Tropical aspects of real polynomials and hypergeometric functions

Jens Forsgård

Academic dissertation for the Degree of Doctor of Philosophy in Mathematics at Stockholm University to be publicly defended on Thursday 4 June 2015 at 13.00 in Sal 14, hus 5, Kräftriket, Roslagsvägen 101.

Abstract

The present thesis has three main topics: geometry of coamoebas, hypergeometric functions, and geometry of zeros.

First, we study the coamoeba of a Laurent polynomial f in n complex variables. We define a simpler object, which we call the *lopsided coamoeba*, and associate to the lopsided coamoeba an *order map*. That is, we give a bijection between the set of connected components of the complement of the closed lopsided coamoeba and a finite set presented as the intersection of an affine lattice and a certain zonotope. Using the order map, we then study the topology of the coamoeba. In particular, we settle a conjecture of M. Passare concerning the number of connected components of the complement of the closed coamoeba in the case when the Newton polytope of f has at most $n+2$ vertices.

In the second part we study hypergeometric functions in the sense of Gel'fand, Kapranov, and Zelevinsky. We define Euler-Mellin integrals, a family of Euler type hypergeometric integrals associated to a coamoeba. As opposed to previous studies of hypergeometric integrals, the explicit nature of Euler-Mellin integrals allows us to study in detail the dependence of A-hypergeometric functions on the homogeneity parameter of the A-hypergeometric system. Our main result is a complete description of this dependence in the case when A represents a toric projective curve.

In the last chapter we turn to the theory of real univariate polynomials. The famous Descartes' rule of signs gives necessary conditions for a pair (p,n) of integers to represent the number of positive and negative roots of a real polynomial. We characterize which pairs fulfilling Descartes' conditions are realizable up to degree 7, and we provide restrictions valid in arbitrary degree.

Keywords: *Amoeba, Tropical Geometry, Hypergeometric function, Geometry of zeros, Discriminant.*

Stockholm 2015
http://urn.kb.se/resolve?urn=urn:nbn:se:su:diva-116358

ISBN 978-91-7649-173-7

Stockholm University

Department of Mathematics

Stockholm University, 106 91 Stockholm

Tropical aspects of real polynomials and hypergeometric functions

Jens Forsgård

Stockholm
University

Tropical aspects of real polynomials and hypergeometric functions

Jens Forsgård

© Jens Forsgård, Stockholm, 2015
Address: Matematiska institutionen, Stockholms universitet, 106 91 Stockholm
E-mail address: jensf@math.su.se

ISBN 978-91-7649-173-7

Printed in Sweden by Holmbergs, Malmö, 2015

Distributor: Department of Mathematics, Stockholm University

Till Margareta

Contents

1. Introduction

Consider a general quintic polynomial

$$f(z) = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + x_4 z^4 + x_5 z^5,$$

with non-vanishing coefficients. The famous Abel–Ruffini theorem asserts that one cannot express the roots ρ of $f(z)$ as an algebraic function of the coefficients $x = (x_0, \ldots, x_5)$. It is less known that it is straightforward to express the roots as convergent power series in x.

Notice first that, as functions of the coefficients x, the roots ρ enjoy a double homogeneity which is captured by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Dehomogenizing, we can fix the value of two coefficients. For example, to set the coefficients of the constant and linear term to plus and minus one respectively, we write A in the block form (A_1, A_2), where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

The columns of the matrix $-A_1^{-1}$ gives the exponents of x_0 and x_1 in the multiplication $f \mapsto x_0^{-1} f$ and the change of variables $z \mapsto -x_0\, x_1^{-1}\, z$. Applying these two operations, we obtain the *reduced form* of f;

$$f(z) = 1 - z + \xi_1 z^2 - \xi_2 z^3 + \xi_3 z^4 - \xi_4 z^5,$$

where

$$\xi_k = \frac{x_{k+1} x_0^k}{x_1^{k+1}}, \quad k = 1, \ldots, 4.$$

The exponents of x_0 and x_1 in the expression for ξ_k is given by the columns of the matrix

$$B = -A_1^{-1} A_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & -5 \end{pmatrix}. \tag{1.1}$$

The Newton polytope $\mathcal{N}$ of f is the interval $[0,5]$. It consists of five copies of the unit interval, one for each root of f. We associate to the matrix A_1 the leftmost unit interval $[0,1]$. Furthermore, to this subinterval of the Newton polytope we associate the root of f of smallest modulus, i.e., the leftmost root in logarithmic scale. We will now express this root as a convergent power series in ξ around the origin.

Firstly, let us show that such a power series exists. Restricting to the polydisc D defined by the inequalities $|\xi_k| < 2^{-k-5}$, $k = 1, \ldots, 4$, we find that

$$2 > 1 + |\xi_1| 2^2 + |\xi_2| 2^3 + |\xi_3| 2^4 + |\xi|_4 2^5.$$

Hence, it follows from Pellet's theorem that for each $\xi \in D$ the polynomial $f(z)$ has exactly one root inside of the circle $|z| = 2$. (In modern terminology, one says that f is *lopsided* at $|z| = 2$ with respect to the monomial of degree one.) In particular, the root $\rho = \rho(\xi)$ of smallest modulus is an analytic function of ξ in the polydisc D. Following Birkeland [7], we obtain the power series expansion of $\rho(\xi)$;

$$\rho(\xi) = \sum \frac{\Gamma(1 + 2k_1 + 3k_2 + 4k_3 + 5k_4)}{\Gamma(2 + k_1 + 2k_2 + 3k_3 + 4k_4)} \frac{\xi_1^{k_1} \xi_2^{k_2} \xi_3^{k_3} \xi_4^{k_4}}{k_1! \, k_2! \, k_3! \, k_4!},$$

where the sum is over $k \in \mathbb{N}^4$. Notice that the rows of the matrix B appear as the coefficients of the indices k in the arguments of the Gamma-functions.

The series $\rho(\xi)$ is a multivariate hypergeometric series in the classical sense. That is, the quotient of subsequent terms with respect to a shift in the indices is a rational function of the summation index k. Hypergeometric functions are ubiquitous in mathematical physics, the most prominent example being Gauss hypergeometric function ${}_2F_1$, in terms of which any solution of a second-order linear ordinary differential equation with three regular singularities can be expressed. We wish not here to further motivate the study of hypergeometric functions. Instead, let us find a more precise description of the convergence domain of the series $\rho(\xi)$. To do so, we turn to the theory of hypergeometric functions in the sense of Gel'fand, Kapranov, and Zelevinsky.

Analytic extension of $\rho(\xi)$ gives a multivalued function analytic outside of the discriminant locus $Z(\Delta) = \{\xi \,|\, \Delta(\xi) = 0\}$. Here, $\Delta(\xi)$ is defined either as the resultant of the polynomials f and f', or as the product

$$\Delta(\xi) = \xi_4^8 \prod_{i<j} (\rho_i - \rho_j)^2,$$

for some enumeration $\rho_1, \ldots, \rho_5$ of the roots of f. It is well known that Δ is a real polynomial in ξ. We conclude that $\rho(\xi)$ will converge up until it meets the surface $Z(\Delta)$. As convergence of the series $\rho(\xi)$ only depends on the modulus $|\xi|$, we are lead to consider the *amoeba* $\mathcal{A}_\Delta$, that is, the image of $Z(\Delta)$ under the componentwise logarithmic map

$$\mathrm{Log}(\xi) = (\log|\xi_1|, \dots, \log|\xi_4|).$$

Intuitively, the series $\rho(\xi)$ should converge in (the inverse image of) a connected component of the complement of the amoeba $\mathcal{A}_\Delta$. To give a precise statement we need to introduce further notation.

The secondary polytope Σ_A of A is a combinatorial object that stores the information of all coherent triangulations of the Newton polytope of f with vertices in A. In our example, when $\mathcal{N}$ is the interval $[0,5]$, a coherent triangulation is a subdivision of $\mathcal{N}$ into intervals of integer length. The set of all such triangulations is in a bijective correspondence with the set of vertices of Σ_A. Remarkably, the Newton polytope of the discriminant Δ coincides with the secondary polytope Σ_A.

One artifact of amoeba theory is the *order map* provided by Forsberg, Passare, and Tsikh in [14]. The order map is an injection from the set of connected components of the complement of the amoeba to the set of integer points in the Newton polytope. For the discriminant Δ, we can associate to each connected component of the complement of the amoeba $\mathcal{A}_\Delta$ an integer point in the secondary polytope. Furthermore, it was proven in [43] that the amoeba $\mathcal{A}_\Delta$ is *solid*. That is, the order map is a bijection between the set of connected components of the amoeba complement and the set of vertices of the Newton polytope Σ_A. In particular, each connected component of the complement of the amoeba has an associated triangulation of the Newton polytope $\mathcal{N}$, and vice versa.

Finally, the convergence domain of the series $\rho(\xi)$, as described in [45], is the inverse image of a Reinhardt domain which contain all connected components of the amoeba complement $\mathbb{R}^4 \setminus \mathcal{A}_\Delta$ whose associated triangulation of the Newton polytope $\mathcal{N}$ contains the leftmost unit interval $[0,1]$.

In the aftermath of the success of amoeba theory, which begun with the discovery of the order map, M. Passare and A. Tsikh defined the *coamoeba* $\mathcal{C}_f$ of the polynomial f. By definition, the coamoeba is the image of the zero locus $Z(f)$ under the componentwise argument mapping. The first and foremost question regarding coamoebas, is to find its order map. This is the main topic of Chapter 3 of this thesis. Replacing the coamoeba by a simplified object, denoted the *lopsided coamoeba*, we can define an *order map for the lopsided coamoeba.* This order map is given in terms of a Gale dual of A. In the example with the quintic, a Gale dual can be formed by adding a block of a 4×4-identity matrix beneath the matrix B. We will then study the geometry and topology of the coamoeba via the order map.

In terms of hypergeometric functions, the transition from amoebas to coamoebas corresponds to a shift of focus to integral representations rather than series representations. We consider only *A-hypergeometric functions*, that is, hypergeometric functions in the sense of Gel'fand, Kapranov, and Zelevinsky. For example, the integral

$$\Phi(\beta;\xi) = \int_C \frac{f(z)^{\beta_1}}{z^{\beta_2}} \frac{dz}{z} \tag{1.2}$$

defines a germ of an analytic function in the coefficients ξ provided that the cycle C is chosen as to ensure convergence. Then, the function $\Phi(\beta;\xi)$ is A-hypergeometric of homogeneity parameter β. To make this example more familiar, consider the parameters $\beta_1 = -1$ and $\beta_2 = 1$, and choose $C = C(\rho)$ as a small positively oriented cycle encircling a simple root ρ of $f(z)$. Then, we obtain the residue integrals $\mathrm{Res}_\rho(\xi)$. If ξ lies outside of the discriminant locus $Z(\Delta)$, then there are five residue integrals, one for each root of f. As we will see, these integrals span the solution space of the A-hypergeometric system in a neighborhood of ξ. Furthermore, a standard computation shows that, if $\Delta(\xi) \neq 0$, then

$$\mathrm{Res}_{\rho_i}(\xi) = \frac{1}{\xi_4} \prod_{j\neq i} \frac{1}{\rho_i - \rho_j}.$$

This illustrates the role of the discriminant locus $Z(\Delta)$ for A-hypergeometric functions; it is the characteristic variety of the A-hypergeometric system, that is, it contains the singularities of all A-hypergeometric functions.

Let us illustrate one further property of amoebas. The single-valued function obtained as the product of the five branches of the residue integrals $\mathrm{Res}_\rho(\xi)$ is given by

$$\prod_{i=1}^{5} \mathrm{Res}_{\rho_i}(\xi) = \frac{1}{\xi_4^3\,\Delta(\xi)}.$$

Now fix a connected component E of the amoeba complement $\mathbb{R}^4 \setminus \mathcal{A}_\Delta$. In this component, there is a basis of that A-hypergeometric system given by Laurent series (see, e.g., [51]). We conclude that the reciprocal of the discriminant Δ has a convergent Laurent series expansion in E. In general, if E is a connected component of the complement of the amoeba $\mathcal{A}_f$, then the reciprocal of f has a convergent Laurent series expansion in E whose coefficients are given by integrals $\Phi(\beta;\xi)$ over the cycle $C = \mathrm{Log}^{-1}(\tau)$ for any $\tau \in E$.

