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Essays on business cycles with liquidity constraints and firm entry-exit dynamics under incomplete information

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Essays on business cycles with liquidity constraints and firm entry-exit dynamics under incomplete information

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To my family

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CHAPTER 1. OVERVIEW

This dissertation addresses two distinct issues. Chapter 2 studies business cycles with asset fire sales under limited commitment in financial markets. Chapters 3 and 4 study firm entry and exit dynamics in a global game with incomplete information. Chapter 3 derives analytical solutions when firms' productivity is uniformly distributed. Chapter 4 extends the analysis to span more general distributions and solves the problem numerically.

The second chapter develops a stochastic over-lapping generations' model to study the intertemporal and intergenerational transmission of productivity shocks. Productivity shocks cause fire sales of capital, which in turn affects the income of future generations. From a constrained-efficiency perspective, competitive equilibria can be inefficient as agents' choices in equilibrium exhibit ex-ante over-borrowing. The inefficiency arises because entrepreneurs cannot get fully financed from outside funds due to limited commitment in financial markets. The fact that the capital prices are determined in competitive markets also contributes to the above inefficiency because agents fail to internalize potential ex-post fire sales. A capital requirement policy can reduce fire sales when adverse productivity shocks occur, and can thus increase the income for all future generations. On the other hand, a lower capital stock even when good productivity shocks occur decreases income for all future generations. Overall, this chapter shows that in the long run, a capital requirement policy can (strictly) increase welfare of agents.

The third chapter develops a static general equilibrium model to study firms' entry and exit decision in a global game with incomplete information. Firms' choices are strategic substitutes. This chapter analytically proves the existence and uniqueness of a monotonic pure strategy equilibrium when the mean productivity and the productivity conditional on the mean are both drawn from uniform distributions. Using numerical examples, it is shown that when the precision of public information increases, the equilibrium switching productivity level increases and, as a result, the aggregate industry productivity increases. By reallocating resources to more productive firms, an increase in the precision of public information leads to a higher welfare.

The fourth chapter extends the problem studied in the third chapter to examine whether and how the shapes of productivity distributions affect the existence of the monotonic pure strategy equilibria. The mean productivity is now drawn from a truncated normal distribution and individual firm's productivity conditional on the mean is drawn from more general (truncated) distributions, such as truncated normal, truncated gamma, and truncated exponential distributions. With numerical examples, it is shown that a unique monotonic pure strategy equilibrium continues to exist when firms' productivity is drawn from non-uniform distributions. As in chapter 3, both the aggregate productivity and the welfare per worker increase with the increase in the precision of public information. However, unlike in chapter 3, the impact of an increase in the precision of private information on aggregate productivity and the welfare depends on the shape of the distribution. In particular, this impact is uncertain when the productivity conditional on the mean is drawn from truncated gamma distribution, which is skewed.

CHAPTER 2. BUSINESS CYCLES WITH ASSET FIRE SALES

2.1 Introduction

Over about the last two decades, the realization that credit booms are often followed by financial crisis with a drop in asset prices and investments has attracted worldwide attention from policymakers as well as academic researchers (see Caballero and Krishnamurthy (2000, 2002), RanciËre and Tornell (2009), Schularick and Taylor (2009), and Claessens et al (2010)). In response, a new set of "macro-prudential" regulations has been proposed that broadly aims to reduce the occurrence and impact of crisis by addressing the problem of "over-borrowing". In order to assess the policy intervention, it is important to understand why private sectors' optimal decisions are sometimes socially inefficient. Equally important is to understand the intertemporal transmission of financial distress to an economy's income and wealth for future generations.

This chapter focuses on the pecuniary externality caused by financial frictions and competitive capital markets for explaining the inefficiency in agents' financial choices relative to the socially optimal level. The key feature of the chapter's model is that adverse productivity shocks cause fire sales, which endogenously generates inefficient liquidation of productive assets and then decrease the capital used in production. By reducing future income, adverse productivity shocks affect the wealth of future generations.

Chapter 2 lays out a stochastic two-period overlapping-generations model of investment and production under financial frictions. In each period two types of agents, the consumer and the entrepreneur, are born. There is a single homogenous perishable consumption good as well as capital in the economy. Young agents work and get wage income in the first period of their lives. The entrepreneur can borrow and invest either fully or partially in a productive asset that yields state-dependent income at the end of the first period. The entrepreneur offers a state-contingent contract to the consumer to borrow and the consumer can accept or reject the offer. Also, for the capital to remain productive in the second period, the entrepreneur is required to incur additional maintenance costs in the first period. In the second period, both types of agents can produce consumption goods. The entrepreneur's second period technology includes both labor and capital and has a higher rate of return, while consumer's technology only involves capital and yields a lower return.

If an adverse productivity shock occurs in the first period of agents' lives, due to limited commitment the entrepreneur cannot fully cover the maintenance costs through outside funding. As a result, fire sales take place and the capital price drops.[1] Although entrepreneurs and consumers are fully rational optimizing agents, in a competitive equilibrium they do not internalize the general equilibrium effect of fire sales of capital on their prices. This generates a pecuniary externality in the sense that there is an initial over-investment, and in the event of fire sales there is a welfare loss due to the reallocation of capital from a highly productive sector to a less productive sector. The welfare is measured by an unconditional expected *ex-ante* welfare of a "representative" generation, followed by Bhattacharya and Singh (2008). Furthermore, the bad shock is dynamically transmitted from current generation to future generations through a decline in wages, which constitute the funds available for future generations. A capital requirement policy can restrict borrowing in the first period of agents' lives and then reduce the investment by the entrepreneur as well as *ex-post* fire sales. Thus, wealth of current generation increases by imposing the capital requirement policy. Although the effects of capital requirement policy on wealth of the next generation depend on the current state, total wealth of all future generations can strictly increase by imposing the capital requirement policy.

[1] Bordo and Jeanne (2002) show that a sharper reduction in investment and output can result from highly leveraged firms when a negative shock hits, offering a similar flavor to the equilibria examined in this chapter.

Among the papers that study inefficiency of equilibria in financially constrained economies, the first ones that focus on pecuniary externality in general equilibrium through asset prices under financial distress are by Shleifer and Vishny (1992) and Kiyotaki and Moore (1997). Jeanne and Korinek (2010) use collateral constraints to feature pecuniary externality in financial crises and they conclude that the source of pecuniary externality is the interaction between debt accumulation and asset prices. Bianchi and Mendoza (2013) extend their work to study optimal time-consistent maro-prudential policy. Both papers assume a representative firm-household agent, and by construction there are no fire sales within the economy. It is through the value of collateral that asset prices work in their model. A lower value of collateral leads to a tighter financial constraint. Caballero and Krishnamurthy (2001) build a time-to-build model with two types of collateral constraints, domestic and international, and they find that fire sales of domestic assets can occur when the international constraint of collateral is binding in aggregate. Chapter 2 takes an alternative approach that relies on agent heterogeneity to generate actual fire sales in the economy and an *ex-post* inefficient reallocation of capital from a second-best perspective.[2] The inefficiency comes from limited commitment in financial contracts on the part of both entrepreneurs and consumers and the fact that capital price is determined in a competitive capital market. Lorenzoni (2008) studies this externality in a static 3-period stochastic model, while the present chapter focuses on the effects of externality in a dynamic model through intertemporal transmission of productivity shocks. Restricting borrowing in current period not only changes the wealth for current generation but also affects wealth for all future generations. Thus, a non state-contingent or timeless policy should consider both the intertemporal and the intergenerational welfare effects.

The basic structure of this chapter is closely related to the static three-period model studied by Lorenzoni (2008). In the present chapter, to ensure dynamic tractability, three-

[2]Krugman (2000) emphasizes the role of asset "fire sales" played during the 1990's financial crisis in his book, while Pulvino (1998) and Aguiar and Gopinath (2005) present large evidence to support the existing of fire sales.

period events have instead been expedited to occur within two periods. Furthermore, to study the business cycle implications of financial frictions and to understand the frequency and persistence of financial distress, the model is extended as an overlapping-generations economy with both labor and capital. As a result, fixed endowments are replaced by wage income. The persistence of business cycle shocks due to financial frictions has been studied by Bernanke and Gertler (1989). In their paper, the financial frictions are the auditing costs and the production shock hits the sector producing the output instead of capital. Negative productivity shocks reduce investment by decreasing entrepreneurial net worth and thus make the investments in the subsequent periods lower. In chapter 2, the productivity shock affects the first period output of the entrepreneur, which is akin to the capital formation in Bernanke and Gertler.

The structure of chapter 2 is as following. In the next section, the basic framework of the stochastic overlapping-generations model is introduced. Section 2.3 constructs individuals' problems and section 2.4 characterizes the competitive equilibrium. The dynamic analysis is derived in section 2.5 and welfare properties are discussed in section 2.6. Section 2.7 concludes.

2.2 The Environment

A stochastic overlapping-generations model is laid out below to study the intertemporal and intergenerational transmission of productivity shocks. Time is discrete and indexed by $t = 0, 1, 2, 3, \dots. +\infty$. The closed economy consists of an infinite sequence of two period lived agents, an initial old generation and an infinitely lived government. Agents born at time t are called generation t. There are two kinds of goods in the economy, consumption goods and capital goods. Consumption goods are perishable immediately at the end of each period. Agents can convert consumption goods into capital 1 to 1 whenever they want, but once the capital is formed, it cannot be transformed back to consumption goods again. Capital goods which are produced by generation t are fully depreciated with the demise of that generation.

Also, if capital is used more than once for production, it needs to be maintained before its reuse. In each period, the good price is normalized to 1 and the capital price is denoted as q_t. The timeline for generation t is showed in Appendix A.1.

In the following analysis, superscripts above variables indicate the type and the date of birth of the agents, while the subscripts show when the actions are executed. For example, $c_{t+1}^{c,t}$ represents the time $t+1$ consumption by the consumer born at time t. Variables without superscripts are market values that cannot be decided by agents; for instance q_t is the capital price in period t as introduced above.

Within each generation, there are two types of agents of equal mass, consumers and entrepreneurs. The population is assumed to be stationary over time and the number of each type of agents is normalized to 1. Consumers are risk neutral and maximize their utility function represented by $E_t\left[c_t^{c,t} + c_{t+1}^{c,t}\right]$, whereas entrepreneurs are risk neutral as well but only care about their last period consumption and the utility function is $E_t\left[c_{t+1}^{e,t}\right]$.

The generation t entrepreneur is endowed with $L^{e,t}$ units of labor at the beginning of period t. She works in period t for labor income as consumption goods. Since the entrepreneur does not value leisure she will work full time and gets labor income $w_t L^{e,t}$ where w_t is the market wage level. The entrepreneur has access to a technology which can give her $a_t(s)k_t^{e,t}$ units of consumption goods within period t if $k_t^{e,t}$ is invested. $a_t(s)$ is a random variable depending on the aggregate state s, which can take two values, good and bad with probability π_h and π_l. In good state h, $a_t(h) = a_h$ and in bad state l, $a_t(l) = a_l$ where $a_h > a_l > 0$. The state variable is identical and independently distributed across periods. The entrepreneur needs to maintain the capital she used in period t before it can be reused in period $t+1$ production process. The per capital maintenance cost is γ units of consumption goods. The capital fully depreciates if the maintenance cost is not paid. The entrepreneur can maintain part of the used capital. If she decides to maintain fraction $\chi_t(s)(\leq 1)$ of capital, then the total maintenance costs are $\gamma\chi_t(s)k_t^{e,t}$ and the undepreciated capital level is $\chi_t(s)k_t^{e,t}$. The entrepreneur can also reinvest at the end of period t. With maintenance and the reinvestment, the total capital used by the entrepreneur in period $t+1$ is $k_{t+1}^{e,t}(s)$. Production of the consumption goods in period $t+1$ is governed by constant return of scale technology $AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))$ where $L_{t+1}^{e,t}(s)$ is the labor used by the entrepreneur and depends on state s for period t. In period $t+1$, the entrepreneur produces, pays wages to workers and then consumes the rest of her output. $k_{t+1}^{e,t}(s)$ fully depreciates at the end of the period. To simplify analysis, assume that the production technology is in the Cobb-Douglas form $A\left(k_{t+1}^{e,t}(s)\right)^{\alpha}\left(L_{t+1}^{e,t}(s)\right)^{1-\alpha}$ from now on.

The generation t consumer is endowed with $L^{c,t}$ units of labor at the beginning of period t. In period t, the consumer supplies inelastic labor in the market and gets $w_t L^{c,t}$ units of consumption goods as labor income. The consumer owns a "riskless" technology and she can invest $k_{t+1}^{c,t}(s)$ capital in this "riskless" technology at the beginning of period $t+1$ and obtains $F_c(k_{t+1}^{c,t}(s))$ units of consumption goods at the end of period $t+1$. The function $F_c(.)$ is increasing, strictly concave, and twice differentiable. It is assumed that $F_c(0)=0$, $F_c'(0)=1$, and $F_c'(k_{t+1}^{c,t}(s))$ is lower bounded by $\underline{q}$.

The infinitely lived government imposes a policy with a capital requirement of the form $\eta k_t^{e,t} \leq w_t L^{e,t}$ in each period t. The initial old entrepreneurs and consumers are endowed with $k_0^{e,-1}$ and $k_0^{c,-1}$ capital at the beginning of period 0, respectively. Within each period, the goods market, capital market and labor market are perfectly competitive.

2.2.1 Financial Contracts with Limited Commitment

In period t, the entrepreneur offers a state-contingent financial contract to the consumer. The contract has 5 variables and is of the form $\left\langle d_t^{e,t}, d_t^{e,t}(s), d_{t+1}^{e,t}(s)\right\rangle$. $d_t^{e,t}$ is the loan from the consumer to the entrepreneur at the beginning of period t. While $d_t^{e,t}(s)$ and $d_{t+1}^{e,t}(s)$ are state-contingent payments from the entrepreneur to the consumer after production processes by the entrepreneur in period t and $t+1$, respectively, for each state s.

The entrepreneur considers the financial frictions when designing the contract. Both entrepreneurs and consumers are lack of commitments to future payments. Think about the no-default conditions for the entrepreneur first. Suppose in period t, the entrepreneur wants to deviate from the repayment $d_t^{e,t}(s)$. She needs to make a take-it-or-leave-it offer to the consumer and the consumer can choose whether to accept it or not. If the consumer denies the offer, then the firm is liquidated. At the end of period t, a firm owns capital stock $k_t^{e,t}$ (without maintenance) and profits $a_t(s)k_t^{e,t}$. After liquidation, $(1-\theta)$ of firm's current profits vanishes ($\theta \in (0,1)$), and the rest of the profits and all the capital stock will go to the consumer. With per unit maintenance cost γ and capital price $q_t(s)$, the net value of a liquidated firm in period t is $(\theta a_t(s) + \max\{q_t(s)-\gamma, 0\})\, k_t^{e,t}$ units of consumption goods. Moreover, if the entrepreneur chooses to default at the end of period $t+1$, the net value of the liquidated firm is $\theta\left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s)L_{t+1}^{e,t}(s)\right)$ instead due to the fully depreciation of capital. With the net value of liquidated firms, the necessary and sufficient conditions for the entrepreneur without default can be expressed as following:

$$d_t^{e,t}(s) + d_{t+1}^{e,t}(s) \leq (\theta a_t(s) + \max\{q_t(s)-\gamma, 0\})\, k_t^{e,t} \tag{2.1}$$

$$d_{t+1}^{e,t}(s) \leq \theta\left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s)L_{t+1}^{e,t}(s)\right) \tag{2.2}$$

for $s = l, h$. Now it is worth explaining why the above two inequalities are both the necessary and the sufficient conditions. If firm's net values of liquidation are greater than the contractual payments, the consumer can always reject the take-it-or-leave-it offer and make benefits. The entrepreneur loses and ends up with nothing in this case. That is, conditions (2.1) and (2.2) are sufficient conditions for never-default. On the other hand, when the contractual payments are greater than the net values of firm, the entrepreneur can provide a take-it-or-leave-it offer with the current and future repayments greater than the net values of liquidated firm but still smaller than the original contractual payments. The consumer will accept this offer and the entrepreneur will always default. That is, conditions (2.1) and (2.2) are necessary conditions for never-default.

Although the liquidation value of the firm can be treated as collateral (Lorenzoni 2008), they are not the source of inefficiency focused on the welfare analysis later.

The consumer can also default contract. There is no reason for her to accept negative repayments from the entrepreneur at any point of time. Thus, the no-default constraints from the lack of commitment on the consumer's side in each period are:

$$d_t^{e,t}(s) + d_{t+1}^{e,t}(s) \geq 0 \tag{2.3}$$

$$d_{t+1}^{e,t}(s) \geq 0 \tag{2.4}$$

for $s = l, h$.

The model only considers the bilateral financial contract between one entrepreneur and one consumer. This type of contract is without loss of generality in the current environment when there are only aggregate uncertainty (Holmström and Tirole 1998) and equal mass of identical entrepreneurs and identical consumers. Cross-holding of financial securities can be created by zero-profit financial intermediations and the limited commitment conditions for those contracts can be converted and aggregated as above conditions (2.1) -(2.4) as well.

2.3 Individuals' Problems

2.3.1 The Entrepreneur's Problem

In period t, generation t entrepreneur can invest her labor income $w_t L^{e,t}$ and the borrowing $d_t^{e,t}$ from the generation t consumer,

$$k_t^{e,t} \leq w_t L^{e,t} + d_t^{e,t} \tag{2.5}$$

After the production process in period t, the entrepreneur receives current revenues $a_t(s)k_t^{e,t}$ and pays back $d_t^{e,t}(s)$ to the consumer. The entrepreneur also needs to cover the maintenance costs by her funds and uses the rest to finance new investment.

The resource constraint at the end of period t is

$$q_t(s)(k^{e,t}_{t+1}(s) - \chi_t(s)k^{e,t}_t) \leq a_t(s)k^{e,t}_t - \gamma\chi_t(s)k^{e,t}_t - d^{e,t}_t(s) \tag{2.6}$$

Finally, in period $t+1$, the entrepreneur produces with constant return of scale technology. She uses the capital return to consume after paying back the debt repayments to the consumer. The period $t+1$ resource constraint is,

$$c^{e,t}_{t+1}(s) \leq AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) - w_{t+1}(s)L^{e,t}_{t+1}(s) - d^{e,t}_{t+1}(s) \tag{2.7}$$

The entrepreneur offers a financial contract that the consumer will always accept. The consumer accepts the contract when her utility for taking the contract is greater than the one when she rejects the contract. In addition, the consumer's consumption in each period should be non-negative. When the consumer accepts the contract, her expected utility $E_t\left[c^{c,t}_t + c^{c,t}_{t+1}\right]$ is

$$\sum \pi_s \left(w_t L^{c,t} - d^{e,t}_t + d^{e,t}_t(s) - q_t(s)k^{c,t}_{t+1}(s) + F_c(k^{c,t}_{t+1}(s)) + d^{e,t}_{t+1}(s)\right)$$

and her consumption profile is

$$c^{c,t}_t(s) = w_t L^{c,t} - d^{e,t}_t + d^{e,t}_t(s) - q_t(s)k^{c,t}_{t+1}(s) \tag{2.8}$$

$$c^{c,t}_{t+1}(s) = F_c(k^{c,t}_{t+1}(s)) + d^{e,t}_{t+1}(s) \tag{2.9}$$

On the other hand, if the consumer rejects the contract, her expected utility is

$$\sum \pi_s \left(w_t L^{c,t} - q_t(s)k^{c,t}_{t+1}(s) + F_c(k^{c,t}_{t+1}(s))\right)$$

Assume that $c_t^{c,t}(s) \geq 0$ and $c_{t+1}^{c,t}(s) \geq 0$ for $s = l, h$. The consumer's participation constraint is then given by

$$d_t^{e,t} \leq \sum \pi_s \left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right) \tag{2.10}$$

Finally, the government imposes a capital requirement for the investment at the beginning of period t, that is,

$$\eta k_t^{e,t} \leq w_t L^{e,t} \tag{2.11}$$

Thus, the generation t entrepreneur's individual problem is to maximize her expected utility $\sum \pi_s c_{t+1}^{e,t}(s)$ by choosing a financial contract $\left\langle d_t^{e,t}, \left\{d_t^{e,t}(s), d_{t+1}^{e,t}(s)\right\}\right\rangle$, investment decisions $\left\langle k_t^{e,t}, \{\chi_t(s), k_{t+1}^{e,t}(s)\}\right\rangle$, labor demands $\{L_{t+1}^{e,t}(s)\}$ and consumption levels $\{c_{t+1}^{e,t}(s)\}$, given wage levels $\{w_t, w_{t+1}(s)\}$ and capital prices $\{q_t(s)\}$. The entrepreneur faces four sets of constraints: the resource constraints (2.5)-(2.7), the consumer's participation constraint (2.10), the no-default constraints (2.1)-(2.4), and the capital requirement constraint (2.11).

2.3.2 The Consumer's Problem

Facing the wage level w_t and capital prices $\{q_t(s)\}$, the consumer can decide whether to accept a contract or not, and then which contract to accept. She maximizes expected utility $E_t\left[c_t^{c,t}(s) + c_{t+1}^{c,t}(s)\right]$ by setting consumptions $\left\{c_t^{c,t}(s), c_{t+1}^{c,t}(s)\right\}$ and the investment $\left\{k_{t+1}^{c,t}(s)\right\}$ with the riskless technology. Since the entrepreneur always makes the consumer to accept the contract, the budget constraints for the consumer are (2.8) and (2.9), and the consumption in each period is non-negative ($c_t^{c,t}(s) \geq 0$ and $c_{t+1}^{c,t}(s) \geq 0$ for $s = l, h$).

The model only considers a multi-period financial contract $\left\langle d_t^{e,t}, d_t^{e,t}(s), d_{t+1}^{e,t}(s)\right\rangle$ with limited commitments. The length of the contract, whether it is multi-period or single period, is irrelevant in this model. Appendix A.2 shows that the problems with two single-period contracts under limited commitments are equivalent to the individuals' problems discussed above.

2.3.3 Assumptions

Before solving individuals' problems, few useful assumptions about parameters are needed to be introduced.

Assumption 2.1. The per unit maintenance cost γ is less than the lower bound $\underline{q}$ of the derivative of the production function for riskless technology.

$$\gamma < \underline{q}$$

The consumer is the buyer in capital market, and her capital demand can be achieved by the profit maximization problem and is characterized by the first order condition $q_t(s) = F_c'(k_{t+1}^{c,t}(s))$. $1 \geq F_c'(k_{t+1}^{c,t}(s)) \geq \underline{q}$ implies that $1 \geq q_t(s) \geq \underline{q}$ and thus $\max\{q_t(s) - \gamma, 0\} = q_t(s) - \gamma$. In other words, assumption 2.1 is the no-scrapping condition for the entrepreneur, $\chi_t(s) = 1$ for $s = l, h$ in any periods.

Assumption 2.2. The marginal return of capital for the constant return of scale technology is greater than 1,

$$AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) > 1 \tag{2.12}$$

with a sufficiently small θ such that

$$\theta AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) < \underline{q} \tag{2.13}$$

Assumption 2.2 ensures that the no-default constraint (2.2) is binding. That is,

$$d_{t+1}^{e,t}(s) = \theta \left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s) L_{t+1}^{e,t}(s) \right) \tag{2.14}$$

At the end of period t, the generation t entrepreneur has three ways to finance her investment. She can (1) convert the consumption goods into capital herself at price 1, (2) buy capital in market at price $q_t(s)(\leq 1)$, or (3) borrow at the marginal cost of $\partial d_{t+1}^{e,t}(s)/\partial k_{t+1}^{e,t}(s)$

from the consumer and buy capital at price $q_t(s)$. Since $q_t(s) \leq 1$ from assumption 2.1, the entrepreneur will always prefer method (2) to (1). Now compare method (2) with (3) for the entrepreneur. When the entrepreneur chooses method (2), the marginal rate of return of the wealth at the end of period t is

$$z_{1s}^{(2)} = \frac{AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))}{q_t(s)}$$

On the other hand, if the entrepreneur chooses method (3), and if equation (2.14) holds, by (2.13)

$$\frac{\partial d_{t+1}^{e,t}(s)}{\partial k_{t+1}^{e,t}(s)} = \theta AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s) < \underline{q} \leq q_t(s)$$

and the marginal rate of return on entrepreneurial wealth at the end of period t is then given by

$$\begin{aligned} z_{1s}^{(3)} &= \frac{AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - \partial d_{t+1}^{e,t}(s)/\partial k_{t+1}^{e,t}(s)}{q_t(s) - \partial d_{t+1}^{e,t}(s)/\partial k_{t+1}^{e,t}(s)} \\ &= \frac{(1-\theta)\, AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))}{q_t(s) - \theta AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))} \end{aligned}$$

By (2.12),

$$z_{1s}^{(3)} > z_{1s}^{(2)}$$

Thus, the entrepreneur will borrow as much as possible to finance her investment at the end of period t, which coincides the equation (2.14) as assumed.

Assumption 2.3.

$$(1-\theta)\, a_h k_t^{e,t} - k_t^{e,t} + \theta\alpha AF(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h)) > 0 \tag{2.15}$$

$$a_l k_t^{e,t} - \gamma k_t^{e,t} + \theta\alpha AF(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l)) < 0 \tag{2.16}$$

The first condition in assumption 2.3 implies positive investment at the end of period t in good state ($k^{e,t}_{t+1}(h) - k^{e,t}_t > 0$), while the second condition indicates negative investment at the end of period t in bad state ($k^{e,t}_{t+1}(l) - k^{e,t}_t < 0$). Since the entrepreneur's utility is strictly increasing, the condition (2.6) is binding. With $\chi_t(s) = 1$, (2.6) can be written as

$$q_t(s)(k^{e,t}_{t+1}(s) - k^{e,t}_t) = a_t(s)k^{e,t}_t - \gamma k^{e,t}_t - d^{e,t}_t(s) \tag{2.17}$$

From the no-default constraint (2.3) and result (2.14),

$$-d^{e,t}_t(s) \leq d^{e,t}_{t+1}(s) = \theta\alpha AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) \tag{2.18}$$

The equality comes from the Cobb-Douglas form of the production function and the labor market clear condition (which will be introduced later). Substituting (2.18) in (2.17) gets,

$$q_t(s)(k^{e,t}_{t+1}(s) - k^{e,t}_t) \leq a_t(s)k^{e,t}_t - \gamma k^{e,t}_t + \theta\alpha AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s))$$

If the right hand side of the above inequality is smaller than 0 in bad state, then $k^{e,t}_{t+1}(l) - k^{e,t}_t < 0$ and the new investment is negative. In addition, from the no-default constraint (2.1), assumption 2.1 and result (2.14), one gets,

$$\begin{aligned} -d^{e,t}_t(s) &\geq d^{e,t}_{t+1}(s) - (\theta a_t(s) + q_t(s) - \gamma)\, k^{e,t}_t \\ &= \theta\alpha AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) - (\theta a_t(s) + q_t(s) - \gamma)\, k^{e,t}_t \end{aligned} \tag{2.19}$$

Replace $-d^{e,t}_t(s)$ in (2.17) with (2.19) and then since $q_t(s) \leq 1$,

$$\begin{aligned} q_t(s)(k^{e,t}_{t+1}(s) - k^{e,t}_t) &\geq (1-\theta)\, a_t(s)k^{e,t}_t - q_t(s)k^{e,t}_t + \theta\alpha AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) \\ &\geq (1-\theta)\, a_t(s)k^{e,t}_t - k^{e,t}_t + \theta\alpha AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) \end{aligned}$$

In good state, if the right hand side of the above inequality is greater than 0, then the new investment is positive.

In sum, assumptions 2.1 to 2.3 together limit individuals' choices: there is no scrapping ($\chi_t(s) = 1$), the entrepreneur maximizes her borrowing ability at the end of period t and the new investment at the end of period t is positive in good state and negative in bad state. In addition, the following assumption 2.4 is used to prove the existence and uniqueness of the competitive equilibrium.

Assumption 2.4.

$$
\begin{aligned}
F_c''(k_{t+1}^{c,t}(l))k_{t+1}^{c,t}(l) + F_c'(k_{t+1}^{c,t}(l)) + \theta\alpha^2 A\left(k_t^{e,t} - k_{t+1}^{c,t}(l)\right)^{\alpha-1} L^{1-\alpha} &> 0 \\
1 - \theta\alpha A\left(k_t^{e,t,CE}\right)^{\alpha-1} L^{1-\alpha} - (a_h + 1 - \gamma) + \frac{d_{t+1}^{e,t,CE}(h) + d_t^{e,t,CE}(h)}{k_t^{e,t,CE}} &< 0
\end{aligned}
$$

where the superscript CE is used to denote the equilibrium values.

2.3.4 Solution to the Individuals' Problems

Below, lemma 2.1 first describes how the entrepreneur chooses the optimal financial contract. Later, other choice variables for individuals' problems are derived. Intuition is provided for the entrepreneur's investment choice, and the idea of a "pecking order" in borrowing, that the entrepreneur will always exhaust her borrowing ability in good state before she can borrow against bad state, is introduced.

Lemma 2.1 *If the capital requirement constraint (2.11) is not binding, the optimal financial contract* $\left\langle d_t^{e,t}, d_t^{e,t}(s), d_{t+1}^{e,t}(s)\right\rangle$ *chosen by the entrepreneur, given wage levels* $\{w_t, w_{t+1}(s)\}$ *and* *capital prices $\{q_t(s)\}$, satisfies following conditions:*

$$
\begin{array}{ll}
d_t^{e,t}(s) + d_{t+1}^{e,t}(s) = 0 & \text{if } z_0 < z_{1s} \\
d_t^{e,t}(s) + d_{t+1}^{e,t}(s) \in \left[0, \left(\theta a_t\left(s\right) + q_t(s) - \gamma\right) k_t^{e,t}\right] & \text{if } z_0 = z_{1s} \\
d_t^{e,t}(s) + d_{t+1}^{e,t}(s) = \left(\theta a_t\left(s\right) + q_t(s) - \gamma\right) k_t^{e,t} & \text{if } z_0 > z_{1s} \\
d_{t+1}^{e,t}(s) = \theta\left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s)L_{t+1}^{e,t}(s)\right) &
\end{array}
$$

for $s = l, h$, where

$$
\begin{aligned}
z_{1s} &= \frac{\left(1-\theta\right) AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))}{q_t(s) - \theta AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))} \\
z_0 &= \frac{\sum_s \pi_s z_{1s}\left(a_t\left(s\right) + q_t(s) - \gamma - \left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right)/k_t^{e,t}\right)}{1 - \sum_s \pi_s\left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right)/k_t^{e,t}}
\end{aligned}
$$

and

$$
d_t^{e,t} = \sum \pi_s\left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right)
$$

When the capital requirement constraint (2.11) binds, then

$$
d_t^{e,t} = k_t^{e,t} - w_t L^{e,t} = \left(\frac{1}{\eta} - 1\right) w_t L^{e,t}
$$

$$
\begin{array}{ll}
\text{if } d_t^{e,t} \le \pi_h\left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t}, & d_t^{e,t}(h) + d_{t+1}^{e,t}(h) = d_t^{e,t}/\pi_h \\
& d_t^{e,t}(l) + d_{t+1}^{e,t}\left(l\right) = 0 \\
\text{if } d_t^{e,t} > \pi_h\left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t}, & d_t^{e,t}(h) + d_{t+1}^{e,t}(h) = \left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t} \\
& d_t^{e,t}(l) + d_{t+1}^{e,t}\left(l\right) = \frac{1}{\pi_l}\left(d_t^{e,t} - \pi_h\left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t}\right) \\
d_{t+1}^{e,t}(s) = \theta\left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s)L_{t+1}^{e,t}(s)\right) &
\end{array}
$$

With the optimal contract choice, the entrepreneur's first time investment is

$$
k_t^{e,t} = d_t^{e,t} + w_t L^{e,t}
$$

and the second time investment is

$$k_{t+1}^{e,t}(s) = \frac{1}{q_t(s)} \left((a_t(s) + q_t(s) - \gamma) k_t^{e,t} - d_t^{e,t}(s) \right) \tag{2.20}$$

the labor demand $L_{t+1}^{e,t}(s)$ satisfies the first order condition

$$AF_2(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) = w_{t+1}(s) \tag{2.21}$$

and the consumptions $c_{t+1}^{e,t}(s)$ in the last period can be obatined by set the inequality (2.7) as equality.

The consumer accepts the contract, chooses $k_{t+1}^{c,t}(s)$ such that

$$F_c'(k_{t+1}^{c,t}(s)) = q_t(s) \tag{2.22}$$

and $c_t^{c,t}(s)$, $c_{t+1}^{c,t}(s)$ satisfying equations (2.8) and (2.9).

The proof of the above lemma 2.1 is provided in Appendix A.3. The variable z_0 can be seen as the marginal rate of return on entrepreneurial wealth at the beginning of period t. The entrepreneur can borrow from the consumer to invest at the beginning of period t and get random return of $\left(a_t(s) + q_t(s) - \gamma - \left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right)/k_t^{e,t}\right)$ at the end of period t. With 1 extra unit of capital invested at the end of period t in state s, the entrepreneur can obtain a return of z_{1s} $\left(= z_{1s}^{(3)} \text{ in assumption 2.2}\right)$. The entrepreneur makes her investment decision by comparing the return of capital before and after the state is revealed. When $z_0 > z_{1s}$, she absorbs outside funds as much as possible by increasing the promised repayment, $d_t^{e,t}(s) + d_{t+1}^{e,t}(s) = (\theta a_t(s) + q_t(s) - \gamma) k_t^{e,t}$. When $z_0 < z_{1s}$, the entrepreneur decreases her promised repayment to 0 in order to shrink the investment. When $z_0 = z_{1s}$, the entrepreneur is indifferent between the two opportunities of investment.

Assumption 2.3 indicates positive reinvestment in good state and negative reinvestment in bad state at the end of period t. If there is positive reinvestment, the entrepreneur is potential buyer in capital market, which implies the consumer is the potential seller of capital. In this case, the capital price $q_t(h) = 1$ since the consumer can only convert consumption goods into capital 1 to 1 necessarily. On the other hand, when there is negative reinvestment, the entrepreneur is the seller in capital market and the consumer is the buyer. By (2.22), $q_t(l) = F_c'(k_{t+1}^{c,t}(l)) < 1$ since $k_{t+1}^{c,t}(l) = -(k_{t+1}^{e,t}(l) - k_t^{e,t}) > 0$ in bad state.

Also from assumption 2.3, it is known that $k_{t+1}^{e,t}(l) < k_t^{e,t} < k_{t+1}^{e,t}(h)$. Since individuals inelastically supply labor, labor market clear implies that $L_{t+1}^{e,t}(s) = L^{e,t+1} + L^{c,t+1} = L$, then

$$
\begin{aligned}
z_{1h} &= \frac{(1-\theta)\, AF_1(k_{t+1}^{e,t}(h), L)}{1-\theta AF_1(k_{t+1}^{e,t}(h), L)} \\
&< \frac{(1-\theta)\, AF_1(k_{t+1}^{e,t}(l), L)}{1-\theta AF_1(k_{t+1}^{e,t}(l), L)} \\
&< \frac{(1-\theta)\, AF_1(k_{t+1}^{e,t}(l), L)}{q_t(l)-\theta AF_1(k_{t+1}^{e,t}(l), L)} = z_{1l}
\end{aligned}
$$

The first inequality comes from decreasing marginal return of capital for the production technology, and the second inequality holds because $q_t(l) < 1$. The above shows that the marginal rate of return on entrepreneurial wealth at the end of period t in bad state is greater than the one in good state. Therefore, the entrepreneur will first exhaust her borrowing capacity in the good state up until $d_t^{e,t}(h) + d_{t+1}^{e,t}(h) = (\theta a_h + 1 - \gamma)\, k_t^{e,t}$ before she can borrow against the bad state.

2.4 Equilibrium

This section first defines the competitive equilibrium with policy intervention, and then introduces the equilibrium financial contract. To illustrate properties of the competitive equilibrium by numerical experiments, parameters for two examples are listed. Moreover, the relationship between wage income and equilibrium choice variables are studied for future use in dynamic analysis.

2.4.1 Equilibrium Definition

A symmetric competitive equilibrium, given initial wage level w_t and capital requirement η, is defined as a sequence of wage levels$\{w_{t+1+\nu}(s)\}_{\nu=0}^{\infty}$ and asset prices $\{q_{t+\nu}(s)\}_{\nu=0}^{\infty}$, optimal financial contracts $\{d_{t+\nu}^{e,t+\nu}, d_{t+\nu}^{e,t+\nu}(s), d_{t+1+\nu}^{e,t+\nu}(s)\}_{\nu=0}^{\infty}$, investment choices $\left\{k_{t+\nu}^{e,t+\nu}, \{\chi_{t+\nu}(s), k_{t+1+\nu}^{e,t+\nu}(s)\}\right\}_{\nu=0}^{\infty}$, labor demands $\left\{L_{t+1+\nu}^{e,t+\nu}(s)\right\}_{\nu=0}^{\infty}$, consumption decisions $\left\{c_{t+1+\nu}^{e,t+\nu}(s)\right\}_{\nu=0}^{\infty}$ made by entrepreneurs, and investments $\left\{k_{t+1+\nu}^{c,t+\nu}(s)\right\}_{\nu=0}^{\infty}$ and consumption decisions $\left\{c_{t+\nu}^{c,t+\nu}(s), c_{t+1+\nu}^{c,t+\nu}(s)\right\}_{\nu=0}^{\infty}$ made by consumers, that solve the entrepreneurs' problems and the consumers' problems introduced in the last section for each period, and goods markets, labor markets, and capital markets clear in all periods and states.

The labor market From the generation t entrepreneur's problem, it is known that the labor demand $L_{t+1}^{e,t}(s)$ satisfies the first order condition (2.21). Since the supply of labor is fixed, the equilibrium labor supply L_{t+1}^{s} in period $t+1$, $L_{t+1}^{e,t} + L_{t+1}^{c,t}$ is the market clearing labor quantity $L_{t+1}^{CE} = L$. Then, by replacing $L_{t+1}^{e,t}(s) = L$ in (2.21), the labor market clearing condition is obtained as:

$$AF_2(k_{t+1}^{e,t}(s), L) = w_{t+1}(s) \tag{2.23}$$

The capital market In the good state, there is no exchange of capital in the market. The capital market clearing requires (followed the analysis in assumption 2.3)

$$\begin{aligned} q_t(h) &= 1 \\ k_{t+1}^{c,t}(h) &= 0 \end{aligned}$$

In bad state, the consumer purchases capital by (2.22), and the entrepreneur sells capital by (2.20). The capital market clearing now requires

$$\begin{aligned} q_t(l) &= F_c'(k_{t+1}^{c,t}(l)) < 1 \\ k_{t+1}^{c,t}(l) &= k_t^{e,t} - k_{t+1}^{e,t}(l) \end{aligned}$$

Note that the capital demand is downward sloping in $q_t(l) - k_{t+1}^{c,t}(l)$ space, and the capital supply

$$k_t^{e,t} - k_{t+1}^{e,t}(l) = -\frac{1}{q_t(l)}\left(\left(a_l - \gamma\right)k_t^{e,t} - d_t^{e,t}(l)\right)$$

is also downward sloping in $q_t(l) - \left(k_t^{e,t} - k_{t+1}^{e,t}(l)\right)$ space due to assumption 2.3. Thus, to have a stable equilibrium in capital market, it is required that the slope of capital supply be greater than the slope of capital demand in absolute values. That is,

$$-\frac{\left(a_l - \gamma\right)k_t^{e,t} - d_t^{e,t}(l)}{\left(k_t^{e,t} - k_{t+1}^{e,t}(l)\right)^2} > -F_c^{"}(k_{t+1}^{c,t}(l))$$

Figure 2.1 shows the capital market equilibrium after a bad productivity shock, for a given $k_t^{e,t}$ and $d_t^{e,t}(l)$.

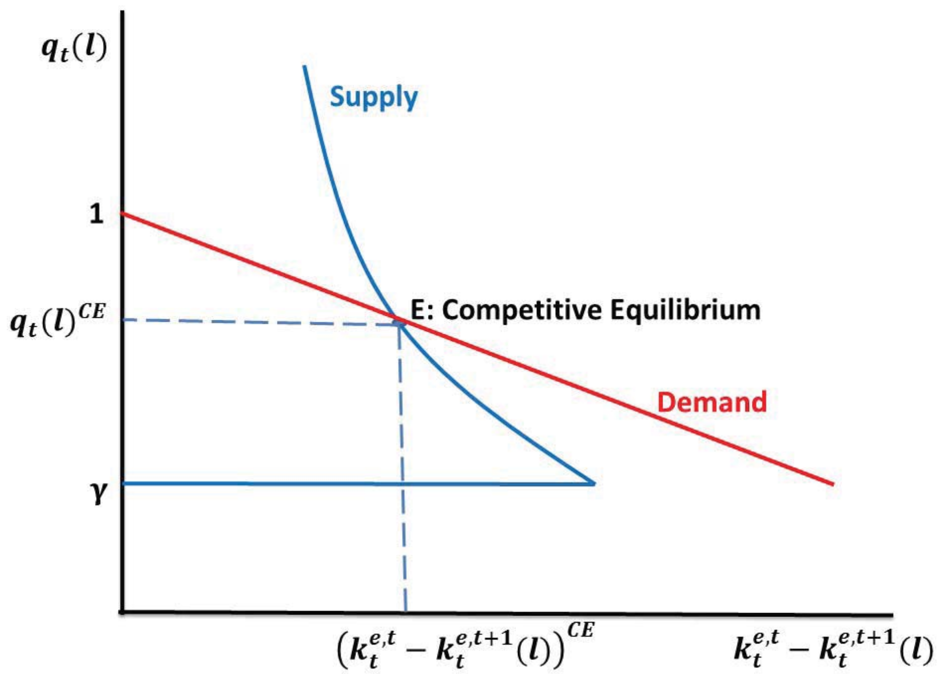


Figure 2.1. Capital market equilibrium

2.4.2 Equilibrium Financial Contract

The sequence of equilibrium financial contracts in the stochastic overlapping-generations model follows the same property as exhibited in Lorenzoni (2008)'s three period model.

Proposition 2.1 *The symmetric competitive equilibrium is unique with capital prices*

$$q_t^{CE}(l) < q_t^{CE}(h) = 1$$

for any t, and the equilibrium financial contract is one of the following three types:

Type 1: $0 \leq b_t^{e,t,CE}(h) < (\theta a_h + 1 - \gamma)$ *and* $b_t^{e,t,CE}(l) = 0$*;*

Type 2: $b_t^{e,t,CE}(h) = (\theta a_h + 1 - \gamma)$ *and* $b_t^{e,t,CE}(l) = 0$*;*

Type 3: $b_t^{e,t,CE}(h) = (\theta a_h + 1 - \gamma)$ *and* $0 < b_t^{e,t,CE}(l) \leq \big(\theta a_l + q_t^{CE}(l) - \gamma\big)$.

where

$$b_t^{e,t,CE}(s) = \frac{d_t^{e,t,CE}(s) + d_{t+1}^{e,t,CE}(s)}{k_t^{e,t}}$$

This proposition coincides with the previous result that the entrepreneur exhausts her borrowing ability in good state first before she borrows in bad state. The proof can be found in Appendix A.4. Below, examples of type 1 and type 3 equilibrium will be introduced and analyzed.

2.4.3 Parameters Used in Experiments

The following Table 2.1 displays parameters used for experiments undertaken to study the two types of equilibrium with. Although the model is very stylized, the parameters in the numerical analysis are set to be as realistic as possible. First, since drop in asset prices (fire sales) is one of the key features of financial crisis in the real world and fire sales occur only after bad productivity shock in the present model, it is assumed that the probability of good shocks is 0.9. This implies that financial crisis takes place once in ten periods on average. Second, the maintenance cost γ can be treated as capital depreciation rate. Nadiri and Prucha (1996) state that the R&D capital depreciation rate is 0.12 in the U.S. total manufacturing sector. The value chosen for $\gamma = 0.2$ here is a bit higher in order to satisfy assumptions 2.3 and 2.4. Recall that parameter values for a_h, a_l and γ together play an important role in determining the type of equilibrium. Similarly, for type 1 equilibrium $\gamma = 0.46$ is chosen to make this type of equilibrium possible. The non-liquidation fraction θ is chosen to be small enough to restrict borrowing ability of the entrepreneur in order to generate fire sales. Since fire sales are less likely to occur in experiment 1, θ for type 1 equilibrium example is smaller than the one for type 3 equilibrium. The capital share α is chosen at its standard value of 0.6 for both types. The labor ratio is set such that the first period consumption for the consumer (equation (2.8)) is non-negative.

Table 2.1. The model parameterizations

Parameters	Type 1	Type 3
Probability of good states (π_h)	0.9	0.9
Productivity of the first time investment in good state (a_h)	0.5	0.9
Productivity of the first time investment in bad state (a_l)	0.01	0.09
Total factor productivity of the second time investment (A)	1	1
Capital share in the second time production function (α)	0.6	0.6
Fraction of firm's profits that goes to the consumer if liquidated (θ)	0.015	0.08
Labor endowment of the entrepreneur (L^e)	1	1
Labor endowment of the consumer (L^c)	15	25
Per unit maintenance cost (γ)	0.46	0.2
Limit inferior of the marginal productivity in riskless sector ($\underline{q}$)	0.5	0.5

For simplicity, the riskless technology used by the consumer is chosen as

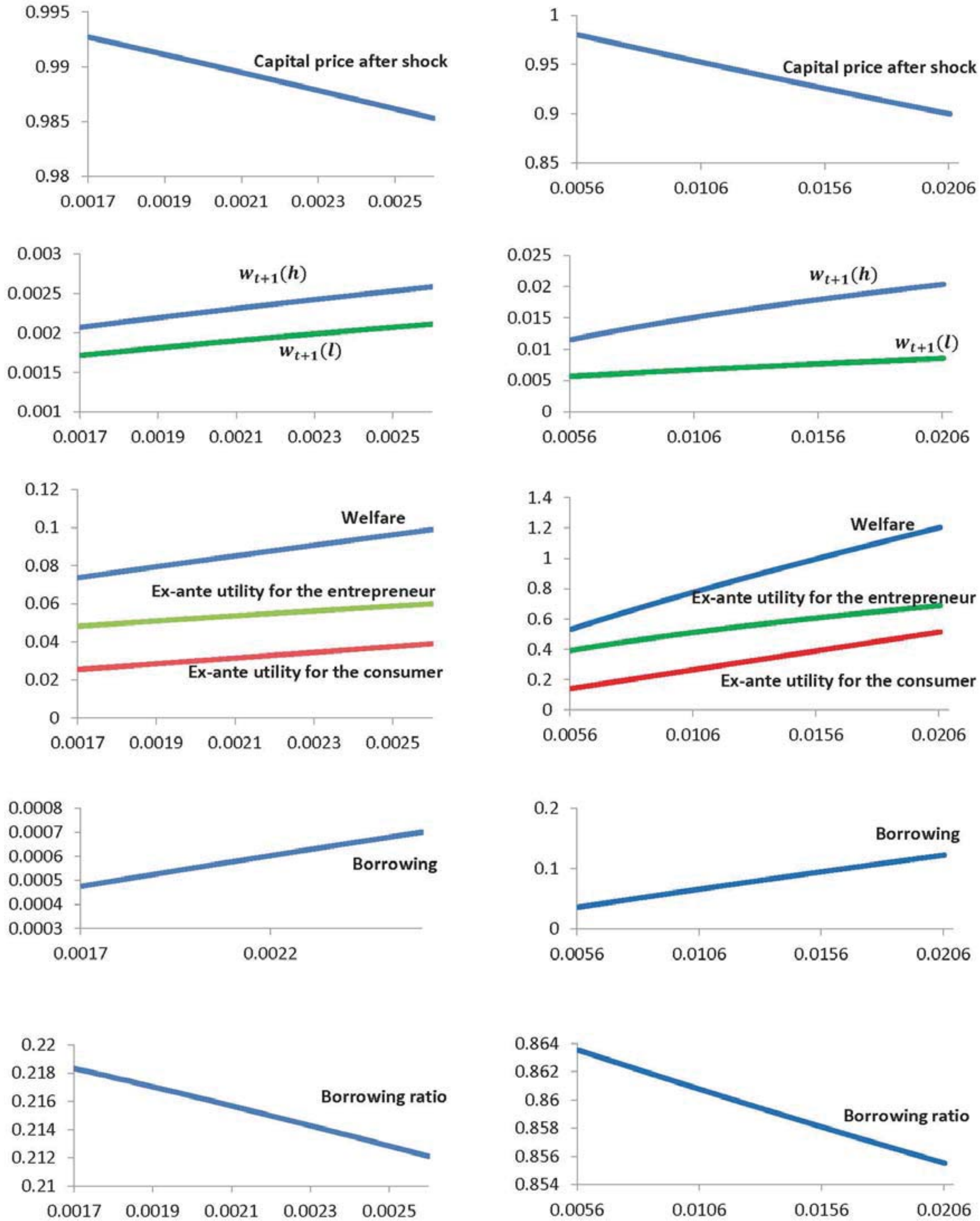
$$F_c(k_{t+1}^{c,t}(s)) = k_{t+1}^{c,t}(s) - m\left(k_{t+1}^{c,t}(s)\right)^2$$

where $m = 10$ for type 1 experiment and $m = 0.5$ for type 3 experiment (The slope of the supply in capital market is lower in type 3 experiment compared with type 1 and thus the capital demand slope is chosen to be smaller). The quadratic form of riskless technology specifies a constant slope of capital demand function, which simplifies algebra required to verify the stability of capital market equilibrium. This production function is increasing, strictly concave and twice differentiable when $k_{t+1}^{c,t}(s) \in (0, 0.025)$ for type 1 experiment and when $k_{t+1}^{c,t}(s) \in (0, 0.5)$ for type 3 experiment.

2.4.4 Intertemporal Effects of Wage Income

As a preparation for the analysis of dynamic transmission of productivity shocks, it is first important to understand the intertemporal effects of movement in wage income. Figure 2.2 shows how equilibrium market prices $\{q_t(l), w_{t+1}(s)\}$, ex-ante indirect utility for generation t agents and borrowing $d_t^{e,t}$ change when the first period wage level w_t for generation t changes. Here the capital requirement constraint does not bind. The horizontal axis is wage level w_t. The left limit $\underline{w}$ of w_t is obtained by imposing a sequence of bad productivity shocks on the economy and the right limit $\overline{w}$ is achieved by imposing continually good states. Since the capital price $q_t(h)$ after good productivity shock is always 1, figure 2.2 only exhibits capital price $q_t(l)$ after the bad shock and wage level $w_{t+1}(s)$ for the following period under both types of productivity shocks as equilibrium market prices.

Figure 2.2 presents a negative relationship between wage level w_t and capital price $q_t(l)$ after the bad shock. With more wage income, the entrepreneur invests more in her capital and thus needs to sell more capital after the shock in order to recover the maintenance costs. Due to the fixed capital demand curve by the consumer, the fire sales prices decrease when their quantities increase.



(a) Type 1 equilibrium (b) Type 3 equilibrium

Figure 2.2. Intertemporal effects of wage income

When w_t increases, the entrepreneur increases her investments $k_{t+1}^{e,t}(h)$ and then $w_{t+1}(h)$ increases. Although the fire sales increase as wage income increases, the positive effects of wage income on $k_{t+1}^{e,t}(l)$ outweighs the negative effects from fire sales. As a result, $w_{t+1}(l)$ increases as well.

With more wage income, the entrepreneur will end up with a higher level of *ex-ante* utility since the choice set is enlarged for her. The consumer's *ex-ante* utility increases first because of a higher wage income and second due to a larger fire sales. Since the marginal productivity in the riskless sector is always greater than the capital price, with more capital used in riskless sector, the consumer earns more profits. The slope of the "*ex-ante* utility of the entrepreneur" curve is smaller than the slop of the "*ex-ante* utility of the consumer" curve in figure 2.2, which indicates that the consumer's utility increases more rapidly than that of the entrepreneur as wage level increases. (Comparing the slope of the "*ex-ante* utility of the entrepreneur" curve with the slop of the "welfare" curve makes the difference more obvious. The welfare is the sum of the *ex-ante* utilities for individuals.)

The borrowing ratio ρ_t (the ratio between the borrowing $d_t^{e,t}$ and the capital investment $k_t^{e,t}$ at the beginning of period t) decreases with the increase of wage level w_t. That is, the "poor" entrepreneur has a high willingness to borrow from outside funding. However, the total borrowing amount $d_t^{e,t}$ increases with wage level w_t, since with more income the entrepreneur has greater ability to repay to the consumer.

2.5 Dynamics

The dynamics of the stochastic overlapping-generations model are derived numerically in this section. A typical path of experiment of each type of equilibrium is showed first. The capital requirement constraint does not bind for these two examples and the effects of policy control will be studied in the next section. Then the dynamics after an adverse productivity shock are studied. The long run distributions of wages are discussed at the end of the section.

