2015

Optimal population and policy implications

Xiying Liu
Iowa State University

Optimal population and policy implications

by

Xiying Liu

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Economics

Program of Study Committee:
Juan Carlos Cordoba, Major Professor
Joydeep Bhattacharya
David M Frankel
Gray Calhoun
Weerasinghe Ananda

Iowa State University

Ames, Iowa

2015

DEDICATION

This dissertation is dedicated to my husband Xin, my daughter Catherine, and my parents whose unconditional love supported me each step of the way.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGMENTS

I would like to take this opportunity to express my thanks to those who helped me with various aspects of conducting research and the writing of this dissertation. I owe my deepest gratitude to my major professor, Dr. Juan Carlos Cordoba, for his guidance, patience and support throughout this research and the writing of this dissertation. His insights and encouragement have often inspired me and renewed my hopes for pursuing an academic career.

I would also like to thank Dr. Joydeep Bhattachraya and Dr. David Frankel for their valuable comments in the earlier stage of my dissertation. I am also indebted to Dr Rajesh Singh, Dr. Gray Calhoun and Dr. Ananda Weerasinghe for their efforts and contributions to this work.

ABSTRACT

This dissertation explores issues of efficient and inefficient population in complete and incomplete market economies with altruistic parents who cares about the welfare of their children. Altruistic models with idiosyncratic risks are central to modern macroeconomics, particularly when studying issues of inequality and redistribution. But this framework seems to fall apart when serious consideration is given to fertility decisions as Barro and Becker (1989) because some of the most appealing conclusions obtained under the exogenous fertility assumption are seriously altered. For example, optimal fertility choice tends to eliminate intergenerational persistence of inequality. A main goal of this dissertation is to recover key features of demographic facts using micro-founded macroeconomic theory and quantitative approaches and derive normative analysis regarding efficiency of population, long run inequality, education, and demographic policies. The consensus of my research is that family decisions made by altruistic parents have substantial aggregate socioeconomic consequences in dynamic environments.

CHAPTER 1. INTRODUCTION

I am interested in the economics of family decisions and their effects at the macroeconomic level. This dissertation focuses on understanding theoretical and empirical predictions of models where fertility decisions are made by fully rational and altruistic parents. Given the significant demographic transitions being experienced and forecasted for many countries, this work employs micro-founded macroeconomic models to explain the key features of data and derives normative implications regarding the efficiency of population, long run inequality, education, and demographic policies.

My dissertation consists of five chapters falling into four categories: (i) population efficiency, including over and under-population; (ii) social mobility and inequality with endogenous fertility, (iii) consequences of demographic policies, such as China's One Child Policy, and (iv) international quantity-quality trade-off related to schooling and fertility choices. The consensus of all chapters is that family decisions made by altruistic parents have substantial aggregate socioeconomic consequences in dynamic environments.

The first chapter, "Efficient Population on a Finite Planet" examines issues of overpopulation in an environment with altruistic parents and fixed resource, specifically, land. I derive the efficient level of population in this benchmark, then compare it with the population level when markets are incomplete. The main finding is that efficient long run population in the standard neoclassical growth model with endogenous fertility choices can be zero or the maximal sustainable level. If the economy starts with low level of population, labor value is high, then children are expensive because they take valuable parents' time. This tends to discourage fertility and leads to lower population in the next generation. Consequently, zero population is possible in the long run. Next my research studies a decentralized model with land market and credit frictions. I find that incomplete market produces under-population compared with the efficient level. The mechanism is that in standard macroeconomics, interest rate is typically low with credit frictions. By standard arbitrage arguments, rent of land in incomplete markets tends to be low. So with a fixed amount of available land, low rent implies a high land-labor ratio and low steady state population. Furthermore, I extend the benchmark model to incorporate uninsurable risk on lifetime earning abilities to evaluate consequence of demographic policies, such as the One Child Policy. Surprisingly, my model suggests that such policies increase long run population.

The second chapter, "Fertility, Social Mobility and Long Run Inequality", joint with Juan Carlos Cordoba and Marla Ripoll, investigates social mobility and long run inequality in the presence of endogenous fertility and intergenerational transfer choices made by altruistic individuals facing uninsurable idiosyncratic risk. Incorporating fertility choices in analyzing long run inequality is important because the differential fertilities among rich and poor families lead to differences in intergenerational wealth transmission, social mobility and long run inequality. Alvarez (1999) finds a counterfactual result that fertility choices by altruistic parents largely reduce intergenerational persistence and increase social mobility. The main contribution of this research is to recover empirically plausible levels of persistence with altruistic models of endogenous fertility. We show that a calibrated version of a Barro-Becker dynastic altruistic model of fertility choice embedded into a Bewley framework of idiosyncratic risk is able to replicate three key aspects of the data: (i) a negative fertility-income relationship; (ii) a negative relationship between family size and savings rates; and (iii) a significant intergenerational persistence or lack of social mobility. We also show that the endogenous fertility model improves upon the exogenous fertility model in a number of other dimensions such as larger wealth dispersion. Our calibration exercise also sheds light on the technology of raising children, the shape of altruism by parents, and the "intergenerational elasticity of substitution".

The third chapter "Accounting for the International Quantity-Quality Trade-off", joint with Juan Carlos Cordoba and Marla Ripoll, provides a quantitative theory accounting for key empirical regularities of fertility and schooling differences across countries. In our model a quantity quality trade-off arises due to financial constraints and endogenous fertility choices. Our model predicts a negative association of fertility with both schooling and income. We find that differences in wages explain most of the cross country dispersion of fertility and years of the schooling dispersion. In addition, schooling differences also come from differences in education policies such as the availability of public schools, school funding, and compulsory schooling.

The fourth chapter, "Altruism, Fertility and Risk", joint with Juan Carlos Cordoba, studies fertility choices and fertility policies when children's earning abilities are random and parents are altruistic. The main contribution of this chapter is to characterize equilibrium allocations arising in endowment economies with complete or incomplete markets. In particular, consumption is proportional to the net financial cost of raising children, which is different from the standard permanent income hypothesis. We find that fertility policies are generally welfare detrimental in our models even when fertility is inefficiently high.

The fifth chapter, "Stochastic Dominance and Demographic Policy Evaluation: A Critique", joint with Juan Carlos Cordoba, is related with stochastic dominance (SD), which is commonly used to rank income distribution and assess social policies. We argue that SD is not a robust criterion for policy evaluation and we show that fertility restrictions are generally detrimental to both individual and social welfare even SD holds.

CHAPTER 2. EFFICIENT POPULATION ON A FINITE PLANET

2.1 Introduction

As natural resource becomes scarcer in the world, the large population in many developing countries have caused concerns on issues of overpopulation since one consequence of overpopulation and scarcity of resource is poverty. For example, Bangladesh has one of the world's highest population density, 1147 persons per square kilometer[1]. In this country a large majority of people engages in low-productivity manual farming and a large fraction suffers from extremely high level of poverty.

In this chapter I want to ask the following questions. Whether a large population is harmful to the social welfare? What is the optimal long run population for the earth with limited resource if welfare of all individuals, those living now and will be born in the future, is taken into account. Whether markets, with and without frictions, produce a socially efficient level of population, and if not, whether they produce higher or lower population than the optimal level. Golosov, Jones and Tertilt (2007) have shown that population is efficient in a dynastic altruist model with endogenous fertility, as in Barro and Becker (1989) and Becker and Barro (1988) (BB henceforth). However, in the altruistic fertility literature, no market force guarantees fertility stabilizing at the replacement rate so that efficient population either typically grows without bound or shrinks toward zero.

In this chapter I seek to answer these questions in a standard neoclassical growth model where parents are purely altruistic toward their children. Fertility is endogenous and costly, particularly in terms of parental time. There exists a fixed resource, land, and a minimum subsistence level of consumption. The model is appealing because it is a version of the standard growth model that allows for endogenous fertility, as in BB. In the economy there is a maximum sustainable level of population, one that would drive everyone's consumption to subsistence, or to immiseration. This economy resembles Malthus' economy but is one with micro-foundation of fertility. I define and characterize the *golden rule* level of population and the

[1] POVERTY AT LARGE : A DARK SPOT IN HUMANITY, cited from http://povertyhci.weebly.com/index.html

modified golden rule level of population, concepts analogous to those used in growth theory to characterize the accumulation of capital. The first concept refers to the level of population that maximizes steady state welfare while the second concept refers to the efficient level of population in steady state. I characterize efficient allocations, and conditions under which a maximum sustainable population is efficient or not. To evaluate the effects of market frictions, bequest constraints, on steady state population, I decentralize the social planner's problem using competitive land markets. A natural notion of over- and under- population arises by comparing population in market economies relative to its efficient population.

The main findings are the following. First, maximum sustainable and zero population can be efficient in the long run, but not always. More specifically, in the pure homothetic Barro and Becker model, efficient steady state population reaches 0 or the maximum sustainable, and the maximum sustainable population is infinite if subsistence consumption is zero. In the more general non-homothetic model efficient population could be less than the maximum sustainable, and consumption is larger than subsistence, but I quantitatively find that this finite level of population is unstable in general.

This is an important benchmark because it shows that Malthusian economies where consumption is driven to subsistence are not necessarily inefficient. The intuition for this result is that if the economy starts with a level of population higher than the steady state, labor value is lower with a fixed resource. When the main opportunity cost of raising children is valuable parental time, children become cheaper. This increases parents' incentive to have children and leads to a higher population growth rate than the steady state growth rate. Population in the next generation would be larger, which leads to a even lower opportunity cost of having children. Then the fertility rate increases, which further raises the population in the next generation. This process repeats overtime and leads to infinite population in the long run. If the population starts with too low, similar mechanism goes through but in the opposite direction, which leads to zero population in the long-term.

If I slightly enrich the model to incorporate goods cost, stability of the well-defined finite population could be obtained and infinite steady state population can be avoided. It is because no matter how cheap children are in terms of parental time, there is a goods cost associated with raising children, and a large population with fixed resource implies low production and low earnings of every individual. A parent with low earnings can not afford too many children due to goods cost, which tends to prevent population from growing without bound.

I then decentralize the social planner's problem using land markets to evaluate the effects of market frictions, bequest constraints, on steady state population. I find that steady state population in incomplete markets is lower than the efficient population. Thus, market frictions similar with credit friction in macroeconomics would produce under-population in the long run rather than over-population. The intuition is that the interest rate is typically low when credit frictions are binding particularly because parents would like to borrow against children's income but are not allowed by the constraint. Low interest rate and arbitrage imply that the rent of land is also lower under credit frictions, and since (i) the rent is equal to the marginal product of land; (ii) the marginal product is inversely related to the land-labor ratio; and (iii) land is fixed, it follows that steady state population is low in the constrained economy.

I also evaluate the consequences of demographic policies, similar with China's One Child Policy in incomplete markets. Since individuals are homogenous in the deterministic model and a binding constraint on fertility choices can not be steady state equilibrium, I extend the benchmark model to incorporate uninsurable risks on earning abilities. Abilities are random and correlated between parents and children. The extended model is essentially a Bewley model with endogenous fertility, credit friction and fixed land. This framework is central to modern macroeconomics, particularly when studying issues of inequality and redistribution. To my surprise, one child policy that aims at restricting people's fertility choices leads to an increase of long run population.

The mechanism through which policies restricting fertility choices increase population is the following. Under such fertility policy, every individual tends to have fewer children than the steady state fertility without any policy, which is 1 child per parent. So under the policy fertility is below 1 if prices of the two economies are the same. At steady state, the wage in the economy under fertility policy must be smaller so that the cost of children is lower and there is a force to induce people to have more children than the other economy. A lower steady state wage of the economy with policies corresponds to a higher population in presence of a fixed land.

Another contribution of this chapter is to show that a negative relationship between fertility and earning ability arises from the stochastic model with uninsurable idiosyncratic risks of earning ability. This negative relationship is not simple to obtain within dynastic altruistic models, as discussed by Jones, Schoonbroodt and Tertilt (2011). In the framework of this chapter, it arises from the interplay of two opposite forces. On the one hand, higher ability individuals face a larger opportunity cost of having a child as children take valuable parental time. On the other hand, higher ability individuals enjoy a larger benefit of having children when abilities are intergenerationally persistent because the utility of a child is positively related to parental ability. I find that the effect of ability on the marginal cost dominates its effect on the marginal benefit if the intergenerational persistence of ability is not perfect. Regression to the mean in abilities means that low ability parents expect their children to be of higher ability on average while high ability parents expect their children to be of lower ability. This explains why poor people have more children and rich ones have fewer children in a fully rational and altruistic environment. Through a similar channel, Cordoba and Liu (2014) and Cordoba, Liu and Ripoll (2014) are able to replicate the inverse correlation between ability and fertility in a homothetic model. Compared with these two papers, this chapter is the first to replicate the feature in a general equilibrium model with production.

In addition to the literature mentioned above, this chapter is closely related to Dasgupta (2005) who studies optimal population in an endowment economy with fixed resource. My model is an extension of his model but is richer in production, altruism and the technology of raising children. Cost of raising children is not considered in Dasgupta's model. Moreover, Dasgupta studies a generation-relative utilitarianism, one close to BB model but not BB. I go further than his paper to find that the steady state allocations are unstable in general and to study the optimal population in a market structure with market imperfection.

Nerlove, Razin and Sadka (1986) is a related paper that proposes competitive equilibrium is efficient under two possible externalities in the context in which parents care about future generations' welfare. First, larger population helps providing more public goods such as national defense. Second, larger population reduces wage rate if there is a fixed amount of land. Eckstein, Stern and Wolpin (1988) show that population can stabilize and non-subsistence consumption arises in the equilibrium when fertility choices is endogenously introduced to a model with fixed amount of land. In a different line of the literature, parents have warm glow altruism in that paper. Peretto and Valente (2011) study the interaction between technological progress, resource scarcity and population dynamics. Their paper focuses on the market allocation while this chapter focuses on the efficient allocation. My normative analysis on the stability of the steady state efficient population under fixed land is consistent with their findings on a special case with Cobb-Douglas production form, but the two papers solve the allocations in different environment since the parental altruism in their model is not pure as in BB (1989). De la Croix (2012) studies sustainable population by proposing non-cooperative bargaining between clans living on an island with limited resource. Children in his model act like an investment good for parents' old-age support.

This chapter is also related to a large literature on the role that demographic transition plays in the economic development and transition. Galor and Weil (2000) develop a unified growth model that captures the historical evolution of population, technology, and output. Parents in their model have warm glow altruism toward children. There is no savings in their model. Issues of over- or under-population are neither the focus of Galor and Weil (2000) nor Doepke (2003). Doepke (2003) develops a growth model to account for the transition from stagnation to growth incorporating the endogenous fertility decisions made by altruistic parents. Land is a public good in that paper so the price of land is not valued. This chapter introduces the limited resource, land, so that a finite level of long run population can be obtained in equilibrium. In Hansen and Prescott (2002) or Kremer (1993), fertility moves mechanically and population grows exogenously.

There has been papers that study inefficiency of fertility in incomplete market models. For example, Schoonbroodt and Tertilt (2014) (ST henceforth) show that under certain assumptions incomplete markets models can result in inefficiently low fertility. Cordoba and Liu (2014) show that under certain different assumptions incomplete markets models can result in inefficiently high fertility. But these papers focus on predictions for fertility, the growth rate of population, not for population size. This chapter focuses on the level of population at steady state.

The rest of the chapter is organized as follows. Section 2.2 studies the basic dynastic altruistic fertility models and the decentralization. I show that complete market model delivers the efficient population. Section 2.3 studies decentralized models: overlapping generation complete and incomplete market models with fixed land. The notion of over- and under-population are formalized as I compare the steady state population in incomplete markets with the efficient level. The fourth section investigates the consequences of demographic policies similar with China's One Child policy in the framework of Bewley model with endogenous fertility and a fixed amount of land. Differential fertility rates arise in this model and are characterized in the quantitative part. Section 2.5 concludes. Propositions are proved in the appendix.

2.2 The Basic Dynastic Altruistic Model

2.2.1 Social Planner's Problem

Benchmark Model

In this section, I study an overlapping generation economy. Parents are altruistic toward children. There is a benevolent social planner who has the same preference with the initial parent and makes all decisions for individuals. I apply a modern altruistic approach to the demand for children along the lines of Becker and

Barro (1988) and Barro and Becker (1989) to study the efficient population of the economy. To obtain a well-defined level of population, there is a fixed amount of land in the environment. Children are expensive since there is an opportunity cost of raising children, the time cost. The model resembles Malthus' model but is different in providing the micro-foundation of the demand for children.

The social planner solves the following problem:

$$\max_{\{N_{t+1},c_t,n_t\}_{t=0}^{\infty}} \sum_t \beta^t N_t^{1-\epsilon} u(c_t)$$

subject to

$$N_{t+1} = n_t N_t, \; N_0 \text{ is given}$$

$$N_t c_t = zF\left(\bar{K}, (1-\lambda n_t) N_t\right)$$

and

$$c_t \geq \underline{c}$$

where $\underline{c}$ is a subsistence level of consumption. Any consumption below this level is not big enough to make a living.

β is the discount rate across generations. c_t and n_t are consumption and number of children of an individual in generation t. For simplicity, I call n_t fertility. Assuming every child costs parents λ percentage of time to raise them. Individuals allocate time between working and raising children. Leisure is not valued in this model. N_t denotes population of generation t and $N_t^{1-\epsilon}$ is the weight that social planner assigns to people of generation t. The weight that social planner puts on generation t is increasing in population at a diminishing rate. When $\epsilon = 0$, the weight is linear in population and it does not decreases as population increases. The first constraint implies that population grows at the rate of fertility, n_t.

$\bar{K}$ is the fixed amount of resource, land. Production of generation t uses land $\bar{K}$ and the total labor supply of the economy, $(1-\lambda n_t) N_t$, which is every individual's labor supply multiplies the population. The fixed land assumption is crucial for obtaining a well-defined population because otherwise population could grow to infinite or shrinks to zeros in the long run. The second constraint is a resource constraint. It requires total consumption of every generation to be equal to the total production. I assume a non-homothetic preference, $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + A$, where A is a non-homothetic term that measures the utility from public goods. The following restrictions on parameters are needed in order to have a well-behaved bounded problem.

Assumption 1 $1-\epsilon > \sigma$, $\sigma > \epsilon$ and $\lambda^{1-\epsilon} > \beta$.

The first two of the assumption are identical to the ones discussed by Barro and Becker (1988) to assure strict concavity of the problem. The third one guarantees bounded utility as the effective discount factor, in which case $\beta f_t^{1-\epsilon} \leq \beta\lambda^{\epsilon-1} < 1$.[2]

Plugging fertility and consumption into the objective, the social planner's problem can be alternatively written as

$$\max_{\{N_{t+1}\}_{t=0}^{\infty}} \sum_t \beta^t N_t^{1-\epsilon} u\left(\frac{zF\left(\bar{K}, N_t - \lambda N_{t+1}\right)}{N_t}\right)$$

subject to

$$\frac{zF\left(\bar{K}, N_t - \lambda N_{t+1}\right)}{N_t} \geq \underline{c}$$

The optimality conditions for the social planner's problem is

$$\begin{aligned} & \beta N_{t+1}^{1-\varepsilon}\left(c_{t+1}\right)^{-\sigma}\left[\frac{zF_2\left(\bar{K}, N_{t+1}-\lambda N_{t+2}\right)}{N_{t+1}} - \frac{zF\left(\bar{K}, N_{t+1}-\lambda N_{t+2}\right)}{N_{t+1}^2}\right] + \beta\left(1-\varepsilon\right) N_{t+1}^{-\varepsilon} u\left(c_{t+1}\right) \\ = \; & \lambda N_t^{1-\varepsilon}\left(c_t\right)^{-\sigma} \frac{zF_2\left(\bar{K}, N_t - \lambda N_{t+1}\right)}{N_t} \end{aligned}$$

[2] An upper bound for the social welfare or the initial old's welfare is $\frac{u(c)}{1-\beta\lambda^{\epsilon-1}}$.

where

$$c_t = \frac{zF\left(\bar{K}, N_t - \lambda N_{t+1}\right)}{N_t}$$

The following proposition characterizes steady state efficient population and consumption.

Proposition 1 Steady state population and consumption are

$$N^* = 0,\ c^* = \infty$$

or

$$N^* = \infty,\ c^* = 0$$

or if $\frac{\lambda-\beta}{\beta}(1-\alpha) > \frac{\sigma-\epsilon}{1-\sigma}(1-\lambda)$, then

$$N^* = \bar{K}(1-\lambda)^{-1-\frac{\sigma}{\alpha(1-\sigma)}}\left(\frac{1}{A(1-\epsilon)}\right)^{\frac{1}{\alpha(1-\sigma)}}\left[\frac{\lambda-\beta}{\beta}(1-\alpha)+\frac{\epsilon-\sigma}{1-\sigma}(1-\lambda)\right]^{\frac{1}{\alpha(1-\sigma)}} z^{\frac{1}{\alpha}}$$

$$c^* = z\left(\bar{K}/N^*\right)^{\alpha}(1-\lambda)^{1-\alpha}.$$

Furthermore, consumption is independent of the technological progress measured by z.

Corollary 1 In the homothetic model, $A = 0$, the nondegenerate efficient population at steady state is zero, or is infinity if the subsistence consumption $\underline{c}$ is 0. If $\underline{c}$ is bigger than 0, then nonzero steady state efficient population is the carrying capacity, which is the maximum sustainable level of population.

$$\bar{N} = \bar{K}\left(\frac{z}{\underline{c}}\right)^{\frac{1}{\alpha}}(1-\lambda)^{\frac{1-\alpha}{\alpha}}$$

In the pure homothetic Barro and Becker model, steady state efficient population reaches the maximum sustainable level or 0. The allocation with maximum sustainable population would drive everyone's consumption to immiseration[3] and population is infinite if subsistence consumption is zero. This is an important benchmark because in a version of the standard growth model that allows for optimal demand for children it shows that two extreme cases are not necessarily inefficient in the long run. First, population is so large that consumption is driven to subsistence or immiseration. Second, the society disappears in the long run.

In the more general non-homothetic model, efficient steady state population could be less than the maximum sustainable and consumption could be larger than the subsistence consumption, e.g. $c^* > \underline{c}$. From Proposition 1, I can see that the existence of a finite efficient population is not guaranteed for all parameter settings. Existence requires the discount rate β to be low, or time cost of raising children λ to be high while both have to be in the interval $(0, 1)$. When λ obtains its upper limit, 1, then the condition $\frac{\lambda-\beta}{\beta}(1-\alpha) > \frac{\sigma-\epsilon}{1-\sigma}(1-\lambda)$ is automatically satisfied because $\sigma > \varepsilon$ and $\beta < 1$ by the assumption.

Proposition 1 also suggests that the BB model with a fixed amount of land is able to replicate the Malthusian trap: population adjusts according to technological improvement but consumption does not adjust. Although this model delivers Malthus property, it has different implication from Malthus model in the sense that infinitely large population and immiseration can be efficient in the long run.

I define *golden rule* level of population N^{gr} and the *modified golden rule* level of population, N^*, analogous to concepts used in growth theory to characterize the accumulation of capital. The first concept refers to the level of population that maximizes steady state welfare. More specifically, N^{gr} maximizes the following problem

$$\max_N N^{1-\epsilon} u \left(\frac{zF\left(\bar{K}, N - \lambda N\right)}{N} \right).$$

[3] Immiseration implies consumption is very low such that it barely sustain individuals' living. We are not the first to find that in principle immiseration could be efficient. The optimal constract literature, in particular Atkeson and Lucas (1992), shows that the optimal allocation of resource with private information leads to immiseration, a senario that every individual's consumption goes to zero and inequality grows to infinity.

The second concept refers to the efficient population at steady state. According to Proposition 3 below, it is the steady state population of the complete market economy.

It is well known that the modified golden rule level of capital is smaller than the golden rule level of capital because of the impatience to save and to consume. With a similar taste, the modified golden rule level of population N^* is smaller than the golden rule level of population N^{gr}. The reason is that children are costly in terms of parental time and the impatience to produce children leads to a lower N^*.

Proposition 2 The modified golden rule level of population is no larger than the golden rule level of population which attains the maximum sustainable level.

As proved in the appendix the golden rule level of population always attains the maximum sustainable. Hence the modified golden rule level of population is smaller than or equal to the golden rule level. This comparison is consistent with Dasgupta (2005). He compares the golden rule level of population with the population that maximizes generation related welfare, one similar with the modified golden rule level in this chapter but is not the same. His paper does not find that the golden rule level of population attains the maximal because his model assumes linear altruism, linear production technology, and cost of raising children is not considered.

Although non-homotheticity opens the possibility of a well-defined finite level of population, I log linearize the system and quantitatively find that the finite population at steady state is typically not stable. The parameters I use for this quantitative exercise are the following: $\beta = 0.15, \lambda = 0.4$, $\sigma = 0.72$, $\alpha = 0.30$, $\varepsilon = 0.57$, $A = 0.01$, and $z = 1$. I set them using the calibrated parameters in stochastic model with earnings risks and adjust λ to guarantee the existence of a finite nonzero steady state population. The value of parameters is not crucial to the instability of the nonzero finite steady state. The numerical exercise shows that the instability is robust to a variation of parameters under which the steady state exists.

Why is the stability of the nonzero finite steady state population hard to obtain in standard growth model with endogenous fertility? If the initial population is lower than the steady state population, with a fixed amount of land, the labor cost of raising a child is more expensive and parents tend to have fewer children than the steady state. This lowers future population which makes children even more expensive to raise in terms of parental time. More expensive children induces parents to have even fewer children, which reinforces a lower population. This process iterates and population becomes 0 in the long run. On the other hand, if the initial population is larger than steady state, children is very cheap since the opportunity cost is low. Similar argument would lead to infinite population in the long run. This result is unappealing to some extent because it implies that a standard Barro-Becker economy of endogenous children with fixed resources would eventually explode or implode, and population would not stabilize in the future.

However, there are several ways to get around this feature and recover stability. One way is to introduce a fixed goods cost to the model as I analyze in the next section.

Social Planner's Problem with Goods Cost Assuming that every child costs not only parental time but also a fixed amount of consumption goods η. In addition to this assumption, everything else is the same with the standard social planner's problem of the previous section and the social planner's problem can be expressed as:

$$\max_{\{N_{t+1},\, n_t,\, c_t\}_{t=1}^{\infty}} \sum_t \beta^t N_t^{1-\varepsilon} u\left(c_t\right)$$

subject to

$$N_{t+1} = n_t N_t$$

and

$$N_t c_t + \eta N_{t+1} = zF\left(\bar{K}, \left(1 - \lambda n_t\right) N_t\right) \tag{1}$$

The optimality conditions are

$$\begin{aligned}
& \beta N_{t+1}^{1-\varepsilon}\left(c_{t+1}\right)^{-\sigma}\left[\frac{zF_2\left(\bar{K},N_{t+1}-\lambda N_{t+2}\right)}{N_{t+1}}-\frac{zF\left(\bar{K},N_{t+1}-\lambda N_{t+2}\right)}{N_{t+1}^2}+\frac{\eta N_{t+2}}{N_{t+1}^2}\right] \\
& +\beta\left(1-\varepsilon\right)N_{t+1}^{-\varepsilon}u\left(c_{t+1}\right) \\
= \; & \lambda N_t^{1-\varepsilon}\left(c_t\right)^{-\sigma}\frac{zF_2\left(\bar{K},N_t-\lambda N_{t+1}\right)}{N_t}+N_t^{-\varepsilon}\left(c_t\right)^{-\sigma}\eta
\end{aligned}$$

for $t \geq 0$ where

$$c_t=\left[zF\left(\bar{K},N_t-\lambda N_{t+1}\right)-\eta N_{t+1}\right]\frac{1}{N_t}$$

Log linearizing the system around the steady state, the numerical exercise shows that a fixed goods cost brings stability back to the finite steady state efficient population, N^*, and prevents infinite population from occurring in the long run. It is because no matter how cheap children are in terms of parental time, there is a goods cost associated with raising children, and a large population with fixed resource implies low production and low earnings of every individual. The zero steady state population is not stable because on one hand, children is expensive in terms of taking parental time, but on the other hand they are cheap as rich parents face a low marginal utility of consumption. If the second effect dominates the first, people tends to have more children which promotes more population in the future.

Similar with the model without goods cost of raising children, the numerical exercise shows that the modified golden rule level of capital per capita is 146.5, much bigger than the golden rule level, 0.0015. The parameters I use for this exercise is the same with the previous model except that $\lambda = 0.21$.

2.2.2 Decentralization

In this section I decentralize the social planner's problem by a complete market economy with fixed resource, land. The purpose is using the decentralized land market to study the effect of market frictions. I show that the complete market economy obtains efficient allocations, in particular the efficient population.

Individuals' Problem Along the line of BB, Individuals live for one period. They maximizes the utility from own consumption and value the welfare of children in an altruistic way.

$$V_t(a_t) = \max_{c_t, b_{t+1}, 0 \leq n_t \leq \bar{n}} u\left(c_t\right) + \Phi(n_t) V_{t+1}(a_{t+1}) \tag{2}$$

subject to

$$c_t + n_t q_t a_{t+1} + n_t \eta \leq \left(q_t + r_t\right) a_t + w_t \left(1 - \lambda n_t\right)$$

$$c_t \geq \underline{c}$$

As specified in the previous section, c_t and n_t denote consumption and fertility. a_t is the transfer of land received from parents when young and a_{t+1} is the transfer of land to every child when parents are old but children are young. The price of land in terms of consumption goods at time t is q_t.

Let $\Phi(n_t) = \beta n_t^{1-\epsilon}$ be the altruism function where β is the intergenerational discount rate. It is the weight that parents place on their n_t children. ϵ is the altruism factor which controls how much parents care about children. When $\epsilon = 0$ parents are perfectly altruistic toward children and they do not exhibit decreasing altruism as more children are born. w_t is wage and R_t is the return on land.

Firms' Problem Firms produce using the Cobb–Douglas production function $F(\bar{K}, L_t) = z\bar{K}^{\alpha} L^{1-\alpha}$ where $\bar{K}$ is the fixed amount of land, L_t is labor and z is total factor productivity. The per-labor production function is $f(k_t) = \frac{F(\bar{K}, L_t)}{L_t}$ where $k_t \equiv \frac{\bar{K}}{L_t}$. Assuming the depreciation rate is 0. In competitive markets rents and wage are determined by

$$r_t = \alpha z k_t^{\alpha-1} \text{ and } w_t = \left(1 - \alpha\right) z k_t^{\alpha}. \tag{3}$$

Definition of Competitive Equilibrium A competitive equilibrium are price sequences $\{w_t, R_t\}_{t=0}^{\infty}$, allocations $\{c_t, n_t, a_{t+1}\}_{t=0}^{\infty}$ and population $\{N_t\}_{t=0}^{\infty}$ such that: (i) allocations solve the individual's problem given prices; (ii) w_t and r_t satisfy (51), (iii) demand for land is equal to the stock of land,

$$\bar{K} = N_{t+1}a_{t+1}$$

and (iv) the evolution of population satisfies

$$N_{t+1} = n_t N_t.$$

Characterize the Steady State Allocations The complete market economy delivers the efficient solution, as formalized in the following proposition.

Proposition 3 Complete market allocations (c_t, n_t, a_{t+1}, N_t) are efficient.

With a well defined efficient population, it would be interesting to investigate how market friction affects long run population. The market friction I consider is similar with credit friction, an important driving force of inequality in macroeconomics. Instead of restricting individuals from borrowing intertemporally, it restricts parents from borrowing intergenerationally against children. We call this type of restriction "bequest constraints". A particularly realistic constraint is the non-negative bequest constraint, which implies that parents can not impose debt on children.

For this purpose, I extend the benchmark model to a dynastic altruistic model in which individuals live for two periods, young and old. The extended model improves upon the dynastic altruistic model with one period life by incorporating the life cycle savings property. The model with one period life is not appropriate to study markets with frictions because a binding nonnegative transfer constraint would not lead to an equilibrium since people are homogenous and everyone transfers zero bequest (land) to children. One alternative dynastic altruistic model is a deterministic model with two periods' life.

2.3 An Overlapping Generation Model

2.3.1 Environment

Individuals' and Firms' Problem To consider the impact of bequest constraints on equilibrium population, I extend the model to an overlapping generation model in which every individual lives for two periods. The fixed goods cost of raising a child is not considered in this section partly because the life-cycle savings property in this richer model could recover stability of the steady state. Also goods cost is taken into account in children's consumption. The fixed cost tends to complicate the model and destroy the inverse relationship between fertility and earning abilities, an empirical regularity in the data.

In this overlapping generation model, people work, save and raise children when young, and retire when old. The individuals' problem for time $t \geq 0$ is:

$$V_t(b_t) = \max_{c_t^y, c_{t+1}^o, a_{t+1}, b_{t+1}, 0 \leq n_t \leq \bar{n}} u\left(c_t^y, c_{t+1}^o\right) + \Phi(n_t) V_{t+1}(b_{t+1}) \tag{4}$$

subject to:

$$c_t^y + q_t a_{t+1} \leq w_t \left(1 - \lambda n_t\right) + b_t \tag{5}$$

$$c_{t+1}^o + n_t b_{t+1} \leq \left(q_{t+1} + r_{t+1}\right) a_{t+1} \tag{6}$$

Let c_t^y and c_{t+1}^o denote the consumption when young and old, respectively. $u\left(c_t^y, c_{t+1}^o\right) = A + \left(c_t^y c_{t+1}^o\right)^\gamma$ is the utility from consumption and public goods. Assuming $2\gamma < 1$. Young individuals at time t purchase an amount of land a_{t+1}. They reap the land rent and sell the land at time $t+1$ when they are old. q_t is the price of land and r_t is the rental rate of land. Following the notation of the previous section, b_t is the bequest received when young from parents while b_{t+1} is the bequest to every child.

For a young agent, consumption and the total value of land purchased is no more than the transfer received from parents and her own labor income. For an old individual, consumption and transfer to children is no more than the summation of rents and the value of land. The two periods' budget constraints can be written as a single one:

$$c_t^y + \left(c_{t+1}^o + n_t b_{t+1}\right)/R_{t+1} \leq b_t + w_t\left(1 - \lambda n_t\right), \text{ and } b_{t+1} \geq 0. \tag{7}$$

where

$$R_{t+1} = \frac{q_{t+1} + r_{t+1}}{q_t}.^4 \tag{8}$$

Notice that the intertemporal elasticity of substitution is -1 while $1/\left(2\gamma\right)$ is the intergenerational elasticity of substitutions. Savings are given by:

$$q_t a_{t+1} = b_t + w_t\left(1 - \lambda n_t\right) - c_t^y. \tag{9}$$

Firms' problem is the same with what's described in the previous section. They pay rents and labor income according to (3).

Since 2γ in the model with two periods' life corresponds to σ in the model with one period life, I assume $2\gamma > \varepsilon$ to guarantee the concavity of individual's problem.

Demographics Let $N_t = N_t^y + N_t^o$ denotes the total population at time t. N_t^y and N_t^o denote the population at time t when young and old, respectively. The demographic structure satisfies the following identities:

$$N_t^y = n_{t-1}N_t^o,\ N_{t+1}^o = N_t^y,\ N_t = \left(1 + n_{t-1}\right)N_t^o \tag{10}$$

so

$$\frac{N_t^y}{N_t} = \frac{n_{t-1}}{1+n_{t-1}}, \; \frac{N_t^o}{N_t} = \frac{1}{1+n_{t-1}}, \; \frac{N_{t+1}}{N_t} = \frac{1+n_t}{1+n_{t-1}} n_{t-1}.$$

Aggregate Resources Aggregate land is in fixed supply and the aggregate labor supply in the economy is $L_t = N_t^y (1 - \lambda n_t)$. The aggregate resource constraint for output is:

$$N_t^y c_t^y + N_t^o c_t^o = F(K, L_t)$$

Alternatively,

$$\frac{N_t^y}{N_t} c_t^y + \frac{N_t^o}{N_t} c_t^o = \frac{L_t}{N_t^y} \frac{N_t^y}{N_t} f(k_t).$$

or

$$n_{t-1} c_t^y + c_t^o = (1 - \lambda n_t) f(k_t) n_{t-1} \tag{11}$$

Additionally, the resource constraint for land is

$$\bar{K} = a_{t+1} N_t^y \tag{12}$$

Definition of Competitive Equilibrium A competitive equilibrium are price sequences $\{q_t, w_t, r_t\}_{t=0}^{\infty}$, allocations $\{c_t^y, c_t^o, n_t, a_{t+1}, b_{t+1}\}_{t=0}^{\infty}$ and a population distribution $\{N_t^y, N_t^o\}_{t=0}^{\infty}$ such that: (i) allocations solves the individual problem given prices, (ii) w_t and r_t satisfy firms' optimality conditions (3), and (iii) q_t clears the land market as defined by (12), (iv) the population distribution satisfies (10), and (v) a resource constraint (11) is satisfied. Note that (12) can be written using (9) as:

$$k_{t+1} = \frac{a_{t+1} N_t^y}{L_{t+1}} = a_{t+1} \frac{1}{n_t} \frac{1}{1 - \lambda n_{t+1}} \tag{13}$$

The market clearing condition for intergenerational transfer b_{t+1} is not written because the amount is transferred from parents to children when both are alive.

2.3.2 Complete Market Model

In this section, I show that steady state population in complete market model is efficient. I then compare the steady state efficient population with the steady state population in incomplete markets when credit constraints bind. The idea is to study whether incomplete markets produce too much or too little population.

Steady State

Proposition 4 The steady state allocations $(n^{ss}, N^{ss}, c^{yss}, k^{ss})$ and prices $(R^{ss}, , r^{ss}, w^{ss})$ of the complete market economy are described by the following system:

$$N^{ss} = \left[\frac{\frac{\gamma}{1-\lambda}\left(\beta^{-1}\lambda - 1\right)\left(1+\beta^{-1}\right)(1-\alpha) - (1-\epsilon-2\gamma)}{1-\epsilon} \right]^{\frac{1}{2\gamma\alpha}} \left[\frac{(1-\lambda)\, z}{\left(1+\beta^{-1}\right)} \right]^{\frac{1}{\alpha}} \beta^{-\frac{1}{2\alpha}} A^{-\frac{1}{2\gamma\alpha}}$$

$$c^{yss} = \left[\frac{1-\epsilon}{\gamma\left(\beta^{-1}\lambda - 1\right)\frac{\left(1+\beta^{-1}\right)}{1-\lambda}(1-\alpha) - (1-\epsilon-2\gamma)} \frac{A}{R^{\gamma}} \right]^{\frac{1}{2\gamma}}$$

$$k^{ss} = \left[\frac{c^{yss}\left(1+\beta^{-1}\right)}{(1-\lambda)\, z} \right]^{\frac{1}{\alpha}}, \; n^{ss} = 1, \; R^{ss} = \frac{1}{\beta}$$

and r^{ss} and w^{ss} follow by (51). If the utility function is homothetic, e.g. $A = 0$, the steady state population goes to 0 or infinite unless a knife edge condition

$$\theta\,(R\lambda - 1)\,(n+R) = n\,(1-\lambda n) \text{ where } \theta \equiv \frac{\gamma\,(1-\alpha)}{\epsilon_n^{\Phi} - 2\gamma}$$

holds. When this condition is satisfied, the steady state population is undetermined.

Since land and technology are fixed at steady state then the only possible balanced growth path is with $n = 1$, which requires zero population growth.

Similar to the model with one period life, the model with two periods' life also suggests the dynastic altruistic model with a fixed land has the Malthusian property: steady state consumption does not respond to technological progress but population does.

Corollary In the frictionless model, steady state consumption c^{yss} is independent of technological progress z. A bigger z leads to more population N.

Proposition 5 shows that in the long run complete market with two periods' life delivers the efficient allocations, the ones solved the social planner's problem with individuals living for two periods.

Proposition 5 In the two periods' problem, the complete market delivers the steady state efficient allocations, the ones solved from the social planner's problem:

$$\max_{N_{t+1}^y, c_t^y, c_{t+1}^o} \sum_{t=0}^{\infty} \beta^t \left(N_t^y\right)^{1-\epsilon} u\left(c_t^y, c_{t+1}^o\right)$$

where

$$u\left(c_t^y, c_{t+1}^o\right) = A + \left(c_t^y c_{t+1}^o\right)^{\gamma}$$

subject to

$$N_t^y c_t^y + N_t^o c_t^o = F\left(\bar{K}, L_t\right)$$

2.3.3 Incomplete Market Model

In this section, I study incomplete markets. I consider a friction similar with credit friction, an important driving force of market incompleteness in macroeconomics. The friction comes from an exogenous restriction on transfer from parents to children, $b_{t+1} \geq \underline{b}$. More specifically, parents are not allowed to transfer an amount lower than $\underline{b}$ to every child. I characterize the steady state allocation with different credit limits $\underline{b}$ and compare the efficient population to the level of population in incomplete markets. A notion of over- and under-population is formalized through this comparison.

Steady State When the credit constraint binds, $b_{t+1} = \underline{b}$. The following proposition characterizes the steady state allocations of the system.

Proposition 6 The dynamics of the system with binding credit constraint is described by the following equations:

$$n_t\left(\underline{b} - n_{t+1}\underline{b}/R_{t+2}\right) + R_{t+1}\left[\left(1-\alpha\right) z k_t^{\alpha}\left(1-\lambda n_t\right) + \underline{b} - n_t \underline{b}/R_{t+1}\right] = \left(1-\lambda n_{t+1}\right) z k_{t+1}^{\alpha} n_t \left(1+\alpha\right) \tag{14}$$

$$2k_{t+1} = \frac{1}{q_t}\left(\left(1-\alpha\right) z k_t^{\alpha}\left(1-\lambda n_t\right) + \underline{b} + n_t \underline{b}/R_{t+1}\right)\frac{1}{n_t}\frac{1}{1-\lambda n_{t+1}} \tag{15}$$

$$\gamma \frac{u_t}{c_t^y}\left(w_t\lambda + \underline{b}/R_{t+1}\right) = \epsilon_n^{\Phi}(n_t)\frac{\Phi(n_t)}{n_t} u_{t+1}\left(1 + \frac{n_{t+1}}{\epsilon_n^{\Phi}}\gamma\frac{1}{c_{t+1}^y}\left(w_{t+1}\lambda + \underline{b}/R_{t+2}\right) + \frac{A}{u_{t+1}}\right) \tag{16}$$

$$R_{t+1} = \frac{q_{t+1} + \alpha z k_{t+1}^{\alpha-1}}{q_t} \tag{17}$$

The nondegenerate steady state allocations (n, k, N, c^y) and prices (r, w, R, q) of the economy with binding credit constraint, $b_{t+1} = \underline{b}$, are given by:

$$n = 1$$

$$\frac{1-\beta}{1-\epsilon}\gamma\frac{1}{c^y}\left(w\lambda + \underline{b}/R\right) = \beta\left(1 + \frac{A}{u}\right)$$

$$k = \frac{1}{2q}\left(w + \underline{b}\left(1 + 1/R\right)\frac{1}{1-\lambda}\right)$$

$$N = \frac{\bar{K}}{k}$$

$$R = 1 + \frac{r}{q}$$

$$u = (c^y)^{2\gamma} R^\gamma, \; c^y = \frac{1}{2}\left(w\left(1-\lambda\right) + \underline{b} - \underline{b}/R\right)$$

and

$$w = z\left(1-\alpha\right)k^\alpha, r = z\alpha k^{\alpha-1}.$$

The Model with Non-negative Credit Constraint A particularly realistic case of the bequest constrained model is the one with non-negative constraint on transfer or credit limit, $b_{t+1} \geq 0$. Non-negative bequests imply that parents cannot legally impose debt obligations on their children. When the constraint is binding, $b_{t+1} = 0$.

Proposition 7 Assume $\frac{1-\beta}{\beta}\frac{\lambda}{1-\lambda}\frac{2\gamma}{1-\epsilon} > 1$. Under the condition $2 < (1-\alpha)\left(1 + \frac{1}{\beta}\right)$, the nonnegative credit constraint binds, and the steady state utility is given by

$$u = \frac{A}{\frac{1-\beta}{\beta}\frac{\lambda}{1-\lambda}\frac{2\gamma}{1-\epsilon} - 1}$$

The restriction $\frac{1-\beta}{\beta}\frac{\lambda}{1-\lambda}\frac{2\gamma}{1-\epsilon} > 1$ guarantees that the utility u is positive. Since $\frac{2\gamma}{1-\epsilon} > 1$ then this requires time cost λ to be not too low. Both conditions require β to be small. The more severe an individual discounts the future, the less he will save through children and the more likely the constraint on transfer binds.

Individual's steady state welfare depends negatively on λ, time cost of raising children. This is because a higher cost of raising children reduces earnings and consumption. Marginal utility of consumption increases as a partial equilibrium effect. Parents care more about the foregone time cost of raising children so fertility tends to go down. On the other hand, the marginal benefit of having a child is lower because at higher cost of raising children the current generation anticipates lower future generations' consumption and welfare. Altruistic parents care about children's welfare. Lower welfare motivates lower fertility. If prices do not adjust steady state fertility when time cost is high tends to be smaller than the one when time cost is low. However the fixed land assumption implies the steady state fertility is 1 as population do not grow at steady state. Hence when the time cost of raising children is higher wage needs to be lower so that parents have stronger incentive to have more children. Higher cost of raising children and lower wage make people worse off.

The credit (borrowing) constraint binds if and only if

$$\frac{u_t}{c_t^y} > R_{t+1} \frac{\Phi(n_t)}{n_t} \frac{u_{t+1}}{c_{t+1}^y}.$$

The parameter restriction to guarantee a binding constraint at steady state is

$$1 > \beta \frac{1+\alpha}{1-\alpha}$$

where $\frac{1+\alpha}{1-\alpha}$ is the value of the gross interest rate at steady state and $\Phi(1) = \beta$.

Using the resource constraint, equation (11),

$$c_t^y = \frac{(1-\alpha)(1-\lambda)}{2} f(k_t)$$

Under the gross interest rate $\frac{1+\alpha}{1-\alpha}$, the consumption that clears the market is exactly the level that is solved from equilibrium.

The model with nonnegative constraint has a property consistent with the Malthus model as well, that is, the technological progress leads to more population but not consumption in the long run.

2.3.4 Simulations of Deterministic Models

Parameters The general equilibrium model with dynastic altruistic fertility choices does not have closed form solutions. I perform quantitative simulation in this section to investigate the steady state property of the deterministic model, and study the effect of market frictions on long-term population by comparing steady state population under and without the friction. For most parameters, I choose the ones calibrated to the stochastic model in the following section. Since people live for two periods in the deterministic model while they live for one period in the stochastic model, the parameters of preference is different. For example, the γ in the deterministic model corresponds to $\frac{\sigma}{2}$ in the stochastic model because the lifetime utility function in deterministic model can be expressed as $u = A + R^{\gamma}\left(c^{y}\right)^{2\gamma}$ at optimal. Parameters are summarized in Table 2.1.

Results Steady state consumption when young c^y, per capita land k, price of land q, and interest rate R in the unconstrained model and the model with binding non-negative credit constraint are summarized in the following table.

Table 2.1 Unconstrained and Constrained Deterministic Models

Variables	Unconstrained	Nonnegative credit constraint
c^y	0.0086	0.0374
k	0.0002	0.0017
q	26.19	31.08
R	5.56	1.84

The quantitative exercise suggests the steady state per capita capital in unconstrained model is lower than the steady state capital in the model with nonnegative constraint on transfer when the constraint binds. Given the environment with a fixed amount of land, it turns out that the model when the constraint binds has more capital per capita and hence fewer population at steady state compared with the model in absence of the credit constraint. Schoonbroodt and Tertilt (2014) show that model when the bequest constraint binds produces fewer fertility than the unconstrained model. The reasoning in ST for why fertility is lower in constrained model is that parents are prevented from extracting resource from children through a negative bequest and children is a normal good. Their model compares fertility rates (the growth rate of population) between complete and incomplete market models while I compare the level of population. However, if I look only at incomplete market models with credit frictions, a surprising result arises: a tighter constraint increases the population at steady state. Figure 2.1 plots how capital and population move as the constraint is relaxed (meaning that the lower bound of transfer from parents to children decreases from 0 to b^{un}).

The left panel plots steady state per capita land k versus bequest lower bound $\underline{b}$ and the right panel plots steady state population N versus $\underline{b}$. In both panels there is a discontinuity at the unconstrained optimal bequest level b^{un}. Other than that point, steady state population increases as the constraint becomes tighter (meaning $\underline{b}$ increases to 0). The mechanism is the following. When the economy originally stays at the steady state, tightening the credit constraint prevents parents from extracting as much resource from children as before, and as a result fertility tends to go down below 1, which pushes the economy away from the original steady state. People have fewer children and spend more time working. The partial equilibrium effect leads to a higher total labor supply of the economy. The wage rate falls as a general equilibrium effects, which makes it cheaper to have children and hence induces people to have more children. Fertility falls as the partial equilibrium effect after the tightening of the constraint and then increases to 1 so that the economy arrives at a new steady state. Under the new steady state wage is lower than that of the original steady state and hence population is higher when total land is fixed. An application of this result is a prediction of lower population after introducing the pay as you go (PAYG) system. PAYG requires current young to pay money for the current old, which is parallel with a more relaxed credit friction. From the perspective of the model, I anticipate the population to go down.

The jump of capital and population at the unconstrained optimal bequest is because the mapping from the degree of friction, described by $\underline{b}$, to steady state interest rate is discontinuous as shown in Figure 2.2. The horizontal axis is bequest lower bound and the vertical axis is interest rate R. When the constraint is very tight, equilibrium interest rate is low. The more relaxed the constraint, the lower the equilibrium interest rate. However, when the constraint is relaxed to the first best level, there are two equilibrium interest rates. One of them is low and the other one is high. The one with high magnitude is exactly the frictionless interest rate. I denote it by point A in the figure.

Stability Analysis of the Steady State In the previous sections, I focus on characterizing the steady state and the comparative static analysis. In this section, I quantitatively investigate the stability of the steady state. For the simplicity of notation, denote the system of equations, (14), (15), (16), and (17) by the following four equations:

$$F^1\left(n_{t+1}, n_t, R_{t+2}, R_{t+1}, k_{t+1}, k_t\right) = 0$$

$$F^2\left(k_{t+1}, k_t, n_t, R_{t+1}, n_{t+1}, q_t\right) = 0$$

$$F^3\left(k_t, k_{t+1}, n_t, n_{t+1}, R_{t+1}, R_{t+2}\right) = 0$$

$$F^4\left(R_{t+1}, q_{t+1}, q_t, k_{t+1}\right) = 0$$

Log linearization around the steady state is done in the appendix. I express the system as $X_{t+1} = A^{-1}BX_t$. Four eigenvalues of the matrix $A^{-1}B$ determine the stability of the four dimensional first order dynamic system. According to the simulated result, all the steady state except for point A, the steady state of the unconstrained model, is saddle point stable. Among the four eigenvalues of the model with credit limit $\underline{b} \in (b^{un}, 0]$, one real eigenvalue is close to 0, another real eigenvalue has absolute value bigger than 1, and the norm of two conjugate complex eigenvalues is bigger than 1. Hence, steady states of credit constraint models are saddle path stable. The four eigenvalues of the complete market model are 0, 0.8123,

$4.2959 \pm 0.3458\ i$. The absolute values of two real roots are smaller than 1 and the norm of the two are bigger than 1. The steady state of the complete market model, point A, is stable.

2.4 Stochastic Model

2.4.1 Environment

In this section, I extend the deterministic model to a stochastic model in which abilities are random and correlated between parents and children. This model is essentially a Bewley model with a fixed amount of land and endogenous population. In this model every individual lives one period. I am able to use a stochastic model with one period life, not necessarily two periods' life, to study the impact of friction on population because with heterogenous agents a binding nonnegative transfer (bequest) constraint can be equilibrium because it does not imply every individual's bequest constraint is tight. In this sense, two periods' life is not as crucial as that in the deterministic case.

In the stochastic model, lifetime earning ability, denoted by ω, is determined at birth and a child's ability ω' is correlated with parent's ability. Assume earning ability follows an AR(1) process

$$\text{ln}\omega' = \rho \text{ln}\omega + \varepsilon, \text{ where } \varepsilon \sim N(0, \sigma_\omega). \tag{18}$$

The lifetime utility of an individual is of the Barro-Becker type:

$$V(a, \omega) = \max_{c,n,a'} u(c) + \Phi(n) E\left[V\left(a', \omega'\right) | \omega\right]$$

where $u(c) = \frac{c^{1-\sigma}}{1-\sigma} + A$. c is consumption and n is the number of children. Let a denotes transfer, or bequest, received from parents. $V(a, \omega)$ is the utility of every individual who inherits an amount of asset a from parents and is endowed with earning ability ω. I call them a type (a, ω) individual in the following.

E is the mathematical expectation operator conditional on parental ability ω. The altruism function is the same with the form in the deterministic model $\Phi(n) = \beta n^{1-\epsilon}$, where $0 \leq \epsilon < 1$.

The resources of an individual endowed with ability ω are labor income and transfers from their parents. Labor income is the multiplication of time devoted to working, $(1 - \lambda n)$, and wage adjusted by ability, ωw. Resources are used to consume and to bequeath to children. The budget constraint of an individual with ability ω is:

$$c + na' \leq (1 + r) a + \omega (1 - \lambda n) w$$

I assume there is a credit constraint, under which parents are restricted from leaving an amount of transfer (bequest) lower than $\underline{a}$,

$$a' \geq \underline{a}.$$

A special case is the nonnegative credit constraint when $\underline{a} = 0$.[5]

The interior optimal fertility condition is

$$u'(c) \left(\omega \lambda w + a'\right) = \beta \Phi'(n) E\left[V\left(a', \omega'\right) |\omega\right]$$

Assume $\sigma \in (0, 1)$. The curvature of the utility function captures the intergenerational elasticity of substitution, not the intertemporal elasticity of substitution. A low σ implies high elasticity of intergenerational substitution. Typically in macroeconomics this parameter is bigger than 1.

2.4.2 Law of Motion of Population and Distribution

Let $N(a, \omega)$ be their population of individuals of type (a, ω), $n(\alpha, \omega)$ be their fertility, and $g(a, \omega)$ be their bequest policy. Children takes parents time, so the effective labor supply of an individual with ability

[5]By setting the per capita land $k \equiv \frac{a}{q}$ and rents $\tilde{r} \equiv \frac{r}{q}$, one can easily show that the model with transferring only asset to children is equivalent with a model in which a parent with type (k, ω) transfer only an amount of land $k'(k, \omega)$ to every child. We show in the appendix that a model in which parents transfer both land and assets to children is equivalent with a Bewley model when parents only transfer land to children.

ω is $\omega\left(1-\lambda n\left(\alpha,\omega\right)\right)$. The total labor L in the economy is the summation of all individuals' labor supply, weighted by population and ability.

$$L=\sum_{(a,\omega)}N\left(a,\omega\right)\omega\left(1-\lambda n\left(\alpha,\omega\right)\right)$$

The law of motion of population is

$$N\left(a',\omega'\right)=\sum_{(a,\omega)}N\left(a,\omega\right)n\left(a,\omega\right)M\left(\omega,\omega'\right)I\left(a',a,\omega\right)$$

where $I\left(a',a,\omega\right)$ is an indicator function

$$I\left(a',a,\omega\right)=\begin{cases}1 & \text{if } a'=g\left(a,\omega\right)\\ 0 & \text{otherwise}\end{cases}$$

$I\left(a',a,\omega\right)$ identifies parents with states (a,ω) who transfer an amount a' to children. Let the unconditional distribution of (a,ω) be $\pi\left(a,\omega\right)\equiv\frac{N(a,\omega)}{N}$. Dividing both sides of the law of motion of population by total population of the economy N,

$$\frac{N'}{N}=\sum_{(a',\omega')}\sum_{(a,\omega)}\frac{N\left(a,\omega\right)}{N}n\left(a,\omega\right)M\left(\omega,\omega'\right)I\left(a',a,\omega\right)=\sum_{(a,\omega)}\pi\left(a,\omega\right)n\left(a,\omega\right)$$

which derives the law of motion of distribution, $\pi\left(a,\omega\right)$:

$$\pi\left(a',\omega'\right)=\frac{\sum_{(a,\omega)}\pi\left(a,\omega\right)n\left(a,\omega\right)M\left(\omega,\omega'\right)I\left(a',a,\omega\right)}{\sum_{(a,\omega)}\pi\left(a,\omega\right)n\left(a,\omega\right)} \tag{19}$$

2.4.3 Equilibrium

Land Market Total supply of asset is the value of land, $q\bar{K}$, which is the price of land q times the stock of land $\bar{K}$. In equilibrium, it must equalize the demand of asset $\sum g(a,\omega) N(a,\omega)$. So asset market clearing condition is

$$q\bar{K} = \sum g(a,\omega) N(a,\omega)$$

It induces the equilibrium condition of land market

$$\bar{K} = \frac{N}{q} \sum g(a,\omega) \pi(a,\omega) \tag{20}$$

Labor Market Population does not change with time at steady state. $N = N'$ if and only if

$$\sum_{(a,\omega)} \pi(a,\omega) n(a,\omega) = 1 \tag{21}$$

holds.

Total labor demand equals the total labor supply, which derives the labor market clearing condition

$$\frac{L}{N} = \sum_{(a,\omega)} \pi(a,\omega) \omega (1 - \lambda n(a,\omega)) \tag{22}$$

Following the notation in deterministic model, wage w and rent r are the marginal product of labor and land, respectively.

$$w = F_L(\bar{K}, L), \ r = F_{\bar{K}}(\bar{K}, L) \tag{23}$$

Definition of Equilibrium at steady state A competitive equilibrium at steady state are price $\{q, w, r\}$, allocations $\{c(a,\omega), n(a,\omega), a'(a,\omega)\}_{(a,\omega)}$, the distribution $\{\pi(\alpha,\omega)\}_{(\alpha,\omega)}$, a population level N, and a level of labor supply L such that: (i) allocations solve the individual problem given prices; (ii) w and r satisfy (23), and q clears the land market as defined by (20); and (iii) population and labor satisfy equations (21) and (22).

This general equilibrium model with heterogenous agents does not have a closed form solution, so the quantitatively solution is solved in the following section.

2.4.4 Simulations of the Stochastic Model

Parameters The following parameters are needed to simulate the model: the Markov process of abilities M, preference parameter σ, nonhomothetic part A of the utility, altruistic parameters β and ϵ, cost of raising children λ, the capital share α in the production function, and the stock of land $\bar{K}$.

I use the Tauchen's Method to discretize the AR(1) process (18) by a three state Markov chain M. Denote by $M(\omega, \omega')$ the probability that a child to be endowed with ability ω' given ω as the parental ability. Setting the intergenerational persistence of log ability ρ to be 0.5 and the variance σ_w to be 0.85.

σ and ε are chosen to be 0.72 and 0.57 respectively according to Cordoba, Liu and Ripoll (2014) (CLR henceforth) who calibrate the dynastic altruistic model to match key features of U.S. inequality. The parameter σ, the curvature of the utility function, is smaller than 1. In the context of dynastic models, it controls the intergenerational elasticity of substitution (EGS) that measures the willingness to substitute consumption across generations (between parents and children). By contrast, typically in macroeconomics σ is bigger than 1 because it controls the intertemporal elasticity of substitution (EIS) that measures the willingness to substitute consumption between different periods of one's life. As Cordoba and Ripoll (2014b) (CR henceforth) investigate, a low EGS is one of the key factors that generate a negative relationship between fertility and income. Time cost of raising children λ is picked to be 0.308, the one calibrated in Cordoba and Ripoll (2014b). Their calibration is based on Folbre (2008) who uses data from the 1997 Child Development Supplement of the PSID and the family income data of USDA (2012).

Other parameters that needs to be calibrated are capital share α, total factor productivity z, intergenerational discount rate β, and the nonhomotheticity A. I calibrate α to match net interest rate which is 2 calibrated in Cordoba, Liu and Ripoll (2014), z to match arable land per person in the United Kingdom, and A to match the income elasticity of fertility. The income elasticity of fertility, estimated in Jones and Tertilt (2006) using U.S. census data from 1826 to 1960, is -0.38. Broadberry, Campbell, Klein, Overton, and Leeuwen (2010) provide data on the population of the United Kingdom in 1700s, which is 5.2 million, and the total arable land during the same period, which is 9.7 millions of acres. Hence the arable land per person is 1.865 acres. I do not match the parameter β to a specific target because the equilibrium of dynastic altruistic model with fixed land is very hard to obtain. Given other parameters, the interest rate that clears the land market tends to be too high to maintain an average fertility rate at 1. The transfer lower limit $\underline{a}$ is set to be 0 and parents face a nonnegative constraint on transfer. There is typically no equilibrium unless the parameter β is low enough to reduce fertility rates of different types. These set of parameters are summarized in Table 2.2.

Table 2.2 Parameters

Parameters	Concept	Values	Targets
β	discount rate	0.15	avg$(n) = 1$
ϵ	altruistic parameter	0.57	CLR (2014)
σ	elasticity of substitution	0.72	CLR (2014)
λ	per child time cost	0.308	CR (2014b)
α	capital share	0.295	interest rate in CLR (2014)
z	TFP	1	per capita land (UK 1700)
A	constant term in utility	0.28	income-fertility elasticity
$\underline{b}$	bequest lower bound	0	exogenous

Results The simulated model shows that high ability individuals choose to have more consumption and fewer children, and bequeath more to every child than low ability individuals. These properties are consistent with the data and are shown from the first three panels of Figure 2.3 in which red dashed line denotes allocations of high ability individuals and blue solid line denotes those of low ability. The upper left panel shows that the amount that high ability individuals' bequeath to every child does not vary a lot as they receive more bequest from parents. The extra resource from parents translates into more demand for children but not more transfer to every child. Notice that in the numerical simulation, I consider a discrete set of number of children. For example, the set can be $\left\{0, 1, 2, \cdots, \frac{1}{\lambda}\right\}$. As the number of children approaches a continuous variable, the bequest policy tends to be flat. This feature is consistent with Alvarez (1999) that considers children and total bequests to all children as two assets in the dynamic portfolio problem. This result crucially depends on the following elements: BB form of homothetic altruism function, linear time cost function and the CES from of the homothetic part of utility function. In Cordoba, Liu and Ripoll (2014), we show that this feature does not necessarily hold for those whose credit constraint are binding, the low type individuals in this chapter. If the transfer received from parents is high enough, low type individuals' optimal fertility could hit the maximum number and is not able to increase as the transfer received from parents rises.