Let Θ denote a connected component of the complement of the closed coamoeba. Then, the reciprocal of f has an integral representation whose kernel is given by the integral $\Phi(\beta;\xi)$ over the cycle $C = \mathrm{Arg}^{-1}(\theta)$ for any

$\theta \in \Theta$. The integral (1.2) over a cycle $\mathrm{Arg}^{-1}(\theta)$ will be called an *Euler–Mellin integral*; they are our primary object of study in Chapter 5. In contrast to previous studies of hypergeometric integrals, the explicitness of our cycles allows a detailed study of the dependence of hypergeometric functions on the parameter β. Our main result in this chapter is a complete description of this dependence in the case when A represents a toric projective curve.

In Chapter 6 we will study of the amoeba $\mathcal{A}_\Delta$ and the coamoeba $\mathcal{C}_\Delta$ of the discriminant Δ. Where previous authors have studied these object as convergence domains of representations of A-hypergeometric functions, we focus instead on the connection to *lopsidedness* and *colopsidedness*.

In the last chapter we apply the theory developed in earlier chapters to the theory of real univariate polynomials. The famous Descartes' rule of signs gives necessary conditions for a pair (p, n) of integers to represent the number of positive and negative roots of a real polynomial. We characterize which of the pairs that fulfills Descartes' conditions that are realizable up to degree 7, and we provide restrictions valid in arbitrary degree.

1.1 Contributions

- Sections 2.2 and 3.1–3.2 is joint work with P. Johansson.
- Section 5.1 is joint work with C. Berkesch Zamaere and M. Passare.
- Sections 5.2–5.3 is joint work with C. Berkesch Zamaere and L. F. Matusevich.
- Chapter 7 is joint work with V. P. Kostov and B. Shapiro.

2. Fundamentals

This chapter consists of a brief introduction to discriminants, A-hypergeometric functions, and amoebas. The main reference is the work [18] by Gel'fand, Kapranov, and Zelevinsky. We introduce here the notation used throughout the thesis.

2.1 Elements of the A-philosophy

Consider a point configuration $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\} \subset \mathbb{Z}^n$ of cardinality N. By abuse of notation, we identify A with the $(1+n) \times N$-matrix

$$A = \begin{pmatrix} 1 & \ldots & 1 \\ \mathbf{a}_1 & \ldots & \mathbf{a}_N \end{pmatrix}. \tag{2.1}$$

The *codimension* of A, denoted m, is the integer $N-1-n$. The configuration A gives rise to a lattice (abelian group) $L_A = \mathbb{Z}A \subset \mathbb{Z}^n$ of index $\iota_A = [\mathbb{Z}^n : L_A]$. Let Vol_n and Vol_A denote the Haar measure on $\mathbb{R}^n = \mathbb{R} \otimes L_A$ normalized so that a minimal simplex with vertices in $\mathbb{Z}^n$ respectively L_A has volume one. For convenience we will write $\mathrm{Vol}(\mathcal{N}) = \mathrm{Vol}_n(\mathcal{N})$ and $\mathrm{Vol}(A) = \mathrm{Vol}_A(\mathcal{N})$. We have that

$$\mathrm{Vol}(\mathcal{N}) = \iota_A \,\mathrm{Vol}(A).$$

We associate to A the family $\mathbb{C}_*^A$ of polynomial functions on $\mathbb{C}^n = \mathbb{C} \otimes L_A$ with support in A;

$$f(z) = \sum_{k=1}^{N} x_k \, z^{\mathbf{a}_k}, \tag{2.2}$$

where the polynomial $f(z)$ is identified with the point $f = (x_1, \ldots, x_N) \in \mathbb{C}_*^A$. Denote by $Z(f)$ the zero locus $\left\{z \in \mathbb{C}_*^n \,|\, f(z) = 0\right\}$. The central idea of the "A-philosophy" is that one should study the whole family $\mathbb{C}_*^A$ rather than a single polynomial f.

We define the Newton polytope of A to be the convex polytope $\mathcal{N} = \mathcal{N}_A = \mathrm{Conv}(A) \subset \mathbb{R}^n$, which equals the Newton polytope of each polynomial $f \in \mathbb{C}_*^A$. The face (poset) lattice of $\mathcal{N}$ is the set of all faces of $\mathcal{N}$ with the partial order $\prec$ induced by inclusion.

For each subset $S \subset A$, there is a natural projection $\mathrm{pr}_S\colon \mathbb{C}_*^A \to \mathbb{C}_*^S$. We write $f_S = \mathrm{pr}_S(f)$, and say that f_S is the *truncated polynomial* of f with respect to S. Of particular interest is the case when $S = \Gamma \cap A$ for some face $\Gamma \prec \mathcal{N}$, in which case we set $f_\Gamma = f_{\Gamma\cap A}$.

An isomorphism $\varphi\colon L_A \to L' \subset \mathbb{Z}^n$ defines a point configuration $A' = \varphi(A)$, such that $L' = L_{A'}$. Such mappings arise, e.g., from a change of coordinates in $\mathbb{C}_*^n$. The mapping φ extends to a linear transformation $\widehat{\varphi}\colon \mathbb{R}\otimes L \to \mathbb{R}\otimes L'$. A composition of φ with a translation $\mathbf{a} \mapsto \mathbf{a} + v$, for $v \in \mathbb{Z}^n$, is called an *integer affine transformation.* In terms of matrices, we multiply A from the left by

$$T = \begin{pmatrix} 1 & 0 \\ v & \widehat{\varphi} \end{pmatrix}.$$

Remark 2.1.1. In greater generality, one consider a family of q point configurations $A_1,\dots,A_q$, which one arranges as a matrix A in a block form by adding a $q\times q$-identity matrix on top of $(A_1,\dots,A_q)$. If N_i denotes the cardinality of A_i, so that $N = N_1+\cdots+N_q$ denotes the number of columns of A, then the codimension m of A is defined as the integer $N-n-q$. We consider here only integer affine transformations that preserve the dimension n and the codimension m. In general, an integer affine transformation of A is a rational matrix T such that TA is an integer matrix. Applying an integer affine transformation to reduce a configuration with $q>1$ to a configuration with $q=1$ is known as a *Cayley trick.* We will say that two point configurations are equivalent if they differ by an integer affine transformation preserving q, and we say that they are Cayley-equivalent if they differ by a general integer affine transformation. ◯

Remark 2.1.2. Notice that $\mathrm{Vol}(\mathcal{N}') = \det(T)\mathrm{Vol}(\mathcal{N})$, which in particular implies the identity $\mathrm{Vol}(A') = \mathrm{Vol}(A)$ of induced volumes. ◯

2.1.1. The A-discriminant. A point $z \in \mathbb{C}_*^n$ is said to be a *critical point* of f if it is a solution of the system of equations

$$\partial_1 f(z) = \cdots = \partial_n f(z) = 0. \tag{2.3}$$

If, in addition, $z \in Z(f)$ then z is said to be a *singular point* of f. The A-discriminant $\Delta(f) = \Delta_A(f)$ is, by definition, an irreducible polynomial with domain $\mathbb{C}_*^A$ which vanishes if and only if f has a singular point in $\mathbb{C}_*^n$. Of equal importance is the principal A-determinant D_A, which can be written as a product

$$D_A(f) = \prod_{\Gamma\prec\mathcal{N}} \Delta_\Gamma(f_\Gamma)^{k_\Gamma}, \tag{2.4}$$

where the multiplicities k_Γ are positive integers, see [19].

Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_N\} \subset \mathbb{Z}^m$ be a point configuration, generating a lattice $L_B = \mathbb{Z}B$ of index $\iota_B = [\mathbb{Z}^m : L_B]$. We identify B with the matrix $(\mathbf{b}_1, \dots, \mathbf{b}_N)^t$, and say that B is a *Gale dual* of A if the columns of B span the kernel of A. That is, B is a Gale dual of A if the matrix B has maximal rank and $AB = 0$. (Notice that we do not require that $L_B = \mathbb{Z}^m$.) To each Gale dual B we associate a zonotope

$$\mathscr{Z}_B = \left\{ \frac{\pi}{2} \sum_{k=1}^{N} \lambda_k \mathbf{b}_k \,\middle|\, |\lambda_k| \le 1 \right\} \subset \mathbb{R}^m. \tag{2.5}$$

For $t \in \mathbb{C}^m$, consider the polynomial

$$f(z) = \sum_{k=1}^{N} \langle \mathbf{b}_k, t \rangle \, z^{\mathbf{a}_k},$$

which has a singular point at $z_1 = \cdots = z_n = 1$. Since B is of full rank we obtain a parametriztion of the discriminant surface $Z(\Delta)$ as

$$(t, w) \mapsto \big(\langle \mathbf{b}_1, t \rangle \, w^{\mathbf{a}_1}, \dots, \langle \mathbf{b}_N, t \rangle \, w^{\mathbf{a}_N} \big), \tag{2.6}$$

provided that $Z(\Delta) \neq \emptyset$. The A-discriminant Δ has $n+1$ homogeneities, one for each row of A. Hence, we can reduce Δ to a polynomial depending on m variables. Such a reduction corresponds to a choice of Gale dual of A, and the induced projection $\mathrm{pr}_B \colon \mathbb{C}_*^A \to \mathbb{C}_*^m$. In coordinates,

$$\xi_j = \prod_{k=1}^{N} x_k^{\mathbf{b}_{kj}}, \quad j = 1, \dots, m. \tag{2.7}$$

Then, there exist a Laurent monomial $M(f)$ and a polynomial $\Delta_B(\xi)$ such that

$$\Delta_B(\xi) = M(f) \Delta_A(f).$$

If we compose (2.7) with the parametrization (2.6), then we obtain the Horn–Kapranov parametrization of the reduced discriminant surface $Z(\Delta_B)$.

Theorem 2.1.3 (Kapranov)**.** *The mapping* $\Phi \colon \mathbb{P}^{m-1} \to Z(\Delta_B)$ *given by*

$$\Phi[t] = \left(\prod_{k=1}^{N} \langle \mathbf{b}_k, t \rangle^{\mathbf{b}_{k1}}, \dots, \prod_{k=1}^{N} \langle \mathbf{b}_k, t \rangle^{\mathbf{b}_{km}} \right) \tag{2.8}$$

is a birational equivalence. □

Remark 2.1.4. In general, Δ_B is a Puiseux polynomial. If $\iota_B = 1$ then Δ_B is a polynomial. However, the converse is not true. As our focus lies on $\mathbb{C}_*^A$, this is a less important issue. We could, as in [44], define the reduced discriminant surface $Z(\Delta_B)$ as the image of the map Φ. ◯

Example 2.1.5 (Circuits)**.** A *circuit* is a point configuration of codimension $m = 1$. A circuit is said to be nondegenerate if it is not a pyramid over a circuit of smaller dimension. That is, if no maximal minor of A vanishes. Since $m = 1$, each Gale dual B is a column vector and the zonotope $\mathcal{Z}_B$ is an interval.

Let $A_k = A\backslash\{\mathbf{a}_k\}$, and let $V_k = \mathrm{Vol}_n(A_k) = |\det(A_k)|$. If A is nondegenerate, so that $V_k > 0$ for all k, then $\mathcal{N}$ admits exactly two coherent triangulations with vertices in A [19]. Denote these two triangulations by T_δ for $\delta \in \{\pm 1\}$. Each simplex $\mathcal{N}_{A_k}$ occurs in exactly one of the triangulations T_δ. That is, there is a well-defined assignment of signs $k \mapsto \delta_k$, where $\delta_k \in \{\pm 1\}$, such that

$$T_\delta = \left\{\mathcal{N}_{A_k}\right\}_{\delta_k=\delta}, \quad \delta = \pm 1,$$

where we identify a triangulation with its set of maximal simplices. As shown in [19, Chapters 7 and 9] a Gale dual of A is given by

$$\mathbf{b}_k = (-1)^k|\det(A_k)| = \delta_k V_k. \tag{2.9}$$

In particular, the zonotope $\mathcal{Z}_B$ is an interval of length $2\pi\,\mathrm{Vol}(\mathcal{N})$. ◯

Remark 2.1.6. It can happen that the codimension of the locus of all $f \in \mathbb{C}_*^A$ such that f has a singular point in $\mathbb{C}_*^n$ is greater than one. In such cases we set $\Delta_A = 1$ and say that A is *dual defect*. Combinatorial criteria of dual defectiveness have been discussed in, e.g., [12]. However, the map (2.6) parametrizes the discriminant locus for arbitrary A. Hence, if A is dual defect, then each maximal minor of the Jacobian matrix of the map (2.6) vanishes for all t and w. Evaluating these maximal minors for a specific choice of a Gale dual (see (3.11)), one obtains algebraic equations in the entries of A such that A is dual defect only if it is contained in the corresponding algebraic set, see [10] ◯

2.1.2. The A-hypergeometric system. A favorite saying of M. Passare was that though the aim Gelf'and, Kapranov, Zelevinsky, and their coauthors, was to unify the theory of multivariate hypergeometric functions, the book [19] does not contain the word "hypergeometric". [1] For the definition of the A-hypergeometric system one should instead consult either one of [17; 18; 20; 21].

