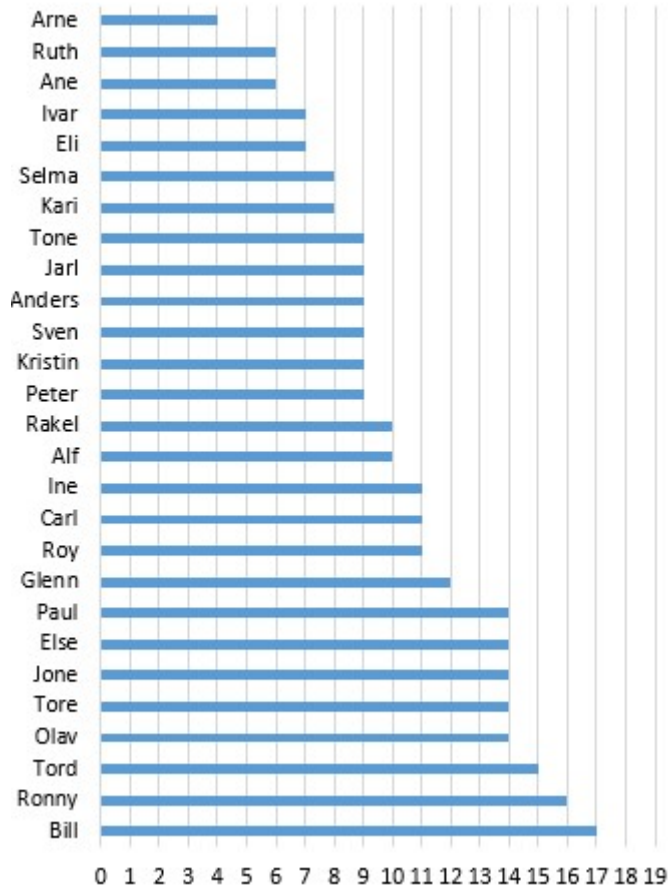
Hvor gamle er brødrene?

The diagram underneath shows the results for each task %.



The next diagram shows students results on the test. There were 19 tasks. The students are ordered according to the results achieved on the test.

September test - results for each student



6.3 December test

This test was given in December as a Semester test. Not all tasks were actual in relation to the focus for this study. The actual tasks are outlined with bold font.

The accepted means were: Computer programs: MathType and TI-interactive, downloaded files: 'The learning book' and solved textbook tasks.

Heldagsprøve i matematikk

Oppgave 1

I denne oppgaven er det deloppgaver med valgfrie alternativer. I hvert tilfelle skal du velge **enten** alternativ 1 med lavest vanskelighetsgrad **eller** alternativ 2 med størst vanskelighets-grad.

Ved sensuren vil du få mer uttelling for riktig løsning av alternativ 2 enn for riktig løsning av alternativ 1.

- a) Skriv så enkelt som mulig:

alternativ 1	alternativ 2
$-5 \cdot (7 - 5) - 2^3 \cdot (4 - 5)$	$\frac{2 \cdot (3 \cdot 4 - 3^2) - (4 + 2^3)}{5^2 - 3 \cdot 7}$

- b) Regn ut:

alternativ 1	alternativ 2
$\frac{2}{3} \left(\frac{1}{2} + \frac{3}{4} \right)$	$1 + \frac{\frac{2}{a}}{\frac{a}{2} - \frac{2}{a}}$

- c) Løs likningene ved regning:

alternativ 1	alternativ 2
$2 + 5x - 1 = x$	$\frac{2}{3} \left(x + \frac{3}{4} \right) = \frac{1}{6}x - 1$

- d) Regn ut uten bruk av tilnæringsverdier:

alternativ 1	alternativ 2
$\frac{x}{4} + \frac{1}{3} > 2$	$2 \left(1 - \frac{y}{2} \right) + 3y < 4 \left(y - \frac{1}{2} \right)$

Oppgave 2

Løs likningssettet grafisk og ved regning.

$$2x + 3y = 5$$

$$-x + y = 5$$

Oppgave 3

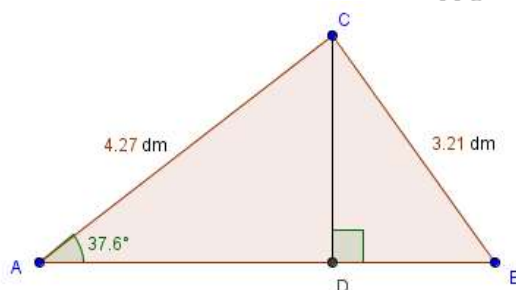
Harry har en stor bil som bruker 1,2 liter bensin på mila. Bensintanken til denne bilen tar 60 liter. Ronny har en mindre bil som bruker 0,7 liter per mil. Denne bilen har en bensintank som rommer 48 liter. Harry og Ronny fyller opp tanken på samme bensinstasjon og kjører etter hverandre på langtur.

- Hvor mye bensin har Harry og Ronny igjen etter 15 mil?
- Finn grafisk når Harry og Ronny har like mye bensin igjen.
- Hvor langt kan Harry kjøre før han må fylle bensin igjen, og hvor langt kan Ronny kjøre før han igjen må fylle?

Oppgave 4

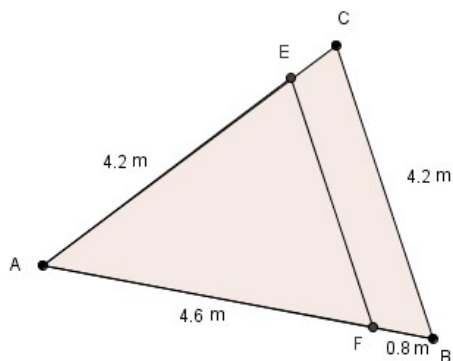
En rød blodcelle har en diameter på $7 \cdot 10^{-6} m$. Et voksent menneske har ca $2 \cdot 10^{13}$ røde blodceller. Hvor langt ville blodcellene rekke hvis de ble lagt etter hverandre?

Oppgave 5



- Regn ut lengden CD.
- Finn $\angle B$
- Regn ut arealet av trekkanten.

Oppgave 6



På figuren under er sidene BC og EF parallelle. AF = 4,6 m, FB = 0,8 m, AE = 4,2 m og BC = 4,2 m.

- Forklar hvorfor $\triangle ABC$ og $\triangle AFE$ er formlike
- Finn sida EF ved regning
- Finn $\angle A$

Oppgave 7

Martin kjører bil med farten 90 km/h på en motorvei. Rett foran seg ser han en hindring og begynner å bremse. t sekunder senere er farten v målt i kilometer per time (km/h) redusert til

$$v(t) = 90 - 3t$$

- Framstill sammenhengen mellom v og t i et koordinatsystem.
- Finn grafisk og ved regning når farten er 45 km/h.
- Gitt $v(t) = 90 - 3t$. Finn en formel for t uttrykt ved v .
- Hvis V er farten i meter per sekund (m/s) i det Martin begynner å bremse og T er tiden i sekunder fra han begynner å bremse til han stopper, er bremsestrekningen S i meter gitt ved

$$S = \frac{1}{2}V \cdot T$$

Finn bremsestrekningen i meter.

Oppgave 8

Funksjonsuttrykket for en lineær funksjon f er gitt ved $f(x) = ax + b$.

Grafen er en rett linje som går gjennom punktene $(1, 4)$ og $(3, -2)$.

- Finn stigningstallet a og konstantleddet b ved regning.
- Tegn grafen til funksjonen.

En annen funksjon g har stigningstallet 2, og den går igjennom punktet $(-1, 0)$.

- Finn skjæringspunktet mellom de to funksjonene f og g .

Oppgave 9

Tabellen viser folkemengden y i millioner i Norge noen år mellom 1900 og 2005.

$x = 0$ svarer til 1900, $x = 30$ svarer til 1930, osv.



x (år)	0	30	60	90	105
y (millioner)	2,10	2,85	3,59	4,23	4,61

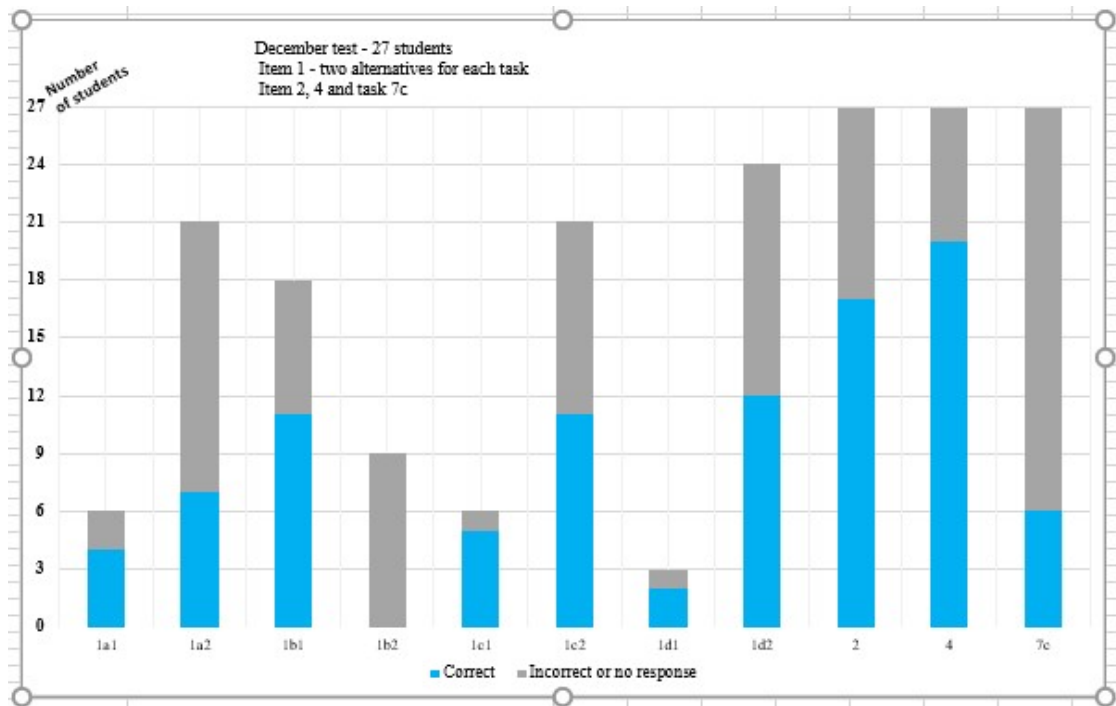
- Hvor mye økte folkemengden i gjennomsnitt per år fra 1900 til 2005?

- b) Bruk lineær regresjon på TI til å vise at funksjonen $f(x) = 0,24x + 2,1$ er den modellen som passer best med dataene i tabellen.
- c) Tegn grafen til $f(x)$ i et passende koordinatsystem og plott inn punktene fra tabellen.
- d) Hva vil folketallet i Norge bli i år 2050 i følge denne modellen?

Oppgave 10

I trekant ABC er vinkel $A = 30^\circ$, $AC = 10$ og $BC = a$.

Finn antall løsninger for ulike verdier av $BC = a$.



Appendix 7: Questionnaire

Spørreundersøkelse om matematikk generelt og algebra spesielt.

Spørsmålene du får, kommer som et ledd i prosjektet, “Lær bedre matematikk”, som skolen din er med på. Hensikten er å få vite mer om hvordan du som elev tenker, for så å ha et grunnlag for å forbedre undervisning og læring.

Husk å skrive på navn. Selv om alt senere blir anonymisert, er det viktig å kunne finne fram til hvem du er for eventuelle intervju senere. I alle andre sammenhenger blir navnet tatt bort, og du får et fiktivt navn om noe fra din besvarelse blir brukt.

Når du svarer på spørsmålene, kan du bruke dine egne ord. Om du ønsker å svare på flere måter, er det helt i orden. Det er bare fint om du gir eksempler.

Navn: _____

1. Hva er matematikk? Forklar kort med egne ord.
2. Hva syns du om matematikk? Du kan godt bruke ord som kjekt - kjedelig, lett - vanskelig osv.
3. Har synet ditt på matematikk forandret seg noen gang (var det annerledes før)? Hvis ja, skriv kort om hvorfor og når det skjedde!
4. Hva er algebra?
5. Forklar med ord hvordan man løser likningen: $4x + 35 = 95 - x$. Tenk deg at du hjelper en som ikke kan løse likninger.
6. Vi skriver $y = x + 5$. hva betyr det? Hvordan henger f. eks. x og y sammen?
Du kan gjerne forklare på mer enn en måte.
7. Er matematikk viktig for deg? Forklar hvorfor eller hvorfor ikke.

Hvis du vil at læreren din skal se hva du har svart, krysse av her.

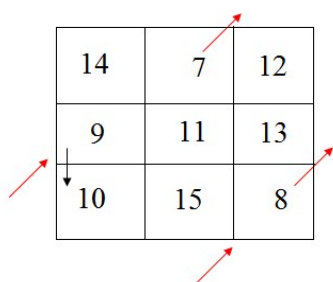
Appendix 8: Textbook – the first sub-chapter

In this appendix, the textbook exposition, examples, and tasks from the start of the textbook are presented. The headlines follow the headlines in the textbook.

8.1 Fun with numbers

Under the headline ‘Fun with numbers’, the authors suggest some activities for students: a game, mind reading, and a magic square. The two first activities can be classified as examples of open texts according to Love and Pimm (1996). The style is authoritative with use of imperatives, however, at the end students are asked not only to do the activity; they are asked to find patterns and to reason about how and why. In the following sub-chapters, such tasks or activities are not present. They were also not used in the classroom.

In the case of the magic square, it seems as if the authors do not expect students to be able to make such squares of their own. Here the authors first describe the square in detail:



14	7	12
9	11	13
10	15	8

We have drawn a square including the whole numbers from 7 to 15.

The numbers are placed so that the sum is the same in all rows as well as in the diagonals. Can we do the same with the nine numbers 1, 2, 3, ..., 9? Alternatively, with nine other consecutive positive numbers? Yes, we can, and the procedure is as follows: ... (p. 9).

The reader is directly addressed with the pronoun singular ‘you’ (‘du’) when asked if he/she can make one such square. The question is immediately answered positively by the authors, and the recipe is given. At the end, it is said that the squares are restricted to contain an odd number of numbers.

While the first two activities ask students to reason and to explain, this last activity is an example of a closed text. The reader is led step by step; only the choice of the sequence of numbers is left to the reader.

8.2 General rules for addition and multiplication

The first sub-chapter starts with the introduction of the set of natural numbers introduced in the context of counting before even and odd numbers are defined. A list of symbols for Roman numerals is offered, and it is stated that the numerals we use today were developed in India.

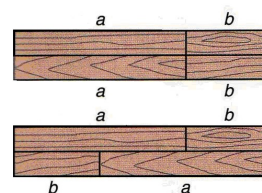
The set $\mathbb{N} = 1, 2, 3, 4, \dots$ is then presented on a number line, emphasising equal intervals between the numbers.

The rules for addition and multiplication are introduced within \square , because that is claimed to be easier, but it is stated that the rules are the same for all sets of numbers.

The commutative property of addition is presented as a rule; firstly by a number example: $4 + 11$ and also $11 + 4$ is equal 15. Then the general rule is presented in a ‘rule box’:

$$a + b = b + a$$

In addition, an imagined ‘real life’ example is offered. A carpenter is building a floor. The length of the floor is the same, no matter how the wood is laid. This is illustrated as in the figure to the right. The word commutative is not used.



The associative law of addition is based on the need of grouping pairs of numbers when one mentally has to add more than two numbers. It is commented that this grouping can be visualised by brackets. A number example is offered before the rule for the associative law of addition is presented in a rule box.

The general rules for commutativity and associativity of multiplication is just presented in rule boxes. It is said that we mostly use these rules without recognising them. The rules are then exemplified in solved example tasks. The examples are set in ‘real life’ contexts.

The distributive law of multiplication over addition is presented by another ‘real life’ example before it is presented in a coloured rule box.

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This law is then applied in a contextual example task in which a school class has been out eating. The students have eaten the same; a main dish, a dessert and a bottle of soft drink, however, they wanted to know what each of the dishes costs for all students in addition to the total bill.

Therefore, they have to apply the distributive law.

Another example: $6 + 3(a + 5) + (1 + b)$ is presented and worked out. Each step in the solution is explained using the inclusive ‘we’.

In this section with the general properties of number, in the textbook called rules, the exposition is written in ordinary language. Only the words addition and multiplication are used from the mathematical vocabulary. Commutativity and distributivity is explained by the authors in ordinary language.

8.3 Negative numbers

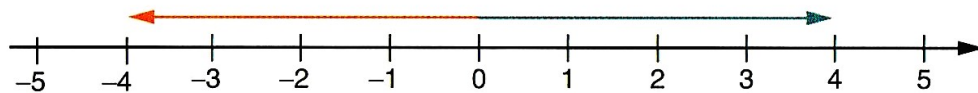
The first short paragraph in the textbook is devoted to the Indian astronomer Brahmagupta who calculated with negative numbers already more than 1300 years ago. He is said to have called the negative numbers *debt*, while the positive numbers then were called *fortune*.

Then the integers are presented as follows:

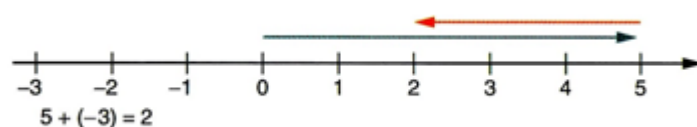
••• The set of whole numbers

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4 \dots\}$$

The negative numbers are introduced in the textbook by the context of an imagined negative bank account. More is used than was contained in this account. Then the owner, the student, owes money to the bank and this amount of money is negative. Negative numbers are illustrated on the number line. It is said that the red arrows symbolising negative numbers are always pointing to the left, while the blue arrows representing positive numbers point to the right (p. 14) as shown in the example below:

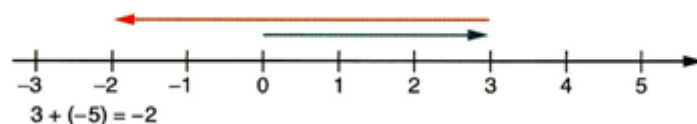


Addition and subtraction including negative numbers and multiplication with one negative factor are explained by arrows on the number line, but first it is said that calculation with negative numbers follows the same rules as the natural numbers. Addition of a negative number is presented on the number line:



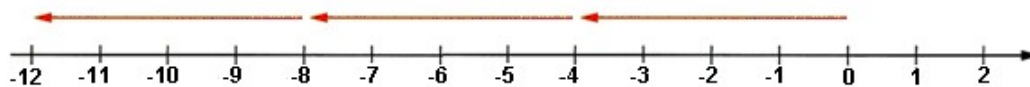
It is commented that to add negative 3 means to subtract 3. Thus $5 + (-3)$ is the same as $5 - 3$. Thus the latter way of writing it, as subtraction is mostly used.

The next illustration has no comments:



Multiplication of a positive number by a negative number is exemplified through the product $3 \cdot (-4)$. This multiplication is illustrated by repeated

addition of the negative number (- 4) on the number line as shown below.



$$(-4)+(-4)+(-4)=-12$$

The commutative law of multiplication is applied to verify that $(-4) \cdot 3$ will be the same as $3 \cdot (-4)$, and thus makes it reasonable to equate $(-4) \cdot 3$ with $3 \cdot (-4)$.

