2012

Institutions and Financial Markets

Ranojoy Basu
Iowa State University

Institutions and Financial Markets

by

Ranojoy Basu

A dissertation submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Major: Economics

Program of Study Committee:

Joydeep Bhattacharya, Co-major Professor

Ananda Weerasinghe, Co-major Professor

David A. Hennessy

Rajesh Singh

John R. Schroeter

Iowa State University

Ames, Iowa

2012

TABLE OF CONTENTS

ACKNOWLEDGEMENTS iv

GENERAL ABSTRACT v

CHAPTER 1. GENERAL INTRODUCTION 1

CHAPTER 2. PUBLIC PROVISION OF SECURITY IN AN INSECURE PROPERTY RIGHTS ENVIRONMENT 3

2.1 Introduction 3

2.2 The model 6

2.2.1 Physical environment 6

2.2.2 Equilibrium 8

2.3 Guard posting: introducing public security 12

2.3.1 Modified environment 12

2.3.2 Equilibrium 13

2.4 Improving property rights 16

2.4.1 Numerical Analysis 17

2.4.2 Discussion 20

2.5 Conclusion 22

2.6 References 24

2.7 Appendix 26

2.7.1 Proof of Proposition 2.2 27

2.7.2 Proof of Corollary 2.1 29

CHAPTER 3. OPTIMAL PORTFOLIO SELECTION WHEN AN INVESTOR'S WEALTH IS SUBJECT TO BANKRUPTCY 31

3.1 Introduction 31

3.2 Model Setup 35

3.2.1 An investor's problem. 35

3.3 Verification lemma 38

3.3.1 Optimal selling Time. 39

3.4 Bankruptcy Model. 43

3.4.1 Problem Formulation. 44

3.4.2 Verification Lemma for the bankruptcy model. 46

3.4.3 Optimal selling time in the presence of bankruptcy. 47

3.5 Comparison of Value Functions 49

3.6 Remark 51

3.7 Optimal Portfolio Selection 51

3.8 Conclusion 57

3.9 Appendix 58

3.9.1 Appendix (a) 58

3.9.2 Appendix (b) 65

3.10 References 71

CHAPTER 4. NUMERICAL ANALYSIS 73

4.1 Numerical Results 73

4.1.1 Effect of a change in volatility σ 73

4.1.2 Effect of a change in mean return μ 76

4.2 Discussion and Future Research 80

4.3 References 81

ACKNOWLEDGEMENTS

This is perhaps the hardest part, for I cannot do justice in so short a passage to all the influences and people who have enriched my life in ways that I had hardly thought possible. My thesis owes a lot to those named below, and any shortcomings that survived did so in spite of them. First, my deepest gratitude to my advisors Professors Joydeep Bhattacharya and Ananda Weerasinghe for their support, patience and encouragement through out my research. Professor Bhattacharya taught me how to express ideas, and Professor Weerasinghe, how to question thoughts. I have throughout tried to emulate them as a researcher, a scholar, and a teacher. Their immense contribution to my knowledge is strewn all over this thesis.

I would like to thank Professor David Hennessy for his extremely useful suggestions on my thesis. I am grateful to Professor Rajesh Singh for his insightful comments that help me understand and enrich my ideas. I admire his enthusiasm and humor. I am indebted to Professor John Schroeter for his generous suggestions on my research.

Life at Economics Department at Iowa State has been full of fun. I would dearly miss the mutual procrastination in the company of good friends.

Finally, there are two persons, my parents, Mr. Subrata Basu and Mrs. Anuradha Basu to whom I don't have the temerity to thank. They might not understand anything in the thesis but everything here owes it to them.

GENERAL ABSTRACT

My thesis deals with Institutions and Financial markets. In chapter 2, of my thesis the role of government in achieving technological progress in an insecure property rights environment is discussed. In such a setting, it is shown that publicly-funded protection of private property rights may successfully support the adoption of frontier technologies as Nash equilibrium which is not possible otherwise.In chapter 3, I study an investment problem faced by a risk averse investor who has the option to invest in a risk free asset (such as a bank account) and a risky asset. The wealth can be transferred between the two assets and there are no transaction costs.The objective is to find an optimal quitting time from the stock market which maximizes the expected discounted utility from terminal wealth. I show the optimal stopping time is of threshold type. Finally in Chapter 4, I discuss numerical results in the context of chapter 3 and future research topics.

CHAPTER 1. GENERAL INTRODUCTION

Insecurity of property rights and allocation of resources are two central issues in economics. My thesis deals with Institutions and Financial markets. In chapter 2, of my thesis (co-authored with Subhra Bhattacharya) the role of government in achieving technological progress in an insecure property rights environment is discussed. In such a setting, it is shown that publicly-funded protection of private property rights may successfully support the adoption of frontier technologies as Nash equilibrium which is not possible otherwise. However, increased security of property rights may not be associated with higher welfare for all adoptees. Indeed it is possible that the poor are made worse off, and may in a political-economy sense, block a movement towards more secure property rights.

In chapter 3, we study an investment problem faced by a risk averse investor who has the option to invest in a risk free asset (such as a bank account) and a risky asset. The wealth can be transferred between the two assets and there are no transaction costs. The proportion of wealth in the risky asset is an a priori chosen deterministic function of wealth. The objective is to find an optimal quitting time from the stock market which maximizes the expected discounted utility from terminal wealth. First, we consider the situation where the wealth process is not subject to bankruptcy and obtain an optimal quitting time. Second, we consider the more realistic scenario when an investor's wealth is subject to default. Here we model bankruptcy via a reduced form model in credit risk theory. We develop necessary mathematical techniques to obtain an optimal selling time in both these circumstances. In both cases, it turned out that the optimal selling time is of the threshold type. We show that higher default intensity leads to a higher optimal exercise boundary and to a higher value function. Moreover, optimal exercise boundary and the value function in the default case are both higher than their counterparts in the no default model. We also show that there exist an optimal portfolio process.

Finally in Chapter 4, we discuss numerical results in the context of chapter 3 and future research topics. We show numerically the impact of change in parameters on optimal threshold and value function. We also verify and illustrate numerically the monotonicity of the optimal exercise boundary and the value function with respect to the default intensity. The results of our numerical analysis provide further insights into the linkages between optimal threshold boundary, value function and relevant policy variables. In particular, we show that there is a positive monotone relationship between optimal threshold boundary and volatility. This leads to future research questions. Can we show theoretically whether optimal threshold increases with volatility. Does value function increases with volatility for a given set of parameter choices. Also we intend to find a sequence of portfolio choices (trading strategies) and try to find out numerically the optimal trading strategy. This would be of considerable importance from a practitioner's point of view as it would determine the optimal buy and hold strategy.

CHAPTER 2. PUBLIC PROVISION OF SECURITY IN AN INSECURE PROPERTY RIGHTS ENVIRONMENT

2.1 Introduction

The term property rights refers to an owner's legal right to use a good/asset for consumption or income generation and also, the right to transfer the good to another party. Property rights have received pride of place in all analyses of the development (and dominance) of the market system in modern societies. Over two centuries ago, Adam Smith and other thinkers expounded on the idea that property rights encourage their holders to develop the property, generate wealth, and efficiently allocate resources via the market mechanism.[1] They noted that the anticipation of profit from "improving one's stock of capital" rests on clear delineation and enforcement of private property rights, which, in turn leads to more wealth and improved standards of living for all.[2]

While the above prescription for material progress and prosperity has been around for over two hundred years, not every country has succeeded in using it to achieve sustained growth and development. Indeed, in most less-developed and transition economies, institutions aimed at defining and preserving property rights are woefully fragile, and as such, property rights are terribly insecure. This insecurity comes at a hefty price – heightened conflict over property and the accompanying dissipation of scarce resources in the creation of effective property rights.[3]

[1] A practical application of this principle can be found in the introduction of the Permanent Settlement System (around 1800) in colonial India. Under this system, the colonizers – the British under Lord Cornwallis, one of the leading British generals in the American War of Independence – granted proprietarial rights to former landholders (would-be zamindars) to the land they occupied. This method of incentivisation of zamindars was intended to encourage improvements of the land, such as drainage, irrigation and the construction of roads and bridges. The land tax was also fixed in perpetuity. Cornwallis successfully argued that "when the demand of government is fixed, an opportunity is afforded to the landholder of increasing his profits, by the improvement of his lands".

[2] Besley (1995) investigates the interconnection between investment and land rights using data from Ghana, when the country was in a state of transition between traditional and modern land rights. His findings for Wassa, a cocoa growing region where most of the land is owned, was supportive of the idea that "better land rights facilitate investment".

[3] In recent times, economists have popularized this line of thinking. De Soto (2000) has brought the argument into a broader public domain. Economic historians such as North (1981), Jones (1986), and Mokyr (2002) have cited evidence to support this view. There is a growing literature that focuses on the links between the security of property and economic behavior at the institutional level in a variety of specific institutional settings. For example Besley (1995), Goldstein and Udry (2008) study the impact of insecure land rights on investment and productivity in rural Ghana. In a related study Field (2007) finds

Our paper studies the consequences of insecure property rights on the mechanics of technological innovation. The work is motivated by a certain "social resistance" to technological change that characterizes many poor economies. For example, Platteau (2000, p.200) documents how fishermen in Congo refused to use a new net technology which was offered to them at no cost. More generally, it has been documented that economic agents in impoverished societies often reject superior technologies – technologies that are on the frontier – even when the cost of adoption appear negligible. In explaining this apparent paradox, Parente and Prescott (1999) make the convincing case that technological innovation is not a Pareto-superior outcome. There are economic winners and losers, and the latter have an incentive to block technology adoption by others because it necessarily influences the expost distribution of wealth. This view finds prominence in Olson (1982), Mokyr (1990), Krusell et al. (1996), among others.

Linked to this, is the view that post-production conflict is inevitable if the property rights are not perfectly enforced. Specifically, output is *contestable* in a society with imperfect property rights and conflict over the output cannot be settled without expending scarce resources in "appropriation" (grabbing the production of other agents or defending it from others). In the last two decades, a growing body of research has tried to explain the consequences of such conflict and appropriation in the process of development. Almost all of this work models conflict as a contest in which a non-cooperative game is played between agents to settle the conflict. A key ingredient of conflict is the use of weapons or defensive means, a composite form of which is termed "appropriative investment". Returns of appropriative investments that accrue to an agent is represented by "technologies of conflict"or "context success functions".

Continuing in this tradition, Gonzalez (2005) argues that the aforementioned paradoxical choice of inferior technologies can be understood as "a strategic response to the anticipation of conflict" over the expost distribution of newly-created wealth especially when property rights over it are insecure. Gonzalez (2005) has in mind a setting in which two agents contemplate adoption of a superior technology in an insecure property-rights environment. While each recognize that such adoption would lead to an increase in future output, each is nevertheless afraid that this newly-created wealth generates an

that issuing of "property titles" in urban Peru has led to a significant increase in labor supply. Johnson et al. (2002) studies the impact of insecure property rights on the investment decisions taken by manufacturing firms in post-communist countries when bank loans were available. A common thread running through these studies is secure property rights facilitates the creation of wealth.

incentive for the rival to engage in a costly game of predation. The expected predatory response discourages adoption of the superior technology in the first place, and thus "... poverty becomes the price of peace." (Bates 2001).[4] The upshot of the Gonzalez (2005) analysis is that adoption of the best-available technology is never sustainable as a Nash equilibrium.

If people are hesitant to adopt superior technologies because of a fear of subsequent conflict, would some sort of external intervention be beneficial? Would it help, if a third party intervenes in this conflict by providing some manner of public protection of rights on private property? To implement this, we introduce a "government" in the framework of Gonzalez (2005). We think of the government as imposing a non-distortionary tax on the initial endowments of each agent at the start of their life. The tax proceeds are utilized to finance the hiring of a "guard". The guard is simply a public security service whose sole aim is to reduce the effectiveness of each agents' predatory activities, without directly interfering in the expost conflict. The posting of a guard is shown to influence agents' decisions on allocation of resources to productive and predatory activities. In sharp contrast to the main result in Gonzalez (2005), we prove that adoption of the frontier technology by each agent can now be supported as a Nash equilibrium.

We go on to extend the analysis by allowing the government to directly influence the nature of the expost conflict. In other words, we allow the government to use its tax-financed resources to alter the existing regime of property rights. Presumably, a government can achieve increased security of property rights by funding the police, the judiciary, and the corrections systems better. We find that adoption of the best-available technology by each agent continues to emerge as a Nash equilibrium. Within this equilibrium, we find that improved property rights, though growth enhancing, is not always socially optimal from an aggregate-welfare point of view. [5]

The paper is organized in the following manner. Section 2.2 describes the benchmark model due to Gonzalez (2005). In section 2.3, we introduce the public security of private property and analyze the equilibrium outcomes. In section 2.4, we endogenize the property rights regime. Section 4 concludes the paper.

[4] Hall and Jones (1999) provide evidence that poor enforcement of property rights can be a serious impediment to technological progress.

[5] In a somewhat-related study, Gonzalez (2007) analyzes the growth-welfare trade-off in an exogenously-specified property rights environment. He showed a symmetric equilibrium allocation associated with more-secure property rights and faster growth can be Pareto dominated by one associated with poorer property rights and slower growth.

2.2 The model

2.2.1 Physical environment

We consider a two-period model of imperfect security of private property and its impact on technology choice. The model economy is inhabited by two agents, named R and P ("rich" and "poor") – these agents can be thought of either as individuals or collectives (such as tribes, nation states, and so on). There is a single good and the aggregate endowment of this good in period 1 is a fixed amount Y. Agent R is endowed with a share $p \in (1/2, 1]$ of Y; correspondingly, Agent P is endowed with the remaining share, $1-p$. Rights to this property in period 1 are perfectly secure for each agent. However, property rights in period 2 are not secure, and all the action in this model derives from this insecurity.

Each agent uses a portion of his property in period 1 and undertakes some productive investment; the latter, via a production technology, produces consumables in period 2. At the start of period 1, each agent costlessly chooses a technology from a set of available technologies, $[A^L,\ A^H]$. A technology is to be interpreted as a blueprint that transforms investment into output in the following period. We assume that each agent has access to the same AK production technology and that productive investments of the agents are decided independently of each other. To be specific, productive investment K_i by agent i, $i \in \{R, P\}$, at period 1 produces output $A_i K_i$ at period 2 where $A_i \in [A^L, A^H]$ is the technology choice of agent i.

In a world with secure property rights, the resources available to agent R in period 2 would be $A_R K_R$, and that to agent P would be $A_P K_P$. Not so here. Here, the total amount of consumables ("common property") available at the start of period 2 is $Y' = (A_R K_R + A_P K_P)$ and property rights over Y' is insecure, that is, it is subject to pillage and appropriation. This insecurity prompts agents to invest in appropriative investments that help convert their claims on production into effective property rights on the common output. Let X_i denote agent i's investment in appropriation, and let p' denote agent R's share of Y'; henceforth p' is labeled the "appropriation function". Then,

$$p' = \frac{(X_R)^m}{(X_R)^m + (X_P)^m} \in [0,1]; \quad m > 0, \tag{2.1}$$

where (2.1) is a share function – taken as a primitive – capturing the technology of conflict over claims on future output. Note p' is increasing in an agent's own appropriative investment and decreasing in that of his rival's. This is the workhorse functional form for the technology of conflict. For future reference, note that p' is symmetric and homogeneous of degree zero in X_R and X_P. This last property is analytically convenient and largely accounts for the widespread use of this functional form in the conflict literature. (For surveys of conflict models and contests see Garfinkel and Skaperdas (2007) and Konard (2009)). As an aside, note that resources allocated to productive investment in period one are not subject to appropriation, only the final output in period two is. Finally, note that if property rights were perfectly secure, agent R's share of Y' would be given by $A_R K_R / Y'$; therefore, as long as p' in (2.1) deviates from this ratio, property rights are insecure. For future use, note that p' in (2.1) can never approach $A_R K_R / Y'$. This last observation will make a major appearance in the penultimate section of this paper.

It is instructive to outline a time-line of events. At the start of period 1, each agent chooses a technology from the aforementioned set of available technologies. Once that is done, and cognizant of his own technology choice but not that of his rival's, an agent makes consumption, appropriation, and productive investment decisions, financing everything from his endowment. Production activity is then initiated. Agents consume and undertake the planned appropriation investments. When period 2 arrives, the common production, Y', is realized and agents receive their share which they consume; agent R gets a share p' and agent P, a share $1 - p'$. Note that p' is determined by *past* appropriation investments of both parties, as is described by (2.1).

The resource constraints in period 1 can be written as

$$pY = C_{1R} + X_R + K_R, \text{ for } i = R \tag{2.2}$$

$$(1-p)Y = C_{1P} + X_P + K_P, \text{ for } i = P \tag{2.3}$$

where C_{1i}, $i \in \{P, R\}$ is consumption by agent i in period 1. The second period constraints are

$$C_{2R} = p'(A_R K_R + A_P K_P), \text{ for } i = R \tag{2.4}$$

$$C_{2P} = (1-p')(A_R K_R + A_P K_P), \text{ for } i = P. \tag{2.5}$$

where C_{2i}, $i \in \{P, R\}$ is consumption by agent i in period 2.

The description of the physical environment is complete once preferences are specified. We assume that agent i has preferences described by the separable utility function, $U_i \equiv \ln C_{1i} + \beta \ln C_{2i}$, $\beta > 0$.

2.2.2 Equilibrium

The aforediscussed time-line of events suggests the following characterization of the game. Period one is characterized by two stages, where in each stage, agents act non-cooperatively to maximize their payoffs without any information on their rivals' strategies. Therefore, we are faced with a two-stage game, where at each stage, agents play a simultaneous-move game, and the outcome of the first stage is not revealed before the actions of the second stage are taken. To find a reasonable solution, we look for the set of subgame-perfect equilibria. In other words, for any choice of technology at stage one, we first find the optimal consumption and investment strategies for each agent which are mutual best responses to each other. These optimal responses are solely a function of the technology choices made in stage one. Then, we incorporate these optimal decisions in the agents' utility maximization problem and find the set of technologies in stage one that produce non-cooperative optima for each agent.

Consider the problem faced by agent R at stage two of period 1. At this point in the game, agent R knows A_R; he takes A_P, X_P and K_P as given, and solves the following problem:

$$\max\ U_R \equiv \ln C_{1R} + \beta \ln C_{2R}$$

subject to

$$pY = C_{1R} + K_R + X_R,$$
$$p'Y' = C_{2R},$$
$$p' = \frac{(X_R)^m}{(X_R)^m + (X_P)^m},$$
$$\text{and } Y' = A_R K_R + A_P K_P.$$

The interior optimality conditions for agent R are given by the following equations:

$$\frac{1}{C_{1R}} = \beta \frac{(X_R)^m}{(X_R)^m + (X_P)^m} A_R \frac{1}{C_{2R}}, \tag{2.6}$$
$$\frac{A_R}{A_R K_R + A_P K_P} = \frac{(X_P)^m}{(X_R)^m + (X_P)^m} \frac{m}{X_R}. \tag{2.7}$$

Equation (2.6) is a standard intertemporal Euler equation equating the marginal rate of substitution (MRS) of consumption between the two time periods with the marginal rate of transformation (MRT). In a standard model with perfect property rights, the MRT for agent R would simply be A_R; here, because of insecure property rights, it is $p'A_R$. The second condition, (2.7) reflects the equality of marginal returns across different the two types of investment activities. An unit of resource can be invested either in productive or in appropriative activities. In equilibrium, these avenues should generate the same return.

Analogously, the reaction functions for agent P are given by

$$\frac{1}{C_{1P}} = \beta \frac{(X_P)^m}{(X_R)^m + (X_P)^m} A_P \frac{1}{C_{2P}}, \tag{2.8}$$

$$\frac{A_P}{A_R K_R + A_P K_P} = \frac{(X_R)^m}{(X_R)^m + (X_P)^m} \frac{m}{X_P}. \tag{2.9}$$

We can use the symmetry of the reaction functions for the two agents to write $(A_R/A_P) = (X_P/X_R)^{m+1}$ and use in (2.1) to get

$$p' = \frac{1}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}}. \tag{2.10}$$

Notice how the appropriation function in (2.1) is transformed to depend solely on the ratio of the technology choices of both agents.

The above formulation of p' highlights the possibility of wealth-ranking reversal in this setup. To see this, suppose the technologies adopted satisfy $A_R > A_P$ (i.e., suppose the initially-wealthier agent adopts the superior technology). Then, (2.10) makes clear that $p' < 1/2$ is possible even when $p > 1/2$ was true. In other words, a wealth-ranking reversal is possible. The fact that there is a scope for redistribution of wealth, from the wealthier and more productive agent to the poorer one, should not come as a surprise. After all, the agent choosing the superior technology has a higher opportunity cost of investing in appropriative activities, which in turn give him a comparative advantage (relative to the other agent) in production. The optimal allocation of saving between different investment activities (or, the equalization of marginal return across productive and appropriative activities) implies that the agent invests more in production and cut back on appropriative investments, and thus end up with less share of future output.

Using (2.6)-(2.10), it is possible to derive the optimal allocation of resources to consumption and appropriation in terms of the stage-one technology choices of both parties. The optimal choices for agent R are given by

$$C_{1R} = \frac{A_P}{A_R} C_{1P} = \frac{\left[(p + (1-p)\frac{A_P}{A_R})Y\right]}{\beta(1+m)+2}, \tag{2.11}$$

$$X_R = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1} \right) \frac{m\beta \left[(p + (1-p)\frac{A_P}{A_R})Y \right]}{2 + \beta(1+m)}, \tag{2.12}$$

and

$$C_{2R} = \frac{\beta}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} . \frac{[(pA_R + (1-p)A_P)Y]}{2 + \beta(1+m)}. \tag{2.13}$$

Analogous expressions for agent P are given by

$$C_{1P} = \frac{A_R}{A_P} C_{1R} = \frac{A_R}{A_P} \frac{\left[(p + (1-p)\frac{A_P}{A_R})Y \right]}{\beta(1+m) + 2}, \tag{2.14}$$

$$X_P = X_R \left(\frac{A_R}{A_P}\right)^{\frac{1}{1+m}} = \left(\frac{A_R}{A_P}\right)^{\frac{1}{1+m}} \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1} \right) \frac{m\beta \left[(p + (1-p)\frac{A_P}{A_R})Y \right]}{2 + \beta(1+m)}, \tag{2.15}$$

and

$$C_{2P} = \left(\frac{A_R}{A_P}\right)^{\frac{m}{1+m}} C_{2R}. \tag{2.16}$$

If the income distribution is highly unequal, we may end up at a corner solution where the poorer agent does not contribute anything to productive investment and invests only in appropriation. Similarly, the richer agent may have absolute advantage in appropriation. Implicitly then, we assume that the initial distribution of income is not very skewed i.e., p is not very close to 1.

From the expressions of (2.11), (2.13), (2.14), (2.16), it is evident that if the initially-wealthier agent adopts a superior technology, he enjoys less consumption in both periods than the poorer agent. Also note that the equilibrium share of output is less for the relatively more-productive agent. These results are invariant to whether the more-productive agent is initially richer or not. This is because equilibrium allocation of resources are determined by comparative advantage. For example, when $A_R > A_P$, agent R has a comparative advantage in production and poor in appropriation. From standard trade theory, it follows that agent P should invest relatively more in appropriation and thus enjoy higher second-period consumption i.e. $C_{2P} > C_{2R}$. On the other hand, agent P is reluctant to sacrifice current consumption to increase the size of the pie as he is relatively less productive, and therefore, he consumes more in the first period i.e., $C_{1P} > C_{1R}$. Similar arguments hold when $A_R < A_P$.

It remains to incorporate these optimal decisions, (2.11)-(2.16), in the agents' utility maximization problem and compute the technology choices (A_R, A_p) in stage one that produce non-cooperative optima for each agent. In other words, we compute U_R as a function of A_R (given A_p) and U_p as a function of A_p

(given A_R). These represent the mutual best-responses. A pure strategy Nash equilibrium is a fixed point of these best-response functions that is consistent with positive levels of productive and appropriative investments, and consumption in each period, by both agents.

Proposition 2.1. *(Gonzalez, 2005) If p is sufficiently close to half and $\frac{A^H}{A^L} \to 1$, then a pure-strategy Nash equilibrium exists. $\left(A_R = A^H, A_P = A^H\right)$ is not a pure-strategy Nash equilibrium, i.e., the equilibrium technology profile cannot involve each agent adopting the best available technology.*

Why might agents not wish to adopt the best available technology even when it is costlessly available? In this environment of insecure property rights, the answer lies in the anticipation of future conflict. While adoption of a better technology by an agent raises tomorrow's common output, the very increase in tomorrow's pie elicits a harmful response from his rival (in the form of an increase in appropriative investment), and this dissuades the agent from adopting superior technologies in the first place. More specifically, the optimality conditions imply that agents allocate resources by equating marginal returns from the two types of investment activities. It follows that adoption of a superior technology raises the opportunity cost of appropriative investments for the adopter, inducing him to shift resources from appropriative to productive activities. Ceteris paribus, this raises future common output. On the flip side, the adoption of a superior technology lures his opponents to specialize in appropriation – appropriative investments act as strategic substitutes – thereby increasing the "expost tax" on the returns to adoption. The upshot is that choosing to adopt a superior technology confers a strategic disadvantage in the subsequent distribution of wealth.

The starting point of our analysis is this striking result in Gonzalez (2005): people are hesitant to adopt superior technologies because of the fear of subsequent heightened conflict. This presents a prima facie case for some sort of external intervention. Would it help, if a third party, say, a government, intervenes in this conflict by providing some manner of public protection of rights on private property? In the next section, we take up a slice of this issue.

2.3 Guard posting: introducing public security

2.3.1 Modified environment

To implement the idea discussed above, we introduce a third party, called "government" in the framework of the benchmark model. We think of the government as imposing a non-distortionary tax on the initial endowments of each agent at the start of their life. The tax proceeds are utilized to finance the hiring of a "guard". In terms of the model economy, the guard is simply a public security service whose sole aim is to reduce the effectiveness of each agents' appropriative investments by a constant amount. Since agents' share of future output depends on their effective appropriative investments, the presence of a guard, in effect, creates a threshold below which all appropriative investments are rendered ineffective. This influences agents' decisions on allocation of resources to various activities, which in turn, affects their marginal returns. The question at hand is: can the presence of a guard induce a reallocation of resources in such a way that adoption of the best-available technology by each agent evolves as a Nash equilibrium? [6]

As discussed above, assume each agent is required by law to pay as a tax, a fixed proportion (τ) of his inherited wealth. Since inherited wealth is exogenously-specified – pY for agent R and $(1-p)Y$ for agent P – the tax is non-distortionary. We denote the total tax revenue by G, where $G = \tau Y$. The government uses the tax proceeds to post a guard whose only job is to equally reduce the effective amounts of the appropriative investments of *each* agent. Specifically, if X_i^e is the effective appropriation investment for agent i, then $X_i^e \equiv X_i - G$ where X_i is the corresponding investment made by agent i in the benchmark model. The technology of conflict, the analog of (2.1), is redefined in the following manner:

$$p'_G = \frac{(X_R^e)^m}{(X_R^e)^m + (X_P^e)^m}. \tag{2.17}$$

The new formulation, which looks a lot like (2.1), maintains the properties of symmetry and homogeneity of degree zero in *effective* appropriative investments; this keeps the model analytically tractable. This formulation requires that each agent invests at least an amount G – the threshold – to get a posi-

[6] By posting a guard, the government can act as a more-effective deterrent against one party capturing more of the final output than is due to that party. A question that legitimately arises at this juncture is, why does the government, via the posting of a guard, get involved in this conflict in the first place? Presumably, the government cares about improving property rights. A fuller discussion of this issue is presented in Section 2.4 below.

tive return from appropriative activities. Since τ can be quite small, the threshold – the restriction that $X_i^e > 0$ has to hold – may not be too onerous for the agents. What is important to note is that diminishing returns in appropriative investments imply that the marginal effect of an extra unit invested in appropriation (over and above the threshold) is much lower than in the benchmark model; additionally, the marginal return on appropriative investments is lower than the marginal utility from consumption or the return to productive activities.

