On complex convexity

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Abstract

This thesis is about complex convexity. We compare it with other notions of convexity such as ordinary convexity, linear convexity, hyperconvexity and pseudoconvexity. We also do detailed study about $\mathbb{C}$-convex Hartogs domains, which leads to a definition of $\mathbb{C}$-convex functions of class C^1. The study of Hartogs domains also leads to characterization theorem of bounded $\mathbb{C}$-convex domains with C^1 boundary that satisfies the interior ball condition. Both the method and the theorem is quite analogous with the known characterization of bounded $\mathbb{C}$-convex domains with C^2 boundary. We also show an exhaustion theorem for bounded $\mathbb{C}$-convex domains with C^2 boundary. This theorem is later applied, giving a generalization of a theorem of L. Lempert concerning the relation between the Carathéodory and Kobayashi metrics.

Keywords: $\mathbb{C}$-convex; Linearly convex, Charathéodory metric, Kobayshi metric
2000 Mathematics Subject Classifications: 32F17; 32F45

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David Jacquet
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Chapter 1

Introduction

1.1 Background

This thesis is about convexity notions in general, and especially complex convexity. Convexity is an important and fundamental notion that appears in most fields of mathematics. One example of how convexity naturally appears in complex analysis is the fact that the domain of convergence for a power series is logarithmically convex. There is also the Paley-Wiener theorem which shows a clear connection between a certain class of entire functions and compact convex sets in $\mathbb{R}^n$. All convex domains are domains of holomorphy, but the converse is not true. Instead there exists the weaker concept pseudoconvexity, which can be shown to be equivalent with being a domain of holomorphy. Some, but not all, theorems which hold for convex domains, remain valid for pseudoconvex ones.

Convexity concepts that are weaker than convexity but stronger than pseudoconvexity have been studied, for example by Benkhe and Peschl [BP35] who introduced linear convexity in $\mathbb{C}^2$ using the term *Planarconvexität*, planar convexity. Linear convexity has also been studied by Martineau [Mar66] in his work on the Fantappiè transform. There has been some discrepancy in the terminology, for example Martineau used the term *linéellement convexe* which translates to *lineally convex* as opposed to *linearly convex* which is the term in [APS04]. We will follow the latter terminology throughout this paper. For a thorough account of the field of convexity, see [Hör94].

The property of ordinary convexity requires that the intersection between the set and a real line should be contractible. A weakening is to require this only for complex lines, which are of real dimension two. This property lies in between of convexity and pseudoconvexity and one calls it complex convexity, usually abbreviated as $\mathbb{C}$-*convexity*. In one complex variable this property is not so interesting because it only specifies whether the set is contractible or not, and some of the theorems about complex convexity apply only in the case of two or more dimensions.

1.2 Summary

In Chapter 2 we exhibit the some of the convexity concepts that occur in complex analysis. We recall some inclusion and exclusion relations among these concepts. However, there seem to remain some open issues regarding pseudoconvexity and local linear convexity. We also recall some properties of defining functions and distance functions of domains satisfying different convexity conditions. These properties reveal some great similarity between the various convexity notions. We also provide a geometrical tool that can be used to determine whether or not Hartogs domains in $\mathbb{C}^2$ are $\mathbb{C}$-convex, see Lemma 2.2.1. The strength this method is that instead of looking at complex affine lines in $\mathbb{C}^2$, we consider cones in $\mathbb{R}^3$. Even though the lemma is very simple, it is the intuitive foundation on which most of our later results are built.

In Chapter 3 we focus on complex convexity and linear convexity. We investigate properties of linearly convex Hartogs domains in $\mathbb{C}^2$ according to the ideas in Chapter 2. In the case of Hartogs domains with a C^1 defining function, we can formulate a property that precisely characterizes the linear convex ones, see Lemmas 3.1.10 and 3.1.13. The former lemma is a global condition on the defining function whereas the latter one is a local condition. The local condition is significantly harder to prove. We also show that if a Hartogs domain in $\mathbb{C}^2$ having a disc as base domain and a defining function of class C^1, then it is linearly convex precisely if it is $\mathbb{C}$-convex. Putting these facts together, we have a definition of $\mathbb{C}$-convex functions of class C^1. This definition fits well with the previously known definition of $\mathbb{C}$-convex functions of class C^2, when viewed from the perspective of C. Kiselman's important theorem in [Kis96], stating the precisely same theorem for C^2 functions. The non linearity of the differential inequality (2.1.2), constituting the C^2 definition of $\mathbb{C}$-convex functions, leads to significant difficulties when working with $\mathbb{C}$-convex functions. Kiselman's theorem can be used to characterize $\mathbb{C}$-convex bounded domains of class C^2. This was done in [APS04] and the methods used where based on work of C. Kiselman and L. Hörmander. We use our corresponding C^1 result in the same manner in the C^1 case, when adding the extra assumption that the domain fullfills the interior ball condition for some positive radius, see Therorem 3.2.2. At the end of the chapter we give an example showing that a theorem of this type cannot hold when considering general domains, see Example 3.2.6. For bounded $\mathbb{C}$-convex domains with C^2 boundary we are able to find a method of exhausting the domain with strictly $\mathbb{C}$-convex domains with C^∞ boundary, see Theorem 3.2.5

In Chapter 4 we find an application of Theorem 3.2.5 regarding the relation between the Carathodory and the Kobayashi metrics. Our result is a generalization of a theorem of L. Lempert stating that the above mentioned metrics are equal on bounded strictly $\mathbb{C}$-convex domains with C^∞ boundary. We are able to show that the statement also holds for bounded $\mathbb{C}$-convex domains with C^2 boundary.

Chapter 2

Some Notions of Convexity

2.1 Definitions and notations

We will refer to several different convexity notions, regarding both functions and sets. For sets we will compare the following notions: convexity, complex convexity, linear convexity, weak linear convexity, local weak linear convexity, hyperconvexity and pseudoconvexity. We will recall how, under certain technical conditions, these notions in the order written above, constitute a chain of weakening notions. Without these technical conditions the situation becomes more complex and also uncertain. For function we will discuss the concepts of convexity, complex convexity and plurisubharmonicity. Most often we will use the abbreviated notation $\mathbb{C}$-convexity insted of complex convexity. If u is a real valued function of class C^1 defined on $\mathbb{C}^n$ we will use the common short notations

$$u_j = \frac{\partial u}{\partial z_j} = \frac{1}{2}\left(\frac{\partial v}{\partial x_j} - i\frac{\partial v}{\partial y_j}\right)$$
$$u_{\bar{j}} = \frac{\partial u}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial v}{\partial x_j} + i\frac{\partial v}{\partial y_j}\right)$$

where the function v goes from $\mathbb{R}^{2n}$ to $\mathbb{R}$ and is defined by

$$v(x_1, y_1, \ldots, x_n, y_n) = u(x_1 + i\,y_1, \ldots, x_n + i\,y_n).$$

If the function u depends of only one complex variable, then we shall often write u_z and $u_{\bar{z}}$ instead of u_1 and $u_{\bar{1}}$.

2.1.1 Functions

Definition 2.1.1. Let u be a real valued functions of class C^2 defined in a neighborhood of $z_0 \in \mathbb{C}$. Then we label the different convexity properties that u may have at z_0 according to the following list:

Convex	$u_{z\bar{z}}(z_0) \geq \lvert u_{zz}(z_0)\rvert$	(2.1.1)
$\mathbb{C}-$**convex**	$u_{z\bar{z}}(z_0) \geq \lvert u_{zz}(z_0) - u_z(z_0)u_z(z_0)\rvert$	(2.1.2)
Plurisubharmonic	$u_{z\bar{z}}(z_0) \geq 0$	(2.1.3)

Assume that the function u is of class C^2 and depends on n complex variables, and that we wish to test one of the convexity concepts in Definition 2.1.1 at a point $z_0 \in \mathbb{C}^n$. Then we consider its restriction to any complex line passing through z_0 and test the concept for this function. More explicitly, for every unit vector $w \in \mathbb{C}^n$ we consider the one variable function

$$f(\lambda) = u(z_0 + \lambda\, w)$$

which at $\lambda = 0$ should satisfy the condition in Definition 2.1.1 corresponding to the convexity notion against which we want to test u. To shorten notation, the plurisubharmonic functions in a domain $\Omega \subset \mathbb{C}^n$, will sometimes be referred to with the standard notation $\mathcal{PSH}(\Omega)$.

2.1.2 Sets

We recall some notions of convexity in the next definition. Later we will also discuss some of these notions in their strict versions, for example strict $\mathbb{C}$-convexity.

Definition 2.1.2. Let Ω be a domain in $\mathbb{C}^n$. Then we label the different convexity properties that Ω may satisfy according to the following list:

Convex – the intersection between Ω and any real line is contractible.

$\mathbb{C}$**-convex** – the intersection between Ω and any complex line is contractible.

Linearly convex – the complement of Ω is a union of complex hyperplanes.

Weakly linearly convex – for every point p on the boundary of Ω, there is a complex hyperplane that intersects p but not Ω.

Locally weakly linearly convex – for every point p on the boundary of Ω, there is a complex hyperplane that intersects p but does not intersect Ω in a neighborhood of p.

Hyperconvex – there exists a continuous negative plurisubharmonic function ϕ defined on Ω such that the set $\Omega_x = \{z \in \Omega : \phi(z) \leq x\}$ is relatively compact to Ω for all real and negative values x.

Pseudoconvex – there exists a continuous plurisubharmonic function ϕ on Ω such that the set $\Omega_x = \{z \in \Omega : \phi(z) \leq x\}$ is relatively compact in Ω for every real value x.

We will sometimes use the self explanatory abbreviations

$$\mathcal{C} - \mathbb{C}\mathcal{C} - \mathcal{LC} - \mathcal{WLC} - \mathcal{LWLC} - \mathcal{HC} - \psi\mathcal{C}$$

for the concepts in Definition 2.1.2.

2.2 Some properties of Hartogs domains in $\mathbb{C}^2$

Here we state some simple properties of Hartogs domains in $\mathbb{C}^2$ that will be important when showing exclusion properties for the concepts in Definition 2.1.2. For example, constructing a domain that is locally weakly linearly convex but not weakly linearly convex. It will also be important later onfor our work with $\mathbb{C}$-convex functions of class C^1.

We will be considering the standard projection of the Hartogs domains, instead of the domains themself. Recall that this projection, which we denote π, is defined as

$$\begin{aligned} \pi : \mathbb{C}^2 &\to \mathbb{C} \times \mathbb{R}_{\geq 0} \\ (z, w) &\mapsto (z, |w|). \end{aligned} \tag{2.2.1}$$

We denote the coordinates of $\mathbb{C} \times \mathbb{R}_{\geq 0}$ by (z, r). The function π sends complex affine lines to cones. By cones we mean sets of the form $\{(z, r) : r = |a + b\,z|\}$ with $a, b \in \mathbb{C}$ or sets of the form $\{(z, r) : z = z_0\}$. The latter ones will be called *degenerate* cones. Note that the inverse image of any cone is a complex affine line in $\mathbb{C}^2$. Fix two distinct points $\alpha, \beta \in \mathbb{C}^2$ and let δ_L be the distance function between affine complex lines in $\mathbb{C}^2$ defined as

$$\delta_L(l_1, l_2) = |(l_1 - \alpha) - (l_2 - \alpha)| + |(l_1 - \beta) - (l_2 - \beta)|.$$

Here $(l - p)$ denotes the unique point $s \in \mathbb{C}^2$ such that $p + s \in l$ and $|s|$ equals the distance between the affine line l and the point p. Note that the topology on the set of affine lines does not depend on the choice of α and β. The set of affine complex lines with this topology will be denoted L. Fix a point $p \in \mathbb{C}^2$, and denote the subset of all lines through p as X. Then δ_L will induce the same topology on X as the Fubini-Study metric, when identifying X with the projetive space $\mathbb{P}(\mathbb{C})$. Now we can define a distance between two cones c_1 and c_2 as

$$\delta_C(c_1, c_2) = \inf\{\delta_L(l_1, l_2) : \pi(l_1) = c_1,\, \pi(l_2) = c_2\}. \tag{2.2.2}$$

The set of cones with the topology inherited by δ_C will be denoted Z. We are now ready for the following lemma.

Lemma 2.2.1. *Let D be a bounded open set in $\mathbb{C}$ and consider the Hartogs domain $H = \{(z, w) : z \in D, |w| < g(z)\}$ where g is a continuous function. Given a point $p \in \mathbb{C} \times \mathbb{R}_{\geq 0}$ let Z_p denote the set of cones that go through p but does not intersect $\pi(H)$. Then the following statements hold:*

1. *H is weakly linearly convex if and only if Z_p is non empty for each point p on the boundary of $\pi(H)$.*
2. *H is $\mathbb{C}$-convex if and only if Z_p is non empty and connected in Z for each p on the boundary of $\pi(H)$.*
3. *H is $\mathbb{C}$-convex if and only if the intersection $\pi(H) \cap C$ is contractable or empty for each cone $C \in Z$.*

Proof.
1. Assume that H is weakly linearly convex. Then take a point $p \in \partial\pi(H)$ and choose a $q = (z, w) \in \pi^{-1}(p)$. Since H is weakly linearly convex there exists a line l_q that goes through q but does not intersect H. Hence the cone

$c_p = \pi(l_q)$ will go through p but not intersect $\pi(H)$. Therefore Z_p is not empty.

Now assume that Z_p is non empty for each point in $\partial\pi(H)$. Take a point q on the boundary of H. Let $p = \pi(q)$. It is clear that $p \in \partial\pi(H)$ and hence we know that there is a cone c_p that passes through p but does not intersect $\pi(H)$. Now consider $\pi^{-1}(c_p)$ which will be a union of affine complex lines, each of them going through p and none of them intersecting H. Precisely one of them will pass through q. Note that if c_p is a degenerate cone, then this union consists of exactly one affine complex line but the statement remains true. Hence statement 1 is proved.

2. Given a point $q = (z, w) \in \mathbb{C}^2$, let L_q denote the set of complex lines that pass through (z, w) but do not intersect H. Assume that H is unbounded, which means that the function g is unbounded. Take a point $z_0 \in D$ let $q_0 = (z_0, g(z_0)) \in \partial H$, then we claim that L_{q_0} is empty and hence is H not even linearly convex and therefore not $\mathbb{C}$-convex. Indeed, let l be a complex line containing the boundary point q_0. If the line l is constant in the first variable, the it must intersect H in all points (z_0, w) where $|w| < g(z_0)$. If l is not constant in the first variable, then it must have a finite slope in the sense that there exist a constant $C > 0$ such that $|w| < C$ for all w such that $z \in D$ and $(z, w) \in l$. Hence l intersects H in, among others, all points (z, w) such that $z \in D$ and $g(z) > C$. This meas that L_{q_0} is empty and hence is $Z_{\pi(q_0)}$ also empty, so the statement is true if H is unbounded.

Therefore we may assume that H is bounded, which according to Theorem 2.5.2 in [APS04], implies that H is $\mathbb{C}$-convex if and only if L_q is non empty and connected in L for every $q \in \partial H$.

Assume that L_q is non empty and connected in L for every $q \in \partial H$. Because H is a Hartogs domain, we have that $\partial\pi(H) = \pi(\partial H)$ and that $Z_{\pi(q)} = \pi(L_q)$. Since π is continuous not only from $\mathbb{C}^2$ to $\mathbb{C} \times \mathbb{R}$, but also from L to Z, it is clear that $Z_{\pi(q)}$ will be non empty and connected if L_q is non empty and connected.

What remains is to show that if L_q is empty or not connected, then $Z_{\pi(q)}$ is empty or not connected. The part about emptiness is obvious and therefore we consider the case when $L_q = A \cup B$ where A and B are two open and disjoint sets. Since L_q is bounded, we may assume that both A and B are bounded. It is clear that the set L_q is closed and that the metric space L is complete. Now let

$$\begin{aligned} LA &= L_q \cap A \\ LB &= L_q \cap B \\ ZA &= \pi(LA) \\ ZB &= \pi(LB). \end{aligned}$$

Since A and B are bounded, all of the sets above must be bounded. We claim that they are also closed. Indeed, since $LA \cup LB = L_q$, we must have that $LA \cup LB$ closed, and since the closures of LA and LB are disjoint, both LA and LB must be closed. Since π is continuous we must have that ZA and ZB are closed. Now assume that ZA and ZB are not disjoint and the intersection between them contains a cone c. Let S_q denotes all affine complex lines intersecting q, which implies that $LA = \pi^{-1}(ZA) \cap S_q$. Hence booth LA and LB would contain the line $\pi^{-1}(c) \cap S_q$ which is a contradiction. Hence ZA and ZB are disjoint. Since they are both closed and bounded there exists two open and disjoint sets ZA' containg ZA and ZB' containg ZB. The union of ZA' and ZB' contains $Z_{\pi(q)}$ which cannot be connected.

3. Since H is a Hartogs domain we have that $H \cap l$ is homeomorphic with $\pi(H) \cap \pi(l)$ for every affine line l that is not constant in its first coordinate. For lines l that are constant in their first coordinate, both $H \cap l$ and $\pi(H) \cap \pi(l)$ are contractible. By definition H is $\mathbb{C}$-convex if the intersection between H and affine complex lines are contractible or empty, hence statement 3 is proved. □

Remark 2.2.2. If the boundedness condition is removed from Lemma 2.2.1, then part 2 becomes false. To see this, let g be a C^∞ non negative concave function on $[1, \infty)$ where $g(x) = x$ for $x \geq 2$, $g(1) = 0$ and $g'(1) = \infty$. Now consider the Hartogs domain in $\mathbb{C}^2$ defined by

$$H = \{(z, w) : |z| > 1, |w| < g(|x|)\}$$

which is illustrated in Figure 2.1. It is clear that ∂H is C^∞ and that it is weakly linearly convex. Therefore Z_p consists of exactly one point for every $p \in \partial\pi(H)$ and is thus obviously connected. However it is clear that H is not $\mathbb{C}$-convex. •

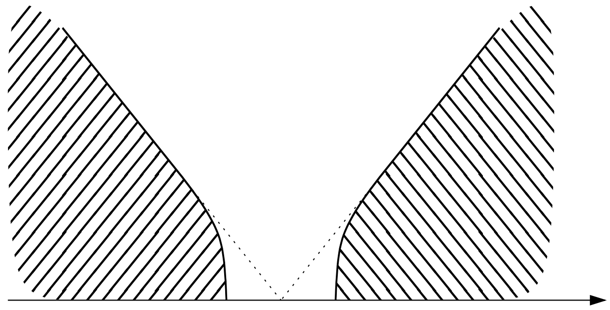


Figure 2.1: A Hartogs domain in $\mathbb{C}^2$ that is weakly linearly convex with C^∞ boundary, which is not $\mathbb{C}$-convex.