The inverse relationship between ability and fertility is not easy to obtain in altruistic fertility models. I replicate this result as shown in the lower left panel of Figure 2.3. The reason why an inverse link is reproduced is due to the following four factors: abilities are persistent but not perfectly persistent across generations, the high elasticity of substitution ($\sigma < 1$), the presence of time cost of raising children, and the market friction. As the earning ability increases, the marginal cost of having a child, $u'(c)(\omega\lambda w + a')$, increases because on one hand children are more costly for high ability parents in terms of time cost. On the other hand, high ability people care less about the time cost because they enjoy lower marginal utility of consumption. When $\sigma < 1$, the time cost dominates and high ability people have higher marginal cost of raising a child. The marginal benefit $\beta\Phi'(n)\,E\left[V(a',\omega')\,|\omega\right]$ increases with earning ability due to the persistence of ability. If the intergenerational persistence of ability is not perfect, the effect of ability on the marginal cost dominates its effect on the marginal benefit. Regression to the mean in abilities means that low ability parents expect their children to be of higher ability on average while high ability parents expect their children to be of lower ability. This explains why poor people have more children and rich ones have fewer children in a fully rational and altruistic environment.

The lower right panel of Figure 2.3 plots the distribution of population of two ability groups. As I can see from the figure, a majority of people are under credit constraint. More specifically 43% of low ability people and 29% of high ability people bequeath nothing to every child. They would like to impose a debt upon children but are prevented from doing so by the nonnegative credit constraint. The huge amount of people under constraint indicates the importance of the credit friction, which is the main driving force of the incomplete market model in this chapter.

Policy Experiments I perform a policy experiment of relaxing the credit constraint to investigate whether steady state population becomes higher or lower. When A is 1 as I use above, the steady state population with different bequest lower bounds are close to 0 and it is hard to capture the trend of population as I relax the credit constraint. As a result, I decrease A to 0.3 to obtain higher population levels. As Figure 2.4 shows, the steady state population drops dramatically from 325 to 1.39 as the bequest lower bound $\underline{a}$ decreases from 0 to -0.11. The result implies that the tighter the credit constraint, the more the steady state population at steady state. Put it differently, the model with the nonnegative credit constraint produce too much population in the long run compared with the one one with loser constraint. Based on this result, I wonder whether a policy aiming at restricting people's fertility choices, such as one child policy, is able to reduce the level of steady state population in the model with nonnegative credit constraint.

I study a demographic policy, $\bar{n}$ children policy that restricts people's fertility choices. $\bar{n}$ is the maximal fertility allowed. I study how population behaves by gradually reducing $\bar{n}$ from $\frac{1}{\lambda}$ to 1.1. Recall that $\frac{1}{\lambda}$ is the maximum number of children one could possibly have since every child needs λ percentage of time from parents and the total amount of time is normalized to 1. When $\bar{n}$ is $\frac{1}{\lambda}$, the policy does not restrict people's fertility choices. $\bar{n}$ has to be no lower than the replacement rate, one. If $\bar{n}$ were below one, the fertility rate of every type of individuals would be below one and the population diminishes over time, in which case equilibrium does not exist with a fixed amount of land.

The result is surprising to some extent. The demographic policies such as one child policy aim at reducing people's fertility, but end up with increasing steady state population. The reason is the following. Assume I impose a policy that further restricts people's fertility choices, e.g. a one child policy, on an economy initially staying at steady state with more children allowed. In the short run when prices are given the policy decreases the utility of everyone living in all generations and decreases the marginal benefit of having a child. Marginal cost of having one more child at every given fertility level is not affected when prices are given. As a result, the policy reduces fertility of every individual. People spend less time raising children and more time working, which increases an economy's labor supply and decrease the wage rage.

The reason why one child policy lowers the utility of every individual is the following. Those who wants to have more than one child is negatively affected because the policy restricts people's fertility choices. Fertility restrictions that only affect a particular group result in lower welfare for all individuals because, regardless of current ability, there is a positive probability that a descendant of the dynasty will fall into the group directly affected in finite time. So those whose fertility choices are not directly affected are also hurt by the policy.

Before the price adjusts, the one child policy shrinks the fertility choice set and lowers every individual's marginal benefit of having a child. People choose to have fewer children than before. Hence population growth become less than 1 and the economy leaves the steady state. For the economy to retrieve a steady state, prices have to adjust to induce fertility rates going up until population growth rate becomes one again. The equilibrium wage rate falls, which decreases the marginal cost of raising a child. It also decreases the marginal benefit. When the decrease of marginal cost is more than that of the marginal benefit, people are induced to have more children. Fertilities of different types of individuals go down immediately after a tighter restriction is imposed and go up later. In the new steady state, average fertility must come back to 1 and the steady state wage is lower than that in the initial steady state. With a fixed amount of land, a lower wage rate implies a higher total labor supply. Total labor supply is the multiplication of population and the weighted average of individuals' labor supply. Average fertility rate is one at steady state, which prevents the average labor supply to move a lot. Hence an increase of total labor supply translate into an increase of total population.

2.5 Conclusion

In this chapter I study the issues of population of an economy with finite resource, land. I find maximum sustainable population can be efficient even in the presence of fixed resources. The long run population is lower in incomplete market economy than the efficient level.

I also study demographic policies in a Bewley model with endogenous fertility and fixed resource. I find that policies that restrict fertility choices increase population in the long run. An empirical regularity, the inverse relationship between fertility and earning ability, is replicated in equilibrium.

This model abstracts a number of factors that potentially interact with population in the long run. For example, I abstract technological progress and the physical capital. Other related factors, such as education, interacts with transfer and affects one's fertility decision and the economy's long run population. Incorporating these factors into the model is left to the future work.

CHAPTER 3. FERTILITY, SOCIAL MOBILITY, AND LONG RUN INEQUALITY

3.1 Introduction

During the last two decades the study on inequality has significantly advanced thanks to the development a fairly unified and tractable framework of analysis known as Bewley models.[6] As explained in Aiyagari (1994), these models build upon the standard growth model of Brock and Mirman (1972) by incorporating precautionary saving motives and liquidity constraints. The connection with the standard growth model is very appealing because a single unified framework can be used to study issues of long term growth, business cycles –as in Kydland and Prescott (1982),– and issues of distribution, or inequality. Implicit in this framework is the idea of dynastic altruism: either individuals are infinitely lived or, more realistically, lives are finite but individuals care about the welfare of their descendants. Dynastic altruism is an important conceptual benchmark because it brings certain level of efficiency, if not full efficiency, to the resulting allocations.

This fairly unified framework, however, seems to fall apart when serious consideration is given to fertility decisions. In particular, Becker and Barro (1988) and Barro and Becker (1989) introduce optimal fertility choices within the optimal growth model and find that some of the most appealing conclusions obtained under the exogenous fertility assumption are seriously altered.[7] On the specific issue of inequality, the optimal fertility choice tends to eliminate any inequality and any persistence of inequality, a result highlighted by Bosi et al. (2011) in the context of a deterministic Barro-Becker model. In contrast, the version of the model with exogenous fertility predicts that any initial inequality is highly persistent, as shown by

[6] Ljungqvist and Sargent (2012, chapter 18) offers a pedagogical exposition. Some of the contributions in this literature include, among many, Loury (1981), Laitner (1992), Aiyagari (1994), Huggett (1996), Krusell and Smith (1998), Castañeda et al. (2003), and Restuccia and Urrutia (2004). See Cagetti and De Nardi (2008) for a comprehensive survey.

[7] Cordoba and Ripoll (2012) discuss some of the counterfactual predictions of the Barro-Becker model. For instance, this model predicts a negative association between individual consumption and individual income. This prediction runs counter to standard consumption theory and a variety of evidence suggesting a positive association between lifetime income and lifetime consumption.

Chatterjee (1994). An analogous result is obtained using Bewley style models. While Bewley models with infinitely lived agents, as in Aiyagari (1994), or with exogenous fertility, as in Castañeda et al. (2003), predict significant and persistent inequality, the analogous version with endogenous fertility predicts lack of persistence (Alvarez, 1999). Section 3.2 derives and discusses in more detail these results.

The key possibility introduced into the growth model when allowing endogenous fertility is that richer individuals can use family size as a way to obtain welfare, an extensive margin, instead of providing more consumption to each descendant, the intensive margin. This turns out to be the optimal solution and, as a result, there is no inequality after the original generation. Although inequality can be recovered when markets are incomplete and shocks are idiosyncratic, Alvarez finds an implausible lack of persistence result, or lack of memory, in this case: there is no persistence in economic status after controlling for innate ability. In other words, social mobility is perfect. Jones et al. (2013) find an analogous result, which they call the "resetting" property, in the context of an optimal contract with private information. We derive a version of these results in section 3.2 below.

Due to some arguably unrealistic predictions of existing altruistic models with endogenous fertility – namely lack of inequality, lack of persistence and/or a positive response of fertility to income– most of the existing literature on inequality either: *(i)* abstracts from endogenous fertility decisions; or *(ii)* departs from the assumption that parents are purely altruistic and exhibit instead certain type of warm glow altruism (e.g., De la Croix and Doepke, 2003; Sholz and Seshadri, 2009). Both approaches are convenient for multiple purposes but unsatisfactory for others. For example, by ignoring issues of fertility the recent literature on inequality is silent about the documented strong association between fertility, inequality and poverty, an association that has been used to support family planning programs around the world (e.g., Chu and Koo, 1990). Furthermore, warm glow altruism is unsatisfactory when addressing issues of policy evaluation and optimal policy design because it introduces, by assumption, inefficiencies at the household level (Kaplow and Shavell, 2001).

Another determinant of inequality is fertility. An older literature on the topic, one that mostly abstracts from savings, inter vivos transfers and bequests, shows that systematic differences in fertility rates among income groups affect the observed distribution of incomes. This literature include authors such as Lam (1986, 1997), and Chu and Koo (1990).

This chapter revisits the relationship between fertility, savings and long run inequality in economies populated by altruistic individuals. Since pure altruism is at the core of modern macroeconomics, a field that builds extensively on the dynastic model, it is natural to wonder if pure altruism is ultimate inconsistent with key stylized facts regarding social mobility, the distribution of earnings, income, wealth, as well as evidence of fertility declining with income (Jones and Tertilt, 2008).[8] We consider various ways to recover inequality and persistence as well as conditions to replicate a negative fertility-income relationship. We are able to show that, under very natural conditions, pure altruism can generate the degree of inequality and persistence as well as the negative fertility income relationship suggested by the data. To the extent of our knowledge, our model is the first altruistic model to get these predictions right.[9] Our analysis implies that altruism is ultimately consistent with empirical evidence of fertility and inequality, and it provides tools for researchers and policy makers to fully incorporate considerations of fertility and family size into the analysis of inequality.

The model we analyze features individuals who live for two periods: as a child and as an adult. While we do not model the childhood period, individuals start adulthood with a level of ability and a level of transfers they receive from their parents. We refer to these transfers as "bequests" although they reflect the present value of all the resources the individual receives from the parent at the beginning of adulthood. We also refer to these bequests as "wealth" as they represent a measure of dynastic wealth. Adults in the model consume and decide on the number of children. Raising children involves a time cost, which endogenously affect the lifetime labor supply of the adult. It also involves a "goods cost" as given by the bequests. While the

[8] Cordoba and Ripoll (2014) address other issues of altruistic models of endogenous fertility besides inequality.
[9] Alvarez (1999) consider some of these possibilities in theory. Our main contribution is quantitative.

level of earnings ability will be drawn from an exogenous distribution, labor supply will be endogenous and will constitute an important channel in explaining the intergenerational persistence of earnings. Similarly, bequests constitute the main channel of intergenerational persistence of wealth.

In addition to deriving some theoretical results regarding intergenerational persistence, we calibrate the model in order to provide a quantitative evaluation of the importance of different modeling assumptions in explaining persistence. Even though the model we analyze is stylized, our calibration follows a heuristic approach in order to better understand the mechanisms at work. We rely on the Panel Study of Income Dynamics in order to compute a number of calibration targets. We also use information provided in other sources, particularly in the Census, the Child Development Survey, and the United States Department of Agriculture to set additional targets. These sources are particularly important to determine the income elasticity of fertility, as well as the costs of raising children.

The analysis yields a number of interesting results regarding intergenerational links and inequality. First, the exogenous persistence of (earnings) ability is not enough to generate persistence of wealth in the model with endogenous fertility. Although the theoretical analysis of Alvarez (1999) suggests this lack of persistence, it is not clear whether this is the case quantitatively. Our calibration exercise confirms the quantitative lack of persistence for the standard fertility model. In particular, the standard modeling assumptions in the literature include a hyperbolic altruistic function, a constant marginal cost of raising children, and a continuous number of children. These assumptions together eliminate the endogenous persistence of inequality because the transfers or "bequests" parents give to their children are only a function of ability, and not of the transfers the parent himself received. Relaxing the assumption that the altruistic function is hyperbolic is enough to recover persistence. In particular, when we replace this function for an exponential one, the model generates significantly more persistence. The reason is that while under hyperbolic altruism the elasticity with respect to the number of children is constant, with exponential altruism this elasticity decreases with the number of children. As we discuss below, in this case wealthier parents have more children, but they also transfer more resources to each of them, increasing persistence. In addition, making the number of children discrete limits the extent to which individuals may freely use children as a margin of adjustment of their wealth portfolios.

Second, in models with endogenous fertility, the exogenous persistence of ability is not enough to generate persistence of earnings. Earnings are computed as the multiplication between ability (exogenous) and labor supply (endogenous). In this respect, the behavior of labor supply, which is directly linked to the cost of raising children, becomes essential in understanding persistence. It turns out that in models with endogenous fertility, labor supply is negatively correlated across generations, which tends to lower persistence. Specifically, low ability individuals would have more children, lower labor supply, lower earnings, and would give lower bequests to their children. In turn, these asset-poor children will have less children and higher labor supply. Breaking the extent of this negative correlation of labor supply across generations is key to recover persistence.

Third, our calibration strongly suggests that the rate at which parents substitute their own consumption with their children's consumption is larger than the rate of substitution of consumption across the individual's life cycle in standard quantitative macro models. In other words, the "elasticity of intergenerational substitution" is larger than the more commonly known elasticity of intertemporal substitution. We calibrate this parameter to match the Gini coefficient of wealth, as intergenerational substitution plays a key role in determining whether or not parents hit the zero-bequest constraint imposed in the model. It turns out that this parameter is also important in guaranteeing a negative relationship between ability (earnings) and fertility.

The remainder of the chapter is organized as follows. Section 3.2 presents the model and derives some analytical results regarding persistence, as well as the relationship between fertility, ability and bequests. Section 3.3 discusses the calibration targets and explains the data used in computing these targets. Section 3.4 presents the results for a number of variations of our benchmark model. In particular, this section highlights how different modeling assumptions on the altruistic function and the cost of raising children affect the quantitative performance of the model. Special attention is given to the comparison of the exogenous and the endogenous fertility models. A policy simulation of an increase in estate taxation in discussed in section 3.5, while section 3.6 briefly concludes the analysis.

3.2 The basic model of dynastic altruism

3.2.1 Preliminaries

The following is a version of the model studied by Alvarez (1999). Individuals live for two periods, one as a child and one as an adult. Children do not consume. Adults have earning ability ω and receive parental transfers b. We also refer to b as bequest or wealth. Lifetime resources are given by $(1+r)\,b+\omega$ where r is a risk free interest rate. Resources can be used to consume, c, or to pay for the cost of raising children. The cost of children includes a time cost, $\Lambda(n)$, and a good cost, nb'. Normalizing total parental time to one, there is maximum feasible number of children, $\overline{n}$, satisfying $\Lambda\left(\overline{n}\right)=1$. Earning abilities are random and drawn from the distribution $F(\omega|\omega_{-1})$, where ω_{-1} is the ability of the parent. Individuals know their own earning ability but not the ability of their children.

Preferences are of the form $V=U(c)+\int_0^n E\left[V_i'|\omega\right]\phi_i di$ where $U(c)$ is the utility flow derived from consumption, $E\left[V_i'|\omega\right]$ is expected lifetime utility of child i, $\phi_i\geq 0$ is the weight that the parent places on the welfare of child i, and n is the mass of children. These preferences are appealing because they describe parents as social planners at the house level. Since weights are non-negative, children are goods to parents only if $V_i'\geq 0$. This requires the restriction $U(c)\geq 0$. We focus on the CRRA case, $U(c)=\frac{c^{1-\sigma}}{1-\sigma}+A$, where $1/\sigma$ is the elasticity of intergenerational substitution (EGS), a parameter that controls the willingness to substitute consumption between parents and children. As discussed in Cordoba and Ripoll (2014), the

EGS is conceptually and quantitatively different from the elasticity of intertemporal substitution (EIS).[10] A positive constant A ensures a positive utility flow in the low curvature case, $\sigma > 1$.

We will focus on the symmetric case, $V_i' = V$.[11] In this case, $\int_0^n E\left[V_i'|\omega\right]\phi_i di = \Phi(n)E\left[V_i'|\omega\right]$ where $\Phi(n) = \int_0^n \phi_i di$ is the weight parents place on their n children. Notice that $\Phi'(n) = \phi_n > 0$. In order to keep utility bounded, it is necessary to assume that parents put more weight on themselves than on all their potential children, $1 > \Phi(\overline{n})$. Assuming further that ϕ_i decreases with i implies that $\Phi(n)$ is concave. Let $\xi(n) = \Phi'(n)\frac{n}{\Phi(n)}$ be elasticity of $\Phi(n)$ with respect to n, an elasticity that plays a central role in fertility choices.

Two functional forms for $\Phi(n)$ are explored below: hyperbolic and exponential child discounting. Hyperbolic discounting is the most common in the literature (e.g., Becker and Barro, 1988). It takes the form $\phi_i = \beta\left(1-\epsilon\right)i^{-\epsilon}$, $0 < \epsilon < 1$, which implies $\Phi(n) = \beta n^{1-\epsilon}$ and a constant elasticity $\xi(n) = 1-\epsilon$. The restriction $0 < \epsilon < 1$ is required for marginal weights to be positive and decreasing. Alvarez (1999) also considers the case $\epsilon > 1$ combined with a negative utility function so that parental utility increases with the number of children. For completeness, we consider this case below but notice that it implies negative marginal weights, $\phi_i < 0$, so that parents are not altruistic toward all their children.

Exponential child discounting takes the form $\phi_i = \beta\mu e^{-\mu i}$, $\mu > 0$, which implies $\Phi(n) = \beta\left(1-e^{-\mu n}\right)$ and a decreasing elasticity $\xi(n) = \frac{\mu n}{e^{\mu n}-1}$ which goes from 1 when $n = 0$ to 0 when $n = \infty$. This type of discounting is the natural counterpart of exponential *time* discounting but applied to individuals. It has the convenient property that $\Phi(\infty) = \beta$ so that $\beta < 1$ ensures the boundedness of parental utility for any positive fertility. This property does not hold in the hyperbolic case.

[10] To see this, one can interpret consumption as a composite good made of consumption flows at various ages:

$$c = \left(\int_0^T e^{-\rho t}c_a^{1-\eta}da\right)^{\frac{1}{1-\eta}}.$$

In this interpreation the EIS is $1/\eta$ while $EGS = 1/\sigma$.

[11] Symmetric treatment is not optimal given that weights are different. However, it may be optimal for strategic reasons as in Bernheim and Severinov (2003).

3.2.2 Recursive formulation

The following is a recursive formulation of the individual's problem:

$$V(b;\omega) = \underset{\overline{n} \geq n \geq 0,\, \overline{b} \geq b' \geq 0}{Max} \left\{ U\left((1+r)\, b + \omega - nb' - \Lambda\,(n)\,\omega \right) + \Phi(n) E\left[V(b';\omega') | \omega \right] \right\}.$$

This problem is not a standard discounted dynamic programming problem due to the endogeneity of the discount factor, $\Phi(n)$, and the non-convexity introduced by the term nb'. As a result standard properties, such as strict concavity of the value function, need to be established. Some properties of the problem are well-known for specific functional forms $U(c)$ and $\Phi(n)$ (Alvarez 1999, Qi and Kanaya 2010). We assume the problem is well-behaved and check numerically that this is in fact the case.

Let $n = N(b,\omega)$ and $b' = B(b,\omega)$ be the optimal solution rules. The optimality conditions for n and b', and the Envelope condition for b are, respectively,

$$b' + \omega\Lambda'(n) = \Phi'(n) \frac{E\left[V(b';\omega')|\omega\right]}{U'(c)}, \tag{24}$$

$$U'(c) \geq \frac{\Phi(n)}{n} E\left[V_b(b';\omega') | \omega \right], \text{ with equality if } b' \geq 0, \text{ and} \tag{25}$$

$$V_b(b;\omega) = (1+r)\, U'(c). \tag{26}$$

The conditions above assume an interior solution for fertility but allow a general solution for transfers. Corner solutions for fertility are discussed below. The left hand side of equation (24) is the marginal cost of a child, including goods and time costs, while the right hand side is the marginal benefit of the n child to a parent. Term $\frac{E[V(b';\omega')|\omega]}{U'(c)}$ is the expected welfare of the child measured in units of parental consumption, while $\Phi'(n) = \phi_n > 0$ is the marginal weight of the n child.

The optimal condition for bequests can be written, using the last two equations, as

$$U'(c) \geq \frac{\Phi(n)}{n}(1+r)\,E\left[U'\left(c'\right)\right]. \tag{27}$$

This version of the Euler Equation describes optimal intergenerational consumption smoothing. An important difference with the traditional Euler Equation is that the average degree of altruism, $\widehat{\beta}(n) \equiv \frac{\Phi(n)}{n}$, takes the place of the discount factor. As a result, family size plays a key role in determining intergenerational savings, and in particular, larger families have less incentives to save since $\widehat{\beta}'(n) < 0$.

Given the policy functions, the wealth-ability distribution can be computed recursively as:

$$p_{t+1}(b',\omega') = \frac{1}{\overline{n}_t}\sum_{\omega}\sum_{\{b:b'=b(b,\omega)\}} p_t(b,\omega)n(b,\omega)F(\omega'|\omega)$$

where $\overline{n}_t = \sum_{\omega,b} p_t(b,\omega)n(b,\omega)$ is average population growth.

Finally, define (lifetime) labor earnings and income as $e = \omega\,(1-\Lambda(n))$ and $i = \omega + rb$. The model does not offer a measure of wealth easily comparable with observed measures of wealth in the data. Variable b' are transfers from parents to children during adulthood and is a measure of dynastic wealth, excluding any life cycle component. Nonetheless, the quantitative exercise we present here will provide insights into the ability of endogenous fertility models to recover certain level of persistence of b. We now discuss two properties of the model regarding persistence and the relationship between fertility, ability and bequests.

3.2.3 Persistence

The most common functional forms of the dynastic altruism model assumes a constant marginal cost of raising children and hyperbolic child discounting. Proposition 1 states that under those assumption the optimal bequest policy is independent of b and therefore there is no endogenous persistence of inequality.

Proposition 1. Suppose $\Lambda'(n) = \lambda$ and $\xi\,(n) = \xi$. Then $b' = B(b,\omega) = B(\omega)$.

Proof. Combining (24) and (47) yields:

$$b' + \omega\Lambda'(n) \leq \xi(n)\,\frac{E\left[V(b';\omega')|\omega\right]}{E\left[V_b(b';\omega')|\omega\right]} \text{ with equality if } b' > 0. \tag{28}$$

Under the stated assumptions, condition (28) is independent of n, and therefore the condition fully describes the solution of b': b' is either 0 or the one that solves equation (28) with equality. Since (28) does not depend on b, the optimal solution takes the form $b' = B(\omega)$.

Proposition 1 states that if the marginal cost of children is constant and the parental weight is an isoelastic function of the number of children, then the optimal bequest policy is independent of b. This result was first obtained by Barro and Becker (1989) for a determinist environment, and later extended by Alvarez (1999) to an stochastic one. Our derivation is novel and more direct.[12] We call this result the lack of (endogenous) persistence property. Proposition 1 is particularly important because it remains the most popular formulation of the Barro-Becker model.

Figure 3.1 illustrates some implications of the lack of persistence property for the deterministic case. Figure 3.1.a. shows, for given ω, the policy function $b' = B(b)$ for the case of exogenous fertility. The figure assumes $n = 1$ and, for simplicity, $(1+r)\beta = 1$. In that case, $b' = b$ is the optimal policy. Thus, if the initial distribution of wealth is described by a vector $\overrightarrow{b}_0$ then financial inequality is perfectly persistent as $\overrightarrow{b}_t = \overrightarrow{b}_0$ for all t. Figure 3.1.b. shows the policy function for the case of endogenous fertility. In that case, $b' = b^*$ regardless of b. As a result, any initial inequality disappears after one generation, a point made transparent in Bosi et al. (2011). The deterministic altruistic model predicts no persistence of economic status.

Figure 3.2 illustrates analogous results for the stochastic case. Figure 3.2a shows the case of exogenous fertility with $(1+r)\beta < 1$ and $n = 1$. The figure follows Aiyagari (1994). In this case, there is inequality

[12] Our derivation uses the household problem, while Alvarez derive the result by aggregating at the dynasty level. His derivation requires to assume that all children have the same ability $\omega\prime$, while our derivation does not impose this assumption.

even in the long run and endogenous persistence of wealth: conditional on ability, richer parents provide more assets to their children except in the region where $b' = B(b, \omega) = 0$. Figure 3.2b illustrates the endogenous fertility case: conditional on ability, asset rich parents do not have asset rich children. Economic status is not persistent beyond any persistence that comes from the exogenous persistence of abilities. Whether this channel of pure exogenous persistence is enough to account for the empirical evidence on persistence is a quantitative question. We explore this possibility in the next section.

Persistence weakens when fertility is endogenous because richer parents can use family size as a way to obtain welfare, the extensive margin, instead of providing more consumption to each descendant, the intensive margin. For the functional forms originally used by Barro and Becker (1989) all (endogenous) persistence disappears.

Proposition 1 suggests that the lack of persistence is an special result obtained for specific but stylized functional forms commonly used in the literature. Equation (28) suggests two ways to recover persistence: an increasing marginal cost of raising children or a decreasing elasticity of altruism.[13] Both alternatives either make more costly or less attractive the use of the family size margin. The second alternative is more plausible since the evidence suggests that the marginal cost of raising children decreases with the number of children due to parental learning by doing. A third channel is to allow a discrete number of children which limits the extent to which parents can use the family size margin.

3.2.4 The fertility-ability-bequest relationship

Consider now the ability of the model to generate a negative relationship between fertility and earnings consistent with the empirical evidence. In the context of a deterministic model, Cordoba and Ripoll (2015) have shown that such pattern can only be obtained if the EGS is larger than one, and Cordoba and Liu (2014) find the same result in the context of an stochastic model with no savings. It turns out that $EGS > 1$ is also

[13] Alvarez (1999) discuss these possibilities.

required in the current model. To see this, it is convenient to rewrite (24) as:

$$\left(b' + \omega\Lambda'(n)\right) U'\left(c\right) = \frac{b' + \omega\Lambda'(n)}{c^{\sigma}} = \Phi'(n) E\left[V\left(b'; \omega'\right)|\omega\right].$$

Notice first that the marginal benefit of having a child, the right hand side of the expression, typically increases with the ability of the parent for two reasons: first, since abilities are intergenerational persistent, the ability of the child also increases; second, since abilities are typically mean reverting, the ability of the child does not increase as much as the ability of the parent inducing parental transfers to increase.

For fertility to decrease with parental ability, ω, the marginal cost must increase more than the marginal benefit. The marginal cost tend to increase both because the time cost increases, $\omega\Lambda'(n)$, and also because transfers are expected to increase. However, parental consumption also increases with ability which reduces the marginal utility of consumption and lowers the cost of raising children. In other words, the diminishing marginal utility of parental consumption makes children more valuable. If σ is sufficiently large this effect would dominate and fertility will increase with ability. Therefore, a negative fertility-earnings relationship requires a low σ, or high EGS. How high? For this purpose, consider the case of poor individuals, those who are constrained and do not leave any transfers. Their marginal cost of raising children is:

$$\frac{\omega\Lambda'(n)}{c^{\sigma}} = \frac{\omega^{1-\sigma}\Lambda'(n)}{\left(1 - \Lambda\left(n\right)\right)^{\sigma}}.$$

For the marginal cost to increase with ability for poor individuals the condition $\sigma < 1$ is needed.

3.3 Data and calibration

We now discuss the data and the calibration exercise. Bringing the model to the data is difficult due to its simplicity and data limitations. In particular, while bequests in the model include total intergenerational transfers, what we can measure in the data is net assets over the life cycle, or wealth. We follow an heuristic approach by assuming that key moments of the distribution of wealth, namely Lorenz curves and degree of persistence, also characterize the distribution of bequests. For our purposes of recovering persistence, this approach turns out to be sufficiently informative as many key results are robust to alternative plausible assumptions. In fact, as we discuss next section, we are able to isolate the role different features (e.g., the type of altruistic discounting, the EGS, the cost function of raising children) play in recovering persistence.

3.3.1 Data

We use a number of data sources to compute the calibration targets, as well as other moments in the data to evaluate the performance of the models.

Panel Study of Income Dynamics The Panel Study of Income Dynamics (PSID) is one of our main data sources. Using data from 1968 to 2011, we are able to obtain and link detailed life cycle observations for two generations of parents and their children who have already grown into adults. As it is well known, this is the only available longitudinal data set in which this can be achieved. We use the PSID to compute the persistence, Gini coefficients, and coefficients of variation of wage earnings, income and wealth, as well as the correlation among these variables.

Although, as discussed in the literature, one of the disadvantages of the PSID is that it does not represent well the very rich, it is the best data set for our purpose for three reasons. First, it is the only data set that allows for linking parents and children, as discussed above. Second, because in our model adults live for only one period, measuring earnings, income and wealth requires that we capture the whole lifetime, not just one observation of a specific year. Alternative data sets such as the Survey of Consumer Finances (SCF) provide a better sampling of the very rich, but its cross-sectional nature would not allow us to measure lifetime statistics for individuals. Last, since our purpose is to compare the extent to which different versions of our model can recover intergenerational persistence, we can still provide a ranking of how these models compare for a given set of targets. Intergenerational persistence can only be computed using PSID data, so even if our Gini coefficients were different from those measured using the SCF, our exercise is still informative of our main purpose.

We follow the methodology in Lee and Solon (2009) in order to exploit all available observations for parents and children over the lifecycle. As in Lee and Solon (2009) we: *(i)* exclude any children born before 1952 to avoid over-representing children who left home at a late age; *(ii)* use income observations no earlier than age 25 to more meaningfully capture long-run income; *(iii)* measure children's adult income in the household in which they have become head of head's spouse; *(iv)* use only the Survey Research Center component of PSID and exclude the sample of the Survey of Economic Opportunities, or "poverty sample" due to representation concerns; and *(v)* exclude income observations imputed by major assignments. The result is an unbalanced panel that uses all available years for each individual.

We use the same econometric specification as in Lee and Solon (2009), except that we update the observations until 2011, while their final year was 2000. Their estimation equation is given by:

$$
\begin{aligned}
y_{ict} = \alpha' D_t + \rho_t X_{ic} + \gamma_1 A_{ic} + \gamma_2 A_{ic}^2 + \gamma_3 A_{ic}^3 + \gamma_4 A_{ic}^4 + \delta_1(t-c-40) + \delta_2(t-c-40)^2 \\
+ \delta_3(t-c-40)^3 + \delta_4(t-c-40)^4 + \phi_1 X_{ic}(t-c-40) + \phi_2 X_{ic}(t-c-40)^2 \\
+ \phi_3 X_{ic}(t-c-40)^3 + \phi_4 X_{ic}(t-c-40)^4 + \varepsilon_{ict}
\end{aligned}
$$

where y_{ict} is the log of family income for individual i in cohort c and time t; D_t is a vector or year dummies; X_{ic} is parental log income measured as the average of log family income over the three years the child was 15 to 17 years old; ρ_t is a time-varying intergenerational elasticity; A_{ic} is the parental age at the time in which parental income is observed; and $(t-c-40)$ is the child's age at the time in which the child's income is observed. The latter implies a normalization such that ρ is the intergenerational income elasticity at age 40.

For calibration purposes, we only need a single ρ value, so we eliminate the time variation of this coefficient. However, controlling for age for both parents and children, as well as for time effects, allows our estimation of intergenerational persistence ρ to include the whole lifetime profile of income of each child who has grown to form his / her own household.

We use the methodology above to compute the intergenerational persistence of both income and wage earnings. Income is measured as PSID variable "total family money", which includes wage earnings of all family members, as well as any other money received by all members of the household. Measuring wage earnings using the PSID is more complicated, as there is not a single variable including wage earning for all family members. Earnings are constructed summing the labor earnings of the head and head's wife, taking into account that after 1994 labor earnings coming from own businesses are reported separately from those coming from employment.

We estimate the specification above separately for sons and daughters, and also for all children including a dummy for daughters. For the case of income, we obtain $\rho_{income} = 0.532$ (standard deviation of 0.013) for the later regression, and the equivalent for labor earnings is $\rho_{earnings} = 0.267$ (s.d. of 0.010).

The PSID provides data on family wealth starting only in 1984. As discussed above, we compute statistics of wealth in the data in order to compare them with their counterparts of bequests in the model. Although the methodology of Lee and Solon (2009) described above could in principle be used to compute the persistence of wealth, we instead follow the methodology in Mulligan (1997) for two reasons. First, if parental wealth is measured when the child is between ages 15 and 17, the oldest cohort that could be included is the one from 1969. This means that even for the oldest possible cohort, wealth data after age 25 would only be available until these individuals turned 42 in 2011, relatively earlier in their life cycle. Regressions following Lee and Solon (2009) would then be heavily biased towards the early part of individual's life cycle, partially defeating the purpose of exploiting the whole life cycle information of parents and children. Second, in contrast with income and earnings, wealth is a stock, so the methodology used in Mulligan (1997) should be good enough to estimate intergenerational persistence of wealth. He measures the average wealth over a five-year period for the parent (head of household) and the child, as well as their average age during that interval. He then regresses the log of the average wealth of the child onto the log of the average wealth of the parent and second-degree polynomials on the average ages of the parent and the child. Given the information available in the PSID we used this methodology for year intervals 1984-1989, 1994-1999 and 2004-2009. Regressions were run separately for sons, daughters, and for all with a dummy for daughter. The intergenerational elasticity of wealth varies slightly across specifications and interval years. For calibration purposes we used the regression including all children for the latest year, which estimates an intergenerational elasticity of wealth $\rho_{wealth} = 0.40$ (s.d. of 0.017).

In addition to intergenerational persistence, we use PSID data to compute Gini coefficients of earnings, wealth and income. In order to exploit the panel structure of the data, we control for time and age effects before computing Gini coefficients. In particular, they are computed over the residuals of the following regression:

$$y_{ict} = \alpha' D_t + \delta_1(t-c-40) + \delta_2(t-c-40)^2 + \delta_3(t-c-40)^3 + \delta_4(t-c-40)^4 + \varepsilon_{ict}$$

where y_{ict} is the income, earnings or wealth of individual i in cohort c and time t. Although we computed year-specific Gini coefficients for each variable, in our calibration we only use the Gini computed over the whole sample of years. We obtained a Gini for income and for earnings of around 0.4, and a Gini for wealth of 0.76.

We also computed the income - earnings correlation over the residuals of the regression above for each of these two variables. We obtained a correlation of about 0.88. The coefficients of variation for income, earnings, and wealth are given by 1.08, 1.19, and 4.02 respectively. Last, the average wealth to average income ratio is 4.02, and the income - wealth correlation is estimated to be 0.338.

Child Development Survey and USDA Time costs or raising children are central for the endogenous fertility models. Given that adults in our model live for one period, the relevant cost is the time costs of raising a child as a fraction of lifetime parental income. Using the 1997 Child Development Supplement of the Panel Study of Income Dynamics, Folbre (2008) estimates the time costs of raising a child by incorporating both primary and secondary time parents spend with children. She concludes that the average amount of both "active" and "passive" parental-care hours per child (not including sleep) is 41.3 per week for a two-parent household with two children ages 0 to 11. Passive care corresponds to the time the child is awake but not engaged in activity with an adult, while active parental care measures the time the child is engaged in activity with at least a parent. In addition to reporting hours spent in child care, Folbre (2008) discusses two alternative ways of computing the monetary value of these hours: one uses a child-care worker's wage and the other the median wage. Folbre (2008) combines this information with the estimates of the goods costs of raising children by the United States Department of Agriculture (USDA, 2012). The latter include direct parental expenses made on children through age 17 such as housing, food, transportation, health care, clothing, child care, and private expenses in education. Folbre concludes that when child-care worker's wages are used to value the hours spend in raising children, then the time cost of raising children is on average around 60% of the total costs (see Table 7.3, p. 135). In addition, since the median wage is around the double of a child-care worker's wage, then using the former time valuation the time cost of raising children increases to 75% of the total costs.

The USDA (2012) computes the present value of the goods costs of raising children ages 0 to 17 for families with low, medium and high income. Using these estimates together with Folbre's scenarios, we can compute the time costs of raising a child as a fraction of lifetime parental income for the average family in each of these income brackets. Since most families in the United States are in the low and middle income brackets, we use the average of these two to compute our target. Specifically, the average family in the USDA (2012) income bracket has an annual income of $43,625 in 2011, which corresponds to a lifetime income $1,217,250.[14] Using the most conservative estimate in Folbre (2008) the present value of the time costs of raising a child for this low-income family is $214,576, about 17.6% of lifetime household income. In the case of the middle-income bracket, the average annual household income is $81,140, lifetime income is $2,264,016, and the time cost of raising a child $297,656, or 13.1% or lifetime income.

Last, Folbre (2008, Table 6.4) suggests strong increasing returns in raising children. Their figures implies that the relative cost of two and three children relative to one child is 1.33 and 1.70, instead of 2 and 3 if there are no increasing returns. This information is used to calibrate the cost function with decreasing costs of raising children.

Census The last set of calibration targets in our model include average fertility and the elasticity of fertility with respect to lifetime income. Although the Childbirth and Adoption History module of the PSID includes a measure of total children born that can be used to approximate completed fertility when measured around age 45, this variable is only available starting in 1985. Unfortunately once this information is merged with the income and wealth panel observations, the sample of individuals for which completed fertility is known is too small, under 3,000 observations, to be a representative sample.

Rather than using the PSID to compute average fertility and the income elasticity of fertility, we rely on estimates already available from Jones and Tertilt (2008). They use US Census data as far back as the 1826 cohort to estimate an income elasticity of fertility of about -0.38. Their analysis is distinct in that they construct a more refined measure of lifetime income by using occupational income and education. Lifetime income and fertility are measured for several cross-sections of five-year birth cohorts from 1826-1830 to 1956-1960. They conclude that most of the observed fertility decline in the US can be explained by the negative fertility-income relationship estimated for each cross-section, together with the outward shift of the income distribution over time. The estimated income elasticity is robust to the inclusion of additional controls such as child mortality and the education of husband and wife, suggesting a strong

[14] This computation uses an interest rate of 2% per year and assumes a 40-year working lifespan.

negative correlation between income and fertility. For the latest cohorts in their data, 1956-1960, the income elasticity of fertility is estimated to be -0.22, and the average fertility is 1.8 children per household. Finally, the coefficient of variation of fertility for the latest cohorts is about 0.6.

3.3.2 Calibration

We now explore the quantitative predictions of various calibrated versions of the dynastic altruistic model. For this purpose we run a horse race between six different versions of the model that differ in: *(i)* whether fertility is endogenous or exogenous; *(ii)* the type of child discounting assumed, either hyperbolic or exponential; *(iii)* the curvature of the utility function, either $EGS > 1$ or $EGS < 1$; and *(iv)* the marginal cost of raising children, either constant or decreasing. We document weaknesses and strengths of each model, and conclude that overall a model with exponential child discounting, $EGS > 1$, and diminishing cost of raising children is the most promising.

3.3.3 Calibration targets

Our calibration strategy is analogous to the one used by Castañeda et al. (2003). Key parameters are chosen to match specific aspects of Lorenz curves for earnings and wealth. The performance of each model is then assessed along various dimensions, in particular regarding their ability to generate realistic persistence as well as to match other features of the Lorenz curves beyond the matching targets.

When comparing to the existent literature, it is important to keep in mind three aspects of the problem that make the calibration non-standard. First, the earning process is not annual but life time. Second, the curvature of the utility function does not reflect the intertemporal elasticity of substitution. In fact, typical calibrations set $\sigma > 1$ but, as discussed in the previous section, a negative fertility-earnings relationship requires $\sigma < 1$. Third, discount factors are family specific and depend on fertility rates. The following are the models considered:

1. Model 1. Exogenous fertility with $EGS = 2/3 < 1$.

2. Model 2. Exogenous fertility with calibrated EGS.

3. Model 3. Alvarez's (1999) model with continuous number of children.

4. Model 4. Alvarez's (1999) model with discrete number of children.

5. Model 5. Exponential child discounting with constant costs of raising children.

6. Model 6. Exponential child discounting with decreasing costs of raising children.

Computing the models require the Markov process $F(\omega|\omega_{-1})$, the function $\Lambda(n)$, and parameters r, σ, β, ϵ and μ. Models 1 to 4 assume hyperbolic child discounting while Models 5 and 6 consider exponential child discounting. A common interest rate, $r = 2$, is assumed for all models.[15] For the labor endowments shocks we approximate the first-order autoregressive process,

$$\ln \omega' = \rho \ln \omega + e, \; e \sim N(0, \sigma_\omega^2),$$

by a 15 states Markov chain using the Tauchen Method. The coefficient ρ is the intergenerational persistence of abilities. A natural calibration procedure would be to calibrate ρ for each model so to match a target for the persistence of earnings, as in Restuccia and Urrutia (2004). However, as we show below, generating enough persistence of earnings when fertility is endogenous is a challenge. Instead, we assume ρ is common across models and report results for $\rho = 0.25$ and discuss, without reporting, results for $\rho = 0.5$. The lower value is consistent with the findings of Urrutia and Restuccia (2004) while the upper value is the intergenerational persistence of (log) hourly wages reported by Mulligan (1997).

The calibration of the remaining parameters is specific to each model. The variance σ_ω^2 is calibrated to match the Gini coefficient of labor earnings. Models 1 and 2, the exogenous fertility models, require only

[15] A net return of 2 is obtained if annual returns are 4.5% for 25 years, or 3.73% for 30 years. 25 or 30 years could be considered the midpoint of adult life.

the extra parameters β and σ since $n = 1$ is assumed. β is identified by targeting the earnings-income correlation. The justification for this target is that β determines the amount of savings, and therefore the earnings-income correlation. For example, the correlation is 100% when there are no savings at all or close to zero if savings are infinite. Model 1 sets $\sigma = 1.5$ as in Castañeda et al. (2003), and Restuccia and Urrutia (2004), a standard value in the literature. Model 2 calibrates σ to match the Gini coefficient of wealth. This is because σ controls the degree of precautionary savings and therefore affects the concentration of wealth. The identification of β and σ in all models, except Model 1, is simultaneous because both parameters affect savings, and therefore the correlation of earnings with income and the concentration of wealth.

Models 3 to 6 require to specify a technology for raising children. We use the function $\Lambda(n) = \lambda\left[(n+\kappa)^{\theta} - \kappa^{\theta}\right]$, $0 < \theta \leq 1$. Notice that $\Lambda(n) = 0$ and $\Lambda'(n) = \theta\lambda(n+\kappa)^{\theta-1}$. A constant marginal cost is obtained when $\theta = 1$, and decreasing when $\theta < 1$. Parameter κ allows to bound the marginal cost of the first (dn) children. We assume a constant marginal cost, λ, for models 3 to 5. To calibrate λ, existent evidence on the time costs of raising children can be used. However, as discussed in Cordoba and Ripoll (2014), the evidence suggests a wide range of possible values for λ. We decided to calibrate λ within each model and then discuss whether the estimates are plausible or not. The target used to identify λ is the average fertility. λ, turns out, have a strong effect on savings too, because it affects the demand for children, and therefore the calibration of (β, σ, λ) is simultaneous. Model 6 assumes a relative small degree of increasing returns to child rearing of around 15%. It assumes that the relative cost of raising two and three children relative to one is 1.84 and 2.59 respectively. The degree of increasing returns suggested by Folbre are much higher but, in order to match an average fertility of 1, they would require an unrealistically high cost for the first child. To avoid this issue, we opted for more moderated degree of increasing returns.

Finally, the curvature parameters of the altruistic functions $\Phi(n)$, ϵ and μ, are calibrated by targeting an income elasticity of fertility of -0.20. Parameters were chosen to minimize the sum of square errors of the model predictions relative to the targets.

3.4 Results

Table 3.1 reports the targets and their model counterparts, Table 3.2 reports the calibrated parameters, and Tables 3.3, 3.4 and 3.5 reports additional moments besides the matching targets. A general issue, made apparent in Table 3.1, is the difficulty to match the targets with precision for various models. The reason is that the models in some cases are too stylized and/or the functional forms are not flexible enough to match all targets. In particular, endogenous fertility models tend to produce too much concentration and too low correlation of earnings and income. A tension in calibrating parameters arises because reducing concentration requires more incentives to save but higher savings would further reduce the earnings-income correlation. The criteria of choosing parameters to minimize a set of equally weighted moments allow us to identify a set of parameters. We tried various other weights and the results discussed below are generally robust to various weights.

Exogenous Fertility Models

Model 1 is our version of a traditional Bewley with a standard curvature for the utility function, $\sigma = 1.5$, or $EGS = 0.66$. This value, used by Castañeda et. al. (2003), is typically estimated using quarterly or annual data. The model assumes $n = 1$, it is analogous to a infinitely lived individual model and serves as baseline for comparison. Model 1 is able to generate significant higher concentration of wealth, $Gini(b) = 0.65$, than of earnings, $Gini(e) = 0.4$, but significantly less than in the data, a finding that is consistent with other results in the literature, e.g. Aiyagari (1994). The model also generates significant persistence, similar to the data, for all variables.[16,17] Significant persistence of earnings is important for the model to generate large savings and wealth concentration.

Model 2 delivers, by construction, more wealth concentration than Model 1. This is achieved by, one the one hand, reducing the need to save for precautionary motives, by reducing σ, which leads more indi-

[16] Persistence is calculated as the coefficient in a regression between the log of the outcome of the children againts the log of the outcome of the parent.

[17] The persistence of wealth would still be high, of 0.47, even if $\rho_\omega = 0$. The focus is to show how persistence is loss and then recovered when fertility is endogenous. Higher persistence of abilities, of 0.5, would result in too much persistence of all variables.

viduals to hit the zero bequest constraint, but on the other hand increasing savings for non-precautionary motives, by increasing β, so that the correlation between earnings and incomes is preserved. The calibrated $EGS = 1/\sigma$ in Model 2 is 2.17 which describes parents as much more willing to substitute consumption intergenerationally than what is traditionally assumed. Notice that the calibrated values of the EGS in all models, except Model 1, are significantly larger than the typical value of 0.66, and likely around 1.4.[18] Moreover, the estimated value of β in Model 2 is significantly higher than what would be required by a more traditional calibration. Model 2 performs similarly to Model 1 but generates slightly higher degree of social mobility that is still consistent with the data.

Alvarez's Endogenous Fertility Model

Model 3 is Alvarez's (1999) model, an intergenerational Bewley-Barro-Becker model.[19] Matching all targets is particularly difficult when fertility is endogenous. The high calibrated time cost of a child, $\lambda = 0.5$, is concerning but required in order to avoid even more concentration of wealth, which is already high, as more individuals, particularly low ability ones, would increase their fertility if the cost is lowered at the expense of reducing bequests per child. Figure 3.3 shows the policy functions and the predicted relationship between average fertility and both bequests and abilities. The model predicts a strong wealth effect, as the elasticity of fertility to bequest is 0.21.

The most striking difference between the exogenous fertility models and Alvarez's endogenous fertility model is in the degree of persistence of all variables, but particularly of bequests and earnings, which now exhibit practically zero persistence. The lack of persistence of earnings, in spite of the 25% persistence of abilities, is surprising and novel. Figure 3.4 illustrates the mechanism at work. The problem is the endogenous determination of labor supply, which lacks intergenerational persistence and exhibits significant dispersion so that it dominates the persistence of earnings. The mechanism explaining the lack of persistence

[18] Expected utility models do not distinguish between risk aversion and aversion to deterministic fluctuations. Our interpreation of a low σ relative to the typical σ means that parents are less risk averse to gambles on their children's earnings than to gambles on their own earnings. Cordoba and Ripoll (2014) calibrate the EGS and the EIS in a model with no risk and also obtain $EGS = 1/\sigma > 1$.

[19] A continuous number of children is approximated by setting the change in the number of children to 1/30.

in labor supply is one in which a high ability individual would have few children, due their high time costs, high labor supply and high labor earnings. The parent endows each child with relatively high bequests. The child, on the other hand, is expected to have significantly more children and lower labor supply than the parent for two reasons: her ability is expected to be lower than her parent's, because abilities are mean reverting, and her initial wealth is higher.

Model 4 restricts the number of children to be discrete, which prevents parents from fully utilizing children as a saving device. The calibrated parameters are similar to the continuous case but the model is able to reduce the cost of raising children. Otherwise, Model 4 performs slightly better than Model 3 but the fundamental issue of lack of persistence remains.

The remaining models also assume a discrete number of children. In this case, n is restricted to lie in the set $\{0, 0.5, 1.5, .., \overline{n}\}$ so that 0.5 child per parent mean 1 child per couple, etc.

Extended Endogenous Fertility Models

Models 5 uses exponential child discounting and maintains the assumption of a constant marginal cost of children. This model is relevant because it uses the channel that Alvarez regarded as the most promising to recover persistence. The model in fact predicts significant amount of persistence for all variables but not enough for earnings. It also reduces significantly a strong wealth present in the previous models as the elasticity of fertility to wealth drops from around 0.2 to 0.1. The mechanism to recover persistence is a strong diminishing marginal benefit of having children due to the assumed exponential child discounting. Two key issues remain: a high time cost of raising children and the low persistence of earnings.

Finally, Model 6 assumes exponential child discounting and increasing returns to raising children. Figure 3.5 shows the policy functions and the predicted relationship between average fertility and both bequests and abilities. Jumps in the policy functions occur at the levels where fertility changes. Similar to the other models, targets cannot be exactly matched, with the main issue being the low predicted value for fertility-income correlations. Model 6 outperforms other endogenous and exogenous fertility models in various key aspects. First, the calibrated time cost function for children implies that on average are less costly. For example, average labor supply is 0.62 rather than 0.5 in Alvarez's model. The calibrated time costs are still high compared with the estimates of time cost reported in the previous section. Second, it predicts similar levels of persistence of bequest as exogenous fertility models, or around 58%, and significantly increases the persistence of earnings. Third, it predicts a relative weak elasticity of fertility to wealth.

Why increasing returns help increase persistence of bequest, if Alvarez expected the opposite, and earnings? The explanation is the following. Model 6 has a much stronger degree of child discounting which itself would raise persistence too much. Increasing returns help reducing persistence from that high level. The gain of increasing returns is that it reduces the dispersion of labor supply because, although additional children are less expensive, parents don't want them. Lower dispersion of labor supply makes the persistence of earnings replicate more closely the persistence of abilities, as illustrated in Figure 3.6. The problem with Alvarez model regarding the lack of persistence of earnings is really the tremendous variation in labor supply due to the linear and high cost needed. The persistence of bequest increases, from Model 5 to Model 6, for the same reason: a combination of a stronger child discounting partly compensated by increasing returns, which overall make the use of the extensive margin less attractive.

3.5 Policy experiments

To illustrate the importance of taking fertility decisions into account we conduct two policy experiments: an increase in estate taxes and a family planning policy limit fertility in the spirit of the one child policy. For this purpose, we compute the steady implications of these policies according to Model 2, our preferred exogenous fertility model, and Model 6, our preferred model of endogenous fertility.

3.5.1 Estate taxes

Consider first the long term effect of introducing a 20% estate tax used to finance an exogenous stream of government expenditures. We don't consider changes in the interest rate so that the results correspond either to partial equilibrium or to a small open economy. Results are reported in Table 3.6. Consider first the effects of the policy in the economy with exogenous fertility. The policy significantly reduces steady state bequests, incomes and consumption, but do not affect earnings.[20] The policy reduces inequality, as measured by standard deviations, but increases according to Gini coefficients and coefficient of variations. Both results are possible because, as intergenerational savings fall, more individuals become constrained which reduces standard deviations but also increase the concentration of bequests.[21] For an utilitarian social planner, means and standard deviations are the two most important moments determining social welfare. We also find significant reductions in the levels of persistence meaning that higher estate taxes increase social mobility. The intuition for this result seems clear from the Euler Equation as estate taxes increase the cost of smoothing consumption. The results on means are consistent with the ones reported by Castañeda et. al. (2003) but they do not find significant changes in Gini coefficients or persistence.

The predictions of the endogenous fertility model differ significantly mostly regarding quantitative prediction and some qualitative predictions. This is because an increase in estate taxes reduces the welfare of children and therefore the incentives to have children. As a result, fertility rates fall around 3%, and labor supply as well as labor earnings increase, instead of being constant. The standard deviation of fertility falls significantly, 42%, mostly due to falling fertility of the very asset rich who disappear from the ergodic set. Average bequest, income and consumption as well as the standard deviation of bequest fall but significantly less, up to 42% less, than in the exogenous fertility case. The results show that endogenizing fertility significantly affect the predicted quantitative effects of policies, mostly dampening the effects. On qualitative

[20] Earnings will fall in a closed economy due to the fall in capital stock.

[21] If distributions were log-normal, for example, standard deviations and Gini's would move together.

differences, while the exogenous fertility model predicts rising inequality of income and consumption, as measured by Gini coefficients, the endogenous fertility model predict lower inequality.

3.5.2 Family Planning Policies

Consider next a family planning policy seeking to reduce fertility rates such as the one child policy. It turns out that such a policy, of 0.5 children per-parent, leads to perpetual growth and eliminates the stationary of the model. This is because, as is well-known, Bewley models with exogenous fertility require the restriction $\frac{\Phi(n)}{n}(1+r) < 1$ in order to have an ergodic set. Otherwise, the incentives to save are too strong and consumption eventually diverges to infinite. This is in fact the case for the calibrated parameters as one obtains $\frac{\Phi(n)}{n}(1+r) \geq \frac{\Phi(0.5)}{0.5}(1+r) > 1$. For this purpose, we consider a less stringent family planning policy.

Table 3.7 shows the results of a two children policy using Model 6. The effect of the policy is a 5% drop in average fertility and, perhaps more significantly, a 52% drop in the standard deviation of fertility explained by the reduction of fertility of very asset rich individuals. Qualitative, the effects of a family planning policy is similar to a reduction in estate taxes because the policy increases the incentives to save as parents average degree of altruism rises. The policy increases mean earnings, income, bequest and consumption, increase inequality as measured by standard deviations but reduce inequality when measured by Ginis or coefficient of variations. Social mobility, particularly of bequests, significantly decreases.

3.6 Concluding comments

We have shown that models of endogenous fertility by dynastic altruistic parents can replicate similar persistence as analogous models with exogenous fertility. Introducing endogenous fertility considerations is important for policy evaluation because the decision to have or not children, and how many, affect most long term economic variables like consumption, savings, income and labor supply. We show, for example, that the long run effects of estate taxes are substantially different when fertility is endogenous.

We recover realistic levels of persistence by combining three novel elements into an otherwise standard Bewley model. An intergenerational elasticity of substitution larger than one, as opposed to the typical intergenerational elasticity of substitution less than one; exponential child discounting instead of hyperbolic discounting; and increasing returns in child rearing.

Table 3.1: Targets (ρ_ω=0.25)

		Exog. Fertility		Alvarez		CLR	
Target	Data	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Interest rate	2.00	2.00	2.00	2.00	2.00	2.00	2.00
Gini bequest	0.76	0.65	0.72	0.80	0.86	0.83	0.81
Gini earnings	0.40	0.40	0.41	0.45	0.44	0.43	0.40
Persistence of abilities	0.25	0.25	0.25	0.25	0.25	0.25	0.25
correlation earnings-income	0.88	0.88	0.89	0.87	0.86	0.88	0.83
average fertility	1.00	------	------	1.01	1.06	1.07	1.03
Income elasticity of fertility	-0.20	------	------	-0.25	-0.22	-0.18	-0.14
cost two children/cost one child	1.33	------	------	------	------	------	1.85
cost three children/cost one child	1.70	------	------	------	------	------	2.59

Model 1: Exogenous fertility (EGS=2/3); Model 2: Exogenous fertility (EGS calibrated); Model 3: Alvarez continuous; Model 4: Alvarez discrete; Model 5: CLR exponential with constant cost; Model 6: CLR exponential with decreasing cost.

Table 3.2: Parameter Values (ρ_ω=0.25)

Parameter	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
r = Interest rate	2.00	2.00	2.00	2.00	2.00	2.00
σ = curvature utility function	1.50	0.46	0.79	0.82	0.68	0.73
β = discount factor	0.23	0.30	0.24	0.24	0.43	0.32
ε=curvature hyperbolic discounting	---	---	0.44	0.57	---	---
μ = curvature exponential discounting	---	---	---	---	1.01	1.93
ρ_ω = persistence log ability	0.25	0.25	0.25	0.25	0.25	0.25
σ_ω = dispersion log ability	0.72	0.73	0.59	0.61	0.67	0.66
λ = parameter cost of children	---	---	0.50	0.45	0.41	3.05
κ = parameter cost of children	---	---	---	---	---	1.84
θ = elasticity cost of children	---	---	---	---	---	0.23
$\Lambda'(1)$ = marginal time cost of one child	---	---	0.50	0.45	0.41	0.31
$\Phi(1)$ = altruism toward first child	0.23	0.30	0.24	0.24	0.27	0.28
EGS	0.67	2.17	1.26	1.22	1.47	1.37

Model 1: Exogenous fertility (EGS=2/3); Model 2: Exogenous fertility (EGS calibrated); Model 3: Alvarez continuous; Model 4: Alvarez discrete; Model 5: CLR exponential with constant cost; Model 6: CLR exponential with decreasing cost.

Table 3.3: Means and Standard Deviations (ρ_ω=0.25)

	Data	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
mean abilities		1.32	1.33	1.21	1.22	1.27	1.27
mean labor supply		1.00	1.00	0.50	0.53	0.56	0.62
mean labor earnings		1.32	1.33	0.64	0.68	0.75	0.80
mean income		1.81	1.70	0.78	0.81	0.90	1.01
mean bequest		0.24	0.18	0.09	0.10	0.10	0.13
mean consumption		1.81	1.70	0.78	0.81	0.89	1.01
mean fertility	1.0	1.00	1.00	1.01	1.06	1.07	1.03
stdev abilities		1.12	1.15	0.81	0.85	0.99	0.97
stdev labor supply		0.00	0.00	0.18	0.15	0.11	0.06
stdev earnings		1.12	1.15	0.59	0.65	0.73	0.71
stdev income		1.38	1.38	0.63	0.71	0.84	0.90
stdev bequest		0.33	0.31	0.18	0.24	0.25	0.30
stdev consumption		1.27	1.28	0.57	0.66	0.78	0.86
stdev fertility	0.6	0.00	0.00	0.36	0.36	0.24	0.19

Model 1: Exogenous fertility (EGS=2/3); Model 2: Exogenous fertility (EGS calibrated); Model 3: Alvarez continuous; Model 4: Alvarez discrete; Model 5: CLR exponential with constant cost; Model 6: CLR exponential with decreasing cost.

Table 3.4: Coefficients of Variation (ρ_{ω}=0.25)

	Data	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Coefficient variation abilities		0.85	0.86	0.67	0.70	0.78	0.76
Coefficient variation labor supply		0.00	0.00	0.36	0.28	0.20	0.10
Coefficient variation earnings	1.19	0.85	0.86	0.92	0.96	0.97	0.89
Coefficient variation income	1.08	0.76	0.81	0.81	0.88	0.93	0.89
Coefficient variation bequest	4.02	1.38	1.72	2.00	2.40	2.50	2.31
Coefficient variation consumption		0.70	0.75	0.73	0.81	0.88	0.85
Coefficient variation fertility	0.6	0.00	0.00	0.36	0.34	0.22	0.18
Gini abilities		0.40	0.41	0.34	0.34	0.38	0.37
Gini labor supply		0.00	0.00	0.18	0.12	0.07	0.03
Gini earnings	0.40	0.40	0.41	0.45	0.44	0.43	0.40
Gini income	0.40	0.38	0.40	0.39	0.41	0.42	0.42
Gini bequest	0.76	0.63	0.72	0.80	0.86	0.83	0.81
Gini consumption		0.36	0.38	0.37	0.39	0.41	0.40
Gini fertility		---	---	0.18	0.14	0.10	0.13

Model 1: Exogenous fertility (EGS=2/3); Model 2: Exogenous fertility (EGS calibrated); Model 3: Alvarez continuous; Model 4: Alvarez discrete; Model 5: CLR exponential with constant cost; Model 6: CLR exponential with decreasing cost.