2.5.1 Dynamic Paths of Two Examples

Figure 2.3 and 2.4 show the dynamics of the economy under different types in 50 periods. Each figure includes the movements of 6 variables, the state of the period s_t ($s_t = 1$ in good state and $s_t = 0$ in bad state), capital price q_t, *ex-ante* utility for the entrepreneur $EU^{e,t}$ and the consumer $EU^{c,t}$, the wage w_t, and the borrowing ratio ρ_t. For all subplots, the horizontal axis is time t. Both experiments are conducted in 1000 periods and the first 100 periods' outcomes are dropped to avoid the influence of starting value. Further analysis on wage distribution is also based on these 900 points. Figure 2.3 and 2.4 only exhibit 50 periods for clear illustration. The figures with $t = 101$ to 1000 are attached in Appendix A.5.

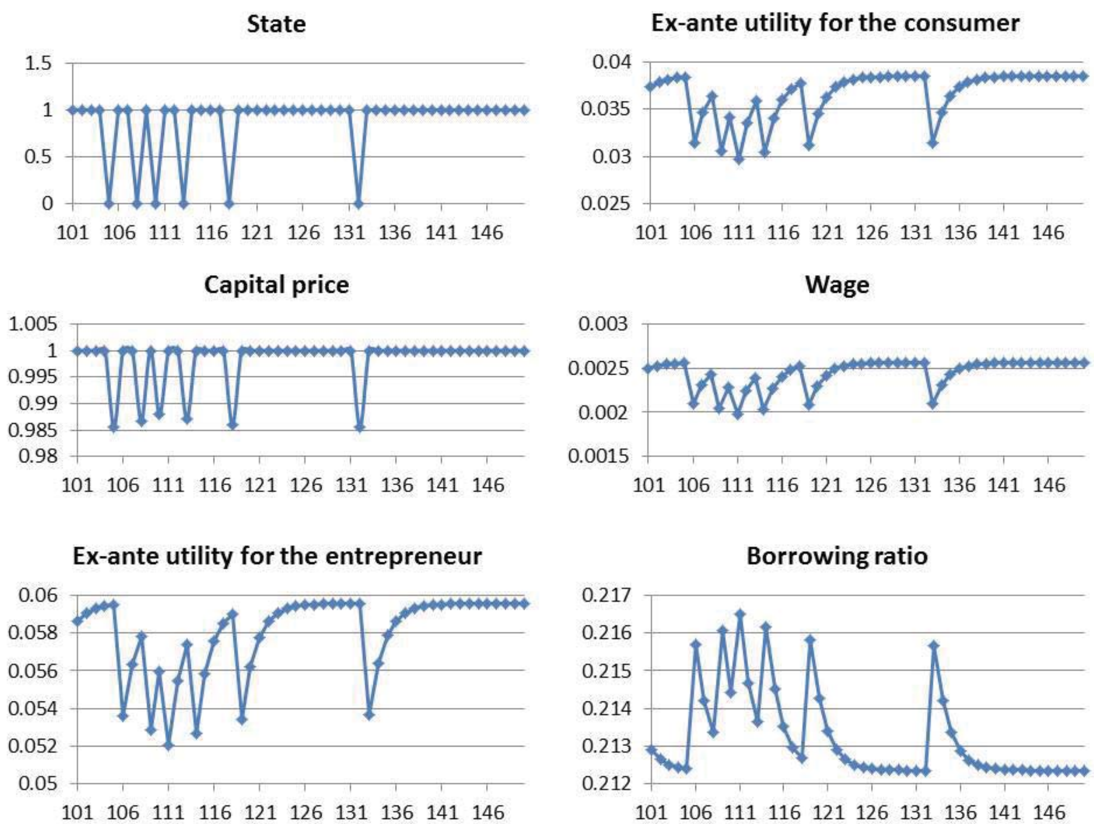


Figure 2.3. Dynamics of type 1 equilibrium

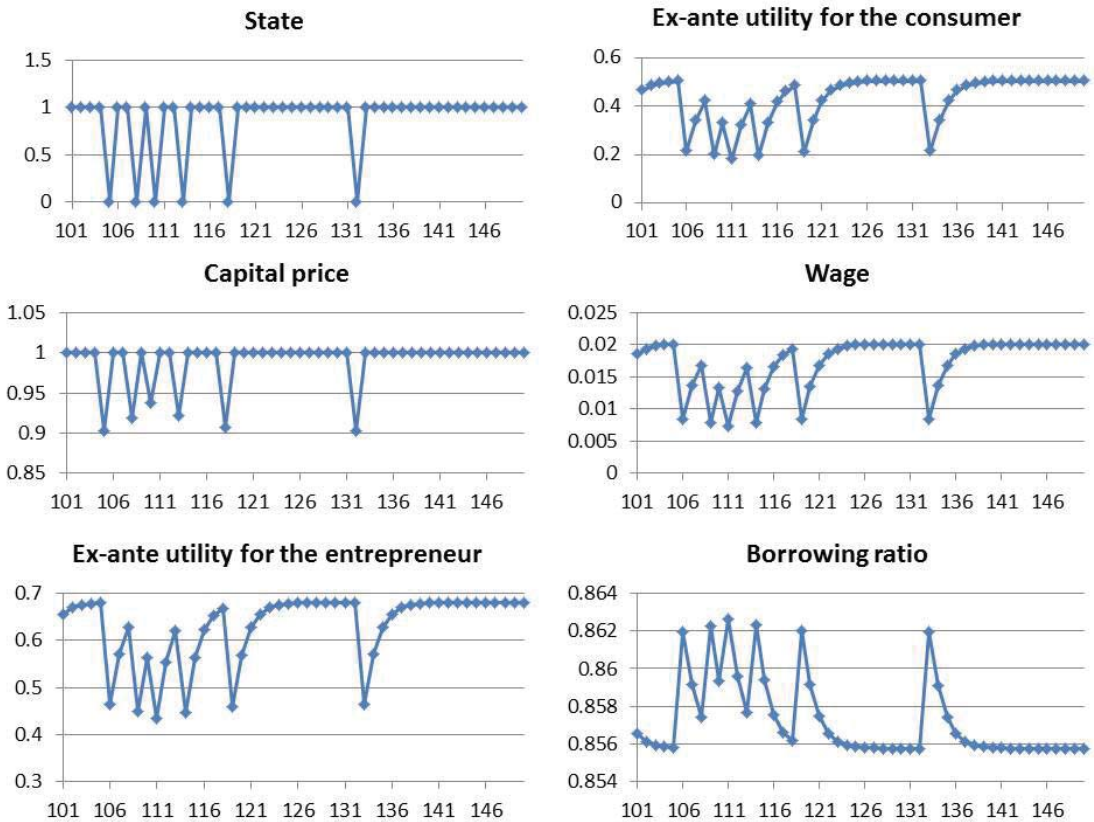


Figure 2.4. Dynamics of type 3 equilibrium

The capital price drops whenever there is a bad productivity shock, and the *ex-ante* utilities for individuals decrease following the period of the bad shock and so does the wage level. With wage level decreases, the borrowing ratio increases.

2.5.2 Dynamics after Realizations of Adverse Productivity Shock

Suppose there is a bad productivity shock in period t, the generation t entrepreneur invests $k_{t+1}^{e,t}(l)$ in her last period production process, earns profits $A\left(k_{t+1}^{e,t}(l)\right)^{\alpha}L^{1-\alpha}-w_{t+1}(l)L$ and pays $d_{t+1}^{e,t}(l)$ to the consumer. The *ex-post* utility for the entrepreneur is

$$c_{t+1}^{e,t}(l)=A\left(k_{t+1}^{e,t}(l)\right)^{\alpha}L^{1-\alpha}-w_{t+1}(l)L-d_{t+1}^{e,t}(l)$$

Replace $d^{e,t}_{t+1}(l)$ by (2.14),

$$\begin{aligned} c^{e,t}_{t+1}(l) &= (1-\theta)\left(A\left(k^{e,t}_{t+1}(l)\right)^{\alpha}L^{1-\alpha} - w_{t+1}(l)L\right) \\ &= (1-\theta)\alpha A\left(k^{e,t}_{t+1}(l)\right)^{\alpha}L^{1-\alpha} \end{aligned}$$

where the second equality comes from the labor market clearing condition. Since the actual realization of bad productivity shock causes fire sales, $k^{e,t}_{t+1}(l) < k^{e,t}_{t} < k^{e,t}_{t+1}(h)$, $c^{e,t}_{t+1}(l) < c^{e,t}_{t+1}(h)$. Note that the repayment $d^{e,t}_{t+1}(l)$ in bad state is smaller than the repayment $d^{e,t}_{t+1}(h)$ in good state. That is, although having less to pay back under the financial contract, the *ex-post* utility for the entrepreneur is smaller after the bad shock than the *ex-post* utility for her in the absence of bad shock.

Now, consider the effects on generation t consumer of bad productivity shock in period t. By (2.8) and (2.9), the *ex-post* utility for the consumer after the bad shock is

$$\begin{aligned} & c^{c,t}_{t}(l) + c^{c,t}_{t+1}(l) \\ = \; & w_t L^{c,t} - d^{e,t}_{t} + d^{e,t}_{t}(l) + d^{e,t}_{t+1}(l) + F_c(k^{c,t}_{t+1}(l)) - q_t(l)k^{c,t}_{t+1}(l) \end{aligned}$$

By proposition 2.1, $d^{e,t}_{t}(l) + d^{e,t}_{t+1}(l) \leq d^{e,t}_{t}(h) + d^{e,t}_{t+1}(h)$. That is, the consumer is hurt by getting less repayments after the bad shock. However, due to fire sales, the consumer will buy capital and earn positive profits in the riskless sector, which will increases her *ex-post* utility. Thus, whether the consumer will be benefit or not by the bad shock is not clear. For the two experiments undertaken here, the consumer's *ex-post* utility is smaller under the bad shock than the one when there is no bad shock.

After the bad shock, because of decreasing marginal productivity of labor, the wage level w_{t+1} in period $t+1$ is smaller than the one without the bad shock. With lower wage income, the wealth of generation $t+1$ entrepreneur and consumer decreases as shown in figure 2.2. The borrowing ratio increases due to a decrease in wage. In period $t+2$, wage w_{t+2} is smaller than that otherwise would be and the transmission of bad shock passes to all future generations. That is, the actual realization of adverse productivity shock causes fire sales with drop in capital price, and then by reducing future productivity, decreases the income and wealth of all future generations.

2.5.3 Long Run Distributions of Wages

Figure 2.5 and 2.6 show the distributions of wages for the two different types of equilibrium. The height of each bar is the relative number of observations (probability), while the sum of the bar heights is 1. The horizontal axis is wage. The analysis below focuses on type 1 equilibrium, while the distribution of wages for type 3 equilibrium follow closely.

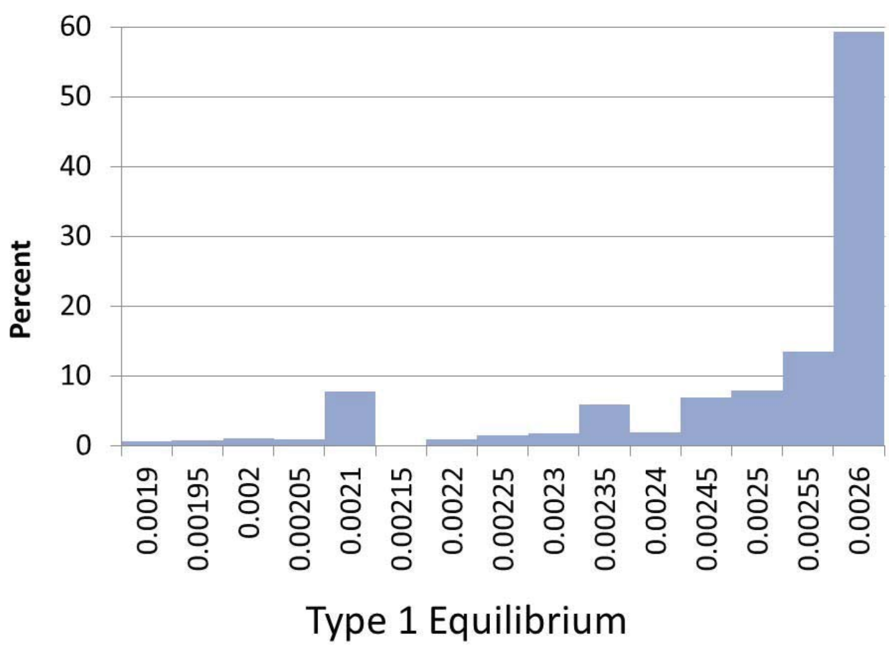


Figure 2.5. Long run distribution of wages (type 1 equilibrium)

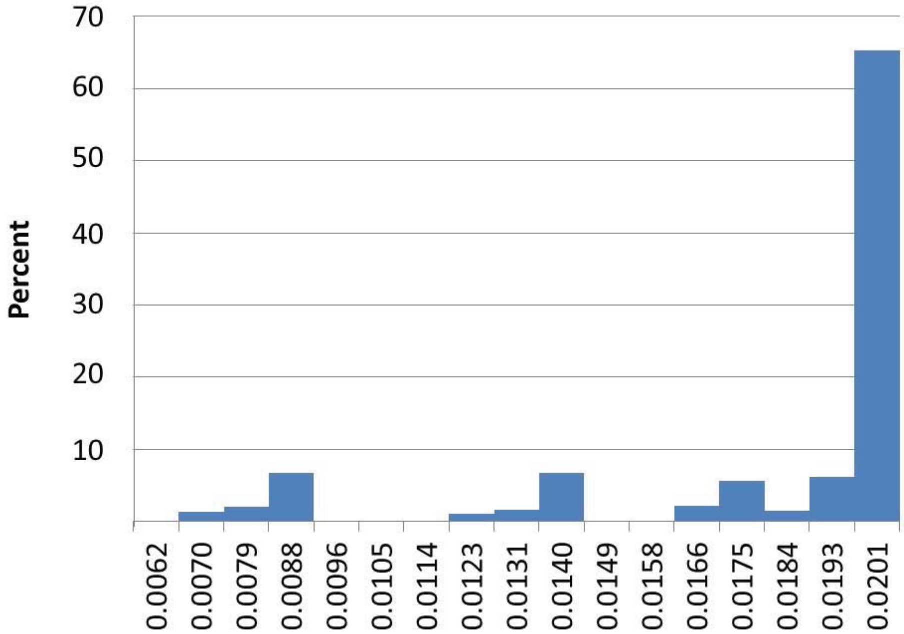


Figure 2.6. Long run distribution of wages (type 3 equilibrium)

The wage distribution is discrete and exhibits mutli-modal property. Suppose the economy start with w_t. Due to uncertain productivity in the first period, the period $t+1$ wage steps either up to $w_{t+1}(h)$ or down to $w_{t+1}(l)$. The wage change is not continuous. For example, with $w_t \approx 0.0026$ to begin, $w_{t+1}(l) \approx 0.0021$. There are some wage levels that cannot be realized and thus the wage distribution is discrete. Furthermore, recall that the probability of good state is 0.9. Thus, a sequence of good states is more likely to happen than a sequence of bad shocks. With a sequence of good states, the wage level converges to its right limit $\overline{w}$. That is, if $w_t = \overline{w}$, then $w_{t+1}(h) = \overline{w}$. On the other hand, with a sequence of bad shocks, the wage level goes to its left limit $\underline{w}$. When $w_t = \underline{w}$, then $w_{t+1}(l) = \underline{w}$. From figure 2.3, $\overline{w} \approx 0.0026$ for the type 1 example. And with $w_t \approx 0.0021$ to begin, $w_{t+1}(h) \approx 0.00235$ and $w_{t+2}(hh) \approx 0.0026$. That's why the wage distribution has peaks around wage levels 0.0021, 0.00235 and 0.0026.

The probability when wage levels are greater than 0.0024 is 89.44%, which is little less than the probability of the occurrence of good states, while the probability when wage levels are below 0.0021 is 11.33%, which is little greater than the probability of the bad states. This is because if the economy starts with the lowest wage level, $w_{t+1}(h)$ is less than 0.0021 even with good state in period t.

2.6 Welfare

This section studies the impact on welfare of the policy with a capital requirement of the form $\eta k_t^{e,t} \leq w_t L^{e,t}$. Following Bhattacharya and Singh (2008), the welfare for this stochastic over-lapping generations model is measured by an unconditional expected ex-ante welfare of a "representative" generation. The welfare expression is:

$$\begin{aligned} & E_{s_0} E_0 \lim_{t\to\infty} \left[E_t \left[c_{t+1}^{e,t} \right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t} \right] \right] \\ = \; & E_{s_0} \lim_{t\to\infty} \left[\sum_{s^t|s_0} \left(E_t \left[c_{t+1}^{e,t} \right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t} \right] \right) \pi\left(s^t|s_0\right) \right] \end{aligned} \tag{2.24}$$

where E_0 is the mathematical expectation calculated at time 0 and thus depends on the initial state s_0. The state variables in period 0 are capital levels held by the initial old generation. E_{s_0} is the expectation operator across all possible initial states. s^t is the history of states up and until period t and $\pi\left(s^t|s_0\right)$ is the probability of observing s^t conditional upon the realization of s_0. The infinitely lived government chooses optimal capital requirement η to maximize welfare (2.24). Other than the capital requirement policy, the government has no direct control of markets. In other words, there are still limited commitment of financial contracts and the capital prices are determined by competitive markets. Agents solve their individuals' problems under the policy as shown in section 2.3.

The government's problem is solved numerically in two steps. First, the expectation conditional on initial state s_0 is formed. Given η, $\left[E_t \left[c_{t+1}^{e,t} \right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t} \right] \right]$ is calculated for 1000 periods with a start wage level w_0. Then the average of the last 900 periods *ex-ante* welfare is used to approximate the period 0 expected welfare:

$$E_0 \lim_{t\to\infty} \left[E_t \left[c_{t+1}^{e,t}\right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t}\right]\right] \approx \frac{1}{900} \sum_{t=101}^{1000} E_t \left[c_{t+1}^{e,t}\right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t}\right] |s_0 \tag{2.25}$$

Next, using the discrete wage distribution shown in Figure 2.5 and 2.6 as the distribution of initial wages, the expectation of (2.25) as the unconditional expectation of welfare is computed:

$$\begin{aligned} & E_{s_0} E_0 \lim_{t\to\infty} \left[E_t \left[c_{t+1}^{e,t}\right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t}\right]\right] \\ \approx & \sum \left(\frac{1}{900} \sum_{t=101}^{1000} E_t \left[c_{t+1}^{e,t}\right] + E_t \left[c_t^{c,t} + c_{t+1}^{c,t}\right] |s_0 \right) \Pr(s = s_0) \end{aligned}$$

As a result, the capital requirement η which gives the highest welfare is the optimal policy choice. For the two experiments conducted in this paper, the starting wage level w_0 has no influence on the average of the last 900 periods *ex-ante* welfare from 1000 periods experiments. Therefore, (2.25) can be used as the approximation of the unconditional expectation of welfare. Below figure 2.7 shows the relationship between different capital requirement level and the percent change between welfare with capital control and without control for type 1 equilibrium.

As can be seen from figure 2.7, when capital requirement $\nu = 78.4\%$, the welfare with capital control is about 0.0002% greater than the welfare without any control. This welfare increase under the policy is quite small since without capital control, the maximum welfare is 19.35% more than the minimum welfare in the business cycles itself. Figure 2.8 shows that the optimal policy restricts borrowing most likely after period of productivity shock when wage income is low (high income implies lower borrowing ratio as showed in figure 2.2 and low wage periods mostly likely happen after productivity shock in previous period).

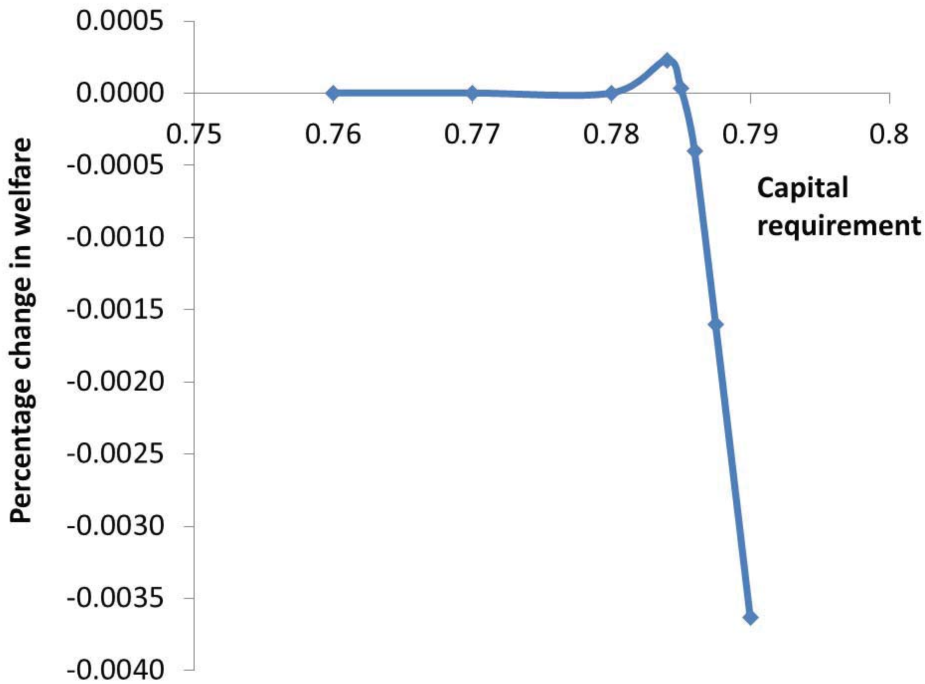


Figure 2.7. Capital requirement and percentage change in welfare (type 1 equilibrium)

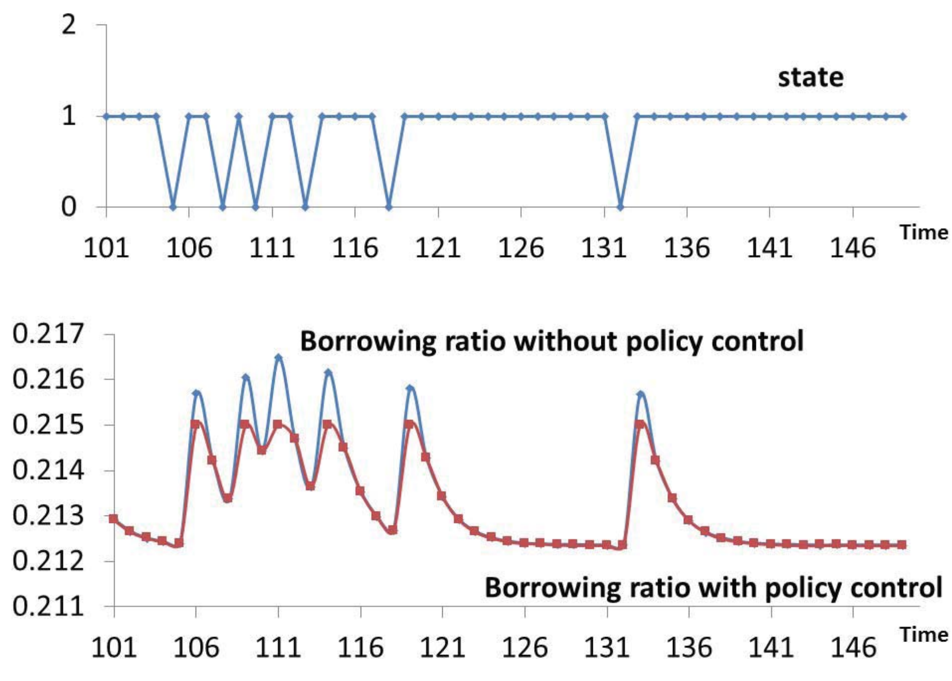


Figure 2.8. Borrowing ratio with and without capital control

Borrowing less can increase the *ex-ante* welfare for the current generation due to pecuniary externality. However, the impact on future generations depends on whether there is a bad shock in the current period. Consider the welfare improvement of current generation from the policy control first. Suppose there is a small decrease $dk_t^{e,t}$ in the first time investment $k_t^{e,t}$ for the entrepreneur. If all market prices (wage and capital prices) are fixed, the change in entrepreneur's utility is 0 due to first order conditions. In particular, the entrepreneur's wealth decreases by $z_0 dk_t^{e,t}$ due to less investment at the beginning of period t and increases by $z_{1h} dk_t^{e,t}$ due to less repayments to the consumer in good state at the end of period t ($z_0 = z_{1h}$ for type 1 equilibrium). The consumer's *ex-ante* welfare is unchanged as well, facing the same prices. However, in general equilibrium, less investments at the beginning can imply less fire sales of used capital. Suppose the capital price after the bad productivity shock increases by $dq_t(l)$. The consumer is hurt by earning less profits $\pi_l k_{t+1}^{c,t}(s) dq_t(l)$, while the entrepreneur's utility increases by $\pi_l z_{1l} k_{t+1}^{c,t}(s) dq_t(l)$. Since $z_{1l} > 1$, the total welfare of agents increases when $k_t^{e,t}$ decreases in a small amount. Figure 2.9 shows the welfare change of current generation with different capital requirement levels when $w_t = 0.0025$. The entrepreneur chooses borrowing ratio at $\rho_t^{CE} = 0.2128$ while the social optimal borrowing ratio $\rho_t^o = 0.2$ which is smaller than the one in competitive equilibrium without policy control. That is, there is ex-ante over-borrowing. And a capital requirement with $\eta = 80\%$ can force the economy to end up with the social optimal level for the current generation.

With binding capital requirement, future generations are hurt by starting with lower wage income when there is a bad productivity shock in period t and they are benefit by earning higher wages if there is no bad shock. With policy control, the fire sales amount decreases (capital price increases) while the *ex-post* capital used in the last period production decreases as well. As a result, the wage income for the next generation declines if there is a bad shock in the current period. This happens because of limited commitment which causes the repayments to the consumer in the first period decline in good state. And that's why a policy which restricts borrowing in all periods may decrease the welfare: only current generation benefits but all following generations are hurt.

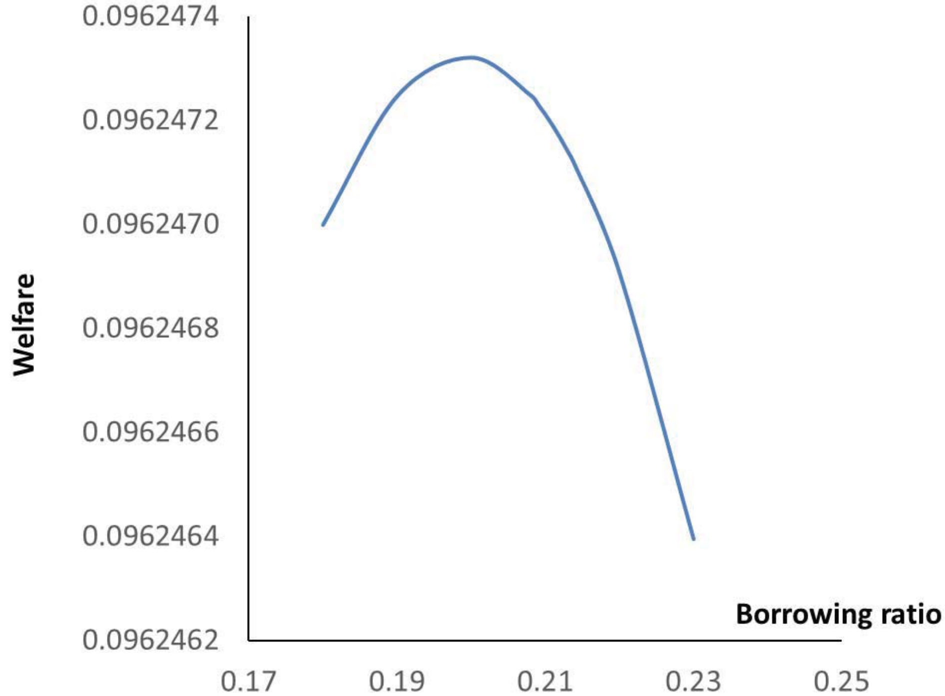


Figure 2.9. Current welfare and the borrowing ratio

The relationship between different capital requirement levels and the percent welfare change with and without capital controls for type 3 equilibrium is shown in figure 2.10. Since for type 3 equilibrium, decreasing borrowing cannot increase (actually decrease) welfare for the current generation, the policy can only benefit agents by increasing future wage income if there is no bad shock. It is no surprise that for the experiment, less borrowing decreases welfare. Because the capital requirement constraint does not always bind, the following proposition follows.

Proposition 2.2 *Under limited commitments in financial contracts and competitive capital market, a capital requirement* η *can always increase welfare.*

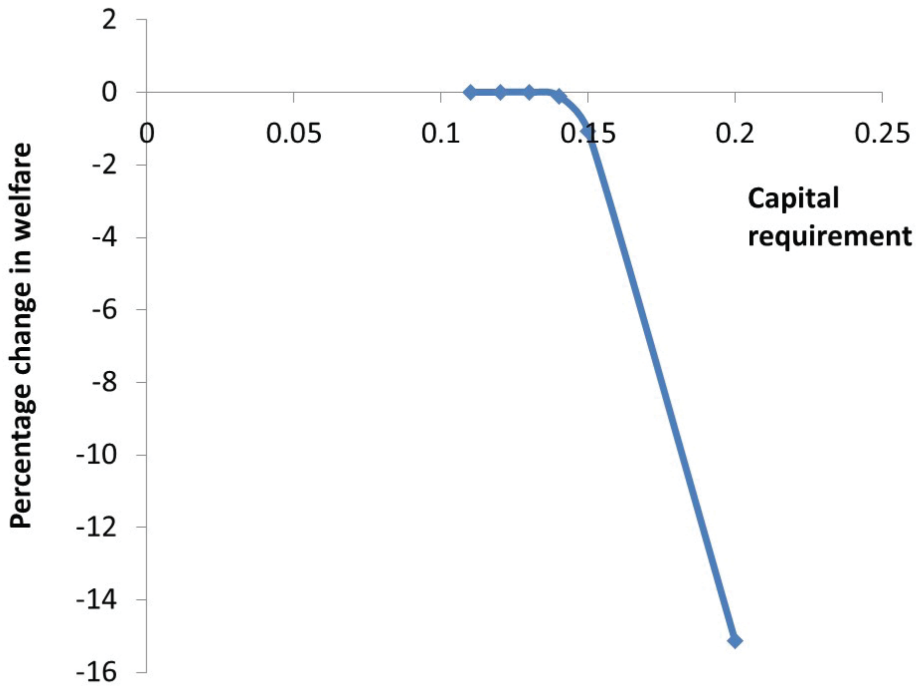


Figure 2.10. Capital requirement and percentage change in welfare (type 3 equilibrium)

2.7 Conclusion

This chapter develops a stochastic overlapping-generations model with financial channel of shock transmission. In the period with adverse productivity shock, the entrepreneur's investment cannot be fully financed with outside funds due to the limited commitment. Fire sales occur after the bad shock and the capital price drops. The wage in the next period decreases since the marginal productivity of labor decreases. The income change for the next generation transmits the impact of the adverse shock to the future. Due to the discrete distribution of the productivity factor of the first time technology $a_t(s)$, the long run wage distribution is also discrete.

As a theoretical exercise, chapter 2 also shows that *ex-ante* over borrowing could occur and a policy to address over borrowing can increase total welfare. By restricting borrowing, the inefficiency caused by the pecuniary externality can be diminished by reducing *ex-post* fire sales. The current generation benefits and future generations profits if there is no bad shock in current period. However, as shown by the examples in the paper, the increase in welfare under the optimal policy control is quite small. On the other hand, if the return to investment is sufficiently high, the individuals' optimal choices would be the socially optimal as well, which limits the possibility of over borrowing *ex-ante*. Whether there is over borrowing in the economy depends on the equilibrium types. A calibration exercise to match the model to data and come up with realistic policy prescriptions is left for future research.

CHAPTER 3. FIRM ENTRY AND EXIT IN A GLOBAL GAME WITH STRATEGIC SUBSTITUES

3.1 Introduction

This chapter extends Melitz's (2003) model of firm entry and exit decision as a global game with incomplete information. In the model presented, firms pay an irreversible entry cost to enter the market before they know their own productivity or the productivity of their competitors. In the second stage, firms realize their own productivity that also acts as a signal for forecasting the mean productivity of its competitors. At this stage, it can choose to exit or stay in the market. If it chooses to stay, it incurs a fixed production cost before it produces and earns market revenues. Firms' choices are strategic substitutes because an incumbent's payoff is decreasing in the mass of competing firms.

Firm heterogeneity plays a critical role in the model since a firm makes stay/exit decision in the second stage based on its realized productivity level.[3] This chapter follows Melitz (2003) to model firm heterogeneity. In Melitz's model, firms with different productivity monopolistically compete with each other (Krugman (1979)), and the representative consumer has a constant elasticity of substitution between any two products (Dixit and Stiglitz (1977)). Despite firm heterogeneity, the supply side of the market is conveniently aggregated and then all aggregates are summarized through a representative firm with an aggregate productivity

[3]If firms have same level of productivities in the second stage, they will all stay or exit the market in a symmetric equilibrium. If all entry firms leave the market, the economy ends up with no producers. If all firms stay in the market, the profit for each firm equals to the irreversible entry costs due to free entry. And the mass of entry/existing firms is the one such that clears the labor market.

level. Here, the aggregate productivity only depends on the distribution of producing firms but not on the number of firms. Thus, Melitz's method helps to make the model tractable enough while still allowing for firm heterogeneity.

In general, global games with incomplete information generate more than one type of equilibrium. In a stylized game of a binary choice between going to a bar versus not going, Karp, Lee, and Mason (2007) characterize a monotonic pure strategy equilibrium, a non-monotonic pure strategy equilibrium, and a mixed strategy equilibrium in their study. In a monotonic pure strategy equilibrium, an individual who receives a signal greater than the switching point will "go" and those with a signal less than the switching point will "don't go". Karp, Lee, and Mason (2007) show that monotonic pure strategy equilibria exist if an individual's signal about the underlying state is imprecise (large variance) and there is a small amount of congestion (small number of people will go to the bar). They also discuss properties of non-monotonic pure strategy equilibria under (unproven) assumption that the equilibria exist. Moreover, Karp, Lee, and Mason (2007) show that the only type of mixed strategy equilibria is the non-monotonic mixed strategy equilibria. The uniqueness of equilibrium has not been addressed in their paper. Since the existences of non-monotonic pure strategy equilibria and mixed strategy equilibria are hard to show, and with firm heterogeneity, a natural conjecture is that firm with higher level of productivity will earn higher profits and then stay in the market, chapter 3 focuses only on monotonic pure strategy equilibria.

A large body of literature focuses on global games under strategic complementarity. When decisions are strategic complements, there can be multiple equilibria or a unique equilibrium.[4] Morris and Shin (1998) show the existence of a unique equilibrium when a small noise is added to the fundamental (state) random variable on the economy. Although the decisions of players in Morris and Shin's (1998) model are strategic complements, the present chapter follows their steps for identifying the existence of a unique equilibrium.

[4] For multiple equilibria, see Diamond and Dybvig (1983) on bank runs and Krugman (1991) on external economics; and for unique equilibrium, see Vives (1990), Morris and Shin (2001) and Milgrom and Roberts (1990).

A number of researchers have also studied global games with strategic substitutes. When studies multimarket oligopoly, Bulow (1985) states that when decisions are strategic substitutes, the relative payoff is always decreasing, and when decisions are strategic complements, the relative payoff is always increasing. In a game with incomplete information, Bulow (1985)'s statement is still true if an individual only receives a signal about a fundamental state variable that reveals no idiosyncratic information. (See Karp, Lee, and Mason (2007), and Morris and Shin (1998)) However, in this chapter, an individual's signal serves two purposes. First, it reveals the true productivity of the firm receiving this signal, and second it serves as a signal of the mean productivity of all other firms. As a result, despite decisions being strategic substitutes, the global game in this paper may still have a monotonic pure strategy equilibrium. With some parametric restrictions, the game in chapter 3 indeed has a unique pure strategy equilibrium. Unlike the present chapter, the decreasing relative payoff property for strategic substitutes is the reason for the absence of monotonic pure strategy equilibrium in Karp, Lee and Mason (2007). Athey (2001) discusses monotonic pure strategy equilibrium in a game with incomplete information that relies on a single cross condition (SCC): once the SCC is satisfied, a pure strategy Nash equilibrium exists in every finite-action game. This SCC is stated by Milgrom and Shannon (1994) as "when choosing between a low action and a high action, if a low type of player weakly prefers the higher action, then all higher types of agent weakly prefer the higher action as well." The present chapter utilizes the SCC property for the equilibrium profile of strategies to prove the existence of a unique equilibrium. Here, the SCC in monotonic pure strategy equilibrium can be specifically stated as: when productivity level is less than a threshold productivity, firms will exit the market. On the other hand, if the threshold productivity firm finds it profitable to stay all higher productivity firms will also stay.

In Melitz's (2003) model on firm entry and exit with complete information, firms know the distribution of productivity of all market entrants before entering the market. Once the firms have entered the industry and realized their productivities from a commonly known distribution, they choose either to stay and produce, or exit the industry. In the present chapter, firms know the distribution of productivities conditional on the mean productivity. However, the mean itself is assumed to be random, and only the distribution of the mean productivity is common knowledge. The distribution of the mean productivity is called public information and the reciprocal of its variance is the precision of public information. The productivity distribution conditional on the mean is called private information and the reciprocal of its variance is the precision of private information.

Chapter 3 analytically proves the existence of a monotonic pure strategy equilibrium, including the uniqueness of the equilibrium. The uniqueness of the equilibrium allows further numerical exploration of comparative statics. The key finding of the chapter is that increasing the precision of public information can improve aggregate productivity and by reallocating resources to more productive firms the welfare also increases. It indicates that the productivity growth can be generated by more precise public information without any changes in the private information. This finding is reinforced in chapter 4 that studies equilibria under more complicated public information (distribution of the mean productivity) and more complicated private information (productivity distribution conditional on the mean).

The structure of chapter 3 is the following. Section 3.2 introduces the model. Section 3.3 first defines the equilibrium and then proves the existence and the uniqueness of the equilibrium. Section 3.4 provides examples of the equilibrium and section 3.5 presents some numerical comparative statics. Section 3.6 concludes.

3.2 The Model

This section first introduces the demand and the supply side of the economy. A set of relevant aggregate relationships are then listed, and finally the timeline of the game with incomplete information is presented.

3.2.1 Consumers' Demand

A representative consumer has a C.E.S. utility function over a continuum of goods indexed by ω:

$$U = \left(\int_{\omega\in\Omega} q\left(\omega\right)^{\rho} d\omega\right)^{1/\rho},$$

where $q\left(\omega\right)$ is the consumption for good ω and Ω is the set of available goods.[5] Here it is assumed that $0 < \rho < 1$ and then the elasticity of substitution between any two goods is $\sigma = \frac{1}{1-\rho} > 1$. The budget constraint for the consumer is

$$\int_{\omega\in\Omega} p\left(\omega\right) q\left(\omega\right) d\omega = R,$$

where R denotes aggregate expenditure and $p\left(\omega\right)$ is the price of good ω. Then the optimal demand for good ω and the expenditure on good ω are:

$$q\left(\omega\right) = Q\left(\frac{p\left(\omega\right)}{P}\right)^{-\sigma}, \tag{3.1}$$

$$r\left(\omega\right) = R\left(\frac{p\left(\omega\right)}{P}\right)^{1-\sigma}, \tag{3.2}$$

where

$$P = \left(\int_{\omega\in\Omega} p\left(\omega\right)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}} \tag{3.3}$$

is the price index and $Q = \frac{R}{P}$ is the aggregate demand. A detailed analysis of the consumer's problem can be found in Appendix B.1.

[5] In fact, U is the aggregate utility of all consumers. That is, suppose each individual has the same utility function and L mass of consumer's aggregate utility function is listed as the utility function of a representative consumer. The representative consumer is used here to follow the definition from Melitz (2003).

3.2.2 Firms' Supply Problem

Firms produce differentiated goods (indexed by ω) by using labor input only. The production technology for firm i is represented by the following cost function

$$l_p(x_i) = f + \frac{q(\omega)}{x_i} \tag{3.4}$$

where x_i is the productivity level for firm i. The fixed cost f is same for all firms and the variable costs $\frac{q(\omega)}{x_i}$ is decreasing in the productivity level. Each firm is a monopoly over its market and the consumer demand curve for good ω has a constant elasticity of σ. Thus, all firms share the same price markup that equals $\sigma/(\sigma-1) = 1/\rho$. That is, the price set by firm i is

$$p(x_i) = \frac{w}{\rho x_i} \tag{3.5}$$

The wage rate w is common for all firms. Expressed in terms of labor, $w = 1$. With price in equation (3.5), firm i's revenue $r(x_i)$, profit $\pi(x_i)$, and the labor used in production $l_p(x_i)$ are, respectively

$$r(x_i) = R(P\rho x_i)^{\sigma-1} \tag{3.6}$$

$$\pi(x_i) = \frac{r(x_i)}{\sigma} - f = \frac{R}{\sigma}(P\rho x_i)^{\sigma-1} - f \tag{3.7}$$

$$l_p(x_i) = \frac{\sigma-1}{\sigma} R(P\rho x_i)^{\sigma-1} + f \tag{3.8}$$

Note that $\frac{r(x_i)}{\sigma}$ is variable profit. A detailed solution of the producer's problem is available in Appendix B.2.

3.2.3 Aggregate Revenue and Profit

Suppose a mass M of firms (and hence M goods) exist in the market with their distribution of productivity levels over a subset of $(0, \infty)$ given by $\mu(x)$. Define the weighted average productivity $\widetilde{x}$ as

$$\widetilde{x} = \left(\int_0^\infty x^{\sigma-1} \mu(x) dx \right)^{\frac{1}{\sigma-1}} \tag{3.9}$$

Then, as in Melitz (2003),[6] the aggregate price P, aggregate quantity Q, aggregate revenue R, and aggregate profit Π can be summarized by this weighted average productivity $\widetilde{x}$:

$$P = M^{\frac{1}{1-\sigma}} p\left(\widetilde{x}\right) \qquad R = PQ = Mr\left(\widetilde{x}\right)$$
$$Q = M^{1/\rho} q\left(\widetilde{x}\right) \qquad \Pi = M\pi\left(\widetilde{x}\right)$$

Thus, the aggregate quantities can be related to those of a representative firm with productivity $\tilde{x}$. This is a result of a constant elasticity of substitution and monopolistic competition, as shown by Melitz (2003). Since aggregate values can be summarized by $\widetilde{x}$ completely, $\widetilde{x}$ can be viewed not only as the weighted average productivity but also the aggregate productivity.

Two properties of the model listed below are used for further analysis. First, the average revenue (profit) of all firms is also the revenue (profit) of firm with productivity levels equal to $\widetilde{x}$.

$$\overline{r} = \frac{R}{M} = r\left(\widetilde{x}\right) \tag{3.10}$$
$$\overline{\pi} = \frac{\Pi}{M} = \pi\left(\widetilde{x}\right) \tag{3.11}$$

Second, the ratio of any two firms' outputs and revenues only depend on the ratio of their productivity levels:

$$\frac{q\left(x_1\right)}{q\left(x_2\right)} = \left(\frac{p\left(x_2\right)}{p\left(x_1\right)}\right)^{\sigma} = \left(\frac{x_1}{x_2}\right)^{\sigma} \tag{3.12}$$
$$\frac{r\left(x_1\right)}{r\left(x_2\right)} = \left(\frac{x_1}{x_2}\right)^{\sigma-1} \tag{3.13}$$

[6] The derivation is available in Appendix B.3.

That is, a more productive firm (higher x_i) will be bigger (larger output and revenues), charge a lower price, and earn higher profits than a less productive firm.

3.2.4 Firm Entry and Exit under Incomplete Information

In Melitz's (2003) model on firm entry and exit with complete information, firms know the distribution of productivity of all market entrants before entering the market. Once they enter and after its own productivity is realized from this known distribution, they choose either to stay and produce, or exit the industry. In chapter 3, firms know the distribution of productivities *conditional* on its mean. The mean itself is assumed to random, and only its distribution is common knowledge.

The entry-exit game occurs in three stages. In the first stage, firms know the distribution of mean productivity Θ[7] and then decide whether to enter the market or not. The distribution of Θ is uniform $U\left[\underline{\theta},\overline{\theta}\right]$. If firms enter the market, each of them needs to pay entry costs f_e in the units of labor. The total mass of entry firms in the first stage is M_e.

In the second stage, the mean productivity Θ is realized as θ, and then firm i gets to know its own productivity level x_i. While θ itself is unknown, it is known that X_i is uniformly distributed as $U\left[\theta-\varepsilon,\theta+\varepsilon\right]$ given $\Theta=\theta$. That is, $X_i|\left(\Theta=\theta\right)=\theta+\Psi_i$ where $\Psi_i\sim U\left[-\varepsilon,\varepsilon\right]$. A firm's productivity level X_i is independently distributed across all firms. Note that x_i is not only the productivity level for firm i, but also its private signal of the mean productivity parameter Θ. With x_i, firm i can form the posterior distribution $J_{\Theta|X_i=x_i}(\theta)$ of Θ conditional on x_i. In Appendix B.4, it is shown that when $\underline{\theta}-\varepsilon<x_i<\overline{\theta}+\varepsilon$,

$$J_{\Theta|X_i=x_i}(\theta)\sim U\left[\underline{\theta}\left(x_i\right),\overline{\theta}\left(x_i\right)\right] \tag{3.14}$$

[7] In what follows, capital letters represent a random variable, its lower case represents one possible realization.

where

$$\underline{\theta}(x_i) = \max\{x_i - \varepsilon, \underline{\theta}\}$$
$$\overline{\theta}(x_i) = \min\{x_i + \varepsilon, \overline{\theta}\}$$

and when $x_i = \underline{\theta} - \varepsilon$, $\Theta = \underline{\theta}$ with probability of 1 while when $x_i = \overline{\theta} + \varepsilon$, $\Theta = \overline{\theta}$ with probability of 1.

The precision of public information Θ is defined as the reciprocal of the variance of Θ in the present paper. That is, the precision of Θ when Θ is drawn from a uniform distribution $U\left[\underline{\theta}, \overline{\theta}\right]$ is $1/Var(\Theta) = 12/\left(\overline{\theta} - \underline{\theta}\right)^2$. The precision of private information is defined as the reciprocal of the variance of $X_i|\left(\Theta = \theta\right)$ and equals to $3/\varepsilon^2$. The relative precision of public information and private information is defined as the ratio between the precision of public information and the precision of private information. If the relative precision of information is greater than 1 (when $2\varepsilon > \overline{\theta} - \underline{\theta}$), the public information is more precise than the private information, and vice versa.

With the productivity level x_i, posterior distribution of Θ conditional on x_i, and a belief of the equilibrium profile of strategies $\Pr(X)$ of all other firms, firm i can calculate its expected payoff of staying in the market, $u(x_i, \Pr(X))$, in the second stage.

$$u(x_i, \Pr(X)) = \int_{-\infty}^{\infty} \pi\left(x_i, \Pr(X), \theta\right) dJ_{\Theta|X_i=x_i}(\theta), \tag{3.15}$$

where $\pi\left(x_i, \Pr(X), \theta\right)$ is the profit, net of fixed cost f, given productivity level x_i, a belief on profile of strategies $\Pr(X)$, and true mean productivity level θ. And the profile of strategies $\Pr(X)$ is defined as

$$\Pr(X) = \left\{pr\left(x_i\right)|x_i \geq 0\right\}, \tag{3.16}$$

where $pr\left(x_i\right)$ is the proportion of firms who stay in the market when their productivity level is x_i. When $u(x_i, \Pr(X)) \geq 0$, firm stays in the market, pays fixed costs f, and enters the last (third) stage to produce output. Otherwise, if $u(x_i, \Pr(X)) < 0$, firm i *exits* the market. Note that when $x_i = \underline{\theta} - \varepsilon$, the expected payoff of staying in the market is degenerate:

$$u(x_i, \Pr(X))|_{x_i = \underline{\theta} - \varepsilon} = \pi\left(\underline{\theta} - \varepsilon, \Pr(X), \underline{\theta}\right) \tag{3.17}$$

and when $x_i = \overline{\theta} + \varepsilon$, the expected payoff of staying is then

$$u(x_i, \Pr(X))|_{x_i = \overline{\theta} + \varepsilon} = \pi\left(\overline{\theta} + \varepsilon, \Pr(X), \overline{\theta}\right). \tag{3.18}$$

After some firms have exited in second stage, the remaining mass of firms M is expressed as

$$M = M_e P_{stay}\left(\Pr(X), \theta\right), \tag{3.19}$$

where

$$P_{stay}\left(\Pr(X), \theta\right) = \int_{-\infty}^{\infty} pr\left(x_i\right) f_{X_i|\Theta=\theta}\left(x_i\right) dx_i \tag{3.20}$$

is the fraction of firms staying in the market given the profile of strategies $\Pr(X)$ and the true mean productivity level θ. Note that given $\Pr(X)$, $P_{stay}\left(\Pr(X), \theta\right)$ is also the *ex ante* first-stage probability of a firm's succeeding in the second stage.

In the last stage, since variable profits $\frac{r(x_i)}{\sigma}$ is always positive, all existing firms will produce and sell in the market. Firm i pays variable costs $\frac{q(\omega)}{x_i}$ and sets price at $p\left(x_i\right)$.

From firm's profit equation (3.7) and the posterior distribution $J_{\Theta|X_i=x_i}(\theta)$ (3.14), the expected payoff expression (3.15) can be simplified as

$$
\begin{aligned}
u(x_i, \Pr(X)) &= \int_{-\infty}^{\infty} \left(\frac{r\left(x_i, \Pr(X), \theta\right)}{\sigma} - f \right) dJ_{\Theta | X_i = x_i}(\theta) \\
&= \int_{\underline{\theta}(x_i)}^{\overline{\theta}(x_i)} \left(\frac{r\left(x_i, \Pr(X), \theta\right)}{\sigma} - f \right) dJ_{\Theta | X_i = x_i}(\theta)) \\
&= \frac{1}{\sigma} \frac{1}{\overline{\theta}\left(x_i\right) - \underline{\theta}\left(x_i\right)} \left(\int_{\underline{\theta}(x_i)}^{\overline{\theta}(x_i)} r\left(x_i, \Pr(X), \theta\right) d\theta \right) - f \qquad (3.21)
\end{aligned}
$$

when $\underline{\theta} - \varepsilon < x_i < \overline{\theta} + \varepsilon$.

3.3 Equilibrium

This section presents the equilibrium of the game with incomplete information. Equilibrium is first defined and then the idea of a monotonic pure strategy equilibrium is introduced. This chapter only focuses on the monotonic pure strategy equilibrium. Some critical assumptions are first required for proving the existence and the uniqueness of equilibrium. The equilibrium is then presented and discussed.

3.3.1 Equilibrium Definition

Although firms with the same productivity level x_i produce differentiated products, they share the same level of revenues and profits. Then, in a symmetric equilibrium, the proportion of firms who stay in the market when their productivity level is x_i, is either equal to 1 or 0. That is, all firms with the same productivity level will either stay in the market or leave the market.

A symmetric equilibrium of the game consists of a profile of strategies $\Pr(X)$, mass of entering firms $M_e\left(\Pr(X)\right)$ in the first stage, mass of continuing firms $M\left(\Pr(X), \theta\right)$ in the second stage; aggregate price $P\left(\Pr(X), \theta\right)$, aggregate quantity $Q\left(\Pr(X), \theta\right)$, aggregate revenue $R\left(\Pr(X), \theta\right)$, and the probability density function $\mu(x_i, \Pr(X), \theta)$ of productivity x_i of existing firms for any realization of Θ, where (1) $pr\left(x_i\right) = 1$ whenever $u(x_i, \Pr(X)) \geq 0$

and $pr\left(x_i\right) = 0$ whenever $u(x_i, \Pr(X)) < 0$, and $\Pr(X)$ is defined in (3.16); (2) $M_e\left(\Pr(X)\right)$ and $M\left(\Pr(X), \theta\right)$ satisfies (3.19) for every θ; (3) $P\left(\Pr(X), \theta\right)$, $Q\left(\Pr(X), \theta\right)$, and $R\left(\Pr(X), \theta\right)$ solve consumers' and producers' problem for each x_i and θ; (4) labor market clears, and (5) the expected payoff of the firms entering the market in the first stage is zero.

The equilibrium is derived backwards. This part first presents the labor market clearing condition and the zero expected profit condition, enumerated as (4) and (5) in the above definition.

Labor Market Equilibrium In the first stage, a mass $M_e\left(\Pr(X)\right)$ of firms enters the market and each of the firms pays entry costs f_e in the units of labor. In the second stage, a mass of $M\left(\Pr(X), \theta\right)$ firms stay in the market and each of them pays fixed producing costs f. In the last stage, existing firm also pays variable producing costs $\frac{q(\omega)}{x_i}$. The sum of fixed producing costs and variable producing costs together must equal the difference between firms' aggregate revenue and profit. From the aggregate equations for revenue, (3.10), and profit, (3.11), and the labor used for production (3.8), the labor market equilibrium after θ is realized is[8]

$$M_e f_e + M_e P_{stay}\left(\Pr(X), \theta\right)\left((\sigma - 1)\,\overline{\pi}\left(\Pr(X), \theta\right) + \sigma f\right) = L \tag{3.22}$$

where $\overline{\pi}\left(\Pr(X), \theta\right)$ is the average profit given equilibrium profile of strategy $\Pr(X)$ and a particular θ.