2.3 Relation of concepts

Here we will investigate the relation between the concepts in Definitions 2.1.1 and 2.1.2. We begin with the case of functions.

2.3.1 Functions

The first thing one may note in Definition 2.1.1 is that the definitions of convex and plurisubharmonic function are linear differential conditions. Hence those inequalities may be interpreted in the sense of distribution theory, which in particular provides immediate definitions of continuous convex and plurisubharmonic functions respectively. This is not the case for the $\mathbb{C}$-convexity definition. The non linearity of inequality (2.1.2) is one of the biggest difficulties that we have to deal with in this thesis.

If a function is convex or $\mathbb{C}$-convex then it is obviously plurisubharmonic. On the other hand there are plenty of functions that are plurisubharmonic but neither convex nor $\mathbb{C}$-convex. The function $u_1(z) = -\log|z|^2$, considered outside the origin, gives equality in inequality (2.1.2) and is hence $\mathbb{C}$-convex. It is clear that u_1 is not convex. Next look at the two variable function $u_2(z_1, z_2) = k(z_1\bar{z}_1 + z_2\bar{z}_2)$. It is clear that u_2 is convex for all $k > 0$. However if k is large enough it will not satisfy inequality (2.1.2), and hence it is not $\mathbb{C}$-convex for large k. The relations between the concepts are illustrated in Figure 2.2.

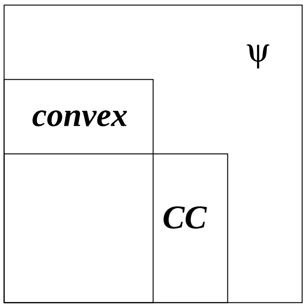


Figure 2.2: The relation between the function convexity concepts regarding convexity, $\mathbb{C}$-convexity and plurisubharmonicity in the C^2 case.

2.3.2 Sets

Here we will present a more hierarchical structure than the one found among the function concepts. Our goal is to understand the logic behind Figure 2.3, which shows the inclusion and exclusion relations between the concepts from Definition 2.1.2. The relations depend on whether we consider bounded C^1 domains or more general ones. We will first investigate the inclusions by giving arguments and the exclusions by giving examples.

Inclusions

We list the inclusions consecutively and provide argument in each case. First we consider the general case when Ω is an arbitrary domain in $\mathbb{C}^n$, and after that we shall add the necessary information for the case when Ω is a bounded domain with C^1 boundary.

$\mathcal{C} \Rightarrow \mathbb{C}\mathcal{C}$

Let Ω be a convex domain and consider the intersection between Ω and a complex line. If the intersection is empty there is nothing to prove, and hence we assume that the intersection is not empty. Since both the complex line and Ω are convex, the intersection is convex. An open convex set is always contractible. Hence Ω is $\mathbb{C}$-convex.

$\mathbb{C}\mathcal{C} \Rightarrow \mathcal{LC}$

Here we follow the proof given by Therorem 4.6.8 in [Hör94]. If the dimension n equals 1 the statement is trivial, since every domain in

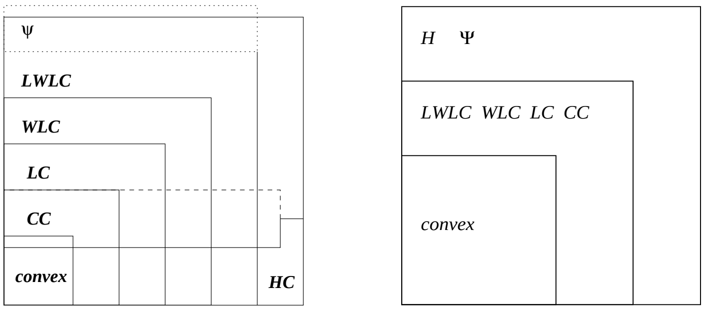


Figure 2.3: The hierarchy among the convexity concepts in Definition 2.1.2. To the left we have the situation for arbitrary domains, and to the right we have the situation for C^1 bounded domains. Note the uncertainty, marked with the dotted line, regarding local weak linear convexity in the general case. In the genereal case, there are also two lines describing hyperconvexity. The filled line is valid when completely arbitrary domains are considered, and the dashed line is valid for domains which does not contain a complex line.

$\mathbb{C}$ is linearly convex, so we may assume that $n \geq 2$. We shall argue by contradiction and assume that the $\mathbb{C}$-convex domain Ω does not contain the origin, but nevertheless any complex hyperplane through the origin has a non empty intersection with Ω.

We begin with the case when $n = 2$ and consider the complex hyperplanes through the origin, parametrized by $\mathbb{P}^1(\mathbb{C})$, here briefly referred to as $\mathbb{P}$. Pick a point $\zeta = (\zeta_1 : \zeta_2) \in \mathbb{P}$ and consider the complex line $L_\zeta = \{v(\zeta_1, \zeta_2) : v \in \mathbb{C}\}$ in $\mathbb{C}^2$ which is independent of the choice of representative (ζ_1, ζ_2) of ζ. Since the intersection $X_\zeta = \Omega \cap L_\zeta$ is non empty and contractible we may define a function

$$A_\zeta : X_\zeta \to \mathbb{R}$$

such that $A_\zeta(\lambda(z_1, z_2)) = A(z_1, z_2) + \arg\lambda$ for every $\lambda \in \mathbb{C}$ close enough to 1. Indeed, such a A_ζ can be constructed by fixing $(a, b) \in X_\zeta$, next defining $B_\zeta(\lambda(a, b)) = \log(\lambda)$ for λ close enough to 1, and then defining B_ζ on the whole of X_ζ by analytic continuation. This can be done since X_ζ is contractible and does not contain the origin. Now we let $A_\zeta(\lambda\,(a, b)) = \mathrm{Im}(B_\zeta(\lambda(a, b))$. No matter how A_ζ is constructed, once the value at (a, b) is prescribed, A_ζ is uniquely determined. Without loss of generality we may assume that $b \neq 0$ and consider the neighborhood around $(a:b)$ given by

$$\omega = \{(\alpha : b) : |\alpha - a| < \epsilon\}$$

for small ϵ. Then we could extend the function A_ζ to a function A_ω defined in $X_\omega = \Omega \cap P^{-1}(\omega)$, where the projection P is defined by $P(z_1, z_2) = (z_1 : z_2)$, by prescribing $A_\omega(\alpha, b) = A_\zeta(a, b)$ for $|\alpha - a| < \epsilon$ and extending A_ω the same way as we did with A_ζ. Note that A_ω is the imaginary part of an analytic function on X_ω, and hence it is a C^∞ function on X_ω. Since $\mathbb{P}$ is compact we may use a partition of unity of $\mathbb{P}$ and construct a C^∞ function A defined on the whole of Ω where $A(\lambda z_1, \lambda z_2) = \arg(\lambda) + A(z_1, z_2)$ for $z_1, z_2 \in \Omega$ and $|1 - \lambda| < \epsilon_z$.

Since $A(L_\zeta)$ is an open interval in $\mathbb{R}$ for every $\zeta \in \mathbb{P}$ we may define a C^∞ function $\phi : \mathbb{P} \to \mathbb{R}$ such that for every $\zeta \in \mathbb{P}$, there is a $z = (z_1, z_2) \in \Omega$ with $(z_1 : z_2) = (\zeta_1 : \zeta_2)$ and $A(z) = \phi(\zeta)$. Moreover, z is uniquely defined up to multiplication with a positive scalar, and hence the unit vector $z/|z|$ is uniquely defined. This construction produces a function from $\mathbb{P}$ to $\mathbb{C}^2 \setminus \{0\}$ which by the implicit function will be C^∞. Hence the function $F(\zeta_1 : \zeta_2) = z/|z|$ produces a non vanishing C^∞ global section on $\mathbb{P}$. This contradicts a well known theorem and hence Ω must be lindn.se early convex.

Now consider the case when $n > 2$ and assume that we know the statement to be true for any lower dimension. By using the same argument as above, one can conclude that there is at least one complex line through the origin that does not intersect Ω. Indeed one of all the complex lines that pass through the origin and lie inside $\{(z_1, z_2, 0, \ldots, 0) : z_1, z_2 \in \mathbb{C}\}$ must have an empty intersection with Ω. We may assume that this line equals $\{z : z_2 = \ldots = z_n = 0\}$. Let T be the projection $T(z_1, \ldots, z_n) = (z_2, \ldots, z_n)$. It is clear $T(\Omega)$ is $\mathbb{C}$-convex. Indeed, since T is surjective, for every complex line $L \subset \mathbb{C}^{n-1}$ there is a line $L' \subset \mathbb{C}^n$ such that the restriction of T to L' is a homeomorphism onto L. Further more we have $L \cap T(\Omega) = T(L' \cap \Omega)$ so $L \cap T(\Omega) \cong L' \cap \Omega$. By induction there is a complex hyperplane $S \subset \mathbb{C}^{n-1}$ that passes through the origin and does not intersect $T(\Omega)$. Hence $T^{-1}(S)$ does not intersect Ω and the statement is proved.

$\mathbb{C}\mathcal{C} \Rightarrow \mathcal{HC}$

Let Ω be a domain that does not contain a complex line, then by Proposition 3 in [NPZ06], we have that Ω is hyperconvex. Without this assumption, this statement is false as we shall sea later on.

$\mathcal{LC} \Rightarrow \mathcal{WLC} \Rightarrow \mathcal{LWLC}$

This is obvious.

$\mathcal{WLC} \Rightarrow \psi\mathcal{C}$

If Ω is the whole of $\mathbb{C}^n$, then it is clearly pseudoconvex, and hence we may assume that Ω is strictly contained in $\mathbb{C}^n$. Since Ω is weakly linearly convex, to any given a point $p \in \partial\Omega$, there is a complex hyperplane L_p containing p but not intersecting Ω. Hence the boundary dinstance function δ may be written as

$$\delta(z) = \inf_{p\in\partial\Omega} |L_p - z|.$$

For fixed $p \in \partial\Omega$, the function $-\log|L_p - z|^2$ is plurisubharmonic. Hence $\phi = -\log\delta^2$ will be plurisubharmonic since it is the maximum of plurisubharmonic functions. Now we define $\psi(z) = \phi(z)+|z|^2$ which is also a plurisubharmonic function. Now the sublevel set $\Omega_x = \{z \in \Omega : \psi(z) \leq x\}$ is relatively compact in Ω for all real values x, hence Ω is pseudoconvex.

$\mathcal{HC} \Rightarrow \psi\mathcal{C}$

Assume that Ω is hyperconvex. Then there exists a continuous negative plurisubharmonic function ϕ defined on Ω such that ϕ tends to zero when approaching the boundary. Since $\phi \in \mathcal{PSH}(\Omega)$ and $\phi < 0$, the function $\psi = -1/\phi$ will be plurisubharmonic. It is clear that $\Omega_x = \{z \in \Omega : \psi(z) \leq x\}$ is relatively compact in Ω for every real x. Hence Ω is pseudoconvex.

Now to the situation when bounded domains of class C^1 are considered. What is left to show is, firstly equality of local weak linear convexity and complex convexity, and secondly, equality between hyperconvexity and pseudoconvexity.

$\mathcal{LWLC} \Rightarrow \mathbb{C}\mathcal{C}$

We follow the proof given by Proposition 4.6.4 in [Hör94]. First we assume only that Ω is a locally weakly linearly convex domain with C^1 boundary but not necessarily bounded. Let L be a complex affine line in $\mathbb{C}^n$. Assume that $X_L = L \cap \Omega$ is non empty. Take a point z that belongs to the boundary of X_L as a subset of L, that is $z \in \overline{X_L} \setminus X_L$. Since Ω is locally weakly linearly convex, L cannot be contained in the tangent plane at z, hence it has to intersect $\partial\Omega$ transversely at z. From now on we add the assumption that Ω is bounded.

Take two points z_0, z_1 in Ω and denote the complex line through them by L_1. Assume that z_0 and z_1 lie in different components of $\Omega \cap L_1$. Since Ω is connected and has C^1 boundary, there is a path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = z_0$ and $\gamma(z_1) = 1$. Let L_t be the complex line through z_0 and $\gamma(t)$ and let $S \subset (0,1]$ be the set where z_0 and $\gamma(t)$ lie in the same component of $\Omega \cap L_t$. It is clear that S is open in $(0,1]$ and that $t \in S$ for t small enough. Since L_t intersects $\partial\Omega$ transversely, any compact subset of the intersection $\partial\Omega \cap L_t$ will transform continuously when we alter t. Since Ω is bounded $\partial\Omega \cap L$ is compact for all L, hence $(0,1] \setminus S$ will also be open in $(0,1]$. Therefore $1 \in S$ and $\Omega \cap L_1$ is connected.

Now take a line L where $\Omega \cap L \neq \emptyset$. Since Ω is bounded and L intersects Ω transversely we must have that $L \cap \Omega$ is homeomorphic to $\Omega \cap L'$ for small perturbations L' of L. The set of complex lines such that $L \cap \Omega \neq \emptyset$ is connected, and any two lines with non empty intersection with Ω can be connected via a compact path in the topology of affine complex lines. Hence, to show that $L \cap \Omega$ is simply connected for all lines intersecting Ω, we only have to show this for one line. Since Ω is bounded, there is a ball B containing Ω such that $\partial B \cap \partial\Omega$ is non empty. Without loss of generality we may assume that the origin lies in $\partial B \cap \partial\Omega$ and that the real tangent plane at the origin is $\{z : \mathrm{Im}(z_n) = 0\}$. Since Ω is C^1 we can introduce the coordinates $z' = (z_1, \ldots, z_{n-1}, \mathrm{Re}(z_n))$ and parametrize the boundary around the origin as

$$\{z : \mathrm{Im}(z_n) = \phi(z'))\}$$

for z' small and some C^1 function ϕ. Since Ω is included in the sphere we know that $\phi(z') > c|z'|^2$ for some positive constant c. It is clear that the complex line $L = \{(z_1, \ldots, z_n) : z_2 = \ldots = z_{n-1} = 0, z_n = i\,\epsilon\}$ has the property that $\Omega \cap L$ is simply connected for small ϵ.

$\psi\mathcal{C} \Rightarrow \mathcal{HC}$

Let Ω be a pseudoconvex domain with C^1 boundary. Then, by Proposition 1 in [KR81], there exists a negative plurisubharmonic continuous function ϕ that tends to zero when approaching the boundary. If Ω is bounded, the subdomains $\{z \in \Omega : \phi(z) \leq x\}$ are compact for all real and negative values x. Hence Ω is hyperconvex.

Remark 2.3.1. As pointed out in Remark 2.2.2 it is essential that the C^1 domain Ω is bounded in order to deduce that if Ω is locally weakly linearly convex, then it is also $\mathbb{C}$-convex. What fails in the proof $\mathcal{LWLC} \Rightarrow \mathbb{C}\mathcal{C}$ above are two things. Firstly, and quite obviously, there is no ball in $\mathbb{C}^2$ of which H is a subset. Secondly, and more interesting, the topology of $H \cap L$ is different for different lines L with $H \cap L \neq \emptyset$. To see this we only have to consider the intersections between H and the two different lines $\{(z,w) : z = 2\}$ and $\{(z,w) : w = 1\}$. This happens even though every complex affine line intersecting H, intersects the boundary of H transversely, a fact that applies for *all* locally weakly linearly complex domains ω with C^1 boundary, as seen in the argument above. To get a picture of what is going on, we may consider the cone $C = \{(z,r) : r = |z-2|\}$. The intersection between the cone and $\pi(H)$ is contractible. However, let C_ϵ be the cone $\{(z,r) : r = (1-\epsilon)|z-2|\}$, then the intersection between $\pi(H)$ and C_ϵ is not simply connected for any positive $\epsilon < 1$. Hence, the statement that small perturbations of the complex affine lines do not alter the topology for the intersection between the line and the weakly linearly complex C^1 domain Ω, is not correct if Ω is unbounded. This can be understood alternatively by noting that the complex line $L = \{(z,w) : w = z-2\}$ has a non transverse intersecting point with the boundary of H at the point at infinity. •

Exclusions

$A = \mathbb{C}\mathcal{C} \setminus \mathcal{C}$

Consider that Hartogs domain

$$H = \{(z,w) : |z| < 1, |w|^2 < h(z) = |2-z|^2\}.$$

Since the base domain of this Hartogs domain is a disc and the defining function h is C^2, a theorem due to Kiselman, see [Kis96], implies that H is $\mathbb{C}$-convex precisely if $u = -\log h$ is $\mathbb{C}$-convex as a C^2 function. Calculating the important derivatives we get

$$\begin{aligned} u_z &= -\frac{1}{z-2}, \\ u_{z\bar{z}} &= 0, \\ u_{zz} &= \frac{1}{(z-2)^2}. \end{aligned}$$

The derivatives above give equality when inserted into inequality (2.1.2). Hence H is $\mathbb{C}$-convex. As claimed in Figure 2.3, there should also be

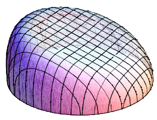
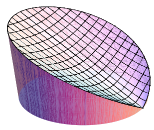
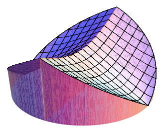
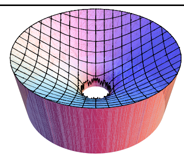
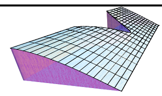
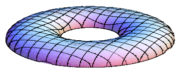
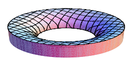
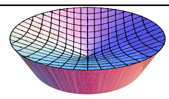
Set	Bounded C^1	Bounded C^0	Unbounded C^1
A			trivial
B	does not exist		
C	does not exist	Hard to draw	?
D	does not exist		?
E			?
F	does not exist		?