It is evident that compared to the benchmark model, the qualitative changes in this section are the imposition of a tax in the first period and the modification of the share function/technology of conflict. The sequence of activities and the information available to each agent at each point of time are exactly the same as that in the baseline model. Therefore, we proceed exactly as before to obtain the set of subgame perfect Nash equilibria (SPNE).

2.3.2 Equilibrium

Analogous to (2.6)-(2.7), the interior optimality conditions for agent R are given by:

$$\frac{1}{C_{1R}} = \frac{\beta p'_G A_R}{C_{2R}}, \tag{2.18}$$

and

$$\frac{m(X_P - G)^m}{(X_R - G)\{(X_R - G)^m + (X_P - G)^m\}} = \frac{A_R}{Y'}. \tag{2.19}$$

The first condition, (2.18), is the familiar intertemporal Euler equation that equates the marginal utility of an unit of consumption across periods. For agent R, an unit of consumption forgone today and invested in the productive technology produces A_R units of future output. Since property rights are insecure, agent R gets to consume only his effective share, $p'_G A_R$. The second optimality condition requires that the marginal returns from both types of investment activities – productive and appropriative – be equated in equilibrium.

It is easy to check that (2.10) continues to hold in this reformulated environment, i.e.,

$$p'_G = \frac{1}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \tag{2.20}$$

holds. Analogous to (2.11)-(2.16), we now have

$$C_{1R} = \frac{Y\left[\left(p + (1-p)\frac{A_P}{A_R}\right)(1-\tau) - \left(1 + \frac{A_P}{A_R}\right)\tau\right]}{\beta(1+m) + 2}, \tag{2.21}$$

$$C_{2R} = \frac{\beta}{1+\left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \cdot \frac{Y\left[(pA_R+(1-p)A_P)(1-\tau)-(A_R+A_P)\tau\right]}{2+\beta(1+m)}, \tag{2.22}$$

$$C_{1P} = \frac{Y\left[\left(\frac{A_R}{A_P}p+(1-p)\right)(1-\tau)-\left(1+\frac{A_R}{A_P}\right)\tau\right]}{\beta(1+m)+2}, \tag{2.23}$$

and

$$C_{2P} = \frac{\beta\left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}}{1+\left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \cdot \frac{Y\left[(A_Rp+(1-p)A_P)(1-\tau)-(A_R+A_P)\tau\right]}{2+\beta(1+m)}. \tag{2.24}$$

Additionally,

$$X_R = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}}+1}\right)\frac{m\beta\Delta}{2+\beta(1+m)} + \tau Y, \tag{2.25}$$

and

$$X_P = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}}+1}\right)\frac{m\beta\Delta}{2+\beta(1+m)}\left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}} + \tau Y \tag{2.26}$$

hold where $\Delta \equiv \left[\left(p+(1-p)\frac{A_P}{A_R}\right)(1-\tau)Y-\left(1+\frac{A_P}{A_R}\right)\tau Y\right]$. It is clear from (2.25)-(2.26) that X_R^e and X_P^e are positive.

What are the main margins on which all the action in this model rests? First, at the margin, a higher tax rate reduces disposable income generating a first order negative effect on utility. However, there may arise a countervailing positive effect since the proceeds from the tax are used to employ a guard, whose actions may help secure property rights, and thereby encourage better technology adoption. How might this happen? Recall that the presence of a guard creates a threshold below which all appropriative investments are rendered ineffective. As a result, the marginal effect of an extra unit invested in appropriation (over and above the threshold) is considerably lowered, raising the corresponding return from productive activities. Both agents now have an incentive to respond to these favorable returns by adopting better technologies. The whole thing turns on the following tension: does the presence of a guard reduce the anticipation of future conflict by so much that the benefit to agents from adopting superior technologies outweighs their contribution to the financing of the guard in the first place? The next proposition argues that for a range of tax rates, the answer may be in the affirmative.

Proposition 2.2. *(Guard-posting) If $p \to 1/2$ and $\frac{A^H}{A^L} \to 1$, a pure strategy equilibrium with positive investment in productive activities exists for $\tau \leq \tau_{inv}$. Moreover for $\tau \in [\tau_H, \tau_{inv}]$, $\left(A_R = A^H,\, A_P = A^H\right)$ can be achieved as a Nash equilibrium technology profile.*

The definitions of τ_{inv} and τ_H – all in terms of underlying parameters – can be found in the appendix. Proposition 2.2 is the central analytical result of our paper. It argues that under the same sorts of parametric restrictions imposed in Proposition 2.1, a publicly-financed guard can significantly improve the equilibrium technology choice. In particular, $[A^H, A^H]$ can now be supported as a Nash equilibrium, something that was not possible in Proposition 2.1 or in Gonzalez (2005). [7]

2.3.2.1 Welfare Analysis

As discussed earlier, there is a tension between utility losses from lower disposable income when young and possible welfare gains from superior technology adoption in the presence of a guard. On net, can we say anything about overall welfare levels with and without public provision of security? To that end, we posit a Benthamite social welfare function:

$$SWF \equiv U_R + U_P. \tag{2.27}$$

Since there are multiple equilibria possible both in the benchmark and in the guard-posting models, indeed the set of equilibria are different, the choice of which equilibria to compare becomes critical. Here we choose to compare social welfare across two symmetric equilibria, $\left(A^L, A^L\right)$ in the benchmark model and $\left(A^H, A^H\right)$ in the guard-posting model.

Corollary 2.1. *If (A^H, A^H) and (A^L, A^L) are equilibrium technology profiles in the guard-posting model and the benchmark model respectively, then aggregate social welfare is higher in the former equilibrium if the following parameter condition holds:*

$$\frac{A^H}{A^L} \geq \left(\frac{1}{(1-3\tau)^{2(\beta+1)}}\right)^{\frac{1}{2\beta}}.$$

Before we close this section, it would be useful to summarize our findings thus far. Gonzalez (2005) argued that a primary reason for technological backwardness is insecurity of property rights. If agents anticipate increased conflict from adoption of a superior technology, they may choose not to. The best-available, and yet free, technologies may never be adopted, with serious consequences for growth and welfare. We introduced the notion of public security of private property rights. In our setup, a

[7] A few words about Proposition 2.2 are in order. When the tax rate lies within the interval $[\tau_H, \tau_{inv}]$, each agent's best response is to choose either the best or the worst available technology. That is, any equilibrium technology profile must be situated in the boundaries of the set of available technologies. If the tax rate lies outside the interval $[\tau_H, \tau_{inv}]$ then emergence of an interior equilibrium in technology choice is possible. In the baseline model, this was never a possibility.

guard is posted by the government with the sole aim of reducing the effectiveness of the appropriative investments of each agent. We find that the best-available technology can now be supported as a Nash equilibrium. This new equilibrium may also exhibit superior welfare.

In the environment studied thus far, the extent of involvement of the government in the post-production conflict was limited to posting a guard. All the guard did was thwart the appropriative activities of each agent, much like a policeman would. As an intuition-building exercise, this thought experiment was useful. What happens if the government takes on a more direct, proactive role in the post-production conflict, and is not restricted to merely impeding the appropriative activities of agents?

2.4 Improving property rights

In this section, we allow the government to utilize the tax proceeds to directly influence the technology of conflict with a view to improving the security of private property rights. This is achieved via the following reformulation of the conflict technology:

$$p'_e = \frac{x_R^{m\theta}(A_R K_R)^{1-\theta}}{x_P^{m\theta}(A_P K_P)^{1-\theta} + x_R^{m\theta}(A_R K_R)^{1-\theta}},\ \theta \in [0,1]. \tag{2.28}$$

In this formulation, p'_e denotes the share of second-period output that accrues to agent R. As is clear from (2.28), p'_e reduces to p' (see (2.1) in the benchmark model) when $\theta = 1$ and to $A_R K_R / Y'$ when $\theta = 0$. In other words, the technology of conflict in (2.28) straddles two extremes, the insecure property-rights regime from the benchmark model and an environment of perfect property rights (where agent R receives his legitimate share, $A_R K_R / Y'$).

We posit that θ is a choice variable for the government albeit not a costless choice. Real resources are diverted to enhance property rights. Specifically, the government can influence θ directly by spending G where $\theta \equiv \Phi(G)$, and $G = \tau Y$. Furthermore, $\Phi(0) = 1$, $\Phi(G^*) = 0$, and $\Omega'(G) < 0$. If the government wishes to improve property rights, it raises τ (and hence, G) and uses the revenue to reduce θ.[8] In the limit, as G approaches a critical level, G^*, a perfect property rights regime is established. In a laissez-faire regime, the government takes no part in post-production conflict and sets $G = 0$. This establishes the polar opposite regime of insecure property rights. Henceforth, θ measures the exact level of insecurity of agents' claims to private property.

[8] This action could be interpreted as improving funding for the police and the judiciary at large.

The rest of the environment is exactly as it is in the benchmark model. Analogous to (2.6)-(2.7), the interior optimality conditions for agents R and P are given by

$$\frac{1}{(1-\tau)pY-X_R-K_R}=\frac{\beta A_R}{A_RK_R+A_PK_P}+\frac{(1-\theta)X_P^{m\theta}(A_PK_P)^{1-\theta}}{[X_P^{m\theta}(A_PK_P)^{1-\theta}+X_R^{m\theta}(A_RK_R)^{1-\theta}]K_R}, \tag{2.29}$$

$$\frac{1}{(1-\tau)pY-X_R-K_R}=\frac{m\theta X_P^{m\theta}(A_PK_P)^{1-\theta}}{[X_P^{m\theta}(A_PK_P)^{1-\theta}+X_R^{m\theta}(A_RK_R)^{1-\theta}]X_R}, \tag{2.30}$$

and

$$\frac{1}{(1-p)(1-\tau)Y-X_P-K_P}=\frac{\beta A_P}{A_RK_R+A_PK_P}+\frac{(1-\theta)X_R^{m\theta}(A_RK_R)^{1-\theta}}{[X_P^{m\theta}(A_PK_P)^{1-\theta}+X_R^{m\theta}(A_RK_R)^{1-\theta}]K_P}, \tag{2.31}$$

$$\frac{1}{(1-p)(1-\tau)Y-X_P-K_P}=\frac{m\theta X_R^{m\theta}(A_RK_R)^{1-\theta}}{[X_P^{m\theta}(A_PK_P)^{1-\theta}+X_R^{m\theta}(A_RK_R)^{1-\theta}]X_P}, \tag{2.32}$$

respectively. The equilibrium technology profile involves solving the above system of equations – (2.29)-(2.32) – for K_R, K_P, X_R and X_P, where $p \in (1/2,1]$, $\beta \in [0,1]$, $\tau \in [0,1]$, $m \in [0,1]$, $\theta \equiv \Phi(G) \in [0,1]$, and $Y > 0$.The nature of non-linearity in the system of equations severely restricts the scope for analytical solutions. We resort to a numerical analysis.

2.4.1 Numerical Analysis

The model economy, and hence, the system eqs. (2.29)-(2.32), has undergone a substantial change over the model described in Section 2.2. There, as Proposition 2.1 had established, all Nash equilibria lay at the boundary of the available technologies set, i.e., either A_P or A_R could take the boundary values A^H or A^L but not an interior value. No such guarantees are available to us in the system, (2.29)-(2.32). Multiple, possibly interior, equilibria are clearly possible here. Since we are ultimately interested in studying changes in θ, matters could get tricky especially if a change in θ takes us from one equilibrium to another. To keep the analysis in this section comparable with Sections 2.2 and 2.3 below, we will restrict the analysis to a single Nash equilibrium, $\left(A_R = A^H,\, A_P = A^H\right)$, even though many others, possibly even a continuum, are possible. Within the confines of this single equilibrium, the one corresponding to both parties choosing the frontier technologies, we will ask, how do various variables of interest vary as θ changes? Specifically, as θ falls (i.e., property rights become more secure), how does growth, inequality, and welfare respond? The question uppermost on our mind is, is government-funded increased security of property rights a good idea always?

We develop the following numerical scheme in order to simulate the system, eqs. (2.29)-(2.32), so as to analyze the effect of property rights improvement (through effective government intervention) on relevant choice variables. Since the model economy is quite stylized, the numerical exercise below is not to be understood as a calibration exercise, rather the exploration of a particular equilibrium using numerical methods.

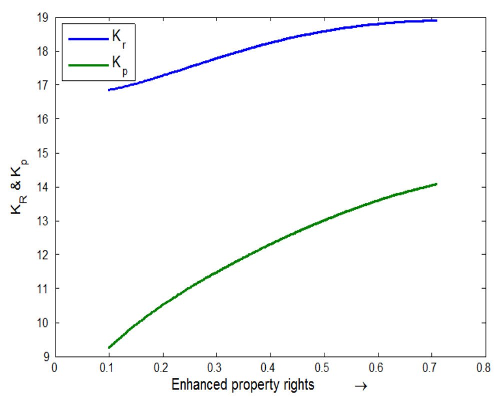
We begin by specifying the values of the parameters, the range of tax rates, and the set from which the technology is chosen: $\tau \in (0, 20\%)$, $Y = 100$, $\beta = 0.8$, $A_i \in [A^L = 18.9, A^H = 20]$, $m = 0.5$, $p = 0.6$, $\Phi(G) \equiv 1 - \frac{(G)^\alpha}{K^\alpha}$ with $K =$, and $\alpha = 0.5$. To stay in line with Propositions 2.1-2.2, we choose $\left(A^H/A^L\right) \approx 1$ and p close to 1/2. Clearly, Y and β are scale parameters and are easily varied without any change in the qualitative properties. The tax rate is kept in a reasonable range of under 20% (indeed, much of the action below happens for tax rates below 10%). α represents the elasticity of effective property rights with respect to government spending.

In steps 1 and 2 below, we summarize the algorithm that we use to identify the set of tax rates that supports the choice of best available technologies as a Nash equilibrium for the poor agent. A similar scheme is developed for the rich agent. Finally, in step 3, we find the interval of tax rates that supports the choice of frontier/best-available technologies for both agents as a Nash equilibrium.

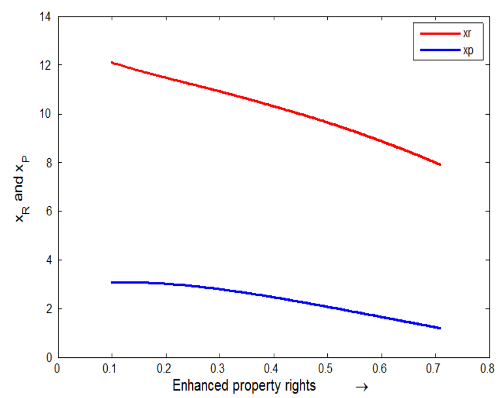
Step 1 : We start by making a grid for τ (the tax rate) and A_P (the technology choice of the poor agent). Given an initial choice of τ at the first grid point, we perform the following analysis: We fix A_R at its highest possible level, $A_R = A^H = 20$, and choose the first grid point of $A_P = A^L = 18.9$. For the given choice of parameters, we simultaneously solve the above-discussed system of non-linear equations (using the Matlab in-built function "simulnonlinear") to get the optimal values of C_{1R}, C_{1P}, X_P and X_R. Using these, we compute K_R, K_P, C_{2R}, and C_{2P}. Next, we evaluate indirect utility of agent P, $U_P \equiv \ln C_{1P} + \beta \ln C_{2P}$, which depends on the initial choice of τ, $A^H = 20$ and $A_P = 18.9$. Keeping the initial choice of τ unchanged, we change A_P along the grid, holding A_R fixed at A^H to see how U_P changes with A_P. This process is iterated twenty times. For the initial choice of τ, the indirect utility curve is plotted as a function of A_P for every iteration.

Step 2 : To check whether agent P has an incentive to choose the best possible technology when agent R has done so, we compute $\partial U_P/\partial A_P$ at $A_P = A_R = A^H$ for this choice of τ. If the slope is positive, we assert that this tax rate supports the best available technology adoption for the poor and consequently, record the value of τ. A negative slope implies an incentive on the part of agent P to deviate from the best technology choice given agent R has chosen it. In that case, we reject that value of τ and proceed to repeat the same exercise for the next point on the grid. This process is repeated for the entire grid of τ and record those τ for which the aforementioned slope is positive. Denote this set by $S_1 = [\tau_1, \tau_2]$.

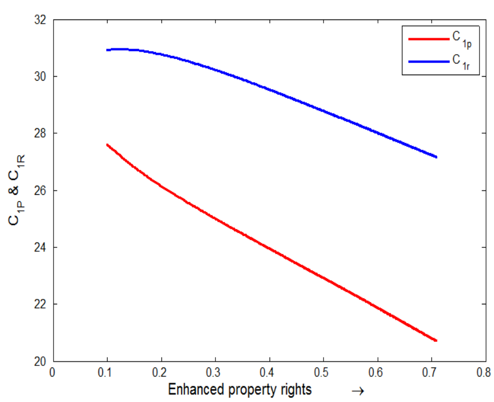
Step 3: An analogous exercise is performed for agent R and a set $S_2 = [\tau_3, \tau_4]$ is found. We denote $S = S_1 \cap S_2$ as the range of tax rates that supports $\{A^H, A^H\}$ as the Nash equilibrium. In our case, $S = [0.0020, 0.1001]$.



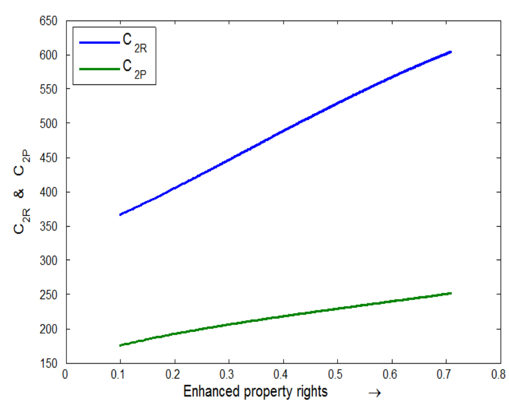
Investment



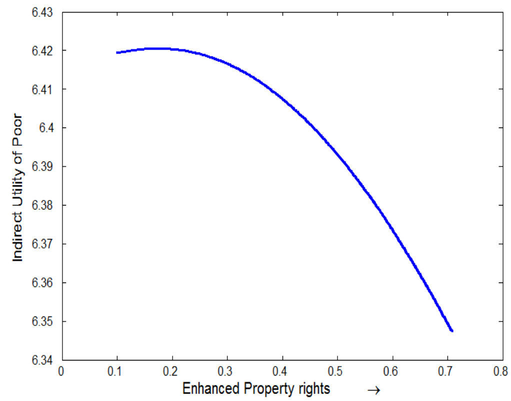
Appropriation.



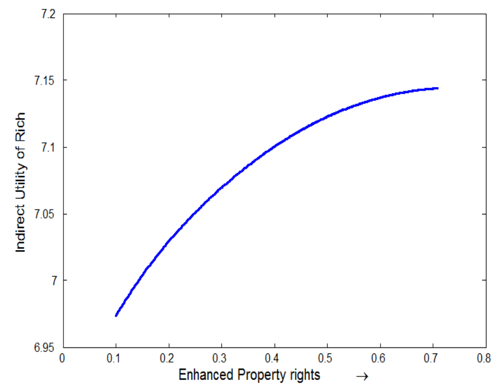
Current Consumption.



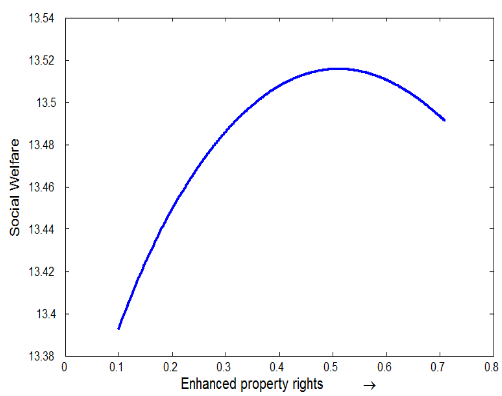
Future Consumption.



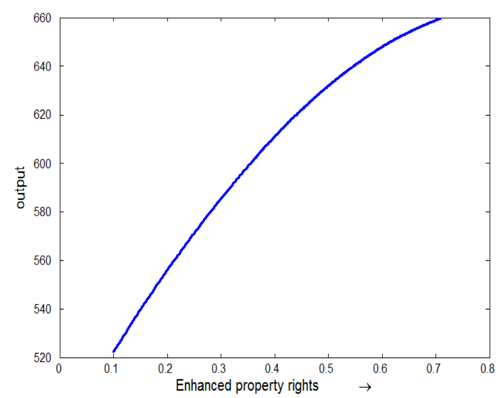
Utility of Poor.



Utility of Rich.



Social Welfare.



Output Growth.

2.4.2 Discussion

Here, we study how an improvement in property rights shapes optimal resource allocations when the best-available technology has already been adopted. Intuition suggests that within this interval, enhanced security of property rights should induce larger productive investments and thereby foster economic growth. Would this benefit come at the cost of lower welfare? Are more secure property rights always desirable? If the government could ensure perfectly secure property rights, would it?

The figures above summarize the movements in resource allocation and other important economic indicators with improvements in property rights. When $1-\theta$ increases, property rights improve, productive investment for both the agents go up and appropriative activities fall. It is evident from the figures that agents sacrifice first period consumption along with appropriative investment and allocate the resources towards productive investment in an anticipation of higher second-period consumption. However, the *rates* at which these changes occur varies significantly across the two agents. Growth of output shows a steady positive relationship with improvement in property rights, which can be logically concluded from the effect on capital accumulation. This also substantiates the empirical findings as documented in International Property Rights Index (IPRI) , 2007, report.[9] However, effect on social welfare is non-monotone. This raises the following question does improvement in property rights always beneficial from a socio economic point of view? Also the fall in the indirect utilty of the poor makes us ponder that poor has an incentive to block a movement towards more secure property rights.

The economic rationale behind the movements in the resource allocation is intuitive and foreseeable from the nature of the problem and the framework considered. The central result here is the growth-welfare trade-off. An improvement in property rights and institutional arrangement induces a reallocation of resources towards productive investment at the expense of appropriation and first period consumption. These effects can be traced back through different avenues for both the individuals and the reason can be attributed to their initial comparative advantage and individual resource allocations. For the poor agent, who had a comparative advantage in appropriation, return from productive investment increases unambiguously which in turn shifts resources towards production form appropriation and current consumption. For the initial rich, this reallocation of resources results from a decrease in the return from appropriation. The effect of improved property rights on the return from production for the rich agent is ambiguous. However, it is insightful to note that, this is a composite of a direct effect of θ and an indirect effect of θ on p' on the return from the productive activities. Similar effects work for the return from appropriation for the poor agent. From the figures, it is clear that investment in production remains the more lucrative option although the rate of capital accumulation varies significantly with improvement in property rights. This is apparent from the curvature of the capital investment curve which is initially increasing and convex and thereafter, concavity sets in. This concavity is a result of

[9] In particular it has been mentioned that there is a positive correlation (89%) between effective property rights and GDP. The study carried out by IPRI covers seventy nations, both industrialized and developing countries. In total, the IPRI country set represents ninety -five percent of world GDP. The index is rated on a scale of 0 (weakest property rights protection) to 10 (strongest protection). The average rank for the whole study's country set is 5.3 . The highest score obtained is 8.3, while the lowest score is 2.2. Countries in the top quartile of the IPRI ranking have an average GDP per capita of $32,924, (Sweeden, Norway, Denmark, etc) more than seven times higher than countries which rank in the bottom quartile (Sub-Saharan African countries , and parts of Latin America). Thus more secure property rights environment fosters growth.

diminishing marginal return, which plays a crucial role in explaining the growth welfare trade-off.

In a Benthamite definition of welfare, both agents are treated equally. In our framework, that simply means the direction of movement in welfare results from the interaction between current and future consumption. Initially, when property rights improves, agents sacrifice appropriative investments and first period consumption in an anticipation of increased second period consumption. Since capital is accumulated at a rapid pace, this anticipation is fulfilled and consequently, we obtain an increasing trend in the welfare. This process continues until we reach a critical value of property right parameter θ, beyond which increase in future consumption is outweighed by the fall in current consumption. This effect can be justified along the following lines: after significant amount of productive investment is undertaken, diminishing returns set in. Though agents still keep reallocating their resources towards production, the increase in second period consumption fail to dominate the loss in utility and thus welfare begins to exhibit a steep decline. In other words, enhanced security of private property promotes productive investment and thus fosters economic growth, which might necessitate a sacrifice in current consumption. The trade off between growth and welfare is central to the policy analysis of this research. An improved property right environment (generated by effective government intervention) that fosters economic growth might not be optimal from a welfare point of view.

2.5 Conclusion

We have considered the role institutions of property rights and conflict management can play in both achieving prosperity and mitigating conflict in developing countries. In the first half of our paper, we consider a scenario where public-funded protection of private property rights may successfully support the adoption of best-available technologies as Nash equilibrium. Such a scheme may even be welfare enhancing. Here the government's role in post production conflict is limited to " posting a guard" who thwarts the appropriative activities each agent much like a policeman Next we try to answer a more pertinent question: what happens if government takes on a more direct, proactive role in post-production conflict? Basically we endogenize the property rights by introducing a new formulation of the conflict technology, where government can explicitly intervene in the existing level of property rights by choosing the tax rate. We allow the government to utilize the tax proceeds to directly influence the technology of conflict with a view to improve the security of private property. With in this set up, we study how an improvement in property rights shapes optimal resource allocations when the best-available technology has already been adopted. This addresses a fundamental question. When the government has the option to choose a tax rate that ensure perfect property rights, is that always desirable? Would such a choice of tax rate be always welfare enhancing? We show that there exists an interval of taxation such that an increase in property security leads to a decrease in welfare. From a policy perspective this surprising result calls for a caution in recommending improved property rights enforcement, particularly when such improvements are to be made incrementally in middle-income countries.

2.6 References

Alesina, A. and Perotti R. (1996). Income distribution, political instability, and investment. European Economic Review 40: 1203-1028.

Bates, R. H. (2001). Prosperity and Violence, New York: Norton & Company.

Besley.T. (1995), Property Rights and Investment Incentives, Theory and Evidence from Ghana. Journal of Political Economy, 103, (5), pp. 903-937.

Benhabib, J., and A. Rustichini. (1996), Social Conflict and Growth. Journal of Economic Growth, Vol. 1, No. 3, March. pp. 125-142.

De Meza, D. and Gould, J. R. (1992), The social efficiency of private decisions to enforce property rights. Journal of Political Economy, vol. 100 (June), pp. 561–80.

Field, E. (2007), Entitled to work: urban property rights and labor supply in Peru, Quarterly Journal of Economics 122, 1561-1602.

Garfinkel, Michelle R. and Stergios Skaperdas. (2007), Economics of Conflict: An Overview, in T. Sandler and K. Hartley (eds.), Handbook of Defense Economics, Vol. II, pp. 649-709.

Gonzalez,.F.M. (2010), “The use of coercion in society: insecure property rights, conflict and economic backwardness”, forthcoming in Michelle Garfinkel and Stergios Skaperdas (editors), Oxford Handbook of the Economics of Peace and Conflict.

Gonzalez, F.M. (2007), Effective Property Rights, Conflict and Growth. Journal of Economic Theory, November 2007,pp. 127-139.

Gonzalez, F.M. (2005), Insecure Property and Technological Backwardness. Economic Journal, July, pp. 703-721.

Goldstein, M. and C. Udry (2008), The profits of power: land rights and agricultural investment in Ghana, Journal of Political Economy 116, 981-1022.

Grossman H. I. (1995), Robin Hood and the redistribution of property income. European Journal of Political Economy, vol. 11, pp. 399-410.

Grossman, H., and M. Kim. (1996), Predation and Accumulation. Journal of Economic Growth, Vol. 1, No. 3, September, pp. 333-351.

International Property Rights Index (IPRI), 2007, report.

Hall, R. E. and Jones, C. I. (1999), Why do some countries produce so much more output per worker than others?. Quarterly Journal of Economics, vol. 114 (February), pp. 83-116.