Free Entry Condition Before entry, a firm's expected profit, based on the prior distribution of mean productivity Θ, equals the entry costs f_e.

$$E_\Theta\left(P_{stay}\left(\Pr(X), \theta\right)\overline{\pi}\left(\Pr(X), \theta\right)\right) - f_e = 0 \tag{3.23}$$

[8] The derivation is in Appendix B.5.1.

As the mean productivity parameter Θ is drawn from the uniform distribution $U\left[\underline{\theta}, \overline{\theta}\right]$, the free entry condition can be simplified as

$$\int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{\overline{\theta}-\underline{\theta}} \left(P_{stay}\left(\Pr(X), \theta\right) \overline{\pi}\left(\Pr(X), \theta\right)\right) d\theta - f_e = 0 \tag{3.24}$$

Some authors, for example Karp et al. (2007), assume that the underlying fundamental parameter (Θ in our model) is drawn from an improper uniform distribution. This simplifies the expression of posterior distribution and facilitates an easy calculation of expected payoff. However, in the present model, not only the expectation based on the posterior distribution needs to be calculated, the expectation based on the prior distribution (3.23) also needs to be derived. As a result, the expectation based on an improper distribution is not well defined. That is why the mean productivity parameter Θ is assumed to be drawn from a uniform distribution $U\left[\underline{\theta}, \overline{\theta}\right]$ instead of an improper uniform distribution.

Once the decision to stay has been chosen in the second stage, firms with higher productivity will reap higher profits. A natural conjecture is that firms will then follow a threshold strategy. A profile of a threshold strategies is defined as

$$\Pr(X) = I_k(x_i) = \begin{cases} 1 & \text{if } x_i \geq k \\ 0 & \text{if } x_i < k \end{cases} \tag{3.25}$$

That is, a firm with productivity level x_i greater than or equal to the switching point k will stay in the market and a firm with productivity level x_i smaller than the switching point k will leave the market. If firms follow the strategy described in (3.25), the equilibrium is termed as a *monotonic pure strategy* equilibrium. This chapter focuses only on the monotonic pure strategy equilibrium, and the proof of existence and uniqueness of this equilibrium is introduced hereafter.

3.3.2 A Monotonic Pure Strategy Equilibrium

Suppose the switching point is x^*, and then firm i believes other firms follow this equilibrium threshold strategy

$$I_{x^*} = \begin{cases} 1 & \text{if } x_i \geq x^* \\ 0 & \text{if } x_i < x^* \end{cases}$$

Firm i's expected payoff of staying in the market in the second stage is

$$u(x_i, I_{x^*}) = \frac{1}{\sigma} \frac{1}{\overline{\theta}(x_i) - \underline{\theta}(x_i)} \left(\int_{\underline{\theta}(x_i)}^{\overline{\theta}(x_i)} r(x_i, I_{x^*}, \theta)\, d\theta \right) - f \tag{3.26}$$

when $\underline{\theta} - \varepsilon < x_i < \overline{\theta} + \varepsilon$, where by (3.13), the revenue for firm i given equilibrium strategy I_{x^*} and mean productivity θ is

$$r(x_i, I_{x^*}, \theta) = \left(\frac{x_i}{\widetilde{x}(I_{x^*}, \theta)} \right)^{\sigma-1} r(\widetilde{x}(I_{x^*}, \theta), I_{x^*}, \theta) \tag{3.27}$$

The average productivity $\widetilde{x}(I_{x^*}, \theta)$ in above equation (3.27) can be obtained by

$$\widetilde{x}(I_{x^*}, \theta) = \left(\int_{\theta-\varepsilon}^{\theta+\varepsilon} (x_i)^{\sigma-1} \mu(x_i, I_{x^*}, \theta) dx_i \right)^{\frac{1}{\sigma-1}} \tag{3.28}$$

where the density function of the productivity distribution is

$$\mu(x_i, I_{x^*}, \theta) = \begin{cases} \frac{1}{\theta+\varepsilon-x^*} & \text{if } \theta - \varepsilon \leq x^* \leq x_i \leq \theta + \varepsilon \\ 0 & \text{if } \theta - \varepsilon \leq x_i < x^* < \theta + \varepsilon \\ \frac{1}{2\varepsilon} & \text{if } x^* < \theta - \varepsilon \\ \text{not defined} & \text{if } x^* \geq \theta + \varepsilon \end{cases} \tag{3.29}$$

Note that given θ, $\theta - \varepsilon \leq x_i \leq \theta + \varepsilon$ for sure.

From (3.10) and (3.7), the revenue $r\left(\widetilde{x}\left(I_{x^*},\theta\right),I_{x^*},\theta\right)$ for firm with productivity level $\widetilde{x}\left(I_{x^*},\theta\right)$ in (3.27) is

$$r\left(\widetilde{x}\left(I_{x^*},\theta\right),I_{x^*},\theta\right)=\overline{r}\left(I_{x^*},\theta\right)=\sigma\left(\overline{\pi}\left(I_{x^*},\theta\right)+f\right)$$

where the average profits $\overline{\pi}\left(I_{x^*},\theta\right)$ given equilibrium strategy I_{x^*} and mean productivity θ can be obtained by the labor market clearing condition and the free entry condition. With threshold strategy I_{x^*}, the labor market equilibrium condition is simplified as

$$L=M_ef_e+M_eP_{stay}\left(I_{x^*},\theta\right)\left(\left(\sigma-1\right)\overline{\pi}\left(I_{x^*},\theta\right)+\sigma f\right) \tag{3.30}$$

and the free entry condition is

$$\int_{\underline{\theta}}^{\overline{\theta}}\frac{1}{\overline{\theta}-\underline{\theta}}\left(P_{stay}\left(I_{x^*},\theta\right)\overline{\pi}\left(I_{x^*},\theta\right)\right)d\theta-f_e=0 \tag{3.31}$$

From (3.30) and (3.31), the average profit can be obtained as:[9]

$$\overline{\pi}\left(I_{x^*},\theta\right)=\frac{\sigma f}{\sigma-1}\left(\frac{P^e_{stay}\left(I_{x^*}\right)}{P_{stay}\left(I_{x^*},\theta\right)}-1\right)+\frac{f_e}{P_{stay}\left(I_{x^*},\theta\right)} \tag{3.32}$$

where $P_{stay}\left(I_{x^*},\theta\right)$ is the probability of staying given θ and the strategy profile I_{x^*}:

$$P_{stay}\left(I_{x^*},\theta\right)=\begin{cases}0 & \text{if } \theta\leq x^*-\varepsilon\\ =P^1_{stay}\left(I_{x^*},\theta\right)=\frac{\theta}{2\varepsilon}+\frac{\varepsilon-x^*}{2\varepsilon} & \text{if } x^*-\varepsilon<\theta\leq x^*+\varepsilon\\ =P^2_{stay}\left(I_{x^*},\theta\right)=1 & \text{if } \theta>x^*+\varepsilon\end{cases} \tag{3.33}$$

[9]The algebra of getting the average profit from labor market clear condition and free entry condition can be found in Appendix B.5.2.

and $P^e_{stay}\left(I_{x^*}\right)$ is the ex-ante probability of staying before entry in the first stage:[10]

$$P^e_{stay}\left(I_{x^*}\right)=\begin{cases} 0 & \text{if } \overline{\theta}\leq x^*-\varepsilon \\ \frac{1}{\overline{\theta}-\underline{\theta}}\frac{\left(\overline{\theta}+\varepsilon-x^*\right)^2}{4\varepsilon} & \text{if } x^*-\varepsilon<\overline{\theta}\leq x^*+\varepsilon \text{ and } \underline{\theta}\leq x^*-\varepsilon \\ =P^{e1}_{stay}\left(I_{x^*}\right)=\frac{\overline{\theta}+\underline{\theta}+2\varepsilon-2x^*}{4\varepsilon} & \text{if } x^*-\varepsilon<\overline{\theta}\leq x^*+\varepsilon \text{ and } x^*-\varepsilon<\underline{\theta}\leq x^*+\varepsilon \\ \frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}-\ x^*\right) & \text{if } \overline{\theta}>x^*+\varepsilon \text{ and } \underline{\theta}\leq x^*-\varepsilon \\ =P^{e2}_{stay}\left(I_{x^*}\right)=\frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}-\ x^*-\frac{\left(\underline{\theta}+\varepsilon-x^*\right)^2}{4\varepsilon}\right) & \text{if } \overline{\theta}>x^*+\varepsilon \text{ and } x^*-\varepsilon<\underline{\theta}\leq x^*+\varepsilon \\ =P^{e3}_{stay}\left(I_{x^*}\right)=1 & \text{if } \overline{\theta}>x^*+\varepsilon \text{ and } \underline{\theta}>x^*+\varepsilon \end{cases} \tag{3.34}$$

In sum, the expected payoff $u(x_i, I_{x^*})$ (3.26) can be rewritten as

$$u(x_i,I_{x^*})=\frac{1}{\overline{\theta}\left(x_i\right)-\underline{\theta}\left(x_i\right)}\left(\int_{\underline{\theta}(x_i)}^{\overline{\theta}(x_i)}\left(\frac{x_i}{\widetilde{x}\left(I_{x^*},\theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^*},\theta\right)+f\right)d\theta\right)-f \tag{3.35}$$

when $\underline{\theta}-\varepsilon<x_i<\overline{\theta}+\varepsilon$, where the average profit is defined in (3.32). Depending on $\underline{\theta}$, $\overline{\theta}$, ε, x^* and θ, the expressions for the probability of staying $P_{stay}\left(I_{x^*},\theta\right)$ and the ex-ante probability of staying $P^e_{stay}\left(I_{x^*}\right)$ in (3.32) follow different parts in equation (3.33) and equation (3.34). The productivity ratio can be expressed as:[11]

$$\left(\frac{x_i}{\widetilde{x}\left(I_{x^*},\theta\right)}\right)^{\sigma-1}=\begin{cases} Ra^1\left(x_i,I_{x^*},\theta\right)=\left(x_i\right)^{\sigma-1}\frac{\sigma\left(\theta+\varepsilon-x^*\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}} & \text{if } \theta-\varepsilon\leq x^*<\theta+\varepsilon \\ Ra^2\left(x_i,I_{x^*},\theta\right)=\left(x_i\right)^{\sigma-1}\frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^{\sigma}-\left(\theta-\varepsilon\right)^{\sigma}} & \text{if } x^*<\theta-\varepsilon \end{cases} \tag{3.36}$$

Equation (3.35) is key to deriving the equilibrium.

3.3.3 Assumptions

Given a mean productivity level, a firm's revenue increases with it's own productivity level. However, firm's productivity also serves as a signal of the mean productivity of all other firms. That is, a higher productivity for firm i means a higher expected mean productivity

[10] The calculation can be found in Appendix B.5.3.
[11] See Appendix B.5.4.

of others and thus relatively low productivity for firm i. This can cause a decrease in revenue. Thus, a few assumptions are needed here to ensure the existence of a monotonic pure strategy equilibrium.

Assumption 3.1. $\underline{\theta} = \varepsilon$.

Assumption 3.1 indicates that when the economy is in the worst situation ($\theta = \underline{\theta}$), the minimum possible productivity level of firms is 0 ($x_i = \underline{\theta} - \varepsilon = 0$).

Assumption 3.2. $\sigma > 3$.

Assumption 3.2 ensures that when x_i increases, the expected payoff of staying for firm i with the equilibrium switching point x_i also increases. This is a sufficient condition of the unique solution for $u(x^*, I_{x^*}) = 0$.

Assumption 3.3. $\lim_{x^* \to \underline{\theta}+\varepsilon^-} u(x^*, I_{x^*}) > 0$.

Assumption 3.3 states that when a firm's productivity level is x_i and the switching point of the believed threshold strategy is also x_i, the expected payoff of staying in the market is greater than 0 as x_i goes to $\underline{\theta} + \varepsilon$ from the left. This assumption guarantees the existence and the uniqueness of solution x^{**} to $u(x^*, I_{x^*}) = 0$.

Assumption 3.4.

$$\underline{g}\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) - f > 0,$$

where the expression for $\underline{g}\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right)$ is defined as following.

When $2\varepsilon > \overline{\theta} - \underline{\theta}$,

$$\underline{g}\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) = \min\left\{g^1\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right), g^2\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right)\right\}$$

where

$$\begin{aligned} g^1\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{2} + 2\varepsilon f_e - \left(\overline{\theta}+\varepsilon\right)\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1) + 2\varepsilon(2-\sigma)} \\ g^2\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{\sigma\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{2} + 2\varepsilon f_e - \left(\overline{\theta}+\varepsilon\right)\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1) + 2\varepsilon(2-\sigma)} \end{aligned}$$

When $2\varepsilon \leq \overline{\theta} - \underline{\theta}$,

$$\underline{g}\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) = \min \left\{ \begin{array}{l} g^3\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right), g^4\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right), g^5\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) \\ g^6\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right), g^7\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right), g^8\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) \end{array} \right\}$$

where

$$\begin{aligned}
g^3\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \left(\overline{\theta} - \varepsilon\right)^{\sigma-1} \int_{\overline{\theta}-2\varepsilon}^{\overline{\theta}} \frac{\sigma}{(\theta+\varepsilon)^{\sigma}} \left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\theta+\varepsilon}{2\varepsilon} \frac{f}{\sigma-1}\right) d\theta \\
g^4\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{\sigma}{(\sigma-1)} \left(\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{(\underline{\theta}+3\varepsilon)}{2\varepsilon} \frac{f}{\sigma-1}\right) - \frac{1}{\left(1+\frac{2\varepsilon}{\underline{\theta}+3\varepsilon}\right)^{\sigma} - 1} (f+f_e) \right) \\
g^5\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{\sigma}{(\sigma-1)} \left(\left(1 - \frac{1}{\left(1+\frac{2\varepsilon}{\overline{\theta}-\varepsilon}\right)^{\sigma} - 1}\right) \left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{f}{\sigma-1}\right) \right) \\
g^6\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\theta+3\varepsilon}{2\varepsilon} \frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1) + \left(\overline{\theta}-\varepsilon\right)(2-\sigma)} \\
g^7\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{2\varepsilon\sigma}{(\underline{\theta}+3\varepsilon)^{\sigma}} \left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\underline{\theta}+3\varepsilon}{2\varepsilon} \frac{f}{\sigma-1}\right) \left(\overline{\theta}+\varepsilon\right)^{\sigma-1} \\
g^8\left(\sigma, \underline{\theta}, \overline{\theta}, \varepsilon, f, f_e\right) &= \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}} \left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{f}{\sigma-1}\right) \left(\overline{\theta}+\varepsilon\right)^{\sigma-1}
\end{aligned}$$

Assumption 3.4 ensures that the profile of strategy $I_{x^{**}}$ obtained from $u(x^{**}, I_{x^{**}}) = 0$ is indeed the profile of strategy of the monotonic pure strategy equilibrium. Note that all assumptions are sufficient but not necessary conditions for the existence of the equilibrium. Thus, parameters that violate above assumptions may still allow the (unique) existence of the equilibrium.

3.3.4 The Existence of Equilibrium

In order to show that a monotonic pure strategy equilibrium exists, it is first proved that there is an unique solution x^{**} to the equation

$$u(x^*, I_{x^*}) = 0 \tag{3.37}$$

where $u(x^*, I_{x^*})$ follows expression (3.35) when $\underline{\theta} - \varepsilon < x_i < \overline{\theta} + \varepsilon$, follows expression (3.17) when $x_i = \underline{\theta} - \varepsilon$, and follows expression (3.18) when $x_i = \overline{\theta} + \varepsilon$.

Second, it needs to be shown that $u(x_i, I_{x^{**}}) \gtreqless 0$, iff $x_i \gtreqless x^{**}$. That is, with the belief of threshold strategy $I_{x^{**}}$ as the equilibrium profile of strategies for all other firms, firm i with productivity level $x_i < x^{**}$, will have a negative expected payoff $(u(x_i, I_{x^{**}}) < 0)$ and will leave the market in the second stage; firm i with productivity level $x_i > x^{**}$ will have a positive expected payoff $(u(x_i, I_{x^{**}}) > 0)$ and will stay in the market; and firm i with productivity level $x_i = x^{**}$, will have a zero expected payoff $(u(x_i, I_{x^{**}}) = 0)$ and will stay in the market. It is assumed that if a firm is indifferent between staying and leaving, it will stay in the market. The following will show that the solution to equation (3.37) is unique and the proof of $u(x_i, I_{x^{**}}) \gtreqless 0$, when $x_i \gtreqless x^{**}$ will be shown thereafter.

Unique solution x^{**} to $u(x^*, I_{x^*}) = 0$ To prove that the solution x^{**} to equation (3.37) exists and is unique, it is first shown that the function $u(x^*, I_{x^*})$ is strictly increasing with x^*. And second, $u(x^*, I_{x^*}) < 0$ when $x^* = \underline{\theta} - \varepsilon$ and $u(x^*, I_{x^*}) > 0$ when x^*approaches $\underline{\theta} + \varepsilon^-$. ($\lim_{x^* \to \underline{\theta} + \varepsilon^-} u(x^*, I_{x^*}) > 0$ is assumed in assumption 3.3.) Thus, the existence and uniqueness of solution x^{**} to $u(x^*, I_{x^*}) = 0$ is proved.

From (3.35), (3.17) and (3.18), note that $u(x^*, I_{x^*})$ is a continuous but piecewise differentiable function. Since a positive derivative is used for showing the increasing property, the possible range of x^* should be divided into different regions such that $u(x^*, I_{x^*})$ is differentiable within each region. Once it is proved that $\frac{d}{dx^*}u(x^*, I_{x^*}) > 0$ and thus $u(x^*, I_{x^*})$ is increasing respect to x^* within each region, by continuity, $u(x^*, I_{x^*})$ is increasing with respect to x^* in the whole domain of possible x^*. When $2\varepsilon > \overline{\theta} - \underline{\theta}$, there are three distinct differentiable regions of x^* and the values of the upper and lower bounds of integration $\overline{\theta}(x^*)$ and $\underline{\theta}(x^*)$ are listed below in table 3.1.

Table 3.1. Region of x^* and integration bounds when $2\varepsilon > \overline{\theta} - \underline{\theta}$

Case	Region of x^*	$\underline{\theta}(x^*)$	$\overline{\theta}(x^*)$
1	$\underline{\theta} - \varepsilon < x^* < \overline{\theta} - \varepsilon$	$\underline{\theta}$	$x^* + \varepsilon$
2	$\overline{\theta} - \varepsilon \leq x^* < \underline{\theta} + \varepsilon$	$\underline{\theta}$	$\overline{\theta}$
3	$\underline{\theta} + \varepsilon \leq x^* < \overline{\theta} + \varepsilon$	$x^* - \varepsilon$	$\overline{\theta}$

Alternatively, when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$, the three differentiable regions and the values of $\overline{\theta}(x^*)$ and $\underline{\theta}(x^*)$ are:[12]

Table 3.2. Region of x^* and integration bounds when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$

Case	Region of x^*	$\underline{\theta}(x^*)$	$\overline{\theta}(x^*)$
4	$\underline{\theta} - \varepsilon < x^* < \underline{\theta} + \varepsilon$	$\underline{\theta}$	$x^* + \varepsilon$
5	$\underline{\theta} + \varepsilon \leq x^* < \overline{\theta} - \varepsilon$	$x^* - \varepsilon$	$x^* + \varepsilon$
6	$\overline{\theta} - \varepsilon \leq x^* < \overline{\theta} + \varepsilon$	$x^* - \varepsilon$	$\overline{\theta}$

Moreover, for a valid monotonic pure strategy equilibrium, the equilibrium switching point x^{**} must be smaller than $\underline{\theta} + \varepsilon$. This condition should be true irrespective of whether 2ε is greater or smaller than $(\overline{\theta} - \underline{\theta})$. Or with mean productivity realization θ such that $\underline{\theta} \leq \theta < x^{**} - \varepsilon$, no firm will have a productivity level greater than x^{**} and then no firm will stay in the market in the second stage. Thus, $x^{**} > \underline{\theta} + \varepsilon$ is not an equilibrium. Assumption 3.3 also rules out the possibility for $x^{**} = \underline{\theta} + \varepsilon$, which is shown in lemma 3.5.[13] As a result, when $2\varepsilon > \overline{\theta} - \underline{\theta}$, the increasing property of $u(x^*, I_{x^*})$ needs to be shown for case 1 and case 2. And when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$, the increasing property needs to be shown for case 4 only. The following lemma 3.1 and lemma 3.2 state that $u(x^*, I_{x^*})$ is increasing within regions 1 and 2, respectively, when $2\varepsilon > \overline{\theta} - \underline{\theta}$, while lemma 3.3 shows that $u(x^*, I_{x^*})$ is increasing in region 4 when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$.

[12] Note that when $2\varepsilon = \overline{\theta} - \underline{\theta}$, the interval for case 5 will degenerate to a single point.

[13] If $x^{**} = \underline{\theta} + \varepsilon$, when the realization of mean productivity is $\underline{\theta}$, the probability of stay $P_{stay}(I_{x^{**}}, \theta) = 0$, which violates the labor market clearing condition (3.30).

Lemma 3.1 *$u(x^*, I_{x^*})$ is strictly increasing in $\left(\underline{\theta} - \varepsilon, \overline{\theta} - \varepsilon\right)$ when $2\varepsilon > \overline{\theta} - \underline{\theta}$.*

Proof. See Appendix B.6. ■

Lemma 3.2 *$u(x^*, I_{x^*})$ is strictly increasing in $\left(\overline{\theta} - \varepsilon, \underline{\theta} + \varepsilon\right)$ when $2\varepsilon > \overline{\theta} - \underline{\theta}$.*

Proof. See Appendix B.7. ■

Lemma 3.1 and lemma 3.2 discuss case 1 and case 2 when $2\varepsilon > \overline{\theta} - \underline{\theta}$. The following lemma 3.3 discusses case 4 when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$.

Lemma 3.3 *$u(x^*, I_{x^*})$ is strictly increasing in $(\underline{\theta} - \varepsilon, \underline{\theta} + \varepsilon)$ when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$.*

The proof of lemma 3.3 is exactly the same as the proof of lemma 3.1. Combining lemma 3.1 through lemma 3.3, the increasing property of $u(x^*, I_{x^*})$ can be proved irrespective of whether 2ε is greater or smaller than $(\overline{\theta} - \underline{\theta})$. The increasing property is then summarized in the following lemma 3.4.

Lemma 3.4 *$u(x^*, I_{x^*})$ is continuous and strictly increasing in x^* for $\underline{\theta} - \varepsilon \leq x^* < \underline{\theta} + \varepsilon$.*

Proof. From (3.35) and (3.17), $u(x^*, I_{x^*})$ is continuous for $\underline{\theta} - \varepsilon \leq x^* < \underline{\theta} + \varepsilon$. When $2\varepsilon > \overline{\theta} - \underline{\theta}$, since $u(x^*, I_{x^*})$ is strictly increasing in $\left(\underline{\theta} - \varepsilon, \overline{\theta} - \varepsilon\right)$ and $\left(\overline{\theta} - \varepsilon, \underline{\theta} + \varepsilon\right)$ by lemma 3.1 and lemma 3.2, $u(x^*, I_{x^*})$ is strictly increasing in $[\underline{\theta} - \varepsilon, \underline{\theta} + \varepsilon)$. Since $u(x^*, I_{x^*})$ is strictly increasing in $(\underline{\theta} - \varepsilon, \underline{\theta} + \varepsilon)$ by lemma 3.3 when $2\varepsilon \leq \overline{\theta} - \underline{\theta}$, $u(x^*, I_{x^*})$ is also strictly increasing in $[\underline{\theta} - \varepsilon, \underline{\theta} + \varepsilon)$. ■

Lemma 3.5 *The solution x^{**} to $u(x^*, I_{x^*}) = 0$ exists and is unique.*

Proof. See Appendix B.8. ■

Proof of $u(x_i, I_{x^{**}}) \gtreqqless 0$, **iff** $x_i \gtreqqless x^{**}$

Lemma 3.6 *If* $x < x^{**}$, $u(x, I_{x^{**}}) < 0$.

Proof. For $2\varepsilon > \overline{\theta} - \underline{\theta}$, the equilibrium switching point $x^{**} \in (\underline{\theta} - \varepsilon, \underline{\theta} + \varepsilon)$ can fall in region 1 and 2; while for $2\varepsilon \leq \overline{\theta} - \underline{\theta}$, x^{**} is in region 4 only. Depending on the region of x^{**}, when $x < x^{**}$, x can fall in different regions. Based on the regions of x^{**} and x, calculate $u(x, I_{x^{**}})$ and shown that

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) < 0$$

Then lemma 3.6 is proved. The details can be found in Appendix B.9. ■

Lemma 3.7 *If* $x > x^{**}$, $u(x, I_{x^{**}}) > 0$.

Proof. As in lemma 3.6, the proof of $u(x, I_{x^{**}}) > 0$ is discussed under different situations when x^{**} falls in different regions. The proof shows the global minimum of $u(x, I_{x^{**}})$ on inteval $\left(x^{**}, \overline{\theta} + \varepsilon\right]$ is positive. It first show $u(x, I_{x^{**}}) > 0$ when $x = \overline{\theta} + \varepsilon$. Second, it shows that the first order derivative of $u(x, I_{x^{**}})$ is continuous and then the local extreme value(s) is(are) positive (by assumption 3.4). The details are in Appendix B.10. ■

Lemma 3.6 and lemma 3.7 together guarantee that the threshold strategy $I_{x^{**}}$ is indeed an equilibrium strategy.

Since there is a unique solution x^{**} to equation (3.37), and $u(x, I_{x^{**}}) \gtreqqless 0$, iff $x \gtreqqless x^{**}$, the existence of a unique equilibrium has been shown. This result is summarized in the following theorem.

Theorem 3.1 *In an entry-exit game with incomplete information,* x^{**} *is the unique switching point of the monotonic pure strategy equilibrium. A firm will stay in the market in the second stage if and only if its productivity level* $x_i \geq x^{**}$.

3.3.5 Equilibrium Values

Once the equilibrium switching point x^{**} is calculated, all other variable values can be easily obtained. The expressions for the productivity distribution at the end of the second stage after firms make their stay/exit decision, equilibrium values of average productivity, price, revenue, and welfare given a realization of Θ are presented below. To facilitate comparative statics, the *ex ante* (or expected) average productivity, price, revenue and welfare per worker are also calculated.

The distribution density function $\mu(x_i, I_{x^{**}}, \theta)$,

$$\mu(x_i, I_{x^{**}}, \theta) = \begin{cases} \frac{1}{\theta+\varepsilon-x^{**}} & \text{if } \theta-\varepsilon \leq x^{**} \leq x_i \leq \theta+\varepsilon \\ 0 & \text{if } \theta-\varepsilon \leq x_i < x^{**} < \theta+\varepsilon \\ \frac{1}{2\varepsilon} & \text{if } x^{**} < \theta-\varepsilon \end{cases}$$

The probability of stay and the ex-ante probability of stay

$$\begin{aligned} P_{stay}\left(I_{x^{**}}, \theta\right) &= \begin{cases} \frac{\theta}{2\varepsilon} + \frac{\varepsilon - x^{**}}{2\varepsilon} & \text{if } x^{**}-\varepsilon < \theta \leq x^{**}+\varepsilon \\ 1 & \text{if } \theta > x^{**}+\varepsilon \end{cases} \\ P^e_{stay}\left(I_{x^{**}}\right) &= \begin{cases} \frac{\overline{\theta}+\underline{\theta}+2\varepsilon-2x^{**}}{4\varepsilon} & \text{if } x^{**}-\varepsilon < \overline{\theta} \leq x^{**}+\varepsilon \text{ and } x^{**}-\varepsilon < \underline{\theta} \leq x^{**}+\varepsilon \\ \frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta} - \ x^{**} - \frac{(\underline{\theta}+\varepsilon-x^{**})^2}{4\varepsilon}\right) & \text{if } \overline{\theta} > x^{**}+\varepsilon \text{ and } x^{**}-\varepsilon < \underline{\theta} \leq x^{**}+\varepsilon \end{cases} \end{aligned} \tag{3.38}$$

The mass of firms at the beginning of the second stage, the mass of firms at the end of the second stage (which is also the mass of goods produced and consumed), and the expected mass of exsiting firms:

$$\begin{aligned} M_e\left(I_{x^{**}}\right) &= \frac{L/\sigma}{f_e + f\ P^e_{stay}\left(I_{x^{**}}\right)} \\ M\left(I_{x^{**}}, \theta\right) &= M_e\left(I_{x^{**}}\right) P_{stay}\left(I_{x^{**}}, \theta\right) \\ M\left(I_{x^{**}}\right) &= E_\Theta\left[M\left(I_{x^{**}}, \theta\right)\right] = M_e\left(I_{x^{**}}\right) P^e_{stay}\left(I_{x^{**}}\right) \end{aligned} \tag{3.39}$$

The average productivity and the expected average productivity:

$$\widetilde{x}\left(I_{x^{**}},\theta\right) = \begin{cases} \left(\frac{(\theta+\varepsilon)^{\sigma}-(x^{**})^{\sigma}}{\sigma(\theta+\varepsilon-x^{**})}\right)^{\frac{1}{\sigma-1}} & \text{if } x^{**}-\varepsilon<\theta\leq x^{**}+\varepsilon \\ \left(\frac{(\theta+\varepsilon)^{\sigma}-(\theta-\varepsilon)^{\sigma}}{2\varepsilon\sigma}\right)^{\frac{1}{\sigma-1}} & \text{if } \theta>x^{**}+\varepsilon \end{cases} \tag{3.40}$$

$$\widetilde{x}\left(I_{x^{**}}\right) = E_{\Theta}\left[\widetilde{x}\left(I_{x^{**}},\theta\right)\right] = \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{\overline{\theta}-\underline{\theta}}\widetilde{x}\left(I_{x^{**}},\theta\right)d\theta$$

The aggregate price:

$$\begin{aligned} P\left(I_{x^{**}},\theta\right) &= \frac{\left(M_e\left(I_{x^{**}}\right)P_{stay}\left(I_{x^{**}},\theta\right)\right)^{\frac{1}{1-\sigma}}}{\rho\widetilde{x}\left(I_{x^{**}},\theta\right)} \\ &= \begin{cases} \rho^{-1}\left(M_e\left(I_{x^{**}}\right)\frac{(\theta+\varepsilon)^{\sigma}-(x^{**})^{\sigma}}{2\varepsilon\sigma}\right)^{\frac{1}{1-\sigma}} & \text{if } x^{**}-\varepsilon<\theta\leq x^{**}+\varepsilon \\ \rho^{-1}\left(M_e\left(I_{x^{**}}\right)\frac{(\theta+\varepsilon)^{\sigma}-(\theta-\varepsilon)^{\sigma}}{2\varepsilon\sigma}\right)^{\frac{1}{1-\sigma}} & \text{if } \theta>x^{**}+\varepsilon \end{cases} \end{aligned} \tag{3.41}$$

and the expected aggregate price

$$P\left(I_{x^{**}}\right) = E_{\Theta}\left[P\left(I_{x^{**}},\theta\right)\right] = \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{\overline{\theta}-\underline{\theta}}P\left(I_{x^{**}},\theta\right)d\theta$$

The aggregate revenue[14]

$$R\left(I_{x^{**}},\theta\right) = M_e\left(I_{x^{**}}\right)\sigma\left(\frac{\sigma f}{\sigma-1}P^{e}_{stay}\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}P_{stay}\left(I_{x^{**}},\theta\right)\right)$$

and by labor market clearing condition, the expected aggregate revenue

$$R\left(I_{x^{**}}\right) = E_{\Theta}\left[R\left(I_{x^{**}},\theta\right)\right] = L$$

Note that when $\left(\overline{\theta}-\underline{\theta}\right)$ goes to 0, the game with incomplete information degenerates to Melitz (2003) model, and as in Melitz model the total revenue is exactly the total labor endowment. When $\overline{\theta}-\underline{\theta}>0$ in this chapter, labor is used for fixed entry costs and the fixed producing costs. By free entry condition, the *ex ante* entry costs is the expected profits

[14] See Appendix B.11 for detail.

once entry. Then the sum of the expected profits and the fix producing costs together is the expected revenue, which equals to the labor endowment.

The aggregate quantity which equals the indirect utility of the aggregate consumer

$$Q\left(I_{x^{**}},\theta\right)=\frac{R\left(I_{x^{**}},\theta\right)}{P\left(I_{x^{**}},\theta\right)} \tag{3.42}$$

and the expected aggregate quantity

$$Q\left(I_{x^{**}}\right)=E_{\Theta}\left[Q\left(I_{x^{**}},\theta\right)\right]=\int_{\underline{\theta}}^{\overline{\theta}}\frac{1}{\overline{\theta}-\underline{\theta}}Q\left(I_{x^{**}},\theta\right)d\theta$$

Lastly, the total welfare (TW)

$$TW\left(I_{x^{**}},\theta\right)=Q\left(I_{x^{**}},\theta\right) \tag{3.43}$$

and the expected total welfare

$$TW\left(I_{x^{**}}\right)=E_{\Theta}\left[TW\left(I_{x^{**}},\theta\right)\right]=\int_{\underline{\theta}}^{\overline{\theta}}\frac{1}{\overline{\theta}-\underline{\theta}}TW\left(I_{x^{**}},\theta\right)d\theta$$

The welfare per worker

$$W\left(I_{x^{**}},\theta\right)=\frac{TW\left(I_{x^{**}},\theta\right)}{L}=\frac{Q\left(I_{x^{**}},\theta\right)}{L} \tag{3.44}$$

and the expected welfare per worker

$$W\left(I_{x^{**}}\right)=E_{\Theta}\left[W\left(I_{x^{**}},\theta\right)\right]=\int_{\underline{\theta}}^{\overline{\theta}}\frac{1}{\overline{\theta}-\underline{\theta}}\left(\frac{1}{L}\frac{R\left(I_{x^{**}},\theta\right)}{P\left(I_{x^{**}},\theta\right)}\right)d\theta$$

Since expectation is a linear operator, $W\left(I_{x^{**}}\right)\neq\left[P\left(I_{x^{**}}\right)\right]^{-1}$, which is the welfare per worker in Melitz model.

3.4 An Example

This section presents an example of the game with incomplete information and shows that a monotonic pure strategy equilibrium exists and is unique. The following table 3.3 exhibits the parameter values that satisfy assumption 3.1 through assumption 3.4. The elasticity of substitution is chosen as 5 to make parameter assumption 3.4 hold even when $\frac{\overline{\theta}-\underline{\theta}}{2\varepsilon} \in (0, 6]$.

Table 3.3. The model parameterizations

Parameters	Value
The elasticity of substitution (σ)	5
Lower bound of the mean productivity θ ($\underline{\theta}$)	1
Higher bound of the mean productivity θ ($\overline{\theta}$)	4
The length of the support of productivity distribution after θ is realized (2ε)	2
The entry costs (f_e)	0.1
The fixed producing costs (f)	0.1

Figure 3.1 shows the graph of $u(x^*, I_{x^*})$ and $u(x, I_{x^{**}})$. From the graph, it can be seen that $u(x^*, I_{x^*})$ is strictly increasing and has a unique solution x^{**} for $u(x^*, I_{x^*}) = 0$. Moreover, $u(x, I_{x^{**}}) \gtreqless 0$, when $x \gtreqless x^{**}$. As a result, the monotonic pure strategy equilibrium exists and is unique. $x^{**} = 1.491$ in this example. (Note that here $\underline{\theta} + \varepsilon = 2$.)

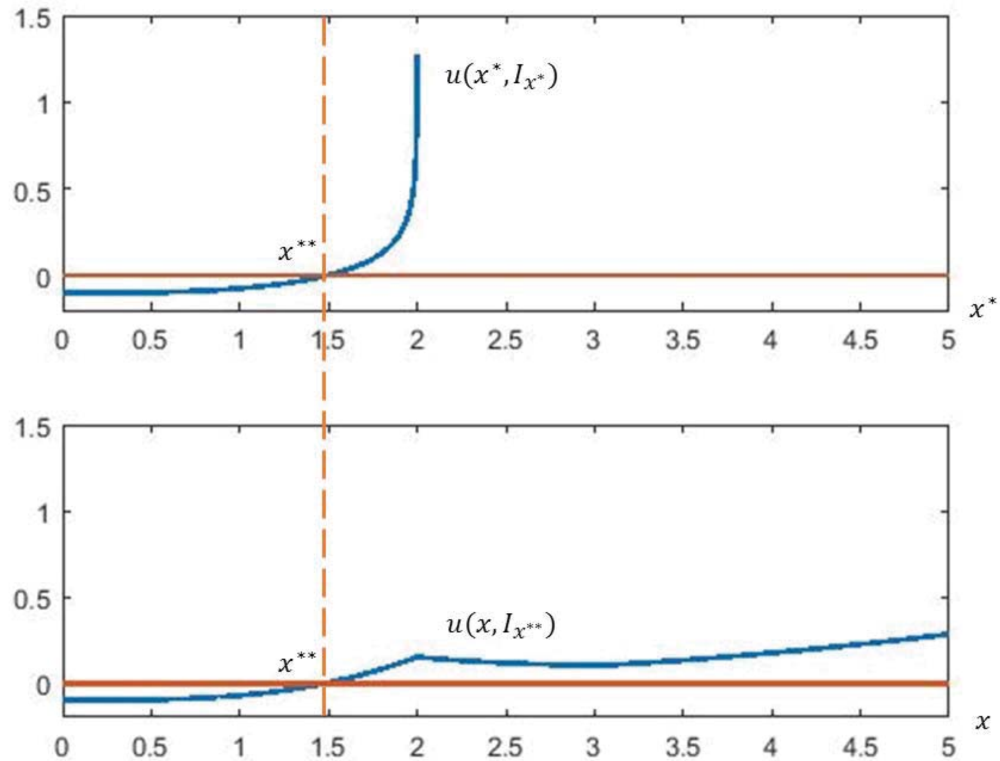


Figure 3.1. $u(x^*, I_{x^*})$ and $u(x, I_{x^{**}})$

In the above example, when $x > 2$, $u(x, I_{x^{**}})$ is not a increasing function with x. In fact, in the monotonic pure strategy equilibrium, given $I_{x^{**}}$, suppose firm i has productivity level x_i and firm j has productivity level x_j. With higher productivity level x_j, firm j will earn higher revenue than firm i for a given θ and thus a fixed productivity distribution. This has a positive effect on firm's expected profit. However, firm's productivity level also serves as a signal of all other firm's productivities. With a higher productivity, firm j believes all other firms will have a higher level of productivity as well. In this situation, firm j will lose its competitive advantage in the market and may end up with a lower revenue. This has a negative effect for a higher x_i given $I_{x^{**}}$. In this example, $\frac{\overline{\theta}-\underline{\theta}}{2\varepsilon} = 1.5 > 1$ and thus x^{**} is in region 4. Consider x in region 5 ($2 \leq x < 3$) for instance. Since $x < 3 < x^{**} + 2\varepsilon = 3.491$,

$$\begin{aligned} u(x, I_{x^{**}}) &= \int_{x-\varepsilon}^{x^{**}+\varepsilon} \frac{1}{2\varepsilon} Ra^1 \left(x, I_{x^{**}}, x+\varepsilon\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\ &\quad + \int_{x^{**}+\varepsilon}^{x+\varepsilon} \frac{1}{2\varepsilon} Ra^2 \left(x, I_{x^{**}}, x+\varepsilon\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta - f \end{aligned}$$

and then

$$\begin{aligned} \frac{d}{dx} u(x, I_{x^{**}}) &= \frac{1}{2\varepsilon} Ra^2 \left(x, I_{x^{**}}, x+\varepsilon\right) \left(\overline{\pi}\left(I_{x^{**}}, x+\varepsilon\right) + f\right) \\ &\quad - \frac{1}{2\varepsilon} Ra^1 \left(x, I_{x^{**}}, x-\varepsilon\right) \left(\overline{\pi}\left(I_{x^{**}}, x-\varepsilon\right) + f\right) \\ &\quad + \int_{x-\varepsilon}^{x+\varepsilon} \frac{d}{dx} \frac{1}{2\varepsilon} Ra \left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \end{aligned}$$

Here, the positive effect is

$$\int_{x-\varepsilon}^{x+\varepsilon} \frac{d}{dx} \frac{1}{2\varepsilon} Ra \left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta > 0$$

and the negative effect is

$$\frac{1}{2\varepsilon} Ra^2 \left(x, I_{x^{**}}, x+\varepsilon\right) \left(\overline{\pi}\left(I_{x^{**}}, x+\varepsilon\right) + f\right) - \frac{1}{2\varepsilon} Ra^1 \left(x, I_{x^{**}}, x-\varepsilon\right) \left(\overline{\pi}\left(I_{x^{**}}, x-\varepsilon\right) + f\right) < 0$$

The relative magnitudes for these two effects in general can not be analytically evaluated and thus the sign of $\frac{d}{dx}u(x, I_{x^{**}})$ cannot be easily determined. The sign of $\frac{d}{dx}u(x, I_{x^{**}})$ depends on the parameters including the shape of the distributions. Some examples with more complex non-uniform distributions will be discussed in chapter 4. The variations in equilibria due to parameters is discussed below in the section on comparative statics. Note, however, that when $x < \underline{\theta} + \varepsilon$ the positive effects dominate the negative effects and $\frac{d}{dx}u(x, I_{x^{**}}) > 0$ can be shown analytically (see the proof of lemma 3.6 and lemma 3.7).

3.5 Comparative Statics

This section undertakes comparative statics by examining how the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ change due to changes in parameters. In particular four parameters (or cases) are considered: (1) the precision of public information, (2) the precision of private information, (3) the elasticity of substitution, and (4) the fixed producing costs. Apart from the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$, some other variables are also included in comparative statics discussion. This is to complete the explanation of how parameter differences affect the expected welfare per worker $W(I_{x^{**}})$ through their effects on the equilibrium switching point x^{**}. For example, the ex-ante probability of stay $P^{e}_{stay}(I_{x^{**}})$, the expected average productivity $\widetilde{x}(I_{x^{**}})$, and so on.

In each graph, values of variables (y-axis number) are standardized. Specifically, the coordinate of a point on the curve is the ratio between true value of the variable and true value of that variable in a particular (standard) case. Then, the variable value of standard case is always 1. Standardization helps to illustrate all variables' movements in one figure. In all exercises, the monotonic pure strategy equilibrium exists and is unique. However, in some exercises, one or some of assumptions on parameters may have to be violated for a complete analysis. Whenever this occurs, an explicit discussion will be provided. In each exercise, parameters used are first introduced, and the graph about equilibrium variables' values affected by parameter change is shown thereafter. At the end of each exercise, some economic intuition is provided.

3.5.1 Changes in the Precision of Public Information

Table 3.4 lists parameters used for the exercise when the precision of public information changes. Other than $\overline{\theta}$, the higher bound of mean productivity θ, parameter values are the same as in the example from last section. The labor endowment L is set to be 1 to compute values for aggregate equilibrium variables (e.g. expected aggregate revenue $R(I_{x^{**}})$). Since variance of a continuous uniform distribution $U\left[\underline{\theta},\overline{\theta}\right]$ is $\frac{\left(\overline{\theta}-\underline{\theta}\right)^2}{12}$, and by assumption 1, $\underline{\theta}=\varepsilon$, then when $\overline{\theta}$ increases, variance increases and then the precision, as reciprocal of variance, decreases. In this exercise, $\overline{\theta}$ varies from 1.001 to 6. That is, precision $\frac{12}{\left(\overline{\theta}-\underline{\theta}\right)^2}$ varies from 0.48 to 1.2×10^7. Since range of precision is big, natural logarithm of precision, $\ln\left(\frac{12}{\left(\overline{\theta}-\underline{\theta}\right)^2}\right)$ is used as the x-axis for figure 3.2, which shows how a change in the precision of public information affects the equilibrium switching point x^{**} and the expected welfare per worker $W\left(I_{x^{**}}\right)$. Note that when $\overline{\theta}-\underline{\theta}=0$, the model degenerates to Melitz (2003) model with the *ex-ante* survival probabilities $\delta=1$ and firms only produce for one period (call it Melitz case). Choose Melitz case as the standard case for this exercise.[15] That is, y-axis is the ratio between true variable values and values for Melitz case when $\overline{\theta}-\underline{\theta}=0$. For $\overline{\theta}\leq 6$, all assumptions are satisfied in this exercise. Below are table 3.4 and figure 3.2.

Table 3.4. Parameterizations when $\overline{\theta}$ changes

Parameters	Value
The elasticity of substitution (σ)	5
Lower bound of the mean productivity θ ($\underline{\theta}$)	1
Higher bound of the mean productivity θ ($\overline{\theta}$)	$[1.001, 6]$
The length of the support of productivity distribution after θ is realized (2ε)	2
The entry costs (f_e)	0.1
The fixed producing costs (f)	0.1
The labor endowment (L)	1

[15] Variable values for Melitz case is computed differently using the zero cutoff profit (ZCP) condition and the free entry (FE) condition by Melitz.

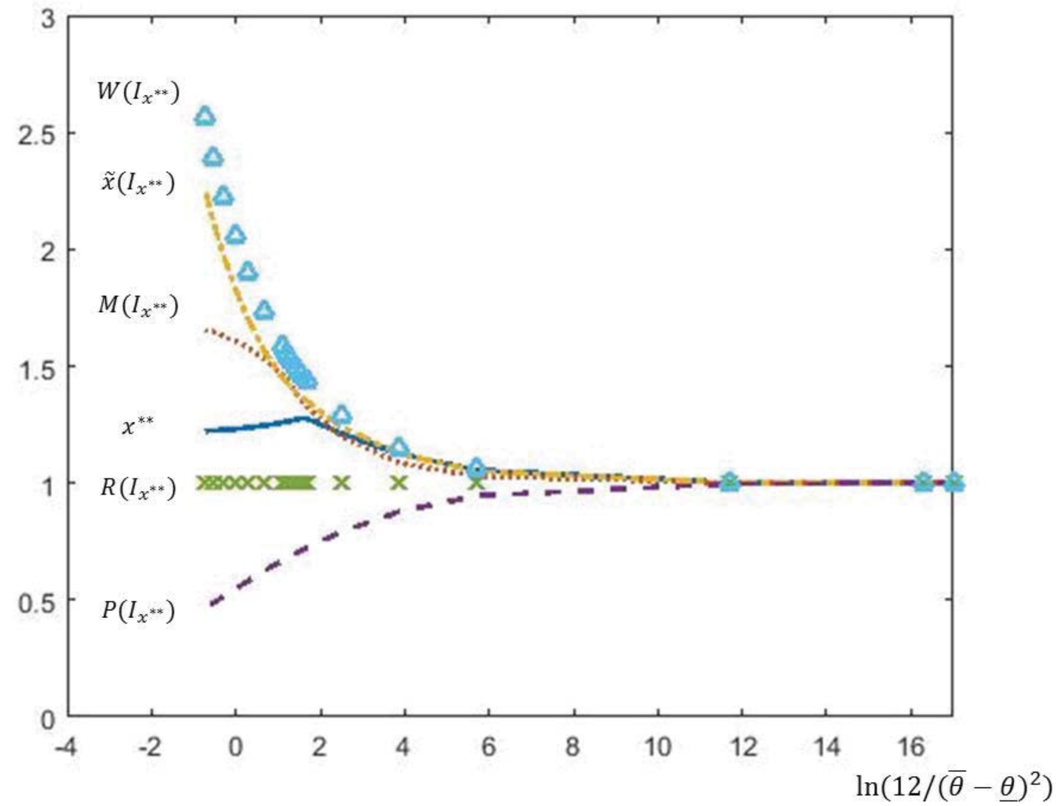


Figure 3.2. The effects of changes in public precision

As $\ln\left(\frac{12}{\left(\overline{\theta}-\underline{\theta}\right)^2}\right)$, the natural logarithm of precision of public information approaches to infinity, the equilibrium switching point x^{**} approaches to the cutoff productivity level of Melitz case. And in Melitz model, the cutoff productivity level is 1.205. When $\ln\left(\frac{12}{\left(\overline{\theta}-\underline{\theta}\right)^2}\right)$ increases, x^{**} first increases and then decreases and the expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases. Note that, to satisfy all parameter assumptions, only $\overline{\theta}$ varies in this exercise. However, when $\overline{\theta}$ changes, not only the precision of Θ changes, the expectation of Θ changes as well. That is, the increasing and decreasing movement of x^{**} and the decreasing of $W\left(I_{x^{**}}\right)$ are caused by a combination effects of $Var(\Theta)$ and $E(\Theta)$.[16] To examine the impact on x^{**} and $W\left(I_{x^{**}}\right)$ by precision only, the expectation of Θ, $E(\Theta)$ should be controlled. A new exercise is conducted here while $E(\Theta)$ is fixed. As a result, assumptions 3.1 and 3.4 are violated simultaneously.

[16] The decreasing of $E(\Theta)$ is the main reason why $W\left(I_{x^{**}}\right)$ decreases.

Table 3.5 shows parameter setting for this new exercise when only the precision of Θ changes but the expectation of Θ stays unchanged. $\varepsilon = 1$ and $E(\Theta) = 3$ in this exercise. Note that the precision of private information given the realization of Θ is $\frac{12}{(2\varepsilon)^2} = 3$. $\overline{\theta}$ varies from 3.001 to 5, and the corresponding precision $\frac{12}{\left(\overline{\theta}-\underline{\theta}\right)^2}$ varies from 0.75 to 3×10^6. In this range, the precision of public information Θ can be greater and smaller than the precision of private information $X|\Theta = \theta$. Figure 3.3 exhibits effects of precision of Θ only on the equilibrium switching point x^{**} and the expected welfare per worker $W\left(I_{x^{**}}\right)$. As in figure 3.2, the x-axis is the natural logarithm of precision. Choose Melitz case as the standard case for this exercise. That is, y-axis is the ratio between true variable values and values for Melitz case when $\overline{\theta} = \underline{\theta} = 3$. Note that since assumption 3.1 and 3.4 are violated, the analytical proof of the unique existence of the monotonic pure strategy equilibrium cannot be applied anymore. However, the unique existence of the equilibrium is examined numerically by checking the unique solution x^{**} to $u(x^*, I_{x^*}) = 0$ and $u(x_i, I_{x^{**}}) \gtreqless 0$, iff $x_i \gtreqless x^{**}$.

Table 3.5. Parameterizations when only precision of Θ changes

Parameters	Value
The elasticity of substitution (σ)	5
Lower bound of the mean productivity θ ($\underline{\theta}$)	$[1, 2.999]$
Higher bound of the mean productivity θ ($\overline{\theta}$)	$6 - \underline{\theta}$
The length of the support of productivity distribution after θ is realized (2ε)	2
The entry costs (f_e)	0.1
The fixed producing costs (f)	0.3
The labor endowment (L)	1

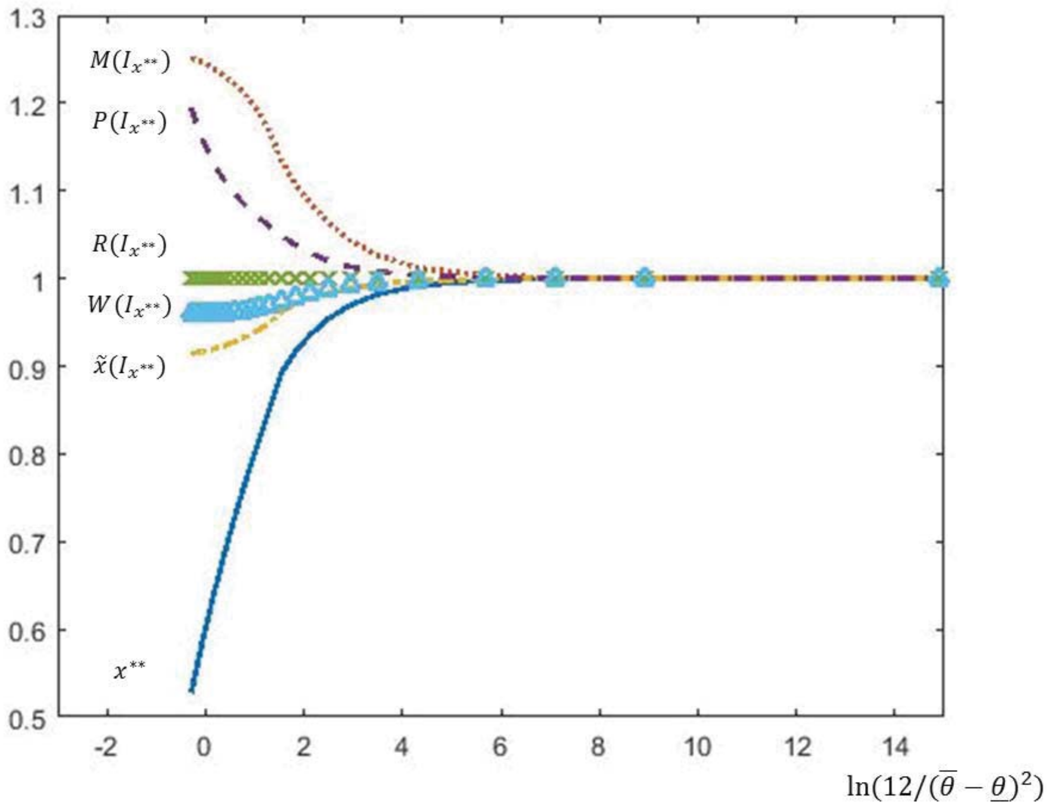


Figure 3.3. The effects of changes in public precision only

As in figure 3.2, as precision of Θ increases, the equilibrium switching point x^{**} increases to the cutoff productivity level (3.1152) of Melitz case when $\overline{\theta} = \underline{\theta} = 3$. Consider a marginal firm with productivity level $x^{**}(\underline{\theta}_1)$, where $x^{**}(\underline{\theta}_1)$ is the equilibrium switching productivity level when $\underline{\theta} = \underline{\theta}_1$. Suppose there is an increase in the precision of public information and thus $\underline{\theta}_1$ increases to $\underline{\theta}_2$. Then the marginal firm loses its chances of earning positive profits since the realization of Θ is less likely to be low (i.e., it is likely to face more competition), and the marginal firm also loses chances of running into loss since the realization of Θ is less likely to be high (i.e., less competition). When the first effects dominate, the expected payoff of the marginal firm decreases and it exits the market. As a result, the equilibrium switching point x^{**} increases as with an increase in the precision of Θ. The following figure 3.4 shows that when $\underline{\theta}_1 = 2.3$ and $x^{**}(\underline{\theta}_1) = 2.8525$, the expected revenue of margin firm with $x^{**}(\underline{\theta}_1)$ decreases when $\underline{\theta}_1$ increases to $\underline{\theta}_2 = 2.4$.

