Abstract

The coamoeba $\mathcal{A}'_V$ of an algebraic variety V on $(\mathbb{C}\backslash\{0\})^n$ is the image on the real torus $\mathbb{T}^n$ of V under the mapping

$$\mathrm{Arg} : (z_1, ..., z_n) \mapsto (\arg z_1, ..., \arg z_n).$$

In this thesis we study the topology and geometry of the coamoeba. Letting $I(V)$ be the ideal defining V, a general result is that the union of the coamoebas $\mathcal{A}'_\omega$ corresponding to the initial ideals I_ω, $\omega \in \mathbb{R}^n$, of $I(V)$, is closed, and furthermore equals the closure of $\mathcal{A}'_V$ when V is a complete intersection. When V is a complete intersection, n is even and $\dim V = n/2$, we show that there is an integer-valued function w on $\mathbb{T}^n \backslash \bigcup_{\omega \neq 0} \mathcal{A}'_\omega$ that is constant on each connected component, such that $\theta \in \mathcal{A}'_V$ whenever $w \neq 0$. When V is an affine space, we give some topological characteristics of $\mathcal{A}'_V$, and more careful descriptions of the coamoeba and amoeba in the special cases when $\dim V = 1$, $\dim V = n-1$ and, when n is even, $\dim V = n/2$.

When V is a hypersurface, it is known that one can derive many striking results on the amoeba of V by use of the *Ronkin function*. We define a function φ for the coamoeba in a similar way as the Ronkin function is defined and give an explicit formula for it. The function φ is affine outside a certain union $\mathcal{H}$ of hyperplanes associated to $\mathcal{A}'_V$ known as the *shell* of $\mathcal{A}'_V$.

When V is a hypersurface and $n = 2$, $\bigcup_{\omega \neq 0} \mathcal{A}'_\omega$ coincide with $\mathcal{H}$. The lines of $\mathcal{H}$ can be given an orientation in a canonical way, and thus $\mathcal{H}$ can be considered as an oriented line arrangement. We give a precise characterization of the shells in $\mathbb{T}^2$ in terms of oriented line arrangements, and show that w as above corresponds to a certain index mapping associated to these arrangements. The connection between $\mathcal{H}$ and oriented line arrangements also allows us to obtain an upper bound for the number of complement components of $\overline{\mathcal{A}}'_V$.

This thesis consists of an introduction together with the following four papers:

[i]
Petter Johansson and Håkan Samuelsson Kalm. A Ronkin type function for coamoebas.

[ii]
Petter Johansson. The argument cycle and the coamoeba. *Complex Var. Elliptic Equ*, 58(3):373-384, 2013.

[iii]
Jens Forsgård and Petter Johansson. Coamoebas and line arrangements in dimension two. To appear in *Mathematische Zeitschrift*.

[iv]
Petter Johansson. Some results on amoebas and coamoebas of affine spaces.

There is an additional paper by Jens Forsgård and me to appear in Arkiv för matematik for which Jens is the main author, while I am the main author of [iii]. Note also that some results in the introduction do not appear in the papers mentioned above.

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On the topology of the coamoeba

Contents

1 Introduction

The branch of mathematics that is studied in this thesis originates from the solving of systems of polynomial equations in several variables. The set of solutions to such a system is known as an *algebraic variety*. The variables can be thought of as coordinates of a vector space over the coefficient field. By $\mathbb{R}^n$ and $\mathbb{C}^n$ we denote the n-dimensional vector spaces over the real and complex numbers respectively. While $\mathbb{R}^n$ seems more familiar to us, it is not algebraically closed like $\mathbb{C}^n$ and as a consequence, algebraic varieties in $\mathbb{R}^n$ do not follow the same rules that hold for complex algebraic varieties.

For a metaphor, think of $\mathbb{R}^n$ as the surface of a lake. Then a real algebraic variety can be thought of as the islands in the lake. These may be positioned in strange formations or not exist at all; however, we know that the islands are just the parts of the sea bed that emerges over the water. The sea bed corresponds to the variety considered in complex space. Its precise shape may be unknown, but at least it exists and we know that it explains the pattern of islands.

But while we actually can see the islands, we cannot see the ground beneath the water clearly. Similarly $\mathbb{R}^n$ resembles physical space that we know well, while $\mathbb{C}^n$ is a much less intuitive construction. How can we visualize complex algebraic varieties?

The obvious suggestion is to study the real and the imaginary parts of the points in the variety V separately. However we will exploit the splitting of $\mathbb{C}^n$ into two real n-spaces in a different way. For $w \in \mathbb{C}^n$, let $\operatorname{Exp} w = e^w := (e^{w_1}, ..., e^{w_n})$. Then Exp maps $\mathbb{C}^n$ surjectively on the complex torus $(\mathbb{C}^*)^n := (\mathbb{C}\backslash\{0\})^n$ and considered on $(\mathbb{C}/2\pi i\mathbb{Z})^n$, Exp is in fact a biholomorphism, that is a bijective complex analytic mapping. The inverse of Exp splits into the mappings Log and Arg that map each coordinate of $z \in (\mathbb{C}^*)^n$ to the logarithm of its absolute value and its argument respectively. We get the following diagram.

$$
\begin{tikzcd}
 & (\mathbb{C}^*)^n \arrow[ddl, "\mathrm{Log}"'] \arrow[dr, "\mathrm{Arg}"] & \\
 & (\mathbb{C}/2\pi i\mathbb{Z})^n \arrow[u, "\mathrm{Exp}", "\cong"'] \arrow[r, "\mathrm{Im}"'] \arrow[dl, "\mathrm{Re}"] & \mathbb{T}^n \\
\mathbb{R}^n & & \\
 & \mathbb{C}^n \arrow[ul, "\mathrm{Re}"] \arrow[uu, "p"'] \arrow[r, "\mathrm{Im}"'] & \mathbb{R}^n \arrow[uu, "p"']
\end{tikzcd}
$$

Here p denotes the natural projection of a space on its given quotient space and $\mathbb{T}^n$ is the real n-torus $(\mathbb{R}/2\pi\mathbb{Z})^n$.

Let us make a minor modification of the premisses and consider V as an algebraic variety in $(\mathbb{C}^*)^n$. It is then natural to let V be defined as the set where all elements of an ideal I of *Laurent polynomials* vanish. A Laurent polynomial is a function $f : (\mathbb{C}^*)^n \to \mathbb{C}$ given by

$$f(z) = \sum_{\alpha \in A} a_\alpha z^\alpha := \sum_{\alpha \in A} a_\alpha z_1^{\alpha_1} z_2^{\alpha_2} ... z_n^{\alpha_n}, \tag{1.1}$$

where A is a finite subset of $\mathbb{Z}^n$ and a_α is a complex non-zero number. The set

$$W = \{w \in \mathbb{C}^n; \forall f \in I : f(e^w) = 0\}$$

is the corresponding analytic variety in $\mathbb{C}^n$ in the diagram above. The natural projection of W on $\mathbb{R}^n$ thus equals $\operatorname{Log} V$ and it is called the *amoeba* of V and denoted by $\mathcal{A}_V$. The corresponding projection of W on $i\mathbb{R}^n$ is periodic with period 2π in each coordinate. Its natural projection on $\mathbb{T}^n$ equals $\operatorname{Arg} V$ and is called the *coamoeba* of V and denoted by $\mathcal{A}'_V$. Sometimes it is convenient to consider the lifting $p^{-1}(\mathcal{A}'_V)$ of the coamoeba to $\mathbb{R}^n$. The relations between the sets we have defined are given by the next diagram.

$$\begin{tikzcd}
& V \arrow[ddl, "\mathrm{Log}"'] \arrow[dr, "\mathrm{Arg}"] & \\
& p(W) \arrow[u, "\cong", "\mathrm{Exp}"'] \arrow[r, "\mathrm{Im}"'] \arrow[dl, "\mathrm{Re}"] & \mathcal{A}'_V \\
\mathcal{A}_V & & \\
& W \arrow[uu, "p"'] \arrow[ul, "\mathrm{Re}"] \arrow[r, "\mathrm{Im}"'] & p^{-1}(\mathcal{A}'_V) \arrow[uu, "p"']
\end{tikzcd}$$

The amoeba was introduced in 1994 by Israel Gelfand, Mikhail Kapranov and Andrei Zelevinsky, see [6]. The term is motivated by the typical appearance of $\mathcal{A}_V$ for varieties V of positive dimension, see Figure 1. Since Log is proper on $(\mathbb{C}^*)^n$, that is $\operatorname{Log}^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact, $\mathcal{A}_V$ is a closed set.

The term coamoeba was coined by Mikael Passare in 2004, see [9], but several mathematicians started independently of each other to study this projection in the mid-00s, see e.g. [11] and [3], and another name that has gained acceptance is "alga", similarly inspired by the typical appearance of $\mathcal{A}'_V$, see again Figure 1. The mapping Arg is not proper, and $\mathcal{A}'_V$ is not necessarily closed.

The overall motivation for this thesis is to describe the coamoeba geometrically and topologically. We are particularly interested in analogies between the coamoeba and the better known amoeba. That the amoeba and the coamoeba carry enough structure to be interesting to study is particularly obvious in the case when V is determined by a single Laurent polynomial f. The polytope obtained by taking the convex hull of the index set A of f in $\mathbb{R}^n$, see (1.1), is known as the *Newton polytope* of f and denoted by Δ_f. This polytope has established itself as one of the key objects in algebraic geometry and its connection to the coamoeba (and amoeba) is a central theme of this thesis.

1.1 Amoebas and coamoebas in dimension one

We begin with some words about the simplest possible amoebas and coamoebas. Let V_g be the zero set of a Laurent polynomial g on $\mathbb{C}^*$ such that $g(w) =$

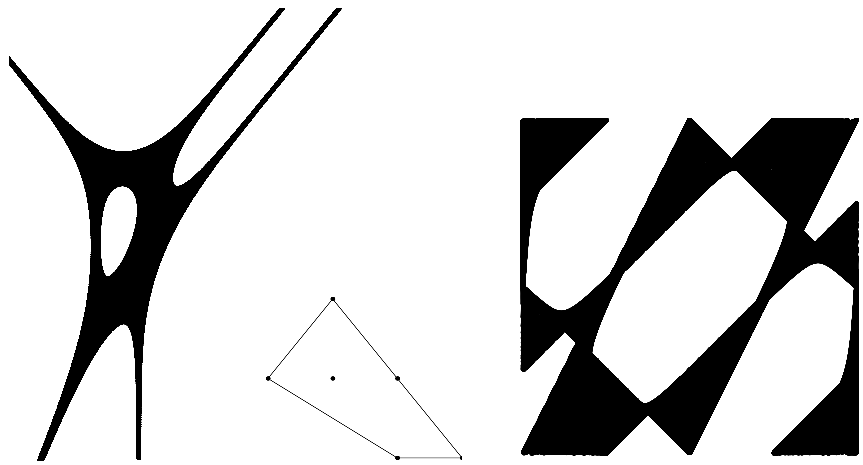

Figure 1: The amoeba of $1-8z_1+z_1z_2+4z_1^2+z_1^2/z_2+z_1^3/z_2$ in $\mathbb{R}^2$ (left), the coamoeba of $1+z_1z_2+z_1^2/z_2+z_1^3/z_2$ in $\mathbb{R}^2/(2\pi\mathbb{Z})^2$ (right) and the common Newton polygon of the defining polynomials (middle).

$h(w)/w^m$ for some $m \in \mathbb{Z}$ and some polynomial h with $h(0) \neq 0$. Define the degree of g as the degree of h. Then by the fundamental theorem of algebra we can write $V_g = \{w_1, w_2..., w_L\}$ where the multiplicity of w_j is D_j and

$$\sum_{j=1}^{L} D_j = \deg g.$$

Each complex number is mapped to a not necessarily unique real number under Log. Thus

$$\mathcal{A}_g := \mathcal{A}_{V_g} = \{a_1, a_2..., a_l\}$$

and similarly

$$\mathcal{A}'_g := \mathcal{A}'_{V_g} = \{b_1, b_2..., b_{l'}\}$$

for integers $1 \leq l, l' \leq L$. Hence in the context of amoebas (or coamoebas) the fundamental theorem of algebra states that the number of connected components of $\mathbb{R}\backslash\mathcal{A}_g$ (or $\mathbb{T}^1\backslash\mathcal{A}'_g$) is at most $1+\deg g$ (or $\deg g$).

Assume that $a_j < a_k$ whenever $j < k$ and let $d_1, ..., d_l$ be positive integers such that $\mathrm{Log}^{-1}(a_j)$ contains d_j points in V_g counted with multiplicity. A classical result in complex analysis is Jensen's formula:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(e^{x+i\theta})|d\theta = \log|h(0)| - mx + \sum_{j=1}^{M(x)} d_j(x-a_j), \tag{1.2}$$

where $M(x)$ is the maximal integer such that $a_{M(x)} < x$. If we fix g, then (1.2) is a function $N_g(x)$ on $\mathbb{R}$ that can be interpreted as the mean value of $\log|g(e^z)|$

on $\mathbb{R} + i\mathbb{T}^1$ for $\operatorname{Re} z$ fixed. The function $\log|g(e^z)|$ is *subharmonic*, see Section 3.2. Thus, $N_g(x)$ is itself a subharmonic function, see e.g. [1], and as it is defined on $\mathbb{R}$, it follows that it is convex. This also follows by (1.2). In fact, letting $a_0 = -\infty$ and $a_{l+1} = +\infty$ we have that $N_g(x)$ is affine on each component $]a_k, a_{k+1}[$ of the complement of $\mathcal{A}_g$ with

$$N_g'(x) = -m + \sum_{j=1}^{k} d_j,\ x \in]a_k, a_{k+1}[.$$

In the sense of distributions, the second derivative of $N_g(x)$ is given by

$$N_g''(x) = \sum_{j=1}^{l} d_j \delta_{a_j}(x),$$

where δ_{a_j} is the dirac measure at a_j. In particular, this means that the support of $N_g''(x)$ equals $\mathcal{A}_g$. For a survey on subharmonic functions on $\mathbb{C}$ including Jensen's formula, see e.g. [1].

As we will see later, the results we have connected to the one-dimensional amoeba and coamoeba are reflections of general results and concepts connected to amoebas and coamoebas of hypersurfaces. The difference is that the Newton polygon plays a substantial role in these matters in the general case, as we will see in Section 2 and 4 respectively. In Section 5 we make some generalizations to algebraic varieties of arbitrary codimension and in Section 3 we pursue a certain strategy to establish an analogue of the *Ronkin function* for the coamoeba. The definition of the Ronkin function and some words about its importance for the amoeba, can be found in Section 2.