Table 3.5: Persistence and Other Statistics (ρ_ω=0.25)

	Data	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
Persistence abilities	0.25	0.25	0.25	0.25	0.25	0.25	0.25
Persistence labor supply		1.00	1.00	0.00	0.10	0.03	0.01
Persistence earnings	0.27	0.25	0.25	-0.05	0.03	0.12	0.20
Persistence income	0.53	0.60	0.55	0.36	0.40	0.45	0.54
Persistence bequest	0.40	0.65	0.59	0.14	0.33	0.38	0.58
Persistence consumption	0.70	0.67	0.61	0.43	0.44	0.49	0.59
Persistence fertility		---	---	-0.12	-0.04	-0.02	-0.01
Correlation earnings-income (e,i)	0.88	0.88	0.89	0.87	0.86	0.88	0.83
Correlation earnings-bequest (e,b)		0.74	0.74	0.89	0.90	0.89	0.81
Correlation income-bequest (i,b)	0.34	0.96	0.95	0.95	0.92	0.94	0.93
labor earnings/income		0.73	0.78	0.82	0.84	0.83	0.79
Fertility elasticity of b'/b (top 2%)	-1.00	---	---	-2.11	-2.11	-1.97	-1.85
Fertility elasticity of b'/b (top 10%)		---	---	-2.40	-2.48	-2.97	-3.47
Income elasticity of fertility	-0.20	---	---	-0.25	-0.22	-0.18	-0.14
wealth elasticity of fertility		---	---	0.21	0.18	0.10	0.05

Model 1: Exogenous fertility (EGS=2/3); Model 2: Exogenous fertility (EGS calibrated); Model 3: Alvarez continuous; Model 4: Alvarez discrete; Model 5: CLR exponential with constant cost; Model 6: CLR exponential with decreasing cost.

Table 3.6: Estate Taxation (+20% tax)

	Exogenous Fertility (M2)			Endogenous Fertility (M6)			
	Benchmark	After Tax	% change	Benchmark	After Tax	% change	diff
Mean earnings	1.33	1.33	0.0%	0.80	0.81	1.3%	1.3%
Mean income	1.70	1.38	-18.8%	1.01	0.87	-13.9%	5.0%
Mean bequest	0.18	0.03	-83.3%	0.13	0.05	-61.5%	21.8%
Mean consumption	1.70	1.37	-19.4%	1.01	0.86	-14.9%	4.6%
Mean fertility	1.00	1.00	0.0%	1.03	1.00	-2.9%	-2.9%
Std earnings	1.15	1.15	0.0%	0.71	0.70	-1.4%	-1.4%
Std income	1.38	1.18	-14.5%	0.90	0.76	-15.6%	-1.1%
Std bequest	0.31	0.10	-67.7%	0.30	0.20	-33.3%	34.4%
Std consumption	1.28	1.12	-12.5%	0.86	0.71	-17.4%	-4.9%
Std fertility	0.00	0.00	0.0%	0.19	0.11	-42.1%	-42.1%
Persistence of earnings	0.25	0.25	1.0%	0.20	0.23	11.9%	10.9%
Persistence income	0.55	0.32	-42.4%	0.54	0.37	-31.7%	10.7%
Persistence bequest	0.59	0.07	-88.6%	0.58	0.31	-46.6%	42.0%
Persistence consumption	0.61	0.34	-44.2%	0.59	0.40	-32.4%	11.7%
Persistence fertility	---	---	---	0.00	-0.04	---	
Gini earnings	0.41	0.41	0.0%	0.40	0.39	-2.5%	-2.5%
Gini income	0.40	0.41	2.5%	0.42	0.40	-4.8%	-7.3%
Gini bequest	0.72	0.92	27.8%	0.81	0.93	14.8%	-13.0%
Gini consumption	0.38	0.40	5.3%	0.40	0.39	-2.5%	-7.8%
Gini fertility	0.00	0.00	0.0%	0.06	0.02	-66.7%	-66.7%
Coefficient variation earnings	0.86	0.86	0.0%	0.88	0.86	-2.3%	-2.3%
Coefficient variation income	0.81	0.86	6.2%	0.89	0.88	-1.1%	-7.3%
Coefficient variation bequest	1.69	3.76	122.5%	2.39	4.12	72.4%	-50.1%
Correlation earnings-bequest (e,b)	0.74	0.83	11.9%	0.81	0.82	0.4%	-11.5%
Correlation income-bequest (i,b)	0.95	0.85	-10.7%	0.93	0.85	-8.5%	2.2%
Average(b)/average(e)	0.14	0.02	-83.3%	0.16	0.06	-62.0%	21.3%
Income elasticity of fertility	---	---	---	-0.13	-0.05	-59.2%	
Bequest elasticity of fertility	---	---	---	0.05	0.04	-11.6%	

Table 3.7: Two Children Policy

	Benchmark	Two Children Policy	% change
Mean earnings	0.80	0.82	2.5%
Mean income	1.01	1.08	6.9%
Mean bequest	0.13	0.15	15.4%
Mean consumption	1.01	1.08	6.9%
Mean fertility	1.03	0.98	-4.9%
Std earnings	0.71	0.71	0.0%
Std income	0.90	0.93	3.3%
Std bequest	0.30	0.32	6.7%
Std consumption	0.86	0.90	4.7%
Std fertility	0.19	0.09	-52.6%
Persistence of earnings	0.20	0.24	19.3%
Persistence income	0.54	0.59	8.7%
Persistence bequest	0.58	0.70	21.4%
Persistence consumption	0.59	0.64	9.0%
Persistence fertility	0.00	-0.03	---
Gini earnings	0.40	0.39	-2.5%
Gini income	0.42	0.41	-2.4%
Gini bequest	0.81	0.79	-2.5%
Gini consumption	0.40	0.40	0.0%
Gini fertility	0.06	0.02	-66.7%
Coefficient variation earnings	0.88	0.87	-1.1%
Coefficient variation income	0.89	0.87	-2.2%
Coefficient variation bequest	2.39	2.15	-10.0%
Correlation earnings-bequest (e,b)	0.81	0.76	-6.1%
Correlation income-bequest (i,b)	0.93	0.94	0.8%
Average(b)/average(e)	0.16	0.18	12.6%
Income elasticity of fertility	-0.13	-0.04	-68.3%
Bequest elasticity of fertility	0.05	0.00	-91.9%

CHAPTER 4. ACCOUNTING FOR THE INTERNATIONAL QUANTITY-QUALITY TRADE-OFF

4.1 Introduction

Fertility rates and schooling attainments differ greatly across countries and exhibit a clear inverse association as illustrated in Figure 4.1. It plots fertility rates against expected years of schooling for a group of 74 countries using UNESCO data for 2005. For example, Niger has the highest fertility rate, 7.2, and the lowest expected schooling, 9.7 years, while Greece has the lowest fertility rate, 1.2, and an average length of education of 22.7 years, one of the highest among 74 countries in the data. What explains the international quality-quantity trade-off illustrated in Figure 4.1?

There is an extensive related literature on the demographic transition but few attempts to *quantitatively* explain the overall international evidence through the lenses of a unified model. An exception is Manuelli and Seshadri (2009) who argue that most of the international differences in schooling and fertility can be explained by differences in total factor productivity (TFP) and taxes. In their theory, there is a Barro-Becker-Ben Porath model and there are no market imperfections and no role for public policies, such as public education, student loans or child labor regulations. Individuals' human capital and fertility decisions are socially optimal, and there is no quality-quantity trade-off at the family level as schooling and educational investments in each child are not constrained by family income or family size. Their explanation is based on introducing a non-homothetic element to a general equilibrium model so that the equilibrium interest rate decreases with income. A lower interest rate rises the optimal amount of human capital and, by arbitrate, reduces the demand for children (see their equation 5). Their proposed explanation suggests that economic growth policies and lower taxes would naturally lead to an efficient demographic transition.

Cordoba and Ripoll (2015) cast doubts on the ability of frictionless Barro-Becker models to account for consumption and fertility data. For optimal fertility to be an interior solution children need to be a net financial costs to parents, as recognized by Barro and Becker. If not, it would be optimal for parents to raise the maximum possible number of children in order to maximize rents. Cordoba and Ripoll document for the U.S. that children are on average a net financial benefit to society as the present value of their earnings substantially surpasses the present value cost of their upbringing. They also show that introducing limits to intergenerational transfers can fix the predictions of the Barro-Becker model. But the efficiency of market allocations, in particular fertility and education, is no longer guaranteed, the demographic transition is not necessarily efficient, and a role for public policies potentially arises. In particular, the model could potentially rationalize the educational revolution of the last century based on the expansion of public education around the world. This chapter uses a calibrated version of the Barro-Becker-Ben Porath model with credit constraints to assess the underlying determinants of fertility and schooling different across countries. We focus on situation of perfect capital mobility across countries so that interest rates differentials play no role in our theory, consistent with the findings of Caselli and Feyrer (2007). In our theory, a quality-quantity trade-off arises at the household level due to binding credit constraints, educational outcomes are determined by parental resources and family size dilutes those resources. We model in some detail the life cycle of individuals, life expectancy, aspects of public education, and wage differential across countries.

Our calibrated model is able to explain 68.3% of the fertility dispersion and 85.1% of the schooling dispersion in the data. We find wage has the most power in explaining the differences in fertility and years of schooling across countries. Regarding the effect of wages on fertility, the channel is a direct effect on the marginal cost of raising children as children take parental time. Higher wages also increase the marginal benefit of having children since they increase the welfare of the child, which parents value, but the presence of non-economic goods weaken this second channel so that the effects of wages on the marginal cost dominates, as discussed in Cordoba and Ripoll (2015). Following wage, adult survival probability ranks the second among all exogenous factors in generating fertility difference across countries. Among two margins of schooling, the intensive margin, the duration of public education, the extensive margin, generates more differences of years of schooling while the annual public educational subsidies, the intensive margin, is influential in explaining part of fertility differentials.

There is a large literature studying fertility and schooling decisions, as well as their interaction. Barro and Becker (1988) is a seminal work of a standard neoclassical growth model with micro-foundation of fertility choices. A main strand of later works in explaining the demographic transition over time and the relationship between fertility with income rely on the time cost of rearing children, e.g. Galor and Weil (GW, 1996), Greenwood and Seshadri (GS, 2002), Cordoba and Ripoll (2013) and many others. Some of them, such as GW (1996) and GS (2002) assumes warm glow altruism and bounded rationality. Doepke (2004) develops a growth model in which children might become skilled or unskilled. His model explains the decline of fertility in the economic transition from stagnation to growth. He argues that the main influencing policy is child labor regulation rather than educational subsidies.

Among all the literature the one closely connected with this chapter is Cordoba and Ripoll (2013), CR (2013) henceforth. They provide a theory explaining the cross-country distribution of average years of schooling, as well as the so called human capital premium puzzle in a Ben Porath model with credit friction. They find fertility plays the most important role in producing schooling difference among all exogenous factor. This chapter goes beyond CR (2013) since we endogenize altruistic fertility choice, instead of taking it as given. Moreover, we quantitatively calibrate parameters relating to the time cost of raising children and parental altruism, which are not so relevant in exogenous fertility models. A difference from the exogenous fertility model is that wage plays key role in generating difference in years of schooling. For example, equating wages of all countries to the U.S. level reduces years of schooling by 63%. The channel through fertility explains this huge response. Higher wage induces people to have fewer children, then parents transfer more to every child not only because of the quantity-quality trade-off but also because of the higher parental altruism on every child.

Concerning the relationship between fertility and years of schooling, there is an early literature on the quantity-quality trade-off by Becker and Lewis (1973) and Becker and Tomes (1976). Becker, Cinnirella, and Woessmann (2009) provide an evidence of the existence of such a trade-off before the demographic transition using census-based dataset.

The rest of the chapter is organized as follows. Section 4.2 sets up the benchmark model and characterizes steady state consumption, bequest, fertility and educational choices, including the length of schooling and private expenditures. Section 4.3 describes the calibration of the model. Section 4.4 assesses the performance of the model and performs counterfactual exercises. Section 4.5 concludes.

4.2 The Benchmark Model

4.2.1 The Model

This section describes the problem faced by a representative individual. The individual maximizes his or her lifetime utility, V, which includes a life cycle utility over the life span of T years and the utility of n children, V' who are born at age F.

$$V\left(b\right)=\frac{C^{1-\eta}}{1-\eta}+\pi\left(F\right)e^{-\rho F}\phi(n,F)V\left(b'\right)$$

where

$$C=\left(\frac{\rho}{1-\rho^{T}}\int_{0}^{T}e^{-\rho t}\pi\left(t\right)^{\frac{1-\sigma}{1-\eta}}c\left(t\right)^{1-\sigma}dt\right)^{\frac{1}{1-\sigma}}+\bar{u}$$

where $V\left(b\right)$ is the value function of an individual with bequest b received from parents when born, C is a composite consumption, $c\left(t\right)$ is the consumption of time t, $\bar{u}$ is utility from non-economic goods, $\pi\left(t\right)$ is the survival probability from being born to age t, and $\phi(n,F)$ is the weight parents attach to their n children. ρ is a time discount rate. $1/\sigma$ is the elasticity of intertemporal substitution (EGS) characterizing the intra-personal willingness to substitute consumptions across periods. $1/\eta$ is the elasticity of intergenerational substitution (EIS) characterizing the inter-personal willingness to substitute consumption across generations. Disentangling EIS from EGS is necessary in models with intergenerational transfers to match imputed values of children and the negative fertility-income relationship well documented in the literature.

Assume

$$\phi(n, F) = \beta e^{-\rho F}\left(1 - e^{-\chi n}\right)$$

And assume the survival probability $\pi(a)$ takes the following form following Cordoba and Ripoll (2013),

$$\pi(t) = \begin{cases} e^{-p_c t} & \text{for } t \leq a_c \\ \pi(a_c)\, e^{-p_s(t-a_c)} & \text{for } a_c \leq t \leq a_s \\ \pi(a_s)\, \frac{e^{-p(t-a_s)}-\xi}{-\xi} & \text{for } a_s \leq t \leq T \end{cases}$$

Individuals are subject to a sequence of budget constraints that could be reduced to a single life-time budget constraint in the absence of credit frictions. However, we assume that only working individuals can borrow and therefore students need to rely on parental resources to finance consumption and education during their schooling years. Working individuals, in particular parents, have full access to credit markets. Parents act as banks for their children but, as we show below, they are imperfect substitutes for financial institutions.

Due to the assumed credit frictions, agents' life span can be divided into two clear phases: student's phase from age 0 to years of schooling s and post-student's phase from age s to T. In the first phase, individuals rely solely on parental transfers, b, to finance expenditures as described by the following budget constraint:

$$\int_0^s \left(c(t) + e_s(t)\right) q(t)\, dt + q(s)\, Q(s) \leq b \tag{29}$$

Private expenditure on education, $\mathbf{E} = \{e_s(t)\}_0^s \geq 0$, complement public education expenses available, e_p every year from age $\underline{s}$ to $\bar{s}$. $q(t)$ is the price of consumption at age t relative to age 0 consumption. With annuity $q(t) = e^{-rt}\pi(t)$ and $Q(s)$ are net savings at the end of schooling. The credit constraint is summarized by the condition:

$$Q(s) \geq 0$$

At the end of the education phase, the agent starts working with an amount of human capital given by $h(s, \mathbf{E})$. The budget constraint for the second phase is described by:

$$\int_s^T c(t)\, q(t)\, dt + q(F)\, nb' \leq q(s)\, Q(s) + W(s,n) \tag{30}$$

$$b' \geq 0$$

where $W(s,n)$ be the present value of life time earnings. The first inequality states that consumption expenditures and transfers to children could not exceed life time income plus savings at the end of student life. The second inequality prevents parents from leaving debts to children. The two life-time budget constraints could be combined into a single one when savings, $Q(s)$, are positive.

Lifetime income $W(s,n)$ is given by

$$W(s,n) = \int_s^R wh(s)\, e^{\nu(a-s)} q(a) l(n) da$$

where r is the risk-free interest rate, F is the age of having children, R is the retirement age, ν is the return to experience, w is the wage rate, $h(s,E)$ is the human capital accumulated after s years of education with an amount E of private education in addition to the public education available. $l(n)$ is the parental effective labor supply.

Assume

$$h\left(s,\mathbf{E}\right)=\left(\int_{0}^{s}\left(\left(e_{p}+e_{s}\left(t\right)\right)/p_{E}\right)^{\mu}dt\right)^{\alpha/\mu}$$

and $l\left(n\right)$ takes the form

$$l(n)=1-\lambda\left[(n+\kappa)^{\theta}-\kappa^{\theta}\right]\qquad\lambda>0,\ \kappa>0,\ 0<\theta<1$$

Parameter κ allows to bound the marginal cost of the first (dn) children. The condition that $\theta<1$ guarantees the convexity of the labor supply function which implies that the marginal time cost of children decreases with the number of children. A simple form of the above labor cost function is the one with linear time cost per child.

$$l\left(n\right)=1-\lambda n$$

Lifetime income $W(s,n)$ is thus a decreasing function of n and an increasing function of ν and w. Absent credit frictions, optimal schooling maximizes lifetime income and therefore $W_s(s,n)$ would be 0. As we show below, $W_s(s,n)>0$ in the presence of credit frictions.

A full characterization of the solution is provided in Appendix. We focus on some key results in what follows.

The optimality choice of schooling must satisfy

$$\underbrace{e^{rs}\pi\left(s\right)^{-1}c^{W}\left(s\right)^{-\sigma}W_{s}\left(s,n\right)}_{\text{net marginal benefit of }s}\;=\;\underbrace{\sigma\Delta u(s)+c^{S}\left(s\right)^{-\sigma}e_{s}\left(s\right)}_{\text{marginal cost of }s}\tag{31}$$

where

$$W_{s}\left(s,n\right)=w\left[-h\left(s\right)q(s)+\left(h_{s}\left(s\right)-\nu h\left(s\right)\right)e^{-vs}\left(\int_{s}^{F}e^{\nu a}q(a)da+l(n)\int_{F}^{R}e^{\nu a}q(a)da\right)\right]$$

The net marginal benefit of one more year of schooling is measured by the percentage increase of the human capital due to an extra year of education. The net marginal cost of an extra year of schooling is comprised of the loss of one year of experience and salary, one year of private educational cost, and one more year of being constrained as a student to consume at a low level compared with the consumption in the second phase.

The relationship between the net marginal benefit and marginal cost of fertility is

$$-\frac{W_n(s,n)}{W(s,n)} = \lambda_2^{-1}\pi\left(F\right)\beta_1 e^{-\rho F}\frac{\phi_n(n,F)}{W(s,n)}V\left(b'\right) - \frac{q(F)b'}{W(s,n)} \tag{32}$$

The right hand side of (32) is the normalized marginal benefit of having one more child net of the bequest to the child. All terms are normalized by dividing the lifetime income $W(s,n)$. The left hand side of (32) characterizes the marginal time cost of having one more child as a percentage of the total labor he supplies. Since we've already subtracted the bequest to every child on the right hand side, the only cost that needs to be taking into account in the LHS is the relative time cost of raising children.

4.2.2 Consumption and Bequest

The agent's consumption at any time can be written as his consumption at age s. There is a jump of consumption at age s because of the credit friction, that is people are not able to borrow as much as they want during student periods by assumption. Let $c^S\left(s\right)$ and $c^W\left(s\right)$ denote the consumption as a student and a worker, respectively.

$$c\left(a\right) = \begin{cases} e^{\frac{(r-\rho)(a-s)}{\sigma}}\left(\frac{\pi(a)}{\pi(s)}\right)^{\frac{1}{\sigma}\frac{\eta-\sigma}{1-\eta}} c^S(s) \text{ for } a \leq s \\ e^{\frac{(r-\rho)(a-s)}{\sigma}}\left(\frac{\pi(a)}{\pi(s)}\right)^{\frac{1}{\sigma}\frac{\eta-\sigma}{1-\eta}} c^W(s) \text{ for } s < a < T \end{cases} \tag{33}$$

Depending on parameter values, the bequest constraint may bind or not. We need to consider both cases. Denote by λ_1 and λ_2 the Lagrange multipliers of the two constraints associated with (74) and (75). Then the first order condition to consumptions become

$$\left(\frac{c^S(s)}{c^W(s)}\right)^{-\sigma} = \frac{\lambda_1}{\lambda_2} = \frac{q(F)n}{\pi\,(F)\,e^{-\rho F}\phi(n,F)} \tag{34}$$

where $c^S(s)$ and $c^W(s)$ are the consumptions at age s in student phase and post-student phase, respectively. A sufficient and necessary condition for $c^S(s) < c^W(s)$, and therefore for the bequest constraint to bind, would be

$$\frac{q(F)n}{\pi\,(F)\,e^{-\rho F}\phi(n,F)} > 1$$

Frictional Case

In the frictional case, we have $c^S(s) < c^W(s)$. The agent would like to borrow but he couldn't get any loan, so the optimal savings at age s, $Q(s)$, would be 0. Define

$$G \equiv \frac{\lambda_1}{\lambda_2} \tag{35}$$

At steady state, G can be solved through (34) and (35). It measures the tightness of the constraint during the first phase of life. The consumption at age s as a worker can be expressed in terms of the consumption at age s as a student by

$$c^W\,(s) = c^S\,(s)\,G^{\frac{1}{\sigma}} \tag{36}$$

Let E^* be the total private expenditure on education as a percentage of GDP during student phase.

$$\mathbf{E} \equiv \int_0^s e_s(t)q(t)da$$

Steady state bequest b can be solved by (74), (75) and (33) as

$$b = \frac{W(s,n)G^{-\frac{1}{\sigma}} + E^{*}\Omega(s)}{\Omega(s) + q(F)\, nG^{-\frac{1}{\sigma}}} \tag{37}$$

where

$$\Omega(s) = \frac{\begin{array}{c}\pi(a_c)^{\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}} e^{p_s a_c \frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}} I\left(\theta + p_s\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}, s, 25\right) \\ +\pi(a_s)^{\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}} \int_{25}^{T} e^{-\theta a}\left[\frac{e^{-p(a-a_s)}-\xi}{1-\xi}\right]^{\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}} da\end{array}}{I\left(\theta + p_c\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}, 0, a_c\right) + \pi(a_c)^{\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}} e^{p_s a_c \frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}} I\left(\theta + p_s\frac{1-\sigma}{\sigma}\frac{\eta}{1-\eta}, a_c, s\right)}$$

$$\theta = r - \frac{r-\rho}{\sigma}$$

and

$$I(\theta, c, d) = \int_{c}^{d} e^{-\theta a} da$$

Frictionless Case

In the frictionless case, the bequest the agent gets in the early life is very high relative to his future income, so consumption at age s as a student and the one as a worker be the same. By (34) and (35), $G = 1$ and the optimal choice of number of children depends solely on exogenous parameters.

$$e^{-(r-\rho)F}\frac{n}{\phi(n,F)} = 1$$

In the frictionless case, $c^{S}(s) = c^{W}(s)$. When $Q(s) = 0$, steady state bequest b' can be solved though (74) and (33).

$$b = \frac{W(s,n) + \Omega(s)E^{*}}{\Omega(s) + q(F)\, n} \tag{38}$$

4.2.3 Human Capital Accumulation

With s years of schooling and $e(a)$ as the total amount of investment on education percentage of GDP at age a, both public and private, the agent accumulates human capital $h(s, E)$ according to the function

$$h(s, \mathbf{E}) = \left(\int_0^s \left(\frac{e(t)}{p_e} \right)^\mu dt \right)^{\frac{\alpha}{\mu}} \tag{39}$$

where p_e is the relative price of education. When the overall annually educational investment, $e(a)$, is a constant, human capital is proportional to the product of an exponential form of s and an exponent of annually total educational investment. In terms of human capital accumulation, a sufficiently long period of education together with a low amount of total educational investment can be the same with a short period of education and a huge amount of total educational investment every year. Ben Porath(1967)'s law of motion of human capital at age a is

$$\dot{h}(t) = z[n(t)h(t)]^{\gamma_1} i(t)^{\gamma_2} - \delta_h h(t) \tag{40}$$

CR (2013) prove that when $n(a) = 1$, $z = 1$, $\delta_h = 1$, $\mu = \gamma_2$, and $\frac{\alpha}{\mu} = \frac{1}{1-\gamma_1}$ holds, (39) is a solution to (40).

4.2.4 Educational Expenditure

Public educational subsidies in the amount e_p per period are available from age $\underline{s}$ to $\bar{s}$. Individuals can make private investments on education, $e_s(t)$, to supplement public education. Let $\hat{e}^*(t)$ be the optimal amount of total expenditure on education, then

$$\hat{e}^*(t) = \begin{cases} e_p + e_s(t) & \text{if } \underline{s} \le t \le \bar{s} \\ e_s(t) & \text{else} \end{cases} \tag{41}$$

Optimal total educational expenditure $e^{*}(t)$ is given by

$$e^{*}(t)=\begin{cases}\max\left\{\hat{e}^{*}\left(t\right),e_{p}\right\} & \text{if } \underline{s}\leq a\leq\bar{s}\\ \hat{e}^{*}\left(t\right) & \text{else}\end{cases}$$

During the range when public education is available, if public educational subsidy is higher than what the individual desires for the total amount of educational investment, the agent chooses pure public education, otherwise, he invests privately to complement the available public education.

As derived in CR (2013), if an agent decides to get s years of schooling, the optimal total educational expenditure at age t is given by:

$$e^{*}\left(t\right)=\begin{cases}\widehat{e}^{*}\left(t\right) & \text{for } t\leq\min(s,\underline{s})\\ e_{p} & \text{for } \min(s,\underline{s})\leq t\leq s_{p}\\ \widehat{e}^{*}\left(t\right) & \text{for } s_{p}\leq t\leq s\end{cases} \tag{42}$$

where

$$s_{p}\equiv\min\left\{s,\bar{s},\max\left[\underline{s},\widehat{a}\right]\right\}. \tag{43}$$

$\widehat{a}$ is defined by

$$\widehat{a}=\sup\left\{t:\widehat{e}^{*}\left(t\right)=e_{p}\right\}$$

When private investment on education is nonzero, the optimal total amount of educational expenditure, $\hat{e}^{*}\left(t\right)$, is

$$\hat{e}^{*}\left(t\right)=\hat{e}^{*}\left(0\right)q(t)^{-\frac{1}{1-\beta}} \tag{44}$$

and the total amount of investment (both public and private) on education at age 0 is

$$\hat{e}^{*}\left(0\right)=\left(\alpha h\left(s\right)^{-\frac{\beta}{\alpha}}p_{E}^{-\beta}W(s,n)/G\right)^{\frac{1}{1-\beta}} \tag{45}$$

$\widehat{a}$ is the supremum age of periods when the total educational investment is pure public. It can be solved using (44) and $e^*(\widehat{a}) = e_p$.

$$\hat{a} = \frac{1}{p_s + r}\left[(1-\beta)\ln\left(\frac{e_p}{\hat{e}(0)}\right) + \ln \pi(a_c) + p_s a_c\right]$$

Using (121), (41), and (44), we get the present value of optimal private expenditures in education, E^*, as:

$$\begin{aligned} E^* &\equiv \int_0^s e_s(a)q(a)da \\ &= \hat{e}^*(0)\left[\int_0^{\min(s,\underline{s})} q(a)^{-\frac{\beta}{1-\beta}} da + \int_{s_p}^{s} q(a)^{-\frac{\beta}{1-\beta}} da\right] - e_p \int_{s_p}^{\min(s,\bar{s})} q(a)da \end{aligned}$$

Total value of the expenditure on education is the summation of private educational expenditure before and after the availability of public education and the total investment on education given total investment exceeds the public educational subsidies, minus the public educational subsidies during the period $[\underline{s}, \bar{s}]$. Similarly, we can also write $h(s)$ as:

$$h(s,E) = \left(\frac{e(0)}{p_e}\right)^{\alpha}\left(\int_0^s q(a)^{-\frac{\beta}{1-\beta}} da\right)^{\frac{\alpha}{\beta}}$$

$$h(s) = \left(\frac{e^*(0)}{p_E}\right)^{\alpha}\left[\left(\int_0^{\min(s,\underline{s})} q(a)^{-\frac{\beta}{1-\beta}} da + \int_{s_p}^{s} q(a)^{-\frac{\beta}{1-\beta}} da\right) + \left(\frac{e_p}{e(0)}\right)^{\beta}(s_p - \min(s,\underline{s}))\right]^{\alpha/\beta} \tag{46}$$

22

[22]Notice that $\min(s,\bar{s}) \leq s_p$.The proof is trivial.
If $\underline{s} \leq \hat{a}$,

$$s_p = \min\{s, \bar{s}, \hat{a}\} \geq \min\{s, \bar{s}, \underline{s}\} = \min\{s, \underline{s}\}$$

Otherwise,

$$s_p = \min\{s, \bar{s}, \underline{s}\} = \min\{s, \underline{s}\}.$$

4.2.5 Steady State Wage

Labor earns the marginal product. Since after tax per capita production y is

$$y = h\left(t\right)^{1-\alpha_1} k^{\alpha_1}$$

so the wage per unit of labor income is

$$w = \left(1-\tau\right)\left(1-\alpha_1\right)\frac{y}{h\left(t\right)}$$

4.3 Calibration

4.3.1 Parameters common across countries

Among all parameters assumed to be common across countries, Table 4.1 shows parameters that are set exogenously from micro evidence and related study, they include interest rate r, returns to experience ν, age parameters regarding survival probability a_c and a_s, age of giving birth to children, F, and age of retirement R. Other than these parameters, we calibrated parameters, including non-consumption goods, $\bar{u}$, altruism parameter, β and χ, intergenerational elasticity of substitution, η, intertemporal elasticity of substitution σ, time preference ρ, labor supply parameters λ, θ, and κ, human capital accumulation α and μ, risk aversion to mortality γ,survival probability p_c, p_s, p, and ξ, life expectancy T, price of education p_e, public education expense e_p, age lower and upper bound of public education, $\underline{s}$, $\bar{s}$, tax rate τ, experience parameter Exp, and per capita GDP y.

Calibration Targets Table 4.2 summarizes the calibrated parameters. We calibrate the model to match key statistics on fertility, education, and bequest in OECD countries and the United States. There are seven parameters that need to be calibrated. Due to the difficulty of exactly matching all of them, we minimize mean square errors between the model's statistics and the targets. The inverse of intergenerational elasticity of substitution η is calibrated to match the mean fertility of OECD countries from the UNESCO data.

The calibrated parameter on intergenerational elasticity of substitution η is 0.68. It quantitatively identifies EGS to be larger than 1, which is different from EIS, smaller than 1. Since EGS captures the easiness of consumption smoothing across generations while EIS captures the extent of consumption smoothing across periods within one's own life, the calibrated parameter indicates that people tolerate the cross-generational consumption fluctuation more than the cross-periods' fluctuation within one's life.

The target to identify the nonhomothetic part of the utility function $\bar{u}$ is the income elasticity of fertility, -0.22, estimated in Jones, Schoonbroodt and Tertilt (2011) (JST (2011) henceforth) using more recent U.S. census data in which cohorts were born between 1956 to 1960. OECD includes relatively richer countries in the sample so we match the income elasticity of fertility across OECD countries to the U.S. level. Higher $\bar{u}$ increases fertility by raising the marginal benefit of having children but it affects more on the poor than the rich as the rich has more consumption and cares less on the utility from nonconsumption goods.

The discount rate on children's consumption β and altruistic parameter χ jointly match the the mean years of schooling across OECD countries and the parental transfer to every child from year 0 to 17 as a percentage of parents' lifetime income in the U.S. calculated in Cordoba and Ripoll (2014). Altruistic parameter χ is chosen to be 2.8 implying for an individual who only have one child, at the age of giving birth to children F, the altruistic level toward the child is 75.3% of what they value themselves. Two children in a household get a total weight around 93.9% from parents, with 47% each one. And each child in a three children household is given a weight around 33% from his/her parent. A decreasing altruistic level can be seen as number of children increases. Up to any arbitrary large number of children, the total weight that every parent puts on all his or her children is less than 1 but close to 1, which means that parents value all their children less than what they value themselves.

The average private expense on education as a percentage of GDP in OECD countries 0.65% and the return to schooling in U.S. 8.28% are used to pin down parameters α and μ in the human capital accumulation function. We choose OECD countries for the fraction of private educational expense instead of the U.S. level 2.1% because US education statistics tend to be somehow atypical among rich countries.

Time cost on every child is an important parameter that governs fertility and time cost, hence we calibrate λ to match time cost as a percentage of lifetime income.

From the calibrated model, mean fertility of the OECD countries is 0.85, mean years of schooling is 21.9, very close to the targets. Transfer from parents to children and time cost of raising children are 15.6% and 19.5%, several percentage points higher than the data. However, since the targets' underestimate goods cost and time cost, in particular, goods transfer does not take into account the transfer from parents to children when children are older than 17. our calibrated parameters deliver statistics in a reasonable range. The fraction of private educational expense is 0.44% and the return of schooling is 9.9%. Our model delivers an income elasticity of fertility in OECD countries, -26.5% which is in between JST (2009)'s estimation using census data in 1960s, -22% and their estimation across the whole sample period -30%.

Table 4.1 Exogenous Parameters Common Across Countries

Parameters	Concept	Value	Target
ν	return to experience	2%	Bils and Klenow (2000a)
$\underline{s}$	starting schooling age	6	UNESCO
F	parenthood age	25	Cordoba and Ripoll (2013)
R	retirement age	65	Cordoba and Ripoll (2013)
α_k	capital share	0.33	Gollin (2002)
σ	inverse EIS	1.5	Cordoba and Ripoll (2014)
r	riskless interest rate	2%	Cordoba and Ripoll(2014)
ρ	time discount	2%	Flat consumption

Table 4.2 Calibrated Parameters			
Parameters	Concept	Value	Targets
η	inverse of EGS	0.68	OECD Mean fertility: 0.86
$\bar{u}$	utility nonconsumption goods	0.14	OECD $\frac{\triangle\text{Fertility}}{\triangle\text{Income}}$: -22%(JST 2011)
β	discount on children	0.2	OECD Mean years of schooling: 22.0
χ	altruism	2.8	US Transfer/Lifetime Income 10.3%
α	human capital	0.4	OECD Private Edu Expense/GDP: 0.65%
μ	human capital	0.16	U.S. Return to schooling at s: 8.28%
λ	time cost per child	0.24	US Time cost/Lifetime Income 15.4%

4.3.2 Parameters specific for different countries

Country-specific parameters are chosen following CR (2013). They includes e_p, the public educational expenditure as a percentage of GDP, price of education P_e, life expectancy T, wage w, the upper bound $\bar{s}$ and the lower bound $\underline{s}$ of the set of age range when public education is available.

4.4 Results

In this section, we report the quantitative predictions of the calibrated model on the steady state levels of fertility and schooling of all the 74 countries in 2005 from statistics of UNESCO.

4.4.1 Model's fit

Figure 4.2 and 4.3 plot the fertility and years of schooling simulated from the model versus that in the data. Our model is able to explain around 68.3% of the fertility dispersion and 85.1% of the schooling dispersion in the data. Model's prediction on fertility rates ranges from 0.63 to 2.64 and fertility rates in data range from 0.58 to 3.87. Model's prediction on years of schooling ranges from 13 to 25 and schooling data ranges from 9.71 to 26.35. The average fertility rate of all the 74 countries predicted by the model is 1.29 with standard deviation 0.53, while the average fertility rate in the data is 1.42 with 0.77 as its standard deviation. The mean value of years of schooling in the model and in the data are 20.04 and 18.96, with the corresponding standard deviations 2.85 and 3.35, respectively. The model slightly underestimates average fertility by 0.13 and overestimates average years of schooling by 0.24. A significant part of the mismatch occurs in poor countries where the predicted fertility is lower and years of schooling is higher than the data.

As Figure 4.4 shows, the model does a good job in generating a negative relationship between fertility and years of schooling. Countries with more education have lower fertility rate. Are these countries richer than others? Figure 4.5 and 4.6 plot fertility and schooling as a function of wage per unit of labor income, respectively. These two figures indicate that people in countries earning higher wage choose to have few children and go to school longer. Why do rich countries have lower fertility rates? There is a tension between the effects of wage on fertility. On one hand higher wage implies higher income, consumption and bequest to children, which leads to a higher utility of every child, which induces parents to have more children as the marginal benefit of having children increases. On the other hand higher wage also increases the cost of having children as children take parental time. When the utility function is nonhomothetic due to the presence of nonconsumption goods, the effect of wage on marginal cost dominates that on marginal benefit so fertility decreases with wage. The model predicts an income elasticity of fertility -0.28 of all the 74 countries in the data. Table 4.3 indicates model's performance.

Table 4.3 Model's Performance for All Countries		
	Model	Data
Mean Fertility	1.29	1.42
Std Fertility	0.53	0.77
Mean Years of Schooling	20.04	18.96
Std Years of Schooling	2.85	3.35
Income Elasticity of Fertility	-0.28	-0.20 to -0.38

4.4.2 Counterfactual Exercises

In this section, we identify key factors in the model that are driving the dispersion of fertility rate and length of education across countries. Table 4.4 shows the effects of experiments on the mean, standard deviation of years of schooling, fertility, and parental transfers across countries. In every counterfactual exercise, we equalize one parameter to the U.S. level and leave others unchanged.

Table 4.4 Counterfactuals (% Change)						
	stds	means	stdn	meann	stdb	meanb
p_c	-6.0	1.2	-10.7	-3.5	-0.1	1.4
p_s	-1.6	0.3	-3.0	-1.0	-0.6	0.0
p	-5.6	1.7	-26.3	-8.3	-7.5	0.0
p_c, p_s, p	-14.4	4.1	-39.2	-13.4	-8.2	5.7
e_p	7.9	-4.8	-21.2	-10.8	4.6	17.5
$\bar{s}$	-35.2	1.1	-14.5	-2.6	-5.1	-7.9
p_e	6.9	-5.5	12.8	10.7	-14.2	-23.0
w	-65.7	22.2	-61.3	-41.5	94.0	411.8

Wage per unit of labor is the most important factor in explaining both the dispersion of fertility and years of schooling across countries. Equating wage to the US level decreases the standard deviation of fertility by 61.3% and decreases cross country years of schooling by 65.7% . Fertility rates of many countries, especially those poor countries, become as low as the U.S. fertility and their schooling hits the maximum level after equating wage to the U.S. level. A higher wage increases the marginal cost of having children since the opportunity cost of raising children is higher. Higher wage rate also increases the marginal benefit at steady state since children's welfare is higher and parents are altruistic toward children. The nonconsumption goods weakens the impact of wage on the marginal benefit. Hence the increase of the marginal cost dominates that of the marginal benefit and raising wage reduces fertility.

Contrary to Cordoba and Ripoll (2013) in which fertility is exogenous and wage is a trivial factor in affecting mean as well as the dispersion of schooling, wage in our model is quantitatively important. Equating wage to the US level decreases the standard deviation of schooling by 61.3% and increases average schooling by 22.2%. There is a channel at work in our model but can not be captured in exogenous fertility models. Higher wage reduces optimal fertility, and it increases the transfer from parent to children by the quantity-quality trade-off and by the higher degree of parental altruism. Table 4.4 shows that the mean of bequest is four hundred times bigger than that without equating wage to the U.S. level. In the model with credit friction, bigger transfer from parents allows young people to invest more resource in education.

In addition to wage, the maximum length of public education $\bar{s}$ also has substantial effect on the dispersion of schooling. Equating the wage of all the 74 countries to that of the U.S. level, the predicted average schooling is raised by 1.1% and the dispersion is shrinked by 35.2%.

Although the the extensive margin of education, $\bar{s}$, has a greater effect on differences in fertility across countries, the intensive margin, the public educational subsidy per pupil e_p, does not play a role in producing the divergence of years of schooling. The reason is that equating e_p to the U.S. level allows people to accumulate high level of human capital, which leads to high earnings per unit of labor and higher consumption when working. This increases the gap between consumption as a student and as a worker and hence increases the marginal cost of schooling and discourages the optimal years of schooling. In short, individuals are less willing to be constrained as students and tend to escape from the constrained student phase with low consumption and eager to enter into the working life earlier. This explains why mean years of schooling is lower by 4.8%.

Regarding the impact on fertility dispersion, adult survival probability is the second important factor after wage. It lowers the fertility dispersion by 26.3% by equating all countries' adult survival probability to the U.S. level. The mechanism at work is that the experiment raises the expected earning by increasing the adult probability. Higher expected earnings per unit of labor increases the marginal cost of having children and this impact dominates the impact on marginal benefit.

Shutting down differences in e_p to the U.S. level lowers the predicted average fertility rate by 10.8% and reduce the predicted standard deviation of fertility by 21.2%. Assigning all countries' e_p to the US level endows a higher human capital accumulation for most countries, which enhances the annual income. A higher time cost of raising children discourages the optimal decision of fertility.

Infant survival probability explains 10.7% of the fertility dispersion. The survival probability of all stages has minor impact on the dispersion of schooling. The standard deviation of years of schooling is merely cut by 3.7% if we shut down the difference of all stages' survival probability by equating all countries' level to that of U.S. But it does play an active role in the dispersion of fertility.

Infant survival probability explains 10.7% of the fertility dispersion. The survival probability of all stages has significant impact on the dispersion of fertility. The standard deviation of years of schooling is cut by 14.4% if we shut down the difference of all stages' survival probability by equating all countries' level to that of U.S.

If we equate the price of public education, p_e, to the U.S. level while keeping other factors unchanged, the dispersion of both fertility and schooling will be even higher than the prediction of the model without counterfactual analysis. It seems that p_e does not produce any diversion, rather, it reduces these dispersions.

4.5 Conclusion

We quantitatively study optimal fertility and years of schooling choices in a dynamic altruistic Barro-Becker context with credit frictions. The model predicts a plausibly negative relationship between steady state fertility and duration of education, consistent with UNESCO data in 2005. Our model performs well in predicting average and dispersion of fertility rate and that of educational length.

Within the altruistic life-cycle model with credit friction, we identify several key forces that drive the diverse steady state fertility and schooling across countries. Our model suggests that wage differentials are the most essential factor in generating differences in fertility and years of schooling. The duration of public education, has substantial effect on the divergence of years of schooling while the intensive margin, the annual public educational subsidy, works better in producing the dispersion of fertility.

In this chapter, we abstract the heterogeneity of individuals in the economy and do not examine the dynastic transition of the economy. Heterogeneity is not considered in this model. In one of the companion chapter of my dissertation, CLR (2015) we quantitatively study U.S. inequality using a heterogeneous model in which people differ in lifetime earning abilities.

CHAPTER 5. ALTRUISM, FERTILITY AND RISK

5.1 Introduction

At least since Malthus it has been recognized that high fertility rates are associated to poverty. Such association motivates many family planning programs around the world that seek to reduce fertility as a way to alleviate poverty.[23] Understanding the link between poverty and fertility, and more generally, between inequality and fertility is an important part of the theory of distribution.

This chapter studies fertility choices and fertility policies in economies where the underlying force driving inequality, poverty, social mobility and fertility differentials is uninsurable idiosyncratic risk. Our model integrates two leading streams of the literature on inequality and fertility. One the one hand, inequality and social mobility is driven by idiosyncratic shocks, as in Huggett (1993) or Aiyagari (1994). On the other hand, fertility is purely motivated by altruistic reasons, as in Barro and Becker (1989) and Becker and Barro (1988). Individuals in our models are fully rational, altruistic toward their descendants, and heterogeneous in their abilities. Earning abilities are randomly determined at birth and potentially correlated with parental abilities. Insurance markets are available but parents cannot leave negative bequests to their children. Altruistic models with idiosyncratic risk are central to modern macroeconomics, particularly when studying issues of inequality and redistribution, but with the important exception of Alvarez (1999), these models assume exogenous fertility. This chapter is the first to provide a characterization of the endowment version of a Bewley economy extended to include endogenous fertility.[24]

The first main contribution of the chapter is to characterize complete markets allocations in the presence of idiosyncratic shocks and endogenous fertility. Surprisingly, while perfect risk sharing and flat consumption are optimal when fertility is exogenous, they are not optimal when fertility is endogenous. Instead

[23] For example, a United Nations Population Fund pamphlet asserts that "effective family planning programmes targeted to meet the needs of poor populations can reduce the fertility gap between rich and poor people, and make a powerful contribution to poverty reduction and the achievement of the Millennium Development Goals." Reported by UNPF at http://www.unfpa.org/rh/planning/mediakit/docs/sheet4.pdf. Retrieved April 2 2014.

[24] See Ljungqvist and Sargent (2012, chapter 18) for a presentation of Bewley models.

consumption and fertility remain stochastic in the complete markets economy. The reason is that when a life is costly to create, optimal consumption is an utilization rate and the higher the net cost of creating a new life the higher its utilization rate, as shown by Barro and Becker (1989). Since the net cost of a child is not constant in our environment, as it is tied to both the parent's and the child's abilities, then neither flat consumption nor flat fertility is optimal. We also show that the complete markets model predicts a negative relationship between fertility and parental ability, inequality and social mobility. However, we also document some key counterfactual implications. For example, the model predicts that all children of the same parent have the same consumption, that the actual ability of an individual does not affects his/her own consumption, or that maximum fertility is often optimal for plausible calibrations of the income process.

We then proceed to characterize the incomplete markets economy, one that arises when the non-negative bequest constraint binds. The model can replicate a number of stylized facts: inequality, persistence as well as social mobility, and fertility decreasing with ability. The equilibrium is characterized by a Markov branching process satisfying the *Conditional Stochastic Monotonicity* property. This means that if a kid from a poor family and a kid from a rich family both fall into one of the poorest classes, it is more likely that the poor kid will be poorer than the rich kid. But the model also requires a significant degree of social mobility in order to rationalize why the poor have more children. Absent social mobility, fertility rates would be equal among the poor and the rich. Furthermore, significant social mobility is a distinguishing feature of the data.

A negative relationship between fertility and ability is not simple to obtain within dynastic altruistic models, as discussed by Jones, Schoonbroodt and Tertilt (2011). Our explanation for the negative relationship is novel and the second main contribution of the chapter. It arises from the interplay of two opposites forces. On the one hand, higher ability individuals face a larger opportunity cost of having children as children take valuable parental time. On the other hand, higher ability individuals enjoy a larger benefit of having children when abilities are intergenerationally persistent because the utility of a child is positively related to parental ability. We find that the effect of ability on the marginal cost dominates its effect on the marginal benefit if the intergenerational persistence of abilities is not perfect. Regression to the mean in abilities means that low ability parents expect their children to be, on average, of higher ability while high ability parents expect their children to be of lower ability. This explains why fertility decreases with ability, and income, in a fully rational, homothetic and altruistic environment.

Golosov, Jones and Tertilt (2007) have shown that allocations are efficient in frictionless Barro-Becker models. Of particular interest is whether fertility is higher or lower when markets are incomplete than when markets are complete. We show analytically that steady state fertility can be higher when markets are incomplete if children are a net financial cost to parents. In that case, binding bequest constrains prevent early generations from extracting resources from later generations making future generations effectively richer. Since fertility is a normal good, the fertility of early generations falls while fertility of later generations, and in particular steady state fertility, increases. Providing conditions under which fertility is higher in incomplete markets economies than in complete market is the third main contribution of the chapter. This chapter complements Schoonbroodt and Tertilt (2014) who have shown that fertility is inefficiently low for early generations when markets are incomplete.

Finally, we study the consequences and optimality of fertility policies, such as family planning programs aiming at reducing fertilities rates. We show that policies restricting the fertility of the poor result in a sequence of income distributions that dominates the original distribution in all periods in the first order stochastic sense. In particular, average income and consumption increase in all periods. This result arises from two forces. First, average ability of (born) individuals increases because the poor has proportionally more low ability children as a result of the assumed conditional stochastic monotonicity property. Second, consumption and income of the poor strictly increases because they spend less time and resources raising children. These results seemingly provide the theoretical support to family planning programs seeking to reduce the fertility of the poor (Chu and Koo 1990).

In spite of these positive implications, the fourth main result of the chapter is to show that fertility restrictions of any type, not only for the poor, unequivocally reduce individual and social welfare in our model, even when fertility is higher than its complete market counterpart. As we show, a policy that restricts fertility reduces the set of feasible choices and invariably reduces welfare of all individuals in all generations, even those whose fertility is not directly affected. This is because altruistic parents care not only about their own consumption and fertility but also care about the consumption and fertility of all their descendants. Regardless of current ability there is a positive probability that a descendant of the dynasty will fall into the group directly affected in finite time. Furthermore, the welfare of those individuals who are not born under the new policy also falls, or at least does not increase. Social welfare falls because the welfare of all individuals, born and unborn, either falls or remain the same. This is the case, for example, if social welfare is defined as classical (Bentham) utilitarianism, a weighted sum of the welfare of all present and future individuals. The result also holds for versions of classical utilitarianism that are consistent with the Barro-Becker concept of diminishing altruism. An interpretation of our results is that the positive effect on welfare of fertility restrictions, namely higher average consumption, is dominated by the negative effect of a smaller dynasty size.

If social welfare is defined as average (Mills) utilitarianism rather than classical utilitarianism, social welfare could increase even if the welfare of all individuals falls if population falls even more. In this case, the net effect of fertility restrictions on social welfare depends on the relative strength of two opposite forces. On the one hand, distributions of abilities and income improve for all periods. On the other hand, welfare of all individuals fall. We present two analytical cases in which the later force dominates and hence social welfare defined as average welfare falls not only in present value but also for all periods. Our quantitative exercises also show that these results hold more generally.

The negative impact of fertility policies on individuals' welfare applies not only to policies aiming at reducing fertilities, but also to policies compelling individuals to increase their fertilities. An example of such policy is the seemingly official Chinese policy of stigmatizing unwed women older than 27 as "leftover women".[25]

We also study the effects on steady state social welfare of taxes or subsidies seeking to increase or reduce the cost of raising children using a calibrated version of the model. The government is required to run a balanced budget. The results suggest steady state average social welfare could be increased but only by a very small tax, say one less than 2% the time cost of raising every child. Out of this range, neither tax nor subsidy would improve social welfare. The consequence on individual welfare varies across ability types and is determined by the magnitude of the tax/subsidy.

In addition to the papers already mentioned, our chapter is related to Alvarez (1999). He studies an economy with idiosyncratic shocks, incomplete markets and endogenous fertility choices by altruistic parents but does not study the high fertility of the poor nor the consequences of fertility policies. Our incomplete markets economy is a version of his model, one with non-negative bequest constraints. In equilibrium no individual leaves positive bequests. This is a stronger degree of market incompleteness than that in Alvarez. Similar degree of market incompleteness is exploited by Krusell et al. (2011) to obtain closed form solution for asset prices in a Huggett (1993) model. Similarly, we are able to derive various closed form solutions and provide analytical proofs of the welfare effects of various fertility policies. Our chapter is also related to Hosseini et al. (2013) who study a related problem using an optimal contracting approach, and to Sommer (2013) who studies fertility in the presence of idiosyncratic shocks but parents are not dynastically altruistic as in Barro and Becker.

There is a related literature that studies fertility policies in general equilibrium. A recent example is Liao (2013) who studies the *One Child Policy* using a calibrated deterministic dynastic altruism model with two types of individuals, skilled and unskilled, in the spirit of Doepke (2004). Although Liao's model can generate fertility differentials, Doepke (2004) documents that this channel alone is relatively weak. Part of

[25]Fincher, Leta Hong (12 October 2012). "OP-ED CONTRIBUTOR; China's 'Leftover' Women". The New York Times. Retrieved 29 March 2014.

the issue is that the model only generates upward mobility in equilibrium. Our model, in contrast, generates significant upward and downward mobility that can lead to significant fertility differentials. The mechanisms are different and therefore complementary. We are also able to derive sharp analytical results. For example, we prove that fertility reducing policies, like the one child policy, decrease every individual's welfare for sure while Liao's calibrated result suggests it's true for almost all generations but not all.[26]

The rest of the chapter is organized as follows. Section 5.2 analyzes a simple deterministic model of fertility. We show that steady state fertility is higher in incomplete markets economies than in complete markets economy when children are a net financial cost to parents. Section 5.3 introduces idiosyncratic shocks into the model. We characterize the optimal fertility as well as its relationship with earning abilities. Section 5.4 studies the effect of fertility restriction policies on individual and social welfare. Numerical simulations, policy experiments and robustness checks are performed in section 5.5. Section 5.6 concludes. Proofs are in the Appendix.

5.2 A deterministic model

It is convenient to consider first a deterministic version of the model. We use the model to show analytically under what conditions fertility may be higher when markets are incomplete. A similar result is obtained for the full model but using numerical simulations. Schoonbroodt and Tertilt (2014) have shown that, under certain assumptions, incomplete markets models can result in inefficiently low fertility. In those cases, policies directed to promote higher fertility may be welfare enhancing. We show that, under different assumptions, incomplete markets models can result in high, not low, fertility relative to the complete markets counterpart. If that is the case then policies limiting fertility could, in principle, be welfare improving. As we later show, this is not the case. Restricting fertility in incomplete markets models, even if fertility is inefficiently high, is generally welfare detrimental.

[26]Another related paper is Moav (2005). In his model, individuals exhibit warm-glow altruism, and comparative advantage in the education of children explains differential fertility rates among rich and poor individuals as well as the persistence of poverty. He does not study fertility policies and social mobility does not occur in equilibrium.

The key assumption explaining why fertility could be lower or higher when markets are incomplete is whether children are a net financial cost or a net financial gain to parents. When children are a net financial gain, as in Schoonbroodt and Tertilt (2014) or Cordoba and Ripoll (2014), then market incompleteness reduces fertility. But when children are a net cost, as in Barro and Becker (1989) and Becker and Barro (1988), then market incompleteness generates increases *steady state* fertility. Short term fertility, on the other hand, in particular the fertility of the first generation, is lower under market incompleteness regardless of whether children are a net financial cost or benefit.

The deterministic models of this section do not generate a negative relationship between fertility and income. Therefore, they do not help to rationalize the negative fertility-income relationship nor the high fertility of the poor. The model in the next section, with idiosyncratic shocks, can generate a negative relationship between fertility and income as well as inefficiently high fertility rates.

5.2.1 Complete markets

Individuals live for two periods, one as a child and one as an adult. Children do not consume. The lifetime utility of an adult individual, or just an individual, at time t is of the Barro-Becker type

$$u\left(c_t\right)+\Phi\left(n_t\right)U_{t+1},\ t=0,1,2,\cdots,$$

where $u\left(c\right)=\frac{c^{1-\sigma}}{1-\sigma}, \sigma\in\left(0,1\right)$, is the utility from consumption, n is the number of children and $\Phi\left(n\right)=\beta n^{1-\varepsilon}$, satisfying $\epsilon\in\left(0,\sigma\right)$, $\Phi\left(\bar{n}\right)<1$, and $n\leq\bar{n}$. $\Phi\left(n\right)$ is the weight that parents attach to the welfare of their n children and U_{t+1} is the welfare of an individual at $t+1$. The upper limit on ϵ is needed for an interior solution of fertility, as discussed in Barro and Becker (1988). The upper limit on $\Phi\left(\overline{n}\right)$ is required to guarantee bounded utility.

Individuals are endowed with one unit of time that is used either to work or to raise children. An adult's labor supply is $l_t=1-\lambda n_t$, where λ is the time cost of raising a child. In additional to labor income, individuals receive transfers, or bequest, from their parents in the amount b_t. Let w be the wage rate per unit of labor supply and r be the interest rate at time. An individual's budget constraint is given by

$$w\left(1-\lambda n_t\right)+\left(1+r\right)b_t \geq c_t + n_t b_{t+1}.$$

The maximum number of children, $\overline{n}$, needs to satisfy two restrictions. First, $\overline{n} \leq 1/\lambda$ is required so that parents labor supply, $1-\lambda n_t$, is nonnegative. Second, $\Phi\left(\overline{n}\right) < 1$ implies $\overline{n} < \beta^{\frac{1}{\varepsilon-1}}$. These restrictions are satisfied if $\lambda^{1-\varepsilon} > \beta$. A final assumption is required so that maximum fertility is not always optimal: $\beta^{\frac{1}{\varepsilon}}\left(1+r\right)^{\frac{1}{\varepsilon}} < \overline{n}$.

Barro and Becker (1989) characterized optimal consumption and fertility allocations, given prices and b_0, of the sequence version of the problem above under the assumption $\left(1+r\right)\lambda w > w$. This assumption requires children to be a net financial burden to parents. The following proposition characterizes optimal allocations.

Proposition 1 *If $\left(1+r\right)\lambda w > w$ then the optimal solutions for $t \geq 1$ are given by*

$$c_t^C = \frac{1-\sigma}{\sigma-\varepsilon}\left[\left(1+r\right)\lambda w - w\right]$$

and

$$n_t^C = \beta^{\frac{1}{\varepsilon}}\left(1+r\right)^{\frac{1}{\varepsilon}}.$$

If $\left(1+r\right)\lambda w < w$ the optimal solutions for $t \geq 1$ satisfy $n_t = \overline{n}$ and $\frac{c_{t+1}}{c_t} = \left(\beta\overline{n}^{-\varepsilon}(1+r)\right)^{1/\sigma}$.

The first part of the Proposition was proven by Barro and Becker (1989) while the second part is proven in the Appendix. According to the proposition, if $\left(1+r\right)\lambda w > w$ then there is no transitional dynamics after the first period, consumption is proportional to the net cost of raising a child, and fertility is proportional to the interest rate but independent of wage income. However, if $w > \left(1+r\right)\lambda w$ then maximum fertility is optimal and the economy is always in transition, except if parameters are such that $\beta\overline{n}^{-\varepsilon}(1+r) = 1$. Maximum fertility is optimal because children earn enough income to compensate their parents for the costs of raising them and such compensation scheme is possible when markets are complete.

5.2.2 Incomplete markets

Consider now an incomplete markets case. Specifically, suppose bequests are constrained to be larger than certain amount, $\underline{b}$. This is a natural restriction because parents are legally unable to leave debts to their children. The first order condition for optimal bequests can be written as a standard Euler equation but across generations:

$$n_t u'(c_t) \geq \Phi(n_t)\, u'(c_{t+1})(1+r) \text{ with equality if } b_{t+1} > \underline{b}. \tag{47}$$

The left hand side of this equation is the cost of endowing n children with additional db bequests per-child while the right hand side is the benefit. Denote n^I the steady state fertility in the incomplete markets model. The following proposition states that fertility is higher in the incomplete markets model when children are a net cost to parents.

Proposition 2 *Suppose $(1+r)\lambda w > w$. Then $n^I \geq n^C$.*

P roof. *In steady state, equation (47) simplifies to $1 \geq \beta(1+r)\left(n^I\right)^{-\varepsilon}$ or*

$$n^I \geq \beta^{\frac{1}{\varepsilon}}(1+r)^{\frac{1}{\varepsilon}} = n^C.$$

■

The intuition for this key result is the following. When bequest constraints bind, early generations cannot extract as much resources from later generations. Since children are normal goods, this redistribution of resources from early to later generations reduces fertility at early times but makes future generations effectively richer and therefore increases fertility at later times. The assumption $(1+r)\,\lambda w > w$ is important because in that case n^C is the relevant fertility when markets are complete. Otherwise, maximum fertility is optimal in the complete markets case and therefore larger or equal than the incomplete markets case.

The results above show that steady fertility may be larger when markets are incomplete than when they are complete. In those instances policies seeking to reduce fertility may be welfare enhancing. The findings also put in context of the results of Schoonbroodt and Tertilt (2014). Their study is related with cases where children are a financial gain to parents and therefore market incompleteness generates too little fertility. It is also the case that even when children are a net financial cost, fertility by early generations may be inefficiently low. But over the longer term, high fertility arises when markets are incomplete and children are a net financial cost.

Whether children are a financial benefit or cost is an open question. Cordoba and Ripoll (2014) find that children in the U.S. are likely a net financial benefit. Lee (2000) estimates intergenerational transfers for different societies and finds that lifetime transfers run from children to parents in relative rich societies, and from parents to children in relative poor societies.

The results of this section take prices as given. In that sense, they correspond to either partial equilibrium or to a small open economy. The Appendix shows that these results can also be obtained in general equilibrium. A limitation of the models so far is that they are silent about why fertility rates are larger among poorer individuals, a key motivating for many family planning policies. The next section shows that an extension of the model that allows for idiosyncratic shocks can explain this regularity.

5.3 Idiosyncratic shocks

This section introduces idiosyncratic shocks into the model of the previous section. We consider complete markets and incomplete markets arrangements. Models with idiosyncratic risk are central in modern macroeconomics, particularly when studying issues of inequality and redistribution but, with the exception of Alvarez (1999), there exists no dynastic altruistic models with endogenous fertility. Given the focus of family planning policies on poverty and inequality reduction, it is natural to study those issues within a standard model where inequality is ultimately the result of idiosyncratic risk.

The remaining of the chapter derives a series of novel results for what can be called Bewley models with endogenous fertility. A key contribution is to show that a negative relationship between fertility and income arises naturally both in the complete and incomplete markets versions of the models. We also show that the complete markets model has some counterfactual predictions, and focus the rest of the discussion in the incomplete markets version.

5.3.1 Environment

Assume there is not aggregate uncertainty, goods are perishable, all information is public, and markets open every period.