Hirshleifer, J. (1995a), Anarchy and its breakdown. Journal of Political Economy 103 pp. 26-52.

Konrad, K.A., 2009. Strategy and Dynamics in Contests. Oxford University Press, Oxford.

Krusell, P. and J.V. Rios-Rull (1996), Vested interests in a theory of growth and stagnation, Review of Economic Studies 63, 301-329.

Loyd-Ellis, H., and N. Marceu. (2003), Endogenous Insecurity and Economic Development. Journal of Development Economics, Vol. 72. pp.1-29.

Mokyr, J. (1990), The lever of riches, New York: Oxford University Press.

Olson, M. (1982). The Rise and Decline of Nations, New Haven: Yale University.

Parente, S.L. and E.C. Prescott (1999), Monopoly rights: a barrier to riches, American Economic Review 89, 1216-1233.

Skaperdas, S. (1992), Cooperation, conflict, and power in the absence of property rights. American Economic Review, vol. 82 (September), pp. 720-39.

2.7 Appendix

Optimal resource allocation of agents R and P

The optimization problem of agent R in the second stage of first period is:

$$\max \ln C_{1R} + \beta \ln C_{2R} \tag{2.33}$$

$$\text{subject to } pY(1-\tau) = C_{1R} + K_R + X_R \tag{2.34}$$

$$p'Y' = C_{2R} \tag{2.35}$$

$$p' = \frac{(X_R - G)^m}{(X_R - G)^m + (X_P - G)^m} \tag{2.36}$$

$$Y' = A_R K_R + A_P K_P \tag{2.37}$$

The interior optimality conditions are:

$$\frac{1}{C_{1R}} = \frac{\beta A_R}{Y'} \tag{2.38}$$

$$\frac{m(X_P - G)^m}{(X_R - G)\{(X_R - G)^m + (X_P - G)^m\}} = \frac{A_R}{Y'} \tag{2.39}$$

Analogous expressions for agent P are given by:

$$\frac{1}{C_{1P}} = \frac{\beta A_P}{Y'} \tag{2.40}$$

$$\frac{m(X_R - G)^m}{(X_P - G)\{(X_R - G)^m + (X_P - G)^m\}} = \frac{A_P}{Y'} \tag{2.41}$$

Denote, $\alpha = (X_R - G)^m + (X_P - G)^m$. Dividing (2.39) by (2.41), we get

$$\frac{m(X_P - G)^m}{(X_R - G)\alpha} \frac{(X_P - G)\alpha}{m(X_R - G)^m} = \frac{A_R}{A_P} \tag{2.42}$$

$$or, \left(\frac{X_P - G}{X_R - G}\right)^{m+1} = \frac{A_R}{A_P} \tag{2.43}$$

$$or, \frac{(X_P - G)}{(X_R - G)} = \left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}}. \tag{2.44}$$

Dividing (2.36) by $(X_R - G)^m$ we get

$$p' = \frac{1}{1 + \left(\frac{X_P - G}{X_R - G}\right)^m} \tag{2.45}$$

Substituting the expression in (2.44) in (2.45) we get

$$p' = \frac{1}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \tag{2.46}$$

Using the resource constraints and above formulation of p', we can reduce the FOC's of agents R and P as a system of linear equations in C_i and $X_i, i \in \{R, P\}$. The unique solution to the linear system is given by:

$$C_{1R} = \frac{1}{\beta(1+m)+2}\left[(p+(1-p)\frac{A_P}{A_R})(1-\tau)Y - (1+\frac{A_P}{A_R})\tau Y\right] \tag{2.47}$$
$$C_{1P} = \frac{1}{\beta(1+m)+2}\left[(\frac{A_R}{A_P}p+(1-p))(1-\tau)Y - (1+\frac{A_R}{A_P})\tau Y\right]$$
$$X_R = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}}+1}\right)\frac{m\beta}{2+\beta(1+m)}\Delta + \tau Y$$
$$X_P = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}}+1}\right)\frac{m\beta}{2+\beta(1+m)}\Delta\left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}} + \tau Y$$

where, $\Delta = [(p+(1-p)\frac{A_P}{A_R})(1-\tau)Y - (1+\frac{A_P}{A_R})\tau Y]$. This concludes the derivation of the optimal consumption and resource allocation.

2.7.1 Proof of Proposition 2.2

We start by proving: If $\tau \le \tau_{inv}$, positive investment equilibrium exists. We need to find a bound on τ such that $X_P - G \ge 0$, $X_R - G \ge 0, K_P \ge 0$, $K_R \ge 0$. From the expressions of appropriative investments from (2.47) we see that $X_R - G \ge 0$ if $\Delta \ge 0$. Now $\Delta \ge 0$ implies

$$\frac{1-\tau}{\tau} \ge \frac{1+\frac{A_P}{A_R}}{p+(1-p)\frac{A_P}{A_R}} \;\forall\; \frac{A_P}{A_R} \in [\frac{A^L}{A^H}, \frac{A^H}{A^L}] \tag{2.48}$$

Taking limit on both sides as $\frac{A^H}{A^L} \to 1$ we have $\frac{1-\tau}{\tau} \ge 2$ this implies $1-\tau \ge 2\tau$, or $3\tau \le 1$, i.e. $\tau \le \frac{1}{3}$. similar reasoning holds good for $X_P - G \ge 0$. Thus for $\tau \in [0, \overline{\tau}]$, where $\overline{\tau} = \frac{1}{3}$, equilibrium effective appropriative investments are positive. We check the conditions under which K_R,$K_P \ge 0$. Substituting the values of X_R, C_{1R} in the expression of K_R we see that K_R reduces to

$$K_R = pY(1-\tau) - [\frac{m\beta}{2+\beta(1+m)}\frac{\Delta}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}}+1} + \tau Y] - \frac{\Delta}{2+(1+m)\beta} \tag{2.49}$$

Upon tedious manipulation we see that $K_R \ge 0$ implies

$$p(1-\tau) - a + b - \tau - \frac{(1-\tau)\Delta}{2+\beta(1+m)} + \frac{(1+z)\tau}{2+\beta(1+m)} \ge 0 \tag{2.50}$$

Where $a = \frac{m\beta(1-\tau)\Delta}{(2+\beta(1+m))(z)^{\frac{m}{m+1}}+1}$, $b = \frac{m\beta(1+z)\tau}{(2+\beta(1+m))(z)^{\frac{m}{m+1}}+1}$, $z = \frac{A_P}{A_R}$. Taking limit on both sides of the above equation as $\frac{A^H}{A^L} \to 1$ we have

$$\tau[-(1-p)+\frac{6+3m\beta}{2(2+\beta(1+m))}] \geq \frac{m\beta+2}{2(2+\beta(1+m))} - p \tag{2.51}$$

If we assume, $[-(1-p)+\frac{6+3m\beta}{2(2+\beta(1+m))}] > 0$, we arrive at a condition that states $p < \frac{1}{2}$, which contradicts our basic assumption. Thus $-(1-p)+\frac{6+3m\beta}{2(2+\beta(1+m))} < 0$. By similar reasoning $\frac{m\beta+2}{2(2+\beta(1+m))} - p < 0$. Rearranging terms we see that $K_R \geq 0$ iff $\tau \leq \frac{\frac{m\beta+2}{2(2+\beta(1+m))}-p}{-(1-p)+\frac{6+3m\beta}{2(2+\beta(1+m))}}$. Let us call $\tau_1 = \frac{\frac{m\beta+2}{2(2+\beta(1+m))}-p}{-(1-p)+\frac{6+3m\beta}{2(2+\beta(1+m))}}$. Again, for $K_P \geq 0$, we substitute the values of X_P and C_P in the expression of K_P, which gives,$K_P = (1-p)(1-\tau)Y - c - \tau Y - d$ where $c = \frac{m\beta\Delta[(1-\tau)}{2+\beta(1+m)} \frac{1}{(1+\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}})} \left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}}$, $d = \frac{\Delta}{2+\beta(1+m)}$. Taking limit on both sides of the expression of K_P as $\frac{A^H}{A^L}$ goes to 1 we get $K_P \geq 0$ iff

$$(1-p)(1-\tau) - \frac{m\beta Y}{2+\beta(1+m)}[\frac{(1-3\tau)}{2} + (1-3\tau)] - \tau \geq 0$$

$$iff,\ \tau \leq \frac{2+m\beta+2(p-1)(2+\beta(1+m))}{-2(1-p)(2+\beta(1+m))+2+m\beta-2\beta}$$

Let $\tau_2 = \frac{2+m\beta+2(p-1)(2+\beta(1+m))}{-2(1-p)(2+\beta(1+m))+2+m\beta-2\beta}$. Thus $K_P \geq 0$ iff $\tau \leq \tau_2$. Thus for $\tau \leq \min\{\overline{\tau}, \tau_1, \tau_2\}$ all the three inequalities are satisfied. We denote $\tau_{inv} = \min\{\overline{\tau}, \tau_1, \tau_2\}$.Thus there exists positive levels of investment for $\tau < \tau_{inv}$ as, $\frac{A^H}{A^L} \longrightarrow 1$

Next, we prove the second part of the proposition. We show that an agent's best response to any choice of technology by the other agent involves in either choosing the best technology or the worst one i.e $A_i \in \{A^L, A^H\}$ for a given interval. Substituting the values of C_{1R} and C_{2R} into the utility function, we get $U_R = U_R(A_R, A_P)$. Differentiating U_R w.r.t A_R we get,

$$\frac{\partial U_R}{\partial A_R} \geq 0 \text{ iff } \frac{\tau}{1-\tau} \geq \Gamma(x) \tag{2.52}$$

provided $(1-\tau)(p+(1-p)x) - (1+x)\tau \geq 0$ and $\phi k(x)(1+x) + x - \beta \geq 0$. Here $\Gamma(x) = \frac{f(x)}{g(x)}$, $x = \frac{A_P}{A_R}$, $f(x) = (p+(1-p)x)[1+\phi k(x)] - (1+\beta)p$, $g(x) = \phi k(x)(1+x) + x - \beta$. Also, $\phi = \frac{m\beta}{m+1} < \beta$,and $k(x) = \frac{1}{1+x^{\frac{m}{m+1}}}$. Now,

$$\Gamma'(x) \geq 0$$

$$iff,\ [\beta(2p-1) + \phi(1-\beta)(1-2p)k(x) + \phi^2(1-2p)k(x)^2 + \phi(1+\beta)(2p-1)k'(x)x \geq 0$$

$$iff,\ \beta x^{\frac{2m}{m+1}} + [(\beta-\phi)(1+\phi) + \beta(1-(\frac{m}{m+1})^2)]x^{\frac{m}{m+1}} + (\beta-\phi)(1+\phi) \geq 0. \tag{2.53}$$

The above is an equation of a parabola where both the roots, say x_1 and x_2, are negative. Therefore, for all $x \geq \max\{x_1, x_2\}$, the $\Gamma(x)$ is positively sloped. For the values of x that satisfy equations (43)−(2.53), we get the best response of agent R is to choose either A^L or A^H. This interval of x implicitly put a restriction on τ. We denote that critical value of $\tau \geq \tau_R = \frac{1-p}{2-p}$. Again, substituting C_{1P} and C_{2P} into the utility function, we get

$$U_P = U_P(A_R, A_P)$$

Following the same steps for the poor agent, we get if ,$(1-\tau)(py+1-p)-\tau(1+y) \leq 0$, and $(1+\beta)(1-p)-(py+1-p)(1+\phi k(y) \leq 0$ then,

$$\frac{\partial U_P}{\partial A_P} \geq 0 \; iff \; \frac{1-\tau}{\tau} \geq G(y) \; where, \tag{2.54}$$

$$G(y) = \frac{\beta - y - \phi k(y)(1+y)}{(1+\beta)(1-p)-(py+1-p)(1+\phi k(y))}, y = \frac{A_R}{A_p}$$

Now, $G'(y) \geq 0$

$$iff \;\; [\beta(2p-1)+\phi(1-\beta)(1-2p)k(y)+\phi^2(1-2p)k(y)^2+\phi(1+\beta)(2p-1)k'(y)y \geq 0$$

$$iff \;\; (\beta-\phi)(1+\phi)x^{\frac{2m}{m+1}} + [(\beta-\phi)(1+\phi)+\beta(1-(\frac{m}{m+1})^2)]x^{\frac{m}{m+1}} + \beta \geq 0 \tag{2.55}$$

Which is again an equation of a parabola, where both the roots (say x_3, x_4) are negative, though different in values. Then, $x \geq \max\{x_3, x_4\}, G(y)$ is positively sloped. Thus, for $x \geq 0$,both $\Gamma(x)$ and $G(y)$ are positively sloped. Therefore, for the values of x that satisfy equations (46)−(2.55), we get the best response for agent P is to choose either A^L or A^H.This interval of x implicitly put a restriction on τ. We denote that critical value of $\tau \leq \tau_P = \frac{p}{1-p}$.

Let $\frac{A^H}{A^L} \to 1$. If $\tau \in [\frac{1-p}{2-p}, \frac{p}{1+p}]$, then both the agents best response is to adopt either A^H or A^L. We denote $\tau_H = \frac{1-p}{2-p}$. From the Lemma 1, we know that positive investment equilibrium exists for $\tau \leq \tau_{inv}$. Thus for $\tau \in [\tau_H, \tau_{inv}]$, $[A^H, A^H]$ can be sustained as a positive investment equilibrium. This completes the proof.

2.7.2 Proof of Corollary 2.1

Given (A^L, A^L) is an equilibrium in the bench-mark model the optimal choices of C_{1i} and C_{2i}, $i \in \{R, P\}$ are given as $C_{1i} = \frac{Y}{2+\beta(1+m)}$,$C_{2i} = \frac{\beta Y A^L}{2(2+\beta(1+m))}$. The SWF in this case is given by

$$SWF(A^L, A^L) = (\ln C_{1P} + \beta \ln C_{2P}) + (\ln C_{1R} + \beta \ln C_{2R})$$

plugging in the values of C_{1i} and C_{2i} into the above equation we have

$$SWF(A^L,A^L) = \ln \frac{Y^2}{(2+\beta(1+m))^2} \left(\frac{\beta A^L Y}{2\,(2+\beta(1+m))} \right)^{2\beta}$$

Similarly when (A^H,A^H) is an equilibrium in the guard posting framework the optimal choices of C_{1i} and C_{2i}, $i \in \{R,P\}$ are $C_{1i} = \frac{Y(1-3\tau)}{2+\beta(1+m)}$ and $C_{2i} = \frac{\beta A^H(1-3\tau)Y}{2(2+\beta(1+m))}$ respectively. Plugging in the expressions of C_{1i} and C_{2i} in the social welfare function $SWF = U_R + U_P$ and rearranging the terms we get

$$SWF(A^H,A^H) = \ln \left[\left(\frac{(1-3\tau)Y}{(2+\beta(1+m))} \right)^2 \left(\frac{\beta A^H Y(1-3\tau)}{2\,(2+\beta(1+m))} \right)^{2\beta} \right]$$

From this it follows that

$$SWF(A^H,A^H) \geq SWF(A^L,A^L)$$

$$if \quad \left(\frac{(1-3\tau)Y}{(2+\beta(1+m))} \right)^2 \left(\frac{\beta A^H Y(1-3\tau)}{2\,(2+\beta(1+m))} \right)^{2\beta} \geq \frac{Y^2}{(2+\beta(1+m))^2} \left(\frac{\beta A^L Y}{2\,(2+\beta(1+m))} \right)^{2\beta}$$

$$\Leftrightarrow \quad (1-3\tau)^2 \left(A^H\right)^{2\beta} (1-3\tau)^{2\beta} \geq \left(A^L\right)^{2\beta} \Leftrightarrow \frac{A^H}{A^L} \geq \left(\frac{1}{(1-3\tau)^{2(\beta+1)}} \right)^{\frac{1}{2\beta}}$$

This completes the proof.

CHAPTER 3. OPTIMAL PORTFOLIO SELECTION WHEN AN INVESTOR'S WEALTH IS SUBJECT TO BANKRUPTCY

3.1 Introduction

Utility maximization problems in mathematical finance are usually of two types : expected utility of consumption on a finite interval or the expected utility of terminal wealth at some future time point. In this article, we are interested in a problem of second category. Research related to expected utility maximization goes back to the seminal articles of Samuelson and Merton (1969), Merton (1971) and in the recent times, for instance in the writings of Pliska (1986), Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989). Although different in approach, these articles share a common setting of an investor who wants to invest a certain portion of his wealth in stocks and the rest in money market account so as to maximize a discounted expected utility of his wealth up to a fixed terminal time.

Here, we consider a variant of these problems by allowing the agent to freely stop at a time of convenience rather than a fixed terminal time. Investor's objective is to maximize the discounted expected utility of his terminal wealth. The discount rate is a known exogenously determined constant. The time chosen to stop is a stopping time and it may depend on the investor's past experience. A typical example of a stopping time is the first time the price of a single share of a stock reaches a certain level. We always know whether that event has already happened or not by knowing the past history (but we do not know when it will happen in future). Such problems are known as optimal stopping problems. For example we can think of a risk averse individual who wants to maximize his expected utility before " quitting" from the stock market and put all his money in bank. Here "quitting" time refers to the time after which an agent only invests in the safe asset and nothing in the risky asset.

Other examples that can fit into this category would be an optimal harvesting strategy for a single species in a random environment whose growth is modeled as a diffusion process. The motive is to look for a harvesting strategy which maximizes the expected discounted income from harvesting up to extinction, (see Chao, Song, Stockbridge, 2010). Several areas in economics draws on such optimal stopping problems. The tree cutting problem analyzed by the Austrian capital theorists also fits into the above framework. Briefly stated, the problem is as follows. Suppose we have a tree whose growth is stochastic. If the tree depletes at a rate $r > 0$, when should the tree be cut down? What is the present value of such a tree? In order to answer when a tree should be felled we are analyzing the optimal timing of cutting a tree to maximize the net present value of a tree when the tree's growth is stochastic rather than deterministic (Brock, Rothschild and J.E Stiglitz, 1979).

We consider an economy consisting of a single risk averse individual and two assets. One is a risky asset such as a stock, and the other is a risk-free asset which can be considered as a bond. At any point in time, the investor wants to allocate his wealth between the two assets. He invests a certain portion of his wealth in stock and the rest in bond. He has the option to quit from the stock market at any random time defined by a stopping rule. The assets available to the agent can be traded continuously, without restrictions. There are no frictions or transaction costs. Against this backdrop, the agent maximizes his discounted expected utility over the class of all stopping times by taking into account that the wealth process is stochastic. It turns out that the optimal stopping time in this scenario is of the threshold type. An optimal policy in this setting involves taking action when the state variable (here the wealth of an investor) exceeds an appropriately chosen threshold and doing nothing when the state variable lies inside the region defined by the threshold.

Using techniques of stochastic calculus and the principle of smooth fit, in optimal control, we derive an optimal threshold of wealth at which the investor quits from the stock market. In the particular case where the agent invest a fixed proportion of his wealth in stocks and bonds, we obtain closed form expressions for optimal exercise boundary as well as for the value function. In the general case, we prove the existence of an optimal threshold but we are unable to derive closed form solution for the threshold. We also analyze the sensitivity of the value function and the optimal threshold with respect to the parameters involved in the model. In the general case, where the proportion of wealth invested in the assets varies with the current wealth we cannot explicitly get a closed form solution of the threshold region, rather we solve it qualitatively.

Once we obtain these results,we are interested in investigating a more realistic problem. What would be the optimal selling time for a financial asset subject to bankruptcy. There are primarily two types of approaches that attempt to model default processes in the credit risk literature: *structural form* and *reduced form* models. Structural models as the name suggests use the evolution of firm's structural variables, such as an asset price and debt values to determine the time of default. Merton's (1974) model was the first to introduce this concept. Here default time is defined as the first time a firm's asset is below it's outstanding debt.

One of the major drawbacks of the structural models is the predictability of default time. Generally, structural models assume complete information about the firm's asset value and threshold. This together with the fact that the asset price movement is continuous makes the default time a predictable stopping time i.e. default does not come as a surprise. The predictability of default makes the models to generate short term credit spreads close to zero. Empirical findings are unable to substantiate the above claim. In fact it is observed in the market that even short-term credit spreads are bounded from below. One of the ways to be out of this predicament is to model the default time in such a manner such that default is unpredictable. Reduced form models accomplish this task. They do not consider the relation between default and firm value in an explicit manner. In contrast to the structural models, the time of default in reduced models is not determined by the firm's economic fundamentals, but it is the first jump of an exogenously given jump process where intensity may depend on the past behavior of the asset price. Hence default is not a predictable event any more, and the default probabilities for short maturity do not go to zero.

Although easy to calibrate, reduced form models lack the link between credit risk and the information regarding firms assets and liabilities. A key element to link both the approaches lies in the model's information assumptions. A detailed discussion that links the two approaches can be found in Jarrow, Protter, Sezer (2007), Jarrow and Protter (2004) Duffie and Lando (2001), Guo, Jarrow and Zeng (2005). A common viewpoint among these articles is "*...it is possible to transform a structural model with a predictable default time into a reduced-form model, with a totally inaccessible default time, by altering the information sets available for modelling purposes"* (Guo, Jarrow and Zeng, 2005, p2). To represent bankruptcy in this article, we use a reduced form model.

Our paper is closely related with the works of Linetsky (2004, 2006), Yor, Elliot and M. Jeanblanc, (2000), Meng (2007). Although the model setup is in line with their work, our problem is different. We consider a scenario where a risk averse individual would like to maximize the discounted expected utility from terminal wealth taking into account the fact that wealth process of the investor is prone to default. The wealth process follows a stochastic differential equation (SDE) with variable coefficients. There are only two assets, namely a stock and a bond. For simplicity, we assume that there are no transaction costs. In contrast to the work of Linetsky (2004,2006), our results are not contingent on negative power hazard functions in the reduced-form model, rather we take a general functional form for the hazard function. When there is no default, the problem reduces to the one mentioned in paragraph four. When an investor's wealth is subject to default risk and the wealth process is represented by a reduced form model, we are able to show that there is an optimal strategy which is of the threshold type.

We also analyze the effect of default risk on the optimal exercise boundary as well as on the value function. Once we incorporate default risk into our model there are two opposing forces: first the default intensity will be added to the drift coefficient of the SDE which is favorable to the investor. Thus the investor has an incentive to keep the asset longer. On the other hand, the discount factor also increases by the same amount and hence the investor tends to sell the asset earlier. Which effect would dominate in an optimal strategy can not be said ex-ante. In this direction, we try to answer the following pertinent questions. Does higher default intensity leads to a higher optimal exercise boundary and to a higher value function? Does the optimal exercise boundary and the value function in the default case are both higher than their counterparts in the no default case? The questions are answered here by some monotonicity results with respect to the default intensity. To be specific, we show that higher default intensity leads to a higher optimal exercise boundary and to a higher value function. Moreover, optimal exercise boundary and the value function in the default case are both higher than their counterparts in the no default model.

Having shown the existence of an optimal stopping time we address the issue of optimal portfolio selection when an investor's wealth is subject to bankruptcy. We show the existence of an optimal portfolio process.

The rest of the article is organized in the following manner: In section 2, we describe the model and utility maximization problem. In section 3, we set up the optimal stopping problem and state a verification lemma associated with the HJB equation of the stopping problem. In sub-section 3.1, we introduce our main theorem assuming the existence of a particular type of solution to the HJB equation. In section 4, we setup the bankruptcy model and proceed in similar lines. In section 5, we derive the monotonicity results. We discuss the existence of an optimal portfolio process in section 6. We intend to do some numerical computations that would reinforce the results we obtained in section 3, 4 and 5. bigskip

A few words about notation is needed. C^0 denotes the space of all continuous functions from R to R. C^1 consists of all differentiable functions whose derivative is continuous. C^2 is the class of functions whose derivative is in C^1. The space of all $\mathscr{F}_t$ measurable stopping times is denoted by Γ. Here $(\mathscr{F}_t)_{t\geq 0}$ is the filtration generated by a standard Brownian motion $B(t)$.

3.2 Model Setup

3.2.1 An investor's problem.

We consider a risk averse individual who is endowed with an initial wealth W_0 and intends to invest it between two assets: a risky asset (a stock) and a riskless asset (say Govt Bonds). The uncertainty is modelled by a probability space $(\Omega, \mathscr{F}, P)$. On this space a one dimensional Brownian motion $\{(W_1(t), \mathscr{F}_t)\}_{t\geq 0}$ is defined and we simply assume that $\{\mathscr{F}_t\}_{t\geq 0}$ to be a right continuous filtration. The σ-algebra $\mathscr{F}_0$ contains all the P-null sets of $\mathscr{F}$, and the Brownian motion $Z(t)$ is $\mathscr{F}_t$ adapted. One of the assets is a risk-free bond which is governed by the differential equation

$$dB(t) = r_0 B(t) dt.$$

with $B(0) = 1$. Here r_0 denotes the fixed market rate of return. The other asset is a stock and stock price is modelled as a Geometric Brownian motion, (GBM). The corresponding Stochastic Differential Equation (SDE) is given by

$$dS(t) = S(t)[\mu dt + \sigma dZ(t)],\, S(0) = s$$

where μ is the drift term and σ is the volatility coefficient (measure of risk). Obviously $\mu > r$, otherwise the investor may not invest in a risky asset. The investment behavior is modelled by a portfolio process $\pi = (p(W(t)), 1 - p(W(t)))$ that is $\mathscr{F}_t$ adapted. Note $\mathscr{F}_t$ contains all the information of the wealth process, to be defined below upto time t. Here $p(W(t))$ represents the fraction of his wealth invested in the stock when the total wealth is $W(t)$. Therefore the quantity $1 - p(W(t))$ is the fraction of total wealth invested in Bonds. The fraction, $p(w)$ is a continuous function which is bounded below by a positive lower bound $\varepsilon > 0$, (i.e. $\varepsilon \leq p \leq 1$) but it need not be differentiable. The investor uses his wealth to buy shares of bond and stock. Out of the amount $p(W(t))W(t)$ invested in the stock, the agent buys $\alpha(t)$ units of stock, each worth $S(t)$ dollars i.e. $\alpha(t)S(t) = p(W(t))W(t)$. The remaining portion of wealth $[1 - p(W(t))]W(t)$ is used to buy bonds, i.e. $\beta(t)B(t) = [1 - p(W(t))]W(t)$, where $\beta(t)$, is the units of bond bought and $B(t)$, is the price of each unit. Having modelled the price and portfolio processes, we now make the standard assumptions of a market model. The two assets that are available to the agent can be traded continuously, without restrictions, frictions or any transaction costs. The investor has to invest in a self financing way, change in the wealth of portfolio is due to gains and losses from investment in the assets. Also we assume the so-called small investor hypothesis i.e. the actions of a single investor should not have any impact on the stock prices. A direct consequence of the assumption of self financing portfolio is that the wealth process of the investor can be written as

$$W(t) = \alpha(t)S(t) + \beta(t)B(t)$$

and, the SDE associated with it is given by $dW(t) = \alpha(t)dS(t) + \beta(t)dB(t)$. Using the equation for $S(t)$ and $B(t)$,

$$dW(t) = \alpha(t)S(t)[\mu dt + \sigma dZ(t)] + \beta(t)[r_0 B(t)]dt$$

Substituting $\alpha(t)S(t) = p(W(t))W(t)$ and $\beta(t)B(t) = (1 - p(W(t))W(t)$ into the above equation we obtain

$$dW(t) = p(W(t))W(t)[\mu dt + \sigma dZ(t)] + (1 - p(W(t))W(t)r_0 dt$$

On rearranging terms we get

$$dW(t) = W(t)[\mu_1(W(t))dt + \widetilde{\sigma}(W(t))dZ(t)]$$

with $W(0) = w_0$, where $\mu_1(w) = \mu p(w) + r_0(1-p(w))$, $\widetilde{\sigma}(w) = \sigma p(w)$ are the drift and diffusion coefficients respectively. Here $Z(\cdot)$ represents a standard Brownian-motion. Note that the non-differentiability of $p(\cdot)$ manifests into the non-differentiability of $\mu_1(\cdot)$ and $\widetilde{\sigma}(w)$. Existence and uniqueness of a solution of such an SDE is guaranteed by Theorem 2.9 (page 289, Karaztaz and Shreve, 1991). The wealth equation can be interpreted as a controlled SDE with the control being the portfolio process π.