Table 2.1: Some domains that show upon exclusion among the concepts in definition 2.1.2. Questionmark indicates that this us unknown to the author. We use the abbreviations $A = \mathbb{C}\mathcal{C} \setminus \mathcal{C}$, $B = \mathcal{LC} \setminus \mathbb{C}\mathcal{C}$, $C = \mathcal{WLC} \setminus \mathcal{LC}$, $D = \mathcal{LWLC} \setminus \mathcal{WLC}$, $E = \mathcal{HC} \setminus \mathcal{LWLC}$, and $F = \mathcal{LC} \setminus \mathcal{HC}$.

bounded C^1 domains in $\mathbb{C}^n$ which are C^1 and $\mathbb{C}$-convex but not convex. One can construct such a domain by smoothing off the edges of the Hartogs domain H, actually this construction was a part of the proof by Kiselman. More precisely, one wants a family $\{h_r : 0 < r < 1\}$ of C^2 functions defined on the closed disc where $h_r(z) > 0$ if $|z| < 1$, $-\log h_r$ is $\mathbb{C}$-convex inside the open unit disc, $h_r(z) \nearrow h(z)$ when $r \to 1$ if $|z| < 1$ and $h_r(z) = 0$ if $|z| = 1$. The last condition guarantees that $H_r = \{(z, w) : |z| < 0, |w|^2 < h_r(z)\}$ is C^1. As shown in Theorem 2.5.16 in [APS04], $\{h(z)\phi_r(|z|)\}$ where

$$\phi_r(x) = \begin{cases} 1 & 0 \le x < r \\ 1 - (x-r)^3/(1-r)^3 & r < x \le 1 \end{cases}$$

is an example of such a family. The figure in Table 2.1 is drawn with $r = 0.9$. It is trivial to construct unbounded C^1 domains in $\mathbb{C}^n$ that are $\mathbb{C}$-convex but not convex. Indeed Let X be any bounded contractible and non convex C^1 domain in $\mathbb{C}$, then $X \times \mathbb{C}$ will be $\mathbb{C}$-convex and C^1 but not convex. One often referes to these kind of product domains as *degenerate* $\mathbb{C}$-convex domains.

$B = \mathcal{LC} \setminus \mathbb{C}\mathcal{C}$

First we give an example of such a bounded domain. Define

$$H = \{(z, w) : |z| < 1, |w| < \min(|1 - z|, |1 + z|)\}$$

which is the intersection of two $\mathbb{C}$-convex Hartogs domains. Looking at the figure in Table 2.1, it seems plausible that H is linearly convex. To show this, fix a point (z_0, r_0) belonging to the complement of $\pi(H)$. From the proof of Lemma 2.2.1, it is clear that we shall find a cone containing the point without intersecting $\pi(H)$. If $|z_0| > 1$ we may take the degenerate cone $\{z_0\} \times \mathbb{R}_{\ge 0}$. Otherwise we may assume that $\mathrm{Re}(z_0) \ge 0$ and the we may take the cone $\{(z, r) : r = r_0|z-1|/|z_0-1|\}$ which clearly does not intersect $\pi(H)$ However, intersecting $\pi(H)$ with the cone $\{(z, r) : r = 11/10\}$ and using Lemma 2.2.1 we conclude that H is not $\mathbb{C}$-convex.

For the C^1 unbounded case, see Remark 2.2.2.

$C = \mathcal{WLC} \setminus \mathcal{LC}$ Here we refer to Example 2 in [Hör94] on page 297. As mentioned later on, if H is a Hartogs domain in $\mathbb{C}^2$, then it is locally linearly convex if it is weakly locally linearly convex. Hence there is no such counterexample in this case. That is why it written, "hard to draw" in Table 2.1.

$D = \mathcal{LWLC} \setminus \mathcal{WLC}$

Here see Remark 3.1.15 and Figure 3.5.

$E = \mathcal{HC} \setminus \mathcal{LWLC}$

Consider the torus $T = \{(z, w) : (2 - |z|)^2 + |w|^2 < 1\}$. Obviously T is not $\mathbb{C}$-convex. Since it is bounded and C^1 it can according to the statements illustrated in Figure 2.3, not be locally weakly linearly convex. However it is hyperconvex. To show this we define $g(z) = (2 - |z|)^2$ and calculate

$$g_{z\bar{z}}(z) = \partial_{z\bar{z}}(|z|^2 + 4|z| + 4) = 1 - 1/|z|$$

which is positive on the domain $|z| > 1$. Hence g is pseudoconvex on T. Let $h(z, w) = g(z) + |w|^2 - 1$. Then h is a continuos and negative pseudoconvex function on T. It it clear that the sublevet sets $\{(z, w) \in T : h(z, w) \leq x\}$ are relatively compact in T for all real and negative valuex x. Hence T is hyperconvex.

The torus T is C^∞. To find an example of a bounded C^0 domain that is hyperconvex but not weakly linearly convex we only have to intersect T with a suitable cylinder as suggested by the figure in Table 2.1.

$F = \mathcal{LC} \setminus \mathcal{HC}$

For the bounded C^0-case we may consider the Hartogs triangle $T = \{(z, w) : |z| < 1, |w| < |z|\}$. From the picture it is clear that T is linearly convex. Indeed take a point (z_0, r_0) in the complement of $\pi(T)$. If $|z_0| < 1$ then the cone $C = \{(z, r) : r = r_0|z|/|z_0|\}$ will intersect (z_0, r_0) but not $\pi(T)$. If $|z_0| \geq 1$ we may take the degenerate cone with z-coordinate equal to z_0. To show that T is not hyperconvex one may use the following well known argumentation. Assume that ϕ is a negative plurisubharmonic function with $\phi(z, w) \to 0$ when $(z, w) \to \partial T$. The function $\psi(z) = \phi(z, 0)$ must be subharmonic on the punctured unit disc. Because of The removable singularities theorem for plurisubharmonic functions, see for example page 373 in [JP93], ψ extends to a subharmonic function on the whole unit disc. By the Maximum principle, it must be constant. This is a contradiction, hence T is not hyperconvex.

$G = \mathcal{C} \setminus \mathcal{HC}$

Let $\Omega = \mathbb{C}$ which is clearly convex. If Ω is hyperconvex there must exist a continuous plurisubharmonic and negative function ϕ on $\mathbb{C}$

such that the sublevel sets $\Omega_x = \{z \in \mathbb{C} : \phi(z) \leq x\}$ are compact for every negative x. By taking a large ball B with center at origin and using that $\phi(0)$ must greater that or eqaul to the mean value inside B, one concludes that $\phi(0) = 0$. This is a contradiction, hence $\mathbb{C}$ cannot be hyperconvex. If Ω is convex but not hyperconvex then, as stated before, Proposition 3 in [NPZ06] implies that Ω must contain a complex line.

2.4 Relation of properties

2.4.1 Intersection preservation

Considering the exclusion example $C = \mathcal{LC} \setminus \mathbb{C}\mathcal{C}$ one realizes that complex convexity is not closed under intersection. Each and one of the other convexity concepts in Definition 2.1.2 are. For convexity, linear convexity and its different weaker relatives, this is obvious. Since the maximum of two plurisubharmonic functions is again plurisubharmonic, the intersection preservation property is clear also for hyperconvexity and pseudoconvexity. This could be viewed as a weakness for the notion of complex convexity as it does not satisfy this rather important convexity property.

2.4.2 Distance functions and defining functions

Despite the differences mentioned above, there are big similarities between the concepts in Definition 2.1.2. The following three propositions show how the boundary distance function and the defining function characterize the convexity concepts in the bounded C^2 case. We will, when we feel necessary, point out the ideas of the proofs, instead of giving the full details. The relevance for us here is not so much the proofs themselves, but rather to exhibit the simmilarity between the properties of the convexity concepts. We shall begin by introducting the following two important quadratic forms, the Hessian and the Levi form:

$$H_\rho(z,w) = \sum \rho_{j\bar{k}}(z) w_j \bar{w}_k - \operatorname{Re}\left(\sum \rho_{jk}(z) w_j w_k\right), \tag{2.4.1}$$

$$L_\rho(z,w) = \sum \rho_{j\bar{k}}(z) w_j \bar{w}_k. \tag{2.4.2}$$

Note that both forms occur in one or more of the definitions of convex functions, $\mathbb{C}$-convex functions and plurisubharmonic functions given in Definition 2.1.1. The Hessian also occurs as the second degree term obtained when Taylor expanding a real valued function ρ depending on n complex variables. More precisely, if $\rho : \mathbb{C}^n \to \mathbb{R}$ is of class C^2 in a neighborhood of z, then we have

$$\rho(z+w) = \rho(z) + 2\operatorname{Re}\sum \rho_j(z) w_j + H_p(z,w) + o(|w|^2). \tag{2.4.3}$$

Proposition 2.4.1. *Let $\Omega = \{z : \rho(z) < 0\}$ be a domain in $\mathbb{C}^n$ where ρ is a C^2 defining function. Then the following properties are equivalent:*

i) The domain Ω is convex.

ii) The Hessian (2.4.1) of ρ is positive semidefinite when restricted to the real tangent plane at any point on $\partial\Omega$.

iii) The function $u = -\log \delta^2$ is convex near the boundary ∂D.

Proof. First we settle the equivalence $i) \iff ii)$. Take a point $z \in \partial\Omega$ and let $T_z^{\mathbb{R}}$ denote the real tangent plane to Ω at z. If we can show that $T_z^{\mathbb{C}} \cap D = \emptyset$ precisely if the Hessian is positive semidefinite at z, we are done. Since the defining function ρ is defined for values slightly bigger than zero we can Taylor expand it at z according to equation (2.4.3). If $z + w \in T_z$ then the first term of the Taylor expansion is zero and, hence the condition on the Hessian is necessary. Now assume that $H_\rho \geq 0$. Then it is possible to find (one can copy the perturbation of the defining function made in the proof of Theorem 3.2.5) a decreasing family of functions $\rho_n \searrow \rho$ with $H_{\rho_n} > 0$ which will define smaller domains D_n. The domains D_n will be strictly convex and increasing and since D is the union of those, it will itself be convex.

Now we turn to the equivalence $i) \iff iii)$. Assume that D is convex. Then at each point $z \in \partial D$ there exists a real tangent plane T_z which does not intersect D. We have

$$\delta(w) \leq \delta_{T_z}(w) = |w - T_z|$$
$$\delta(w) = \inf_{z \in \partial D} \delta_{T_z}(w).$$

The boundary function δ_{T_z} is affine and linear. The infimum of such functions is concave. Hence δ is concave. Since log is increasing and concave, $\log \delta$ is concave, and therefore $-\log \delta^2 = -2 \log \delta$ is convex. Now to the converse. Assume that there exists a non constant convex function $f : D \to \mathbb{R}$ such that

$$D_t = \{z \in D : f(z) \leq t\}$$

is compact for all $t < m = \sup_{z \in D} f(z)$. Now take $z_1, z_2, z_3 \in D$ and assume $[z_1, z_2]$, $[z_2, z_3] \subseteq D$. Let $X = \{z \in [z_1, z_2] : [z, z_3] \subseteq D\}$ and let t_0 be the supremum of f on the union of the line segments $[z_1, z_2]$, $[z_2, z_3]$. Note that t_0 is smaller than the supremum of f on D. Indeed, since D is open and f is convex and non constant, it cannot attain its supremum at any point of D. Otherwise it would be constant on every line going through that point and since D is assumed to be connected f would be constant on the whole of D. Since D is open, X is relatively open in $[z_1, z_2]$. But if $z \in [z_1, z_2]$ then

$$[z, z_3] \subseteq D \Rightarrow [z, z_3] \subseteq D_{t_0}$$

since t_0 was chosen to be the supremum of f on $[z_1, z_2]$, $[z_2, z_3]$, and f attains its maximum value on line segments at one of the endpoints. But D_{t_0} is by assumption compact, which implies that X is also relatively closed in $[z_1, z_2]$. Of course $z_2 \in X$, so X is not empty which implies $X = [z_1, z_2]$ and D is convex. If D is such that $-\log \delta^2$ is convex then $f = -\log \delta^2 + |z|^2$ is also convex. Since its level sets stay away from the boundary, D_t is closed. The $|z|^2$ term makes D_t bounded. Hence the function $f = \max(-\log \delta^2(z), N)$ has the properties described above, and therefore Ω must be convex. □

Proposition 2.4.2. *Let $D = \{z : \rho(z) < 0\}$ be a bounded domain in $\mathbb{C}^n, n > 1$, where ρ is a C^2 defining function whose gradient is non zero on ∂D. Then the following properties are equivalent:*

i) D is $\mathbb{C}$-convex.

ii) The Hessian (2.4.1) of ρ is positive semidefinite when restricted to the complex tangent plane at any point on ∂D.

iii) The function $u = -\log \delta^2$ is $\mathbb{C}$-convex near the boundary.

Proof. First we do $i) \Rightarrow ii)$. From the inclusion stated before and illustrated in Figure 2.3, we know that D is weakly linearly convex. Hence for every $z \in \partial D$ there is a complex tangent plane T_z that does not intersect D. If $z + w \in T_z$ with w such that $H_\rho(z, w) < 0$ we would according to equation (2.4.3) have $\rho(z + rw) < 0$ for small $r > 0$.

$ii) \Rightarrow iii)$: Let $d = \delta^2$ and fix a point $p \in D \cap N_D$, where the open set N_D containing the boundary of D is so small that $\delta \in C^2(N_D)$, and every point in N_D has a unique closest point on the boundary. Let V be a neighborhood of p which is contained in $N_D \cap D$ and let q be the closest boundary point to p. We intend to show that $-\log d$ is $\mathbb{C}$-convex in V. Without any loss of generality we may assume that $p = (0, iy_n)$ with $y_n > 0$ and $q = 0$. As D is C^1 we can locally parametrize ∂D as a graph over the real tangent plane $\{y_n = 0\}$. Since the condition on the Hessian is independent of the choice of ρ as long as $\nabla \rho$ is non zero on the boundary, see Remark 2.4.6, we may assume that

$$\rho(z) = f(z', \mathrm{Re}(z_n)) - \mathrm{Im}(z_n), \qquad \forall z \in V,$$

where $z' = (z_1, ..., z_{n-1})$, f is C^2 and $f(0) = f_j(0) = 0$. Now take a small w with $p + w \in V$. Of course $(w', i\, f(w', 0)) \in \partial D$ and hence, with $w_n = u_n + i\, v_n$, we get

$$\begin{aligned} d(p + w) &\leq |(p + w) - (w', i\, f(w', 0)|^2 = |i\, y_n + w_n - i\, f(w', 0)|^2 \\ &\leq y_n^2 + 2y_n v_n + u_n^2 + f(w', 0)^2 \\ &\leq d(z) + 2\mathrm{Re}(\sum d_j(z) w_j) + |\sum d_j(z) w_j|^2/d(z) + o(|w|^2). \end{aligned}$$

Now we can compare the last line with equation (2.4.3) and conclude that

$$H_d(z, w) \leq |\sum d_j(z) w_j|^2/d(z),$$

which combined with equations (2.4.1) and (2.1.2) implies that $-\log d$ is $\mathbb{C}$-convex.

$iii) \Rightarrow i)$: Since D is C^1 and bounded it is, by the inclusion stated before and illustrated in Figure 2.3, enough to prove that D is weakly linearly convex. For this we take a point $q \in \partial D$ and let $p \in N_D$ have q as its closest boundary point. We may assume that p and q are as before. Let w be such that $p + \lambda w \in V$ for all $|\lambda| \leq 1$. Then we define $u(\lambda) = -\log d(p + \lambda w)$ which implies that u is $\mathbb{C}$-convex in the unit disc U, and hence the corresponding Hartogs domain

$$H = H_U^u = \{(\lambda, \sigma) : |\sigma|^2 < d(z + \lambda w)\}$$

is $\mathbb{C}$-convex. Then we define the following linear transformation

$$\begin{aligned} T &: H \to D \\ (\lambda, \sigma) &\mapsto z + \lambda w + (0, ..., 0, \sigma). \end{aligned}$$

This image domain $\tilde{H} = T(H)$ will be embedded in D and since $0 \in \partial \tilde{H}$, D and $\tilde{H}$ must be tangential, so $L = \{z : z_n = 0\}$ must be the complex tangent plane to both D and $\tilde{H}$ at the origin. Since $p+w \in \partial \tilde{H}$ and $d(p+w) \leq |p+w|^2$ we must have a point $a \in \partial \tilde{H}$ with $d(p+w) = |p+w-a|^2$. But $\tilde{H}$ is $\mathbb{C}$-convex, and hence by the inclusions illustrated in Figure 2.3, lineally convex so $L \cap \tilde{H} = \emptyset$. Therefore

$$|p + w - a|^2 = d(p + w) \leq \inf_{b \in L} |p + w - b|.$$

This means that L cannot intersect D near the origin and we are done. □

Proposition 2.4.3. *Let $D = \{z : \rho(z) < 0\}$ be a domain in $\mathbb{C}^n, n > 1$ where ρ is a C^2defining function whose gradient is different from zero on ∂D. Then the following are equivalent:*

i) The domain D is pseudoconvex.

ii) The Levi form (2.4.2) of ρ is positive semidefinite when restricted to complex tangent plane at any point on ∂D.

iii) The function $u = -\log \delta^2$ is plurisubharmonic near the boundary.