2 The duality of $\mathcal{A}_f$ and Δ_f

In this section, we will relate the shape of the amoeba of a Laurent polynomial f to the Newton polytope Δ_f. To this end, we will consider three different strategies, of which the first is purely algebraic and the second involves complex analysis. The third strategy also uses complex analysis but in a more conceptual way.

2.1 Normal cones and the complement of the amoeba

Let f be as in (1.1). Given $\alpha \in A$, assume that $x \in \mathbb{R}^n$ is such that

$$\forall \alpha' \in A \backslash \{\alpha\} : (\alpha - \alpha') \cdot x > r \tag{2.1}$$

for some $r \in \mathbb{R}$. If r is large enough we have that

$$|a_\alpha| e^{\alpha \cdot x} > \sum_{\alpha' \in A \backslash \{\alpha\}} |a_{\alpha'}| e^{\alpha' \cdot x}, \tag{2.2}$$

and thus $f(z) \neq 0$ whenever $\operatorname{Log} z = x$ for x as in (2.1). This means that $x \notin \mathcal{A}_f$. Let r_α be the infimum the numbers r for which (2.1) implies (2.2), given α, and let D_α be the set of $x \in \mathbb{R}^n$ satisfying (2.1) for $r = r_\alpha$. Then

$$\mathcal{A}_f \subseteq \left(\bigcup_{\alpha \in A} D_\alpha \right)^c.$$

For a face Γ of the Newton polytope Δ_f, the *normal cone* N_Γ of Γ is defined as

$$N_\Gamma = \{x \in \mathbb{R}^n; \forall y \in \Gamma, y' \in \Delta : (y - y') \cdot x \geq 0\}.$$

It is easy to check that N_Γ is a cone of dimension $n - \dim \Gamma$ in the normal space of Γ, directed outwards from Δ_f when its vertex is placed at Γ. Furthermore, if $x \in D_\alpha$, then the translation of N_Γ by x is contained in D_α whenever $\alpha \in \Gamma$. In particular, if α is a vertex of Δ_f, then D_α is clearly non-empty and thus contains a translate of the n-dimensional cone N_α. This explains the shape of the complement components of $\mathcal{A}_f$ as in Figure 1. In [14], this approach to a description of the amoeba is thoroughly investigated.

2.2 The tentacles and vacuoles of the amoeba

We will now define, given a Laurent polynomial f as in (1.1), certain associated Laurent polynomials that are of great importance in this work. For a face Γ of Δ_f, let

$$f_\Gamma(z) = \sum_{\alpha \in \Gamma \cap A} a_\alpha z^\alpha.$$

If Γ is an edge, that is a one-dimensional face, let α, α' be its two vertices. Furthermore, let

$$\beta = (\alpha' - \alpha)/d, \tag{2.3}$$

where d is the maximal integer for which the right hand side of (2.3) becomes an integer vector. Thus d is the *integer length* of Γ and we set $|\Gamma| = d$. One can write

$$A \cap \Gamma = \{\alpha, \alpha + k_1\beta, ..., \alpha + k_l\beta\},$$

where $0 < k_1 < ... < k_l = d$. Furthermore,

$$f_\Gamma(z) = z^\alpha g(z^\beta) \tag{2.4}$$

for some polynomial g on $\mathbb{C}$. Using the notation from Section 1.1 connected to g, we have that

$$\mathcal{A}_{f_\Gamma} = \bigcup_{j=1}^{l} \{x \in \mathbb{R}^n; \beta \cdot x = a_j\}, \tag{2.5}$$

that is, $\mathcal{A}_{f_\Gamma}$ equals a union of l hyperplanes orthogonal to Γ.

For two sets $S, U \subset \mathbb{R}^n$ and a scalar λ, let

$$S + U := \{s + u; s \in S,\ u \in U\},\ \lambda S = \{\lambda s; s \in S\}.$$

The former operation is called the *Minkowski sum* of S and U.

Lemma 2.1. *Given $M > 0$ large enough and $\varepsilon > 0$, there is a set $D_M^\varepsilon \subset \mathbb{R}^n$ that approximates $\{s\beta; |s| < M\} + N_\Gamma$ and a finite-to-one branched covering*

$$\psi : V_f \cap \operatorname{Log}^{-1} D_M^\varepsilon \to V_{f_\Gamma} \cap \operatorname{Log}^{-1} D_M^\varepsilon$$

such that

1. *the multiplicity of $z \in V_{f_\Gamma}$ equals the sum of the multiplicities in V_f of the points in $\psi^{-1}(z)$,*
2. *for $z \in V_f$, $\operatorname{Log} z - \operatorname{Log} \psi(z) = s\beta$ for some $s \in \mathbb{R}$ with $|s| < \varepsilon$.*

Lemma 2.1 is a consequence of Rouché's theorem and Lemma 4.6 in [i], where the exact meaning of the approximation is specified. Note however that D_M^ε as used here corresponds to a translate of the equally denoted set in [i]. In view of (2.5), the second point of Lemma 2.1 implies the existence of the "tentacles" of $\mathcal{A}_f$ stretching out in the normal directions of the edges of Δ_f, see Figure 1.

In [4], Forsberg, Passare and Tsikh showed that the complement of $\mathcal{A}_f$ is a union of pairwise disjoint connected components $\{E_\alpha\}_{\alpha \in A}$ such that $D_\alpha \subseteq E_\alpha$, thus also putting the "vacuole" in Figure 1 into context. The next theorem is an immediate consequence of this result (see Theorem 2.8 in [4]).

Theorem 2.2 (Forsberg, Passare, Tsikh)**.** *The number of components of the complement of $\mathcal{A}_f$ is at most $|\Delta_f \cap \mathbb{Z}^n|$.*

This bound was later shown to be sharp in a very strict sense by Rullgård, see [16], Corollary 6. Note that since

$$|\Delta_g \cap \mathbb{Z}| = 1 + \deg g$$

when g is a Laurent polynomial in one variable, Theorem 2.2 is a generalization to arbitrary dimension of the amoeba version of the fundamental theorem of algebra from Section 1.

2.3 The spine of the amoeba

The suggested duality in Figure 1 is elegantly uncovered to an even larger extent by use of a somewhat surprising tool. The function $N_g(x)$ that we defined in Section 1 is naturally generalized to $\mathbb{R}^n$ when viewed as the mean value of the function $\log|f(z)|$ for $\operatorname{Log} z \in \mathbb{R}^n$ fixed:

$$N_f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \log|f(e^{x+i\theta})| d\theta,$$

where $d\theta = d\theta_1 \wedge ... \wedge d\theta_n$. This is known as the *Ronkin function* corresponding to f. The following theorem is essentially due to Ronkin, see [15], but translated to amoebas by Passare and Rullgård in [13].

Theorem 2.3 (Ronkin)**.** *The function N_f is convex on $\mathbb{R}^n$ and it is affine on an open set U if and only if U is contained in a component E_α of the complement of $\mathcal{A}_f$. The gradient of N_f at an arbitrary point $x \in E_\alpha$ equals α.*

The indexation of the components of the complement of $\mathcal{A}_f$ can be viewed as just one aspect of an even stronger duality between $\mathcal{A}_f$ and Δ_f implied by the Ronkin function. Note that we can approximate N_f by a piecewize affine function

$$S(x) = \max\{c_\alpha + \alpha \cdot x; \exists \text{ component of } \mathcal{A}_f^c \text{ of order } \alpha\},$$

where $c_\alpha = N_f(x) - \alpha \cdot x$ for any $x \in E_\alpha$. The function $S(x)$ is an example of what is known as a *tropical polynomial*, see e.g. [18]. Let $\mathcal{S}_f \subset \mathbb{R}^n$ be the set where S is non-smooth. Then $\mathcal{S}_f$ is a polyhedral complex, or a tropical hypersurface, that is known as the *spine* of $\mathcal{A}_f$. The term is motivated by the next result.

Theorem 2.4 (Passare, Rullgård)**.** *The spine $\mathcal{S}_f$ is contained in, and is a deformation retract of, the amoeba $\mathcal{A}_f$.*

For a deeper investigation of the structure of the spine of the amoeba, see [13].

3 Ronkin type functions

We hinted at in the previous section that many properties of the amoeba that appear to be ad hoc when studied by straightforward methods, are put into context by the introduction of the Ronkin function. This motivates us to establish the frameworks of *plurisubharmonic* functions and *closed positive currents*. For details, see, e.g., [8]. Thereafter we will investigate the possibility to define a function similar to N_f that corresponds to the coamoeba.

3.1 Differential calculus in $\mathbb{C}^n$

We start with some words about differential calculus in the complex coordinates $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$. By definition we have that

$$\begin{aligned} dz_j &= dx_j + idy_j, \\ d\bar{z}_j &= dx_j - idy_j. \end{aligned} \tag{3.1}$$

Furthermore the transition between derivatives of complex and real coordinates are given by the formulas

$$\begin{aligned} \frac{\partial}{\partial z_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \\ \frac{\partial}{\partial \bar{z}_j} &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right). \end{aligned} \tag{3.2}$$

A (p, q)-form on $\mathbb{C}^n$ is a sum of terms

$$\varphi_{IJ}(z) dz_I \wedge d\bar{z}_J := \varphi_{IJ}(z) dz_{j_1} \wedge ... \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge ... \wedge d\bar{z}_{k_q}$$

where $I = \{j_1, ..., j_p\}$, $J = \{k_1, ..., k_q\}$ and $\varphi_{IJ} \in C^\infty$. The differential operators ∂ and $\bar{\partial}$ act on each term of a (p, q)-form as follows.

$$\begin{aligned} \partial(\varphi_{IJ} dz_I \wedge d\bar{z}_J) &= \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J, \\ \bar{\partial}(\varphi_{IJ} dz_I \wedge d\bar{z}_J) &= \sum_{j=1}^{n} \frac{\partial \varphi}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J. \end{aligned} \tag{3.3}$$

Thus if u is a (p, q)-form, then ∂u is a $(p + 1, q)$-form and $\bar{\partial} u$ is a $(p, q + 1)$-form. Set $d = \partial + \bar{\partial}$ and $d^c = (\partial - \bar{\partial})/(2\pi i)$. Then d is the standard exterior differential operator and we say that a (p, q)-form u is *closed* if $du = 0$. One can verify that $\partial^2 = \bar{\partial}^2 = 0$. This implies that $dd^c = (\pi i)^{-1} \bar{\partial}\partial$. The case when $\varphi \in C^\infty$ is independent of $\operatorname{Im} z$, is of particular interest in this work. Then we have by (3.1) and (3.2) that

$$dd^c \varphi = \frac{1}{2\pi} \sum_{1 \le j,k \le n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} dx_j \wedge dy_k. \tag{3.4}$$

3.2 Plurisubharmonic functions

Let φ be an upper semicontinuous function on $\mathbb{C}$. Then φ is said to be *subharmonic* if

$$\varphi(z) \le \frac{1}{2\pi} \int_0^{2\pi} \varphi(z + re^{i\theta}) d\theta$$

for $r > 0$ small enough. Subharmonicity of functions in one complex variable is a central concept in complex analysis that can be thought of as the notion corresponding to convexity of functions in one real variable. Plurisubharmonicity is the corresponding notion for upper semicontinuous functions φ in arbitrarily many complex variables. Similarly to the definition of convexity in several real variables, we say that φ is *plurisubharmonic* if its restriction to any complex line is subharmonic. The analogy to convexity is quite concrete: If φ is plurisubharmonic and independent of $\operatorname{Im} z$, then φ is a convex function on $\mathbb{R}^n$.

A typical example of a plurisubharmonic function is $\log |f|$, where f is a holomorphic function on $\mathbb{C}^n$. One can use Fubini's theorem to show that for a compact subset K of $\mathbb{R}^n$, the function

$$z \mapsto \int_{r \in K} \varphi(z + ir) dr$$

is plurisubharmonic whenever it is upper semicontinuous and φ is plurisubharmonic. Letting $\varphi = \log |f(e^z)|$ for a Laurent polynomial f and $K =]-\pi, \pi]^n$, we have in particular that the Ronkin function N_f is plurisubharmonic, if considered as a function on $\mathbb{C}^n$ that is independent of $\operatorname{Im} z$. Thus the convexity of the Ronkin function follows, see Theorem 2.3.

3.3 Closed positive $(1,1)$-currents

A (p,q)-*test form* ξ on $\mathbb{C}^n$ is a (p,q)-form with compact support. The space of (p,q)-test forms is denoted by $\mathcal{D}^{p,q}$. Given a non-empty compact set $K \subset \mathbb{C}^n$ and $k \in \mathbb{N}$ we can define the seminorm

$$s_K^k(\xi) = \max_{z\in K} \max_{\substack{|I|=p\\|J|=q}} \max_{\substack{\alpha,\beta\in\mathbb{N}^n,\\|\alpha|+|\beta|\le k}} \left| \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} \frac{\partial^{\beta_1}}{\partial \bar{z}_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial \bar{z}_n^{\beta_n}} \xi_{IJ}(z) \right|$$

on $\mathcal{D}^{p,q}$ and consider the topology on $\mathcal{D}^{p,q}$ induced by the union of all such seminorms.

A (p,q)-*current* is a continuous linear functional $T : \mathcal{D}^{n-p,n-q} \to \mathbb{C}$ that acts like a distribution T_{IJ} on ξ_{IJ} for every monomial $\xi_{IJ} dz_I \wedge d\bar{z}_J$ of the test form $\xi \in \mathcal{D}^{n-p,n-q}$. This means that the space $\mathcal{D}'_{p,q}$ of (p,q)-currents is the dual vector space of $\mathcal{D}^{n-p,n-q}$. The space $\mathcal{D}'_{p,q}$ is equipped with the weak topology:

$$T_j \to 0 \ \text{ if } \ \forall \xi \in \mathcal{D}^{n-p,n-q} : T_j.\xi \to 0.$$

By differentiating T_{IJ} in the sense of distributions, the operators ∂ and $\bar{\partial}$, and thus also the notion of closedness, can be defined on $\mathcal{D}'_{p,q}$. The support of T is defined as the complement of the maximal open set $U \subseteq \mathbb{C}^n$ for which $T.\xi = 0$ whenever $\operatorname{supp}\xi \subseteq \emptyset$.

If $\xi \in \mathcal{D}^{n-1,n-1}$, then ξ is *positive* if each of its terms can be written as

$$\lambda i dz_{j_1} \wedge d\bar{z}_{j_1} \wedge ... \wedge i dz_{j_{n-1}} \wedge d\bar{z}_{j_{n-1}}$$

for some non-negative test function λ. A $(1,1)$-current T is positive if $T.\xi \geq 0$ whenever $\xi \in \mathcal{D}^{n-1,n-1}$ is positive. A plurisubharmonic function φ can be considered as the $(0,0)$-current

$$\mathcal{D}^{n,n} \ni \xi \mapsto \varphi.\xi := \int \varphi\xi.$$

In this sense, the class of plurisubharmonic functions coincides with the class of $(0,0)$-currents φ for which $dd^c\varphi$ is positive. In fact one can use the Poincaré lemma to show that $T \in \mathcal{D}'_{1,1}$ is closed and positive if and only if T equals $dd^c\varphi$ for some plurisubharmonic function φ.