Evolution of abilities Consider an endowment economy populated by a large number of dynastic altruistic individuals who live for two periods, one as a child and one as an adult. Children do not consume. Individuals differ in their labor endowments, or earning abilities. Let $\Omega \equiv \{\omega_1, \omega_2, ..., \omega_K\}$ be the set of possible earning abilities, where $0 < \omega_1 < ... < \omega_K$. Earning abilities are drawn at the beginning of the adult life from the Markov chain $M(\omega', \omega) = \Pr(\omega_{t+1} = \omega' | \omega_t = \omega)$ where ω_t is the ability of the parent and ω_{t+1} is the ability of the child. We use the following assumption, due to Chu and Koo (1990), to guarantee intergenerational persistence of abilities in the first order stochastic sense.

Assumption 1 Conditional Stochastic Monotonicity (CSM):

$$\frac{\sum_{i=1}^{I} M_{i1}}{\sum_{j=1}^{J} M_{j1}} \geq \frac{\sum_{i=1}^{I} M_{i2}}{\sum_{j=1}^{J} M_{j2}} \geq ... \geq \frac{\sum_{i=1}^{I} M_{iK}}{\sum_{j=1}^{J} M_{jK}},\ 1 \leq I \leq J \leq K.$$

Assumption 1 means that if a low ability kid and a high ability kid both fall into one of the lowest ability classes, it is more likely that the kid born by a low ability parent be endowed with a lower ability than the kid born by a high ability parent. CSM implies first order stochastic dominance. To see this notice that when $J = K$, the condition becomes:

$$\sum_{i=1}^{I} M_{i1} \geq \sum_{i=1}^{I} M_{i2} \geq ... \geq \sum_{i=1}^{I} M_{iK}, 1 \leq I \leq K.$$

Two examples of Markov chains satisfying Assumption 1 are an i.i.d. process and quasi-diagonal matrices of the form:

$$M = \begin{bmatrix} a+b & c & 0 & 0 & .. & 0 & 0 \\ a & b & c & 0 & .. & 0 & 0 \\ 0 & a & b & c & .. & 0 & 0 \\ .. & .. & .. & .. & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 & a & b+c \end{bmatrix}.$$

where $(a, b, c) \gg 0$, $a + b + c = 1$ and $b > 0.5$.

Assume further that M has a unique invariant distribution, μ, satisfying:

$$\mu(\omega_j) = \sum_{\omega_i \in \Omega} \mu(\omega_i) M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega. \tag{48}$$

Let $\omega^t = [\omega_0, \omega_1, ..., \omega_t] \in \Omega^{t+1}$ denote a particular history of ability realizations up to time t. We call ω^t a family history or a family branch. With some abuse of notation, let $\omega^t = \left(\omega^{t-1}, \omega_t\right)$ for $t > 1$ and $\omega^0 = \omega_0$.

Individual resources The technology of production is linear in ability: one unit of labor produces one unit of perishable output. The total resources available to an individual of ability ω_t at time t are labor income

$\omega_t(1-\lambda n_t)$ and transfers from their parents. Let $c_t(\omega^t)$ and $n_t(\omega^t)$ denote consumption and fertility of an individual whose family history is ω^t. Let $b_t(\omega^t)$ denote transfers, or bequests, received from parents.[27] Resources are used to consume and to leave transfers to children. Insurance market exists as parents can leave bequests contingent on the ability of their children. Let $q_t(\omega^t, \omega_i)$ be the time$-t$ price of a contingent asset $b_{t+1}(\omega^t, \omega_i)$ that delivers one unit of consumption at $t+1$ contingent on $\omega_{t+1} = \omega_i$ given family history ω^t. The budget constraint of an individual at time t with history ω^t is:

$$c_t(\omega^t) + n_t(\omega^t)\sum_{i=1}^{K} q_t(\omega^t, \omega_i) b_{t+1}(\omega^t, \omega_i) \leq \omega_t\left(1-\lambda n_t(\omega^t)\right) + b_t(\omega^t). \tag{49}$$

Aggregate resources Let $N_0(\omega^0) \equiv 1$. $N_t(\omega^t) = N_{t-1}(\omega^{t-1}) n_{t-1}(\omega^{t-1}) M(\omega_t, \omega_{t-1})$ is the population size at time t of a particular branch of the family tree, the one described by ω^t. Assume goods are perishable, total consumption cannot exceed the total endowment of the economy at any point in time:

$$\sum_{\omega^t \in \Omega^{t+1}} N_t(\omega^t)\left[\omega_t\left(1-\lambda n_t(\omega^t)\right) - c_t(\omega^t)\right] \geq 0 \text{ for all } t.$$

Preferences The lifetime utility of an individual born at time t is of the Barro-Becker expected-utility type:

$$u\left(c_t(\omega^t)\right) + \Phi\left(n_t(\omega^t)\right) E_t\left[U_{t+1}(\omega^{t+1}) | \omega^t\right], t = 0, 1, 2, \cdots \tag{50}$$

where $u(c)$ and $\Phi(n)$ satisfy the same properties of the previous section, E_t is the mathematical expectation operator conditional on the information up to time t and $E_t U_{t+1}$ is the expected utility of individuals in the next generation.

Market arrangements All markets are competitive and insurance prices are actuarially fair. Let $p_t = \sum_i q_t(\omega^t, \omega_i)$ be the price of a riskless bond. Under the assumption of no aggregate uncertainty, $p_t = p$.

[27] We denote b_t bequest for short. This bequest includes all net transfers from a parent to each of his/her adult child. In particular, it includes inter vivos transfers.

The riskless interest rate is then given by $1+r=1/p$. Furthermore, actuarially fair prices must satisfy the arbitrage condition

$$q_t(\omega^t,\omega_i)=pM\left(\omega_i,\omega_t\right). \tag{51}$$

so that $q_t(\omega^t,\omega_i)=q(\omega_t,\omega_i)$.

We study two different market arrangements. We first consider a complete markets economy in which families are only subject to natural debt limits, limits that prevent Ponzi schemes but that do not bind in equilibrium. In particular, parents can leave negative bequests to their children in a complete markets economy. We also consider an incomplete markets economy in which parents cannot leave negative bequests to their children. In particular, in this case we assume:

$$b_{t+1}\left(\omega^t,\omega_{t+1}\right)\geq 0 \text{ for all } \omega^t\in\Omega^{t+1},\omega_{t+1}\in\Omega \text{ and all } t\geq 0. \tag{52}$$

5.3.2 Complete markets

Optimal consumption and fertility Consider first the complete markets economy. We find convenient to write the problem recursively. Let $V_t\left(b_t,\omega^t\right)$ be the maximum lifetime utility of a time$-t$ adult with family history ω^t. Then

$$V_t\left(b_t,\omega^t\right)=\max_{n_t\in[0,\overline{n}],b_{t+1}(\omega^t,\omega_i)}u\left(c_t\right)+\beta n_t^{1-\varepsilon}E_t\left[V_{t+1}\left(b_{t+1}\left(\omega^{t+1}\right),\omega^{t+1}\right)|\omega_t\right]$$

subject to

$$c_t+n_t\sum_{i=1}^{K}q(\omega^t,\omega_i)b_{t+1}(\omega^t,\omega_i)\leq\omega_t\left(1-\lambda n_t\right)+b_t \text{ for all } t \text{ and } \left(\omega^t,\omega_i\right). \tag{53}$$

The following Proposition generalizes Proposition 1 for the case of stochastic earning abilities and children being a net financial costs to parents. Proofs are in the Appendix.

Proposition 3 *Suppose* $(1+r)\lambda\omega_t > E(\omega_{t+1}|\omega_t)$ *for all* ω_t*. Then the optimal interior solutions for consumption and fertility are given by:*

$$c_{t+1} = c(\omega_t) = \frac{1-\sigma}{\sigma-\varepsilon}\left[(1+r)\lambda\omega_t - E(\omega_{t+1}|\omega_t)\right] \text{ for all } t \geq 0 \tag{54}$$

and

$$n_t = n(\omega_{t-1},\omega_t) = \beta^{\frac{1}{\varepsilon}}(1+r)^{\frac{1}{\varepsilon}}\left(\frac{(1+r)\lambda\omega_{t-1} - E(\omega_t|\omega_{t-1})}{(1+r)\lambda\omega_t - E(\omega_{t+1}|\omega_t)}\right)^{\frac{\sigma}{\varepsilon}} \text{ for all } t \geq 1. \tag{55}$$

If $(1+r)\lambda\omega_t > E(\omega_{t+1}|\omega_t)$ *for some* t*, then the optimal fertility at time* t *is* $\bar{n}$*.*

According to the proposition, consumption of all generations, except the first one, is proportional to the net expected financial cost of raising a child, $(1+r_t)\lambda\omega_t - E(\omega_{t+1}|\omega_t)$. Equation (54) implies that individual consumption is a random variable, one that increases with the ability of the parent and falls with the expected ability of the individual. The randomness of consumption is perhaps surprising because with exogenous fertility consumption it is not random but constant. In that case, individuals are able to insure all idiosyncratic risk, and consumption obeys the permanent income hypothesis.

But consumption in the endogenous fertility case obeys a completely different logic. In contrast to the exogenous fertility case, the existence of a next period consumer, the child, is not guaranteed nor costless. When a life is costly to create, optimal consumption becomes an utilization rate and the higher the net cost of creating a new variety, a new child, the higher the utilization rate. Furthermore, if costs and/or benefits of raising a child is random so is consumption. The randomness of consumption also implies that there exists social mobility in the endogenous-fertility complete-markets model. Equation (54) also implies that all children of the same parent has the same consumption, and that the actual ability of an individual does not affects his/her own consumption. These two predictions are particularly problematic. Evidence shows that there exists significant consumption inequality among siblings and that consumption increases with earnings (e.g., Mulligan 1997 and Gaviria 2002).

The solution for optimal fertility described by equation (55) is a restatement of the intergenerational Euler equation, $u'(c_t) = \beta n_t^{-\varepsilon} (1 + r_{t+1}) E_t u'(c_{t+1})$, where $\beta n_t^{-\varepsilon}$ is the average degree of altruism. Fertility, rather than consumption, now plays the key role of smoothing family welfare. If parental consumption is high while children consumption is low, then high fertility is required to smooth family utility intertemporally. More precisely, equation (55) shows that fertility depends negatively on the ability of the parent, and positively on the ability of the grandparent. The model thus provides a novel explanation for the negative fertility income relationship, an explanation that relies purely on the random nature of abilities. Conditional on grandparents' abilities, the model predicts that high ability parents have fewer children because children are more costly to those parents.

Notice that optimal fertility is the lowest for high ability parents with low ability grandparents. This is due to the fact that, with complete markets, unlucky grandparents can borrow against the income of their high ability children which reduces the wealth of those children, and therefore their fertility. As we show below, when markets are incomplete unlucky grandparents cannot borrow against their children's income and therefore the fertility of high ability parents with low ability grandparents is higher under incomplete markets.

Corner solutions are not only possible but likely in complete markets models with idiosyncratic shocks. First, if children are a net financial benefit to parents, then maximum fertility is optimal, $n_t = \overline{n}$. Second, equation (55) can easily imply $n_t > \overline{n}$ for parents whose ability is far below that of their grandparents. As we document below, using a calibrated version of the model, this feature of the complete markets model is problematic because it leads to unrealistic high fertility rates for many families.

General equilibrium Given optimal fertility rules $n(\omega_{t-1}, \omega_t)$, let $N_{t+1}(\omega_t, \omega_{t+1})$ be the population at time $t+1$ with own type ω_{t+1} and parental type ω_t. Assuming a law of large number holds, aggregate population evolves according to:

$$N_{t+1}\left(\omega_t,\omega_{t+1}\right)=\sum_{\omega_{t-1}}N_t\left(\omega_{t-1},\omega_t\right)n\left(\omega_{t-1},\omega_t\right)M\left(\omega_{t+1},\omega_t\right).$$

Let $N_t\equiv\sum_{\omega_{t-1}}\sum_{\omega_t}N_t\left(\omega_{t-1},\omega_t\right)$ be total population at time, $\pi_t\left(\omega_{t-1},\omega_t\right)\equiv\frac{N_t(\omega_{t-1},\omega_t)}{N_t}$ be the fraction of population of type ω_t and parental type ω_{t-1},and $1+g_t\equiv\frac{N_{t+1}}{N_t}$ the gross growth rate of population. Then, the previous expression can be written as:

$$\pi_{t+1}\left(\omega_t,\omega_{t+1}\right)=\frac{1}{1+g_t}\sum_{\omega_{t-1}}\pi_t\left(\omega_{t-1},\omega_t\right)n\left(\omega_{t-1},\omega_t\right)M\left(\omega_{t+1},\omega_t\right)\tag{56}$$

Adding across $\left(\omega_t,\omega_{t+1}\right)$ pairs, it can be seen that $1+g_t=\sum_{\omega_t}\sum_{\omega_{t-1}}\pi_t\left(\omega_{t-1},\omega_t\right)n\left(\omega_{t-1},\omega_t\right)$. The stationary distribution $\pi\left(\omega_{t-1},\omega_t\right)$ is the invariant distribution that solves (56).

Given that production is perishable, the equilibrium interest rate has to equate aggregate consumption to aggregate endowments. Using the stationary distribution, the equilibrium condition can be expressed as

$$\sum_{\omega'}\sum_{\omega}c\left(\omega\right)\pi\left(\omega,\omega'\right)=\sum_{\omega'}\sum_{\omega}\left[1-\lambda n\left(\omega,\omega'\right)\right]\omega'\pi\left(\omega,\omega'\right).$$

This equation pins down the steady state interest rate r since both $c\left(\omega\right)$ and $n\left(\omega,\omega'\right)$ are functions of the interest rate r.

5.3.3 Incomplete markets

Recursive formulation Consider next the case of bequests constraints of the type (52). Assume output is perishable, aggregate consumption must equal aggregate production. Alternatively, aggregate savings are zero. Moreover, savings are equal to the total amount of bequests left by parents. Since all bequests are non-negative then aggregate savings are zero if and only if all bequests are zero. Therefore, in any equilibrium with bequest constraints the budget constraint (74) simplifies to:

$$c_t\left(\omega^t\right) \leq \omega_t\left(1-\lambda n_t\left(\omega^t\right)\right) \text{ for all } \omega^t \in \Omega^{t+1} \text{ and all } t \geq 1. \tag{57}$$

This is balanced budget constraint for every period and state. The lack of intergenerational transfers significantly simplifies the problem. To study the incomplete markets problem, we first show that the principle of optimality holds. Standard arguments cannot be used because the discount factor is endogenous. Alvarez (1999) shows that the principle of optimality holds for a dynastic version of this problem, while we show that it holds for the household version of the problem.[28]

Let $\hat{N}_0\left(\omega^{-1}\right)=1$, $\hat{N}_t\left(\omega^{t-1}\right)=\prod_{j=0}^{t-1} n_j\left(\omega^j\right)$ for $t \geq 1$. The problem can be written in sequential form, using (75) and (77) recursively, as follows:

$$V_0^*\left(\omega_0\right)=\sup_{\left\{\hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right)\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \hat{N}_t\left(\omega^{t-1}\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda \frac{\hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right)}{\hat{N}_t\left(\omega^{t-1}\right)}\right)\right) \tag{58}$$

subject to

$$0 \leq \hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right) \leq \hat{N}_t\left(\omega^{t-1}\right)/\lambda \text{ for all } \omega^{t-1} \in \Omega^t, \omega_t \in \Omega \text{ and } t \geq 0.$$

The recursive formulation of this problem is:

$$U(\omega)=\max_{n \in[0,\bar{n}]} u(\omega(1-\lambda n))+\beta n^{1-\varepsilon} E\left[U\left(\omega'\right)|\omega\right]. \tag{59}$$

The next proposition states that the principle of optimality holds for this problem.

[28] The analogous dynastic problem is:

$$U(N,\omega)=\max_{N' \in[0,\bar{n}]} u\left(\omega-\lambda N'/N\right) N^{1-\epsilon}+\beta E\left[U\left(N',\omega'\right)|\omega\right].$$

In this problem the number of family members is a state variable, N, all members have the same ability, ω, and make the same choices. The household problem does not impose these constraints. On problem is simpler than Alvarez's because there are no savings. But the problem is still dynamic.

Proposition 4 *The functional equation (79) has a unique solution,* $U(\omega)$*. Moreover* $U(\omega) = V_0^*(\omega)$ *for* $\omega \in \Omega$*.*

Optimal fertility The optimality condition for fertility choices is

$$\lambda \omega u'\left(\left(1-\lambda n^*\right)\omega\right) = \beta\left(1-\varepsilon\right) n^{*-\varepsilon} E\left[U\left(\omega'\right)|\omega\right]. \tag{60}$$

The left hand side of (80) is the marginal cost of an additional child while the right hand side is the marginal benefit. The marginal cost is the product of the opportunity cost of raising a child, $\lambda\omega$, times the marginal utility of consumption. The marginal benefit to the parent is the discounted expected welfare of a child, $\beta E\left[U\left(\omega'\right)|\omega\right]$, times the parental weight associated to the last child, $\left(1-\varepsilon\right)n(\omega)^{-\varepsilon}$. Let $n^* = n(\omega)$ be the optimal fertility rule and $c^* = c\left(\omega\right) \equiv \left(1-\lambda n(\omega)\right)\omega$ the optimal consumption rule.

In contrast to the complete markets case, corner solutions are not optimal in the incomplete markets case under the assumed functional forms. Having no children is not optimal because the marginal benefit of a child is infinite while the marginal cost is finite. In particular, $E\left[U\left(\omega'\right)|\omega\right] > 0$ for all ω while $\lim_{n\to 0} n^{-\epsilon} = \infty$. Having the maximum number of children is also sub-optimal because the marginal cost is infinite when parental consumption is zero, while the marginal benefit is finite.

Consider now the relationship between fertility, n^*, and parental earning ability, ω. According to (80), both marginal benefits (MB) and marginal costs (MC) are affected by abilities. MB increase with ω because of the postulated intergenerational persistence of abilities: high ability parents are more likely to have high ability children. Regarding MC, there are two effects. On one hand, MC tends to rise with ω because higher ability increases the opportunity cost of the parental time required to raise children. On the other hand, MC tends to fall because the larger the ability the smaller the marginal utility of consumption. Given that $\sigma \in (0,1)$, the first effect dominates the second one so MC increases with ω. The need for $\sigma \in (0,1)$ suggests a tension between the theory and the empirics since estimates of the intertemporal elasticity of substitution are typically lower than 1. But the correct interpretation of $1/\sigma$ is an intergenerational elasticity of substitution (IGES), one controlling inter-personal consumption smoothing rather than intra-personal consumption smoothing (Cordoba and Ripoll 2011).

Since both MB and MC increase with ω, it is not clear in principle whether fertility increases or decreases with ability. The following proposition provides the answer in three cases: i.i.d. abilities across generations, perfect intergenerational persistence of abilities without uncertainty and random walk (log) abilities.[29]

In order to analyze the case of perfect persistence of ability, it is instructive to write the first order condition in an alternative way. Using equation (80) to express (79) as:

$$U_t = u\left(c\left(\omega\right)\right) + \frac{1}{1-\varepsilon} n\left(\omega\right)\lambda\omega u'\left(c\left(\omega\right)\right) \tag{61}$$

Then use (81) to rewrite (80) as:

$$u'\left(c\left(\omega\right)\right) n\left(\omega\right)^{\varepsilon} = \beta E\left[u'\left(c\left(\omega'\right)\right)\frac{\omega'}{\omega}\left(\frac{1}{\lambda} + \frac{\sigma-\varepsilon}{1-\sigma}\left(\frac{1}{\lambda} - n\left(\omega'\right)\right)\right)\middle|\omega\right] \tag{62}$$

This equation is useful because it only requires marginal utilities, rather than total utility as in equation (80), and corresponds to the Euler equation of the problem describing the optimal consumption rule. Although savings are zero in equilibrium, fertility allows individuals to smooth consumption across generations.[30]

[29] Although a random walk does not satisfy some of the assumptions above, it helps to develop some intuition.

[30] Equation (80) can also be written in the form of a more traditional Euler equation. Let $1+r'$ be the gross return of "investing" in a child. It is given by $1+r' \equiv \frac{U(\omega')/u'(c')}{\lambda\omega}$. In this expression, $U(\omega')/u'(c')$ is the value of a new life, in terms of goods, while $\lambda\omega$ is the cost of creating a new individual. Then (80) can be written as:

$$u'(c) = \beta\left(1-\epsilon\right) n^{*-\epsilon} E\left[u'(c')\left(1+r'\right)|\omega\right]. \tag{63}$$

This is an Euler equation with a discount factor $\beta\left(1-\epsilon\right) n^{*-\epsilon}$. It suggests that optimal fertility choices are similar to saving decisions and that children are like an asset, as pointed out by Alvarez (1999). However, two important differences with the traditional Euler Equation are that the individual controls the discount factor and the gross return.

Proposition 5 ***Persistence and the fertility-ability relationship.*** *(i) Fertility decreases with ability if abilities are i.i.d. across generations. In this case* $n(\omega)$ *satisfies the equation* $\frac{n(\omega)^{\varepsilon}}{(1-\lambda n(\omega))^{\sigma}} = A\omega^{\sigma-1}$ *where* A *is a constant. Furthermore, fertility is independent of ability in one of the following two cases: (ii)* M *is the identity matrix (abilities are perfectly persistent and deterministic); or (iii)* $\ln \omega_t = \ln \omega_{t-1} + \varepsilon_t$ *where* $\varepsilon_t \sim N(0, \sigma_{\varepsilon}^2)$.

According to Proposition 5, fertility decreases with ability when abilities are i.i.d. The intuition is that without intergenerational persistence, a higher ability of the parent only affects her or his marginal cost but not marginal benefit as $E[U(\omega')|\omega] = E[U(\omega')]$ for all $\omega \in \Omega$. On the other extreme, fertility is independent of ability when abilities are perfectly persistent across generations (cases ii and iii). This is because in those cases both the marginal cost and marginal benefit are proportional to $\omega^{1-\sigma}$. Given that fertility becomes independent of ability only in the extreme case of perfect persistent, it is natural to conjecture that fertility decreases with ability when persistence is less than perfect. We were able to confirm this conjecture numerically but analytical solutions were not obtained.

General equilibrium Denote $N_t(\omega)$ the mass of population with ability ω, $N_t \equiv \sum_{\omega \in \Omega} N_t(\omega)$ total population, $\pi_t(\omega) = \frac{N_t(\omega)}{N_t}$ the fraction of population with ability $\omega \in \Omega$ at time t, and $1 + g_t = \sum_{\omega \in \Omega} n(\omega) \pi_t(\omega)$ the gross growth rate of population. The initial population of different ability types $\{N_0(\omega_i)\}_{i=1}^K$ is given.

As in section 5.3.2.2., it can be shown that the law of motion for $\pi_t(\omega_i)$ satisfies

$$\pi_{t+1}(\omega_j) = \frac{1}{1+g_t} \sum_{\omega_i \in \Omega} n(\omega_i) \pi_t(\omega_i) M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega \tag{64}$$

Let $\pi^*(\omega) \equiv \lim_{t \to \infty} \pi_t(\omega)$ represents the limit distribution of abilities where $\omega \in \Omega = \{\omega_i\}_{i=1}^N$. Given the Assumption 1 and that $n(\omega_1) \geq ... \geq n(K) > 0$, a unique limit distribution exists (see Chu and Koo 1990).

We now provide some analytical results characterizing the distribution π_t, the limit distribution π^* and their relationship to fertility for some special cases. Calibrated results are provided in section 5.5. The following proposition provides a simple but important benchmark. The first part states that when fertility is identical across types then $\pi^* = \mu$: the endogenous limit distribution of abilities equals the exogenous invariant distribution of abilities described by equation (48). In other words, the endogenous distribution of abilities just reflects the genetic distribution of abilities, what can be termed nature rather than nurture. The second part of the proposition shows that this results also holds when fertility rates are different across types but there is not persistence of abilities.

Proposition 6 ***When $\boldsymbol{\pi}^*$ equals $\boldsymbol{\mu}$***. *Suppose one of the following two conditions holds: (i) $n(\omega) = n$ for all $\omega \in \Omega$; or (ii) $M(\omega', \omega)$ is independent of ω for all $\omega' \in \Omega$. Then $\pi^*(\omega) = \mu(\omega)$ for all $\omega \in \Omega$. Moreover, if (ii) holds then $\pi_t(\omega) = M(\omega, :)$ for all $\omega \in \Omega$ and all $t \geq 1$.*

The following proposition uses propositions 5 and 6 to characterize fertility and the limiting ability distribution in cases of either no persistence or perfect persistence of abilities.

Proposition 7 ***Persistence, fertility and ability distribution***. *(i) If $M(\omega', \omega)$ is independent of ω then $n(\omega)$ decreases with ω and $\pi_t(\omega) = \mu(\omega) = M(\omega, :)$ for all $\omega \in \Omega$ and $t \geq 1$; (ii) if M is the identity matrix then $n(\omega) = n$ for all $\omega \in \Omega$ and $\pi_t(\omega) = \pi^*(\omega) = \pi_0(\omega)$ for all $\omega \in \Omega$ and all t; (iii) if $\ln \omega$ follows a Gaussian random walk then $n(\omega) = n$, and given ω_0 the variance of abilities diverges to ∞.*

In words, if children's abilities are independent of parental abilities, then fertility decreases with ability but the observed distribution of abilities is independent of fertility choices and determined by the Markov chain M from the second period on. Furthermore, with certainty and perfect intergenerational persistence of abilities the observed distribution of abilities in any period is identical to the initial distribution of abilities. Finally, if (log) abilities follow a random walk then there is not limit distribution of abilities since its variance goes to infinite.

As for intermediate case of some persistence, the following Proposition is an application of Chu and Koo's (CK, 1990) Theorem 2. It states that if the fertility of the poor, meaning the group with lowest income, is higher than the fertility of the rest of the population then π^* is different from μ, and moreover, μ dominates π^* in the first order stochastic sense.

Proposition 8 *Suppose M satisfies Assumption 1 and $n(\omega_1) > n(\omega_i) = n$ for all $i > 1$. Then $\sum_{i=1}^{I} \pi^*(\omega_i) > \sum_{i=1}^{I} \mu(\omega_i)$ for all $1 \leq I \leq K$.*

P roof. See Chu and Koo (1990, pp.1136). ■

5.4 Welfare analysis of fertility policies

We now consider simple types of fertility policies that directly set constraints on fertility choices. In particular, let $\underline{n}(\omega) \geq 0$ and $\overline{n}(\omega) \leq \overline{n}$ be the lower and upper bounds on fertility set by the policy. Bounds could potentially depend on individual abilities. The motivation for the upper bound is the one child policy. Although fertility policies are significantly more complex than simple bounds, the bounds capture a key essence of the intent of these policies which is to limit or, more recently, promote fertility.

5.4.1 Fertility policies and income

A key aspect of fertility policies such as family planning policies is their ability to increase the income of the poor and also, the average income of the economy. Since income equals $\omega(1 - \lambda n(\omega))$ then restricting fertility in fact increases individual income because it increases the effective labor supply. Furthermore, average earning abilities and average income are defined as:

$$E_t = \sum_{\omega \in \Omega} \omega \pi_t(\omega)\,; \; I_t = \sum_{\omega \in \Omega} \omega(1 - \lambda n(\omega)) \pi_t(\omega)\,.$$

These expressions shows that, in addition to increasing individual income, fertility policies also have the potential to increase average income if those policies also result in better distributions of abilities, $\pi_t(\omega)$, in the first order stochastic sense. In fact, a corollary of Proposition 6 (i) and Proposition 8, is that reducing the fertility of the poor to the same level as that of others results in a limit distribution that dominates the original distribution. More generally, CK show that if fertility decreases with income, M satisfies Assumption 1, and the initial distribution of incomes is at its steady state level, $\pi_0^*(\omega_i)$, then a reduction in the fertility of the poor results in a sequence of income distributions that first order stochastically dominate $\pi_0^*(\omega_i)$, that is, $\sum_{i=1}^{I} \pi_t(\omega_i) < \sum_{i=1}^{I} \pi_0^*(\omega_i)$ for all $1 \leq I \leq K$ and $t > 0$.

The positive effects of family planning policies on the income of the poor, average income and average ability are often used to provide support for these policies. We next show that, in spite of its positive effects on income, fertility policies are welfare detrimental in our incomplete markets economies.

5.4.2 Fertility policies and individual welfare

The indirect utility $U^r(\omega)$ of the problem with fertility constraints solves the Bellman equation:

$$U^r(\omega) = \max_{n \in [\underline{n}(\omega), \overline{n}(\omega)]} u((1 - \lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U^r(\omega') |\omega\right]. \tag{65}$$

Let $n^r(\omega)$ denotes the corresponding optimal fertility rule. The following proposition states that binding fertility restrictions in at least one state reduce the indirect utility, or welfare, of all individuals even those whose fertility is not directly affected. The proposition also states that fertility restrictions of any type (weakly) reduce the fertility of all individuals except perhaps those whose fertility rates are at or below the lower bound.

Proposition 9 *$U^r(\omega) \leq U(\omega)$ for all ω and it holds with strict inequality for all ω if $n(\hat{\omega}) > \overline{n}(\hat{\omega})$ or $n(\hat{\omega}) < \underline{n}(\hat{\omega})$ for at least one $\hat{\omega} \in \Omega$. Furthermore $n^r(\omega) = \underline{n}(\omega)$ if $n(\omega) \leq \underline{n}(\omega)$ and $n^r(\omega) \leq n(\omega)$*

otherwise. In particular, $n^r(\omega) < n(\omega)$ *for all* ω *if* $n(\omega) > \underline{n}(\omega)$ *for all* ω *and* $n(\hat{\omega}) > \overline{n}(\hat{\omega})$ *for at least one* $\hat{\omega} \in \Omega$.

Proposition 9 implies that policies such as the *One Child Policy*, which limits fertility of all individuals, or policies that compel individuals to increase their fertility such as the "leftover women" stigma in China, are detrimental to all individuals' welfare in our incomplete markets model.

Although fertility is possibly inefficiently high for some ability types and there might be potential room for policies to improve individual welfare, the proposition shows that policies restricting fertility choices do not help because it restricts individual's choices without providing any compensation. Furthermore, fertility restrictions that only affect a particular group result in lower welfare for all individuals because, regardless of current ability, there is a positive probability that a descendant of the dynasty will fall into the group directly affected in finite time. Given that welfare of every individual falls with fertility restrictions, the marginal benefits of having children also falls while the marginal cost remains the same. Thus fertility must fall for all types except perhaps for those who are constrained by the policy to increase their fertility.

5.4.3 Fertility policies and social welfare

Given that fertility policies reduce the welfare of all individuals, as shown in Proposition 9, it is natural to infer that social welfare should also fall. The answer, however, depends on how social welfare is defined and whether the policy reduces or increases population. In this section we focus on fertility policies that impose upper limits on fertility rates such as limiting the fertility of the poor or the *One Child Policy*. Other fertility policies like coercing the rich to have more children and fertility related taxes/subsidies involve more competing factors and are postponed to the quantitative exercise.

Total Utilitarian Social Welfare Classical (Bentham) utilitarianism defines social welfare as the total discounted welfare of all (born) individuals:[31]

$$W = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) N_t(\omega). \tag{66}$$

In this formulation $\beta_p(t) \geq 0$ is the weight the social planner assigns to generation t, $U(\omega)$ is the utility of an individual with ability ω, and $N_t(\omega)$ is the size of population of generation t endowed with ability ω. Since individuals are altruistic toward their descendants, $\beta_p(t) > 0$ means that the planner gives additional weight to generation t on top of what is implied by parental altruism. A particular case in which the planner weights only the original generation and therefore adopts its altruistic weights is the one with $\beta_p(0) = 1$ and $\beta_p(t) = 0$ for $t > 0$:

$$W_0 = \sum_{\omega \in \Omega} U(\omega) N_0(\omega) \tag{67}$$

According to Proposition 9, upper limits affecting the fertility of at least one ability group reduce fertility of all ability groups. Therefore, upper limits on fertility unequivocally reduce population of all ability groups at all times after time 0. Given that both population and individual welfare fall for all ability types, we are able to show that fertility limits unequivocally decrease social welfare if social welfare is of the classical or Benthamite utilitarian form.

Proposition 10 *Imposing upper limits on fertility choices reduces social welfare as defined by (66).*

An identical result is obtained if the planner exhibits positive but diminishing returns to population, say if $N_t(\omega)$ in expression (66) is replaced by $N_t(\omega)^{1-\varepsilon_p}$ where $\varepsilon_p \in (0,1)$. This formulation seems a natural extension of the Barro-Becker preferences for a planner. The result does not necessarily hold if we define social welfare as the average, or Mills, utilitarian form as discussed next.

[31] The results in this section are similar if the welfare of the unborn is explicitly considered as long as the unborn enjoy lower utility than the born.

Average Utilitarian Social Welfare Define average, or Mills, utilitarian social welfare as

$$\overline{W} = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) \pi_t(\omega) \tag{68}$$

This definition of welfare is analogous to (66) but uses population shares, $\pi_t(\omega)$, rather than population, $N_t(\omega)$ as weights. A particular case is one where the planner cares only about steady state welfare so that $\beta_p(t) = 0$ for all t and $\lim_{t \to \infty} \beta_p(t) = 1$, the steady state welfare function is given as

$$\overline{W}^* = \sum_{\omega \in \Omega} U(\omega) \pi^*(\omega). \tag{69}$$

This definition of social welfare is the one used by CK to argue in favor of family planning policies. It is also a commonly used criteria in social choice theory because it avoids Parfit's repugnant conclusion. Notice that average social welfare could increase even if the welfare of all individuals fall when fertility limits are enacted. The net effect depends on the relative strength of two potentially opposite forces: on the one hand individual welfare falls but on the other hand the distribution of abilities, π, may improve. CK assume that $U(\omega)$ is invariant to the policy in place and therefore the only consequence of the policy is to change π. In that case, Proposition 8 states that limiting the fertility of the poor improves the distribution of abilities, in a stochastic dominance sense, and therefore increases average welfare, given U. The following corollary summarizes CK main result.

Corollary 11 *Suppose social welfare is defined by (68) where $U(\omega)$ is a non-decreasing function of ability. Furthermore, suppose M satisfies CSM and fertilities are exogenously given and satisfy $n(\omega_1) > n(\omega_i) = n$ for all $1 < i \leq N$. Then (i) reducing the fertility of the lowest ability group increases social welfare; (ii) fertility policies that do not change the distribution of abilities do not change social welfare.*

This result by CK assumes, however, that U is invariant to the policy in place which is not true in our endogenous fertility model. As shown above, welfare of all individuals falls with the policy. Such reduction in individual welfare may be strong enough to offset the positive effects of the policy. The following proposition shows that in fact this is possible.

Proposition 12 *Suppose $M(\omega', \omega)$ is independent of ω for all $\omega' \in \Omega$. Then upper limits on fertility choices reduce social welfare as defined by (68).*

Proposition 12 relies on the earlier finding in Proposition 6 that, when abilities are i.i.d., the distribution of abilities among the population is independent of fertility choices and thus fully determined by the Markov matrix M. Therefore, in the i.i.d. case the effect of any fertility policy on social welfare, as defined by (68), is only determined by its effect on individual welfare, U.

The following proposition for a deterministic case of perfect persistence of abilities states that average utilitarian welfare unequivocally falls with "uniform" fertility restrictions such as the one child policy.

Proposition 13 *Suppose M is the identity matrix. Then a uniform fertility restriction $\bar{n}(\omega) = \bar{n}$ reduces social welfare as defined by (68).*

Proposition 13 provides another example in which fertility restriction do not affect π. Since in the deterministic case all ability groups have the same fertility choices and the fertility restriction affects all ability groups equally, it follows that $\pi_t = \pi_0$ for all t so that the effect of the policy on social welfare is only determined by the effect on individual welfare U.

Proposition 12 and 13 formally provide analytical examples in which social welfare falls when fertility is restricted even under the notion of average social welfare. We now turn to quantitative exercise to investigate more general cases in which abilities are correlated but less than perfect persistent across generations. The calibrated results suggest that fertility restricting policies decrease social welfare in general, even when social welfare is defined as average welfare.

5.5 Quantitative Exploration

We now explore some of the quantitative implications of the model. While the computation of the incomplete markets model is relatively simple, the computation of the complete markets model is not. The reason is that corner solutions easily arise for standard calibrations of the income process. In particular, equation (55) would easily result in fertility larger than the maximum if ω_{t-1} and ω_t are sufficiently different.

To avoid corner solutions we first consider an example that sets a relatively narrow range for the set of abilities. Although the example produces only limited inequality, it allows us to illustrate that fertility in the incomplete markets model may be inefficiently high relative to the complete markets version. The rest of the section uses a more realistic calibration of the income process but provides only results for the incomplete markets model.

5.5.1 Example

Suppose abilities follow a two-state Markov process characterized by $[\omega_H, \omega_L] = [0.85, 1.18]$ and $\Pr(\omega_H|\omega_H) = \Pr(\omega_L|\omega_L) = 0.9$, and $\Pr(\omega_H|\omega_L) = \Pr(\omega_L|\omega_H) = 0.1$. Assume the following parameters values: $\sigma = 0.5$, $\beta = 0.2$, $\varepsilon = 0.3$, and $\lambda = 0.3$. These values are similar to the ones calibrated in the next section. The risk free interest rate that clears the asset market in the complete markets model is 3.44. Table 5.1 reports steady state fertilities for the complete and incomplete markets cases according to formulas (55) and (80). Notice that fertility in the first environment is a function of both grandparents and parental abilities while in the second environment is only a function of parental ability.

Table 5.1 Fertility in Complete and Incomplete Markets

$n_t = n(\omega_{t-1}, \omega_t)$	$n(\omega_H, \omega_H)$	$n(\omega_H, \omega_L)$	$n(\omega_L, \omega_H)$	$n(\omega_L, \omega_L)$
Complete Markets	0.67	1.60	0.28	0.67
Incomplete Markets	0.58	0.69	0.58	0.69

Table 5.1 illustrates the result that, in presence of idiosyncratic shocks, steady state fertility can be inefficiently high when markets are incomplete. For instance, the fertility rate of high ability parents with low ability grandparents under complete markets is 0.28 while the fertility of the same individual under incomplete markets is 0.58. As discussed above, when markets are complete unlucky grandparents can borrow against the income of their high ability children which reduces the wealth of those children, and therefore their fertility. However, when markets are incomplete unlucky grandparents cannot borrow against their children's income and therefore the fertility of high ability parents with low ability grandparents is higher under incomplete markets. Thus, the result that policies restricting fertility rates are welfare detrimental is not due to fertility being always inefficiently low in incomplete markets models. Even if fertility is inefficiently high, those policies still reduce welfare in our models.

5.5.2 Benchmark Calibration

In this section, we use Brazilian data to calibrate the model and evaluate the welfare effects of fertility restriction policies. The use of Brazilian data takes advantage of data assembled by Lam (1986) on the intergenerational income process, and it corresponds to a relevant case of a developing economy with high fertility rates. We are also interested in evaluating the potential effects of the one child policy in China, but given data limitations, Brazil in 1986 perhaps provides a reasonable approximation for China at the time of the one child policy.

The following parameters are needed: the Markov chain of abilities M, ability vector $\vec{\omega}$, curvature of the utility function σ, discount factor β, parent's degree of altruism ε, and time cost of raising every child λ. For social welfare calculations we also need a social planner's weight on every generation $\beta_p(t)$.

Data on different income groups, fertility of each group, and the Markov chain are taken from Lam (1986) who provides estimates for Brazil. Average incomes of each of five income groups are $\overrightarrow{I} = [553, 968, 1640, 2945, 10991]$. They describe income classes of Brazilian male household heads aged from

40 to 45 in 1976. Average fertility of each income group are $\overrightarrow{n} = [6.189, 5.647, 5.065, 4.441, 3.449]/2$. We divide fertility by two to obtain fertility per-adult. Using income and fertility data, we calculate earning abilities of different groups as $\omega_i = \frac{I(\omega_i)}{1-\lambda n(\omega_i)}$ and normalize the lowest ability to be 1. The Markov chain provided by Lam is:

$$M = \begin{bmatrix} 0.50 & 0.25 & 0.15 & 0.10 & 0.05 \\ 0.25 & 0.40 & 0.20 & 0.20 & 0.10 \\ 0.15 & 0.20 & 0.35 & 0.20 & 0.20 \\ 0.05 & 0.10 & 0.20 & 0.35 & 0.25 \\ 0.05 & 0.05 & 0.10 & 0.15 & 0.40 \end{bmatrix}$$

This chain does not satisfy the CSM property required by Assumption 1, but it implies certain level of earning persistency across generations since its diagonal elements dominate their corresponding off-diagonal elements. We also considered a Markov chain provided by Chu and Koo (1990) that satisfies CSM, and obtain similar results.

Initial population is normalized to 1 and the initial distribution of abilities, π_0, is approximated by the stationary distribution implied by M and $\overrightarrow{n}$.

Our discount factor β and altruistic parameter, ε, are calibrated to match altruism function in Manuelli and Seshadri (2009) (MS henceforth). Their altruistic function in a life cycle model takes the form

$$e^{-\rho B} e^{-\alpha_0 + \alpha_1 \ln n}$$

where α_0, α_1, and ρ are set to be 0.24, 0.65, 0.04, respectively. The child-bearing age B is 25. So the proper mappings from our parameters to theirs are $\beta = e^{-\rho B} e^{-\alpha_0}$ and $1 - \varepsilon = \alpha_1$, which solve our β and ε as 0.29 and 0.35, respectively.

Other two key parameters are the curvature of the utility function σ and the time cost of raising every child λ. We originally set σ to be 0.62, the one used by MS and calibrate λ to match the mean of fertility data because time costs are not provided by MS. But the simulated dispersion of fertility at steady state is way below that of the Brazilian fertility data. To better fit the data, we calibrate σ and λ to jointly match the mean and standard deviation of the fertility data. The calibrated values are $\sigma = 0.526$ and $\lambda = 0.243$. σ is below but not far away from the one used in MS. Under this values, the mean and standard deviation of steady state fertility in the model are 2.653 and 0.433 respectively, which are close to the targets 2.648 and 0.416 in the data.

Our parameter of time cost of raising a child approximately prescribes a maximum number of 8 children per couple, or that each parent spends around 12% of their time on every child. For the social planner's weights we assume $\beta_p(t) = \delta^t$ with $\delta = 0.1$. Remember that $\delta = 0.0$ means that the planner values future generations just as much as the original generation does. The set of parameters chosen for the benchmark exercises are summarized in Table 5.2.

Table 5.2 Parameters

Parameters	Concept	Values
β	individual discount factor	0.29
ε	altruistic parameter	0.35
σ	elasticity of substitution	0.526
λ	per child time cost	0.243
δ	weight of social planner	0.1

5.5.3 Results

The simulated model replicates a negative relationship between fertility and ability similar to the Brazilian data[32]. As shown in Figure 5.1, fertility per household falls from around 6 to 2.8 as earning abilities increase from 1 to 8.5. This negative relationship arises from the interplay of two opposite forces. On the one hand, individuals with higher abilities have a larger opportunity cost of raising children. On the other hand, they enjoy a larger benefit of having children when abilities are intergenerationally persistent. The effect of ability on the marginal cost dominates as long as the intergenerational persistence of ability is less than perfect and the IGES is larger than 1.

We now use the model to perform policy experiments. First, we study policies that directly restrict or encourage fertility. Second, we consider the effects of taxes and/or subsidies on family size.

5.5.4 Policy Experiments

Restricting Fertility Consider first the effects of policies limiting the fertility rate to be no more than $\widehat{n}$ children, where $\widehat{n} < \bar{n}$. We have proved that these policies reduce total utilitarian social welfare in Proposition 10 and reduce average social welfare in certain cases, in Propositions 12 and 13. We now consider average social welfare in an empirically plausible case.

The first panel of Figure 5.2 shows the effect of limiting fertility on steady state average ability, $\overline{\omega}$, and average income, $\overline{y}$. These two variables increase as the upper limit on fertility decreases. As predicted by CK, tighter fertility limits, which affect lower income groups more severely, increase average income and ability. These results seemingly provide support to family planning programs. However, they ignore the negative welfare consequences of limiting family size. The second panel of Figure 5.2 shows that steady state average social welfare, $\overline{W}^*$, average social welfare of all generations, $\overline{W}$, total social welfare of the

[32] By construction, our calibration targets the mean and dispersion of fertilities but not the sign of the relationship between fertility and income.

initial generation, W_0, and total social welfare of all generations, W, consistently increase as the upper limit on fertility is relaxed.

Knowing that policies limiting fertility reduce social welfare, we also simulate policies that impose lower limits on fertility rates. An example of this policy is recent efforts by the Chinese government to induce educated unwed women older than 27 to marry by stigmatizing them as "leftover women". The objective of these efforts seems to be twofold. To reduce pressures due to sex imbalances brought about by the One Child Policy, and to improve the quality of the workforce. These type of policies disproportionately affect the rich, or high ability individuals, because their unconstrained fertility is typically lower. The first panel of Figure 5.3 shows that this policy improves average ability since high ability individuals have proportionally more high ability children. On the other hand, the policies reduce average income because individuals, especially those with high abilities, spend more time raising children and this effect dominates the effect of an improved ability distribution.

The second panel of Figure 5.3 illustrates the impact of setting lower limits on four social welfare measures. In general all four welfare measures exhibit a decreasing trend as the lower bound increases although there are certain ranges in which total welfare increase slightly. For example, total social welfare has a tiny increase by 0.08 as the lower bound increases from 5 to 6. A further increase in the lower bound, say above 6, results in all welfare measures eventually plunging.

These results confirm the main message of the chapter: in general fertility restrictions, on the poor or other groups, do not have strong theoretical support for improving social welfare.

Taxes and Subsidies on Fertility We next study the effect on social welfare of taxes and/or subsidies on fertility while preserving a government balanced budget. Specifically, consider a subsidy, s, that helps parents offset the costs of raising children as is the spirit of recent policies in Europe. We also allow s to be negative in order to consider policies deterring parents from having children. The subsidy (tax) is financed by a lump sum tax (transfer) $T(s)$ imposed on every individual. Under this policy, individual's budget constraint becomes

$$c + (\lambda - s)\,\omega n \leq \omega - T(s)\,.$$

The governments' budget constraint is

$$T(s) = s \sum_{\omega} \omega n(\omega)\,\pi^{*}(\omega)\,.$$

Figure 5.4 shows the effects of this policy on fertility and individual welfare. The horizontal axis of every panel is the subsidy, s, ranging from -10% to 10%. The range of subsidies is such that they do not fully compensate for the cost of raising children, $s < \lambda$. The first panel shows that fertility rates decrease as taxes increase, say as s drops from 0 to -0.1, while the effect is ambiguous for subsidies.

The ambiguous impact of a subsidy on fertility is because of the income effects of the lump sum transfer. In presence of taxes or subsidies the marginal cost of an additional child becomes

$$(\lambda - s)\,\omega^{1-\sigma}\left(1 - (\lambda - s)\,n - T(s)\,\omega^{-1}\right)^{-\sigma}$$

while the marginal benefit is $\beta(1-\varepsilon)\,n^{-\varepsilon}E\left[U(\omega')\,|\omega\right]$. Marginal benefit also depends on s since $U(\omega)$ does although it is not clear from the expression how it moves with s. For marginal cost, when $s > 0$, an increase in s decreases the marginal cost if $T(s)$ is unchanged. But the lump sum tax $T(s)$ imposed on everyone increases with s, which causes marginal cost to go up. The effect of $T(s)$ on marginal cost is small for high ability types, so for them the effect of s on MC tends to dominate that of $T(s)$ while the inverse tends to be true for low ability individuals. This explains what happens in the first panel of Figure 5.4 where fertility increases for the two highest ability types and decreases for other types as s increases. This is because MC of the two highest ability types decrease much faster than all other types' MC as subsidy increases or possibly other types' MC increase with the subsidy. Likewise, when $s < 0$, s becomes more negative as taxes increase, MC of high ability types increase more than that of low ability types. The numerical results show that everyone's fertility falls as taxes increase. Panel 2 shows a mixed policy consequences on individual welfare. A high subsidy benefits high ability individuals while harms other types, especially low ability individuals.

Figure 5.5 shows the effects on social welfare, defined as steady state average social welfare. Social welfare could be improved only when there is a very low tax, e.g. up to 2% of the time cost of raising every child. Otherwise, it is neither improved by taxes nor by subsidies.

Robustness Checks We now report the results of various robustness checks. For this purpose we change one parameter at a time while keeping all the other parameters at their benchmark values and study the policy effects of reducing fertility on various welfare measures. We find that the qualitative results obtained above are mostly robust for reasonable parameters although there exist parameters for which average steady state welfare, $\overline{W}^*$, improves with fertility restrictions. The set of parameters studied is further restricted by the need to have finite utility and concavity.

We find that the results are robust to setting σ anywhere in the range 0.45 to 1[33]. When $\sigma \in [0.35, 0.45]$, fertility restrictions could moderately improves steady state average welfare $\overline{W}^*$ but only when the limit is at a very high level, as illustrated in the first panel of Figure 5.6 for $\sigma = 0.4$. The intuition for the increase in $\overline{W}^*$ is the following. First, a small σ implies a high IGES and therefore a small gain of smoothing consumption through fertility choices. Thus fertility restrictions when σ is small are less harmful to individual welfare in which case people's incentive to smooth consumption through fertility is relatively weak. Second, $\overline{W}^*$ could be slightly increased by a tighter fertility restriction only in the area near the unconstrained fertility choice of the low ability group. In that case, only fertility choices of low ability individuals are directly affected. Due to their high fertility rate and low welfare, tighter restrictions could significantly improve

[33]To guarantee the concavity of the utility function and nonnegative utility, $1 > \sigma > \varepsilon$ is needed.

distribution without hurting other types. So the effect on distribution dominates that on individual welfare. In summary, when σ is low fertility restrictions starting from a high level may increase average steady state welfare because they have a minor impact on individual welfare but relatively large effect on distribution.

The results are robust to setting β in the range $[0.27, 0.4]$[34]. Similar with σ, as β is below 0.27, $\overline{W}^*$ may increase when the limit on fertility is large enough as illustrated in the second panel of Figure 5.6 for the case $\beta = 0.2$. A low β means that parents care little about future generations. As a response, they would have fewer kids, more consumption and lower marginal utility of consumption. In this case, fertility restrictions have a minor effect on individual welfare and, as a result, the change of the ability distribution is the dominant effect determining social welfare. However, this low degree of altruism also implies that the model predicts counterfactually low fertility rates. In particular, the simulated unrestricted fertility range is between 1.14 to 3.12 per household which is below the minimum of Brazilian's fertility data, 3.449. A similar result is obtained when ε is particularly large, as illustrated in the third panel for $\varepsilon = 0.52$.

We also performed robustness checks for the cost of raising children λ over the feasible range $(0.15, 0.32)$. The lower bound is required by the concavity of utility function while the upper bound is required to guarantee the labor supply to be nonnegative. The result is robust to all $\lambda \in (0.19, 0.32)$. When λ is lower than 0.19, fertility of everyone is too high and population becomes infinite in the long run and total social welfare is not well defined.

5.6 Conclusion

This chapter studies optimal fertility choices and fertility policies when children's earnings are random and parents are altruistic. We characterize equilibrium allocations in endowment economies with complete and incomplete markets. In the complete markets case, consumption and fertility are not deterministic as is the case when fertility is exogenous. This novel result is a natural consequence of a key insight provided by

[34]Boundedness of utility requires $\beta < \lambda^{1-\varepsilon}$. Given $\lambda = 0.243$ and $\varepsilon = 0.35$, the upper bound for β is 0.4 .

Barro and Becker (1989): when fertility is endogenous consumption is proportional to the net cost of raising a child. We generalize this result to the case of idiosyncratic shocks.

Our analysis shows that the incomplete markets models can replicate various features of the evidence on fertility and income distribution. For example, fertility decreases with ability and social mobility occurs in equilibrium. The negative relationship between ability and fertility arises in this framework due to the combination of four factors: incomplete markets, time cost of raising children, less than perfect persistence of abilities and an intergenerational elasticity of substitution larger than 1.

We further show that incomplete markets could lead to inefficiently high fertility rates when children are a net financial burden to parents. However, this feature of the equilibrium allocation does not imply that restricting fertility is welfare improving. On the contrary, we find that fertility restrictions are detrimental to every individual's welfare, even to those whose fertility choices are not directly affected, and therefore detrimental to social welfare if welfare is defined as classical (Bentham) utilitarianism. If social welfare is defined as average (Mills) utilitarianism, then fertility restrictions may improve social welfare but only if the distribution of abilities improves strongly. We also perform policy experiments using calibrated version of the model. These experiments suggest that, in general, fertility policies such as taxes or subsidies that affect the cost of raising children do not increase social welfare.

Our models abstracts from a number of aspects that are potentially important for fertility decisions such as bequests and human capital accumulation. Liao (2013) provides a model with human capital and find similar results using a calibrated deterministic model. We are extending our results to production economies in ongoing work (Cordoba et al. 2015).

CHAPTER 6. STOCHASTIC DOMINANCE AND DEMOGRAPHIC POLICY EVALUATION: A CRITIQUE

6.1 Introduction

A classical literature on the measurement of inequality claims that stochastic dominance provides a robust criterion to rank income distributions. This literature originated in papers by Kolm (1969) and Atkinson (1970), and was extended by Dasgupta et al. (1973), Rothschild and Stiglitz (1973), Saposnik (1981, 1983), and Foster and Shorrocks (1988a, 1988b) among many.[35] As summarized by Foster and Shorrocks (1988a), first order stochastic dominance (FSD) "can be regarded as the welfare ordering that corresponds to unanimous agreement among all monotonic utilitarian functions." As such, FSD seemingly provides a robust criterion for policy evaluation because it only requires minimal knowledge of the social welfare function. A natural prescription of this literature would be to look for policies that improve the distribution of income in the FSD sense.

An important application of stochastic dominance is the one by Chu and Koo (1990) (CK henceforth). They use FSD to evaluate the consequences of changing the reproduction rate of a particular income group. Using a Markovian branching framework with differential fertility among income groups, they show that an exogenous reduction in the fertility of the poor results in a sequence of income distributions that conditionally first-degree stochastically dominate (CFSD) the original distribution. CFSD implies FSD. CK argue that stochastic dominance "provides us with very strong theoretical support in favor of family-planning programs that encourage the poor in developing countries to reduce their reproductive rate (pp. 1136)." Numerical simulations of CK's model further confirm that more general fertility reduction programs that disproportionately targets lower income groups, such as the *One Child Policy*, or policies that promote fer-

[35] A more complete list of references can be found in Davidson and Duclos (2000) and Atkinson and Brandolini (2010). A more precise terminology is "welfare dominance" as used by Foster and Shorrocks (1988b). We use stochastic dominance because this is the term used in the paper that is the focus of our critique.

tility of high income groups, should increase social welfare.[36] These policies generally result in a sequence of income distributions that dominates the distribution without the program in the first order stochastic sense.

CK's results are nevertheless puzzling. Basic economic principles suggest that absent externalities or market failures individuals' decisions should be efficient. In fact, various authors have shown that fertility choices made by altruistic parents, i.e. parents who care about the number and welfare of their children, are socially optimal under certain conditions. Early papers in this category include Pazner and Razin (1980), Willis (1985), Becker (1983), Eckstein and Wolpin (1985). Golosov et al. (2007) further show that market allocations are Pareto optimal in a variety of models of endogenous fertility. These findings suggest that family planning programs aiming at reducing the fertility of the poor do not necessarily have the strong theoretical support claimed by CK. Lam (1993, pp 1043) expresses similar skepticism.

Unfortunately CK do not fully spell out the decision problem of individuals, a common feature of the literature cited in the first paragraph. However their two main assumptions, grounded on empirical evidence, are in fact hard to rationalize by frictionless models of fertility. First, they assume intergenerational mobility across income and consumption groups but complete market models, such as the Barro-Becker model, predict no mobility.[37] Second, they assume that fertility decreases with individual income, a feature that is also difficult to rationalize by efficient models of fertility (see Cordoba and Ripoll, 2015). It is possible that behind these two assumptions there are some implicit frictions explaining why fertility is suboptimal in CK's model and intervention is welfare enhancing.

This chapter revisits the question of optimality of family planning programs, as envisioned by CK, but explicitly takes into consideration the household decision problem. For this purpose we use a version of the Barro-Becker fertility model enriched to study issues of income distribution. Individuals in our model differ

[36] On policies seeking to increase the fertility of high income groups, the New York Times reports about the Chinese policy of "upgrading" the quality of their population in order to increase its international competitiveness. It suggests a strategy that includes stigmatizing unmarried women older than 28, who are typically highly educated, as "leftover" women. See http://www.nytimes.com/2012/10/12/opinion/global/chinas-leftover-women.html? Last accessed 3/15/2013

[37] Mobility is still hard to obtain by models of incomplete markets. For example, Alvarez (1999) finds lack of mobility in the Barro-Becker model even in the face of uninsurable idiosyncratic risk. Using a non-altruistic framework, Raut (1990) also finds that the economy reaches a steady state, with no mobility, in two periods.

in their innate abilities, are altruistic toward their descendants, and choose their own fertility optimally. Abilities are random, determined at birth and correlated with parental abilities. Insurance markets are available but parents cannot leave negative bequests to their children. Due to the assumed market incompleteness, mobility arises in equilibrium and fertility differs across ability groups.

The equilibrium of the model satisfies the two assumptions postulated by CK. First, fertility decreases with ability in the presence of uncertainty about children's abilities. To the extent of our knowledge, this result is novel and of independent interest by itself. Although there is a literature documenting and studying a negative relationship between fertility and ability[38], obtaining such negative relationship within a fully dynamic altruistic model with uncertainty is novel. The negative relationship arises from the interplay of two opposite forces. On the one hand, higher ability individuals face a larger opportunity cost of having children due to the time cost of raising children. On the other hand, higher ability individuals enjoy a larger benefit of having children when abilities are intergenerationally persistent. We find that the effect of ability on the marginal cost dominates its effect on the marginal benefit if the intergenerational persistence of abilities is not perfect. This explains why fertility decreases with ability. Second, the equilibrium of the model exhibits mobility. In particular, the equilibrium is characterized by a Markov branching process satisfying the *Conditional Stochastic Monotonicity* property. This requirement means that if a kid from a poor family and a kid from a rich family both fall into one of the poorest classes, it is more likely that the poor kid will be poorer than the rich kid.

Given that the equilibrium of our model satisfies the assumptions postulated by CK, direct application of their Theorem 2 implies that a reduction in the fertility of the poor generates a sequence of income distributions that dominates the original distribution in all periods in the first order stochastic sense. In particular, average income and consumption increase for all periods. This result comes from two forces. First, average ability of (born) individuals increases because the poor have proportionally more low ability

[38] See for example Becker (1960), Jones and Tertilt (2006), Kremer (1993), Hansen and Prescott (2002), Cordoba and Ripoll (2010).

children as a result of the assumed conditional stochastic monotonicity property. Second, consumption and income of the poor strictly increases because they spend less time and resources raising children. However, contrary to CK's claim, we find that individual and social welfare fall. Our main results, Propositions 7 and 8 show that fertility restrictions of any type, not only for the poor, unequivocally reduce individual and social welfare in our model, in spite of the strong degree of market incompleteness. Hence we conclude that stochastic dominance alone is not a sound criterion to rank social welfare as claimed by Chu and Koo in particular, and by a larger literature in general.

The primary reason why stochastic dominance fails to rank welfare properly is because it does not take into account the fact that indirect utility functions are not invariant to the policies in place. As we show, a policy that restricts fertility in our model reduces the set of feasible choices and invariably reduces welfare of all individuals in all generations, even those whose fertility is not directly affected. This is because altruistic parents care not only about their own consumption and fertility but also care about the consumption and fertility of all their descendants. Furthermore, the welfare of those individuals who are not born under the new policy also falls, or at least does not increase. Social welfare falls because the welfare of all individuals, born and unborn, either falls or remain the same. This is the case, for example, if social welfare is defined as classical (Bentham) utilitarianism, a weighted sum of the welfare of all present and future individuals. The result also holds for versions of classical utilitarianism that are consistent with the Barro-Becker concept of diminishing altruism. An interpretation of our results is that the positive effect on welfare of fertility restrictions, namely higher average consumption, is dominated by the negative effect of an smaller dynasty size.

CK define social welfare as average (Mills) utilitarianism rather than classical utilitarianism. Under this definition, social welfare can increase even if the welfare of all individuals falls if population falls even more. The net effect of fertility restrictions on social welfare depends in this case on the relative strength of two opposite forces. On the one hand, the distribution of abilities and incomes improves for all periods, as stressed by CK. On the other hand, the welfare of all individuals falls. Propositions 9 and 10 provide two examples in which the later force dominates and social welfare, defined as average welfare, falls not only in present value but also for all periods. These are counterexamples to the claim that stochastic dominance is a sufficient condition to rank social welfare, even when welfare is defined as average utilitarianism. We further provide a variety of numerical simulations to illustrate that our results are more general, not just extreme examples.

Our results challenge the policy implications of CK's paper but also the broader literature, mentioned in the first paragraph, claiming that stochastic dominance alone provide robust normative implications. We show that carefully modeling the microfoundations of the problem makes a difference and can reverse the conclusions obtained by simple stochastic dominance criteria. Our findings are an application of the Lucas' critique. CK's results are based on the assumption that reduced form parameters and indirect utility functions are invariant to policy changes. Specifically, fertility rates as well as indirect utility functions of individuals are assumed to be invariant to the policies in place. However these are not structural parameters but function of deeper parameters, those governing preferences, technologies and policies in place. Policy evaluations based on the assumed constancy of the parameters may be misleading. In his classic critique, Lucas argued that the observed negative relationship between unemployment and inflation cannot be exploited by policymakers to systematically reduce unemployment. The analogous argument in our context is that the observed negative relationship between fertility and income cannot be exploited by policymakers to improve social welfare.