Proof. We sketch some basic steps in the proof. $i) \Rightarrow ii)$: Assume that $ii)$ is false. Then there exists a point z_0 on the boundary of D and a quadratic polynomial q such that $\{z : |z_0 - z| < \epsilon, q(z) < 0\}$ is contained in D and there is a non zero w with $\sum q_j(z_0)w_j = 0$. By Theorem 4.1.25 in [Hör94] D is pseudoconvex. $ii) \Rightarrow iii)$: Assume that $iii)$ is false. By Theorem 4.1.19 in [Hör94] we know that $i)$ is false. However, if $i)$ is false, know by Theorem 4.1.25 in [Hör94] that there is a quadratic polynomial as the one described above. By the same theorem we deduce that D cannot be pseudoconvex. $iii) \Rightarrow i)$: Obvious. □

Remark 2.4.4. As pointed out in the introduction, one example on how important convexity is for complex analysis is the fact that a domain D is a domain of holomorphy precisely if it is pseudoconvex. To show this, take a compact subset K of D. Then define the analytic convex hull and the pseudoconvex hull:

$$\text{ach}(K,D) = \{z \in D : |f(z)| \leq \sup_{z' \in K} |f(z')| \ \forall f \in \mathcal{A}(D)\},$$
$$\psi\text{ch}(K,D) = \{z \in D : f(z) \leq \sup_{z' \in K} f(z') \ \forall f \in \mathcal{PSH}(\text{D})\}.$$

We claim that the following two statements are equivalent:

$$\text{ach}(K,D) \Subset D \qquad \forall K \Subset D, \tag{2.4.4}$$
$$\psi\text{ch}(K,D) \Subset D \qquad \forall K \Subset D. \tag{2.4.5}$$

That (2.4.4) implies (2.4.5) is trivial, because if $f \in \mathcal{A}(D)$ then $\log|f| \in \mathcal{PSH}(D)$. The converse is by far more difficult and the details can be found in Corollary 4.2.8[Hör94]. The property (2.4.4) is quite easily shown to be equivalent with $-\log\delta$ being plurisubharmonic (see Theorem 4.1.19 in [Hör94]) so it is by Figure 2.3 equivalent with D being pseudoconvex. On the other hand it is also quite straightforward to prove that equation (2.4.5) is equivalent with D being a domain of holomorphy, for details see Theorem 2.5.5 in [Hör73]. •

2.4.3 Strict convexity conditions

The three propositions above give us a hint about how to define the strict counterparts of the convexity notions in Definition 2.1.2. We shall do this in the C^2 case, and according to the inclusions illustrated in Figure 2.3, we only have to consider convexity, $\mathbb{C}$-convexity and pseudoconvexity.

Definition 2.4.5. Let Ω be the domain $\{z \in \mathbb{C}^n : \rho(z) < 0\}$ where the defining function ρ is of class C^2. We label the different convexity properties that Ω may satisfy according to the following list:

> **Strict convexity**, the Hessian $H_\rho(z,w)$ is positive for every $z \in \partial\Omega$ and every non zero w such that $z+w$ lies in the real tangent plane at z.
>
> **Strict $\mathbb{C}$-convexity**, the Hessian $H_\rho(z,w)$ is positive for every $z \in \partial\Omega$ and every non zero w such that $z+w$ lies in the complex tangent plane at z.

Strict pseudoconvexity, the Levi form $L_\rho(z, w)$ is positive for every $z \in \partial\Omega$ and every non zero w such that $z + w$ lies in the complex tangent plane at z.

Remark 2.4.6. In order for the definitions to make sense, we need to show that the different conditions on ρ are independent of which defining function ρ we choose. Consider two defining functions ρ_1 and ρ_2. Then there will be a positive function h such that $\rho_1 = h\,\rho_2$ in a neighborhood of the boundary. Now take a point z at the boundary, if $z + w$ lies in the complex tangent plane at z then we have

$$H_{\rho_1}(z, w) = h(z)\, H_{\rho_2}(z, w)$$
$$L_{\rho_1}(z, w) = h(z)\, L_{\rho_2}(z, w).$$

Hence the two Hessians will have the same sign and simmilarly for the Levi forms. •

Looking at equation (2.4.3) we realize that the domain $\Omega = \{z : \rho(z) < 0\}$ is strictly $\mathbb{C}$-convex precisely if

$$\rho(z + w) \geq c_z |w|^2, \tag{2.4.6}$$

for some positive constant c_z, all $z \in \partial\Omega$, and all w small enough and such that $z + w$ lies in the complex tangent plane at z. Since this definition remains valid if ρ is only a defining function of class C^1 we can make the following definition.

Definition 2.4.7. Let $\Omega = \{z : \rho(z) < 0\}$ be a domain in $\mathbb{C}^n$ where ρ is a defining function of class C^1. If ρ satisfies inequality (2.4.6), then Ω is called *strictly* $\mathbb{C}$-convex.

Chapter 3

Complex Convexity

3.1 $\mathbb{C}$-convex functions

Here we look at some patterns that show up for $\mathbb{C}$-convex functions and sets in the C^2 case. We shall extract some conditions that do not at first sight seem to have anything to do with C^2, and then use them to make the following definition.

Definition 3.1.1. Let u be a real valued C^1 function defined in a neighborhood of $z \in \mathbb{C}$. Then u is called $\mathbb{C}$-*convex* at z if there is a positive number δ_z such that

$$u(w) \geq u(z) - \log|(w-z)u_z(z) - 1|^2 \qquad (3.1.1)$$

for every w with $|w - z| < \delta_z$.

If there exist two positive constants δ_z and c_z such that

$$u(w) \geq u(z) - \log|(w-z)u_z - 1|^2 + c_z|z-w|^2 \qquad (3.1.2)$$

for every w with $|w - z| < \delta_z$, then u is called *strictly* $\mathbb{C}$-*convex* at z. A function is called (strictly) $\mathbb{C}$-convex in an open set $\Omega \subset \mathbb{C}$ if it is (strictly) $\mathbb{C}$-convex at each point of Ω. A function depending on several complex variables is called (strictly) $\mathbb{C}$-convex if the restriction to any complex line is (strictly) $\mathbb{C}$-convex.

3.1.1 Some basic comparisons with the C^2 case

A basic requirement on Definition 3.1.1 is that it does not contradict the standard definition for $\mathbb{C}$-convex functions of class C^2. The definitions, see Definition 2.5.15 in [APS04], of a C^2 $\mathbb{C}$-convex function, and strictly $\mathbb{C}$-convex function respectively, from $\mathbb{C}$ to $\mathbb{R}$ can be stated as the differential conditions

$$u_{z\bar{z}} \geq |u_{zz} - u_z\, u_z|, \tag{3.1.3}$$
$$u_{z\bar{z}} > |u_{zz} - u_z\, u_z|, \tag{3.1.4}$$

whose non linearity provides substantial difficulties in generalizing $\mathbb{C}$-convexity to lower order of smoothness. The following lemma builds a first bridge between the two definitions.

Lemma 3.1.2. *Let u be a real valued C^2 function defined in a neighborhood of $z \in \mathbb{C}$. If u is $\mathbb{C}$-convex at z in the sense of (3.1.1), then it is $\mathbb{C}$-convex at z in the sense of (3.1.3). Furthermore, u is strictly $\mathbb{C}$-convex at z in the sense of (3.1.2) if and only if it is strictly $\mathbb{C}$-convex at z in the sense of (3.1.4).*

Proof. First we consider the non strict case. Let $A = u(w) - u(z)$ and $B = -\log|u_z(z)(w-z) - 1|^2$. Note that whenever the quantity

$$x = |u_z(z)(w-z)|^2 - 2\mathrm{Re}[u_z(z)(w-z)]$$

is small, we may write

$$B = -\log(1+x) = -x + x^2/2 + O(x^3).$$

Expanding A and B in Taylor series and using the above mentioned expansion for B, we get

$$\begin{aligned}
A &= 2\mathrm{Re}[u_z(z)(w-z)] + u_{z\bar{z}}(z)|w-z|^2 + \mathrm{Re}[u_{zz}(z)(w-z)^2] + o|w-z|^2 \\
B &= -|u_z(z)(w-z)|^2 + 2\mathrm{Re}[u_z(z)(w-z)] \\
&\quad + 2\mathrm{Re}[u_z(z)(w-z)]^2 + O|w-z|^3 \\
&\quad = 2\mathrm{Re}[u_z(z)(w-z)] + \mathrm{Re}[(u_z(z)(w-z))^2] + O(|w-z|^3)
\end{aligned}$$

Note that the last equality for B is just a messy reformulation of the triviality $(a^2 + \bar{a}^2)/2 = (a + \bar{a})^2/2 - a\bar{a}$. It is clear that (3.1.1) is equivalent with $A - B \geq 0$. Since the condition (3.1.1) has to be valid for all w close enough to z, we can rotate $w - z$ and write $A - B \geq 0$ in its strongest way. More precisely, if $|z - w|$ is fixed then $A - B$ attains its minimum when $(u_z z(z) - u_z(z)^2)(w - z)^2$ is purely real and negative. Hence we end up with

$$u_{z\bar{z}}(z) \geq |u_z(z)^2 - u_{zz}(z)| + \frac{o(|w - z|^2)}{|w - z|^2}.$$

Letting $w \to z$ proves the non strict case. Now assume that (3.1.2) holds for w close enough to z. As before we can Taylor expand $A - B$. However in this case we can remove the $o(|w - z|^2)$ and $O(|w - z|^3)$ terms that we got above when we started out from inequality (3.1.1). This is due to the $c_z|w - z|^2$ term that dominates both the remainder terms. Hence, inequality (3.1.2) holding for w close enough to z, is equivalent with the inequality

$$u_{z\bar{z}}(z) \geq |u_z(z)^2 - u_{zz}(z)| + c_z,$$

and the second statement of the lemma is proved. □

Remark 3.1.3. It is clear that one cannot get equivalence in the non strict statement in Lemma 3.1.2. To see this one may consider the function $u = 1 - |z|^4$ at $z = 0$. Since all derivatives of order two and lower vanish at the origin, u will be ℂ-convex at $z = 0$ in the sense of (3.1.3). However, u is not ℂ-convex at the origin in the sense of (3.1.1). We will later show that the two definitions coincide for C^2 functions on open sets, rather than at single points, see Corollary 3.1.14. •

In the C^2 case one often works with the function $\exp(-u)$ instead of u for ℂ-convex functions u. One reason for this is the important theorem of Kiselman which states that if u is a C^2-function then the Hartogs domain $\{(z, w) : |z| < r, |w|^2 < \exp(-u(z))\}$ is ℂ-convex precisely if u is ℂ-convex. We will later show the same thing for C^1 functions. In order to do this we need the following lemma.

Lemma 3.1.4. *Let u be a real valued C^1 function, $h = \exp(-u)$ and $g =$*

$\sqrt{h}$. *Pick two points $z, w \in \mathbb{C}$, then the following are equivalent*

$$u(w) \geq u(z) - \log|(w-z)u_z(z) - 1|^2 \tag{3.1.5}$$
$$h(w) \leq |h(z) + h_z(z)(w-z)|^2/h(z) \tag{3.1.6}$$
$$g(w) \leq |g(z) + 2g_z(z)(w-z)|. \tag{3.1.7}$$

Proof.
We prove the equivalence by showing the following three implications.

(3.1.5) $\Rightarrow$ (3.1.6)
Plugging in $-\log(h) = u$ into (3.1.5) we get the following equivalent inequalities

$$\begin{aligned} -\log(h(w)) &\geq -\log(h(z)) - \log|-h_z(z)(w-z)/h(z) - 1|^2 \\ h(w) &\leq h(z)| - h_z(z)(w-z)/h(z) - 1|^2 = \frac{|h_z(z)(w-z) + h(z)|^2}{h(z)} \end{aligned}$$

which proves the statement.

(3.1.6) $\Rightarrow$ (3.1.7)
Plugging in $g^2 = h$ into (3.1.6) and one gets

$$g(w)^2 \leq \frac{|2g_z(z)\, g(z)\, (w-z) + g(z)^2|^2}{g(z)^2} = |2g_z(z)(w-z) + g(z)|^2.$$

Taking the square root of the left hand side and right hand side respectively proves the statement.

(3.1.7) $\Rightarrow$ (3.1.5)
Plugging in $\exp(-u/2) = g$ into inequality (3.1.7) we get the following equivalent inequalities

$$\begin{aligned} \exp(-u(w)/2) &\leq |\exp(-u(z)/2) - \exp(-u(z)/2)\, u_z(z)(w-z)| \\ \exp(u(z)/2 - u(w)/2) &\leq |1 - u_z(z)(w-z)| \\ u(w) &\geq u(z) - \log|u_z(z)(w-z) - 1|^2 \end{aligned}$$

which proves the statement.

□

We also need the same kind of lemma for the strict versions of the inequalities.

Lemma 3.1.5. *Let u be a real valued C^1 function from $\mathbb{C}$ and let $h = -\exp(u)$ and $g = \sqrt{h}$. Fix a point $z \in \mathbb{C}$. Then for w close enough to z, the following inequalities are equivalent*

$$u(w) \geq u(z) - \log|(w-z)u_z(z) - 1|^2 + c_1|w-z|^2 \tag{3.1.8}$$
$$h(w) \leq |h(z) + h_z(z)(w-z)|^2/h(z) - c_2|w-z|^2. \tag{3.1.9}$$
$$g(w) \leq |g(z) + 2g_z(z)(w-z)| - c_3|w-z|^2 \tag{3.1.10}$$

Here c_1, c_2, c_3 are some positive real constants.

Proof. We prove the equivalence by showing the following three implications.

(3.1.8) ⇒ (3.1.9)

Again, from the proof of Lemma 3.1.4 we know that (3.1.8) implies that

$$\log(h(w)) \leq \log(h(z)) + \log|-h_z(z)(w-z)/h(z) - 1|^2 - c_1|w-z|^2$$

but this clearly implies that

$$\exp(c_1|w-z|^2) \leq h(z)\ |-h_z(z)(w-z)/h(z) - 1|^2/h(w)$$

which implies that

$$1 + c_1|w-z|^2 + O(|w-z|^4) \leq h(z)\ |-h_z(z)(w-z)/h(z) - 1|^2/h(w)$$

which proves the statement.

(3.1.9) ⇒ (3.1.10)

Put in $g^2 = h$ into (3.1.9). Then we get the inequality

$$g^2(z) \leq \underbrace{|g(z) + 2g_z(z)(w-z)|}_{a}{}^2 - c_2\underbrace{|w-z|}_{b}{}^2.$$

which gives us that

$$g(z) \leq \sqrt{a^2 - c_2b^2}.$$

Now take a positive number t that is smaller than $1/(2a)$. Then we will have

$$(a - tc_2b^2)^2 = a^2 - 2tc_2ab^2 + t^2c_2^2b^4 \geq a^2 - c_2b^2.$$

If we denote $c_3 = tc_2$ we can write

$$g(z) \leq a - tc_2b^2 = |g(z) + 2g_z(z)(w-z)| - c_3|w-z|^2$$

which proves the statement.

(3.1.10) $\Rightarrow$ (3.1.8)

From the proof of Lemma 3.1.4 we get that (3.1.10) implies

$$\exp(\frac{u(z)-u(w)}{2}) \leq |1 - u_z(z)(w-z)| - (c_3|w-z|^2)\exp(u(z)/2).$$

If we divide both sides with $|1 - u_z(z)(w-z)|$ and apply log on both sides we get the inequality

$$\frac{u(z)-u(w)}{2} - \log|1-u_z(z)(w-z)| \leq \log\left(1 - \frac{c_3|w-z|^2\exp(u(z)/2)}{|1-u_z(z)(w-z)|}\right).$$

Denoting the right hand side of the inequality above by $R_1(w)$, we get by Taylor expanding in the w variable that

$$R_1(w) = -c_3|w-z|^2\exp(u(z)/2) + f_1(w)|w-z|^3$$

where f_1 is a continuous real valued function. Denoting $c_1 = c_3\exp(u(z)/2)$ the statement is proved.

□

In the C^2 case it is trivial to show that if a function $u : \mathbb{C} \to \mathbb{R}$ is $\mathbb{C}$-convex then it is also plurisubharmonic. Indeed if if u fulfills inequality (3.1.3) then obviously $u_{z\bar{z}} \geq 0$. In the C^1 case it is almost as easy. Assume that $u : \mathbb{C} \to \mathbb{R}$ is C^1 and u fulfills inequality (3.1.1). Then we consider the mean value of u according to

$$\begin{aligned} u(z) &= \frac{1}{|B_r(z)|}\int_{B_r(z)} u(z)dw \leq \\ &\leq \frac{1}{|B_r(z)|}\int_{B_r(z)} u(w) + \log|(w-z)u_z(z) - 1|^2 dw = \\ &= \frac{1}{|B_r(z)|}\int_{B_r(z)} u(w)dw. \end{aligned}$$

The last equality is due to the harmonicity of $f(w) = \log|(w-z)u_z(z) - 1|$. In neither of the situations described it is a problem that we only considered

ℂ-convex functions of one variable. In both the C^1 and C^2 cases a function u dependent of n complex variables is called ℂ-convex precisely if its restriction to any complex line is ℂ-convex. Hence if a function u is C^1 and ℂ-convex then it is plurisubharmonic.

3.1.2 Strictly ℂ-convex C^1 functions.

Here we discuss the concept of a strictly ℂ-convex C^1-function, which will be important in the connection between ℂ-convex functions and weakly linearly convex Hartogs domains.

Lemma 3.1.6. *Let $\Omega \subset \mathbb{C}$ be a bounded open set and let $u : \Omega \to \mathbb{R}$ be a bounded C^1 ℂ-convex function. Then there exists a sequence of strictly ℂ-convex C^1 functions $u^\epsilon : \Omega \to \mathbb{R}$ that decreases uniformly to u on Ω.*

Proof. Let $h = \exp(-u)$. From Lemma 3.1.4 we know that there exists a positive function δ on Ω such that

$$h(w) \leq |h_z(z)(z-w) - h(z)|^2/h(z) \tag{3.1.11}$$

for every z and w such that $|w-z| < \delta(z)$. From Lemma 3.1.5 we know that it will be enough to find a sequence of positive functions h^ϵ that increases uniformly to h on Ω and a sequence of positive functions c^ϵ for which

$$h^\epsilon(w) \leq |h^\epsilon_z(z)(w-z) + h^\epsilon(z)|^2/h^\epsilon(z) - c^\epsilon(z)|w-z|^2$$

for every z and w where $|w - z| < \delta(z)$. To achieve this, consider the sequence $h^\epsilon(z) = h(z) - \epsilon(1 + |z|^2)$. Since u was bounded we know that h^ϵ will be positive. Now plug in $h(z) = h^\epsilon(z) + \epsilon(1 + |z|^2)$ into (3.1.11) whose left hand side L and right hand side R then become

$$\begin{aligned}
L &= h(w) + \epsilon\left(1 + |w|^2\right) \\
R &= \frac{|(h_z(z) + \epsilon\bar{z})(w-z) + h^\epsilon(z) + \epsilon(1 - |z^2|)^2}{h^\epsilon(z) + \epsilon(1 + |z|^2)} = \\
&= \frac{|h^\epsilon(z)(w-z) + h^\epsilon(z) + \epsilon\left(\bar{z}w + 1\right)|^2}{h^\epsilon(z) + \epsilon(1 + |z|^2)} \leq \\
&\leq \frac{|h^\epsilon(z)(w-z) + h^\epsilon(z)|^2}{h^\epsilon(z)} + \epsilon\frac{|\bar{z}w + 1|^2}{1 + |z|^2}.
\end{aligned}$$

The last inequality is a consequence of the fact that $|\alpha + \beta|/(r_1 + r_2) \leq |\alpha|/r_1 + |\beta|/r_2$ for all complex α, β and all real and positive r_1, r_2. What remains is therefore to show that there exists a real positive function $c(z)$ such that

$$c(z)\,|w - z|^2 \leq (1 + |w|^2) - \frac{|\bar{z}w + 1|^2}{1 + |z|^2}.$$

The right hand side of this inequality can be rewritten as

$$\frac{1 + z\bar{z} + w\bar{w} + z\bar{z}w\bar{w} - 1 - z\bar{w} - w\bar{z} - z\bar{z}w\bar{w}}{1 + |z|^2} = \frac{(z - w)(\bar{z} - \bar{w})}{1 + |z|^2} = \frac{|w - z|^2}{1 + |z|^2}$$

Hence we can choose $c(z) = 1/(1 + |z|^2)$. Note that since Ω is bounded then we may in fact choose $c(z)$ to be the positive constant $\inf_\Omega 1/(1 + |z|^2)$. □

The result has a higher dimensional counterpart which we formulate in following corollary.