We will now look at two examples of corresponding positive currents and plurisubharmonic functions. Consider the *integration current*

$$\mathcal{D}^{n-1,n-1} \ni \xi \mapsto [V_f].\xi := \int_{V_{f\,\mathrm{reg}}} \xi,$$

where V_f is the zero set of a holomorphic function f. It is clear that $[V_f]$ is positive, but it is harder to determine wether it is closed or not. However, if $df(z) \neq 0$ whenever z is a regular point, it is known that

$$[V_f] = dd^c \log|f|,$$

and thus $[V_f]$ is closed. More generally, whenever f is holomorphic, $dd^c \log |f|$ is the integration current of f counted with multiplicity, that is the *Lelong current*, which thus is closed.

For the second example, let $f(z) := g(e^z)$ for some Laurent polynomial g and consider the Ronkin function $N_f := N_g$ as a plurisubharmonic function on $\mathbb{C}^n$ independent of $\operatorname{Im} z$. A computation gives that the closed positive current corresponding to N_f is given by

$$\mathcal{D}^{n-1,n-1} \ni \xi \mapsto dd^c N_f.\xi = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} T_f^{is}.\xi ds, \tag{3.5}$$

where $T_f^t := dd^c \log |f(\cdot + t)|$ for $t \in \mathbb{C}^n$. Clearly the support of the right hand side of (3.5) equals $\mathcal{A}_g$. Hence (3.5) implies that N_f in affine on the complement of $\mathcal{A}_g$, cf. Theorem 2.3.

3.4 Ronkin type functions on the coamoeba

As above, let $f = g(e^z)$, where g is a Laurent polynomial. For convenience, we let the amoeba $\mathcal{A}_f$ and the coamoeba $\mathcal{A}'_f$ refer to the real and imaginary parts respectively of the zero set of f. This means that $\mathcal{A}_f = \mathcal{A}_g$ and that $\mathcal{A}'_f$ is a periodic subset of $\mathbb{R}^n$ with $p(\mathcal{A}'_f) = \mathcal{A}'_g$, see Section 1. Moreover, we set $\Delta_f := \Delta_g$.

An intriguing question is if there exists a convex function that corresponds to the coamoeba $\mathcal{A}'_f$ in a similar way as the Ronkin function corresponds to $\mathcal{A}_f$. Let us make this question more specific: Assume that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex, denote by $\mathcal{F}_f$ the set of components of the complement of $\overline{\mathcal{A}}'_f$ and set $\check{T} = dd^c\varphi$ where φ is considered as a plurisubharmonic function on $\mathbb{C}^n$ independent of $\operatorname{Re} z$. Then we say that φ is a *Ronkin type function* associated to the coamoeba of f if it fulfills the following assertions.

1. The current $\check{T}$ vanishes outside of $\overline{\mathcal{A}}'_f$.
2. The gradient $\nabla\varphi$ does not take the same value in two different components $E, E' \in \mathcal{F}_f$.
3. For any $\lambda \in 2\pi\mathbb{Z}^n$, $\check{T}(z + i\lambda) = \check{T}(z)$.

The first two points correspond to properties of the Ronkin function and guarantee that $\nabla\varphi$ induces an injective mapping on $\mathcal{F}_f$. To consider this mapping on the set $\mathcal{E}_g$ of components of the complement of $\overline{\mathcal{A}}'_g \subseteq \mathbb{T}^n$ instead of on $\mathcal{F}_f$, see below, we will use the last point.

Before we attempt to construct a Ronkin type function as characterized above, let us study some properties that such a function and its corresponding closed positive current must have. It follows from (3.4) and the definition of derivatives of distributions that

$$\check{T} = \frac{1}{2\pi} \sum_{1\leq j,k\leq n} \frac{\partial^2\varphi}{\partial y_j \partial y_k} dx_j \wedge dy_k. \tag{3.6}$$

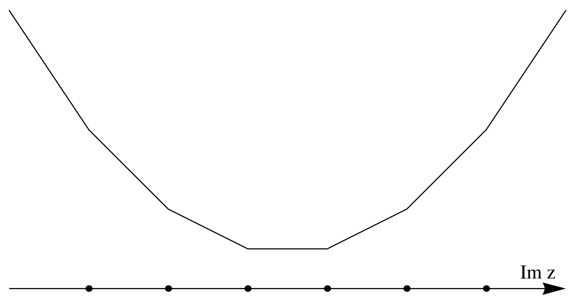

Figure 2: A Ronkin type function associated to the coamoeba of $f = 1 + e^z$. The points on the $\operatorname{Im} z$-axis marks $\mathcal{A}'_f$.

Let e_k be the k:th vector of the standard basis of $\mathbb{R}^n$ and let

$$c_{jk}(y) = \frac{\partial \varphi}{\partial y_j}(y + 2\pi e_k) - \frac{\partial \varphi}{\partial y_j}(y) = \int_0^{2\pi} \frac{\partial^2 \varphi}{\partial y_j \partial y_k}(y + se_k)ds. \tag{3.7}$$

By (3.6) and the third point above,

$$\frac{\partial c_{jk}}{\partial y_l}(y) = \frac{\partial^2 \varphi}{\partial y_l \partial y_j}(y + 2\pi e_k) - \frac{\partial^2 \varphi}{\partial y_l \partial y_j}(y) = 0$$

for every $1 \leq l \leq n$, that is c_{jk} is a constant. Notice that

$$\begin{aligned} 2\pi c_{jk} &= \int_0^{2\pi} \frac{\partial \varphi}{\partial y_j}(y + te_j + 2\pi e_k) - \frac{\partial \varphi}{\partial y_j}(y + te_j)dt \\ &= \int_0^{2\pi} \int_0^{2\pi} \frac{\partial^2 \varphi}{\partial y_j \partial y_k}(y + te_j + se_k)dsdt. \end{aligned} \tag{3.8}$$

We apply Fubini's Theorem and then make the computation of (3.8) backwards to show that $c_{jk} = c_{kj}$. We denote the symmetric matrix (c_{jk}) by S.

For $\lambda \in 2\pi\mathbb{Z}^n$, consider the function

$$h_\lambda(y) = \varphi(y + \lambda) - \varphi(y).$$

Lemma 3.1. *For some vector* $u \in \mathbb{R}^n$,

$$h_\lambda(y) = \lambda S y^t + \lambda S \lambda^t/2 + u \cdot \lambda. \tag{3.9}$$

Proof. If we first assume that $\lambda = \pm 2\pi e_k$ for some k, then it follows by (3.7) that $\partial h_\lambda(y)/\partial y_j = \pm c_{jk}$ for every j. Hence

$$h_\lambda(y) = \lambda S y^t + \lambda S \lambda^t/2 \pm u_k$$

for some $u_k \in \mathbb{R}$. Set $u = (u_1, ..., u_n)$ and assume that h_ν and h_μ are given by (3.9) for $\lambda = \nu$ and $\lambda = \mu$ respectively. Then

$$\begin{aligned} h_{\nu+\mu}(y) &= h_\nu(y+\mu) + h_\mu(y) \\ &= \nu S(y+\mu)^t + \nu S\nu^t/2 + u \cdot \nu + \mu S y^t + \mu S \mu^t/2 + u \cdot \mu \\ &= (\nu+\mu)Sy^t + (\nu+\mu)S(\nu+\mu)^t/2 + u \cdot (\nu+\mu), \end{aligned}$$

that is, $h_{\nu+\mu}$ is given by (3.9) for $\lambda = \nu + \mu$. It follows by induction that (3.9) holds for every $\lambda \in 2\pi\mathbb{Z}^n$. □

We will now use Lemma 3.1 to define an injective mapping on $\mathcal{E}_g$. To this end, consider the torus $\mathbb{R}^n/\mathbb{Z}[2\pi S]$, where $\mathbb{Z}[2\pi S]$ denotes the lattice in $\mathbb{R}^n$ generated by the rows of $2\pi S$ over $\mathbb{Z}$. Let $k_\varphi : \mathcal{E}_g \to \mathbb{R}^n/\mathbb{Z}[2\pi S]$ be the mapping given by

$$k_\varphi(E) = [\nabla\varphi(y)], \ y \in p^{-1}(E).$$

Proposition 3.2. *If φ is a Ronkin type function, then k_φ is well-defined and injective.*

Proof. We have to show that $[\nabla\varphi(y+\lambda)] = [\nabla\varphi(y)]$, or equivalently that $[\nabla h_\lambda] = 0$ for every $\lambda \in 2\pi\mathbb{Z}^n$. But this follows immediately from Lemma 3.1. By the characterization of a Ronkin type function, $\nabla\varphi(y)$ is furthermore independent of $y \in p^{-1}(E)$. Hence k_φ is well-defined.

It follows by the second point in the characterization of a Ronkin type function that k_φ does not take the same value on two different components in $\mathcal{F}_f$. Assume now that $[\nabla\varphi(y)] = [\nabla\varphi(y')]$ for $y \in F, y' \in F'$, $F, F' \in \mathcal{F}_f$. Then there is a $\lambda \in 2\pi\mathbb{Z}^n$ such that

$$\nabla\varphi(y') = \nabla\varphi(y) + \lambda S = \nabla\varphi(y) + \nabla h_\lambda(y) = \nabla\varphi(y+\lambda)$$

and thus $y + \lambda \in F'$, that is, $[y]$ and $[y']$ are contained in the same component $E \in \mathcal{E}_g$. This means that k_φ is injective. □

3.5 An example of a Ronkin type function on the coamoeba

In Section 3.4 we started from the characteristic properties of the Ronkin function to set up characteristics for a Ronkin type function for the coamoeba. To construct such a function, we will instead start from the definition of the Ronkin function and try to find an analogous plurisubharmonic function that is connected to the coamoeba.

Let f be as in Section 3.4. Since f is periodic in $\operatorname{Im} z$ with period 2π in each coordinate, the Ronkin function $N_f(x)$ can be considered as the mean value of $\log|f(x+iy)|$ over $y \in \mathbb{R}^n$. A first naive approach to construct a Ronkin type function associated to $\mathcal{A}'_f$ would be to take the mean value of $\log|f(x+iy)|$ over x. However, this mean value is infinite for every y.

Let us instead see if it is possible to connect a closed positive $(1,1)$-current to $\mathcal{A}'_f$ in a similar way as $dd^c N_f(x)$ is connected to $\mathcal{A}_f$. Let $B^k_R \subset \mathbb{R}^k$ be the

k-dimensional ball with center in the origin and radius R, and set $B_R = B_R^n$. Furthermore, let $\mathrm{Vol}_k A$ be the k-dimensional volume of a set A of real dimension k and set $\mathrm{Vol} = \mathrm{Vol}_n$. By (3.5), $dd^c N_f$ is the mean value of the currents T_f^{is} for $s \in \mathbb{R}^n$. In order to define a corresponding "mean value" of T_f^r over $\mathbb{R}^n$, we need to determine the order of growth of $\mathrm{Vol}_{2n-2}(V_f \cap (B_R + i[-\pi,\pi]^n))$ when $R \to \infty$. Clearly we can here exchange B_R with $\mathcal{A}_f \cap B_R$.

To this end we make two observations. Recall the sets $D_\alpha \subset \mathcal{A}_f^c$, the constants $r_\alpha \geq 0$ and the concept of normal cones N_Γ from Section 2.1. For an edge Γ of Δ_f and $M \geq 0$, let

$$N_\Gamma^M = B_M + N_\Gamma$$

and let

$$N^M = \bigcup_{\Gamma \text{ edge of } \Delta_f} N_\Gamma^M.$$

Then $\mathrm{Vol}(N^M \cap B_R) \leq c\,\mathrm{Vol}_{n-1}(B_R^{n-1})$ for some $c > 0$ that is independent of $R > 0$. But if $M = \max_{\alpha \in A} r_\alpha$, then N^M contains the complement of $\bigcup_{\alpha \in A} D_\alpha$, that is $\mathcal{A}_f \subseteq N^M$. Thus also $\mathrm{Vol}(\mathcal{A}_f \cap B_R) \leq c\,\mathrm{Vol}_{n-1}(B_R^{n-1})$.

Second, the function on $\mathbb{R}^n$ given by

$$r \mapsto \mathrm{Vol}_{2n-2}(V_f \cap (\{x; |x-r| < 1\} + i]-\pi,\pi]^n)) \tag{3.10}$$

is uniformly bounded by some constant, see Theorem 3.1 (ii) in [i]. We thus have for $\xi \in \mathcal{D}^{n-1,n-1}$ and M large enough that

$$\begin{aligned} \frac{1}{\mathrm{Vol}_{n-1}(B_R^{n-1})} \int_{|r|<R} T_f^r.\xi dr =& \frac{1}{\mathrm{Vol}_{n-1}(B_R^{n-1})} \int_{\substack{|r|<R \\ r \in N^M}} T_f^r.\xi dr \\ \leq& C \frac{\mathrm{Vol}(\operatorname{supp}\xi)}{\mathrm{Vol}_{n-1}(B_1^{n-1})} \sup|\xi| \end{aligned} \tag{3.11}$$

for some $C > 0$ that is independent of $R > 0$. Here the equality follows from the first and the inequality from the second observation. Letting $\check{T}_f^R \in \mathcal{D}'_{1,1}$ be the current defined by the left hand side of (3.11), the family $\{\check{T}_f^R\}_{R>0}$ has by (3.11) at least one accumulation point $\check{T}_f \in \mathcal{D}'_{1,1}$. Since T_f^r is closed, positive and periodic in $\mathrm{Im}\, z$ and has support on $\mathbb{R}^n + i\overline{\mathcal{A}}'_f$, the same is true for $\check{T}_f^R$ for every $R > 0$ and thus also for $\check{T}_f$. Furthermore, since

$$T^r.\xi(x+iy) = T^{r+x-x'}.\xi(x'+iy),$$

it follows that the coefficients T_{IJ} of $\check{T}_f$ are independent of x. So far $\check{T}_f$ seems to be the current we are looking for.

When $\dim \Delta_f = 1$, there is a polynomial g on $\mathbb{C}$ with $g(0) \neq 0$ such that $f(z) = e^{\alpha\cdot z} g(e^{\beta\cdot z})$, where α is a vertex of Δ_f and β is an integral tangent vector of Δ_f, see (2.4). Writing $g(w) = \prod_j (w - a_j)^{d_j}$ for distinct numbers $a_j \in \mathbb{C}^*$, we have that T_f^r is just the Lelong current over the hyperplanes $\{\beta \cdot z = \log a_j\}$ with multiplicities d_j. A calculation of $\check{T}_f$ gives us the following result, cf. Proposition 4.3 in [i].