In addition to the papers already mentioned, our work is related to Alvarez (1999). He studies an economy with idiosyncratic shocks, incomplete markets and endogenous fertility choices by altruistic parents. Our endowment economy is a version of his model, one with non-negative bequest constraints. In equilibrium no individual leaves positive bequests. This is a stronger degree of market incompleteness than that in

Alvarez and it explains why mobility arises in the equilibrium of our model but not in his. As a result, our model maps exactly into CK's Markovian model.

Golosov et al. (2007) proves that equilibrium outcomes are efficient in fertility models of Barro and Becker, but their first welfare theorem does not apply to our model in which fertility is inefficient due to the constraint on intergenerational transfer. Schoonbroodt and Tertilt (2014) have shown that, under certain assumptions, incomplete markets models result in inefficiently low fertility. In those cases it is natural to expect that policies seeking to reduce fertility even more, such as One Child Policy, would reduce social welfare. We show in a companion paper, Cordoba and Liu (2014), that under different assumptions incomplete markets models can result in too much fertility compared with the complete market fertility. In those cases it is not obvious that policies limiting fertility are welfare reducing. We show that restricting fertility in incomplete markets models, even if fertility is inefficiently high, is generally welfare detrimental.

There is a related literature that studies fertility policies in general equilibrium. A recent example is Liao (2013) who studies the One Child Policy using a calibrated deterministic dynastic altruism model with two types of individuals, skilled and unskilled, in the spirit of Doepke (2004). Although Liao's model can generate fertility differentials, Doepke (2004) documents that this channel alone is relatively weak. Part of the issue is that the model only generates upward mobility in equilibrium. Our model, in contrast, generates significant upward and downward mobility that can lead to significant fertility differentials. The mechanisms are different and therefore complementary. In addition, we are able to derive sharp analytical results. For example, we prove that fertility policies, like the one child policy, decrease every individual's welfare for sure while Liao's calibrated result suggests it is true for almost all generations but not all.

The rest of the paper is organized as follows. Section 6.2 revisits the basic connection between fertility, distribution of income and social welfare in models with exogenous fertility. The section reviews the result of CK and provides further analysis. Section 6.3 endogenizes fertility and shows that fertility generally decreases with ability and income. Section 6.4 studies social policies. It shows the basic limitation of CK's assumptions and argues that fertility policies typically reduce social welfare. Numerical simulations and robustness checks are performed in this section. Section 6.5 concludes. Proofs are in the Appendix.

6.2 Distribution and Social Welfare with Exogenous Fertility

Consider an economy populated by a large number of individuals who live for one period. Individuals differ in their labor endowments, or earning abilities. Let $\Omega \equiv \{\omega_1, \omega_2, ..., \omega_n\}$ be the set of possible abilities, where $0 < \omega_1 < ... < \omega_n$. The technology of production is linear in ability: one unit of labor produces one unit of perishable output. In this section, the income of an individual is equal to his/her ability. Let $f(\omega)$ be the fertility rate of an individual with ability ω. It satisfies the following assumption.

Assumption 1. $f(\omega_i)$ is decreasing in ability.

6.2.1 Abilities

Ability is determined at birth and correlated with the ability of the parent. Ability is drawn from the Markov chain M where $M_{ij} = \Pr(\omega_{\text{child}} = \omega_i | \omega_{\text{parent}} = \omega_j)$ for ω_i and $\omega_j \in \Omega$. As in CK, assume that M satisfies the following condition:

Assumption 2. Conditional Stochastic Monotonicity (CSM):

$$\frac{\sum_{i=1}^{I} M_{i1}}{\sum_{j=1}^{J} M_{j1}} \geq \frac{\sum_{i=1}^{I} M_{i2}}{\sum_{j=1}^{J} M_{j2}} \geq ... \geq \frac{\sum_{i=1}^{I} M_{in}}{\sum_{j=1}^{J} M_{jn}},\ 1 \leq I \leq J \leq n$$

Assumption 2 means that if a poor kid and a rich kid both fall into one of the poorest classes, it is more likely that the poor kid will be poorer than the rich kid. Assumption 2 assures intergenerational persistence of abilities: higher ability parents are more likely to have higher ability children. CSM implies first order stochastic dominance. To see this notice that when $J = n$ the condition becomes:

$$\sum_{i=1}^{I} M_{i1} \geq \sum_{i=1}^{I} M_{i2} \geq ... \geq \sum_{i=1}^{I} M_{in}, 1 \leq I \leq n.$$

Two examples of Markov chains satisfying Assumption 2 are an i.i.d. process and quasi-diagonal matrices of the form:

$$M_1' = \begin{bmatrix} a+b & c & 0 & 0 & .. & 0 & 0 \\ a & b & c & 0 & .. & 0 & 0 \\ 0 & a & b & c & .. & 0 & 0 \\ .. & .. & .. & .. & .. & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 & a & b+c \end{bmatrix}.$$

where $(a, b, c) \gg 0$, $a + b + c = 1$ and $b > 0.5$.

We further assume that M has a unique invariant distribution, μ, where μ satisfies:

$$\mu(\omega_j) = \sum_{\omega_i \in \Omega} \mu(\omega_i) M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega.$$

6.2.2 Fertility and the distribution of abilities

Let $P_t(\omega)$ be the size of population with ability ω at time $t = 0, 1, 2, ...$, and $P_t \equiv \sum_{\omega \in \Omega} P_t(\omega)$ be total population at time t. The initial distribution of population, $\{P_0(\omega_i)\}_{i=1}^{n}$, is given. Assuming that a law of large number holds, the size of population in a particular income group evolves according to:

$$P_{t+1}(\omega_j) = \sum_{\omega_i \in \Omega} f(\omega_i) P_t(\omega_i) M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega. \tag{70}$$

Let $\pi_t(\omega) \equiv P_t(\omega)/P_t$ be the fraction of population with ability $\omega \in \Omega$ at time t. Since income is equal to ability, π also characterizes the income distribution of the economy. The law of motion of π is given by:

$$\pi_{t+1}(\omega_j) = \frac{P_t}{P_{t+1}} \sum_{\omega_i \in \Omega} f(\omega_i)\, \pi_t(\omega_i)\, M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega \tag{71}$$

Let $\pi^*(\omega) = \lim_{t \to \infty} \pi_t(\omega)$. As shown by CK, the limit is well defined.

A central topic of the chapter is to characterize π_t and π^* as well as their relationship to fertility. The following proposition provides a simple but important benchmark. The first part states that when fertility is identical across types the limit distribution of incomes is equal to μ, the invariant distribution associated to M. This result provides a baseline distribution in absence of fertility differences. In that case, the distribution of income just reflects the genetic distribution of abilities, what can be termed nature rather than nurture. The second part of the Proposition shows that fertility differences alone does not necessarily affect the long-run distribution of income, π. In particular, fertility differences are irrelevant for the income distribution when abilities are i.i.d.

Proposition 1. When $\boldsymbol{\pi}^*$ equals $\boldsymbol{\mu}$. Suppose one of the following two assumptions hold: (i) $f(\omega) = f$ for all $\omega \in \Omega$; or (ii) $M(\omega', \omega)$ is independent of ω for all $\omega' \in \Omega$. Then $\pi^*(\omega) = \mu(\omega)$ for all $\omega \in \Omega$.

Fertility differences affect the distribution of incomes when abilities are persistent. The following Proposition is an application of CK's Theorem 2. It states that if the fertility of the poor is higher than the fertility of the rest of the population then π^* is different from μ, and moreover, μ dominates π^* in the first order stochastic sense.

Proposition 2. Suppose M satisfies Assumption 1 and $f(\omega_1) > f(\omega_i) = f$ for all $i > 1$. Then $\sum_{i=1}^{I} \pi^*(\omega_i) > \sum_{i=1}^{I} \mu(\omega_i)$ for all $1 \leq I \leq n$.

Proof. See Chu and Koo (1990, pp.1136).

Comparing Propositions 1 and 2, it follows that a reduction in the fertility of the poor results in a limit distribution that dominates the original distribution. More generally, CK show that if fertility decreases with income and the initial distribution of incomes is at its steady state level, $\pi_0^*(\omega_i)$, then a reduction in the fertility of the poor results in a sequence of income distributions that first order stochastically dominates $\pi_0^*(\omega_i)$, that is, $\sum_{i=1}^{I} \pi_t(\omega_i) < \sum_{i=1}^{I} \pi_0^*(\omega_i)$ for all $1 \leq I \leq n$ and $t > 0$.

6.2.3 Social Welfare

CK consider average utilitarian welfare functions of the form:

$$\overline{W} = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) \pi_t(\omega) \tag{72}$$

where $U(\omega)$ is the utility of an individual with ability ω and $\beta_p(t)$ is the weight of generation t in social welfare. A particular case emphasized by CK is one where the planner cares only about steady state welfare: $\beta_p(t) = 0$ for all t and $\lim_{t \to \infty} \beta_p(t) = 1$. In that case,

$$\overline{W}^* = \sum_{\omega \in \Omega} U(\omega) \pi^*(\omega) \tag{73}$$

The following corollary of Proposition 2 provides the theoretical support to family planning programs for the poor, as claimed by CK.

Corollary 3. Suppose social welfare is defined by (72) where $U(\omega)$ is a non-decreasing function of ability. Furthermore, suppose M satisfies Assumption 2 and $f(\omega_1) > f(\omega_i) = f$ for all $i > 1$. Then (i) reducing the fertility of the poor increases social welfare; (ii) fertility policies that do not change the observed distribution of abilities, π_t, does not change social welfare.

Corollary 3 holds because reducing fertility of the poor improves the observed distribution of abilities but does not alter $U(\cdot)$. In the next two sections we show counterexamples to Corollary 3 when fertility is endogenous. This is an crucial consideration because if fertility of the poor is to be restricted, one needs to rationalize why the poor choose to have more children in the first place. As a preview of the results, we show a case in which fertility is restricted by fertility policies, the observed distribution of incomes, π_t, does not change in any period but social welfare as well as individual welfare decreases for all individuals in all periods compared to the unrestricted case. The reason why the previous Corollary fails to account for this possibility is that it presumes that $U(\omega)$ is invariant to policies, it lacks microfoundations. However, $U(\omega)$ is in fact an indirect utility function and therefore it is not invariant to policies.

6.3 An Economic Model of Fertility

We now consider the endogenous determination of fertility. Assumptions are the same as in the previous section. In particular, the initial distribution of population across abilities, $\{P_0(\omega_i)\}_{\omega_i \in \Omega}$, is given, abilities are random, determined at birth and described by a Markov chain M satisfying Assumption 1, and having a unique invariant distribution, μ. The technology of production is linear in labor: one unit of labor produces one unit of perishable output. Let $\omega^t = [\omega_0, \omega_1, ..., \omega_t] \in \Omega^{t+1}$ denote a particular realization of ability history up to time t, for a particular family line. There is neither capital nor aggregate risk.

6.3.1 Individual and aggregate constraints

Markets open every period. The resources of an individual of ability ω_t at time t are labor income and transfers from their parents. Labor income equals $\omega_t(1 - \lambda f_t)$ where λ is the time cost of raising a child. Let $b_t(\omega_t)$ denote transfers, or bequests, received from parents. Resources are used to consume and to leave bequests to children. Insurance market exists as parents can leave bequest contingent on the ability of their children. Let $q_t(\omega^t, \omega_{t+1})$ be price of an asset that delivers one unit of consumption to a child of ability ω_{t+1}

given that the history up to time t is ω^t.[39] The budget constraint of an individual at time t with history ω^t is:

$$c_t\left(\omega^t\right)+f_t\left(\omega^t\right)\sum_{i=1}^{n}q_t(\omega^t,\omega_i)b_{t+1}\left(\omega^t,\omega_i\right)\leq\omega_t\left(1-\lambda f_t\left(\omega^t\right)\right)+b_t\left(\omega^t\right). \tag{74}$$

We assume that parents cannot leave negative bequests to their children:

$$b_{t+1}\left(\omega^t,\omega_i\right)\geq 0 \text{ for all } \omega^t\in\Omega^{t+1},\omega_i\in\Omega \text{ and all } t>0.$$

Furthermore, suppose $b_0\left(\omega_i\right)=0$ for all $\omega_i\in\Omega$.

Since output is perishable, aggregate consumption must be equal to aggregate production. Alternatively, aggregate savings must be zero. Savings are equal to the total amount of bequests left by parents. Since bequests are non-negative then aggregate savings are zero if and only if all bequests are zero. Therefore, in any equilibrium the budget constraints (74) simplifies to:

$$c_t\left(\omega^t\right)\leq\omega_t\left(1-\lambda f_t\left(\omega^t\right)\right) \text{ for all } \omega^t\in\Omega^{t+1} \text{ and all } t\geq 0. \tag{75}$$

This is balanced budget constraint for every period and state. The lack of intergenerational transfers significantly simplifies the problem and explains why social mobility arises in the equilibrium. Otherwise, as shown by Alvarez (1999), parents will use family size to buffer against shocks and use transfers to smooth consumption across time and states regardless of ability which prevents any social mobility. Absent transfers, ability becomes the key determinant of consumption and fertility, as we see below.

In addition to budget constraints, individuals must satisfy time constraints. In particular, the time spent in raising children cannot exceed the time available to an individual, which is normalized to 1. Thus,

$$0\leq f_t\left(\omega^t\right)\leq\frac{1}{\lambda}. \tag{76}$$

[39] The price also depends on the aggregate distribution of abilities at time t.

6.3.2 Individual's Problem

The lifetime utility of an individual born at time t is of the Barro-Becker type (Barro and Becker 1989 and Becker and Barro 1988):

$$U_t = u(c_t) + \beta f_t^{1-\epsilon} E_t U_{t+1}, t = 0, 1, 2, ... \tag{77}$$

where $u(c) = \frac{c^\sigma}{\sigma}$, $\sigma \in (0,1)$, is the utility from consumption, f_t is the number of children, U_{t+1} is the utility of the time $t+1$ generation, and E_t is the mathematical expectation operator conditional on the information up to time t. The term $\beta f_t^{1-\epsilon}$ is the weight that parents place on their f_t children. When $\epsilon = 0$ parents are perfectly altruistic toward children. We assume $0 \leq \epsilon < 1$.

The following restrictions on parameters are needed in order to have a well-behaved bounded problem.

Assumption 3. $1 - \epsilon > \sigma$ and $\lambda^{1-\epsilon} > \beta$.

The first part of the assumption is identical to the one discussed by Barro and Becker (1988) to assure strict concavity of the problem. The second part guarantees bounded utility as the effective discount factor in that case satisfies $\beta f_t^{1-\epsilon} \leq \beta \lambda^{\epsilon-1} < 1$.[40]

The individual's problem is to choose a sequence $\left\{f_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$ to maximize U_0 subject to (75) and (76). The problem can be written in sequence form, by recursively using (77), to obtain:

$$U_0^*(\omega_0) = \sup_{\{P_{t+1}(\omega^{t-1},\omega_t)\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t P_t\left(\omega^{t-1}\right)^{1-\varepsilon} u\left(\omega_t\left(1 - \lambda \frac{P_{t+1}\left(\omega^{t-1},\omega_t\right)}{P_t\left(\omega^{t-1}\right)}\right)\right) \tag{78}$$

subject to

$$0 \leq P_{t+1}\left(\omega^{t-1},\omega_t\right) \leq P_t\left(\omega^{t-1}\right)/\lambda \text{ for all } \omega^{t-1} \in \Omega^t, \omega_t \in \Omega \text{ and } t \geq 0,\ P_0 > 0.$$

[40] An upper bound for U_t is $\frac{u(\omega_n)}{1-\beta\lambda^{\epsilon-1}}$.

In this formulation, $P_0\left(\omega^{-1}\right)=1$. $P_{t+1}\left(\omega^t\right)=\prod_{j=0}^{t} f_j\left(\omega^j\right)$. With a little abuse of notation, let $P_{t+1}\left(\omega^{t-1},\omega_t\right)=P_{t+1}\left(\omega^t\right)$ where $\omega^t=\left(\omega^{t-1},\omega_t\right)$. Fertility rates can be recovered as $f_t\left(\omega^t\right)=\frac{P_{t+1}\left(\omega^{t-1},\omega_t\right)}{P_t\left(\omega^{t-1}\right)}$.

An alternative way to describe the household problem is by the following functional equation:

$$U\left(\omega\right)=\max_{f\in[0,\frac{1}{\lambda}]} u\left(\omega\left(1-\lambda f\right)\right)+\beta f^{1-\epsilon}E\left[U\left(\omega'\right)|\omega\right] \tag{79}$$

The next proposition states that the principle of optimality holds for this problem. This result is novel because the functional equation is not standard due to the endogeneity of fertility. In particular the discount factor is endogenous. Alvarez (1999) shows that the principle of optimality holds for a dynastic version of this problem while we show that it holds for the household version of the problem.[41] Our household problem is simpler because of the lack of intergenerational transfers in equilibrium.

Proposition 4. The functional equation (79) has a unique solution, $U\left(\omega\right)$. Moreover $U\left(\omega\right)=U_0^*\left(\omega\right)$ for $\omega\in\Omega$.

6.3.3 Optimal Fertility

The optimality condition for an interior fertility choice is:

$$\lambda\omega u'\left(\left(1-\lambda f^*\right)\omega\right)=\beta\left(1-\epsilon\right)f^{*-\epsilon}E\left[U\left(\omega'\right)|\omega\right] \tag{80}$$

Let $f^*=f(\omega)$ be the optimal fertility rule and $c^*=c\left(\omega\right)\equiv\left(1-\lambda f(\omega)\right)\omega$ the optimal consumption rule. The left hand side of this expression is the marginal cost of a child while the right hand side is the marginal benefit. The marginal cost is the product of the cost per child, $\lambda\omega$, and the marginal utility of consumption.

[41] The analogous dynastic problem is:

$$V\left(N,\omega\right)=\max_{N'\in[0,\frac{1}{\lambda}N]} u\left(\omega-\lambda N'/N\right)N^{1-\epsilon}+\beta E\left[U\left(N',\omega'\right)|\omega\right].$$

In this problem the number of family members is a state variable, N, all member have the same ability, ω, and make the same choices. The household problem does not impose these constraints.

The marginal benefit to the parent is the expected welfare of a child, $E\left[U\left(\omega'\right)|\omega\right]$, times the parental weight associated to the last child, $\beta\left(1-\epsilon\right)f(\omega)^{-\epsilon}$.

Corner solutions are not optimal in the incomplete market case under the assumed functional forms because the marginal benefit of a child is infinite while the marginal cost is finite. In particular, $E\left[U\left(\omega'\right)|\omega\right] > 0$ for all ω while $\lim_{f\to 0} f^{-\epsilon} = \infty$. Having the maximum number of children is also sub-optimal because the marginal cost is infinite when parental consumption is zero, while the marginal benefit is finite.

Consider now the relationship between fertility, f^*, and parental earning ability, ω. According to (80), both marginal benefits (MB) and marginal costs (MC) are affected by abilities. MB increase with ω because of the postulated intergenerational persistence of abilities: high ability parents are more likely to have high ability children. Regarding MC, there are two effects. On one hand, MC tends to rise with ω because high ability parents have high opportunity cost of raising children as their wage rate is high. On the other hand, MC tends to fall because higher ability implies more consumption and lower marginal utility of consumption. When $\sigma \in (0, 1)$, the first effect dominates the second one so MC increases with ω. The need for a small curvature, $\sigma \in (0, 1)$, different from the one typically used in Macroeconomics, suggests a tension between the theory and the empirical since estimates of the elasticity of intertemporal substitution are typically lower than 1. But the correct interpretation of $1/(1-\sigma)$ is an elasticity of intergenerational substitution (EGS), one controlling inter-personal consumption smoothing rather than intra-personal consumption smoothing (Cordoba and Ripoll 2011, 2014).

Since both MB and MC increase with ω when σ is smaller than 1, it is not clear in principle whether fertility increases or decreases with ability. The following Proposition consider three cases: i.i.d abilities across generations, perfect intergenerational persistence of abilities with no uncertainty and random walk (log) abilities[42]

[42] Although a random walk does not satisfy some of the assumptions above, it helps to develop some intuition.

Proposition 5. Persistence and the fertility-ability relationship. (i) Fertility decreases with ability if abilities are i.i.d across generations. In this case $f(\omega)$ satisfies the equation $\frac{f(\omega)^{\epsilon}}{(1-\lambda f(\omega))^{1-\sigma}} = A\omega^{-\sigma}$ where A is a constant. Furthermore, fertility is independent of ability in one of the following two cases: (ii) M is the identity matrix (abilities are perfectly persistent and deterministic); or (iii) $\ln \omega_t = \ln \omega_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim N(0, \sigma^2)$.

According to Proposition 5, fertility decreases with ability when abilities are i.i.d. The intuition is that without intergenerational persistence, the ability of the parent only affects her/his marginal cost but not her/his marginal benefit as $E[U(\omega')|\omega] = E[U(\omega')]$ for all $\omega \in \Omega$. On the other extreme, fertility is independent of ability when abilities are perfectly persistent across generations (cases ii and iii). This is because in those cases both the marginal cost and the marginal benefit are proportional to ω^{σ}.

To understand the second extreme, it is instructive to write the first order condition in an alternative way. First, use equation (80) to express (79) as:

$$U_t = u(c(\omega)) + \frac{1}{1-\epsilon} f(\omega)\lambda\omega u'(c(\omega)) \tag{81}$$

Then use (81) to rewrite (80) as:

$$u'(c(\omega))(f(\omega))^{\epsilon} = \beta E\left[u'\left(c\left(\omega'\right)\right)\frac{\omega'}{\omega}\left(\frac{1}{\lambda} + \frac{1-\epsilon-\sigma}{\sigma}\left(\frac{1}{\lambda} - f\left(\omega'\right)\right)\right)\middle|\omega\right]. \tag{82}$$

This equation is useful because it only requires marginal utilities, rather than total utility as in equation (80), and corresponds to the Euler Equation of the problem describing the optimal consumption rule. Although savings are zero in equilibrium, fertility allows individuals to smooth consumption across generations.[43] If $\omega' = \omega$, (136) becomes one equation with one unknown and $f(\omega) = f$.

Given that fertility becomes only independent of ability in the extreme case of perfect persistent, it is natural to conjecture that fertility decreases with ability when persistence is less than perfect. We were able to confirm this conjecture numerically but analytical solutions were not obtained.

6.3.4 Dynamics of the Income Distribution

Given the optimal fertility rule $f(\cdot)$, initial distribution $\pi_0(\cdot)$ of population across abilities, and initial population P_0, distributions of income for all periods can be obtained using equations (70) and (71). Furthermore, average earning abilities and average income are given by:

$$E_t = \sum_{\omega \in \Omega} \omega \pi_t(\omega)\,; \; I_t = \sum_{\omega \in \Omega} \omega \left(1 - \lambda f(\omega)\right) \pi_t(\omega)$$

In the next section we use the microfounded model to perform welfare evaluations of family planing programs. The model also allows us to assess whether Assumption 1 and Assumption 2 are somewhat associated. We show they are. A mobility matrix with less than perfect persistence of intergenerational abilities can give rise to a negative relationship between fertility and ability. The following Proposition revisits Proposition 1 at the light of the micro-founded model. It plays an important role in section 6.4 when providing counter-examples to CK's claims.

Proposition 6. Persistence, fertility and ability distribution. (i) If $M(\cdot, \omega)$ is independent of ω then $f(\omega)$ decreases with ω and $\pi_t(\omega) = \mu(\omega)$ for $\omega \in \Omega$ and $t > 0$; (ii) if M is the identity matrix then

[43] Equation (80) can also be written in the form of a more traditional Euler Equation. Let $1 + r'$ be the gross return of "investing" in a child. It is given by $1 + r' \equiv \frac{U(\omega')/u'(c')}{\lambda \omega}$. In this expression, $U(\omega')/u'(c')$ is the value of a new life, in terms of goods, while $\lambda \omega$ is the cost of creating a new individual. Then (80) can be written as:

$$u'(c) = \beta (1 - \epsilon) f^{*-\epsilon} E\left[u'(c')\left(1 + r'\right)|\omega\right]. \tag{83}$$

This is an Euler Equation with a discount factor $\beta (1 - \epsilon) f^{*-\epsilon}$. It suggests that optimal fertility choices are similar to saving decisions and that children are like an asset, as pointed out by Alvarez (1999). However, two important differences with the traditional Euler Equation are that the individual controls both the discount factor and the gross return.

$f(\omega) = f$ for all $\omega \in \Omega$ and $\pi_t(\omega) = \pi^*(\omega) = \pi_0(\omega)$ for all $\omega \in \Omega$ and all t; (iii) if $\ln \omega$ follows a Gaussian random walk then $f(\omega) = f$, and given ω_0 the variance of abilities diverges to ∞.

In words, if abilities are i.i.d. across generations, then fertility decreases with ability but the observed limit distribution of abilities is independent of fertility choices and equal to $\mu(\omega)$. Furthermore, with certainty and perfect intergenerational persistence of abilities the observed distribution of abilities in any period is identical to the initial distribution of abilities. Finally, if (log) abilities follow a random walk then there is not limit distribution of abilities since its variance goes to infinite.

6.3.5 Fertility Policies and Individual Welfare

Consider now a family planning policy that sets lower and/or upper bounds on fertility choices. Let $\underline{f}(\omega) \geq 0$ and $\overline{f}(\omega) \leq 1/\lambda$ be the lower and upper bound respectively. Bounds potentially depend on individual abilities. The indirect utility $U^r(\omega)$ of the constrained problem is described by the following Bellman equation:

$$U^r(\omega) = \max_{f \in [\underline{f}(\omega), \overline{f}(\omega)]} u((1 - \lambda f)\omega) + \beta f^{1-\epsilon} E\left[U^r(\omega') \,|\omega\right]. \tag{84}$$

Let $f^r(\omega)$ denotes the optimal fertility rule. The following Proposition is one of the main results of the chapter. It states that binding fertility restrictions for at least one state reduces the indirect utility, or welfare, of all individuals even those whose fertility is not directly affected. The Proposition also states that fertility restrictions of any type (weakly) reduce the fertility of all individuals except perhaps those whose fertility rates are at or below the lower bound.

Proposition 7. $U^r(\omega) \leq U(\omega)$ with strict inequality if $f(\omega) > \overline{f}(\omega)$ or $f(\omega) < \underline{f}(\omega)$ for at least one $\omega \in \Omega$. Furthermore, $f^r(\omega) = \underline{f}(\omega)$ if $f(\omega) \leq \underline{f}(\omega)$ and $f^r(\omega) \leq f(\omega)$ otherwise.

Fertility restrictions reduce welfare because it restricts individuals' choices without providing any compensation. Furthermore, fertility restrictions that only affects a particular group, say the lowest ability individuals, results in lower welfare for all individuals because, regardless of current ability, there is a positive probability that a descendant of the dynasty will fall into the group directly affected in finite time. Proposition 7 implies that policies such as the *One Child Policy*, which imposes a uniform bound on all ability levels, or policies that compel individuals to increase their fertility, such as the "leftover" women stigma in China, are detrimental to individual welfare, according to our model. Given that welfare of all individuals falls, the marginal benefits of having children also falls while the marginal cost remains the same. As a result, fertility must fall for all types except perhaps for those who are constrained by the policy to increase their fertility. We next study the consequences of fertility restrictions on social welfare.

6.4 Family Planning and Social Welfare Reconsidered

Given that fertility policies reduces the welfare of all individuals, as stated in Proposition 7, it is natural to infer that social welfare should also fall. The answer, however, depends on how social welfare is defined and whether the policy reduces or increases population. In this section we focus on fertility policies that impose upper limits on fertility rates such as limiting the fertility of the poor or the *One Child Policy*.

6.4.1 Analytical Results

According to Proposition 7, upper limits on the fertility of any ability group reduce fertility of all ability groups. Therefore, upper limits on fertility unequivocally reduce population of all ability groups at all times after time 0. Given that both population and individual welfare fall for all ability types, we are able to show that fertility limits unequivocally decrease social welfare if social welfare is of the classical, or Bentham, utilitarian form. Classical utilitarianism defines social welfare as the total discounted welfare of all (born) individuals:[44]

$$W = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) P_t(\omega) . \tag{85}$$

In this formulation $\beta_p(t) \geq 0$ is the weight the social planner assigns to generation t. Since individuals are altruistic toward their descendants, $\beta_p(t) > 0$ means that the planner gives additional weight to generation t on top of what is implied by parental altruism. A particular case in which the planner weights only the original generation, and therefore adopts its altruistic weights, is the one with $\beta_p(0) = 1$ and $\beta_p(t) = 0$ for $t > 0$:

$$W_0 = \sum_{\omega \in \Omega} U(\omega) P_0(\omega) \tag{86}$$

The following Proposition states the main conclusion of the chapter: restricting fertility decreases classical utilitarian welfare.

Proposition 8. Imposing upper limits on fertility choices reduces social welfare as defined by (85).

An identical result is obtained if the planner exhibits positive but diminishing returns to population, say if $P_t(\omega)$ in expression (85) is replaced by $P_t(\omega)^{1-\epsilon_p}$ where $\epsilon_p \in (0,1)$. This formulation seems the natural extension of the Barro-Becker preferences for a planner.

An alternative definition of social welfare is average, or Mills, utilitarianism as defined by equation (72) for the general case, and with (73) as a special case. This definition of welfare, the one used by CK, is analogous to (85) but uses population shares, $\pi_t(\omega)$, rather than population, $P_t(\omega)$. Under this definition, social welfare could increase even if the welfare of all individuals falls. The net effect depends on the relative strength of two potentially opposite forces: on the one hand individual welfare falls but on the other hand the distribution of abilities, π, may improve, as in CK. We next show analytical examples in which social welfare falls even under this definition. These are formal analytical counterexamples to CK's claims that fertility limits on the poor are welfare enhancing.

[44] The results are similar if the welfare of the unborn is explicitly considered as long as the unborn enjoy lower utility than the born.

The following Proposition states that fertility restrictions of any type reduce average utilitarian welfare if abilities are i.i.d.

Proposition 9. Suppose $M(\cdot,\omega)$ is independent of ω for all $\omega \in \Omega$. Then upper limits on fertility choices reduce social welfare as defined by (72).

Proposition 9 relies on the earlier finding in Proposition 1 that, when abilities are i.i.d, the observed distribution of abilities, π_t, is independent of fertility choices, even if the poor have more children, and the limit distribution of abilities is the invariant distribution of M. We show in the appendix that $\pi_t(\omega_i) = \mu(\omega_i)$ for all t if $\pi_0(\omega_i) = \mu(\omega_i)$. Therefore, in the i.i.d case the effect of any fertility policy on social welfare, as defined by (72), is only determined by its effect on individual welfare, U.

A particular implication of Proposition 9 is that limiting the fertility of the poor reduces welfare which contradicts CK's claim stated in Corollary 3. The i.i.d case in Proposition 9 satisfies CK's Assumptions 1 and 2 since fertility rates are decreasing, as stated in Proposition 5, and i.i.d abilities satisfies conditional stochastic monotonicity. Corollary 3 fails to properly describe the effect of the policy on social welfare because it implicitly assumes that U is unaffected by the policy change.

The following is a deterministic example showing that average utilitarian welfare unequivocally falls with "uniform" fertility restrictions such as the one child policy.

Proposition 10. Suppose M is the identity matrix and $\overline{f}(\omega) = \overline{f}$. Then fertility restrictions reduces social welfare as defined by (72).

Proposition 10 provides another example in which fertility restriction do not affect π. Since in the deterministic case all ability groups have the same fertility choices, and the fertility restriction affect all ability groups equally, then it follows that $\pi_t = \pi_0$ for all t so that the effect of the policy on social welfare is only determined by the effect on individual welfare U.

We now turn to numerical simulations to investigate more generally the effects of policies restricting fertility choices on social welfare.

6.4.2 Calibration and Simulations

Benchmark Calibration The following parameters are needed to simulate the model: the Markov process of abilities M, preference parameter σ, altruistic parameters β and ϵ, cost of raising children λ, and social planner weight $\beta_p(t)$.

Income groups, fertilities of different income groups, and the Markov chain are taken from Lam (1986) who provides estimates for Brazil. Average incomes for five income groups are

$$\overrightarrow{I} = [553, 968, 1640, 2945, 10991].$$

They describe income classes of Brazilian male household heads aged from 40 to 45 in 1976. Average fertility of each income group are $\overrightarrow{f} = [6.189, 5.647, 5.065, 4.441, 3.449]/2$. We divide fertility by two to obtain fertility per-adult. Using income and fertility data, we calculate earning abilities of different groups as $\omega_i = \frac{I(\omega_i)}{1-\lambda f(\omega_i)}$ and normalize the lowest ability to be 1. The Markov chain provided by Lam is:

$$M = \begin{bmatrix} 0.50 & 0.25 & 0.15 & 0.10 & 0.05 \\ 0.25 & 0.40 & 0.20 & 0.20 & 0.10 \\ 0.15 & 0.20 & 0.35 & 0.20 & 0.20 \\ 0.05 & 0.10 & 0.20 & 0.35 & 0.25 \\ 0.05 & 0.05 & 0.10 & 0.15 & 0.40 \end{bmatrix}$$

This chain does not satisfy conditional stochastic monotonicity property although its diagonal elements dominate other elements implying certain level of earning persistency across generations. We also consider the Markov chain provided by CK, which satisfies CSM, and obtain similar results. Initial population is normalized to 1. The initial distribution of abilities, π_0, is approximated by the stationary distribution implied by M and $\overrightarrow{f}$.

Our altruistic function, $\beta f^{1-\epsilon}$, is calibrated following Manuelli and Seshadri (2009) (MS henceforth).[45] For σ we initially used MS's parameter of 0.38. However, the fertility rates implied by the calibrated model were too high and the range of fertilities too small compared to Brazilian fertility data. We set $\sigma = 0.68$ to better fit the fertility data. Another key parameter of the model is the time cost of raising a child, λ. We choose $\lambda = 0.2$ which implies a maximum number of 10 children per couple, or that each parent spend 10% of their time on every child. We perform robustness checks for this and other parameters. For the social planner weights we assume $\beta_p(t) = \delta^t$ with $\delta = 0.1$. The set of parameters used for the benchmark exercises are summarized in Table 6.1.

Table 6.1 Parameters Setting

Parameters	Concept	Values
β	individual discount factor	0.29
σ	elasticity of substitution	0.68
ϵ	altruistic parameter	0.35
λ	per child time cost	0.2
δ	weight of social planner	0.1

Results The simulated model reproduces a negative relationship between fertility and ability similar to the Brazilian data.[46] Because abilities are persistent but not perfectly persistent across generations, the increase of the marginal cost of children dominates that of the marginal benefit as ability increases. As shown in the first panel of Figure 6.1, fertility per household falls from 9 to 2 as earning abilities increase from 1 to 12.

[45] Their altruistic function takes the form $e^{-\rho B}e^{-\alpha_0+\alpha_1 \ln f}$ where $B = 25$ is the age of fertility. So the proper mapping is β=$e^{-\rho B}e^{-\alpha_0}$ and $1-\epsilon = \alpha_1$.

[46] By construction, our calibration targets the dispersion of fertilities but not the sign of the relationship between fertility and income.

This inverse relationship between fertility and ability in dynastic altruistic model with stochastic earning abilities is very hard to obtain as documented by Jones et al. (2010). Four factors are crucial in our model to fulfill this task. They are imperfect persistence of abilities across generations, EGS larger than 1 (low curvature of the utility function), time cost of raising children and incomplete markets.

The second panel plots average ability, $\overline{\omega}$, and average income, $\overline{y}$, as the upper bound of fertility increases. As predicted by CK, tighter fertility limits, which affect lower income groups more severely, increase average income and ability.

The remaining panels in Figure 6.1 illustrate the effect of fertility limits on various welfare measures. On the horizontal axis is the uniform fertility upper limit imposed on all ability groups, a limit that goes from 0 to 10 children per household. It shows that steady state average welfare, $\overline{W}^{*}$, average welfare of all generations, $\overline{W}$, welfare of the initial generation, W_0, and total welfare of all generations, W, all increase as the upper bound on fertility is relaxed. These results confirm the main message of the chapter: fertility restrictions, on the poor or other groups, do not have strong theoretical support for improving people's welfare.

We also study the welfare effects of imposing lower bounds on fertility rates. This type of restrictions disproportionately affects the rich, or high ability individuals, because their unconstrained fertility is typically lower. Figure 6.2 shows that this policy increases average ability since high ability individuals have proportionally more high ability children. On the other hand, the policy reduces average income because individuals, especially those with high ability, spend more time raising children and this effect dominates the effect of an improved ability distribution. All four welfare measures unanimously decrease as the lower bound on fertility increases.

In summary, the results above show that fertility restrictions, on the poor and on the rich, do not result into higher social welfare although they may improve the distribution of abilities and income.

Robustness Checks We now report the results of various robustness checks. For this purpose we change one parameter at a time while keeping all the other parameters at their benchmark values and study the effect on the various welfare measures of imposing an upper limit on fertility. We find that the qualitative results obtained above are mostly robust although there exists a set of parameters for which average steady state welfare, $\overline{W}^*$, improves with fertility restrictions. The set of parameters studied is further restricted by the need to have finite utility.

The results are robust to setting σ below 0.74. If σ is larger than 0.74, relaxing fertility restrictions slightly reduce steady state average welfare, $\overline{W}^*$, but only when there is a tight upper limit on fertility, of between 1 and 2, as illustrated in the first panel of Figure 6.3 for $\sigma = 0.9$. A high elasticity of intergenerational substitution significantly reduces the gains of smoothing consumption through fertility choices. In addition, low fertility allows higher consumption and low marginal utility of consumption. As a result, a relaxation of fertility restrictions has a minor impact on individual utility and the change in the distribution of abilities determines the change in social welfare. However, further relaxation of the upper limit increases $\overline{W}^*$.

We also find that if β is sufficiently low, a tighter fertility restriction may increase $\overline{W}^*$ as illustrated in the second panel of Figure 6.3 for the case $\beta = 0.2$. A low β means that parents care little about future generations, have fewer kids, higher consumption and lower marginal utility of consumption. In this case, fertility restrictions have a minor effect on individual welfare and, as a result, the change of the ability distribution is the dominant effect determining social welfare. However, this low degree of altruism also implies that the model predicts counterfactually low fertility rates. A similar result is obtained when ϵ is particularly large, as illustrated in the third panel of Figure 6.3 for $\epsilon = 0.53$.

Finally, if the cost of raising children, λ, is sufficiently large then a tighter fertility restriction may increase $\overline{W}^*$ as shown in the last panel of Figure 6.3 for the case $\lambda = 0.28$. In this case the high cost of raising children itself prevents households from having many children and therefore fertility restrictions are not very harmful for individual welfare. The change in social welfare is therefore primarily determined by the change in the distribution of abilities.

6.5 Conclusion

Stochastic dominance, or welfare dominance, seemingly provides a robust criterion for policy evaluation. It allows ranking policies by simply looking at the resulting income distribution without requiring much knowledge of individuals' preferences and constraints, or knowledge of the social welfare function. Chu and Koo (1990) exploit such apparent generality to provide a striking policy recommendation. They assert that stochastic dominance "provides us with very strong theoretical support in favor of family-planning programs that encourage the poor in developing countries to reduce their reproductive rate (pp 1136)." Such fundamental claim has surprisingly remained unchallenged. In this chapter we show that stochastic dominance alone does not provide the strong theoretical support claimed by CK. Our findings challenge not only CK's main normative conclusion but also the larger classical literature on the topic of welfare dominance which is the foundation of such conclusion.

Our main contribution is to provide explicit micro-foundations to CK's model. The key features are altruism, random abilities, labor costs of raising children, non-negative bequest constraints, and an endowment economy. The model is particularly useful because its equilibrium exactly maps into the Markov branching framework of CK. It also successfully replicates two basic features of the evidence on fertility and income distribution: fertility decreases with ability and social mobility occurs in equilibrium. These features are not easily obtained by altruistic models of fertility.

Although fertility policies do not directly address the underlying frictions leading to inefficient fertility, these policies could in principle increase social welfare much in the same way as monetary policy could increase social welfare even if the policy by itself does not address price rigidities.

The welfare effect of family planning policies can potentially depend on how social welfare is defined. When social welfare is defined as the classical utilitarianism, in spite of the fact that policies and family planning programs directed toward reducing the fertility of the poor may result in superior income distributions in the first order stochastic sense, we find first order stochastic dominance does not provide a strong theoretical support to these policies, contrary to CK and to a larger literature mentioned in the introduction. The main reason for this failure is that stochastic dominance does not account for the fact that indirect utility functions are not invariant to fertility policies.

When social welfare is defined as average utilitarianism, policies restricting fertility choices could increase social welfare under certain parameters which require people to discount future very heavily or the degree of altruism toward children to be very low. Average utilitarianism social welfare is not well micro-founded as parents acting as the social planner at the household level would maximize a weighted average of the total utility of the family but not average utility.

In this chapter we evaluate policies restricting fertility choices. We perform other policy experiments, such as fertility related taxes and subsidies, in Cordoba and Liu (2014) and find those policies in general do not increase social welfare as well. Our model abstracts from a number of aspects that are potentially important to fertility decisions such as bequests and wealth inequality. We study these extensions in Cordoba et al. (2015). The models are significantly more complicated, and do not map into a simple Markov branching framework, but our early results confirm the findings that policies restricting fertility typically do not increase social welfare.

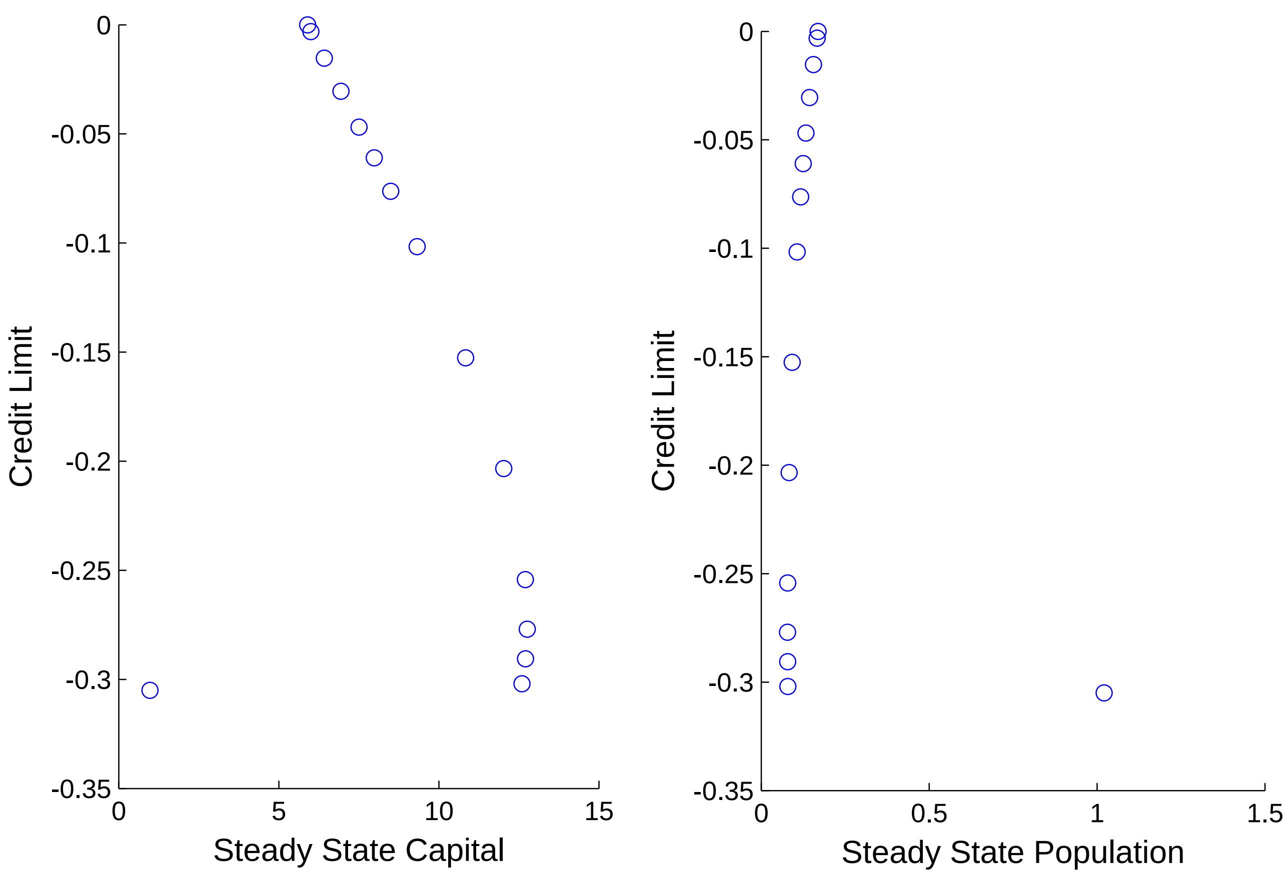

Figure 2.1: Credit Limit versus S.S. Capital and Population

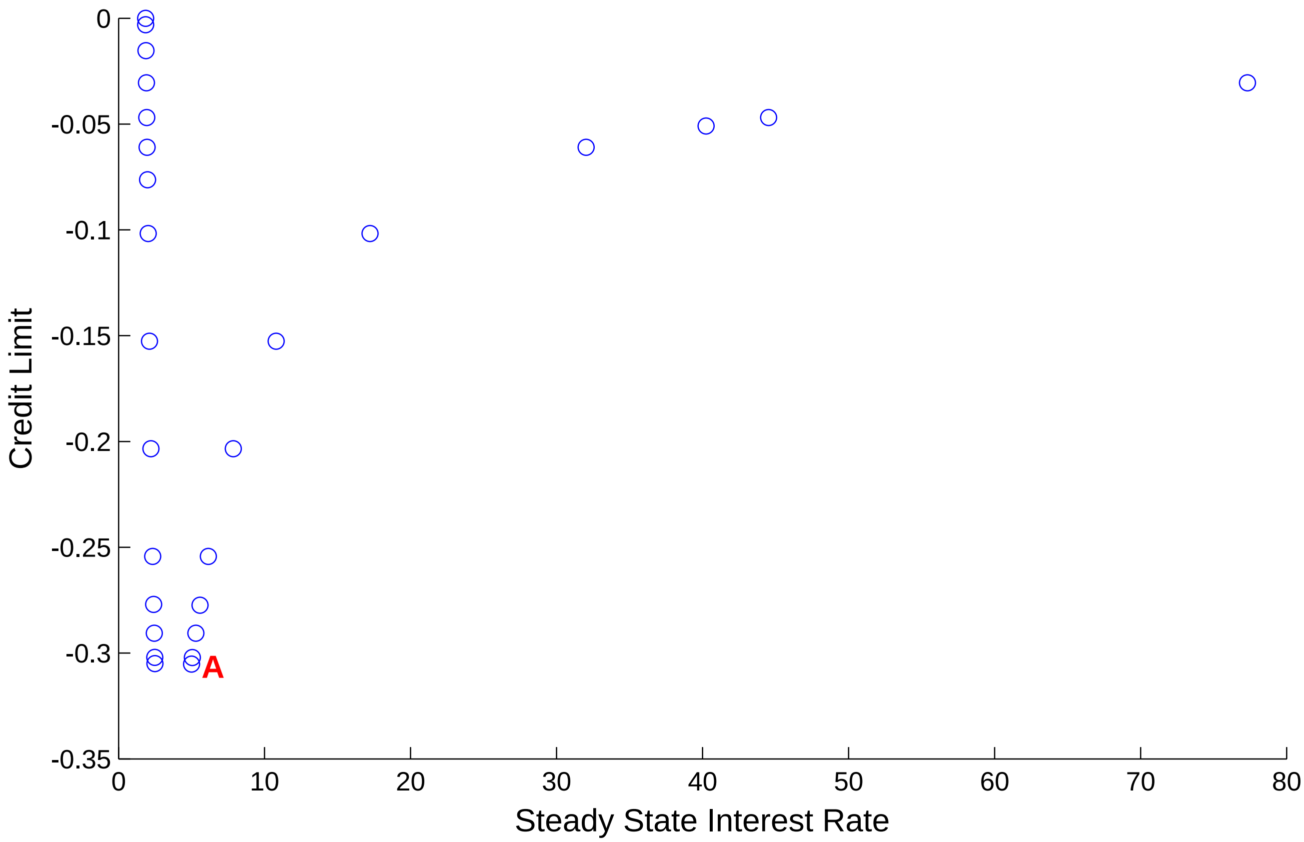

Figure 2.2: Mapping from Credit Limit to Interest Rate

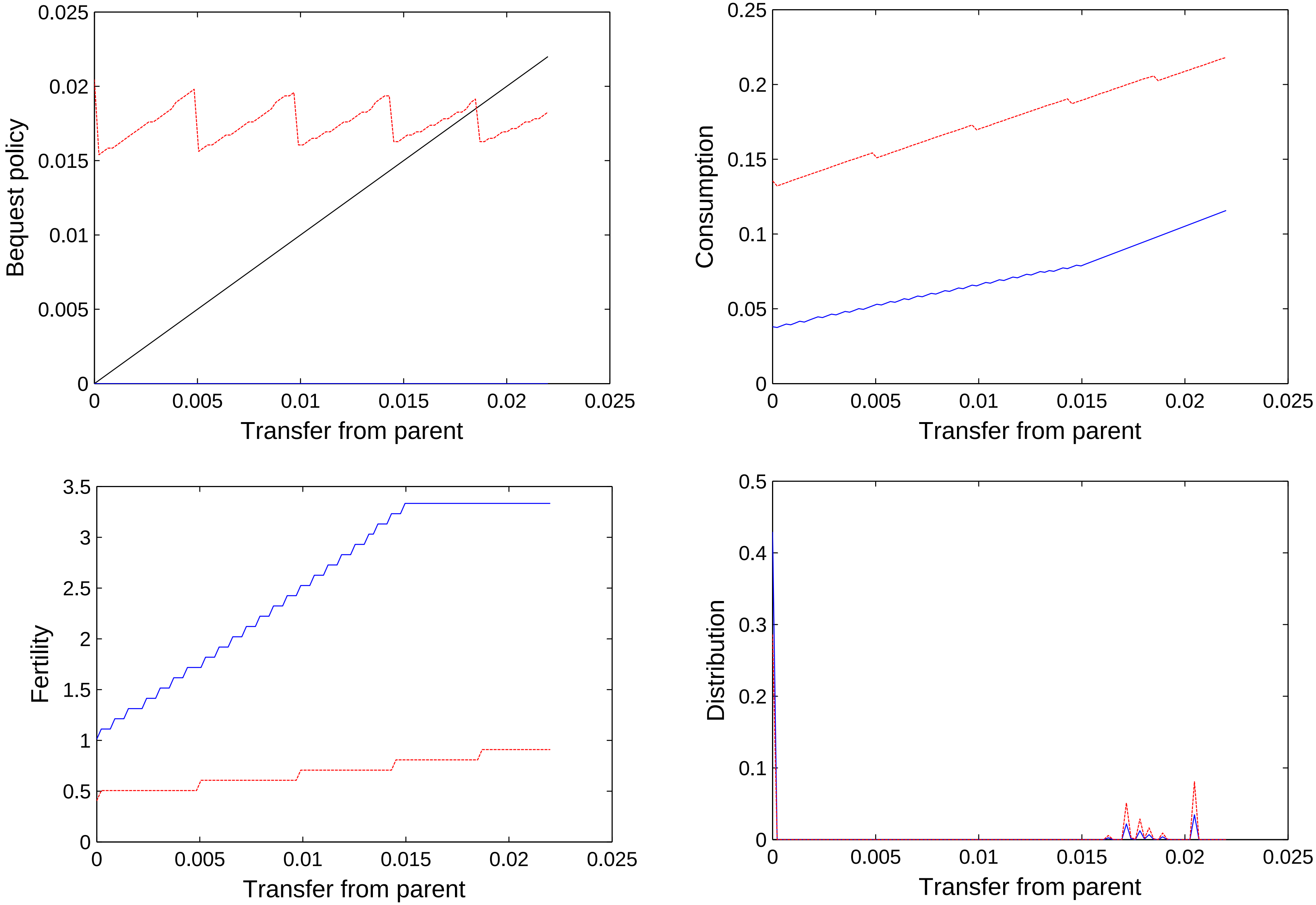

Figure 2.3: Bequest Policy, Consumption, Fertility and Distribution

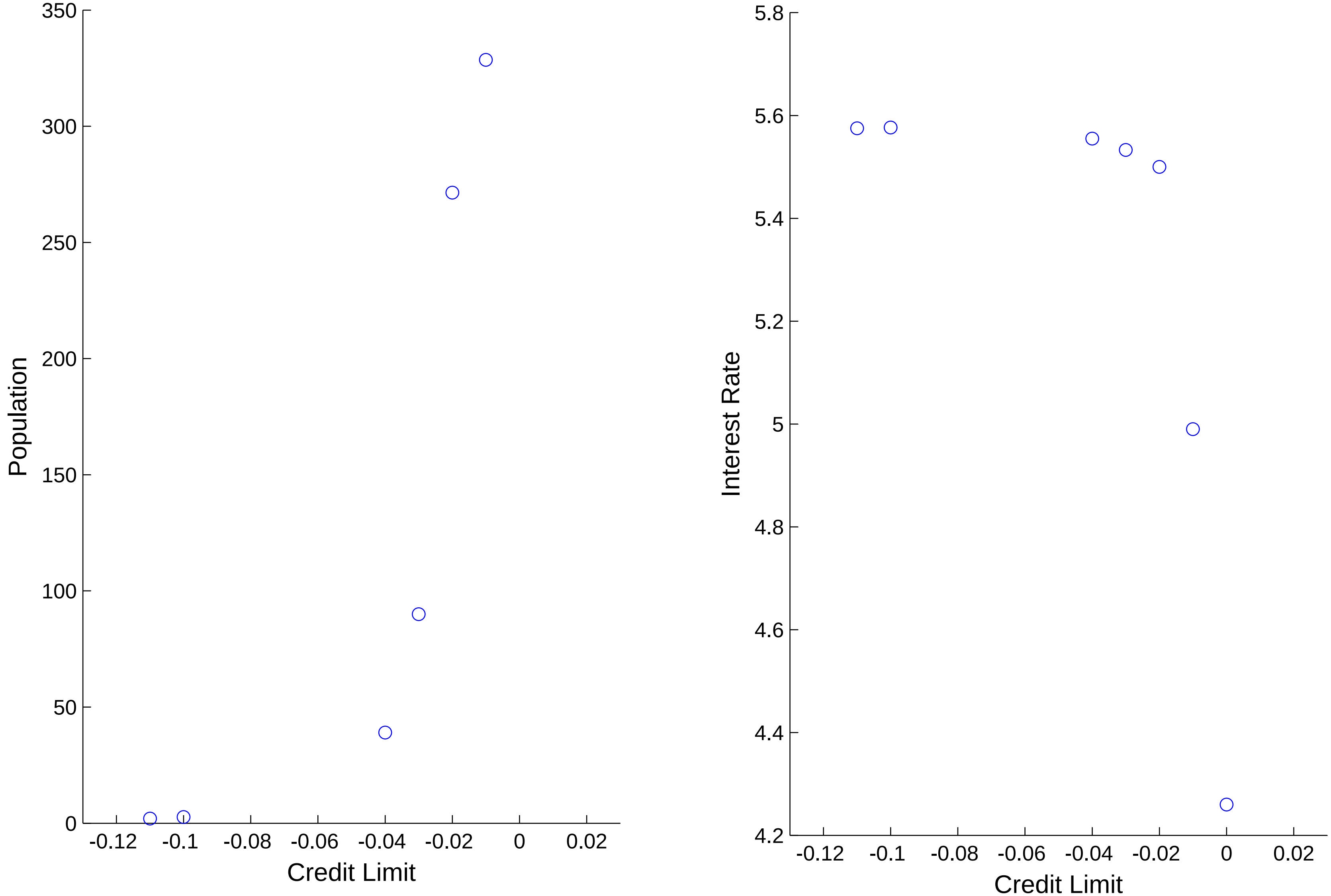

Figure 2.4: Population and Interest Rate at Stationary Distribution

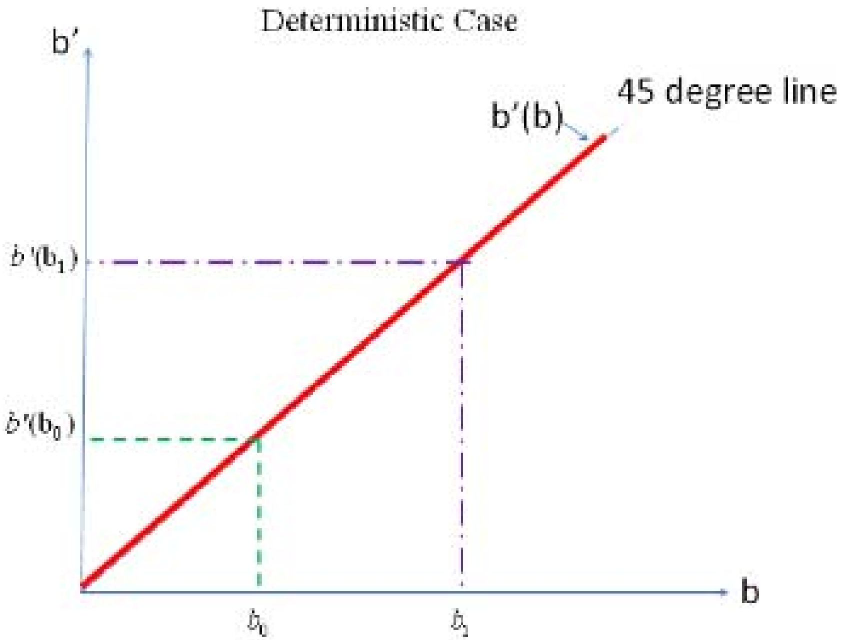

Figure 3.1a: Optimal Bequest Policy (Exog Fertility)

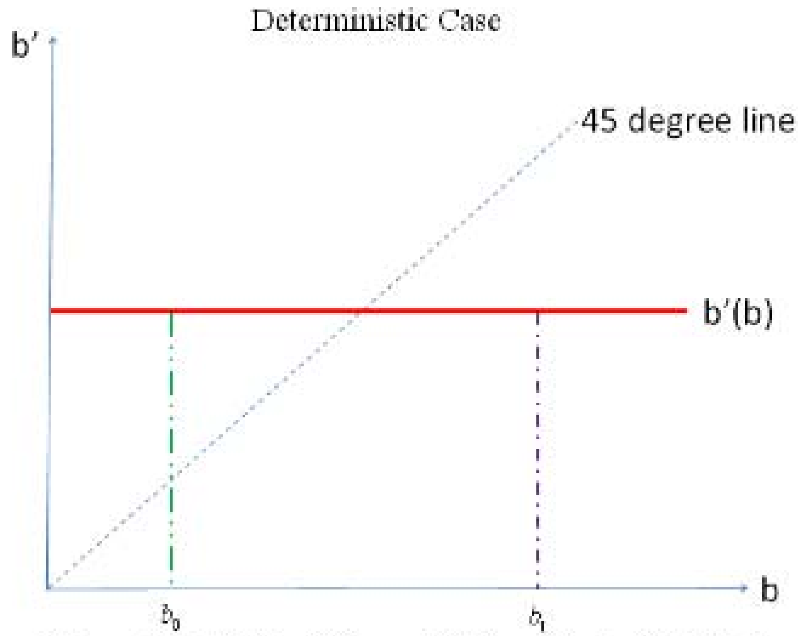

Figure 3.1b: Optimal Bequest Policy (Endog Fertility)

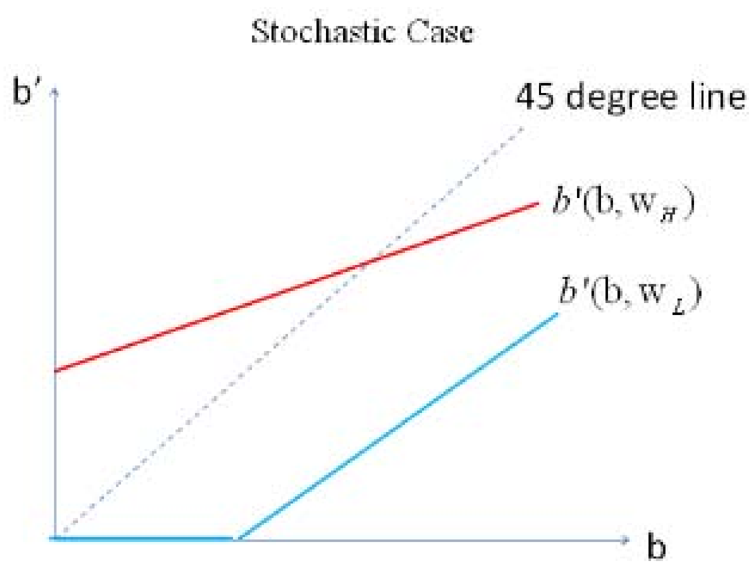

Figure 3.2a: Optimal Bequest Policy (Exog Fertility)