The rule of signs is then presented:

The product of a positive and a negative number is negative.
The product of two negative numbers is positive.

As a note, it is said that the rule concerning two negative numbers will not be proved.

Three example tasks are worked out without any explanation. The tasks are:

a) $(-7 \cdot 4)$ b) $(-7) \cdot (-4)$ c) $(-7) \cdot (-4) \cdot (-5)$

Then it is stated:” Often we meet a plus sign or a minus sign before an expression within a bracket. Then we follow the rules below” (p. 16):

When we are going to open a bracket with a plus sign or no sign in front, we just remove the bracket.
If we are going to open a bracket with a minus sign in front, we have to change the sign for all terms in the bracket.

Two examples are then shown in which brackets are to be subtracted:

Multiply, open the brackets and simplify:

a) $5 - 3(2a - b + 5) - (2b - 5a)$

b) $2u(3u + 4) - 6(u^2 + u - 2)$

The second example task is solved in this way (p. 16):

$$2u(3u + 4) - 6(u^2 + u - 2)$$

← We multiply into the brackets.

$$= (6u^2 + 8u) - (6u^2 + 6u - 12)$$

← We remove the brackets and change the signs, when there is a minus in front.

$$= 6u^2 + 8u - 6u^2 - 6u + 12$$

← We collect like terms.

$$= 2u + 12$$

In both examples, there is a guideline to each line about what to do. Example task b includes u to the power of 2. There is, however, no comment on this.

8.4 Powers and scientific notation

Powers are introduced in the context of large and small numbers. It is told that those can be written shorter by scientific notation; with powers of 10.

The number example 1000 000 000 is said to be 10 multiplied by itself 9 times: $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^9$

“It is not a presentation of a new number; it is just another way of writing the number”. (p. 17). Then a power is defined as repeated multiplication:

$a^n = \overbrace{a \cdot a \cdot \dots \cdot a}^{n \text{ factors}}$. n is a natural number and a is an arbitrary number. a is the base and n the exponent.

This is followed by three worked out example tasks:

$$a) 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81 \quad b) (-5)^3 = (-5) \cdot (-5) \cdot (-5) = (-5) \cdot 25 = -125$$

$$c) 3,14^5 = 3,14 \cdot 3,14 \cdot 3,14 \cdot 3,14 \cdot 3,14 = 305,24$$

Negative indexes are introduced in relation to powers of 10, and it is stated that zero as a base cannot be raised to a negative power, because it is not possible to divide by zero.

Some powers of 10 with negative exponents are converted into fractions with this equality in the margin: $a^{-n} = \frac{1}{a^n}$. This equality is not

commented upon. The example tasks are:

$$a) 10^{-1} = \frac{1}{10^1} = \frac{1}{10}, \quad b) 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \quad c) 10^{-6} = \frac{1}{10^6} = \frac{1}{1000\ 000}$$

A general formula for a number a written in scientific notation is presented: $a = \pm k \cdot 10^n$. Here k is a number with one digit in front of the decimal sign, and n is a whole number. Two decimal numbers, one large and one small, are then converted into scientific numbers. The explanation is about how many steps the comma has to be moved. This is not connected to the number system in any way.

At the end, a recipe is given for which buttons to push on the calculators (Texas and Casio) in order to write numbers in scientific notation. This section ends with an example task in which the transformation of numbers from normal form to scientific form is demonstrated. In addition, a task from a ‘real life’ context is applied to demonstrate multiplication and division of numbers written in scientific notation.

8.5 Order of operations

The first sub-chapter in the textbook ends with a recipe for the orders of operations:

- Simplify the expressions in the brackets
- Calculate the powers
- Execute the operations of multiplication and division
- Execute addition and subtraction (p. 21).

Three example tasks are then solved; no comments or hints are made to the solutions:

Calculate and write the answers as simple as possible:

a) $2 + 3 \cdot 2^3$ c) $(2a - b)b - a(a - 2b + a)$

b) $(2 + 3) \cdot 2^3$

One question is posed at the end: “How can we interpret the expression: $100:10 \cdot 2$?” In the answer, it is emphasised that multiplication and division have the same priority according to the rules, and in situations such as the one above, one has to use brackets. Not much space is devoted to powers and order of operations in the textbook. Powers are defined, but then they are only presented as powers of 10 in the exposition part, in relation to scientific notation. Some few tasks however, are offered with powers of numbers different from 10.

8.6 Work plan and tasks

The work plan for the first two weeks was:

Table 8-1: Work plan for the students. Adapted from the web site above:

Time	Aims	Textbook pages	Items
Week 35	To know the rules $a + b = b + a$ and $(a + b) + c = a + (b + c)$ for addition, and the rules $a \cdot b = b \cdot a$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for multiplication	11–12	1.1.7, 1.1.8
	To use the rule $a(b + c) = ab + ac$ multiplying brackets	12–13	1.1.9, 1.1.12, 1.1.13
	To calculate with negative numbers	14–16	1.1.10, 1.1.11
	To define powers	17	1.1.14, 1.1.15, 1.1.16
	To know scientific notation and calculate with such numbers	18–20	1.1.17, 1.1.18, 1.1.20
	To know the order of operations	20–21	1.1.20

This plan was made by the textbook authors. The plan included learning aims for the period, and the plan showed where to find the exposition

and example tasks (textbook pages). The assigned tasks were placed at the end of the first sub-chapter; 20 numbered items including 67 tasks. The students were given the plan for the first two sub-chapters unchanged from the publisher.

All tasks are numbered according to the chapter, sub-chapter and number within the sub-chapter. Task 1.1.17b, is task b in item number 17 in subchapter 1 in the main chapter 1 (see the table below).

Table 8-2: Numbering of tasks in the textbook

Chapter	Sub-chapter	Numbered item within sub-chapter	Number in the item
1	1	17	b

Thus, it can be seen in the work plan above that some tasks in the textbook are left out on the plan. The aims, according to the plan, are for most of the exercises to practice explicitly stated rules or procedures. It is also stated where these rules are to be found. For the students, this means that they should know exactly where to find examples to follow in their solution process.

On the plan, it is explained that aims written in bold are the most central and important aims for learning, while numbered items written with bold fonts are said to be the easiest or most important items. Students' saved computer files, and the observations in the classroom confirm that students mostly skipped the items not written with bold fonts.

Two items including six tasks are questionnaires requesting students to reply to facts from the exposition part of the sub-chapter. None of them were listed in the work plan. Two numbered items including 7 tasks might have been 'potential problems' according to Niss (1993). The tasks are different from the examples and there is from the outset no known strategy to solve them. The authors, however, have posed questions that lower the cognitive demand and lead the students through small steps to the solutions. One example is item 1.1.7. No student had solved it (p. 22):

1.1.7 a: Make a new order of the numbers $1 + 2 + 3 + 4 + 5 + 6$ and group them in such a way that you will have $7 + 7 + 7$.

1.1.7 b What will the sums be if you perform the same rearrangement of the numbers in the sums $1 + 2 + 3 + 4 + 5 + 6 + 7$ and $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$?

1.1.7 c Could you suggest a rearrangement of the sum $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \dots + 100$? Use this rearrangement to calculate the sum.

The table below gives an overview over type of tasks offered. Tasks from the textbook, not present on the work plan are left out. The column

headed fonts, gives the number of tasks in each category written with bold fonts.

Table 8-3: Examples of tasks on the work plan for sub-chapter 1.1

Themes	Number of tasks	Examples of tasks	Bold fonts
Grouping and adding sequences of numbers (no examples in textbook)			
	3	<i>Order the numbers</i> $1+2+3+4+5+6$ <i>so that you get the sum</i> $7+7+7$	0
Negative numbers			
Multiplication	2	$(-3)(-2) \cdot 4$	2
Addition/subtraction	2	$12 - (-7 + 2)$	2
Powers			
Notation	1	<i>Write the product</i> $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ <i>in a simpler way</i>	1
Calculation	7	$(-5)^3$; 10^{-4}	7
Scientific notation			
Notation	7	<i>Write the number</i> $12\,300\,000$ <i>in scientific notation</i>	6
Calculation	2	<i>Word problem from chemistry</i>	0
Multiplying single expressions			
	3	$2s \cdot 5 \cdot 3b$	0
Collecting like terms in polynomial with:			
Brackets preceded by +	3	$(-3x+7y)+(4y-6x)$	0
Brackets preceded by -	1	$8a-3b+(7b-3a+6)$ $+13-(-3a+2b)-8a$	0
Brackets with a 'positive' pre-multiplier	7	$a+2a(3-2b)$	4
Brackets with a 'negative' pre-multiplier	2	$3x-2(4x-3)+10$	0
Brackets with a negative post-multiplier	3	$3(2a+2)-2(3a-1)+(8a+3)(-3)$	2
Sum of tasks	43		24

Only one task on the work plan is a task not set up in mathematical notation. It is from the context of chemistry:

The diameter of an atom of iron has the length of $2,0 \cdot 10^{-10} m$ and the mass of $9,3 \cdot 10^{-26} kg$. One paper clip contains 0,50g iron.

a) Transform 0,50g into kg and write the answer in scientific notation.

- b) How many atoms of iron does the iron clip contain?
- c) How long would a row of the atoms of iron contained in the paper clip be, if one was to place all the atoms in a row? (p. 24)

Although the students had to decide which operations to apply, the operations should be routine operations. The challenge might be the small numbers, or perhaps the context of chemistry. This task is on the work plan categorised as a task to practice order of operations (item 1.1.20 on the plan above), and it is written with bold fonts. It seems to be an error in the plan. This item does not have anything to do with the order of operations³⁷. Another item with 8 tasks seems to have been conducted in order to practice those rules, but is left out.

The tasks are variations of examples in the textbooks, and thus routine tasks for practicing learned rules and skills, the only exception on the plan is the task above. All the tasks are of low level of procedural complexity with less than 5 steps using a common solution method.

16 students solved at least some of the tasks on this plan (data from 18 students). Three students solved more than 30 tasks, one of them more than 40. Five had solved more than 20 tasks, seven between 10 and 20, one only 7 tasks, and 2 had solved none of the tasks.

³⁷ The work plan was still (2011) the same for students to load down from the publisher's web-site. The item numbered 1.1.20 was still listed twice.

Appendix 9: Students' work with numbers and letters

9.1 Understanding of literal symbols

One task, task 3 in the algebra test, tested if students accepted letters as general numbers. The task was: Write down the expression/expressions underneath which represent a number that is:

$$k + 5; \quad k + 2; \quad 2k; \quad k + k; \quad 0,5k; \quad \frac{k}{2}; \quad \frac{k}{5}; \quad 5k$$

- a) Double the value of k
- b) Half the value of k
- c) 2 more than k

All except for four students had picked out correct expressions. One student did not solve this task. She had solved all the others. However, not all 20 students, who picked out correct expressions, included both expressions in the tasks a) and b). The results are presented below.

Table 9-1: Algebra test - item 3 with correct results (%)

Task	Expression(s)	Autumn			Spring		
		Both	Only the first	Only the second	Both	Only the first	Only the second
3a	Double the value of k	52	36	0	75	21	0
3b	Half the value of k	52	20	24	71	17	8
3c	2 more than k	84			96		

32 % or 8 students presented only one expression in both tasks in the autumn. In task 3a all of those had chosen $2k$; the first expression in the sequence of the expressions to choose, while only two of those eight had chosen the first correct expression in task 3b. This might indicate that at least six students (24 %) had a problem to see that the expressions were equivalent since they had chosen the first expression in the first task and the second expression in the next task. In addition, four students equated non-equivalent expressions. It is visible that there is a considerable improvement from autumn to spring.

From the task, it might be inferred that all, except for one student, accepted letter symbols representing numbers.

The next task on the algebra test was the ‘student professor problem’:
In a school there are 10 students to each teacher. Which of the expression(s) represent the correct relation?

L = number of teachers, E = number of students.³⁸ Mark the, or those correct expression(s).

$$10L = E \quad 10E = L \quad L = 10E \quad E = 10L \quad 10L + E \quad 11LE$$

Table 9-2: Algebra test, item 4 - results and alternative responses (%)

	Autumn	Spring
Both expressions correct	16	9
One correct expression	0	0
Opposite relation - both	60	78
Opposite relation – one of them	8	13
Other answers	16	

Only two students or 8 %, chose only one expression in the autumn. 76% chose two equivalent expressions, although 60 % chose equivalent expressions representing the opposite relation to the one asked for. In this task, a larger group of the students picked out two expressions than in the foregoing task above, which can be taken as an indication that many students had problems to find equivalent expressions in item 3.

Only 16 % (9 % in the spring) chose the expressions representing the correct relation between teacher and students. Most of the students seemed to rely on the syntax rather than on the semantics; the meaning of the statement. This is reported to be a common error.

9.2 Simplification of algebraic expressions

Tasks given to test simplification of expressions (algebra test). The tasks are presented in the table below:

Table 9-3: Algebra test, item 1-Results (%) and alternative answers 25 students

Item	Task	Correct	Alternatives	No response
1a)	$2x + 5x$	96	$6x$ *	
1b)	$x + x + 2x$	88	$3x$; $5x$; $x^2 + 2x$;	
1e	$a - 3a + 2a$	44	$4a$ (48%)	8
1f	$10x + 3(4 - 3x) + 8$	84	$x = 20$ (A); $19x + 20$ (B); $x + 15$ (C)	4
1g	$5a - 2(7 - a) + 6$	68	$3a - 8$ (A)(16 %); $3a + 20$ (B); $6a - 8$ (C); $7a - 20$ (D)(8%)	
1h	$8x + 15 + 4x - 5$	80	$6x + 5$; $12x - 10$ (12 %)	

*When no percent is given for the alternative solutions, each solution was presented by one student (4 %).

³⁸ In Norwegian, teacher is 'lærer' (L), and student is elev (E).

In task 1b three students came to the solutions $3x$, $5x$, and $x^2 + 2x$. Two of them showed that they also had problems deciding which expressions were equivalent in task 3c in the foregoing section (Table 9-1). In the spring only Olav wrote the same; the quadratic expression as in the autumn. Then the others solved the task correctly.

Task 1e is discussed in the thesis.

In task 1f, Kari presented an equality (A). She did not explain her result. Eli added 3 and 4 and presented result (C). Selma changed the sign in the bracket, result (B).

In task 1g, the term $3a$, in solutions (A) and (B), is the result of not changing sign in the bracket (Rakel, Kristin, Ane, and Selma). The solutions (B) and (D) included the number 20. Those students have either seen the minus sign as a splitting factor; -20 (Olav, Kari), or calculated from right to left; $+20$ (Ane). The student in solution C has not followed the law of distribution (Sven).

The solution $12x - 10$ in 1h is probably due to calculation from right to left. Carl, Bill and Jone were observed doing that at other occasions, Jone succeeded though in task 1e). Kari came to a correct result but divided both terms by 2, resulting in $6x + 5$.

The textbook items 1.1.9 and 1.1.12 is presented below:

Table 9-4: Textbook, item 1.1.9 and item 1.1.12 (number of students)

Item	Task	Correct	Solutions	Alternative solutions (Number of solutions in the brackets)
1.1.9a	$3(a+2)$	15	15	
1.1.9b	$5(2b-3a-5)$	10	15	Detachment of terms from indicated operations (3) * $2b-3a-5$ (2)
1.1.9c	$2(2a+3)+3(5-4a)$	13	15	The letter disappears (1) Only multiplication; no simplification (1)
1.1.12a	$2(a+3)-(3+3a)$	15	15	
1.1.12b	$3x-2(4x-3)+10$	15	16	$5x+16$ (1) from right to left
1.1.12c	$3(2a-5)-(6-2a)+3(a+1)2$	10	16	$2a-19$ (1) Detachment of terms from indicated operations Multiplication error (3) Law of distribution (1) Do not change sign (2)
1.1.12d	$3(2a+2)-2(3a-1)+(8a+3)(-3)$	6	14	Wrong sign in last bracket (2) Law of distribution (4) Multiplication error (2)

* Numbers within brackets show how many students had the actual solution

The table above gives an overview of solutions and alternative solutions of the two textbook items. Since not all students solved the tasks, each task is given with the number of students who solved it (number of solutions) and the number of students with correct solution (Correct solutions). Dataset 18 students.

9.3 Negative numbers and the minus sign

One reason for the errors in the foregoing table is related to the minus sign. The table below presents the textbook tasks explicitly stated to be related to negative numbers. Dataset 18 students.

Table 9-5: Negative numbers - Textbook item 1.1.10 (Number of students)

Item	Task	Correct	Solutions	Alternative solutions
1.1.10a	$(-3)(-2) \cdot 4$	13	14	$(-3)(-2) \cdot 4 = (-6) \cdot 4 = -24$
1.1.10b	$(-1)(-2)(-3)(-4)$	10	13	$(-1)(-2)(-3)(-4) = 2 + 12 = 14$ (A) $(-1)(-2)(-3)(-4) = 3 \cdot 12 = \underline{\underline{36}}$ (B) $(-1)(-2)(-3)(-4) = (-10) = \underline{\underline{-10}}$ (C)
1.1.10c	$12 - (7 - 2)$	12	12	
1.1.10d	$12 - (-7 + 2)$	9	13	$12 - (-7 + 2) = 12 - (-5) = 7$ (D) $12 - (-7 + 2) = 12 + 7 + 2 = \underline{\underline{21}}$ (E) $12 - (-7 + 2) = 12 + 7 - 2 = \underline{\underline{-21}}$ (F)

1.1.10c was solved correctly by those who solved it. Else had problems with all tasks except for task 1.1.10c. Five other students solved one of the tasks incorrectly.

The solutions in task 1.1.10b show that some students mix addition and multiplication. Tord (A) split the task into two terms, each with two factors, before he added the products. Kari (B) added the two first factors, before she multiplied the sum by the product of the two last factors. Else (C) added the factors as if they were negative terms. This happened although the students had put the negative numbers within brackets.

The task 1.1.10d includes a double minus sign. The first minus sign is an operation sign while the second is a symbol for negative 7. Else added $-7 + 2$ correctly with the result negative 5, but then $12 - (-5)$ becomes positive 7 (D). Jone and Ane, solution (E), wrote correctly positive 7 but did not change the sign in front of the number 2 and therefore the result was positive 21. Solution (F) is partly correct, having $12 + 7 - 2$. Then the result is written as negative 21. If Jarl calculated from the right, grouping 12 and 7 resulting in -19 before subtracting -2,

this would be the result. However, he had not presented his strategy or thinking.

Only three tasks in the textbook had two successive minus signs. One was on the work plan, the task 1.1.10d shown in the table above. Another was on the work plan, but written with normal fonts see table 8-3. Only 3 students solved it, and they solved it correctly (dataset 18 students).

Two tasks with double minus signs were given in the ordinary test 2 weeks after the work with the textbook tasks. One of them was: $-3(a+b)-(-b)\cdot 3$. Only three students failed in this task because of the double minus sign. In another task in the same test, the situation was different. The task was: $2(2a-3)+a(2-b)-5(-b+3)$. In this task, 9 students or 33 % wrote $-5b$ instead of $+5b$.