Corollary 3.1.7. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and let $u : \Omega \to \mathbb{R}$ be a bounded $\mathbb{C}$-convex function class C^1. Then there exists a sequence of strictly $\mathbb{C}$-convex C^1 functions that decreases uniformly to u on Ω.*

Proof. As in the proof of Lemma 3.1.6 we will work with $h = \exp(-u)$ instead of u itself. The converging sequence will in this case be

$$h^\epsilon(z_1, \ldots, z_n) = h(z_1, \ldots, z_n) - \epsilon(1 + |(z_1, \ldots, z_n)|^2).$$

To prove that $u^\epsilon = -\log h^\epsilon$ is strictly $\mathbb{C}$-convex we have to show that h^ϵ fulfills inequality (3.1.9) on every complex line. Therefore pick a $z \in \Omega$ and a point $w \in \mathbb{C}^n$ with $|w| = 1$ and consider

$$f(\lambda) = h(z + \lambda w)$$
$$f^\epsilon(\lambda) = h^\epsilon(z + \lambda w) = f(\lambda) - \epsilon|w|^2(1/|w|^2 + |\lambda|^2).$$

Now the statement is a direct consequence of the proof of Lemma 3.1.6. □

One is sometimes interested in approximating $\mathbb{C}$-convex functions with strictly $\mathbb{C}$-convex function of class C^∞. If the original function is of class C^2 this is not very hard as shown by the following theorem.

Theorem 3.1.8. *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and let $u : \Omega \to \mathbb{R}$ be a bounded $\mathbb{C}$-convex function of class C^2. Given any relatively compact and open set E in Ω, then there exists a sequence of strictly $\mathbb{C}$-convex C^∞ functions that decreases uniformly to u on E.*

Proof. From Lemma 3.1.6 we know that there exists a function v defined in a neighborhood E' of E such that v is strictly $\mathbb{C}$-convex in E' and $|u - v| < \epsilon$ on E'. Now we can regularize v to be C^∞. Hence take a $\phi \in C_0^\infty(\mathbb{C}^n)$ with total mass one and define $\phi_k(z) = \phi(kz)$ for $k > 0$. From Theorem 1.3.2 in [Hör03] we know that the function

$$v_k(z) = (\phi_k * v)(z) = \int_{\mathbb{C}^n} \phi(z - w)v(w)dw$$

converges uniformly to v in C^2-norm on E. Hence it is clear that for k large enough, we will have that $|v_k - u| < 2\epsilon$ on E and that v_k is strictly $\mathbb{C}$-convex on E. □

Remark 3.1.9. It would of course be nice if such a theorem could be shown for $\mathbb{C}$-convex functions of class C^1. More precisely the aim should be, given a C^1 function ϕ on a bounded open set E in $\mathbb{C}^n$ that is strictly $\mathbb{C}$-convex and a realitively compact subset $E' \subset E$, then for any C^∞ function v approximating the Dirac measure well enough, the convolution $\phi * v$ is strictly $\mathbb{C}$-convex on E'. However it seems to be much harder to show the preservation of the semi differential condition (3.1.1) for C^1 functions rather than the differential condition (3.1.3) for C^2 functions, under the operation of convolution. •

3.1.3 Background to the C^1 definition

Let H_u be the Hartogs domain consisting of all $(z, w) \in \mathbb{C}^2$ such that z lies in a disc and $|w| < \exp(-u(z))$. From [Kis96] we know that if u is twice continuously differentiable then H_u is $\mathbb{C}$-convex if and only if u is $\mathbb{C}$-convex in the sense of (3.1.3). Therefore we could use this information to declare a continuous or C^1 function to be $\mathbb{C}$-convex if all its circular Hartogs domains are $\mathbb{C}$-convex. However, it is quite difficult to determine whether a Hartogs domain is $\mathbb{C}$-convex or not. Furthermore, it is more desirable to have a formula as the definition of a $\mathbb{C}$-convex function instead of a geometrical condition related to Hartogs domains.

However, one could try to do the calculations backwards. That is, try to see which properties of the function u control whether H_u is $\mathbb{C}$-convex or not. Since the theorem of Kiselman is pretty hard, one can assume that this will be quite difficult. Having the great similarity between $\mathbb{C}$-convexity and weak linear convexity in mind, another approach is to consider weakly linearly convex Hartogs domains. It turns out that this is quite feasible in the C^1 case.

3.1.4 $\mathbb{C}$-convex and weakly linearly convex Hartogs domains

In genereal weak linear convexity does not imply linear convexity, see for instance Example 2, p. 297 in [Hör94]. However, a Hartogs domain $H = \{(z, w) : z \in \Omega, |w| < g(z)\}$ in $\mathbb{C}^2$ is weakly linearly convex if and only if it is linearly convex. This is almost obvious, indeed take a point $(z_0, w_0) \in H$, if $z_0 \notin \Omega$ then the line $z_0 \times \mathbb{C}$ goes through the point but does not intersect H. Otherwise, since H is assumed to be weakly linearly convex, there is a cone $C(z)$ that goes through $(z_0, g(z))$ but does not intersect $\pi(H)$. Hence the cone $|w_0|C(z)/g(z_0)$ will go through the point $(z_0, |w_0|)$ without intersecting $\pi(H)$. This makes it clear that H is linearly convex. Having this equivalence in mind we will work with the term weakly linearly convex rather than linearly convex for Hartogs domains.

The following lemma gives a precise answer to the question when a Hartogs domain in $\mathbb{C}^2$ defined by a C^1 function is weakly linearly convex.

Lemma 3.1.10. *Let Ω be an open set in $\mathbb{C}$ and $g : \Omega \to \mathbb{R}$ a positive C^1 function. Then the open Hartogs set $H = \{(z, z') : z \in \Omega, |z'| < g(z)\}$ is weakly linearly convex if and only if*

$$g(w) \leq |2g_z(z)\,(w - z) + g(z)| \tag{3.1.12}$$

for every pair $z, w \in \Omega$.

Proof. According to Lemma 2.2.1, we must show that condition (3.1.12) is equivalent with Z_p being non empty for each point p on the boundary of $\pi(H)$. Therefore pick a boundary point $p = (z_0, r)$. If $z_0 \in \partial\Omega$ then Z_p is obviously non empty. Hence we may assume that $z_0 \in \Omega$ and $r = g(z)$. Since g is C^1, Z_p can consist of at most one point, the tangential cone at p which we denote C_p. Since C_p is tangential to $\pi(H)$ at p, it cannot be

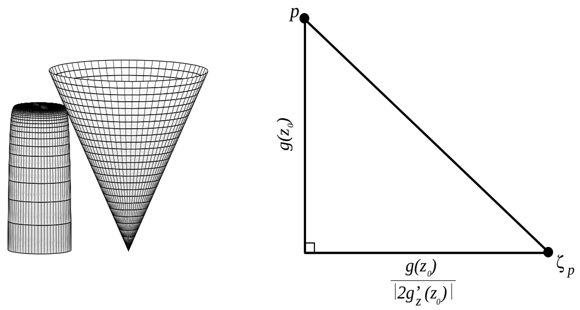


Figure 3.1: The tangential cone to the left and computational triangle to the right.

a degenerate cone. Hence it is a function of z whose gradient at z_0 must be parallel with the gradient of g at z_0. If $\partial_z g(z_0) = 0$ then it is clear that Z_p is empty precisely if (3.1.12) is violated. Therefore we may assume that $\nabla C_p \neq 0$ which tells us that the cone $C_p(w)$ has a slope k and a vertex ζ_p which can be calculated as

$$\begin{aligned} C_p(w) &= k|w - \zeta_p| \\ k &= |2g_z(z_0)| \\ \zeta_p &= z_0 - \frac{\overline{g_z(z_0)}}{|g_z(z_0)|} \frac{g(z_0)}{|2g_z(z_0)|} \end{aligned}$$

The calculation of ζ_p and C_p is illustrated in Figure 3.1. Writing the function C_p in plain text we end up with

$$C_p(w) = |2g_z(z)\,(w - z_0) + g(z_0)|$$

and the lemma is proved. □

Remark 3.1.11. Lemma 3.1.10 is strongly inspired by Proposition 1.5 in [Hör08]. This proposition is based on the both simple and brilliant idea of characterizing weakly linearly convex domains by a formula involving only δ and $\nabla\delta$. If the domain Ω is C^1, then the formula is derived using nothing else but completely elementary linear algebra. •

It is well known, see for example Corollary 4.6.9 in [Hör94], that $\mathbb{C}$-convexity and weak linear convexity coincide for bounded C^1 domains. The following theorem says the same thing for Hartogs domains in $\mathbb{C}^2$ where the base domain is a disc and the defining function is C^1.

Theorem 3.1.12. *Let D be an open disc in $\mathbb{C}$ and $g : D \to \mathbb{R}$ a positive C^1 function. Then the Hartogs domain $H = \{(z, w) : z \in D, |w| < g(z)\}$ is weakly linearly convex if and only if it is $\mathbb{C}$-convex.*

Proof. Since every $\mathbb{C}$-convex domain is weakly linearly convex, see for example Theorem 4.6.8 in [Hör94], all we have to prove is that if H is weakly linearly convex, then it is also $\mathbb{C}$-convex. Assume therefore that H is weakly linearly convex. It will be enough to show that $H' = \{(z, w) : z \in D', |w| < g(z)\}$ is $\mathbb{C}$-convex for every compactly contained disc $D' \Subset D$, for then one can exhaust H with an increasing sequence of $\mathbb{C}$-convex domains H'. Hence we may assume that g is greater than some positive constant and of class C^1 in an open set containing the closure of D. According to Lemma 2.2.1, 2, it is enough to show that Z_p is connected and non empty for every $p = (p_z, r) \in \partial\pi(H)$. Therefore consider such a point $p \in \partial\pi(H)$. If $p_z \in D$ or $r < g(p_z)$ then Z_p consists of exactly one point and will hence be connected. Therefore we may assume that $p_z \in \partial D$ and $r = g(p_z)$. Without loss of generality we may assume that D is centred at the origin and that $p_z = R_1$, where R_1 is the radius of D. The calculations will be more easy if we identify $\mathbb{C} \cong \mathbb{R}^2$ and consider g as a function of the two real variables (x, y) and hence $p_z = (R_1, 0)$. We want to find all cones that pass through $(R_1, 0, g(R_1, 0))$ but do not intersect $\pi(H)$. One obvious such cone is the degenerate one consisting of the points $(R_1, 0, r)$ where $r \in [0, \infty)$. This means that Z_p is not empty. All other cones will have finite slope and hence we may calculate their partial derivatives. Take a non degenerate cone C that is passing through $(R_1, 0, g(R_1, 0))$ but not intersecting $\pi(H)$. Then, identifying C with the function from $\mathbb{R}^2$ to $\mathbb{R}$ whose graph is equal to C, we must have $C'_y(R_1, 0) = g'_y(R_1, 0)$. Let $v = g'_y(R_1, 0)$ and assume first that $v \neq 0$. We will parametrize all such cones C by the angle which is given by the arc starting at the positive x-axis and going counter clockwise to the line segment connecting the origin and the vertex of the cone. Denoting this angle β, we define the corresponding cone function

$$C_\beta(x, y) = \frac{g(R_1, 0)}{d(\beta)} \big|(x, y) - d(\beta)\,(\cos(\beta), \sin(\beta))\big|$$

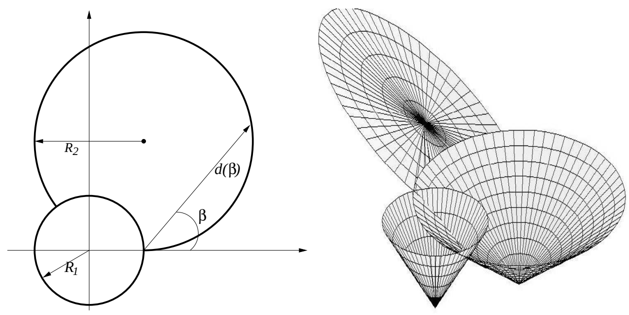


Figure 3.2: To the left we have the calculation setup. To the right we have two tangential cones for the set $H = \{(z, w) : |z| < 1/10, |w| < 1/10 + \mathrm{Im}(z)\}$. We have drawn the cones for $\beta = \pi/6, \pi/2$. Note that the latter angle is the largest one in Γ in this case.

where $d(\beta)$ is the distance to from the vertex of the cone to the origin. Calculating the partial derivative of C_β with respect to y at the point $(R_1, 0)$, one concludes that it equals $g'_y(R_1, 0)$ precisely if

$$d(\beta) = \frac{g(R_1, 0)\sin(\beta)}{v}.$$

Note that if $v < 0$ then β has to be in the interval $[0, \pi]$, and if $v > 0$ then $\beta \in [\pi, 2\pi]$. Neither of the alternatives will cause any difference in the calculations compared with the other one, and we may hence assume that $v < 0$. The case when $v = 0$ will be dealt with later. The vertex of the cone C_β will move along the closed curve

$$\gamma = \{d(\beta)(\cos(\beta), \sin(\beta)) + (R_1, 0) : \beta \in [0, \pi]\}$$

which will be a circle with center in $(R_1, g(R_1, 0)/(2v))$ and radius $R_2 = g(R_1, 0)/(2v)$. The circle γ will intersect the boundary of D at precisely two points, one when $\beta = 0$ and the other one when

$$\beta = \beta_m = \pi - \arctan(R_1/R_2).$$

Let $\Gamma = \{\beta : C_\beta(x, y) \geq g(x, y)\,\forall\,(x, y) \in D\}$. We want to show that Γ is connected. It is clear that $\Gamma \subset [0, \beta_m]$. Writing the points in polar coordinates $(x, y) = (s\cos(\theta), s\sin(\theta))$ and putting $F_\beta = C_\beta^2$ we can write

$$F_\beta(s, \theta) = v^2 \frac{(s\cos(\theta) - R_1 - 2R_2\sin(\beta)\cos(\beta))^2 + (s\sin(\theta) - 2R_2\sin^2(\beta))^2}{\sin^2(\beta)}.$$

We will now fix $(s,\theta) \in [0,R_1) \times [0,2\pi]$ and consider $f(\beta) = F_\beta(s,\theta)$. If we can show that $f'(\beta)$ is non positive when $\beta \in [0,\beta_m]$, we are done.

A somewhat involved derivation yields

$$\begin{aligned} f'(\beta) &= u_1(\beta)\, u_2(\beta), \qquad \text{where} \\ u_1(\beta) &= \frac{2v^2}{\sin(\beta)(\cos^2(\beta)-1)}, \\ u_2(\beta) &= -2\cos(\beta)\cos(\theta)\, s\, R_1 + \cos(\beta)R_1^2 + s^2\cos(\beta) \\ &\quad - 2R_2\, s\, \cos(\theta)\sin(\beta) + 2R_1R_2\sin(\beta). \end{aligned}$$

It is clear that $u_1(\beta) < 0$ when $\beta \in [0,\pi]$ and hence we need to show that $u_2(\beta) \geq 0$ for $\beta \in [0,\beta_m]$. Solving the equation $u_2(\beta) = 0$ with the additional condition that $\beta \in [0,\pi]$ yields

$$\beta_{root} = \pi - \arctan\left(\frac{R_1^2 + s^2 - 2\cos(\theta)R_1 s}{2R_2(R_1 - s\cos(\theta))}\right).$$

Now we want to compare β_{root} and β_m which is best done by looking at

$$\begin{aligned} &\frac{R_1^2 + s^2 - 2\cos(\theta)R_1 s}{2R_2(R_1 - s\cos(\theta))} = \frac{s^2 - R_1^2}{2R_2(R_1 - s\cos(\theta))} + \frac{2R_1(R_1 - s\cos(\theta))}{2R_2(R_1 - s\cos(\theta))} = \\ =&\frac{R_1}{R_2} - \frac{R_1^2 - s^2}{2R_2(R_1 - s\cos(\theta))} < R_1/R_2 \end{aligned}$$

and hence we have that $\beta_{root} > \beta_m$. The setup of the calculations together with a twin tangent cone situation is illustrated in Figure 3.2.

If $v = 0$ then it is almost obvious that Z_p is connected. Indeed, in that case the vertex will move along the outward pointing normal, instead of along a circle. This means that γ will be the x-axis and one may use $\partial_x C$ to parametrize the cones in Z_p. □

Recall that Lemma 3.1.10 tells us when a Hartogs domain in $\mathbb{C}^2$ with a C^1 defining function is weakly linearly convex. The answer is a global condition on the defining function g. If the base domain is a disc, we now use Theorem 3.1.12 to sharpen the lemma so that the condition only has to apply locally.

Lemma 3.1.13. *Let D be an open disc in $\mathbb{C}$ and $g : D \to \mathbb{R}$ a positive function of class C^1, then the Hartogs domain*

$$H = \{(z, z') : z \in D, |z'| < g(z)\}$$

is weakly linearly convex if and only if there is a positive function δ on D such that

$$g(w) \leq |2g_z(z)(w - z) - g(z)|$$

for every $z, w \in D$ where $|z - w| < \delta(z)$.