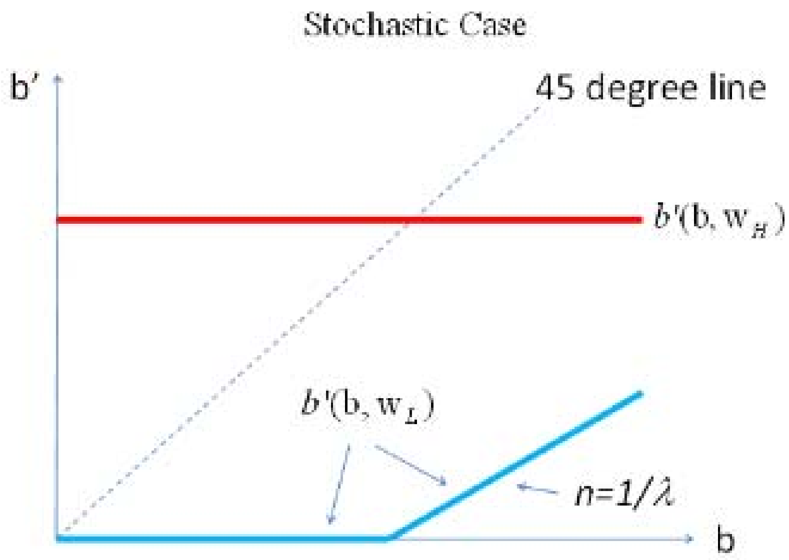

Figure 3.2b: Optimal Bequest Policy (Endog Fertility)

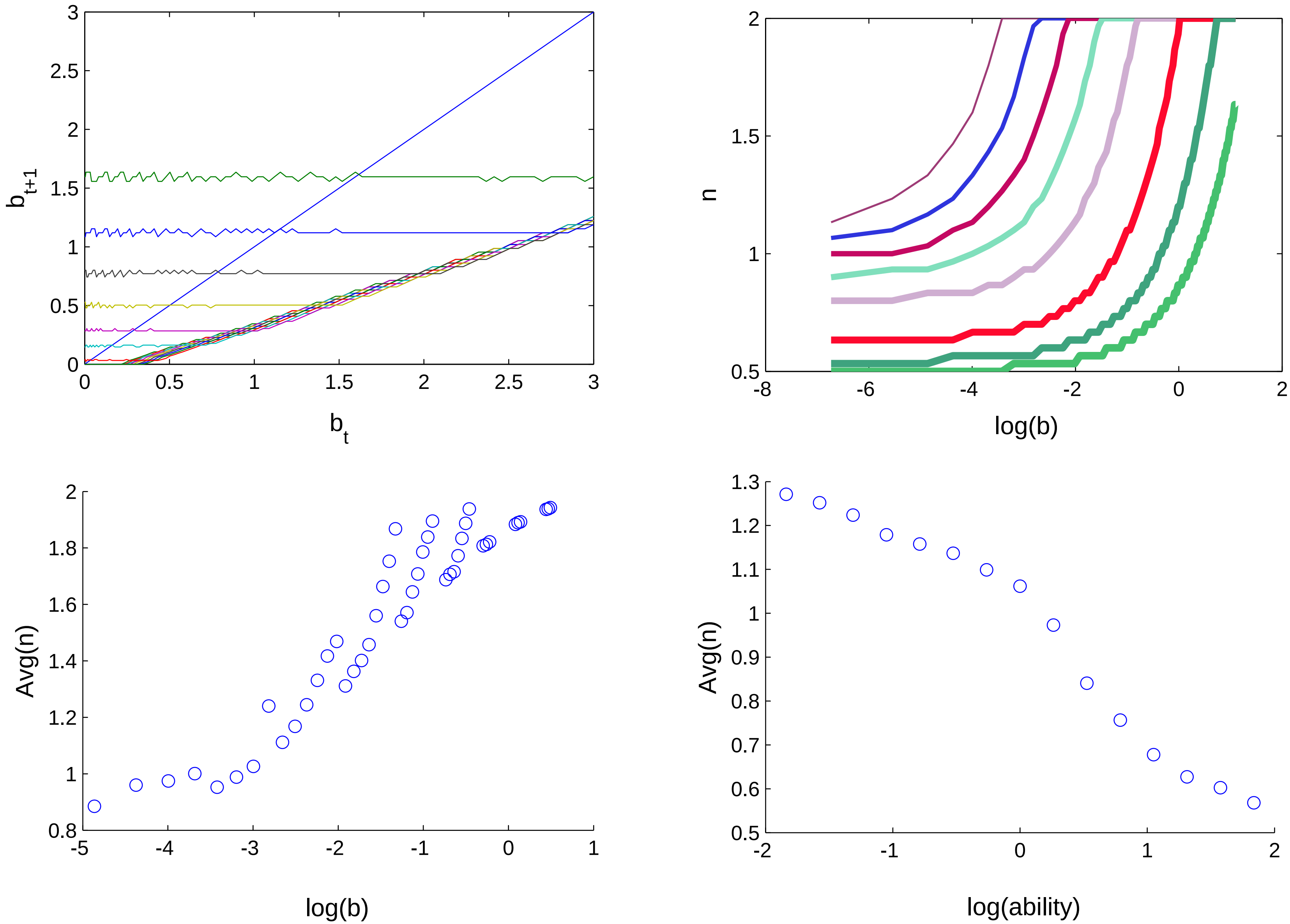

Figure 3.3: Model 3 (Alvarez), Policy Functions and Some Predictions

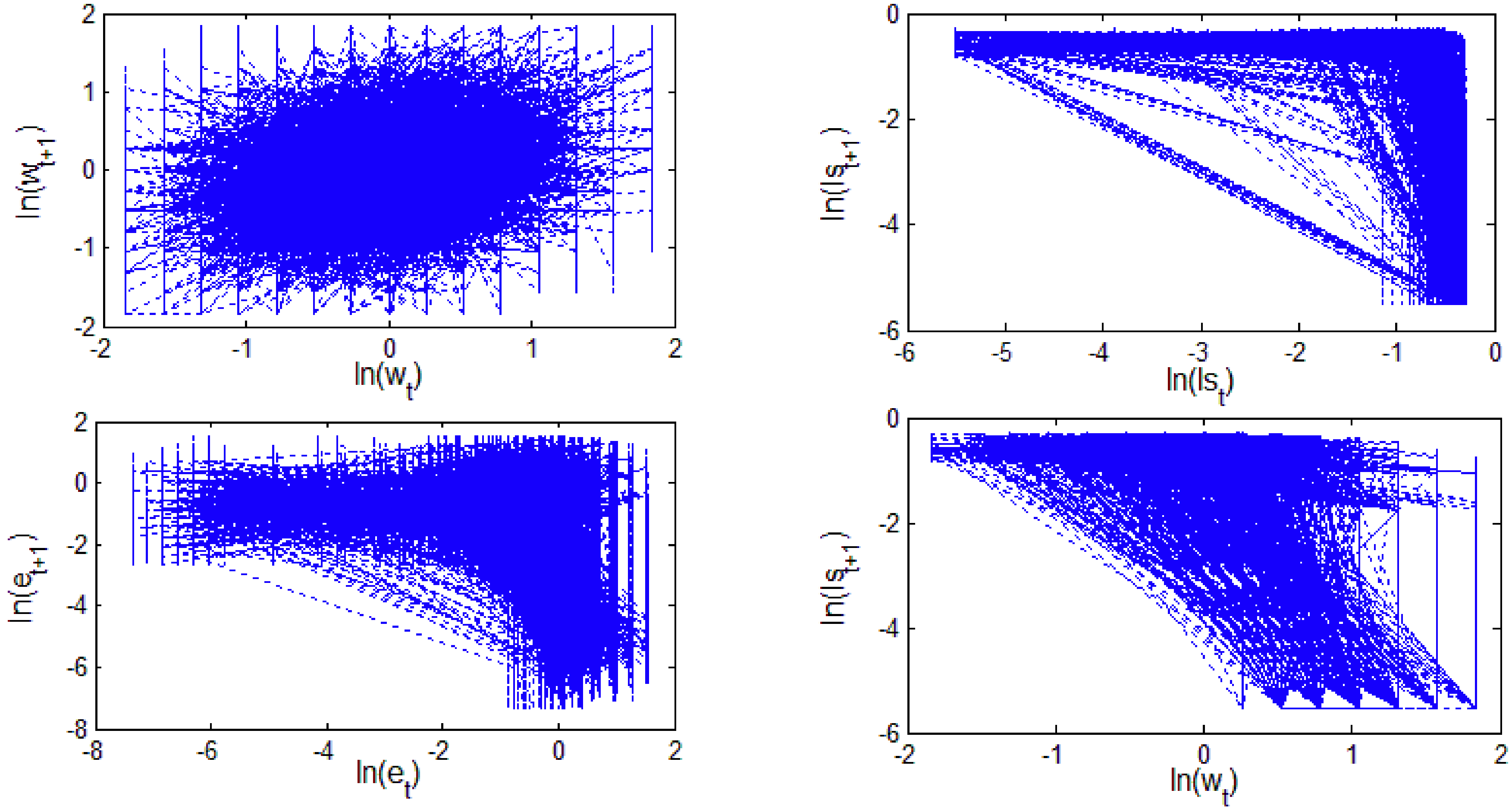

Figure 3.4: Model 3 (Alvarez), Persistence of Abilities, Labor Supply and Earnings

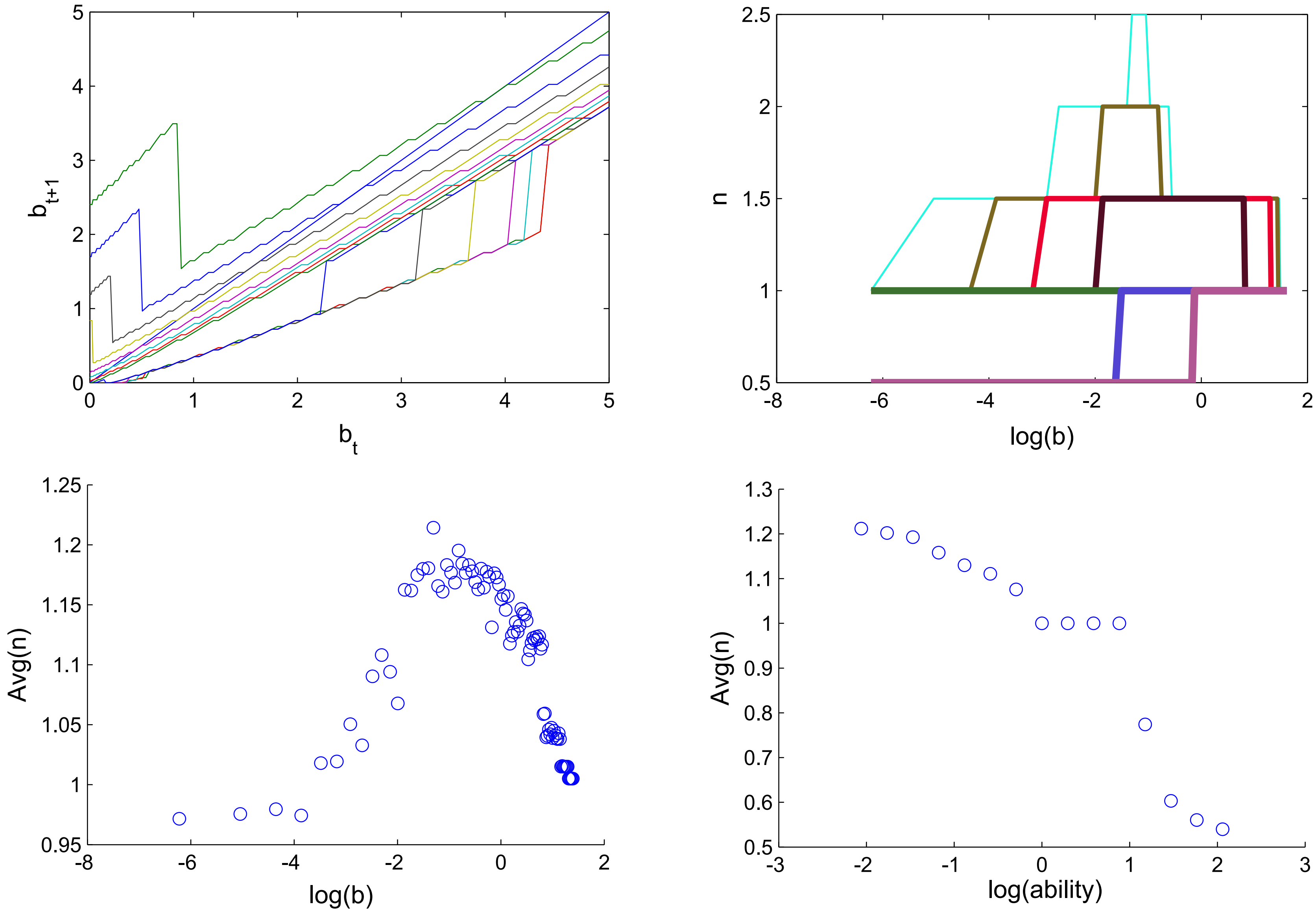

Figure 3.5: Model 6 (Exp. Disc + Dimin. Costs),
Policy Functions and Some Predictions

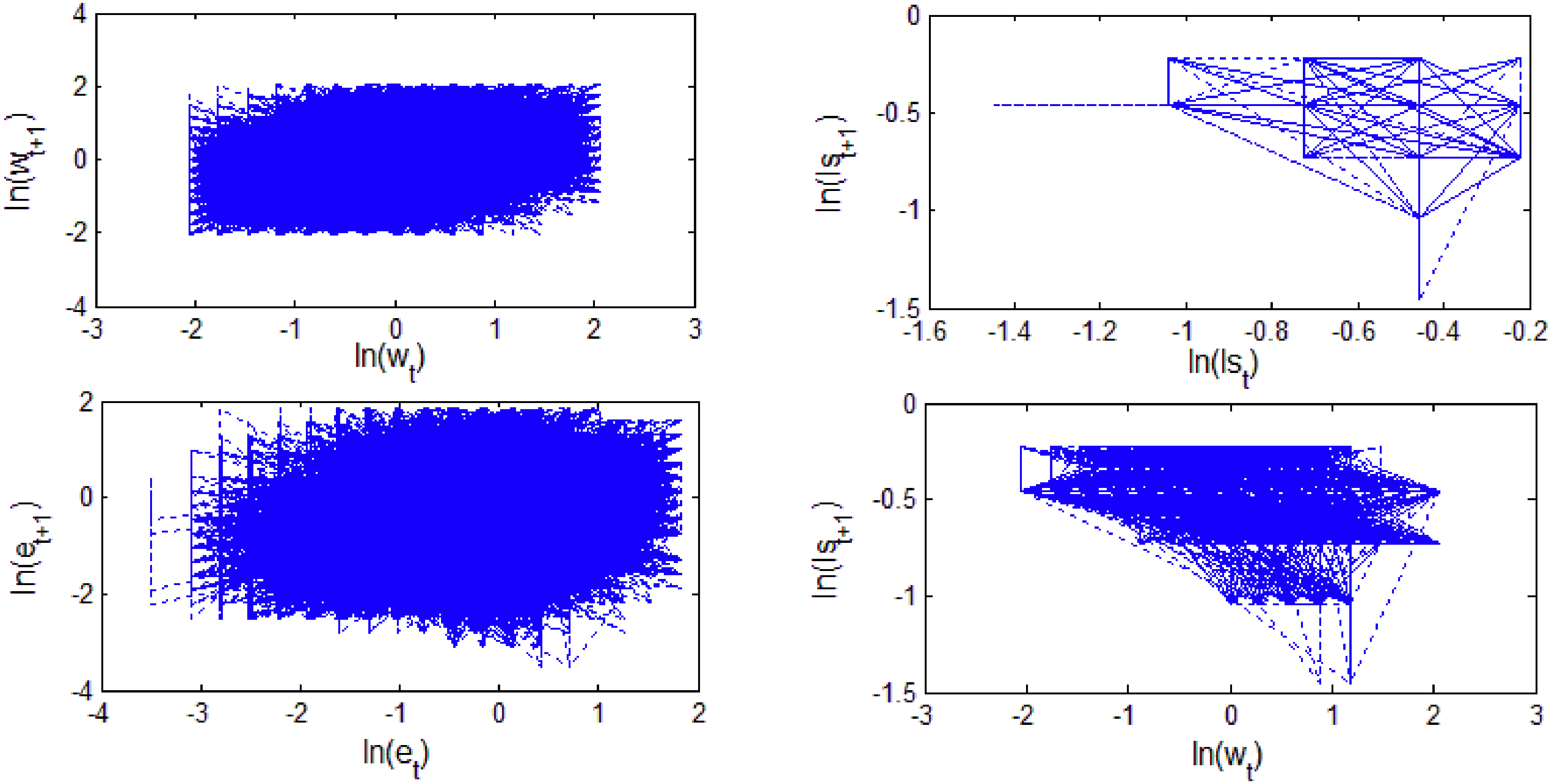

Figure 3.6: Model 6 (Exp. Disc + Dimin. Costs),
Persistent of Abilities, Labor Supply and Earnings

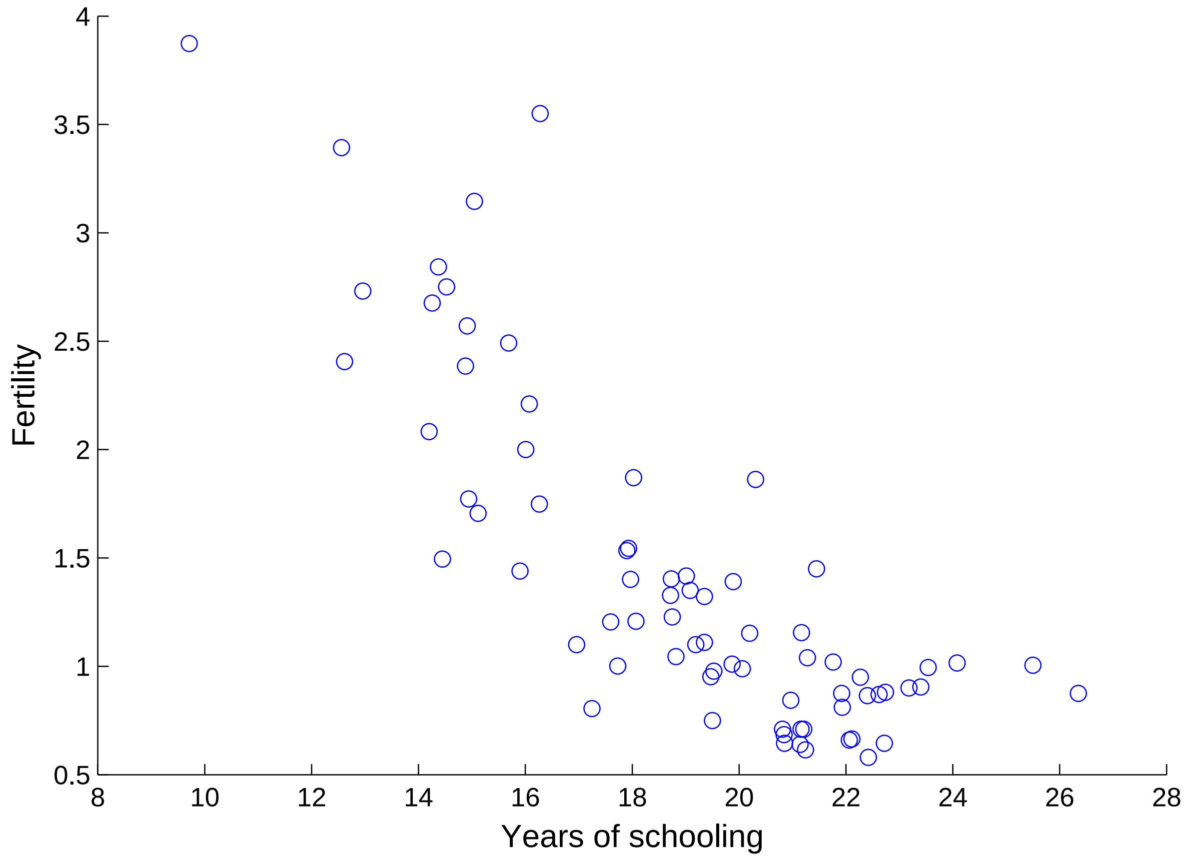

Figure 4.1: Schooling Data versus Fertility Data - 2005

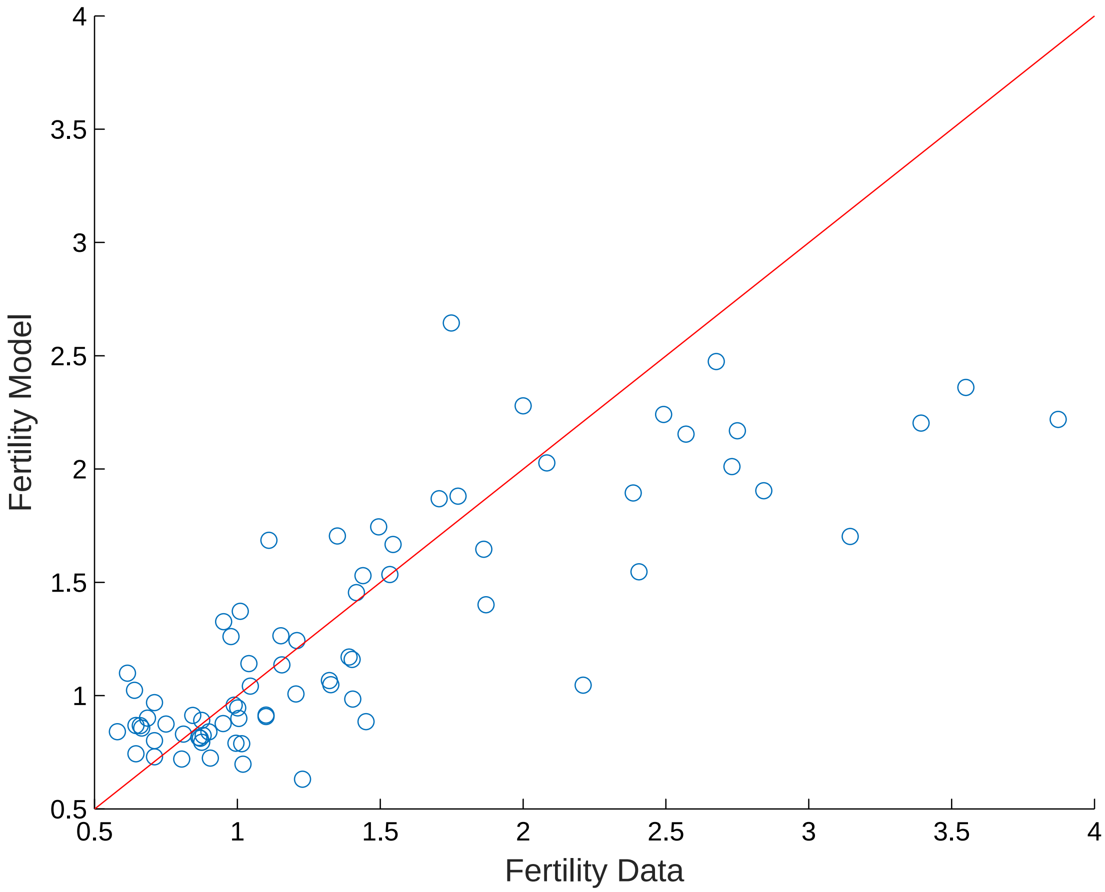

Figure 4.2: Fertility in the Model and the Data - 2005

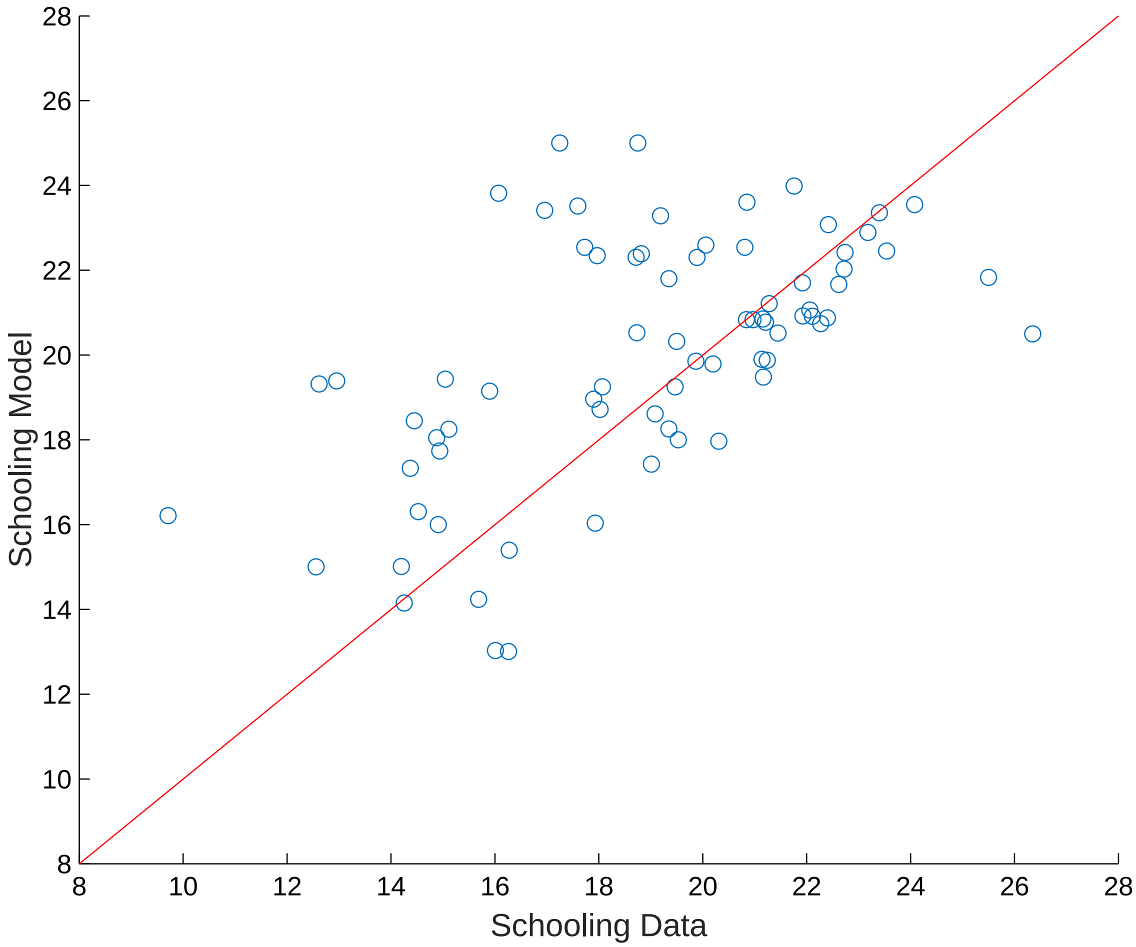

Figure 4.3: Years of Schooling in the Model and the Data - 2005

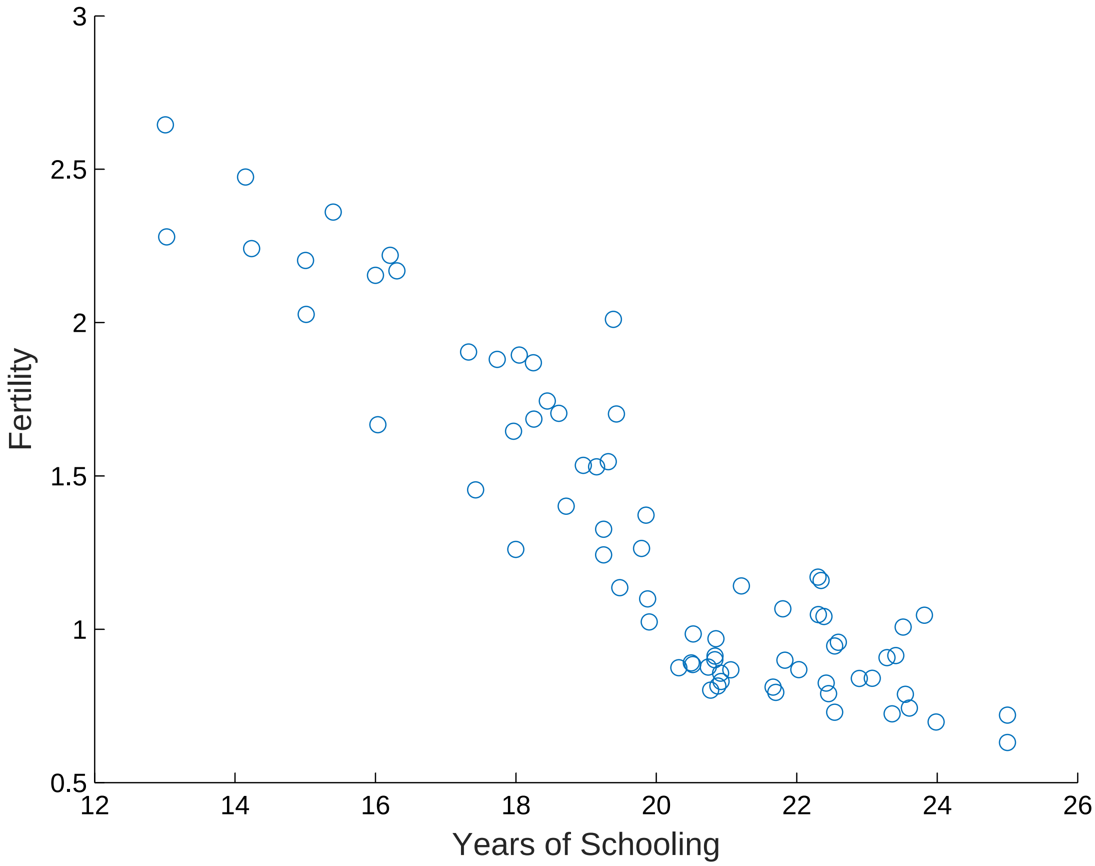

Figure 4.4: Fertility versus Years of Schooling

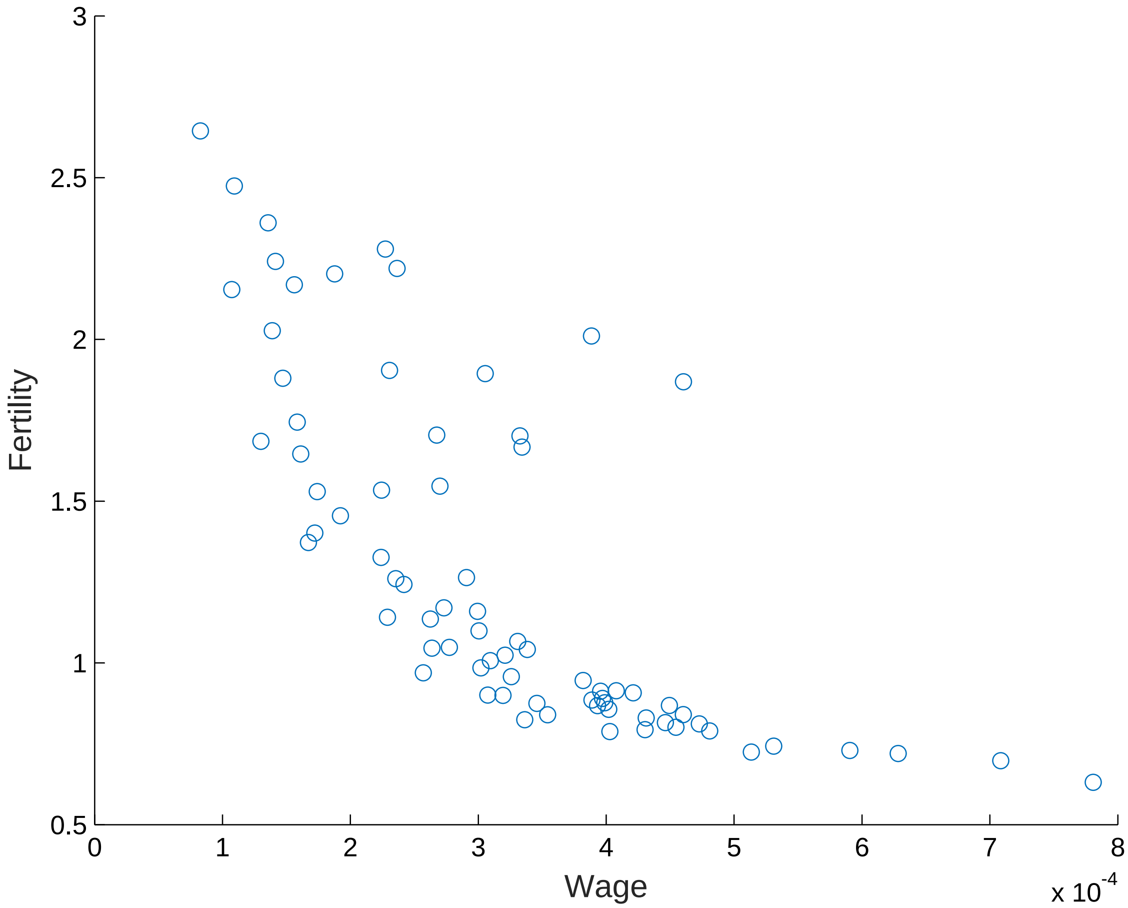

Figure 4.5: Fertility versus Wage

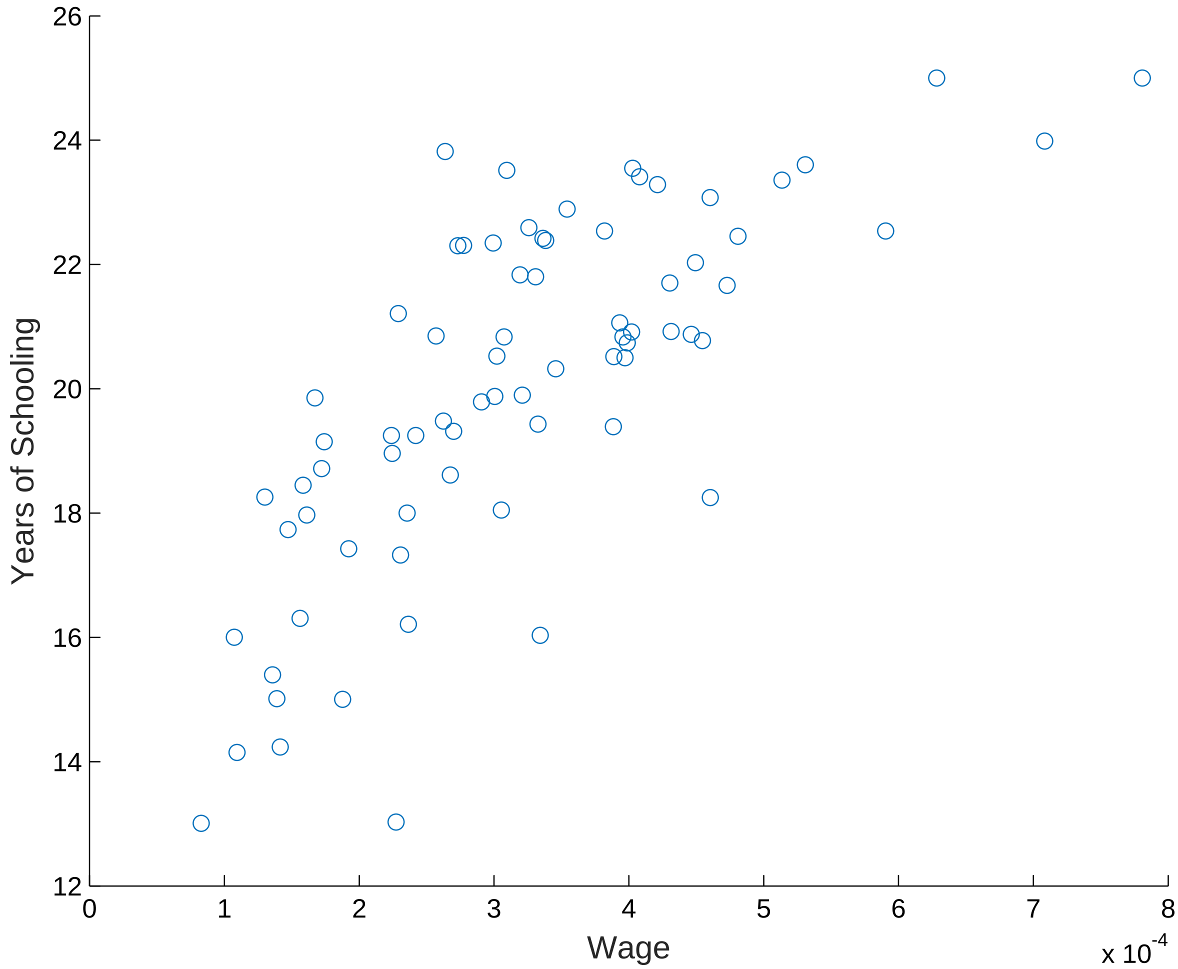

Figure 4.6: Years of Schooling versus Wage

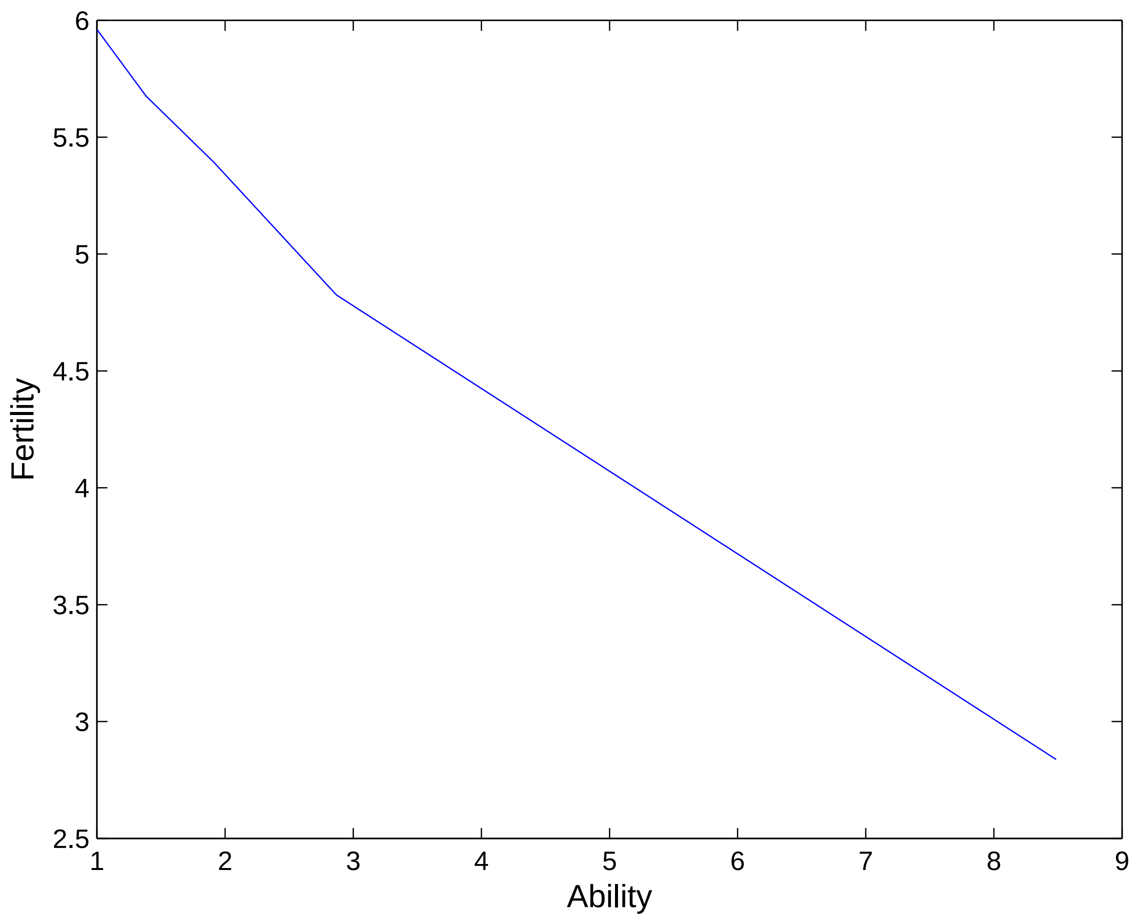

Figure 5.1: Fertility versus Earning Abilities

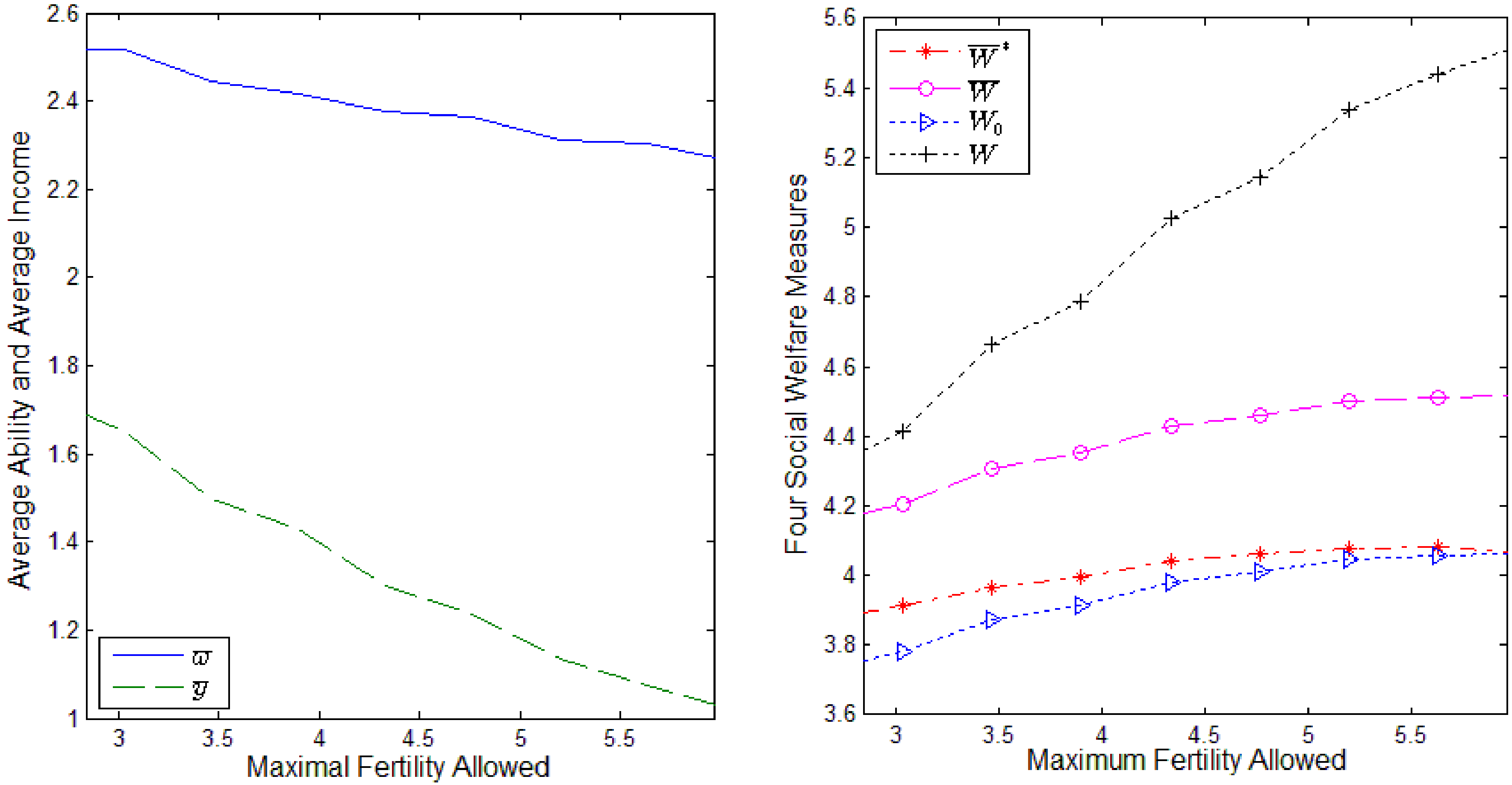

Figure 5.2: Effects of Limiting Fertility

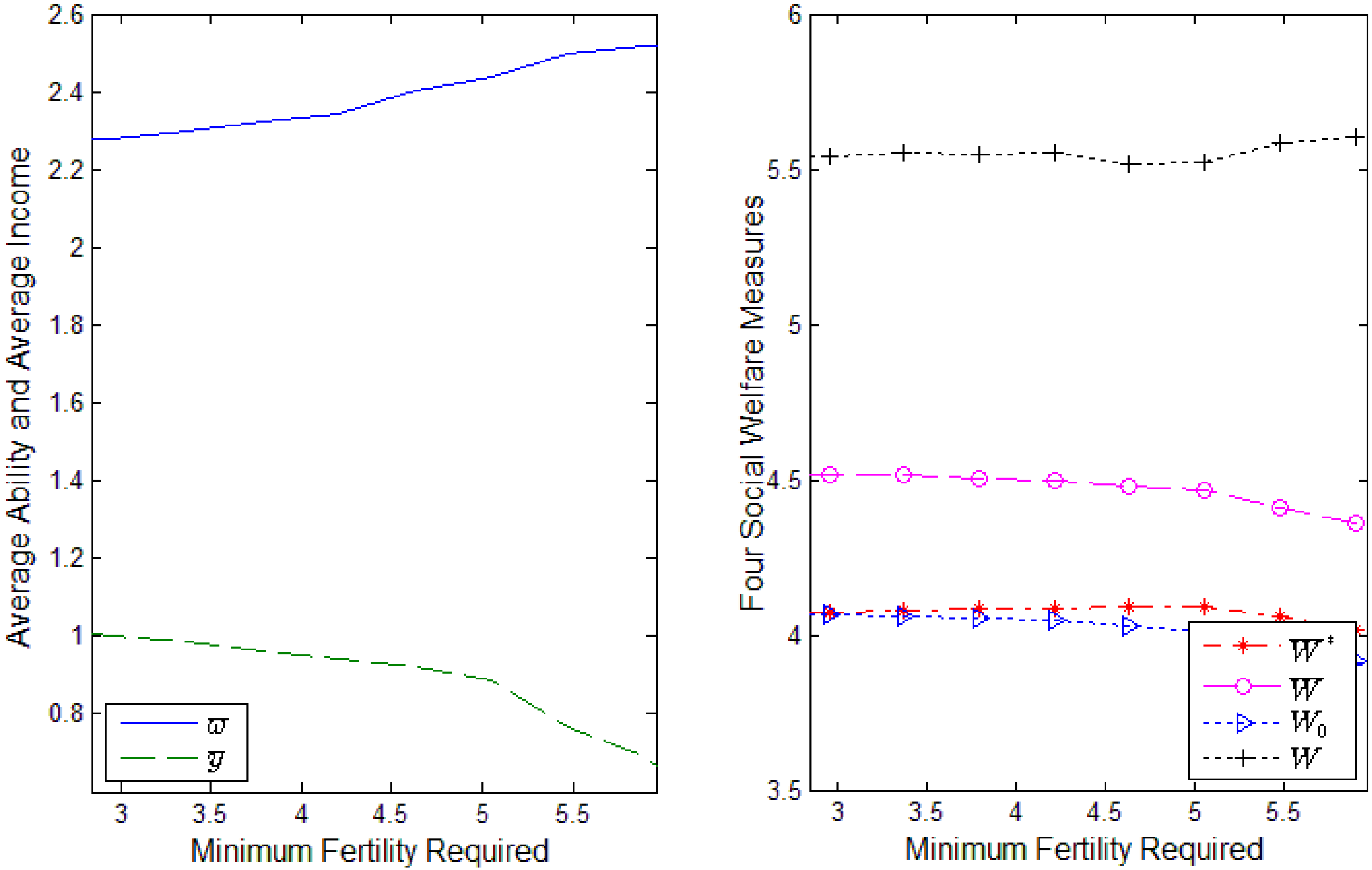

Figure 5.3: Effects of Raising Fertility

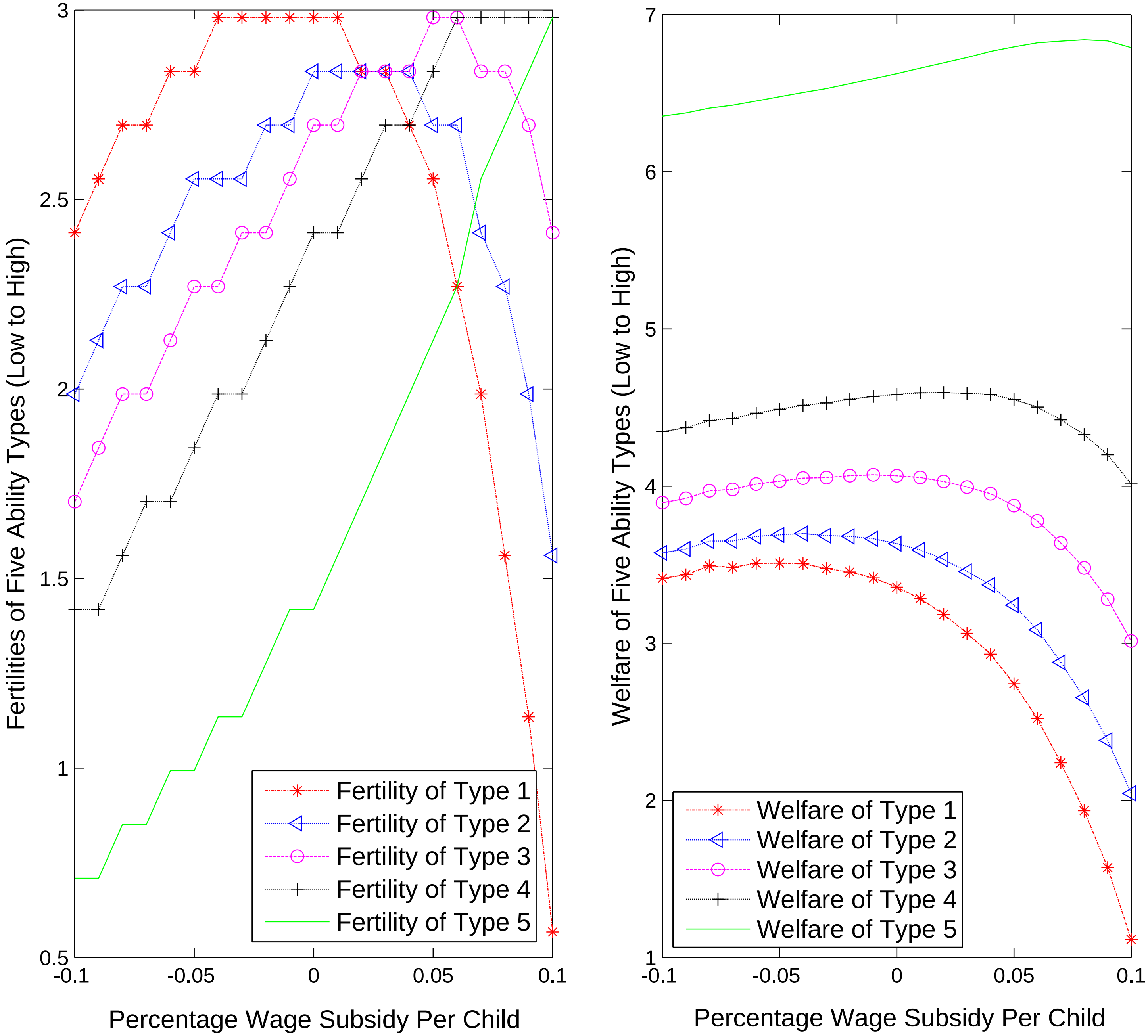

Figure 5.4: Policy Effects of Fertility Related Wage Taxes and Subsidies

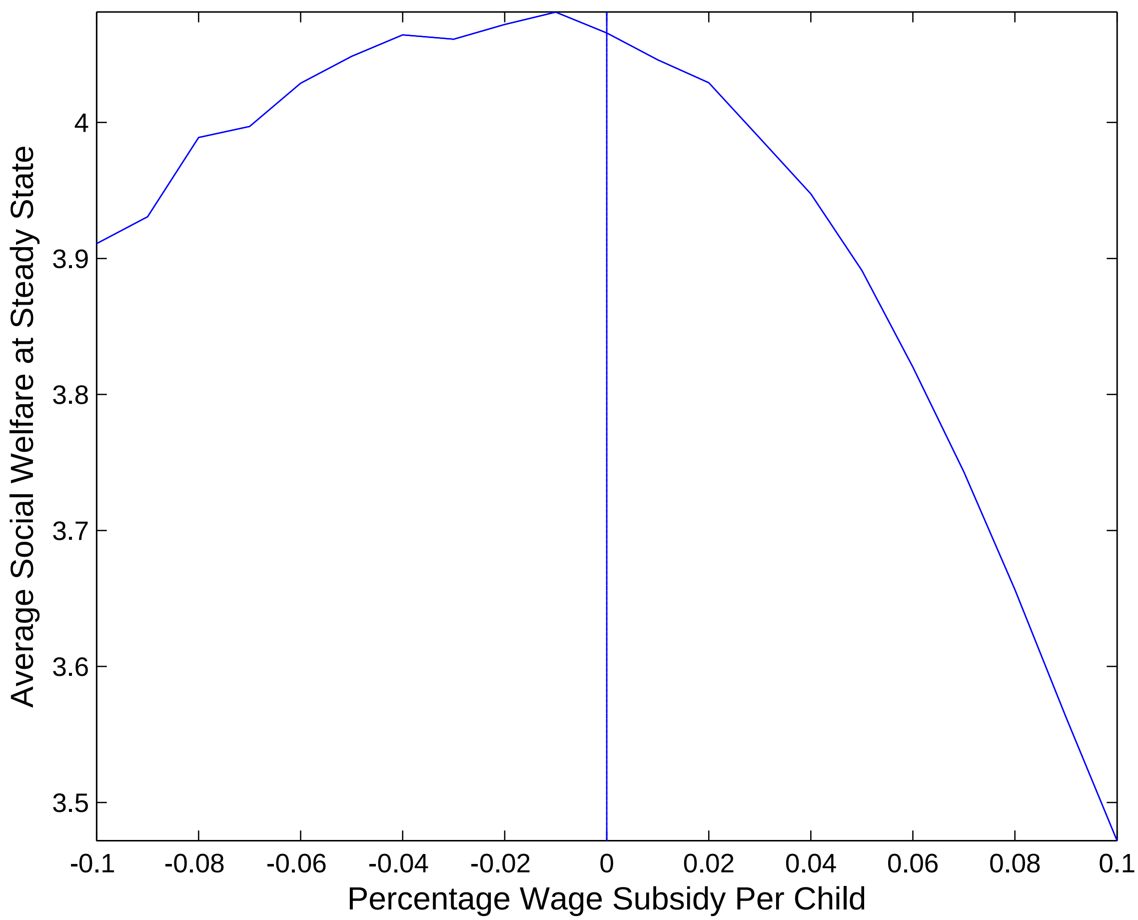

Figure 5.5: Effects of Fertility Related Taxes and Subsidies on Steady State Average Social Welfare

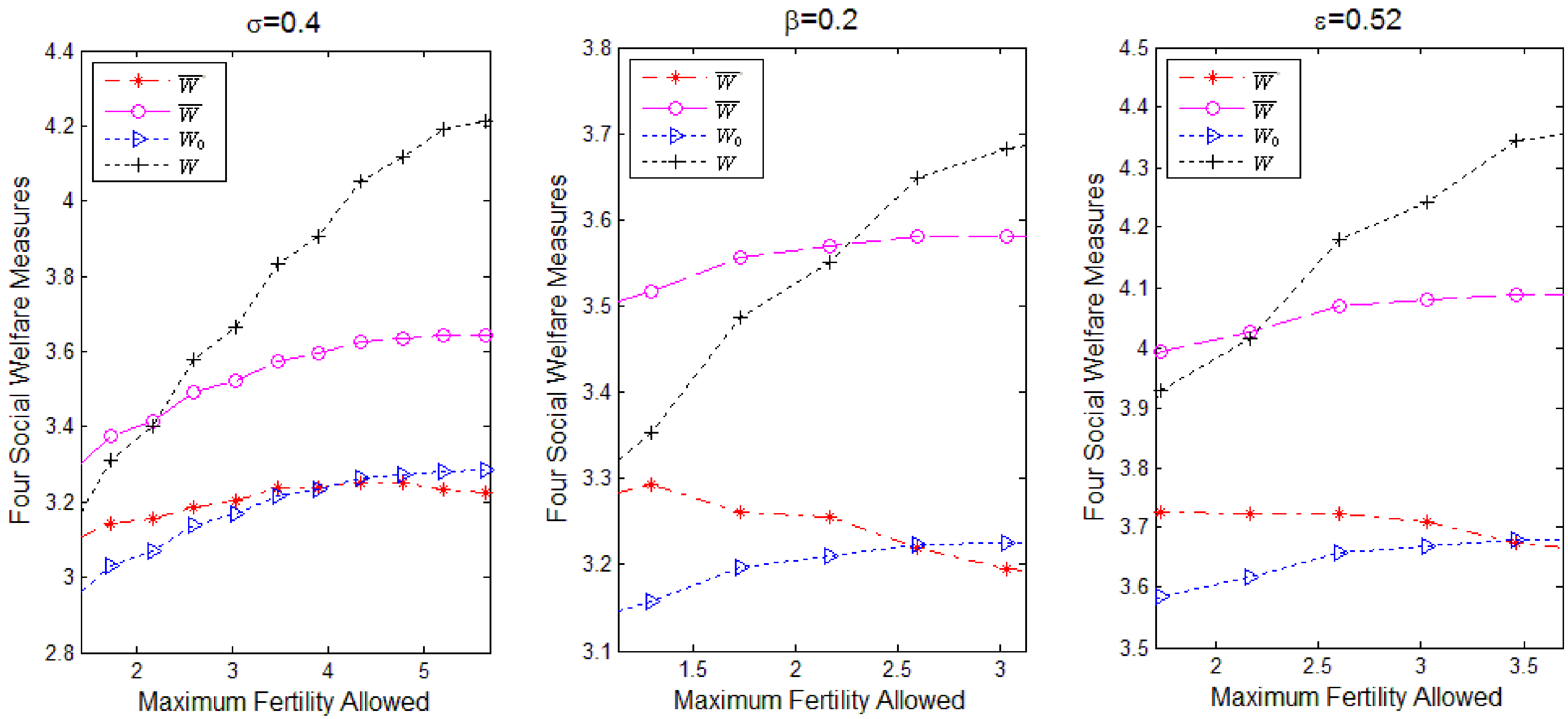

Figure 5.6: Robustness Checks

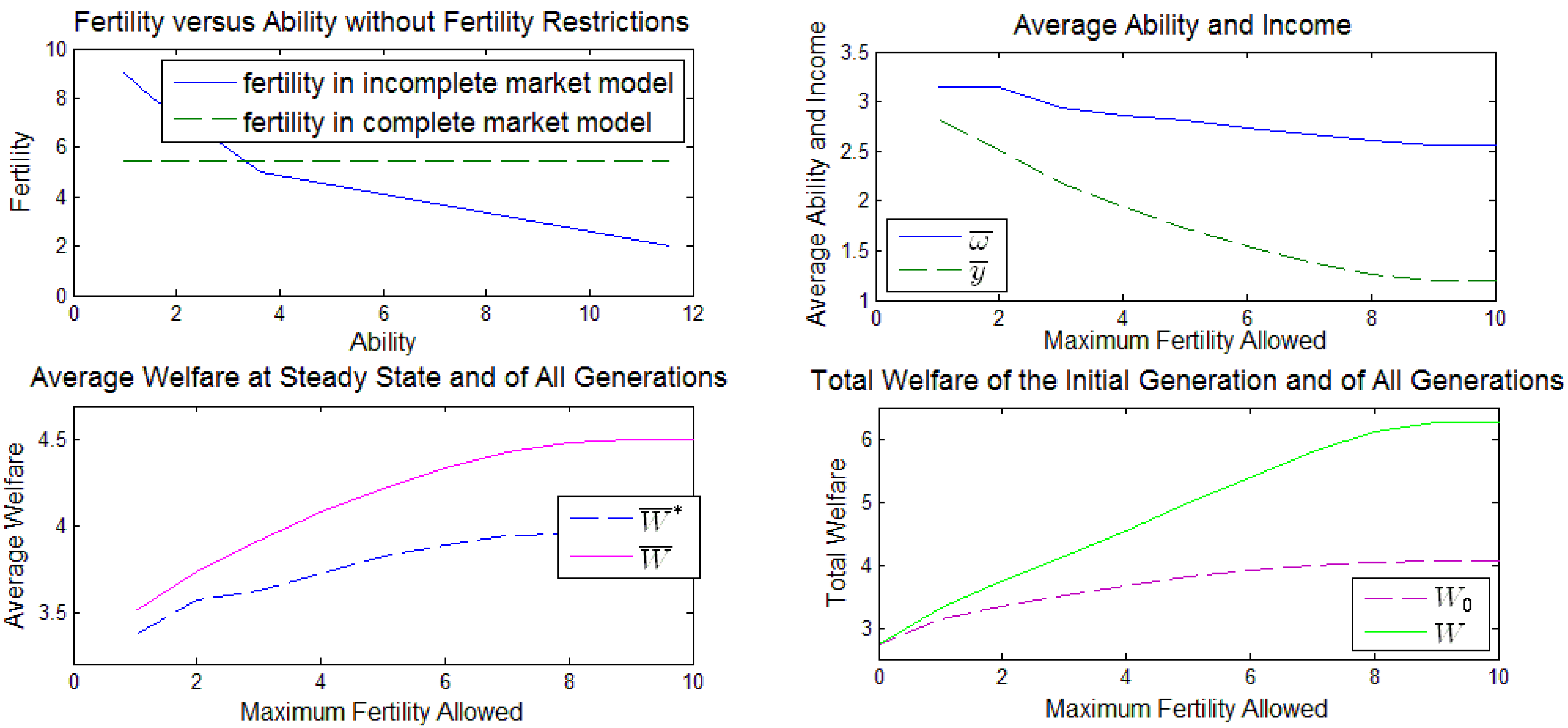

Figure 6.1: Effects of Fertlity Restrictions: Upper Bound

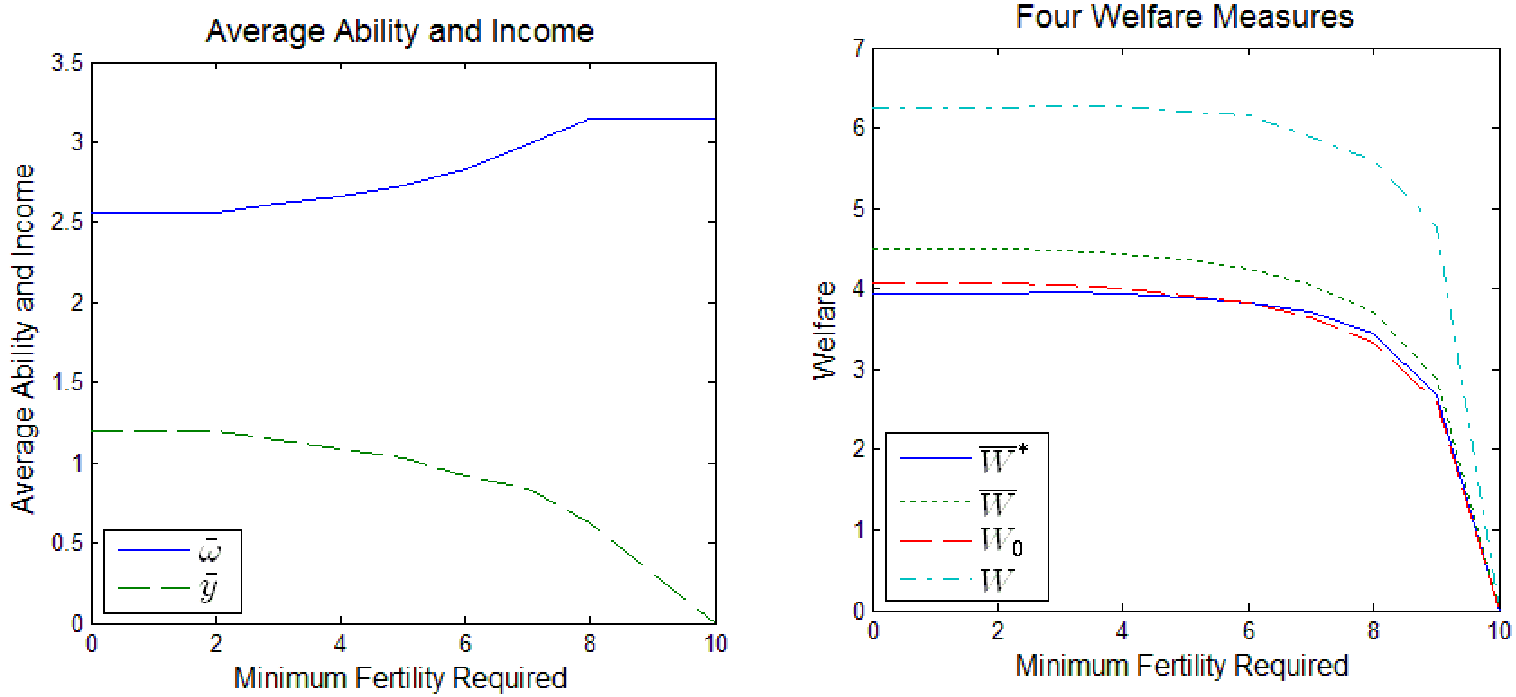

Figure 6.2: Effects of Fertility Restrictions: Lower Bound

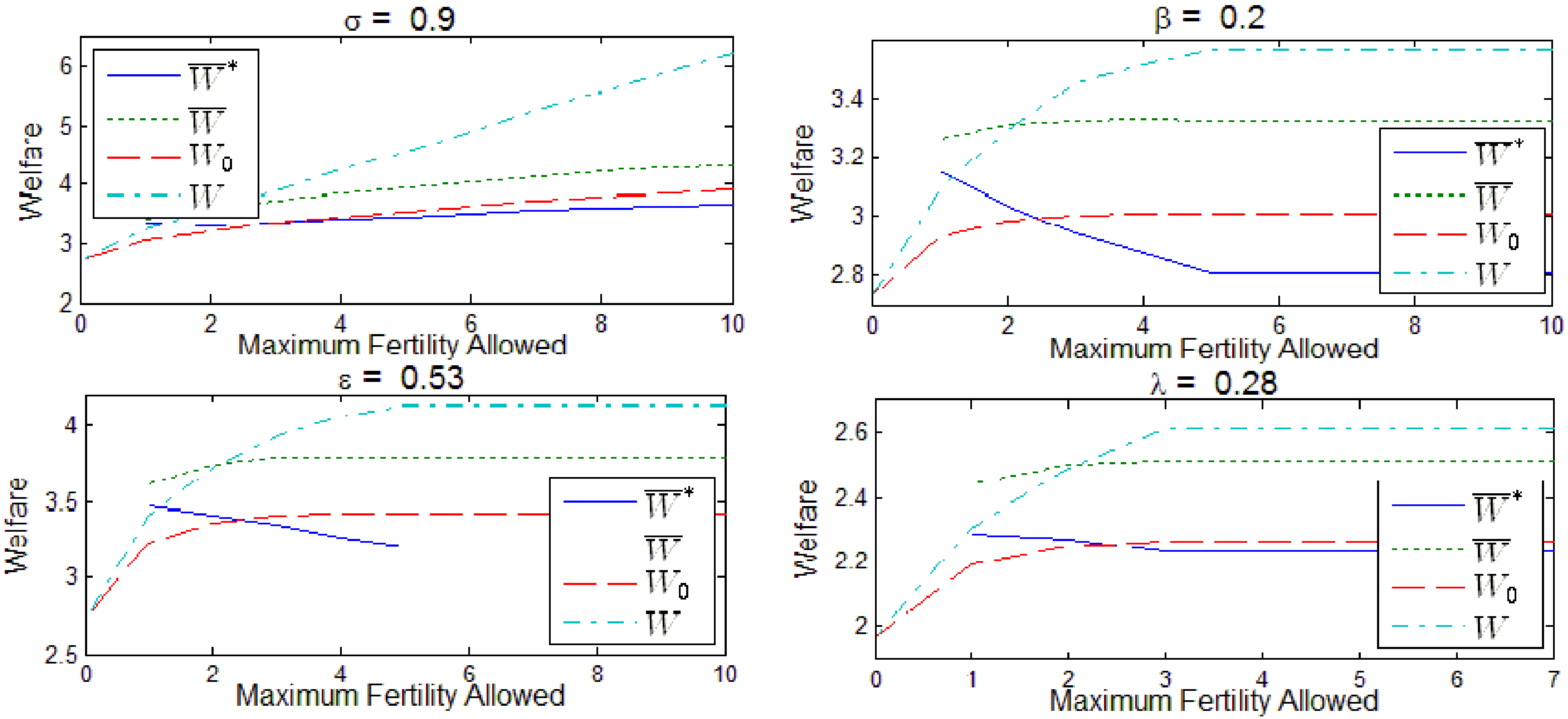

Figure 6.3: Robustness Checks

BIBLIOGRAPHY

[1] Atkinson, A. B. (1970), "On the Measurement of Inequality", Journal of Economic Theory, Vol. 2, pp. 244-263.