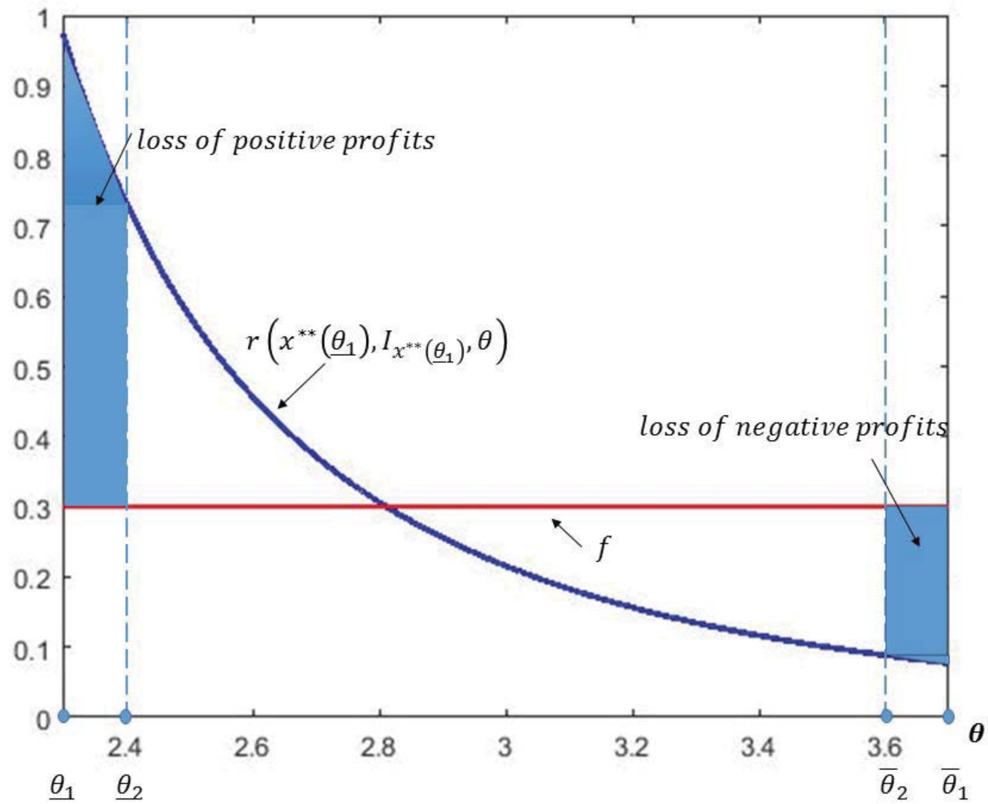


Figure 3.4. The effects of changes in public precision on margin firm

In figure 3.4, the x-axis is θ, and the y-axis is the revenue $r(x^{**}(\underline{\theta}_1), I_{x^{**}(\underline{\theta}_1)}, \theta)$ for firm with productivity level $x^{**}(\underline{\theta}_1)$, facing the equilibrium strategy $I_{x^{**}(\underline{\theta}_1)}$, and the true realization of Θ as θ. When θ increases from $\underline{\theta}_1\left(x^{**}(\underline{\theta}_1)\right) (= \max(x^{**}(\underline{\theta}_1) - \varepsilon, \underline{\theta}_1) = \underline{\theta}_1 = 2.3)$ to $\overline{\theta}_1\left(x^{**}(\underline{\theta}_1)\right) (= \min(x^{**}(\underline{\theta}_1) + \varepsilon, \overline{\theta}_1) = \overline{\theta}_1 = 3.7)$, the revenue $r(x^{**}(\underline{\theta}_1), I_{x^{**}(\underline{\theta}_1)}, \theta)$ decreases since more firms compete in the labor market when θ increases. The expectation of the revenue $r(x^{**}(\underline{\theta}_1), I_{x^{**}(\underline{\theta}_1)}, \theta)$ equals the fixed producing costs f. That is why $x^{**}(\underline{\theta}_1)$ is the equilibrium switching point when $\theta = \underline{\theta}_1$. Now, suppose $\underline{\theta}_1$ increases to $\underline{\theta}_2 = 2.4$. Then the lower bound $\underline{\theta}_2\left(x^{**}(\underline{\theta}_1)\right) = 2.4$ and the higher bound $\overline{\theta}_2\left(x^{**}(\underline{\theta}_1)\right) = 3.6$. The marginal firm loses more of its positive profits than it gains from its reduction in losses. As a result, the marginal firm's expected payoff decreases and it exits the market.

Moreover, when x^{**} increases, the expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases. Since $\widetilde{x}\left(I_{x^{**}}\right)$ increases, the expected aggregate price $P\left(I_{x^{**}}\right)$ decreases due to the fixed markup in monopolistic competition. As a result, the expected welfare per worker $W(I_{x^{**}})$ increases. $W(I_{x^{**}})$ increases due to the decreases in $P(I_{x^{**}})$. In sum, when precision of public information increases, the equilibrium switching point x^{**} increases and then the expected average productivity increases. By reallocating resource to more productive firms, the expected welfare per work also increases. This conclusion of welfare increases is the main finding for this chapter, and it will be stressed in chapter 4 when more examples using complicated distributions are studied.

3.5.2 Changes in the Precision of Private Information

This part examines how the precision of private information affects the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$. Table 3.6 shows parameter settings. ε varies from 0.3 to 1, and the corresponding precision $\frac{12}{(2\varepsilon)^2}$ varies from 3 to 33.3. In this range, the precision of private information can be greater or smaller than the precision of public information $\frac{12}{\left(\overline{\theta}-\underline{\theta}\right)^2}$, which is 12. Figure 3.5 shows how the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ change as the precision of private information changes. The x-axis is the natural logarithm of precision of private information. The standard case is when the precision of public information and the precision of private information are the same ($\overline{\theta} - \underline{\theta} = 2\varepsilon = 1$). And then, y-axis is the ratio between true variable values and values for standard case when $ln\left(\frac{12}{(2\varepsilon)^2}\right) = 2.485$. Since $\varepsilon \neq \underline{\theta}$ for all ε, assumptions 3.1 and 3.2 are violated and then equilibrium is checked by the unique solution x^{**} to $u(x^*, I_{x^*}) = 0$ and $u(x_i, I_{x^{**}}) \gtreqless 0$, iff $x_i \gtreqless x^{**}$. When $\varepsilon < 0.3$, a monotonic pure strategy equilibrium does not exist as $u(x, I_{x^{**}}) \ngtr 0$ when $x > x^{**}$.

Table 3.6. Parameterizations when (ε) changes

Parameters	Value
The elasticity of substitution (σ)	5
Lower bound of the mean productivity θ ($\underline{\theta}$)	1
Higher bound of the mean productivity θ ($\overline{\theta}$)	2
The length of the support of productivity distribution after θ is realized (2ε)	$[0.6, 2]$
The entry costs (f_e)	0.1
The fixed producing costs (f)	0.3
The labor endowment (L)	1

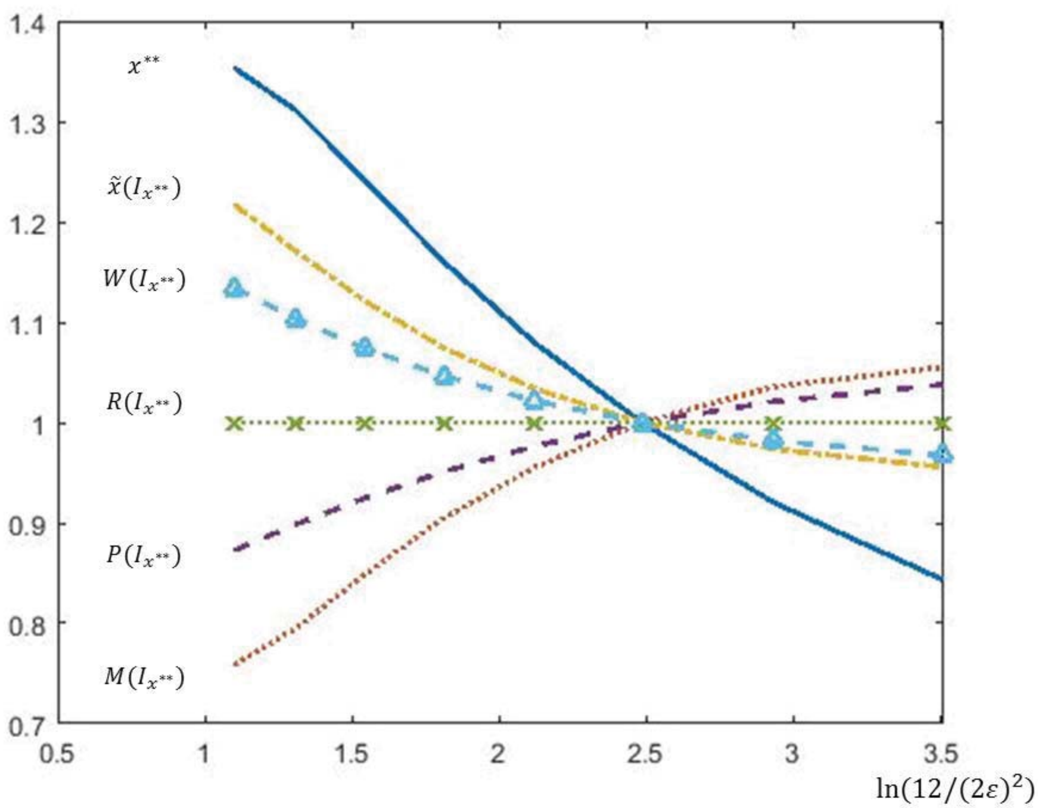


Figure 3.5. The effects of changes in private precision

From figure 3.5, it can be seen that as the precision of private information increases, the equilibrium switching point x^{**} decreases and the expected welfare per worker $W(I_{x^{**}})$ decreases as well. A decrease in the equilibrium switching point x^{**} causes a decrease in the expected average productivity $\widetilde{x}(I_{x^{**}})$, and an increase in the expected mass of existing firms $M(I_{x^{**}})$. The decrease in $\widetilde{x}(I_{x^{**}})$ causes an increase in the expected aggregate price $P(I_{x^{**}})$ while the increase in $M(I_{x^{**}})$ causes a decrease in $P(I_{x^{**}})$. This examples shows that the expected aggregate price $P(I_{x^{**}})$ increases as the precision of private information increases, indicating the effects of $\widetilde{x}(I_{x^{**}})$ dominates the effects of $M(I_{x^{**}})$ on $P(I_{x^{**}})$. As a result of increasing $P(I_{x^{**}})$, the expected welfare per worker $W(I_{x^{**}})$ decreases.

Figure 3.6 below shows how the precision of private information affects the equilibrium cutoff point x^{**} and welfare per worker $W(I_{x^{**}})$ for the Melitz case when $\overline{\theta} = \underline{\theta} = \theta$. All parameters are the same as in table 3.6 except $\theta = 1.5$. The x-axis is the natural logarithm of the precision of private information. The standard case is when $2\varepsilon = 1$, the same standard case for figure 3.5.

In figure 3.6, as the precision of private information increases, the equilibrium cutoff point x^{**} decreases and the welfare per worker $W(I_{x^{**}})$ decreases as well. When the precision of private information increases while θ is fixed, a bigger precision means a smaller ε, indicating a smaller portion of firms with high productivities. With less severe competition in the market, firms expect to earn larger profits and thus the cutoff point x^{**} decreases. With a decrease in aggregate productivity, the welfare per worker decreases as well. The analysis of why an increase in the precision of private information can cause a decrease in cutoff point and why the standardized cutoff curve and standardized welfare curve coincide with each other are derived in Appendix B.12.

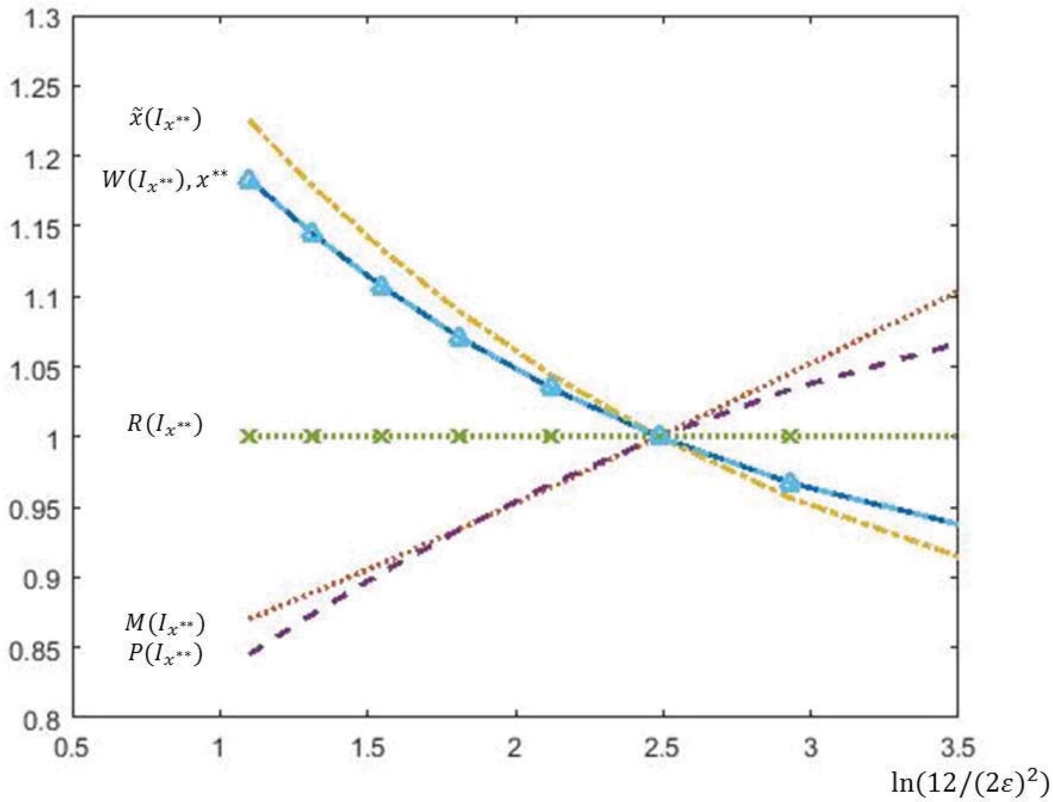


Figure 3.6. The effects of changes in private precision for Melitz case

3.5.3 Changes in the Elasticity of Substitution

Table 3.7 shows parameter setting for exercise when the elasticity of substitution σ changes. The elasticity of substitution σ varies from 3.2 to 6.6. In the exercise, the precision of public information is chosen to be greater than the precision of private information such that all parameter assumptions are satisfied for all $\sigma \in [3.2, 6.6]$. Figure 3.7 shows how the change in the elasticity of substitution σ affects the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$. The x-axis is the elasticity of substitution σ, while the y-axis is the ratio between true variable values and values for the case when $\sigma = 3.2$.

Table 3.7. Parameterizations when (σ) changes

Parameters	Value
The elasticity of substitution (σ)	$[3.2, 6.6]$
Lower bound of the mean productivity θ ($\underline{\theta}$)	1
Higher bound of the mean productivity θ ($\bar{\theta}$)	2
The length of the support of productivity distribution after θ is realized (2ε)	2
The entry costs (f_e)	0.1
The fixed producing costs (f)	0.1
The labor endowment (L)	1

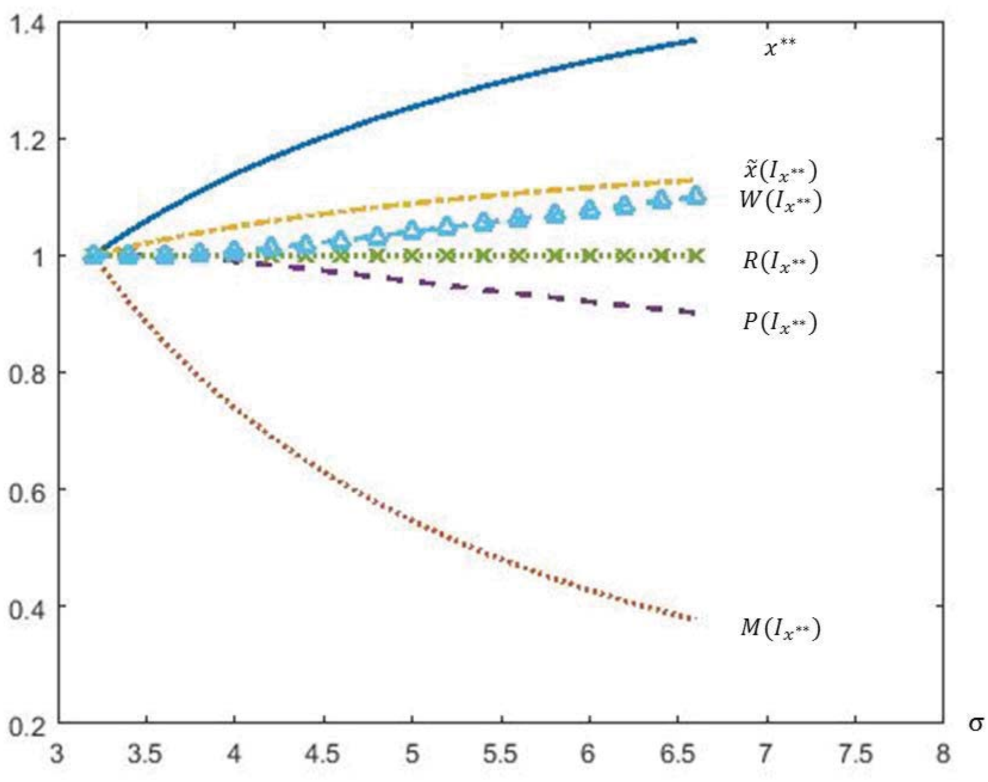


Figure 3.7. The effects of elasticity of substitution σ

When the elasticity of substitution σ increases, the equilibrium switching point x^{**} increases and the expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases and then increases. With higher level of σ, firms with lower level of productivity lose revenues further and therefore exit the market. The expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases due to the increase in x^{**}. The expected aggregate price $P\left(I_{x^{**}}\right)$ increases and then decreases owing to the combination impact by increasing expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ and decreasing mass of existing firms $M\left(I_{x^{**}}, \theta\right)$ (for a given θ). Then, the corresponding expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases and then increases as σ increases.

3.5.4 Changes in the Fixed Production Costs

Parameter setting for exercise when the fixed producing costs f changes is shown in table 3.8. The fixed producing costs f varies from 0.01 to 1 and thus the ratio between the fixed producing costs f and the entry costs f_e changes from 0.1 to 10. Figure 3.8 shows how the change in fixed producing costs f affects the equilibrium switching point x^{**} and the expected welfare per worker $W\left(I_{x^{**}}\right)$. Since the equilibrium switching point x^{**} only be impacted by the ratio between f and f_e, no comparative statics analysis is conduct for changes in entry costs f_e and the x-axis for figure 6 is set to be $\frac{f}{f_e}$ instead of f. The y-axis is the ratio between true variable values and values for the case when $f = f_e$. Note that when $\frac{f}{f_e} > 2$, assumption 3.4 is violated. However, the monotonic pure strategy equilibrium still exists and is unique. This can be proved by showing the uniqueness of the solution to $u(x^*, I_{x^*}) = 0$ and $u(x_i, I_{x^{**}}) \gtreqless 0$, iff $x_i \gtreqless x^{**}$.

Table 3.8. Parameterizations when $(\frac{f}{f_e})$ changes

Parameters	Value
The elasticity of substitution (σ)	5
Lower bound of the mean productivity θ ($\underline{\theta}$)	1
Higher bound of the mean productivity θ ($\bar{\theta}$)	2
The length of the support of productivity distribution after θ is realized (2ε)	2
The entry costs (f_e)	0.1
The fixed producing costs (f)	$[0.01, 1]$
The labor endowment (L)	1

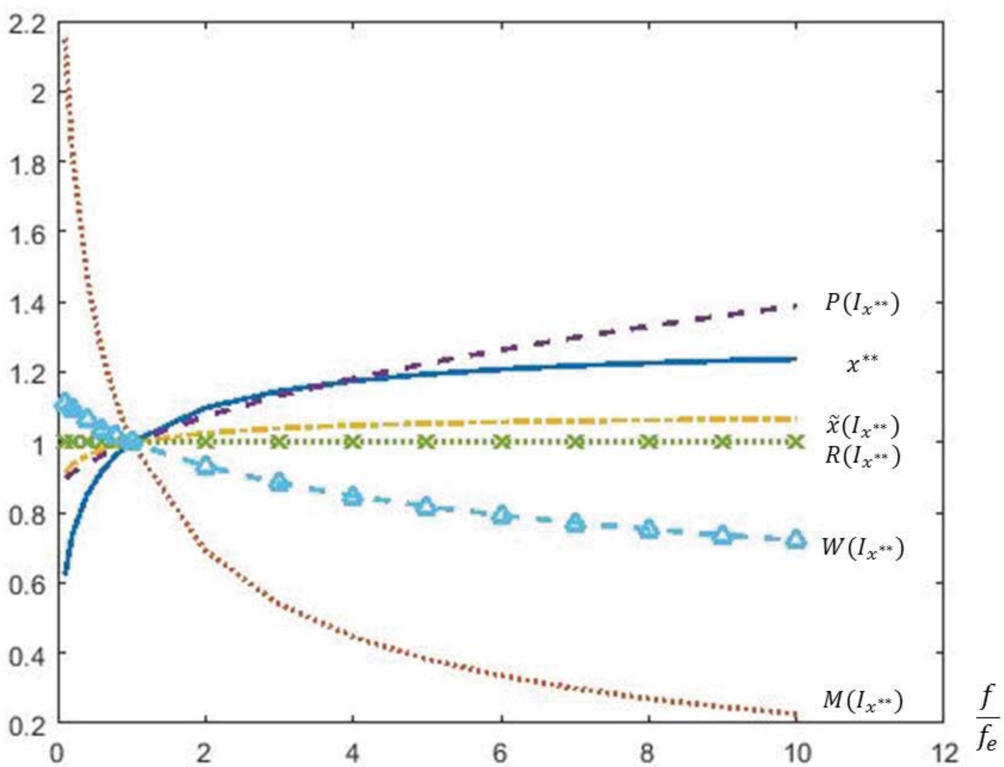


Figure 3.8. The effects of $\frac{f}{f_e}$

From figure 3.8, it can be seen that when the fixed production cost f increases, the equilibrium switching point x^{**} increases and the expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases. The increase in x^{**} is caused by the decrease of the expected profits since f increases. Although the expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases, the expected aggregate price $P\left(I_{x^{**}}\right)$ increases since the positive impact of mass of existing firms $M\left(I_{x^{**}},\theta\right)$ (for a given θ) on $P\left(I_{x^{**}}\right)$ dominates the negative impact of $\widetilde{x}\left(I_{x^{**}}\right)$ on $P\left(I_{x^{**}}\right)$. As a result, the corresponding expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases as fixed producing cost f increases.

3.6 Conclusion

This chapter discusses firms' entry and exit decision in a global game with incomplete information. Firms entry/exit choices are strategic substitutes. It is proved that a monotonic pure strategy equilibrium exists and is unique. Specifically, there is a switching productivity level x^{**} such that firms stay in the market if their productivity levels are greater than x^{**}, and firms leave the market if their productivity levels are less than x^{**}.

Comparative statics exercises show that when the precision of public information increases, the equilibrium switching productivity level increases, and consequently the expected average productivity increases. The upshot is that more precise information about the mean productivity leads to inter-firm reallocations toward more productive firms. Finally, welfare per worker increases as the precision of public information is improved.

In this chapter, it is assumed that productivity is (conditional) uniformly distributed given the realization of mean productivity. Simple uniform distribution assumption facilitates analytical proof of the existence of a unique monotonic pure strategy equilibrium. Parameter assumptions 3.1 through 3.4 are specific to this uniform distribution case and they guarantee the existence of a unique equilibrium. To understand how distribution shapes can affect the existence of monotonic pure strategy equilibrium, examples with more general distributions need to be examined. The numerical examples with firms' productivities drawn from conditional normal distribution, conditional gamma distribution, and conditional exponential distribution are discussed in chapter 4. The conclusions that (1) unique monotonic pure strategy equilibrium exists, and (2) increasing precision of public information reallocates resources to more productive firms and thus increases welfare are validated in chapter 4. Chapter 3 and chapter 4 only consider monotonic pure strategy equilibria. In future, they can be extended to include for non-monotonic pure strategy equilibrium and mixed strategy equilibrium, as in Karp, Lee, and Mason (2007).

CHAPTER 4. FIRM ENTRY AND EXIT IN A GLOBAL GAME WITH STRATEGIC SUBSTITUES: EQUILIBRIA UNDER NON-UNIFORM DISTRIBUTIONS

4.1 Introduction

This chapter extends the study undertaken in chapter 3 on firm's entry and exit decision in a global game with incomplete information. It studies whether and how the shapes of conditional productivity distributions affect the equilibria studied in chapter 3. In this chapter, the mean productivity Θ is drawn from a truncated normal distribution. Truncated distribution for Θ is used for computational convenience. Given mean productivity $\Theta = \theta$, firm's productivity level $X|\Theta = \theta$ is drawn from more general (truncated) distributions, such as truncated normal, truncated gamma, and truncated exponential distributions. Truncated distributions for $X|\Theta = \theta$ are chosen to guarantee that productivity levels are always positive. As in chapter 3, only monotonic pure strategy equilibria are discussed. Since analytical proof of the existence of a unique equilibrium is not easy to derive for cases with non-uniform distributions, the existence and the uniqueness is established numerically. Using numerical plots, it is shown that there exists only one solution x^{**} to $u(x^*, I_{x^*}) = 0$ and that $u(x_i, I_{x^{**}}) \gtreqless 0$, iff $x_i \gtreqless x^{**}$.

Chapter 4 validates conclusions derived in chapter 3. Specifically, these are: (1) a monotonic pure strategy equilibrium exists and it is unique; (2) as public information becomes more precise, the aggregate productivity as well as economy's welfare increases.

However, there are some differences as well. Unlike the result in chapter 3, when private information becomes more precise, the aggregate productivity and welfare are not always decreasing when $X|\Theta = \theta$ is drawn from truncated gamma distribution.

Since the model examined below is the same as the one in chapter 3, the details are skipped. Essentials of the model and equilibrium definitions are introduced briefly in Section 4.2 and 4.3. In section 4.4, the numerical results for various productivity distributions are presented, and section 4.5 provides comparative statics. Section 4.6 presents some concluding remarks.

4.2 The Model

Consumers' demand, firms' supply, and their aggregation are the same as in chapter 3, which relies on monopolistic firms with different productivities competing in the market while facing a constant elasticity of product demand.

4.2.1 The Timeline

In the first stage, firms pay a fixed entry costs f_e to enter the market with knowledge of the distribution of mean productivity Θ but not the true θ. Θ is drawn from a truncated normal distribution $N\left(\theta; \mu_\theta, \sigma_\theta^2, \underline{\theta}, \overline{\theta}\right)$ restricting the original normal distribution $N\left(\theta; \mu_\theta, \sigma_\theta^2\right)$ to a closed domain of $\left[\underline{\theta}, \overline{\theta}\right]$. In the second stage, Θ is realized as θ and the firm receives its private productivity $X = x_i|\Theta = \theta$. The mean productivity θ is unknown to the public and $X|\Theta = \theta$ is drawn from a distribution $F\left(x; \theta, a, b, \lambda_1, \lambda_2, ...\right)$ given θ. Where $F\left(x; \theta, a, b, \lambda_1, \lambda_2, ...\right)$ is a truncated distribution of X from an original distribution $F\left(x; \theta, \lambda_1, \lambda_2, ...\right)$ on the closed domain of $[a, b]$. The probability density function (pdf) of the original distribution is denoted as $f\left(x; \theta, \lambda_1, \lambda_2, ...\right)$. Here $\lambda_1, \lambda_2, ...$ are distribution parameters other than the mean θ. The support of $F\left(x; \theta, \lambda_1, \lambda_2, ...\right)$ covers the interval $[a, b]$ to whatever the value of θ is. After observing their productivities x, firms decide whether to stay in the market and pay the fixed production cost f, or exit the market. In the last stage, an existing firm pays variable cost

$\frac{q}{x}$, sets price at $p\left(x\right)$ and then sells outputs in the market. To ensure positive productivities, it is assumed that $a \geq 0$.

Further, the equilibria focus on $X|\Theta = \theta$ drawn from non-uniform distributions. A normal distribution relates to a case in which productivities are drawn from a symmetric distribution, while exponential distribution covers a case when high productivity levels are less likely than low productivity levels. The gamma distribution presents the case where the shape of productivity distribution is between normal and exponential.

4.3 Equilibrium

In equilibrium, consumer maximizes utility and firm maximizes profits. The labor market clears and a free entry condition at the first stage is satisfied. Since only monotonic pure strategy equilibrium is considered, all the following expressions relate to such equilibria.

With threshold strategy I_{x^*}, firm i's expected payoff from staying is

$$u(x_i, I_{x^*}) = \int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x_i}{\widetilde{x}\left(I_{x^*}, \theta\right)} \right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^*}, \theta\right) + f\right) dJ_{\Theta|X_i=x_i}(\theta) - f \tag{4.1}$$

where x_i is the productivity level for firm i, $\widetilde{x}\left(I_{x^*}, \theta\right)$ is the average productivity given threshold strategy I_{x^*} and mean productivity θ, $\overline{\pi}\left(I_{x^*}, \theta\right)$ is the average profit given threshold strategy I_{x^*} and mean productivity θ, and f is the fixed production cost. $J_{\Theta|X_i=x_i}(\theta)$ is the conditional distribution of Θ by firm i (with productivity level x_i). Equations for each variable are shown below.

- The average productivity is

$$\widetilde{x}\left(I_{x^*}, \theta\right) = \left(\int_{x^*}^{b} \left(x_i\right)^{\sigma-1} \mu(x_i, I_{x^*}, \theta) dx_i \right)^{\frac{1}{\sigma-1}}$$

where the density function of the productivity distribution given I_{x^*} and θ is

$$\mu(x_i, I_{x^*}, \theta) = \frac{f(x_i; \theta, \lambda_1, \lambda_2, ...)}{F(b; \theta, \lambda_1, \lambda_2, ...) - F(x^*; \theta, \lambda_1, \lambda_2, ...)}$$

for $a \leq x_i \leq b$ and $\mu(x_i, I_{x^*}, \theta) = 0$ otherwise.As a result,

$$\widetilde{x}(I_{x^*}, \theta) = \left(\int_{x^*}^{b} (x_i)^{\sigma-1} \frac{f(x_i; \theta, \lambda_1, \lambda_2, ...)}{F(b; \theta, \lambda_1, \lambda_2, ...) - F(x^*; \theta, \lambda_1, \lambda_2, ...)} dx_i \right)^{\frac{1}{\sigma-1}}$$

- From the labor market condition and free entry condition, the average profit is

$$\overline{\pi}(I_{x^*}, \theta) = \frac{\sigma f}{\sigma - 1} \left(\frac{P^e_{stay}(I_{x^*})}{P_{stay}(I_{x^*}, \theta)} - 1 \right) + \frac{f_e}{P_{stay}(I_{x^*}, \theta)}$$

where $P_{stay}(I_{x^*}, \theta)$ is the probability of staying given θ and the strategy profile I_{x^*}

$$\begin{aligned} P_{stay}(I_{x^*}, \theta) &= \int_{x^*}^{b} f(x_i; \theta, a, b, \lambda_1, \lambda_2, ...) \, dx_i \\ &= \int_{x^*}^{b} \frac{f(x_i; \theta, \lambda_1, \lambda_2, ...)}{F(b; \theta, \lambda_1, \lambda_2, ...) - F(a; \theta, \lambda_1, \lambda_2, ...)} dx_i \end{aligned}$$

and $P^e_{stay}(I_{x^*})$ is the *ex-ante* probability of staying before entry

$$\begin{aligned} P^e_{stay}(I_{x^*}) &= E_{\Theta}(P_{stay}(I_{x^*}, \theta)) \\ &= \int_{\underline{\theta}}^{\overline{\theta}} P_{stay}(I_{x^*}, \theta) \frac{f_N(\theta; \mu_\theta, \sigma_\theta^2)}{F_N(\overline{\theta}; \mu_\theta, \sigma_\theta^2) - F_N(\underline{\theta}; \mu_\theta, \sigma_\theta^2)} d\theta \end{aligned}$$

where $f_N(\theta; \mu_\theta, \sigma_\theta^2)$ is the pdf of $N(\theta; \mu_\theta, \sigma_\theta^2)$ and $F_N(\overline{\theta}; \mu_\theta, \sigma_\theta^2)$ is the cdf.

- Consider the posterior distribution $dJ_{\Theta|X_i=x_i}(\theta)$. By Bayes theorem, the probability density function of Θ when $X_i = x_i$ is

$$f_{\Theta|X_i=x_i}(\theta) = \frac{f_{X_i|\Theta=\theta}(x_i)\, f_\Theta(\Theta=\theta)}{\int_{-\infty}^{\infty} f_{X_i|\Theta=\theta}(x_i)\, f_\Theta(\Theta=\theta)\, d\theta}$$

$$= \frac{\frac{f(x_i;\theta,\lambda_1,\lambda_2,\ldots)}{F(b;\theta,\lambda_1,\lambda_2,\ldots)-F(a;\theta,\lambda_1,\lambda_2,\ldots)} \frac{f_N\left(\theta;\mu_\theta,\sigma_\theta^2\right)}{F_N\left(\overline{\theta};\mu_\theta,\sigma_\theta^2\right)-F_N\left(\underline{\theta};\mu_\theta,\sigma_\theta^2\right)}}{\int_{\underline{\theta}}^{\overline{\theta}} \frac{f(x_i;\theta,\lambda_1,\lambda_2,\ldots)}{F(b;\theta,\lambda_1,\lambda_2,\ldots)-F(a;\theta,\lambda_1,\lambda_2,\ldots)} \frac{f_N\left(\theta;\mu_\theta,\sigma_\theta^2\right)}{F_N\left(\overline{\theta};\mu_\theta,\sigma_\theta^2\right)-F_N\left(\underline{\theta};\mu_\theta,\sigma_\theta^2\right)} d\theta}$$

for $\underline{\theta} \leq \theta \leq \overline{\theta}$, and $f_{\Theta|X_i=x_i}(\theta) = 0$, otherwise.

If it can be shown that $u(x^*, I_{x^*}) = 0$ has a unique solution and $u(x_i, I_{x^*}) \lesseqqgtr 0$ when $x_i \lesseqqgtr x^*$, the existence of a unique monotonic pure strategy equilibrium is proved. The proof is obtained numerically. The equilibrium payoff graph is displayed in the next section. Once the equilibrium value of the switching point x^* is obtained, the equilibrium values of all other variables can be easily derived using the above equations.

4.4 Numerical Examples

This section uses three examples (one each for normal, gamma and exponential distributions) to show the existence of a unique monotonic pure strategy equilibrium. Table 4.1 exhibits parameter values. Since gamma distribution is defined on positive support only, the mean of gamma distribution is positive so the lower bound of the mean, $\underline{\theta}$ is set to be $0.01 (> 0)$ instead of 0. For the exponential distribution case, the lower bound of the mean productivity θ, $\underline{\theta}$ is set to be 1 to let the monotonic pure strategy equilibrium exist. Moreover, since exponential distribution only has one parameter, there is nothing such as an exogenous variance parameter for $X|\Theta = \theta$ when $X|\Theta = \theta$ is drawn from an exponential distribution.

Table 4.1. The model parameterizations

	Value		
Parameters	Normal	Gamma	Exponential
The elasticity of substitution (σ)	3	3	3
Lower bound of the mean productivity θ ($\underline{\theta}$)	0	0.01	1
Higher bound of the mean productivity θ ($\bar{\theta}$)	5	4.99	4
Mean of the mean productivity θ (μ_θ)	2.5	2.5	2.5
Variance of the mean productivity θ (σ_θ^2)	16	16	16
Lower bound of productivity x (a)	0	0	0
Higher bound of productivity x (b)	5	5	5
Variance of productivity x (σ_x^2)	1	1	N/A
The entry costs (f_e)	0.1	0.1	0.1
The fixed producing costs (f)	1	1	1
The labor endowment (L)	1	1	1

Figure 4.1 shows the graph of $u(x^*, I_{x^*})$ and $u(x, I_{x^{**}})$ for the three cases: normal, gamma, and exponential productivity. From the graph, it can be seen that $u(x^*, I_{x^*})$ is strictly increasing and has a unique solution x^{**} for $u(x^*, I_{x^*}) = 0$. Moreover, $u(x, I_{x^{**}}) \gtreqless 0$, when $x \gtreqless x^{**}$. As a result, the monotonic pure strategy equilibrium exists and is unique. $x^{**} = 1.555$ for the normal distribution case, $x^{**} = 1.875$ for gamma distribution case, and $x^{**} = 2.894$ for exponential distribution case. Note that $u(x, I_{x^{**}})$ is not an increasing function with x.

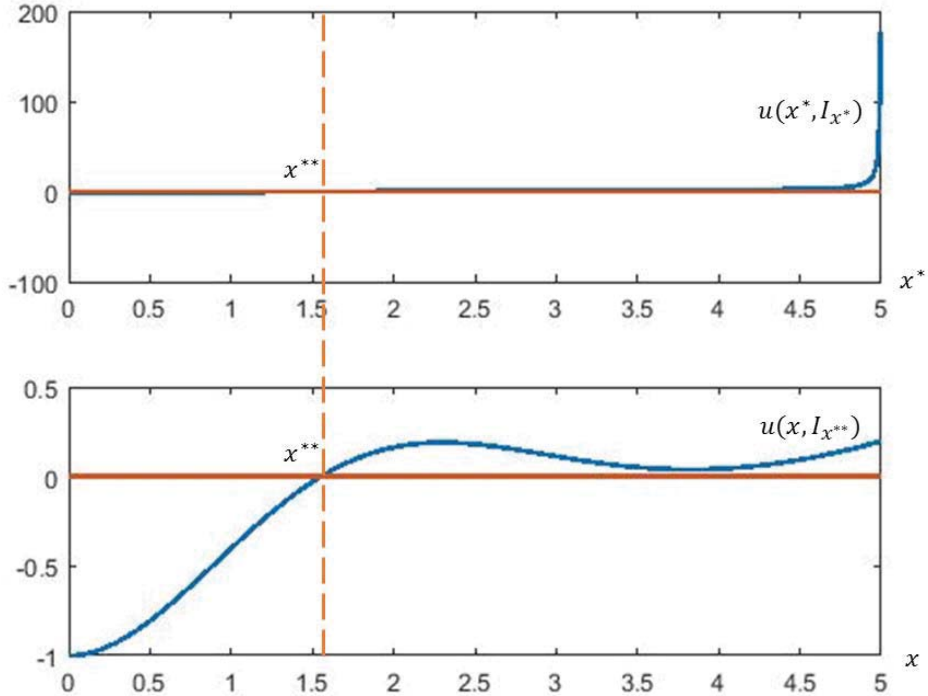


Figure 4.1.a. $u(x^*, I_{x^*})$ and $u(x, I_{x^{**}})$ for normal distribution

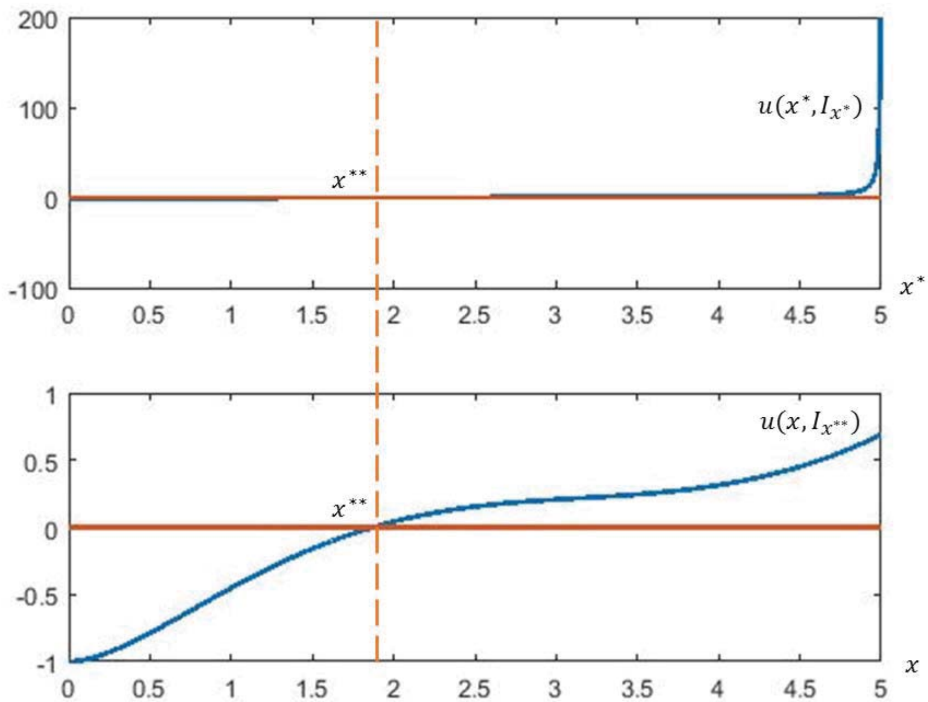


Figure 4.1.b. $u(x^*, I_{x^*})$ and $u(x, I_{x^{**}})$ for gamma distribution

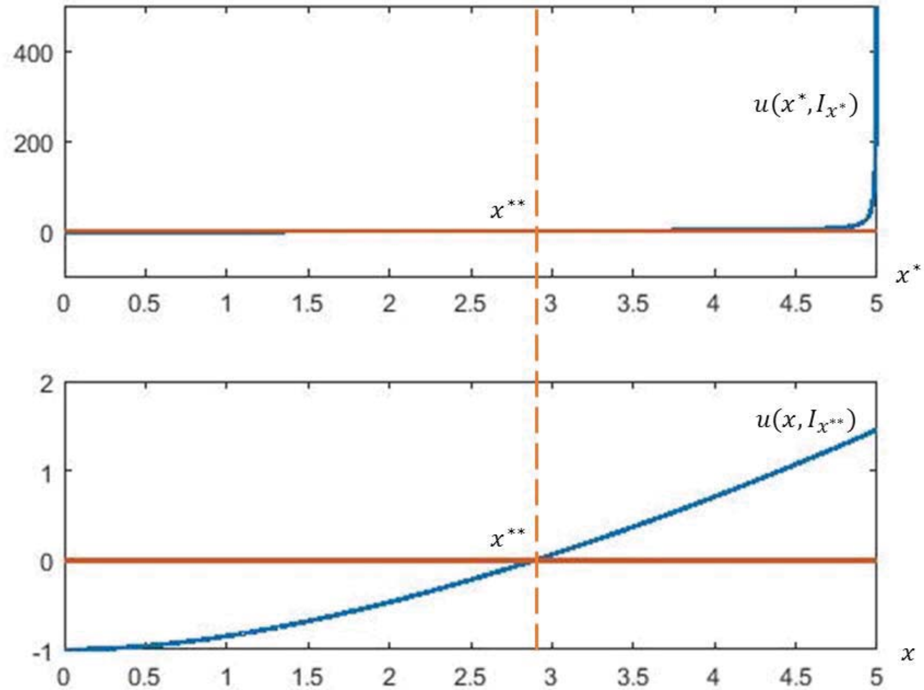


Figure 4.1.c. $u(x^*, I_{x^*})$ and $u(x, I_{x^{**}})$ for exponential distribution

4.5 Comparative Statics

This section performs comparative statics. As in chapter 3, it examines how the changes in the following parameters affect the economy: changes in the precision of public information $1/\sigma_\theta^2$, changes in the precision of private information $1/\sigma_x^2$, changes in the elasticity of substitution σ, and changes in the fixed producing costs f. In all exercises, a unique monotonic pure strategy equilibrium is first shown to exist. The numerical results for proving the unique existence of equilibrium are not included in the text. Apart from the equilibrium switching point x^{**} and the expected welfare per worker $W\left(I_{x^{**}}\right)$, the expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$, the expected mass of existing firms $M\left(I_{x^{**}}\right)$, the expected aggregate price $P\left(I_{x^{**}}\right)$ and the expected aggregate revenue $R\left(I_{x^{**}}\right)$ are also included in the graph to complete the analysis.

4.5.1 Changes in the Precision of Public Information

Table 4.2 shows parameter values for the exercise when the precision of Θ, $1/\sigma_\theta^2$ changes. σ_θ^2 varies from 0.04 to 16, and the corresponding precision $1/\sigma_\theta^2$ varies from 0.0625 to 25. In this range, the precision of public information Θ can be greater or smaller than the precision of private information $X|\Theta = \theta$, which is 1 for normal and gamma cases. Note that for the exponential distribution case, the precision of private information $1/\sigma_x^2 = 1/\theta$ varies as the mean productivity θ changes. The expectation of Θ remains unchanged and equals 2.5. When the precision of public information goes to infinity, the game with incomplete information degenerates to Melitz case.

Table 4.2. Parameterizations when precision of public information changes

	Value		
Parameters	Normal	Gamma	Exponential
The elasticity of substitution (σ)	3	3	3
Lower bound of the mean productivity θ ($\underline{\theta}$)	0	0.01	1
Higher bound of the mean productivity θ ($\overline{\theta}$)	5	4.99	4
Mean of the mean productivity θ (μ_θ)	2.5	2.5	2.5
Variance of the mean productivity θ (σ_θ^2)	$[0.04, 16]$	$[0.04, 16]$	$[0.04, 16]$
Lower bound of productivity x (a)	0	0	0
Higher bound of productivity x (b)	5	5	5
Variance of productivity x (σ_x^2)	1	1	N/A
The entry costs (f_e)	0.1	0.1	0.1
The fixed producing costs (f)	1	1	1
The labor endowment (L)	1	1	1

Figure 4.2 shows how the precision of public information affects the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$. The x-axis is the natural logarithm

of precision. The y-axis is the ratio between true variable values and values for Melitz case when $\overline{\theta} = \underline{\theta} = 2.5$.

First of all, as can be seen from figure 4.2, when the precision of public information increases, the equilibrium switching point x^{**} increases to the cutoff point for Melitz model. In addition, as precision of Θ increases, the expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases as result of increasing x^{**}. The mass of entry firm $M_e\left(I_{x^{**}}\right)$ increases while the ex-ante expected probability of stay $P^e\left(I_{x^{**}}\right)$ decreases. By the changes through $\widetilde{x}\left(I_{x^{**}}\right)$, $M_e\left(I_{x^{**}}\right)$ and $P^e\left(I_{x^{**}}\right)$, the expected aggregate price $P\left(I_{x^{**}}\right)$ decreases. And then the expected welfare per worker $W\left(I_{x^{**}}\right)$ increases due to the price drop. In sum, when the precision of public information increases, the equilibrium switching point x^{**}, the expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ and the expected welfare per worker $W\left(I_{x^{**}}\right)$ increase. This result holds for examples of all three distribution and replicates the results of the comparative statics related with the precision of public information in chapter 3.

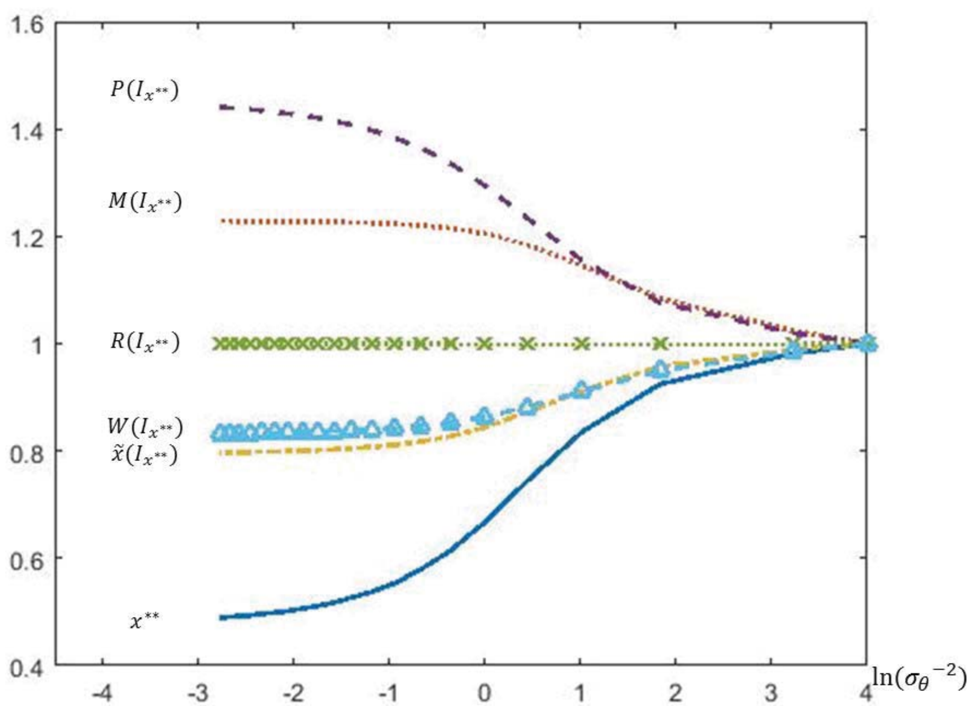


Figure 4.2.a. The effects of precision of public information (normal distribution)

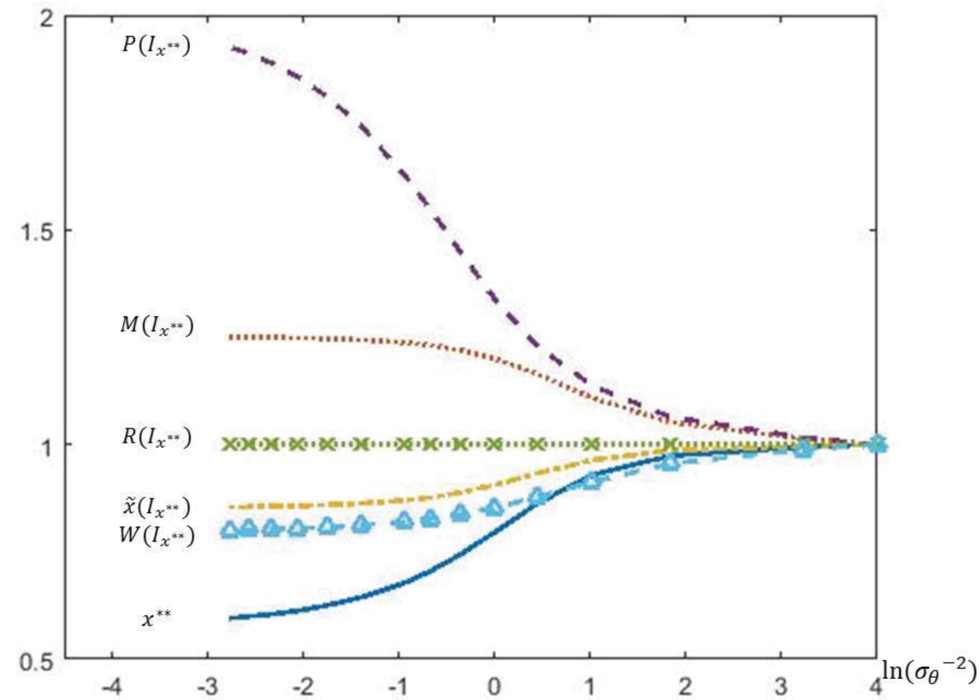


Figure 4.2.b. The effects of precision of public information (gamma distribution)

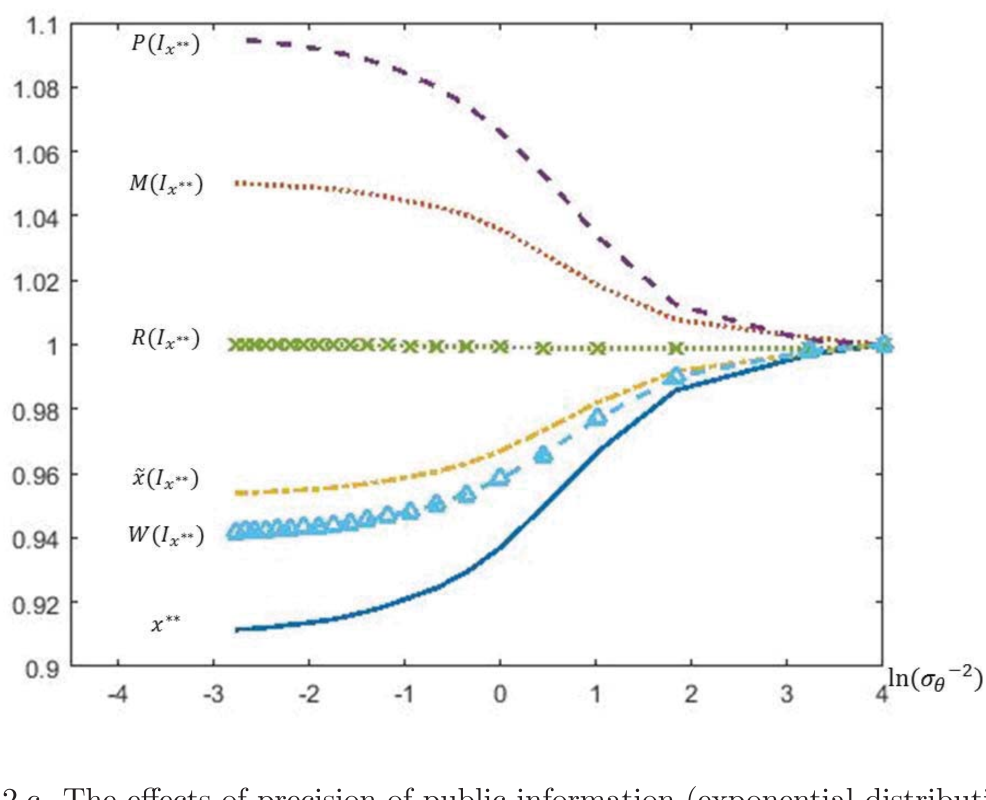


Figure 4.2.c. The effects of precision of public information (exponential distribution)

4.5.2 Changes in the Precision of Private Information

This part examines how the precision of private information affects the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$. Table 4.3 shows the parameter values. The variance of productivity x, σ_x^2 varies from 0.36 to 16, and the corresponding precision $1/\sigma_x^2$ varies from 0.0625 to 2.78. In this range, the precision of private information can be greater or smaller than the precision of public information $1/\sigma_\theta^2$, which is 1. When $\sigma_x^2 < 0.36$, a monotonic pure strategy equilibrium does not exist. Note that for exponential distribution case, the precision of private information $1/\sigma_x^2 = 1/\theta$ and varies as the mean productivity θ changes. As a result, exponential distribution case is excluded from the comparative statics analysis here.