Let D denote the Weyl algebra in the variables x_k and the partials $\partial_k = \partial/\partial x_k$ for $k = 1,\dots,N$, and write $\partial = (\partial_1,\dots,\partial_N)$. Denote the components of the matrix A by $\mathbf{a}_{jk}$, for $j = 0,\dots,n$ and $k = 1,\dots N$. For a vector $u \in \mathbb{Z}^N$,

[1] A claim which is not entirely true; the word "hypergeometric" appears on eight out of totally 506 pages.

let u_+ and u_- be the unique vectors in $\mathbb{N}^N$ with disjoint support such that $u = u_+ - u_-$. Define the differential operators $\Box_u$ and E_j by

$$\Box_u = \partial^{u_+} - \partial^{u_-} \quad \text{and} \quad E_j = \sum_{k=1}^{N} \mathbf{a}_{jk}\partial_k.$$

Definition 2.1.7. For $\beta \in \mathbb{C}^{1+n}$, the *A-hypergeometric ideal* $H_A(\beta)$ of homogeneity parameter β is the left ideal in D generated by the operators $E_j - \beta_j$, for $j = 0, \ldots, n$, and $\Box_u$, for $u \in \ker A$. The *A-hypergeometric system* is the quotient $D/H_A(\beta)$. ⬡

A *solution* Φ of $H_A(\beta)$ in a domain $U \subset \mathbb{C}^N$ is a (possibly multivalued) analytic function on U, such that $P \bullet \Phi = 0$ for each $P \in H_A(\beta)$. We will denote by $\mathrm{Sol}_U(H_A(\beta))$ the solution space of $H_A(\beta)$ over U. Solutions of the A-hypergeometric system can be represented as series [9; 21; 51], Euler-type integrals [18], or Barnes-type integrals [6; 35]; each representation having different advantages and drawbacks.

The singular locus of $H_A(\beta)$ is the hypersurface $Z(D_A)$ defined as the zero locus of the principal A-determinant. In particular, this set is independent of β [20; 21].

Definition 2.1.8. By abuse of notation we denote by $\mathcal{N}$ also the convex hull of the columns of A in $\mathbb{R}^{1+n}$. Let $\mathrm{Cone}(\mathcal{N}) \subset \mathbb{R}^{1+n}$ denote the cone generated by $\mathcal{N}$, and let $\Gamma \prec \mathrm{Cone}(\mathcal{N})$ be a proper face. A parameter $\beta \in \mathbb{C}^{1+n}$ is said to be *resonant with respect to* Γ if $\beta \in \mathbb{Z}^{1+n} + \mathbb{C}\Gamma$. Further, β is *resonant* if it is resonant with respect to some proper face of $\mathrm{Cone}(\mathcal{N})$. Let $\mathscr{R}_A$ denote the set of resonant parameters for $H_A(\beta)$. ⬡

Resonance is linked to the behavior of the *rank* of $H_A(\beta)$, that is, the dimension of its solution space. If $\beta \notin \mathscr{R}_A$, then by [1; 17; 21] $\mathrm{rk}(H_A(\beta)) = \mathrm{Vol}(A)$. Let $\mathscr{E}_A$ denote the set of *rank-jumping* parameters of $H_A(\beta)$, that is

$$\mathscr{E}_A = \{\beta \in \mathbb{C}^{1+n} \mid \mathrm{rk}\, H_A(\beta) > \mathrm{Vol}(A)\}.$$

A combinatorial description of $\mathscr{E}_A$ is available in [5]. In particular, if $\mathbb{C}^{1+n}$ is stratified by all possible intersections of hyperplanes in $\mathscr{R}_A$, then the rank of the solution space of $H_A(\beta)$ is constant along each stratum.

2.2 Projections of the complex torus

The complex exponential map defines a group isomorphism $\mathbb{C}_*^n \simeq \mathbb{R}^n \oplus \mathbf{T}^n$, with projections Log and Arg onto the first and second term respectively. Given an algebraic set $Z(f) \subset \mathbb{C}_*^n$, its images under these projections are known as the *amoeba* $\mathcal{A}_f$ and *coamoeba* $\mathcal{C}_f$ respectively. We will drop the index f when there is no risk of confusion. It is sometimes beneficial to consider the multivalued argument mapping, which gives the coamoeba as a multiply periodic subset of $\mathbb{R}^n$.

It is natural to also consider the projections Log and Arg as mappings on the complex torus $\mathbb{C}_*^A$. If $f = f_{\mathbf{a}}$ is a monomial, then it defines an affinity $f_{\mathbf{a}}\colon \mathbb{C}_*^n \to \mathbb{C}_*$ (i.e., a group homomorphism composed with a translation). In this case, we obtain unique maps $|f_{\mathbf{a}}|$ and $\hat{f}_{\mathbf{a}}$ such that the following diagram of short exact sequences commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R}_+^n \arrow[r] \arrow[d, "|f_{\mathbf{a}}|"'] & \mathbb{C}_*^n \arrow[r] \arrow[d, "f_{\mathbf{a}}"'] & (S^1)^n \arrow[r] \arrow[d, "\hat{f}_{\mathbf{a}}"'] & 0 \\ 0 \arrow[r] & \mathbb{R}_+ \arrow[r] & \mathbb{C}_* \arrow[r] & S^1 \arrow[r] & 0. \end{tikzcd}$$

That is, $\hat{f}_{\mathbf{a}}(\theta) = \arg(f_{\mathbf{a}} e^{i\langle \mathbf{a},\theta\rangle})$ and $|f_{\mathbf{a}}|(\tau) = |f_{\mathbf{a}} e^{\langle \mathbf{a},\tau\rangle}|$. If, however, f is not a monomial, then there is no obvious interpretation of, for example, $\mathrm{Log}(f)$, as there is no natural algebraic identification of the map $\log(f(z))$ with the point $(\log(f_1),\ldots,\log(f_N)) \in \mathbb{R}^A$. However, such an identification is allowed within the framework provided by the *hyperfield approach to tropical geometry*. In fact, one should consider Log as a hyperfield morphism. (We will briefly discuss this aspect in Section 3.4.) For an arbitrary polynomial, let us denote by $\hat{f}(\theta)$ the vector with components $\hat{f}_{\mathbf{a}}(\theta)$, and denote by $|f|(\tau)$ the vector with components $|f_{\mathbf{a}}|(\tau)$.

Let T be an integer affine transformation of A preserving the dimension n. Employing the induced change of coordinates in $\mathbb{C}_*^n$, we conclude the following relation previously described in [39].

Proposition 2.2.1. *In $\mathbb{R}^n$, we have that $\mathcal{C}_{T(f)}$ (respectively $\mathcal{A}_{T(f)}$) is the image of $\mathcal{C}_f$ (respectively $\mathcal{A}_f$) under the linear transformation $(T^{-1})^t$.* □

Corollary 2.2.2. *In $\mathbf{T}^n$, we have that $\mathcal{C}_{T(f)}$ consists of $|\det(T)|$ copies of $\mathcal{C}_f$.*

Proof. The transformation $(T^{-1})^t$ acts with a scaling factor $1/|\det(T)|$ on $\mathbb{R}^n$. Hence, it maps $|\det(T)|$-many copies of a fundamental domain of $2\pi\mathbb{Z}^n$ onto one fundamental domain of $2\pi\mathbb{Z}^n$. □

If T is an integer affine transformation which alters n, then there are still relations between the (co)amoeba of f and that of $T(f)$. However, there is no "equivalence" in any meaning of the word. For example, we will see in Section 3.2.2 that the maximal number of complement components of coamoebas of polynomials in $\mathbb{C}_*^A$ and $\mathbb{C}_*^{TA}$ need not be equal. In that regard, the maximal number of connected component of the (co)amoeba is our first example of an "invariant" of equivalent point configurations which is not invariant under Cayley-equivalence.

In [14] the map $\mathrm{ord}_{\mathcal{A}}\colon \mathbb{R}^n \setminus \mathcal{A} \to \mathbb{Z}^n$, defined componentwise by

$$\mathrm{ord}_{\mathcal{A}}(\tau)_j = \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(\tau)} \frac{z_j \partial_j f(z)}{f(z)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n},$$

was considered. There it was proven that $\mathrm{ord}_{\mathcal{A}}$ induces an injective map from the set of connected components of the complement $\mathbb{R}^n \setminus \mathcal{A}$ to the finite set $\mathbb{Z}^n \cap \mathcal{N}$. For this reason $\mathrm{ord}_{\mathcal{A}}$ is known as the *order map* of the amoeba.

For $\mathbf{a} \in \mathrm{Im}(\mathrm{ord}_{\mathcal{A}})$, the connected component of $\mathbb{R}^n \setminus \mathcal{A}$ of order $\mathbf{a}$ is denoted by $E_{\mathbf{a}}$, and the set of all $f \in \mathbb{C}_*^A$ such that $\mathbf{a} \in \mathrm{Im}(\mathrm{ord}_{\mathcal{A}})$ is denoted by $U_{\mathbf{a}}$. It is known that, for any $\mathbf{a} \in \mathrm{Im}(\mathrm{ord}_{\mathcal{A}})$, the normal cone $N_{\mathbf{a}}\mathcal{N}$ coincides with the recession cone of the component $E_{\mathbf{a}}$ of the complement of the amoeba $\mathcal{A}$. Note that $\mathrm{Im}(\mathrm{ord}_{\mathcal{A}})$ need not be a subset of A; it suffices to consider the case when f is a univariate polynomial to realize that the behavior of $\mathrm{Im}(\mathrm{ord}_{\mathcal{A}})$ as a set valued function of f is nontrivial.

Evaluating $\mathrm{ord}(\tau)$ in the univariate case corresponds, by the argument principle, to counting zeros of f inside the circle $\log^{-1}(\tau)$. With the analogous interpretation of $\mathrm{ord}_{\mathcal{A}}$ for multivariate polynomials in mind, it is not hard to see that the vertex set $\mathrm{vert}(\mathcal{N})$ is always contained in the image of $\mathrm{ord}_{\mathcal{A}}$. Furthermore, it was shown in [50] that any subset of $\mathbb{Z}^n \cap \mathcal{N}$ that contains $\mathrm{vert}(\mathcal{N})$ appears as the image of the order map for some polynomial with the given Newton polytope. Thus, even though the image of $\mathrm{ord}_{\mathcal{A}}$ is non-trivial to determine, this map gives a good understanding of the structure of the set of connected components of the complement of the amoeba $\mathcal{A}$. In particular, we have the sharp lower and upper bounds on the cardinality of this set given by $|\mathrm{vert}(\mathcal{N})|$ and $|\mathbb{Z}^n \cap \mathcal{N}|$ respectively. (See [33] and [46] for an overview of amoeba theory.)

The coamoeba $\mathcal{C}$ is in general not closed, since the map Arg is not proper. It is natural to focus on $\overline{\mathcal{C}}$ rather than on $\mathcal{C}$, the main reason being that the components of the complement of $\overline{\mathcal{C}}$ in $\mathbb{R}^n$ are convex. To see this, we give the following argument due to Passare. If $\Theta \subset \mathbb{R}^n$ is a connected component of the complement of $\overline{\mathcal{C}}$, then the function $g(w) = 1/f(e^{iw})$ is holomorphic on the tubular domain $\Theta + i\mathbb{R}^n$. As it cannot be extended to a holomorphic function on any larger tubular domain, convexity follows from Bochner's tube theorem [8]. It was shown in [26] and [40] that

$$\overline{\mathcal{C}}_f = \bigcup_{\Gamma \prec \mathcal{N}} \mathcal{C}_{f_\Gamma}. \tag{2.10}$$

Consider a binomial

$$f(z) = x_1\, z^{\mathbf{a}_1} + x_2\, z^{\mathbf{a}_2},$$

whose coamoeba $\mathcal{C}$ is the set of $\theta \in \mathbb{R}^n$ such that

$$\langle \theta, \mathbf{a}_1 - \mathbf{a}_2 \rangle = \pi + \arg_\pi(x_2) - \arg_\pi(x_1) + 2\pi k, \quad k \in \mathbb{Z}.$$

Hence, $\mathcal{C}$ consists of a family of parallel hyperplanes, whose normal vector $\mathbf{a}_1 - \mathbf{a}_2$ is parallel to the Newton polytope $\mathcal{N}$. By the fundamental theorem of algebra, any polynomial whose Newton polytope is a line segment factors into a product of binomials, and hence its coamoeba consists of a family of parallel hyperplanes.