Jarl's solution can be an example. He solved the task in the way shown below:

$$\begin{aligned}
 &2(2a-3)+a(2-b)-5(-b+3)= \\
 &(4a-6)+(2a-ab)-(-5b+15)= \quad \text{Here he has multiplied the last bracket by 5} \\
 &4a-6+2a-ab-5b-15= \quad \text{The bracket is removed.} \\
 &\underline{\underline{6a-5b-ab-21}}
 \end{aligned}$$

It might be that he had $+5b-15$ after changing the signs in the bracket, but the negative sign in front of the bracket might have caused him to change again.

9.3.1 Result and solution numbers

In the work plans for the first chapter there were three tasks with negative result numbers, and four tasks with negative solution numbers. In the table below the tasks with negative result numbers are listed with number of correct solutions and alternative solutions (Dataset 18 students):

Table 9-6: Tasks with negative result numbers (Numbers of students)

Item	Task	Correct	Solutions	Alternative solutions
1.2.2c	$\frac{5}{3}-\frac{10}{3}+\frac{2}{3}$	5	10	$\frac{7}{3}; \frac{-3}{3}$
1.2.17e	$\frac{2}{b}-\frac{1}{3b}-\left(\frac{1}{2}+\frac{5}{3b}\right)$	8	13	$\frac{1}{2}; \frac{3b}{6b}$
1.2.18b	$\frac{a}{12}+\frac{a-2}{4}-\frac{a}{3}$	7	10	$\frac{1}{2}; \frac{-6}{12}$

A large group of students solving the tasks, made errors. In the first task, the reason for the errors was not related to a lack of acceptance of a negative result number. The reason for the positive solution $\frac{7}{3}$ is due to cal-

calculation from right to left, or a grouping of the two last terms before subtracting the first term. The alternative solution $\frac{-3}{3}$ is correct, but the result is not simplified.

In the next task 1.2.17e, three students ignored the minus sign in their result, writing a positive solution. Two had come to a positive solution because of erroneous calculations. One student came to a positive solution, but after checking the textbook answer, the result was changed. Only one student ignored the sign in two of the tasks.

The next table shows the tasks with negative solution numbers.

Table 9-7: Tasks with negative solution numbers

Item	Task	Correct	Solutions	Alternative solutions
1.3.3f	$x - 2(1 - x) = 8x + 1$	13	13	
1.3.4f	$\frac{-2}{3} = \frac{4x}{5}$	14	15	
1.3.14c	$2x - \frac{1}{3}(x + \frac{3}{4}) = 1 + \frac{1}{2}(\frac{5}{6}x - 5)$	6	6	
1.3.18f	$8x - 2(x - 3) \geq 3x - (5 - x)$	13	14	

The tasks were mostly solved correctly after the students had been helped. There were many questions related to the minus sign in all tasks. In 1.3.3f the students seem not to regard it possible to divide by ‘minus’. Rakel said: “If we divide by minus 5, the minus sign will come down there?” In 1.3.4f Kari and Tord made sign errors. Tord corrected his solution, after checking it. In task 1.3.14c, Tone and Ruth asked for help and were told to solve the task once again, and they succeeded. In task 1.3.18.f. Tone, Ane, and Arne asked for help about sign and the brackets.

9.4 Powers

The table below (dataset 18 students) shows that nearly all students solving the tasks in the items 1.14, 1.15, and 1.16, solved them correctly. Three students made an error in 1.1.16b but they solved task a and task c correctly. Olav came to a negative result in task 1.15.b.

Table 9-8: Powers and calculation with powers – textbook tasks

Item	Task	Correct	Solutions	Alternative solutions
1.1.14	Write $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ in a different way	15	15	
1.1.15a	2^3	13	13	
1.1.15b	$(-3)^4$	11	12	-81
1.1.15c	$(-5)^3$	12	12	
1.1.15d	10^6	11	11	
1.1.16a	10^{-2}	11	11	
1.1.16b	10^{-4}	7	10	$\frac{1}{1000}$
1.1.16c	10^{-1}	11	11	

The next items 1.1.17 and 1.1.18 are presented in the table underneath:

Table 9-9: Powers and calculation with powers – textbook tasks

Item	Convert to scientific numbers	Correct	Solutions	Alternatives
1.1.17a	12 300 000	8	10	$1,23^7$; $1,2300000^7$
1.1.17b	0,0123	5	9	$1,23 \cdot 10^2$ (2students); $1,23^{-2}$; $1,23 \cdot 10^4$
1.1.17c	432,1	7	8	$4,321^2$
1.1.17d	0,000 043 21	6	8	$4,321^{-5}$ (2students)
1.1.18a	6 380 000 m	8	10	$6,38^6 m$ (2students)
1.1.18b	0,000 000 000 282m	9	10	$2,82^{-10}$

One student, Else, does not include the base 10, indicating that the power does not mean anything to her. Jone does the same in the few tasks he solved, while Tone did that in one of the tasks. Tore, Ronny, and Jarl seem not to have understood the use of a negative exponent (1.1.17b). 10 students solved one or more tasks; three of them made no error.

Two of the tasks related to powers in the algebra test are presented in the next table:

Table 9-10: Algebra test, item 1 with results (%) and alternative answers

Item	Task	Correct	Alternatives	No respons
1c	$t \cdot t \cdot t$	100		
1d	$2y \cdot y^2$	60	(4%); $2y^2$ (12%); $3y^3$ (4%); $3y$ (4%)	16

The first task, similar to 1.1.14a, was solved correctly by all 25 students. In the next task, task 1d: Simplify the expression $2y \cdot y^2$, 60 % gave a correct expression and 16 % did not answer. An invisible exponent might cause the problem since 12 % regarded it to be equal to $2y^2$.

Some textbook tasks required some knowledge about powers. Some of them were solved by very few students. The tasks, number of students solving the tasks, number of correct solutions and alternative solution are presented in the table below. (Dataset 18 students).

Table 9-11: Textbook tasks related to powers (Number of students)

Item	Task	Correct	Solutions	Alternatives
1.1.13d	$u + u(2u - 1)$	1	1	
1.1.13e	$-u(2 - u) + u^2 - 2(2u + 3)$	0	1	$-6u + u^4 - 6$
1.2.17c	$\frac{2}{3} + \frac{5}{a} - \frac{a}{2}$	9	12	$\frac{4 + 30 - 3a^2}{6a}$
1.2.17d	$\frac{3}{a} + \frac{5}{b} - \frac{a}{2b} - \frac{2}{ab}$	13	13	
1.2.18c	$\frac{a-b}{a} - \frac{b-a}{b}$	9	13	Sign errors
1.2.21c	$-3x + \frac{x}{2}(x-3) - \frac{x}{3}(4-2x)$	3	3	

Only in task 1.1.13e, the error is related to powers. Rakel who made the error, stated in the classroom, that she was unsure about powers.

Two students, Kristin and Arne, were observed during seatwork writing ordinary multiplication as exponentiation, and asked for help. Arne continued making this error both in September and in the spring.

In December on the semester test, six of 27 students made errors related to powers in the first task. Most of them were caused by not following the correct order of operations, and in addition some students multiplied base and exponents in that they equated 2^3 with 6. The table shows the distribution of correct and alternative solutions:

Table: 9-12: December test - order of operations (21 students chose this task)

Task 1a, alternative 2	$\frac{2 \cdot (3 \cdot 4 - 3^2) - (4 + 2^3)}{5^2 - 3 \cdot 7}$
Correct	24 %
View 3 and 4 as different terms	39 %
Power errors	24 % (half of those multiplied the new base by the exponent)

On the test there were two alternatives of item 1a, and this task was announced to be the more difficult of the two. 27 students were present on the test and 21 students chose alternative 2. Thus, the dataset for the table includes 21 students. Five made power errors, Alf, Tore, Peter, Ane, and Kari. (Roy made a 'power error' in alternative 1).

Appendix 10: Fractions

10.1 Numerical fractions - the textbook

In the textbook fractions are presented as the result of dividing whole numbers; as quotients. It is said that dividing one whole number by another whole number, the answer will not always be a whole number, as is the case for the other arithmetical operations. The example given is 2 divided by 5. The calculator gives the answer 0,4 or 4 tenths. The answer can also be written as a fraction $2 : 5 = \frac{2}{5}$ which is read 2 fifths.

It continues; fractions can be interpreted as part-whole situations: “the numerator tells how many parts we have, and the denominator tells what parts they are.” (p. 25), and this is illustrated by a pie chart. A pie chart is also applied to illustrate equivalent fractions. The expression ‘equivalent fractions’ is not applied; however, it is said that after expanding a fraction it will keep its value. To expand is explained to be to multiply numerator and denominator by the same number.

Simplification is shown as division by the same number in numerator

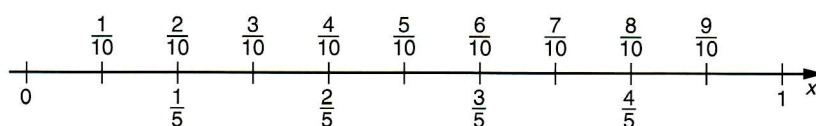
and denominator: $\frac{15}{25} = \frac{15:5}{25:5} = \frac{3}{5}$ mostly written: $\frac{15}{25} = \frac{3 \cdot \cancel{5}}{5 \cdot \cancel{5}} = \frac{3}{5}$.

The rule is: “Before simplifying a fraction, one has to factorise numerator and denominator. Then numerator and denominator must be divided by the same factor” (p.26).

To illustrate the need of having a common denominator before adding, an example is given: A tourist wants to add 250 Norwegian kroner, 420 Euros, and 40 US dollars. In order to execute the calculation, he has to convert all amounts to the same currency. The same is said to be the case with fractions. In order to add them, they have to be converted into the same type of fractions; finding a common denominator and expand the fractions.

For multiplication and division of fractions, the rules are simply given without reasons for the rules. Whole numbers are represented by a fraction with 1 as the denominator.

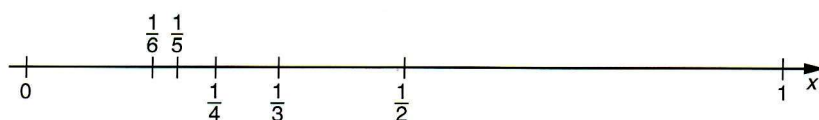
In order to present the set of rational numbers, the number line is used as the mode of representation. Negative numbers are not included:



The interval $[0,1]$ is divided into ten parts, and it is stated if it is divided into 5 parts, each part will be one fifth. It is stated that fractions larger than one, and negative fractions can also be located on the fraction line. The authors conclude that all whole numbers, \mathbb{Z} , and all fractions with whole numbers in numerator and denominator have their locations on the number line. Those numbers constitute the set \mathbb{Q} ; the rational numbers.

The exception with zero in the denominator is not mentioned, although zero is included in \mathbb{Z} , the integers, on page 14.

At the end, there is one paragraph in which the number line is applied to illustrate that it is always possible to divide any interval on the number line into smaller intervals, and it ends with the proposition: Any interval on the number line includes an infinite number of rational numbers.



The number π is used as an example of irrationals, and with this, the set \mathbb{R} is introduced: “On the number line we find whole numbers, rational numbers and irrationals. Together they constitute the set \mathbb{R} of real numbers” (p. 30). How to write intervals with set notation is introduced, and the meaning of those symbols is explained at the end.

The example tasks are tasks just to illustrate the procedures of following the algorithmic fraction rules. In the solution processes, the numerators and the denominators are factorised into prime factors before cancelling common factors.

10.2 Ordinary fractions - ‘the learning book’

The file is a condensed version of the rules in the textbook. Expansion, simplification, the operations of multiplication and division are shown by examples. Simplification is explained in this way:

We can simplify fractions by dividing numerator and denominator by the same number: $\frac{12}{27} = \frac{12:3}{27:3} = \frac{4}{9}$

Prime factorisation is not shown.

Addition and subtraction is explained in this way:

In order to add or subtract two fractions, they must have the same denominators. When the denominators are equal, we add (or subtract) the numerators, while we

keep the denominator: $\frac{7}{8} - \frac{4}{8} = \frac{3}{8}$.

If the denominators are not equal, we have to find a common denominator, this we find by simplifying or expanding at least one of the fractions.

The operations of multiplication and division are given:

We multiply two fractions by multiplying numerator-by-numerator and denominator-by-denominator:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

We divide two fractions by multiplying the first by the second, after turning the

second fraction upside down $\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$ Ex: $\frac{2}{5} : \frac{1}{3} = \frac{2}{5} \cdot \frac{3}{1} = \frac{6}{5}$

All number sets actual for this course and examples and interpretation of intervals in set notation are presented:

Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Whole numbers: $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational numbers: \mathbb{Q} – All numbers that can be written as a fraction with both numerator and denominator as whole numbers.

Real numbers: \mathbb{R} – All whole numbers, rational numbers and irrational numbers.

$\{2, 3, 4, 5\}$ - the four numbers 2, 3, 4 and 5.

$[3, 6]$ - all real numbers from 3 to 6; 3 and 6 included.

$\langle 3, 6 \rangle$ - all real numbers between 3 and 6.

$[3, \rightarrow)$ - all real numbers equal to or larger than 3.

$\langle \leftarrow, 5 \rangle$ - all real numbers less than 5.

10.3 Work plan and tasks - ordinary fractions

The work plan for this part is, as for the first plan, identical with the plan from the publisher. The plan is presented below:

Table 10-1: Work plan - ordinary fractions

Aims	Textbook pages	Items
To expand, simplify, add, multiply and divide fractions	24–28	1.2.1, 1.2.2, 1.2.3, 1.2.6, 1.2.7, 1.2.9
To know the whole numbers, the rational numbers, irrationals, and intervals on the number line	29–30	1.2.10

Only one task, a word problem, might be categorised as demanding a high level of reasoning, the others are similar to the example tasks. The word problem, item 1.2.9 (not bold) is: “One advertisement says: Take three plants, pay for two. The other says: Take five plants and pay for three. The question is which offer is the best”. This is a ‘real life’ situation, and the challenge might be to set up the correct fractions to be compared.

The other two items, not bold, asked students to write intervals in formal set notation, and can be solved by just imitating the examples.

The table below shows the items and how many students solved all tasks in each of them, how many who solved some of the tasks, and how many who solved none, according to the data set of 18 students.

Table 10-2: Number of solved tasks in each item

Number of students who:	Item 1.2.1	Item 1.2.2	Item 1.2.3	Item 1.2.6	Item 1.2.7	Item 1.2.9	Item 1.2.10
Solved all	12	7	5	5	2	3	3
Picked out some	5	7	7	3	1		
Solved none	1	4	6	10	15	15	15

The table shows that the higher the numbering of the items, the higher the number of students not solving any of the tasks.

In the table below the first item is presented. The number of correct solutions, number of saved solutions, and alternative answers are included:

Table 10-3: Ordinary fractions – textbook items 1.2.1 (18 students)

Item	Task	Correct	Solutions	Alternative answers
1.2.1a	$\frac{10}{15}$	16	16	
1.2.1b	$\frac{14}{49}$	16	16	
1.2.1c	$\frac{42}{630}$	8	16	<p>(A) $\frac{\cancel{4}2^6}{\cancel{6}30^{90}} = \frac{\cancel{6}^2}{90^{30}} = \frac{\cancel{2}}{\cancel{3}0^{15}} = \underline{\underline{15}}$</p> <p>(B) $\frac{\cancel{4}2^6}{\cancel{6}30^{90}} = \frac{6}{90}$</p> <p>(C) $\frac{42}{630} = \frac{42:6}{630:6} = \frac{8}{\underline{\underline{105}}}$</p>
1.2.1d	$\frac{24}{39}$	11	12	<p>(D) $\frac{24:3}{39:3} = \frac{8}{\underline{\underline{11}}}$</p>

Most students wrote remaining factors as indexes (A and B). Some few showed the division in both numerator and denominator (C).

Jone, Selma and Tone (solution A), dividing by 2 in the numerator, did not write the result, 1. No number is left in the numerator, and the solution is equalled to 15.

Paul, Tore, Ronny, and Kari did not simplify the fraction to its simplest form (the solutions B and C). It might have been better if they had

prime factorised the numbers before cancelling. Jarl made a division error (solution D).

The next item on the work plan includes addition and subtraction of fractions. Data set 18 students.

Table 10-4: Ordinary fractions – textbook items 1.2.2

Number	Task	Correct	Solutions	Alternative answers
1.2.2a	$\frac{3}{5} + \frac{4}{5}$	12	12	
1.2.2b	$\frac{2}{5} - \frac{3}{5} + \frac{4}{5}$	6	9	(A) $\frac{2}{5} - \frac{3}{5} + \frac{4}{5} = \frac{5}{5}$
1.2.2c	$\frac{5}{3} - \frac{10}{3} + \frac{2}{3}$	5	10	(B) $\frac{5}{3} - \frac{10}{3} + \frac{2}{3} = \frac{7}{3}$ (C) $\frac{5}{3} - \frac{10}{3} + \frac{2}{3} = \frac{5-10+2}{3} = \frac{-3}{3} = \underline{\underline{-1}}$

The minus sign and the size of the numbers involved, might have caused Tord, Ruth, and Tone to ‘detach some terms from their indicated operations’ (A and B). They also did not simplify at the end.

The minus sign might have caused Jone and Paul (C) not to simplify.

Table 10-5: Ordinary fractions – textbook items 1.2.3 (Data set 18 students)

Number	Task	Correct	Solutions	Alternative answers
1.2.3a	$\frac{1}{2} + \frac{1}{3}$	9	9	
1.2.3b	$\frac{1}{2} + \frac{2}{3} + \frac{3}{4}$	5	6	(D) $\frac{1 \cdot 6}{2 \cdot 6} + \frac{2 \cdot 4}{3 \cdot 4} + \frac{3 \cdot 3}{4 \cdot 3} = \frac{6}{12} + \frac{8}{12} + \frac{9}{12} = 2\frac{1}{12}$
1.2.3c	$\frac{4}{9} + \frac{2}{15}$	7	8	(E) $\frac{4 \cdot 10}{9 \cdot 10} + \frac{2 \cdot 6}{15 \cdot 6} = \frac{\cancel{18}^3}{\cancel{90}^{15}} = \frac{3}{15}$
1.2.3d	$\frac{1}{2} - \frac{2}{3} + 1 - \frac{5}{6}$	6	7	(F) $\frac{1 \cdot 3}{2 \cdot 3} - \frac{2 \cdot 2}{3 \cdot 2} + \frac{1 \cdot 6}{1 \cdot 6} - \frac{5}{6} = \frac{3}{6} - \frac{4}{6} + \frac{6}{6} - \frac{5}{6} = \frac{2}{6} = \underline{\underline{\frac{1}{3}}}$

Oscar’s solution (D) is either due to an addition error, or it is a problem to transform improper fractions into a mixed number. The solution (E) is due to a calculation error (Tone). The result fractions (E) and (F), are not reduced to their simplest form, and Ruth ‘detaches the terms from indicated operation’ (F).

The next item on the plan 1.2.6 is shown in the two tables below. Data set 18 students.