Table 4.3. Parameterizations when precision of $X|\Theta = \theta$ changes

	Value	
Parameters	Normal	Gamma
The elasticity of substitution (σ)	3	3
Lower bound of the mean productivity θ ($\underline{\theta}$)	0	0.01
Higher bound of the mean productivity θ ($\overline{\theta}$)	5	4.99
Mean of the mean productivity θ (μ_θ)	2.5	2.5
Variance of the mean productivity θ (σ_θ^2)	1	1
Lower bound of productivity x (a)	0	0
Higher bound of productivity x (b)	5	5
Variance of productivity x (σ_x^2)	$[0.36, 16]$	$[0.36, 16]$
The entry costs (f_e)	0.1	0.1
The fixed producing costs (f)	1	1
The labor endowment (L)	1	1

Figure 4.3 shows how the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ changes with the change in precision of private information. The x-axis is

the natural logarithm of precision of private information. And the y-axis is the ratio between true variable values and values for standard case when $\sigma_x^2 = \sigma_\theta^2 = 1$, and then $\ln(1/\sigma_x^2) = 0$.

From figure 4.3, it can be seen that as the precision of private information increases, movements of the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ for the normal distribution case and the gamma distribution case are different. For the normal distribution case, i.e., when productivity distribution is symmetric, the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ decrease as the precision of private information increases. This is the same result as in chapter 3 when productivity distribution is uniform and thus is symmetric. However, for the gamma distribution case, the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ increase and then decrease as the precision of private information increases. This difference indicates that the shapes of productivity distribution do affect the comparative statics result on precision of private information. Note that as in chapter 3, the welfare per worker and the equilibrium switching point move in the same direction: they either increase or decrease at the same time.

Figure 4.4 shows how the precision of private information affects the equilibrium cutoff point x^{**} and welfare per worker $W(I_{x^{**}})$ for Melitz case when $\overline{\theta} = \underline{\theta} = \theta$. For this exercise, $\theta = 2.5$ and all other parameters are the same as in table 4.3. The x-axis is the precision of private information, and the y-axis is the ratio between true variable values and values for the case when $\sigma_x^2 = 1$.

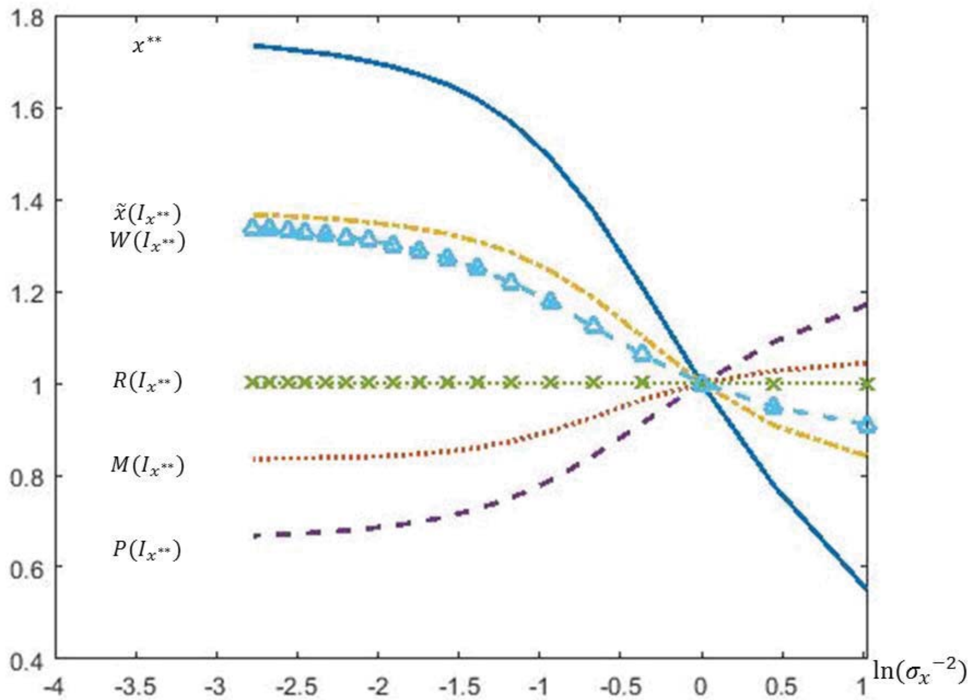


Figure 4.3.a. The effects of precision of private information (normal distribution)

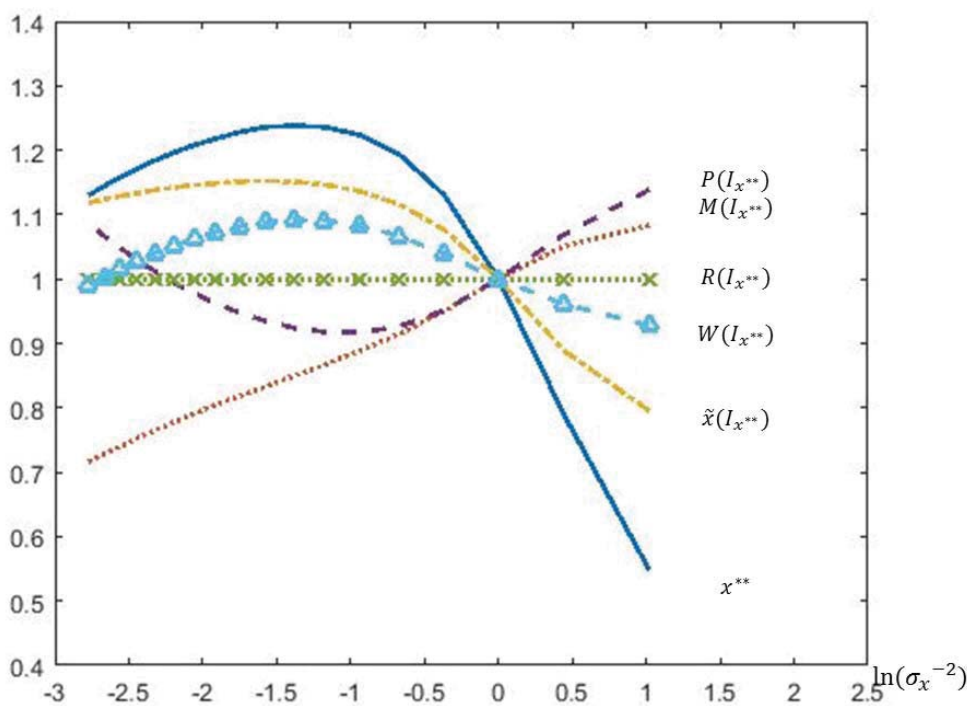


Figure 4.3.b. The effects of precision of private information (gamma distribution)

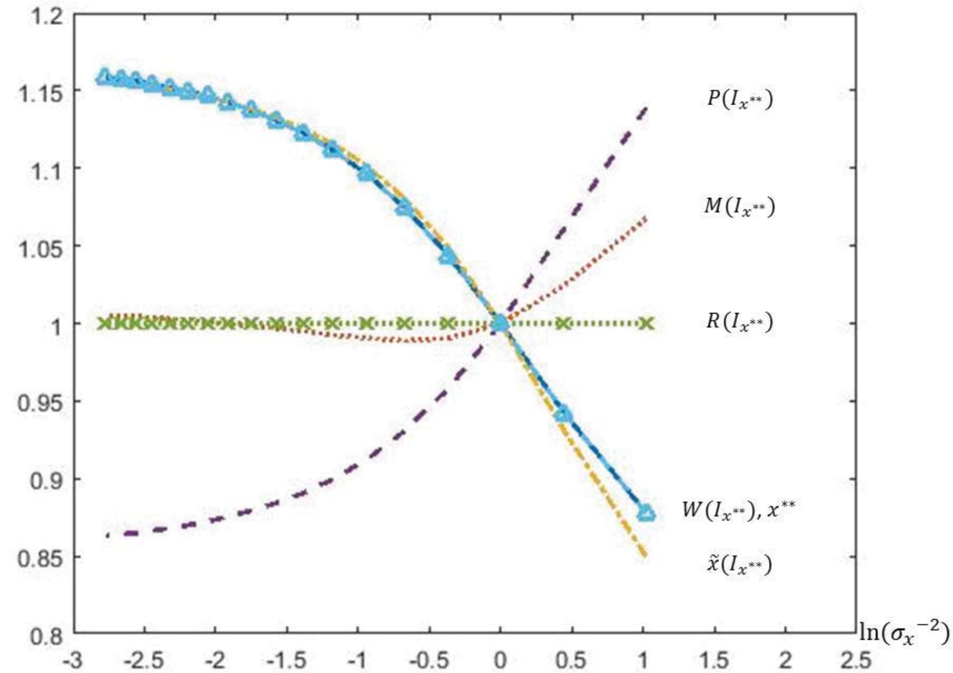


Figure 4.4.a. The effects of precision of private information for Melitz case (normal distribution)

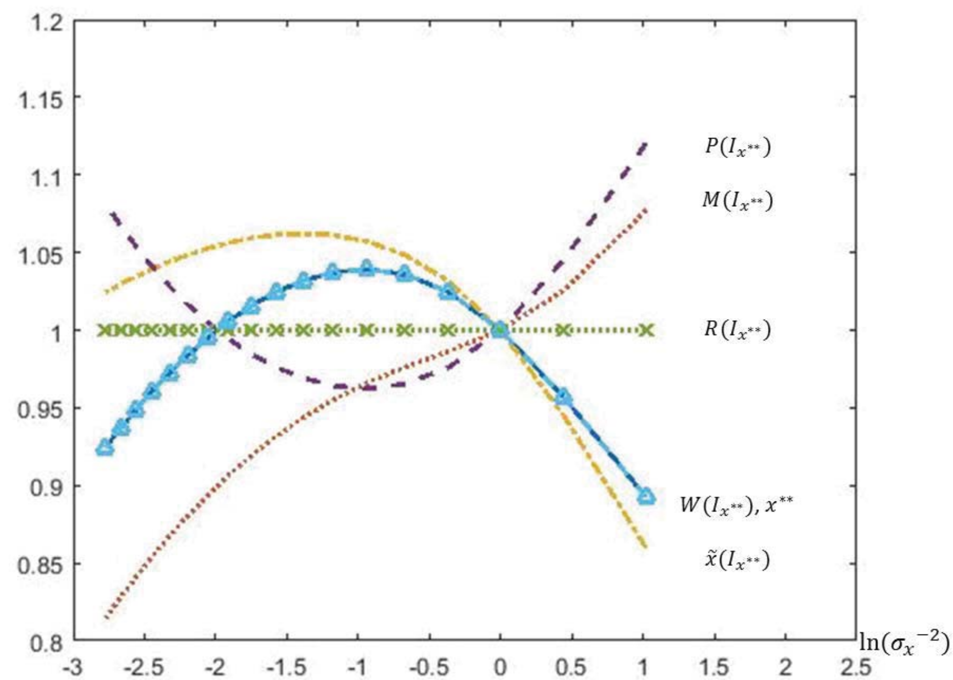


Figure 4.4.b. The effects of precision of private information for Melitz case (gamma distribution)

From figure 4.4, it can be seen that as the precision of private information increases for the normal distribution case, the equilibrium cutoff point x^{**} decreases. For the gamma distribution case, the equilibrium cutoff point x^{**} increases and then decreases. There are differences of the effects of changing precision of private information under symmetric distribution vis-à-vis skewed distribution, as shown in figure 4.3. Moreover, the standardized welfare curve and the standardized cutoff productivity curve coincide with each other as shown in chapter 3. Note that when the productivity is drawn from a gamma distribution, although in some range, increasing private information precision will increase equilibrium switching point and increase welfare, increasing the precision of public information will always increase welfare irrespective of the precision of private information. The following exercise considers $\sigma_x^2 = 16$ (precision $= -2.77$), and for a small neighborhood around this private precision level, increasing the private precision can increase the welfare as σ_θ^2 changes from 0 to 7. The x-axis is the precision of public information and the y-axis the ratio between equilibrium values and the value for the case when $\sigma_\theta^2 = 0$.

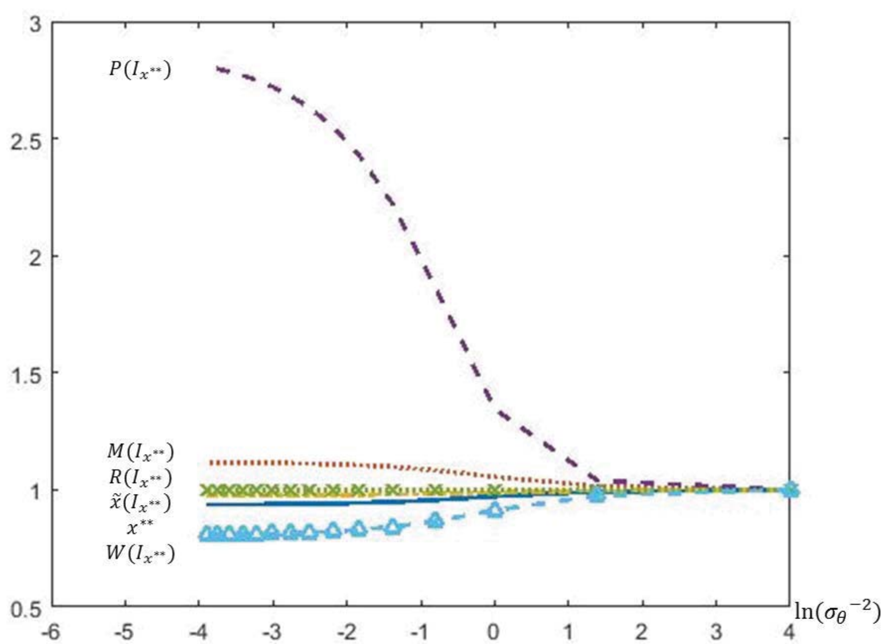


Figure 4.5. The effects of precision of public information when $\sigma_x^2 = 16$ (gamma distribution)

4.5.3 Changes in the Elasticity of Substitutions

Table 4.4 shows parameter setting for exercise when the elasticity of substitution σ changes. The elasticity of substitution σ varies from 2 to 6 for all three distribution cases. When $\sigma < 2$, a monotonic pure strategy equilibrium does not exist.

Figure 4.6 shows how the equilibrium switching point x^{**} and the expected welfare per worker $W(I_{x^{**}})$ changes with the changes in σ. The x-axis is the elasticity of substitution σ, while the y-axis is the ratio between true variable values and values for the case when $\sigma = 2$.

Table 4.4. Parameterizations when σ changes

	Value		
Parameters	Normal	Gamma	Exponential
The elasticity of substitution (σ)	$[2, 6]$	$[2, 6]$	$[2, 6]$
Lower bound of the mean productivity θ ($\underline{\theta}$)	0	0.01	1
Higher bound of the mean productivity θ ($\overline{\theta}$)	5	4.99	4
Mean of the mean productivity θ (μ_θ)	2.5	2.5	2.5
Variance of the mean productivity θ (σ_θ^2)	0.5	0.5	0.5
Lower bound of productivity x (a)	0	0	0
Higher bound of productivity x (b)	5	5	5
Variance of productivity x (σ_x^2)	1	1	N/A
The entry costs (f_e)	0.1	0.1	0.1
The fixed producing costs (f)	1	1	1
The labor endowment (L)	1	1	1

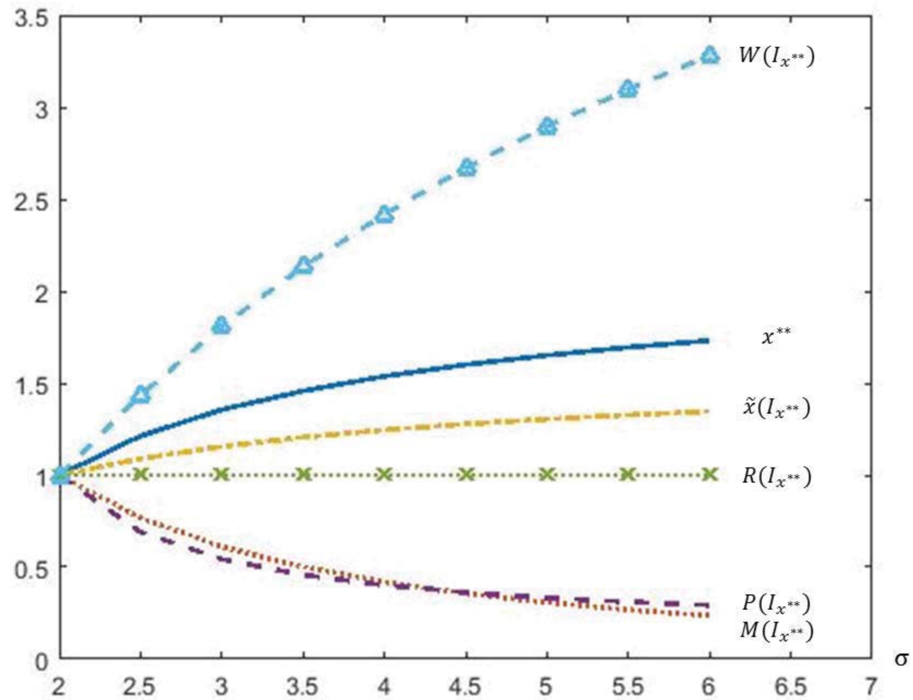


Figure 4.6.a. The effects of the elasticity of substitutions (normal distribution)

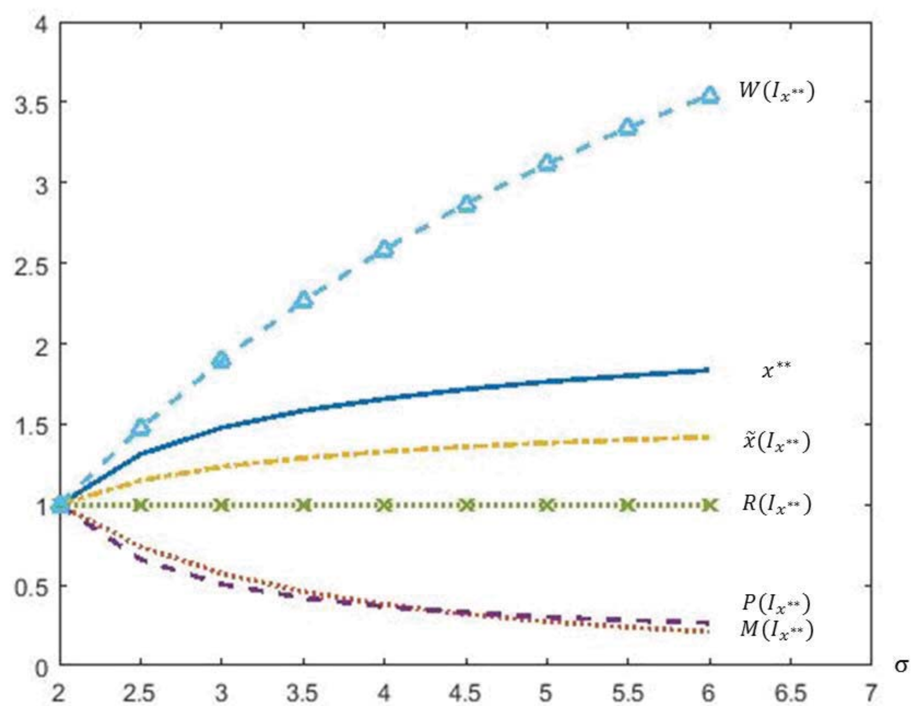


Figure 4.6.b. The effects of the elasticity of substitutions (gamma distribution)

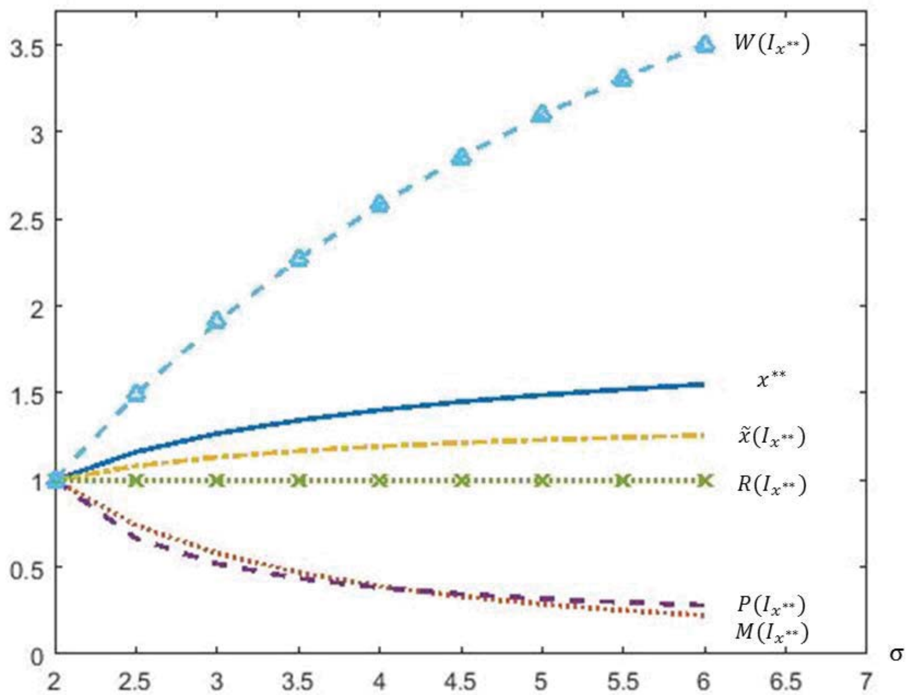


Figure 4.6.c. The effects of the elasticity of substitutions (exponential distribution)

As can be seen from figure 4.6, when the elasticity of substitution σ increases, the equilibrium switching point x^{**} increases since with higher level of σ, firms with lower level of productivity lose revenues further and therefore exit the market. The impact of elasticity of substitution σ on the equilibrium switching point x^{**} is the same for uniform distribution case in chapter 3 and the non-uniform distribution cases here in chapter 4. When the equilibrium switching point x^{**} increases, the corresponding expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases as a result. At the same time, the expected mass of existing firm $M\left(I_{x^{**}}\right)\left(= M_e\left(I_{x^{**}}\right)P^e\left(I_{x^{**}}\right)\right)$ decreases. Then the welfare per worker changes due to two effects: the positive effect is caused by a decrease in the expected aggregate price $P\left(I_{x^{**}}\right)$ as expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases, and the negative effect is caused by a decreasing of expected mass of existing firm which offers less variety to consumer. When the postive effect dominates the negative effect, welfare per worker increases, as can be seen for non-uniform distribution cases for $\sigma \in [2, 6]$ in chapter 4. However, in chapter 3, the welfare per work decreases and then increases as the negative effect is stronger when the elasticity of substitution is low.

4.5.4 Changes in Fixed Production Costs

Parameter values for the exercise when the fixed producing costs f changes is shown in table 4.5. The fixed production cost f varies from 0.01 to 1 and thus the ratio between the fixed production cost f and the entry cost f_e changes from 0.1 to 10. Figure 4.7 shows how the change in fixed production cost f affects the equilibrium switching point x^{**} and the expected welfare per worker $W\left(I_{x^{**}}\right)$. Since the equilibrium switching point x^{**} is only impacted by the ratio between f and f_e, no comparative statics analysis is conducted for changes in entry costs f_e and the x-axis for figure 6 is set to be $\frac{f}{f_e}$ instead of f. The y-axis is the ratio between true variable values and those for the case when $f = f_e$.

Table 4.5. Parameterizations when f changes

	Value		
Parameters	Normal	Gamma	Exponential
The elasticity of substitution (σ)	3	3	3
Lower bound of the mean productivity θ ($\underline{\theta}$)	0	0.01	1
Higher bound of the mean productivity θ ($\overline{\theta}$)	5	4.99	4
Mean of the mean productivity θ (μ_θ)	2.5	2.5	2.5
Variance of the mean productivity θ (σ_θ^2)	0.5	0.5	0.5
Lower bound of productivity x (a)	0	0	0
Higher bound of productivity x (b)	5	5	5
Variance of productivity x (σ_x^2)	1	1	N/A
The entry costs (f_e)	0.1	0.1	0.1
The fixed producing costs (f)	$[0.01, 1]$	$[0.01, 1]$	$[0.01, 1]$
The labor endowment (L)	1	1	1

From figure 4.7 it can be seen that when the fixed producing cost f increases, the equilibrium switching point x^{**} increases and the expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases. The increasing of x^{**} is caused by the decrease of the expected profits since f increases. Although the expected average productivity $\widetilde{x}\left(I_{x^{**}}\right)$ increases, the expected aggregate price $P\left(I_{x^{**}}\right)$ increases since the impact of the expected mass of existing firm $M\left(I_{x^{**}}\right)$ on $P\left(I_{x^{**}}\right)$ dominates the impact of $\widetilde{x}\left(I_{x^{**}}\right)$ on $P\left(I_{x^{**}}\right)$. As a result, the corresponding expected welfare per worker $W\left(I_{x^{**}}\right)$ decreases as fixed producing costs f increases. The result for this exercise is the same as the one for chapter 3.

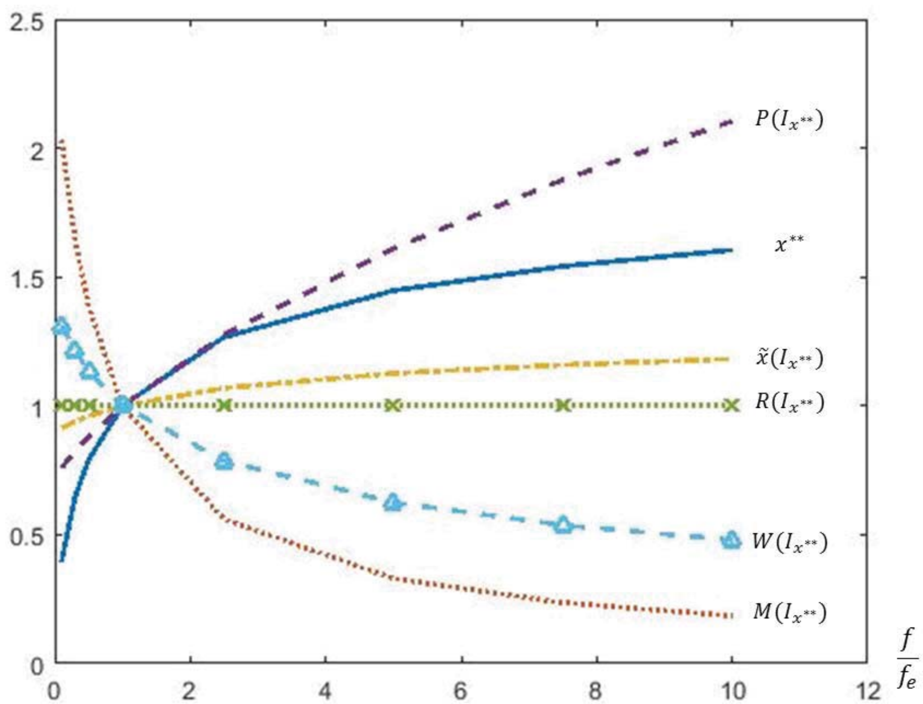


Figure 4.7.a. The effects of the fixed producing costs (normal distribution)

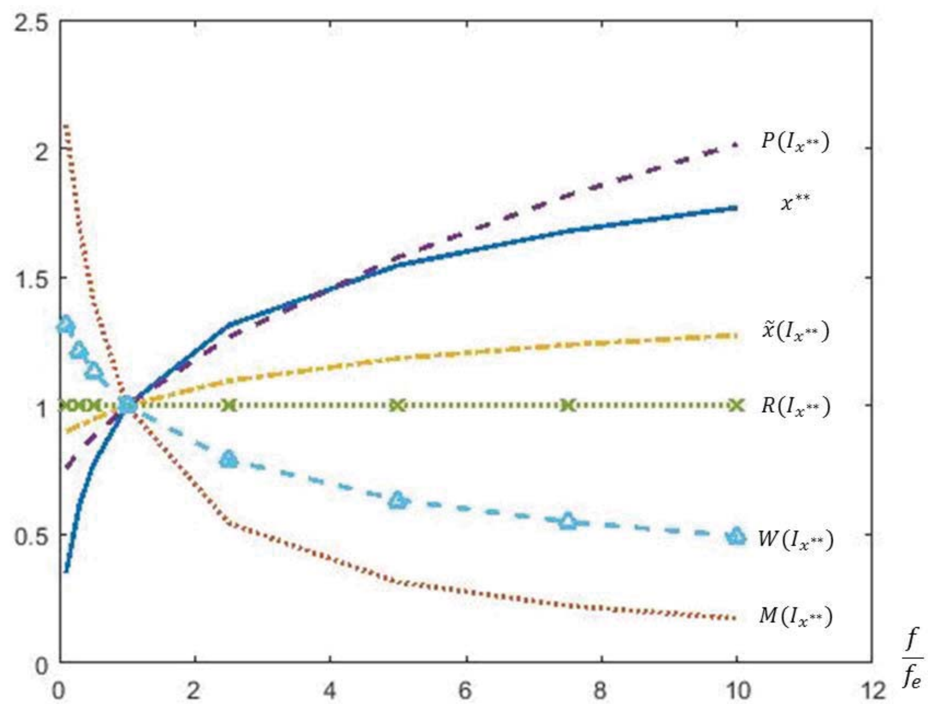


Figure 4.7.b. The effects of the fixed producing costs (gamma distribution)

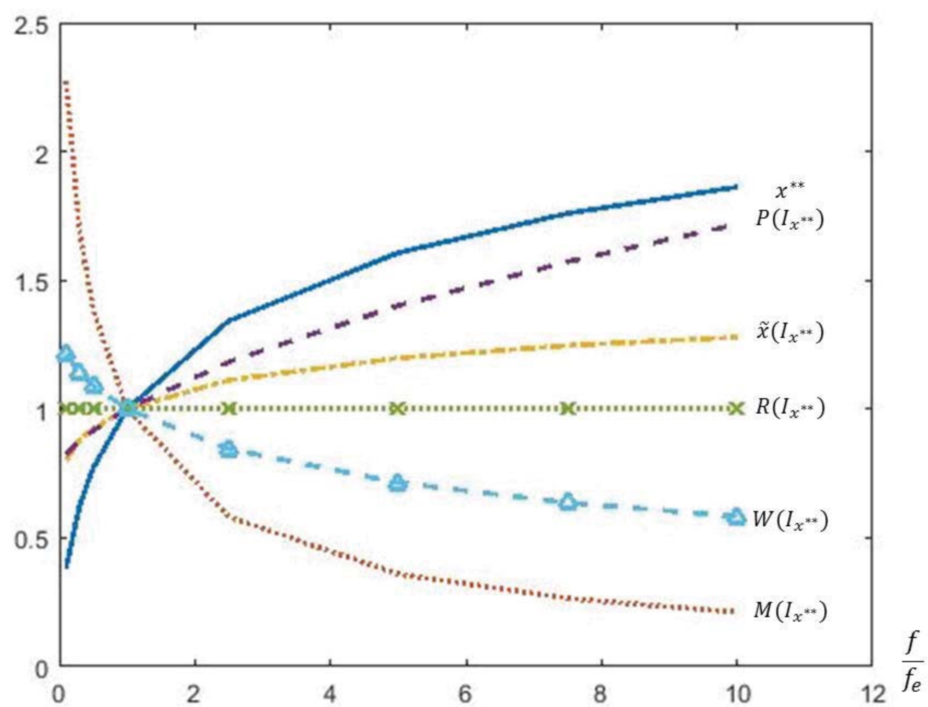


Figure 4.7.c. The effects of the fixed producing costs (exponential distribution)

4.6 Conclusion

This chapter generalizes the analysis undertaken in chapter 3 with more general distributions. It verifies that (1) a unique monotonic pure strategy equilibrium exists in a global game with incomplete information; (2) with more precise public information, the aggregate productivity and the welfare increase. However, some results depend on the shape of the distribution. In chapter 3, with more precise private information, the aggregate productivity and the welfare decrease monotonically. Here in chapter 4, when the conditional productivity is drawn from normal distribution, the same result prevails. However, when the conditional productivity is drawn from gamma distribution, which is skewed, the response of aggregate productivity and the welfare to an increase in the precision of private information is non-monotonic: It first increases and then begins to decrease. The precision of private information however has less to do with the information asymmetry and more to do with the ex-post dispersion of firms' productivity. The response of welfare and aggregate productivity with respect to the precision of productivity distribution essentially follows from Melitz's (2003) model with complete information.

In chapter 4, individual firm's productivity $X|\Theta = \theta$ is drawn from truncated normal, truncated gamma, and truncated exponential distributions. While in chapter 3, $X|\Theta = \theta$ is drawn from a uniform distribution. Since the monotonic pure strategy equilibrium unique exists for cases in chapter 3 and chapter 4, it can be concluded that the shapes of conditional productivity distributions does not preclude the existence of a unique monotonic pure strategy equilibrium. Moreover, because the aggregate productivity and the welfare increases with more precise public information for all cases in chapter 3 and chapter 4, a clear conclusion can be drawn that an increase in the precision of public information invariably makes the economy better off, and this result is independent of the dispersion of firms' productivities.

CHAPTER 5. CONCLUDING REMARKS

This dissertation discusses two topics: business cycles with asset fire sales and firm entry and exit dynamics in global games with incomplete information. For both topics, the key focus is to identify policies and/or information structure that can improve welfare. In the second chapter, a capital requirement policy can increase welfare in the long run, and in chapter 3 and chapter 4, more precise public information can increase both the aggregate productivity and the welfare per worker.

Since all the three chapters undertake theoretical exercises, corresponding models need to be empirically calibrated before any policy advice can be conclusively offered. In chapter 2, for example, whether there is over-borrowing from the second best point of view in the economy depends on the equilibrium types and thus without calibration one cannot conclude that the optimal capital control policy can always strictly increase welfare. It is possible that the equilibrium type is such that any binding capital control policy will reduce welfare and thus hurt the economy.

Chapter 4 generalizes the study in chapter 3 on firm entry and exit dynamics in global games with more complicated distributions. The assumption that mean productivity is drawn from a (truncated) normal distribution is more realistic due to the central limit theorem. Furthermore, whether the conditional productivity distribution when the mean is given is symmetric such as normal distribution, or skewed like gamma or exponential distribution, can only be examined by calibrating to the firm-level productivity data. If the theoretical models developed in this dissertation can be linked to empirical studies, the conclusions will be more valuable for policy advice. This is left for future research.

APPENDIX A. CHAPTER 2 APPENDIX

A.1 The Timeline for Generation t

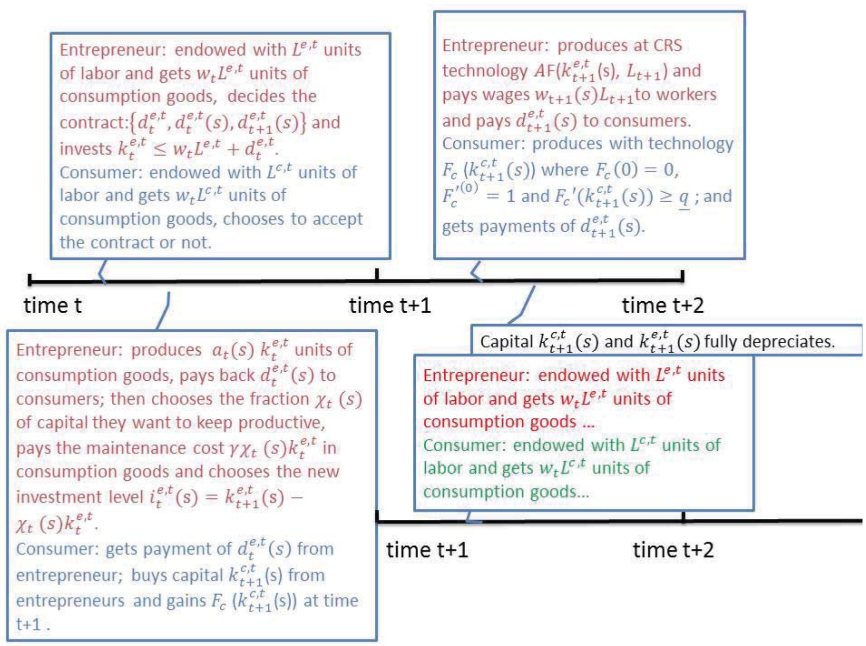


Figure A.1. The timeline for generation t

A.2 Equivalence of Multi-period Financial Contract and Single-period Contracts

Consider the entrepreneur's problem with two single period contracts $\langle z_t^{e,t}, z_t^{e,t}(s)\rangle$ and $\langle x_t^{e,t}(s), x_{t+1}^{e,t}(s)\rangle$. The contract $\langle z_t^{e,t}, z_t^{e,t}(s)\rangle$ specifies a loan $z_t^{e,t}$ from the consumer to the entrepreneur before entrepreneur's period t production, and then state-contingent payments $z_t^{e,t}(s)$ from the entrepreneur to the consumer after production. The contract $\langle x_t^{e,t}(s), x_{t+1}^{e,t}(s)\rangle$

characterizes a loan $x_t^{e,t}(s)$ from the consumer to the entrepreneur at the end of period t and repayments $x_{t+1}^{e,t}(s)$ from the entrepreneur to the consumer after entrepreneur's period $t+1$ production.

Given wage levels $\{w_t, w_{t+1}(s)\}$ and capital prices $\{q_t(s)\}$, the entrepreneur's individual problem is to maximize her expected utility $\sum \pi_s c_{t+1}^{e,t}(s)$ by choosing the first financial contract $\langle z_t^{e,t}, z_t^{e,t}(s)\rangle$ at the beginning of period t, the second financial contract $\langle x_t^{e,t}(s), x_{t+1}^{e,t}(s)\rangle$ at the end of period t, investment decisions $\langle k_t^{e,t}, \{\chi_t(s), k_{t+1}^{e,t}(s)\}\rangle$, labor demands $\{L_{t+1}^{e,t}(s)\}$ and consumption levels $\{c_{t+1}^{e,t}(s)\}$. As the case with multi-period contract, the entrepreneur faces four sets of constraints: the resource constraints, the consumer's participation constraints, the no-default constraints, and the capital requirement constraint. The capital requirement constraint here is exactly same as (2.11) by definition. The first three sets of constraints are introduced below.

The resource constraints are:

$$
\begin{aligned}
k_t^{e,t} &\leq w_t L^{e,t} + z_t^{e,t} &\text{(A.1)}\\
q_t(s)(k_{t+1}^{e,t}(s) - \chi_t(s)k_t^{e,t}) &\leq a_t(s)k_t^{e,t} - \gamma\chi_t(s)k_t^{e,t} - z_t^{e,t}(s) + x_t^{e,t}(s) &\text{(A.2)}\\
c_{t+1}^{e,t}(s) &\leq AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s)L_{t+1}^{e,t}(s) - x_{t+1}^{e,t}(s) &\text{(A.3)}
\end{aligned}
$$

Since the entrepreneur offers two contracts, she needs to make the consumer accept both of them. Consumer's expected utility when accepts the first contract $\langle z_t^{e,t}, z_t^{e,t}(s)\rangle$ without considering the second contract is

$$\sum \pi_s \left(w_t L^{c,t} - z_t^{e,t} + z_t^{e,t}(s) - q_t(s)k_{t+1}^{c,t}(s) + F_c(k_{t+1}^{c,t}(s))\right)$$

and the utility when the contract is rejected is

$$\sum \pi_s \left(w_t L^{c,t} - q_t(s)k_{t+1}^{c,t}(s) + F_c(k_{t+1}^{c,t}(s))\right)$$

Then, the consumer's participation constraint for the first contract is given by

$$z_t^{e,t} \leq \sum \pi_s z_t^{e,t}(s) \tag{A.4}$$

At the end of period t, because there is no uncertainty, the consumer accepts the second contract when

$$x_t^{e,t}(s) \leq x_{t+1}^{e,t}(s) \tag{A.5}$$

If she accepts the two contracts, the consumption for each period is

$$c_t^{c,t}(s) = w_t L^{c,t} - z_t^{e,t} + z_t^{e,t}(s) - x_t^{e,t}(s) - q_t(s) k_{t+1}^{c,t}(s) \tag{A.6}$$

$$c_{t+1}^{c,t}(s) = F_c(k_{t+1}^{c,t}(s)) + x_{t+1}^{e,t}(s) \tag{A.7}$$

Assume that $c_t^{c,t}(s) \geq 0$ and $c_{t+1}^{c,t}(s) \geq 0$ for $s = l, h$. The consumer's participation constraints are (A.4) and (A.5).

The no-default constraints for the entrepreneur are:

$$z_t^{e,t}(s) \leq (\theta a_t(s) + \max\{q_t(s) - \gamma, 0\})\, k_t^{e,t} \tag{A.8}$$

$$x_{t+1}^{e,t}(s) \leq \theta\left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s) L_{t+1}^{e,t}(s)\right) \tag{A.9}$$

and the no-default constraints for the consumer are:

$$z_t^{e,t}(s) \geq 0 \tag{A.10}$$

$$x_{t+1}^{e,t}(s) \geq 0 \tag{A.11}$$

Call individuals' problems (the entrepreneur's problem and the consumer's problem) with multi-period contract $\left\langle d_t^{e,t}, d_t^{e,t}(s), d_{t+1}^{e,t}(s)\right\rangle$ problem 1, and individuals' problems with two single period contracts $\left\langle z_t^{e,t}, z_t^{e,t}(s)\right\rangle$ and $\left\langle x_t^{e,t}(s), x_{t+1}^{e,t}(s)\right\rangle$ problem 2. Now, show the equivalence of problem 1 and problem 2. First consider the entrepreneur's problem in problem

2. Since the utility function is strictly increasing with $c^{e,t}_{t+1}(s)$ and the entrepreneur is the creator of contracts, by (A.5), $x^{e,t}_{t+1}(s) = x^{e,t}_{t}(s)$. Define the following variables

$$
\begin{aligned}
d^{e,t}_{t} &= z^{e,t}_{t} \\
d^{e,t}_{t+1}(s) &= x^{e,t}_{t+1}(s) \\
d^{e,t}_{t}(s) &= z^{e,t}_{t}(s) - x^{e,t}_{t}(s) = z^{e,t}_{t}(s) - x^{e,t}_{t+1}(s) = z^{e,t}_{t}(s) - d^{e,t}_{t+1}(s)
\end{aligned}
$$

Then the resource constraints (A.1)-(A.3) in problem 2 can be rewritten as the resource constraints (2.5)-(2.7) in problem 1, the consumer's participation constraints (A.4) and (A.5) can be summarized as one single constraint (2.10), and the no-default constraints (A.8)-(A.11) in problem 2 can be simplified as (2.1)-(2.4) in problem 1. Moreover, because the per-period consumption in problem 2 (equations (A.6) and (A.7)) can be written as the consumption in equations (2.8) and (2.9) in problem 1, the non-negative consumption conditions are the same for the two problems. In sum, with the same constraints and the same objective function, the entrepreneur's problem in problem 1 and problem 2 are identical. In addition, since the entrepreneur always makes the consumer accept contracts, the consumer's problem in problem 1 and problem 2 are identical as well. That is, individuals' problems are identical irrespective of the length of the contract.

A.3 Proof of Lemma 2.1

By (2.14) and strictly increasing of entrepreneur's utility function, the entrepreneur's problem can be rewritten as

$$
\max \sum\nolimits_s \pi_s \left(\begin{array}{c} AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) - w_{t+1}(s) L^{e,t}_{t+1}(s) \\ -\theta \left(AF(k^{e,t}_{t+1}(s), L^{e,t}_{t+1}(s)) - w_{t+1}(s) L^{e,t}_{t+1}(s) \right) \end{array} \right)
$$

such that

$$
\begin{aligned}
w_t L^{e,t} - \eta k_t^{e,t} &\geq 0 \\
w_t L^{e,t} + \pi_h b_t^{e,t}(h) k_t^{e,t} + \pi_l b_t^{e,t}(l) k_t^{e,t} - k_t^{e,t} &\geq 0 \\
\left(\begin{array}{c} a_t(s) k_t^{e,t} + q_t(s) k_t^{e,t} - \gamma k_t^{e,t} - b_t^{e,t}(s) k_t^{e,t} \\ +\theta \left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s) L_{t+1}^{e,t}(s) \right) - q_t(s) k_{t+1}^{e,t}(s) \end{array} \right) &\geq 0 \\
(\theta a_t(s) + q_t(s) - \gamma) - b_t^{e,t}(s) &\geq 0 \\
b_t^{e,t}(s) &\geq 0
\end{aligned}
$$

where

$$
b_t^{e,t}(s) = \frac{d_t^{e,t}(s) + d_{t+1}^{e,t}(s)}{k_t^{e,t}}
$$

is the net present value of promised repayments per unit of capital.

The constraint set is non-empty and compact, and then a solution exists. Moreover, the solution of the above problem is the solution of the original entrepreneur's problem.

$$
\begin{aligned}
L \;=\; & \pi_h \left(\begin{array}{c} AF(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h)) - w_{t+1}(h) L_{t+1}^{e,t}(h) \\ -\theta \left(AF(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h)) - w_{t+1}(h) L_{t+1}^{e,t}(h) \right) \end{array} \right) \\
& + \pi_l \left(\begin{array}{c} AF(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l)) - w_{t+1}(l) L_{t+1}^{e,t}(l) \\ -\theta \left(AF(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l)) - w_{t+1}(l) L_{t+1}^{e,t}(l) \right) \end{array} \right) \\
& + g \left(w_t L^{e,t} - \eta k_t^{e,t} \right) + z_0 \left(w_t L^{e,t} + \pi_h b_t^{e,t}(h) k_t^{e,t} + \pi_l b_t^{e,t}(l) k_t^{e,t} - \tau - k_t^{e,t} \right) \\
& + \pi_h z_{1h} \left(\begin{array}{c} a_h k_t^{e,t} + q_t(h) k_t^{e,t} - \gamma k_t^{e,t} - b_t^{e,t}(h) k_t^{e,t} \\ +\theta \left(AF(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h)) - w_{t+1}(h) L_{t+1}^{e,t}(h) \right) - q_t(h) k_{t+1}^{e,t}(h) \end{array} \right) \\
& + \pi_l z_{1l} \left(\begin{array}{c} a_l k_t^{e,t} + q_t(l) k_t^{e,t} - \gamma k_t^{e,t} - b_t^{e,t}(l) k_t^{e,t} \\ +\theta \left(AF(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l)) - w_{t+1}(l) L_{t+1}^{e,t}(l) \right) - q_t(l) k_{t+1}^{e,t}(l) \end{array} \right) \\
& + \lambda_1 \left((\theta a_h + q_t(h) - \gamma) - b_t^{e,t}(h) \right) + \lambda_2 b_t^{e,t}(h) \\
& + \lambda_3 \left((\theta a_l + q_t(l) - \gamma) - b_t^{e,t}(l) \right) + \lambda_4 b_t^{e,t}(l)
\end{aligned}
$$

Then, the first order conditions are:

$$\frac{\partial L}{\partial k_t^{e,t}} = \begin{pmatrix} -g\eta - z_0\left(1-\pi_h b_t^{e,t}(h) - \pi_l b_t^{e,t}(l)\right) \\ +\pi_h z_{1h}\left(a_h + q_t(h) - \gamma - b_t^{e,t}(h)\right) \\ +\pi_l z_{1l}\left(a_l + q_t(l) - \gamma - b_t^{e,t}(l)\right) \end{pmatrix} \leq 0 \tag{A.12}$$

$$\frac{\partial L}{\partial k_{t+1}^{e,t}(h)} = \begin{pmatrix} \pi_h\left(1-\theta\right) AF_1(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h)) \\ -\pi_h z_{1h}\left(q_t(h) - \theta F_1(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h))\right) \end{pmatrix} \leq 0 \tag{A.13}$$

$$\frac{\partial L}{\partial k_{t+1}^{e,t}(l)} = \begin{pmatrix} \pi_l\left(1-\theta\right) AF_1(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l)) \\ -\pi_l z_{1l}\left(q_t(l) - \theta F_1(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l))\right) \end{pmatrix} \leq 0 \tag{A.14}$$

$$\frac{\partial L}{\partial b_t^{e,t}(h)} = z_0\pi_h k_t^{e,t} - \pi_h z_{1h} k_t^{e,t} - \lambda_1 + \lambda_2 \leq 0 \tag{A.15}$$

$$\frac{\partial L}{\partial b_t^{e,t}(l)} = z_0\pi_l k_t^{e,t} - \pi_l z_{1l} k_t^{e,t} - \lambda_3 + \lambda_4 \leq 0 \tag{A.16}$$

$$\frac{\partial L}{\partial L_{t+1}^{e,t}(h)} = \pi_h\left(1-\theta\right)\left(AF_2(k_{t+1}^{e,t}(h), L_{t+1}^{e,t}(h)) - w_{t+1}(h)\right) \leq 0 \tag{A.17}$$

$$\frac{\partial L}{\partial L_{t+1}^{e,t}(l)} = \pi_l\left(1-\theta\right)\left(AF_2(k_{t+1}^{e,t}(l), L_{t+1}^{e,t}(l)) - w_{t+1}(l)\right) \leq 0 \tag{A.18}$$

By (A.17) and (A.18),

$$w_{t+1}(s) = AF_2(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)$$

By (A.15) and (A.16),

$$\begin{aligned} &\text{If } z_0\pi_s k_t^{e,t} - \pi_s z_{1s} k_t^{e,t} < 0, \quad \text{then } b_t^{e,t}(s) = 0 \\ &\text{If } z_0\pi_s k_t^{e,t} - \pi_s z_{1s} k_t^{e,t} = 0, \quad \text{then } b_t^{e,t}(s) \in [0, (\theta a_t\left(s\right) + q_t(s) - \gamma)] \\ &\text{If } z_0\pi_s k_t^{e,t} - \pi_s z_{1s} k_t^{e,t} > 0, \quad \text{then } b_t^{e,t}(s) = (\theta a_t\left(s\right) + q_t(s) - \gamma) \end{aligned}$$

By (A.13), (A.14) and $k_{t+1}^{e,t}(s) > 0$, one gets

$$z_{1s} = \frac{\left(1-\theta\right) AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))}{q_t(s) - \theta AF_1(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s))}$$

By (A.12) and $k_t^{e,t} > 0$, when the capital requirment constraint does not bind,

$$z_0 = \frac{\sum_s \pi_s z_{1s} \left(a_t\left(s\right) + q_t(s) - \gamma - \left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right)/k_t^{e,t}\right)}{1 - \sum_s \pi_s \left(d_t^{e,t}(s) + d_{t+1}^{e,t}(s)\right)/k_t^{e,t}}$$

The above proves the first part of lemma 2.1. Now consider the case when the capital requirement constraint binds. With binding capital requirement constraint,

$$k_t^{e,t} = \frac{1}{\eta} w_t L^{e,t}$$

and

$$d_t^{e,t}(\eta) = k_t^{e,t} - w_t L^{e,t} = \left(\frac{1}{\eta} - 1\right) w_t L^{e,t}$$

Since $z_{1l} > z_{1h}$, the entrepreneur will exhaust her borrowing in good state first before she can borrow against bad state. That is, if $d_t^{e,t}(\eta) \le \pi_h \left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t}$, the entrepreneur only repays in good state and $d_t^{e,t}(h) + d_{t+1}^{e,t}(h) = d_t^{e,t}(\eta)/\pi_h$, $d_t^{e,t}(l) + d_{t+1}^{e,t}\left(l\right) = 0$. If $d_t^{e,t}(\eta) > \pi_h \left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t}$, the entrepreneur also repays in bad state since the marginal return of investment is pretty hgih. In this case, the repayment limit in good state binds $(d_t^{e,t}(h) + d_{t+1}^{e,t}(h) = \left(\theta a_h + q_t(h) - \gamma\right) k_t^{e,t})$, and the entrepreneur repays the rest in bad state, $d_t^{e,t}(l) + d_{t+1}^{e,t}\left(l\right) = \frac{1}{\pi_l}\left(d_t^{e,t}(\eta) - \pi_h \left(\theta a_h + 1 - \gamma\right) k_t^{e,t}\right)$. The entrepreneur still maximizes her borrowing ability in the last period because $z_{1s} > 1$. Thus, $d_{t+1}^{e,t}(s) = \theta\left(AF(k_{t+1}^{e,t}(s), L_{t+1}^{e,t}(s)) - w_{t+1}(s) L_{t+1}^{e,t}(s)\right)$.