Now let $f \in \mathbb{C}_*^A$. The *shell* $\mathcal{H} = \mathcal{H}_f$ of the coamoeba $\mathcal{C}$ is defined as the union (2.10) taken over the set of edges of the Newton polytope $\mathcal{N}$, see [25; 38]. As edges are one-dimensional, the shell $\mathcal{H}$ is a hyperplane arrangement in $\mathbf{T}^n$ (or $\mathbb{R}^n$). Its importance can be seen in the following lemma.

Lemma 2.2.3 (Fundamental lemma of the shell)**.** *Let $l \subset \mathbb{R}^n$ be a line segment with endpoints in the complement of $\overline{\mathcal{C}}_f$, such that l intersects $\overline{\mathcal{C}}_f$. Then l intersects $\mathcal{C}_{f_\Gamma}$ for some edge $\Gamma \prec \mathcal{N}$. In particular, each cell of the hyperplane arrangement $\mathcal{H}_f$ contains at most one connected component of the complement of $\overline{\mathcal{C}}_f$.*

The polynomial f, and the point configuration A, is said to be *maximally sparse* if $A = \operatorname{vert}(\mathcal{N})$. If in addition $\mathcal{N}$ is a simplex, then $Z(f)$ is known as a *simple hypersurface*, and we will say that f is a *simple polynomial.*

Example 2.2.4. The coamoeba of $f(z) = 1 + z_1 + z_2$, as described in e.g. [40], can be seen in Figure 2.1, where it is drawn in the fundamental domains $[-\pi,\pi]^2$ and $[0,2\pi]^2$. The shell $\mathcal{H}_f$ consists of the hyperplane arrangement shown in black. In this case, the shell is equal to the boundary of $\mathcal{C}_f$. The Newton polytope $\mathcal{N}_A$. and its outward normal vectors, are drawn in the rightmost picture. If $\mathcal{H}_f$ is given orientation in accordance with the outward normal vectors of $\mathcal{N}_A$, then the interior of the coamoeba consists of the oriented cells.

Consider now the case when $\mathcal{N}$ is the standard n-simplex in $\mathbb{R}^n$, that is, $f(z) = 1 + z_1 + \cdots + z_n$. Let $P(A)$ denote the power set of A, and let $P_l(A)$ denote the set of all subsets of A of cardinality l. It was shown in [25], that we have the identity

$$\overline{\mathcal{C}}_f = \bigcup_{S \in P_3(A)} \overline{\mathcal{C}}_{f_S}, \tag{2.11}$$

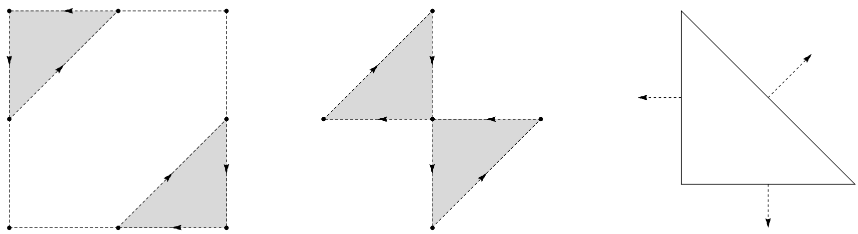

Figure 2.1: The coamoeba of $f(z) = 1 + z_1 + z_2$ in two fundamental regions, and the Newton polytope $\mathcal{N}_A$.

which holds without taking closures if $n \neq 3$. As any simple polynomial can be transformed to the above case by the action of an integer affine transformation, the identity (2.11) holds for all simple hypersurfaces.

The complement of the closed coamoeba of $f(z) = 1 + z_1 + \cdots + z_n$, in the fundamental domain $[-\pi, \pi)^n$ in $\mathbb{R}^n$, consists of the convex hull of the open cubes $(0,\pi)^n$ and $(\pi, 0)^n$. In particular the complement of $\overline{\mathcal{C}}_f$ has exactly one connected component in $\mathbf{T}^n$. Thus, in this case, the number of connected components of the complement of $\overline{\mathcal{C}}_f$ equals the normalized volume $\mathrm{Vol}(\mathcal{N}) = 1$. It follows from Remark 2.1.2 that for any simple hypersurface, the number of connected components of the complement of its coamoeba will be equal to $\mathrm{Vol}(\mathcal{N})$. ◯

It has been conjectured that the number of connected components of the complement of $\overline{\mathcal{C}}$ is at most $\mathrm{Vol}(\mathcal{N})$.[1] A proof in arbitrary dimension has been proposed by Nisse in [38], and an independent proof in the case $n = 2$ was given in [16].

The remaining part of this section is devoted to the proof of Lemma 2.2.3.

Proof of Lemma 2.2.3. Part 1: Let us present a modification of the argument given in [25, Lemma 2.10] when proving the left to right inclusion of (2.10). Assume that $\mathcal{N}$ has full dimension and that the sequence $\{z(j)\}_{j=1}^{\infty} \subset Z(f)$ is such that

$$\lim_{j\to\infty} z(j) \notin \mathbb{C}_*^n \quad \text{and} \quad \lim_{j\to\infty} \mathrm{Arg}(z(j)) = \theta \in \mathbf{T}^n.$$

We claim that $\theta \in \mathcal{C}_{f_\Gamma}$ for some strict subface $\Gamma \subset \mathcal{N}_A$. As $Z(f)$ is invariant under multiplication of f with a Laurent monomial, we can assume that

[1]This conjecture has commonly been attributed to Mikael Passare, however, it seems to originate from a talk given by Mounir Nisse at Stockholm University in 2007.

the constant 1 is a monomial of f. We can also choose a subsequence of $\{z(j)\}_{j=1}^{\infty}$ such that, possibly after reordering A,

$$|z(j)^{\mathbf{a}_1}| \geq \cdots \geq |z(j)^{\mathbf{a}_N}|, \quad j = 1,2,\ldots$$

and in addition

$$\lim_{j\to\infty} \frac{|z(j)^{\mathbf{a}_k}|}{|z(j)^{\mathbf{a}_1}|} \to d_k$$

for some $d_k \in [0,1]$. It is shown in the proof of [25, Lemma 2.10] that $\Gamma = \{\mathbf{a}_k \mid d_k > 0\}$ is a face of $\mathcal{N}_A$, and furthermore that $\theta \in \mathcal{C}_{f_\Gamma}$. With the above ordering of A, assume that the constant 1 is the pth monomial. We need to show that Γ is a strict subface of $\mathcal{N}_A$. Assuming the contrary, we find that $d_k > 0$ for each k, and hence

$$\lim_{j\to\infty} |z(j)^{\mathbf{a}_k}| = \lim_{j\to\infty} \frac{|z(j)^{\mathbf{a}_k}|}{|z(j)^{\mathbf{a}_1}|} |z(j)^{\mathbf{a}_1}| = \frac{d_k}{d_p},$$

which in particular is finite and nonzero. As $\mathcal{N}_A$ has full dimension, this implies that that $\lim_{j\to\infty} |z(j)_m|$ is finite and nonzero for each $m = 1,\ldots,n$. As $\mathrm{Arg}(z(j)) \to \theta$ when $j \to \infty$, we find that $\lim_{j\to\infty} z(j) \in \mathbb{C}_*^n$, a contradiction. Hence, $d_N = 0$, and Γ is a strict subface of $\mathcal{N}_A$.

Part 2: We now claim that if $n \geq 2$, then the set

$$P = \{z \in Z(f) \mid \mathrm{Arg}(z) \in N(l) \cap \mathcal{C}_f\},$$

where $N(l)$ is an arbitrarily small neighborhood of l in $\mathbb{R}^n$, is such that $\mathrm{Log}(P)$ is unbounded. To see this, consider the function $g(w) = f(e^w)$, where $w_k = x_k + i\theta_k$. Notice that the w-space $\mathbb{C}^n$ is identified with the image of the z-space $\mathbb{C}_*^n$ under the multivalued complex logarithm. That is, the coamoeba $\mathcal{C}_f$ and the line l are considered as subsets of $\mathbb{R}^n$, which is the image of the w-space $\mathbb{C}^n$ under taking coordinatewise imaginary parts.

We can assume that l is parallel to the θ_1-axis and, by a translation of the coamoeba, that there are $\rho_1,\ldots,\rho_n > 0$ such that the set

$$S = [-\rho_1, \rho_1] \times \cdots \times [-\rho_n, \rho_n]$$

fulfills $l \subset S \subset N(l)$. Furthermore we can choose $0 < r < \rho_1$ such that, with

$$\tilde{S} = [-r, r] \times [-\rho_2, \rho_2] \times \cdots \times [-\rho_n, \rho_n],$$

the set $S \setminus \tilde{S}$ consists of two n-cells that are neighborhoods of the endpoints of l. Hence, we can assume that $S \setminus \tilde{S} \subset (\mathbb{R}^n \setminus \overline{\mathcal{C}}_f)$. If we assume that $\mathrm{Log}(P)$ is bounded, then there exists a sufficiently large $R \in \mathbb{R}$ such that if

$$D = \{x \in \mathbb{R}^n \mid |x| > R\},$$

then $g(w)$ has no zeros in $D+iS \subset \mathbb{C}^n$. Let w' denote the vector $(w_2,\dots,w_n)$, and let $(D+iS)'$ be the projection of $D+iS$ onto the last $n-1$ components. Then in particular, $g(w)$ has no zeros when $w' \in (D+iS)'$ and w_1 lies in the domain given by $\{w_1 \,|\, r < |\Im(w_1)| < \rho_1\} \cup (\{w_1 \,||\Re(w_1)| > R\} \cap \{w_1 \,||\Im(w_1)| < \rho_1\})$, see Figure 2.2. Consider a curve γ as in Figure 2.2, and the integral

$$k(w') = \frac{1}{2\pi i}\int_\gamma \frac{g_1'(w_1,w')}{g(w_1,w')}dw_1, \quad w' \in (D+iS)'.$$

By the argument principle, for a fix w_1, the integral $k(w')$ counts the number of roots of $g(w)$ inside the box in Figure 2.2. As it is continuous in w' in the domain $(D+iS)'$, it is constant. By considering w' with $|x'| > R$ (here it is essential that $n \geq 2$) we conclude that it is zero. However, this is a contradiction to the assumption that l intersects $\mathcal{C}_f$. Hence, $\mathrm{Log}(P)$ is unbounded.

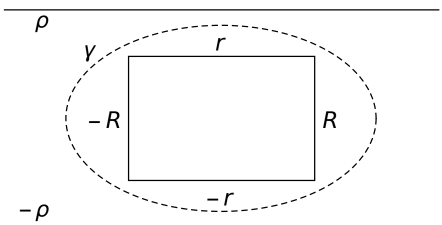

Figure 2.2: The curve $\gamma \subset \mathbb{C}$.

Part 3: We will now prove Lemma 2.2.3 by induction on the dimension d of $\mathcal{N}$. If $d = 1$, then there is nothing to prove. Assume that $d > 1$, and that the statement is proven for all smaller dimensions. Notice that f has $n-d$ homogeneities, and hence it is essentially a polynomial in d variables. Dehomogenizing f corresponds to a projection $\mathrm{pr}\colon \mathbf{T}^n \to \mathbf{T}^d$ such that $\mathcal{C}_f = \mathrm{pr}^{-1}(\mathcal{C}_{\mathrm{pr}(f)})$. The line segment $\mathrm{pr}(l)$ will intersect the shell $\mathcal{H}_{\mathrm{pr}(f)}$ if and only if l intersects the shell $\mathcal{H}_f$. Hence, it is enough to prove the statement under the assumption that $d = n$. In particular, $n \geq 2$.