Proposition 3.3. *If* $\dim \Delta_f = 1$*, then for a test form* $\xi \in \mathcal{D}^{n-1,n-1}$*,*

$$\check{T}_f.\xi = \sum_j d_j \int_{\beta\cdot y = \arg a_j} \xi(x+iy) \wedge \frac{\beta \cdot dx}{|\beta|}, \tag{3.12}$$

where we consider the multi-valued function arg.

Also when $\dim \Delta_f > 1$, we have an explicit formula for $\check{T}$. The next result corresponds to Theorem 1.1 in [i].

Theorem 3.4. *For an exponential polynomial* f *as above,*

$$\check{T}_f = \sum_{\Gamma \text{ edge of } \Delta_f} \frac{\mathrm{Vol}_{n-1}\left(N_\Gamma \cap B_1\right)}{\mathrm{Vol}(B_1)} \check{T}_{f_\Gamma}. \tag{3.13}$$

Note that Proposition 3.3 applies for $\check{T}_{f_\Gamma}$. A consequence of Proposition 3.3 and Theorem 3.4 is that the choice of $\check{T}_f$ does not matter, i.e., $\lim_{R\to\infty} \check{T}_f^R$ exists.

We will motivate Theorem 3.4 in a heuristic way; for a rigorous proof see [i]. First of all we note that if $\Gamma \neq \Gamma'$ are edges of Δ_f and Σ is the face of lowest dimension that contains both Γ and Γ', then $\mathrm{Vol}(N_\Gamma^M \cap N_{\Gamma'}^M \cap B_R)$ grows with order $R^{n-\dim\Sigma} \leq R^{n-2}$. Since the function given by (3.10) is uniformly bounded, this means that we can write

$$\begin{aligned}
&\lim_{R\to\infty} \frac{1}{\mathrm{Vol}_{n-1}(B_R^{n-1})} \int_{\substack{|r|<R \\ r\in N^M}} T_f^r.\xi \\
&\qquad = \sum_{\Gamma \text{ edge of } \Delta} \lim_{R\to\infty} \frac{1}{\mathrm{Vol}_{n-1}(B_R^{n-1})} \int_{\substack{|r|<R \\ r\in N_\Gamma^M}} T_f^r.\xi.
\end{aligned}$$

The key step is to show that T_f^r can be replaced by $T_{f_\Gamma}^r$ in the integral over $N_\Gamma^M \cap B_R$, cf. Proposition 4.5 in [i]. Morally, this follows from Lemma 2.1. Assume for simplicity that Γ is parallel to the vector $(0,...,0,1)$ and that $\mathcal{A}'_{f_\Gamma}$ is the single hyperplane $\{z; z_n = 0\}$. Fix a test form ξ and denote its support by K. Writing

$$\xi = \xi_{II} dz_I \wedge d\bar{z}_I + \xi_3 \wedge dz_n + \xi_4 \wedge d\bar{z}_n \tag{3.14}$$

for $I = \{1, ..., n-1\}$ and test forms ξ_3, ξ_4, we make the Taylor expansion

$$\xi_{II} dz_I \wedge d\bar{z}_I = \xi_0 + z_n \xi_1 + \bar{z}_n \xi_2 \tag{3.15}$$

in the z_n-direction of ξ_{II} at 0, where ξ_0 is independent of z_n. Then ξ_3, ξ_4 have support on K while ξ_0, ξ_1, ξ_2 have support on $\{z + (0, ..., 0, s); z \in K, s \in \mathbb{C}\}$. Choose $\varepsilon > 0$. By Lemma 2.1 there is a set D_M^ε approximating N_Γ^M such that

$$\int_{\mathrm{Re}\, z\in D_M^\varepsilon} T_f^r.\xi_0(z) = \int_{\substack{z_n=0, \\ \mathrm{Re}\, z\in D_M^\varepsilon}} \xi_0(z-r) = \int_{\mathrm{Re}\, z\in D_M^\varepsilon} T_{f_\Gamma}^r.\xi(z) \tag{3.16}$$

and

$$\int_{\operatorname{Re} z \in D_M^\varepsilon} T_f^r \cdot (z_n \xi_1(z)) \leq \varepsilon \sup |\xi_1| \operatorname{Vol}\left(K \cap \{f(\cdot + r) = 0\}\right). \tag{3.17}$$

The number $\sup|\xi_1|$ is bounded by $s_K^1(\xi)$. Thus we can once again use the boundedness of the function given by (3.10) to show that the right hand side of (3.17) is bounded by ε times a constant that only depends on ξ and we can obtain similar bounds for the remaining terms in (3.14) and (3.15). It remains to show that N_Γ^M and D_M^ε are interchangeable in (3.16) and that regardless of $\varepsilon > 0$,

$$\lim_{R \to \infty} \frac{1}{\operatorname{Vol}_{n-1}(B_R^{n-1})} \int_{\substack{|r|<R \\ r \in N_\Gamma^M}} \int_{\operatorname{Re} z \notin D_M^\varepsilon} dd^c \log|f(z+r)| \wedge \xi(z) = 0.$$

Since ξ has compact support and D_M^ε and N_Γ^M are approximatively the same sets, this seems plausible. However the actual proof of Theorem 3.4 in [i] uses a computation of the trace mass of $\check{T}_f$ to avoid these two estimates.

By (3.6), the Ronkin type functions corresponding to $\check{T}_f$ are straightforwardly derived from (3.13) and (3.12). For an explicit formula, see (7.5) in [i]; here we just note that any such function φ is continuous and piecewise affine, with its non-smooth locus corresponding to the lifting of the *shell* of $\mathcal{A}'_f$ to $\mathbb{R}^n$, see Section 4.

4 The duality of $\mathcal{A}'_f$ and Δ_f

We saw in Section 2 that the amoeba of f_Γ is a union of hyperplanes orthogonal to Γ whenever Γ is an edge of Δ_f. Similarly we have that the coamoeba of f_Γ is a union of hyperplanes on $\mathbb{T}^n$ orthogonal to Γ. Let $\mathcal{H}_f$ be the union of all hyperplanes L such that $L \subseteq \mathcal{A}'_{f_\Gamma}$ for some edge Γ of Δ. We call $\mathcal{H}_f$ the *shell* of $\mathcal{A}'_f$.

The shell corresponds to the spine in the obvious sense that both sets are polyhedral complexes, or more specifically, tropical varieties. While the spine is contained in the amoeba, the shell is contained in the closure of the coamoeba, see e.g. Theorem 4.2 in [ii]. In particular one have that

$$\overline{\mathcal{A}'_f} = \mathcal{A}'_f \cup \mathcal{H}_f$$

whenever $n = 2$. In nice examples, the shell makes such a large imprint on the appearance of the coamoeba that it can arguably be given the honor for the alternative name alga, just like the spine explains the most striking aspect of the amoeba. We remark that the shell is not analogous to the spine in the sense that it tells us where the complement components of the coamoeba are placed. However, combined with a related concept that we soon will define, it provides a structure that at least tells us where there cannot be any complement components. We will concentrate on the case $n = 2$ where this structure can be established quite elegantly.

4.1 The argument cycle

For the rest of Section 4, we will refer to a continuous mapping $\gamma : [a,b] \to \mathbb{C}$ for $a,b \in [-\infty,+\infty]$, as a *path*. Furthermore, if $\gamma(a) = \gamma(b)$, then we call γ a *cycle*. We will also use some standard notation concerning any two-dimensional Newton polygon that will occur if nothing else is stated: Thus $\Gamma_1, ..., \Gamma_m$ is an anti-clockwise enumeration of its edges, α_k is the common vertex of Γ_k and Γ_{k+1}, $1 \leq k < m$, and $\alpha_0 = \alpha_m$ is the common vertex of Γ_m and Γ_1. Furthermore we let a_k be the coefficient of the monomial in f with index α_k, $\omega_k \in \mathbb{R}^n$ the unit normal vector of Γ_k directed outward from the Newton polygon when placed at Γ_k and β_k the unit vector for which the pair (ω_k, β_k) is a positively oriented ON-basis.

We will begin with a simple observation. Given f as in (1.1) and $\theta \in \mathbb{T}^2$, define the function F_θ on $\mathbb{R}^2$ by setting

$$F_\theta(x) = f(e^{x+i\theta}) / \sum_{\alpha \in A} |a_\alpha| e^{\alpha \cdot x}. \tag{4.1}$$

Notice that $F_\theta(\mathbb{R}^2)$ is contained in the closure of the complex unit disc D. Furthermore $\theta \in \mathcal{A}'_f$ if and only if $0 \in F_\theta(\mathbb{R}^2)$.

To obtain an approximation of $F_\theta(\mathbb{R}^2)$, consider an open neighborhood U of the origin in $\mathbb{R}^2$ and let U^r be the dilation of U by $r \geq 0$. Let $\mathcal{K}^r_\theta$ be the oriented cycle given by

$$[a,b] \ni t \mapsto F_\theta(\rho^r(t)),$$

where $\rho^r : [a,b] \to \partial U^r$ is an anti-clockwise parametrization of ∂U^r. We denote the winding number of $\mathcal{K}^r_\theta$ around $0 \in \mathbb{C}$ by $w^r_f(\theta)$ whenever $0 \notin \mathcal{K}^r_\theta$. Notice that $\theta \mapsto w^r_f(\theta)$ is constant on the connected components in $\mathbb{T}^2$ where it is defined. The cycle $\mathcal{K}^r_\theta$ deforms continuously to a point when $r \to 0$ and hence if $w^r_f(\theta) \neq 0$ there must be at least one $r' < r$ such that $0 \in \mathcal{K}^{r'}_\theta$. This means that $f(e^{x+i\theta}) = 0$ for some $x \in \partial U^{r'}$ and hence, $\theta \in \mathcal{A}'_f$.

Since $\overline{D}$ is a compact set, one can show that $\mathcal{K}^r_\theta$ converges to a cycle $\mathcal{K}_\theta$ when $r \to \infty$, see Proposition 3.5 in [ii] or see below. Let $w_f(\theta)$ be the winding number of $\mathcal{K}_\theta$ around 0 whenever $0 \notin \mathcal{K}_\theta$. In view of the previous discussion, the next result is not surprising.

Proposition 4.1. *For a generic $\theta \in \mathbb{T}^2$, the number of points $z \in V_f$ that map to θ under* Arg *is at least $|w_f(\theta)|$. In particular, $\theta \in \mathcal{A}'_f$ whenever $w_f(\theta) \neq 0$.*

This is a special case of Theorem 5.1 in [ii], where the genericity condition is specified.

We will now define $\mathcal{K}^r_\theta$ and $\mathcal{K}_\theta$ for a particular choice of neighborhood U^r that only depends on Δ_f, which we assume to have full dimension. Notice that one can write

$$\Delta_f = \{x; \omega_1 \cdot x \leq r_1\} \cap ... \cap \{x; \omega_m \cdot x \leq r_m\}$$

for certain $r_1, ..., r_m \in \mathbb{R}$. Hence

$$\tilde{\Delta}_f^r = \bigcap_{k=1}^{m} \{x; \omega_k \cdot x \leq r\},\ r \in \mathbb{R}^+,$$

is a polygon. Furthermore $\operatorname{int} \tilde{\Delta}_f^1$ is an open neighborhood of the origin and $\tilde{\Delta}_f^r$ is the dilation of $\tilde{\Delta}_f^1$ by r. Since $\omega_j \cdot \omega_k < 1$ whenever $j \neq k$, there is an edge $\tilde{\Gamma}_k^r \subset \{\omega_k \cdot x = r\}$ of $\tilde{\Delta}_f^r$ for which $r\omega_k$ is an interior point. Let $\tilde{\alpha}_k$ be the common vertex of $\tilde{\Gamma}_k^1$ and $\tilde{\Gamma}_{k+1}^1$, $1 \leq k < m$, and let $\tilde{\alpha}_0 = \tilde{\alpha}_m$ be the common vertex of $\tilde{\Gamma}_m^1$ and $\tilde{\Gamma}_1^1$. Then clearly

$$\tilde{\alpha}_{k-1} \cdot \omega_k = \tilde{\alpha}_k \cdot \omega_k = 1 \tag{4.2}$$

and

$$\tilde{\alpha}_{k-1} \cdot \beta_k < 0 < \tilde{\alpha}_k \cdot \beta_k.$$

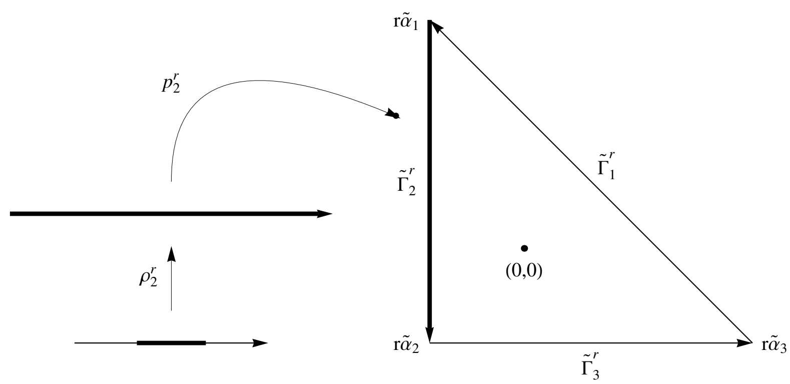

Figure 3: The parametrization of $\partial\tilde{\Delta}_f^r$ in the case $f = 1 + z_1 + z_2$.