A.4 Proof of Proposition 2.1

The characterization part is proved first and then the existence and uniqueness are shown.

A.4.1 The Characterization

By assumption 2.3, fire sales occur if and only if there is a bad productivity shock. That is, $q_t(l) < 1$ and $q_t(h) = 1$. When solving the individuals' problems, it is already showed that

$z_{1h} < z_{1l}$. Therefore, if the capital requirement constraint is not binding, one of the following three cases applies (1) $z_0 \leq z_{1h} < z_{1l}$, (2) $z_{1h} < z_0 < z_{1l}$, and (3) $z_{1h} < z_{1l} \leq z_0$. Applying the first part in lemma 2.1, these three cases give the equilibrium financial contracts of types 1-3. When the capital requirement constraint binds, it is still true that $z_{1h} < z_{1l}$. Although $z_0(\eta) \neq z_0$, entrepreneurs still borrow first against good state before they can borrow against bad state. Thus, the above three cases hold irrespective of the policy control.

A.4.2 The Existence and Uniqueness

Here a new variable ρ is introduced as the ratio of outside borrowing to total capital invested at the beginning of period t. The existence and uniqueness of ρ^{CE} are firstly proved and then the existence and uniqueness of the corresponding competitive equilibrium are shown thereafter. In a competitive equilibrium with binding capital requirement, $\rho^{CE} = 1 - \eta$ by definition and thus is unique. If the capital requirement constraint does not bind, ρ^{CE} can be chosen from $[0, \overline{\rho}]$. The following step 1 and step 2 are combined together to show that ρ^{CE} exists and is unique if the capital requirement constraint does not bind.

The existence and uniqueness of ρ^{CE} **Step 1.** To show that the equilibrium capital price in bad state $q_t(l)$ is a continuous and decreasing function with ρ.

With ρ chosen,

$$d_t^{e,t} = \rho k_t^{e,t}$$

By (2.20),

$$k_t^{e,t} - k_{t+1}^{e,t}(l) = -\frac{1}{q_t(l)}\left((a_l - \gamma)\, k_t^{e,t} - d_t^{e,t}(l)\right)$$

and then by (2.14) and labor market clear condition (2.23),

$$k_t^{e,t} - k_{t+1}^{e,t}(l) = -\frac{1}{q_t(l)}\left((a_l - \gamma)\, k_t^{e,t} - \theta\alpha A \left(k_{t+1}^{e,t}(l)\right)^{\alpha} L^{1-\alpha}\right)$$

Moreover, by capital market clear condition,

$$k_t^{e,t} - k_{t+1}^{e,t}(l) = -\frac{1}{F_c'(k_t^{e,t} - k_{t+1}^{e,t}(l))}\left(\left(a_l - \gamma\right)k_t^{e,t} - \theta\alpha A\left(k_{t+1}^{e,t}(l)\right)^{\alpha} L^{1-\alpha}\right)$$

Now, set

$$H = F_c'(k_t^{e,t} - k_{t+1}^{e,t}(l))\left(k_t^{e,t} - k_{t+1}^{e,t}(l)\right) + \left(\left(a_l - \gamma\right)k_t^{e,t} - \theta\alpha A\left(k_{t+1}^{e,t}(l)\right)^{\alpha} L^{1-\alpha}\right) = 0$$

Assume the production function in riskless sector is

$$F_c(k_{t+1}^{c,t}(s)) = k_{t+1}^{c,t}(s) - m\left(k_{t+1}^{c,t}(s)\right)^2$$

Then H can be written as

$$H = k_{t+1}^{c,t}(l)\left(1 - 2mk_{t+1}^{c,t}(l)\right) + \left(\left(a_l - \gamma\right)k_t^{e,t} - \theta\alpha A\left(k_t^{e,t} - k_{t+1}^{c,t}(l)\right)^{\alpha} L^{1-\alpha}\right)$$

and

$$\begin{aligned}
\frac{\partial H}{\partial k_{t+1}^{c,t}(l)} &= 1 - 4mk_{t+1}^{c,t}(l) + \theta\alpha^2 A\left(k_t^{e,t} - k_{t+1}^{c,t}(l)\right)^{\alpha-1} L^{1-\alpha} > 0 \\
\frac{\partial H}{\partial k_t^{e,t}} &= \left(a_l - \gamma\right) - \theta\alpha^2 A\left(k_t^{e,t} - k_{t+1}^{c,t}(l)\right)^{\alpha-1} L^{1-\alpha} < 0
\end{aligned}$$

By implicit function theorem,

$$\frac{\partial k_{t+1}^{c,t}(s)}{\partial k_t^{e,t}} = -\frac{\partial H/\partial k_t^{e,t}}{\partial H/\partial k_{t+1}^{c,t}(s)} > 0$$

which implies

$$\frac{\partial q_t(l)}{\partial \rho} < 0$$

since

$$\begin{aligned} F_c'(k_{t+1}^{c,t}(s)) &= 1 - 2mk_{t+1}^{c,t}(s) = q_t(l) \\ k_t^{e,t} &= \frac{1}{1-\rho} w_t L^{e,t} \end{aligned}$$

That is, the equilibrium capital price in bad state $q_t(l)$ is a continuous and decreasing function with ρ for $\rho \in [0, \overline{\rho}]$.

Define $q_t(l) = f_t(\rho)$ for future use. By the characterization, the equilibrium financial contract takes the form:

$$\begin{aligned} b_{t+1}^{e,t}(l) &= \frac{1}{\pi_l} \max\{\rho - \widehat{\rho}, 0\} \\ b_{t+1}^{e,t}(h) &= \frac{1}{\pi_h} \min\{\rho, \widehat{\rho}\} \end{aligned} \tag{A.19}$$

where

$$\widehat{\rho} = \pi_h (\theta a_h + 1 - \gamma)$$

Step 2. To show the existence and uniqueness of ρ^{CE} when the capital requirement constraint does not bind.

Define a function $\varsigma : [0, \overline{\rho}] \to R$ as following:

$$\varsigma(\rho) = \begin{cases} z_0 - z_{1h}, \text{ if } \rho \in [0, \widehat{\rho}] \\ z_0 - z_{1l}, \text{ if } \rho \in (\widehat{\rho}, \overline{\rho}] \end{cases}$$

The function $\varsigma(\rho)$ is continuous and differentiable except at $\widehat{\rho}$. Now, it is time to show that $\varsigma(\rho)$ is a decreasing function with ρ.

When $\rho \in [0, \widehat{\rho})$,

$$\begin{array}{rcl}\varsigma(\rho) & = & z_0 - z_{1h} \\ & = & \left(\begin{array}{c} \frac{\sum_s \pi_s z_{1s}\left(a_t(s)+q_t(s)-\gamma-\left(d_t^{e,t}(s)+d_{t+1}^{e,t}(s)\right)/k_t^{e,t}\right)}{1-\sum_s \pi_s\left(d_t^{e,t}(s)+d_{t+1}^{e,t}(s)\right)/k_t^{e,t}} \\ -\frac{(1-\theta)AF_1(k_{t+1}^{e,t}(h),L)}{1-\theta AF_1(k_{t+1}^{e,t}(h),L)} \end{array} \right) \\ & = & \left(\begin{array}{c} \frac{\pi_h z_{1h}\left(a_t(h)+1-\gamma-\frac{\rho}{\pi_h}\right)+\pi_l z_{1l}(a_t(l)+f_t(\rho)-\gamma)}{1-\sum_s \pi_s b_t^{e,t}(s)} \\ -\frac{(1-\theta)AF_1(k_{t+1}^{e,t}(h),L)}{1-\theta AF_1(k_{t+1}^{e,t}(h),L)} \end{array} \right) \end{array}$$

And then,

$$\frac{\partial\varsigma(\rho)}{\partial\rho} = \frac{1}{1-\sum_s \pi_s b_t^{e,t}(s)}\left(-z_{1h} + \pi_l z_{1l} f_t'(\rho)\right) < 0$$

When $\rho \in (\widehat{\rho}, \overline{\rho}]$,

$$\begin{array}{rcl}\varsigma(\rho) & = & z_0 - z_{1l} \\ & = & \left(\begin{array}{c} \frac{\sum_s \pi_s z_{1s}\left(a_t(s)+q_t(s)-\gamma-\left(d_t^{e,t}(s)+d_{t+1}^{e,t}(s)\right)/k_t^{e,t}\right)}{1-\sum_s \pi_s\left(d_t^{e,t}(s)+d_{t+1}^{e,t}(s)\right)/k_t^{e,t}} \\ -\frac{(1-\theta)AF_1(k_{t+1}^{e,t}(l),L)}{q_t(l)-\theta AF_1(k_{t+1}^{e,t}(l),L)} \end{array} \right) \\ & = & \left(\begin{array}{c} \frac{\pi_h z_{1h}\left(a_t(h)+1-\gamma-\frac{\widehat{\rho}}{\pi_h}\right)+\pi_l z_{1l}\left(a_t(l)+f_t(\rho)-\gamma-\frac{\rho-\widehat{\rho}}{\pi_l}\right)}{1-\sum_s \pi_s b_t^{e,t}(s)} \\ -\frac{(1-\theta)AF_1(k_{t+1}^{e,t}(h),L)}{f_t(\rho)-\theta AF_1(k_{t+1}^{e,t}(h),L)} \end{array} \right) \end{array}$$

Thus,

$$\frac{\partial\varsigma(\rho)}{\partial\rho} = \frac{\pi_l z_{1l}}{1-\sum_s \pi_s b_t^{e,t}(s)}\left(f_t'(\rho) - \frac{1}{\pi_l}\right) + \frac{(1-\theta)AF_1(k_{t+1}^{e,t}(h),L)}{\left(f_t(\rho)-\theta AF_1(k_{t+1}^{e,t}(h),L)\right)^2} f_t'(\rho) < 0$$

In addition, $\varsigma(\widehat{\rho}) > \lim_{\rho\to\widehat{\rho}+}\varsigma(\rho)$ since $z_{1h} < z_{1l}$. In other words, $\varsigma(\rho)$ is a continuous and decreasing function of ρ except at $\widehat{\rho}$, where it has a downward jump. This implies that there exists one and only one $\rho^{CE} \in [0, \overline{\rho}]$ that satisfies one of the following conditions:

1. $\rho^{CE} = 0$, $\varsigma\left(\rho^{CE}\right) \leq 0$ where $z_0^{CE} \leq z_{1h}^{CE} < z_{1l}^{CE}$;
2. $\rho^{CE} \in (0, \widehat{\rho})$, $\varsigma\left(\rho^{CE}\right) = 0$ where $z_0^{CE} = z_{1h}^{CE} < z_{1l}^{CE}$;
3. $\rho^{CE} = \widehat{\rho}$, $\varsigma\left(\widehat{\rho}\right) \geq 0 \geq \lim_{\rho \to \widehat{\rho}+} \varsigma\left(\widehat{\rho}\right)$ where $z_{1h}^{CE} < z_0^{CE} < z_{1l}^{CE}$;
4. $\rho^{CE} \in (\widehat{\rho}, \overline{\rho})$, $\varsigma\left(\rho^{CE}\right) = 0$ where $z_{1h}^{CE} < z_0^{CE} = z_{1l}^{CE}$;
5. $\rho^{CE} = \overline{\rho}$, $\varsigma\left(\rho^{CE}\right) \geq 0$ where $z_{1h}^{CE} < z_{1l}^{CE} \leq z_0^{CE}$.

The existence and uniqueness of equilibrium For each of above five cases, it is possible to find out an equilibrium with ρ^{CE}. Given ρ^{CE}, the first time investment and the borrowing are

$$
\begin{aligned}
k_t^{e,t} &= \frac{1}{1-\rho^{CE}} w_t L^{e,t} \\
d_t^{e,t} &= \frac{\rho^{CE}}{1-\rho^{CE}} w_t L^{e,t}
\end{aligned}
$$

Following step 1 in above section,

$$
q_t^{CE}(l) = f_t\left(\rho^{CE}\right)
$$

and by capital market clear, $k_{t+1}^{e,t,CE}(l)$ can be found by solving

$$
F_c'(k_t^{e,t,CE} - k_{t+1}^{e,t,CE}(l)) = q_t^{CE}(l)
$$

and then by (A.19), the repayments in bad state are

$$
\begin{aligned}
d_{t+1}^{e,t,CE}(l) &= \theta \alpha A \left(k_{t+1}^{e,t,CE}(l)\right)^{\alpha} L^{1-\alpha} \\
d_t^{e,t,CE}(l) &= \frac{1}{\pi_l} \max\left\{\rho^{CE} - \widehat{\rho}, 0\right\} k_t^{e,t} - d_{t+1}^{e,t,CE}(l)
\end{aligned}
$$

Now, consider the repayments in good state.

$$k_{t+1}^{e,t,CE}(h) = (a_h + 1 - \gamma)\, k_t^{e,t,CE} - d_t^{e,t,CE}(h)$$

where

$$\begin{aligned} d_t^{e,t,CE}(h) &= b_t^{e,t,CE}(h) k_t^{e,t,CE} - d_{t+1}^{e,t,CE}(h) \\ &= \frac{1}{\pi_h} \min\left\{\rho^{CE}, \widehat{\rho}\right\} k_t^{e,t,CE} - \theta\alpha A \left(k_{t+1}^{e,t,CE}(h)\right)^{\alpha} L^{1-\alpha} \end{aligned}$$

Define

$$J = k_{t+1}^{e,t,CE}(h) - \theta\alpha A \left(k_{t+1}^{e,t,CE}(h)\right)^{\alpha} L^{1-\alpha} - (a_h + 1 - \gamma)\, k_t^{e,t,CE} + \frac{1}{\pi_h} \min\left\{\rho^{CE}, \widehat{\rho}\right\} k_t^{e,t,CE}$$

and J is an increasing function with $k_{t+1}^{e,t,CE}(h)$ by assumption 2.2. With assumption 2.4, $J\left(k_{t+1}^{e,t}(h)\right) = 0$ has an unique solution $k_{t+1}^{e,t,CE}(h)$ such that $k_{t+1}^{e,t,CE}(h) \geq k_t^{e,t,CE}$. And thus, the ρ^{CE} can be used to find out an unique equilibrium.

A.5 Dynamics with $t = 101$ to 1000

A.5.1 Type 1 Equilibrium

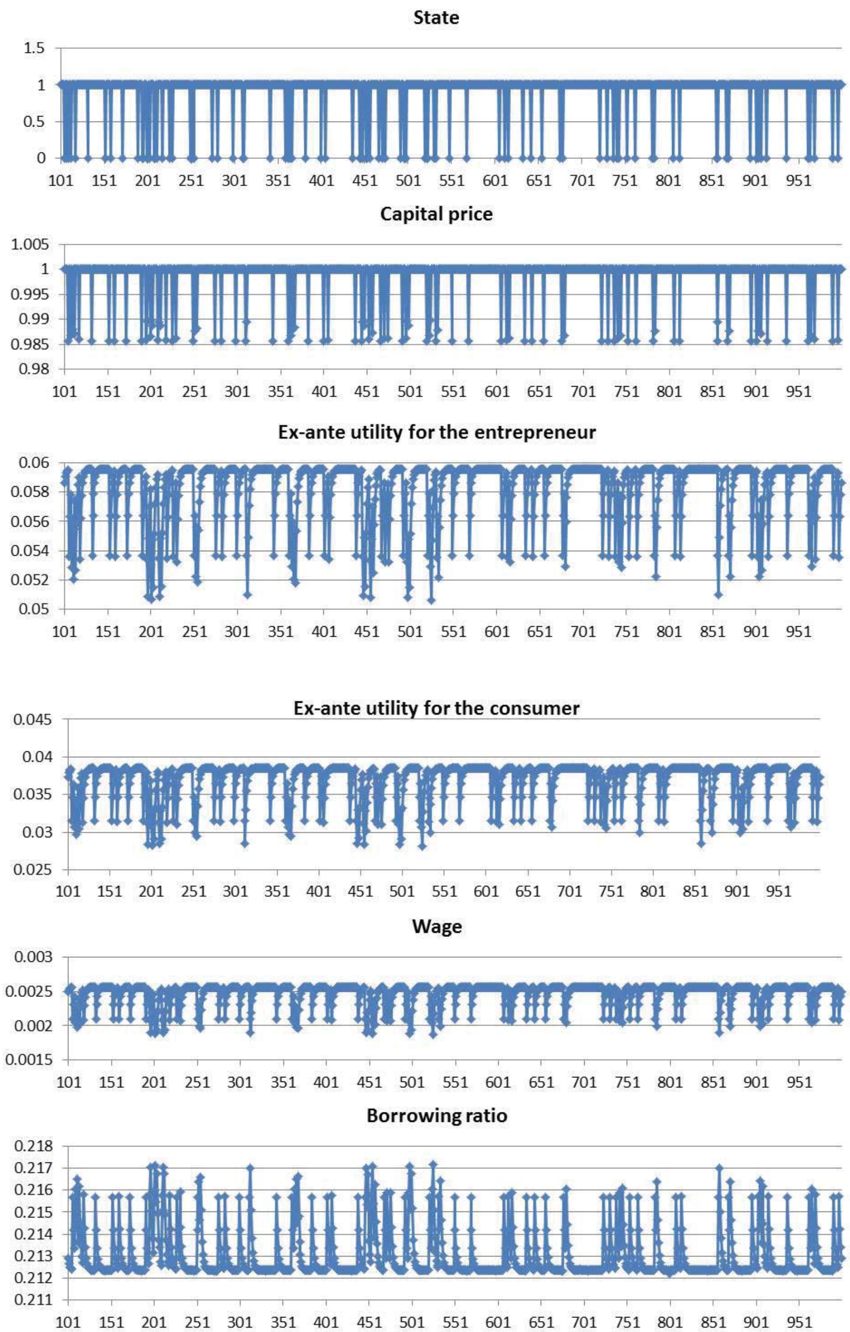


Figure A.2. Dynamics of type 1 equilibrium ($t = 101$ to 1000)

A.5.2 Type 3 Equilibrium

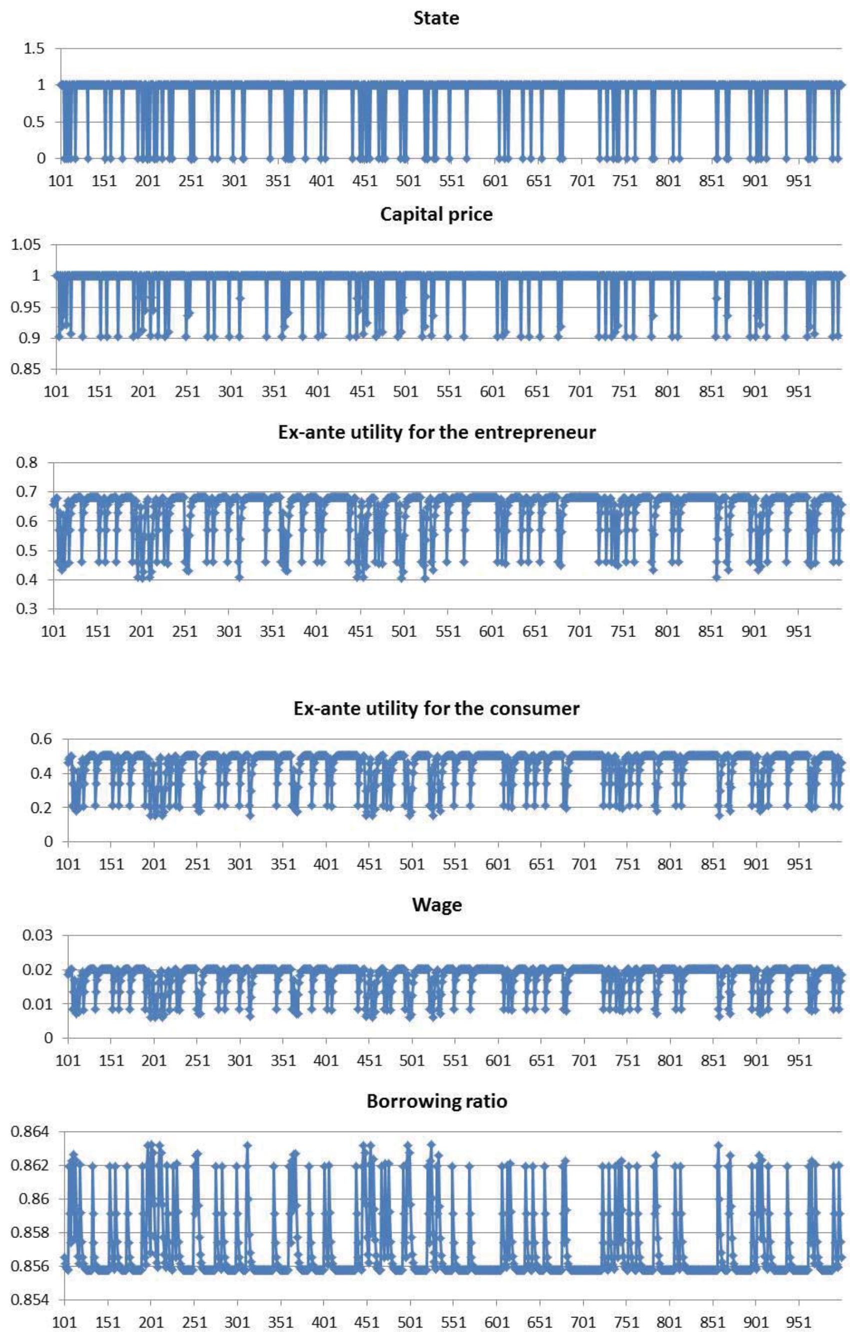


Figure A.3. Dynamics of type 3 equilibrium ($t = 101$ to 1000)

APPENDIX B. CHAPTER 3 APPENDIX

B.1 The Consumer's Problem

First show that the demand $q(\omega)$ of good ω and the price P of the aggregate good $Q(\equiv U)$, satisfy equations (3.1) and (3.3).

Define $Y = \int_{\omega\in\Omega} q(\omega)^{\rho} \, d\omega$ so that $Q = Y^{1/\rho}$. The demand of good ω is determined by equating the market price $p(\omega)$ to the marginal benefit of buying the good:

$$p(\omega) = PY^{1/\rho-1} q(\omega)^{\rho-1} \tag{B.1}$$

Solving for $q(\omega)$:

$$q(\omega) = \left(p(\omega)^{-1} PY^{1/\rho-1}\right)^{\frac{1}{1-\rho}} = p(\omega)^{-\frac{1}{1-\rho}} P^{\frac{1}{1-\rho}} Y^{1/\rho} = p(\omega)^{-\frac{1}{1-\rho}} P^{\frac{1}{1-\rho}} Q$$

which gives equation (3.1). Moreover, from this equation one also obtains (after multiplying by $p(\omega)$ and integrating):

$$\int_{\omega\in\Omega} p(\omega) q(\omega) \, d\omega = P^{\frac{1}{1-\rho}} Q \int_{\omega\in\Omega} p(\omega)^{1-\sigma} \, d\omega \tag{B.2}$$

Note also from (B.1), that

$$\begin{aligned} \int_{\omega\in\Omega} p(\omega) q(\omega) \, d\omega &= \int_{\omega\in\Omega} PY^{1/\rho-1} q(\omega)^{\rho} \, d\omega = PY^{1/\rho-1} \int_{\omega\in\Omega} q(\omega)^{\rho} \, d\omega \\ &= PY^{1/\rho} = PQ \end{aligned}$$

Equating this result to (B.2) it follows that

$$P^{\frac{1}{1-\rho}} Q \int_{\omega\in\Omega} p\left(\omega\right)^{1-\sigma} d\omega = PQ$$

and solving for P results in the equation (3.3).

Then, the expenditure of good ω is

$$r\left(\omega\right) = p\left(\omega\right) q\left(\omega\right) = p\left(\omega\right) Q \left(\frac{p\left(\omega\right)}{P}\right)^{-\sigma} = PQ \left(\frac{p\left(\omega\right)}{P}\right)^{1-\sigma} = R \left(\frac{p\left(\omega\right)}{P}\right)^{1-\sigma}$$

where $R = PQ = \int_{\omega\in\Omega} r\left(\omega\right) d\omega$ denotes aggregate expenditure.

B.2 The Producer's Problem

The producer which produces good ω with productivity level x_i is a monopoly in good ω market, and the demand it faces is

$$q\left(\omega\right) = Q \left(\frac{p\left(\omega\right)}{P}\right)^{-\sigma}$$

Then, the producer's problem is

$$\max_{p(\omega)} q\left(\omega\right) p\left(\omega\right) - \left(f + \frac{q\left(\omega\right)}{x_i}\right) w = Q \left(\frac{p\left(\omega\right)}{P}\right)^{-\sigma} p\left(\omega\right) - \left(f + \frac{Q\left(\frac{p(\omega)}{P}\right)^{-\sigma}}{x_i}\right) w$$

where w is the wage rate.

The F.O.C is

$$Q \left(\frac{1}{P}\right)^{-\sigma} \left(1-\sigma\right) \left(p\left(\omega\right)\right)^{-\sigma} + \sigma Q \frac{1}{x_i} \left(\frac{1}{P}\right)^{-\sigma} \left(p\left(\omega\right)\right)^{-\sigma-1} w = 0$$

Solve the first order condition and then get

$$p(\omega) = \frac{\sigma}{\sigma - 1}\frac{w}{x_i} = \frac{w}{\rho x_i}$$

The S.O.C is

$$-\sigma Q\left(\frac{1}{P}\right)^{-\sigma}(1-\sigma)(p(\omega))^{-\sigma-1} + (-\sigma-1)\,\sigma Q\frac{1}{x_i}\left(\frac{1}{P}\right)^{-\sigma}(p(\omega))^{-\sigma-2}w < 0$$

Since each firm produces different good and firm sets price based on its own productivity level,

$$p(x_i) = p(\omega) = \frac{w}{\rho x_i}$$

The optimal quantity is then

$$q(\omega) = q(x_i) = Q\left(\frac{p(x_i)}{P}\right)^{-\sigma} = Q\left(\frac{w}{P\rho x_i}\right)^{-\sigma} = Q(P\rho x_i)^{\sigma}$$

The last equality holds since the wage ratio is normalized to 1. And firm's revenue is

$$r(x_i) = q(x_i)\,p(x_i) = Q(P\rho x_i)^{\sigma}\frac{1}{\rho x_i} = (PQ)\,P^{\sigma-1}(\rho x_i)^{\sigma-1} = R(P\rho x_i)^{\sigma-1}$$

The labor used for producing is

$$l_p(x_i) = f + \frac{q(x_i)}{x_i} = f + Q(P\rho x_i)^{\sigma}\frac{1}{x_i} = f + \rho r(x_i) = f + \frac{\sigma-1}{\sigma}R(P\rho x_i)^{\sigma-1}$$

and thus the profit of the firm is

$$\pi(x_i) = r(x_i) - l_p(x_i) = \frac{R}{\sigma}(P\rho x_i)^{\sigma-1} - f$$

B.3 Aggregation

Here are the derivation of aggregate variables.

$$\begin{aligned}
P &= \left(\int_0^\infty p(x)^{1-\sigma} M\mu(x)dx\right)^{\frac{1}{1-\sigma}} = M^{\frac{1}{1-\sigma}}\left(\int_0^\infty \left(\frac{1}{\rho x}\right)^{1-\sigma}\mu(x)dx\right)^{\frac{1}{1-\sigma}} \\
&= M^{\frac{1}{1-\sigma}}\frac{1}{\rho}\left(\int_0^\infty x^{\sigma-1}\mu(x)dx\right)^{\frac{1}{1-\sigma}} = M^{\frac{1}{1-\sigma}}p(\widetilde{x}) \\
Q &= \left(\int_0^\infty q(x)^\rho M\mu(x)dx\right)^{1/\rho} = \left(\int_0^\infty q(\widetilde{x})^\rho \left(\frac{x}{\widetilde{\widetilde{x}}}\right)^{\sigma\rho} M\mu(x)dx\right)^{1/\rho} \\
&= M^{1/\rho}q(\widetilde{x})\left(\frac{1}{\widetilde{\widetilde{x}}}\right)^{\sigma}\left(\int_0^\infty (x)^{\sigma\rho}\mu(x)dx\right)^{\frac{\sigma}{\sigma-1}} = M^{1/\rho}q(\widetilde{x}) \\
R &= PQ = \int_0^\infty r(x)M\mu(x)dx = \int_0^\infty r(\widetilde{x})\left(\frac{x}{\widetilde{\widetilde{x}}}\right)^{\sigma-1}M\mu(x)dx \\
&= Mr(\widetilde{x})\left(\frac{1}{\widetilde{\widetilde{x}}}\right)^{\sigma-1}\left(\int_0^\infty x^{\sigma-1}\mu(x)dx\right)^{\frac{\sigma-1}{\sigma-1}} = Mr(\widetilde{x}) \\
\Pi &= \int_0^\infty \pi(x)M\mu(x)dx = \frac{1}{\sigma}\int_0^\infty r(x)M\mu(x)dx - Mf \\
&= M\left(\frac{r(\widetilde{x})}{\sigma} - f\right) = M\pi(\widetilde{x})
\end{aligned}$$

B.4 Posterior Distribution of θ

Now consider the probability density function of Θ when $X_i = x_i$ where $\underline{\theta} - \varepsilon < x_i < \overline{\theta} + \varepsilon$ first.

$$\begin{aligned}
f_{\Theta|X_i=x_i}(\theta) &= \frac{f_{X_i,\Theta}(x_i,\theta)}{f_{X_i}(x_i)} \\
&= \frac{f_{X_i,\Theta}(x_i,\theta)}{\int_{-\infty}^{\infty} f_{X_i,\Theta}(x_i,\theta)\,d\theta} \\
&= \frac{f_{X_i|\Theta=\theta}(x_i)\,f_\Theta(\Theta=\theta)}{\int_{-\infty}^{\infty} f_{X_i|\Theta=\theta}(x_i)\,f_\Theta(\Theta=\theta)\,d\theta}
\end{aligned} \tag{B.3}$$

where the probability density function $f_\Theta(\Theta = \theta)$ is

$$f_\Theta(\Theta = \theta) = \begin{cases} \frac{1}{\overline{\theta} - \underline{\theta}} & \text{if } \underline{\theta} \leq \theta \leq \overline{\theta} \\ 0 & \text{otherwise} \end{cases}$$

and the conditional probability density function $f_{X_i|\Theta=\theta}(x_i)$ is

$$f_{X_i|\Theta=\theta}(x_i) = \begin{cases} \frac{1}{2\varepsilon} & \text{if } \theta - \varepsilon \leq x_i \leq \theta + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

since $X_i|\,(\Theta = \theta) = \theta + \Psi_i$ where $\Psi_i \sim U[-\varepsilon, \varepsilon]$.

Define set A as

$$\begin{aligned} A &= \{\theta | f_\Theta(\Theta = \theta) > 0 \text{ and } f_{X_i|\Theta=\theta}(x_i) > 0\} \\ &= \{\theta | \underline{\theta}(x_i) \leq \theta \leq \overline{\theta}(x_i)\} \end{aligned}$$

where

$$\begin{aligned} \underline{\theta}(x_i) &= \max\{x_i - \varepsilon, \underline{\theta}\} \\ \overline{\theta}(x_i) &= \min\{x_i + \varepsilon, \overline{\theta}\} \end{aligned}$$

Note that since $\underline{\theta} - \varepsilon < x_i < \overline{\theta} + \varepsilon$, $\underline{\theta}(x_i) \neq \overline{\theta}(x_i)$.

Then,

$$\begin{aligned} f_{X_i}(x_i) &= \int_{-\infty}^{\infty} f_{X_i|\Theta=\theta}(x_i) f_\Theta(\Theta = \theta)\, d\theta \\ &= \int_A f_{X_i|\Theta=\theta}(x_i) f_\Theta(\Theta = \theta)\, d\theta \\ &= \int_{\underline{\theta}(x_i)}^{\overline{\theta}(x_i)} \frac{1}{2\varepsilon} \frac{1}{\overline{\theta} - \underline{\theta}} d\theta \\ &= \frac{1}{2\varepsilon} \frac{1}{\overline{\theta} - \underline{\theta}} \left(\overline{\theta}(x_i) - \underline{\theta}(x_i)\right) \end{aligned} \tag{B.4}$$

Thus, when $\theta \in A$ ($\underline{\theta}(x_i) \leq \theta \leq \overline{\theta}(x_i)$), from (B.4), the conditional density function (B.3) can be simplified as

$$f_{\Theta|X_i=x_i}(\theta) = \frac{\frac{1}{2\varepsilon}\frac{1}{\overline{\theta}-\underline{\theta}}}{f_{X_i}(x_i)} = \frac{\frac{1}{2\varepsilon}\frac{1}{\overline{\theta}-\underline{\theta}}}{\frac{1}{2\varepsilon}\frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}(x_i)-\underline{\theta}(x_i)\right)} = \frac{1}{\overline{\theta}(x_i)-\underline{\theta}(x_i)}$$

and when $\theta \notin A$ ($\theta < \underline{\theta}(x_i)$ or $\theta > \overline{\theta}(x_i)$),

$$f_{\Theta|X_i=x_i}(\theta) = \frac{0}{f_{X_i}(x_i)} = 0$$

In sum, the posterior distribution $J_{\Theta|X_i=x_i}(\theta)$ of Θ conditional on x_i is $U\left[\underline{\theta}(x_i), \overline{\theta}(x_i)\right]$ when $\underline{\theta} - \varepsilon < x_i < \overline{\theta} + \varepsilon$.

When $X_i = x_i = \underline{\theta} - \varepsilon$, firm i knows that the only possible realization of Θ is $\underline{\theta}$ because if $\Theta = \theta > \underline{\theta}$, $X_i|\Theta = \theta \geq \theta - \varepsilon > \underline{\theta} - \varepsilon$. That is, the posterior distribution $J_{\Theta|X_i=x_i}(\theta)$ of Θ conditional on $X_i = \underline{\theta} - \varepsilon$ is $\Theta = \underline{\theta}$ for probability of 1. Similarly, when $X_i = x_i = \overline{\theta} + \varepsilon$, the posterior distribution $J_{\Theta|X_i=x_i}(\theta)$ of Θ conditional on x_i is $\Theta = \overline{\theta}$ for probability of 1.

B.5 Expressions with Threshold Strategy

B.5.1 Labor Market Clear Condition

$$\begin{aligned}
L &= M_e f_e + R(\Pr(X), \theta) - \Pi(\Pr(X), \theta) \\
&= M_e f_e + M(\Pr(X), \theta)\, r(\widetilde{x})(\Pr(X), \theta) - M(\Pr(X), \theta)\, \pi(\widetilde{x})(\Pr(X), \theta) \\
&= M_e f_e + M(\Pr(X), \theta)\left((\sigma - 1)\pi(\widetilde{x})(\Pr(X), \theta) + \sigma f\right) \\
&= M_e f_e + M(\Pr(X), \theta)\left((\sigma - 1)\overline{\pi}(\Pr(X), \theta) + \sigma f\right) \\
&= M_e f_e + M_e P_{stay}(\Pr(X), \theta)\left((\sigma - 1)\overline{\pi}(\Pr(X), \theta) + \sigma f\right)
\end{aligned}$$

B.5.2 Average Profit

From labor market clear condition (3.30) obtain

$$P_{stay}(I_{x^*},\theta)\,\overline{\pi}(I_{x^*},\theta)=\frac{\frac{L}{M_e}-f_e-P_{stay}(I_{x^*},\theta)\,\sigma f}{\sigma-1}$$

Now use the above in free entry condition (3.31) to obtain

$$M_e(x^*)=\frac{L/\sigma}{f_e+\frac{f}{\overline{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{\overline{\theta}}P_{stay}(I_{x^*},\theta)\,d\theta}=\frac{L/\sigma}{f_e+fP^e_{stay}(I_{x^*})} \tag{B.5}$$

Substituting (B.5) in (3.30) now gives the average profits

$$\begin{aligned}\overline{\pi}(I_{x^*},\theta) &= \frac{1}{P_{stay}(I_{x^*},\theta)}\frac{1}{\sigma-1}\left(\sigma\left(f_e+fP^e_{stay}(I_{x^*})\right)-f_e-P_{stay}(I_{x^*},\theta)\,\sigma f\right)\\ &= \frac{1}{P_{stay}(I_{x^*},\theta)}\frac{1}{\sigma-1}\left((\sigma-1)\,f_e+\sigma f\left(P^e_{stay}(I_{x^*})-P_{stay}(I_{x^*},\theta)\right)\right)\\ &= \frac{\sigma f}{\sigma-1}\left(\frac{P^e_{stay}(I_{x^*})}{P_{stay}(I_{x^*},\theta)}-1\right)+\frac{f_e}{P_{stay}(I_{x^*},\theta)}\end{aligned}$$

B.5.3 Ex-ante Probability of Staying

The ex-ante probability of staying given I_{x^*}:

$$\begin{aligned}P^e_{stay}(I_{x^*}) &= E_{\Theta}\left(P_{stay}(I_{x^*},\theta)\right)\\ &= \int_{\underline{\theta}}^{\overline{\theta}}\frac{1}{\overline{\theta}-\underline{\theta}}P_{stay}(I_{x^*},\theta)\,d\theta\\ &= \frac{1}{\overline{\theta}-\underline{\theta}}\left(\int_{-\infty}^{\overline{\theta}}P_{stay}(I_{x^*},\theta)\,d\theta-\int_{-\infty}^{\underline{\theta}}P_{stay}(I_{x^*},\theta)\,d\theta\right)\end{aligned}$$

where

$$\int_{-\infty}^{\theta}P_{stay}(I_{x^*},\theta)\,d\theta=\begin{cases}0 & \text{if } \theta\le x^*-\varepsilon\\ \frac{(\theta+\varepsilon-x^*)^2}{4\varepsilon} & \text{if } x^*-\varepsilon<\theta\le x^*+\varepsilon\\ \theta-\ x^* & \text{if } \theta>x^*+\varepsilon\end{cases}$$

Thus,

$$P^{e}_{stay}\left(I_{x^*}\right)=\begin{cases} 0 & \text{if } \overline{\theta}\leq x^*-\varepsilon \\ \frac{1}{\overline{\theta}-\underline{\theta}}\ \frac{\left(\overline{\theta}+\varepsilon-x^*\right)^2}{4\varepsilon} & \text{if } x^*-\varepsilon<\overline{\theta}\leq x^*+\varepsilon \text{ and } \underline{\theta}\leq x^*-\varepsilon \\ =P^{e1}_{stay}\left(I_{x^*}\right)=\frac{\overline{\theta}+\underline{\theta}+2\varepsilon-2x^*}{4\varepsilon} & \text{if } x^*-\varepsilon<\overline{\theta}\leq x^*+\varepsilon \text{ and } x^*-\varepsilon<\underline{\theta}\leq x^*+\varepsilon \\ \frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}-\ x^*\right) & \text{if } \overline{\theta}>x^*+\varepsilon \text{ and } \underline{\theta}\leq x^*-\varepsilon \\ =P^{e2}_{stay}\left(I_{x^*}\right)=\frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}-\ x^*-\frac{(\underline{\theta}+\varepsilon-x^*)^2}{4\varepsilon}\right) & \text{if } \overline{\theta}>x^*+\varepsilon \text{ and } x^*-\varepsilon<\underline{\theta}\leq x^*+\varepsilon \\ =P^{e3}_{stay}\left(I_{x^*}\right)=1 & \text{if } \overline{\theta}>x^*+\varepsilon \text{ and } \underline{\theta}>x^*+\varepsilon \end{cases}$$

B.5.4 Productivity Ratios

From (3.28) and (3.29), when $\theta-\varepsilon\leq x^*<\theta+\varepsilon$, the average productivity is

$$\begin{aligned}\widetilde{x}\left(I_{x^*},\theta\right) &= \left(\int_{\theta-\varepsilon}^{\theta+\varepsilon}\left(x_i\right)^{\sigma-1}\mu(x_i,I_{x^*},\theta)dx_i\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{\theta+\varepsilon-x^*}\int_{x^*}^{\theta+\varepsilon}\left(x_i\right)^{\sigma-1}dx_i\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{\theta+\varepsilon-x^*}\left(\frac{1}{\sigma}\left(\theta+\varepsilon\right)^{\sigma}-\frac{1}{\sigma}\left(x^*\right)^{\sigma}\right)\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{\sigma\left(\theta+\varepsilon-x^*\right)}\left(\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}\right)\right)^{\frac{1}{\sigma-1}} \end{aligned} \tag{B.6}$$

and when $x^*<\theta-\varepsilon$, then average productivity is

$$\begin{aligned}\widetilde{x}\left(I_{x^*},\theta\right) &= \left(\int_{\theta-\varepsilon}^{\theta+\varepsilon}\left(x_i\right)^{\sigma-1}\mu(x_i,I_{x^*},\theta)dx_i\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{2\varepsilon}\int_{\theta-\varepsilon}^{\theta+\varepsilon}\left(x_i\right)^{\sigma-1}dx_i\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{2\varepsilon}\left(\frac{1}{\sigma}\left(\theta+\varepsilon\right)^{\sigma}-\frac{1}{\sigma}\left(\theta-\varepsilon\right)^{\sigma}\right)\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{2\varepsilon\sigma}\left(\left(\theta+\varepsilon\right)^{\sigma}-\left(\theta-\varepsilon\right)^{\sigma}\right)\right)^{\frac{1}{\sigma-1}} \end{aligned} \tag{B.7}$$

Then, the productivity ratio is

$$\left(\frac{x_i}{\widetilde{x}\left(I_{x^*},\theta\right)}\right)^{\sigma-1}=\begin{cases} Ra^1\left(x_i,I_{x^*},\theta\right)=\left(x_i\right)^{\sigma-1}\frac{\sigma(\theta+\varepsilon-x^*)}{(\theta+\varepsilon)^{\sigma}-(x^*)^{\sigma}} & \text{if } \theta-\varepsilon\leq x^*<\theta+\varepsilon \\ Ra^2\left(x_i,I_{x^*},\theta\right)=\left(x_i\right)^{\sigma-1}\frac{2\varepsilon\sigma}{(\theta+\varepsilon)^{\sigma}-(\theta-\varepsilon)^{\sigma}} & \text{if } x^*<\theta-\varepsilon \end{cases}$$

B.6 Proof of Lemma 3.1

This section shows that $\frac{d}{dx^*}u(x^*,I_{x^*})>0$ for $\underline{\theta}-\varepsilon<x^*<\overline{\theta}-\varepsilon<\underline{\theta}+\varepsilon$. Note that assumption 3.2 assumes $\sigma>3$.

Define

$$A\left(x^*,\theta\right)=\frac{1}{\overline{\theta}\left(x^*\right)-\underline{\theta}\left(x^*\right)}\left(\frac{x^*}{\widetilde{x}\left(I_{x^*},\theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^*},\theta\right)+f\right)$$

when $\underline{\theta}-\varepsilon<x_i<\overline{\theta}+\varepsilon$. Thus, within each differentiable region,

$$\begin{aligned} & \frac{d}{dx^*}u(x^*,I_{x^*}) \\ = & \frac{d}{dx^*}\left(\int_{\underline{\theta}(x^*)}^{\overline{\theta}(x^*)}A\left(x^*,\theta\right)d\theta-f\right) \\ = & \frac{d}{dx^*}\int_{\underline{\theta}(x^*)}^{\overline{\theta}(x^*)}A\left(x^*,\theta\right)d\theta \\ = & A\left(x^*,\overline{\theta}\left(x^*\right)\right)\frac{d}{dx^*}\overline{\theta}\left(x^*\right)-A\left(x^*,\underline{\theta}\left(x^*\right)\right)\frac{d}{dx^*}\underline{\theta}\left(x^*\right)+\int_{\underline{\theta}(x^*)}^{\overline{\theta}(x^*)}\frac{d}{dx^*}A\left(x^*,\theta\right)d\theta \end{aligned} \quad \text{(B.8)}$$

Since

$$\underline{\theta}-\varepsilon<x^*<\overline{\theta}-\varepsilon<\underline{\theta}+\varepsilon$$

then,

$$x^*-\varepsilon<\underline{\theta}\leq\theta\leq x^*+\varepsilon<\overline{\theta}$$

and thus the integration bounds are

$$\begin{aligned} \underline{\theta}\left(x^*\right) &= \underline{\theta} \\ \overline{\theta}\left(x^*\right) &= x^*+\varepsilon \end{aligned}$$

The productivity ratios is

$$\left(\frac{x^*}{\widetilde{x}\left(I_{x^*},\theta\right)}\right)^{\sigma-1} = Ra^1\left(x^*, I_{x^*}, \theta\right)$$

and the probability of staying and the *ex ante* probability of staying are respectively,

$$\begin{aligned} P_{stay}\left(I_{x^*},\theta\right) &= P^1_{stay}\left(I_{x^*},\theta\right) \\ P^e_{stay}\left(I_{x^*}\right) &= P^{e2}_{stay}\left(I_{x^*}\right) \end{aligned}$$

As a result, (B.8) can be simplified as

$$\frac{d}{dx^*}u(x^*, I_{x^*}) = A\left(x^*, x^*+\varepsilon\right) + \int_{\underline{\theta}}^{x^*+\varepsilon} \frac{d}{dx^*}A\left(x^*,\theta\right)d\theta \qquad \text{(B.9)}$$

where

$$A\left(x^*,\theta\right) = \frac{1}{x^*+\varepsilon-\underline{\theta}}Ra^1\left(x^*, I_{x^*},\theta\right)\left(\frac{\sigma f}{\sigma-1}\left(\frac{P^{e2}_{stay}\left(I_{x^*}\right)}{P^1_{stay}\left(I_{x^*},\theta\right)}-1\right)+\frac{f_e}{P^1_{stay}\left(I_{x^*},\theta\right)}+f\right) \qquad \text{(B.10)}$$

To show that $\frac{d}{dx^*}u(x^*, I_{x^*}) > 0$, try to prove that $A\left(x^*, x^*+\varepsilon\right) > 0$ and $\int_{\underline{\theta}}^{x^*+\varepsilon} \frac{d}{dx^*}A\left(x^*,\theta\right)d\theta > 0$ for (B.9).

Step 1. Consider $A\left(x^*, x^*+\varepsilon\right)$.

From (B.10),

$$A\left(x^*, x^*+\varepsilon\right) = \frac{1}{x^*+\varepsilon-\underline{\theta}}Ra^1\left(x^*, I_{x^*}, x^*+\varepsilon\right)\left(\left(\overline{\pi}\left(I_{x^*}, x^*+\varepsilon\right)+f\right)\right)$$

Since $\underline{\theta}-\varepsilon < x^*$, $\frac{1}{x^*+\varepsilon-\underline{\theta}} > 0$. Moreover, by definition, $Ra^1\left(x^*, I_{x^*}, x^*+\varepsilon\right)$ is the ratio between two productivity levels, and thus is positive. Note that from producer's problem,

$$\left(\overline{\pi}\left(I_{x^*}, x^*+\varepsilon\right)+f\right) = \frac{1}{\sigma}\overline{r}\left(I_{x^*}, x^*+\varepsilon\right)$$

Because $\overline{r}\left(I_{x^*}, x^*+\varepsilon\right)$ is the average revenue which is positive, $\left(\overline{\pi}\left(I_{x^*}, x^*+\varepsilon\right)+f\right)>0$. Thus, $A\left(x^*, x^*+\varepsilon\right)>0$.

Step 2. Consider $\int_{\underline{\theta}}^{x^*+\varepsilon} \frac{d}{dx^*} A\left(x^*, \theta\right) d\theta$.

Define

$$\begin{aligned} A^{11}\left(x^*, \theta\right) &= \frac{1}{x^*+\varepsilon-\underline{\theta}} Ra^1\left(x^*, I_{x^*}, \theta\right) \\ &= \frac{1}{x^*}\left(x^*\right)^{\sigma-1} \frac{\sigma\left(\theta+\varepsilon-x^*\right)}{(\theta+\varepsilon)^\sigma-\left(x^*\right)^\sigma}=\left(x^*\right)^{\sigma-2} \frac{\sigma\left(\theta+\varepsilon-x^*\right)}{(\theta+\varepsilon)^\sigma-\left(x^*\right)^\sigma}>0 \end{aligned}$$

$$\begin{aligned} A^{12}\left(x^*, \theta\right) &= \left(\frac{\sigma f}{\sigma-1}\left(\frac{P_{stay}^{e2}\left(I_{x^*}\right)}{P_{stay}^1\left(I_{x^*}, \theta\right)}-1\right)+\frac{f_e}{P_{stay}^1\left(I_{x^*}, \theta\right)}+f\right) \\ &= \left(\frac{\sigma f}{\sigma-1}\left(\frac{\frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}-\ x^*-\frac{\left(\theta+\varepsilon-x^*\right)^2}{4 \varepsilon}\right)}{\frac{\theta}{2 \varepsilon}+\frac{\varepsilon-x^*}{2 \varepsilon}}-1\right)+\frac{f_e}{\frac{\theta}{2 \varepsilon}+\frac{\varepsilon-x^*}{2 \varepsilon}}+f\right)>0 \end{aligned}$$

then

$$\begin{aligned} \frac{d}{dx^*} A\left(x^*, \theta\right) &= \frac{d}{dx^*}\left(A^{11}\left(x^*, \theta\right) A^{12}\left(x^*, \theta\right)\right) \\ &= A^{12}\left(x^*, \theta\right) \frac{d}{dx^*} A^{11}\left(x^*, \theta\right)+A^{11}\left(x^*, \theta\right) \frac{d}{dx^*} A^{12}\left(x^*, \theta\right) \end{aligned} \tag{B.11}$$

and thus

$$\int_{\underline{\theta}}^{x^*+\varepsilon} \frac{d}{dx^*} A\left(x^*, \theta\right) d\theta=\int_{\underline{\theta}}^{x^*+\varepsilon} A^{12}\left(x^*, \theta\right) \frac{d}{dx^*} A^{11}\left(x^*, \theta\right) d\theta+\int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}\left(x^*, \theta\right) \frac{d}{dx^*} A^{12}\left(x^*, \theta\right) d\theta$$

Now, it is going to show that $\int_{\underline{\theta}}^{x^*+\varepsilon} A^{12}\left(x^*, \theta\right) \frac{d}{dx^*} A^{11}\left(x^*, \theta\right) d\theta>0$ and that $\int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}\left(x^*, \theta\right) \frac{d}{dx^*} A^{12}\left(x^*, \theta\right) d\theta>0$.