Choose a decreasing sequence $\{\varepsilon(k)\}_{k=1}^\infty$ of positive real numbers, such that $\lim_{k\to\infty}\varepsilon(k) = 0$, and consider the family of neighborhoods of l given by

$$N(l,k) = \left\{\theta \in \mathbb{R}^n \,\middle|\, \inf_{x\in l}|\theta - x| < \varepsilon(k)\right\},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Define

$$P(k) = \{z \in Z(f) \,|\, \mathrm{Arg}(z) \in N(l,k) \cap \mathcal{C}_f\}.$$

As $n \geq 2$, Part 2 shows that for each k, the set $\mathrm{Log}(P(k))$ is unbounded. That is, for each k, we can find a sequence $\{z(k,m)\}_{m=1}^{\infty}$ such that $z(k,m) \in Z(f)$, with

$$\mathrm{Arg}(z(k,m)) \in N(l,k) \cap \mathcal{C}_f \subset \overline{N(l,k) \cap \mathcal{C}_f},$$

but $\lim_{m\to\infty} z(k,m) \notin \mathbb{C}_*^n$. Since $\overline{N(l,k) \cap \mathcal{C}_f}$ is compact, we can choose a subsequence such that $\mathrm{Arg}(z(k,m))$ converges to some $\theta(k) \in \overline{N(l,k) \cap \mathcal{C}_f}$ when $m \to \infty$. Then, Part 1 gives a strict subface $\Gamma(k)$ of $\mathcal{N}_A$ such that $\theta(k) \in \mathcal{C}(f_{\Gamma(k)})$. Since $\mathcal{N}_A$ has only finitely many strict subfaces, we can choose a subsequence of $\{\theta(k)\}_{k=1}^{\infty}$ such that $\Gamma = \Gamma(k)$ does not depend on k. As $\{\theta(k)\}_{k=1}^{\infty} \subset \overline{N(l,1)}$, which is compact, we can also choose this subsequence such that $\theta(k)$ converges to some $\theta \in \overline{N(l,1)}$ when $k \to \infty$. On the one hand, we have that $\theta \in l$ by construction of the sets $N(l,k)$. On the other hand, that $\theta(k) \in \mathcal{C}(f_\Gamma)$ implies that $\theta \in \overline{\mathcal{C}}(f_\Gamma)$. In particular, $\theta \in l \cap \overline{\mathcal{C}}_{f_\Gamma}$.

The identity (2.10) shows that the endpoints of l are contained in the complement of $\overline{\mathcal{C}}_{f_\Gamma}$. As the dimension of Γ is strictly less than the dimension of $\mathcal{N}$, the induction hypothesis shows that l intersects the coamoeba of an edge of Γ. As each edge of Γ is an edge of $\mathcal{N}_A$, the lemma is proven. $\square$

3. Pellet's criterion

The following corollary of Rouche's Theorem was published by Pellet in 1881 [47]. Let $f \in \mathbb{C}_*^A$ be a univariate polynomial and consider the auxiliary real polynomial

$$F_\kappa(z) = |x_\kappa| z^{\mathbf{a}_\kappa} - \sum_{k \neq \kappa} |x_k| z^{\mathbf{a}_k}.$$

If F_κ has two positive roots z_1 and z_2, with $z_1 \le z_2$, then f has exactly κ roots inside the circle $|z| \le z_1$ and no roots in the annulus $z_1 < |z| < z_2$. Notice that F_κ has two positive roots, counting multiplicities, if and only if there exist a $z \in \mathbb{C}_*^n$ such that

$$\left| x_\kappa z^{\mathbf{a}_\kappa} \right| \ge \sum_{k \neq \kappa} \left| x_k z^{\mathbf{a}_k} \right|. \tag{3.1}$$

With f fixed, the fulfillment of (3.1) only depends on $\tau = \log|z|$. Furthermore, the conclusion of Pellet's Theorem is equivalent to that

$$(\tau_1, \tau_2) \subset E_\kappa \subset \mathbb{R} \setminus \mathcal{A}_f.$$

The n-variate version of Pellet's Theorem was given by Rullgård in [50], and reads as follows. If z is such that (3.1) holds, then $\tau = \operatorname{Log}|z| \in E_\kappa$. Later, Purbhoo [48] named (3.1) the *lopsidedness criterion*, and defined the *lopsided amoeba* $\mathcal{L} = \mathcal{L}_f$ as the set of all $\tau \in \mathbb{R}^n$ such that (3.1) does not hold for any $z \in \operatorname{Log}^{-1}(\tau)$ and any κ. The name can be misleading as the lopsided amoeba is not, per se, an amoeba. It is, however, equal to the intersection of the amoeba of the $(N+n)$-variate polynomial

$$F(w, z) = \sum_{k=1}^{N} w_k z^{\mathbf{a}_k} \tag{3.2}$$

and the affine space (in logarithmic coordinates) defined by the equations $\operatorname{Log}|w| = \operatorname{Log}|f|$.

Pellet's Theorem implies that there is an inclusion $\mathcal{A} \subset \mathcal{L}$. This inclusion allows us to define a map $\operatorname{ord}_\mathcal{L}$ by

$$\operatorname{ord}_\mathcal{L}(\tau) = \operatorname{ord}_\mathcal{A}(\tau), \quad \tau \in \mathbb{R}^n \setminus \mathcal{L}.$$

Then, Rullgårds Theorem implies that

$$\operatorname{ord}_\mathcal{L} \colon \mathbb{R}^n \setminus \mathcal{L} \to A.$$

Furthermore, as the connected components of the complement of $\mathcal{L}$ are convex, a fact which follows from the convexity of the dittos of the amoeba of $F(w,z)$, we find that $\mathrm{ord}_{\mathcal{L}}$ induces an injective map on the set of connected components of $\mathcal{L}$. For this reason we call $\mathrm{ord}_{\mathcal{L}}$ the *order map of the lopsided amoeba.*

The criterion (3.1) has made several appearances throughout history. Most notable is its applications within the analysis of of local growth of power series, which is known as Wiman–Valiron Theory, see [24] and the references therein. Generalizations of Pellet's Theorem for univariate polynomials can be found in, e.g., [30].

3.1 Colopsidedness

Definition 3.1.1. The polynomial f is said to be *colopsided* at $\theta \in \mathbb{R}^n$ (or $\mathbf{T}^n$) if there exist a phase φ such that

$$\Re\left(e^{i\varphi} f_{\mathbf{a}}(e^{i\theta})\right) \geq 0, \quad \forall\, \mathbf{a} \in A, \tag{3.3}$$

with at least one of the inequalities (3.3) being strict. We define the lopsided coamoeba, denoted $\mathcal{D} = \mathcal{D}_f$, as the set of all θ such that (3.3) does not hold for any φ. ○

The main result of this section is the following proposition, describing the relation between $\mathcal{D}$ and $\mathcal{C}$.

Proposition 3.1.2. *Each connected component of the complement of $\overline{\mathcal{C}}$ contains at most one connected component of the complement of $\overline{\mathcal{D}}$.*

We will begin with a number of lemmas describing equivalent definitions of colopsidedness.

Lemma 3.1.3. *Consider $\mathbb{C} \simeq \mathbb{R}^2$ as an $\mathbb{R}$-vector space. Then, the polynomial f is colopsided at θ if and only if there exist a nonzero vector $\mathbf{n} \in \mathbb{R}^2$ such that $\hat{f}(\theta)$ is contained in the halfspace $H = \{z \,|\, \langle \mathbf{n}, z\rangle \geq 0\}$ but it is not contained in the vector subspace $\ell = \{z \,|\, \langle \mathbf{n}, z\rangle = 0\}$.*

Proof. If (3.3) is fulfilled at θ, then we can choose $\mathbf{n} = (\cos(\varphi), -\sin(\varphi))$. Conversely, if $\zeta \in \ell$ with $\zeta \neq 0$, then (3.3) is fulfilled at θ for either $\varphi = \arg(\zeta)$ or $\varphi = \arg(-\zeta)$. □

Let $\mathrm{Cone}\, \hat{f}(\theta)$ denote the open cone consisting of all positive linear combinations of the points $\hat{f}_k(\theta)$, $k = 1, \ldots, N$.

Lemma 3.1.4. *We have that $\theta \in \mathcal{D}$ if and only if $0 \in \mathrm{Cone}\, \hat{f}(\theta)$.*

Proof. If $\theta \in \mathbf{T}^n \setminus \mathcal{D}$, then $\operatorname{Cone}\hat{f}(\theta) \subset \operatorname{int}(H)$, where $H \subset \mathbb{C}$ is a halfspace from Lemma 3.1.3. Conversely, if $0 \notin \operatorname{Cone}\hat{f}(\theta)$ then the fact that $\operatorname{Cone}\hat{f}(\theta)$ is convex implies that there exists a halfspace H containing $\operatorname{Cone}\hat{f}(\theta)$ in its interior. □

Corollary 3.1.5. *We have the inclusion* $\mathcal{C} \subset \mathcal{D}$.

Proof. If $f(re^{i\theta}) = 0$ then $0 \in \operatorname{Cone}\hat{f}(\theta)$. □

Corollary 3.1.6. *If A is simple, then $\mathcal{C} = \mathcal{D}$ for all $f \in \mathbb{C}_*^A$.*

Proof. By applying an integer affine transformations we can reduce to the case when $\mathcal{N}$ is the standard n-simplex in $\mathbb{R}^n$. That is, it is enough to prove the statement for the polynomial $f(z) = 1 + z_1 + \cdots + z_n$. We have that $0 \in \operatorname{Cone}\hat{f}(\theta)$ if and only if we can find positive numbers $r_0, \ldots, r_n$ such that $r_0 + r_1 e^{i\theta_1} + \cdots + r_n e^{i\theta_n} = 0$, which is equivalent to that $\theta \in \mathcal{C}$. □

Remark 3.1.7. There are non-simple polynomials for which the identity $\mathcal{C} = \mathcal{D}$ holds. Take, for example, any polynomial f such that $\mathcal{C} = \mathbf{T}^n$. Such examples can be constructed by taking products of polynomials, which corresponds to taking unions of coamoebas. We will encounter less trivial examples in Section 4.3. ○

Lemma 3.1.8. *We have that*

$$\mathcal{D}_f = \bigcup_{g \in \mathbb{R}_+^A \cdot f} \mathcal{C}_g,$$

where $\mathbb{R}_+^A \cdot f$ denotes the orbit of f under the action of $\mathbb{R}_+^A$ on $\mathbb{C}_*^A$.

Proof. If $\theta \in \mathcal{C}_g$ for $g \in \mathbb{R}_+^A$ then $0 \in \operatorname{Cone}\hat{g}(\theta) = \operatorname{Cone}\hat{f}(\theta)$. Conversely, if $0 \in \operatorname{Cone}\hat{f}(\theta)$, then there exists an $r \in \mathbb{R}_+^A$ such that $r \cdot f(e^{i\theta}) = 0$. □

Lemma 3.1.9. *Let $f \in \mathbb{C}_*^A$, and let $F(w,z)$ be as in* (3.2). *Then, the lopsided coamoeba $\mathcal{D}_f$ is equal to the intersection of the coamoeba $\mathcal{C}_F$ with the affine space in $\mathbf{T}^{N+n}$ defined by* $\operatorname{Arg}(w) = \operatorname{Arg}(f)$.

Proof. We have that $\operatorname{Cone}\hat{f}(\theta) = \operatorname{Cone}\widehat{F}(\operatorname{Arg}(f), \theta)$. Since F is a simple polynomial, the statement follows from Corollary 3.1.6 and Lemma 3.1.4. □

Proposition 3.1.10. *Let $n \geq 2$. Then,*

$$\overline{\mathcal{D}}_f = \bigcup_{S \in P_3(A)} \overline{\mathcal{C}}_{f_S}. \tag{3.4}$$

Figure 3.1: Above: the coamoeba and lopsided coamoeba of $f(z_1,z_2) = z_1^3 + z_2 + z_2^2 - z_1 z_2$. Below: dittos of $f(z_1,z_2) = 1 + z_1 + z_2 + i z_1 z_2$.

Proof. By Lemma 3.1.9 we have that $\mathcal{D}_f = \mathcal{C}_F \cap H$, where H is the sub n-torus of $\mathbf{T}^{N+n}$ defined by $\mathrm{Arg}(w) = \mathrm{Arg}(f)$. Since F is a simple polynomial, the identity follows from (2.11). $\square$

As was the case for (2.11), the identity (3.4) holds without taking closures if $N \neq 4$. Lopsided coamoebas first appeared under this disguise in [25]. Proposition 3.1.10 is the motivation for the following definition, in which we consider the union of the shells of the coamoebas appearing in the right hand side of (3.4).

Definition 3.1.11. The *shell* $\mathcal{K}_f$ of the lopsided coamoeba $\mathcal{D}_f$ is defined as the union

$$\mathcal{K}_f = \bigcup_{S \in P_2(A)} \mathcal{C}_{f_S}.$$

That is, $\mathcal{K}_f$ is the union of all coamoebas of truncated binomials of f. $\hexagon$

Proposition 3.1.12. *The shell $\mathcal{K}$ contains the boundary of $\mathcal{D}$.*

Proof. If $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}$, then we can choose φ such that

$$\Re\left(e^{i\varphi} f_{\mathbf{a}}(e^{i\theta})\right) > 0, \quad \forall\, \mathbf{a} \in A.$$

We conclude that the boundary of $\mathcal{D}$ is contained in the hyperplane arrangement consisting of all θ such that two components of $\hat{f}(\theta)$ are antipodal. That is, it is contained in $\mathcal{K}$. $\square$

We are now ready to prove the main result of this section.