We are now ready to construct an anti-clockwise parametrization of $\partial\tilde{\Delta}_f^r$. For $k = 1, ..., m$ and $r > 0$, let $p_k^r : \mathbb{R} \to \mathbb{R}^2$ be given by

$$p_k^r(s) = s\beta_k + r\omega_k$$

and let $\{\rho_k^r\}_{r=1}^{\infty}$ be a sequence of orientation-preserving diffeomorphisms such that ρ_k^r maps $[k-1, k]$ onto $[r\tilde{\alpha}_{k-1} \cdot \beta_k, r\tilde{\alpha}_k \cdot \beta_k]$ in such a way that $\lim_{r\to\infty} \rho_k^r(x)$ converges for every $x \in]k-1, k[$. Notice that in view of (4.2), and since (β_k, ω_k) is an ON-basis,

$$p_k^r \circ \rho_k^r(k) = (r\tilde{\alpha}_k \cdot \beta_k)\beta_k + (r\tilde{\alpha}_k \cdot \omega_k)\omega_k = r\tilde{\alpha}_k,$$

and similarly $p_k^r \circ \rho_k^r(k-1) = r\tilde{\alpha}_{k-1}$. Hence we have the parametrization

$$[0, m] \ni t \mapsto \begin{cases} p_1^r \circ \rho_1^r(t) & \text{if } 0 \leq t \leq 1, \\ ... \\ p_m^r \circ \rho_m^r(t) & \text{if } m-1 \leq t \leq m. \end{cases}$$

of $\partial\tilde{\Delta}_f^r$ that we suggestively will denote by $\partial\tilde{\Delta}_f^r(t)$. Notice that $\omega_k \cdot \alpha = \omega_k \cdot \alpha_k$ whenever $\alpha \in \Gamma_k$. Hence for $s \in \mathbb{R}$ and $\theta \in \mathbb{T}^2$,

$$f_k(\exp(s\beta_k + r\omega_k + i\theta)) = e^{r\omega_k \cdot \alpha_k} f_k(\exp(s\beta_k + i\theta)).$$

where $f_k = f_{\Gamma_k}$. Letting $F_{k\theta}$ correspond to f_k as F_θ corresponds to f, we thus have that

$$F_{k\theta} \circ p_k^r(s) = \frac{e^{r\omega_k \cdot \alpha_k} F_{k\theta}(s\beta_k + i\theta)}{e^{r\omega_k \cdot \alpha_k} \sum_{\alpha \in A \cap \Gamma_k} |a_\alpha e^{s\beta_k}|} = F_{k\theta}(s\beta_k), \tag{4.3}$$

that is, $F_{k\theta} \circ p_k^r$ is independent of r. The maximum of $\omega_k \cdot \alpha$ for $\alpha \in \Delta$ is attained if and only if $\alpha \in A \cap \Gamma_k$. In view of this, it is straightforward to verify that for every $s \in \mathbb{R}$,

$$F_\theta \circ p_k^r(s) \to F_{k\theta}(s\beta_k), \ r \to \infty.$$

Hence the two paths $F_{k\theta} \circ p_k^r \circ \rho_k^r$ and $F_\theta \circ p_k^r \circ \rho_k^r$ converges uniformly on $[k-1, k]$ to the same path $\mathcal{K}_{k\theta} \subset \overline{D}$ when r goes to infinity and we may define the cycle

$$\mathcal{K}_\theta(t) = \lim_{r \to \infty} F_\theta\left(\partial\tilde{\Delta}_f^r(t)\right).$$

Clearly Proposition 4.1 holds for $\mathcal{K}_\theta$. We call $\mathcal{K}_\theta$ the *argument cycle* of f at θ. The next lemma follows by the construction of $\mathcal{K}_\theta$.

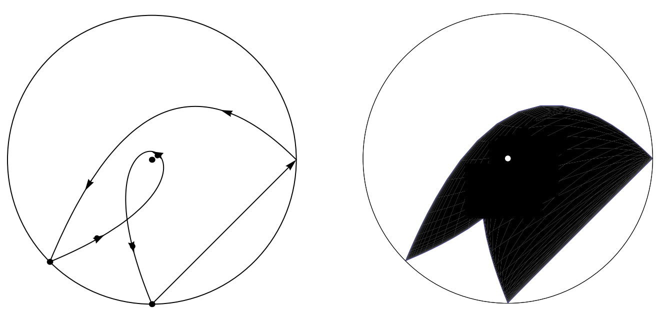

Figure 4: The image of the argument cycle $\mathcal{K}_\theta$ (left) and the set $F_\theta(\mathbb{R}^2)$ (right) when $f = 1 + 2iz_1 + \frac{-1-i}{\sqrt{2}}(z_1^2 + 3z_1^2 z_2 + 3z_1^2 z_2^2 + z_1^2 z_2^3)$ and $\theta = (0, 2\pi/3)$. The winding number of $\mathcal{K}_\theta$ around the origin is two and consequently the multiplicity of θ in $\mathcal{A}'_f$ is by Proposition 4.1 at least two. One have that $(\alpha_k - \alpha) \cdot \tilde{\alpha}_k > 0$ for every $\alpha \in A \backslash \{\alpha_k\}$ and hence it is a general fact that $\mathcal{K}_\theta(k) = \exp(i(\arg a_k + \alpha_k \cdot \theta))$. One can also check that if f_k is a binomial, then $\mathcal{K}_{k\theta}([k-1, k])$ is always the line segment between $\exp(i(\arg a_{k-1} + \alpha_{k-1} \cdot \theta))$ and $\exp(i(\arg a_k + \alpha_k \cdot \theta))$.

Lemma 4.2. *For every $\theta \in \mathbb{T}^2$ we have, for $k-1 \leq t \leq k$, that $\mathcal{K}_\theta(t) = \mathcal{K}_{k\theta}(t)$.*

Lemma 4.2 is the connection between the coamoeba and its shell. Since β_k, ω_k is a basis for $\mathbb{R}^2$, it follows from (4.3) that

$$F_{k\theta}(\mathbb{R}^2) = F_{k\theta}(\beta\mathbb{R}) = \mathcal{K}_{k\theta}(]k-1,k[).$$

Hence Lemma 4.2 implies that $\theta \in \mathcal{H}_f$ if and only if $0 \in \mathcal{K}_\theta$. This means that $w_f(\theta)$ is constant on each component E of the complement of $\mathcal{H}_f$. By Proposition 4.1 one thus obtain an approximation of the coamoeba just by checking w_f for one point in every component of the complement of $\mathcal{H}_f$. But we can say more than so, as we soon will see.

Remark 1. It should be possible to obtain all important results in this thesis connected to the argument cycle without defining the mappings F_θ that may look unnecessarily complicated, but I choose this approach since this is what I did in [ii] and since winding numbers are intuitive to work with.

4.2 Balanced arrangements on $\mathbb{T}^2$

With an *arrangement* we will refer to a finite set $\mathcal{H}$ of oriented lines with integral slope on the torus $\mathbb{T}^2$, where every $L \in \mathcal{H}$ is equipped with a weight $\mu(L) \in \mathbb{Z}^+$. The set of components of the complement of the union of lines $L \in \mathcal{H}$ is denoted by $\mathcal{E}$. For two paths γ, σ on $\mathbb{T}^2$, we say that γ intersects σ *positively (negatively)* at θ if both paths are smooth at θ and the pair $(\nabla\sigma(\theta), \nabla\gamma(\theta))$ of tangent vectors is positively (negatively) oriented. Given an arrangement $\mathcal{H}$ and a smooth path γ that only intersects $\mathcal{H}$ transversely, we define the following function over the lines in $\mathcal{H}$ and the intersection points of $\mathcal{H}$ and γ:

$$k_\gamma(L,x) = \begin{cases} 1 & \text{if } \gamma \text{ intersects } L \text{ positively at } x \\ -1 & \text{if } \gamma \text{ intersects } L \text{ negatively at } x. \end{cases}$$

We say that $\mathcal{H}$ is *balanced* if

$$\sum_{L\in\mathcal{H}} \sum_{x \in L\cap\gamma} k_\gamma(L,x)\mu(L) = \sum_{L\in\mathcal{H}} \sum_{x \in L\cap\sigma} k_\sigma(L,x)\mu(L) \tag{4.4}$$

whenever γ, σ are paths as above with starting points in the same component E and endpoints in the same component F of the complement of $\mathcal{H}$. Thus if $\mathcal{H}$ is balanced, we may define an integer-valued mapping i on $\mathcal{E}$ in the following way: Fix $i(E) \in \mathbb{Z}$ for some $E \in \mathcal{E}$. For every $F \in \mathcal{E}$ set $i(F)$ as the sum of $i(E)$ and either side of (4.4) for any γ with starting point in E and endpoint in F. Such a mapping i is called an *index mapping* on $\mathcal{E}$.

A Newton polytope Δ in $\mathbb{R}^2$ of dimension 2 has an infinite class of *dual arrangements*, see [iii]. Such an arrangement consists of lines orthogonal to the edges Γ of Δ and with tangent vectors pointing outward from Δ when placed at Γ. The criterion on the weights $\mu(L)$ is that the sum of these over the lines correspondning to a fixed Γ, must equal $|\Gamma|$, see Section 2.2.

Proposition 4.3. *An arrangement is dual to some Newton polygon if and only if it is balanced.*

Proposition 4.3 implies that every balanced arrangement has a *dual Newton polygon*. For a complete proof of Proposition 4.3 we refer to Proposition 2.2 in [iii]. Note here that the equivalent property for balancy of an arrangement given by Proposition 4.3 was chosen as the definition for balancy in [iii]. The key to Proposition 4.3 is Poincaré duality, see e.g. [7].

Let f be a Laurent polynomial on $(\mathbb{C}^*)^2$ with $\dim \Delta_f = 2$ and let f_k be as in Section 4.1. With the *multiplicity* of a line $\{\beta_k \cdot \theta = v\} \subseteq \mathcal{A}'_{f_k}$, we refer to the sum of the multiplicities of the points in the fiber over $v \in \mathcal{A}'_g$, where g is as in (2.4). The shell $\mathcal{H}_f$ is generally a dual arrangement of Δ_f, if we orient the lines in $\mathcal{A}'_{\Gamma_k}$ as described above and let $\mu(L)$ be the multiplicity of L in $\mathcal{A}'_{\Gamma_k}$. However, there is a subtlety here: If there are two parallel edges Γ and Σ, there might be a line L that is contained both in $\mathcal{A}'_\Gamma$ and in $\mathcal{A}'_\Sigma$. In this case, let d_Γ and d_Σ be the multiplicity of L in $\mathcal{A}'_\Gamma$ and $\mathcal{A}'_\Sigma$ respectively. If $d_\Gamma - d_\Sigma = 0$ we exclude L from $\mathcal{H}_f$. If say $d_\Gamma > d_\Sigma$, we orient L so that its tangent points outwards from Δ_f when placed at Γ and set $\mu(L) = d_\Gamma - d_\Sigma$.

By shortening Γ and Σ appropriately whenever this happens, it follows recursively that $\mathcal{H}$ either is the empty set or still has a dual polygon, even if it does not equal Δ_f, cf. Figure 5.

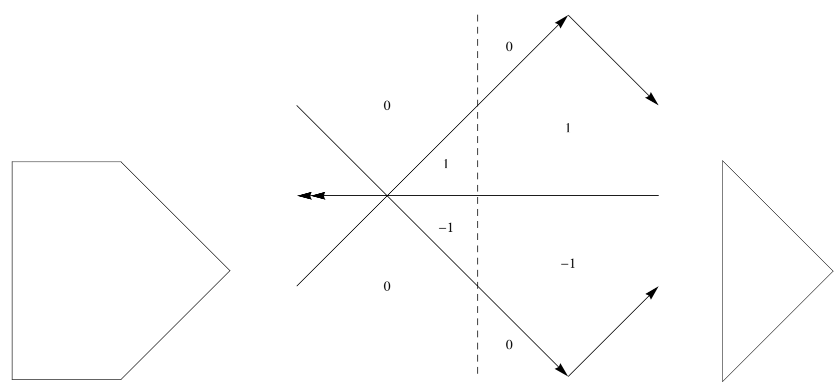

Figure 5: The Newton polygon Δ_f, the shell $\mathcal{H}_f$ and the dual polygon of $\mathcal{H}_f$ considered as an arrangement, for $f = 1 - z - 3w + w^2 + iz^2w - zw^2$. The numbers in the cells of the complement of $\mathcal{H}_f$ denote the values of the index mapping of $\mathcal{H}_f$ with weighted mean value zero. The dashed line is contained in $\mathcal{H}_f$ considered as a shell, but not in $\mathcal{H}_f$ considered as an arrangement; thus the dual polygon of $\mathcal{H}_f$ does not equal Δ_f. The other lines in $\mathcal{H}_f$ have weight one, except for the horisontal line which has weight two.

With the *weighted mean value* of an index mapping i, we refer to the number

$$\sum_{E \in \mathcal{E}} i(E)\,\mathrm{Area}(E).$$

If i and i' are index mappings on $\mathcal{E}$, it is easy to check that $i - i'$ is constant.

Thus there is at most one index mapping with a specified weighted mean value, given $\mathcal{H}$.

Theorem 4.4. *A balanced arrangement $\mathcal{H}$ is the shell $\mathcal{H}_f$ of $\mathcal{A}'_f$ for some Laurent polynomial f if and only if there is an index mapping i on $\mathcal{E}$ with weighted mean value zero. In this case, $i(E)$ is given by $w_f(\theta)$ for any $\theta \in E$.*

This result is not included in [i]-[iv] or any other paper of which I know, and thus we will go through the proof carefully in Section 4.3 and Section 4.6, although the second part is more or less a computation.

The next result is an immediate consequence of Theorem 4.4 and Proposition 4.1.

Corollary 4.5. *Assume that $\mathcal{H}_f$ is a non-empty arrangement. Then the number of points in V_f that map to $\theta \in E \in \mathcal{E}_f$ under* Arg*, is generically at least $|i_f(E)|$, where i_f is the index mapping on $\mathcal{E}_f$ with weighted mean value zero. In particular, $E \subset \mathcal{A}'_f$ whenever $i_f(E) \neq 0$.*

4.3 The index mapping with mean value zero and w_f

In this section, we will show that w_f induces an index mapping on $\mathcal{E}_f$ of weighted mean value zero, that is one of the implications of Theorem 4.4. To this end, fix $\theta \in \mathbb{T}^n \backslash \mathcal{H}_f$. Considering arg as a multi-valued function, let $\arg_\gamma : [a,b] \to \mathbb{R}$ be the branch of $\arg \circ \gamma(t) - \arg \circ \gamma(a)$ along the path $[a,b] \ni t \mapsto \gamma(t) \subset \mathbb{C}\backslash\{0\}$ chosen so that $\arg_\gamma(a) = 0$. Furthermore we let from now on arg (without subscript) take values in $]-\pi,\pi]$.

Consider a polynomial $g = \sum_{k=0}^{M} a_k w^k$ on $\mathbb{C}$, where $a_M = 1$ and $a_0 \neq 0$. For fixed $t \in \mathbb{R}$ we can define the mappings $g_t : \mathbb{R}^{\geq 0} \to \mathbb{C}$ and $G_t : \mathbb{R}^{\geq 0} \to \overline{D}$ by setting $g_t(r) = g(re^{it})$ and

$$G_t(r) = g(t) / \left(\sum_{k=0}^{M} |a_k| r^k \right), \tag{4.5}$$

cf. (4.1). It is straightforward to verify the limit

$$G_t(+\infty) := \lim_{s \to +\infty} G_t(s) = e^{iMt},$$

and hence we may consider G_t as a path over $[0,+\infty]$. For $R > 0$, let g_t^R be the path given by the restriction of g_t to the interval $[0,R]$. Whenever $0 \notin G_t([0,+\infty])$, we have that

$$\begin{aligned} \arg_{G_t}(R) = \arg_{g_t^R}(R) &= \operatorname{Im}\left(\log \circ g(re^{it}) - \log \circ g(0)\right) \\ &= \operatorname{Im} \int_0^R \frac{dg(re^{it})}{g(re^{it})}, \end{aligned} \tag{4.6}$$

if we consider the appropriate branch of the function log.