Table 10-6: Ordinary fractions – Textbook item 1.2.6 tasks a, b, c, d

Number	Task	Correct	Solutions	Alternative solutions
1.2.6a	$\frac{2}{3} \cdot \frac{6}{7}$	6	7	Result not simplified
1.2.6b	$\frac{3}{4} \cdot \frac{16}{9}$	4	7	Result not simplified (3)
1.2.6c	$\frac{3}{8} : \frac{9}{4}$	5	6	(A) $\frac{3}{8} : \frac{9}{4} = \frac{3 \cdot 4}{8 \cdot 9} = \frac{7}{\underline{\underline{72}}}$
1.2.6d	$\frac{15}{16} : \frac{5}{8}$	4	5	(B) $\frac{15}{16} : \frac{5}{8} = \frac{15 \cdot 8}{16 \cdot 5} = \frac{120^{15}}{80^{10}} = \frac{15}{\underline{\underline{10}}}$

There is one clear tendency in the tasks above; all the students multiplied before simplifying the fractions, although the numbers and the structure of the tasks invite simplification.

Table 10-7: Ordinary fractions – Textbook item 1.2.6 tasks e, f, g and h

Number	Task	Correct answers	Solutions	Alternative solutions
1.2.6e	$6 : \frac{2}{3}$	5	6	(C) $6 : \frac{2}{3} = \frac{12}{\underline{\underline{3}}} = 9\frac{3}{3} = \underline{\underline{9}}$
1.2.6f	$\frac{2}{3} : 6$	6	7	(D) $\frac{2}{3} : 6 = \frac{2 \cdot \cancel{6}^2}{\cancel{3}} = \underline{\underline{4}}$
1.2.6g	$\left(\frac{3}{5} + \frac{4}{5}\right) \cdot \frac{3}{14}$	3	5	(E) $\left(\frac{3}{5} + \frac{4}{5}\right) \cdot \frac{3}{14} = \frac{7}{5} \cdot \frac{3}{14} = \frac{21^7}{70^{10}} = \frac{7}{\underline{\underline{10}}}$ (F) $\left(\frac{3+4}{5+5}\right) \cdot \frac{3}{14} = \frac{7}{10} \cdot \frac{3}{14} = \frac{21}{\underline{\underline{140}}}$
1.2.6h	$\left(\frac{1}{2} + \frac{2}{3}\right) : \frac{10}{9}$	6	7	(G) $\left(\frac{1}{2} + \frac{1}{3}\right) : \frac{10}{9} = \frac{2}{6} \cdot \frac{9}{10} = \frac{18}{\underline{\underline{60}}}$

In task 1.2.6a and 1.2.6b, Tone, Ruth, and Oscar did not simplify the fractions at the end.

Four students had solved all tasks correctly. Tone and Ruth represent all the alternative solutions listed. Tone's problem seems to be that she had problems handling whole numbers in relation to fractions (C) and (D). In task 1.2.6e (C), she does not follow the rule for division, and it might be that she has subtracted 3 from the numerator 12, giving the answer 9, which she writes as a whole number. The rest, 3, is written as a

fraction with the denominator 3. The result is $\frac{3}{3}$, perhaps equated with 1.

If she multiplies 9 by 1, the result is 9. This is one interpretation, another might be that she has seen the answer in the textbook and then has adjusted her own answer in accordance with that. In task 1.2.6g (E) she does not simplify correctly.

Ruth in task 1.2.6c (A) turns the last fraction upside down, but when going to multiply, she adds the numerators and multiplies the denominators. In task 1.2.6g (F) she adds the fractions by adding the numerators and the denominators before she executes the multiplication. Here she performs the multiplication properly. She seems to regard the numerator and the denominator as separate whole numbers. In task 1.2.6h (G) she finds the least common denominator; however, she does not expand the fractions properly, then she adds the numerators and multiplies the denominators.

It seems as if Ruth has no stable conception of the fraction operations. At this point in time, she seems not to be consistent at all.

The other tasks on the work plan were not written with bold fonts, however, four students had solved some of them. Task 1.2.7 is about placing two sets of fractions on a number line (none of them were equivalent). Ruth, Rakel, Else, and Olav ordered them in two sequences from least to the largest with no explanation, as in the textbook key.

Task 1.2.9 was the word problem (see appendix 10.3). Three students solved the task, two by writing the solution as it was written in the textbook. Only one student, Jone, seemed to have found the fractions before comparing them.

Task 1.2.10 asked students to write sets of numbers in formal set notation, imitating the examples in the textbook. Rakel, Jone, and Olav, had done that; correctly.

10.4 Algebraic fractions in the textbook

It is stated that the same rules are valid for algebraic fractions as for numerical fractions. When simplifying fractions with binomials, however, one has to factorise by setting common factors outside a bracket, and then simplify by dividing numerator and denominator by the same expression. It is said to be the opposite of expanding the bracket. This is told in a few sentences, and two tasks are solved as examples:

$$\frac{8x^2y}{24xy} = \frac{\cancel{8} \cdot \cancel{x} \cdot x \cdot \cancel{y}}{3 \cdot \cancel{8} \cdot x \cdot y} = \frac{x}{3} \quad \text{and} \quad \frac{3a+6}{6a-9} = \frac{\cancel{3}(a+2)}{\cancel{3}(2a-3)} = \frac{a+2}{2a-3}$$

Then the book states that to add or subtract fractions, the denominators must be equal, and when not, the fractions must be expanded. Two ex-

ample tasks are worked out to exemplify this: a) $\frac{a+3}{4} - \frac{a-1}{6}$ and

b) $\frac{2}{x+2} - \frac{1}{3x+6}$. During the solution, brackets are written around each

binomial, but it is not commented upon. A piece of advice is given in the margin, that it is wise to put the binomials on one fraction line before the brackets are removed.

In example b), the readers are just told to first factorise the denominators. Then the solution is shown.

Two more tasks are worked out as examples of multiplication and division. It is claimed that one has to follow the same rules as when operating on numerical fractions. The tasks are solved:

$$\text{a) } \frac{6x^2}{5} \cdot \frac{15}{2x} = \frac{\cancel{2} \cdot 3 \cdot \cancel{x} \cdot x \cdot 3 \cdot \cancel{5}}{\cancel{5} \cdot \cancel{2} \cdot \cancel{x}} = 9x$$

$$\text{b) } \frac{3}{a+1} : \frac{4}{3a+3} = \frac{3}{a+1} \cdot \frac{3a+3}{4} = \frac{3}{(a+1)} \cdot \frac{3(a+1)}{4} = \frac{9}{4}$$

10.5 Algebraic fractions –in the learning book

There is one difference between the textbook and the learning book, which is worth mentioning. The way to write the simplification is different. In the ‘learning book’ the simplification is presented in this way:

$$\frac{8x^2y}{24xy} = \frac{\cancel{8}x^{\cancel{2}}\cancel{y}}{\cancel{24}^3\cancel{x}\cancel{y}} = \frac{x}{\underline{\underline{3}}}$$

In the textbook, the fraction is factorised and written like this:

$$\frac{8x^2y}{24xy} = \frac{\cancel{8} \cdot \cancel{x} \cdot x \cdot \cancel{y}}{3 \cdot \cancel{8} \cdot \cancel{x} \cdot \cancel{y}} = \frac{x}{\underline{\underline{3}}}$$

The same example tasks are presented as in the textbook.

10.6 Algebraic fractions - introduction

The teacher points to the day before when they worked with ordinary fractions, and introduces the new lesson by saying:

We are going to work with the same as yesterday. We are going to simplify fractions. Yesterday we worked only with numbers; today we are going to bring in letters. Do you think that is difficult?

There are some few responses. Then she presents six examples one after another, all are example tasks from the textbook. The students are expected not to write since they can find the examples and their solutions in the textbook or on the web site, in the ‘learning book’.

Simplifying fractions

The first example is: $\frac{8x^2y}{24xy}$

When she writes the fraction on the whiteboard, she says what she writes. This she does every time she presents an example on the whiteboard. She points to a student and asks if she can start. The first student she asks to join her, Ane, says that she does not know how to do it, the next, Ine answers, and the teacher writes on the whiteboard:

- Ine: Can't we cancel some that are similar?
T: Yes, tell me which of them.
Ine: We can cancel the y-s.
T: Yes, I can simplify by cancelling the y-s. What am I doing then?
Ine: We cancel both.
T: What operation is it?
Ine: We divide.
T: We divide. y divided by y is equal to? (*The teacher points to the numerator*)
Ine: One
T: And y divided by y is equal to? (*She points to the denominator*)
Ine: One
T: Yes, so then we just cancel them. Normally we do not write the number 1. (*She draws a straight line over each y.*) Ok? Are we going to continue?

Ine knows what to do, and when the teacher prompts her, she explains that the operation she executes, is division. The teacher writes, revoices Ine's response, and comments that the remaining factor 1, is normally not written. Instead, the y -s are just cancelled by overwriting them. Tord then asks a question:

- Tord: Can't we cancel the x -es as well? $2x$ up there and x down there? (*He points to the numerator and then to the denominator*)
T: Are there two x -es up there?
Jone: No, that is x times x .
Tord: Yes, it is x times x .

Tord talks about $2x$ but supposedly he means x squared. The teacher reacts by asking if there are two x -es, and Jone and Tord respond by giving the correct notion.

She then asks what to cancel.

- Roy: You have to cancel 2 (*the index*)
T: Yes, cancel this against? (*She draws a straight line over the index.*)
Roy: The x .
T: This one? (*She points to the x in the denominator*)
Tord: Then it will be one, sort of.

At the end when working with this example, Else raises her hand and says that 8 times 3 is 24, and thus the numbers can be simplified as well. The teacher revoices what Else said and shows the cancelling on the

whiteboard, saying that 8 divided by 8 is 1 and 24 divided by 8 is 3. The number 1, the remaining factor, she writes as an index over the number 8 in the numerator, to let the cancelling be visible, the same is done with the number three above 24. She has drawn a straight line over the two

initial numbers. On the whiteboard, she has written $\frac{1 \cancel{8} x^{\cancel{2}}}{3 \cancel{24} \cancel{3}}$ and asks:

T: And that will be?

Some students: x divided by 3.

T: Yes, x divided by 3 (she writes $\frac{x}{3}$), and then we are finished. (*She turns towards the students*) Okay? Then we have found those factors that are equal.

The transcripts above describe the style of interaction normally occurring in the classroom. The teacher has a distinct goal and engages the students. Students are willing to participate, although the first student she asked here, said she did not know what to do. The teacher accepted her reluctance.

The questions posed, require only short answers. The teacher expresses no need for a wider explanation, although she asks follow up questions in order to emphasise what operation was applied. However, the teacher opens up, then waits patiently for the responses, and the students have a possibility to expand their suggestions.

The next example task is: $\frac{3a+6}{6a-9}$

Some students suggest cancelling the a 's, Ronny, however, states it is impossible because there is a minus sign. Roy suggests that division will give a divided by $2a$. It seems that many students ignore the fact that the numerator and the denominator are binomials. The teacher stands at the front patiently listening to the suggestions, telling them that she does not agree with them.

To illustrate the need of factorising, the teacher applies the example $\frac{25-5}{5}$, writing it to the right on the whiteboard. From this example, the responses show that students agree that to cancel the 5-s will not give the correct solution. However, on the question why it will not work, the students only point to the fact that the answer will be wrong, or that there is a minus sign in the numerator. The teacher wants them to expand on this and waits for more. After many hints and a period of silence Jarl answers 'factors' to the question of what makes a product. Nobody, however, came up with the notion of 'term' and the teacher introduces this notion herself.

She uses the first example, $\frac{8x^2y}{24xy}$, to focus on invisible operation

signs, saying that although there are no visible signs between letters and numbers, the operation is multiplication.

She then goes back to the second example and emphasises by gestures that the expressions in the numerator and in the denominator both are entities and must be handled as such. The terms within them cannot be cancelled. She points to each term, and again she circles around the numerator and around the denominator to make clear that each of them is to be seen as one entity. In order to simplify in this case, it is therefore needed to factorise. Only factors can be cancelled.

Jone is the first to try to solve the problem. He suggests to write the factors as 2 times 3 times a . The teacher points to the fact that he still is operating on one only of the terms. Tord suggests changing the “signs in front of the ‘things’ in the numerator and denominator”. He stops saying he cannot remember how it should be, but mentions something about things that should be equal.

The teacher then draws the attention to the expression $3(a + 2)$, she writes it to the right on the whiteboard, and comments that this should be familiar, and asks what to do. Many of the students respond, and the teacher performs the operation led by the students. She writes $3(a + 2) = 3a + 6$. She points to the left and to the right of the equal sign asking if the two expressions are the same. The students agree. She asks:

T: But that is what is here as well? (*She points to the initial binomial. The students agree*). Then what we have to do, is to do the opposite of multiplying the bracket, when we factorise. We are going to find a factor common for the two terms in the binomial, (*she circles with her pen around each term*) and then we put this factor outside a bracket. We have to think opposite of multiplication. Here 3 is a factor in both terms, and 3 is to be set outside the bracket. You see that? (*There are clear yes-responses from the students*). And now Ivar, what is the common factor in the denominator?

Ivar: 3

T: Yes, it is 3 also in the denominator. Because 3 goes in 6 and 3 goes in 9. What is left then?

More students: 2a

T: And?

Several students: minus 3

T: and minus 3. Is it possible now to simplify?

Peter: I think you can cancel the 3s beside the brackets.

T: Yes, that's nice. Then the result is $\frac{(a + 2)}{(2a - 3)}$ (*she says what she writes*)

The teacher asks if it was new for them to factorise binomials. Some of the students answer it should not be, but they admit they had forgotten. Finally, she says:

T: You have to remember the difference between, terms and factors. It is so important. Immediately, when you see that you have two terms, not factors, then you actually now should know what to do.

Adding and subtracting fractions

After a break the teacher continues. She presents two examples of subtraction of fractions, both are examples from the textbook. And she writes : $\frac{a+3}{4} - \frac{a-1}{6}$ on the white board, expressing what she writes.

She asks what should be done, and Roy says they have to find a common denominator. The students, who respond, tell the teacher that 12 is the least common denominator. She looks around in the classroom and asks if all agree. They express agreement. Carl tells her to “time up and down” She revoices his utterance by emphasising the word ‘expand’. She does what he says, and writes on the whiteboard: $\frac{a+3 \cdot 3}{4 \cdot 3}$. She asks if

it is okay, but the students tell her to put in a bracket. The real problem comes, however, when the last fraction is to be subtracted. Both fractions are expanded. There is not agreement among the students if the sign should be a plus or a minus sign in the last binomial. The teacher gives them time to discuss before she gives them the didactical advice to write the binomials in brackets and not perform more than one operation at a time; in order to keep control.

Else interrupts and asks why the sign is a minus sign in the last binomial. She thought it should be a plus sign because the post-multiplier was positive. The teacher points to the preceding minus sign.

The next example causes more trouble for the students. The binomial, however, in the second denominator is identical except for the letter symbol, to the numerator in the example with the fraction simplified above. Thus. it is not new to the students to factorise this binomial. The example task is: $\frac{2}{x+2} - \frac{1}{3x+6}$.

The teacher asks for the least common denominator. There is silence, but Eli has a suggestion after a while: $3x + 6$ should be the common denominator. The teacher asks if she is going to factorise the expression, which Eli denies.

By pointing to the two denominators, the teacher states that they seem to be very different, and suggests to write them underneath each other on the right side of the whiteboard and to factorise them. She asks

if they have applied that method when working with arithmetical fractions. Some of the students confirm this. She writes the binomials asking if the first denominator can be factorised. There is no response. She says that it is not possible, and asks if it is possible to factorise the next:

T: Do we have a common factor?

Tord: 3

T: Yes, it is 3. 3 is a factor in both $3x$ and in 6. Then we factorise as in the foregoing example. What is left in the bracket? (She points to the binomial).

Tord: $x + 2$

T: $x +$ and 2, because 6 divided by 3 is 2. Then we can find the common denominator. What will that be? What must be included in the common denominator?

Tord: 3 and x

T: Yes we must have 3 and

Tord: And x .

T: And? (*she has written 3 and a bracket*)

Tord: Oh?

Ronny: $x + 2$

T: $x + 2$. Yes, that's a factor. Both are factors (*the denominators*). Do I have to write that factor twice?

Else: No.

T: No, we must not. And now we have to check the other denominator. I have to find that denominator also within the common denominator. $x + 2$, is included, 3 multiplied by $x + 2$, is also there. We have to find all the denominators. This is not easy, but what factor do I have to use to expand the first fraction?

Tore: 3

T: Yes, by 3 (*she multiplies both numerator and denominator by 3*). And we put $x + 2$ in a bracket, to the right (*on the whiteboard*) where we have the factorised denominators. And the second fraction?

One student: We do nothing.

T: Yes, nothing, and then we write both fractions on the same fraction line. Then I have $6 - 1$, Here we do not need any bracket, there is only one term in each numerator. And in the denominator?

Roy: (murmurs) $3x + 6$

T: $3(x + 2)$. 6 minus 1 is 5, can we simplify?

Bill: We can multiply.

T: Yes, I can multiply or I can let it be $3(x+2)$.

Ronny: Is there any point in executing the multiplication?

T: No, actually not. We can let it be like it is. (*She goes to the computer*) This solution is on the web site. (*She shows the computer file*). I have written only the common denominator, not shown how we find it.

Multiplication of fractions

In the example the rule for multiplication: numerator multiplied by numerator and denominator by denominator is emphasised. In addition, the

teacher draws the attention to the benefit of simplifying before multiplication. The example is $\frac{6x^2}{5} \cdot \frac{15}{2x}$. Some students have suggested multiplying first.

- T: Do we just multiply now? (*she has to wait for an answer*)
 Else: No, we can simplify.
 T: Yes, we can simplify. Was Else the only who noticed that? No? Yes, we will simplify.
 Tord: I just didn't dare to say it (smiling).
 T: You just didn't dare to say it? (*she smiles at him*). Okay we simplify, can you, Else do it? Take one factor at a time.
 Else: Okay, 2 goes, 2 times 3 is 6.
 T: Yes, then we have?
 Else: 3
 T: (To the class) We have 3; 2 goes 3 times into 6, you see?
 Else: And then x and x squared.
 T: Okay. Then I cancel that x (*in the numerator, she makes a line over the index 2*), I cancel one of the two x -es, because x squared is x times x and one must be left.
 Else: Yes.
 T: Up there we now have $3x$ and 3 (*she points to the numbers left*). $3x$ and 3 . That will be?
 Else: $9x$
 T: Is something left down there?
 Else: No

This last question and answer was not commented upon, but the teacher emphasises that the rules are the same for algebraic fractions as for ordinary fractions. Ronny comments that it seems more complicated.

Division of fractions

The example is: $\frac{3}{a+1} : \frac{4}{3a+3}$ The teacher asks for the rule for the division

of fractions, and Tord expresses it. The fractions are written on the white board once again both on the same fraction line, the last fraction turned

upside down: $\frac{3(3a+3)}{(a+1)4}$. The teacher has put $3a+3$ within a bracket

without mentioning it. When she comes to $a+1$, she explains that this is a factor and has to be written within a bracket. The teacher asks:

- T: Can we simplify now?
 Ronny: Yes, the a -s.
 T: Can we?