Proof. First we note that if $u = -\log(g^2)$ then Lemma 3.1.4 tells us that the condition on g above is equivalent with u being $\mathbb{C}$-convex. If H is weakly linearly convex then Lemma 3.1.10 together with Lemma 3.1.4 immediately gives that u is $\mathbb{C}$-convex. Hence we have to show that if u is $\mathbb{C}$-convex then H is weakly linearly convex. In fact it will be enough to show this implication when u is strictly $\mathbb{C}$-convex. Indeed from Lemma 3.1.6 we know that we can approximate u with a decreasing sequence of strictly $\mathbb{C}$-convex C^1 functions that converges uniformly to u on any relatively compact subset of D. These functions would then induce an increasing sequence of Hartogs domains, all of which are $\mathbb{C}$-convex. The union of these sets would itself be $\mathbb{C}$-convex.

Following the notation and spirit of Lemma 2.2.1, we want to show that Z_p is non empty for every $p \in \partial\pi(H)$. Let $p = (p_z, p_r) \in \mathbb{C} \times \mathbb{R}_{\geq 0}$. If $p_z \in \partial D$ then Z_p is certainly not empty and hence we may assume that $p_z \in D$. For any $p \in \partial\pi(H)$ with $p_z \in D$, which gives that $p_r = g(p_z)$, we need to show that the tangential cone to g at p does not intersect $\pi(H)$. More precisely, the unique cone C_{p_z} with $C_{p_z}(p_z) = p_r$ and $\nabla C_{p_z}(p_z) \parallel \nabla g(p_z)$ fulfills the inequality

$$C_{p_z}(z) \geq g(z) \quad \forall z \in D.$$

Since u is strictly $\mathbb{C}$-convex, Lemma 3.1.5 gives us that there is a positive function δ on D and a positive constant c such that

$$C_{p_z}(z) \geq g(z) + c|z - p_z|^2 \quad \forall z : |z - p_z| < \delta(p_z). \tag{3.1.13}$$

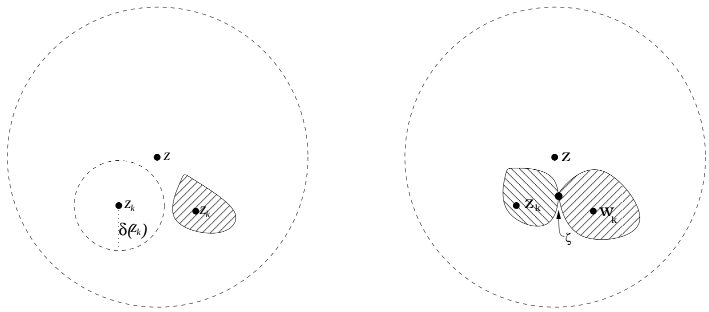


Figure 3.3: The lowering of the cone in the case when $\partial A_k \cap \partial B_k \neq \emptyset$. To the left we have the intersection set before lowering the cone, that is $C_{z_k} \geq h$. To the right we have the intersection set after lowering the cone, that is $t_k\, C_{z_k} \geq h$.

Without loss of generality, we may assume that D is centered at the origin. Denote $D_\lambda = \{\lambda\, z : z \in D\}$ where $\lambda \in (0, 1]$. Let S be the set of λ for which D_λ is weakly linearly convex. We want to show that $S = (0, 1)$. It is clear that if $x \in S$ then $(0, x] \subset S$. If we can show that S is non empty, open and closed in $(0, 1]$ we are done.

S is non empty. Denote by $r(z)$ the radius of the largest open disc centered at z and contained in D such that the corresponding Hartogs domain is weakly linearly convex. Let us argue by contradiction and assume that $r(z) = 0$ for some $z \in D$. This would imply that there is a sequence $\{(z_k, w_k)\}$ with the property that $z_k \neq w_k$, $z_k, w_k \to z \in D$ and

$$C_{z_k}(w_k) < g(w_k).$$

For $t \in [0, 1]$, let $C^t_{z_k}$ be the cone $C^t_{z_k} = tC_{z_k}$. Now denote

$$\Omega^t_k = \{w \in D : g(w) < C^t_w(w)\}$$

which will be an open set. Note that $C_{z_k} \to C_z$ in the topology of cones and that $C_z(w) - g(w) \geq c|z - w|^2$ for $|w - z| = \delta(z)$. This gives that if k is large enough, there will be a smallest $t_k \in (0, 1)$ such that z_k and w_k will be in different components of $\Omega^{t_k}_k$. Let the component which contain z_k be denoted A_k and let the component which contain w_k be denoted B_k. First we assume that $\partial A_k \cap \partial B_k \neq \emptyset$. Therefore take a point $\zeta \in \partial A_k \cap \partial B_k$. It is clear that $C_{z_k}(\zeta) = g(\zeta)$. Since the the zero set of the function $C_{z_k} - g$ is non connected around ζ we must have $\nabla C_{z_k}(\zeta) = \nabla g(\zeta)$. Therefore C_{z_k} is the unique tangential cone also to ζ. But $C_\zeta = C_{z_k}$ does not fulfill inequality (3.1.13) for $p_z = \zeta$.

Now to the case if $\partial A_k \cap \partial B_k = \emptyset$. Recall that z_k and w_k will be in the same component of Ω_k^t for any positive t with $t < t_k$. Thus there will be a sequence of numbers $\{\zeta_j\}$, for large j enough, with the properties that $C_{z_k}(\zeta_j) - g(\zeta_j) \leq 0$ and that the distance from ζ_j to A_k equals $1/j$. Let ζ be a limit point of $\{\zeta_j\}$, which will belong to the boundary of A_k. Now assume that $\nabla(C_{z_k} - g)(\zeta) \neq 0$. Then, by the implicit function theorem, the zero set of $C_{z_k} - g$ near the point ζ will be a C^1 curve that locally constitutes the boundary of A_k. For some small disc centered at ζ, the C^1 curve would divide the disc into two open sets, one which belongs to A_k where $C_{z_k} - g < 0$ and one that belongs to the complement of A_k where $C_{z_k} - g > 0$. This contradicts the assumptions made on the sequence. Hence we must have $\nabla C_{z_k}(\zeta) = \nabla g(\zeta) = 0$. But then, we would have $C_\zeta = C_{z_k}$ which does not fulfill inequality (3.1.13) for $p_z = \zeta$. Hence we know that $r(z) > 0$ for all $z \in D$. The idea with the lowering of the cone C_{z_k} is illustrated in Figure 3.3.

S is open in $(\mathbf{0}, \mathbf{1}]$. Again we argue by contradiction and assume that there is a largest $\lambda \in S$ with $\lambda < 1$. This implies that there are two sequences $\{z_k\}$ and $\{w_k\}$ with $C_{z_k}(w_k) < g(w_k)$ which has limit points $\zeta \in \partial D_\lambda$ and $\omega \in \overline{D_\lambda}$. Since $r(\zeta) > 0$ we have that $\zeta \neq \omega$. But g fulfills inequality (3.1.13) and hence the open set

$$X = \{z : z \in D', t\, C_\zeta(z) < g(z)\}$$

will consist of at least two components for $0 \ll t < 1$. Indeed one component will have ζ in its closure and one will have ω in its closure, and since we have $c|z - w|^2$ term in inequality (3.1.13), ζ and ω cannot be joined in $\bar{X}$ if t is close enough to one. This would imply that H' is not ℂ-convex. But according to Theorem 3.1.12 H' must be ℂ-convex since H' is weakly linearly convex and g is C^1.

S is closed in $(\mathbf{0}, \mathbf{1}]$. This is clear since the union of an increasing sequence of ℂ-convex sets is itself ℂ-convex. □

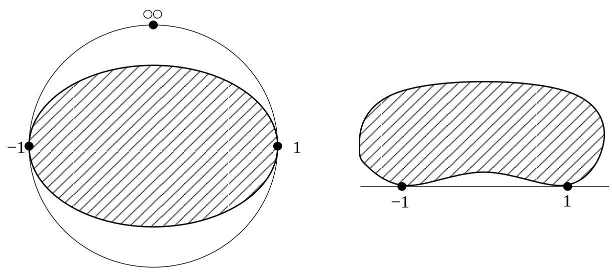


Figure 3.4: A Moebius transformation taking Ω to a non convex set.

By combining Theorem 3.1.12, Lemma 3.1.4 and Lemma 3.1.13 we get the following corollary.

Corollary 3.1.14. *Let D be an open disc and $u : D \to \mathbb{R}$ a positive C^1 function, then the Hartogs set $H=\{(z, w) : z \in D, |w|^2 < \exp(-u(z))\}$ is $\mathbb{C}$-convex if and only if u is $\mathbb{C}$-convex on D.*

The corollary above is the final evidence that the definition of a $\mathbb{C}$-convex C^1 function given by (3.1.5) is sound. This is because the corollary has a complete analogue for C^2 functions u. The C^2 theorem is due to Kiselman, see [Kis96], and becomes formulated as above when combining Example 2.5.13 and Theorem 2.5.16 in [APS04]. Hence if $u : \mathbb{C}^n \to \mathbb{R}$ is a C^2 function, then it is $\mathbb{C}$-convex in an open set Ω according to (3.1.1) precisely if it is $\mathbb{C}$-convex in Ω according to (3.1.3).

Remark 3.1.15. In Theorem 3.1.12, Lemma 3.1.13 and Corollary 3.1.14 it is essential that the base domain is a disc. To be precise we consider a Hartogs domain $H = \{(z, w) : z \in \Omega, |w| < g(z)\}$ where g is a positive C^∞ function and Ω is not a disc, but it is contractable and equals the interior of its closure. By Lemma 8.2 in [Kis96] there is a disc D that contains Ω and such that there are two points $a, b \in \partial\Omega \cap \partial D$ that cannot be joined in $\partial\Omega \cap \partial D$. Hence there is a Moebius transformation m, mapping Ω to a non convex bounded domain as suggested in Figure 3.4.

As a direct consequence of Theorem 2.3.6 i [APS04] we know that the set $H_m = \{(z, w) : z \in m(\Omega), |w| < g(m^{-1}(z))\}$ is $\mathbb{C}$-convex precisely if H is $\mathbb{C}$-convex. This has the implication that $g_m = g \circ m^{-1}$ satisfies condition (3.1.7) precisely if g does. To show the importance of Ω being a disc we have to do it in the following two cases:

1. *Theorem 3.1.12 and Corollary 3.1.14:* Here we can let $g_m(z) = c -$

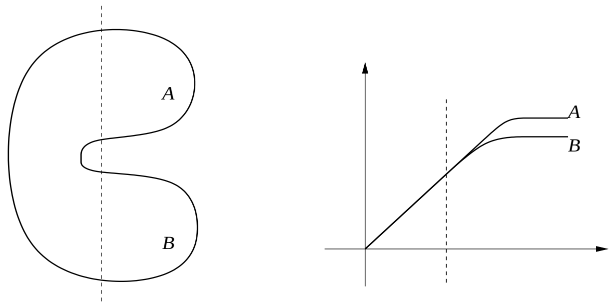


Figure 3.5: To the left we have the non convex base domain $m(\Omega)$ of H_m. To the right we indicate the defining graph g_m. This will make H_m non weakly linearly convex. However, it is *locally* weakly linearly convex. The domain is illustrated as a three dimensional picture in Table 2.1.

Im(z) where c is a positive constant so big that $g_m > 0$ on Ω, then H_m will not be $\mathbb{C}$-convex but it will be weakly linearly convex. We will also have that $u_m = -\log g_m^2$ is $\mathbb{C}$-convex.

2. *Lemma 3.1.13:* Here we let g_m be as in Figure 3.5. Since g_m is concave it is obvious that g_m satisfies condition (3.1.7). It is also clear that H_m is not weakly linearly convex. Indeed, take a horizontal tangent cone at area B, then it will intersect H below area A. Now look at H which also will have two horizontal areas with different altitude, hence H is not weakly linearly convex. However, as mentioned above, $g = g_m \circ m$ satisfies condition (3.1.7).

•

3.1.5 Extremal functions

It is well known that the extremal $\mathbb{C}$-convex C^2 functions are the ones of the form $u(z) = -\log|a+bz|^2$. By extremal we mean the ones who results in equality when tested against inequality (3.1.3). However this is not very easy to prove. One has to solve the non linear differential equation which is produced when changing the '$\geq$' symbol to the '$=$' symbol in the previous mentioned inequality. To show that functions of the form $-\log|a+bz|^2$ satisfy the equality is of course trivial, but to show that they are the only solutions is not. When it comes to proving that these functions are the extremal also to the C^1 definition 3.1.1 is even harder. However, instead of using the partial differential equation approach, there is a shortcut. Indeed assume that u is extremal with respect to (3.1.5) in an open set Ω. Then $g = \exp(-u/2)$ is also extremal, but with respect to inequality (3.1.7). Now take a point $z_0 \in \Omega$ and let D be a disc that contains z and is contained in Ω. Consider the Hartogs domain $H = \{(z,w) : z \in D, |w| < g(z)\}$ and the tangent cone $C_{z_0}(z)$ to H at the point $(z_0, g(z_0))$. Since g is extremal with respect to (3.1.7), Lemma 3.1.10 gives us that $C_{z_0}(z) = g(z)$ in a neighborhood of z_0. Hence for every compact $K \subset \Omega$ we must have $C_{z_0}(z) = g(z)$ for all $z \in K$ and hence $g = C_{z_0}$. Since C_{z_0} was a tangent cone to H and $z_0 \in \Omega$ it must be of the form $g(z) = |a + b\,z|$.

3.2 $\mathbb{C}$-convex sets

3.2.1 A characterization theorem in the C^1-case.

Let Ω be a bounded domain in $\mathbb{R}^n$. One says that Ω satisfies the *interior ball condition* with radius r if Ω is a union of balls of radius r. If Ω is a bounded domain of class C^k, $k \geq 2$, then Ω automatically satisfies the interior ball condition for some $r > 0$, however this is not the case if $k = 1$. When $k \geq 2$, the boundary distance function δ is of class C^k near the boundary, see for example [KP80]. For the C^1 case we need the following lemma.

Lemma 3.2.1. *Let X be a bounded domain in $\mathbb{R}^n$ of class C^1 that satisfies the interior ball condition for some positive radius r. Then the boundary distance function δ is of class C^1 near ∂X inside X.*

Proof. Assume that X is a domain that satisfies the interior ball condition for $r > 0$, and δ is the boundary distance function. In the proof of Lemma 2.1.29 in [Hör94] it is stated that δ is differentiable at $x \in X$ precisely if there is a unique $y \in \partial X$ such that $|x - y| = \delta(x)$. Let X_s be all x in X such that $\delta(x) < s$. Assume that there is a point x in X_r such that there exist distinct points y_1, y_2 on ∂X where

$$\delta(x) = |x - y_1| = |x - y_2| < r$$

Now, for every p in the line segment $[x, y_1]$ there is a ball B_p with radius r that contains p and is contained in X. By letting $p \to y_1$, there must be a ball B_{y_1} that is contained in X but is tangential to ∂X at y_1; here it is important that X is C^1. However B_{y_1} must contain y_2 which is a contradiction. Hence every point in X_r has a unique closest point on the boundary. Therefore δ is differentiable in X_r. We only need to show that the gradient $\nabla\delta$ is continuous in X_r. It is clear that $|\nabla d| = 1$, therefore we need only to check the directions. More precisely take $x \in X_r$ whose closest point on the boundary is denoted y and assume that there is a sequence $\{x_n\}$ converging to x whose corresponding closest points $\{y_n\}$ do not converge to y. Then there exists a subsequence y_{n_k} that converges to a point y' on ∂X that is distinct from y. This contradicts the fact that y was the unique closest point to x on ∂X. □

From the proof it is also clear that Ω satisfying the interior ball condition for $r > 0$, is equivalent with $\delta(z)$ being differentiable for all $z \in \Omega$ where $\delta(z) < r$.

Theorem 3.2.2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^1 boundary that satisfies the interior ball condition for some positive radius. Then Ω is $\mathbb{C}$-convex precisely if $-\log(\delta^2)$ is $\mathbb{C}$-convex at all points in Ω close to the boundary.*

Proof. Note that Lemma 3.2.1 implies that δ is C^1 near the boundary. We will first assume that $u = -\log\delta^2(z)$ is $\mathbb{C}$-convex near the boundary and show that this implies that Ω is $\mathbb{C}$-convex. Since Ω is C^1, Proposition 2.5.8 in [APS04] implies that we only need to show that for every $p \in \partial\Omega$, the complex tangent plane to Ω at p, L_p, avoids Ω in a neighborhood of p. Therefore fix a $p \in \partial\Omega$ and consider the tangent plane L_p. Without loss of generality we may assume that p is the origin and the inward pointing normal to $\partial\Omega$ at p is $(0, \ldots, 0, i)$. This gives that the real tangent plane at p consists of all $z = (z_1, \ldots, z_n)$ where z_n is purely real. Hence L_p consists of all z where $z_n = 0$. Now let $q = (0, \ldots, 0, i\,y_n)$ where y_n is real and positive but so small that δ is C^1 in a neighborhood of q. Hence we can find a positive ϵ such that for any $\xi \in \mathbb{C}^n$ with $|\xi| = 1$, the one variable function

$$f(\lambda) = \delta(q + \lambda\xi)$$

is C^1 inside the disc $D = \{\lambda : |\lambda| < 1\}$. Since u is assumed to be $\mathbb{C}$-convex, f will satisfy inequality (3.1.7) in Lemma 3.1.4 for all $\lambda \in D$. Now Lemma 3.1.13 gives us that the Hartogs domain

$$H = \{(\lambda, \sigma) : |\lambda| < 1, |\sigma| < f(\lambda)\}$$

is weakly linearly convex. Next consider the affine linear mapping

$$\begin{aligned} &T : \mathbb{C}^2 \to \mathbb{C}^n \\ &(\lambda, \sigma) \mapsto q + \lambda\xi + (0, \ldots, 0, \sigma). \end{aligned}$$

By construction we must have $T(H) \subset \Omega$. Since T is affine and $T(\mathbb{C}^2) \nsubseteq L_p$, $T^{-1}(L_p)$ is an affine complex line in $\mathbb{C}^2$. By construction $T^{-1}(L_p)$ is tangential to H at $(0, f(0))$, but H is weakly linearly convex and therefore we must have that $H \cap T^{-1}(L_p) = \emptyset$ which together with the injectivity of T implies that $T(H) \cap L_p = \emptyset$. Hence the distance from $q + \epsilon\xi$ to the tangent plane L_p cannot be smaller than the distance from $q + \epsilon\xi$ to the boundary of Ω. Since ξ was an arbitrary unit vector we can state that

$$|\zeta - L_p| \geq \delta(\zeta) \tag{3.2.1}$$

for all ζ in the ball centered at q with radius ϵ. Now assume that there is a sequence $\{p_k\}$ where $p_k \in \Omega \cap L_p$ and $p_k \to p$. Let p'_k be the unique point on $\partial\Omega$ which is the closest one to p_k. Now consider the ray r_k that starts at p'_k and goes through p_k. Since the boundary of Ω is C^1 the ray r_k will approximate $r = \{(0, \ldots, 0, iy) : y \geq 0\}$ for large k. The ray r passes through q and hence for k large enough, the ray r_k will intersect the ball centred at q with radius ϵ. Let k be large and take a point ζ that lies both on such a ray r_k and in the ball centred at q with radius ϵ. It follows that

$$|\zeta - L_p| \leq |\zeta - p_k| < |\zeta - p'_k| = \delta(\zeta)$$

which contradicts (3.2.1). This proves the first part of the theorem. The geometry around the embedded Hartogs domain is illustrated in Figure 3.6.