In our problem, we take the portfolio process as given. Since the agent is a risk averse individual, we intend to use a concave utility function to represent his preferences. Our methods work for any non-negative concave utility function which satisfies $\lim\limits_{x \longrightarrow \infty} U(x) = \infty$. For analytical simplicity, we choose the utility function to be $U(w) = \ln w$. In this setting if an investor wants to sell the stock at an $(\mathscr{F}_t)$ stopping time τ, the present value utility is given by $e^{-\rho\tau}(\ln W(\tau))$ where $\rho > r_0$ is a discount factor. Thus the agent's problem is to find an $\mathscr{F}_t$ stopping time τ^* which maximizes $E[e^{-\rho\tau}(\ln W(\tau))]$ over the class of all $\mathscr{F}_t$ stopping times τ. We denote this class of stopping times by Γ. Notice that if the investor waits too long to withdraw from the stock market then his discounted utility goes to zero $(\rho > r_0)$. On the contrary, if he starts with a very low initial wealth, it may be prudent to wait for sometime rather than quit immediately from the stock market. With these considerations in mind, we guess that an optimal selling strategy is determined by some threshold: when the price is below the threshold we hold our asset, but once the price is above the threshold we sell it. Now we are in a position to formally formulate the optimization problem:

$$U(w) = \underset{\tau \in \Gamma}{Sup}\, E[e^{-\rho\tau} \ln W(\tau) \mid W(0) = w] \tag{3.1}$$

with

$$dW(t) = W(t)[\mu_1(W(t))dt + \widetilde{\sigma}(W(t))dZ(t)] \tag{3.2}$$

and $W(0) = w_0$. For analytical tractability, we make the following substitution $Y(t) = \ln W(t)$. Applying Ito's lemma to the process $Y(t)$ we get the SDE for $Y(t)$ as

$$dY(t) = \widetilde{\mu}(Y(t))dt + \widetilde{\sigma}(Y(t))dZ(t)] \tag{3.3}$$

with $Y(0) = y$, $\widetilde{\mu}(y) = \mu_1(e^y) - \frac{1}{2}[\widetilde{\sigma}(e^y)]^2 = \mu h(y) + r_0(1-h(y)) - \frac{1}{2}[\sigma h(y)]^2$, where $h(y) = p(e^y)$ and $\widetilde{\sigma}(y) = \sigma h(y)$. Furthermore we assume $K_1 < |\widetilde{\mu}(y)| < K_2$ and $\varepsilon_1 < \widetilde{\sigma}(y) < \sigma \in R$. Also $\widetilde{\mu}(y)$ is a Lipschtiz continuous function with Lipschitz constant $K < \rho$. With the above transformation, the reformulated problem can be stated as

$$V(y) = \underset{\tau \in \Gamma}{Sup}\, E[e^{-\rho\tau} Y(\tau) \mid Y(0) = y] \tag{3.4}$$

where $Y(t)$ follows the SDE given in equation (3.3). As with any optimal stopping problem (see EL Karoui (1981)), we introduce two disjoint sets, usually called the stopping region E and continuation region C. These regions are defined by $E = \{y > 0 : V(y) = y\}$, $C = \{y > 0 : V(y) < y\}$. They describe the optimal threshold in terms of E and C as their common boundary point. The approach for deriving the optimal threshold follows from (Ghosh, A., and Weerasinghe, A., 2007), (Weerasinghe, A., 2005) and (Ocone,D., and Weerasinghe, A., 2005).

3.3 Verification lemma

Following the theory of stochastic control, we can formulate the Hamilton-Jacobi-Bellman (HJB) equation associated with the reformulated optimal stopping problem in (3.4) as

$$Max\ \{\frac{1}{2}\widetilde{\sigma}^2(y)Q''(y) + \widetilde{\mu}(y)Q'(y) - \rho Q(y),\ y - Q(y)\}\ = 0\text{ , for all } y \text{ a.s on R} \tag{3.5}$$

The following, verification lemma shows that any smooth solution to (3.5) is an upper bound for the value function. The proof of the following Lemma closely follows that of Meng (2007).

Lemma 3.1. *Let $Q(y)$ be a non-negative function satisfying the following conditions:*

(i) $Q(y)$ is a C^1 function on R and piecewise twice continuously differentiable.

(ii) $Q(y)$ a solution to the HJB equation

(ii) $\lim_{c^-} Q''(y)$ and $\lim_{c^+} Q''(y)$ exits and are finite for all c on R.

Then $Q(y) \geq V(y)$ for all $y \in R$, where V is the value function defined above.

Proof. The second order differential operator associated with the SDE in (3.3) is given by

$$\mathscr{L} = \frac{1}{2}\widetilde{\sigma}^2(y)\frac{d^2}{dy^2} + \widetilde{\mu}(y)\frac{d}{dy} - \rho \tag{3.6}$$

for all $y \in R$. Let us introduce a sequence of stopping times τ_n with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$, for every $n > y$ by

$$\tau_n = \begin{cases} \inf\{t \geq 0 : Y(t) \notin (-\infty, n] \\ \infty, \text{ if the above set is empty} \end{cases}$$

where $Y(t)$ satisfies (3.3). Define $s = \tau_n \wedge \tau \wedge N$. Clearly $s \to \tau$ a.e P as $n \to \infty$ and $N \to \infty$. Applying Ito's Lemma, to $e^{-\rho t} Q(Y(t))$ where $Q(y)$ is C^2 we obtain

$$\begin{aligned} E_y[e^{-\rho s}Q(Y(s))] &= Q(y) + E_y[\int_0^s e^{-\rho u}LQ(Y(u))du] + E_y[\int_0^s e^{-\rho u}Q'(Y(u))\widetilde{\sigma}(u)dZ(u)] \\ &= Q(y) + E_y[\int_0^s e^{-\rho u}LQ(Y(u))du] \leq Q(y) \end{aligned}$$

Since $Y(u)$ is bounded on $[0, \tau \wedge \tau_n \wedge N]$ and $\widetilde{\sigma}$ and $Q^{'}$ are continuous the integrand in the stochastic integral is bounded and hence the integral is a mean zero martingale. The last inequality follows from the fact that $\mathscr{Q} \leq 0$ as Q satisfies (3.5). Since Q and $\exp(\cdot)$ are both nonnegative functions applying Fatou's Lemma we obtain

$$E_y[e^{-\rho\tau}Q(Y(\tau))] \leq \liminf_{n\to\infty} E_y[e^{-\rho(\tau\wedge\tau_n\wedge N)}Q(Y(\tau \wedge \tau_n \wedge N))] \leq Q(y)$$

for each $y \in R$. Taking supremum over all $\mathscr{F}_t$ stopping times we obtain $V(y) \leq Q(y)$ for all $y \in R$.

3.3.1 Optimal selling Time.

In this section we want to show that the optimal selling time of the stock is of threshold type i.e. the investor "quits" from the stock market when the $Y(t)$ process reaches for the first time a threshold $[y^*, \infty)$. This y^* separates the continuity region C and the stopping region E and hence it is the free boundary of (3.5). The proof of the above statement runs along the following lines. First we assume that there exists a free boundary point y^* and a function $\widetilde{Q}(y)$ which satisfies (I) given below. Then we use the pair $(y^*, \widetilde{Q})$ to construct a function Q^* that is a smooth solution to the HJB equation. It turned out that the the point y^* is indeed the free boundary of the HJB equation (3.5). We would then use it to construct our optimal selling time. In the appendix, we establish the existence of the pair $(y^*, \widetilde{Q})$ satisfying (I).

From now on let us assume the existence of a point y^* and a non-negative function $\widetilde{Q}(y)$ which satisfy the following conditions :

$$\left.\begin{array}{l} (i)\ \widetilde{Q}: R \to (0,\infty) \ and\ \mathscr{L}\widetilde{Q}(y) = 0\ , for\ all\ y \in (-\infty,\infty),\ \widetilde{Q}(y^*) = y^*,\ \widetilde{Q}(-\infty) = 0. \\ (ii)\ \widetilde{Q}(y) > y\ , for\ all\ y < y^*\ ,\ \widetilde{Q}'(y^*) = 1,\ \widetilde{Q}''(y^*) > 0. \end{array}\right\} .....(I)$$

Note $\widetilde{Q}(y)$ is C^1 on R. Existence of such a pair $(\widetilde{Q}, y^*)$ will be proved in appendix. We can describe our main result depending on the existence of this pair $(\widetilde{Q}, y^*)$.

Next, we introduce the function $Q^*(\cdot)$ by

$$Q^*(y) = \begin{cases} \widetilde{Q}(y) \text{ for } y \le y^* \\ y^* \text{ for } y \ge y^* \end{cases} \tag{3.7}$$

We are now in a position to introduce one of the main theorems in this article that shows $Q^*(y) = V(y)$.

Theorem 3.1. *Assume the existence of the function $\widetilde{Q}$, and the point y^* satisfying (I). Let $Q^*(y)$ be as defined above . Then we claim the following results hold.*

(i) $Q^*(y)$ is a C^2- function on the set $R/\{y^*\}$ and C^1- function on R that satisfies the HJB equation.

(ii) if $\tau^* = \begin{cases} \inf\{t \ge 0 : Y(t) \notin (-\infty, y^*] \\ \infty, \textit{ if the above set is empty} \end{cases}$

Then τ^* is an optimal stopping time and $V(y) = Q^*(y)$ for all $y \in R$, where V is the value function.

Proof. (i) By construction $Q^* = \widetilde{Q}(y)$ on $y \in (-\infty, y^*]$ (since $\widetilde{Q}(y)$ is C^1 it follows Q^* is C^1 on $(-\infty, y^*]$) and $Q^* = y$ on $[y^*, \infty)$. Thus Q^* is C^1. Now $\lim_{y^*_-} Q^{*''}(y) = \widetilde{Q}''(y^*) > 0$ and $\lim_{y^*_+} Q''^*(y) = 0$. Thus $Q^*(y^*)$ is well defined at y^* proving $Q^*(y)$ is twice continuously differentiable on $R \backslash \{y^*\}$.

In order to show that $Q^*(y)$ satisfies the HJB equation, we split R into two open intervals $(-\infty, y^*)$, and (y^*, ∞). On the open interval $(-\infty, y^*)$, $Q^* = \widetilde{Q}$ therefore $\mathscr{L}_Y Q^* = L_Y \widetilde{Q} = 0$ and by construction $y - \widetilde{Q}(y) < 0$. Hence Q^* satisfies the HJB on the open interval $(-\infty, y^*)$. To verify it on (y^*, ∞) we see that at y^*_-, $\mathscr{L}_Y Q^*(y^*_-) = \frac{1}{2}\widetilde{\sigma}^2(y^*_-)\widetilde{Q}''(y^*_-) + \widetilde{\mu}(y^*_-)\widetilde{Q}'(y^*_-) - \rho \widetilde{Q}(y^*_-) = 0$. Since $\widetilde{Q}''(y^*) > 0$, $\widetilde{Q}''(y^*_-) > 0$ (by continuity), $\widetilde{Q}'(y^*_-) = 1$ and $\widetilde{Q}(y^*_-) \simeq y^*_-$. Incorporating these facts in $\mathscr{L}_Y Q^*(y^*_-) = 0$ it follows that $\widetilde{\mu}(y^*_-) - \rho y^*_- < 0$, at y^*_-. By continuity of $\widetilde{\mu}(y)$ it follows that $\widetilde{\mu}(y^*) - \rho y^* < 0$ at y^*. For $y > y^*$, we need to show that $\mathscr{L}_Y Q^*(y) = \widetilde{\mu}(y) - \rho y < 0$. If we can show $\widetilde{\mu}(y) - \rho y \leq \widetilde{\mu}(y^*) - \rho y^*$ we are done. But this amounts to showing $\widetilde{\mu}(y) - \widetilde{\mu}(y^*) \leq \rho(y - y^*)$, for $y > y^*$. Since $\widetilde{\mu}(y)$ is Lipchitz continuous with a Lipchitz constant less than ρ this immediately follows. Thus for $y > y^*$, $y = Q^*(y)$ and $\mathscr{L}_Y Q^*(y) < 0$. This completes the proof of part (i).

(ii) We can use the proof just outlined along with Lemma 1 to conclude that $Q^*(y) \geq V(y)$ for all $y \in R$. It remains to show that $Q^*(y) \leq V(y)$ for all $y \in R$ to conclude the proof of part (ii).

We first consider the domain $(-\infty, y^*)$. The problem occurs on the boundary of C. Within C evolution is governed by diffusion and is therefore smooth. However, once the boundary ∂C is reached the smooth evolution comes to an abrupt end. The classical Ito's rule can not be applied but luckily we have generalizations of Ito's rule that do not require functions to be C^2 everywhere. Applying the extended Ito's rule (page 218 Karatzas and Shreve, 1991) to $e^{-\rho t} Q^*(Y(t))$ we have

$$E_y[Q^*(Y(k))e^{-\rho(N \wedge \tau^*)}] = Q^*(y) + E_y \int_0^{N \wedge \tau^*} e^{-\rho t} L Q^*(Y(t))dt + E_y \int_0^{N \wedge \tau^*} e^{-\rho u} Q^{'}(Y(u))\widetilde{\sigma}(u) dZ(u)$$

Clearly as N tends to ∞, $k \to \tau^*$ a.e P. Since $Y(t)$ is bounded on $[0, N \wedge \tau^*]$ and both Q' and $\widetilde{\sigma}$ are continuous it follows that the last term in the above equation is a mean 0 martingale. Now for $y < y^*$, $\mathscr{L} Q^*(y) = 0$ and $Q^*(y^*) = y^*$. Letting N going to infinity we have

$$Q^*(y) = E_y[e^{-\rho \tau^*} Y(\tau^*)]$$

Since τ^* is a $\mathscr{F}_t$ stopping time and $V(y)$ is obtained by taking maximum over Γ it follows that $Q^*(y) \leq V(y)$ for all $y < y^*$. For the case $y \geq y^*$, $Q^*(y) = y$ and by definition $\tau^* = 0$. We can still write $Q^*(y) = Q^*(y) = E_y[e^{-\rho \tau^*} Y(\tau^*)]$. Hence $Q^*(y) \leq V(y)$ for all $y \in R$. This completes the proof that $Q^*(y) = V(y)$ for all y and τ^* is an optimal stopping time.

The existence of a function $\widetilde{Q}$ and it's associated free boundary y^* is proved in Theorem 8 (see Appendix (a)). In fact we also show that such a pair $(\widetilde{Q}, y^*)$ is unique. Having established the existence of an optimal threshold in full generality, we now close the section with an example.

Example.

Let us consider a simple case where our risk averse investor invests a fixed proportion of his wealth in stocks and bonds i.e. the portfolio process is a constant vector. In the light of this change the problem becomes analytically tractable. As a bargain we get a closed form solution of the value function. The optimization problem now becomes

$$V = \underset{\tau \in \Gamma}{Sup}\, E_{Y(0)=y}[e^{-\rho \tau} Y(\tau)] \tag{3.8}$$

where $Y(t)$ follows the process $dY(t) = \overline{\mu} dt + \overline{\sigma} dZ(t)$. The HJB equation associated with this problem is given by

$$Max\{\frac{1}{2}\overline{\sigma}^2 Q''(y) + \overline{\mu} Q'(y) - \rho Q(y),\ y - Q(y)\} = 0 \text{ , for all } y \text{ a.s on R} \tag{3.9}$$

where $\overline{\sigma} = \sigma p$, $\overline{\mu} = \mu p + r_0(1-p) - \frac{1}{2}[\sigma p]^2$. The closed form solution of the value function is given by

$$V(y) = \begin{cases} \frac{1}{\alpha_1} e^{\alpha_1 (y - \frac{1}{\alpha_1})} \text{ for } y \in (-\infty, \frac{1}{\alpha_1}] \\ y \text{ for } y \geq \frac{1}{\alpha_1} \end{cases} \tag{3.10}$$

i.e. the optimal threshold is given by $\frac{1}{\alpha_1}$ where α_1 is given by

$$\alpha_1 = \frac{-[\overline{\mu}] + \sqrt{\overline{\mu}^2 + 2\rho\overline{\sigma}^2}}{\overline{\sigma}^2} = \frac{-[\mu p + r_0(1-p) - \frac{1}{2}\sigma^2 p^2] + \sqrt{\overline{\mu}^2 + 2\rho\overline{\sigma}^2}}{\overline{\sigma}^2}.$$

We now analyze how the optimal threshold change as we change the parameters of the problem. The effect of r_0 on the optimal threshold is studied by looking at the sign of the derivative $\frac{\partial \alpha_1}{\partial r_0}$, where $\frac{\partial \alpha_1}{\partial r_0} = -\frac{(1-p)}{\overline{\sigma}^2} + \frac{1}{2\sqrt{\overline{\mu}^2 + 2\rho\overline{\sigma}^2}}(2\overline{\mu}(-p))$. Note that as r_0 increases, the 1st term in the numerator decreases and simultaneously the term under the square root increases. The net effect of a change in r_0 on α_1 depends on which effect dominates. We can not say a priori what would be the effect of a change in interest rate on the optimal threshold. Similar argument holds for the effect of p, μ, σ, on α_1.

3.4 Bankruptcy Model.

3.4.0.1 Jump to default diffusion process.

To accommodate a defaultable risky asset in our problem it is necessary to use new mathematical techniques.

In this section, we consider the problem of an investor who wants to invest his wealth in stocks and bonds in such a manner so as to maximize his expected utility before "quitting" from the stock market taking into account the fact that his wealth can be subject to bankruptcy. First, we formally introduce the information structure and the SDE associated with the pre-default process. As usual, we begin with a probability space $(\Omega, \mathscr{F}, \mathbb{Q})$ carrying a standard Brownian motion $\{Z(t), t \geq 0\}$ and an exponential random variable e with parameter 1 ($e \sim \exp(1)$) independent of $Z(t)$. We assume frictionless markets, no arbitrage, and take an equivalent martingale measure (EMM) $\mathbb{Q}$ as given. The pre-default wealth process under the risk neutral measure is modelled as a diffusion process $\{W_t, t \geq 0\}$ solving a SDE given by

$$dW(t) = W(t)[(\mu_1(W(t)) + h(W(t))dt + \sigma_1(W(t))dZ(t)] \tag{3.11}$$

where $W_0 = w$. where $\mu_1(\cdot) + h(\cdot)$ can be interpreted as the intensity adjusted drift, $\sigma_1(\cdot)$ the diffusion coefficient and $h(\cdot)$ the default intensity (hazard intensity) that satisfies the following assumption. Existence and uniqueness of a solution to (3.11) is guaranteed by Theorem 2.9 in Karatzas and Shreve (page 289, 1991).

Assumption 1. We assume that $h(\cdot) \in C^1(0,\infty)$ is a non-negative Borel measurable function that is strictly decreasing and satisfies the following conditions:

$$\lim_{w \to 0^+} h(w) = \infty, \ \lim_{w \to \infty} h(w) = 0.$$

The assumption on the hazard rate seems to be counter-intuitive. As the wealth process tends towards zero the hazard rate increases to infinity. When the wealth process increases the default intensity goes to zero. The above assumption on $h(\cdot)$ generates a non-exploding solution to (3.11) on $(0,\infty)$ with both 0 and ∞ inaccessible boundaries.

We model the random time of default τ_0 as the first time when the process $\int_0^t h(W(u))du$ is greater than or equal to the independent random variable e where $e \sim \exp(1)$. Formally, it is captured by

$$\tau_0 = \begin{cases} \inf\{t \geq 0 : \int_0^t h(W(u))du \geq e\} \\ \infty, \text{ if the above set is empty} \end{cases} \tag{3.12}$$

Note τ_0 need not be $\mathscr{F}_t$ measurable since e is independent of $\mathscr{F}_t$. It turned out τ_0 is an unpredictable stopping time with respect to the enlarged filtration $\mathscr{G}_t$ defined below. At the time of default τ_0 the wealth process jumps to the default state Δ, in the terminology of Markov processes it is known as a cementary state. In otherwords in order to model the wealth process subject to bankruptcy as a diffusion process $\{W_t^{\triangle}, t \geq 0\}$ we extend the state space to $E^{\triangle} = (0,\infty) \cup \{\Delta\}$ where

$$W_t^{\triangle} = \begin{cases} W_t \text{ for } t < \tau_0 \\ \triangle \text{ for } t \geq \tau_0 \end{cases} \tag{3.13}$$

In order to explain how information is modelled, we follow (Elliot et al 2000) to introduce a bankruptcy jump indicator process $\{N_t, t \geq 0\}$, $N_t = 1_{[t \geq \tau_0]}$. We denote $H = \{H_t,\ t \geq 0\}$, the filtration generated by the Brownian motion, $Z(t)$, $t \geq 0$. We introduce by $\mathscr{G}_t = \mathscr{F}_t \vee H_t = \sigma\langle \mathscr{F}_t \cup H_t \rangle$ the enlarged filtration. It contains all the information in both $W(t)$ and the jump process. Following Linetsky (2004, 2006) we can write the wealth process subject to bankruptcy as

$$dW_t^{\triangle} = W_{t-}^{\triangle}[(\mu_1(W(t)) + h(W(t))dt + \widetilde{\sigma}(W(t))dZ(t) - dM_t]$$

where

$$M_t = N_t - \int_0^t h(W(u))1_{[\tau_0 > u]}du$$

is a $\mathscr{G}_t$ martingale. It is clear that $(e^{-\rho t}M_t)$ is a $\mathscr{G}_t$ martingale.

3.4.1 Problem Formulation.

We consider our risk averse individual to invest in stocks and bonds in such a manner that his expected utility at the chosen quitting time is maximized over all such available quitting times. The information regarding the wealth process available to the investor at time $t > 0$ is contained in the $\sigma-$algebra $\mathscr{F}_t$. Thus τ_0 is a stopping time but not predictable with respect to $\mathscr{G}_t$. We assume there is no transaction costs.

From the above discussion, it follows that the investor's optimization exercise is to find an $\mathscr{F}_t$ stopping time τ^* which maximizes $E[e^{-\rho\tau}\ln W_t^{\triangle} 1_{[\tau<\tau_0]}]$ over all $\mathscr{F}_t$ stopping times τ. Formally we have

$$\overline{U}(w) = \underset{\tau\in\Gamma}{Sup}\, E_{W(0)=w}[e^{-\rho\tau}\ln W_\tau^{\triangle} 1_{[\tau<\tau_0]}] \tag{3.14}$$

where $W_t^{\triangle}$ is the wealth process with default and Γ is the space of all $\mathscr{F}_t$ stopping time τ. Note when the wealth process is not subject to default we have solved the problem qualitatively in section 1. In order to facilitate the optimization exercise we invoke a result from Meng (2007) that

Proposition 3.1 (page 6). For any finite ($\mathscr{F}_t$) stopping time τ we have $E[1_{[\tau<\tau_0]} \mid \mathscr{F}_t] = e^{-\int_0^\tau h(W(u))du}$ where τ_0 is the default time. Using this we can derive an alternative expression of the value function in the following manner. Note we can write $E_w[e^{-\rho\tau}\ln W_t^{\triangle} 1_{[\tau<\tau_0]}] = E_w[e^{-\rho\tau}\ln W_\tau 1_{[\tau<\tau_0]}] = E_w[e^{-\rho\tau}\ln W_\tau E_w[1_{[\tau<\tau_0]} \mid \mathscr{F}_t] = E_w[e^{-\int_0^\tau(\rho+h(W(u))du}\ln(W(\tau))]$. Thus we have reduced the problem from an optimal stopping problem with default to an optimal stopping problem for processes without default. This implies that our optimization exercise becomes

$$\overline{U}(w) = \underset{\tau\in\Gamma}{Sup}\, E_w[e^{-\int_0^\tau(\rho+h(W(u))du}\ln(W(\tau))]$$

For analytical simplicity we make the following transformation $Y(t) = \ln(W(t))$. Applying Ito's Lemma to the process $Y(t)$ we get the SDE for $Y(t)$ as

$$dY(t) = [\widetilde{\mu}(Y(t)) + \psi(Y(t))]dt + \widetilde{\sigma}(Y(t))dZ(t)] \tag{3.15}$$

with $Y(0) = \ln(W(0))$ where $\lim_{y\to\infty}\psi(y) = 0$, and $\lim_{y\to-\infty}\psi(y) = \infty$. The reformulated problem is stated below

$$V(y) = \underset{\tau\in\Gamma}{Sup}\, E_y[e^{-\int_0^\tau(\rho+\psi(u))du}Y(\tau)] \tag{3.16}$$

where the transformed pre-default process $Y(t)$ satisfies (3.15). Next, we introduce a verification lemma. It can be effectively used to calculate an achievable lower bound for the value function.

3.4.2 Verification Lemma for the bankruptcy model.

Following the theory of stochastic control, we can formulate the Hamilton-Jacobi-Bellman equation associated with the reformulated optimal stopping problem with bankruptcy in (3.15) as

$$Max\{\frac{1}{2}\widetilde{\sigma}^2(y)Q''(y)+(\widetilde{\mu}(y)+\psi(y))Q'(y)-(\rho+\psi(y))Q(y),\ y-Q(y)\}\ =0 \text{ , for all } y \text{ a.s on R} \quad (3.17)$$

It turned out that value function is the only function that is a " smooth solution" to the HJB equation. In the following proofs we use $Q(\cdot)$ to represent the solution to the ODE.

Lemma 3.2. *Let $Q(y)$ be a non-negative function satisfying the following conditions:*

(i) Q is continuously differentiable and Q'' also continuous except at finitely many points.

(ii) Q is a solution to the HJB equation

(ii) $\lim_{c^-} Q''(y)$ *and* $\lim_{c^+} Q''(y)$ *exits and are finite for all c on R.*

Then,

$Q(y) \geq V(y)$ for all $y \in R$, where V is the value function defined above.

Proof. The differential operator associated with the pre default process $(Y(t))$ is given by

$$\mathscr{L} = \frac{1}{2}\widetilde{\sigma}^2(y)\frac{d^2}{dy^2}+(\widetilde{\mu}(y)+\psi(y))\frac{d}{dy}-(\rho+\psi(y)) \quad (3.18)$$

for each $y \in R$. For each n we introduce a sequence of stopping times τ_n by

$$\tau_n = \begin{cases} \inf\{t \geq 0 : Y(t) \notin (-\infty, n], \\ \infty, \text{ if the above set is empty} \end{cases}$$

Define $\theta = \tau_n \wedge \tau \wedge N$. Clearly $\theta \to \tau$ a.e P as $n \to \infty$ and N tends to ∞. Applying Ito's Lemma, to $e^{-\int_0^t(\rho+\psi(u))du}Q(Y(t))$ where $Q(y)$ is C^2 we obtain

$$\begin{aligned} E_y[e^{-\rho\int_0^\theta((\rho+\psi(Y(u))du)}Q(Y(s))] &= Q(y)+E_y[\int_0^\theta e^{-\int_0^u(\rho+\psi(Y(s))ds}\mathscr{L}Q(Y(u))du] \\ &\leq Q(y) \end{aligned}$$

Since $Y(u)$ is bounded on $[0, \tau \wedge \tau_n \wedge N]$ and $\widetilde{\sigma}$ and $Q^{'}$ are continuous functions, the integrand in the stochastic integral is bounded and hence it is a mean zero martingale. The last inequality follows from the fact that $\mathscr{Q} \leq 0$ as Q satisfies (3.17). Since Q is a nonnegative function applying Fatou's Lemma we obtain

$$E_y[e^{-\rho\int_0^\tau((\rho+\psi(Y(s))ds)}Q(Y(\tau))] \leq \liminf_{n\to\infty} E_y[e^{-\rho\int_0^\theta((\rho+\psi(Y(u))du)}Q(Y(\tau\wedge\tau_n\wedge N))] \leq Q(y)$$

for each $y \in R$. Taking supremum over all $\mathscr{F}_t$ stopping times we obtain $V(y) \leq Q(y)$ for all $y \in R$. Note if $\psi(y)$ is identically zero , then we are back to the case of optimal stopping problem without bankruptcy.