[2] Atkinson, A. B. (2012), "Optimum Population, Welfare Economics, and Inequality", Oxford Martin School Seminar Paper.

[3] Atkinson, A. B. and Brandolini, A. (2010), "On Analyzing the World Distribution of Income", World Bank Economic Review, Vol. 24, pp. 1-37.

[4] Atkeson, A. and Lucas, R. (1992), "On Efficient Distribution with Private Information", Review of Economic Studies, Vol. 59 (3), pp. 427–453.

[5] Alvarez, F. (1999), "Social Mobility: The Barro-Becker Children Meet the Laitner-Loury Dynasties", Review of Economic Dynamics, Vol. 2, pp. 65-103.

[6] Aiyagari, R. (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving", Quarterly Journal of Economics, Vol. 109 (3), 659-684.

[7] Barro, Robert J. and Becker, Gary S (1989), "Fertility Choice in a Model of Economic Growth", Econometrica, 57 (2): 481-501.

[8] Becker, G. S. (1960), "An Economic Analysis of Fertility, in Demographic and Economic Change in Developed Countries", Conference of the Universities, National Bureau Committee for Economic Research, Princeton University Press, pp. 209-240.

[9] Becker, G. S. (1983), "Some Notes on Population Growth and Economic Growth", Unpublished manuscript.

[10] Becker, Gary S and Barro, Robert J.(1988), "A Reformulation of the Economic Theory of Fertility", The Quarterly Journal of Economics, Page [1] of 1-25

[11] Becker, G. S and Lewis, H. Gregg (1973), "On the Interaction between the Quantity and Quality of Children", The Journal of Political Economy 81: pp. S279–S288.

[12] Bernheim, B. D. and S. Severinov. (2003). "Bequests as Signals: An Explanation for the Equal Division Puzzle", Journal of Political Economy, Vol. 111 (4), 733-764.

[13] Benhabib, J. and K. Nishimura (1993), "Endogenous Fertility and Growth", in General Equilibrium, Growth, and Trade II, in R. Becker et al., eds., Academic Press, San Diego, 237-247.

[14] Ben-Porath, Y. (1967), "The Production Function of Human Capital and the Life Cycle of Earnings", The Journal of Political Economy, 75 (4, Part 1), 352-365.

[15] Bils, M., Klenow, P. (2000a), "Does Schooling Cause Growth?", American Economic Review, 90(5), 1160-1183.

[16] Bosi, S., R. Boucekkine, and T. Seegmuller. (2011), "The Dynamics of Wealth Inequality under Endogenous Fertility: A Remark on the Barro-Becker Model with Heterogenous Endowments", Theoretical Economics Letters, Vol. 1(1), 3-7.

[17] Brock, W. and L. Mirman. (1972), "Optimal Economic Growth and Uncertainty: The Discounted Case", Journal of Economic Theory, Vol. 4 (3), 479-513.

[18] Cagetti, M. and M. De Nardi. (2008). "Wealth Inequality: Data and Models", Macroeconomic Dynamics, Vol. 2, Supplement 2, 285-313.

[19] Caselli, F. and J. Feyrer (2007), "The Marginal Product of Capital", The Quarterly Journal of Economics, Vol 122 (2): 537-568.

[20] Castaneda, A. J. Diaz-Gimenez, and J.V. Rios-Rull. (2003). "Accounting for the US Earnings and Wealth Inequality", Journal of Political Economy, Vol. 111 (4), 818-857.

[21] Chatterjee, S. (1994). "Transitional Dynamics and the Distribution of Wealth in a Neoclassical Growth Model", Journal of Public Economics, Vol. 54, 97-119.

[22] Chu, C. Y. C. and Koo, H.-W. (1990), "Intergenerational Income-Group Mobility and Differential Fertility", American Economic Review, Vol. 80, pp. 1125-1138.

[23] Cordoba, J. C. and Liu, X. (2014), "Altruism, Fertility and Risk", Working Paper No. 14010, Department of Economics, Iowa State University.

[24] Cordoba, J. C., Liu, X. and Ripoll, M. (2014), "Cross Country Differences in Fertility and Schooling: An Explanation", manuscript.

[25] Cordoba, J. C., Liu, X. and Ripoll, M. (2015), "Endogenous Fertility, Social Mobility and Long Run Inequality", forthcoming in Journal of Monetary Economics, Vol 63, 2016.

[26] Cordoba, J. C. and M. Ripoll. (2011), "A Contribution to the Economic Theory of Fertility", Working Paper No. 11011, Department of Economics, Iowa State University.

[27] Cordoba, J. C. and M. Ripoll. (2012), "Barro-Becker with Credit Frictions", Working Paper No. 12019, Department of Economics, Iowa State University.

[28] Cordoba, J. C. and Ripoll, M. (2014), "Intergenerational Transfers and the Fertility-Income Relationship", Economic Journal, forthcoming.

[29] Cordoba, J. C. and Ripoll, M. (2014), "The Elasticity of Intergenerational Substitution, Parental Altruism, and Fertility Choice", Iowa State University working paper, No. 14015.

[30] Cordoba, J. C. and Ripoll, M., (2013), "What explains schooling differences across countries?", Unpublished manuscript.

[31] Dasgupta, P. (2005), "Regarding Optimum Population", The Journal of Political Philosophy, Vol. 13, No. 4, pp. 414-442.

[32] Dasgupta, P., Sen, A. and Starrett, D. (1973), "Notes on the Measurement of Inequality", Journal of Economic Theory, Vol. 6, pp. 180-187.

[33] Davidson, R. and Duclos, J.-Y. (2000), "Statistical Inference for Stochastic Dominance and for the Measurement of Poverty and Inequality", Econometrica, Vol. 68, pp. 1435-1464.

[34] De la Croix, D. (2012), "Fertility, Education, Growth and Sustainability", Cambridge University Press.

[35] De la Croix, D. and M. Doepke. (2003). "Inequality and Growth: Why Differential Fertility Matters", American Economic Review, Vol. 93 (4), 1091-1113.

[36] Doepke, Matthias (2004), "Accounting for Fertility Decline During the Transition to Growth", Journal of Economic Growth, Springer, vol. 9(3), pages 347-383, 09.

[37] Eckstein, Z., Stern, S., and Wolpin, K.I., (1988), "Fertility Choices, Land, and the Malthusian Hypothesis", International Economic Review, Vol 29, No. 2, pp. 353-361.

[38] Eckstein, Z. and Wolpin, K. I. (1985), "Endogenous Fertility and Optimal Population Size", Journal of Public Economics, Vol. 27, pp. 93-106.

[39] Fincher, L. H. (2012), "Op-ed Contributor; China’s ’Leftover’ Women", The New York Times. Retrieved 19 March 2014.

[40] Folbre. N. (2008), "Valuing Children: Rethinking the Economics of the Family", Cambridge, MA.

[41] Foster, J. E. and Shorrocks, A. F. (1988a), "Poverty Orderings", Econometrica, Vol. 56, pp. 173-77.

[42] Foster, J. E. and Shorrocks, A. F. (1988b), "Poverty Orderings and Welfare Dominance", Social Choice and Welfare, Vol. 5, pp. 179-198.

[43] Galor, O and Weil, D. N., (1996), "The Gender Gap, Fertility, and Growth", American Economic Review, pp. 374-87.

[44] Galor, O. and Weil, D. N. (2000), "Population, Technology, and Growth: From Malthusian Stagnation to the Demographic Transition and Beyond", American Economic Review, Vol. 90, No. 4, pp 806-828.

[45] Gaviria, A. (2002), "Intergenerational mobility, sibling inequality and borrowing constraints", Economics of Education Review 21: 331-340.

[46] Golosov, M., Jones, L. E. and Tertilt, M. (2007), "Efficiency with Endogenous Population Growth", Econometrica, Vol. 75, pp. 1039-1071.

[47] Greenwood J. and Seshadri A. (2002), "The U.S. Demographic Transition", American Economic Review, 92 (2): 153–159.

[48] Hansen, G. D. and Prescott, E.C.. (2002), "Malthus to Solow", American Economic Review, Vol. 92, No. 4, pp. 1205-1217.

[49] Hosseini, R., Jones, L. E., and Shourideh, A. (2013), "Optimal contracting with dynastic altruism: Family size and per capita consumption", Journal of Economic Theory, Vol 148, pp. 1806-1840.

[50] Huggett, M. (1993), "The Risk-free Rate in Heterogeneous-agent Incomplete-insurance Economies", Journal of Economics Dynamics and Control, Vol. 17, pp. 953-969.

[51] Huggett, M. (1996). "Wealth distribution in life-cycle economies", Journal of Monetary Economics, Vol. 38, 469-494.

[52] Jones, L. E., Schoonbroodt, A. and Tertilt, M. (2011), "Fertility Theories: Can They Explain the Negative Fertility-Income Relationship?", Demography and the Economy, Ch. 2, John B. Shoven, ed., Univ. Chicago Press.

[53] Jones, Larry E. and Schoonbroodt (2010), "A. Baby Busts and Baby Booms: The Response of Fertility to Shocks in Dynastic Models", NBER Working Paper, w16596.

[54] Jones, L. E. and Tertilt, M. (2008), "An Economic History of Fertility in the U.S.: 1826-1960", In Frontiers of Family Economics, Vol. 1, P. Rupert (eds.), Bingley, UK: Emerald Press, pp. 165– 230.

[55] Kaplow, L. and Shavell, S. (2001), "Any Non-welfarist Method of Policy Assessment Violates the Pareto Principle", Journal of Political Economy, 109(2), pp. 281-286.

[56] Kolm, S. (1969), "The Optimal Production of Justice", Public Economics, pp. 145-200.

[57] Kremer, M. (1993), "Population Growth and Technological Change: One Million B.C. to 1990", Quarterly Journal of Economics, Vol. 108, pp. 681-716.

[58] Krusell, P. and A. Smith. (1998), "Income and Wealth Heterogeneity in the Macroeconomy", Journal of Political Economy, Vol. 106 (5), 867-896.

[59] Krusell, P., Mukoyama, T. and Smith, A. A. (2011), "Asset prices in a Huggett economy", Journal of Economic Theory, Vol. 146, pp. 812-844.

[60] Kydland, F. and E. Prescott. (1982), "Time to build and aggregate fluctuations", Econometrica, 50 (6), 1345-1371.

[61] Laitner, J. (1992), "Random Earnings Differences, Lifetime Liquidity Constraints, and Altruistic Intergenerational Transfers," Journal of Economic Theory, Vol. 58, 135-170.

[62] Lam, D. A. (1986), "The Dynamics of Population Growth, Differential Fertility, and Inequality", American Economic Review, Vol. 76, pp. 1103-1116.

[63] Lam, D. A. (1997), "Demographic Variables and Income Inequality", in M. R. Rosenzweig and O. Stark, eds, Handbook of Population and Family Economics, Vol. 1, Part B, Elsevier, Chapter 18, 1015-1059.

[64] Lee, D. R. (2000), "Intergenerational Transfers and the Economic Life Cycle: A Cross-cultural Perspective", In Sharing the Wealth, A. Mason and G. Tapinos Editors, Oxford: Oxford University Press.

[65] Liao, P. J. (2013), "The One-Child Policy: A Macroeconomic Analysis", Journal of Development Economics, Vol. 101, pp. 49-62.

[66] Ljungqvist, L. and Sargent, T. (2012), "Recursive Macroeconomic Theory, 3rd edition", MIT Press, Cambridge MA.

[67] Loury, G. C. (1981), "Intergenerational Transfers and the Distribution of Earnings", Econometrica, Vol. 49, 843-867.

[68] Lucas, R. Jr. (1976), "Econometric Policy Evaluation: A Critique", Carnegie-Rochester Conference Series on Public Policy, Vol. 1(1), pp. 19-46.

[69] Manuelli, R. E. and Seshadri, A. (2009), "Explaining International Fertility Differences", The Quarterly Journal of Economics, Vol. 124, pp. 771-807.

[70] Moav, O. (2005), "Cheap Children and the Persistence of Poverty", The Economic Journal, Vol 115, pp. 88-110.

[71] Mulligan, C. B. (1997), "Parental Priorities and Economic Inequality", Chicago: University of Chicago Press.

[72] Nerlove, M., Razin, A. and Sadka, E. (1986), "Endogenous Population with Public Goods and Malthusian Fixed Resources: Efficiency or Market Failure", International Economic Review. Vol. 27, No. 3, pp. 601-609.

[73] Pazner, E. A. and Razin, A. (1980), "Competitive Efficiency in an Overlapping-generation Model with Endogenous Population", Journal of Public Economics, Vol. 13, pp. 249-258.

[74] Peretto, P. F. and Valente, S. (2011), "Growth on a Finite Planet: Resources, Technology and Population in the Long Run", Manuscript.

[75] Qi, L. and S. Kanaya. (2010), "The concavity of the value function of the extended Barro–Beckermodel", Journal of Economic Dynamics and Control, Vol. 34, 314-329.

[76] Raut, L. K. (1990), "Capital Accumulation, Income Distribution and Endogenous Fertility in an Overlapping Generations General Equilibrium Model", Journal of Development Economics, Vol. 34, pp. 123-150.

[77] Restuccia, D. and C. Urrutia. (2004), "Intergenerational Persistance of Earnings: The Role of Early and College Education", American Economic Review, Vol. 94 (5), 1354-1378.

[78] Rothschild, M. and Stiglitz, J. E. (1973), "Some Further Results on the Measurement of Inequality", Journal of Economic Theory, Vol. 6, pp. 188-204.

[79] Saposnik, R. (1981), "Rank-dominance in Income Distributions", Public Choice, Vol. 36, pp. 147-151.

[80] Saposnik, R. (1983), "Ranking Income Distributions", Econometrica, Vol. 50, pp. 3-17.

[81] Schoonbroodt, A. and Tertilt, M. (2014), "Property rights and efficiency in OLG models with endogenous fertility", Journal of Economic Theory, Vol. 150(C), pp. 551-582.

[82] Sholz, J. K. and A. Seshadri. (2009), "Children and Household Wealth", Manuscript.

[83] Sommer, K. (2013), "Fertility Choice in a Life Cycle Model with Idiosyncratic Uninsurable Earnings Risk", mimeo.

[84] Wang, C. (2012), "History of the Chinese Family Planning program: 1970-2010", Contraception, Vol 85, pp. 563-569.

[85] Willis, R. J. (1985), "Externalities and Population", in D. Gale Johnson and Ronald D. Lee (eds.), Population Growth and Economic Development: Issues and Evidence, Madison, WI: University of Wisconsin Press.

Proof of Proposition 1 The first order condition with respect to N_{t+1} is

$$\begin{aligned}&\frac{\partial V}{\partial N_{t+1}} = \beta N_{t+1}^{1-\epsilon}\left(c_{t+1}\right)^{-\sigma}\left[\frac{F_2\left(\bar{K}, N_{t+1}-\lambda N_{t+2}\right)}{N_{t+1}} - \frac{F\left(\bar{K}, N_{t+1}-\lambda N_{t+2}\right)}{N_{t+1}^2}\right]\\ &+\beta\left(1-\epsilon\right) N_{t+1}^{-\epsilon} u\left(c_{t+1}\right)\\ &-\lambda N_t^{1-\epsilon}\left(c_t\right)^{-\sigma}\frac{F_2\left(\bar{K}, N_t-\lambda N_{t+1}\right)}{N_t}\end{aligned}$$

(i) If the constraint on consumption is binding, $c = \underline{c}$, and by the constraint $\underline{N} = \left(\frac{z}{\underline{c}}\right)^{\frac{1}{\alpha}} \bar{K}\left(1-\lambda\right)^{\frac{1-\alpha}{\alpha}}$. The first two terms are the marginal benefit of an additional individual to the economy, denoted by MB, while the third term is the marginal benefit, denoted by MC. At steady state,

$$\begin{aligned} MB|_{N=\underline{N},\, c=\underline{c}} &= \beta N^{-\epsilon} c^{-\sigma}\left[F_2\left(\bar{K}, N-\lambda N\right) - \frac{F\left(\bar{K}, N-\lambda N\right)}{N}\right]\\ &\quad +\beta\frac{1-\epsilon}{1-\sigma} c N^{-\epsilon} c^{-\sigma} + \beta\left(1-\epsilon\right) N_{t+1}^{-\epsilon} A\\ &= \beta N^{-\epsilon} c^{-\sigma}\left(\frac{1-\alpha}{1-\lambda} + \frac{\sigma-\epsilon}{1-\sigma}\right)\frac{F\left(\bar{K}, N-\lambda N\right)}{N} + \beta\left(1-\epsilon\right) N_{t+1}^{-\epsilon} A\end{aligned}$$

using $F_2\left(\bar{K}, N-\lambda N\right) = \frac{F\left(\bar{K}, N-\lambda N\right)}{N}\frac{1-\alpha}{1-\lambda}$.

$$MC|_{N=\underline{N},\, c=\underline{c}} = N^{-\epsilon} c^{-\sigma}\frac{F\left(\bar{K}, N-\lambda N\right)}{N}\lambda\frac{1-\alpha}{1-\lambda}$$

When $A = 0$,

$$N^* = 0,\ c^* = \infty$$

or

$$N^* = \infty,\ c^* = 0 \text{ if } \frac{\sigma-\epsilon}{1-\sigma} > \frac{\lambda-\beta}{\beta}\frac{1-\alpha}{1-\lambda}$$

When $A > 0$, if $\frac{\sigma-\epsilon}{1-\sigma} > \frac{\lambda-\beta}{\beta}\frac{1-\alpha}{1-\lambda}$, the marginal benefit of an additional individual in the economy exceeds that of the marginal cost and the social planner prefers more population. Hence $N^* = \infty$ and $c^* = 0$.

If $\frac{\sigma-\epsilon}{1-\sigma} < \frac{\lambda-\beta}{\beta}\frac{1-\alpha}{1-\lambda}$, finite population is possible. If $N^* \neq \infty$ or 0, the first order condition at steady state is reduced to

$$\begin{aligned}&-\beta\frac{F\left(\bar{K}, N-\lambda N\right)}{N} + \beta\left(1-\epsilon\right)\left[\frac{1}{1-\sigma}\frac{F\left(\bar{K}, N-\lambda N\right)}{N} + A\left(\frac{F\left(\bar{K}, N-\lambda N\right)}{N}\right)^{\sigma}\right]\\ &= \left(\lambda-\beta\right) F_2\left(\bar{K}, N-\lambda N\right)\end{aligned}$$

This simplification focuses on the non-degenerate population when population is nonzero. If population is zero, it may not hold. Plugging functional form of production function, we get

$$\begin{aligned} & -\beta z\bar{K}^{\alpha}\left(1-\lambda\right)^{1-\alpha}+\left[\beta\frac{\left(1-\epsilon\right)}{1-\sigma}z\bar{K}^{\alpha}\left(1-\lambda\right)^{1-\alpha}+\beta\left(1-\epsilon\right)N^{\alpha(1-\sigma)}A\left(z\bar{K}^{\alpha}\left(1-\lambda\right)^{1-\alpha}\right)^{\sigma}\right] \\ = & \left(\lambda-\beta\right)z\bar{K}^{\alpha}\left(1-\alpha\right)\left(1-\lambda\right)^{-\alpha} \end{aligned}$$

Manipulating terms, population is solved as

$$N^{*}=\bar{K}\left(1-\lambda\right)^{-\frac{(1-\alpha)\sigma+\alpha}{\alpha(1-\sigma)}}\left(\frac{1}{A\left(1-\epsilon\right)}\right)^{\frac{1}{\alpha(1-\sigma)}}\left[\frac{\lambda-\beta}{\beta}\left(1-\alpha\right)+\frac{\epsilon-\sigma}{1-\sigma}\left(1-\lambda\right)\right]^{\frac{1}{\alpha(1-\sigma)}}z^{\frac{1}{\alpha}}$$

and consumption is

$$c^{*}=z\left(\bar{K}/N^{*}\right)^{\alpha}\left(1-\lambda\right)^{1-\alpha}$$

Obviously, z cancels out once population N^* is plugged into the formula of steady state consumption. As a result, consumption does not respond to TFP.

Proof of Proposition 2 To solve the golden rule level of population, the derivative of steady state social welfare with respect to population is

$$\begin{aligned} \frac{\partial V}{\partial N} &= N^{1-\epsilon}u'\left(\frac{F\left(\bar{K},N-\lambda N\right)}{N}\right)\frac{F_2\left(\bar{K},N-\lambda N\right)\left(1-\lambda\right)N-F\left(\bar{K},N-\lambda N\right)}{N^2} \\ &\quad +\left(1-\epsilon\right)N^{-\epsilon}u\left(\frac{F\left(\bar{K},N-\lambda N\right)}{N}\right) \\ &= N\left(c^{-\sigma}\right)\left[\frac{F_2\left(\bar{K},N-\lambda N\right)\left(1-\lambda\right)}{N}-\frac{F\left(\bar{K},N-\lambda N\right)}{N^2}\right]+\left(1-\epsilon\right)\left[\frac{c^{1-\sigma}}{1-\sigma}+A\right] \end{aligned}$$

where

$$c=\frac{F\left(\bar{K},N-\lambda N\right)}{N}=z\bar{K}^{\alpha}N^{-\alpha}\left(1-\lambda\right)^{1-\alpha}$$

so

$$\begin{aligned} \frac{\partial V}{\partial N} &= -\alpha z\bar{K}^{\alpha}N^{-1-\alpha}\left(1-\lambda\right)^{1-\alpha}N\left(c^{-\sigma}\right)+\left(1-\epsilon\right)\left[\frac{c^{1-\sigma}}{1-\sigma}+A\right] \\ &= \left(\frac{1-\epsilon}{1-\sigma}-\alpha\right)\left(zk^{\alpha}\left(1-\lambda\right)^{1-\alpha}\right)^{1-\sigma}+\left(1-\epsilon\right)A \end{aligned}$$

$\frac{1-\epsilon}{1-\sigma}>1>\alpha$, so $\frac{\partial V}{\partial N}>0$ for all k, so it is optimal to have the highest sustainable population.

Proof of Proposition 3: First order conditions:

$$a_{t+1}:u'\left(c_t\right)n_tq_t=\Phi(n_t)V'_{t+1}(a_{t+1})$$

$$n_t:u'\left(c_t\right)\left(q_ta_{t+1}+w_t\lambda\right)=\Phi'(n_t)V_{t+1}(a_{t+1})$$

Envelop condition

$$V_t'(a_t) = u'(c_t)(q_t + r_t)$$

so foc with respect to a_{t+1} becomes

$$u'(c_t) = \frac{\Phi(n_t)}{n_t} u'(c_{t+1}) \left(\frac{q_{t+1} + r_{t+1}}{q_t} \right)$$

so

$$\frac{q_{t+1} + r_{t+1}}{q_t} = \frac{1}{\beta} \left(\frac{F\left(\bar{K}, N_{t+1} - \lambda N_{t+2}\right)}{F\left(\bar{K}, N_t - \lambda N_{t+1}\right)} \right)^{\sigma} \left(\frac{N_{t+1}}{N_t} \right)^{\varepsilon - \sigma}$$

Plugging foc with respect to n_t into the objective function, postponing one period and plugging it into foc with respect to n_t and use Euler equation,

$$\begin{aligned} & \frac{\Phi(n_t)}{n_t} u'(c_{t+1}) \left(\frac{q_{t+1} + r_{t+1}}{q_t} \right) (q_t a_{t+1} + w_t \lambda) \\ = \;& \Phi'(n_t) \left[u(c_{t+1}) + \frac{1}{1-\varepsilon} u'(c_{t+1}) \left(w_{t+1} + (q_{t+1} + r_{t+1}) a_{t+1} - c_{t+1} \right) \right] \end{aligned}$$

Manipulate terms,

$$\frac{q_{t+1} + r_{t+1}}{q_t} w_t \lambda = \frac{\sigma - \varepsilon}{1 - \sigma} c_{t+1} + (1 - \varepsilon) \frac{A}{c_{t+1}^{-\sigma}} + w_{t+1}$$

Plugging wage and the resource constraint into the first order condition of population, and use the Euler equation,

$$\begin{aligned} & F_2\left(\bar{K}, N_{t+1} - \lambda N_{t+2}\right) + \frac{\sigma - \varepsilon}{1 - \sigma} \frac{F\left(\bar{K}, N_{t+1} - \lambda N_{t+2}\right)}{N_{t+1}} + (1 - \varepsilon) A \left(\frac{F\left(\bar{K}, N_{t+1} - \lambda N_{t+2}\right)}{N_{t+1}} \right)^{\sigma} \\ = \;& \lambda \left(\frac{F\left(\bar{K}, N_{t+1} - \lambda N_{t+2}\right)}{F\left(\bar{K}, N_t - \lambda N_{t+1}\right)} \right)^{\sigma} \frac{1}{\beta} \left(\frac{N_{t+1}}{N_t} \right)^{\varepsilon - \sigma} F_2\left(\bar{K}, N_t - \lambda N_{t+1}\right) \end{aligned}$$

which is the optimality condition of the social planner's problem.

Proof of Proposition 4 Fixed amount of land and the time-independent equilibrium prices at steady state imply total labor L is time independent, so the steady state fertility n is always 1. The first order conditions are:

$$c_t^o : \left(c_t^y\right)^{-1} = R_{t+1} \left(c_{t+1}^o\right)^{-1}. \tag{87}$$

$$n_t : \gamma \frac{u_t}{c_t^y} \left(w_t \lambda + b_{t+1}/R_{t+1} \right) = \epsilon_n^{\Phi}(n_t) \frac{\Phi(n_t)}{n_t} V_{t+1}(b_{t+1}). \tag{88}$$

$$b_{t+1} : \gamma \frac{u_t}{c_t^y} / R_{t+1} = \frac{\Phi(n_t)}{n_t} V_{t+1}'(b_{t+1}). \tag{89}$$

By Envelop condition (89) can be written as:

$$\frac{u_t}{c_t^y} = R_{t+1}\frac{\Phi(n_t)}{n_t}\frac{u_{t+1}}{c_{t+1}^y}. \tag{90}$$

Using (87), it follows that:

$$c_{t+1}^o = R_{t+1}c_t^y. \tag{91}$$

The budget constraint and u_t can be written as

$$w_t\left(1-\lambda n_t\right) + b_t - b_{t+1}/R_{t+1} = c_t^y + c_{t+1}^o/R_{t+1} = 2c_t^y. \tag{92}$$

and

$$u_t = \left(\frac{1}{2}\right)^{2\gamma}\left(w_t\left(1-\lambda n_t\right) + b_t - b_{t+1}/R_{t+1}\right)^{2\gamma} R_{t+1}^{\gamma} \tag{93}$$

Note that from (88) and the budget constraint:

$$\Phi(n_t)V_{t+1} = \frac{\gamma}{\epsilon_n^{\Phi}(n_t)}\frac{u_t}{c_t^y}\left(b_t + w_t - c_t^y - c_{t+1}^o/R_{t+1}\right) \tag{94}$$

Lagging the previous equation one period and using (90),

$$\frac{\gamma}{\epsilon_n^{\Phi}(n_{t-1}^y)}\frac{u_t}{c_t^y}\left(R_t w_{t-1}\lambda + b_t\right) = u_t + A + \gamma\frac{1}{\epsilon_n^{\Phi}(n_t)}\frac{u_t}{c_t^y}\left(b_t + w_t - c_t^y - c_{t+1}^o/R_{t+1}\right)$$

$\epsilon_n^{\Phi}(n_{t-1}^y) = 1-\epsilon$. Collecting terms,

$$c_t^y + c_{t+1}^o/R_{t+1} = \frac{2\gamma}{\epsilon_n^{\Phi} - 2\gamma}\left(R_t w_{t-1}\lambda - w_t\right) - \frac{2\epsilon_n^{\Phi}}{\epsilon_n^{\Phi} - 2\gamma}\frac{A\left(c_t^y\right)^{1-2\gamma}}{R_{t+1}^{\gamma}} \tag{95}$$

(i) If $A = 0$ or a rich economy so that $A\left(c_t^y\right)^{1-2\gamma} \simeq 0$:

$$c_t^y + c_{t+1}^o/R_{t+1} = \frac{2\gamma}{\epsilon_n^{\Phi} - 2\gamma}\left(R_t w_{t-1}\lambda - w_t\right) \tag{96}$$

The solution for present value consumption is

$$c_t^y = \frac{\gamma}{\epsilon_n^{\Phi} - 2\gamma}\left(R_t w_{t-1}\lambda - w_t\right) \tag{97}$$

At steady state,

$$c_t^y = \frac{\gamma}{1-\epsilon-2\gamma}\left(R\lambda - 1\right)\left(1-\alpha\right)f\left(k\right) \tag{98}$$

and by resource constraint for output (11)

$$c_t^y (1+R) = (1-\lambda) f(k) \tag{99}$$

(90) and (93) becomes:

$$\left(c_t^y\right)^{2\gamma-1} R_{t+1}^{\gamma} = R_{t+1} \frac{\Phi(n_t)}{n_t} \left(c_{t+1}^y\right)^{2\gamma-1} R_{t+2}^{\gamma}. \tag{100}$$

This equation resembles the Euler equation in Barro and Becker (1988) (BB henceforth) but is different from BB. In particular, the terms R^{γ} do not appear in BB. This is because the model now adds life-cycle features while in BB individuals live only one period. Plugging the functional form of the altruism function, the steady state condition of (100) can be reduced to $R = \frac{1}{\beta}$. If steady state per capita capital k is not zero, then the formula of consumption (98) and the resource constraint (100) require the following condition to hold at steady state.

$$\frac{\gamma(1-\alpha)}{1-\epsilon-2\gamma}\left(\frac{\lambda}{\beta}-1\right) = \frac{1-\lambda}{1+1/\beta}$$

Otherwise the equilibrium population has to be infinite. (ii) If $A \neq 0$, the equilibrium solution for n and R still holds except that consumption is not determined by (97). (95) and the resource constraint (99) give consumption as

$$c^y = \left[\frac{1-\epsilon}{\gamma\left(\beta^{-1}\lambda-1\right)\frac{(1+\beta^{-1})}{1-\lambda}(1-\alpha)-(1-\epsilon-2\gamma)}\frac{A}{R^{\gamma}}\right]^{\frac{1}{2\gamma}}$$

Proof of Corollary At steady state, c^y is independent of z and k is decreasing in z. Since $k_t = \frac{K}{L_t}$, a smaller steady state capital k implies a bigger population of labor L. By $L_t = N_t^y(1-\lambda n_t)$, N^y and $N\ (=2N^y)$ are bigger at steady state.

Proof of Proposition 5 The social planner's allocations $\left\{c_t^y, c_{t+1}^o, n_t\right\}$ are characterized by

$$\begin{cases} \beta \frac{c_{t+1}^o}{c_{t+1}^y}\frac{u_{t+1}}{u_t} n_t^{-\varepsilon} = 1 \\ N_t^y c_t^y + N_{t-1}^y c_t^o = F\left(\bar{K}, N_t^y - \lambda N_{t+1}^y\right) \\ (1-\varepsilon)\left[u_t + A\right] + \frac{\gamma u_t}{c_t^y}\left[F_2\left(\bar{K}, N_t^y - \lambda N_{t+1}^y\right) - c_t^y\right] - \gamma u_t \\ = \beta^{-1}\frac{\gamma u_{t-1}}{c_{t-1}^y} n_{t-1}^{\varepsilon}\left[\lambda F_2\left(\bar{K}, N_{t-1}^y - \lambda N_t^y\right) + \eta\right] \\ \frac{u_t}{c_t^y} = R_{t+1}\frac{\Phi(n_t)}{n_t}\frac{u_{t+1}}{c_{t+1}^y} \end{cases}$$

In the complete market problem, the Euler equation (derived from foc w.r.t b and Envelop Thm) and the focs with respect to c_t^y and c_{t+1}^o give

$$c_{t+1}^o = R_{t+1} c_t^y$$

Combine these two equations,

$$\beta \frac{c_{t+1}^o}{c_{t+1}^y}\frac{u_{t+1}}{u_t} n_t^{-\varepsilon} = 1$$

Foc with respect to n and Euler equation imply

$$\frac{\gamma}{\epsilon_n^\Phi(n_{t-1}^y)}\frac{u_t}{c_t^y}R_t w_{t-1}\lambda = u_t + A + \frac{\gamma}{\epsilon_n^\Phi(n_t)}\frac{u_t}{c_t^y}w_t - 2\frac{\gamma}{\epsilon_n^\Phi(n_t)}u_t$$

By Euler equation, $\frac{u_t}{c_t^y} = R_{t+1}\frac{\Phi(n_t)}{n_t}\frac{u_{t+1}}{c_{t+1}^y}$,

$$\gamma\frac{u_t}{c_t^y}R_t F_2\left(\bar{K}, N_{t-1}^y - \lambda N_t^y\right)\lambda = \gamma R_t^{-1}\frac{n_{t-1}}{\Phi(n_{t-1})}\frac{u_{t-1}}{c_{t-1}^y}R_t F_2\left(\bar{K}, N_{t-1}^y - \lambda N_t^y\right)\lambda$$

so

$$\begin{aligned} & \gamma\frac{n_{t-1}}{\Phi(n_{t-1})}\frac{u_{t-1}}{c_{t-1}^y}F_2\left(\bar{K}, N_{t-1}^y - \lambda N_t^y\right)\lambda \\ = \quad & (1-\varepsilon)\left[u_t + A\right] + \gamma\frac{u_t}{c_t^y}F_2\left(\bar{K}, N_t^y - \lambda N_{t+1}^y\right) - 2\gamma u_t \end{aligned}$$

which is the third equation in social planner's problem. In equilibrium of both models, resource constraint needs to be satisfied.

Proof of Proposition 6 When credit constraint binds, $b_{t+1} = \underline{b}$. Plugging (94) into the value function and use the lifetime budget constraint (7). Postpone V_t by one period and plug it back into (88),

$$\gamma\frac{u_t}{c_t^y}\left(w_t\lambda + \underline{b}/R_{t+1}\right) = \epsilon_n^\Phi(n_t)\frac{\Phi(n_t)}{n_t}u_{t+1}\left(1 + \frac{n_{t+1}}{\epsilon_n^\Phi}\gamma\frac{1}{c_{t+1}^y}\left(w_{t+1}\lambda + \underline{b}/R_{t+2}\right) + \frac{A}{u_{t+1}}\right) \quad (101)$$

where by (87) and (7)

$$c_t^y = \frac{1}{2}\left(w_t\left(1-\lambda n_t\right) + \underline{b} - n_t\underline{b}/R_{t+1}\right)$$

and by (93)

$$u_t = \left(c_t^y\right)^{2\gamma}R_{t+1}^\gamma.$$

Plugging consumptions into the resource constraint (11) and manipulate terms, we get

$$n_t\left(\underline{b} - n_{t+1}\underline{b}/R_{t+2}\right) + R_{t+1}\left[(1-\alpha)zk_t^\alpha\left(1-\lambda n_t\right) + \underline{b} - n_t\underline{b}/R_{t+1}\right] = \left(1-\lambda n_{t+1}\right)zk_{t+1}^\alpha n_t\left(1+\alpha\right).$$

Every young individual's budget constraint (5) and (13) imply

$$k_{t+1} = \frac{1}{q_t}\left(\frac{1}{2}w_t\left(1-\lambda n_t\right) + \frac{1}{2}\underline{b} + \frac{1}{2}n_t\underline{b}/R_{t+1}\right)\frac{1}{n_t}\frac{1}{1-\lambda n_{t+1}} \quad (102)$$

By (8) and (51),

$$R_{t+1} = \frac{q_{t+1} + \alpha z k_{t+1}^{\alpha-1}}{q_t}.$$

At steady state, fertility rate n is still 1. The rest of the system that describes the steady state with binding credit constraint consists of equation (101), (102), (8) and (51). At steady state, the system can be simplified as

$$\frac{1-\beta}{1-\epsilon}\gamma\frac{1}{c^y}\left(w\lambda+\underline{b}/R\right)=\beta\left(1+\frac{A}{u}\right)$$

where

$$c^y=\frac{1}{2}\left(w\left(1-\lambda\right)+\underline{b}-\underline{b}/R\right)$$

and

$$u=\left(c^y\right)^{2\gamma}R^{\gamma}$$

$$k=\frac{1}{2q}\left(w+\underline{b}\left(1+1/R\right)\frac{1}{1-\lambda}\right)$$

$$R=1+\frac{r}{q}$$

and

$$w=z\left(1-\alpha\right)k^{\alpha},r=z\alpha k^{\alpha-1}$$

Proof of Proposition 7 Consider the budget constraint in steady state:

$$b=\frac{c^y+c^o/R-w\left(1-\lambda n\right)}{\left(1-n/R\right)}=\frac{Rc^y+c^o-wR\left(1-\lambda n\right)}{R-n}$$

A nonpositive first best bequest $b\leq 0$ requires $c^y+c^o/R\geq w\left(1-\lambda n\right)$. At steady state,

$$w\left(1-\lambda\right)=\left(1-\alpha\right)\left(1+R\right)c^y$$

and $c^o=Rc^y$, so this condition implies

$$2<\left(1-\alpha\right)\left(1+\frac{1}{\beta}\right)$$

The steady state versions of equations are:

$$\frac{\lambda n}{1-\lambda n}=\Phi(n)\left(\frac{\epsilon_n^{\Phi}}{2\gamma}+\frac{\lambda n}{1-\lambda n}+\frac{\epsilon_n^{\Phi}}{2\gamma}\frac{A}{u}\right). \tag{103}$$

$$1=\left(1-\alpha\right)zk^{\alpha-1}\frac{1}{2q}. \tag{104}$$

$$u=\left(\left(1-\alpha\right)zk^{\alpha}\left(1-\lambda n\right)/2\right)^{2\gamma}R^{\gamma}$$

$$R=\frac{q+r}{q} \tag{105}$$

$$r=f'\left(k\right)=\alpha zk^{\alpha-1} \tag{106}$$

By (104), (105) and (106), the steady state interest rate R is a constant.

$$R = \frac{1+\alpha}{1-\alpha}$$

and utility is

$$u = \frac{A}{\frac{1-\beta}{\beta}\frac{\lambda}{1-\lambda}\frac{2\gamma}{1-\epsilon} - 1} \tag{107}$$

Proof of the Corollary Equations

$$u = \left(c^y\right)^{2\gamma}\left(\frac{1+\alpha}{1-\alpha}\right)^{\gamma}. \tag{108}$$

By the equation of consumption and the expression of w,

$$c^y = (1-\alpha)\, z k^{\alpha}\,(1-\lambda)/2 \tag{109}$$

(107) and (108) indicate both the homothetic part of utility u, and steady state consumption of a young agent c^y are independent of z. (109) then implies k is decreasing in z.

Log Linearization of the Dynamic System

Log linearize around the steady state to express the system as $X_{t+1} = A^{-1}BX_t$ where $A = \begin{bmatrix} A_{11} & 0 & A_{13} & A_{14} \\ A_{21} & 0 & A_{23} & 0 \\ A_{31} & 0 & A_{33} & A_{34} \\ A_{41} & A_{42} & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & 0 & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & 0 & B_{33} & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{bmatrix}$ and $X_t = \begin{bmatrix} k_t \\ q_t \\ n_t \\ R_{t+1} \end{bmatrix}$. The components of the matrix are

$$\begin{aligned}
A_{11} &= -\alpha \\
A_{13} &= -\text{b}\frac{n^{*2}}{R^*}\frac{1}{(1-\lambda n^*)\, zk^{*\alpha}n^*\,(1+\alpha)} + \frac{\lambda n^*}{1-\lambda n^*} \\
A_{14} &= \frac{1}{(1-\lambda n^*)\, zk^{*\alpha}n^*\,(1+\alpha)}\text{b}\frac{n^{*2}}{R^*} \\
B_{11} &= -\frac{(1-\alpha)\,\alpha}{1+\alpha}\frac{R^*}{n^*} \\
B_{13} &= 1 - \frac{1}{(1-\lambda n^*)\, zk^{*\alpha}n^*\,(1+\alpha)}\left((1-\alpha)\, zk^{*\alpha}\,(-\lambda n^*)\, R^* - \frac{n^{*2}}{R^*}\text{b}\right) \\
B_{14} &= -\frac{1}{(1-\lambda n^*)\, zk^{*\alpha}n^*\,(1+\alpha)}R^*\left[(1-\alpha)\, zk^{*\alpha}\,(1-\lambda n^*) + \text{b}\right]
\end{aligned}$$

$$\begin{aligned}
A_{21} &= 1 \\
A_{23} &= -\frac{\lambda n^*}{1-\lambda n^*} \\
B_{21} &= \frac{\alpha(1-\alpha) z k^{*\alpha}(1-\lambda n^*)}{(1-\alpha) z k^{*\alpha}(1-\lambda n^*)+\underline{b}+n^*\underline{b}/R^*} \\
B_{22} &= -1 \\
B_{23} &= -1+\frac{1}{(1-\alpha) z k^{*\alpha}(1-\lambda n^*)+\underline{b}+n^*\underline{b}/R^*}\left(\frac{n^*}{R^*}\underline{b}-\lambda(1-\alpha) z k^{*\alpha} n^*\right) \\
B_{24} &= -\frac{1}{(1-\alpha) z k^{*\alpha}(1-\lambda n^*)+\underline{b}+n^*\underline{b}/R^*}\frac{n^*}{R^*}\underline{b}
\end{aligned}$$

$$\begin{aligned}
A_{31} = & \left(-\frac{\frac{1}{u^*}}{\frac{1}{1+\frac{1}{1-\epsilon}\gamma\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)+\frac{A}{u^*}}}\frac{A}{u^{*2}}\right)\gamma\left(\frac{1}{2}\left(\begin{array}{c} w^*(1-\lambda) \\ +\underline{b}-\underline{b}/R^* \end{array}\right)\right)^{2\gamma-1} R^{*\gamma}(1-\alpha) z k^{*\alpha}\alpha(1-\lambda) \\
& +\frac{1}{1+\frac{1}{1-\epsilon}\gamma\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)+\frac{A}{u^*}}\frac{\gamma}{1-\epsilon}(1-\alpha) z\alpha k^{*\alpha}\frac{1}{c^{y*}}\left(\lambda-\frac{1}{2c^{y*}}\left(w^*\lambda+\frac{\underline{b}}{R^*}\right)(1-\lambda)\right)
\end{aligned}$$

$$\begin{aligned}
A_{33} = & -\left(-\frac{\frac{1}{u^*}}{\frac{1}{1+\frac{1}{1-\epsilon}\gamma\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)+\frac{A}{u^*}}}\frac{A}{u^{*2}}\right)\gamma\left(\frac{1}{2}\left(\begin{array}{c} w^*(1-\lambda) \\ +\underline{b}-\underline{b}/R^* \end{array}\right)\right)^{2\gamma-1} R^{*\gamma}\left(\lambda w^*+\frac{\underline{b}}{R^*}\right) \\
& +\frac{1}{1+\frac{1}{1-\epsilon}\gamma\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)+\frac{A}{u^*}}\frac{\gamma}{1-\epsilon}\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)\left(1+\frac{1}{2c^{*y}}\left(\lambda w^*+\frac{\underline{b}}{R^*}\right)\right)
\end{aligned}$$

$$\begin{aligned}
A_{34} = & \left(\frac{1}{u^*}-\frac{1}{1+\frac{1}{1-\epsilon}\gamma\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)+\frac{A}{u^*}}\frac{A}{u^{*2}}\right)\gamma\left(\frac{1}{2}\left(\begin{array}{c} w^*(1-\lambda) \\ +\underline{b}-\underline{b}/R^* \end{array}\right)\right)^{2\gamma} R^{*\gamma} \\
& -\frac{1}{1+\frac{1}{1-\epsilon}\gamma\frac{1}{c^{*y}}(w^*\lambda+\underline{b}/R^*)+\frac{A}{u^*}}\frac{\gamma}{1-\epsilon}\frac{1}{c^{y*}}\frac{\underline{b}}{R^*}\left(\frac{1}{2c^{*y}}\left(w^*\lambda+\frac{\underline{b}}{R^*}\right)+1\right)
\end{aligned}$$

$$B_{31}=(1-\alpha) z\alpha k^{*\alpha}\left[\begin{array}{c} \frac{1}{u^*}\gamma\left(\frac{1}{2}(w^*(1-\lambda)+\underline{b}-\underline{b}/R^*)\right)^{2\gamma-1} R^{*\gamma}(1-\lambda) \\ -\frac{1}{w^*(1-\lambda)+\underline{b}-\underline{b}/R^*}(1-\lambda)-\frac{1}{w^*\lambda+\underline{b}/R^*}\lambda \end{array}\right]$$

$$\begin{aligned}
B_{33} = & -\left(\lambda w^*+\frac{\underline{b}}{R^*}\right)\frac{1}{u^*}\gamma\left(\frac{1}{2}(w^*(1-\lambda)+\underline{b}-\underline{b}/R^*)\right)^{2\gamma-1} R^{*\gamma} \\
& +\frac{1}{w^*(1-\lambda)+\underline{b}-\underline{b}/R^*}\left(\lambda w^*+\frac{\underline{b}}{R^*}\right)+\epsilon
\end{aligned}$$

$$
\begin{aligned}
B_{34} &= \frac{1}{2}\frac{1}{u^*}\gamma\left(\frac{1}{2}\left(w^*\left(1-\lambda\right)+\underline{b}-\underline{b}/R^*\right)\right)^{2\gamma-1}R^{*\gamma}\left(w^*\left(1-\lambda\right)+\underline{b}+\underline{b}/R^*\right) \\
&\quad -\frac{1}{w^*\left(1-\lambda\right)+\underline{b}-\underline{b}/R^*}\frac{\underline{b}}{R^*}-\frac{1}{w^*\lambda+\underline{b}/R^*}\underline{b}\frac{1}{R^*}
\end{aligned}
$$

$$
\begin{aligned}
A_{41} &= \frac{1}{q^*+\alpha z k^{*\alpha-1}}\alpha z\left(\alpha-1\right)k^{*\alpha-1} \\
A_{42} &= \frac{1}{q^*+\alpha z k^{*\alpha-1}}q^* \\
B_{42} &= 1 \\
B_{44} &= 1
\end{aligned}
$$

APPENDIX B. ACCOUNTING FOR THE INTERNATIONAL QUANTITY-QUALITY TRADE-OFF

$$\max V(b) = \max_{c(t),b',n} \frac{1}{1-\eta}\{U(c)+\bar{u}\}^{1-\eta} + \pi(F)\beta_1 e^{-\rho F}\phi(n,F)V(b')$$

where

$$U(c) = \left[\frac{\rho}{1-\rho^T}\int_0^T e^{-\rho t}\pi(t)^{\frac{1-\sigma}{1-\eta}} c(t)^{1-\sigma}\,dt\right]^{\frac{1}{1-\sigma}}$$

subjecting to

$$b \geq q(s)\varpi(s) + \int_0^s (c(a)+e_s(a))\,q(a)\,da$$

$$q(s)\varpi(s) + W(s,n) \geq \int_s^T c(a)\,q(a)\,da + q(F)\,nb'$$

$$\left(\int_0^s ((e_p+e_s(a))/p_E)^\beta\,da\right)^{\alpha/\beta} \geq h(s)$$

$$e_s(a) \geq 0, \varpi(s) \geq \underline{\omega}$$

$$W(s,n) = \int_s^R wh(s)\,e^{\nu(a-s)}q(a)l(n)da$$

$$h(s) = \left(\int_0^s (e_p+e_s(a))^\beta\,da\right)^{\alpha/\beta} p_E^\alpha$$

$$\phi(n,F) = 1-e^{-\chi_1 n}$$

$$\phi_n(n,F) = \chi_1 e^{-\chi_1 n}$$

and

$$q(F) = e^{-rF}\pi(F)$$

To solve the individual's problem consider the associated Lagrangian:

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{1-\eta}\{U(c)+\bar{u}\}^{1-\eta} + \pi(F)\beta_1 e^{-\rho F}\phi(n,F)V(b') \\ &+\lambda_1\left[b - q(s)\varpi(s) - \textstyle\int_0^s (c(a)+e_s(a))\,q(a)\,da\right] \\ &+\lambda_2\left[q(s)\varpi(s) + W(s,n) - \textstyle\int_s^T c(a)\,q(a)\,da - q(F)\,nb'\right] \\ &+\lambda_3\left[\left(\textstyle\int_0^s ((e_p+e_s(a))/p_E)^\beta\,da\right)^{\alpha/\beta} - h(s)\right] + \lambda_4 e_s(a) + \lambda_5\left[\varpi(s) - \underline{\omega}\right].\end{aligned}$$

where

$$M \equiv \int_0^T e^{-\rho t}\pi(t)^{\frac{1-\sigma}{1-\eta}} c(t)^{1-\sigma}\,dt$$

When the borrowing constraint binds, $Q(s) = 0$. We focus on this case.

First order condition to $c(a)$,

$$\begin{aligned} \frac{\partial U(c)}{\partial c(a)} &= \lambda_1 q(a) \quad \text{for } a \leq s \\ \frac{\partial U(c)}{\partial c(a)} &= \lambda_2 q(a) \quad \text{for } s < a < T \end{aligned} \tag{110}$$

When $a \leq s$

$$\begin{aligned} c(a) &= \{U(c)+\bar{u}\}^{-\frac{\eta}{\sigma}} M^{\frac{1}{1-\sigma}} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{\sigma}\frac{1}{1-\sigma}} e^{\frac{(r-\rho)a}{\sigma}} \pi(a)^{\frac{1}{\sigma}\frac{\gamma-\sigma}{1-\gamma}} \lambda_1^{-\frac{1}{\sigma}} \quad \text{for } a \leq s \\ c(a) &= \{U(c)+\bar{u}\}^{-\frac{\eta}{\sigma}} M^{\frac{1}{1-\sigma}} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{\sigma}\frac{1}{1-\sigma}} e^{\frac{(r-\rho)a}{\sigma}} \pi(a)^{\frac{1}{\sigma}\frac{\gamma-\sigma}{1-\gamma}} \lambda_2^{-\frac{1}{\sigma}} \quad \text{for } s \leq a < T \end{aligned} \tag{111}$$

$$c^S(a) = e^{\frac{(r-\rho)a}{\sigma}} \pi(a)^{\frac{1}{\sigma}\frac{\eta-\sigma}{1-\eta}} c(0) \tag{112}$$

$$\frac{c^W(a)}{c(0)} = e^{\frac{(r-\rho)a}{\sigma}} \pi(a)^{\frac{1}{\sigma}\frac{\gamma-\sigma}{1-\gamma}} G^{\frac{1}{\sigma}} \tag{113}$$

First order condition to b' and the Envelop Theorem,

$$\pi(F)\beta_1 e^{-\rho F}\phi(n,F)\lambda_1^{child} = \lambda_2 q(F) n$$

At steady state,

$$G \equiv \frac{\lambda_1}{\lambda_2} = \frac{1}{\beta_1} e^{-(r-\rho)F} \frac{n}{1-e^{-\chi_1 n}} = \left(\frac{c^S(s)}{c^W(s)}\right)^{-\sigma} \tag{114}$$

First order condition to $h(s)$ gives

$$\frac{\lambda_3}{\lambda_2} = \frac{W(s,n)}{h(s)} = \int_s^R w e^{\nu(a-s)} q(a) l(n) da \tag{115}$$

and

$$\frac{\lambda_1}{\lambda_3} = \frac{Gh(s)}{W(s,n)} = \frac{G}{\int_s^R w e^{\nu(a-s)} q(a) l(n) da} \tag{116}$$

First order condition to $e_s(a)$,

$$\lambda_3 \frac{\partial h(s)}{\partial e_s(a)} + \lambda_4 = \lambda_1 q(a) \tag{117}$$

When $e_s(a) > 0$, (117) implies

$$\lambda_3 p_E^{-\beta} \alpha h(s)^{\frac{\beta}{\alpha}\left(\frac{\alpha}{\beta}-1\right)} \hat{e}(a)^{\beta-1} = \lambda_1 q(a)$$

so

$$\hat{e}^*(a) = \hat{e}^*(0)\, q(a)^{-\frac{1}{1-\beta}} \tag{118}$$

When $e_s(a) > 0$, plugging $\frac{\partial h(s)}{\partial e_s(a)}$and $\frac{\lambda_1}{\lambda_3}$ into (117),

$$e^*(0) = \left(\alpha h(s)^{-\frac{\beta}{\alpha}} p_E^{-\beta} W(s,n)/G\right)^{\frac{1}{1-\beta}}.$$

6.5.1 Value function

In this section, we simiplify the objective function into a two-period model by substituting consumptions.

$$V = \frac{\{U(c) + \bar{u}\}^{1-\eta}}{1-\eta} + \bar{u}_0 + \pi(F)\beta_1 e^{-\rho F}\phi(n,F)V',\ 0 < \eta < 1.$$

At steady state, the value function can be written as

$$V(b) = \frac{1}{1-\pi(F)\beta_1 e^{-\rho F}\phi(n,F)}\left[\frac{\{U(c)+\bar{u}\}^{1-\eta}}{1-\eta} + \bar{u}_0\right] \tag{119}$$

Following Cordoba and Ripoll (JME), let

$$\pi(a) = \begin{cases} e^{-p_c a} & \text{for } a \le a_c \\ \pi(a_c)\, e^{-p_s(a-a_c)} & \text{for } a_c \le a \le a_s \\ \pi(a_s)\, \frac{e^{-p(a-a_s)}-\xi}{-\xi} & \text{for } a_s \le a \le T \end{cases}$$

$a_c = 5$, $a_s = F = 25$. Then we require $a_s = F \geq s \geq \underline{s} = 6$.
Derive terms,

$$U(c) = \left[\frac{\rho}{1-\rho^T} M\right]^{\frac{1}{1-\sigma}}$$

and

$$M = c(0)^{1-\sigma}\left[\begin{array}{c} I\left(\theta + p_c\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}, 0, a_c\right) \\ +\pi(a_c)^{\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}} e^{p_s a_c \frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}} I\left(\theta + p_s\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}, a_c, s\right) \\ +G^{\frac{1-\sigma}{\sigma}}\left[\begin{array}{c} \pi(a_c)^{\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}} e^{p_s a_c \frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}} I\left(p_s\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma} + \theta, s, a_s\right) \\ +\pi(a_s)^{\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}} \int_{a_s}^{T}\left(\frac{e^{-p(t-a_s)}-\xi}{1-\xi}\right)^{\frac{1-\sigma}{1-\eta}\frac{\eta}{\sigma}} e^{-\theta t}dt \end{array}\right] \end{array}\right] \tag{120}$$

6.5.2 Schooling

$$e^*(a) = \begin{cases} \widehat{e}^*(a) & \text{for } a \le \min(s,\underline{s}) \\ e_p & \text{for } \min(s,\underline{s}) \le a \le s_p \\ \widehat{e}^*(a) & \text{for } s_p \le a \le s \end{cases} \tag{121}$$

where

$$s_p \equiv \min\{s, \bar{s}, \max[\underline{s}, \widehat{a}]\}. \tag{122}$$

$\widehat{a}$ is defined by

$$\widehat{a} = \sup \{a : \widehat{e}^* (a) = e_p\} \tag{123}$$

First order condition to s gives

$$\begin{aligned} & \{U (c) + \bar{u}\}^{-\eta} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{1-\sigma}} \left(\frac{1}{1-\sigma}\right) M^{\frac{\sigma}{1-\sigma}} e^{-\rho s} \pi (s)^{\frac{1-\sigma}{1-\eta}} \left[c^S (s)^{1-\sigma} - c^W (s)^{1-\sigma}\right] \\ & -\lambda_1 \left(c^S (s) + e_s (s)\right) q (s) + \lambda_2 \frac{\partial W(s,n)}{\partial s} + \lambda_2 c^W (s) q (s) \\ = \; & 0 \end{aligned}$$

Then

$$\begin{aligned} & - \{U (c) + \bar{u}\}^{-\eta} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{1-\sigma}} \left(\frac{1}{1-\sigma}\right) M^{\frac{\sigma}{1-\sigma}} e^{-\rho s} \pi (s)^{\frac{1-\sigma}{1-\eta}} c^W (s)^{1-\sigma} \\ & +\lambda_2 c^W (s) q (s) \\ = \; & -\frac{\sigma}{1-\sigma} \{U (c) + \bar{u}\}^{-\eta} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{1-\sigma}} M^{\frac{\sigma}{1-\sigma}} e^{-\rho s} \pi (s)^{\frac{1-\sigma}{1-\eta}} c^W (s)^{1-\sigma} \end{aligned}$$

Plugging formulas of λ_1 and λ_2 and combining terms, first order condition to s becomes

$$\begin{aligned} & e^{rs} \pi (s)^{-1} c^W (s)^{-\sigma} w \left[-h (s) q(s) + (h_s (s) - \nu h (s)) e^{-vs} \left(\begin{array}{c} \int_s^F e^{\nu a} q(a) da \\ +l(n) \int_F^R e^{\nu a} q(a) da \end{array} \right)\right] \\ = \; & = \sigma \Delta u(s) + c^S (s)^{-\sigma} e_s (s) \end{aligned} \tag{124}$$

where

$$\Delta u(s) \equiv \frac{1}{1-\sigma} \left(c^W (s)^{1-\sigma} - c^S (s)^{1-\sigma}\right) \tag{125}$$

$h(s,E) = \left(\int_0^s \left(\frac{e(a)}{p_e}\right)^{\beta} da\right)^{\frac{\alpha}{\beta}}$.

$$\underbrace{e^{rs} \pi (s)^{-1} c^W (s)^{-\sigma} \frac{\partial}{\partial s} \left[\int_s^R wh (s) e^{v(a-s)} q(a) l(n) da\right]}_{\text{net marginal benefit of } s} = \underbrace{\sigma \Delta u(s) + c^S (s)^{-\sigma} e_s (s)}_{\text{marginal cost of } s}$$

so that the optimal years of schooling equates its marginal benefit to marginal costs. LHS of (124) is the marginal return to schooling consists of four parts of cost: an additional year of private educational cost, one year of return to experience, one year of wage, and one more year of being constrained to consume a low amount as students.