Now assume that Ω is $\mathbb{C}$-convex, which implies that it is weakly linearly convex. Then we have to show that $u = -\log \delta^2$ is $\mathbb{C}$-convex near the

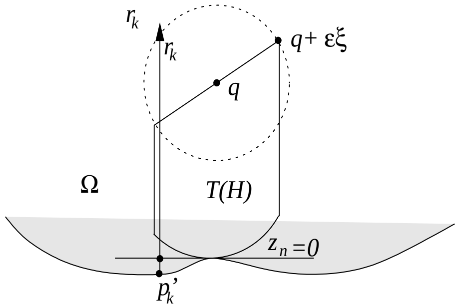


Figure 3.6: The embedded Hartogs domain $T(H)$. Note that the figure contradicts inequality (3.2.1).

boundary. More precisely this means that for every point q near the boundary and every complex line l_q through such a point q, the restriction of u to l_q must be $\mathbb{C}$-convex near q. For this we take a point $q \in \Omega$ which is so close to the boundary that there is a unique point p on the boundary that is the closest one. As before we may assume that p is the origin and the complex tangent plane at p is $L_p\{z : z_n = 0\}$. Take a point $\xi \in \mathbb{C}^n$ with $|\xi| = 1$ and define $f(\lambda) = \delta(q + \xi\lambda)$. According to 3.1.4 we have to show that $f(\lambda) \leq |2f_z(0)(\lambda - 0) + f(0)|$ for λ close enough to zero. We know that $L_p \cap \Omega = \emptyset$ which means that for λ small enough we must have

$$f(\lambda) = \delta(q + \xi\lambda) \leq |q + \xi\lambda - L_p| = |q_n + \lambda\xi_n|.$$

However, we have that $2f_z(0) = \bar{q}_n\xi_n/|q_n|$ which implies that

$$|2f_z(0)\lambda + f(0)| = \left|\frac{\bar{q}_n\xi_n}{|q_n|}\lambda + |q_n|\right| = |q_n + \lambda\xi_n|.$$

This proves the theorem. □

3.2.2 Defining functions

Consider a bounded open domain $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, where ρ is a defining function of class C^1 whose gradient is non vanishing on the boundary of Ω.

Proposition 3.2.3. *Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a bounded C^1 domain. If ρ is $\mathbb{C}$-convex in a neighborhood of $\partial\Omega$, then Ω is $\mathbb{C}$-convex.*

Proof. Pick a point z in the boundary of Ω. As mentioned before, it is enough to show that the complex tangent plane at z, denoted T_z, locally avoids Ω. Take a point $w \in T_z$ with $|w-z| = 1$ and consider the one variable function $f(\lambda) = \rho(z + \lambda w)$. Since ρ is $\mathbb{C}$-convex in a neighborhood of $\partial\Omega$, inequality (3.1.5) implies that there is a constant r such that

$$f(\lambda) \geq f(0) - \log|\lambda f'_\lambda(0) - 1|^2 = 0 - \log|1|^2 = 0,$$

for every λ with $|\lambda| < r$. Hence we know that $\rho(z + \lambda w) \geq 0$ which means that $z + \lambda\, w \notin \Omega$. □

3.2.3 Exhaustion of $\mathbb{C}$-convex sets

We will now consider the problem how to exhaust $\mathbb{C}$-convex domain with strictly $\mathbb{C}$-convex subdomains. Much of the work has already been done when we managed to approximate a $\mathbb{C}$-convex function with strictly $\mathbb{C}$-convex functions.

Theorem 3.2.4. *Let Ω be a $\mathbb{C}$-convex bounded domain with C^1 boundary that satisfies the interior ball condition for some positive radius. Then there exist a sequence of strictly $\mathbb{C}$-convex domains $\{\Omega_n\}$ with C^1 boundary such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \cup\Omega_n$.*

Proof. Let $u(z) = -\log\delta(z)^2$. Theorem 3.2.2 tells us that u is $\mathbb{C}$-convex for all $z \in \Omega$ close to the boundary. According to Corollary 3.1.7, there is a sequence of C^1 functions $\{u_n\}$ that decreases uniformly to u and such that every u_n is strictly $\mathbb{C}$-convex wherever u is $\mathbb{C}$-convex. Let $v_n = u_n - n$ and consider the domain $\Omega_n = \{z \in \Omega : v_n(z) < 0\}$. Note that Ω_n lies compactly contained in Ω and if n is large enough, then v_n is strictly $\mathbb{C}$-convex in a neighborhood of $\partial\Omega_n$. Because of Sard's theorem, the level sets $u_n < c$ are C^1 for almost all c in the range of u_n. By letting n' be a real number very close to n and redefining $v_n = u_n - x$ we may assume that $\Omega_n = \{z \in \Omega : v_n(z) < 0\}$ has C^1 boundary and u_n is strictly $\mathbb{C}$-convex in a neighborhood of $\partial\Omega_n$.

Now take a $z \in \partial\Omega_n$ and fix a unit vector w such that $z + w$ lies in the complex tangent plane to Ω_n at z. Consider the one variable function

$$f_n(\lambda) = v_n(z + \lambda\, w).$$

Since v_n is strictly ℂ-convex f will satisfy the inequality

$$f(\lambda) \geq f(0) - \log|\lambda f'_\lambda(0) - 1|^2 + c_z|w|^2 = c_{n,z,w}|w|^2.$$

for λ small enough. However, according to the proof of Lemma 3.1.6, the constant $c_{z,n,w}$ is independent of both z and w. Hence there is a positive and global constant c_n such that

$$v_n(z, w) \geq c_n|w|^2$$

for all w small enough such that $z + w$ lies in the complex tangent plane at z. Hence, according to Definition 2.4.7, Ω_n is strictly ℂ-convex with C^1 boundary. It is clear that $\Omega = \cup\Omega_n$. Since Ω_n was compactly contained in Ω there is a subsequence $\{\Omega_{n_k}\}$ satisfying $\Omega_{n_k} \subset \Omega_{n_{k+1}}$ for all k. □

If the boundary of the domain that we want to exhaust is of class C^2, then we can regularize the boundary by methods of convolution. Whether or not this is possible in the C^1 case remains an open issue.

Theorem 3.2.5. *Let Ω be a ℂ-convex bounded domain with C^2 boundary. Then there exists a sequence of strictly ℂ-convex domains $\{\Omega_n\}$ with C^∞ boundary such that $\Omega_n \subset \Omega_{n+1}$ and $\Omega = \cup\Omega_n$.*

Proof. Here the proof is precisely as the one for Theorem 3.2.4 but according to Theorem 3.1.8 the sequence v_n can be taken to be C^∞. □

3.2.4 A counterexample in the C^0-case

Here we show that a theorem like the one in subsection 3.2.1 is not possible in the C^0 case. For this we consider the following example.

Example 3.2.6. We will construct two Hartogs domains H_1 and H_2 where H_1 is not $\mathbb{C}$-convex but H_2 is. However, for every point $p_1 \in H_1$ there is a positive ϵ, a $\xi \in \mathbb{C}$ with $|\xi| = 1$ and a point p_2 such that

$$\delta_1(p_1 + (z, w)) = \delta_2(p_2 + (\xi z, w)) \tag{3.2.2}$$

for every $(z, w) \in \mathbb{C}^2$ with $|(z, w)| < \epsilon$. Hence whatever local rotationally invariant property that δ_1 violates, must be violated also by δ_2. Hence there cannot be any rotationally invariant local condition so that a domain is $\mathbb{C}$-convex if and only if the distance function satisfies the property.

Let $C_r(z) = |z - r|/r$ for positive r and let $C(z) = \inf C_r(z)$. Now define

$$H_1 = \{(z, w) : 0 < |z| < 1, |w| < 1\},$$
$$H_2 = \{(z, w) : 0 < |z| < 1, |w| < C(z)\}.$$

It it obvious that H_1 is not $\mathbb{C}$-convex. What is left to show is that equation (3.2.2) holds and that H_2 is $\mathbb{C}$-convex. If we write the function C in polar coordinates (s, θ), $z = \!\!\mapsto s \cos\theta + is \sin\theta$, a calculation yields

$$C(s, \theta) = \begin{cases} \sin\theta & \theta \in [-\pi/2, \pi/2] \\ 1 & \text{otherwise}, \end{cases}$$

for $s \in (0, 1)$. Now we want to show that equation (3.2.2) holds. Therefore take a point $p_1 = (z_1, w_1) \in H_1$. Let $p_2 = (-|z_1|, w_1)$. From the polar coordinate version of C above, it is clear that $p_2 \in H_2$. Note that the level set

$$\{C(s, \theta) \geq |w_1|\} = \{(s, \theta) : 0 < s < 1, \theta \in [\arcsin w_1, 2\pi - \arcsin w_1]\}$$

is the PacMan like figure on the left in Figure 3.7. It is clear that there is some positive $d(|w_1|)$ such that

$$\delta_2\left(\frac{-\overline{z_1}}{|z_1|} z, w_1\right) = \min(1 - w_1, |z|, 1 - |z|) = \delta_1(z, w_1) \tag{3.2.3}$$

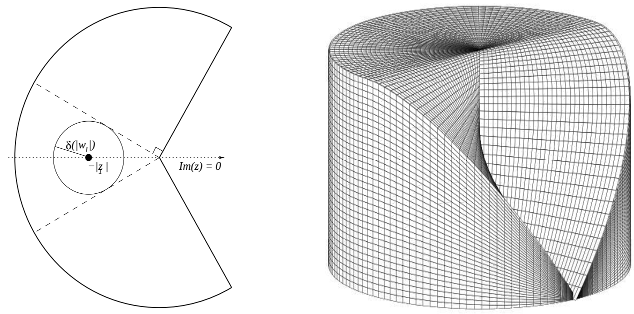


Figure 3.7: To the left we have the level set $L = \{(z, w) \in H_2 : w = w_1\}$. This has the property that points $(z, w_1) \in L$ where z lies in the disc in the figure, will fulfill equation (3.2.3). To the right we have $\pi(H_2)$.

for every z with $|z - z_1| < d(|w_1|)$. It is also clear that the function d will be continuous and hence there is a positive ϵ such that (3.2.2) holds with $p_2 = -|p_1|$, $\xi = -\bar{z}_1/|z_1|$, and for w such that $|w| < \epsilon$. The geometry of the situation is illustrated in Figure 3.7.

Now to the somewhat more tricky task of showing that H_2 is ℂ-convex. According to Lemma 2.2.1 we have to show that Z_p is connected in Z, the set of all cones, for every point on $\pi(\partial H_2)$. It is clear that $\pi(\partial H_2)$ consists of the union of the five sets

$$\begin{aligned}
V_1 =& \{(s, \theta, l) : s \in (0, 1], \theta = 0, l = 0\} \\
V_2 =& \{(s, \theta, l) : s = 0, l \in (0, 1]\} \\
V_3 =& \{(s, \theta, l) : s \in (0, 1), \theta \in (0, 2\pi), l = C(s, \theta)\} \\
V_4 =& \{(s, \theta, l) : s = 1, \theta \in (0, 2\pi), 0 \le l < C(s, \theta)\} \\
V_5 =& \{(s, \theta, l) : s = 1, \theta \in (0, 2\pi), l = C(s, \theta)\}
\end{aligned}$$

Now take a point $p \in \pi(\partial H_2)$ for which we want to show that Z_p is non empty and connected. That Z_p is non empty is obvious since H_2 by construction is weakly linearly convex. We now go through each and one of the five sets V_i.

V_1 If $p = (r, 0, 0)$ then the set Z_p consists of all cones of the form $c|z - r|$

where $c \geq 1/r$ and the degenerate cone $r \times \mathbb{R}_{\geq 0}$. It is clear that Z_p is connected in Z.

V_2 Here Z_p consists of exactly one point so it is connected.

V_3 Here Z_p again consists of exactly one point so it is connected.

V_4 Here Z_p again consists of exactly one point so it is connected.

V_5 Since the base domain of the Hartogs domain H_2 is a disc minus a zero set, we can use the same argument as in Theorem 3.1.12 (where the vertex of the cones lie on circle) which shows that Z_p is connected.

Hence H_2 is $\mathbb{C}$-convex and the statement above is showed.

In this example it might be interesting to examine what δ_2 looks like at one of these special points where it behaves like δ_1. Hence fix a real $a \in (1/2, 1)$ and $\lambda \in \mathbb{C}$ with modulus small enough. We have

$$g(/\lambda) = \delta_2((1-a, 1-a) + (\lambda, 0)) = \delta_1((1-a, 1-a) + (\lambda, 0)).$$

It is clear that $g(\lambda) = \min(|\lambda - a|, a)$. However if we consider the Hartogs domain

$$H = \{(\lambda, \sigma) : |\lambda| < 1-a, |\sigma| < g(\lambda)\}$$

we see that it is not $\mathbb{C}$-convex. Indeed, the intersection between $\pi(H)$ and the cone $\{(\lambda, r) : r = |\lambda - 2a|/2\}$ is not connected. This could not happen if H_2 had C^1 boundary and satisfied the interior ball condition for some positive radius as shown by Theorem 3.2.2. The projection of the Hartogs domain H is displayed in Figure 3.8. •

3.3 Open Issues

For the study of complex convexity, there remain many open issues. Here we give a brief list of such unsolved problems.

Are locally weakly linearly convex domains always pseudoconvex?

Let E be a locally weakly linearly convex domain in $\mathbb{C}^n$. We want to show that $-\log \delta$ is pseudoconvex where δ is the boundary distance function. If

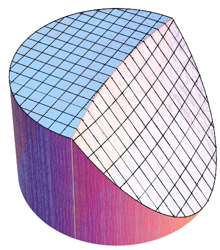


Figure 3.8: The projected Hartogs domain $\pi(H)$ showing the behavior of the distance function at one of the critical values.

E is weakly linearly convex, the argument is easy as shown in the proof of the implication $\mathcal{WLC} \Rightarrow \psi\mathcal{C}$ in Section 2.3.2. This argument does not work in the case when E is just locally weakly linearly convex and the failure is due to the following. Let $z \in \partial E$, Γ_z be the set of complex hyperplanes containing z but locally avoiding E, and let

$$r(z) = \sup\{|z - \partial E \cap L| : L \in \Gamma_z\}.$$

If $\inf r(z)$ for $z \in \partial E$ is positive, then E is pseudoconvex and the same proof as in the case for weakly linearly convex domains can be applied. However, as illustrated by the example when E equals the Hartogs domain

$$H = \{(z, w) : |z| < 1, |w| < g(z)\}$$

where

$$g(z) = \begin{cases} \min(|1-z|, |1+z|) & \text{if} \quad \mathrm{Re}(z) > 0 \\ \min(|1-z|, |1+z|, 1) & \text{otherwise,} \end{cases}$$

we have that $\lim r(-s\,i, 1) = 0$ when $s \searrow 0$. Hence the method used for a weakly linearly convex domain cannot be applied for H.

Can $\mathbb{C}$-convex C^1 functions be approximated by strictly $\mathbb{C}$-convex C^∞ functions?

For C^2 functions we know that the answer is yes and the precise formulation can be found in Theorem 3.1.8. However, as noted in Remark 3.1.9, the question remains open in the C^1 case. One reason for this is that it is hard to see how the C^1 condition for $\mathbb{C}$-convex function transforms under the operation of convolution. This illustrates an important difference between the C^1 condition (3.1.1) and the C^2 condition (3.1.3).

Given a bounded $\mathbb{C}$-convex domain, is it possible to find a sequence of smooth strictly $\mathbb{C}$-convex domains exhausting the original domain?

This has been an open issue for a long time. The question was first asked by L. Aizenberg in [HE94a], Problem 1.18. Some other open issues regarding $\mathbb{C}$-convexity are stated in problem 17.10 in [HE94b]. This problem seems to be a hard one.

Chapter 4

An Applications

In this chapter we shall apply our knowledge about $\mathbb{C}$-convex sets in the field of hyperbolic metrics. More precisely we are going to generalize some of the results due to L. Lempert regarding the relationship between the Carathéodory and Kobayashi metrics.

4.1 Hyperbolic metrics

Let U denote the unit disc in $\mathbb{C}$ and take two points $\zeta, \omega \in U$ and define the associated real number

$$\delta_h(\zeta, \omega) = 2 \tanh^{-1} \frac{\zeta - \omega}{1 - \zeta\bar{\omega}} \tag{4.1.1}$$

where one should recall that $\tanh^{-1}(x) = -\frac{1}{2} \log \frac{1+x}{1-x}$ for $x \in (-1, 1)$. The function δ_h actually constitutes a metric on U and is called the *Poincaré hyperbolic metric* on U. Below we will describe two metrics that both extend δ_h to any open set. We shall however only be considering bounded domains.