3.4.3 Optimal selling time in the presence of bankruptcy.

In this section, we provide a road map of what to do. We want to show that the optimal selling time of the stock is of the threshold type $[y^*,\infty)$ when the wealth process is subject to bankruptcy. This y^* determines the free boundary for (3.17). The proof of the above claim can be approached via the following steps. First we assume that there exists a point y^* and a function $\widehat{Q}(y)$ satisfying the conditions in (3.19) below. Then we construct a function Q^* which is a smooth fit solution to the HJB equation. We further show that y^* is the optimal threshold. From now on we assume the existence of a point y^* and a non-negative function $\widehat{Q}(y)$ which satisfy the following conditions

$$\left.\begin{aligned} &(i)\ \widehat{Q}: R \to (0,\infty), \mathscr{L}\widehat{Q}(y) = 0, forall\ y \in R, \widehat{Q}(y^*) = y^*, \widehat{Q}(-\infty) = 0 \\ &(ii)\ \widehat{Q}(y) > y, \text{ for all } y < y^*,\ \widehat{Q}'(y^*) = 1, \widehat{Q}''(y^*) > 0 \end{aligned}\right\} \tag{3.19}$$

Existence of such a pair $(\widehat{Q}, y^*)$ will be proved in the appendix. We describe the main result here depending on the existence of this pair $(\widehat{Q}, y^*)$.

Next we introduce the function $Q^*(y)$ by

$$Q^*(y) = \begin{cases} \widehat{Q}(y) \text{ for } y \leq y^* \\ y^* \text{ for } y \geq y^* \end{cases} \tag{3.20}$$

The following theorem shows that $Q^*(y) = V(y)$.

Theorem 3.2. *Let $Q^*(y)$ be as defined above. Then we claim the following to hold.*

(i) $Q^*(y)$ is a C^1 function on R and C^2 on $R/\{y^*\}$ that satisfies the HJB equation.

(ii) if $\tau_y^* = \begin{cases} \inf\{t \geq 0 : Y(t) \notin (-\infty, y^*] \\ \infty, \text{ if the above set is empty} \end{cases}$

Then τ^* *is an optimal stopping time and* $V(y) = Q^*(y)$ *for all* $y \in R$, *where* V *is the value function.*

Proof. The proof of part (i) that $Q^*(y)$ is C^1-function which is C^2-function everywhere except at y^* is straight forward. To show that $Q^*(y)$ satisfies the HJB equation, we perform the following trick. We divide R into two disjoint intervals $(-\infty, y^*)$, and (y^*, ∞). Then separately show that $Q^*(y)$ satisfies the HJB equation on both the intervals. On the set $(-\infty, y^*)$, $Q^* = \widehat{Q}$ by (3.19) therefore $\mathscr{L}_Y Q^* = L_Y \widehat{Q} = 0$ and by construction $y - \widehat{Q}(y) < 0$. Thus on the open interval $(-\infty, y^*)$, $Max\ \{\mathscr{L}_Y Q^*, y - \widehat{Q}(y)\} = 0$, hence Q^* satisfies the HJB equation on this interval. To verify it on (y^*, ∞) we see that at y^*_-, $\mathscr{L}_Y Q^*(y^*_-) = \frac{1}{2}\widetilde{\sigma}^2(y^*_-)\widehat{Q}''(y^*_-) + (\widetilde{\mu}(y^*_-) + \psi(y^*_-))\widehat{Q}'(y^*_-) - (\rho + \psi(y^*_-))y^*_- = 0$. Since $\widehat{Q}''(y^*_-) > 0$ therefore $\widehat{Q}''(y^*_-) > 0$ (by continuity), $\widehat{Q}'(y^*_-) = 1$ and $\widetilde{Q}(y^*_-) \simeq y^*_-$. Incorporating these facts it follows from $\mathscr{L}_Y Q^*(y^*_-) = 0$ that $(\widetilde{\mu}(y^*_-) + \psi(y^*_-)) - (\rho + \psi(y^*_-))y^*_- < 0$, at y^*_-. By continuity it follows that $[\widetilde{\mu}(y^*) + \psi(y^*)] - (\rho + \psi(y^*))y^* < 0$ at y^*. For $y > y^*$, we need to show that $\mathscr{L}_Y Q^*(y) = (\widetilde{\mu}(y) + \psi(y)) - (\rho + \psi(y))y < 0$. If we can show that $(\widetilde{\mu}(y) + \psi(y)) - (\rho + \psi(y))y < [\widetilde{\mu}(y^*) + \psi(y^*)] - (\rho + \psi(y^*))y^*$ then $\mathscr{L}_Y Q^*(y) < 0$ follows. Rearranging, we get $[\widetilde{\mu}(y) - \widetilde{\mu}(y^*)] + \psi(y)[1-y] - \psi(y^*)[1-y^*] - \rho(y - y^*)$. We have to show it is less than 0. Note since $\psi(\cdot)$ is a decreasing function and $\widetilde{\mu}(\cdot)$ is Lipshcitz continuous we are done. Thus for $y > y^*$, $y = Q^*(y)$ and $\mathscr{L}_Y Q^*(y) < 0$. Thus $Q^*(y)$ satisfies the HJB equation on the interval (y^*, ∞). This completes the proof of part (i).

We can use the proof just outlined along with Lemma 3 to conclude that $Q^*(y) \geq V(y)$ for all $y \in R$. It remains to show that $Q^*(y) \leq V(y)$ for all $y \in R$ there by completing the proof of part (ii). We first consider the domain $(-\infty, y^*)$. Now Q^* is a C^1, function on R and it is C^2 everywhere except at y^*. But $Q^{*\prime\prime}(y^*_-)$ and $Q^{*\prime\prime}(y^*_+)$ exists and are finite. The classical Ito's rule can not be applied, but luckily we have generalizations of Ito's rule that do not require functions to be C^2 everywhere. Applying the extended Ito's rule (page 218, Karatzas-Shreve, 1991) to $e^{-\rho \int_0^t (\rho + \psi(s))ds} Q^*(Y(t))$, we have

$$E_y[Q^*(Y(N \wedge \tau^*))e^{-\int_0^{N \wedge \tau^*}(\rho + \psi(u))du}] = Q^*(y) + E_y[\int_0^{N \wedge \tau^*} L_Y Q^*(Y(u))e^{-\int_0^{N \wedge \tau^*}(\rho + \psi(s))ds} du] \tag{3.21}$$

Define $k = N \wedge \tau^*$, clearly as N tends to ∞, $k \to \tau^*$ a.e P. Since $Y(t)$ is bounded on $[0, N \wedge \tau^*]$ and both Q' and $\widetilde{\sigma}$ are continuous functions it follows that the integrand in the stochastic integral is bounded and hence it is a mean zero martingale. Now for $y < y^*$, $\mathscr{L}Q^*(y) = 0$ and $Q^*(y^*) = y^*$. Letting N tend, to infinity we have

$$Q^*(y) = E_y[Y(\tau^*)e^{-\int_0^{\tau^*}(\rho+\psi(u))du}]$$

Since τ^* is a $\mathscr{F}_t$ stopping time and $V(y)$ is obtained by taking maximum over Γ it follows that $Q^*(y) \le V(y)$ for all $y < y^*$. For the case $y \ge y^*$, $Q^*(y) = y$ and by definition $\tau^* = 0$. We can still write $Q^*(y) = Q^*(y) = E_y[e^{-\rho\tau^*}Y(\tau^*)]$. Hence $Q^*(y) \le V(y)$ for all $y \in R$. This completes the proof that $Q^*(y) = V(y)$ for all y and τ^* is an optimal stopping time.

Our next objective is to show the very existence of a function $\widehat{Q}(y)$ and an associated free boundary point y^* which satisfy (3.19). This is shown in Theorem 12 (Appendix b). We also show that such a pair $(\widehat{Q}(y), y^*)$ exists.

3.5 Comparison of Value Functions

In this section we trace out the effect of a change in the default intensity function $\psi(\cdot)$ on the optimal threshold y^* and in the value function $V(\cdot)$. To identify this dependency, corresponding to the default intensity ψ we label the optimal threshold as y^*_ψ and the value function as $V_\psi(\cdot)$ respectively. When the asset is not subject to bankruptcy , i.e. when the default intensity is identically zero, the optimal exercise boundary and the value functions are denoted by y^*_0 and $V_0(\cdot)$ respectively.

Theorem 3.3. *Let us suppose that there are two default intensity functions such as $\psi_1(y)$ and $\psi_2(y)$ such that $\lim_{y\to\infty}\psi_i(y) = 0$ and $\lim_{y\to-\infty}\psi_i(y) = \infty$, further let $y^*_{\psi_i}$ and $V_{\psi_i}(\cdot)$ be the optimal exercise boundary and the corresponding value function respectively. Then the following results hold:*

(i) If $\psi_1(y) > \psi_2(y)\ \forall\ y \in R$, then $y^*_{\psi_2} < y^*_{\psi_1}$ and $V_{\psi_2} \le V_{\psi_1}$ for all $y \in R$. *Moreover, $V_{\psi_2} < V_{\psi_1}$ in the open interval $(-\infty, y^*_{\psi_1})$.*

(ii) Let y^*_0 and $V_0(\cdot)$ be the optimal exercise boundary and the value function when there is no default (i.e $\psi(\cdot)$ is identically zero). Then $y^*_0 < y^*_\psi$ and *$V_0(y) \le V_\psi(y)$ for y and $V_0(y) < V_\psi(y)$ in the open interval $(0, y^*_\psi)$ where $\psi(y)$ is any default intensity function which satisfies $\lim_{y\to\infty}\psi(y) = 0$ and*

$\lim_{y\to-\infty}\psi(y)=\infty$. *Here y^*_ψ and $V_\psi(\cdot)$ represents the corresponding optimal exercise boundary and the value function respectively.*

Proof. Corresponding to each default intensity $\psi_i(\cdot)$ where $i=1,2$, let $\widehat{Q}_{\psi_i}(\cdot)$ be the function used in (3.18) and part (ii) of Theorem 13. We make the following transformation $y=\log z$ and define $\widehat{Q}_{\psi_i}(z)=Q_{\psi_i}(\log z)$. Note $Q_{\psi_i}\geq y$ for all $y\in R$ and $Q_{\psi_i}=y^*_{\psi_i}$. Thus it follows that $\widehat{Q}_{\psi_i}(z)\geq \log z$ for all $z\in(0,\infty)$ and $\widehat{Q}_{\psi_i}(z^*_{\psi_i})=\log z^*_{\psi_i}$ for $i\in\{1,2\}$. We prove part (i) by contradiction.

Let us assume $y^*_{\psi_2}\geq y^*_{\psi_1}$. This implies that $z^*_{\psi_2}\geq z^*_{\psi_1}$. Define $\eta(z)=\widehat{Q}_{\psi_1}(z)-\widehat{Q}_{\psi_2}(z)$. Since $\widehat{Q}_{\psi_1}(z^*_{\psi_1})=\ln z^*_{\psi_1}\leq \widehat{Q}_{\psi_2}(z^*_{\psi_1})$ and $\widehat{Q}_{\psi_2}(z^*_{\psi_2})=\ln z^*_{\psi_2}\leq \widehat{Q}_{\psi_1}(z^*_{\psi_2})$. Thus $\eta(z^*_{\psi_1})\leq 0$ at $z^*_{\psi_1}$ and $\eta(z^*_{\psi_2})\geq 0$. It follows from intermediate value theorem that there exist a $\xi\in[z^*_{\psi_1},z^*_{\psi_2}]$ such that $\eta(\xi)=0$. We intend to obtain a contradiction by proving that there does not exist such a ξ.

Case I Suppose $\xi>z^*_{\psi_1}$. Let $\widehat{\mathscr{L}_{\psi_i}}(\cdot)$ be the differential operator associated with the function $\psi_i(\cdot)$ after we make the transformation $y=\log z$. It can be easily shown that it has the following expression.

$$\widehat{\mathscr{L}_{\psi_i}}(\cdot)=\frac{1}{2}\widehat{\sigma}^2(z)z^2\frac{d^2}{dz^2}+(\frac{1}{2}\widehat{\sigma}^2(z)+\widehat{\mu}(z)+\widehat{\psi}_i(z))z\frac{d}{dz}-(\rho+\widehat{\psi}_i(z)) \tag{3.22}$$

where $\widehat{\sigma}^2(z)=\widetilde{\sigma}^2(\log z)$, $\widehat{\mu}(z)=\widetilde{\mu}(\log z)$, $\widehat{\psi}_i(z)=\psi_i(\log z)$. Clearly $\widehat{\mathscr{L}_{\psi_i}}\widehat{Q}_{\psi_i}(z)=0$ for all $z\in[0,\xi]$

and

$$\widehat{\mathscr{L}_{\psi_2}}\widehat{Q}_{\psi_1}(z)=\widehat{\mathscr{L}_{\psi_1}}\widehat{Q}_{\psi_1}(z)+(\psi_1-\psi_2)(\widehat{Q}_{\psi_1}-z\widehat{Q}'_{\psi_1}) \tag{3.23}$$

$$=(\psi_1-\psi_2)(\widehat{Q}_{\psi_1}-z\widehat{Q}'_{\psi_1}) \tag{3.24}$$

We know that $\psi_1(z)>\psi_2(z)$ but to determine the sign of $(\widehat{Q}_{\psi_1}-z\widehat{Q}'_{\psi_1})$ we do the following trick. Perform the transformation $z=e^y$ and we are back to the operator $\mathscr{L}_{\psi_i}$ as in (3.18). Note $\widehat{\mathscr{L}_{\psi_2}}\widehat{Q}_{\psi_1}(z)=\mathscr{L}_{\psi_2}Q_{\psi_1}(y)=(\psi_2-\psi_1)(Q'_{\psi_1}(y)-Q_{\psi_1}(y))$. Thus the sign of $\widehat{\mathscr{L}_{\psi_2}}\widehat{Q}_{\psi_1}$ depends on the sign of $(Q'_{\psi_1}(y)-Q_{\psi_1}(y))$. It can be shown via Lemma 19 (see appendix (b)) that if $\rho>\mu_1$, then $(Q'_{\psi_1}(y)-Q_{\psi_1}(y))$ is strictly positive. Thus it follows that $\widehat{\mathscr{L}_{\psi_2}}\widehat{Q}_{\psi_1}(z)$ is <0 for all $z\in[0,\xi]$. By applying $\widehat{\mathscr{L}_{\psi_2}}$ on $\eta(z)$ on the interval $[0,\xi]$ we see that $\mathscr{L}_{\psi_2}\eta(z)<0$. Thus it follows from maximum principle that $\widehat{Q}_{\psi_1}(z)>\widehat{Q}_{\psi_2}(z)$ on the open interval $(0,\xi)$. If $\xi>z^*_{\psi_1}$ we arrive at a contradiction since $\widehat{Q}_{\psi_1}(z^*_{\psi_1})=\ln z^*_{\psi_1}\leq\widehat{Q}_{\psi_2}(z^*_{\psi_1})$.

Case 2. If $\xi = z^*_{\psi_1}$, then $\widehat{Q}_{\psi_1}(\xi) = \widehat{Q}_{\psi_2}(\xi) = \ln\xi$ and $\widehat{Q}'_{\psi_2}(\xi) = \frac{1}{\xi}$ and $\widehat{Q}'_{\psi_1}(\xi) = \frac{1}{\xi}$. Therefore $\widehat{\mathscr{L}_{\psi_2}}(\widehat{Q}_{\psi_1} - \widehat{Q}_{\psi_2})(\xi) = \frac{\widehat{\sigma}^2}{2}\xi^2(\widehat{Q}_{\psi_1} - \widehat{Q}_{\psi_2})''(\xi) < 0$. This implies $\widehat{Q}_{\psi_1}(z) < \widehat{Q}_{\psi_2}(z)$ on the open interval $(\xi - \delta, \xi)$ for some δ. This is a contradiction to the fact that $\widehat{Q}_{\psi_1}(z) > \widehat{Q}_{\psi_2}(z)$ on $(0, \xi)$. Thus we can say that $z^*_{\psi_2} < z^*_{\psi_1}$ and there does not exist a $\xi \geq z^*_{\psi_1}$ for which $\widehat{Q}_{\psi_1}(\xi) < \widehat{Q}_{\psi_2}(\xi)$. By Theorem 18 (see appendix (b)), it follows that $V_{\psi_2} = Q_{\psi_2}(y)$ in the interval $(-\infty, z^*_{\psi_2}]$ and $V_{\psi_1} = Q_{\psi_1}(y)$ in the interval $(-\infty, z^*_{\psi_1}]$. Thus $V_{\psi_2}(y) \leq V_{\psi_1}(y)$ This completes the proof of part (i).

The proof of part (ii) is essentially the same with the relevant differential operator being $\mathscr{L}_0 = \frac{1}{2}\sigma^2(y)\frac{d^2}{dy^2} + \widehat{\mu}(y)\frac{d}{dy} - \rho$ instead of $\widehat{\mathscr{L}_{\psi_2}}$ above. Thus, we omit the details. This concludes the proof of the theorem.

3.6 Remark

The economic interpretation of the above result is important from a practitioner's point of view. Let us consider an investor who has the option to invest in stocks with different intensities of default. Then the optimal strategy for the investor would be to invest in the stock which has a higher intensity of default, since the value associated with it is much higher than the one with less probability of default. The incorporation of default intensity adds linearly to the drift coefficient of the SDE governing the wealth process of the investor there by increasing the expected earnings from the stock. Thus the investor has an incentive to keep the asset longer. On the contrary, the discount factor is replaced by the intensity-adjusted discount factor $(\rho + h(W(s)))$ and hence the investor tends to sell the asset earlier. The above Theorem 5 asserts that a stock with a higher intensity of default, the positive effect accruing from the increase in the drift coefficient more than offsets the negative effect arising from the rise in discount factor. This explains why a higher default intensity leads to a higher optimal exercise boundary and to a higher value function. Also, optimal exercise boundary and the value function in the default case are both higher than their counterparts in the no default model.

3.7 Optimal Portfolio Selection

In this section, we consider the portfolio problem of an investor trading in different (financial) assets when his wealth may be subject to bankruptcy. To be more precise, an investor, with a given initial wealth w_0, has to decide how many shares of which asset he should hold at each instant to maximize his expected utility from terminal wealth. Our goal is to deliver a solution method for the portfolio problem. The objective of the agent is to maximize expected utility over the class of all stopping times and admissible self-financing portfolio processes. Thus the optimization exercise of the agent can be formulated as

$$\widetilde{V}(y) = \underset{p \in U}{Sup}\, V_p(y) \tag{3.25}$$

$$\text{where } V_p(y) = \underset{\tau \in \Gamma}{Sup}\, E_{Y(0)=y}[e^{-\int_0^\tau (\rho + \psi(u))du} Y(\tau)] \tag{3.26}$$

subject to the SDE governing the wealth process (see 3.15). Here

$$U = \{p_n : R \longrightarrow [\varepsilon, 1] \text{ such that } p_n\text{'s satisfy } \mid p_n(x) - p_n(y) \mid \leq c \mid x - y \mid\} \tag{3.27}$$

where $c > 0$ is a priori given fixed constant. $p_n(y)$ represent the investment strategy of the investor. The space U is endowed with the norm $\| p_n(y) \| = \underset{y \in R}{Sup}\; p_n(y)$. We show that there exist an optimal portfolio process $p^* \in U$ that achieves the supremum in (3.26). In order to prove this result we introduce Lemma 6, Lemma 7 and Lemma 8. We first show that the problem in (3.26) is well-defined i.e $\widetilde{V}(y) < \infty$. This is captured in Lemma 6.

Lemma 3.3. *$\widetilde{V}(y) < \infty$ for all $y \in R$ and $M = \underset{p \in U}{Sup}\; y_p^* < \infty$, where y_p^*, satisfies $y_p^* = Q_p(y_p^*)$, $Q_p'(y_p^*) = 1$, $\mathscr{L}_p Q_p(y_p^*) = 0$.*

Proof. Since $\widetilde{\mu}(y)$ is bounded therefore it follows $\widetilde{\mu}(y) \leq \mu_0$ for all $y \in R$. The SDE corresponding to μ_0 is given by

$$dY_0(t) = \mu_0 dt + \widetilde{\sigma}(Y_0(t))dZ(t) \tag{3.28}$$

By a Comparison Theorem in Karaztaz- Shreve (page 293, 1991) it follows that $Y(t) \leq Y_0(t)$. Now, for each fixed τ, we have $E_y[e^{-\int_0^\tau (\rho+\psi(u))du} Y(\tau)] \leq E_y[e^{-\int_0^\tau (\rho+\psi(u))du} Y_0(\tau)]$. Taking supremum over all τ it follows $\underset{\tau \in \Gamma}{Sup}\; E_y[e^{-\int_0^\tau (\rho+\psi(u))du} Y(\tau)] \leq \underset{\tau \in \Gamma}{Sup}\; E_y[e^{-\int_0^\tau (\rho+\psi(u))du} Y_0(\tau)]$. We thus have $V_p(y) \leq V_p^0(y)$. If we can show $V_p^0(y) < \infty$ then we are done. It is well known that , via a random time change method (see Theorem 4.6 Karatzas and Shreve, 1991) one can write the process $Y_0(t)$ as

$$Y_0(t) = y_0 + \mu_0 t + Z(\int_0^t \sigma_0^2(p^2(Y_0(s)))ds)$$
$$= y_0 + \mu_0 t + \widehat{Z}(t)$$

It follows that

$$\underset{[0,c_0\tau]}{Sup}\ Y_0(t) \leq \underset{[0,c_0\tau]}{Sup}\ (y_0 + \mu_0 t + \widehat{Z}(t))$$
$$= y_0 + \mu_0 c_0 \tau + \underset{[0,c_0\tau]}{Sup}\ \widehat{Z}(t)$$

Taking expectations over both sides we obtain

$$E_y[e^{-\int_0^\tau(\rho+\psi(u))du}\underset{[0,c_0\tau]}{Sup}\ Y_0(t)] \leq y_0 + \mu_0 c_0 E_y[e^{-\int_0^\tau(\rho+\psi(u))du}\tau] + E_y[e^{-\int_0^\tau(\rho+\psi(u))du}\underset{[0,c_0\tau]}{Sup}\ |\ \widehat{Z}(t)\ |] \quad (3.29)$$

We intend to show that the right hand side of (3.29) is finite. Clearly the first two terms are finite. For the last term notice that

$$E_y[e^{-\int_0^\tau(\rho+\psi(u))du}\underset{[0,c_0\tau]}{Sup}\ |\ \widehat{Z}(t)\ |] = E_y[\sum_{n=0}^{\infty} 1_{[n\leq\tau\leq n+1]} e^{-\int_0^n(\rho+\psi(u))du}\underset{[0,c_0\tau]}{Sup}\ |\ \widehat{Z}(t)\ |]$$
$$\leq \sum_{n=0}^{\infty} e^{-\int_0^n(\rho+\psi(u))du}\sqrt{E_y[1_{[n\leq\tau\leq n+1]}]E_y[\underset{[0,c_0(n+1)]}{Sup}\ |\ \widehat{Z}(t)\ |^2]}$$

Where the last inequality follows from Cauchy -Scwartz inequality. By Dobb's L^2 inequality, it follows that $E_y[\underset{[0,c_0(n+1)]}{Sup}\ |\ \widehat{Z}(t)\ |^2] \leq [c_0(n+1)]^2$. Using this the right hand side becomes $\sum_{n=0}^{\infty} e^{-\rho n}[c_0(n+1)]^2 < \infty$. Thus $V_p^0(y) < \infty$. Thus $V_p(y) \leq V_p^0(y) < \infty$. Taking supremum over all p's we get $\widetilde{V}(y) < \infty$ for all $y \in R$. By definition $V_p(y_p^*) = y_p^*$. Since $\underset{p\in U}{Sup}\ V_p(y) < \infty$ we get $\underset{p\in U}{Sup}\ V_p(y_p^*) < \infty$. This implies $\underset{p\in U}{Sup}$ $y_p^* < \infty$. Thus $M < \infty$, where $M = \underset{p\in U}{Sup}\ y_p^*$. This completes the proof.

The next Lemma guarantees the continuity of $V_p(y)$ in p.

Lemma 3.4. *Let* $\{Q_{p_n}\}_{n\geq1}$ *and* $\{p_n\}_{n\geq1} \in U$ *be sequences such that*

(i) $\lim_{n\longrightarrow\infty} p_n = p_0$

(ii) $\mathscr{L}_n Q_{p_n} = 0$, $Q_{p_n}(y_{p_n}) = y_{p_n}$ *and* $\lim_{y \longrightarrow -\infty} Q_{p_n}(y) = 0$,

(iii) $\lim_{k \longrightarrow \infty} y_{p_{n_k}} = y_{p_0}$

where $\mathscr{L}_n = \frac{1}{2}\widetilde{\sigma}_n^2(y)\frac{d^2}{dy^2} + (\widetilde{\mu}_{p_n}(y) + \psi(y))\frac{d}{dy} - (\rho + \psi(y))$ *then*

a) $\lim_{n \longrightarrow \infty} Q_{p_n} = Q_{p_0}$.

b) $\mathscr{L}_0 Q_{p_0} = 0$, $Q_{p_0}(y_{p_0}) = y_{p_0}$ *and* $\lim_{y \longrightarrow -\infty} Q_{p_0}(y) = 0$, $Q'_{p_0}(y_{p_0}) = 1$.

$\mathscr{L}_0 = \frac{1}{2}\widetilde{\sigma}_0^2(y)\frac{d^2}{dy^2} + (\widetilde{\mu}_{p_0}(y) + \psi(y))\frac{d}{dy} - (\rho + \psi(y))$ *where* $\widetilde{\sigma}_0^2(y) = \lim_{n \longrightarrow \infty} \widetilde{\sigma}_{p_n}^2(y)$ *and* $\widetilde{\mu}_0(y) = \lim_{n \longrightarrow \infty} \widetilde{\mu}_{p_n}(y)$.

Proof. We begin by giving a proof of part (a). Notice $\mathscr{L}_n Q_{p_n}(y) = 0$ implies $\frac{1}{2}\widetilde{\sigma}_n^2(y)Q''_n(y) + (\widetilde{\mu}_n(y) + \psi(y))Q'_n(y) - (\rho + \psi(y))Q_n(y) = 0$. Rearranging terms this can be rewritten as

$$Q''_{p_n}(y) + \beta_n(y)Q'_n(y) = \gamma_n Q_n(y) \tag{3.30}$$

where $\beta_n(y) = \frac{2(\widetilde{\mu}_n + \psi)}{\widetilde{\sigma}_n^2}$, $\gamma_n(y) = \frac{2(\rho+\psi)}{\widetilde{\sigma}_n^2}$. Now (3.30) can be written as $\frac{d}{dy}(e^{\int_0^x \beta_n(y)dy} Q'_{p_n}(y)) = \gamma_n(y)Q_{p_n}(y)e^{\int_0^x \beta_n(y)dy}$. Integrating out we get

$$e^{\int_0^x \beta_n(y)dy} Q'_{p_n}(y) = \int_{-\infty}^{x} \gamma_n(y)Q_{p_n}(y)e^{\int_0^x \beta_n(y)dy} du\,, \tag{3.31}$$

since $\lim_{z \longrightarrow -\infty} e^{\int_0^z \beta_n(y)dy} Q'_{p_{n_k}}(z) = 0$. By Helly's selection criterion there exist a sub-sequence $\{Q_{p_{n_k}}\}_{k \geq 1}$ such that $\lim_{k \longrightarrow \infty} Q_{p_{n_k}} = \widetilde{Q}_{p_0}$. We would later show that $\widetilde{Q}_{p_0} = Q_{p_0}$. We now prove the continuity of $\widetilde{Q}_{p_0}$. It follows from (3.31) that

$$e^{\int_0^x \beta_{n_k}(y)dy} Q'_{p_{n_k}}(y) = \int_{-\infty}^{x} \gamma_{n_k}(y)Q_{p_{n_k}}(y)e^{\int_0^x \beta_{n_k}(y)dy} du$$

As k tends to ∞, by Helly's Theorem and the convergence of β_{n_k} and γ_{n_k} to β_0 and γ_0 respectively it follows that $\{Q'_{p_{n_k}}(y)\}_{k \geq 1}$ converges to a function pointwise. We denote this function by $g(y)$ where $\lim_{k \longrightarrow \infty} Q'_{p_{n_k}}(y) = g(y)$. The limit exists and is finite. We would show $g(y) = \widetilde{Q}'_{p_0}(y)$.