Lemma 4.6. *The following assertions are true.*

1. *If $v \in \mathcal{A}'_g$ and $\varepsilon > 0$, then $\arg_{G_{v-\varepsilon}}(+\infty) - \arg_{G_{v+\varepsilon}}(+\infty)$ converges to 2π times the multiplicity of v in $\mathcal{A}'_g$ when $\varepsilon \to 0$.*

2. *The mean value of $t \mapsto \arg_{G_t}(+\infty)$ over $]-\pi, \pi]\backslash\mathcal{A}'_g$ is zero.*

Proof. Let $w_1, ..., w_L$ be an enumeration of the distinct zeroes of g where the multiplicity of w_k is D_k. Let $-\pi < v_1 < ... < v_l \leq \pi$ be the images of $w_1, ..., w_L$ under arg. The multiplicity d_k of v_k is the sum of multiplicities D_j over j such that $\arg w_j = v_k$.

To show the first point, fix $1 \leq k \leq l$ and let $\varepsilon > 0$ be such that $|v_k - v_j| > \varepsilon$ for every $j \neq k$ and $v_1 + \pi - v_l > \varepsilon$. Let C be the circular sector in $\mathbb{C}$ of the disk of radius R whose boundary is given by the union of the sets

$$\begin{aligned}
C_1 &= \{re^{i(v_k-\varepsilon)}; r \in [0, R]\}, \\
C_2 &= \{re^{i(v_k+\varepsilon)}; r \in [0, R]\}, \\
C_3 &= \{Re^{it}; t \in [v_k - \varepsilon, v_k + \varepsilon]\}.
\end{aligned}$$

Then for R large enough,

$$2\pi d_k = \mathrm{Im} \int_{\partial C} \frac{dg}{g} = \arg_{G_{v_k-\varepsilon}}(R) - \arg_{G_{v_k+\varepsilon}}(R) + \mathrm{Im} \int_{C_3} \frac{dg}{g}, \tag{4.7}$$

where the first equality follows from the argument principle and the second equality follows from (4.6). The last term of (4.7) is uniformly bounded by ε times a constant that is independent of R. Hence the first point of the lemma follows if we let $R \to \infty$ and $\varepsilon \to 0$.

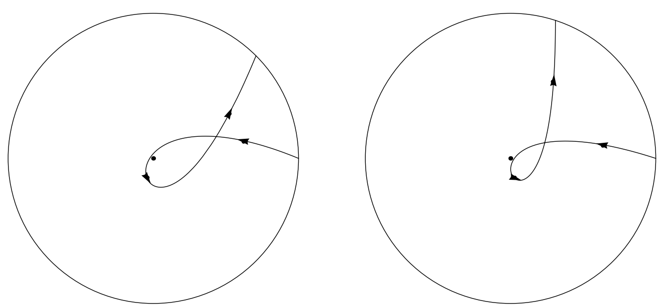

Figure 6: The cycle G_t for $t = 3\pi/4$ and $t = 4\pi/5$, where $g = 1 + 3z + 4z^2 + z^3$. The values of $\arg_{G_t}(+\infty)$ are $9\pi/4$ and $2\pi/5$ respectively, meaning that the integers p in the proof of Lemma 4.6 are 0 and -1 respectively.

For the second point we first notice that

$$\arg(G_t(+\infty)) = Mt + 2q\pi$$

for some integer q, whenever $t \notin \{v_1, ..., v_l\}$. We have that $\arg_{G_t}(+\infty)$ is congruent to $\arg(G_t(+\infty)) - \arg(G_t(0))$ modulo 2π. Thus, for each connected component C of $]-\pi, \pi]\backslash\mathcal{A}'_g$ there is a $p \in \mathbb{Z}$ such that

$$\forall t \in C : \arg_{G_t}(+\infty) = -\arg a_0 + 2\pi p + Mt.$$

In particular, $|\arg_{G_t}(+\infty)|$ is uniformly bounded on $]-\pi, \pi]\backslash\mathcal{A}'_g$. Furthermore there is an $R > 0$ such that $|G_t(+\infty) - G_t(r)| < 1$ for every $t \in \mathbb{T}^1$ whenever $r > R$, cf. r_α in Section 2.1. Since $|G_t(+\infty)| = 1$, this means that $|\arg_{G_t}(r)|$ is uniformly bounded for $r > R$ and $t \in \mathbb{T}^1\backslash\mathcal{A}'_g$. Thus it follows by dominated convergence that the mean value of $\arg_{G_t}(+\infty)$ over $]-\pi, \pi]\backslash\mathcal{A}'_g$ is given by $(2\pi)^{-1}$ times

$$\int_{-\pi}^{\pi} \arg_{G_t}(+\infty)dt = \lim_{R\to+\infty} \int_{-\pi}^{\pi} \arg_{G_t}(R)dt. \tag{4.8}$$

To prove the second point of the lemma, it suffices to show that the integral on the right hand side of (4.8) is zero for any $R \geq 0$. To this end we will use the following equality that is verified straightforwardly.

$$\frac{dg(re^{it})}{g(re^{it})} \wedge dt = id\log|g(re^{it})| \wedge (\frac{dr}{r} - idt). \tag{4.9}$$

Using (4.6) in the first step and (4.9) in the second, we have that

$$\begin{aligned}
\int_{-\pi}^{\pi} \arg_{G_t}(R)dt &= \int_{-\pi}^{\pi} \operatorname{Im} \int_0^R \frac{dg(re^{it})}{g(re^{it})} \wedge dt \\
&= \int_{-\pi}^{\pi} \operatorname{Im} \int_0^R id\log|g(re^{it})| \wedge (\frac{dr}{r} - idt) \\
&= \int_{-\pi}^{\pi} \int_0^R d\log|g(re^{it})| \wedge \frac{dr}{r} \\
&= \int_0^R \int_{-\pi}^{\pi} \frac{\partial}{\partial t} \log|g(re^{it})|dt \wedge \frac{dr}{r} \\
&= \int_0^R (\log|g(-r)| - \log|g(-r)|) \wedge \frac{dr}{r} \\
&= 0.
\end{aligned} \tag{4.10}$$

□

Recall from Section 4.1 the paths $\mathcal{K}_{k\theta} : [k-1, k] \to \overline{D}$, where $0 \notin \mathcal{K}_{k\theta}([k-1, k])$ if and only if $\theta \notin \mathcal{A}'_{f_k}$. We will now reformulate Lemma 4.6 for these paths.

Lemma 4.7. *For $1 \leq k \leq m$ the following points hold.*

1. *If $\varepsilon > 0$, L is a line contained in $\mathcal{A}'_{f_k}$ and $\theta \in L$, then $\arg_{\mathcal{K}_{k(\theta-\varepsilon\beta_k)}}(k) - \arg_{\mathcal{K}_{k(\theta+\varepsilon\beta_k)}}(k)$ converges to 2π times the multiplicity of L in $\mathcal{A}'_{f_k}$ when $\varepsilon \to 0$.*

2. The mean value of $\theta \mapsto \arg_{\mathcal{K}_{k\theta}}(k)$ *over* $\mathbb{T}^2 \backslash \mathcal{A}'_{f_k}$ *is zero.*

Proof. Assume for convenience that $\beta_k \in \mathbb{Z}^n$ with integer length one. First notice that since ρ_k^r converges pointwise on $]k-1, k[$ while $\rho_k^r(k-1)$ and $\rho_k^r(k)$ converges monotonously to $-\infty$ and $+\infty$ respectively, we may as well consider $\mathcal{K}_{k\theta}$ as the path $r \mapsto F_{k\theta}(\log r\beta_k)$ over $[0, +\infty]$, where we set

$$\mathcal{K}_{k\theta}(0) := \lim_{r\to-\infty} F_{k\theta}(r\beta_k) = \exp(\arg a_{k-1} + \alpha_{k-1}\cdot\theta),$$
$$\mathcal{K}_{k\theta}(+\infty) := \lim_{r\to+\infty} F_{k\theta}(r\beta_k) = \exp(\arg a_k + \alpha_k\cdot\theta).$$

Second one can check that the division of f_k by a monomial does not affect $\arg_{\mathcal{K}_{k\theta}}(+\infty)$. Thus we may assume that $\alpha_{k-1} = 0$ and $a_k = 1$. Finally notice that in this case $F_{k(s\omega+t\beta_k)}$ is independent of $s \in \mathbb{R}$. Thus, setting $g(w) = f_k(w^{\beta_k})$, we have that

$$\mathcal{K}_{k(s\omega_k+t\beta_k)} = \mathcal{K}_{k(t\beta_k)} = G_t,$$

where G_t corresponds to a monic polynomial g with non-zero constant term as in (4.5). Clearly $\mathcal{A}'_g = \{t; t\beta_k \in \mathcal{A}'_{f_k}\}$. By letting s vary over $]-\pi,\pi]$, the result hence follows from Lemma 4.6.

□

We are now ready to show that the mapping given by $E \mapsto w_f(\theta)$ for any $\theta \in E$, is an index mapping on $\mathcal{H}_f$ with weighted mean value zero.

Partial proof of Theorem 4.4. We have by Lemma 4.2 and induction over m that for $\theta \notin \mathcal{A}'_f$,

$$w_f(\theta) = \arg_{\mathcal{K}_\theta}(m)/2\pi = \sum_{k=1}^{m} \arg_{\mathcal{K}_{k\theta}}(k)/2\pi.$$

That the mean value of w_f over $\mathbb{T}^2\backslash\mathcal{H}$ is zero, follows from the second point in Lemma 4.7. To show that w_f is an index mapping on $\mathcal{E}_f$, let γ be a path on $\mathbb{T}^2$ that intersects $L \subseteq \mathcal{A}'_{f_k}$ transversely, or equivalently orthogonally, and negatively at x. Then L has the same slope and orientation as β_k at x. Since $\mu(L)$ is defined as the multiplicity of L in $\mathcal{A}'_{f_k}$, the first point in Lemma 4.7 is hence the exact criterion for w_f to be an index mapping. □

Remark 2. A *tiling* of $\mathbb{T}^2$ is a subdivision of $\mathbb{T}^2$ into polygons or tiles. Thus a shell defines a tiling of $\mathbb{T}^2$. A *Dimer model* is a special type of tiling that includes the tilings defined by shells. A dimer model has a certain dual graph or *quiver*. This quiver can be considered as an alternative representation of the argument cycle. The connection between the coamoeba and the shell considered as a dimer model and its dual quiver was observed by Feng, He, Kennaway and Vafa in [3].

4.4 A bound for the number of components of the complement of $\overline{\mathcal{A}}'_f$

An arrangement $\mathcal{H}$ is called *degenerate* if some $\theta \in \mathbb{T}^2$ is contained in more than two lines in $\mathcal{H}$ or $\mu(L) > 1$ for some $L \in \mathcal{H}$. Assuming that $\mathcal{H}$ is non-degenerate and balanced, fix some index mapping i on $\mathcal{H}$ and let $\mathcal{E}^0$ be the set of cells in the complement of $\mathcal{H}$ for which i vanishes. The key result (Theorem 2.4) of [iii] can be formulated as follows.

Theorem 4.8. *For $\mathcal{H}$ as above, $|\mathcal{E}^0| \le 2 \operatorname{Area} \Delta$, where Δ is a dual polygon of $\mathcal{H}$.*

Assume that f is a Laurent polynomial with $\dim \Delta_f = 2$ for which $\mathcal{H}_f$ is a non-degenerate (balanced) arrangement dual to Δ_f. Furthermore, let $\mathcal{E}^0$ be defined as above when i is the index mapping with weighted mean value zero and let $\mathcal{C}_f$ be the set of components of the complement of $\overline{\mathcal{A}}'_f$. Then by Corollary 4.5, there is a mapping from $\mathcal{C}_f$ to $\mathcal{E}^0$ taking $C \in \mathcal{C}_f$ to the cell $E \in \mathcal{E}^0$ that contains C. In [5] (Lemma 2.3) it was shown that this mapping is injective, and thus $|\mathcal{E}^0|$ is an upper bound for $|\mathcal{C}_f|$. This means that Theorem 4.8 implies an analogue of Theorem 2.2 for coamoebas as well as an analogue for dimension two of the "coamoeba version" of the fundamental theorem of algebra given in Section 1.

Theorem 4.9. *For f as above, $|\mathcal{C}_f| \le 2 \operatorname{Area} \Delta_f$.*

Actually, one can show by continuity arguments that the only necessary criterion on f for this result to hold is that $\dim \Delta_f = 2$, see Theorem 1.2 in [iii].

We will now outline the proof of Theorem 4.8. What is the connection between the number of certain cells in the complement of a balanced arrangement and the area of its dual polygon? The answer is: Angles. Let us give some definitions: On one hand there are *internal* and *external* angles of a convex polygon E, where the internal angle at a vertex x of E is the angle on the "inner" side of E between the edges meeting at x, and the external angle is its supplementary angle. It is well-known that the sum of external angles of E equals 2π. In particular

$$|\mathcal{E}^0| = \frac{1}{2\pi} \sum_{E \in \mathcal{E}^0} \sum_{\substack{v \text{ external} \\ \text{angle of } E}} v. \tag{4.11}$$

On the other hand, we can define *inner* and *outer* angles at the intersection point of two oriented line segments, where the outer angle is the non-oriented angle between outgoing (or ingoing) lines and the inner angle is its supplementary angle. It is possible to express the area of Δ as a weighted sum of inner angles of all pairs (β_j, β_k) of tangent vectors of the edges of Δ, see Lemma 3.1 in [iii]. This formula is in Lemma 3.2 in the same paper, by use of Poincaré duality, translated to inner angles at the intersection points of $\mathcal{H}$:

Lemma 4.10. *The area of Δ equals $1/2\pi$ times the sum of the inner angles of $\mathcal{H}$ (counting one per vertex).*

Thus the external angles of the cells in $\mathcal{E}^0$ are by (4.11) tied to the number of cells in the complement of $\mathcal{H}$, while the inner angles of $\mathcal{H}$ by Lemma 4.10 are connected to the area of Δ. The main part of the proof of Theorem 4.8 is the establishing of a relation between these two sets of angles.

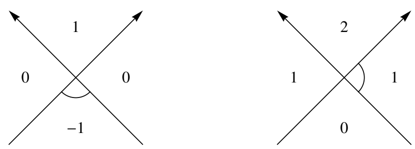

Figure 7: To the left a vertex of $\mathcal{H}$ whose inner angle is the external angle of two different cells in $\mathcal{E}^0$ and to the right a vertex of $\mathcal{H}$ whose outer angle is the external angle of a unique cell in $\mathcal{E}^0$. The two vertices are included in $\mathcal{V}_0$ and $\mathcal{V}_1$ respectively.