Let $\Omega \subseteq C^n$ be a bounded domain, then the Carathéodory metric on Ω, denoted c_Ω, is defined as

$$c_\Omega(z, w) = \sup\{\delta_h(F(z), F(w)) : F \in \mathcal{A}(\Omega, U)\}$$

where $\mathcal{A}(U, \Omega)$ denotes the space of holomorphic functions from U to Ω. One can also define another, similar function,

$$k_\Omega(z,w) = \inf\{\delta_h(\zeta,\omega) : f \in \mathcal{A}(U,\Omega), f(\zeta) = z, f(\omega) = w\}.$$

The function k_Ω does not always constitute a metric since it for some Ω does not satisfy the triangle inequality, see Example 4.1.1. However, one can create a metric out of k_Ω by introducing *the Kobayashi metric* defined as

$$k'_\Omega(z,w) = \inf\{k_\Omega(z,a_1) + k_\Omega(a_1,a_2) + \cdots + k_\Omega(a_n,w) : a_i \in \Omega, n \in \mathbf{N}\}.$$

Now take $f \in \mathcal{A}(U,\Omega)$ and $F \in \mathcal{A}(\Omega,U)$ and let

$$\begin{aligned} c_F(z,w) &= \delta_k(F(z),F(w)),\\ k_f(z,w) &= \inf\{\delta_h(\zeta,\omega) : f(\zeta) = z, f(\omega) = w\}. \end{aligned}$$

Note that the composed function $g = F \circ f$ belongs to $\mathcal{A}(U,U)$ which means that $c_F(z,w) = \delta_h(g(\zeta),g(\omega))$. By the well known Schwarz lemma, see for example [Kob70], we have that that $\delta_h(g(\zeta),g(\omega)) \leq \delta_h(\zeta,\omega)$ for every $g \in \mathcal{A}(U,U)$. Since c_Ω satisfies the triangle inequality we have

$$c_\Omega(z,w) \leq k'_\Omega(z,w) \leq k_\Omega(z,w)$$

for all domains Ω. If Ω is a strictly $\mathbb{C}$-convex domain with C^∞ boundary, then Theorem 1 in [Lem84] gives that

$$c_\Omega(z,w) = k'_\Omega(z,w) = k_\Omega(z,w)$$

which of course in particular implies that k_Ω really is a metric. The equality does not hold for general pseudoconvex bounded domains. One way to see this is to consider the following example, see Remark 3.1.11 in [JP93].

Example 4.1.1. Consider the Hartogs domain

$$H = \{(z_1,z_2) : h(z_1,z_2) = \max(|z_1|,|z_2|,\sqrt{|z_1 z_2|/\epsilon}) < 1\}$$

where $\epsilon \in (0,1/4)$. The defining function h is non negative plurisubharmonic with $h(s\,\zeta) = |s|\,h(z)$ and H is therefore called a balanced pseudoconvex domain. Proposition 3.1.10 in [JP93] tells us that $k_H(0,z) = \delta_h(0,h(z))$.

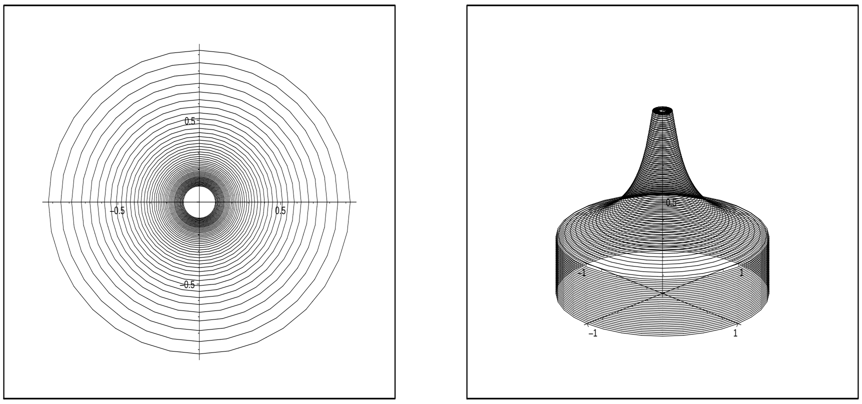


Figure 4.1: The projected Hartogs domain $\pi(H)$ where $\epsilon = 0.1$. The left picture displays the level sets.

Fix $\epsilon < t < \sqrt{\epsilon}$, so that $k(0,(t,t)) = \delta_h(0, t/\sqrt{\epsilon})$. Now we estimate k'_H with

$$\begin{aligned} k'_H(0,(t,t)) &\le k_H(0,(0,t)) + k_H((0,t),(t,t)) \\ &\le \delta_h(0,t) + \delta_h(0,t^2/\epsilon) \\ &\le \frac{1}{2}\log\left(\frac{1+t}{1-t}\frac{\epsilon+t^2}{\epsilon-t^2}\right) \\ &< \frac{1}{2}\log\frac{\sqrt{\epsilon}+t}{\sqrt{\epsilon}-t} = k_H(0,(t,t)). \end{aligned}$$

Hence k_H does not satisfy the triangle inequality. This domain H is clearly hyperconvex, but it is not $\mathbb{C}$-convex. To realize this we consider its projections $\pi(H)$, see equation (2.2.1), which is displayed in Figure 4.1. If we intersect $\pi(H)$ with the cone line going through $(0, 1-a)$ and $(1-a, \sqrt{\epsilon}-a)$ for some small positive a, the intersection is not contractible. Hence H is not $\mathbb{C}$-convex. •

To get a greater family of sets for which $k_\Omega = c_\Omega$ we need the following:

Lemma 4.1.2. *If* $\Omega_1 \subseteq \Omega_2 \subset \cdots \subseteq \Omega$ *are domains with* $\cup\Omega_n = \Omega$*, then we have*

$$\begin{aligned} c_{\Omega_n}(z,w) &\searrow c_\Omega(z,w) \\ k_{\Omega_n}(z,w) &\searrow k_\Omega(z,w) \end{aligned}$$

for all pairs of points z, w in Ω.

Proof. First we assume that we have two fixed points z, w in Ω. Let us first consider the function k_Ω. We know that $k_\Omega \leq k_{\Omega_n}$since k_Ω is an infimum over a larger set. Take a $f \in \mathcal{A}(U, \Omega)$ which almost realizes the infimum for the pair of points z, w. Now for all $r \in (0, 1)$, define $f_r(\zeta) = f(r\zeta)$ which is a holomorphic function such that the closure of $f_r(U)$ lies in Ω. Hence there exists an N such that $f_r \in \mathcal{A}(U, \Omega_n)$ for all $n \geq N$. By choosing r close enough to 1 we conclude that k_Ω must be the limit of k_{Ω_n}

Now we turn to c_Ω. We know that $c_\Omega \leq c_{\Omega_n}$ because c_Ω is a supremum of a smaller set. Now take a $F_n \in \mathcal{A}(\Omega_n, U)$ which almost realizes the supremum for the pair of points z, w. Then take an open relatively compact subset Ω' of Ω, with $z, w \in \Omega'$. We can assume that $\Omega' \subseteq \Omega_1$. So we have a sequence of holomorphic functions F_n from Ω' to the unit disc, which obviously makes the sequence uniformly bounded. Therefore we can use Vitali's theorem and conclude that there exists a subsequence F_{n_k} and a holomorphic function, which is the limit function of the subsequence, see page 168 in [Tit60]. Since this argument is valid for any open relatively compact subset Ω', we can extend the function by analytic continuation to the whole of Ω. Then we get a holomorphic function $\overline{F} \in \mathcal{A}(\Omega, \overline{U})$ which is the limit function of F_{n_k}. But now we can take $r \in (0, 1)$ and define $F_r(z) = r\overline{F}(z) \in \mathcal{A}(\Omega, U)$. By choosing r close enough to one we conclude that c_Ω must be the limit of c_{Ω_n}. □

Combining Theorem 3.2.5 and Lemma 4.1.2 we get the following result.

Theorem 4.1.3. *If a domain Ω is a union of increasing strictly $\mathbb{C}$-convex C^∞ domains, for example if Ω is $\mathbb{C}$-convex with C^2 boundary, then*

$$c_\Omega = k'_\Omega = k_\Omega.$$

If Ω is strictly $\mathbb{C}$-convex with C^∞ boundary, then for all $z, w \in \Omega$, there will be a so called extremal function $f \in \mathcal{A}(U, \Omega)$, realizing the infimum. Since the metric δ_h is invariant under projective automorphisms of U we may assume that $f(0) = z$ and $f(r) = w$, where r is a positive real number. As we now make this a requirement, the extremal function becomes unique. One calls the image $f(U)$ an extremal disc in Ω. For any two points in Ω there will be a unique extremal disc passing trough the points. An extremal mapping f can be extended into a C^∞ mapping $\tilde{f}$ which will be a C^∞ embedding of $\bar{U}$ onto $\tilde{f}(\bar{U})$ with $\tilde{f}(\partial U) \subseteq \partial\Omega$. Moreover, any two extremal discs will meet in at most one point and f will be extremal with respect to any two points in the extremal disc. These facts are due to L. Lempert, see Theorems 2, 3 and 4 in [Lem84].

We will now show almost this in the following more general setting.

Theorem 4.1.4. *Let Ω be a bounded domain such that there exists a sequence of subdomains $\Omega_n \subset \Omega_{n+1}$ whose union equals Ω with each Ω_n being strictly $\mathbb{C}$-convex domain with C^∞ boundary (for example Ω may be* **C***-convex with C^2 boundary). Then for all $z, w \in \Omega$, there exists an extremal mapping with respect to the Kobayashi metric. Moreover, this mapping is extremal with respect to any two points in its extremal disc.*

Proof. First we conclude that Ω, since it is a union of an increasing sequence of $\mathbb{C}$-convex domain, it is itself $\mathbb{C}$-convex. Since Ω is bounded the statements illustrated in Figure 2.3, implies that Ω is hyperconvex. Now pick two points z and w which can be assumed to belong to Ω_1. Then for each n there will be an extremal function $f_n \in \mathcal{A}(U, \Omega_n)$ with $f(r_n) = w$. Since Ω is bounded, we can as in the proof Lemma 4.1.2, use Vitali's theorem and conclude that there is a subsequence f_{n_k} converging to a function $f \in \mathcal{A}(U, \bar{\Omega})$. To show that this is an extremal function for $z, w \in \Omega$, we must first show that the image does not intersect the boundary of Ω. Since Ω is hyperconvex, it exists a negative plurisubharmonic function Φ on Ω which is tends to zero when approaching the boundary. Since f is holomorphic, it follows that $g = \Phi \circ f$ is a non positive subharmonic function. By the maximum principle we conclude that, since U is open, g will not attain zero, unless it is constantly zero. This cannot be the case because $g(\zeta) < 0$, since $f(\zeta) = z \in \Omega$. Therefore we must have $f \in \mathcal{A}(U, \Omega)$. Now we use Lemma 4.1.2 and conclude that $k(z, w) = \delta_h(0, r)$, where r is the limit point of $\{r_n\}$. Since $f_n' \to f'$ and $|f'(r)| < \infty$, we get the implication

$$\lim f_n(r) = f(r) \Rightarrow \lim f_n(r_n) = f(r).$$

Therefore f is an extremal function and $f(U)$ is an extremal disc. Now take two points $\zeta, \omega \in U$ with $f(\zeta) = z$ and $f(\omega) = w$. We intend to show that f is maximal with respect to these points. For this we look at the two sequences

$$f_n(\zeta) = z_n \to z$$
$$f_n(\omega) = w_n \to w,$$

and calculate

$$
\begin{aligned}
&|k(z,w) - k_n(z_n,w_n)| \leq \\
&|k(z,w) - k_n(z,w)| + |k_n(z,w) - k_n(z_n,w_n)| \leq \\
&|k(z,w) - k_n(z,w)| + |\pm(k_n(z,z_n) + k_n(z,w) + k_n(w,w_n) - k(z,w))| \leq \\
&2\,|k(z,w) - k_n(z,w)| + |k_1(z,z_n) + k_1(w,w_n)| \to 0.
\end{aligned}
$$

Since f_n is extremal with respect to k_n, we have $k_n(z_n,w_n) = \delta_h(\zeta,\omega)$ Hence we also have $k(z,w) = \delta_h(\zeta,\omega)$. □

The questions about uniqueness and the number of points in the intersection of two extremal discs are tricky. This is shown by the following example.

Example 4.1.5. Consider the following three sets

$$
\begin{aligned}
\Omega_1 &= U \times U \\
\Omega_2 &= U \times U \setminus \{(p,0)\} \\
\Omega_3 &= B_2 \setminus \{(p,0)\}
\end{aligned}
$$

where $p \in (0,1)$ and B_2 is the unit ball in $\mathbb{C}^2$. Now let $z = (0,0)$ and $w = (p/2,0)$. We wish to determine the three real numbers

$$k_1(z,w) \leq k_2(z,w) \leq k_3(z,w).$$

We know that Ω_1 is convex, hence it satisfies the assumptions of Theorem 4.1.4. Therefore there exists an extremal function $f = (f_1,f_2) \in \mathcal{A}(U,\Omega_1)$ with

$$
\begin{aligned}
f_1(0) = f_2(0) = f_2(r) = 0 \\
f_1(r_1) = p/2.
\end{aligned}
$$

Since $f_1 \in \mathcal{A}(U,U)$ we can use the Schwarz lemma and conclude that $r \geq p/2$. If f_1 is the identity map then $r = p/2$ and, by the Schwarz lemma again, if f_1 is not the identity map then $r > p/2$. We conclude that $k_1(z,w) = \delta_h(0,p/2)$ and that the extremal functions are precisely those of the form (id,h) where $h \in \mathcal{A}(U,U)$ with $h(0) = h(p/2) = 0$. Hence there exist extremal functions, but they are not unique. This gives us immediately that $k_2(z,w) = p/2$ as well, and that the extremal functions exist and are of the form (id,h), where h is the same as before with the additional property that $h(p) \neq 0$. For example $h(\zeta) = c\,\zeta(\zeta - p/2)$ where c is a small constant. Now to Ω_3. By considering function of the form

$$f_t(\zeta) = \left(t\,\zeta, \frac{\zeta - \frac{p}{2t}}{1-t} \right)$$

for $0 \ll t < 1$, see Remark 3.2.5 in [JP93], one concludes that $k_3(z,w)$ also equals $p/2$. Assume that the distance infimum is realized for some $f = (f_1, f_2) \in \mathcal{A}(\Omega_3, U)$, hence $f_1(p/2) = p/2$. But this implies that $f_1 = id$ and since $f \in \mathcal{A}(B_2, U)$ we must have $f_2(\zeta) \to 0$ when $|\zeta| \to 1$. But then f_2 is identically zero by the maximum principle and hence $f = (id, 0)$, but then $(p, 0)$ must be in the image of f which is a contradiction. Thus we have the following situation

$$k_1(z,w) = k_2(z,w) = k_3(z,w) = k_U(z,w),$$

where Ω_1 has a lot of extremal functions, Ω_2 a little fewer and Ω_3 none. •

The puncturing of a set can be made in general. Take a domain $\Omega \subseteq \mathbb{C}^n$ and two points $z, w \in \Omega$. Assume that there exists a unique extremal mapping f with respect to these points. Now choose a third point $z_0 = f(\zeta_0)$ that is different from z and w. Let $\Omega' = \Omega \setminus \{z_0\}$ which implies that $k_\Omega(z,w) \leq k_{\Omega'}(z,w)$. Fix $0 \ll t < 1$ and let ϵ be so small that $f(t\,\zeta) + v \in \Omega$ for all $\zeta \in U$ and $|v| \leq \epsilon$. Now consider

$$\begin{aligned} g :& U \to \mathbb{C}^n \\ &\zeta \mapsto f(\zeta) - z_0. \end{aligned}$$

Since the g is analytic and the dimension of B_n is greater than one, there must exist a non zero w_0 with modulus less than ϵ that does not lie in the image of g. Hence we can define

$$f_t(\zeta) = f(t\,\zeta) + w_0$$

where $f_t \to f$ and $f \in \mathcal{A}(U, \Omega')$. Therefore $k_\Omega(z,w) = k_{\Omega'}(z,w)$ but there is no extremal function for Ω'.

Bibliography

[APS04] M. Andersson, M. Passare, and R. Sigurdsson, *Complex convexity and analytic functionals*, Progress in Mathematics, vol. 225, Birkhäuser, Berlin, 2004.

[BP35] H. Behnke and E. Peschl, *Zur Theorie der Funktionen mehrerer komplexer Verändlichen. Konvexität in bezug auf analytische Ebenen im kleinen und großen.*, Math. Ann. **111** (1935), 158–177.

[HE94a] V. P. Havin and N. K. Nikolski (Eds.), *Linear and complex analysis problem book 3, part I*, Lecture Notes in Mathematics, vol. 1574, Springer-Verlag, Berlin Heidelberg, 1994.

[HE94b] ______, *Linear and complex analysis problem book 3, part II*, Lecture Notes in Mathematics, vol. 1574, Springer-Verlag, Berlin Heidelberg, 1994.

[Hör73] L. Hörmander, *An introduction to complex analysis in severeal varibles*, 2nd ed., North-Holland Mathematical Library, vol. 7, North-Holland, Amsterdam, 1973.

[Hör94] ______, *Notions of convexity*, Progress in Mathematics, vol. 127, Birkhäuser, Cambridge, USA, 1994.

[Hör03] ______, *The analysis of linear partial differential operators i*, second ed., Springer, Berlin, 2003.

[Hör08] ______, *Weak linear convexity and a related notion of concavity*, Mathematica Scandinavica **102:1** (2008), 73–100.

[JP93] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*, de Gruyter Expositions in Mathematics, vol. 9, de Gruyter, Berlin, 1993.

[Kis96] C. O. Kiselman, *Lineally convex Hartogs domains*, Acta Math. Vietnamica **21** (1996), 69–94.

[Kob70] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Applied Mathematics, vol. 2, M. Decker, New York, 1970.

[KP80] S. G. Krantz and H. R. Parks, *Distance to C^k hyperfaces*, Journal of Differential Equations **40** (1980), 116–120.

[KR81] N. Kerzmann and J. Rosay, *Fonctions plurisubharmoniques d'exhaustion bornées et domaines taut*, Math. Ann. **257** (1981), no. 2, 171–184.

[Lem84] L. Lempert, *Intrinsic distances and holomorphic retracts*, Complex Analysis and Applications **81** (1984), 341–364.

[Mar66] A. Martineau, *Sur la topologie des espaces de fonctions holomorphes*, Math. Ann. **163** (1966), 62–88.

[NPZ06] N. Nikolov, P. Pflug, and W. Zwonek, *An example of a bounded $\mathbb{C}$-convex bounded domain which is not biholomorphic to a convex domain*, arXiv (2006), 1–7, arXiv:math/0608662V2 [math.CV] 19 Sep 2006.

[Tit60] E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford University Press, Oxford, 1960.