Integrating (3.30) between (y, M) we get

$$Q'_{p_{n_k}}(M) - Q'_{p_{n_k}}(y) = \int_{y}^{M} (\gamma_{n_k}(u)Q_{p_{n_k}}(u) - \beta_{n_k}(u)Q'_{p_{n_k}}(u))du \tag{3.32}$$

Taking limit both sides as k tends to ∞ we get

$$1-g(y)=\int_y^M(\gamma_0(u)\widetilde{Q}_{p_0}(u)-\beta_0(u)g(u))du \tag{3.33}$$

The L.H.S of the above equation is finite. This implies that R.H.S is Riemann Integrable and by differentiating (3.32), we get $-g(y)=-(\gamma_0(y)\widetilde{Q}_{p_0}(y)-\beta_0(y)g(y))$. By integrating (3.32) on $[y,M]$ we obtain

$$Q'_{p_{n_k}}(M)\int_y^M du-\int_y^M dQ_{p_{n_k}}(u)=\int_y^M\int_u^M(\gamma_{n_k}(r)Q_{p_{n_k}}(r)-\beta_{n_k}(r)Q'_{p_{n_k}}(r))dr.$$

Using (3.33), it follows that

$$Q'_{p_{n_k}}(M)(M-y)-[Q_{p_{n_k}}(M)-Q_{p_{n_k}}(y)]=\int_y^M(1-g(u))du$$

Using $Q_{p_{n_k}}(M)=M$ and $Q'_{p_{n_k}}(M)=1$ and taking limit we get $(M-y)-[M-\widetilde{Q}_{p_0}(y)]=\int_y^M(1-g(u))du$. Since the L.H.S is finite, by differentiation, it follows that $-1+\widetilde{Q}'_{p_0}(y)=-1+g(y)$. Thus we obtain $g(y)=\widetilde{Q}'_{p_0}(y)$. Hence $\widetilde{Q}_{p_0}(y)$ is continuous. Next we intend to show, $\lim_{k\longrightarrow\infty}Q'_{p_{n_k}}(y)=\widetilde{Q}'_{p_0}(y)$. By letting n tend to infinity it follows that $1-\widetilde{Q}'_{p_0}(y)=\int_y^M(\gamma_0(u)\widetilde{Q}_{p_0}(u)-\beta_0(u)\widetilde{Q}'_{p_0}(u))du$. Since the left hand side is finite it follows $\widetilde{Q}''_{p_0}(y)=\gamma_0(y)\widetilde{Q}_{p_0}(y)-\beta_0(y)\widetilde{Q}'_{p_0}(u)$. From (3.30), we have $Q''_{p_{n_k}}(y)+\beta_{n_k}(y)Q'_{n_k}(y)=\gamma_{n_k}Q_{n_k}(y)$. Taking limit as k tends to ∞ , and by Helly's selection criterion, we get $f(y)+\beta_0(y)\widetilde{Q}'_{p_0}(y)=\gamma_0\widetilde{Q}_{p_0}(y)$. From this we deduce $f(y)=\widetilde{Q}''_{p_0}(y)$. We now show $\mathscr{L}_0\widetilde{Q}_{p_0}=0$, $\widetilde{Q}_{p_0}(y_{p_0})=y_{p_0}$ and $\lim_{y\to-\infty}\widetilde{Q}_{p_0}(y)=0$. From (3.30) and the fact that $\lim_{k\longrightarrow\infty}Q_{p_{n_k}}(y)=\widetilde{Q}_{p_0}(y)$, $\lim_{k\longrightarrow\infty}Q'_{p_{n_k}}(y)=\widetilde{Q}'_{p_0}(y)$ and $\lim_{k\longrightarrow\infty}Q''_{p_{n_k}}(y)=\widetilde{Q}''_{p_0}(y)$ we infer $\mathscr{L}_0\widetilde{Q}_{p_0}=0$. From continuity of $Q'_{p_{n_k}}$ and $Q_{p_{n_k}}$ one can conclude that $\widetilde{Q}'_{p_0}(y_{p_0})=1$ and $\widetilde{Q}_{p_0}(y_{p_0})=y_{p_0}$ respectively. $\lim_{y\longrightarrow-\infty}\widetilde{Q}_{p_0}(y)=0$ follows from part (iii) of proposition (16) (see appendix (b)). By part (b) of theorem 18 and convexity of $\widetilde{Q}_{p_0}(y)$, it follows that $\widetilde{Q}_{p_0}(y)$ is a C^2 function which satisfies $\mathscr{L}_0y=0$, $\widetilde{Q}_{p_0}(y_{p_0})$, $\widetilde{Q}'_{p_0}(y_{p_0})=1$.

Finally we show $\widetilde{Q}_{p_0}=Q_{p_0}$. We define Q_{p_0} to be a solution of $\mathscr{L}_0Q_{p_0}=0$ such that $Q_{p_0}(y_{p_0})=y_{p_0}$, $Q'_{p_0}(y_{p_0})=1$ and $\lim_{y\longrightarrow-\infty}Q_{p_0}(y)=0$. By the uniqueness of solutions it follows that $Q_{p_0}(y)=\widetilde{Q}_{p_0}(y)$ establishing our claim.

Let U be as in (3.27). Define

$$S(y)=\{y_p:p\in U\}. \tag{3.34}$$

The next Lemma shows that $S(y)$ is a closed set.

Lemma 3.5. *The functional $\Lambda : U \longrightarrow R$, such that $\Lambda(p_n) = y_{p_n} = Q_{p_n}(y_{p_n})$ is continuous in U. Note U is endowed with the sup-norm $\|p(y)\| = \underset{y\in R}{Sup}\ p_n(y)$.*

Proof. Let $\lim\limits_{n\longrightarrow\infty} p_n = p_0$ we need to show $\lim\limits_{n\longrightarrow\infty} \Lambda(p_n) = \Lambda(p_0)$. We need the following facts to prove the claim. From Monotonicity of $(Q_{p_n})_{n\geq 1}$, in y variable it follows $Q_{p_n}(y_1) \geq Q_{p_n}(y_2)$ whenever $y_1 \geq y_2$. Hence $\lim\limits_{n\longrightarrow\infty} \widetilde{Q}_{p_n}(y_1) \geq \lim\limits_{n\longrightarrow\infty} \widetilde{Q}_{p_n}(y_2)$ and thus $\widetilde{Q}_{p_0}(y_1) \geq \widetilde{Q}_{p_0}(y_2)$ if $y_1 \geq y_2$, proving monotonicity of Q_{p_0} in y. Also Q_{p_0} is a smooth solution to the boundary value problem as established in Lemma 8. Since $(y_{p_n})_{n\geq 1}$ is bounded (as $0 < y_{p_n} \leq M$), by Bolzano-Weistressas theorem there exist a convergent subsequence $(y_{p_{n_k}})_{n\geq 1}$ such that $\lim\limits_{k\longrightarrow\infty} y_{p_{n_k}} = y_{p_0}$. If $(y_{\acute{p}_{n_k}})_{n\geq 1}$ be another sequence such that $\lim\limits_{k\longrightarrow\infty} y_{\acute{p}_{n_k}} = \acute{y}_{p_0}$ then $y_{p_0} = \acute{y}_{p_0}$ for the following reasons. Note $y_{\acute{p}_{n_k}} = Q_{p_{n_k}}(y_{p_{n_k}})$. By continuity of $Q_{p_{n_k}}$ it follows $\acute{y}_{p_0} = Q_{p_0}(y_{p_0})$. From monotonicity and smooth pasting property of Q_{p_0}, we infer $y_{p_0} = \acute{y}_{p_0}$. Since $\lim\limits_{k\longrightarrow\infty} y_{p_{n_k}} = y_{p_0}$ we conclude from the definition of Λ that $\lim\limits_{k\longrightarrow\infty} \Lambda(p_{n_k}) = \Lambda(p_0)$.

Corollary 3.1. *The set $S(y)$ in (3.34) is closed.*

Proof. Suppose $\lim\limits_{n\longrightarrow\infty} y_{p_n} = c$ and $p_n \in U$ then we need to show that there exist a $p_0 \in U$ such that $c = y_{p_0}$. By definition of Λ we see $\lim\limits_{k\longrightarrow\infty} \Lambda(p_n) = c$. By Arezala -Ascoli theorem (page 208, Royden) there exist a subsequence $\{p_{n_k}\}_{k\geq 1}$ such that $\lim\limits_{k\longrightarrow\infty} p_{n_k} = p_0$ and $p_0 \in U$. From continuity of Λ it follows that $\lim\limits_{n\longrightarrow\infty} \Lambda(p_n) = \Lambda(p_0)$. Hence $c = y_{p_0}$, which completes the proof.

Theorem 3.4. *The set $\Psi = \{V_p(y) : p \in U\}$ is closed. Let $M = Sup\ \{y_p : p \in U\}$, then $M = y_{p_*}$.*

Proof. We first prove that Ψ is closed. Let $\lim\limits_{n\longrightarrow\infty} V_{p_n}(y) = z$ and $p_n \in U$. We need to show there exist a $p_* \in U$ such that $z = V_{p_*}(y)$. By Arezala -Ascoli theorem there exist a subsequence $\{p_{n_k}\}_{k\geq 1}$ such that $\lim\limits_{k\longrightarrow\infty} p_{n_k} = p_*$ and $p_* \in U$. By continuity of $V_p(y)$, in p, $\lim\limits_{k\longrightarrow\infty} V_{p_{n_k}}(y) = V_{p_*}(y)$. Since the original sequence converges to a limit and the subsequence also has a limit it follows from elementary analysis that $z = V_{p_*}(y)$. This completes the proof that Ψ is closed. Now we prove $M = y_{p_*}$. By definition of supremum we have $M \geq y_{p_*}$. We need to show $M \leq y_{p_*}$. Note by definition $y_{p_*} = V_{p_*}(y_{p_*}) \geq V_p(y_{p_*}) \geq V_p(y_p) = y_p$. The first inequality follows from definition of supremum. Thus $y_{p_*} \geq y_p$ for any p. Taking supremum over both sides we get $y_{p_*} \geq M$. This completes the proof.

It immediately follows from the above theorem along with monotonicity of the value function that their exist an optimal portfolio p^*.

3.8 Conclusion

- We considered the continuous time portfolio optimization problem when an investor's wealth is subject to bankruptcy. We assumed that the price dynamics of the stock market is governed by a geometric Brownian motion. We have shown that the utility maximization problem may be separated into no-default and a with-default optimization subproblems, and proven verification theorems for both cases taking into account that the portfolio process is given. We showed that the optimal stopping time is of the threshold type for both the cases. We obtain closed form solution of the optimal threshold region and the value function when an agent invests a fixed proportion of his wealth in stock and rest in money market account.

- We also analyze the effect of default risk on the optimal exercise boundary as well as on the value function. Once we incorporate default risk into our model there are two opposing effects: first the default intensity will be added to the drift coefficient of the SDE which is favorable to the investor. Thus the investor has an incentive to keep the asset longer. On the other hand, the discount factor also increases by the same amount and hence the investor tends to sell the asset earlier. In this context, we establish some monotonicity results with respect to the default intensity. To be specific we show that higher default intensity leads to a higher optimal exercise boundary and to a higher value function. Moreover, optimal exercise boundary and the value function in the default case are both higher than their counterparts in the no-default model. Additionally, we established the existence of an optimal portfolio process.

3.9 Appendix

3.9.1 Appendix (a)

3.9.1.1 Existence of free boundary without bankruptcy.

In this section we present the technical results and details that would help us to prove the existence of a function $\widetilde{Q}(y)$ and an associated free boundary point y^* which satisfy (I) (see subsection 3.1). The existence and the uniqueness of the pair $(\widetilde{Q}(y), y^*)$ is established in Theorem 14. The free boundary y^* turned out to be the optimal threshold boundary for our problem. In order to accomplish our goal we follow the following steps:

(i) For a large d, the boundary value problem $\mathscr{L}Q_d(y) = 0$, $Q_d(y) = d$ and $\lim_{y\to-\infty} Q_d(y) = 0$ has a unique solution that has a stochastic representation given by (3.8).

(ii) Second we extend the solution to this differential equation over the entire real line.

(iii) Lastly, we show that $Q_d'(y) > 1$ for large $d > 0$ and the graph of $Q_d(y)$ intersects the line $y = x$ at least twice in the open interval $(0, d)$.

Once we have found such a function $Q_d(y)$ we take constant multiples of it to get a smooth solution which meets the line $y = x$ tangentially. In order to achieve our target for each $d > 0$ we introduce the function

$$Q_d(y) = dE_y[e^{-\rho\tau_d}] \tag{3.35}$$

where

$$\tau_d = \begin{cases} \inf\{t \geq 0 : Y(t) \notin (-\infty, d] \\ \infty, \text{ if the above set is empty} \end{cases} \tag{3.36}$$

To prove theorem 14, we introduce Lemma 11, Propositions 12 and 13. In the following Lemma we show that $Q_n(y)$ is the unique solution to the boundary value problem $\mathscr{L}Q_n(y) = 0$, for all y in the interval $(-n,\, d)$ $Q_n(d) = d$ and $Q_d(-n) = 0$.

Lemma 3.6. *Let $d > 0$ and $Y(t)$ satisfies (3). We now introduce the stopping times τ_n by*

$$\tau_n = \begin{cases} \inf\{t \geq 0 : Y(t) \notin [-n, d] \\ \infty, \textit{ if the above set is empty.} \end{cases} \tag{3.37}$$

Define

$$Q_n(y) = dE_y[e^{-\rho\tau_n} 1_{\{\omega : Y(\tau_n) = d\}}] \tag{3.38}$$

for each $y \leq d$. Then we get the following results:

(i) $Q_n(y)$ is the unique solution to the boundary value problem $\mathscr{L}Q_n(y) = 0$ for all $y \in (-n, d)$, $Q_n(d) = d$ and $Q_n(-n) = 0$.

(ii) $Q_n(y)$ has no local extrema and, $Q_n'(y) > 0$ on the interval $(-n, d)$.

(iii) for any fixed $y < d$, $\{Q_n(y)\}_{n=1}^{\infty}$ is strictly increasing in n.

(iv) $\lim_{n\to\infty} Q_n(y) = Q_d(y)$ for each $y \leq d$ where $Q_d(y)$ is given by $Q_d(y) = dE_y[e^{-\rho\tau_d}]$.

Proof. *(i)* Notice that since $\widetilde{\sigma}^2(y)$ and $\widetilde{\mu}(y)$ are both positive and continuous, we know from the theory of differential equations that there exists a unique solution to the boundary value problem $\mathscr{L}Q_n(y) = 0$, $Q_n(d) = d$, $Q_n(-n) = 0$. To show that the unique solution is given by $Q_n(y) = dE_y[e^{-\rho\tau_n} 1_{\{\omega : Y(\tau_n) = d\}}]$, we apply Ito's lemma to $e^{-\rho t} Q_n(Y(t))$ to get $E_y[e^{-\rho\tau_n} Q_n(Y(\tau_n))] = Q_n(y)$. This completes the proof of part (i).

(ii) It's clear from part (i) that $Q_n(y) \geq 0$ on $(-n, d)$. Continuity of $Q_n(y)$ is guaranteed since it is a solution to a second order ODE. Suppose there exist a local maximum at $\widehat{y}$ such that $Q_n'(\widehat{y}) = 0$, $\widehat{y} \in (-n, d)$. From $\mathscr{L}Q_n(y) = 0$ it follows that $\frac{1}{2}\widetilde{\sigma}^2(\widehat{y}) Q_d''(\widehat{y}) - \rho Q(\widehat{y}) = 0$, since ρ, $Q(\widehat{y})$, $\widetilde{\sigma}^2(\widehat{y})$ are strictly positive, it follows that $Q_d''(\widehat{y}) > 0$, contradicting the fact that $\widehat{y}$ is a local maximum. Similar argument hold for the case of a local minimum. Since $Q_n(-n) = 0$ and $Q_n(d) = d$ and $Q_n(y)$ is continuous with $Q_n(y) > 0$ (there doesn't exist any local max or min) it follows that $Q_n(y)$ is increasing over $(-n, d)$. Also note it is bounded above by d. This completes the proof of part (ii).

(iii) To show that the sequence $\{Q_n(y)\}_{n\geq 1}$ is strictly increasing on the open interval $(-\infty, d)$ it suffices to show it is true for any $y \in (-n, d)$. By induction, we can show that this holds for any $y < d$. By construction $Q_{n+1}(d) = Q_n(d) = d$. Note $Q_{n+1}(-n) > Q_n(-n) = 0$. Suppose not then $Q_{n+1}(-n) \leq 0$. This is not possible, since $Q_{n+1}(y) > 0$ for $y \in (-(n+1), d)$. Now $\mathscr{L}(Q_{n+1} - Q_n)(y) = 0$

on $y \in (-n, d)$. Invoking the Maximum principle of differential equations (Weinberger and Porter, 1984) we have $Q_{n+1} > Q_n$ for all $y \in (-n, d)$. Since n is arbitrary, it follows by induction that $Q_{n+1} > Q_n$, for $y < d$. This completes the part of (iii).

(iv) In order to complete the proof of part (iv) we do the following trick. We introduce the scale function associated with the diffusion $Y(t)$. Namely

$$S(y) = \int_0^y e^{\frac{-2}{\tilde{\sigma}^2}\int_0^z \tilde{\mu}(u)du} dz$$

It is well defined since $\tilde{\mu}(y)$ is bounded and $\tilde{\sigma}^2(y) > 0$. Now

$$P_y[Y_{\tau_n} = d] = \frac{S(y) - S(-n)}{S(d) - S(-n)} \tag{3.39}$$

Note $S(-n) = -\int_{-n}^0 e^{\frac{2}{\tilde{\sigma}^2}\int_z^0 \tilde{\mu}(u)du} dz$. Now $\tilde{\mu}(y) = \mu h(y) + r_0(1 - h(y)) - \frac{1}{2}[\sigma h(y)]^2$. Since $h(y)$ is bounded below by $\varepsilon > 0$ on $(-\infty, \infty)$, for large negative value of y, $\tilde{\mu}(y) \simeq \mu\varepsilon + r_0(1-\varepsilon) - \frac{1}{2}[\sigma\varepsilon]^2$. We assume $\tilde{\mu}(y) > 0$ Implicitly, this means $\varepsilon(\mu - r_0) + r_0 > \frac{1}{2}[\sigma\varepsilon]^2$. Thus for large n we have $S(-n) \simeq -\int_{-n}^0 e^{-\alpha z} dz$, $\alpha > 0$. Clearly as $\lim_{n\longrightarrow\infty} S(-n) = -\infty$. Hence $P_y[Y_{\tau_n} = d] \to 1$ as n tends to ∞. Thus it follows that $1_{[Y_{\tau_n}=d]} \to 1$ a.e P_y. Define $\phi_n = e^{-\rho\tau_n} 1_{\{\omega: Y(\tau_n)=d\}}]d$. Clearly the sequence $(\phi_n)_{n\geq 1}$ is increasing to ϕ, where $\phi = E_y[e^{-\rho\tau_d}]d$ and $0 \leq \phi_n \leq 1$. Thus by bounded convergence theorem it follows that $\lim_{n\rightharpoonup\infty} Q_n(y) = Q_d(y)$ for all $y \in R$.

The following proposition states some qualitative results regarding the function $Q_d(y)$.

Proposition 3.1. *Given $Q_d(y) = E_y[e^{-\rho\tau_d}]d$ for each $y \in (-\infty, d]$ the following results hold*

(i) $Q_d(y)$ satisfies $\mathscr{L}Q_d(y) = 0$ for all $y < d$.

(ii) $Q_d(y) > 0$ and $Q'_d(y) > 0$ for all $y < d$. Also $Q_d(y)$ is bounded on $(-\infty, d]$.

(iii) $Q_d(y)$ is the unique bounded solution to the boundary value problem $\mathscr{L}Q_d(y) = 0$ for all $y < d$ with the boundary conditions $Q_d(d) = d$ and $\lim_{y\to-\infty} Q_d(y) = 0$.

Proof. (i) We consider the sequence $\{Q_n(y)\}_{n\geq 1}$ introduced in Lemma 11 where $Q_n(y)$ has the stochastic representation given (3.38). By part (iv) of Lemma 11 we have

$$Q_\infty(y) \equiv \lim_{n\longrightarrow\infty} Q_n(y) = Q_d(y) \tag{3.40}$$

To obtain part (i), we write the differential equation for $Q_n(y)$ in integral form and show that $Q_n'(\cdot)$ as well as $Q_n''(\cdot)$ are convergent. Then we use (3.40) to establish that $Q_d(\cdot)$ also satisfy the same integral equation which leads to the conclusion of (i). Let us consider the family of solutions (Q_n) 's to second order linear ODE's indexed by n as $\frac{1}{2}\widetilde{\sigma}^2(y)Q_n''(y)+\widetilde{\mu}(y)Q_n'(y)-\rho Q_n(y)=0$ with boundary conditions $Q_n(-n)=0$, $Q_n(d)=d$. The above differential equation can be rewritten as

$$Q_n''(y)+b(y)Q_n'(y)-\alpha(y)Q_n(y)=0 \tag{3.41}$$

where $b(y)=\frac{2\widetilde{\mu}(y)}{\widetilde{\sigma}^2(y)}>0$, $\alpha(y)=\frac{2}{\widetilde{\sigma}^2(y)}>0$ and are bounded by $\overline{K}$ and K_0 respectively. If we can show that as n tends to ∞, $\lim\limits_{n\longrightarrow\infty}Q_n''(y)=Q_\infty''(y)$, $\lim\limits_{n\longrightarrow\infty}Q_n'(y)=Q_\infty'(y)$, $\lim\limits_{n\longrightarrow\infty}Q_n(y)=Q_\infty$ for all $y<d$, then we are done since $Q_\infty(y)=Q_d(y)$ by part (iv) of Lemma 11 (see Appendix (a)). We can write (3.41) in the following manner: $\frac{d}{dy}(e^{\int_c^y b(u)du}Q_n'(y))=\alpha(y)e^{\int_c^y b(u)du}Q_n(y)$. Integrating between y to d we have

$\int_y^d d(e^{\int_c^u b(z)dz}Q_n'(z))=\int_y^d \alpha(z)e^{\int_c^z b(u)du}Q_n(z)dz$ and this implies $e^{\int_c^d b(z)dz}\lim\limits_{n\to\infty}Q_n'(d)-e^{\int_c^y b(z)dz}\lim\limits_{n\to\infty}Q_n'(y)=\int_y^d \alpha(z)e^{\int_c^z b(u)du}\lim\limits_{n\to\infty}Q_n(z)dz$. Since $\{Q_n'(d)\}_{n=1}^\infty$ is a monotone decreasing sequence, which is bounded below, it converges and has a finite limit say λ_0. We know $Q_n(z)$ goes to $Q_d(z)$ as n goes to infinity, by part (iv) of Lemma 11. Thus $Q_n'(y)$ converges to a finite limit, say $f(y)$. Integrating (3.41) between (y,d), we have $Q_n'(d)-Q_n'(y)=\int_y^d(\alpha(u)Q_n(u)-b(u)Q_n'(u))du$. By letting n tends to ∞ we have $\lambda_0-f(y)=\int_y^d(\alpha(u)Q_\infty(u)-b(u)f(u))du$. Since the left hand side is finite by differentiation we have $-f'(y)=-(\alpha(y)Q_\infty(y)-b(y)f(y))$. Hence $Q_n'(d)-Q_n'(u)=\int_u^d(\alpha(r)Q_n(r)-b(r)Q_n'(r))dr$. By integrating this equation, we obtain

$$Q_n'(d)\int_y^d du-\int_y^d dQ_n(u)=\int_y^d\int_u^d(\alpha(r)Q_n(r)-b(r)Q_n'(r))drdu$$

Consequently, $Q_n'(d)(d-y)-[Q_n(d)-Q_n(y)]=\int_y^d(\lambda_0-Q_\infty'(y))du$. Letting n tending to infinity and using $Q_n(d)=d$, we have $Q_\infty(y)-d+\lambda_0(d-y)=\int_y^d(\lambda_0-f(y))du$. Thus $Q_\infty(y)$ is differentiable and we obtain $Q_\infty'(y)-\lambda_0=-\lambda_0+f(y)$, implying $f(y)=Q_\infty'(y)$ for all y. To see that $\lim\limits_{n\longrightarrow\infty}Q_n''(y)=Q_\infty''(y)$, we integrate (3.41) and let n tend to infinity to obtain $Q_\infty'(d)-Q_\infty'(y)=\int_y^d(\alpha(r)Q_\infty(r)-b(r)Q_\infty'(r))dr$. Consequently $Q_\infty''(y)=\alpha(y)Q_\infty(y)-b(y)Q_\infty'(y)$. Rearranging (3.41), we get $Q_n''(y)=\alpha(y)Q_n(y)-b(y)Q_n'(y)$. As n goes to infinity, we have $\lim\limits_{n\longrightarrow\infty}Q_n''(y)=g(y)$ the limit exists and is finite since $Q(y)$ and $Q_\infty'(y)$ are finite. Thus $g(y)=\alpha(y)Q_\infty(y)-b(y)Q_\infty'(y)$. Hence $g(y)=Q_\infty''(y)$ and this implies that $\lim_{n\to\infty} Q_n''(y) = Q_\infty''(y)$. From (3.41) upon taking limit we get $Q_\infty''(y) + b(y)Q_\infty'(y) - \alpha(y)Q_\infty(y) = 0$, proving $\mathscr{L}Q_d(y) = 0$ for all $y < d$. This completes the proof of part (i).

From expression (3.8), it is clear that $Q_d(y) > 0$. We claim $Q_d'(y) \geq 0$, i.e $Q_d(y_1) \geq Q_d(y_2)$ for $y_1 \geq y_2$. Suppose not, then $Q_d(y_1) < Q_d(y_2)$. Hence, by part (ii) (iii) and (iv) of Lemma 11 it follows that $Q_n(y_1) < Q_n(y_2)$ for large n. This contradicts the monotone increasing property of $Q_n(y)$. Suppose $Q_d'(\eta) = 0$ for some $\eta < d$. Then $\mathscr{L}Q_d(\eta) = 0$ implies that $Q_d''(\eta) > 0$ thus η is a local minimum for $Q_d(y)$. By Lemma 11, it follows that there exists a n, such that $Q_n(y)$ has a local minimum in the interval $(\eta - \delta, \eta + \delta)$. This contradicts with the part (ii) of Lemma 11. Hence $Q_d'(y) > 0$ for $y < d$. It's clear that $0 < Q_n(y) \leq Q_d(y) < Q_d(d) = d$. Thus, part (ii) is complete.