Proof of Proposition 3.1.2. It suffices to show that any line segment $l \subset \mathbb{R}^n$ that intersect $\mathcal{D}$ and has its endpoints in $\mathbf{T}^n \setminus \overline{\mathcal{D}}$ also intersect $\overline{\mathcal{C}}$.

Consider first the case when $f(z)$ is a univariate polynomial. Let $l = [\theta_1, \theta_2] \subset \mathbb{R}$, and assume that there exist a $\theta \in (\theta_1, \theta_2)$ such that $\theta \in \mathcal{D}$. By Lemma 3.1.4 there exists an $r \in \mathbb{R}_+^A$ such that $\theta \in \mathcal{C}_{r \cdot f}$. Let γ be the path

$$\gamma(t)_k = r_k^{1-t} x_k, \quad t \in [0,1],$$

from $r \cdot f$ to f in $\mathbb{C}_*^A$, and let f_t denote the polynomial with coefficients $\gamma(t)$. By Lemma 3.1.4 it holds that $\mathcal{C}_{f_t} \subset \mathcal{D}_f$ for all $t \in [0,1]$. In particular, $\theta_1, \theta_2 \notin \mathcal{C}_{f_t}$. Let $z \in \mathbb{C}_*$ denote a root of $f_0(z) = (r \cdot f)(z)$ such that $\arg(z) = \theta$. By continuity of roots, there is a continuous path $t \mapsto z(t) \in \mathbb{C}_*$ such that $z(t)$ is a root of the polynomial $f_t(z)$. We conclude that the path $t \mapsto \arg(z(t))$ is continuous, which in turn implies that $\arg(z(t)) \in (\theta_1, \theta_2)$ for all t. In particular, $\arg(z(1)) \in (\theta_1, \theta_2)$, which proves the proposition in this case.

Consider now the case when $\mathcal{N} \subset \mathbb{R}^n$ is one-dimensional. To dehomogenize f corresponds to a projection $\mathrm{pr}\colon \mathbf{T}^n \to \mathbf{T}$ such that, if g denotes the dehomogenization of f, then $\mathcal{C}_f = \mathrm{pr}^{-1}(\mathcal{C}_g)$. As such a projection maps a line segment to a line segment, this case follows from the univariate case.

Now consider an arbitrary multivariate polynomial $f(z)$. By Lemma 3.1.4 there exists an $r \in \mathbb{R}_+^A$ such that the line segment l intersects the coamoeba of $r \cdot f$, whose closure is contained in $\overline{\mathcal{D}}$. It follows that the endpoints of l are contained in the complement $\mathbb{R}^n \setminus \overline{\mathcal{C}}_{r \cdot f}$. By Lemma 2.2.3, there is an edge $\Gamma \prec \mathcal{N}$ such that l intersects the coamoeba of the truncated polynomial $(r \cdot f)_\Gamma$. It follows that l intersects the lopsided coamoeba of $(r \cdot f)_\Gamma$, which coincides with the lopsided coamoeba of f_Γ. Furthermore, we can conclude from (2.10) that the endpoints of l are contained in the complement of the lopsided coamoeba of f_Γ. Since Γ is one dimensional, we conclude from the previous case that l intersects coamoeba of f_Γ. Finally, we conclude from (2.10) that the line segment l intersects $\overline{\mathcal{C}}_f$. $\square$

Let us end this section with a characterization of colopsidedness that in its phrasing is similar to lopsidedness.

Lemma 3.1.13. *Let $\sigma \in \mathfrak{S}_N$ be such that the points $\hat{f}_{\sigma(k)}(\theta)$, $k = 1, \ldots N$, are cyclically ordered in S^1. Define*

$$\psi_k = \arg_{2\pi}\left(\frac{\hat{f}_{\sigma(k)}(\theta)}{\hat{f}_{\sigma(k-1)}(\theta)}\right),$$

so that ψ_k denotes the kth intermediate angle of adjacent points. Then, f is colopsided at θ if and only if there exists a unique index κ such that

$$\psi_\kappa \geq \sum_{k \neq \kappa} \psi_k. \tag{3.5}$$

Proof. We have that $\sum_k \psi_k = 2\pi$. Hence, the inequality (3.5) holds if only if κ is the unique index such that $\psi_\kappa \geq \pi$. We find that $\psi_\kappa \geq \pi$ if and only if (3.3) is fulfilled for

$$\varphi = \arg\left(\hat{f}_{\sigma(\kappa-1)}(\theta)\right).$$

Furthermore, one inequality of (3.3) is strict if and only if the index κ is unique. □

Remark 3.1.14. Using the construction from Lemma 3.1.13, we can define a (in general nonconvex) polytopal complex in $\mathbf{T}^n$ as the set of all θ such that the maximum $\max_k \psi_k$ is attained at least twice. In several examples, this definition yields the *dimer model* related to the coamoeba $\mathcal{C}$ that was defined ad hoc in, e.g., [13] and [56]. For generic polynomials f, computing this polytopal complex is notoriously difficult, however, they deserve further attention. ⬡

3.2 The order map of the lopsided coamoeba

The aim of this section is to define the order map for the lopsided coamoeba. Let $f \in \mathbb{C}_*^A$, and fix both an index κ and a Gale dual B. For each index k, consider the function $p_\kappa^k \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$p_\kappa^k(\theta) = \arg_\pi\left(\frac{x_k e^{i\langle \mathbf{a}_k, \theta\rangle}}{x_\kappa e^{i\langle \mathbf{a}_\kappa, \theta\rangle}}\right) - \arg_\pi(x_k) + \arg_\pi(x_\kappa) - \langle \mathbf{a}_k - \mathbf{a}_\kappa, \theta\rangle.$$

We let the functions p_κ^k, $k = 1, \ldots, N$, be the components of a vector valued function

$$p_\kappa(\theta) = \left(p_\kappa^1(\theta), \ldots, p_\kappa^N(\theta)\right).$$

Finally, we define the map $\operatorname{ord}_{\mathcal{D}} \colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$\operatorname{ord}_{\mathcal{D}}(\theta) = \left(\operatorname{Arg}_\pi(f) + p_\kappa(\theta)\right) B. \tag{3.6}$$

Theorem 3.2.1. *Assume that $\iota_A = 1$. Then,* (3.6) *defines a map*

$$\operatorname{ord}_{\mathcal{D}} \colon \mathbf{T}^n \setminus \overline{\mathcal{D}} \to \operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}_\pi(f) B + 2\pi L_B) \tag{3.7}$$

which does not depend on κ. Furthermore, the map (3.7) *induces a bijective map between the set of connected components of the lopsided coamoeba complement $\mathbf{T}^n \setminus \overline{\mathcal{D}}$ and the finite set* $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}_\pi(f) B + 2\pi L_B)$.

Definition 3.2.2. The map $\operatorname{ord}_{\mathcal{D}}$ from Theorem 3.2.1 is called the *order map* of the lopsided coamoeba $\mathcal{D}$. ⬡

Remark 3.2.3. In general, when considered as a map from the set of connected components of the complement of the closed coamoeba, the map $\operatorname{ord}_{\mathcal{D}}$ will be ι_A to one. Thus, if one considers $\operatorname{ord}_{\mathcal{D}}$ as a map from the set of components of the complement of $\overline{\mathcal{D}}$ into the full affine lattice $\operatorname{Arg}_\pi(f)B + 2\pi\mathbb{Z}^m$, then injectivity is measured in terms of ι_A, while surjectivity is measured in terms of ι_B. ⬡

Lemma 3.2.4. *For each $k = 1, \ldots N$, the map p_κ^k is locally constant on the complement of the coamoeba of the binomial $x_\kappa z^{\mathbf{a}_\kappa} + x_k z^{\mathbf{a}_k}$. Furthermore, its image is contained in the lattice $2\pi\mathbb{Z}$.*

Proof. We have that

$$\arg_\pi\left(\frac{x_k e^{i\langle \mathbf{a}_k, \theta\rangle}}{x_\kappa e^{i\langle \mathbf{a}_\kappa, \theta\rangle}}\right) = \arg_\pi(x_k) - \arg_\pi(x_\kappa) + \langle \mathbf{a}_k - \mathbf{a}_\kappa, \theta\rangle + 2\pi j(\theta),$$

where $j(\theta) \in \mathbb{Z}$. Hence, $p_\kappa^k(\theta) = 2\pi j(\theta)$, proving the last claim. Finally, the function $j(\theta)$ is locally constant on the complement of the locus where

$$\arg_\pi\left(\frac{x_k e^{i\langle \mathbf{a}_k, \theta\rangle}}{x_\kappa e^{i\langle \mathbf{a}_\kappa, \theta\rangle}}\right) = \pi,$$

and this locus is equal to the coamoeba of the binomial $x_\kappa z^{\mathbf{a}_\kappa} + x_k z^{\mathbf{a}_k}$. □

Corollary 3.2.5. *The map $p_\kappa(\theta)$ is constant on each cell of the hyperplane arrangement $\mathcal{K} \subset \mathbb{R}^n$. In particular, it is constant on each connected component of the complement $\mathbb{R}^n \setminus \overline{\mathcal{D}}$.* □

Lemma 3.2.6. *Let* $\operatorname{ord}_{\mathcal{D}}\colon \mathbb{R}^n \to \mathbb{R}^m$ *denote the map* (3.6). *Then,*

i) $\operatorname{ord}_{\mathcal{D}}$ *maps* $\mathbb{R}^n$ *into the affine lattice* $\operatorname{Arg}_\pi(f)B + 2\pi L_B$,

ii) $\operatorname{ord}_{\mathcal{D}}$ *is well-defined on* $\mathbf{T}^n$*, i.e., it is periodic in each* θ_i *with period* 2π,

iii) $\operatorname{ord}_{\mathcal{D}}$ *is invariant under multiplication of f by a Laurent monomial, and*

iv) if $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}$, then $\operatorname{ord}_{\mathcal{D}}(\theta) \in \operatorname{int}(\mathcal{Z}_B)$.

Proof. The first claim follows from the definition of $\operatorname{ord}_{\mathcal{D}}$ and Lemma 3.2.4. To prove the second and third claim, we note that $\operatorname{Arg}_\pi(f) + p_\kappa(\theta)$ equals

$$\left(\arg_\pi\left(\frac{x_1 e^{i\langle \mathbf{a}_1, \theta\rangle}}{x_\kappa e^{i\langle \mathbf{a}_\kappa, \theta\rangle}}\right), \ldots, \arg_\pi\left(\frac{x_N e^{i\langle \mathbf{a}_N, \theta\rangle}}{x_\kappa e^{i\langle \mathbf{a}_\kappa, \theta\rangle}}\right)\right) + (\arg_\pi(x_\kappa)\langle \mathbf{a}_\kappa, \theta\rangle, \theta_1, \ldots, \theta_n)A,$$

where A is the matrix (2.1). It follows that

$$\left(\operatorname{Arg}_\pi(f)+p_\kappa(\theta)\right)B=\left(\arg_\pi\left(\frac{x_1e^{i\langle\mathbf{a}_1,\theta\rangle}}{x_\kappa e^{i\langle\mathbf{a}_\kappa,\theta\rangle}}\right),\dots,\arg_\pi\left(\frac{x_Ne^{i\langle\mathbf{a}_N,\theta\rangle}}{x_\kappa e^{i\langle\mathbf{a}_\kappa,\theta\rangle}}\right)\right)B. \tag{3.8}$$

We conclude that $\operatorname{ord}_{\mathcal{D}}$ is well-defined on $\mathbf{T}^n$, and that it is invariant under multiplication of f by a Laurent monomial.

Let us now turn to the last claim. Given a point θ in the complement of $\overline{\mathcal{D}}$, the components of $\hat{f}(\theta)$ are contained in an open half-space $H\subset\mathbb{C}$. As $\operatorname{ord}_{\mathcal{D}}$ is invariant under multiplication of f with a Laurent monomial, we can assume that $\kappa=0$ and that $H=H_0$ is the right half space. That is

$$\arg_\pi\left(x_ke^{i\langle\mathbf{a}_k,\theta\rangle}\right)=\frac{\pi}{2}\mu_k,\quad k=1,\dots,N,$$

where $\mu_k\in(-1,1)$. Since $\arg_\pi(\zeta_1\zeta_2)=\arg_\pi(\zeta_1)+\arg_\pi(\zeta_2)$ for any two elements $\zeta_1,\zeta_2\in H_0$, we find that

$$p_0^k(\theta)=\arg_\pi\left(x_ke^{i\langle\mathbf{a}_k,\theta\rangle}\right)-\arg_\pi(x_k)-\langle\mathbf{a}_k,\theta\rangle.$$

Thus, the following identities hold.

$$\left\{\begin{array}{rcl}\arg_\pi(x_1)+\langle\mathbf{a}_1,\theta\rangle+p_0^1(\theta)&=&\frac{\pi}{2}\mu_1\\ &\vdots&\\ \arg_\pi(x_N)+\langle\mathbf{a}_N,\theta\rangle+p_0^N(\theta)&=&\frac{\pi}{2}\mu_N.\end{array}\right. \tag{3.9}$$

Hence,

$$\left(\operatorname{Arg}_\pi(f)+p_\kappa(\theta)\right)B=\left(\frac{\pi}{2}\mu-(0,\theta_1,\dots,\theta_n)A\right)B=\frac{\pi}{2}\mu B\in\operatorname{int}(\mathcal{Z}_B).\qquad\square$$

Lemma 3.2.7. *The map* $\operatorname{ord}_{\mathcal{D}}$ *from* (3.7) *is independent of the choice of* κ.