At a first glance, this connection does not look very far-fetched. Let $\mathcal{V}_k$ be the set of vertices of $\mathcal{H}$ with two adjacent cells of index k and let u_x be the inner angle and v_x the outer angle at x. If $x \in \mathcal{V}_0$, then u_x is the external angle of two different cells $E, E' \in \mathcal{E}^0$, see Figure 7. If $x \in \mathcal{V}_{\pm 1}$, then v_x is the external angle of exactly one cell $E \in \mathcal{E}^0$, and if $x \in \mathcal{V}_k$ for $|k| > 1$, then neither u_x nor v_x is an external angle such a cell. Thus it seems favorable for the number of complement components, that there are no cells of indices other than -1, 0 or 1. Note that in this case it follows from (4.11) and Lemma 4.10 that the inequality of Theorem 4.8 is an equality. However the equality can be attained also in other situations, see e.g. Figure 7 in [iii].

At least, we have reduced the problem to the following: Show that

$$\sum_{x \in \mathcal{V}_{-1} \cup \mathcal{V}_1} v_x \leq 2 \sum_{|k|>0} \sum_{x \in \mathcal{V}_k} u_x.$$

A way to establish this inequality is to construct certain oriented cycles $\sigma_1, ..., \sigma_l$ on $\mathcal{H}$ whose union contains $\mathcal{V}_{\pm 1}$, that only intersect each other transversely, and turn monotonously and only at vertices in $\mathcal{V}_{\pm 1}$. Then

$$\sum_{x \in \mathcal{V}_{-1} \cup \mathcal{V}_1} v_x = \sum_{j=1}^{k} \sum_{x \in \sigma_j} v_x = 2\pi \sum_{j=1}^{l} d_j,$$

where d_j is the winding number of σ, see Section 4 in [iii]. It then suffices to show that

$$2\pi d_j \leq 2 \sum_{x \in \sigma_j \cap \mathcal{V}_{\pm 1}} u_x + \sum_{|k|>1} \sum_{x \in \sigma_j \cap \mathcal{V}_k} u_x,$$

since $x \in \mathcal{V}_{\pm 1}$ means that x is contained in a unique cycle σ_j and $x \in \mathcal{V}_k$ for $|k| > 1$ means that x is contained in at most two cycles σ_i, σ_j. To see how this can be done, see Section 4 in [iii].

4.5 Ronkin type functions revisited

By Theorem 2.3, the Ronkin function is affine exactly on the complement of the amoeba. The function φ as defined in Section 3.5 is instead affine exactly on the complement of the shell. Furthermore it follows from Theorem 3.4 and Proposition 3.3 that $dd^c\varphi$ is not affected by the orientation of the shell, and thus a given φ may correspond to two very different coamoebas.

For an example, consider the two polynomials

$$\begin{aligned} f &= (1-z_1)(1+z_1)(1+z_2), \\ g &= (1-z_1)^2 + (1+z_1)^2 z_2. \end{aligned} \tag{4.12}$$

We have that $\Delta_f = \Delta_g$ and that

$$\mathcal{H}_f = \mathcal{H}_g = \{\theta_1 = 0\} \cup \{\theta_1 = \pi\} \cup \{\theta_2 = \pi\} =: L_1 \cup L_2 \cup L_3.$$

By Theorem 3.4 and Proposition 3.3, the multiplicity of the integral over L_j is two for $j = 1, 2, 3$ both in $\check{T}_f$ and in $\check{T}_g$, and thus $\varphi_f = \varphi_g$. However, considered as oriented lines, see Section 4.2, the multiplicities of L_1 and L_2 are zero in $\mathcal{H}_f$ and plus minus two in $\mathcal{H}_g$. One can check, using Corollary 4.5, that the coamoeba of g covers the whole torus, possibly except for $\mathcal{H}_g$, while the coamoeba of f clearly equals $\mathcal{H}_f$.

A challenge would be to refine φ in some way so that it depends on the orientation of $\mathcal{H}_f$, and thereby link the results in this section to the results of Section 3.

4.6 The remaining part of the proof of Theorem 4.4

We will now prove the remaining direction of Theorem 4.4, that is the implication that $\mathcal{H}$ is a shell of some Laurent polynomial whenever $\mathcal{E}$ has an index mapping of weighted mean value zero. To this end, consider a fixed balanced arrangement $\mathcal{H}$ given by

$$\begin{aligned} \mathcal{H} = &\bigcup_{k=1}^{m} \bigcup_{j=1}^{m_k} L_{kj}, \\ L_{kj} = &\{\theta \in \mathbb{T}^2; p_k\theta_1 + q_k\theta_2 \equiv \nu_{kj} \mod 2\pi\} \end{aligned} \tag{4.13}$$

for some $\nu_{kj} \in \mathbb{R}$, where

$$\sum_{j=1}^{m_k} \mu(L_{kj}) = |\Gamma_k| \tag{4.14}$$

and p_k, q_k are the relatively prime integers for which

$$\alpha_{k-1} + |\Gamma_k|(p_k, q_k) = \alpha_k \tag{4.15}$$

for $k = 1, 2, ..., m$, where $\alpha_0 := \alpha_m$. We begin with a lemma.

Lemma 4.11. *The balanced arrangement $\mathcal{H}$ has an index mapping with weighted mean value zero if and only if*

$$\sum_{k=1}^{m}\sum_{j=1}^{m_k}(1-\nu_{kj}/\pi)\mu(L_{kj}) \equiv 0 \mod 2. \tag{4.16}$$

Proof. Introduce a coordinate system on $\mathbb{T}^2$ such that $0 \notin \mathcal{H}$ and construct an index mapping on $\mathbb{T}^2\backslash\mathcal{H}$ in the following way: fix a line L_{kj}, let $\theta \in \mathbb{T}^2\backslash\mathcal{H}$ and let $p(\theta)$ be the shortest line segment between the origin and θ, oriented towards θ. Now let $i_{kj}(\theta)$ be the number of positive intersections of the line L_{kj} by $p(\theta)$ minus the number of negative ones. Letting mv denote the mean value of a function on $\mathbb{T}^2\backslash\mathcal{H}$, one can verify that $\mathrm{mv}(i_{kj}) = (1+\nu_{kj}/\pi)/2$. Set $i = \sum_{k,j}\mu(L_{kj})i_{kj}$. Then i must be a balanced index mapping. Furthermore we have

$$\mathrm{mv}(i) = \sum_{k=1}^{m}\sum_{j=1}^{m_k}\mu(L_{kj})\,\mathrm{mv}(i_{kj}) = \sum_{k=1}^{m}\sum_{j=1}^{m_k}(1+\nu_{kj}/\pi)\mu(L_{kj})/2. \tag{4.17}$$

The right hand side of (4.17) is an integer if and only if (4.16) holds. Since any balanced index mapping of $\mathcal{H}$ is attained by adding an integer to i and vice versa, the result follows. □

Remaining proof of Theorem 4.4. One can verify that the inner union in (4.13) coincide with the coamoeba of

$$g_k(z) = \prod_{j=1}^{m_k}(1-e^{-i\nu_{kj}}z^{(p_k,q_k)})^{\mu(L_{kj})}.$$

Fix a dual polygon Δ of $\mathcal{H}$ with vertices $\alpha_0, \alpha_1, ..., \alpha_m$ where $\alpha_0 = \alpha_m = 0$. We can assume that the indices of the integers p_k, q_k are chosen so that the Newton polygon of $z^{\alpha_{k-1}}g_k(z)$ equals the face Γ_k with vertices α_{k-1} and α_k. Set

$$f(z) = \sum_{k=1}^{m}\exp\left(i\sum_{l=1}^{k-1}\sum_{j=1}^{m_l}(\pi-\nu_{lj})\mu(L_{lj})\right)z^{\alpha_{k-1}}(g_k(z))-1). \tag{4.18}$$

When $2 \le k \le m$, we have that f truncated to Γ_k equals the k:th term on the right hand side of (4.18) plus

$$\begin{aligned}&\exp\left(i\sum_{l=1}^{k-2}\sum_{j=1}^{m_l}(\pi-\nu_{lj})\mu(L_{lj})\right)z^{\alpha_{k-2}}\\ &\quad\cdot\prod_{j=1}^{m_{k-1}}\exp\left((i(\pi-\nu_{(k-1)j})\mu(L_{(k-1)j})\right)z^{\mu(L_{(k-1)j})(p_{k-1},q_{k-1})}\\ &=\exp\left(i\sum_{l=1}^{k-1}\sum_{j=1}^{m_l}(\pi-\nu_{lj})\mu(L_{lj})\right)z^{\alpha_{k-1}},\end{aligned} \tag{4.19}$$

that is a constant times $z^{\alpha_{k-1}} g_k(z)$. The equality in (4.19) follows from (4.14) combined with (4.15). Furthermore, Lemma 4.11 implies that

$$\frac{1}{\pi}\sum_{l=1}^{m}\sum_{j=1}^{m_l}(\pi-\nu_{lj})\mu(L_{lj})\equiv 0 \mod 2,$$

meaning that f truncated to Γ_1 similarly equals $z^{\alpha_0}(g_1-1)+z^{\alpha_m}=z^{\alpha_0}f_1$. Hence $\mathcal{A}'_{f_k}=\mathcal{A}'_{g_k}$ for $1\le k\le m$, that is $\mathcal{H}_f=\mathcal{H}$. The theorem follows. □

5 Generalizations to higher codimension

In the previous sections, our results on the coamoeba are to a large extent built around the Newton polytope. An algebraic variety of codimension greater than one does not correspond to a Newton polytope in the same way as a hypersurface and demands a new framework. For our purposes, *initial ideals* will work well.

Given a Laurent polynomial f on $(\mathbb{C}^*)^n$ and a vector $\omega\in\mathbb{R}^n$, let f_ω be the sum of all monomials $a_\alpha z^\alpha$ of f for which $\alpha\cdot\omega$ is maximal. Then f_ω is called the *initial form* of f with respect to ω. Given an ideal I we can now define the *initial ideal* I_ω by setting

$$I_\omega=\langle f_\omega; f\in I\rangle, \tag{5.1}$$

see [17] p. 4. If V is the set of points for which every polynomial in I vanishes, then we write V_ω for the *initial variety* of V, that is the set of points where every polynomial in I_ω vanishes.

For f and ω are as above, let Γ be the face of Δ_f of maximal dimension for which $\omega\in N_\Gamma$. Then $f_\omega=f_\Gamma$ and in particular $f_0=f$. If V is a hypersurface determined by the Laurent polynomial f, it follows that for every $\omega\in\mathbb{R}^n$,

$$\mathcal{A}'_{V_\omega}=\mathcal{A}'_{f_\Gamma}$$

for some face Γ of Δ_f. Thus the natural generalization of Theorem **??** in the context of initial ideals would be the formula

$$\overline{\mathcal{A}}'_V=\bigcup_{\omega\in\mathbb{R}^n}\mathcal{A}'_{V_\omega}. \tag{5.2}$$

where we note that since $V_0=V$, $\mathcal{A}'_V$ is contained in the union on the right hand side.

To see if (5.2) has credibility, let $f^1,\dots,f^p$ be Laurent polynomials that define V. We generalize the definitions of F_θ and $F_{k\theta}$ from Section 4 by letting

$$F_{\omega\theta}:\mathbb{R}^n\to\overline{D}^p$$

equal $F^j_{\omega\theta}$ in its j:th coordinate, where $F^j_{\omega\theta}$ is obtained from f^j_ω as F_θ from f in (4.1). Set $F_\theta=F_{0\theta}$. Clearly we have that $\theta\in\mathcal{A}'_V$ if and only if $0\in F_\theta(\mathbb{R}^n)$. Furthermore,

$$\overline{F_\theta(\mathbb{R}^n)}=\bigcup_{\omega\in\mathbb{R}^n}F_{\omega\theta}(\mathbb{R}^n). \tag{5.3}$$

This formula, that is relatively straightforward to verify (see the proof of Proposition 3.5 in [ii]), mirrors (5.2) in a striking way. For the inclusion $\subseteq$ in (5.2), (5.3) is indeed useful as we will now see.

Given V, it is well-known that we can choose $f = (f^1, ..., f^p)$ such that $f^1_\omega, ..., f^p_\omega$ determines V_ω for every $\omega \in \mathbb{R}^n$ – such a set of Laurent polynomials is called a *universal Gröbner basis*, see e.g. [17]. Let F_θ be obtained from such a mapping f. It is straightforward to verify that any partial derivative of the function $\theta \mapsto |F_\theta(x)|$ is uniformly bounded by some constant $C > 0$ that does not depend on x.

Assume that $\theta \in \overline{\mathcal{A}}'_V$ and let $\{\theta^k\}$ be a sequence in $\mathcal{A}'_V$ converging to θ with $F_{\theta^k}(x^k) = 0$ for some $x^k \in \mathbb{R}^n$. Then

$$|F_\theta(x^k)| = |F_\theta(x^k) - F_{\theta^k}(x^k)| < C|\theta - \theta_k|.$$

The right hand side converges to zero as $k \to \infty$, and hence $0 \in \overline{F_\theta(\mathbb{R}^n)}$. By (5.3), this means that $0 \in F_{\omega\theta}(\mathbb{R}^n)$ for some $\omega \in \mathbb{R}^n$. Since the components of f is a universal Gröbner basis, the desired inclusion follows.

Also the other inclusion of (5.2) holds, but we will not go further into this. Both directions are shown by Nisse and Sottile in [12] and the $\supseteq$-inclusion is also shown for complete intersections in [ii]. All four single-directed proofs use quite different techniques.

5.1 Complete intersections of codimension $n/2$

Assume that n is even and that V is an $n/2$-dimensional complete intersection of hypersurfaces cut out by $f = (f_1, ..., f_{n/2})$. Then a generalization $\mathcal{K}_\theta : S^{n-1} \to \overline{D}^{n/2}$ of the argument cycle from Section 4.1 is given by

$$\mathcal{K}_\theta(x) := \lim_{R\to\infty} F_\theta(Rx).$$

If $0 \notin \mathcal{K}_\theta(S^{n-1})$, then $\mathcal{K}_\theta$ can be considered as a continuous mapping between the $(n-1)$-sphere S^{n-1} and $\overline{D}^{n/2}\setminus\{0\}$. As such a mapping, it has a *degree*, namely a number d_θ such that the image of S^{n-1} under $\mathcal{K}_\theta$ is homologous to $d_\theta S^{n-1}$. For further reading, see e.g. [2].