Since $Q_d(y)$ is non-negative and strictly increasing it follows that $\lim_{y\to-\infty} Q_d(y)$ exists and is finite. In the next proof we show $\lim_{y\to-\infty} Q_d(y) = 0$. (iii) Let us consider the boundary value problem, $\mathscr{L}_n Q_n(y) = 0$ where $\mathscr{L}_n = \frac{d^2}{dy^2} + b(y)\frac{d}{dy} - \alpha(y)$, $Q_n(-n) = 0$, $Q_n(d) = d$. Integrating on the interval $(-n, x)$, we have or,

$$Q_n'(x) - Q_n'(-n) + \int_{-n}^{x} b(y)Q_n'(y)dy - \rho \int_{-n}^{x} \alpha(y)Q_n(y)dy = 0.$$

Since $Q_n'(-n) \geq 0$, it follows that

$$Q_n'(x) + \int_{-n}^{x} b(y)Q_n'(y)dy \geq \rho \int_{-n}^{x} \alpha(y)Q_n(y)dy.$$

This implies $Q_n'(x) + \int_{-n}^{x} b(y)Q_n'(y)dy \geq \rho K_0 \int_{-n}^{x} Q_n(y)dy$, since $\alpha(y) \leq K_0$. Consequently,

$$\rho K_0 \int_{-n}^{x} Q_n(y)dy \leq Q_1'(x) + \overline{K} \int_{-n}^{x} Q_n'(y)dy$$

since $\{Q_n'(y)\}_{n=1}^{\infty}$ and $b(y)$ are bounded above by $Q_1'(x)$ and $\overline{K}$ respectively. Thus it follows that $\rho K_0 \int_{-n}^{x} Q_n(y)dy \leq Q_1'(d) + \overline{K}d < \infty$, (since $Q_n(x)$ is bounded by d). The LHS can be rewritten as $\rho K_0 \int_{-\infty}^{x} Q_n(y)1_{[-n,d]}dy < \infty$.Since $Q_n(y)1_{[-n,d]}$ is non-negative, and $\lim_{n\longrightarrow\infty} Q_n(y) = Q_\infty(y)$ by Fatou's lemma we obtain

$$\liminf_{n\to\infty} \int_{(-\infty,d]} \rho K_0 Q_n(y)1_{[-n,d]}dy \geq \int_{(-\infty,d]} \rho K_0 Q_\infty(y)dy \tag{3.42}$$

$Q_d(y) \equiv Q_\infty(y)$ by part (iv) of Lemma 11, it follows that $Q_1'(d) + \overline{K}d \geq \int_{(-\infty,d]} \rho K_0 Q_\infty(y)dy$ and thus $\lim_{y\longrightarrow-\infty} Q_d(y) = 0$. In order to prove the uniqueness, let $\widehat{Q}_d(y)$ be another bounded solution to the boundary value problem $\mathscr{L}Q_d(y) = 0$, $Q_d(d) = d$, $\lim_{y \to -\infty} Q_d(y) = 0$. Applying Ito's Lemma to it we get the same stochastic representation for $\widehat{Q}_d(y)$ as (3.8). Thus $\widehat{Q}_d(y) = Q_d(y)$ for all $y \in (-\infty, d]$. Q.E.D.

We now extend each $Q_d(y)$ to $(-\infty, \infty)$ so that it satisfies the differential equation $\mathscr{L}Q_d(y) = 0$ for all $y \in R$. Each such $Q_d(y)$ is positive and strictly increasing over R. Also $Q_d(d) = d$ and $\lim_{y \to -\infty} Q_d(y) = 0$. The proofs of these facts runs along similar lines to that in part (ii) of Lemma (11). In the next proposition, we show that there exists a curve $Q_{d_1}(y)$ among the family of solutions to $\mathscr{L}Q_d(y) = 0$ such that it intersects the line $y = x$ at least twice.

Proposition 3.2. *There exists a function $Q_{d_1}(y)$ such that it intersects the line $y = x$ at least twice in the interval $(0, d_*]$. Moreover, $Q'_{d_1}(d_*) > 1$, $Q_{d_1}(d_1) = d_1$, and $Q_{d_1}(y) < y$ for all y in the interval $(0, d_*]$.*

Proof. Let us consider the ODE, $\frac{1}{2}\widetilde{\sigma}^2(y)Q''_d(y) + \widetilde{\mu}(y)Q'_d(y) - \rho Q_d(y) = 0$,where $\widetilde{\mu}(y) = \mu h(y) + r_0(1 - h(y)) - \frac{1}{2}[\sigma h(y)]^2$. This can be written as $\widetilde{\mu}(\cdot) = \mu z + r_0(1 - z) - \frac{1}{2}[\sigma z]^2$. This is a quadratic in z. Now $\widetilde{\mu}(\cdot)$ is bounded above by K_2 and since $Q'_d(y) > 0$ it follows that

$$\frac{1}{2}\sigma^2 h^2(y)Q''_d(y) + K_2 Q'_d(y) - \rho Q_d(y) \geq 0$$

or $Q''_d(y) + \frac{2K_2}{\sigma^2 h^2(y)}Q'_d(y) - \frac{2\rho}{\sigma^2 h^2(y)}Q_d(y) \geq 0$ for all $y \in R$, implying $Q''_d(y) + C_3 Q'_d(y) - C_4 Q_d(y) \geq 0$, where $C_3 = \frac{2K_2}{\sigma^2 \varepsilon_0^2}$, $C_4 = \frac{2\rho}{\sigma^2 \varepsilon_0^2}$, and $\varepsilon_0 = \inf_{y \in R} h(y)$. Multiplying both sides with the integrating factor $e^{C_3 y}$ we get $\frac{d}{dy}(Q'_d(y)e^{C_3 y}) \geq C_4 e^{C_3 y} Q_d(y)$. Integrating between 0 to y we have $Q'_d(y)e^{C_3 y} - Q'_d(0) \geq C_4 \int_0^y e^{C_3 s} Q_d(s) ds$. Since $Q_d(s)$ is increasing therefore $Q_d(s) > Q_d(0)$. Thus $Q'_d(y)e^{C_3 y} - Q'_d(0) \geq C_4 Q_d(0)[\frac{e^{C_3 y}}{C_3} - \frac{1}{C_3}]$. Rearranging terms we get $Q'_d(y)e^{C_3 y} \geq Q'_d(0) + \frac{C_4}{C_3}Q_d(0)(e^{C_3 y} - 1)$. Thus $Q'_d(y) \geq e^{-C_3 y}[Q'_d(0) - \frac{C_4}{C_3}Q_d(0)] + \frac{C_4}{C_3}Q_d(0)$ for all y. As y goes to infinity we have $Q'_d(y) \geq \frac{C_4}{C_3}Q_d(0)$. We can choose a d say d_1, such that $Q_{d_1}(0) > \frac{C_3}{C_4}$ and thus $Q'_{d_1}(y) > 1$. Thus there exists a point d_* such that $Q_{d_1}(y) = d_1$. and $Q'_{d_1}(d_*) > 1$. Since slope of $Q_{d_1}(y)$ at d_1 is less than 1 and $Q_{d_1}(y) > 0$, $Q'_{d_1}(y) > 0$ it follows that the graph of $Q_{d_1}(y)$ intersects the line $y = x$ at least twice in the interval $(0, d_*]$. This completes the proof.

Once we have established the existence of a curve like $Q_{d_1}(y)$ that intersects the line y at least twice we are almost done. We now take constant multiples of it until we reach a point of tangency between one such curve and the line $y = x$. This determines the optimal threshold or the free boundary. The next proposition aptly demonstrates this fact.

Theorem 3.5. *(i) There exists a point $y^* \in (d_1, d_*)$ and a strictly positive function $Q_{y^*}(y)$ defined on R such that $Q_{y^*}(y^*) = y^*$, $Q_{y^*}(y) > y$ for all $y \in R/\{y^*\}$ and $Q'_{y^*}(y) = 1$, $\mathscr{L}Q_{y^*}(y) = 0$ for all $y \in R$.*

(ii) *There exists a function $\widetilde{Q} : R \to R_+$ that satisfies all the conditions of (I) (see subsection 3.1). Moreover the threshold y^* and the function $\widetilde{Q}$ are unique.*

Proof. (i) Step1: Here we intend to show that there exists a curve $Q_{y_1}(y)$ that lies above $Q_{d_1}(y)$ and intersects the $y = x$ line twice.

Let us consider the family of curves, $\{tQ_{d_1}(y) : t \geq 1\}$. By continuity of $Q_{d_1}(y)$ there exists a $\delta > 0$ such that for all $d_* - \delta < y < d_*$, we have $Q_{d_1}(y) < y$. From here it follows that there exists a $t_1 > 1$ and a point $y_2 \in (d_1, d_*)$ such that $t_1 Q_{d_1}(y_2) = y_2$ and $t_1 Q'_{d_1}(y_2) > 1$. Also since $Q_{d_1}(d_1) > 0$ and $Q_{d_1}(y) > y$, for all y less than d_1, therefore there exist a δ_0 and a point $y_1 \in (d_1, d_*)$ such that $t_1 Q_{d_1}(y_1) = y_1$, where $y_1 = d_1 + \delta_0$.Thus the graph of $Q_{y_1}(y)$ intersects the line $y = x$ straight line at lest twice in the interval (d_1, d_*).

Step 2: Here we show the existence of the optimal threshold y^* and the function $Q_{y^*}(y)$ that is tangent to the straight line $y = x$. Let's define $t^* = \inf\{t \geq 1$ such that $tQ_{d_1}(y) > y$ for all $y \in (d_1, d_*)\}$. The infimum is well defined as the set is bounded below by 1 and $t_2 = \frac{d_*}{d_1}$ is greater than 1 clearly belongs to the set. Note $t_1 < t < t_2$. We claim now $t^* Q_{d_1}(y^*) = y^*$ where $y^* = \inf\{y : y \in (d_1, y_2)$ and $t^* Q_{d_1}(y) = y\}$. Suppose not, then either $t^* Q_{d_1}(y^*) > y^*$, or $t^* Q_{d_1}(y^*) > y^*$ (not possible by construction). Since $t^* Q_{d_1}(y^*) > y^*$ therefore there exist $\widetilde{t} < t^*$ such that $\widetilde{t} Q_{d_1}(y^*) > y^*$, violating the definition of infimum, hence we have proved $t^* Q_{d_1}(y^*) = y^*$. (b) $t^* Q'_{d_1}(y^*) = 1$. Suppose not then $t^* Q'_{d_1}(y^*) < 1$ or greater than 1. If less than 1 then the function $t^* Q_{d_1}(y)$ intersects the line $y = x$ at y^* and dips down. By continuity, there exists an interval $(y^*, \delta_2) \subset (d_1, d_*)$ such that $t^* Q_{d_1}(y) < y$, (contradiction to the definition of t^*). Similar argument negates the other case. Thus we have shown the existence of the function $Q_{y^*}(y) = t^* Q_{d_1}(y)$ and the optimal threshold y^*. To show that $\mathscr{L}Q_{y^*}(y) = 0$. Note that $\mathscr{L}Q_{y^*}(y) = \mathscr{L}(t^* Q_{d_1}(y)) = t^* \mathscr{L}Q_{d_1}(y) = 0$, since $\mathscr{L}Q_{d_1}(y) = 0$. This completes the proof of part (i).

(ii) In order to complete the proof of part (ii) we simply define $\widetilde{Q}(y) = Q_{y^*}(y)$ for $y \in R$ and y^* as defined above. Clearly $Q_{y^*}(y)$ and y^* satisfies all the conditions in (I) (see subsection 3.1). The uniqueness of $\widetilde{Q}(y)$ and y^* are straightforward. Let there be two thresholds y_1^* and y_2^* corresponding to the two functions $\widetilde{Q}_1(y)$ and $\widetilde{Q}_2(y)$ such that $y_1^* < y_2^*$. Clearly $\widetilde{Q}_2(y_1^*) > \widetilde{Q}_1(y_1^*)$ since $Q(y)$ is monotone increasing and solutions of ODE don't intersect more than twice. By Theorem 2 and (3.7), it follows that $V(y_1^*) = \widetilde{Q}_2(y_1^*)$, $V(y_1^*) = \widetilde{Q}_1(y_1^*)$, contradicting the initial assumption $\widetilde{Q}_2(y_1^*) > \widetilde{Q}_1(y_1^*)$ Q.E.D.

3.9.2 Appendix (b)

3.9.2.1 Existence of free boundary for the bankruptcy model.

In this section,we present the technical details that would help us to prove the very existence of a function $\widehat{Q}(y)$ and an associated free boundary point y^* which satisfy (3.7) (see subsection 2.3). The existence and the uniqueness of the pair $(\widetilde{Q}(y), y^*)$ is established in Theorem 18 where $\widetilde{Q}(\cdot)$ satisfies the differential equation. The free boundary point y^* which satisfy (3.19) would evolve as an optimal threshold boundary for our problem. In order to accomplish our goal, we follow the following steps:

(i) We show that for large b, the boundary value problem $\mathscr{L}Q_b(y) = 0$, $Q_b(y) = b$ and $\lim_{y \to -\infty} Q_b(y) = 0$ has a unique solution that has a stochastic representation given by (3.43) .

(ii) We extend the solution $Q_b(y)$ over the entire real line.

(ii)Here we show that $Q_b'(y) > 1$ for large $b > 0$ and the graph of $Q_b(y)$ intersects the line $y = x$ at least twice in the open interval $(0, d_*)$.

Once we have found out our candidate $Q_b(y)$ we take constant multiples of it to get a smooth fit solution. In order to achieve our target for each $b > 0$ we introduce the function $Q_b(y)$ defined on $(-\infty, b]$ as

$$Q_b(y) = bE_y[e^{-\int_0^{\tau_b}(\rho+\psi(u))du}] \tag{3.43}$$

where

$$\tau_b = \begin{cases} \inf\{t \geq 0 : Y(t) \notin (-\infty, b] \\ \infty, \text{ if the above set is empty} \end{cases} \tag{3.44}$$

In the next Lemma we show some qualitative properties of the function $Q_b(y)$. We also show that $Q_b(y)$ is the unique solution to the boundary value problem $\mathscr{L}Q_b(y) = 0$, for all $y < b$, $Q_b(b) = b$ and $\lim_{y \to -\infty} Q_b(y) = 0$.

Lemma 3.7. *Let $b > 0$ and $Y(t)$ satisfies (3.15). We now introduce the stopping times τ_n by*

$$\tau_n = \begin{cases} \inf\{t \geq 0 : Y(t) \notin [-n, b] \\ \infty, \textit{ if the above set is empty} \end{cases} \tag{3.45}$$

Define

$$Q_n(y) = bE_y[e^{-\rho \int_0^{\tau_n}(\rho+\psi(Y(s))ds} 1_{\{\omega : Y(\tau_n)=b\}}] \tag{3.46}$$

for each $y \leq b$. Then we get the following results:

(i) $Q_n(y)$ is the unique solution to the boundary value problem $\mathscr{L}Q_n(y) = 0$ for all $y \in (-n, b)$, $Q_n(b) = b$ and $Q_n(-n) = 0$.

(ii) $Q_n(y)$ has no local extrema and, $Q_n'(y) > 0$ on the interval $(-n, b)$.

(iii) for any fixed $y < b$, $\{Q_n(y)\}_{n=1}^{\infty}$ is strictly increasing in n.

(iv) $\lim_{n\to\infty} Q_n(y) = Q_b(y)$ for each $y \leq b$ where $Q_b(y)$ is given by (3.43).

Proof. Since $\widetilde{\sigma}^2(y)$, $(\widetilde{\mu}(y) + \psi(y))$ are continuous, and $\rho + \psi(y) > 0$, we know from the theory of differential equations that there exists a unique solution to the boundary value problem $\mathscr{L}Q_n(y) = 0$, $Q_n(b) = b$, $Q_n(-n) = 0$. This unique solution is given by $Q_n(y) = bE_y[e^{-\rho \int_0^{\tau_n}(\rho+\psi(Y(s))ds} 1_{\{\omega : Y(\tau_n)=b\}}]$. It can be verified by applying Ito's lemma to $e^{-\rho \int_0^t(\rho+\psi(Y(s))ds} Q_n(Y(t))$. This completes the proof of part (i).

(ii) It's clear from part (i) that $Q_n(y) \geq 0$ on $(-n, b)$. Continuity of $Q_n(y)$ is guaranteed since it is a solution to a second order ODE. Suppose there exist a local maximum at $\widetilde{y}$ such that $Q_n'(\widetilde{y}) = 0$, $\widetilde{y} \in (-n, b)$. From $\mathscr{L}Q_n(y) = 0$ it follows that $\frac{1}{2}\widetilde{\sigma}^2(\widetilde{y})Q_b''(\widetilde{y}) - (\rho + \psi(\widetilde{y}))Q(\widetilde{y}) = 0$, since ρ, $Q(\widetilde{y})$, $\widetilde{\sigma}^2(\widetilde{y})$ and $\psi(\widetilde{y})$ are all strictly positive it follows that $Q_b''(\widetilde{y}) > 0$, contradicting the fact that $\widetilde{y}$ is a local maximum. Similar argument holds for the case of a local minimum. Since $Q_n(-n) = 0$ and $Q_n(b) = b$ and $Q_n(y)$ is continuous with $Q_n(y) > 0$, (since there doesn't exist any local max or min) it follows that $Q_n(y)$ is increasing over $(-n, b)$. Also note it is bounded above by b. This completes the proof of part (ii).

(iv) To show that $Q_n(y)$'s are strictly increasing on the open interval $(-\infty, b)$ it suffices to show it is true for any $y \in (-n, b)$. By induction we can show that this holds for any $y < b$. By construction $Q_{n+1}(b) = Q_n(b) = b$. Let us verify $Q_{n+1}(-n) > Q_n(-n) = 0$. We prove by contradiction. Suppose $Q_{n+1}(-n) \leq 0$ This is not possible, since $Q_{n+1}(y) > 0$ for $-(n+1) < y < b$. Now $\mathscr{L}(Q_{n+1} - Q_n)(y) = 0$

on $y \in (-n,b)$. Invoking the Maximum principle, we have $Q_{n+1} > Q_n$ for all $y \in (-n,b)$. Since n was arbitrary by induction it follows that $Q_{n+1} > Q_n$, for $y < b$.

(iv) In order to complete the proof of part (iv) we do the following trick. We introduce the scale function $S(\cdot)$ associated with the diffusion $Y(t)$. Namely

$$S(y) = \int_0^y e^{\frac{-2}{\widetilde{\sigma}^2}\int_0^z (\widetilde{\mu}(u)+\psi(u))du} dz$$

thus

$$P_y[Y_{\tau_n} = b] = \frac{S(y) - S(-n)}{S(b) - S(-n)} \tag{3.47}$$

Since $\lim_{y\to-\infty} \psi(y) = \infty$ it follows that $\lim_{n\to\infty} S(-n) = -\infty$. and thus by (3.47) we have $\lim_{n\to\infty} P_y[Y_{\tau_n} = b]$. Let $\varphi_n = be^{-\rho\int_0^{\tau_n}(\rho+\psi(Y(s))ds} 1_{\{Y(\tau_n)=b\}}$. Then clearly φ_n increases to φ a.e P_y, where $\varphi = e^{-\int_0^{\tau_b}(\rho+\psi(u))du} b$. Also $0 \le \varphi_n \le 1$. Thus by bounded convergence theorem we have $Q_n(y)$ goes to $Q_b(y)$ for every $y < b$. In the next proposition, we derive some properties of $Q_b(y)$.

Proposition 3.3. *Given $Q_d(y) = bE_y[e^{-\int_0^{\tau_b}(\rho+\psi(u))du}]$ for each $y \in (-\infty,b]$, $y > 0$ then the following results hold*

(i) $Q_b(y)$ satisfies $\mathscr{L}Q_b(y) = 0$ for all $y < b$.

(ii) $Q_b(y) > 0$ and $Q_b'(y) > 0$ for all $y < b$. Also $Q_b(y)$ is bounded on $(-\infty,b]$.

(iii) $Q_b(y)$ is the unique bounded solution to the boundary value problem $\mathscr{L}Q_b(y) = 0$ for all $y < b$ with the boundary conditions $Q_b(b) = b$ and $\lim_{y\to-\infty} Q_b(y) = 0$.

Proof: The proof of part (i) is almost similar to part(i) of proposition 12.

(ii) This is similar to that in proposition 12 and thus omitted.

(iii) Note $Q_b(y)$ is non-negative and strictly increasing thus $\lim_{y\to-\infty} Q_b(y)$ exists and is finite, say ℓ. We have to show that $l = 0$. In order to show $\lim_{y\to-\infty} Q_b(y) = 0$, we do the following trick. Note $\mathscr{L}Q_b(y) = 0$ can be written as $Q''(y) + \frac{2(\widetilde{\mu}(y)+\psi(y))}{\widetilde{\sigma}^2(y)} Q'(y) - \frac{2(\rho+\psi(y))}{\widetilde{\sigma}^2(y)} Q(y) = 0$. Since $\widetilde{\mu}(y)$ and $\widetilde{\sigma}^2(y)$ are bounded and $\psi(y)$ is strictly increasing it follows that $\frac{2(\widetilde{\mu}(y)+\psi(y))}{\widetilde{\sigma}^2(y)} \ge M > 0$, for all $y \le y_0$. Hence it follows that $Q''(y) + MQ'(y) \le \frac{2(\rho+\psi(y))}{\widetilde{\sigma}^2(y)} Q(y)$, for all $y \le y_0$. Since $\widetilde{\sigma}^2(y) \ge \delta_0$, (follows from the boundedness criterion) it follows that $Q''(y) + MQ'(y) \le \delta_0(\rho + \psi(y))Q(y)$, for all $y \le y_0$. Since $\widetilde{\mu}(y) < K_2$ we have $Q''(y) + [K_2 + \psi(y)]Q'(y) \ge \delta_0(\rho + \psi(y))Q(y)$ or $\frac{Q''(y)}{[K_2+\psi(y)]} + Q'(y) \ge \delta_0 \frac{(\rho+\psi(y))}{[K_2+\psi(y)]} Q(y)$. By integration, we see that $\int_y^{y_0} \frac{Q''(x)}{[K_2+\psi(x)]}dx + Q(y_0) \geq l + l(y_0 - y)$. Here we used the fact that as y takes very large negative value $Q(y) \simeq l$. Note that

$$\begin{aligned}
&Q(y_0) + \int_y^{y_0} \frac{Q''(x)}{[K_2+\psi(x)]}dx \\
&= Q(y_0) + \frac{Q'(y_0)}{[K_2+\psi(y_0)]} - \frac{Q'(y)}{[K_2+\psi(y)]} - \int_y^{y_0} Q'(u)\frac{d}{du}(\frac{1}{[K_2+\psi(u)]})du \\
&\geq l + l(y_0 - y)
\end{aligned}$$

Consequently,

$$Q(y_0) + \frac{Q'(y_0)}{[K_2+\psi(y_0)]} \geq \frac{Q'(y)}{[K_2+\psi(y)]} + \int_y^{y_0} Q'(u)\frac{d}{du}(\frac{1}{[K_2+\psi(u)]})du + l + l(y_0 - y)$$

Note the left hand side is a finite positive number, as y goes to $-\infty$ the right hand side tends to infinity, a contradiction. Thus $\ell = 0$. This implies that $\lim_{y\to-\infty} Q_b(y) = 0$. Q.E.D.

Our next goal is to extend each $Q_b(y)$ to R so that it satisfies the differential equation $\mathscr{L}Q_b(y) = 0$ for all $y \in R$. Each such $Q_b(y)$ is positive and strictly increasing over R. Also $Q_b(b) = d$ and $\lim_{y\to-\infty} Q_b(y) = 0$. The proofs of these facts runs along similar lines to that in part (ii) of Lemma 11. In the next proposition we show that there exists a curve $Q_{b_1}(y)$ among the family of solutions to $\mathscr{L}Q_b(y) = 0$ such that it intersects the line $y = x$ at least twice in the interval $(0, b_*)$ where b_* is defined below.

Proposition 3.4. *There exists a function $Q_{b_1}(y)$ such that it intersects the line $y = x$ at least twice in the interval $(0, b_*]$ where $Q'_{b_1}(b_*) > 1$, $Q_{b_1}(b_*) = b_*$, $y \in (b_1, b_*)$, and $Q_{b_1}(y) < y$.*

Proof. We first give a brief outline of how to proceed. Since $Q_b(b) = b$, each function $Q_b(\cdot)$ has one common point with the line $y = b$. Next we intend to find a point $b_* = \inf\{y : Q_b(y) \geq y$ and $Q'_b(y) > 1\}$. Having obtained a point like b_* we show that the curve $Q_b(y)$ intersects the line $y = x$ twice in the interval $(0, b_*]$

Step 1: Let b_1 be the first point of intersection of the curve $Q_{b_1}(y)$ with the $y = x$ line. Since $Q_{b_1}(y)$ is increasing therefore at b_* slope of $Q_{b_1}(y)$ is less than 1. By continuity there exists a point y_0 to be defined below such that $Q_{b_1}(y_0) < y_0$. Let us choose a function $F(y) = C_1 e^{C_2 y}$ such that it satisfies the following conditions (i) $F'(y_0) = C_1C_2e^{C_2y_0} = k < Q'_{b_1}(y_0)$ (ii) $Q_b(y_0) = F(y_0)$. We now state the methods by which y_0 and the constants C_1, C_2 are chosen.

Rule 1 : Since $h(y)$ is bounded above on $(0,\infty)$, and $\psi(\cdot)$ is a decreasing function we can choose $y_0 > 0$ such that

$$\frac{\sigma^2}{2}h^2(y) + \widetilde{\mu}(y) + \psi(y) = \mu h(y) + (1-h(y))r_0 + \psi(y) \leq M$$

for all $y \in [y_0,\infty)$. We now choose C_2 small enough such that

$$\frac{\sigma^2}{2}h^2(y)C_2^2 + (\widetilde{\mu}(y) + \psi(y))C_2 \leq \rho < (\rho + \psi(y))$$

such a choice of C_2 is possible since $\frac{\sigma^2}{2}h^2(y) + \widetilde{\mu}(y) + \psi(y)$ is bounded. $|\ \frac{\sigma^2}{2}h^2(y) + \widetilde{\mu}(y) + \psi(y)\ | \leq M$. Call C_5 the C_2 that satisfies this condition. Also C_2 must satisfy $F'(y_0) = C_2Q_b(y_0)$. Call it C_6. For $C_2 < \min\{C_5,C_6\} = C_2^*$ both of them holds simultaneously.

Rule 2: Let C_1 be such that

$$C_1e^{C_2y_0} = Q_b(y_0)$$

Step 2: Introduce the differential operator $\mathscr{L} = \frac{\sigma^2}{2}h^2(y)\frac{d^2}{dy^2} + (\widetilde{\mu}(y) + \psi(y))\frac{d}{dy} - (\rho + \psi(y))$. Now $\mathscr{L}F = [\frac{\sigma^2}{2}h^2(y)C_2^2 + (\widetilde{\mu}(y) + \psi(y))C_2 - (\rho + \psi(y))]F(y) < 0$. This follows from the boundedness criterion mentioned in Rule 1 and the fact that $F(y)$ is greater than 0. Now $\mathscr{L}(Q_{b_1} - F) = -\mathscr{L}F > 0$ for all $y \geq y_0$. Define $\eta(y) = Q_{b_1}(y) - F(y)$. Clearly at y_0, $\eta(y_0) = 0$. We claim that for all $y \geq y_0$, $Q_{b_1}(y) > F(y)$ holds. Once we have shown this we are done. Since $Q_{b_1}(y)$ lies above an exponential curve and it intersected the $y = x$ line where it's slope was less than 1, it follows that thereafter if it intersects $y = x$ line then it's slope would be greater than 1.

Now we prove $Q_{b_1}(y) > F(y)$, for all $y > y_0$. Suppose not, then there exist, y_1 such that $Q_{b_1}(y_1) \leq F(y_1)$ and $y_1 > y_0$. Since at y_0, $Q_{b_1}(y_0) = F(y_0)$ and $Q'_{b_1}(y_0) > F(y_0)$ therefore for $y_0 < y < y_0 + \delta$ we have $F(y) < Q_{b_1}(y)$, thus $y_1 > y_0 + \delta$. Define $z = \inf\{y : y > y_0 : F(y) \geq Q_{b_1}(y)\}$. Clearly $y_0 + \delta < z \leq y_1$. By continuity of $F(y)$ and $Q_{b_1}(y)$, $F(z) = Q(z)$. Also $F(y) < Q(y)$ on the interval (y_0,z). Thus we have $\eta(y_0) = 0$ and $\eta(z) = 0$ and $\mathscr{L}\eta > 0$, for all $y \geq y_0$. Hence we conclude $\mathscr{L}\eta > 0$ for all $y \in (y_0,z)$. Thus by the Maximum Principle $\eta < 0$ on the interval (y_0,z). Hence $Q(y) < F(y)$ on (y_0,z), a contradiction to the fact that $Q(y) > F(y)$ on $(y_0, y_0+\delta)$. Thus $Q(y) > F(y)$ for all $y \geq y_0$. Thus $Q(y)$ intersects the straight line twice in the interval (b_1, b_*). This completes our proof.