Several students are to be heard; their comments go about that it is not possible. The teacher asks why not, and Tore answers that it is not possible since there are terms. Tord at the same time suggests carrying out the multiplication. It is not clear if the teacher ignores Tord's suggestion or if she does not hear it, but she replies to Tore's utterance. She repeats

Tore's claim about the terms, and Tore continues to say that they are not factors. The teacher agrees and emphasises that the a 's are within the brackets, and that it is the whole brackets that are the factors. The brackets are entities and must be considered as such. She asks if it is possible to simplify. Ruth expresses the same suggestion as Tord did, to multiply, and now the teacher listens.

T: To multiply here, like this you mean? And then multiply there?

She points to the numerator and then the denominator on the whiteboard before she repeats what Ruth said. She then says that she first will check if it is possible to simplify the expression. She waits for other suggestions and Tord comes up with a new one:

Tord: Can't we just take away the a down there and then sort of bringing the $2a$ -s up?

T: Take this away and then two of them? We cannot do that. It is, as he (Tore) said, the two of them there, they belong to each other (*she circles around the bracket with her pen*). If a had the value of 4 for example, then this would have been 5, (*she points to the first bracket*) and if we had 4 there, (*she points to the other bracket*) it would have had the value of 3 times 4, which is equal to 12 and ...

The teacher tries to exemplify by substituting the letter symbol by a number, in order to let it be obvious that Tord's suggestion will not lead to an acceptable solution. But she is interrupted by Tore, who asserts that if something should be done, one has to operate on the numbers 3 and 4 because they are outside the brackets. She now focuses on Tore, and leaves the number example.

Actually what Tord suggested, was to subtract. Peter who was listening to Tord, follows up on his suggestion, taking the whole brackets into account:

Peter: Can't we take both the numbers within the bracket?

T: Certainly.

Peter: Then you can do like this; $2a$ and 2 in the upper bracket.

T: No, we can't.

Peter: We cannot?

T: No, how is it possible to factorise then?

She patiently waits for responses. Then she explains carefully by circling around the brackets, that they are to be handled as entities. They cannot be split. She does not go into Peter's idea which seems to be to take $3a - a$ and $3 - 1$ resulting in $2a - 2$ in the numerator. He applies subtraction.

The teacher continues, asking how it is possible to factorise. She circles around the bracket in the numerator, and then Jone says:

Jone: 3 times, and then a bracket with $a + 1$.

T: Nice! What have you done then?

He replies, and the teacher repeats his reply for the whole class:

- T: He said that he saw that he had a common factor here.
Some of the students seemed surprised.
- T: Yes, he did, and then we could place that factor outside the bracket. So in that way he factorised that expression.
The teacher then performs the factorising under Jone's guidance.
- T: Now we will see. You put 3 outside the bracket (*she writes*). What was left then?
- Jone: $a + 1$.
- T: Yes, you all have to remember that it will be 1 left, because we divide 3 by 3. Good! And in the denominator I have $a + 1$ multiplied by 4 (*she writes $a + 1$ in a bracket*) And then?
- Ronny: (*whispers*) Then we cancel the brackets.
- T: Yes, now we can see that we have two equal factors. That one is equal to that one (*she points to the two brackets*). Now they are identical and after cancelling we have 9 fourth.

She ends the session by stating that they had used a pretty long time to come to the solution. Once again she emphasises the need to factorise, and that binomials in numerator and denominator have to be handled as entities.

Immediately after the plenary session, Tone, Ruth and Selma asked the teacher to come to their desk and explain once again what was done in the worked out example $\frac{3a+6}{6a-9}$ from the textbook. The teacher points to the whiteboard while she explains:

- T: You see that 3 is a factor both in the first and the second term in the numerator?
- Selma: Yes?
- T: Yes and then I put that factor outside the bracket.
- Selma: Okay, so then you remove it from the a ?
- T: Yes, I am factorising, I put 3 outside the bracket. That was exactly the same as we did on the first task today (*they are focused on the white board*)
- Selma: Yes?
- T: And now we will see. If we look into the textbook, then you will see. *Selma's question about removing the 3 from a , and the teacher's answer does not include which operation to execute. However, the conversation goes on. Ruth asks where the example is to be found in the textbook and the three students look at the book, while the teacher explains.*
- T: Do you remember this, where we factorised and put a number outside? That was exactly what we did there (*She points to the white board*). In order to find two equal factors, we had to factorise the numerator as much as possible. You see that 3 is a common factor?
- Selma: Oh, yes.
- T: Yes, do you see the relationship?
- Selma: Yes
- T: Fine! (*It might be that the teacher is not convinced, that Selma understands, at least she repeats:*)

T: You see that example? You see that the number 3 is common in those two (*terms*).

She points at the whiteboard and also at each term in the textbook, and Selma nods.

T: Those terms there you see? I write 3 outside the bracket, and when I have divided those (*terms*) by 3, then I have *a* there, and after dividing there by 3, I have 1 left. (*She is pointing to each of the terms while she explains.*)

Selma: Yes.

T: Do you understand?

Selma: Yes.

Although the teacher just had worked out the example on the white board, she patiently describes once again what has to be done, and at the end she emphasises that factorising is about division.

10.7 Algebraic fractions work plan and tasks

The work plan for algebraic fractions:

Table 10-8: Algebraic fractions – work plan

Aims	Textbook	Items	
Calculate with fractions including letters	33–34	1.2.12, 1.2.13, 1.2.14, 1.2.15	Blue track
Calculate with fractions including letters	35–36	1.2.16, 1.2.17, 1.2.18, 1.2.19, 1.2.21	Red track

The students should follow the red track. Those items are listed in the table below with number of students solving the tasks (18 students). Also here it is evident that the further out in the plan, the smaller is the number of students solving them:

Table 10-9: Number of students solving tasks in each item

Number of students who:	Item 1.2.16	Item 1.2.17	Item 1.2.18	Item 1.2.19	Item 1.2.21
Solved all	13	11	9	7	1
Picked out some	3	5	5	4	2
Solved none	2	2	4	7	15

The items written with bold fonts should be home work if they had not been solved during the lessons. The data from the classroom show that 6 of the 11 students who solved one or more of the tasks in item 1.2.19 had done that during the lessons. Five of them must have solved them at home. The others had not solved that task.

10.7.1 Simplification, multiplication and division of fractions

The following tables are organised with item number in the first row, and the tasks in the second row. The category ‘solutions’ gives the number of students solving the tasks from the data set of 18 students (saved computer files). In the last row in each table are responses related to each task from the whole class, presented in keywords.

Table 10-10: Algebraic fractions – Simplification (textbook item 1.2.16)

1.2.16a	1.2.16b	1.2.16c
$\frac{20a^2}{4b}$	$\frac{56x^3y}{21xy^2}$	$\frac{4a+8}{5a+10}$
13 solutions (of 18)	16 solutions (of 18)	16 solutions (of 18)
<p>Selma invisible sign, factor and terms</p> <p>Jone and Paul - discuss and solve the task</p> <p>Tone- needs confirmation</p>	<p>Sven, Glenn, Ronny, Kristin, and Tore- do not understand how to come from 56/21 to 8/3. they ask for help</p> <p>Sven – factor?</p> <p>Jone, Glenn, and Olav-ask about powers</p>	<p>Kristin, Kari, and Rakel - how to factorise?</p> <p>Eli - cancels term by term within the brackets</p> <p>Ine and Ane – factorise by dividing only the first term. Teacher asks them to multiply after factorising to control their result.</p> <p>Ronny and his peers partly cancelling; no sense to factorise, correct answer anyway.</p> <p>Tord – computer search for a way to solve the task.</p> <p>Tone, Ruth, Sven, and Glenn – ask how to factorise brackets</p> <p>Else- partly cancelling</p>

Selma although saying in the foregoing section that she understood how to factorise, is still stuck when solving task 1.2.16a. The teacher asks her to make the invisible operation sign visible, and emphasises the difference between factor and terms.

Five students asked for help about factorising the numbers in task 1.2.16b after being surprised by the answer in the textbook key. Ronny and Jarl used the calculator in the TI program. Sven, Kristin, and Glenn were challenged by the teacher to find the factors in each of the numbers.

Olav was videotaped when he worked on the same task. He was asked how he would start. He answered:

Olav: Well yes, with the numbers perhaps. And then for example x , you cancel the x and then you cancel the 3. (*The x in the denominator and the index in the numerator*).

R1: What is left when you cancel the 3 (*the index*)?

Olav: Then I have x .

R1: Hum

Olav: Let me see. Yes.

Presumably since I did not give him a clear positive response, he checked his suggestion once again and seemed to be satisfied, saying yes.

R1: What does the number 3 mean in relation to the x ?

Olav: It means that, or that x (he points to the x in the numerator on the screen). It must, it must be one x left.

He makes a line over the x in the denominator and a line over the index 3 in the numerator to visualise the cancelling. From his utterance, it seems that he means that there is one x left in the numerator. The conversation continues:

R1: But when we have x to the power of 3?

Olav: Yes, that means x times x times x .

R1: Okay. So when you then divide by?

Olav: No... (In his computer file later his solution was correct.)

Task 1.2.16c $\frac{4a+8}{5a+10}$ is discussed in the thesis.

The next item on the work plan involved fractions with binomial numerators. The tasks are presented in the table below. The row with number of solutions, gives the number of students solving the task (dataset 18 students). The last row is based on the whole class.

Table 10-11: Algebraic fractions – Simplification (textbook item 1.2.17a, b, c)

Task 1.2.17a	Task 1.2.17b	Task 1.2.17c
$\frac{x+3}{5} - \frac{2x-1}{5}$	$\frac{x+1}{2} - \frac{x-2}{3}$	$\frac{2}{3} + \frac{5}{a} - \frac{a}{2}$
13 solutions of 18	14 solutions of 18	11 solutions of 18
<p>Else - minus sign error</p> <p>Alf - problems both minus sign and fractions</p> <p>Peder - tells he does not write brackets</p>	<p>Rakel asks for confirmation</p> <p>Ine and Ane - changes sign in binomial too early</p> <p>Tord: $-3x - 2x = -x$</p> <p>Ronny - multiplication as exponentiation</p> <p>Glenn - multiplier as index</p>	<p>Kari and Rakel - discuss the writing of powers. Fractions formally incorrect</p> <p>Ronny - asks about powers, is surprised how he has to write expansion.</p> <p>Selma and Atle – problem LCD, ask the teacher</p> <p>Jone - sloppy talk about powers</p> <p>Peter and Erik - need confirmations</p>

In the first tasks above the errors and the questions are related to the minus sign and to powers. Else in task 1.2.17a had not changed the sign in the last binomial. In the next task, when she had to expand the fractions before subtraction, her solution was correct. The last tasks in the item are presented below (Number of solutions from dataset 18 students; the last row observed from all students):

Table 10-12: Algebraic fractions – Simplification (textbook item 1.2.17d, e, f)

Task 1.2.17d	Task 1.2.17e	Task 1.2.17f
$\frac{3}{a} + \frac{5}{b} - \frac{a}{2b} - \frac{2}{ab}$	$\frac{2}{b} - \frac{1}{3b} - (\frac{1}{2} + \frac{5}{3b})$	$\frac{4}{2a} + \frac{2}{3b} + \frac{a-2b}{ab}$
13 solutions of 18	13 solutions of 18	14 solutions of 18
<p>Olav – mixes multiplication/ exponentiation</p> <p>Selma – how to expand after finding the LCD</p>	<p>Rakel, Ruth, Bill, Ronny and Kari - saved positive solutions</p> <p>Ine and Ane ask about bracket and sign, and about simplification</p> <p>Jarl - positive solution, but is helped by his peers.</p> <p>Glenn- problems with common denominator, and with expansion; is helped by peers.</p> <p>Sven - problem with signs, asks teacher for help</p>	<p>Bill, Carl, Ronny, Tore, Glenn, and Jone - do not reduce to its simplest form.</p> <p>Ruth asks where to write brackets.</p> <p>Kristin mixes multiplication/exponentiation</p> <p>Eli- simplifies after checking the textbook key</p>

The data from the classroom, show that for some few students it is not so easy to work with fractions. Alf's solution of task 1.2.17a will be presented. He had solved the tasks in item 1.2.16. When the observation starts he has written:

$$\frac{x+3}{5} - \frac{2x-1}{5}$$

$$\frac{(x+3) \cdot 2}{5 \cdot 2} - \frac{(2x-1)}{5 \cdot 3}$$

$$\frac{(x+3) \cdot 2}{5 \cdot 2} - \frac{(2x-1) \cdot 3}{5 \cdot 3}$$

He has copied line 3 and is about to perform the operations. He is asked if he thinks this is easy:

- Alf: No, I have no clue about what I am doing really. I just act as if I have. *He is observed when he performs the operations, and he comes to the result: $\frac{2x+6}{10} - \frac{6x-3}{15}$. He works when the conversation continues.*
- R1: I see that you have used the number 2 in the first fraction and 3 in the other. What thoughts do you have about what you are doing?
- Alf: Actually I have no thoughts; I just try to multiply the numbers by each other. *(He points to the screen)*. That's what I have done up through the whole elementary school. So now I'm just doing the same over again.
- R1: Okay?
- Alf: It tends to become somewhat easy. But it...
- R1: But why did you for example choose the number 2 in the fraction to the left?

Alf: Because there is a 2 there (*he points to the number 2 in the numerator in the next fraction*). It is as simple as that.

R1: Because you have a 2 there in front of the x ?

Alf: Yes, and now, then I time with 2 once again, and we will see what happens then. (*He writes*). And then I time with ... No, that one we time by 3 (*substitutes the number 2 by 3*). Then we try to get the two (*denominators*) there to be equal.

R1: Okay.

Alf: It's not just a stroke of genius, but ...oh damn.

R1: But the number 3, from where did you choose that? You just took it?

Alf: I just pulled it out of thin air.
Although he said it was accidentally, he expanded the second fraction by 2 in order to have the denominator 30 in both fractions. He was then asked if this was his purpose:

Alf: Yes, they should be equal (*he points to the two denominators*). Then I get them on one fraction line and there will be some extra lines, but I don't care.

R1: How were they in the beginning? Were they then different?

Alf: (*He watches the screen.*) Then they were actually equal.
He deletes what he has written, except for the first line.

Alf: Thanks for the hint.
He starts from new. The first line he keeps, and in the next line he writes only the numerators. He goes to the first line again saying:

Alf: Perhaps I should write a line over them.
He writes in the first line: $\frac{x+3}{\cancel{5}} - \frac{2x-1}{\cancel{5}}$

R1: What happened with the numbers 5?

Alf: The denominators were equal and now at least I hope it will be correct.

R1: Hm. Did they disappear now?

Alf: Yes, didn't they? I really don't have a clue.

R1: Hum?

Alf: I really don't have a clue. That's why I am just trying things out.

R1: Yes?

Alf: I should really think I was a bit clever here, but now it didn't work.
He continues writing: $(x+3) - (2x-1)$ before he removes the brackets saying:

Alf: It will be minus anyway.
He writes his final result $x+3-2x-1 = \underline{\underline{-x+2}}$

R1: I was a bit surprised about the numbers 5.

Alf: I think it became very nice, I really do.

R1: Hum?

Alf: It is just the same, isn't it? (*he points back to the denominators in the first line*). Then it was time to bring all on one fraction line.

R1: What if you had no letters, what would you do then? If you for instance had the fractions $\frac{3}{5} - \frac{1}{5}$? (*He then puts in 5 as denominator in line 2*)

R1: You probably think fractions are difficult?

Alf: I have no idea what I do anyway.

The final result he had not changed when I left. He was still working. Alf said openly that he had problems, however, did not ask for help or advice. He was working continually while we were talking together.

When I left, he was still working, and I thought he was going to put in the denominator also in the final answer, but it might be that he did not. Alf is not among the 18 students from which I have the saved computer files. On the test in September he solved a similar task:

$$\begin{aligned} & \frac{3}{2a} + \frac{a}{4b} - \frac{a-2b}{ab} \\ & (2b \cdot \frac{3}{2a}) + (a \cdot \frac{a}{4b}) - (4 \cdot \frac{a-2b}{ab}) \\ & \frac{6b}{4ab} + \frac{a^2}{4ab} - \frac{4a+8b}{4ab} \\ & 6b + a^2 - 4a + 8b \\ & \underline{\underline{= 2a^2 + 14b - 4a}} \end{aligned}$$

In the task above he wrote the expansion of the fractions in a formally correct way, but here (to the left) he has written the expansion as ordinary multiplication. He, however, performed the expansion correctly. He then cancels the denominators before he adds the numerators. The sign in the binomial numerator he changed when the binomial still was the numerator in the last fraction. In the same test, he kept the denominator when solving ordinary fractions.

Another student, Ronny, worked with task 1.2.17c. He correctly found the least common denominator, but asked how to write the expansion. He was surprised when the teacher told him to multiply in both numerator and denominator.

In the classroom, few students were observed asking for help because of signs and binomials, but some asked the teacher to check their solutions.

Atle asked the teacher how to find the common denominator in task 1.2.17c. Selma had found it. Her problem was how to find the factors by which she should expand the fractions. She says:

Selma: I have to find the common denominator. It will be $6a$, but how am I going to do it, I mean the multiplication?

T: You can just think like this: What do I have and what do I miss?

Selma: Then I have to multiply by 2 and a in the first term.

From this it seems as if she was helped in her thinking. Selma performs the task correctly.

Glenn in task 1.2.17e had multiplied only the denominators, when performing expansion. He was helped by his classmates.

Again, in the next table, the number of solutions refers to the number of students from the data set of 18 students, and responses in the last row are from the whole class.

Table 10-13: Algebraic fractions – Addition (textbook item 1.2.18)

1.2.18a	1.2.18b	1.2.18c
$\frac{2a}{3} - \frac{3+a}{6}$	$\frac{a}{12} + \frac{a-2}{4} - \frac{a}{3}$	$\frac{a-b}{a} - \frac{b-a}{b}$
10 solutions of 18	11 solutions of 18	13 solutions of 18
<p>Rakel and Kari -sign errors and partly cancelling.</p> <p>Tord and Kari -divide each term by 3. Do not factorise.</p> <p>Ruth –has calculation error and partly cancelling.</p> <p>Ronny – “it can’t be simplified”, but Jarl helped</p> <p>Tone - conjoins $3 + a = 3a$</p>	<p>Ruth – asks about brackets, has sign error, and does not simplify</p> <p>Bill- does not simplify</p> <p>Tone - forgets the first term.</p> <p>Kari- did not end her solution</p> <p>Sven-asks about negative result</p>	<p>Rakel, Ruth, Ane and Tone -sign error. Rakel ends with correct solution.</p> <p>Ronny, Eli, Victor and Ine - ask about the solution in the textbook:</p> <p>$\frac{a^2 - b^2}{ab} = \frac{a}{b} - \frac{b}{a}$. The teacher explains how to split into two fractions.</p>

The main problems in these tasks were: the minus sign, to simplify the results, and partly cancelling.

In item 1.2.19, students were asked to multiply, divide and simplify fractions. The tasks are presented in the table below. Numbers of solutions from 18 students, and responses from the whole class in the last row.