Step 2.1. Show that $\int_{\underline{\theta}}^{x^*+\varepsilon} A^{12}(x^*,\theta)\frac{d}{dx^*}A^{11}(x^*,\theta)\,d\theta > 0$.

$$
\begin{aligned}
& \frac{d}{dx^*}A^{11}(x^*,\theta) \\
= \;& \frac{d}{dx^*}\left((x^*)^{\sigma-2}\frac{\sigma(\theta+\varepsilon-x^*)}{(\theta+\varepsilon)^{\sigma}-(x^*)^{\sigma}}\right) \\
= \;& \frac{(x^*)^{\sigma-3}}{((\theta+\varepsilon)^{\sigma}-(x^*)^{\sigma})^2}\left(-(x^*)^{\sigma+1}+2(x^*)^{\sigma}(\theta+\varepsilon)+(1-\sigma)(x^*)(\theta+\varepsilon)^{\sigma}+(\sigma-2)(\theta+\varepsilon)^{\sigma+1}\right) \\
= \;& \frac{(x^*)^{\sigma-3}}{((\theta+\varepsilon)^{\sigma}-(x^*)^{\sigma})^2}\frac{1}{(\theta+\varepsilon)^{\sigma+1}}\left(-\left(\frac{x^*}{\theta+\varepsilon}\right)^{\sigma+1}+2\left(\frac{x^*}{\theta+\varepsilon}\right)^{\sigma}+(1-\sigma)\left(\frac{x^*}{\theta+\varepsilon}\right)+(\sigma-2)\right)
\end{aligned}
$$

Define

$$F^{11}(m) = -m^{\sigma+1}+2m^{\sigma}+(1-\sigma)m+(\sigma-2)$$

where

$$m = \frac{x^*}{\theta+\varepsilon} \in (0,1)$$

and then

$$
\begin{aligned}
\frac{d}{dm}F^{11}(m) &= 2m^{\sigma-1}\sigma - m^{\sigma} - \sigma - m^{\sigma}\sigma + 1 \\
\frac{d}{dm}\frac{d}{dm}F^{11}(m) &= -m^{\sigma-2}\sigma(m-2\sigma+m\sigma+2)
\end{aligned}
$$

When $\sigma > 3$,

$$m-2\sigma+m\sigma+2 = \sigma(m-2)+m+2 < 3(m-2)+m+2 = 4m-4 < 0$$

then $\frac{d}{dm}\frac{d}{dm}F^{11}(m) > 0$ and

$$\frac{d}{dm}F^{11}(m) < \frac{d}{dm}F^{11}(m=1) = 2\sigma-1-\sigma-\sigma+1 = 0$$

As a result,

$$F^{11}(m) > F^{11}(m=1) = -1 + 2 + (1-\sigma) + (\sigma - 2) = 0$$

Thus, $\frac{d}{dx^*}A^{11}(x^*, \theta) > 0$ and then $\int_{\underline{\theta}}^{x^*+\varepsilon} A^{12}(x^*, \theta) \frac{d}{dx^*}A^{11}(x^*, \theta)\, d\theta > 0$.

Step 2.2. Show that $\int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*, \theta) \frac{d}{dx^*}A^{12}(x^*, \theta)\, d\theta > 0$.

With assumption 3.1 that $\underline{\theta} = \varepsilon$, $A^{12}(x^*, \theta)$ can be simplified as

$$A^{12}(x^*, \theta) = \frac{1}{2}\frac{\sigma f}{\sigma - 1}\frac{1}{\overline{\theta} - \underline{\theta}}\frac{-(x^*)^2 + 4\varepsilon\overline{\theta} - (2\varepsilon)^2}{\theta + \varepsilon - x^*} + \frac{2\varepsilon f_e}{\theta + \varepsilon - x^*} - \frac{f}{\sigma - 1}$$

and

$$\frac{d}{dx^*}\left(\frac{1}{\theta + \varepsilon - x^*}\right) = \frac{1}{(\theta + \varepsilon - x^*)^2} > 0$$

$$\begin{aligned} & \frac{d}{dx^*}\left(\frac{-(x^*)^2 + 4\varepsilon\overline{\theta} - (2\varepsilon)^2}{\theta + \varepsilon - x^*}\right) \\ = & -\frac{1}{(\theta - x^* + \varepsilon)^2}\left(-(x^*)^2 + 2(x^*)\varepsilon + 2\theta(x^*) + 4\varepsilon^2 - 4\overline{\theta}\varepsilon\right) \\ = & \frac{1}{(\theta - x^* + \varepsilon)^2}\left((x^*)^2 - 2(x^*)(\theta + \varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) \end{aligned}$$

Then,

$$\begin{aligned} & \int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*, \theta)\frac{d}{dx^*}A^{12}(x^*, \theta)\, d\theta \\ = & \int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*, \theta)\left(\frac{1}{2}\frac{\sigma f}{\sigma - 1}\frac{1}{\overline{\theta} - \underline{\theta}}\frac{1}{(\theta - x^* + \varepsilon)^2}\left((x^*)^2 - 2(x^*)(\theta + \varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)\right) d\theta \\ & + \int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*, \theta)\left(\frac{1}{(\theta + \varepsilon - x^*)^2}\right) d\theta \end{aligned}$$

It is easy to see that the second term is positive. Consider the first term now.

- Rewrite the first term as following

$$\int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*,\theta)\left(\frac{1}{2}\frac{\sigma f}{\sigma-1}\frac{1}{\overline{\theta}-\underline{\theta}}\frac{1}{(\theta-x^*+\varepsilon)^2}\left((x^*)^2-2(x^*)(\theta+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon\right)\right)d\theta$$
$$= \frac{1}{2}\frac{\sigma^2 f}{\sigma-1}\frac{(x^*)^{\sigma-2}}{\overline{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{x^*+\varepsilon}\frac{(\theta+\varepsilon-x^*)}{(\theta+\varepsilon)^\sigma-(x^*)^\sigma}\frac{1}{(\theta-x^*+\varepsilon)^2}\left((x^*)^2-2(x^*)(\theta+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon\right)d\theta$$
$$= \frac{1}{2}\frac{\sigma^2 f}{\sigma-1}\frac{(x^*)^{\sigma-2}}{\overline{\theta}-\underline{\theta}}\int_{\underline{\theta}}^{x^*+\varepsilon}\frac{1}{(\theta+\varepsilon)^\sigma-(x^*)^\sigma}\frac{1}{(\theta-x^*+\varepsilon)}\left((x^*)^2-2(x^*)(\theta+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon\right)d\theta$$

Define

$$F^{12}(x^*,\theta)=\frac{1}{(\theta+\varepsilon)^\sigma-(x^*)^\sigma}\frac{1}{(\theta-x^*+\varepsilon)}>0$$

and then

$$\frac{d}{d\theta}F^{12}(x^*,\theta)=\frac{1}{((\theta+\varepsilon)^\sigma-(x^*)^\sigma)^2(\theta-x^*+\varepsilon)^2}\left((x^*)^\sigma+x^*\sigma(\theta+\varepsilon)^{\sigma-1}-(\theta+\varepsilon)^\sigma(1+\sigma)\right)$$

Since

$$(x^*)^\sigma+x^*\sigma(\theta+\varepsilon)^{\sigma-1}-(\theta+\varepsilon)^\sigma(1+\sigma)<(\theta+\varepsilon)^\sigma+(\theta+\varepsilon)\sigma(\theta+\varepsilon)^{\sigma-1}-(\theta+\varepsilon)^\sigma(1+\sigma)=0$$

$$\frac{d}{d\theta}F^{12}(x^*,\theta)<0$$

- Now consider $(x^*)^2-2(x^*)(\theta+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon$.

Define θ^* as the value of θ such that

$$(x^*)^2-2(x^*)(\theta^*+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon=0$$

And then $(x^*)^2-2(x^*)(\theta^*+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon>0$ when $\theta<\theta^*$, while $(x^*)^2-2(x^*)(\theta^*+\varepsilon)-4\varepsilon^2+4\overline{\theta}\varepsilon<0$ when $\theta>\theta^*$.

Define

$$A^{13}(x^*) = \int_{\underline{\theta}}^{x^*+\varepsilon} F^{12}(x^*,\theta)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta$$

If $\theta^* \leq \underline{\theta}$, $(x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon < 0$,

$$\begin{aligned}
F^{12}(x^*,\theta) &< F^{12}(x^*,\underline{\theta}) \\
&\Rightarrow F^{12}(x^*,\theta)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) \\
&> F^{12}(x^*,\underline{\theta})\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)
\end{aligned}$$

then,

$$\begin{aligned}
A^{13}(x^*) &> \int_{\underline{\theta}}^{x^*+\varepsilon} F^{12}(x^*,\underline{\theta})\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
&= F^{12}(x^*,\underline{\theta})\int_{\underline{\theta}}^{x^*+\varepsilon}\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta
\end{aligned}$$

where

$$\begin{aligned}
&\int_{\underline{\theta}}^{x^*+\varepsilon}\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
=\ &\int_{\underline{\theta}}^{x^*+\varepsilon} -2x^*\theta + \left((x^*)^2 - 2x^*\varepsilon - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
=\ &\left(-x^*\theta^2 + \left((x^*)^2 - 2x^*\varepsilon - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)\theta\right)|_{\underline{\theta}}^{x^*+\varepsilon} \\
=\ &-x^*\left((x^*+\varepsilon)^2 - \underline{\theta}^2\right) + \left((x^*)^2 - 2x^*\varepsilon - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)(x^*+\varepsilon-\underline{\theta}) \\
=\ &-(x^*)^3 - 2\varepsilon(x^*)^2 + (x^*)^3 - 2\varepsilon(x^*)^2 - 4\varepsilon^2(x^*) + 4\overline{\theta}\varepsilon(x^*) \\
=\ &4\varepsilon x^*\left(\overline{\theta} - \varepsilon - x^*\right)
\end{aligned}$$

since $x^* < \overline{\theta} - \varepsilon$,

$$\left(\overline{\theta} - \varepsilon - x^*\right) > 0$$

Thus,

$$\int_{\underline{\theta}}^{x^*+\varepsilon} \left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta > 0 \tag{B.12}$$

and $A^{13}(x^*) > 0$ for $\theta^* \leq \underline{\theta}$.

If $\underline{\theta} < \theta^* \leq x^* + \varepsilon$, for $\underline{\theta} < \theta < \theta^*$, $(x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon > 0$,

$$\begin{aligned}
F^{12}(x^*,\theta) &> F^{12}(x^*,\theta^*) \\
&\Rightarrow F^{12}(x^*,\theta)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) \\
&> F^{12}(x^*,\theta^*)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)
\end{aligned}$$

and for $\theta^* < \theta \leq x^* + \varepsilon$, $(x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon < 0$,

$$\begin{aligned}
F^{12}(x^*,\theta) &< F^{12}(x^*,\theta^*) \\
&\Rightarrow F^{12}(x^*,\theta)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) \\
&> F^{12}(x^*,\theta^*)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)
\end{aligned}$$

then,

$$\begin{aligned}
A^{13}(x^*) &= \int_{\underline{\theta}}^{\theta^*} F^{12}(x^*,\theta)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
&\quad + \int_{\theta^*}^{x^*+\varepsilon} F^{12}(x^*,\theta)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
&\geq \int_{\underline{\theta}}^{\theta^*} F^{12}(x^*,\theta^*)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
&\quad + \int_{\theta^*}^{x^*+\varepsilon} F^{12}(x^*,\theta^*)\left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta \\
&= F^{12}(x^*,\theta^*)\int_{\underline{\theta}}^{x^*+\varepsilon} \left((x^*)^2 - 2(x^*)(\theta+\varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta
\end{aligned}$$

From (B.12) and $F^{12}(x^*,\theta^*) > 0$, $A^{13}(x^*) > 0$ when $\underline{\theta} < \theta^* \leq x^* + \varepsilon$.

If $\theta^* > x^* + \varepsilon$, $(x^*)^2 - 2(x^*)(\theta + \varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon > 0$,

$$A^{13}(x^*) = \int_{\underline{\theta}}^{x^*+\varepsilon} F^{12}(x^*, \theta)\left((x^*)^2 - 2(x^*)(\theta + \varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right) d\theta > 0$$

In sum, $A^{13}(x^*) > 0$. Then,

$$\int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*, \theta)\left(\frac{1}{2}\frac{\sigma f}{\sigma - 1}\frac{1}{\overline{\theta} - \underline{\theta}}\frac{1}{(\theta - x^* + \varepsilon)^2}\left((x^*)^2 - 2(x^*)(\theta + \varepsilon) - 4\varepsilon^2 + 4\overline{\theta}\varepsilon\right)\right) > 0$$

and thus

$$\int_{\underline{\theta}}^{x^*+\varepsilon} A^{11}(x^*, \theta)\frac{d}{dx^*}A^{12}(x^*, \theta)\, d\theta > 0$$

Combine with $\int_{\underline{\theta}}^{x^*+\varepsilon} A^{12}(x^*, \theta)\frac{d}{dx^*}A^{11}(x^*, \theta)\, d\theta > 0$ and $A(x^*, x^* + \varepsilon) > 0$, it is proved that $\frac{d}{dx^*}u(x^*, I_{x^*}) > 0$. That is, $u(x^*, I_{x^*})$ is increasing when $\underline{\theta} - \varepsilon < x^* < \overline{\theta} - \varepsilon < \underline{\theta} + \varepsilon$.

B.7 Proof of Lemma 3.2

This section shows that $\frac{d}{dx^*}u(x^*, I_{x^*}) > 0$ for $\overline{\theta} - \varepsilon < x^* < \underline{\theta} + \varepsilon$.

Since

$$\underline{\theta} - \varepsilon < \overline{\theta} - \varepsilon \leq x^* < \underline{\theta} + \varepsilon$$

it can be shown that

$$x^* - \varepsilon < \underline{\theta} \leq \theta \leq \overline{\theta} \leq x^* + \varepsilon$$

and then the integration bounds are

$$\begin{aligned} \underline{\theta}(x^*) &= \underline{\theta} \\ \overline{\theta}(x^*) &= \overline{\theta} \end{aligned}$$

the productivity ratio is

$$\left(\frac{x^*}{\widetilde{x}\left(I_{x^*},\theta\right)}\right)^{\sigma-1} = Ra^1\left(x^*, I_{x^*}, \theta\right)$$

and the probability of staying and the *ex ante* probability of staying are

$$\begin{aligned} P_{stay}\left(I_{x^*},\theta\right) &= P^1_{stay}\left(I_{x^*},\theta\right) \\ P^e_{stay}\left(I_{x^*}\right) &= P^{e1}_{stay}\left(I_{x^*}\right) \end{aligned}$$

Thus, the expression (B.8) can be simplified as

$$\frac{d}{dx^*}u(x^*, I_{x^*}) = \int_{\underline{\theta}}^{\overline{\theta}} \frac{d}{dx^*}A\left(x^*,\theta\right)d\theta$$

where

$$\begin{aligned} A\left(x^*,\theta\right) &= \frac{1}{\overline{\theta}-\underline{\theta}}Ra^1\left(x^*, I_{x^*},\theta\right)\left(\frac{\sigma f}{\sigma-1}\left(\frac{P^{e1}_{stay}\left(I_{x^*}\right)}{P^1_{stay}\left(I_{x^*},\theta\right)}-1\right)+\frac{f_e}{P^1_{stay}\left(I_{x^*},\theta\right)}+f\right) \\ &= \frac{1}{\overline{\theta}-\underline{\theta}}\left(x^*\right)^{\sigma-1}\frac{\sigma\left(\theta+\varepsilon-x^*\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\left(\frac{\frac{\overline{\theta}+\underline{\theta}+2\varepsilon-2x^*}{4\varepsilon}}{\frac{\theta}{2\varepsilon}+\frac{\varepsilon-x^*}{2\varepsilon}}-1\right)+\frac{f_e}{\frac{\theta}{2\varepsilon}+\frac{\varepsilon-x^*}{2\varepsilon}}+f\right) \\ &= \frac{1}{\overline{\theta}-\underline{\theta}}\left(x^*\right)^{\sigma-1}\frac{\sigma\left(\theta+\varepsilon-x^*\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\left(\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)}\right)+\frac{2\varepsilon f_e}{\left(\theta+\varepsilon-x^*\right)}+f\right) \end{aligned}$$

Define

$$\begin{aligned} A^{21}\left(x^*,\theta\right) &= \left(x^*\right)^{\sigma-1}\frac{\sigma\left(\theta+\varepsilon-x^*\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}} > 0 \\ A^{22}\left(x^*,\theta\right) &= \left(\frac{\sigma f}{\sigma-1}\left(\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)}\right)+\frac{2\varepsilon f_e}{\left(\theta+\varepsilon-x^*\right)}+f\right) = \overline{\pi}\left(I_{x^*},\theta\right) > 0 \end{aligned}$$

and then

$$\frac{d}{dx^*}A\left(x^*,\theta\right)=\frac{1}{\overline{\theta}-\underline{\theta}}\left(A^{22}\left(x^*,\theta\right)\frac{d}{dx^*}A^{21}\left(x^*,\theta\right)+A^{21}\left(x^*,\theta\right)\frac{d}{dx^*}A^{22}\left(x^*,\theta\right)\right) \tag{B.13}$$

$$\frac{d}{dx^*}u(x^*,I_{x^*})=\int_{\underline{\theta}}^{\overline{\theta}}\frac{d}{dx^*}A\left(x^*,\theta\right)d\theta \tag{B.14}$$

$$=\frac{1}{\overline{\theta}-\underline{\theta}}\left(\int_{\underline{\theta}}^{\overline{\theta}}\left(A^{22}\left(x^*,\theta\right)\frac{d}{dx^*}A^{21}\left(x^*,\theta\right)\right)d\theta+\int_{\underline{\theta}}^{\overline{\theta}}\left(A^{21}\left(x^*,\theta\right)\frac{d}{dx^*}A^{22}\left(x^*,\theta\right)\right)d\theta\right)$$

Step 1. Show that $\int_{\underline{\theta}}^{\overline{\theta}}\left(A^{22}\left(x^*,\theta\right)\frac{d}{dx^*}A^{21}\left(x^*,\theta\right)\right)d\theta>0$.

$$\begin{aligned}
&\frac{d}{dx^*}A^{21}\left(x^*,\theta\right)\\
=\;&\frac{d}{dx^*}\left(\left(x^*\right)^{\sigma-1}\frac{\sigma\left(\theta+\varepsilon-x^*\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}}\right)\\
=\;&\frac{\sigma\left(x^*\right)^{\sigma-2}}{\left(\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}\right)^2}\left(\left(x^*\right)^{\sigma}\left(\theta+\varepsilon\right)-\left(x^*\right)\sigma\left(\theta+\varepsilon\right)^{\sigma}+\left(\sigma-1\right)\left(\theta+\varepsilon\right)^{\sigma+1}\right)\\
=\;&\frac{\sigma\left(x^*\right)^{\sigma-2}}{\left(\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}\right)^2}\left(\theta+\varepsilon\right)^{\sigma+1}\left(\left(\frac{x^*}{\theta+\varepsilon}\right)^{\sigma}-\sigma\frac{x^*}{\theta+\varepsilon}+\left(\sigma-1\right)\right)
\end{aligned}$$

Define

$$F^{21}\left(m\right)=m^{\sigma}-\sigma m+\left(\sigma-1\right)$$

where

$$m=\frac{x^*}{\theta+\varepsilon}<\frac{\underline{\theta}+\varepsilon}{\theta+\varepsilon}<1$$

and get

$$\frac{d}{dm}F^{21}\left(m\right)=\sigma\left(m^{\sigma-1}-1\right)<0$$

then

$$F^{21}\left(m\right)>F^{21}\left(1\right)=0 \tag{B.15}$$

Thus,

$$\frac{d}{dx^*}A^{21}\left(x^*,\theta\right) > 0 \tag{B.16}$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(A^{22}\left(x^*,\theta\right) \frac{d}{dx^*}A^{21}\left(x^*,\theta\right)\right) d\theta > 0 \tag{B.17}$$

Step 2. Show that $\int_{\underline{\theta}}^{\overline{\theta}} \left(A^{21}\left(x^*,\theta\right) \frac{d}{dx^*}A^{22}\left(x^*,\theta\right)\right) d\theta > 0$.

$$\begin{aligned}
& \frac{d}{dx^*}A^{22}\left(x^*,\theta\right) \\
= \; & \frac{d}{dx^*}\left(\frac{\sigma f}{\sigma-1}\left(\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)}\right)+\frac{2\varepsilon f_e}{\left(\theta+\varepsilon-x^*\right)}+f\right) \\
= \; & \frac{\sigma f}{\sigma-1}\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)^2}+\frac{2\varepsilon f_e}{\left(\theta+\varepsilon-x^*\right)^2}
\end{aligned}$$

then

$$\begin{aligned}
& \int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^*,\theta\right)\frac{d}{dx^*}A^{22}\left(x^*,\theta\right) d\theta \\
= \; & \int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^*,\theta\right)\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)^2}+\frac{2\varepsilon f_e}{\left(\theta+\varepsilon-x^*\right)^2}\right) d\theta \\
= \; & \int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^*,\theta\right)\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)^2} d\theta + \int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^*,\theta\right)\frac{2\varepsilon f_e}{\left(\theta+\varepsilon-x^*\right)^2} d\theta
\end{aligned}$$

the second term is positive since $A^{21}\left(x^*,\theta\right) > 0$, and now it is needed to show that $\int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^*,\theta\right)\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)^2} d\theta \geq 0$. Note that

$$\begin{aligned}
& A^{21}\left(x^*,\theta\right)\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)^2} \\
= \; & \left(x^*\right)^{\sigma-1}\frac{\sigma\left(\theta+\varepsilon-x^*\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}}\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^*\right)^2} \\
= \; & \left(x^*\right)^{\sigma-1}\frac{\sigma}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}}\frac{1}{2\left(\theta+\varepsilon-x^*\right)}\left(\overline{\theta}+\underline{\theta}-2\theta\right)
\end{aligned}$$

Define

$$F^{22}\left(x^*,\theta\right)=\left(x^*\right)^{\sigma-1}\frac{\sigma}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^*\right)^{\sigma}}\frac{1}{2\left(\theta+\varepsilon-x^*\right)}=\left(x^*\right)^{\sigma-1}\frac{\sigma}{2}F^{12}\left(x^*,\theta\right)>0$$

and since

$$\frac{d}{d\theta}F^{12}\left(x^*,\theta\right)<0$$

it is easy to see that

$$\frac{d}{d\theta}F^{22}\left(x^*,\theta\right)<0$$

For $\underline{\theta}<\theta<\frac{\overline{\theta}+\underline{\theta}}{2}$,

$$\begin{aligned}F^{22}\left(x^*,\theta\right) &> F^{22}\left(x^*,\frac{\overline{\theta}+\underline{\theta}}{2}\right)\\ F^{22}\left(x^*,\theta\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right) &> F^{22}\left(x^*,\frac{\overline{\theta}+\underline{\theta}}{2}\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)\end{aligned}$$

and for $\frac{\overline{\theta}+\underline{\theta}}{2}<\theta<\overline{\theta}$,

$$\begin{aligned}F^{22}\left(x^*,\theta\right) &< F^{22}\left(x^*,\frac{\overline{\theta}+\underline{\theta}}{2}\right)\\ F^{22}\left(x^*,\theta\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right) &> F^{22}\left(x^*,\frac{\overline{\theta}+\underline{\theta}}{2}\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)\end{aligned}$$

then

$$\begin{aligned}
& \int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^{*},\theta\right)\frac{\overline{\theta}+\underline{\theta}-2\theta}{2\left(\theta+\varepsilon-x^{*}\right)^{2}}d\theta \\
= & \int_{\underline{\theta}}^{\overline{\theta}} F^{22}\left(x^{*},\theta\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)d\theta \\
= & \int_{\underline{\theta}}^{\frac{\overline{\theta}+\underline{\theta}}{2}} F^{22}\left(x^{*},\theta\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)d\theta+\int_{\frac{\overline{\theta}+\underline{\theta}}{2}}^{\overline{\theta}} F^{22}\left(x^{*},\theta\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)d\theta \\
> & \int_{\underline{\theta}}^{\frac{\overline{\theta}+\underline{\theta}}{2}} F^{22}\left(x^{*},\frac{\overline{\theta}+\underline{\theta}}{2}\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)d\theta+\int_{\frac{\overline{\theta}+\underline{\theta}}{2}}^{\overline{\theta}} F^{22}\left(x^{*},\frac{\overline{\theta}+\underline{\theta}}{2}\right)\left(\overline{\theta}+\underline{\theta}-2\theta\right)d\theta \\
= & F^{22}\left(x^{*},\frac{\overline{\theta}+\underline{\theta}}{2}\right)\int_{\underline{\theta}}^{\overline{\theta}}\left(\overline{\theta}+\underline{\theta}-2\theta\right)d\theta \\
= & 0
\end{aligned}$$

and thus

$$\int_{\underline{\theta}}^{\overline{\theta}} A^{21}\left(x^{*},\theta\right)\frac{d}{dx^{*}}A^{22}\left(x^{*},\theta\right)d\theta>0$$

Combine with $\int_{\underline{\theta}}^{\overline{\theta}} A^{22}\left(x^{*},\theta\right)\frac{d}{dx^{*}}A^{21}\left(x^{*},\theta\right)d\theta>0$, it is proved that $\frac{d}{dx^{*}}u(x^{*},I_{x^{*}})>0$. That is, $u(x^{*},I_{x^{*}})$ is increasing when $\overline{\theta}-\varepsilon<x^{*}<\underline{\theta}+\varepsilon$.

B.8 Proof of Lemma 3.5

Since $u(x^{*},I_{x^{*}})$ is continuous and strictly increasing, once it can be shown that $u(x^{*},I_{x^{*}})<0$ when $x^{*}=\underline{\theta}-\varepsilon$ and $\lim_{x^{*}\to\underline{\theta}+\varepsilon^{-}}u(x^{*},I_{x^{*}})>0$, then the solution x^{**} to $u(x^{*},I_{x^{*}})=0$ exists and is unique. Now consider $u(x^{*},I_{x^{*}})$ when $x^{*}=\underline{\theta}-\varepsilon$. By (3.17),

$$
\begin{aligned}
& u(x^*, I_{x^*})|_{x_i=\underline{\theta}-\varepsilon} \\
= \; & \pi\left(\underline{\theta}-\varepsilon, I_{\underline{\theta}-\varepsilon}, \underline{\theta}\right) \\
= \; & Ra\left(\underline{\theta}-\varepsilon, I_{\underline{\theta}-\varepsilon}, \underline{\theta}\right)\left(\overline{\pi}\left(I_{\underline{\theta}-\varepsilon}, \underline{\theta}\right)+f\right)-f \\
= \; & \left(\underline{\theta}-\varepsilon\right)^{\sigma-1} \frac{2\varepsilon\sigma}{\left(\underline{\theta}+\varepsilon\right)^{\sigma}-\left(\underline{\theta}-\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\left(\frac{E_{stay}\left(I_{\underline{\theta}-\varepsilon}\right)}{P_{stay}\left(I_{\underline{\theta}-\varepsilon}, \underline{\theta}\right)}-1\right)+\frac{f_e}{P_{stay}\left(I_{\underline{\theta}-\varepsilon}, \underline{\theta}\right)}+f\right)-f \\
= \; & (0)^{\sigma-1} \frac{2\varepsilon\sigma}{\left(\underline{\theta}+\varepsilon\right)^{\sigma}-(0)^{\sigma}}\left(f_e+f\right)-f \\
= \; & -f<0
\end{aligned}
$$

Furthermore, with assumption 3.3 that $\lim_{x^* \to \underline{\theta}+\varepsilon^-} u(x^*, I_{x^*}) > 0$, lemma 3.5 can be obtained.

B.9 Proof of Lemma 3.6

B.9.1 x^{**} is in Region 1

Here exhibit the analysis when x^{**} is in region 1, and x is in region 1 as well.

First consider the case when $x = \underline{\theta} - \varepsilon \, (= 0)$. By (3.17),

$$
u(x, I_{x^{**}})|_{x=\underline{\theta}-\varepsilon} = \pi\left(\underline{\theta}-\varepsilon, I_{x^{**}}, \underline{\theta}\right) = \left(\frac{\underline{\theta}-\varepsilon}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right)-f = -f < 0
$$

Note that $u(x, I_{x^{**}})|_{x=\underline{\theta}-\varepsilon} = -f < 0$ holds no matter the relative precision of public and private information or the value of switching point x^{**}.

Now consider the case when $\underline{\theta} - \varepsilon < x < x^{**} < \overline{\theta} - \varepsilon < \underline{\theta} + \varepsilon$. That is, when x^{**} is region 1, and x is also in region 1. (3.35) can be written as

$$
u(x, I_{x^{**}}) = \frac{1}{x+\varepsilon-\underline{\theta}}\left(\int_{\underline{\theta}}^{x+\varepsilon}\left(\frac{x}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right)d\theta\right)-f
$$

and by assumption 3.1,

$$u(x, I_{x^{**}}) = \int_{\underline{\theta}}^{x+\varepsilon} x^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta - f$$

On the other hand, when $x = x^{**}$,

$$u(x^{**}, I_{x^{**}}) = \int_{\underline{\theta}}^{x^{**}+\varepsilon} (x^{**})^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta - f \tag{B.18}$$

thus

$$\begin{aligned}
& u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) \\
= & \int_{\underline{\theta}}^{x+\varepsilon} x^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\
& - \int_{\underline{\theta}}^{x+\varepsilon} (x^{**})^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\
& - \int_{x+\varepsilon}^{x^{**}+\varepsilon} (x^{**})^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\
= & \int_{\underline{\theta}}^{x+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2}\right) \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\
& - \int_{x+\varepsilon}^{x^{**}+\varepsilon} (x^{**})^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta
\end{aligned}$$

Since $x < x^{**}$ and

$$\left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) = \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \frac{1}{\sigma} r\left(\widetilde{x}\left(I_{x^{**}}, \theta\right), I_{x^{**}}, \theta\right) > 0$$

It can be shown that

$$\begin{aligned}
\int_{\underline{\theta}}^{x+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2}\right) \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta &< 0 \\
\int_{x+\varepsilon}^{x^{**}+\varepsilon} (x^{**})^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta &> 0
\end{aligned}$$

As a result,

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) < 0$$

That is,

$$u(x, I_{x^{**}}) < u(x^{**}, I_{x^{**}}) = 0$$

B.9.2 x^{**} is in Region 2

Now consider the cases when x^{**} is in region 2 or region 4. When x^{**} is in region 2, depending on whether x is in region 1 or 2, the expression of $u(x, I_{x^{**}})$ is different. Then, the proof is derived based on the regions of x. And when x^{**} is in region 4, the proof is exactly the same as the case when x^{**} is in region 1 and $x < x^{**}$.

x is in Region 1 Show that when $\sigma > 2$, if $\underline{\theta} - \varepsilon < x < \overline{\theta} - \varepsilon \leq x^{**} < \underline{\theta} + \varepsilon$, then $u(x, I_{x^{**}}) < 0$.

From (3.35),

$$\begin{aligned}
u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x) - \underline{\theta}(x)} \left(\int_{\underline{\theta}(x)}^{\overline{\theta}(x)} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\
&= \frac{1}{x + \varepsilon - \underline{\theta}} \left(\int_{\underline{\theta}}^{x+\varepsilon} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\
&= \int_{\underline{\theta}}^{x+\varepsilon} x^{\sigma-2} \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta - f
\end{aligned}$$

and

$$\begin{aligned}
u(x^{**}, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x^{**}) - \underline{\theta}(x^{**})} \left(\int_{\underline{\theta}(x^{**})}^{\overline{\theta}(x^{**})} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\
&= \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\
&= \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{x+\varepsilon} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \\
&\quad + \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{x+\varepsilon}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\
&\geq \frac{1}{x^{**}} \left(\int_{\underline{\theta}}^{x+\varepsilon} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \\
&\quad + \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{x+\varepsilon}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f
\end{aligned}$$

The inequality holds since

$$0 < \overline{\theta} - \varepsilon = \overline{\theta} - \underline{\theta} \leq x^{**}$$

Then,

$$\begin{aligned}
& u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) \\
\leq & \int_{\underline{\theta}}^{x+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2} \right) \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \\
& - \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{x+\varepsilon}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right)
\end{aligned}$$

Since $x < x^{**}$ and $\left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f) > 0$,

$$\begin{aligned}
\int_{\underline{\theta}}^{x+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2} \right) \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta &< 0 \\
\frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{x+\varepsilon}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) &> 0
\end{aligned}$$

and thus,

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) < 0$$

That is,

$$u(x, I_{x^{**}}) < u(x^{**}, I_{x^{**}}) = 0$$

x **is in Region 2** Show that for $\overline{\theta} - \varepsilon \leq x < x^{**} < \underline{\theta} + \varepsilon$, $u(x, I_{x^{**}}) < 0$. From (3.35),

$$\begin{aligned} u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x) - \underline{\theta}(x)} \left(\int_{\underline{\theta}(x)}^{\overline{\theta}(x)} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\ &= \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \end{aligned}$$

and then

$$\begin{aligned} & u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) \\ = & \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \\ & - \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \\ = & \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(x^{\sigma-1} - (x^{**})^{\sigma-1} \right) \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \end{aligned}$$

Since $x < x^{**}$ and $\left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f) > 0$,

$$\frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(x^{\sigma-1} - (x^{**})^{\sigma-1} \right) \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) < 0$$

and thus,

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) < 0$$

That is,

$$u(x, I_{x^{**}}) < u(x^{**}, I_{x^{**}}) = 0$$

B.10 Proof of Lemma 3.7

This section shows that when x^{**} is in region 1, 2 or 4, for $x > x^{**}$, $u(x, I_{x^{**}}) > 0$. When x^{**} is in region 1, x can fall in region 1, 2 or 3. When x^{**} is in region 2, x can fall in region 2 or 3. When x^{**} is in region 4, x can fall in region 4, 5 or 6. The proof is done for different cases separately.

B.10.1 x^{**} is in Region 1

When x^{**} is in region 1, depending on the value of x, the expression of $u(x, I_{x^{**}})$ is different. Consider the cases when x is in region 1, 2 or 3, separately. The proof of $u(x, I_{x^{**}}) > 0$ when $x = \overline{\theta} + \varepsilon$ will be included in the part when x is in region 3.

x is in Region 1 First show that if $\underline{\theta} - \varepsilon < x^{**} < x < \overline{\theta} - \varepsilon < \underline{\theta} + \varepsilon$, then $u(x, I_{x^{**}}) > 0$. (Both x^{**} and x are in region 1.) From (3.35),

$$\begin{aligned} u(x, I_{x^{**}}) &= \frac{1}{x+\varepsilon-\underline{\theta}} \left(\begin{array}{c} \int_{\underline{\theta}}^{x^{**}+\varepsilon} Ra^1 \left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\ + \int_{x^{**}+\varepsilon}^{x+\varepsilon} Ra^2 \left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \end{array} \right) - f \\ &= \frac{1}{x+\varepsilon-\underline{\theta}} \int_{\underline{\theta}}^{x^{**}+\varepsilon} \left(\frac{x}{\widetilde{x}\left(I_{x^{**}}, \theta\right)} \right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\ &\quad + \frac{1}{x+\varepsilon-\underline{\theta}} \int_{x^{**}+\varepsilon}^{x+\varepsilon} Ra^2 \left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta - f \end{aligned}$$

and by assumption 3.1,

$$\begin{aligned} u(x, I_{x^{**}}) &= \int_{\underline{\theta}}^{x^{**}+\varepsilon} x^{\sigma-2} \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)} \right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\ &\quad + \frac{1}{x+\varepsilon-\underline{\theta}} \int_{x^{**}+\varepsilon}^{x+\varepsilon} Ra^2 \left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta - f \end{aligned}$$

From (B.18),

$$
\begin{aligned}
& u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) \\
= \; & \int_{\underline{\theta}}^{x^{**}+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2}\right) \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta \\
& + \frac{1}{x+\varepsilon-\underline{\theta}} \int_{x^{**}+\varepsilon}^{x+\varepsilon} Ra^2\left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta
\end{aligned}
$$

Since $x > x^{**}$ and $\left(\frac{1}{\widetilde{x}(I_{x^{**}},\theta)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) > 0,\ Ra^2\left(x, I_{x^{**}}, \theta\right) > 0$,

$$
\begin{aligned}
\int_{\underline{\theta}}^{x^{**}+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2}\right) \left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1} \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta &> 0 \\
\frac{1}{x+\varepsilon-\underline{\theta}} \int_{x^{**}+\varepsilon}^{x+\varepsilon} Ra^2\left(x, I_{x^{**}}, \theta\right) \left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) d\theta &> 0
\end{aligned}
$$

and then,

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) > 0$$

That is,

$$u(x, I_{x^{**}}) > u(x^{**}, I_{x^{**}}) = 0$$

x is in Region 2 Second, show that if $\underline{\theta} - \varepsilon < x^{**} < \overline{\theta} - \varepsilon \leq x < \underline{\theta} + \varepsilon$, then $u(x, I_{x^{**}}) > 0$. (x^{**} is in region 1 and x in region 2.) From (3.35),

$$
\begin{aligned}
u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}-\underline{\theta}}\left(\begin{array}{c} \int_{\underline{\theta}}^{x^{**}+\varepsilon} Ra^1\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\ +\int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \end{array}\right)-f \\
&= \frac{1}{\overline{\theta}-\underline{\theta}} \int_{\underline{\theta}}^{x^{**}+\varepsilon}\left(\frac{x}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
&\quad +\frac{1}{\overline{\theta}-\underline{\theta}} \int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta-f \\
&\geq \frac{1}{x} \int_{\underline{\theta}}^{x^{**}+\varepsilon}\left(\frac{x}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
&\quad +\frac{1}{\overline{\theta}-\underline{\theta}} \int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta-f \\
&= \int_{\underline{\theta}}^{x^{**}+\varepsilon} x^{\sigma-2}\left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
&\quad +\frac{1}{\overline{\theta}-\underline{\theta}} \int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta-f
\end{aligned}
$$

The inequality comes from the fact that

$$
0<\overline{\theta}-\underline{\theta}=\overline{\theta}-\varepsilon \leq x
$$

From (B.18),

$$
\begin{aligned}
& u(x, I_{x^{**}})-u(x^{**}, I_{x^{**}}) \\
\geq\ & \int_{\underline{\theta}}^{x^{**}+\varepsilon} x^{\sigma-2}\left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
& +\frac{1}{\overline{\theta}-\underline{\theta}} \int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
& -\int_{\underline{\theta}}^{x^{**}+\varepsilon}\left(x^{**}\right)^{\sigma-2}\left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
=\ & \int_{\underline{\theta}}^{x^{**}+\varepsilon}\left(x^{\sigma-2}-\left(x^{**}\right)^{\sigma-2}\right)\left(\frac{1}{\widetilde{x}\left(I_{x^{**}}, \theta\right)}\right)^{\sigma-1}\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta \\
& +\frac{1}{\overline{\theta}-\underline{\theta}} \int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2\left(x, I_{x^{**}}, \theta\right)\left(\overline{\pi}\left(I_{x^{**}}, \theta\right)+f\right) d\theta
\end{aligned}
$$

Since $x > x^{**}$ and $\left(\frac{1}{\widetilde{x}(I_{x^{**}},\theta)}\right)^{\sigma-1}(\overline{\pi}(I_{x^{**}},\theta)+f) > 0,\ Ra^2(x, I_{x^{**}},\theta) > 0$,

$$\begin{aligned} \int_{\underline{\theta}}^{x^{**}+\varepsilon} \left(x^{\sigma-2} - (x^{**})^{\sigma-2}\right)\left(\frac{1}{\widetilde{x}(I_{x^{**}},\theta)}\right)^{\sigma-1}(\overline{\pi}(I_{x^{**}},\theta)+f)\,d\theta &> 0 \\ \frac{1}{\overline{\theta}-\underline{\theta}}\int_{x^{**}+\varepsilon}^{\overline{\theta}} Ra^2(x, I_{x^{**}},\theta)(\overline{\pi}(I_{x^{**}},\theta)+f)\,d\theta &> 0 \end{aligned}$$

and then,

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) > 0$$

That is,

$$u(x, I_{x^{**}}) > u(x^{**}, I_{x^{**}}) = 0$$

x is in Region 3 Lastly, show that if $\underline{\theta}-\varepsilon < x^{**} < \overline{\theta}-\varepsilon < \underline{\theta}+\varepsilon \leq x \leq \overline{\theta}+\varepsilon$, then $u(x, I_{x^{**}}) > 0$. (x^{**} is in region 1 and x in region 3.) It is first shown that $u(x, I_{x^{**}})$ is a continuous function with continuous first order derivative and the end points of $u(x, I_{x^{**}})$ in interval $\left[\underline{\theta}+\varepsilon, \overline{\theta}+\varepsilon\right]$ are positive. Then, to show $u(x, I_{x^{**}}) > 0$ for $\underline{\theta}+\varepsilon \leq x \leq \overline{\theta}+\varepsilon$ can be simplified as to show the local extrema is (are) positive. Write the expected payoff for firm with productivity level x. For $\underline{\theta}+\varepsilon \leq x < \overline{\theta}+\varepsilon$,

$$\begin{aligned} u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x)-\underline{\theta}(x)}\left(\int_{\underline{\theta}(x)}^{\overline{\theta}(x)}\left(\frac{x}{\widetilde{x}(I_{x^{**}},\theta)}\right)^{\sigma-1}(\overline{\pi}(I_{x^{**}},\theta)+f)\,d\theta\right) - f \\ &= \frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\int_{x-\varepsilon}^{\overline{\theta}} g(I_{x^{**}},\theta)\,d\theta - f \end{aligned}$$

where

$$\begin{aligned} g(I_{x^{**}},\theta) &= \left(\frac{1}{\widetilde{x}(I_{x^{**}},\theta)}\right)^{\sigma-1}(\overline{\pi}(I_{x^{**}},\theta)+f) \\ &= \begin{cases} g^1(I_{x^{**}},\theta) & \text{when } \theta \leq x^{**}+\varepsilon \\ g^2(I_{x^{**}},\theta) & \text{when } \theta > x^{**}+\varepsilon \end{cases} \end{aligned}$$

Here

$$\begin{aligned} g^1\left(I_{x^{**}},\theta\right) &= \frac{\sigma\left(\theta+\varepsilon-x^{**}\right)}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^{**}\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\left(\frac{P^e_{stay}\left(I_{x^{**}}\right)}{\frac{\theta}{2\varepsilon}+\frac{\varepsilon-x^{**}}{2\varepsilon}}-1\right)+\frac{f_e}{\frac{\theta}{2\varepsilon}+\frac{\varepsilon-x^{**}}{2\varepsilon}}+f\right) \\ &= \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^{\sigma}-\left(x^{**}\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}E_{stay}\left(I_{x^{**}}\right)+f_e+\frac{\left(x^{**}-\left(\theta+\varepsilon\right)\right)}{2\varepsilon}\frac{f}{\sigma-1}\right) \end{aligned}$$

and

$$\begin{aligned} g^2\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^{\sigma}-\left(\theta-\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\left(\frac{P^e_{stay}\left(I_{x^{**}}\right)}{1}-1\right)+\frac{f_e}{1}+f\right) \\ &= \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^{\sigma}-\left(\theta-\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right) \end{aligned}$$

If $x < x^{**}+2\varepsilon$,

$$u(x,I_{x^{**}}) = \frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\left(\int_{x-\varepsilon}^{x^{**}+\varepsilon} g^1\left(I_{x^{**}},\theta\right)d\theta+\int_{x^{**}+\varepsilon}^{\overline{\theta}} g^2\left(I_{x^{**}},\theta\right)d\theta\right)-f$$

and

$$\begin{aligned} \frac{d}{dx}u(x,I_{x^{**}}) &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\right)\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta+\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\frac{d}{dx}\left(\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta\right) \\ &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\right)\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta-\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}g^1\left(I_{x^{**}},x-\varepsilon\right) \end{aligned}$$

If $x \geq x^{**}+2\varepsilon$,

$$u(x,I_{x^{**}}) = \frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\left(\int_{x-\varepsilon}^{\overline{\theta}} g^2\left(I_{x^{**}},\theta\right)d\theta\right)-f$$

and

$$\begin{aligned} \frac{d}{dx}u(x,I_{x^{**}}) &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\right)\int_{x-\varepsilon}^{\overline{\theta}} g^2\left(I_{x^{**}},\theta\right)d\theta+\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\frac{d}{dx}\left(\int_{x-\varepsilon}^{\overline{\theta}} g^2\left(I_{x^{**}},\theta\right)d\theta\right) \\ &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\right)\int_{x-\varepsilon}^{\overline{\theta}} g^2\left(I_{x^{**}},\theta\right)d\theta-\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}g^2\left(I_{x^{**}},x-\varepsilon\right) \end{aligned}$$

for $x > x^{**} + 2\varepsilon$. Since

$$\lim_{x\to x^{**}+2\varepsilon^-} \frac{d}{dx}u(x, I_{x^{**}}) = \lim_{x\to x^{**}+2\varepsilon^+} \frac{d}{dx}u(x, I_{x^{**}})$$

and $u(x, I_{x^{**}})$ is continuous, $\frac{d}{dx}u(x, I_{x^{**}})$ is continuous at point $x = x^{**} + 2\varepsilon$. Summarize $\frac{d}{dx}u(x, I_{x^{**}})$ in the following expression

$$\frac{d}{dx}u(x, I_{x^{**}}) = \frac{d}{dx}\left(\frac{x^{\sigma-1}}{\overline{\theta} - (x-\varepsilon)}\right)\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}}, \theta\right) d\theta - \frac{x^{\sigma-1}}{\overline{\theta} - (x-\varepsilon)} g\left(I_{x^{**}}, x-\varepsilon\right)$$

Now, consider

$$\lim_{x\to\overline{\theta}+\varepsilon} u(x, I_{x^{**}}) = \lim_{x\to\overline{\theta}+\varepsilon} \frac{x^{\sigma-1}}{\overline{\theta} - (x-\varepsilon)}\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}}, \theta\right) d\theta - f$$

Consider $g\left(I_{x^{**}}, \theta\right)$ when $\theta \in \left[x-\varepsilon, \overline{\theta}\right]$ now. Since $P^e_{stay}\left(I_{x^{**}}\right)$ is a decreasing function with x^{**}, and $x^{**} < \overline{\theta} - \varepsilon$,

$$P^e_{stay}\left(I_{x^{**}}\right) > P^e_{stay}\left(I_{x^{**}}\right)|_{x^{**}=\overline{\theta}-\varepsilon} = \frac{\overline{\theta} + \underline{\theta} + 2\varepsilon - 2\left(\overline{\theta} - \varepsilon\right)}{4\varepsilon} = \frac{5\varepsilon - \overline{\theta}}{4\varepsilon}$$

then,

$$\begin{aligned}
g^1\left(I_{x^{**}}, \theta\right) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma - (x^{**})^\sigma}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right) + f_e + \frac{(x^{**} - (\theta+\varepsilon))}{2\varepsilon}\frac{f}{\sigma-1}\right) \\
&> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon} + f_e - \frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right) \\
&\geq \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon} + f_e - \frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)
\end{aligned}$$

and

$$\begin{aligned} g^2\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^{\sigma}-(\theta-\varepsilon)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}P^{e}_{stay}\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right) \\ &> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+f_e-\frac{f}{\sigma-1}\right) \\ &> \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right) \end{aligned}$$

That is,

$$g\left(I_{x^{**}},\theta\right) > \underline{g}$$

where $\underline{g}$ is a constant and is defined as

$$\underline{g} = \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)$$

By assumption 3.4, $\underline{g} > 0$. And then,

$$\begin{aligned} \lim_{x\to\overline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) &\geq \lim_{x\to\overline{\theta}+\varepsilon^-}\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\int_{x-\varepsilon}^{\overline{\theta}}\underline{g}\, d\theta - f \\ &= \lim_{x\to\overline{\theta}+\varepsilon^-}\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\underline{g}\left(\overline{\theta}-(x-\varepsilon)\right)-f \\ &= \underline{g}\left(\overline{\theta}+\varepsilon\right)^{\sigma-1}-f \end{aligned}$$

where

$$\begin{aligned} & \underline{g}\left(\overline{\theta}+\varepsilon\right)^{\sigma-1}-f \\ =\; & \left(\overline{\theta}+\varepsilon\right)^{\sigma-1}\frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)-f \\ =\; & \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)}\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)-f \\ >\; & \frac{\sigma\left(\frac{\sigma f}{\sigma-1}\frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+2\varepsilon f_e-\left(\overline{\theta}+\varepsilon\right)\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+2\varepsilon(2-\sigma)}-f > 0 \end{aligned}$$

by assumption 3.4. That is,

$$\lim_{x \to \overline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) > 0$$

Moreover, by left continuous of $u(x, I_{x^{**}})$ at point $x = \overline{\theta} + \varepsilon$,

$$u(x, I_{x^{**}})|_{x=\overline{\theta}+\varepsilon} = \lim_{x \to \overline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) > 0$$

From the definition of limitation, for $\epsilon > 0$, there exists a $\delta > 0$, such that for all x that satisfy $\left(\overline{\theta} + \varepsilon\right) - x < \delta$, the inequality $|u(x, I_{x^{**}}) - 0| < \epsilon$ holds. That is, a x' can be found such that $u(x, I_{x^{**}}) > 0$ for $x \in [x', \overline{\theta} + \varepsilon)$. That is, $u(x', I_{x^{**}}) > 0$. Moreover, since $u(x, I_{x^{**}})$ is continuous and $\lim_{x \to \underline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) > 0$, $u(\underline{\theta} + \varepsilon, I_{x^{**}}) > 0$. Then, to show $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 1 and x belongs to region 3 can be simplified as to show that all local extrema is (are) positive. Suppose x_1 satisfies the first order condition

$$\begin{aligned}
\frac{d}{dx} u(x, I_{x^{**}})|_{x=x_1} &= 0 \\
\int_{x_1-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}}, \theta\right) d\theta &= \frac{\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)} g\left(I_{x^{**}}, x_1 - \varepsilon\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)}
\end{aligned}$$

And then

$$\begin{aligned}
u(x_1, I_{x^{**}}) &= \frac{x_1^{\sigma-1}}{\overline{\theta} - (x_1 - \varepsilon)} \int_{x_1-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}}, \theta\right) d\theta - f \\
&= \frac{\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)^2 g\left(I_{x^{**}}, x_1 - \varepsilon\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)} - f \\
&= \frac{x_1^{\sigma} g\left(I_{x^{**}}, x_1 - \varepsilon\right)}{\left(\overline{\theta} + \varepsilon\right)(\sigma - 1) + x_1 (2 - \sigma)} - f
\end{aligned}$$

Consider $x_1^\sigma g\left(I_{x^{**}}, x_1-\varepsilon\right)$ now.

$$\begin{aligned}
& x_1^\sigma g^1\left(I_{x^{**}}, x_1-\varepsilon\right) \\
= \; & x_1^\sigma \frac{2\varepsilon\sigma}{x_1^\sigma-\left(x^{**}\right)^\sigma}\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e+\frac{\left(x^{**}-x_1\right)}{2\varepsilon} \frac{f}{\sigma-1}\right) \\
> \; & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e+\frac{\left(x^{**}-x_1\right)}{2\varepsilon} \frac{f}{\sigma-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
& x_1^\sigma g^2\left(I_{x^{**}}, x_1-\varepsilon\right) \\
= \; & x_1^\sigma \frac{2\varepsilon\sigma}{x_1^\sigma-\left(x_1-2\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right) \\
> \; & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right)
\end{aligned}$$

Since $\underline{\theta}+\varepsilon<x_1<\overline{\theta}+\varepsilon$,

$$\begin{aligned}
x^{**}-x_1 &> -x_1>-\left(\overline{\theta}+\varepsilon\right) \\
\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+x_1(2-\sigma) &< \left(\overline{\theta}+\varepsilon\right)(\sigma-1)+\left(\underline{\theta}+\varepsilon\right)(2-\sigma)
\end{aligned}$$

As a result, when $x<x^{**}+2\varepsilon$,

$$u(x_1, I_{x^{**}})>\frac{\sigma\left(\frac{\sigma f}{\sigma-1} \frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+2\varepsilon f_e-\left(\overline{\theta}+\varepsilon\right) \frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+2\varepsilon(2-\sigma)}-f>0$$

and when $x \geq x^{**}+2\varepsilon$,

$$\begin{aligned}
u(x_1, I_{x^{**}}) &> \frac{\sigma\left(\frac{\sigma f}{\sigma-1} \frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+2\varepsilon f_e-2\varepsilon \frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+2\varepsilon(2-\sigma)}-f \\
&> \frac{\sigma\left(\frac{\sigma f}{\sigma-1} \frac{5\varepsilon-\overline{\theta}}{4\varepsilon}+2\varepsilon f_e-\left(\overline{\theta}+\varepsilon\right) \frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+2\varepsilon(2-\sigma)}-f>0
\end{aligned}$$

by assumption 3.4. In sum, $u(x_1, I_{x^{**}}) > 0$ for $\frac{d}{dx}u(x, I_{x^{**}})|_{x=x_1} = 0$ and thus $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 1, x belongs to region 3. Combined with the fact that $u(x, I_{x^{**}})|_{x=\overline{\theta}+\varepsilon} > 0$, the result is $u(x, I_{x^{**}}) > 0$ for $\underline{\theta} + \varepsilon \leq x \leq \overline{\theta} + \varepsilon$.