Once we have established the existence of a curve like $Q_{b_1}(y)$ that intersects the line y at least twice we are almost done. We give a road map of what we intend to do. In step1 we show that there exists a curve that lies above $Q_{b_1}(y)$ that intersects the $y = x$ at least twice. In step 2 we take constant multiples of it until we reach a point of tangency between one such curve and the line $y = x$. This determines the optimal threshold or the free boundary. The next proposition aptly demonstrates this fact.

Theorem 3.6. *(i) There exists a point $y^* \in (b_1, b_*)$ and a strictly positive function $Q_{y^*}(y)$ defined on R such that $Q_{y^*}(y^*) = y^*$, $Q_{y^*}(y) > y$ for all $y \in R/\{y^*\}$ and $Q'_{y^*}(y) = 1$, $\mathscr{L}Q_{y^*}(y) = 0$ for all $y \in R$.*

(ii) *There exists a function $\widehat{Q} : R \to R_+$ that satisfies all the conditions of (19). Moreover the threshold y^* and the function $\widehat{Q}$ are unique.*

Proof. The proof of this is identical with Theorem 14 and hence omitted.

Lemma 3.8. *If $b > 0$ and consider the function $Q_b(\cdot)$ extended to $(-\infty, \infty)$. If $\rho > \mu_1$. Where $\mu_1 = \frac{1}{2}\widetilde{\sigma}^2 + \widetilde{\mu}$. Then*

$$Q'_b - Q_b \geq \int_{-\infty}^{y} e^{-\int_z^y (1+\frac{2}{\widetilde{\sigma}^2}\widetilde{\mu}+\frac{2}{\widetilde{\sigma}^2}\psi)du} \frac{2}{\widetilde{\sigma}^2}(\rho-\mu_1)Q_b(z)dz \text{ for every } y \in R \tag{3.48}$$

Proof. The differential operator associated with Q_b is given by $\mathscr{L}_\psi = \frac{1}{2}\widetilde{\sigma}^2(y)\frac{d^2}{dy^2} + (\widetilde{\mu}(y) + \psi(y))\frac{d}{dy} - (\rho + \psi(y))$ with $Q_b(-\infty) = 0$ and $Q_b(b) = b$. Introduce the function $H_b(y) = Q'_b(y) - Q_b(y)$. Substituting this in (3.48) and rearranging terms we get

$$H'_b + (1 + \frac{2}{\widetilde{\sigma}^2}\mu + \frac{2}{\widetilde{\sigma}^2}\psi)H_b = \frac{2}{\widetilde{\sigma}^2}(\rho-\mu_1)Q_b(y)$$

Multiplying the above equation by the integrating factor $e^{-\int_c^y (1+\frac{2}{\widetilde{\sigma}^2}\widetilde{\mu}+\frac{2}{\widetilde{\sigma}^2}\psi)du}$ we obtain

$$d(H_b(y))e^{\int_c^y (1+\frac{2}{\widetilde{\sigma}^2}\mu+\frac{2}{\widetilde{\sigma}^2}\psi)du} = e^{\int_c^y (1+\frac{2}{\widetilde{\sigma}^2}\mu+\frac{2}{\widetilde{\sigma}^2}\psi)du}[\frac{2}{\widetilde{\sigma}^2}(\rho-\mu_1)Q_b(y)]du$$

Integrating we have

$$H_b(y) = H_b(c)e^{-\int_c^y (1+\frac{2}{\widetilde{\sigma}^2}\mu+\frac{2}{\widetilde{\sigma}^2}\psi)du} + \int_c^y e^{-\int_z^y (1+\frac{2}{\widetilde{\sigma}^2}\mu+\frac{2}{\widetilde{\sigma}^2}\psi)du}[\frac{2}{\widetilde{\sigma}^2}(\rho-\mu_1)Q_b(y)]du$$

Now, $H_b(c) = Q_b'(c) - Q_b(c)$. Thus $H_b(c) + Q_b(c) > 0$. This implies $H_b(c) > -Q_b(c)$. By letting c tend to $-\infty$, it follows

$$\lim_{c \longrightarrow -\infty} e^{-\int_c^y ((1+\frac{2}{\tilde{\sigma}^2}\mu+\frac{2}{\tilde{\sigma}^2}\psi)du} H_b(c) \geq -\lim_{c \longrightarrow -\infty} Q_b(c) e^{-\int_c^y (1+\frac{2}{\tilde{\sigma}^2}\mu+\frac{2}{\tilde{\sigma}^2}\psi)du} = 0$$

As c tends to $-\infty$ we have $Q_b(-\infty) = 0$. Thus it follows that

$$H_b(y) = Q_b' - Q_b \geq \int_c^y e^{-\int_z^y (1+\frac{2}{\tilde{\sigma}^2}\mu+\frac{2}{\tilde{\sigma}^2}\psi)du} [\frac{2}{\tilde{\sigma}^2}(\rho - \mu_1) Q_b(y)] du$$

From the above equation it follows that $Q_b' - Q_b \geq 0$ if $\rho > \mu_1$. This completes the proof.

3.10 References

L.H.R. Alvarez and L.A. Shepp, Optimal harvesting of stochastically fluctuating populations, J. Math. Biol., 37 (1998), 155-177.

Brock, William A., Michael Rothschild and Joseph Stiglitz, 1979, Notes on stochastic capital theory, Mimeo. (University of Wisconsin,Madison, WI)

P.Carr and V.Linetsky, A jump to default extended CEV model: an application of Bessel processes. Finance and Stochastics,10 (2006) pp 303-330.

Zhu, C., Song, Q., and Stockbridge, R. H, (2010) : On optimal harvesting problems in random enviornments (working paper).

Cox,J. & Huang, C.F. (1989) : Optimal consumption and portfolio policies when asset proces follow a diffusion process. J.Econ.Theory 49, 33-83.

R. J. Elliot, M.Jeanblanc and M.Yor : On models of default risk, Math. finance,10 (2000) pp 179-195.

Guo, X., Jarrow, R., and Zeng, Y., 2005, " Information Reduction in credit risk Models".

A.P Ghosh and A.P Weerasinghe : An optimal buffer size for a stochastic processing network in heavy traffic, Queuing systems, 55 (2007) pp 147-159.

R.A. Jarrow and P.Protter, Structural versus reduced form models: a new information based perspective, Journal of Investment Management, 2 (2004) pp 1-10.

R.A. Jarrow and P.Protter and A. Deniz Sezer, Information reduction via level crossings in a credit risk model, Finance and Stochastics, 11 (2007) pp 195-212.

Karatzas, I., Lehoczky, J.P. & Shreve, S.E. [KLS] (1987) Optimal portfolio and consumption decisions for a "small investor" on a finite horizon. SIAM J. Control & Optim.25, 1557-1586.

Karatzas, I. & Shreve, S.E. (1991) Brownian Motion and Stochastic Calculus. Second Edition, Springer-Verlag, New York.

R. J. Elliott, M. Jeanblanc and M. Yor, On models of default risk, Math Finance, 10 (2000) pp 179-195.

Meng, Q.(2007), Topics in pricing American type financial contracts. Thesis (PhD) Iowa State University.

Merton, R.C. (1971) Optimum consumption and portfolio rules in a continuous-time model.J. Econom. Theory 3, 373-413. Erratum: ibid. 6 (1973), 213-214.

D. Ocone and A. Weerasinghe : Degenarate variance control of a one dimensional diffusion process, SIAM J. Control and Opt. 39 (2000) pp. 1-24.

Pliska, S.R. (1986) :A stochastic calculus model of continuous trading: optimal portfolios.Math. Oper. Research 11, 371-382.

Samuelson, P.A. & Merton, R.C. (1969) A complete model of warrant-pricing that maximizes utility. Industr. Mangmt. Review 10, 17-46.

Weerasinghe, A : A bounded variation control problem for diffusion processes, SIAM J. Control and Opt., 44 (2005), pp 389-417.

CHAPTER 4. NUMERICAL ANALYSIS

4.1 Numerical Results

In this chapter, we show numerically the impact of change in parameters on optimal threshold and value function. We also verify and illustrate numerically the monotonicity of the optimal exercise boundary and the value function with respect to the default intensity. The theory of dynamic programming for optimal stopping problems suggests that the optimal stopping problem is a smooth solution of HJB equation given in (3.17). This is the starting point of our numerical method. The optimal exercise boundary y^* and the value function $V(y)$ are calculated numerically by solving the following boundary value problem.

$$\frac{1}{2}\widetilde{\sigma}^2(y)Q''(y) + (\widetilde{\mu}(y) + \psi(y))Q'(y) - (\rho + \psi(y))Q(y) = 0$$

$$Q(y) = y \text{ and } \lim_{y \longrightarrow -\infty} Q(y) = 0$$

We use the Matlab boundary value problem solver (bvp4c or bvp5c) to solve the problem. These solvers decompose the differential equation into a system of first order ODE's and use the collocation method to solve the differential equations and are quite robust in solving both linear and non-linear ODE's. In the following set of examples unless otherwise stated, we assume volatility parameter $\sigma = 0.5$, $r_0 = 0.05$ (risk free interest rate), $\mu = 0.25$ (mean rate of return), and the rate of discount $\rho = 1$. The portfolio process is chosen $h(y) = \frac{1}{2}(\frac{2}{\pi}\arctan(\exp(y)) + 1)$. Our initial interest is to study the effect of a change in volatility of the stock price on the optimal threshold and value function.

4.1.1 Effect of a change in volatility σ

Here we summarize the effect of change in volatility of stock price on the optimal threshold and the value function. Keeping other things fixed numerical tests shows that as volatility increases the optimal threshold also increases. This is captured in the Table 1 below.

σ	r_0	μ	ρ	y^*
0.5	0.05	0.25	1	0.530253
0.9	0.05	0.25	1	0.605261
3.0	0.05	0.25	1	0.848685
4.0	0.05	0.25	1	0.894289

Table 1

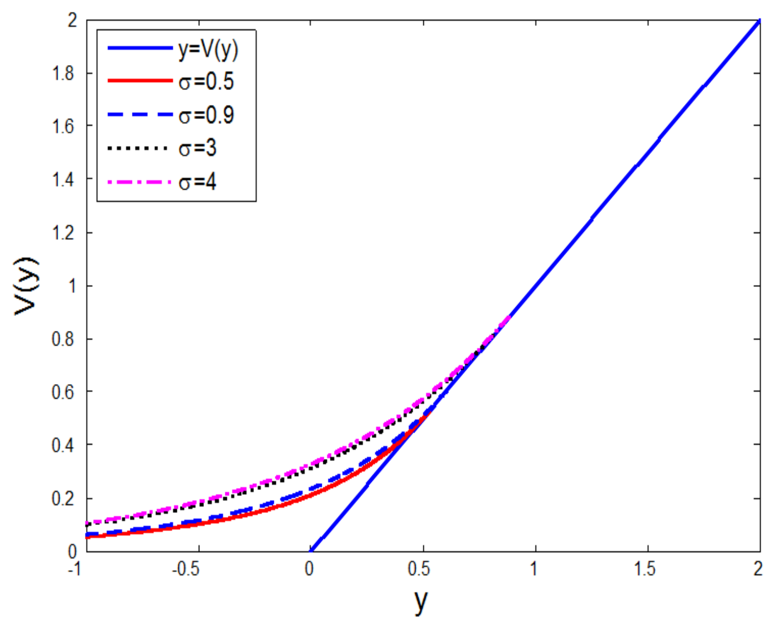


Figure 1. Effect of σ on optimal threshold.

Figure 1 plots the value functions for the corresponding values of volatility. We see that both the optimal threshold and the value function monotonically increases with volatility. The economic interpretation of the above findings is the following : as volatility of stock price increases agents want to quit from the stock market early as a result stock price tends to fall which in turn increases the default intensity $\psi(\cdot)$. The rise in intensity of default increases the drift coefficient of the SDE governing the wealth process. On this count agents wants to hold their position in stock market a little longer but the discount factor also increases by the same factor propelling the agents to quit from the stock market early. This is what makes the intensity modelling interesting. It turns out that in the parameter range that we have chosen the effect of a rise in *effective drift coefficient* $\widetilde{\mu}(\cdot)+\psi(\cdot)$ more than offsets the rise in *effective discount rate* $\rho+\psi(\cdot)$. This explains why optimal threshold and value function increases with volatility. The next example gives a completely different picture. Here we measure the effect of volatility on optimal threshold taking into consideration that the mean return of the stock increased from 0.25 to 4. Table 2 presents optimal thresholds for various levels of volatility.

σ	r_0	μ	ρ	y^*
0.5	0.05	2	1	1.782778
0.9	0.05	2	1	1.649165
3	0.05	2	1	1.151115
4	0.05	2	1	1.090309

Table 2

To our surprise we see a diametrically opposite picture. As volatility increases optimal threshold falls. Again this can be explained via the counteractive forces. The rise in mean return of stock price coupled with an increase in volatility causes the stock price to be highly volatile than the previous scenario. This causes a fall in stock price which in turn increases the default intensity thereby causing a forward drift. It turns out in this case that the rise in discount rate more than offsets the positive effect from the drift term. Fig 2 shows an inverse relationship between value function and volatility.

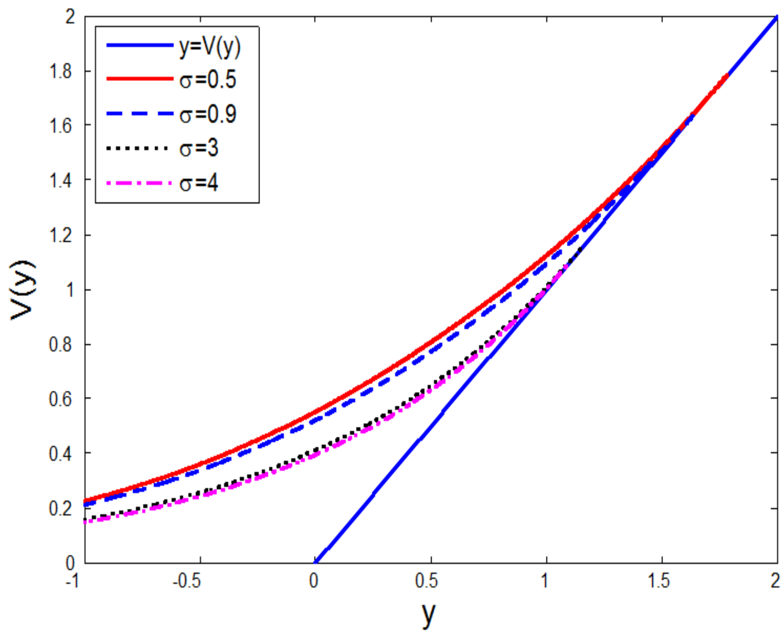


Figure 2. Effect of σ on threshold given $\mu = 2$.

4.1.2 Effect of a change in mean return μ

In this section we address the issue of an increase in mean rate of return μ of the stock on optimal threshold and pay-off. As mean return increases we see from Table 3 that optimal threshold increases.

σ	r_0	μ	ρ	y^*
0.5	0.05	0.25	2	0.371437
0.5	0.05	0.8	2	0.50185
0.5	0.05	1.5	2	0.711871
0.5	0.05	2	2	0.894889

Table 3

This is quite intuitive. For as mean return of the stock increases keeping other things fixed $\widetilde{\mu}(\cdot)$ increases thus agents tend to hold their position in the stock market for a longer time. Consequently there is a positive monotone relationship between value function and mean rate of return. Fig 3.illustrates this view.

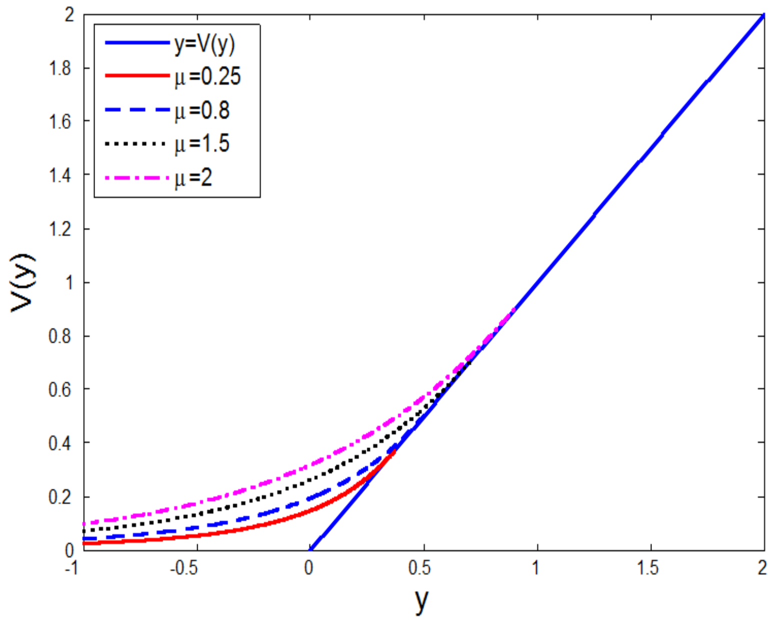


Figure 3. Effect of μ on threshold.

4.1.2.1 Effect of a change in interest rate r_0 and discount factor ρ

These effects are pretty much counterintuitve and we sum them up in the findings of Table 4 and Table 5.

σ	r_0	μ	ρ	y^*
0.5	0.05	0.25	1	0.530053
0.5	1	0.25	1	0.609861
0.5	2	0.25	1	0.690544
0.5	4	0.25	1	0.839134

Table 4

σ	r_0	μ	ρ	y^*
0.5	0.05	0.25	0.09	1.429143
0.5	0.05	0.25	0.2	1.044904
0.5	0.05	0.25	1	0.530253
0.5	0.05	0.25	1.5	0.433043

Table 5

We see from Table 4 that as interest rate increases optimal threshold also increases. This is obvious since as interest rate rises the drift term $\widetilde{\mu}(\cdot)$ increases which allows the agent to wait a little longer so that one can reap the benefits of higher expected return. This is illustrated in figure 4 below. On the contrary when ρ increases the mirror opposite holds true. From Table 5 we observe that there is an inverse relation between the discount factor and the optimal threshold. We can argue this along the following lines. As ρ increases the discounted present value of utility falls and also the mean return from the wealth process (as ρ increases $\widetilde{\mu}(\cdot)$ falls). Thus agents don't have a tendency to wait long as waiting is costly and hence we observe a fall in optimal threshold as the discount factor increases. Figure 5 illustrates this graphically.

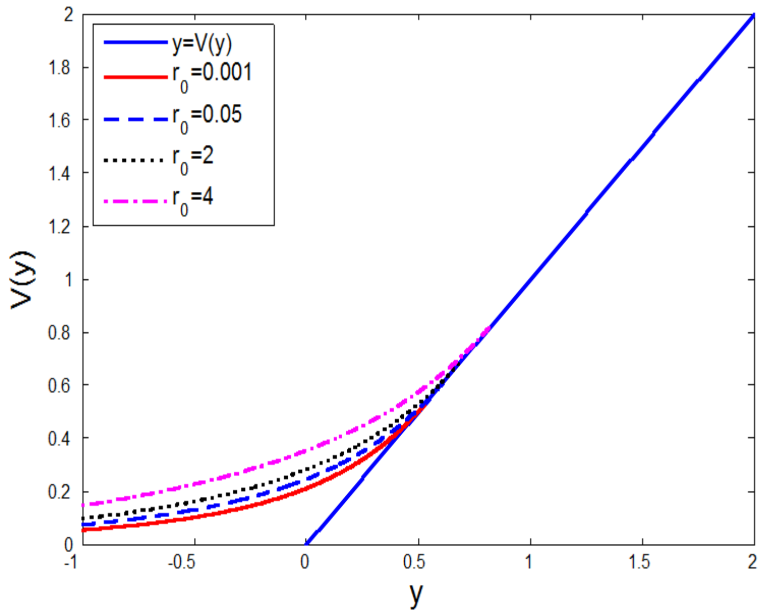


Figure 4. Effect of r_0 on threshold.

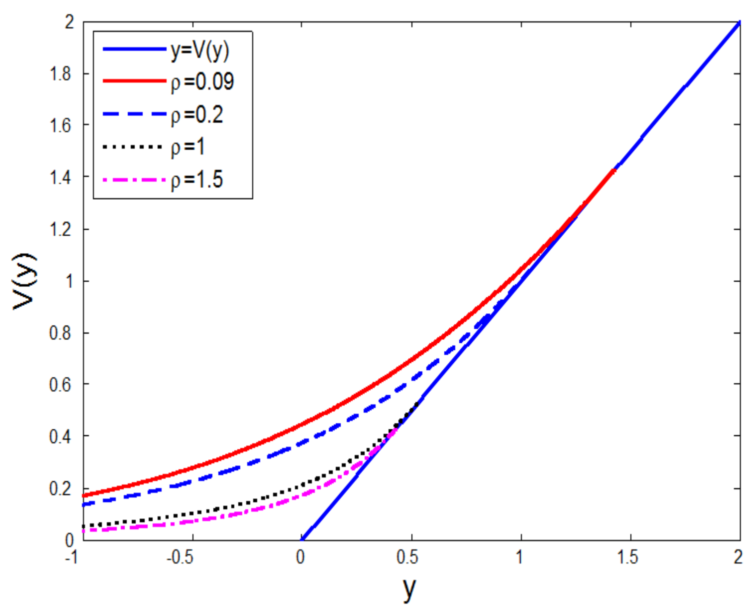


Figure 5. Effect of ρ on threshold.

4.1.2.2 Effect of p on optimal threshold.

In this section, we numerically verify the monotonicity result obtained in section 5. We consider a family $\{e^{-py}\}$ of default intensity functions parametrized by p and investigate the influence of it on the boundary y^*. We intend to answer the following question. How does the default intensity function $\psi(x)$ affect the optimal threshold y^* and the value function $V(y)$. From Table 6 we see that y^* obtained from any jump-to default model is always higher than the no-default model. Also we see that for $\psi_3(y) > \psi_2(y)$ we have $y^*_{\psi_3} > y^*_{\psi_2}$ and $V_{\psi_3}(y) > V_{\psi_2}(y)$. This reassures the validity of the monotonicity results numerically.

Default Intensity	y^*
$\psi_1(y) = 0$	0.513051
$\psi_2(y) = e^{-0.6y}$	0.68146
$\psi_3(y) = e^{-y}$	1.391539

Table 6

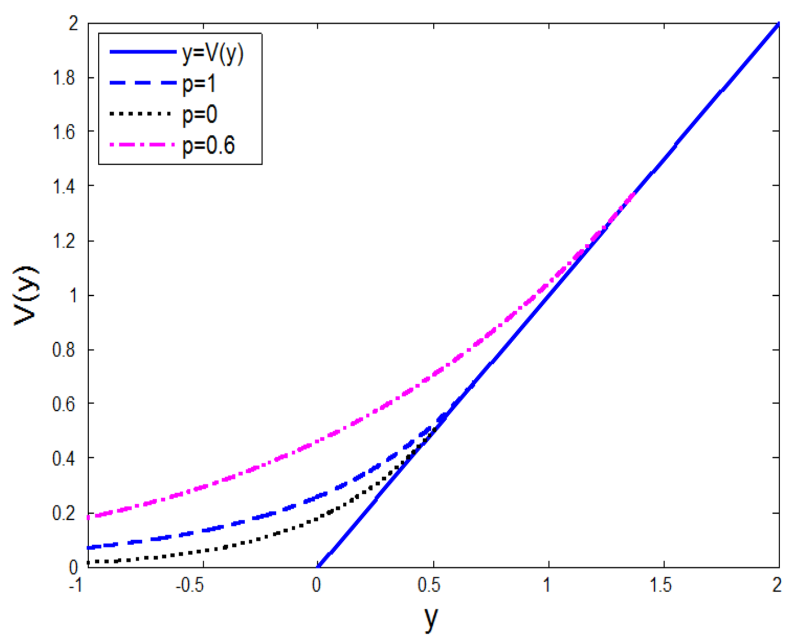


Figure 6. Effect of $\psi(\cdot)$ on threshold.

4.2 Discussion and Future Research

- The results of our numerical analysis provide further insights into the linkages between optimal threshold boundary, value function and relevant policy variables. In particular, we show that there is a positive monotone relationship between optimal threshold boundary and volatility. This leads to future research questions. Can we show theoretically whether optimal threshold increases with volatility. Does value function increases with volatility for a given set of parameter choices. Also we intend to find a sequence of portfolio choices $\{p_n(y)\}_{n\geq 1}$ and try to find out numerically the optimal trading strategy. This would be of considerable importance from a practitioner's point of view as it would determine the optimal buy and hold strategy.

- Throughout our analysis chapter 3, we have taken the market rate of interest to be fixed. One can relax the assumption of a fixed rate of interest and instead allow the interest rate to follow a short rate model. In particular, we can assume that interest rate evolves according to Ho-Lee model in mathematical finance. In financial mathematics, the Ho–Lee model is a short rate model widely used in the pricing of bond options, swaptions and other interest rate derivatives, and in modeling future interest rates. Under this model, the short rate follows a normal process $dr(t) = \theta(t)dt + \sigma dW(t)$. Here $\theta(t)$ represents time dependent drift term. An optimal portfolio process in this framework with finite time horizon has been explicitly dealt by Korn and Kraft (2001). One can try to extend it in the infinite time horizon incorporating default in an investor's wealth.

- Another direction for future work would be to find an optimal stopping rule among the class of all stopping times under a regime switching model. Regime switching models refers to the case where we model stock price movements as a geometric Brownian motion coupled by a finite state space Markov chain. For a detailed analysis of regime switching models one can refer to Yin and Zhu [2010], X.Guo [2001]. Such models can incorporate the behavior of stock-prices during market cycles and thus making it more realistic. We plan to work on optimal portfolio investment problems related to this model. We are interested in addressing this issue since, in finance literature generally stock price movements are modeled as geometric Brownian motion (GBM) with deterministic coefficients. But this has serious limitations due to its insensitivity to random parameter changes such as changes in market trends.

- A passport option grants its holder the right to engage in a short/long trading strategy of his own choice, while obligating the option writer to cover any net losses on the strategy. Passport option has been studied by Hyer et.al. [1997], Andersen et.al. [1998], Henderson and Hobson [1999], Delbaen and Yor [1999] and Nagayama [1999]. Heyer et. [1997] provide a closed form solution of the value function in absence of transaction costs. In contrast to previous papers Shreve and Vecer [2000] using probabilistic techniques found the value of the option, the optimal strategy of the buyer in absence of transaction costs. Our interest stems from the fact that such an optimal strategy derived in Shreve and Vecer [2000] cannot be implemented in the presence of transaction costs since there will be infinite no of transactions. Hence such a strategy is not optimal in the presence of transaction costs. We would like to consider how the optimal strategy is in presence of proportional transaction cost.

4.3 References

Andersen, L., Andreasen, J., Brotherton-Ratcliffe, R., The passport option, The Journal of Computational Finance Vol.1. No. 3, Springer 1998, 15-36.

Delbaen, F., and Yor, M., Passport options, Mathematical Finance, Vol.12, pp.299-328, 2002.

Guo., X., An explicit solution to an optimal stopping problem with regime switching, Journal of Applied Probability, 38(2):464-481, 2001.

Henderson, V., Hobson, D., Local time, coupling and the passport option, Finance and Stochastics, Vol.4, No.1,1999.

Hyer,T., Lipton-Lifschitz, A., Pugachevsky, D., Passport to success, Risk, Vol. 10, No.9, September 1997, 127-131.

Ralf Korn, Holger Kraft.,A stochastic control approach to portfolio problems with stochastic interest rates,SIAM Journal on Control and Optimization 40, 2001, p. 1250-1269.

Nagayama, I., Pricing of passport option, J. Math. Sci. Univ. Tokyo 5, 747-785, 1999.

Yin, G., and Zhu ,C., Hybrid Switching Diffusions: Properties and Applications, Springer, New York, 2010.