Theorem 5.1. *The multiplicity of $\theta \in \mathbb{T}^n$ in $\mathcal{A}'_V$ is at least $|d_\theta|$. Furthermore, $|d_\theta|$ is constant on each component of the complement of $\bigcup_{\omega\neq 0} \mathcal{A}'_\omega$.*

This is the correct statement of Theorem 5.1 in [ii] and a generalization of Proposition 4.1. Notice that if $0 \in \mathcal{K}_\theta$, then it follows from Proposition 3.5 in the same paper that $\theta \in \bigcup_{\omega\neq 0} \mathcal{A}'_\omega$. Hence the mapping $\theta \mapsto d_\theta$ is constant on every component of the complement of $\bigcup_{\omega\neq 0} \mathcal{A}'_\omega$.

5.2 The affine case

Let $m \leq n$ and let $C = (C_0, C_1, ..., C_n)$ be an $m \times (n+1)$-matrix of rank m for which every column is contained in the linear span of the remaining columns.

Then the linear system $C(1, z_1, ..., z_n)^t = 0$ defines a non-empty variety, or an *affine space*, L on $(\mathbb{C}^*)^n$.

Given $\theta \in \mathbb{T}^n$ and $1 \leq j \leq n$, consider the real $2m \times (n+1)$-matrix C^θ whose odd and even rows are given by

$$\mathrm{Re}\,(C_0, e^{i\theta_1}C_1, ..., e^{i\theta_n}C_n)$$

and

$$\mathrm{Im}\,(C_0, e^{i\theta_1}C_1, ..., e^{i\theta_n}C_n)$$

respectively. Notice that $\theta \in \mathcal{A}'_L$ if and only if there is an $r \in (\mathbb{R}^+)^n$ such that $C^\theta(1, r_1, ..., r_n)^t = 0$. Thus C^θ and F_θ are related concepts and if F_θ was our main tool earlier in Section 5, our results in this section will largely be built upon the study of C^θ.

Consider the n:th unit simplex

$$\Delta_n = \{x \in [0, \infty[^{n+1}; \sum_{j=1}^{n} x_j = 1\}$$

and set

$$L^\theta = \{s \in \Delta_n; C^\theta s = 0\}.$$

Then L^θ is the fiber over θ in the *compactified amoeba* of L, see e.g. [6]. Indeed, consider the diffeomorphism $\psi : \mathrm{int}\,\Delta_n \to \mathbb{R}^n$ given by

$$\psi(s) = (\log(s_1/s_0), ..., \log(s_n/s_0)).$$

It is easy to verify that for every $\theta \in \mathbb{T}^n$, the fiber in $\mathcal{A}_L$ over θ is given by $\psi(\mathrm{int}\,L^\theta)$. Since furthermore $\Delta_n \cap K$ is a polygon of dimension $\dim K - 1$ whenever K is a subspace of $\mathbb{R}^{n+1}$ that intersects the interior of Δ_n, we have the following result.

Proposition 5.2. *The fiber in $\mathcal{A}_L$ over $\theta \in \mathcal{A}'_L$ is diffeomorphic to the interior of a polygon of dimension $n - \mathrm{rank}\,C^\theta$. In particular, the fiber is a single point whenever* $\mathrm{rank}\,C^\theta = n$.

5.3 The contour in the affine case

In Section 1 we declared that giving a description of the coamoeba was the main motivation for this thesis. So far, this description has been made in terms of the shell and the coamoebas of the initial varieties, but there is another object that is equally crucial in this matter. It is called the *contour* of the coamoeba.

If z is a singular point of an algebraic variety V or if z is a regular point but the Jacobian of $\mathrm{Arg} : V \to \mathbb{T}^n$ does not have full rank at z, then z is called a *critical* point of V with respect to Arg. The contour $\mathcal{C}'_V$ of $\mathcal{A}'_V$ is the image of all critical points of V under Arg. When $m \leq n/2$, the intersection of the coamoeba and its boundary is contained in the contour. In view of (5.2), this means that the coamoeba can be sketched whenever the proper initial coamoebas and the contour is known. See Theorem 6.4 in [iv] for an example of what role the contour could play when $m > n/2$.

There are several distinctive types of critical points (e.g. singular points or points where Arg "folds" the variety. See [10] for the related situation of the amoeba) and thus $\mathcal{C}'_V$ is not a suitable object for general statements. However, when the variety is an affine space, there is essentially just one type of critical points, which is indicated by the following result.

Proposition 5.3. *Let $\theta \in \mathcal{A}'_L$. When $2m \geq n$, $\theta \in \mathcal{C}'_L$ if and only if C^θ has rank strictly less than n. When $2m \leq n$, $\theta \in \mathcal{C}'_L$ if and only if the rank of C^θ is not maximal.*

The proof of this is rather straightforward in view of Proposition 5.2. See the proof of Proposition 3.3 in [iv] for details. Also note that the second statement is a special case of Proposition 2.2 in [ii].

The initial varieties, or initial spaces, of L, are of certain interest for the contour of L as we soon will see. But first we give a characterization of the initial spaces. The faces of Δ_n can be indexed by the proper subsets of $\{0, 1, ..., n\}$, letting

$$\Gamma_N = \{s \in \Delta_n; s_j = 0, j \in N\}.$$

Let $\tilde{\Gamma}_N$ be the image of Γ_N under the projection $(s_0, s_1...s_n) \mapsto (s_1...s_n)$. Then the Newton polygon of a generic affine polynomial is given by $\tilde{\Gamma}_\emptyset$. Indeed, every initial form of a affine polynomial f is given by $f_{\tilde{\Gamma}_N}$ for some $N \subset \{0, 1, ..., n\}$. This fact can be transferred to ideals generated by affine polynomials: Let C_N be the matrix obtained by setting all entries to zero in the columns of C with indices in $N \subset \{0, 1, ..., n\}$. In view of e.g. Proposition 1.6 in [17] it is straightforward to verify that any initial space of L is given by

$$L_N := \{C_N(1, z)^t = 0\}$$

for some N. Furthermore any space L_N is an initial space of L if L satisfies a certain genericity condition, see Lemma 4.1 in [iv].

Let us look at the relation between $\mathcal{C}'_L$ and the coamoebas $\mathcal{A}'_{L_N}$. It is easy to verify that $\theta \in \mathcal{A}'_{L_N}$, that is $C^\theta_N s = 0$ for some $s \in \operatorname{int} \Delta_n$, if and only if $C^\theta s' = 0$ for some $s' \in \operatorname{int} \Gamma_N$. Notice that the faces of L^θ are precisely the sets $L^\theta \cap \Gamma_N$ that are non-empty. Hence we can use the indices N also for the corresponding faces of L^θ, and obtain the following result, cf. Lemma 4.2 in [iv].

Lemma 5.4. *The faces Γ^θ of the $(n - \operatorname{rank} C^\theta)$-dimensional polygon L^θ can be indexed with subsets of $\{0, 1, ..., n\}$ such that*

1. *if M and N are indices of faces and $M \subset N$, then $\Gamma^\theta_M \supset \Gamma^\theta_N$,*
2. *one has that $\theta \in \mathcal{A}'_{L_N}$ if and only if N is an index of some face of L^θ.*

Lemma 5.4 lies behind the most general results on coamoebas of affine spaces that we prove in this thesis. A consequence of the second point follows here (for a more specific statement, see Corollary 4.4 in [iv]).

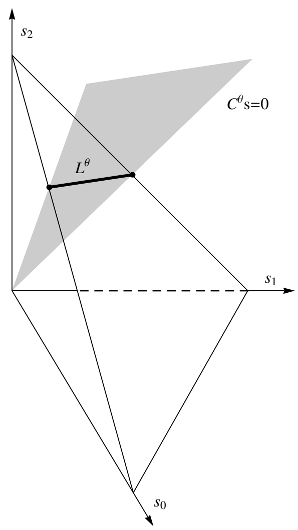

Figure 8: The polygon L^θ equals the intersection of the subspace $\{C^\theta s = 0\}$ and the unit simplex Δ_n. In the picture, the two facets of L^θ correspond to the two faces $\Gamma_{\{0\}}$ and $\Gamma_{\{1\}}$ of Δ_2.

Proposition 5.5. *The contour $\mathcal{C}'_L$ is given by a union of sets $\bigcap_{N\in\mathcal{N}} \mathcal{A}'_{L_N}$, where $\bigcup_{N\in\mathcal{N}} N = \emptyset$ and $|M|, |N| > \max\{0, n-2m\}$.*

Sketch of proof. We will concentrate on the case when $2m \geq n$, noting that the other case is shown in a similar way. Assume that $\theta \in \mathcal{C}'_L$. Then by Proposition 5.3, $\operatorname{rank} C^\theta \leq n-1$, that is $\dim L^\theta \geq 1$. We check the case when equality holds, that is when L^θ has 2 vertices. Let Γ_M, Γ_N be the faces of Δ_n of smallest dimension containing these vertices. Since $\theta \in \mathcal{A}'_L$, L^θ intersects the interior of Δ_n and hence $M \cap N = \emptyset$. Furthermore it follows by the second point of Lemma 5.4 that θ is contained in $\mathcal{A}'_{L_M} \cap \mathcal{A}'_{L_N}$.

If θ' is contained in the intersection above, then there are points $s_M \in \operatorname{int}\Gamma_M$, $s_N \in \operatorname{int}\Gamma_N$ such that $C^{\theta'} s_M = C^{\theta'} s_N = 0$. The convex hull of $\{s_M, s_N\}$ is a line segment that is contained in $L^{\theta'}$ and intersects the interior of Δ_n. Thus $\theta' \in \mathcal{A}'_L$ and $\operatorname{rank} C^{\theta'} \leq n-1$, that is, by Proposition 5.3, $\theta' \in \mathcal{C}'_L$. The proposition follows. □

If $\mathcal{C}'_L$ is contained in the closure of the non-critical values of Arg, then $\overline{\mathcal{A}}'_L$ can be considered as a closed, differentiable manifold of real dimension $\min\{n, 2m\}$. Notice that according to Proposition 5.5 and Lemma 5.4, all values of Arg considered on L_N are critical if $|N| > \max\{0, n-2m\}$. However, letting $K_N \subset (\mathbb{C}^*)^{n-|N|}$ be the projection of L_N on the coordinates of which it is dependent, $\overline{\mathcal{A}}'_{L_N}$ is given by the Cartesian product of $\overline{\mathcal{A}}'_{K_N}$ and a subtorus of $\mathbb{T}^n$. One can thus check that $\overline{\mathcal{A}}'_{L_N}$ is a closed manifold of dimension at most $\min\{2n -$

$2m-1, n-1)\}$. One can also check that for Lgeneric, $\mathcal{A}'_{L_M}$ and $\mathcal{A}'_{L_N}$ intersects transversely almost everywhere when also $|M| > \max\{0, n-2m\}$. In view of Proposition 5.5, the following result is expected.

Theorem 5.6. *For a generic affine spaces L, there is no open set U such that $U \cap \partial\mathcal{A}'_L \subseteq \mathcal{A}'_L$.*

This result is part of Theorem 4.8 in [iv], where also a genericity condition is given.

We conclude this section with some examples of how the contour can affect the appearance of the coamoeba of an affine space.

Example 5.7. Assume that L is a hyperplane. Then by Proposition 5.5, the contour of $\mathcal{A}'_L$ is given by intersections of n transversal hyperplanes, and is hence a set of points. These hyperplanes are clearly the hyperplanes of the shell $\mathcal{H}$ of $\mathcal{A}'_L$ and it is easy to check that the boundary of $\mathcal{A}'_L$ is contained in $\mathcal{H}$. We conclude that $\overline{\mathcal{A}}'_L$ is a polyhedral complex. For details, see Section 5 in [iv].

Example 5.8. Assume that L is a line, that is $m = n-1$. The initial space $L_{\{k\}}$ of L is just the Cartesian product of $(\mathbb{C}^*)^{n-1}$ and a point, given that it is non-empty, and thus $\mathcal{A}'_{L_{\{k\}}}$ is a line. If $n > 2$, then two such lines generally do not intersect, and thus by Proposition 5.5, $\mathcal{C}'_L = \emptyset$. This means that $\mathcal{A}'_L$ is homeomorphic to L, which in turn is homeomorphic to the Riemann sphere minus a finite number of points.

However we may choose entries of C such that the coamoebas of two distinct proper initial spaces of L intersect. In this case, which is called the *real* case, all such coamoebas intersect pairwise, and the projection of $\mathcal{A}'_L$ on any three coordinates equals the coamoeba of a line in $(\mathbb{C}^*)^2$. For details and pictures in the case $n = 3$, we refer to Section 6 in [iv].

Example 5.9. Assume that L is a plane in $(\mathbb{C}^*)^4$. Generally, the contour of $\mathcal{A}'_L$ is given by ten disjoint components $\mathcal{C}'_{jk} := \mathcal{A}'_{L_{\{j\}}} \cap \mathcal{A}'_{L_{\{k\}}}$, $j \neq k \in \{0, 1, ..., n\}$, cf. Proposition 5.5. But for certain so called real planes L, $\mathcal{C}'_L$ is instead connected. For more details, see Example 7.4 in [iv].

References

[1] Mats Andersson. *Topics in complex analysis.* Universitext. Springer-Verlag, New York, 1997.

[2] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.

[3] Bo Feng, Yang-Hui He, Kristian D. Kennaway, and Cumrun Vafa. Dimer models from mirror symmetry and quivering amoebae. *Adv. Theor. Math. Phys.*, 12(3):489–545, 2008.

[4] Mikael Forsberg, Mikael Passare, and August Tsikh. Laurent determinants and arrangements of hyperplane amoebas. *Adv. Math.*, 151(1):45–70, 2000.

[5] Jens Forsgård and Petter Johansson. On the order map for hypersurface coamoebas. arXiv:1205.2014 [math.AG], to appear in Ark. Mat., 2013.

[6] I. M. Gel'fand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1994.

[7] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[8] Lars Hörmander. *An introduction to complex analysis in several variables*, volume 7 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[9] Christer O. Kiselman. Questions inspired by Mikael Passare's mathematics. To appear in Afrika Mat., 2012.

[10] Grigory Mikhalkin. Real algebraic curves, the moment map and amoebas. *Ann. of Math. (2)*, 151(1):309–326, 2000.

[11] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18(2):313–377, 2005.

[12] Mounir Nisse and Frank Sottile. The phase limit set of a variety. *Algebra Number Theory*, 7(2):339–352, 2013.

[13] Mikael Passare and Hans Rullgård. Amoebas, Monge-Ampère measures, and triangulations of the Newton polytope. *Duke Math. J.*, 121(3):481–507, 2004.

[14] Kevin Purbhoo. A Nullstellensatz for amoebas. *Duke Math. J.*, 141(3):407–445, 2008.

[15] L. I. Ronkin. On zeros of almost periodic functions generated by functions holomorphic in a multicircular domain. In *Complex analysis in modern mathematics (Russian)*, pages 239–251. FAZIS, Moscow, 2001.

[16] Hans Rullgård. *Topics in geometry, analysis and inverse problems*. PhD thesis, Stockholm University, Stockholm, 2003.

[17] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

[18] Bernd Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2002.