Solve for two integrals:

$$\int_s^F e^{(v-r)a} \pi (a) da = \pi (a_c) e^{p_s a_c} I (r + p_s - v, s, F)$$

$$\int_F^R e^{(v-r)a}\pi(a)\,da = \pi(a_s)\left[\frac{1}{1-\xi}e^{pa_s}I(p-v+r,F,R) - \frac{\xi}{1-\xi}I(r-v,F,R)\right]$$

which can be plugged in (124) and be used to solve for $W(s,n)$.

$$W(s,n) = wh(s)\,e^{-vs}\left[\begin{array}{c}\pi(a_c)\,e^{p_s a_c}I(r+p_s-v,s,F)\\ +l(n)\,\pi(a_s)\left[\frac{1}{1-\xi}e^{pa_s}I(p-v+r,F,R) - \frac{\xi}{1-\xi}I(r-v,F,R)\right]\end{array}\right] \quad (126)$$

$h(s)$ is solved as

$$h(s,E) = \left(\frac{e^*(0)}{p_E}\right)^{\alpha}\left[\begin{array}{c}\int_0^{\min(s,\underline{s})} q(a)^{-\frac{\beta}{1-\beta}}da + \int_{s_p}^{s} q(a)^{-\frac{\beta}{1-\beta}}da\\ +\left(\frac{e_p}{e(0)}\right)^{\beta}(s_p - \min(s,\underline{s}))\end{array}\right]^{\alpha/\beta}$$

If $s > \underline{s}$, since $\hat{e}^*(a) = \hat{e}^*(0)\,q(a)^{-\frac{1}{1-\beta}}$,

$$\hat{e}(s) = \hat{e}^*(0)\left[e^{-rs}\pi(a_c)\,e^{-p_s(s-a_c)}\right]^{-\frac{1}{1-\beta}}.$$

In the numerical exercise, we have $s > \underline{s}$, so

$$\begin{aligned}&\int_0^{\min(s,\underline{s})} q(a)^{-\frac{\beta}{1-\beta}}da\\ =\ & I\left(-(r+p_c)\frac{\beta}{1-\beta},0,a_c\right) + \pi(a_c)^{-\frac{\beta}{1-\beta}}\,e^{-\frac{\beta}{1-\beta}p_s a_c}I\left(-(r+p_s)\frac{\beta}{1-\beta},a_c,\underline{s}\right)\end{aligned}$$

$$\int_{s_p}^{s} q(a)^{-\frac{\beta}{1-\beta}}da = \left[\pi(a_c)\,e^{p_s a_c}\right]^{-\frac{\beta}{1-\beta}} I\left(-(r+p_s)\frac{\beta}{1-\beta},s_p,s\right)$$

$$h(s,E) = \left(\frac{e^*(0)}{p_E}\right)^{\alpha}\left[\begin{array}{c}I\left(-(r+p_c)\frac{\beta}{1-\beta},0,a_c\right)\\ +\pi(a_c)^{-\frac{\beta}{1-\beta}}\,e^{-\frac{\beta}{1-\beta}p_s a_c}I\left(-(r+p_s)\frac{\beta}{1-\beta},a_c,\underline{s}\right)\\ +\left[\pi(a_c)\,e^{p_s a_c}\right]^{-\frac{\beta}{1-\beta}} I\left(-(r+p_s)\frac{\beta}{1-\beta},s_p,s\right)\\ +\left(\frac{e_p}{e(0)}\right)^{\beta}(s_p-\min(s,\underline{s}))\end{array}\right]^{\alpha/\beta} \quad (127)$$

From the solution of $h(s)$ solved above, we have

$$\int_0^s \left(\frac{\hat{e}(a)}{p_e}\right)^{\beta} da = h(s,E)^{\frac{\beta}{\alpha}}$$

$$= \left(\frac{e^*(0)}{p_E}\right)^{\beta} \left[\begin{array}{c} I\left(-(r+p_c)\frac{\beta}{1-\beta},0,a_c\right) \\ +\pi(a_c)^{-\frac{\beta}{1-\beta}} e^{-\frac{\beta}{1-\beta}p_s a_c} I\left(-(r+p_s)\frac{\beta}{1-\beta},a_c,\underline{s}\right) \\ +\left[\pi(a_c)e^{p_s a_c}\right]^{-\frac{\beta}{1-\beta}} I\left(-(r+p_s)\frac{\beta}{1-\beta},s_p,s\right) \\ +\left(\frac{e_p}{e(0)}\right)^{\beta}(s_p-\min(s,\underline{s})) \end{array}\right]$$

6.5.3 Consumption and bequests

Substituting (111) into the first budget constraint:

$$b = \int_0^s (c(a)+e_s(a))\,q(a)\,da = \int_0^s c(a)\,q(a)\,da + \int_0^s e_s(a)\,q(a)\,da$$

Using (111),

$$\int_0^s c^S(a)\,q(a)\,da = \int_0^s e^{\frac{(r-\rho)a}{\sigma}} \pi(a)^{\frac{1}{\sigma}\frac{\gamma-\sigma}{1-\gamma}} c(0)\,q(a)\,da$$

$$= c(0)\left[\begin{array}{c} I\left(\theta+p_c\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma},0,a_c\right) \\ +\left(\pi(a_c)e^{p_s a_c}\right)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} I\left(\theta+p_s\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma},a_c,s\right) \end{array}\right]$$

E^* is the present value of the optimal private expenditures in education as given by:

$$E^* = \int_0^s e_s(a)\,q(a)\,da$$

Plug (118) into E^*, $\hat{e}^*(a) = \hat{e}^*(0)\,q(a)^{-\frac{1}{1-\beta}}$ if $e_s(a) > 0$.

$$\begin{aligned} E^* &= e^*(0)\left[\begin{array}{c} I\left(-(r+p_c)\frac{\beta}{1-\beta},0,a_c\right)+\pi(a_c)^{-\frac{\beta}{1-\beta}} e^{-\frac{\beta}{1-\beta}p_s a_c} I\left(-(r+p_s)\frac{\beta}{1-\beta},a_c,\underline{s}\right) \\ +\pi(a_c)^{-\frac{\beta}{1-\beta}} e^{-\frac{\beta}{1-\beta}p_s a_c} I\left(-(r+p_s)\frac{\beta}{1-\beta},s_p,s\right) \end{array}\right] \\ &\quad -e_p\pi(a_c)e^{p_s a_c} I\left((r+p_s),s_p,\min(s,\overline{s})\right) \end{aligned} \tag{128}$$

Substitute $c^S(a)$ in terms of $c(0)$ into the first budget constraint,

$$b - E^* = c(0)\left[\begin{array}{c} I\left(\theta+p_c\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma},0,a_c\right) \\ +\left(\pi(a_c)e^{p_s a_c}\right)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} I\left(\theta+p_s\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma},a_c,s\right) \end{array}\right] \tag{129}$$

By (129) and (111),

$$c(0) = \{U(c) + \bar{u}\}^{-\frac{\eta}{\sigma}} M^{\frac{1}{1-\sigma}} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{\sigma}\frac{1}{1-\sigma}} \lambda_1^{-\frac{1}{\sigma}}$$

$$\lambda_1^{-\frac{1}{\sigma}} \tag{130}$$

$$= \frac{b - E^*}{\{U(c) + \bar{u}\}^{-\frac{\eta}{\sigma}} M^{\frac{1}{1-\sigma}} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{\sigma}\frac{1}{1-\sigma}} \left[\begin{array}{c} I\left(\theta + p_c \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}, 0, a_c\right) \\ +\left(\pi(a_c) e^{p_s a_c}\right)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} I\left(\theta + p_s \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}, a_c, s\right) \end{array}\right]}$$

6.5.4 Lifetime earnings $W(s,n)$

$$W(s,n) = \int_s^T c^W(a)\, q(a)\, da + q(F)\, nb'$$

By (111),

$$\begin{aligned} & \left[W(s,n) - q(F)\, nb'\right] \lambda_2^{\frac{1}{\sigma}} \{U(c) + \bar{u}\}^{\frac{\eta}{\sigma}} \\ = \; & M^{\frac{1}{1-\sigma}} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{\sigma}\frac{1}{1-\sigma}} \left[\begin{array}{l} \pi(a_c)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} e^{p_s a_c \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} I\left(\theta + p_s \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}, s, 25\right) \\ +\pi(a_s)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} \int_{25}^T e^{-\theta a}\left[\frac{e^{-p(a-a_s)} - \xi}{1-\xi}\right]^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} da \end{array}\right] \end{aligned}$$

$$\lambda_2^{-\frac{1}{\sigma}} \tag{131}$$

$$= \frac{W(s,n) - q(F)\, nb'}{\{U(c) + \bar{u}\}^{-\frac{\eta}{\sigma}} M^{\frac{1}{1-\sigma}} \left(\frac{\rho}{1-\rho^T}\right)^{\frac{1}{\sigma}\frac{1}{1-\sigma}} \left[\begin{array}{l} \pi(a_c)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} e^{p_s a_c \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} I\left(\theta + p_s \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}, s, 25\right) \\ +\pi(a_s)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} \int_{25}^T e^{-\theta a}\left[\frac{e^{-p(a-a_s)} - \xi}{1-\xi}\right]^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} da \end{array}\right]}$$

Since we consider only steady state situations, let $b = b'$ in the two previous equations, (130) and (131). Dividing one by the other, we derive the following optimal level of transfers:

$$\Omega(s) \tag{132}$$

$$= \frac{\begin{array}{c} \pi(a_c)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} e^{p_s a_c \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} I\left(\theta + p_s \frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}, s, 25\right) \\ +\pi(a_s)^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} \int_{25}^T e^{-\theta a}\left[\frac{e^{-p(a-a_s)} - \xi}{1-\xi}\right]^{\frac{1-\sigma}{\sigma}\frac{\gamma}{1-\gamma}} da \end{array}}{I\left(\theta + p_c \frac{\gamma}{\sigma}\frac{1-\sigma}{1-\gamma}, 0, a_c\right) + \pi(a_c)^{\frac{\gamma}{\sigma}\frac{1-\sigma}{1-\gamma}} e^{p_s a_c \frac{\gamma}{\sigma}\frac{1-\sigma}{1-\gamma}} I\left(\theta + p_s \frac{\gamma}{\sigma}\frac{1-\sigma}{1-\gamma}, a_c, s\right)}$$

$$b = \frac{W(s,n) G^{-\frac{1}{\sigma}} + E^* \Omega(s)}{\Omega(s) + q(F)\, nG^{-\frac{1}{\sigma}}} \tag{133}$$

Once b is obtained, one can use them to solve for $\lambda(a)$ and $c(a)$. In particular,

$$\frac{c^W(s)}{c^S(s)} = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{\sigma}} = G^{\frac{1}{\sigma}} \tag{134}$$

$$\lambda_2 = \frac{\lambda_1}{G} \tag{135}$$

6.5.5 Fertility

First order condition of fertility can be written as

$$\pi(F)^{\frac{1-\eta}{1-\gamma_c}} \beta_1 e^{-\rho F} \frac{\phi_n(n,F)V(b')}{W(s,n)} - \lambda_2 \frac{q(F)b'}{W(s,n)} = -\lambda_2 \frac{W_n(s,n)}{W(s,n)}$$

$$\frac{W_n(s,n)}{W(s,n)} = \frac{\pi(a_s)\left[\frac{1}{1-\xi}e^{pa_s}I(p-v+r,F,R) - \frac{\xi}{1-\xi}I(r-v,F,R)\right]l'(n)}{\pi(a_c)e^{p_s a_c}I(r+p_s-v,s,F) + l(n)\pi(a_s)\left[\frac{1}{1-\xi}e^{pa_s}I(p-v+r,F,R) - \frac{\xi}{1-\xi}I(r-v,F,R)\right]}$$

Plug steady state V, (4) into the left hand side of the above equation,

$$-\frac{W_n(s,n)}{W(s,n)} = \lambda_2^{-1}\pi(F)^{\frac{1-\eta}{1-\gamma_c}} \beta_1 e^{-\rho F} \frac{\phi_n(n,F)}{W(s,n)} V(b') - \frac{q(F)b'}{W(s,n)}$$

where

$$l(n) = 1 - \lambda\left[(n+\kappa)^\theta - \kappa^\theta\right]$$
$$l_n(n) = -\lambda\theta(n+\kappa)^{\theta-1} < 0$$

6.5.6 Integrals

Define s_p such that it lies between $\underline{s}$ and $\overline{s}$.

$$I_1(d,t_1,t_2) = \int_{t_1}^{t_2} e^{-da}da = \left[\frac{e^{-da}}{d}\right]_{t_2}^{t_1}$$

6.5.7 Solution Algorithm

We solve the model by first assuming an initial value of s, n and e(0) given e_p, p_E, $\underline{s}$, $\overline{s}$ and F. The optiaml educational expenditure when public education is available, $\widehat{e}^*(s)$, optimal public expenditure $e^*(s)$, $e_s(s)$ and G can be gotten immediately by (44), (41), (114).

$$\hat{e}^*(s) = \hat{e}^*(0)\, q(s)^{-\frac{1}{1-\beta}} = \hat{e}^*(0)\left[e^{-rs}\pi(a_c)\, e^{-p_s(s-a_c)}\right]^{-\frac{1}{1-\beta}}$$

$$e_s(a) = \begin{cases} \hat{e}^*(a) - e_p & \text{if } \underline{s} \le a \le \bar{s} \\ \hat{e}^*(a) & \text{else} \end{cases}$$

Then by (44), $\widehat{a}$ satisfying

$$e_p = \hat{e}(0)q(\hat{a})^{-\frac{1}{1-\beta}}$$

Since $\underline{s} \le \hat{a} \le \bar{s}$,

$$e_p = \hat{e}(0)\left[\pi(a_c)\, e^{-(p_s+r)\hat{a}+p_s a_c}\right]^{-\frac{1}{1-\beta}}$$

$$\Rightarrow \hat{a} = \frac{1}{p_s + r}\left[(1-\beta)\ln\left(\frac{e_p}{\hat{e}(0)}\right) + \ln\pi(a_c) + p_s a_c\right]$$

After $\widehat{a}$ is solved, s_p, $h(s)$, E^*, $W(s,n)$, $\frac{\lambda_3}{\lambda_2}$, Ω, b, $c^W(s)$ can be derived through (122), (46),ll (128), (126), (115), (132), (133), (134) successively. After all these variables are available, we are able to update s, n and e(0) by iterating (45),

$$\hat{e}^*(0) = \left(\gamma h(s)^{-\frac{\beta}{\gamma}}\, p_E^{-\beta} W(s,n)/G\right)^{\frac{1}{1-\beta}}$$

(32), and (124).

Inefficiently high fertility in a closed economy

Section 5.2 shows that fertility could be inefficiently high in a small-open incomplete-markets economy. We now show that a similar could be obtained in a closed economy. Suppose the production function is $f(k,l)$ where k is capital and l is labor. In a closed economy, interest rates and wages are given by

$$r_t = f_1\left(b_t, 1-\lambda n_t\right) - \delta \text{ and } w_t = f_2\left(b_t, 1-\lambda n_t\right).$$

where δ is the rate of depreciation. Substituting out prices, the first order condition with respect to fertility becomes

$$u'\left(c_t\right)\left[f_2\left(b_t, l_t\right)\lambda + b_{t+1}\right] \leq \Phi'\left(n_t\right) U_{t+1}\left(b_{t+1}\right) \tag{136}$$

Furthermore, the corresponding Euler equation is

$$u'\left(c_t\right) n_t \geq \Phi\left(n_t\right) u'\left(c_{t+1}\right)\left(f_1\left(b_{t+1}, l_{t+1}\right) + 1 - \delta\right)$$

If bequest constraints do not bind, then steady state fertility is determined by

$$1 = \beta\left(f_1\left(b, 1-\lambda n\right) + 1 - \delta\right) n^{-\varepsilon} \tag{137}$$

If bequest constraint binds, then interior solution of fertility is determined by (136). The following proposition provides a case in which steady state fertility is larger when bequest constraints binds.

Proposition D.1. Assume the economy is closed and $f\left(k,l\right) = Ak + Bl$ where $A, B > 0$. If the marginal productivity of capital A is large enough such that $\frac{\beta(1-\varepsilon)}{1-\sigma}\left(A+1-\delta\right)\lambda^{\varepsilon} > 1 + \frac{\sigma-\varepsilon}{1-\sigma}\beta\lambda^{\varepsilon-1}$, then steady state fertility when the constraint $b \geq \underline{b}$ binds is higher than the unconstrained fertility level.

Proof Equation (136) at steady state, together with the budget constraint and equilibrium prices results in

$$u'\left(c\right)\left(f_2\left(b,l\right)\lambda + b\right) = \frac{\Phi'\left(n\right)}{1-\Phi\left(n\right)}\left[f\left(b, 1-\lambda n\right) + \left(1-\delta\right)b - nb\right]\frac{u'\left(c\right)}{1-\sigma}.$$

Using the specific production function, utility function and altruistic function specified above and collect terms, this equation becomes

$$\begin{aligned} B\lambda &= \frac{\beta\left(1-\varepsilon\right)}{1-\sigma}\left[B + \left(A+1-\delta\right)b\right]n^{-\varepsilon} - b - \left(b + B\lambda\right)\frac{\sigma-\varepsilon}{1-\sigma}\beta n^{1-\varepsilon} \\ &= \frac{\beta\left(1-\varepsilon\right)}{1-\sigma}Bn^{-\varepsilon} + b\left[\frac{\beta\left(1-\varepsilon\right)}{1-\sigma}\left(A+1-\delta\right)n^{-\varepsilon} - 1 - \frac{\sigma-\varepsilon}{1-\sigma}\beta n^{1-\varepsilon}\right] \\ &\quad - B\lambda\frac{\sigma-\varepsilon}{1-\sigma}\beta n^{1-\varepsilon}. \end{aligned} \tag{138}$$

Denote the right hand side of this equation by $RHS\left(n\right)$. Notice that $RHS'\left(n\right) < 0$, $\lim_{n\to 0} RHS\left(n\right) = \infty$, and $\lim_{n\to\infty} RHS\left(n\right) = -\infty$. An interior solution for fertility exists if and only if $RHS\left(\frac{1}{\lambda}\right) < B\lambda$, e.g.

$$1+\frac{\sigma-\varepsilon}{1-\sigma}\beta\lambda^{\varepsilon-1}>\frac{\beta\left(1-\varepsilon\right)}{1-\sigma}\left(A+1-\delta\right)\lambda^{\varepsilon}-B\lambda\frac{1-\beta\lambda^{\varepsilon-1}}{b}.$$

By the assumption, the term in the square bracket of (138) is positive. As a result, an exogenous increase in b increases the right hand side but does not affect the left hand side, and thus leads to a bigger steady state fertility. Hence if b is restricted to be higher than the unconstrained optimal choice b^*, e.g. $b^* < \underline{b}$, then the steady state fertility in the unconstrained case (complete markets) is smaller than that in the constrained case (incomplete markets).

Proof of Proposition 1 For the case when $(1+r)\lambda\omega>\omega$, see Barro and Becker (1989). In this proof, we focus on the case $(1+r)\lambda\omega<\omega$. First order conditions for bequests and fertility are:

$$u'\left(c_t\right)n_t=\Phi\left(n_t\right)U'_{t+1}\left(b_{t+1}\right)$$

$$u'\left(c_t\right)\left(\lambda\omega+b_{t+1}\right)\leq\Phi'\left(n_t\right)U_{t+1}\left(b_{t+1}\right)\quad\text{with equality if } n_t<\bar{n}.$$

$\Phi'(0)=\infty$ excludes the possibility of zero children. Using the envelope condition $U'_t\left(b_t\right)=u'\left(c_t\right)(1+r)$, the first condition with respect to bequest becomes:

$$u'\left(c_t\right)n_t=\Phi\left(n_t\right)u'\left(c_{t+1}\right)(1+r)$$

Therefore,

$$\begin{aligned}U_t\left(b_t\right)&=u\left(c_t\right)+\Phi\left(n_t\right)U_{t+1}\left(b_{t+1}\right)\\&\geq u\left(c_t\right)+\frac{\Phi\left(n_t\right)}{\Phi'\left(n_t\right)}u'\left(c_t\right)\left(\lambda\omega+b_{t+1}\right)\\&=u\left(c_t\right)+\frac{\Phi\left(n_t\right)}{\Phi'\left(n_t\right)n_t}u'\left(c_t\right)\left(\omega+(1+r)b_t-c_t\right)\end{aligned}$$

Forward this inequality one period and use the specific functional forms for utility and altruistic functions to obtain

$$\begin{aligned}U_{t+1}\left(b_{t+1}\right)&\geq u\left(c_{t+1}\right)+\frac{\Phi\left(n_{t+1}\right)}{\Phi'\left(n_{t+1}\right)n_{t+1}}u'\left(c_{t+1}\right)\left(\omega+(1+r)b_{t+1}-c_{t+1}\right)\\&=u'\left(c_{t+1}\right)\frac{1}{1-\varepsilon}\left[\frac{\sigma-\varepsilon}{1-\sigma}c_{t+1}+(1+r)b_{t+1}+\omega\right]\\&>u'\left(c_{t+1}\right)\frac{1}{1-\varepsilon}(1+r)\left(\lambda\omega+b_{t+1}\right)\\&=u'\left(c_t\right)\frac{1}{\Phi'\left(n_t\right)}\left(\lambda\omega+b_{t+1}\right)\end{aligned}$$

for all n_t. The last inequality sign is due to the assumption that $(1+r)\lambda\omega<\omega$. Hence

$$U_{t+1}\left(b_{t+1}\right)\Phi'\left(n_t\right)>u'\left(c_t\right)\left(\lambda\omega+b_{t+1}\right)$$

for all $n_t \in [0, \bar{n}]$ implies $n_t = \bar{n}$ is optimum. Consumption growth follows from the Euler equation.

Proof of Proposition 3 When contingent assets are available, first order condition to fertility n_t is

$$\left[\lambda\omega_t + \sum_{i=1}^{K} q(\omega_t, \omega_i) b_{t+1}(\omega^t, \omega_i)\right] u'(c_t) = \beta(1-\varepsilon) n_t^{-\varepsilon} E_t \left[V_{t+1}\left(b_{t+1}; \omega^{t+1}\right) |\omega_t\right] \tag{139}$$

where consumption is given by (53)

$$c_t = \omega_t (1 - \lambda n_t) + b_t - n_t \sum_{i=1}^{K} q(\omega_t, \omega_i) b_{t+1}(\omega^t, \omega_i)$$

first order condition to $b_{t+1}(\omega^t, \omega_i)$:

$$u'(c_t) q(\omega_t, \omega_i) n_t = \beta n_t^{1-\varepsilon} M(\omega_i, \omega_t) \frac{\partial V_{t+1}\left(b_{t+1}; \omega^{t+1}\right)}{\partial b_{t+1}}$$

which together with envelop condition $\frac{\partial V_t\left(b_t; \omega^t\right)}{\partial b_t} = u'(c_t)$ and the actuarially fair price of $b_{t+1}(\omega^t, \omega_i)$ gives the Euler equation

$$u'(c_t) = \beta n_t^{-\varepsilon} (1+r) u'(c_{t+1}) \tag{140}$$

Notice that all children from the same family enjoy the same consumption which is independent of ω_{t+1}. Substituting (139) into the objective function and use the budget constraint,

$$V_t\left(b_t; \omega^t\right) = u'(c_t) \left[\frac{c_t(\sigma - \varepsilon)}{(1-\sigma)(1-\varepsilon)} + \frac{1}{1-\varepsilon}(\omega_t + b_t)\right]$$

Forward this equation by one period, then use it and (27) to rewrite (139) as,

$$\lambda\omega_t(1+r) + (1+r)\sum_{i=1}^{K} q(\omega_t, \omega_i) b_{t+1}(\omega^t, \omega_i) = \frac{\sigma - \varepsilon}{1-\sigma} c_{t+1} + E\left[b_{t+1}(\omega^t, \omega_i)|\omega_t\right] + E(\omega_{t+1}|\omega_t)$$

After some manipulations, the consumption of every child endowed with ability ω' given parental ability ω is

$$c_{t+1} = c(\omega_t) = \frac{1-\sigma}{\sigma-\varepsilon}\left[\lambda\omega_t(1+r) - E(\omega_{t+1}|\omega_t)\right]$$

for all t and ω_t. Furthermore, using (27) fertility can be solved as:

$$n_t = n(\omega_{t-1}, \omega_t) = \beta^{\frac{1}{\varepsilon}} (1+r)^{\frac{1}{\varepsilon}} \left(\frac{\lambda\omega_{t-1}(1+r) - E(\omega_t|\omega_{t-1})}{\lambda\omega_t(1+r) - E(\omega_{t+1}|\omega_t)}\right)^{\frac{\sigma}{\varepsilon}}$$

When $\lambda\omega_t(1+r) < E(\omega_{t+1}|\omega_t)$, the proof follows the same logic with the second part of Proposition 1.

Proof of Proposition 4 We first show that there exists a solution $U(\cdot)$ that solves the functional equation (79). Define a set of mappings.

$$S = \{U : \Omega \to \mathbb{R} |\, \|U\| \leq M\}$$

where $M = \frac{u(\omega_K)}{1-\beta\lambda^{\varepsilon-1}}$, and $\|\cdot\|$ is the sup norm. S is a complete metric space. Define operator T as

$$TU(\omega) \equiv \max_{0 \leq n \leq \frac{1}{\lambda}} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U(\omega')|\omega\right] \tag{141}$$

for all $\omega \in \Omega$ and $U \in S$. Given $U(\cdot)$ and ω, the right hand side of (141) has a solution that attains the maximum. First show that T is a contraction. It suffices to show that T satisfies two properties, monotonicity and discounting. Standard argument can show that given U and $\tilde{U} \in S$ satisfying $U(\omega) \leq \tilde{U}(\omega)$ for all $\omega \in \Omega$, $TU(\omega) \leq T\tilde{U}(\omega)$ for all $\omega \in \Omega$. The following arguments prove discounting property holds. For any given constant b,

$$\begin{aligned} T(U(\omega)+b) &= \max_{0 \leq n \leq \frac{1}{\lambda}} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U(\omega') + b|\omega\right] \\ &\leq \max_{0 \leq n \leq \frac{1}{\lambda}} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U(\omega')|\omega\right] + \beta b\left(\frac{1}{\lambda}\right)^{1-\varepsilon} \\ &= TU(\omega) + \beta\lambda^{\varepsilon-1} b \end{aligned}$$

$\beta\lambda^{\varepsilon-1} < 1$ by assumption. By Contraction Mapping Theorem, there exists a unique fixed point $U : \Omega \to \mathbb{R}$ that solves the functional equation $TU = U$. The existence of a solution $U(\cdot)$ has been proved. Next we show $U(\omega_0) = V_0^*(\omega_0)$ for all $\omega_0 \in \Omega$, that is to show $U(\omega_0)$ is the supremum of the sequential problem (58) for any given $\omega^0 = \omega_0$.

$$\begin{aligned} U(\omega_0) &= \max_{n \in [0, \frac{1}{\lambda}]} u(\omega_0(1-\lambda n)) + \beta n^{1-\varepsilon} E_0\left[U(\omega_1)|\omega_0\right] \\ &\geqslant u(\omega_0(1-\lambda n_0(\omega_0))) + E_0\left(\begin{array}{c} \beta n_0(\omega_0)^{1-\varepsilon} u\left(\omega_1\left(1-\lambda n_1\left(\omega^1\right)\right)\right) \\ +\beta^2 n_0(\omega_0)^{1-\varepsilon} E_0 n_1\left(\omega^1\right)^{1-\varepsilon} E_1\left[U(\omega_2)|\omega_1\right] \end{array}\right) \\ &\geqslant \cdots \\ &\geqslant E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n_j\left(\omega^j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n_t\left(\omega^t\right)\right)\right) + \beta^{T+1} E_0 \prod_{j=0}^{T} n_j\left(\omega^j\right)^{1-\varepsilon} U(\omega_{T+1}) \end{aligned}$$

for all feasible plan $\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$. Let $\prod_{j=0}^{-1} n_j\left(\omega^j\right)^{1-\varepsilon} = 1$. The last term

$$\beta^{T+1} \prod_{j=0}^{T} n_j\left(\omega^j\right)^{1-\varepsilon} U(\omega_{T+1}) \leq \left(\beta\lambda^{\varepsilon-1}\right)^{T+1} \frac{u(\omega_K)}{1-\beta\lambda^{\varepsilon-1}}$$

The right hand side of this inequality converges to 0 as T goes to infinite. Hence for all feasible plan $\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$

$$U\left(\omega_0\right) \geqslant E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} n_j\left(\omega^j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n_t\left(\omega^t\right)\right)\right) \tag{142}$$

Given $\varepsilon > 0$, choosing a sequence of positive real numbers $\{\delta_t\}_{t=1}^{\infty}$ such that $\sum_{t=0}^{\infty}\left(\beta\lambda^{\varepsilon-1}\right)^t \delta_t \leq \varepsilon$. Let $n^*\left(\omega_t\right)$ be the solution that attains $U\left(\omega_t\right)$, then for all t

$$U\left(\omega_t\right) < u\left(\omega_t\left(1-\lambda n^*\left(\omega_t\right)\right)\right) + \beta n^*\left(\omega_t\right)^{1-\varepsilon} E_t\left[U\left(\omega_{t+1}\right)|\omega_t\right] + \delta_t$$

Starting from period 0, iteratively substituting the value function $U\left(\omega_{t+1}\right)$ into the above inequality shows that for all ω_0,

$$\begin{aligned} U\left(\omega_0\right) < {} & E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n^*\left(\omega_j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n^*\left(\omega_t\right)\right)\right) + \beta^{T+1} E_0 \prod_{j=0}^{T} n^*\left(\omega_j\right)^{1-\varepsilon} U\left(\omega_{T+1}\right) \\ & + E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n^*\left(\omega_j\right)^{1-\varepsilon} \delta_t \end{aligned}$$

The choice of $\{\delta_t\}$ guarantees that the last term is no more than ε as $T \to \infty$. We have shown that

$$\lim_{T\to\infty} \beta^{T+1} E_0 \prod_{j=0}^{T} n^*\left(\omega_j\right)^{1-\varepsilon} U\left(\omega_{T+1}\right) = 0$$

Hence for any given $\varepsilon > 0$, there exists a feasible plan $\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty} = \{n^*\left(\omega_t\right)\}_{t=0}^{\infty}$ such that

$$U\left(\omega_0\right) < E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} n^*\left(\omega_j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n^*\left(\omega_t\right)\right)\right) + \varepsilon \tag{143}$$

By (142) and (143),

$$\begin{aligned} U\left(\omega_0\right) &= \sup_{\left\{n_t\left(\omega^t\right)\right\}_{t=0}^{\infty} \in \left[0, \frac{1}{\lambda}\right]} E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} n_j\left(\omega^j\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda n_t\left(\omega^t\right)\right)\right) \\ &= \sup_{\left\{\hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right)\right\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta_t^t \hat{N}\left(\omega^{t-1}\right)^{1-\varepsilon} u\left(\omega_t\left(1-\lambda \frac{\hat{N}_{t+1}\left(\omega^{t-1},\omega_t\right)}{\hat{N}_t\left(\omega^{t-1}\right)}\right)\right) \end{aligned}$$

Therefore

$$U\left(\omega_0\right) = V^*\left(\omega_0\right)$$

Proof of Proposition 5 (i) In this case, equation (80) can be written as $\frac{n(\omega)^{\varepsilon}}{(1-\lambda n(\omega))^{\sigma}} = A\omega^{\sigma-1}$ where $A = \frac{\beta(1-\varepsilon)}{\lambda} E\left[u\left(\omega'\right)\right]$ is a constant. Using the implicit function theorem, it follows that

$$n'(\omega) = -\frac{(1-\sigma)/\omega}{\frac{\varepsilon}{n(\omega)} + \frac{\lambda\sigma}{1-\lambda n(\omega)}} < 0.$$

(ii) In deterministic case, $\omega' = \omega$. Equation (62) simplifies to:

$$n^{*\varepsilon} = \beta\left(\frac{1}{\lambda} + \frac{\sigma-\varepsilon}{1-\sigma}\left(\frac{1}{\lambda} - n^*\right)\right) \tag{144}$$

The left hand side of equation (151) is strictly increasing in n^* while the right hand side is strictly decreasing in n^*. Obviously $n^* > 0$. An interior solution with $n^* < 1/\lambda$ exists since $\lambda^{1-\varepsilon} > \beta$. (iii) Let n^* denotes the optimal fertility given ω. Plug functional form of $u(\cdot)$ into equation (81)

$$U(\omega) = h(n^*)\,\omega^{1-\sigma} \tag{145}$$

where

$$h(n^*) \equiv \frac{1}{1-\sigma}(1-\lambda n^*)^{1-\sigma} + \frac{1}{1-\varepsilon}\lambda n^*(1-\lambda n^*)^{-\sigma} \tag{146}$$

We make a guess on the value function and let it take the form: $U(\omega) = A\omega^{1-\sigma}$ where A is a constant, independent of ω. Equating this guess with (152) results in:

$$A = h(n^*) \tag{147}$$

Thus, in order for A to be independent of ω, we must verify that the results n^* is independent of ω. Notice that,

$$E\left[U\left(\omega'\right)|\omega\right] = E\left[A\omega'^{1-\sigma}|\omega\right] = A\omega^{1-\sigma}e^{\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}}$$

The last equality holds because the assumption that ω' is lognormal distributed with $\ln\omega$ and σ_ε as the mean and variance of $\ln\omega'$. Plug this equality into (80) to obtain:

$$\lambda(1-\lambda n^*)^{-\sigma}\omega^{1-\sigma} = A\beta(1-\varepsilon)\,n^{*-\varepsilon}e^{\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}}\omega^{1-\sigma}$$

ω cancels out of this equation and therefore n^* is independent of ω confirming our guess. This expression together with (153) and (154) gives a rule to solve the optimal fertility n^*.

$$\frac{\lambda(1-\sigma)}{\beta(1-\varepsilon)\,n^{*-\varepsilon}}e^{-\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}} = 1 - \lambda n^* + \frac{n^*}{1-\varepsilon}\lambda(1-\sigma)$$

Manipulate terms

$$\frac{\lambda(1-\sigma)\,n^{*\varepsilon}}{\beta(1-\varepsilon)}\left[e^{-\frac{(1-\sigma)^2\sigma_\varepsilon^2}{2}} - \beta n^{*1-\varepsilon}\right] = 1 - \lambda n^*$$

The solution of n^* does not depend on ω which confirms the guess on $U(\omega)$. In case (ii) and (iii), fertility is independent of ability.

Proof of Proposition 6 (i) If fertility is exogenously the same for every individual,

$$N_{t+1} = N_t \sum_{\omega_i \in \Omega} n\pi_t(\omega_i) = N_t n$$

By equation (71),

$$\pi_{t+1}(\omega_j) = \frac{nN_t}{N_{t+1}} \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, \omega_i) = \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, \omega_i)$$

Taking limit to both sides of the expression with π, we get

$$\pi^*(\omega_j) = \lim_{t \to \infty} \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, \omega_i) = \sum_{\omega_i \in \Omega} \pi^*(\omega_i) M(\omega_j, \omega_i)$$

Hence $\pi^*(\cdot) = \mu(\cdot)$ is the invariant distribution of M. (ii) $M(\cdot\ , \omega_i)$ is independent of ω_i implies $M(\omega_j, \omega_i) = M(\omega_j, :)$ for every $\omega_j \in \Omega$. By (71),

$$\begin{aligned}\pi_{t+1}(\omega_j) &= \frac{M(\omega_j, :)}{N_{t+1}} \sum_{\omega_i \in \Omega} n(\omega_i) \pi_t(\omega_i) N_t \\ &= M(\omega_j, :) = \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, :)\end{aligned}$$

for all $t \geq 0$, these equalities imply $\pi_{t+1}(\omega_j) = \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, \omega_i)$ and hence $\pi^*(\omega_j) = \pi_{t+1}(\omega_j) = \mu(\omega_j) = M(\omega_j, :)$ for all ω_j.

Proof of Proposition 7 Part (i) directly applies Proposition 5(i) and Proposition 6 (ii). For part (ii), we can apply Proposition 5 (ii), in which fertility is independent of ability when M is identity. We use this result to prove the distribution of every period as well as the limit distribution is the same with the initial one.

$$\begin{aligned}\pi_{t+1}(\omega_j) &= \frac{N_t}{N_{t+1}} \sum_{\omega_i \in \Omega} n(\omega_i) \pi_t(\omega_i) M(\omega_j, \omega_i) \\ &= \frac{N_t n}{N_{t+1}} \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, \omega_i) \\ &= \sum_{\omega_i \in \Omega} \pi_t(\omega_i) M(\omega_j, \omega_i) = \pi_t(\omega_j)\end{aligned}$$

The last equality holds because M is identity matrix. Therefore $\pi^*(\omega) = \pi_t(\omega) = \pi_0(\omega)$ for all ω and all t. Part (iii) follows Proposition 5 (iii). The conditional variance of $\ln \omega_t$ diverges to infinite because $\ln \omega_t = \ln \omega_0 + \sum_{i=1}^{t} \varepsilon_i,\ E(\ln \omega_t | \omega_0) = \ln \omega_0,\ Var(\ln \omega_t | \omega_0) = t^2 \sigma_\varepsilon^2$ and $\lim_{t\to\infty} Var(\ln \omega_t | \omega_0) = \infty.$

Proof of Proposition 9 Notice that

$$
\begin{aligned}
U(\omega) &= \max_{n_t \in [0,1/\lambda]} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U(\omega')|\omega\right] \\
&\geq \max_{[\underline{n}(\omega),\overline{n}(\omega)]} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U(\omega')|\omega\right] := U^1(\omega') \\
&\geq \max_{[\underline{n}(\omega),\overline{n}(\omega)]} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U^1(\omega')|\omega\right] := U^2(\omega') \\
&.. \\
&\geq \max_{[\underline{n}(\omega),\overline{n}(\omega)]} u((1-\lambda n)\omega) + \beta n^{1-\varepsilon} E\left[U^r(\omega')|\omega\right] = U^r(\omega')
\end{aligned}
$$

where the first inequality is strict if a constraint is binding for any particular ω, the remaining inequalities follow from the contraction mapping recursion, and the final inequality uses the contraction mapping theorem. Furthermore, a strict inequality for a particular ω translates into a strict inequality for all $\omega' s$ since M is a regular Markov chain meaning that, regardless of initial ability there is positive probability that someone in the dynasty will reach a binding state in finite time. The second part of the proposition follows because fertility restrictions do not change the marginal costs of having children but it decreases the marginal benefits by reducing $U(\omega)$ for all ω (see equation (80)). Hence an upper bound of fertility makes people have fewer children than the unrestricted case as long as fertility upper bounds affect at least one of those types.

Proof of Proposition 10 Let $N_t^r(\omega)$ be the size of population with ability ω at time t in presence of fertility upper limits. By Proposition 9, for all ω_j

$$
\begin{aligned}
N_1(\omega_j) &= \sum_{\omega_i \in \Omega} n(\omega_i) N_0(\omega_i) M(\omega_j, \omega_i) \\
&\geq \sum_{\omega_i \in \Omega} n^r(\omega_i) N_0^r(\omega_i) M(\omega_j, \omega_i) = N_1^r(\omega_j)
\end{aligned}
$$

where initial population is not affected by policies $N_0(\omega_i) = N_0^r(\omega_i)$. Given the inequality $N_1(\omega_j) \geq N_1^r(\omega_j)$, an inductive argument shows

$$
N_{t+1}(\omega_j) = \sum_{\omega_i \in \Omega} n(\omega_i) N_t(\omega_i) M(\omega_j, \omega_i) \geq \sum_{\omega_i \in \Omega} n^r(\omega_i) N_t^r(\omega_i) M(\omega_j, \omega_i) = N_{t+1}^r(\omega_j)
$$

for all ω_j and all $t \geq 0$. Apply this result,

$$\begin{aligned} W^r\left(\beta_p\right) &= \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U^r(\omega) N_t^r\left(\omega\right) \\ &\leq \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) N_t^r\left(\omega\right) \\ &\leq \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) N_t\left(\omega\right) = W\left(\beta_p\right) \end{aligned}$$

Proof of Proposition 12 When $M\left(\cdot,\omega\right)$ is independent of ω, the proof of Proposition 6 (ii) gives

$$\pi_{t+1}\left(\omega_j\right) = M(\omega_j,:) \quad \text{for all } t \geq 0$$

Initial ability distribution $\pi_0\left(\omega\right)$ is given and not affected by the policy in place. So restriction of a fertility upper bound only reduces individual utility by Proposition 9 but does not affect the ability distribution of any period. It decreases social welfare defined by (68).

Proof of Proposition 13 This Proposition relies on Proposition 7 (ii)'s results, $n\left(\omega\right) = n$ and $\pi_t\left(\omega\right) = \pi^*\left(\omega\right) = \pi_0\left(\omega\right)$ when M is an identity. Similar with Proposition 12, restriction does not alter distribution, which together with Proposition 9, finishes the proof.

APPENDIX D. STOCHASTIC DOMINANCE AND DEMOGRAPHIC POLICY EVALUATION: A CRITIQUE

Proof of Proposition 1 (i) If fertility is exogenously the same for every individual, divide both sides of (70) by P_{t+1}.

$$\frac{P_{t+1}(\omega_j)}{P_{t+1}} = \frac{P_t}{P_{t+1}} \sum_{\omega_i \in \Omega} f(\omega_i) \frac{P_t(\omega_i)}{P_t} M(\omega_j, \omega_i) \text{ for all } \omega_j \in \Omega.$$

Using the definition of π_t,

$$\begin{aligned}\pi_{t+1}(\omega_j) &= \frac{P_t}{P_{t+1}} \sum_{\omega_i \in \Omega} f(\omega_i)\, \pi_t(\omega_i)\, M(\omega_j, \omega_i) \\ &= \frac{f P_t}{P_{t+1}} \sum_{\omega_i \in \Omega} \pi_t(\omega_i)\, M(\omega_j, \omega_i) = \sum_{\omega_i \in \Omega} \pi_t(\omega_i)\, M(\omega_j, \omega_i).\end{aligned}$$

The last equality holds because

$$P_{t+1} = P_t \sum_{\omega_i \in \Omega} f \pi_t(\omega_i) = P_t f.$$

Taking limit to both sides of the expression with π, we get

$$\pi^*(\omega_j) = \underset{t\to\infty}{lim} \pi_{t+1}(\omega_j) = \underset{t\to\infty}{lim} \sum_{\omega_i \in \Omega} \pi_t(\omega_i)\, M(\omega_j, \omega_i) = \sum_{\omega_i \in \Omega} \pi^*(\omega_i)\, M(\omega_j, \omega_i).$$

Hence $\pi^*(\cdot) = \mu(\cdot)$ is the invariant distribution of M.

Proof of Proposition 1 (ii) $M(\cdot\,, \omega_i)$ is independent of ω_i implies $M(\omega_j, \omega_i) = \mu(\omega_j)$ for every $\omega_j \in \Omega$. By (70),

$$\pi_{t+1}(\omega_j) = \frac{1}{P_{t+1}} \sum_{\omega_i \in \Omega} f(\omega_i)\, M(\omega_j, \omega_i) \pi_t(\omega_i)\, P_t = \mu(\omega_j) \text{ for } t \geq 0.$$

Proof of Proposition 4 We first show that there exists a solution $U(\cdot)$ that solves the functional equation (79). Define a set of functions.

$$S = \{ f : \Omega \to \mathbb{R} |\, \|f\| \leq M\}$$

where $M = \frac{u(\omega_n)}{1-\beta\lambda^{\epsilon-1}}$, and $\|\cdot\|$ is the sup norm. We can show that S is a complete metric space. Define operator T as

$$TU(\omega) \equiv \max_{0 \leq f \leq \frac{1}{\lambda}} u((1-\lambda f)\,\omega) + \beta f^{1-\varepsilon} E\left[U\left(\omega'\right)|\omega\right] \tag{148}$$

for all $\omega \in \Omega$ and $U \in S$. Given $U(\cdot)$ and ω, the right hand side of (141) has a solution that attains the maximum by the Weierstrass Theorem. We first show that T is a contraction. It suffices to show that T satisfies two properties, monotonicity and discounting. Standard arguments can show that given U

and $\tilde{U} \in S$ satisfying $U(\omega) \leq \tilde{U}(\omega)$ for all ω, $TU(\omega) \leq T\tilde{U}(\omega)$ for all ω. The following arguments prove discounting property holds. For any given constant b,

$$\begin{aligned} T(U(\omega)+b) &= \max_{0\leq f \leq \frac{1}{\lambda}} u((1-\lambda f)\omega) + \beta f^{1-\varepsilon} E\left[U\left(\omega'\right) + b|\omega\right] \\ &\leq \max_{0\leq f \leq \frac{1}{\lambda}} u((1-\lambda f)\omega) + \beta f^{1-\varepsilon} E\left[U\left(\omega'\right)|\omega\right] + \beta b \left(\frac{1}{\lambda}\right)^{1-\varepsilon} \\ &= TU(\omega) + \beta\lambda^{\epsilon-1} b. \end{aligned}$$

By Contraction Mapping Theorem, there exists a unique fixed point $U : \Omega \rightarrow \mathbb{R}$ that solves the functional equation $TU = U$. The existence of a unique solution $U(\cdot)$ has been proved.
Next we show $U(\omega_0) = U_0^*(\omega_0)$ for all $\omega_0 \in \Omega$, that is to show $U(\omega_0)$ is the supremum in problem (78) for any ω_0. Define $\prod_{j=0}^{-1} f_j\left(\omega^j\right)^{1-\epsilon} = 1$. Then for any feasible plan $\left\{f_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$,

$$\begin{aligned} U(\omega_0) &= \max_{f\in[0,\frac{1}{\lambda}]} u(\omega_0(1-\lambda f)) + \beta f^{1-\epsilon} E_0\left[U(\omega_1)|\omega_0\right] \\ &\geqslant u\left(\omega_0\left(1-\lambda f_0\left(\omega^0\right)\right)\right) + \beta f_0\left(\omega^0\right)^{1-\epsilon} E_0\left[U(\omega_1)|\omega_0\right] \\ &\geqslant u\left(\omega_0\left(1-\lambda f_0\left(\omega^0\right)\right)\right) + \beta f_0\left(\omega^0\right)^{1-\epsilon} E_0\left(\begin{array}{c} u\left(\omega_1\left(1-\lambda f_1\left(\omega^1\right)\right)\right) \\ +\beta f_1\left(\omega^1\right)^{1-\epsilon} E_1\left[U(\omega_2)|\omega_1\right] \end{array}\right) \\ &\geqslant \cdots \\ &\geqslant E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} f_j\left(\omega^j\right)^{1-\epsilon} u\left(\omega_t\left(1-\lambda f_t\left(\omega^t\right)\right)\right) + \beta^{T+1} E_0 \prod_{j=0}^{T} f_j\left(\omega^j\right)^{1-\epsilon} U(\omega_{T+1}). \end{aligned}$$

Notice that the last term satisfies

$$\beta^{T+1} \prod_{j=0}^{T} f_j\left(\omega^j\right)^{1-\epsilon} U(\omega_{T+1}) \leq \left(\beta\lambda^{\epsilon-1}\right)^{T+1} \frac{u(\omega_n)}{1-\beta\lambda^{\varepsilon-1}}.$$

The right hand side of this inequality converges to 0 as T goes to infinite. Hence for all feasible plan $\left\{f_t\left(\omega^t\right)\right\}_{t=0}^{\infty}$

$$U(\omega_0) \geqslant E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} f_j\left(\omega^j\right)^{1-\epsilon} u\left(\omega_t\left(1-\lambda f_t\left(\omega^t\right)\right)\right). \tag{149}$$

We now show that $U(\omega_0)$ is the smallest upper bound. Given $\varepsilon_1 > 0$, choose a sequence of positive real numbers $\{\delta_t\}_{t=1}^{\infty}$ such that $\sum_{t=0}^{\infty}\left(\beta\lambda^{\epsilon-1}\right)^t \delta_t \leq \frac{\varepsilon_1}{2}$. Let $f^*(\omega_t)$ be the solution that attains $U(\omega_t)$, then for all t

$$U(\omega_t) < u(\omega_t(1-\lambda f^*(\omega_t))) + \beta f^*(\omega_t)^{1-\epsilon} E_0\left[U(\omega_{t+1})|\omega_t\right] + \delta_t.$$

Starting from period 0, iteratively substituting the value function $U(\omega_{t+1})$ into the above inequality shows that for all ω_0

$$U(\omega_0) < E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} f^*(\omega_j)^{1-\epsilon} u(\omega_t(1-\lambda f^*(\omega_t))) + \beta^{T+1} E_0 \prod_{j=0}^{T} f^*(\omega_j)^{1-\epsilon} U(\omega_{T+1})$$
$$+ E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} f^*(\omega_j)^{1-\epsilon} \delta_t.$$

The choice of $\{\delta_t\}$ guarantees that the last term is no more than $\frac{\varepsilon_1}{2}$ as $T \to \infty$. We also have shown that

$$\lim_{T\to\infty} \beta^{T+1} \prod_{j=0}^{T} f^*(\omega_j)^{1-\epsilon} U(\omega_{T+1}) = 0.$$

So for any given $\varepsilon_1 > 0$, there exists a feasible plan $\left\{f^*\left(\omega^t\right)\right\}_{t=0}^{\infty}$ such that

$$U(\omega_0) < E_0 \sum_{t=0}^{T} \beta^t \prod_{j=0}^{t-1} f^*(\omega_j)^{1-\epsilon} u(\omega_t(1-\lambda f^*(\omega_t))) + \frac{\varepsilon_1}{2}. \tag{150}$$

By (142) and (143),

$$U(\omega_0) = \sup_{\{f_t(\omega^t)\}_{t=0}^{\infty} \in \left[0, \frac{1}{\lambda}\right]} E_0 \sum_{t=0}^{\infty} \beta^t \prod_{j=0}^{t-1} f_j\left(\omega^j\right)^{1-\epsilon} u\left(\omega_t\left(1-\lambda f_t\left(\omega^t\right)\right)\right).$$

Therefore

$$U(\omega_0) = U^*(\omega_0).$$

Proof of Proposition 5 (i) In this case, equation (80) can be written as $\frac{f(\omega)^{\epsilon}}{(1-\lambda f(\omega))^{1-\sigma}} = A\omega^{-\sigma}$ where A is a constant. Using the implicit function theorem, it follows that $f'(\omega) = -\frac{\sigma/\omega}{\frac{\epsilon}{f(\omega)} + \frac{\lambda(1-\sigma)}{1-\lambda f(\omega)}} < 0.$

Proof of Proposition 5 (ii) In deterministic case, $\omega' = \omega$, $c(\omega') = c(\omega)$ for all $\omega \in \Omega$ and equation (136) simplifies to:

$$f^{*\epsilon} = \beta\left(\frac{1}{\lambda} + \frac{1-\epsilon-\sigma}{\sigma}\left(\frac{1}{\lambda} - f^*\right)\right). \tag{151}$$

The left hand side of equation (151) is strictly increasing in f^* while the right hand side is strictly decreasing in f^*. Obviously $f^* > 0$. An interior solution with $f^* < 1/\lambda$ exists since $\lambda^{1-\epsilon} > \beta$.

Proof of Proposition 5 (iii) Let $f^*(\omega)$ denotes the optimal fertility given ω. Plug functional form of $u(\cdot)$ into equation (81),

$$U(\omega) = h(f^*(\omega))\,\omega^{\sigma}. \tag{152}$$

where

$$h(f^*(\omega)) \equiv \frac{1}{\sigma}(1-\lambda f^*(\omega))^{\sigma} + \frac{1}{1-\epsilon}\lambda f^*(\omega)(1-\lambda f^*(\omega))^{\sigma-1}. \tag{153}$$

We make a guess on the value function and let it take the form: $U(\omega) = A\omega^{\sigma}$ where A is a constant, independent of ω. Equating this guess with (152) results in:

$$A = h\left(f^*\left(\omega\right)\right). \tag{154}$$

Thus, in order for A to be independent of ω, we must verify that the results $f^*(\omega)$ is independent of ω. Notice that,

$$E\left[U\left(\omega'\right)|\omega\right] = E\left[A\omega'^{\sigma}|\omega\right] = A\omega^{\sigma}e^{\frac{\sigma^2\sigma_\epsilon^2}{2}}.$$

The last equality holds because the assumption that ω' is lognormal distributed with $\ln\omega$ and σ_ε as the mean and variance of $\ln\omega'$. Plug this equality into (80) to obtain

$$\lambda\left(1-\lambda f^*\left(\omega\right)\right)^{\sigma-1}\omega^{\sigma} = A\beta\left(1-\epsilon\right)f^*\left(\omega\right)^{-\epsilon}e^{\frac{\sigma^2\sigma_\epsilon^2}{2}}\omega^{\sigma},$$

where ω cancels out of this equation and therefore $f^*(\omega)$ is independent of ω confirming our guess. This expression together with (153) and (154) gives a rule to solve the optimal fertility f^*, as given by

$$\frac{\lambda\sigma f^{*\epsilon}}{\beta\left(1-\epsilon\right)}\left[e^{-\frac{\sigma^2\sigma_\epsilon^2}{2}} - \beta f^{*1-\epsilon}\right] = 1-\lambda f^*.$$

Proof of Proposition 6 Part (i) follows from Proposition 1 and Proposition 5. As for part (ii), when M is the identity matrix, fertility is independent of ability and $\pi_{t+1}\left(\omega_j\right) = \sum_{\omega_i\in\Omega}\pi_t\left(\omega_i\right)M(\omega_j,\omega_i) = \pi_t\left(\omega_j\right).$ The first equality uses the proof of Proposition 1. More generally, $\pi_t(\omega) = \pi_0(\omega)$ for all ω and all t and $\pi^*(\omega) = \pi_0(\omega)$. Part (iii) follows Proposition 5 (iii). The conditional variance of $\ln\omega_t$ diverges to infinite because $\ln\omega_t = \ln\omega_0 + \sum_{i=1}^{t}\epsilon_i$, $E\left(\ln\omega_t|\omega_0\right) = \ln\omega_0$, $Var\left(\ln\omega_t|\omega_0\right) = t^2\sigma_\epsilon^2$ and $\lim_{t\to\infty} Var\left(\ln\omega_t|\omega_0\right) = \infty$.

Proof of Proposition 7. Notice that

$$\begin{aligned}
U(\omega) &= \max_{f_t\in[0,1/\lambda]} u\left(\left(1-\lambda f\right)\omega\right) + \beta f^{1-\epsilon}E\left[U\left(\omega'\right)|\omega\right] \\
&\geq \max_{[\underline{f}(\omega),\overline{f}(\omega)]} u\left(\left(1-\lambda f\right)\omega\right) + \beta f^{1-\epsilon}E\left[U\left(\omega'\right)|\omega\right] := U^1\left(\omega'\right) \\
&\geq \max_{[\underline{f}(\omega),\overline{f}(\omega)]} u\left(\left(1-\lambda f\right)\omega\right) + \beta f^{1-\epsilon}E\left[U^1\left(\omega'\right)|\omega\right] := U^2\left(\omega'\right) \\
&.. \\
&\geq \max_{[\underline{f}(\omega),\overline{f}(\omega)]} u\left(\left(1-\lambda f\right)\omega\right) + \beta f^{1-\epsilon}E\left[U^r\left(\omega'\right)|\omega\right] = U^r\left(\omega'\right)
\end{aligned}$$

where the first inequality is strict if a constraint is binding for any particular ω, the remaining inequalities follow from the contraction mapping recursion, and the final inequality uses the contraction mapping theorem. Furthermore, a strict inequality for a particular ω translates into a strict inequality for all $\omega's$ since M is a regular Markov chain meaning that, regardless of initial ability there is positive probability that someone in the dynasty will reach a binding state in finite time. The second part of the proposition follows because fertility restrictions do not change the marginal costs of having children but it decreases (or has no effect on) the marginal benefits by reducing $U(\omega)$ for all ω (see equation (80)). Hence an upper bound of fertility makes people have fewer children than (or the same number of children with) the unrestricted case.

Proof of Proposition 8 Let $P_t^r(\omega)$ be the size of population with ability ω at time t when there are restrictions on fertility. By Proposition 7

$$\begin{aligned} P_1(\omega_j) &= \sum_{\omega_i \in \Omega} f(\omega_i) P_0(\omega_i) M(\omega_j, \omega_i) \\ &\geq \sum_{\omega_i \in \Omega} f^r(\omega_i) P_0^r(\omega_i) M(\omega_j, \omega_i) = P_1^r(\omega_j) \end{aligned}$$

where $P_0(\omega_i) = P_0^r(\omega_i)$. The following inductive argument guarantees that if $P_t^r(\omega_i) \leq P_t(\omega_i)$ then $P_{t+1}^r(\omega_i) \leq P_{t+1}(\omega_i)$ for all ω_j and all $t \geq 0$,

$$P_{t+1}(\omega_j) = \sum_{\omega_i \in \Omega} f(\omega_i) P_t(\omega_i) M(\omega_j, \omega_i) \geq \sum_{\omega_i \in \Omega} f^r(\omega_i) P_t^r(\omega_i) M(\omega_j, \omega_i) = P_{t+1}^r(\omega_j).$$

Therefore,

$$\begin{aligned} W^r &= \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U^r(\omega) P_t^r(\omega) \\ &\leq \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) P_t^r(\omega) \\ &\leq \sum_{t=0}^{\infty} \sum_{\omega \in \Omega} \beta_p(t) U(\omega) P_t(\omega) = W(\beta_p) \end{aligned}$$

Proof of Proposition 9 When $M(\cdot, \omega)$ is independent of ω, the proof of Proposition 1(ii) shows that

$$\pi_{t+1}(\omega_j) = \mu(\omega_j) fort \geq 0.$$

Moreover, $\pi_0(\cdot)$ is invariant to policy changes. So restrictions on fertility only reduces individual utility, by Proposition 7, but does not affect the ability distribution of any generation. Therefore, it decreases social welfare as defined by (72).

Proof of Proposition 10 This Proposition relies on Proposition 6 (ii)'s results, $f(\omega) = f$ and $\pi_t(\omega) = \pi^*(\omega) = \pi_0(\omega)$ when M is the identity matrix. Similar to Proposition 9, fertility restrictions do not alter the distribution of abilities, which together with Proposition 7, finishes the proof.