Table 10-14: Algebraic fractions – (textbook item 1.2.19)

1.2.19a	1.2.19b	1.2.19c
$\frac{15a^2}{7b} \cdot \frac{21b^2}{5a}$	$\frac{3x+9}{4} \cdot \frac{12}{2x+6}$	$\frac{2a+6}{5} \cdot \frac{a+3}{15}$
10 solutions of 18	7 solutions of 18	10 solutions of 18
<p>Only Rakel simplifies before multiplication.</p> <p>Tore and Ronny - cancel all letters</p>	<p>Rakel, Olav and Tone - solve the task correctly</p> <p>Tore, Ronny and Carl - cancel the letters symbols</p> <p>Ruth transforms all operations signs into multiplication signs</p>	<p>Rakel, Olav, Paul, Jone, Tone, Ine, and Ane – correct results</p> <p>All students flip the last fraction correctly</p> <p>Tore, Ronny and Carl - multiply before partly cancelling</p> <p>Ruth- transforms all operations signs into multiplication signs, but changes some of them back again</p>

Tore and Ronny eliminated the letter symbols. Six of the 10 students, had followed the teacher’s advice and looked for the structure. They solved the tasks b and c correctly. Not all solved both tasks.

Tore's solution is an example of not factorising:

$$\frac{3x+9}{4} \cdot \frac{12}{2x+6} = \frac{36x+108}{8x+24} = \frac{\cancel{36}^9 \cdot \cancel{x} + \cancel{108}^{27}}{\cancel{8}^2 \cdot \cancel{x} + \cancel{24}^6} = \frac{9+27}{8} = \frac{\cancel{36}^9}{\cancel{8}^2} = \underline{\underline{\frac{9}{2}}}$$

Ronny did the same. He was observed earlier, arguing that factorisation was not needed. Tore stated in plenary that binomials could not be partly cancelled.

Carl simplified the first terms in both numerator and denominator by 4 and the second terms by 12: $\frac{3x+9}{4} \cdot \frac{12}{2x+6} = \frac{\cancel{36}x^9 + \cancel{108}^9}{\cancel{8}x^2 + \cancel{24}^2} = \frac{9x+9}{2x+2} = \underline{\underline{\frac{9}{2}}}$

Ruth was the fourth student giving incorrect solutions to the two tasks. Her two solutions were:

$$1.2.19b \quad \frac{3x+9}{4} \cdot \frac{12}{2x+6} = \frac{3x+9 \cdot 12}{4 \cdot 2x+6} = \frac{3 \cdot \cancel{x} \cdot 3 \cdot 3 \cdot 3 \cdot \cancel{2} \cdot 2}{2 \cdot 2 \cdot 2 \cdot \cancel{x} \cdot \cancel{2} \cdot 3} = \frac{27}{6} = \underline{\underline{\frac{9}{2}}}$$

$$1.2.19c \quad \frac{2a+6}{5} : \frac{a+3}{15} = \frac{2a+6}{5} \cdot \frac{15}{a+3} = \frac{(2a \cdot 6) \cdot 15}{5 \cdot (a+3)} = \frac{2 \cdot \cancel{(a+3)} \cdot 15}{5 \cdot \cancel{(a+3)}} = \frac{30}{5} = \underline{\underline{6}}$$

Seven students were observed solving the task 1.2.19c correctly. Olav

was one of them: $\frac{2a+6}{5} : \frac{a+3}{15} = \frac{2 \cdot (a+3)}{5} \cdot \frac{15}{(a+3)} = \frac{2 \cdot \cancel{(a+3)} \cdot 3 \cdot \cancel{5}}{\cancel{5} \cdot \cancel{(a+3)}} = \underline{\underline{6}}$

He factorised the binomial in the numerator and put the other binomial in a bracket, clearly signalling that he considered it to be a factor. After that he simplified the fraction.

Ronny and Tore simplified term by term as in task 1.2.19b, Ruth's, and Carl's solutions are shown above, and Bill who solved only this task in the item, left out the binomial $a+3$.

It was possible in this task, as in two of the tasks in item 1.2.16, to reach the same solution as the ones in the textbook, without factorising and following the rules for simplification.

On the ordinary test in September, 4 weeks later, one similar task was given: $\frac{3a+15}{6} : \frac{2a+10}{8}$. The result is presented in the table below:

Table 10-15: task 4c from ordinary test September 2007 (27 students)

	Solutions of task 4c – September test	Number of students
A	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{(3a+15)8}{6(2a+10)} = \frac{\cancel{(a+5)} 8 \cdot 3}{6 \cdot 2 \cancel{(a+5)}} = \frac{24:12}{12:12} = \underline{\underline{2}}$	5
B	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a+15}{6} \cdot \frac{8}{2a+10} = \frac{(24a+120):12}{(12a+60):12} = \frac{2a+10}{\underline{\underline{a+5}}}$	1
C	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a+15}{6} \cdot \frac{8}{2a+10} =$ $\frac{3a+15}{6} \cdot \frac{8}{2a+10} = \frac{\cancel{24}^{12} \cancel{a} + \cancel{120}^{12}}{\cancel{12}^{12} \cancel{a} + \cancel{60}^{12}} = \frac{2+10}{1+5} = \frac{12}{6} = \underline{\underline{2}}$	4
D	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{8(3a+15)}{6(2a+10)} = \frac{\cancel{24}^2 \cancel{a} + \cancel{120}^2}{\cancel{12} \cancel{a} + \cancel{60}} = \underline{\underline{4}}$	9
E	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a+15}{6} \cdot \frac{8}{2a+10} = \frac{3 \cdot \cancel{a} + \cancel{3} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{2} \cdot \cancel{2}}{\cancel{2} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{a} \cdot \cancel{2} \cdot \cancel{3}} = \underline{\underline{3}}$	1
F	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a+15 \cdot 8}{6 \cdot 2a+10} = \frac{\cancel{3a} + \cancel{120}^{12}}{\cancel{12a}^{4a} + \cancel{10}} = \frac{12}{\underline{\underline{4a}}}$	1
G	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{8(3a+15)}{6 \cdot 8} : \frac{6(2a+10)}{8 \cdot 6} = \frac{24a+120}{48} : \frac{12a+60}{48} =$ $\frac{24a+120}{48} \cdot \frac{48}{12a+60} = \frac{1152a+5760}{576a+2880} = 1162a+2880$	1
H	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{(3a+15) \cdot 4}{6 \cdot 4} : \frac{(2a+10) \cdot 3}{8 \cdot 3} = \frac{12a+60}{24} : \frac{6a+30}{24} =$ $\frac{(12a+60) \cdot 24}{24(6a+30)} = \frac{288a+1440}{144a+720} = \frac{\cancel{2a}(144+720)}{\cancel{2a}(72+360)} = \frac{864:432}{432:432} = \underline{\underline{2}}$	1
I	$\frac{3a+15}{6} : \frac{2a+10}{8} = \frac{3a-2a}{6} \cdot \frac{8}{10-15} = \frac{a}{6} \cdot \frac{8}{-5} = \frac{8a:2}{-30:2} = \underline{\underline{-15}}$	1
	No solution	3

In solution E, Else, performs operations as if there are only factors involved. Solution F is Eli's solution. She was the one, who during lessons factorised binomials but simplified term by term within the brackets. Here she has not factorised and she performs a form of partly cancelling.

In the solutions (G) and (H) the students find a common denominator, before they divide the fractions. Tone, (G) did the same when solving an ordinary fraction (section 0). Paul, in solution (H), make an erroneous factorisation.

The last solution, solution I, was written by Glenn. He collects like terms as if he should simplify strings of algebraic terms, before he divides the fractions.

One item, 1.2.20 on the work plan, was written with normal fonts. Only two students solved the task:

Table 10-16: Fractions – word problem (textbook item 1.2.20)

Task 1.2.20.a and b	
Problem	Jan ate one third of a pizza. Jonas ate 2 thirds of what was left. a) Which one ate the most? b) How much of the pizza was left?
Paul	He answers the first question correctly, without showing how
Jone	Correct answers in both a) and b)

Another task was item 1.2.21, solved by three students; Jone, Rakel, and Paul. Jone solved one of the tasks; correctly. Rakel and Paul had problems with whole numbers and fractions. Rakel put the whole number on the nearest fraction line, while Paul asked the teacher for help. She let Jone tell Paul that 3 could be written as $\frac{3}{1}$. Paul had suggested to write $\frac{3}{9}$. This was the only task on the work plan including both fractions and whole numbers when working on algebraic fractions. There was one task related to addition in the section with numerical fractions, and two related to multiplication/division see section 0 task 1.2.6 e and f.

Appendix 11: Equations

In the introduction of equations in the textbook, ‘change side- change sign’ is equated with performing the same operation on both sides of the equal sign.

11.1 Equations – the algebra test

This test was carried out before the work with equations in class. One task was designed to test students’ creation of equations from a given solution. Two solutions were given: $x=10$ and $y=4$. In the table underneath it is shown how students’ solutions were distributed:

Table 11-1: Equations created - solutions given (algebra test item 2)

Item	2a		2b	
	$x = 10$		$y = 4$	
	<i>Correct</i>	<i>Incorrect</i>	<i>Correct</i>	<i>Incorrect</i>
The unknown on one side	56 %	8 %	48 %	16 %
The unknown on both sides	20 %	8 %	20 %	4 %
Expression only		4 %		4 %
No response		4 %		8 %

Most students in both tasks wrote equations with the unknown on only one side. One student did not write equations in any of the tasks, but expressions; clearly not considering the equal sign to belong to the equation. Two students applied x as the unknown also in task 2b. One student created an equation in 2a, but not in 2b, which might be an indication of not accepting y as the unknown.

Tone had these two equations: $2x=10$, and $2y=10$. Both are equations but none of them satisfies the conditions given in the task.

Kristin presented her solution although that was not asked for:
 $2x + 6 = 4 - 2x \rightarrow 2x - 2x = 6 + 4 \rightarrow x = 10$.

The equation is acceptable but not with the solution $x=10$. In the questionnaire, she said she used the rule ‘move over - change sign’. Here it seems that she has not changed sign. In addition, she subtracts $2x - 2x$ which she equals to x . She treated the other equation in the same way.

Six students wrote second grade equations. They were categorised as correct if one of the solutions were the same as given in the task. Selma, wrote a non-accepted second grade equation; $4 \cdot x^2 = 104$ in 2a (see the table above), which might indicate that she has ‘removed’ the number 4 from both sides of the equation. In the questionnaire, she explained that

in order to ‘remove’ the unknown’s coefficient at the end of the solution, the operation was division on both sides of the equal sign.

11 students answered the questionnaire and 10 of them explained the solution of equation by giving the rules: ‘move over - change sign’ and then at the end, in order to have one x ‘you have to divide both sides by the number of x -es’. Selma did not include the changing of signs.

It seems from the questionnaires that one way of thinking is triggered when the operations addition and subtraction are involved, while another way of thinking is activated when it comes to multiplication and division.

The tasks presented in the table below were included in the algebra test and were solved before equations had been a topic in class.

Table 11-2: Equation tasks with results – algebra test

Item	9a	9b	9c	9d
	$4x - 15 = 75$	$123 + 2x = 195 - x$	$\frac{x+1}{x+4} = \frac{4}{5}$ *	$9x = 45$
Correct solution	12 (48%)	18 (72%)	0	23 (92%)
Calculation based error	7 (28%)	3 (12%)	0	1 (4%)
‘Move over’ do not change sign	5 (20%)	3 (12%)		
Not solved	1 (4%)	1 (4%)	8 (32%)	1 (4%)

* all students failed because of the binomials in numerator and denominator

One category is ‘move over’ do not change sign. It seems that also many of the students in the correct category are applying the same rule but they change signs. At least three of them have written an arrow from the number 15 pointing over to the other side.

Two of the students clearly ignoring equivalence were Kristin and Olav. However, when it comes to division Kristin divides by the same number on both sides of the equal sign. She succeeds in the last task, 9d, where the solution is found by dividing both sides by 9.

Olav has a correct answer, but the way he solves the task is not in line with formal mathematics:

$$4x - 15 = 75$$

$$75 + 15 = \frac{90}{4x} = \frac{\cancel{4x}}{\cancel{4x}} = 22,50$$

He correctly adds 75 and 15 but equals this to 90 before he divides by $4x$ and gets $\frac{90}{4x}$ which he again equals to $\frac{\cancel{4x}}{\cancel{4x}}$ which he claims is equal to

22,50. He has partly applied the equal sign as an announcement of a result, and at the same time he has cancelled both 4 and x before he presents the result. He is able to find the value of the unknown, however, by ignoring equivalence and cancelling the symbol for the unknown in his writing.

The task 9c is new to the students. None of them, however, tried to follow the rule of performing the same operation on both sides of the equal sign. Most students have operated as if they had two equations, one for the numerators and one for the denominators, resulting in the solution $\frac{x}{x} = \frac{3}{1}$;, or $x = \frac{3}{1}$, or $x = 3$, $x = 1$.

In task 9d all except for Olav, and Ivar who did not solve the task, succeeded.

11.2 Equations presented in the digital learning book

The document starts with a description of equations and the purpose of solving them:

An equation always includes an unknown (often this is x) and an equal sign. We solve equations in order to find the value of the unknown. We apply the four arithmetic operations with the aim to have x alone on one of the sides of the equal sign, then we have found the value of the unknown.

After this description the same rules as in the textbook are listed. Two simple tasks are solved to exemplify this:

$$\begin{array}{l} x - 7 = 11 \\ x - 7 + 7 = 11 + 7 \\ \underline{x = 18} \end{array} \qquad \begin{array}{l} 4 \cdot x = 24 \\ \frac{4 \cdot x}{4} = \frac{24}{4} \\ \underline{x = 6} \end{array}$$

A reminder is given:

If you execute an operation on one side of the equal sign, you must remember to execute the same operation on the other side as well.

A piece of advice is given:

It is always smart to control the value you find, by substituting the unknown in the initial equation with the found value.

After this introduction three example tasks from the textbook are solved.

$$\begin{array}{l} 2(3x - 8) = 5 - 3(x + 1) \\ 2 + \frac{3}{2}x = \frac{1}{3} + x \\ \frac{4x}{3} - \frac{1}{5}(2x + 1) = \frac{2}{3}(x - \frac{3}{5}) \end{array}$$

Related to equations with fractions, students are told to multiply both sides in the equation by the common denominator, in order to get rid of

the denominators. In a bracket it is explained: “multiplication is the opposite of division”.

Cross multiplication is introduced: “If you have only one fraction on both sides of the equal sign, we solve it by cross multiplying”.

The example task is:

Trude is going to mix juice with 20 l of water in the ratio 3:10. How much juice must she add? (p. 41)

Solution:

x – unknown volume of juice

$$\frac{x}{20} = \frac{3}{10}$$

$$\frac{x}{20} \times \frac{3}{10}$$

$$x \cdot 10 = 3 \cdot 20$$

$$10x = 60$$

$$\frac{10x}{10} = \frac{60}{10}$$

$$x = 6$$

She has to add 6
liters of juice

Cross multiplication is not shown in the textbook, and this task is solved by multiplying both sides with the least common denominator in the textbook.

11.3 Equations introduction and tasks

The introduction and the first example task is presented in the thesis.

The second example task was: $2 + \frac{3}{2}x = \frac{1}{3} + x$. The teacher starts by asking

if the x , standing after the fraction, belongs to the numerator, to the denominator or to the whole fraction.

Arne: I think it belongs to the whole fraction.

T: Then it doesn't matter if it is written in the numerator or in the denominator?

Else: No, it is for the whole fraction.

T: For the whole?

Else: Yes, it belongs both to the numerator and to the denominator.

T: To both?

Else: Yes.

T: Do you agree?

More students answer no.

T: No, where is it actually to be placed?

Jarl: In the numerator.

T: Yes, it could actually be placed up in the numerator.

To illustrate this, the teacher applies a numerical example and writes

$$\frac{1}{3} \cdot 2. \text{ The discussion in class is loud.}$$

T: One third taken twice, what will that be, Else?

Else: Yes, that is two thirds. Nevertheless, if we have whole numbers, then it matters where we write it.

T: We can write it behind the fraction, but we have to know that it belongs to the numerator. It doesn't belong to the denominator.

She points to the denominator and writes: $\frac{1}{3} \cdot 2 = \frac{1 \cdot 2}{3 \cdot 2}$ saying that this will not be okay. Afterwards she deletes this example.

Here the teacher focuses on the concept of, and operations on fractions. Alf suggests after a while to multiply all terms by 6, because as he said, to “keep the balance”. The teacher shows this by putting the whole left and right sides in brackets multiplied by 6: $(2 + \frac{3}{2}x) \cdot 6 = (\frac{1}{3} + x) \cdot 6$.

She emphasises that they have to perform the same operation on both sides. Arne interrupts by asking why they multiplied by 6 and not 4 for example. The teacher asks Alf to explain, and he says the reason is that 6 is the least common denominator for the fractions. He suggests then to execute the multiplication, while the teacher asks for more suggestions. She wants them to simplify before multiplying. However, the students are reluctant to do that. She had written: $2 \cdot 6 + \frac{3}{2}x \cdot 6 = \frac{1}{3} \cdot 6 + x \cdot 6$. None seem to see that it is possible and desirable to simplify before multiplying. They argued against her. Some wanted her to write 6 in the numerator. All uttering wanted to multiply before simplifying.

The last example task was worked out on the computers. She advised the students to apply the function ‘align at =’ which placed the equal sign in the centre of the solution. The notion of term, was mentioned again, and the difference between expansion and fraction multiplication was emphasised. She prompted students to look for the possibilities to simplify before multiplication.

11.4 Equations –work plan and tasks

The work plan for equations differs from publisher’s plan; some items are left out, one item is added. The work plan is:

Table 11-3: Equations work plan and tasks

Aims	Textbook pages	Tasks - hard version
Apply the rules for equation solving, linear equations with one unknown	40-41	1.3.3a, b, e-h 1.3.4d-f
Write and solve equations about ratios	42-42	1.3.6
Solve equations including brackets and fractions	42-43	1.3.13 1.3.14
Transform word problems into equations and solve them		1.3.2, 1.3.5, 1.3.9 1.3.10

Although there are different fonts on the aims, all the items are written with bold fonts.

In the table below the items are listed in the same order as in the work plan with number of students who solved the tasks in each item. The last items 1.3.2, 1.3.5, 1.2.3.9, and 1.3.10 are word problems, and include only one task in each item. Dataset 18 students.