Proof. Let $\theta\in\mathbf{T}^n\setminus\overline{\mathcal{D}}$. Since $\operatorname{ord}_{\mathcal{D}}$ is invariant under multiplication of f with a Laurent monomial we can assume that $\mathbf{a}_1=0$, and that the right half space $H=H_0$ contains $\hat{f}(\theta)$. Then, the difference

$$p_1^k(\theta)-p_\kappa^k(\theta)=\arg_\pi\left(x_\kappa e^{i\langle\mathbf{a}_\kappa,\theta\rangle}\right)-\arg_\pi(x_\kappa)-\langle\mathbf{a}_\kappa,\theta\rangle$$

is independent of k, and hence $\left(p_1(\theta)-p_\kappa(\theta)\right)B=0$. $\square$

For convenience, let us impose assumptions on the matrix A and the Gale dual B. After multiplication with a Laurent monomial, an operation with leaves both the map $\operatorname{ord}_{\mathcal{D}}$ and the lopsided coamoeba $\mathcal{D}$ unaffected, we can assume that A is of the form

$$A=\begin{pmatrix}1&1&1\\0&A_{\mathbf{1}}&A_{\mathbf{2}}\end{pmatrix}, \tag{3.10}$$

where $A_{\mathbf{1}}$ is a nonsingular $n \times n$ matrix. We can also assume that $x_0 = 1$, i.e., that the constant 1 is a monomial of f. Any Gale dual of A can be presented in the form

$$B = \begin{pmatrix} \mathbf{b}_0 \\ -A_{\mathbf{1}}^{-1} A_{\mathbf{2}} \\ I_m \end{pmatrix} T, \tag{3.11}$$

where $\mathbf{b}_0 \in \mathbb{Q}^m$ is such that each column of B sums to zero, and $T \in \mathrm{GL}_m(\mathbb{Q})$.

Lemma 3.2.8. *Let A be under the above imposed assumptions. Let $x_{\mathbf{1}}$ and $x_{\mathbf{2}}$ denote the vectors $(x_1, \dots, x_n)$ and $(x_{n+1}, \dots, x_{n+m})$ respectively, and use similar notation for $l \in \mathbb{Z}^N$ and $\mu \in \mathbb{R}^N$. Consider the system*

$$\left\{ \begin{array}{rcl} \mathrm{Arg}_\pi(x_{\mathbf{1}}) + \theta A_{\mathbf{1}} + 2\pi l_{\mathbf{1}} & = & \dfrac{\pi}{2}\mu_{\mathbf{1}} \\ \mathrm{Arg}_\pi(x_{\mathbf{2}}) + \theta A_{\mathbf{2}} + 2\pi l_{\mathbf{2}} & = & \dfrac{\pi}{2}\mu_{\mathbf{2}}. \end{array} \right. \tag{3.12}$$

Then, $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}$ if and only if θ solves (3.12) for some integers l and some numbers $\mu_0, \dots, \mu_{n+m}$ such that $\mu_0, \mu_1 + \mu_0, \dots, \mu_{n+m} + \mu_0 \in (-1, 1)$.

Proof. Let $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}$. Then, there is a halfplane H_φ containing $\hat{f}(\theta)$. As the constant 1 is a monomial of f, we can choose $\varphi \in (-\pi/2, \pi/2)$. It follows that there are numbers $\lambda_1, \dots, \lambda_{n+m} \in (-1, 1)$ and integers $l_1, \dots, l_{n+m}$ such that

$$\arg_\pi(x_k) + \langle \theta, \mathbf{a}_k \rangle + 2\pi l_k = \frac{\pi}{2}\lambda_k + \varphi, \quad k = 1, \dots, n+m.$$

This shows that θ fulfills (3.12) with l as above, $\mu_0 = -2\varphi/\pi$ and $\mu_k = \lambda_k + 2\varphi/\pi$ for $k = 1, \dots, n+m$. Conversely, if θ fulfills (3.12) for such l and μ, then $\hat{f}(\theta) \subset H_\varphi$, where $\varphi = -\pi\mu_0/2$. □

Proof of Theorem 3.2.1. It follows from Lemma 3.2.6 that the map $\mathrm{ord}_{\mathcal{D}}$ from (3.6) induces a well defined map on $\mathbf{T}^n \setminus \overline{\mathcal{D}}$, whose image is contained in the finite set $(\mathrm{Arg}_\pi(f)B + 2\pi L_B)$. Furthermore, by Lemma 3.2.7 we find that the given map is independent of κ. It only remains to prove that $\mathrm{ord}_{\mathcal{D}}$ is a bijection.

Let A be under the above imposed assumptions. Solve the first equation of (3.12) for θ by multiplying from the left with $A_{\mathbf{1}}^{-1}$, and proceed by elimination of θ in the second equation. After multiplication by T, we arrive at the equivalent system

$$\left\{ \begin{array}{rcl} \theta & = & \dfrac{\pi}{2}\mu_{\mathbf{1}} A_{\mathbf{1}}^{-1} - \mathrm{Arg}(x_{\mathbf{1}}) A_{\mathbf{1}}^{-1} - 2\pi l_{\mathbf{1}} A_{\mathbf{1}}^{-1} \\ \mathrm{Arg}_\pi(f)B + 2\pi(0, l_{\mathbf{1}}, l_{\mathbf{2}})B & = & \dfrac{\pi}{2}(0, \mu_{\mathbf{1}}, \mu_{\mathbf{2}})B. \end{array} \right. \tag{3.13}$$

To see that $\mathrm{ord}_{\mathcal{D}}$ is a surjection, consider a point $\mathrm{Arg}_\pi(f)B + 2\pi lB = \pi\lambda B/2 \in \mathrm{int}(\mathcal{Z}_B)$. We can assume that $l_0 = 0$. Define μ by $\mu_k = \lambda_k - \lambda_0$ for $k = 0,\ldots,n+m$. It follows that the pair (l,μ) fulfills the second equation of (3.13). Let $\theta \in \mathbb{R}^n$ be defined by the first equation of (3.13). It then follows that the triple (θ, l, μ) fulfills (3.12), and thus by Lemma 3.2.8 we have that $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}$. By tracing backwards we find that the order of the connected component of the complement of $\overline{\mathcal{D}}$ containing θ is $\mathrm{Arg}_\pi(f)B + 2\pi lB$. Hence, the map $\mathrm{ord}_{\mathcal{D}}$ is surjective.

To see that $\mathrm{ord}_{\mathcal{D}}$ is an injection, consider a point $p \in \mathrm{int}(\mathcal{Z}_B)$. The set of all $\mu \in \mathbb{R}^N$ such that $2\pi\mu B = p$, is an affine space, hence convex. It follows that the set of all $\mu \in (-1,1)^N$ such that $2\pi\mu B = p$, being the intersection of two convex sets, is also convex. This implies that for fix integers l, the set of $\theta \in \mathbb{R}^n$ such that (3.12) is fulfilled with $\mu_0, \mu_1 - \mu_0, \ldots, \mu_N - \mu_0 \in (-1,1)$ is in turn also convex, as it is the image of a convex set under an affine transformation. As the right hand side of (3.6) is constant on each cell of $\mathcal{K}$, this set is exactly one connected component of the complement of $\overline{\mathcal{D}}$ in $\mathbb{R}^n$. Thus, if we consider two points θ and $\tilde{\theta}$ in $\mathbb{R}^n$ which both maps to $\mathrm{Arg}(f)B + 2\pi lB$, then we can assume that θ and $\tilde{\theta}$ fulfills (3.12) for the same numbers μ, however possibly for different integers l. Under this assumption there are integers $s_1, \ldots, s_N$ such that

$$\langle \mathbf{a}_k, \theta\rangle = \langle \mathbf{a}_k, \tilde{\theta}\rangle + 2\pi s_k, \quad k = 1,\ldots,N.$$

Since $\iota_A = 1$, we have that $L_A = \mathbb{Z}^n$, and hence for each vector e_i of the standard basis there are integers $r_i = (r_{i1},\ldots,r_{iN})$ such that $e_i = \sum_k r_{ik}\mathbf{a}_k$. Hence,

$$\theta_i = \langle e_i, \theta\rangle = \sum_{k=1}^{N} r_{ik}\langle \mathbf{a}_k, \theta\rangle = \sum_{k=1}^{N} r_{ik}\langle \mathbf{a}_k, \tilde{\theta}\rangle + 2\pi r_{ik}s_k = \tilde{\theta}_i + 2\pi\langle r_i, s\rangle,$$

which shows that $\theta = \tilde{\theta}$ in $\mathbf{T}^n$. □

Remark 3.2.9. To evaluate the order of a component Θ of the complement of $\overline{\mathcal{D}}$ it is convenient to use the right hand side of (3.8). In particular, if $\mathbf{a}_0 = \mathbf{0}$ and $x_0 = 1$, then

$$\mathrm{ord}_{\mathcal{D}}(\theta) = \mathrm{Arg}_\pi\left(\hat{f}(\theta)\right)\cdot B,$$

where $\theta \in \Theta$ is arbitrary. ◯

Example 3.2.10. Let us determine the map $\mathrm{ord}_{\mathcal{D}}$ explicitly in the first example shown in Figure 3.1, that is we consider the polynomial $f(z_1, z_2) = z_1^3 + z_2 + z_2^2 - z_1z_2$. We have that

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix},$$

and a Gale dual of A is given by

$$B = (-1,-1,-1,3)^t.$$

The zonotope $\mathcal{Z}_B$ is the interval $[-3\pi, 3\pi]$. As the translation $\operatorname{Arg}_\pi(f)B = 3\arg_\pi(-1) = 3\pi$, the image of the map $\operatorname{ord}_{\mathcal{D}}$ will be the doubleton $\{-\pi, \pi\}$. We find that

$$\begin{array}{rcrcr} \operatorname{ord}_{\mathcal{D}}(\theta_1) & = & (0,-2\pi,-2\pi,-\pi)B & = & \pi \\ \operatorname{ord}_{\mathcal{D}}(\theta_2) & = & (0,2\pi,2\pi,\pi)B & = & -\pi, \end{array}$$

where $\theta_1 = (-2\pi/3, 0)$ and $\theta_2 = (2\pi/3, 0)$. ◯

Example 3.2.11. Let us also consider a univariate case of codimension 1, namely

$$f(z) = 1 + z^3 + iz^5.$$

A Gale dual of A is given by $B = (2,-5,3)^t$. Hence, the zonotope $\mathcal{Z}_B$ is the interval $[-5\pi, 5\pi]$. We have that $\operatorname{Arg}_\pi(f)B = (0,0,\pi/2)B = 3\pi/2$, and hence the image of $\operatorname{ord}_{\mathcal{D}}$ is the set $\{-9\pi/2, -5\pi/2, -\pi/2, 3\pi/2, 7\pi/2\}$. The lopsided coamoeba $\overline{\mathcal{D}}$ can be seen in Figure 3.2. We choose one point from each

Figure 3.2: $\overline{\mathcal{D}}$ in the fundamental domain $[-\pi, \pi]$.

connected component of its complement, namely

$$\theta_1 = -\frac{7\pi}{8}, \quad \theta_2 = -\frac{\pi}{2}, \quad \theta_3 = 0, \quad \theta_4 = \frac{5\pi}{16}, \quad \theta_5 = \frac{3\pi}{4},$$

and find that

$$\begin{array}{rcrcr} \operatorname{ord}_{\mathcal{D}}(\theta_1) & = & (0,-5\pi/8,\pi/8)B & = & 7\pi/2 \\ \operatorname{ord}_{\mathcal{D}}(\theta_2) & = & (0,\pi/2,0)B & = & -5\pi/2 \\ \operatorname{ord}_{\mathcal{D}}(\theta_3) & = & (0,0,\pi/2)B & = & 3\pi/2 \\ \operatorname{ord}_{\mathcal{D}}(\theta_4) & = & (0,15\pi/16,\pi/16)B & = & -9\pi/2 \\ \operatorname{ord}_{\mathcal{D}}(\theta_5) & = & (0,\pi/4,\pi/4)B & = & -\pi/2. \end{array}$$

Note that the orders does not reflect the (circular) ordering of the connected components of $\mathbf{T} \setminus \mathcal{C}$. ◯

3.2.1. Cayley configurations. The image of the order map $\mathrm{ord}_{\mathcal{D}}$ depends only on a Gale dual B of A. As Gale duals are invariant under general integer affine transformations, we expect also the number of connected components of the complement of a lopsided coamoeba to be invariant under such transformations. To formulate such a statement in a precise manner, we need to extend the definition of the order map to an arbitrary Cayley configuration. That is, we consider a polynomial

$$f(z) = f_1(z)\cdots f_q(z) \in \mathbb{C}_*^{A_1} \times \cdots \times \mathbb{C}_*^{A_q}.$$

We define the lopsided coamoeba of f by

$$\mathcal{D}_f = \bigcup_{i=1}^{q} \mathcal{D}_{f_i}.$$

To define the map $\mathrm{ord}_{\mathcal{D}}$, fix a point $\mathbf{a} = (\mathbf{a}_1, \ldots, \mathbf{a}_q) \in A_1 \times \cdots \times A_q$. For each $\mathbf{a}_i$ we obtain a vector valued function $p_{\mathbf{a}_i} : \mathbb{R}^n \to \mathbb{R}^{N_i}$, through which we define the function $p_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^N$ by

$$p_{\mathbf{a}}(\theta) = \big(p_{\mathbf{a}_1}(\theta), \ldots, p_{\mathbf{a}_q}(\theta)\big).$$

Finally, we define $\mathrm{ord}_{\mathcal{D}}$ as in (3.6).