B.10.2 x^{**} is in Region 2

x is in Region 2 Show that when $\overline{\theta} - \varepsilon \leq x^{**} < x < \underline{\theta} + \varepsilon$, $u(x, I_{x^{**}}) > 0$. From (3.35),

$$\begin{aligned} u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x) - \underline{\theta}(x)} \left(\int_{\underline{\theta}(x)}^{\overline{\theta}(x)} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\ &= \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \end{aligned}$$

and then

$$\begin{aligned} & u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) \\ = & \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \\ & - \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{x^{**}}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \\ = & \frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(x^{\sigma-1} - (x^{**})^{\sigma-1} \right) \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) \end{aligned}$$

Since $x > x^{**}$ and $\left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f) > 0$,

$$\frac{1}{\overline{\theta} - \underline{\theta}} \left(\int_{\underline{\theta}}^{\overline{\theta}} \left(x^{\sigma-1} - (x^{**})^{\sigma-1} \right) \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) > 0$$

and thus,

$$u(x, I_{x^{**}}) - u(x^{**}, I_{x^{**}}) > 0$$

That is,

$$u(x, I_{x^{**}}) > u(x^{**}, I_{x^{**}}) = 0$$

x **is in Region 3** When $\underline{\theta} - \varepsilon < \overline{\theta} - \varepsilon \leq x^{**} < \underline{\theta} + \varepsilon$, and $\underline{\theta} + \varepsilon \leq x < \overline{\theta} + \varepsilon$,

$$x < \overline{\theta} + \varepsilon = \left(\overline{\theta} - \varepsilon\right) + 2\varepsilon \leq x^{**} + 2\varepsilon$$

then,

$$u(x, I_{x^{**}}) = \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \left(\int_{x-\varepsilon}^{x^{**}+\varepsilon} g^1 \left(I_{x^{**}}, \theta\right) d\theta + \int_{x^{**}+\varepsilon}^{\overline{\theta}} g^2 \left(I_{x^{**}}, \theta\right) d\theta \right) - f$$

and

$$\frac{d}{dx} u(x, I_{x^{**}}) = \frac{d}{dx} \left(\frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \right) \int_{x-\varepsilon}^{\overline{\theta}} g \left(I_{x^{**}}, \theta\right) d\theta - \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} g^1 \left(I_{x^{**}}, x - \varepsilon\right)$$

Now, consider

$$\lim_{x \to \overline{\theta}+\varepsilon} u(x, I_{x^{**}}) = \lim_{x \to \overline{\theta}+\varepsilon} \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \int_{x-\varepsilon}^{\overline{\theta}} g \left(I_{x^{**}}, \theta\right) d\theta - f$$

Consider $g\left(I_{x^{**}}, \theta\right)$ when $\theta \in \left[x - \varepsilon, \overline{\theta}\right]$ now. Since $P^e_{stay}\left(I_{x^{**}}\right)$ is a decreasing function with x^{**}, and $x^{**} < \underline{\theta} + \varepsilon$,

$$P^e_{stay}\left(I_{x^{**}}\right) > P^e_{stay}\left(I_{x^{**}}\right)|_{x^{**}=\underline{\theta}+\varepsilon} = \frac{\overline{\theta} + \underline{\theta} + 2\varepsilon - 2\left(\underline{\theta} + \varepsilon\right)}{4\varepsilon} = \frac{\overline{\theta} - \varepsilon}{4\varepsilon}$$

then,

$$\begin{aligned}
g^1\left(I_{x^{**}}, \theta\right) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^{\sigma} - (x^{**})^{\sigma}} \left(\frac{\sigma f}{\sigma - 1} P^e_{stay}\left(I_{x^{**}}\right) + f_e + \frac{\left(x^{**} - (\theta + \varepsilon)\right)}{2\varepsilon} \frac{f}{\sigma - 1} \right) \\
&> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^{\sigma}} \left(\frac{\sigma f}{\sigma - 1} \frac{\overline{\theta} - \varepsilon}{4\varepsilon} + f_e - \frac{\theta + \varepsilon}{2\varepsilon} \frac{f}{\sigma - 1} \right) \\
&\geq \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}} \left(\frac{\sigma f}{\sigma - 1} \frac{\overline{\theta} - \varepsilon}{4\varepsilon} + f_e - \frac{\overline{\theta} + \varepsilon}{2\varepsilon} \frac{f}{\sigma - 1} \right)
\end{aligned}$$

and

$$
\begin{aligned}
g^2\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma-(\theta-\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right)\\
&> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+f_e-\frac{f}{\sigma-1}\right)\\
&> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+f_e-\frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)\\
&\geq \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)
\end{aligned}
$$

That is,

$$
g\left(I_{x^{**}},\theta\right) > \underline{g}
$$

where $\underline{g}$ is a constant and is defined as

$$
\underline{g} = \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)
$$

By assumption 3.4, $\underline{g} > 0$. And then,

$$
\begin{aligned}
\lim_{x\to\overline{\theta}+\varepsilon^-} u(x,I_{x^{**}}) &\geq \lim_{x\to\overline{\theta}+\varepsilon^-}\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\int_{x-\varepsilon}^{\overline{\theta}}\underline{g}d\theta - f\\
&= \lim_{x\to\overline{\theta}+\varepsilon^-}\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\underline{g}\left(\overline{\theta}-(x-\varepsilon)\right)-f\\
&= \underline{g}\left(\overline{\theta}+\varepsilon\right)^{\sigma-1}-f
\end{aligned}
$$

where

$$\begin{aligned}
& \underline{g}\left(\overline{\theta}+\varepsilon\right)^{\sigma-1}-f \\
= & \left(\overline{\theta}+\varepsilon\right)^{\sigma-1}\frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)-f \\
= & \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+f_e-\frac{\overline{\theta}+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)-f \\
> & \frac{\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{4\varepsilon}+2\varepsilon f_e-\left(\overline{\theta}+\varepsilon\right)\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+2\varepsilon(2-\sigma)}-f>0
\end{aligned}$$

by assumption 3.4. That is,

$$\lim_{x\to\overline{\theta}+\varepsilon^-} u(x,I_{x^{**}})>0$$

Moreover, by left continuous of $u(x,I_{x^{**}})$ at point $x=\overline{\theta}+\varepsilon$,

$$u(x,I_{x^{**}})|_{x=\overline{\theta}+\varepsilon}=\lim_{x\to\overline{\theta}+\varepsilon^-} u(x,I_{x^{**}})>0$$

From the definition of limitation, for $\epsilon>0$, there exists a $\delta>0$, such that for all x that satisfy $\left(\overline{\theta}+\varepsilon\right)-x<\delta$, the inequality $|u(x,I_{x^{**}})-0|<\epsilon$ holds. That is, a x' can be found such that $u(x,I_{x^{**}})>0$ for $x\in[x',\overline{\theta}+\varepsilon)$. That is, $u(x',I_{x^{**}})>0$. Moreover, since $u(x,I_{x^{**}})$ is continuous and $\lim_{x\to\underline{\theta}+\varepsilon^-}u(x,I_{x^{**}})>0$, $u(\underline{\theta}+\varepsilon,I_{x^{**}})>0$. Then, to show $u(x,I_{x^{**}})>0$ when x^{**} belongs to region 2 and x belongs to region 3 can be simplified as to show that all local extrema is(are) positive. Suppose x_1 satisfie(s) the first order condition

$$\begin{aligned}
\frac{d}{dx}u(x,I_{x^{**}})|_{x=x_1} &= 0 \\
\int_{x_1-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta &= \frac{\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}g^1\left(I_{x^{**}},x_1-\varepsilon\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)}
\end{aligned}$$

And then

$$
\begin{aligned}
u(x_1, I_{x^{**}}) &= \frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\int_{x_1-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta - f \\
&= \frac{\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)^2 g^1\left(I_{x^{**}}, x_1-\varepsilon\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)} - f \\
&= \frac{x_1^{\sigma} g^1\left(I_{x^{**}}, x_1-\varepsilon\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+x_1(2-\sigma)} - f
\end{aligned}
$$

Consider $x_1^{\sigma} g^1\left(I_{x^{**}}, x_1-\varepsilon\right)$ now.

$$
\begin{aligned}
& x_1^{\sigma} g^1\left(I_{x^{**}}, x_1-\varepsilon\right) \\
= \ & x_1^{\sigma}\frac{2\varepsilon\sigma}{x_1^{\sigma}-(x^{**})^{\sigma}}\left(\frac{\sigma f}{\sigma-1}P^{e}_{stay}\left(I_{x^{**}}\right)+f_e+\frac{(x^{**}-x_1)}{2\varepsilon}\frac{f}{\sigma-1}\right) \\
> \ & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1}P^{e}_{stay}\left(I_{x^{**}}\right)+f_e+\frac{(x^{**}-x_1)}{2\varepsilon}\frac{f}{\sigma-1}\right)
\end{aligned}
$$

Since $\underline{\theta}+\varepsilon < x_1 < \overline{\theta}+\varepsilon$,

$$
\begin{aligned}
x^{**}-x_1 &> -x_1 > -\left(\overline{\theta}+\varepsilon\right) \\
\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+x_1(2-\sigma) &< \left(\overline{\theta}+\varepsilon\right)(\sigma-1)+\left(\underline{\theta}+\varepsilon\right)(2-\sigma)
\end{aligned}
$$

As a result,

$$
u(x_1, I_{x^{**}}) > \frac{\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-\varepsilon}{2}+2\varepsilon f_e-\left(\overline{\theta}+\varepsilon\right)\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+2\varepsilon(2-\sigma)} - f > 0
$$

by assumption 3.4. In sum, $u(x_1, I_{x^{**}}) > 0$ for $\frac{d}{dx}u(x, I_{x^{**}})|_{x=x_1} = 0$ and thus $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 2, x belongs to region 3. Combine with the fact that $u(x, I_{x^{**}})|_{x=\overline{\theta}+\varepsilon} > 0$, the result is $u(x, I_{x^{**}}) > 0$ for $\underline{\theta}+\varepsilon \leq x \leq \overline{\theta}+\varepsilon$.

B.10.3 x^{**} is in Region 4

x is in Region 4 The proof of this case is the same as the case when x^{**} belongs to region 1 and x belongs to region 4.

x is in Region 5 When $\underline{\theta} - \varepsilon < x^{**} < \underline{\theta} + \varepsilon \leq \overline{\theta} - \varepsilon$, and $\underline{\theta} + \varepsilon \leq x < \overline{\theta} - \varepsilon$,

$$\begin{aligned} u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x) - \underline{\theta}(x)} \left(\int_{\underline{\theta}(x)}^{\overline{\theta}(x)} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\ &= \frac{x^{\sigma-1}}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} g(I_{x^{**}}, \theta)\, d\theta - f \end{aligned}$$

where

$$\begin{aligned} g(I_{x^{**}}, \theta) &= \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f) \\ &= \begin{cases} g^1(I_{x^{**}}, \theta) & \text{when } \theta \leq x^{**} + \varepsilon \\ g^2(I_{x^{**}}, \theta) & \text{when } \theta > x^{**} + \varepsilon \end{cases} \end{aligned}$$

Here

$$\begin{aligned} g^1(I_{x^{**}}, \theta) &= \frac{\sigma(\theta + \varepsilon - x^{**})}{(\theta+\varepsilon)^\sigma - (x^{**})^\sigma} \left(\frac{\sigma f}{\sigma - 1} \left(\frac{P^e_{stay}(I_{x^{**}})}{\frac{\theta}{2\varepsilon} + \frac{\varepsilon - x^{**}}{2\varepsilon}} - 1 \right) + \frac{f_e}{\frac{\theta}{2\varepsilon} + \frac{\varepsilon - x^{**}}{2\varepsilon}} + f \right) \\ &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma - (x^{**})^\sigma} \left(\frac{\sigma f}{\sigma-1} P^e_{stay}(I_{x^{**}}) + f_e + \frac{(x^{**} - (\theta+\varepsilon))}{2\varepsilon} \frac{f}{\sigma - 1} \right) \end{aligned}$$

and

$$\begin{aligned} g^2(I_{x^{**}}, \theta) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma - (\theta-\varepsilon)^\sigma} \left(\frac{\sigma f}{\sigma - 1} \left(\frac{P^e_{stay}(I_{x^{**}})}{1} - 1 \right) + \frac{f_e}{1} + f \right) \\ &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma - (\theta-\varepsilon)^\sigma} \left(\frac{\sigma f}{\sigma-1} P^e_{stay}(I_{x^{**}}) + f_e - \frac{f}{\sigma - 1} \right) \end{aligned}$$

If $x < x^{**} + 2\varepsilon$,

$$u(x, I_{x^{**}}) = \frac{x^{\sigma-1}}{2\varepsilon}\left(\int_{x-\varepsilon}^{x^{**}+\varepsilon} g^1\left(I_{x^{**}}, \theta\right) d\theta + \int_{x^{**}+\varepsilon}^{x+\varepsilon} g^2\left(I_{x^{**}}, \theta\right) d\theta\right) - f$$

and

$$\begin{aligned}\frac{d}{dx}u(x, I_{x^{**}}) &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{2\varepsilon}\right)\int_{x-\varepsilon}^{x+\varepsilon} g\left(I_{x^{**}}, \theta\right) d\theta + \frac{x^{\sigma-1}}{2\varepsilon}\frac{d}{dx}\left(\int_{x-\varepsilon}^{x+\varepsilon} g\left(I_{x^{**}}, \theta\right) d\theta\right) \\ &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{2\varepsilon}\right)\int_{x-\varepsilon}^{x+\varepsilon} g\left(I_{x^{**}}, \theta\right) d\theta + \frac{x^{\sigma-1}}{2\varepsilon}\left(g^2\left(I_{x^{**}}, x+\varepsilon\right) - g^1\left(I_{x^{**}}, x-\varepsilon\right)\right)\end{aligned}$$

If $x \geq x^{**} + 2\varepsilon$,

$$u(x, I_{x^{**}}) = \frac{x^{\sigma-1}}{2\varepsilon}\left(\int_{x-\varepsilon}^{x+\varepsilon} g^2\left(I_{x^{**}}, \theta\right) d\theta\right) - f$$

and

$$\begin{aligned}\frac{d}{dx}u(x, I_{x^{**}}) &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{2\varepsilon}\right)\int_{x-\varepsilon}^{x+\varepsilon} g^2\left(I_{x^{**}}, \theta\right) d\theta + \frac{x^{\sigma-1}}{2\varepsilon}\frac{d}{dx}\left(\int_{x-\varepsilon}^{x+\varepsilon} g^2\left(I_{x^{**}}, \theta\right) d\theta\right) \\ &= \frac{d}{dx}\left(\frac{x^{\sigma-1}}{2\varepsilon}\right)\int_{x-\varepsilon}^{x+\varepsilon} g^2\left(I_{x^{**}}, \theta\right) d\theta + \frac{x^{\sigma-1}}{2\varepsilon}\left(g^2\left(I_{x^{**}}, x+\varepsilon\right) - g^2\left(I_{x^{**}}, x-\varepsilon\right)\right)\end{aligned}$$

for $x > x^{**} + 2\varepsilon$. Since

$$\lim_{x\to x^{**}+2\varepsilon^-}\frac{d}{dx}u(x, I_{x^{**}}) = \lim_{x\to x^{**}+2\varepsilon^+}\frac{d}{dx}u(x, I_{x^{**}})$$

and $u(x, I_{x^{**}})$ is continuous, $\frac{d}{dx}u(x, I_{x^{**}})$ is continuous at point $x = x^{**} + 2\varepsilon$. Summarize $\frac{d}{dx}u(x, I_{x^{**}})$ in the following expression

$$\frac{d}{dx}u(x, I_{x^{**}}) = \frac{d}{dx}\left(\frac{x^{\sigma-1}}{2\varepsilon}\right)\int_{x-\varepsilon}^{x+\varepsilon} g\left(I_{x^{**}}, \theta\right) d\theta + \frac{x^{\sigma-1}}{2\varepsilon}\left(g\left(I_{x^{**}}, x+\varepsilon\right) - g\left(I_{x^{**}}, x-\varepsilon\right)\right)$$

Now, consider

$$\lim_{x\to\overline{\theta}-\varepsilon^-} u(x, I_{x^{**}}) = \lim_{x\to\overline{\theta}-\varepsilon^-} \frac{x^{\sigma-1}}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} g\left(I_{x^{**}},\theta\right)d\theta - f$$

Consider $g\left(I_{x^{**}},\theta\right)$ when $\theta \in [x-\varepsilon, x+\varepsilon]$ now. Since $P^e_{stay}\left(I_{x^{**}}\right)$ is a decreasing function with x^{**}, and $x^{**} < \underline{\theta} + \varepsilon$,

$$P^e_{stay}\left(I_{x^{**}}\right) > P^e_{stay}\left(I_{x^{**}}\right)|_{x^{**}=\underline{\theta}+\varepsilon} = \frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta} - \ x^* - \frac{\left(\underline{\theta}+\varepsilon-x^*\right)^2}{4\varepsilon}\ \right)|_{x^{**}=\underline{\theta}+\varepsilon} = \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}$$

then,

$$\begin{aligned} g^1\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^\sigma - \left(x^{**}\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right) + f_e + \frac{\left(x^{**}-\left(\theta+\varepsilon\right)\right)}{2\varepsilon}\frac{f}{\sigma-1}\right) \\ &> \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right) \end{aligned}$$

and

$$\begin{aligned} g^2\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^\sigma - \left(\theta-\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right) + f_e - \frac{f}{\sigma-1}\right) \\ &> \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{f}{\sigma-1}\right) \\ &> \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right) \end{aligned}$$

Then,

$$\begin{aligned} \lim_{x\to\overline{\theta}-\varepsilon^-} u(x, I_{x^{**}}) &= \lim_{x\to\overline{\theta}-\varepsilon^-} \frac{x^{\sigma-1}}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} g\left(I_{x^{**}},\theta\right)d\theta - f \\ &\geq \lim_{x\to\overline{\theta}-\varepsilon^-} \frac{x^{\sigma-1}}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} \frac{2\varepsilon\sigma}{\left(\theta+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)d\theta - f \\ &= \left(\overline{\theta}-\varepsilon\right)^{\sigma-1}\int_{\overline{\theta}-2\varepsilon}^{\overline{\theta}} \frac{\sigma}{\left(\theta+\varepsilon\right)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)d\theta - f \\ &> 0 \end{aligned}$$

by assumption 3.4. That is,

$$\lim_{x \to \overline{\theta} - \varepsilon^-} u(x, I_{x^{**}}) > 0$$

Moreover, by left continuous of $u(x, I_{x^{**}})$ at point $x = \overline{\theta} - \varepsilon$,

$$u(x, I_{x^{**}})|_{x=\overline{\theta}-\varepsilon} = \lim_{x \to \overline{\theta} - \varepsilon^-} u(x, I_{x^{**}}) > 0$$

From the definition of limitation, for $\epsilon > 0$, there exists a $\delta > 0$, such that for all x that satisfy $\left(\overline{\theta} - \varepsilon\right) - x < \delta$, the inequality $|u(x, I_{x^{**}}) - 0| < \epsilon$ holds. That is, a x' can be found such that $u(x, I_{x^{**}}) > 0$ for $x \in [x', \overline{\theta} - \varepsilon)$. That is, $u(x', I_{x^{**}}) > 0$. Moreover, since $u(x, I_{x^{**}})$ is continuous and $\lim_{x \to \underline{\theta} + \varepsilon^-} u(x, I_{x^{**}}) > 0$, $u(\underline{\theta} + \varepsilon, I_{x^{**}}) > 0$. Then, to show $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 4 and x belongs to region 5 can be simplified as to show that all local extrema is(are) positive. Suppose x_1 satisfie(s) the first order condition

$$\begin{aligned}
\frac{d}{dx} u(x, I_{x^{**}})|_{x=x_1} &= 0 \\
\int_{x_1-\varepsilon}^{x_1+\varepsilon} g\left(I_{x^{**}}, \theta\right) d\theta &= \frac{\frac{x_1^{\sigma-1}}{2\varepsilon}\left(g\left(I_{x^{**}}, x_1 - \varepsilon\right) - g\left(I_{x^{**}}, x_1 + \varepsilon\right)\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{2\varepsilon}\right)}
\end{aligned}$$

And then

$$\begin{aligned}
u(x_1, I_{x^{**}}) &= \frac{x_1^{\sigma-1}}{2\varepsilon} \int_{x_1-\varepsilon}^{x_1+\varepsilon} g\left(I_{x^{**}}, \theta\right) d\theta - f \\
&= \frac{\left(\frac{x_1^{\sigma-1}}{2\varepsilon}\right)^2 \left(g\left(I_{x^{**}}, x_1 - \varepsilon\right) - g\left(I_{x^{**}}, x_1 + \varepsilon\right)\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{2\varepsilon}\right)} - f \\
&= \frac{x_1^{\sigma}\left(g\left(I_{x^{**}}, x_1 - \varepsilon\right) - g\left(I_{x^{**}}, x_1 + \varepsilon\right)\right)}{2\varepsilon\left(\sigma - 1\right)} - f
\end{aligned}$$

Consider $x_1^\sigma \left(g\left(I_{x^{**}}, x_1-\varepsilon\right)-g\left(I_{x^{**}}, x_1+\varepsilon\right)\right)$ now. Since $\underline{\theta}+\varepsilon<x_1<\overline{\theta}-\varepsilon$,

$$\begin{aligned}
& x_1^\sigma\left(g^1\left(I_{x^{**}}, x_1-\varepsilon\right)-g^2\left(I_{x^{**}}, x_1+\varepsilon\right)\right) \\
= \; & x_1^\sigma\left[\begin{array}{c}
\frac{2\varepsilon\sigma}{(x_1)^\sigma-(x^{**})^\sigma}\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e+\frac{(x^{**}-(x_1))}{2\varepsilon} \frac{f}{\sigma-1}\right) \\
-\frac{2\varepsilon\sigma}{(x_1+2\varepsilon)^\sigma-(x_1)^\sigma}\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right)
\end{array}\right] \\
> \; & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{(\underline{\theta}+3\varepsilon)}{2\varepsilon} \frac{f}{\sigma-1}\right)-\frac{2\varepsilon\sigma}{\left(1+\frac{2\varepsilon}{x_1}\right)^\sigma-1}\left(\frac{\sigma f}{\sigma-1}+f_e-\frac{f}{\sigma-1}\right) \\
\geq \; & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{(\underline{\theta}+3\varepsilon)}{2\varepsilon} \frac{f}{\sigma-1}\right)-\frac{2\varepsilon\sigma}{\left(1+\frac{2\varepsilon}{\underline{\theta}+3\varepsilon}\right)^\sigma-1}\left(f+f_e\right)
\end{aligned}$$

and

$$\begin{aligned}
& x_1^\sigma\left(g^2\left(I_{x^{**}}, x_1-\varepsilon\right)-g^2\left(I_{x^{**}}, x_1+\varepsilon\right)\right) \\
= \; & x_1^\sigma\left[\begin{array}{c}
\frac{2\varepsilon\sigma}{(x_1)^\sigma-(x_1-2\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right) \\
-\frac{2\varepsilon\sigma}{(x_1+2\varepsilon)^\sigma-(x_1)^\sigma}\left(\frac{\sigma f}{\sigma-1} P_{stay}^e\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right)
\end{array}\right] \\
> \; & x_1^\sigma\left(\frac{2\varepsilon\sigma}{(x_1)^\sigma-(x_1-2\varepsilon)^\sigma}-\frac{2\varepsilon\sigma}{(x_1+2\varepsilon)^\sigma-(x_1)^\sigma}\right)\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{f}{\sigma-1}\right) \\
> \; & \left(2\varepsilon\sigma-\frac{2\varepsilon\sigma}{\left(1+\frac{2\varepsilon}{\overline{\theta}-\varepsilon}\right)^\sigma-1}\right)\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{f}{\sigma-1}\right)
\end{aligned}$$

As a result, when $x<x^{**}+2\varepsilon$,

$$u(x_1, I_{x^{**}})>\frac{\sigma}{(\sigma-1)}\left(\begin{array}{c}
\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{(\underline{\theta}+3\varepsilon)}{2\varepsilon} \frac{f}{\sigma-1}\right) \\
-\frac{1}{\left(1+\frac{2\varepsilon}{\underline{\theta}+3\varepsilon}\right)^\sigma-1}\left(f+f_e\right)
\end{array}\right)-f>0$$

and when $x \geq x^{**}+2\varepsilon$,

$$u(x_1, I_{x^{**}})>\frac{\sigma}{(\sigma-1)}\left(\left(1-\frac{1}{\left(1+\frac{2\varepsilon}{\overline{\theta}-\varepsilon}\right)^\sigma-1}\right)\left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{f}{\sigma-1}\right)\right)-f>0$$

by assumption 3.4.

In sum, $u(x_1, I_{x^{**}}) > 0$ for $\frac{d}{dx}u(x, I_{x^{**}})|_{x=x_1} = 0$ and thus $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 4, x belongs to region 5.

x is in Region 6 When $\underline{\theta} - \varepsilon < x^{**}\underline{\theta} + \varepsilon \leq \overline{\theta} - \varepsilon$, and $\overline{\theta} - \varepsilon \leq x < \overline{\theta} + \varepsilon$,

$$\begin{aligned} u(x, I_{x^{**}}) &= \frac{1}{\overline{\theta}(x) - \underline{\theta}(x)} \left(\int_{\underline{\theta}(x)}^{\overline{\theta}(x)} \left(\frac{x}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f)\, d\theta \right) - f \\ &= \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \int_{x-\varepsilon}^{\overline{\theta}} g(I_{x^{**}}, \theta)\, d\theta - f \end{aligned}$$

where

$$\begin{aligned} g(I_{x^{**}}, \theta) &= \left(\frac{1}{\widetilde{x}(I_{x^{**}}, \theta)} \right)^{\sigma-1} (\overline{\pi}(I_{x^{**}}, \theta) + f) \\ &= \begin{cases} g^1(I_{x^{**}}, \theta) & \text{when } \theta \leq x^{**} + \varepsilon \\ g^2(I_{x^{**}}, \theta) & \text{when } \theta > x^{**} + \varepsilon \end{cases} \end{aligned}$$

Here

$$\begin{aligned} g^1(I_{x^{**}}, \theta) &= \frac{\sigma(\theta + \varepsilon - x^{**})}{(\theta + \varepsilon)^\sigma - (x^{**})^\sigma} \left(\frac{\sigma f}{\sigma - 1} \left(\frac{P^e_{stay}(I_{x^{**}})}{\frac{\theta}{2\varepsilon} + \frac{\varepsilon - x^{**}}{2\varepsilon}} - 1 \right) + \frac{f_e}{\frac{\theta}{2\varepsilon} + \frac{\varepsilon - x^{**}}{2\varepsilon}} + f \right) \\ &= \frac{2\varepsilon\sigma}{(\theta + \varepsilon)^\sigma - (x^{**})^\sigma} \left(\frac{\sigma f}{\sigma - 1} P^e_{stay}(I_{x^{**}}) + f_e + \frac{(x^{**} - (\theta + \varepsilon))}{2\varepsilon} \frac{f}{\sigma - 1} \right) \end{aligned}$$

and

$$\begin{aligned} g^2(I_{x^{**}}, \theta) &= \frac{2\varepsilon\sigma}{(\theta + \varepsilon)^\sigma - (\theta - \varepsilon)^\sigma} \left(\frac{\sigma f}{\sigma - 1} \left(\frac{P^e_{stay}(I_{x^{**}})}{1} - 1 \right) + \frac{f_e}{1} + f \right) \\ &= \frac{2\varepsilon\sigma}{(\theta + \varepsilon)^\sigma - (\theta - \varepsilon)^\sigma} \left(\frac{\sigma f}{\sigma - 1} P^e_{stay}(I_{x^{**}}) + f_e - \frac{f}{\sigma - 1} \right) \end{aligned}$$

If $x < x^{**} + 2\varepsilon$,

$$u(x, I_{x^{**}}) = \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \left(\int_{x-\varepsilon}^{x^{**}+\varepsilon} g^1 \left(I_{x^{**}}, \theta \right) d\theta + \int_{x^{**}+\varepsilon}^{\overline{\theta}} g^2 \left(I_{x^{**}}, \theta \right) d\theta \right) - f$$

and

$$\begin{aligned} \frac{d}{dx} u(x, I_{x^{**}}) &= \frac{d}{dx} \left(\frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \right) \int_{x-\varepsilon}^{\overline{\theta}} g \left(I_{x^{**}}, \theta \right) d\theta + \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \frac{d}{dx} \left(\int_{x-\varepsilon}^{\overline{\theta}} g \left(I_{x^{**}}, \theta \right) d\theta \right) \\ &= \frac{d}{dx} \left(\frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \right) \int_{x-\varepsilon}^{\overline{\theta}} g \left(I_{x^{**}}, \theta \right) d\theta - \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} g^1 \left(I_{x^{**}}, x - \varepsilon \right) \end{aligned}$$

If $x \geq x^{**} + 2\varepsilon$,

$$u(x, I_{x^{**}}) = \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \left(\int_{x-\varepsilon}^{\overline{\theta}} g^2 \left(I_{x^{**}}, \theta \right) d\theta \right) - f$$

and

$$\begin{aligned} \frac{d}{dx} u(x, I_{x^{**}}) &= \frac{d}{dx} \left(\frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \right) \int_{x-\varepsilon}^{\overline{\theta}} g^2 \left(I_{x^{**}}, \theta \right) d\theta + \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \frac{d}{dx} \left(\int_{x-\varepsilon}^{\overline{\theta}} g^2 \left(I_{x^{**}}, \theta \right) d\theta \right) \\ &= \frac{d}{dx} \left(\frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} \right) \int_{x-\varepsilon}^{\overline{\theta}} g^2 \left(I_{x^{**}}, \theta \right) d\theta - \frac{x^{\sigma-1}}{\overline{\theta} - (x - \varepsilon)} g^2 \left(I_{x^{**}}, x - \varepsilon \right) \end{aligned}$$

for $x > x^{**} + 2\varepsilon$. Since

$$\lim_{x \to x^{**}+2\varepsilon^-} \frac{d}{dx} u(x, I_{x^{**}}) = \lim_{x \to x^{**}+2\varepsilon^+} \frac{d}{dx} u(x, I_{x^{**}})$$

and $u(x, I_{x^{**}})$ is continuous, $\frac{d}{dx} u(x, I_{x^{**}})$ is continuous at point $x = x^{**} + 2\varepsilon$. Summarize

$\frac{d}{dx}u(x, I_{x^{**}})$ in the following expression

$$\frac{d}{dx}u(x, I_{x^{**}}) = \frac{d}{dx}\left(\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\right)\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta - \frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}g\left(I_{x^{**}}, x-\varepsilon\right)$$

Now, consider

$$\lim_{x\to\overline{\theta}+\varepsilon} u(x, I_{x^{**}}) = \lim_{x\to\overline{\theta}+\varepsilon}\frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)}\int_{x-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta - f$$

Consider $g\left(I_{x^{**}},\theta\right)$ when $\theta \in \left[x-\varepsilon, \overline{\theta}\right]$ now. Since $P^e_{stay}\left(I_{x^{**}}\right)$ is a decreasing function with x^{**}, and $x^{**} < \underline{\theta}+\varepsilon$,

$$P^e_{stay}\left(I_{x^{**}}\right) > P^e_{stay}\left(I_{x^{**}}\right)|_{x^{**}=\underline{\theta}+\varepsilon} = \frac{1}{\overline{\theta}-\underline{\theta}}\left(\overline{\theta}- \quad x^* - \frac{\left(\underline{\theta}+\varepsilon-x^*\right)^2}{4\varepsilon}\quad\right)|_{x^{**}=\underline{\theta}+\varepsilon} = \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}$$

then,

$$\begin{aligned}
g^1\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma-(x^{**})^\sigma}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right)+f_e+\frac{(x^{**}-(\theta+\varepsilon))}{2\varepsilon}\frac{f}{\sigma-1}\right)\\
&> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{\theta+\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)\\
&\geq \frac{2\varepsilon\sigma}{(\underline{\theta}+3\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{\underline{\theta}+3\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
g^2\left(I_{x^{**}},\theta\right) &= \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma-(\theta-\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right)\\
&> \frac{2\varepsilon\sigma}{(\theta+\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{f}{\sigma-1}\right)\\
&> \frac{2\varepsilon\sigma}{(\overline{\theta}+\varepsilon)^\sigma}\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{f}{\sigma-1}\right)
\end{aligned}$$

That is,

$$g\left(I_{x^{**}},\theta\right) > \underline{g}$$

where $\underline{g}$ is a constant and is defined as

$$\underline{g} = \min \left\{ \begin{array}{c} \frac{2\varepsilon\sigma}{(\underline{\theta}+3\varepsilon)^{\sigma}} \left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\underline{\theta}+3\varepsilon}{2\varepsilon} \frac{f}{\sigma-1} \right), \\ \frac{2\varepsilon\sigma}{\left(\overline{\theta}+\varepsilon\right)^{\sigma}} \left(\frac{\sigma f}{\sigma-1} \frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{f}{\sigma-1} \right) \end{array} \right\}$$

By assumption 3.4, $\underline{g} > 0$. And then,

$$\begin{aligned} \lim_{x \to \overline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) &\geq \lim_{x \to \overline{\theta}+\varepsilon^-} \frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)} \int_{x-\varepsilon}^{\overline{\theta}} \underline{g} d\theta - f \\ &= \lim_{x \to \overline{\theta}+\varepsilon^-} \frac{x^{\sigma-1}}{\overline{\theta}-(x-\varepsilon)} \underline{g} \left(\overline{\theta} - (x-\varepsilon) \right) - f \\ &= \underline{g} \left(\overline{\theta} + \varepsilon \right)^{\sigma-1} - f > 0 \end{aligned}$$

by assumption 3.4. That is,

$$\lim_{x \to \overline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) > 0$$

Moreover, by left continuous of $u(x, I_{x^{**}})$ at point $x = \overline{\theta} + \varepsilon$,

$$u(x, I_{x^{**}})|_{x=\overline{\theta}+\varepsilon} = \lim_{x \to \overline{\theta}+\varepsilon^-} u(x, I_{x^{**}}) > 0$$

From the definition of limitation, for $\epsilon > 0$, there exists a $\delta > 0$, such that for all x that satisfy $\left(\overline{\theta} + \varepsilon\right) - x < \delta$, the inequality $|u(x, I_{x^{**}}) - 0| < \epsilon$ holds. That is, a x' can be found such that $u(x, I_{x^{**}}) > 0$ for $x \in [x', \overline{\theta} + \varepsilon)$. That is, $u(x', I_{x^{**}}) > 0$. In addition, as showed in the above part, $u(\underline{\theta} - \varepsilon, I_{x^{**}}) > 0$. Then, to show $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 4 and x belongs to region 6 can be simplified as to show that all local extrema is(are) positive.

Suppose x_1 satisfie(s) the first order condition

$$\begin{aligned} \frac{d}{dx}u(x,I_{x^{**}})|_{x=x_1} &= 0 \\ \int_{x_1-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta &= \frac{\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)} g\left(I_{x^{**}},x_1-\varepsilon\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)} \end{aligned}$$

And then

$$\begin{aligned} u(x_1,I_{x^{**}}) &= \frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\int_{x_1-\varepsilon}^{\overline{\theta}} g\left(I_{x^{**}},\theta\right)d\theta - f \\ &= \frac{\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)^2 g\left(I_{x^{**}},x_1-\varepsilon\right)}{\frac{d}{dx_1}\left(\frac{x_1^{\sigma-1}}{\overline{\theta}-(x_1-\varepsilon)}\right)} - f \\ &= \frac{x_1^{\sigma} g\left(I_{x^{**}},x_1-\varepsilon\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1)+x_1(2-\sigma)} - f \end{aligned}$$

Consider $x_1^{\sigma} g\left(I_{x^{**}},x_1-\varepsilon\right)$ now.

$$\begin{aligned} & x_1^{\sigma} g^1\left(I_{x^{**}},x_1-\varepsilon\right) \\ = \ & x_1^{\sigma}\frac{2\varepsilon\sigma}{x_1^{\sigma}-(x^{**})^{\sigma}}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right)+f_e+\frac{(x^{**}-x_1)}{2\varepsilon}\frac{f}{\sigma-1}\right) \\ > \ & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{\underline{\theta}+3\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right) \end{aligned}$$

and

$$\begin{aligned} & x_1^{\sigma} g^2\left(I_{x^{**}},x_1-\varepsilon\right) \\ = \ & x_1^{\sigma}\frac{2\varepsilon\sigma}{x_1^{\sigma}-(x_1-2\varepsilon)^{\sigma}}\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right)+f_e-\frac{f}{\sigma-1}\right) \\ > \ & 2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon}+f_e-\frac{f}{\sigma-1}\right) \end{aligned}$$

As a result, when $x < x^{**} + 2\varepsilon$,

$$u(x_1, I_{x^{**}}) > \frac{2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\underline{\theta}+3\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1) + \left(\overline{\theta}-\varepsilon\right)(2-\sigma)} - f > 0$$

and when $x \geq x^{**} + 2\varepsilon$,

$$\begin{aligned} u(x_1, I_{x^{**}}) &> \frac{2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1) + \left(\overline{\theta}-\varepsilon\right)(2-\sigma)} - f \\ &> \frac{2\varepsilon\sigma\left(\frac{\sigma f}{\sigma-1}\frac{\overline{\theta}-2\varepsilon}{\overline{\theta}-\varepsilon} + f_e - \frac{\underline{\theta}+3\varepsilon}{2\varepsilon}\frac{f}{\sigma-1}\right)}{\left(\overline{\theta}+\varepsilon\right)(\sigma-1) + \left(\overline{\theta}-\varepsilon\right)(2-\sigma)} - f > 0 \end{aligned}$$

by assumption 3.4. In sum, $u(x_1, I_{x^{**}}) > 0$ for $\frac{d}{dx}u(x, I_{x^{**}})|_{x=x_1} = 0$ and thus $u(x, I_{x^{**}}) > 0$ when x^{**} belongs to region 4, x belongs to region 6. Combine with the fact that $u(x, I_{x^{**}})|_{x=\overline{\theta}+\varepsilon} > 0$, the result is that $u(x, I_{x^{**}}) > 0$ for $\underline{\theta} + \varepsilon \leq x \leq \overline{\theta} + \varepsilon$. This completes the proof.

B.11 Equilibrium Values

B.11.1 The Aggregate Revenue

$$\begin{aligned} & R\left(I_{x^{**}}, \theta\right) \\ = \;& M\left(I_{x^{**}}, \theta\right)\overline{r}\left(I_{x^{**}}, \theta\right) \\ = \;& M\left(I_{x^{**}}, \theta\right)\sigma\left(\overline{\pi}\left(I_{x^{**}}, \theta\right) + f\right) \\ = \;& M_e\left(I_{x^{**}}\right)P_{stay}\left(I_{x^{**}}, \theta\right)\sigma\left(\frac{\sigma f}{\sigma-1}\left(\frac{P^e_{stay}\left(I_{x^{**}}\right)}{P_{stay}\left(I_{x^{**}}, \theta\right)} - 1\right) + \frac{f_e}{P_{stay}\left(I_{x^{**}}, \theta\right)} + f\right) \\ = \;& M_e\left(I_{x^{**}}\right)\sigma\left(\frac{\sigma f}{\sigma-1}P^e_{stay}\left(I_{x^{**}}\right) + f_e - \frac{f}{\sigma-1}P_{stay}\left(I_{x^{**}}, \theta\right)\right) \end{aligned}$$

B.11.2 The Expected Aggregate Revenue

$$
\begin{aligned}
R(I_{x^{**}}) &= E_\Theta\left[R(I_{x^{**}},\theta)\right] \\
&= E_\Theta\left[L - M_e(I_{x^{**}}) f_e + M(I_{x^{**}},\theta)\,\pi(\widetilde{x})(I_{x^{**}},\theta)\right] \\
&= L - M_e(I_{x^{**}}) f_e + M_e(I_{x^{**}}) E_\Theta\left[P_{stay}(I_{x^{**}},\theta)\,\pi(\widetilde{x})(I_{x^{**}},\theta)\right] \\
&= L
\end{aligned}
$$

where the last equality comes from the free entry condition.

B.12 Comparative Statics for Melitz Model

This part discusses the comparative statics for Melitz case when the precision of private information changes. Following the game with incomplete information, there are three stages for Melitz case. In the first stage, firm pays fixed entry costs f_e to enter the market with a common knowledge of the mean productivity θ. In the second stage, firm knows its own productivity x and decides whether to stay in the market or not. Firm pays fixed costs f if it chooses to stay. In the last stage, existing firm pays variable costs $\frac{q}{x}$, sets price p, and sells outputs in the market. The equilibrium cutoff productivity level x^* can be obtained through the following two conditions: zero cutoff profit condition and free entry condition. Only key steps are shown here, and the detailed derivations can be found in Melitz (2003).

B.12.1 Zero Cutoff Profit Condition

x^* is the cutoff productivity level, and in the second stage the profit for firm with productivity level x^* is

$$
\begin{aligned}
\pi(x^*) &= \frac{1}{\sigma} r(x^*) - f = 0 \\
&\Rightarrow r(x^*) = \sigma f
\end{aligned}
$$

Then, the average profit is

$$\overline{\pi} = \pi\left(\widetilde{x}\left(x^*\right)\right) = \frac{1}{\sigma} r\left(\widetilde{x}\left(x^*\right)\right) - f = \frac{1}{\sigma}\left(\frac{\widetilde{x}\left(x^*\right)}{x^*}\right)^{\sigma-1} r\left(x^*\right) - f = \left(\left(\frac{\widetilde{x}\left(x^*\right)}{x^*}\right)^{\sigma-1} - 1\right) f \tag{B.19}$$

where

$$\begin{aligned} \widetilde{x}\left(x^*\right) &= \left(\int_{x^*}^{\theta+\varepsilon} \frac{1}{\theta+\varepsilon-x^*}\left(x\right)^{\sigma-1} dx\right)^{\frac{1}{\sigma-1}} \\ &= \left(\frac{1}{\sigma}\frac{1}{\theta+\varepsilon-x^*}\left(\left(\theta+\varepsilon\right)^{\sigma} - \left(x^*\right)^{\sigma}\right)\right)^{\frac{1}{\sigma-1}} \end{aligned}$$

Note it is assumed that $\theta - \varepsilon \leq x^* < \theta + \varepsilon$.

B.12.2 Free Entry Condition

The net value v_e of entry is zero.

$$\begin{aligned} v_e &= Pstay\overline{\pi} - f_e = 0 \\ &\Rightarrow \overline{\pi} = \frac{f_e}{Pstay} = \frac{2\varepsilon f_e}{\theta+\varepsilon-x^*} \end{aligned} \tag{B.20}$$

As a result, in equilibrium,

$$\left(\left(\frac{\widetilde{x}\left(x^*\right)}{x^*}\right)^{\sigma-1} - 1\right) f = \frac{2\varepsilon f_e}{\theta+\varepsilon-x^*}$$

B.12.3 Equilibrium Variables

With equilibrium cutoff productivity level x^*, equilibrium value of other variables can be found as following.

- Mass of successful entrants $M = PstayM_e$

By (B.20),

$$L_e = M_e f_e = \frac{M}{Pstay} f_e = M\overline{\pi} = \Pi$$

then,

$$\begin{aligned} R &= L_p + \Pi = L_p + L_e = L \\ \Rightarrow \quad M &= \frac{R}{r\left(\widetilde{x}\left(x^*\right)\right)} = \frac{L}{\sigma\left(\pi\left(\widetilde{x}\left(x^*\right)\right) + f\right)} \end{aligned}$$

- Mass of entrants M_e

$$M_e = \frac{M}{Pstay}$$

- Total revenue $R = Mr\left(\widetilde{x}\left(x^*\right)\right)$

$$R = L$$

- Aggregate price P

$$P = M^{\frac{1}{1-\sigma}} p\left(\widetilde{x}\left(x^*\right)\right) = M^{\frac{1}{1-\sigma}} \frac{1}{\rho\widetilde{x}\left(x^*\right)}$$

- Aggregate quantity $Q(\equiv U)$

$$Q = \frac{R}{P} = \frac{L\rho\widetilde{x}\left(x^*\right)}{M^{\frac{1}{1-\sigma}}}$$

- Welfare per worker

$$W = \frac{U}{L} = \frac{Q}{L} = \frac{R}{P}\frac{1}{L} = \frac{1}{P}$$

B.12.4 How Changes in ε Affect the Cutoff Point x^*

The cutoff productivity level x^* is such that

$$\left(\left(\frac{\widetilde{x}\left(x^*\right)}{x^*}\right)^{\sigma-1} - 1\right) f - \frac{2\varepsilon f_e}{\theta + \varepsilon - x^*} = 0$$

Define

$$\begin{aligned} & F\left(x^{*},\varepsilon\right) \\ = \; & \left(\left(\frac{\widetilde{x}\left(x^{*}\right)}{x^{*}}\right)^{\sigma-1}-1\right)f-\frac{2\varepsilon f_{e}}{\theta+\varepsilon-x^{*}} \\ = \; & \left(\left(\frac{1}{\sigma}\frac{1}{\theta+\varepsilon-x^{*}}\left(\left(\theta+\varepsilon\right)^{\sigma}-\left(x^{*}\right)^{\sigma}\right)\right)\left(\frac{1}{x^{*}}\right)^{\sigma-1}-1\right)f-\frac{2\varepsilon f_{e}}{\theta+\varepsilon-x^{*}} \end{aligned}$$

Then,

$$\frac{\partial x^{*}}{\partial\varepsilon}=-\frac{\partial F\left(x^{*},\varepsilon\right)/\partial\varepsilon}{\partial F\left(x^{*},\varepsilon\right)/\partial x^{*}}$$

Consider $\frac{\partial F(x^{*},\varepsilon)}{\partial\varepsilon}$ first.

$$\begin{aligned} & \frac{\partial F\left(x^{*},\varepsilon\right)}{\partial\varepsilon} \\ = \; & \frac{f}{\sigma}\left(\frac{1}{x^{*}}\right)^{\sigma-1}\frac{1}{\left(\theta-x^{*}+\varepsilon\right)^{2}}\left(x^{*\sigma}-x^{*}\sigma\left(\theta+\varepsilon\right)^{\sigma-1}+\left(\sigma-1\right)\left(\theta+\varepsilon\right)^{\sigma}\right)+2f_{e}\frac{x^{*}-\theta}{\left(\theta-x^{*}+\varepsilon\right)^{2}} \\ = \; & \frac{1}{\left(\theta-x^{*}+\varepsilon\right)^{2}}\left(\frac{f}{\sigma}\left(\frac{1}{x^{*}}\right)^{\sigma-1}\left(x^{*\sigma}-x^{*}\sigma\left(\theta+\varepsilon\right)^{\sigma-1}+\left(\sigma-1\right)\left(\theta+\varepsilon\right)^{\sigma}\right)+2f_{e}\left(x^{*}-\theta\right)\right) \end{aligned}$$

Define

$$F^{1}\left(x^{*},\varepsilon\right)=\frac{f}{\sigma}\left(\frac{1}{x^{*}}\right)^{\sigma-1}\left(x^{*\sigma}-x^{*}\sigma\left(\theta+\varepsilon\right)^{\sigma-1}+\left(\sigma-1\right)\left(\theta+\varepsilon\right)^{\sigma}\right)+2f_{e}\left(x^{*}-\theta\right)$$

When $F^{1}\left(x^{*},\varepsilon\right)>0$, $\frac{\partial F(x^{*},\varepsilon)}{\partial\varepsilon}>0$ and when $F^{1}\left(x^{*},\varepsilon\right)<0$, $\frac{\partial F(x^{*},\varepsilon)}{\partial\varepsilon}<0$. Note that

$$x^{*\sigma}-x^{*}\sigma\left(\theta+\varepsilon\right)^{\sigma-1}+\left(\sigma-1\right)\left(\theta+\varepsilon\right)^{\sigma}>0$$

and then if $\left(x^{*}-\theta\right)\geq0$, $\frac{\partial F(x^{*},\varepsilon)}{\partial\varepsilon}>0$ for sure.

Consider $\frac{\partial F(x^*,\varepsilon)}{\partial x^*}$ now.

$$\begin{aligned}
\frac{\partial F\left(x^*,\varepsilon\right)}{\partial x^*} &= \frac{\partial}{\partial x^*}\left(\left(\frac{f}{\sigma}\frac{1}{\theta+\varepsilon-x^*}\left(\left(\theta+\varepsilon\right)^\sigma-\left(x^*\right)^\sigma\right)\right)\left(\frac{1}{x^*}\right)^{\sigma-1}-\frac{2\varepsilon f_e}{\theta+\varepsilon-x^*}\right) \\
&= \left(\frac{1}{x^*}\right)^{\sigma-1}\frac{\partial}{\partial x^*}\left(\frac{f}{\sigma}\frac{1}{\theta+\varepsilon-x^*}\left(\left(\theta+\varepsilon\right)^\sigma-\left(x^*\right)^\sigma\right)\right) \\
&\quad +\left(\frac{f}{\sigma}\frac{1}{\theta+\varepsilon-x^*}\left(\left(\theta+\varepsilon\right)^\sigma-\left(x^*\right)^\sigma\right)\right)\frac{\partial}{\partial x^*}\left(\frac{1}{x^*}\right)^{\sigma-1}-\frac{\partial}{\partial x^*}\left(\frac{2\varepsilon f_e}{\theta+\varepsilon-x^*}\right) \\
&= -\left(\frac{1}{x^*}\right)^{\sigma-1}\frac{f}{\sigma}\frac{1}{\left(\theta-x^*+\varepsilon\right)^2}\left(x^{*\sigma}-x^*\sigma\left(\theta+\varepsilon\right)^{\sigma-1}+\left(\sigma-1\right)\left(\theta+\varepsilon\right)^\sigma\right) \\
&\quad -\left(\frac{f}{\sigma}\frac{1}{\theta+\varepsilon-x^*}\left(\left(\theta+\varepsilon\right)^\sigma-\left(x^*\right)^\sigma\right)\right)\left(\sigma-1\right)\left(\frac{1}{x^*}\right)^\sigma-2\varepsilon\frac{f_e}{\left(\theta-x^*+\varepsilon\right)^2} \\
&< 0
\end{aligned}$$

As a result, when $F^1\left(x^*,\varepsilon\right)>0$, $\frac{\partial x^*}{\partial\varepsilon}>0$ and when $F^1\left(x^*,\varepsilon\right)<0$, $\frac{\partial x^*}{\partial\varepsilon}<0$. For the exercise in change of precision of private information for Melitz case, $F^1\left(x^*,\varepsilon\right)>0$, and then $\frac{\partial x^*}{\partial\varepsilon}>0$. That is, the cutoff productivity level decreases when the precision of private information increases (ε decreases).

B.12.5 The Welfare Ratio When the Precision of Private Information Changes

$$\begin{aligned}
\frac{W\left(\varepsilon_1\right)}{W\left(\varepsilon_2\right)} &= \frac{P\left(\varepsilon_2\right)}{P\left(\varepsilon_1\right)} \\
&= \frac{M\left(\varepsilon_2\right)^{\frac{1}{1-\sigma}}\rho\widetilde{x}\left(x^*\left(\varepsilon_1\right)\right)}{M\left(\varepsilon_1\right)^{\frac{1}{1-\sigma}}\rho\widetilde{x}\left(x^*\left(\varepsilon_2\right)\right)} \\
&= \frac{\left(\frac{L}{\sigma\left(\pi\left(\widetilde{x}\left(x^*\left(\varepsilon_2\right)\right)\right)+f\right)}\right)^{\frac{1}{1-\sigma}}\rho\widetilde{x}\left(x^*\left(\varepsilon_1\right)\right)}{\left(\frac{L}{\sigma\left(\pi\left(\widetilde{x}\left(x^*\left(\varepsilon_1\right)\right)\right)+f\right)}\right)^{\frac{1}{1-\sigma}}\rho\widetilde{x}\left(x^*\left(\varepsilon_2\right)\right)} \\
&= \frac{\widetilde{x}\left(x^*\left(\varepsilon_1\right)\right)\left(\pi\left(\widetilde{x}\left(x^*\left(\varepsilon_1\right)\right)\right)+f\right)^{\frac{1}{1-\sigma}}}{\widetilde{x}\left(x^*\left(\varepsilon_2\right)\right)\left(\pi\left(\widetilde{x}\left(x^*\left(\varepsilon_2\right)\right)\right)+f\right)^{\frac{1}{1-\sigma}}}
\end{aligned}$$

Since

$$\begin{aligned}
& \widetilde{x}\left(x^{*}\left(\varepsilon_{1}\right)\right)\left(\pi\left(\widetilde{x}\left(x^{*}\left(\varepsilon_{1}\right)\right)\right)+f\right)^{\frac{1}{1-\sigma}} \\
= & \widetilde{x}\left(x^{*}\left(\varepsilon_{1}\right)\right)\left(\left(\left(\frac{\widetilde{x}\left(x^{*}\left(\varepsilon_{1}\right)\right)}{x^{*}\left(\varepsilon_{1}\right)}\right)^{\sigma-1}-1\right) f+f\right)^{\frac{1}{1-\sigma}} \\
= & \widetilde{x}\left(x^{*}\left(\varepsilon_{1}\right)\right)\left(\frac{\widetilde{x}\left(x^{*}\left(\varepsilon_{1}\right)\right)}{x^{*}\left(\varepsilon_{1}\right)}\right)^{-1} \\
= & x^{*}\left(\varepsilon_{1}\right)
\end{aligned}$$

then,

$$\frac{W\left(\varepsilon_{1}\right)}{W\left(\varepsilon_{2}\right)}=\frac{x^{*}\left(\varepsilon_{1}\right)}{x^{*}\left(\varepsilon_{1}\right)}$$

That is, the ratio between welfare per worker when the precision of private information changes equals the ratio between the respective cutoff productivity levels. In other words, when the cutoff productivity level increases, the welfare per worker increases. Note that this equalization of ratios between welfare and ratios between cutoff productivities holds for any distribution of productivities. In chapter 4, one can see that the standardized welfare curve and the standardized cutoff productivity curve coincides with each other for the Melitz case when the precision of private information changes.

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