Table 11-4: Work plan - equations

	Item 1.3.3	Item 1.3.4	Item 1.3.6	Item 1.3.13	Item 1.3.14	Item 1.3.2	Item 1.3.5	Item 1.3.9	Item 1.3.10
Solved all	12	9	15	6	5	9	7	7	6
Picked out some	5	8		5	4				
Solved none	1	1	3	7	9	9	11	11	12

The first tasks on the work plan, the tasks in item 1.3.3, did not involve fractions, but task f and task h involved brackets to be subtracted. Data set 18 students (number of solutions). The data set in the last row includes all students:

Table 11-5: Equations – textbook tasks (item 1.3.3 a, b, c)

1.3.3a	1.3.3b	1.3.3c	1.3.3d
$3x - 5 = 10$	$8x - 12 = 5x$	$4 - x = 2$	$16 - 3x = 2x - 9$
16 solutions of 18	13 solutions of 18	6 solutions of 18	6 solutions of 18
Olav - cancels the x . Selma - shows by arrows that she moves terms, and writes extra equal signs.	Rakel - does not end her solution Selma - has extra equal signs. Olav - cancels the x .	Selma - shows that she moves over Olav - cancels the x	Oscar -writes $\frac{5}{5} = 4$ Olav - cancels the x

Table 11-6: Equations – textbook tasks (item 1.3.3 e, f, g, h)

1.3.3e	1.3.3f	1.3.3g	1.3.3h
$2(x+4)+3=15$	$x-2(1-x)=8x+1$	$(x+1)\cdot 2=5(1-x)+2x$	$3(x-3)-2=x-(4-x)$
15 solutions of 18	13 solutions of 18	12 solutions of 18	18 solutions of 18
<p>Selma- shows the movements of the terms by arrows</p> <p>Olav- cancels x</p> <p>Tord- does not follow the distributive law</p> <p>Oscar – $\frac{2x}{2} = 4$</p>	<p>Many questions about the minus sign</p> <p>Selma- signal movements of the terms</p> <p>Olav- cancels x, does not follow the distributive law</p> <p>Rakel and Kari – asks about dividing by minus</p> <p>Eli-says she avoids negative coefficients</p>	<p>Olav- cancels the x, was helped with the minus sign and distributivity</p> <p>Ronny- about the invisible sign.</p> <p>Ruth and Tone- do not understand the symmetric property</p>	<p>Arne- asks about brackets and sign</p>

Oscar and Olav wrote the equations in an informal incorrect way. The minus sign appearing in front of the bracket, and the minus sign at the end of the solution process in task f, both caused many students to ask for help. Ronny, Else, Selma, Rakel and Kari aligned their result to the one in the textbook. Rakel and Kari expressed that they did not understand why the result should be negative. Rakel explained the problem: “If we divide by minus 5, the minus sign will come down there” (in the denominator). Eli told she collected the x -es on the right side.

In task h, Arne asked if the signs in all brackets should be changed. According to the data, this was his first task with a bracket preceded by a minus sign. He solved few tasks.

Only Oscar and Olav solved the first tasks in the next item:

Table 11-7: Equations – textbook tasks (item 1.3.4 a, b, c)

1.3.4a	1.3.4b	1.3.4c
$\frac{x}{2} = 3$	$\frac{x}{3} = \frac{5}{6}$	$\frac{2}{x} = \frac{1}{3}$
Both correct	<p>Oscar- Long fraction line</p> <p>Olav- cancels the x, has extra equal signs</p>	<p>Olav- cancels the x, has extra equal signs</p>

The last tasks in this item were solved by more students. Most students performed cross multiplication. Kari, Rakel, and Tord multiplied by a common denominator. Ruth found her own informal way because of the cross-multiplication, but she found the correct values of the unknown.

The results are shown below. Number of solutions from data set 18 students and responses in row 4 are from the whole dataset:

Table 11-8: Equations – textbook tasks (item 1.3.4 d, e and f)

1.3.4d	1.3.4e	1.3.4f
$\frac{2}{3} = \frac{6}{x}$	$\frac{5x}{9} = \frac{1}{8}$	$\frac{-2}{3} = \frac{4x}{5}$
15 solutions of 18	11 solutions of 18	16 solutions of 18
<p>Kari and Rakel- multiply by the common denominator</p> <p>The others- cross multiply</p> <p>Ruth- multiplies only to the left ending in an expression not an equation</p> <p>Olav- cancels the x, has extra equal signs</p>	<p>Kari, Tord, and Rakel- multiply by the common denominator</p> <p>The others- cross multiply</p> <p>Oscar- long fraction line</p> <p>Ruth- multiplies only to the left ending in an expression not an equation</p> <p>Olav- cancels the x, has extra equal signs</p>	<p>Kari and Tord- multiply by the common denominator, and makes a sign error.</p> <p>Ruth- multiplies only to the left ending in an expression not an equation</p> <p>Olav- cancels the x, has extra equal signs</p> <p>Bill, Carl, Jone, Tore and Paul- do not simplify at the end</p> <p>Ane- changes sides, but do not change sign for the x-term</p>

The problems seem to be the simplification at the end, the minus sign, and the sign rules. Oscar and Olav write their solutions not formally correct. Olav started in this item to write equal signs at the end of each line.

Ane, solved task f correctly before she changed the side of the x in the result. The result was positive. It might be that she does not regard equations as relations with the property of symmetry, or she thought that the rule ‘change side-change sign’ does not yield the unknown term.

The next item 1.3.13 includes fractions. Number of solutions from data set 18 students and responses in row 4, are from the whole dataset:

Table 11-9: Equations – textbook tasks (item 1.3.13)

1.3.13a	1.3.13.b	1.3.13.c
$\frac{3x}{2} - 4 = \frac{x}{2} + 1$	$\frac{5x}{2} - \frac{2}{5} = \frac{x}{10} + \frac{1}{2}$	$\frac{15x}{4} - \frac{1}{2} = \frac{1}{2}(3x+1)$
8 solutions of 18	7 solutions of 18	9 solutions of 18
<p>Ruth and Olav- whole numbers/fractions</p> <p>Olav- cancels the x, has extra equal signs</p>	<p>Olav- cancels the x, has extra equal signs</p> <p>Tord- makes a sign error</p>	<p>Olav- cancels the x, has extra equal signs</p>

The students with correct solutions in task a solved it by multiplying each term by the common denominator 2. Tord seems to change side, not sign.

Ruth's solution is presented to the right. She multiplied by 2, the common denominator, but multiplied only the fractions. Actually she multiplied by 4 (line 3). Then she collected the numbers on the right side and the x -terms on the left side. Then in line 5, she divides both sides by 4, and she writes her solution as $x = 5$, in line with the correct solution in the textbook. She was seen eager to check her solutions.

$$\begin{aligned} \frac{3x}{2} - 4 &= \frac{x}{2} + 1 \\ \frac{3x}{\cancel{2}} \cdot \cancel{2} - 4 &= \frac{x}{\cancel{2}} \cdot \cancel{2} + 1 \\ 6x - 4 &= 2x + 1 \\ 6x - 2x &= 1 + 4 \\ \frac{4x}{4} &= \frac{5}{4} \\ x &= \underline{\underline{5}} \end{aligned}$$

Olav resolved the problem with the whole numbers by regarding them as part of the numerators in the fractions (line 2). He then multiplies by 4 on both sides, before he violates the law of distribution. This is the reason why he reaches the same result as in the textbook. In addition, he still cancels the unknown. His use of more than one equal sign, signals an operational view of the sign.

$$\begin{aligned} \frac{3x}{2} - 4 &= \frac{x}{2} + 1 = \\ \frac{3x-4}{\cancel{2}} \cdot \cancel{4}^2 &= \frac{x+1}{\cancel{2}} \cdot \cancel{4}^2 = \\ 3x-8 &= x+2 \\ 3x-x &= 2+8 \\ \frac{\cancel{2}x}{\cancel{2}x} &= \frac{10}{2} = \underline{\underline{5}} \end{aligned}$$

The tasks in item 1.3.14 were similar to the foregoing tasks, however, in addition to fractions and brackets, some of these tasks also involve fractions with binomials in the numerator. Some brackets include both whole numbers and fractions. It is evident from the classroom that some of the students found it too difficult to solve them at home. Others were seen to start but then gave up.

The tasks are presented in the tables below. Number of solutions are from data set 18 students. The responses in row 4 are from the whole dataset 27 students:

Table 11-10: Equations – textbook item 1.3.14 a, b, c

1.3.14a	1.3.14b	1.3.14c
$\frac{1}{2}(x+3) + \frac{2}{3} = x - \frac{1}{3}$	$\frac{2}{5} - \frac{1}{3}(2x+5) = \frac{1}{2}(5-2x) - \frac{3}{5}x$	$2x - \frac{1}{3}(x + \frac{3}{4}) = 1 + \frac{1}{2}(\frac{5}{6}x - 5)$
7 solutions of 18	7 solutions of 18	6 solutions of 18
Ruth- makes sign error, but aligns the result to the result in the textbook Olav- cancels the x	Kari- gives up Olav- cancels the x	Olav cancels the x , has extra equal signs Tone- asks for help sign error

Table 11-11: Equations – textbook item 1.3.14 d, e and f

1.3.14d	1.3.14e	1.3.14f
$\frac{1-x}{5} + \frac{x}{20} = \frac{x}{5} - \frac{x+1}{4}$	$2x - \frac{4x-3}{3} = \frac{1}{2}(3x-1)$	$3 - 2(x + \frac{1}{4}) = \frac{1}{3}(x + \frac{1}{2})$
5 solutions of 18	5 solutions of 18	5 solutions of 18
<p>Olav- asks for help, violates distributive law, has sign error, cancels the x, extra equal signs</p> <p>Ruth- problem to collect the x-es</p>	<p>Jone - sign error asks for help</p> <p>Tone- asks for help: multiplication error, sign error in binomial</p> <p>Ruth, Ine and Ane- ask for help to find what is wrong</p>	<p>Ruth – sign error</p> <p>Teacher tries to convince her to perform multiplication before expansion</p>

Olav continues cancelling the unknown, but the transformations are correct after checking with the textbook key and with help from the teacher. He has problems with the signs in the binomials, and in following the distributive law. He has one equal sign in each line, except for one occasion in one of the tasks.

From the table it is seen that even fewer students are solving these tasks than in the foregoing item. Many of the students had not solved the tasks during lesson, nor at home, and the teacher asked them to solve them after finishing tasks related to inequalities. Some students went back to these tasks, and solved them the day inequalities had been the focus for the plenary session. Although they were encouraged to do so, only about one third of the 18 students solved one or more of the tasks.

Kari, Ruth, and Jone, who saved erroneous solutions, made errors related to the minus sign. From the observation in the classroom it is evident that more students struggled when deciding if the sign should be positive or negative when opening brackets, when placing binomials as numerators on a common fraction line, and even when simplifying strings of fractions, numbers or literal terms.

Ruth, in task a regarded $-3x - 6x$ equal to $3x$. Ruth was one of the students seemingly adding from right to left in earlier tasks. At the end, she just aligned her result to that in the textbook.

Task 1.3.14b caused more students to ask for help, probably because the least common denominator was 30, and it seemed more difficult to come to a correct result when the numbers were larger.

Both in task 1.3.14d and in task 1.3.14e some students were observed struggling with the binomials. The teacher once again explained about binomials as entities.

Jone and Paul asked for help solving tasks e. Jone had changed the sign in the numerator, when multiplying all terms by the least common denominator. Paul had not:

Paul: We disagree a bit, because our results differ.

T: It is okay to disagree.

Jone: It is a plus sign there?

Paul: Is there something wrong here?

T: Yes, this is a unity. It is standing on a fraction line, so you have to take that as a whole. There it is correct, but you will have a plus sign, because the sign in the front is a minus sign.

Paul: Then I have to change that one?

T: Yes, you should remember this. It is so easy to make that mistake if you don't carefully write the bracket after writing the product. You should not make such a mistake, since you seem to have control. You should avoid this.

Paul and Jone had consequently chosen one or two tasks in each item, and had solved only one subtraction task with binomials in the numerators. They felt confident in their work, but their computer files, show they have problems deciding whether the signs should be positive or negative, and it seemed during observation that they relied on each other, instead of checking the textbook. At the end of this episode, Jone said:

Jone: We discussed if it should be a minus sign up there as well (*he points to the bracket on the right side of the equal sign*).

T: No, there is no minus, because you just multiply by 2. It's correct what you have done. Then you just have to write the sum.

Paul: Can I move the x over to the other side?

The teacher says it does not matter where the x is located. These two boys gave the impression that they were mathematically proficient, and especially Jone gave important inputs in the plenary sessions, working hard. This conversation illustrates that also the most competent students felt unsure about the signs.

Paul was seen during the observed lessons that he consequently collected the x -es on that side of the equal sign where he avoided negative coefficients. Eli asked for help in relation to the two last tasks, it seemed that she struggled with the signs as well.

Tone and Rebekka discussed how to handle binomials in the numerators. They solved some inequalities earlier in the lessons. However, when solving the equations in item 1.3.14 they both realise that they have added strings of numbers and expressions erroneously in almost all tasks. The result in the textbook key caused them to go back and search for the error. However, as seen in earlier tasks, they sometimes just changed their results after checking the textbook answers.

11.5 Equations in tests

On the ordinary test in September, one week after they had worked with equations in the class, three equations were given for students to solve. The results are presented in the table below. 27 students participated.

Table 11-12: Equations – test tasks, ordinary test September. Results (%)

Equation test tasks	Task a	Task b	Task c
Item 6	$\frac{-6}{9} = \frac{4x}{3}$	$2 \cdot \frac{7}{x} = \frac{5}{25}$	$\frac{x}{2} - x = 2\left(\frac{x}{2} - \frac{2}{3}\right)$
Correct	44	44	48
Solution not simplified	18		
Sign problem	15	4	4
Term/factor problem		11	
Fraction problem	11	15	15
Equivalence	11	7	18
Calculation error			7
Not solved	4	22	18

The sums are more than 100 %. Some students had more problems, and all are listed. One student had expanded the fractions on the right side by 2 instead of having multiplied by 2, in addition he had multiplied by 6 only on the right side, violating the equivalence relation. His solution is thus categorised both as showing a ‘fraction problem’ and as a problem handling the equation as an ‘equivalence’.

Only five students (19 %) solved all three tasks correctly, Ine, Else, Tord, Tore and Ronny.

In task a, 12 students or 44 % had solved the equation correctly. Five more had followed the rules, but not simplified the result. Three of those, Kari, Rakel, and Paul, ended their solutions by writing either $x = \frac{-18}{36}$

or $x = \frac{18}{-36}$. This might be caused by uncertainty about where to put the minus sign. It was observed that some students were not sure about where the sign belonged (see appendix 0). Those solutions are, however, categorised as ‘not simplified’ in the table.

Four students presented a positive solution; categorised as sign problem.

Three students had made other errors. They can be said not to have a formal concept image of equations. Below are those solutions presented:

Kristin

$$\frac{-6}{9} \cdot 2 = \frac{4x}{3} \cdot 6$$

$$\frac{-12}{18} = 8x$$

?

Selma

$$\frac{-6}{9} \times \frac{4x}{3}$$

$$\frac{-18}{36x}$$

Olav

$$\frac{-6}{9} = \frac{4x}{3}$$

$$\frac{-6}{\cancel{9}} \cdot \cancel{9} = \frac{4x}{\cancel{3}} \cdot \cancel{9} =$$

$$\frac{-6}{12} = \frac{12\cancel{x}}{\cancel{12}x} =$$

$$\underline{\underline{x = -0,5}}$$

Kristin did not execute the same operation on both sides of the equal sign, and she expanded the fraction on the left side although written as multiplication. In the lessons, she seemed to work correctly, although she in the algebra test before the observation started, was seen not to treat equations as binary relations. She did not solve many tasks, and was quiet in the plenary sessions. However, she was observed when solving a similar task: $\frac{-2}{3} = \frac{4x}{5}$. She solved it correctly by cross multiplication.

Selma was observed during lessons solving the equations properly, except for having extra equal signs at the end of the lines in the first equation tasks. On the test, she presented her solution as an algebraic expression. She seemingly tried to cross multiply but was ending up with the expression on the left side ignoring that this equals to 1.

Olav is still cancelling the unknowns, and here he writes extra equal signs at the end of the lines, presumably signalling here comes more. He was asked before the test what the result would be if he divided $2x$ by $2x$. He meant the result would be x . He responded correctly on a similar number example, and thus the teacher thought he had understood what was incorrect in his thinking. He then left his habit for a while.

Ruth's, Atle's, and Jone's solutions are presented:

Ruth

$$\frac{-6}{9} = \frac{4x}{3}$$

$$\frac{-6}{9} : 3 = \frac{4x}{3} : 3$$

$$\frac{-2}{3} : 4 = 4x : 4$$

$$\frac{-8}{3} = x$$

$$\underline{\underline{-x = -\frac{8}{3}}}$$

Atle

$$\frac{-6}{9} = \frac{4x}{3} \quad | \cdot 9$$

$$\frac{-54}{36} = \frac{36x}{36}$$

$$\frac{-54}{36} = x$$

$$\underline{\underline{x = -1,5}}$$

Jone

$$\frac{-6}{9} = \frac{4x}{3}$$

$$-18 = 36x$$

$$\frac{36x}{36} = \frac{-18}{36}$$

$$\underline{\underline{x = -2}}$$

Ruth was seen to struggle both with the minus sign and with fractions during observation in class. She was mostly working together with her classmate, Tone, who was also struggling not feeling confident in what she did. They were eager to ask the teacher to help. She sometimes sat down with them for rather long periods. Tone seems to have progressed, while Ruth seems still to have the same problems.

Atle is multiplying by the common denominator, but he just multiplies, probably forgetting to simplify. At least he seemingly ignores the denominators.

Jone has performed all operations correctly until he in the last line equals $\frac{-18}{36}$ to negative 2. Jone was one of three students who in task

1.2.1c equalled the fraction $\frac{1}{15}$ to the number 15 (see appendix 10.3), which might indicate that this error is not a coincidence, although he during the time of observation was regarded both by the teacher and by himself to be confident in his mathematical work.

The second equation on the test, $2 \cdot \frac{7}{x} = \frac{5}{25}$ was solved correctly by 12 students (44%), while 3 students, Roy, Sven, and Peter, (11 %) performed the operations as if the fraction had to be expanded by 2 instead of being multiplied by 2, as in Sven's solution presented below.

Three other students, Eli, Bill, and Olav multiplied both sides by the least common denominator, but did not differentiate between terms and factors. One example is Eli's solution presented below:

$$\begin{array}{l} \text{Sven} \\ 2 \cdot \frac{7}{x} = \frac{5}{25} \\ \frac{14}{2x} = \frac{5}{25} \\ \frac{10x}{10} = \frac{350}{10} \\ \underline{\underline{x = 35}} \end{array}$$

$$\begin{array}{l} \text{Eli} \\ 2 \cdot \frac{7}{x} = \frac{5}{25} \\ 2 \cdot 25x \cdot \frac{7 \cdot 25x}{x} = \frac{5 \cdot 25x}{25} \\ 50x \frac{175 \cancel{x}}{\cancel{x}} = \frac{125^5 x}{25} \\ 50x - 5x = 175 \\ \frac{45x}{45} = \frac{175}{45} \\ \underline{\underline{x = \frac{35}{9}}} \end{array}$$

Eli's solution demonstrates skills in solving equations, however, the concepts of terms and factors seem still not well-developed. The problems for the six students solving the equations as Eli or Sven do, has to do

with the formal writing of expansion/multiplication of fractions and the concepts of terms and factors.

Ruth correctly cross multiplied, but at the end she equated $\frac{1}{x} = \frac{1}{70}$,

with $x = \frac{1}{70}$. Olav does not differentiate between factors and terms, and he multiplies by different factors, 25 and 25x. In addition, he ends some of the lines with an equal sign. In one line, the unknown disappeared, but appeared in the last line.

The last equation on the test $\frac{x}{2} - x = 2(\frac{x}{2} - \frac{2}{3})$ was solved correctly by 13 students (48%), and two had minor multiplication errors. Some students had more problems and all are listed. Five students act as if the equation is not an equivalence relation. Sven expands the fractions on the right side, then he ‘moves’ the *x-es* over to the left side without changing signs. It is a question if his solution should be interpreted to be a sign problem, because he turns all terms into positive terms. However, here his solution is regarded as well to be an evidence of him not executing the same operations on both sides of the equal sign.

Alf multiplies the equations on the right side correctly by 2 before he expands the fractions; written as ordinary multiplication. At the end, he violates the equivalence relation by multiplying only the right side by 6.

Olav continues cancelling the unknown as in the foregoing tasks. In addition, he continues writing equal signs at the end of most of the lines in his solution, and he does not apply the distributive law.

$$\begin{aligned} \frac{x}{2} - x &= 2(\frac{x}{2} - \frac{2}{3}) = \\ \frac{x}{2} - x &= \frac{2x}{2} - \frac{4}{3} = \\ \frac{x}{2} \cdot 6 - x \cdot 6 &= \frac{2x}{2} \cdot 6 - \frac{4}{3} \cdot 6 = \\ \frac{x \cdot \cancel{6}^3}{\cancel{2}} - 6x &= \frac{2x \cdot \cancel{6}^3}{\cancel{2}} - \frac{4 \cdot \cancel{6}^2}{\cancel{3}} \\ 3x - 6x &= 6x - 8 = \\ 3x - 6x - 6x &= 8 = \\ \cancel{9}x &= 8 \\ \cancel{9} &= -9 \\ x &= -\frac{8}{9} \end{aligned}$$

Carl’s solution is shown to the left. It is tempting to categorise his error only as a sign error. The fraction in the end solution has a minus sign, it should have been positive. However, he writes an extra equal sign at the end of each line, which is not in line with having a perception of equations as equivalence relations. The equal signs are there to signal that more will follow. In addition, it might be that this extra equal sign causes him to change the sign before the number 8. He collects the *x-es* on the left side, while the number 8 is left alone in between two equal signs but now the number is changed from being negative to positive. Carl was not seen solving equations in this way during classroom observation. He had

solved only few of the simplest equations.

Glenn seems to have problems with the minus signs. He solved the task correctly, ends, however, with the solution $x = \frac{-8}{-9}$, which is categorised as ‘problem minus sign’.

Tone wrote extra equal signs in the same way as Carl and Olav. She was not observed doing that during seat work. She correctly multiplied the fractions on the right side by 2 before she expands them in order to let all fractions have 6 as the common denominator. Her mistake seems to be that she multiplies the alone-standing x by 3. The fractions she expanded. Then she equals $\frac{x \cdot 3}{2 \cdot 3} - x \cdot 3$ with $\frac{0}{6}$, subtracting as if $3x$ was a fraction with denominator 6. In the next line she presents only an expression, before she in the last line equates this with $\frac{0}{6}$. She does not present any solution of the equation. Her main problem seems to be that she does not know how to handle the x , standing as a whole number. She worked hard, asked the teacher for help and relied heavily on the textbook key. On the test, she did not have this possibility.

Ruth, Selma, and Peter, expanded the fractions by 2 instead of multiplying as was intended by the symbolisation of the task. Peter stopped his solution process after this expansion. Selma followed the rules for equations, ending with an erroneous result because of a fraction error. Her solutions in task a and b might have been avoided if she had multiplied by a common denominator instead of performing cross multiplication. Except for the fraction error, she performed the operations according to the rules for equation solving in task c.

Ruth shows the same lack of understanding of the alone-standing x as Tone and Olav. She puts it in as part of the numerator of the first fraction. As for Tone, her left side of the equation resulted in zero. She managed to go further on by ‘correctly moving’ the x -es she had on the right side over to the left side. However, then she writes that she will divide by 12 on both sides. What she actually does, is to multiply by 12 in order to cancel the denominators. Her problems are related to the fractions in this task.

One item in the December test included two alternative equations to be solved. The students had to choose one of them. The distribution of students and results are shown below (data set 27) students.

Table 11-13: December test - Equations

December test	Alternative 1	Alternative 2
	$2x + 5x - 1 = x$	$\frac{2}{3}(x + \frac{3}{4}) = \frac{1}{6}x - 1$
Number of students	6	21
Correct	83 % (5 students)	52 %
Sign error		5 %
Equivalence problems		14 %
Fraction error	17 % (1 student)	
Calculation error		5 %
Factor/term problem		10 %
Other problems		14 %

Six students, Kristin, Atle, Peter, Glenn, Eli, and Ivar chose the simplest one. The three first had no correctly solved equations in the September test. The others had one out of three. Kristin was the one having an error in the December test. She equalled $-\frac{1}{4}$ with -4 .

In the alternative 2 task, 21 students solved the task and 11 succeeded. Olav is still cancelling the unknown.

Only one had a sign error. However, the design of the task was such that it was not likely that some should have such an error. Ine detached the term from its indicated operation; $-12 - 6 = -6$. She was not seen to have done that earlier.

Tord and Kari changed side but not sign, and the error is categorised as an equivalence problem. Tord was observed doing that in the classroom, but on the September test he solved all equations perfectly. Kari was not observed doing it earlier.

Else had solved the task correctly until she at the end did a minor calculation error, resulting in zero on the right side. Then she carried out some acrobatics against the properties of fractions in order to reach an answer that she could accept. Alf had only a minor calculation problem. He solved none of the equations in the September test correctly. This is thus an indication that he has made progression.

Ane and Tone did not differentiate between factor and term. Both avoided that in the September test, since they carried out the fraction multiplication before they cross multiplied. The others in this test followed the teacher's advice this time and multiplied the bracket by the pre-multiplier before they multiplied by the common denominator.

In the category 'other problems' are Carl's, Tore's and Olav's solutions. Carl came to the result $\frac{-1}{3} = \frac{1}{2}x$ and stopped there. Tore ended up

with an expression not a value for the unknown. Olav violated the distributive law. He does this in two out of five brackets in this test. It means that it is not a consistent error. This he was seen to do occasionally in the lessons.

Appendix 12: Inequalities

Inequalities were in focus for the plenary session the day after the concept of equation was recapitulated.

12.1 Inequalities in the textbook

Inequalities are presented in the same sub-chapter as equations, treated as a sub-topic of equations.

The signs denoting inequality are introduced and translated into ordinary language. In addition, they are applied in connection to daily life contexts of distances between two Norwegian towns, and differences between prices.

The inequalities $3 > -5$ and $-8 < -3$ are presented on the number line. A question is posed: “For which values of x , is the expression $x + 3$ larger than 5?”. This question is then transformed into the mathematical notation: $x + 3 > 5$. To solve this inequality, it is said, means “to find those values of x that makes the left side larger than 5. Then we see that x has to be a number larger than 2”.

The rules for solving equations are applied on an inequality to check if they hold. The conclusion is: The sign must be flipped when multiplying or dividing by a negative number.

Three context free example tasks are solved. In the last task, if the x -terms are kept on the left side of the inequality sign, the sign must be flipped:

$$5x + 7 > -3 \quad 3x + 5 > 4 - 2(x + 3) \quad \frac{1}{3}x + \frac{1}{2} \leq \frac{7}{6} + x$$

The introduction is short and the rules are presented in a textbox:

12.2 Inequalities in the ‘learning book’

One inequality example is applied to check which operations are allowed. The example is $4 > 2$. The following is presented:

We check a number example:

$$4 > 2$$

We divide by 2

$$4 > 2$$

We divide by negative 2

$$\frac{4}{2} > \frac{2}{2}$$

$$\frac{4}{-2} > \frac{2}{-2}$$

$$2 > 1 \quad \text{This is true}$$

$$-2 > -1$$

This is not true, we have to flip the inequality sign in order to let it be true.

$$-2 < -1 \quad \text{Now it is true.}$$

The same rules are said to be applied as for equations, but the inequality sign has to be flipped, when division or multiplication by a negative

number are executed on both sides of the sign. At the end, one example task is solved:

$$\begin{aligned}
 4 - 2x &> 12 \\
 -2x &> 12 - 4 \\
 -2x &> 8 \\
 \frac{-2x}{-2} &< \frac{8}{-2} \\
 \underline{\underline{x}} &< \underline{\underline{-4}}
 \end{aligned}$$

The solution of an inequality as a set of numbers, is not mentioned, but this was in focus in the classroom.

12.3 Inequalities work plan and tasks

On the work plan only two items were set to be solved. The last, 1.3.19, was written with normal fonts. However, the students were told explicitly to solve also this item.

Table 12-1: Inequalities – work plan

	Aims	Textbook	Items
	Solve linear inequalities	44–47	1.3.18, 1.3.19

The textbook offers only those two items. All tasks in both items are in pure mathematical language. The tables below presents the tasks in item 1.3.18 and the number of students who solved the tasks (solutions). The dataset is the saved computer files from 18 students. In the next row are all students' observed responses taken in.

Table 12-2: Inequalities – textbook tasks (item 1.3.18)

1.3.18a	1.3.18b	1.3.18c	1.3.18d
$x + 7 > 10$	$5x + 2 > 2x + 8$	$3x - 4 < 8 + x$	$10 + x < 3x - 4$
6 solutions of 18	4 solutions of 18	4 solutions of 18	4 solutions of 18
Alf – correctly interprets the solution correctly to be a set of numbers	Ane – equal sign	Ane – equal sign Jone – calculation error	Jone – sign error Ane – wrote inequality sign

Not many students solved the first tasks in this item. Alf was asked about the solution and clearly interpreted it to mean all numbers larger than 3. In addition he told that he did not like these kinds of task. Ane wrote equal signs and Jone made calculation and sign errors.

Table 12-3: Inequalities – textbook tasks (item 1.3.18)

1.3.18e	1.3.18f	1.3.18g	1.3.18h
$3-2x \leq 5(3-x)$	$8x-2(x-3) \geq 3x-(5-x)$	$x+2 < 3x-4$	$3(4-2x) \geq 10-5x$
14 solutions of 18	14 solutions of 18	14 solutions of 18	15 solutions of 18
<p>Ane- writes $x=4$ Arne- $x \leq 4$ Roy- does not find the solution for one x, has a sign error</p>	<p>Ane- flips the sign incorrectly, and shows problem with the negative sign Arne- struggled for a long time, multiplies the bracket by $3x$, made a sign error, and waited for the teacher who was busy with others. Afterwards he managed the other tasks. Tone- struggled, would give up, but Ruth found a sign error in her work</p>	<p>Carl- does not flip the sign Ane- writes equal sign Olav- flips the sign incorrectly Most students- flip the sign after having performed the division Arne – sign error</p>	<p>Ane- does not perform the same operations on both sides. Ane and Selma - Thought it was not possible to divide by a neg. number Most students- flip the sign after having performed the division The others-keep the x-es on the right side.</p>

Ane wrote equal signs in most solutions. Arne was seen struggling with both the equal signs and the solutions of the tasks. He asked for help, and after that he solved the tasks seemingly without problems.

Task d, g, and h resulted in negative x -coefficients on the left side of the inequality sign. In task d all four students solved the task correctly.

Ane in task f, and Olav in task g flipped the sign although having a positive coefficient. Ane and Carl did not flip the sign although dividing by a negative number in task g and task h. Ane and Kristin were two of the students asking about the inequality signs and when to flip them. When coming to the statement $-x \geq -2$ Ane asked the teacher how to continue. Her problem was the division of a negative number.

Roy stored his solutions before they were finished in the right sense, before coming to the solution set for one x . The pattern seems to be that he avoids fractions in the final solution.

The second item on the work plan is presented in the table below. The data set which it is based on, is the same as in the foregoing table (18 students). The responses in the last row is from the whole data set, 27 students.

Task a and d were solved correctly by those who solved those tasks. Two students saved solutions with equality sign, Ane in task b and Tone in task f.

Only in task c it was necessary to flip the sign, if keeping the x -es on the left side. Ane did not flip the sign while Ruth and Jarl did it only in the end solution; a bit late, but with correct solutions.

Most problems are not related to the object of inequality. However, in task 1.3.19b the task could be solved by cross multiplication. For a couple of students this caused trouble for them.

Table 12-4: Inequalities – textbook tasks (item 1.3.19 a, b, c)

1.3.19a	1.3.19b	1.3.19c
$\frac{x}{3} \leq 5$	$\frac{3x}{4} \geq \frac{4}{3}$	$2 > x + \frac{1}{3}$
14 solutions of 18	14 solutions of 18	13 solutions of 18
	Tore and Ronny –problems because of cross multiplication. Ane - equal sign	Ane - does not flip the sign Those who flip - do it after the division is performed.

Table 12-5: Inequalities – textbook tasks (item 1.3.19 a, b, c)

1.3.19d	1.3.19e	1.3.19f
$\frac{x}{3} + \frac{1}{4} > 2$	$\frac{4x-5}{6} - \frac{2-x}{3} > \frac{1}{2}$	$\frac{1}{4}(x-3) \leq \frac{2-x}{3}$
13 solutions of 18	14 solutions of 18	12 solutions of 18
Olav - extra equal signs at the end of each line (again) Roy - does not find the solution for one x Tore -wants to cross-multiply on the left side	Ane and Else - sign error Ane and Jarl - do not follow the distributive law Glenn - asks about the minus sign	Ane - does not follow the distributive law Tone - writes equal sign Olav - is told not to cancel the letter symbol Ronny – how to multiply fraction and brackets

Oscar and Tord did not solve any of the tasks, 11 students had solved all. The others had picked out some tasks.

12.4 Inequality and test tasks

On the test in September less than one week after students had worked in class with inequalities, one inequality task $-3x + 6 < 2(x + 8)$ was included. There were no demands of interpreting or commenting on the result.

The table underneath includes the results; numbers of students and percentages (27 students).

Table 12-6: Inequalities – 6 d September test 27 students (%)

Inequality test task	Frequency *	Category
$-3x + 6 < 2(x + 8)$		
Correct	19	
Wrote another inequality, but correctly solved	7	A
Incorrect or no flipping of the inequality sign	33	B
Calculation errors	7	C
Solution presented as specific unknown (equals sign)	29	D
Do not perform same operation on both sides	7	E
Problem negativity	33	F
Not solved	4	

* The sum is higher than 100%. Some solutions are located in more than one category.

Peter and Carl wrote a different inequality (A). This eased the task, but they solved it correctly.

$-3x + 6 < 2(x + 8)$ Roy made only a flipping error in that he flipped the sign from the second line to the third line (to the left).
 $-3x - 2x > 16 - 16$

In category B, seven students had not flipped the sign, two had divided by negative 5, without flipping the sign immediately. They waited till the last line. A reason for this may be that they think that it is no division before the fraction is simplified. This means that this: $\frac{-5x}{-5} < \frac{10}{-5}$ is not seen as a division before the division is ‘executed’ as in $x > -2$ and therefore the sign is changed here and not in the first case. For Tord this was the only mistake, and his solution is thus categorised as correct. Then five students (19 %) are regarded to have solved the task correctly.

Three students (category C), Else, Jone, and Ane made calculation errors. Else made just a minor calculation error. Ane simplified the fraction in this way: $\frac{10}{-5} = -5$. She must have subtracted ending up with a negative result. Tord had $16 - 6 = 12$.

The next category (D) includes solutions where the task is treated as an inequality from the outset, but when the solution is reached, it is presented as a specific unknown with an equal sign. Six students were in this group, Ane who did the same in the textbook tasks was among them. Arne wrote $x \Rightarrow (-2)$ at the end, but had correctly solved the task so far. He had gained control with the conventional notation after struggling for a while. There is also the possibility that he thinks orally: x is equal to greater than minus 2; interpreting the equal sign to be to announce a result, however, his reasoning is not known. The others were Jarl, Tone, Glenn, Ruth, and Olav.

Category E includes two students, Tore and Ronny, they had not performed the same operation on both sides of the inequality sign, resulting in erroneous solutions.

The students in category G showed problems related to the minus sign. Ine, Atle, and Kristin detached one term from its intended operation at the end of their solutions. Ine was not seen to have done that earlier, but in the December test she also made an error related to the minus sign. Peter and Kristin ignore the unary minus sign in the start of the task. Alf ends with $-x < 2$, probably not knowing how to handle this. Several students did not think that they could divide by a negative number.

Tord, Selma, Kristin, Olav, and Jarl were asked about their strategy and thoughts when starting their solution process, and about the meaning of the result.

All five students went directly on to the solution process telling what actions they undertook. Tord first says that he just starts working, carrying out the operations, but after being prompted to come with his thoughts about this specific task, he says:

Tord: I regard it to be an equation, but during the solution process I have to think... how was it then? If I times by negatives or something similar, then I have to change the fraction or that sign (he points to the inequality sign). That is sort of what has to be remembered. Otherwise it has to be treated as an equation.

Tord is very clear about the close connection to equations. During the solution process he talked about 'equal to' although he wrote the inequality sign, the same was the case with the others.

All told that they did not think about the tasks and what to find. They just went on solving it. In the conversation with Olav he extended his explanation. After having solved the task he was asked about his thoughts when doing the tasks:

R1: How are your thoughts when you solve the tasks? Do you think about what you are aiming at?

Olav: No, first I think about what topic has recently been the focus in class, and what we have done within that topic. Then it is easier to solve the tasks.

He relates the tasks to what has recently been done and worked with in class, and this guides his actions. He also tells that when he has reached his solution, he is satisfied. The work is done. I interpret his utterances to indicate that he himself left alone with the task, had no strategy to check his results. Otherwise I assume that he would relate his result to the initial task and check it. None of the interviewees expressed that they checked their results in any of the tasks.

In the interviews it was asked about the interpretation of the solution $x > -2$. Tord answered:

Tord: It means that x is larger than minus 2? It means that the number you are searching for, is a number larger than negative 2. It is larger than, it is such a sequence, right? You have negative 2, zero and then the numbers further on.

R1: Are all numbers larger than 2 within the solution?

Tord: Yes.

R1: Then it might be?

Tord: Any number larger than minus 2.

Although he compares inequalities with equations (see above) and solves the inequality as an equation except for the case when he has to multiply with negatives, he seems to have grasped the concept of inequalities and interprets the solution correctly.

Jarl was also clear about the interpretation of the result. He said:

Jarl: It means it can be all (*numbers*) larger than minus 2.

R1: And then, are all numbers included, or is it a specific number larger than minus 2?

Jarl: It is really all numbers.

Kristin said she did not know the meaning of the result. Selma said she could not know which number it should be, she just knew that it had to be larger than negative 2. From her utterances it seems that she regarded the solution to be one specific number, not a set of numbers.

The same was the case with Olav. He as well found the correct solution and had flipped the sign correctly. He talked about 'x is' during the solution process. When asked about the end solution, he asserted:

Olav: It is minus 1, minus 2, zero, 1, 2... They are lying there, and then it is minus 2, but at the same time you don't know the x .

R: You don't know the x ?

Olav: No, but you know that it has to be larger than minus 2.

R1: What does that mean?

Olav: It means that it is larger.

R1: Some specific number which is larger?

Olav: Yes.

From the interview it seems that he knows that the solution is lying on the number line 'they are lying there', but that it should be a specific number, not the set of numbers larger than negative 2.