Theorem 3.2.12. *Let A be a Cayley configuration, and assume that $\iota_A = 1$. Then, all claims of Theorem 3.2.1 holds.*

Proof. Compare the two Cayley equivalent configurations

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ A_1 & A_2 & \ldots & A_q \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ A_1 & A_2 & \cdots & A_q \end{pmatrix},$$

whose kernels coincide. The first configuration is associated to products $f(z)$, while the second configuration is associated to polynomials

$$F(z, w) = f_1(z) + w_1 f_2(z) + \cdots + w_{q-1} f_q(z).$$

We have that $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}_f$ if and only if there exist $\varphi_1, \ldots, \varphi_q$ such that $\hat{f}_i(\theta) \subset H_{\varphi_i}$. This is equivalent to that $\widehat{F}(\theta, \varphi_1 - \varphi_2, \ldots, \varphi_1 - \varphi_q) \subset H_{\varphi_1}$. Further more we have that $\mathrm{ord}_F(\theta, \varphi_1 - \varphi_2, \ldots, \varphi_1 - \varphi_q) = \mathrm{ord}_f(\theta)$, where ord_F and ord_f denotes order maps for $\mathcal{D}_F$ and $\mathcal{D}_f$ respectively. Thus, we have reduced to the case covered by Theorem 3.2.1. □

3.2.2. On the number of components of $\mathbf{T}^n \setminus \overline{\mathcal{C}}$. In this section we will discuss the current status of the conjectures regarding the number of connected components of the complement of a closed hypersurface coamoeba $\overline{\mathcal{C}}$. Our first result is based on the theory of Mellin–Barnes integrals, which we will touch upon in Section 5.1.2.

Corollary 3.2.13. *Let A be a Cayley configuration. Then, the number of connected components of the complement $\mathbf{T}^n \setminus \overline{\mathcal{D}}$ is at most* $\mathrm{Vol}(\mathcal{N})$.

Proof. It is enough to give the proof in the case when $\iota_A = 1$, so that $\mathrm{Vol}(\mathcal{N}) = \mathrm{Vol}(A)$. Choose a Gale dual B such that $\iota_B = 1$. It was proven in [6] that the set $\mathrm{int}(\mathcal{Z}_B) \cap (\mathrm{Arg}_\pi(f) + 2\pi\mathbb{Z}^m)$ is in a bijective correspondence with a family of, for sufficiently generic parameters β, linearly independent Mellin–Barnes hypergeometric integrals. As the rank of $\mathrm{Sol}_x(H_A(\beta))$ is equal to $\mathrm{Vol}(A)$ for generic parameters, the result follows. □

With Corollary 3.2.13 in mind, we suggest the following sharpened version of the $\mathrm{Vol}(\mathcal{N})$-bound on the number of connected components of the complement of a closed coamoeba.

Conjecture 3.2.14. Lat A be a Cayley configuration with $\iota_A = 1$. Then, the number of connected components of the complement $\mathbf{T}^n \setminus \overline{\mathcal{C}}$ is at most $\mathrm{Vol}(A)$.

The number of connected components of the complement of a closed lopsided coamoeba is invariant under integer affine transformations. Therefor it might, at a first glance, seem like the statement of Conjecture 3.2.14 is merely a reformulation of the $\mathrm{Vol}(\mathcal{N})$-bound. However, the number of connected components of the complement $\mathbf{T}^n \setminus \overline{\mathcal{C}}$ is not invariant under general integer affine transformations.

Example 3.2.15. Let $\mathcal{N}$ be the three-dimensional unit cube. That is,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We associate to A the family of generic multiaffine trivariate polynomials

$$f(z_1, z_2, z_3) = 1 + z_1 + z_2 + z_3 + x_3 z_1 z_2 + x_2 z_1 z_3 + x_1 z_2 z_3 + x_0 z_1 z_2 z_3.$$

In particular, we have that $\mathrm{Vol}(A) = 6$. Note that A is Cayley equivalent the point configuration associated to the product of two bivariate multiaffine polynomials

$$f_1(z_1, z_2) = 1 + z_1 + z_2 + x_3 z_1 z_2 \quad \text{and} \quad f_2(z_1, z_2) = 1 + x_2 z_1 + x_2 z_2 + x_0 z_1 z_2.$$

By examining the lopsided coamoeba of the product $f_1 f_2$, one can conclude that its complement has at most four connected components. Hence, the number of connected components of the complement of the closed lopsided coamoeba of f is at most four.

A computer aided computation performed in [15] showed that the complement $\mathbf{T}^3 \setminus \overline{\mathcal{C}}_f$ has at most four connected components. However, this is not the case for the product $f_1 f_2$, as seen in Figure 3.3. ⬡

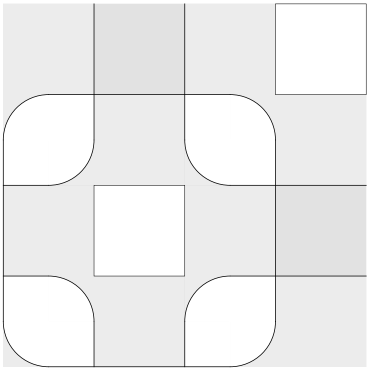

Figure 3.3: The coamoeba of $f(z_1, z_2) = (1 + z_1 + z_2 + i z_1 z_2)(1 - z_1 - z_2 + i z_1 z_2)$.

Let us now turn to the question of whether a lopsided coamoeba $\mathcal{D}$ with $\mathrm{Vol}(A)$-many connected components of its complement can be constructed.

Theorem 3.2.16. *Let A be a circuit. Then, the complement of the lopsided coamoeba $\overline{\mathcal{D}}$, and hence also the complement of the coamoeba $\overline{\mathcal{C}}$, has $\mathrm{Vol}(\mathcal{N})$-many connected components for generic coefficients.*

Proof. It is enough to give the proof under the assumption that $\iota_A = 1$, so that $\mathrm{Vol}(\mathcal{N}) = \mathrm{Vol}(A)$. Choose B such that $\iota_B = 1$. We noted in Example 2.1.5 that the zonotope $\mathcal{Z}_B$ is an interval of length $2\pi \mathrm{Vol}(A)$. Thus, the image of the order map $\mathrm{ord}_{\mathcal{D}}$ is of cardinality $\mathrm{Vol}(A)$ for generic coefficients. □

It was conjectured by Passare [29, Conjecture 8.1] that if A is maximally sparse, then the maximal number of connected components of the complement of the closed coamoeba is obtained for generic coefficients. In general this conjecture is false, with counterexamples given already in the text [29]. However, we can conclude that the conjecture is true in the following special case.

Corollary 3.2.17. *If the Newton polytope $\mathcal{N}$ has $n+2$ vertices, then the upper bound $\mathrm{Vol}(\mathcal{N})$ on the number of connected components of the complement of* *the coamoeba $\overline{\mathcal{C}}$ is obtained for maximally sparse polynomials with generic coefficients.*

Proof. Using the Theorem 3.2.16, it is enough to show that if f is maximally sparse, then $A = \operatorname{supp}(f)$ is a non-degenerate circuit. Indeed, as all points in A are vertices of $\mathcal{N}$, we find that any choice of $n+1$ points will span a simplex of full dimension, implying that the corresponding determinant is nonvanishing. □

3.3 The intersection theorem

The main theorem of [48], where the term "lopsidedness" was coined, is the existence of a sequence of polynomials $\{f_K\}_{K=0}^{\infty} \subset \langle f \rangle$ such that the lopsided amoebas of f_r converges to the amoeba $\mathcal{A}_f$ in Hausdorff distance as $r \to \infty$. In particular, we have that

$$\mathcal{A}_f = \bigcap_{g \in \langle f \rangle} \mathcal{L}_g.$$

In particular, the approximation of $\mathcal{A}_f$ by lopsidedness is fine enough for applications. This section is devoted to the corresponding problem for coamoebas. Our main theorem is the following.

Theorem 3.3.1. *Let $\mathcal{N}$ be a simplex. Then, for any $f \in \mathbb{C}_*^A$, it holds that*

$$\overline{\mathcal{C}}_f = \bigcap_{g \in \langle f \rangle} \overline{\mathcal{D}}_g.$$

In our proof of Theorem 3.3.1 we will, for each $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{C}}_f$, construct an explicit sequence $\{g_K^\theta\}_{K=0}^{\infty} \subset \langle f \rangle$ such that g_K^θ is colopsided at θ for K sufficiently large.

Let us consider a homogenous real polynomial

$$F(z) = \sum_{\mathbf{a} \in A} x_{\mathbf{a}} \frac{z^{\mathbf{a}}}{\mathbf{a}!}$$

of degree d and such that $\mathcal{N}_F$ is a dilation (by a factor of d) of the standard n-simplex in $\mathbb{R}^{1+n}$. Let $L \subset \mathbb{Z}^{1+n}$ be a lattice, and define the monoid

$$L^+ = L \cap \mathbb{R}_{\geq 0}^{1+n}.$$

The homogenous part of L^+ of degree k is, by definition,

$$L_k^+ = \{\beta \in L^+ \,|\, |\beta| = k\}.$$

Theorem 3.3.2. *Let $F(z)$ be a homogenous real polynomial whose Newton polytope $\mathcal{N}$ is a dilation of the standard n-simplex. Let A and d denote the support and degree of F respectively. Assume that $F(z)$ is positive on $z \geq 0$ and $\sum z > 0$. Let L by any lattice containing L_A, and define the polynomials*

$$G_k(z) = (dk)! \sum_{\beta \in L^+_{dk}} \frac{z^\beta}{\beta!} = \sum_{\beta \in L^+_{dk}} \binom{dk}{\beta} z^\beta.$$

Then, there exists a positive integer K such that, for each $k \geq K$, all non-zero coefficients of the polynomial $G_K(z)F(z)$ are positive.

If we choose L as the full lattice $L = \mathbb{Z}^{n+1}$, then $G_k(z)$ is the dk-fold product $(z_0 + z_1 + \cdots + z_n)^{dk}$. In this version, Theorem 3.3.2 is due to Pólya, with roots tracing back to Poincaré and Meissner. Our proof of Theorem 3.3.2 is based on Pólyas proof, as presented in [23].

Proof of Theorem 3.3.2. The region $z \geq 0$ and $\sum z = d$, coincides with the Newton polytope $\mathcal{N}$. By assumption, F is positive and continuous on this region. Thus, since $\mathcal{N}$ is compact, the polynomial F obtains its minimum μ, which is positive, in this region. For $t > 0$, consider the function

$$\varphi(z;t) = t^d \sum_{\mathbf{a} \in A} x_{\mathbf{a}} \prod_{j=1}^{n} \binom{z_j t^{-1}}{\mathbf{a}_j},$$

where $\binom{z_j t^{-1}}{\mathbf{a}_j}$ denotes the generalized binomial coefficient. That is,

$$\binom{z_j t^{-1}}{0} = 1 \quad \text{and} \quad t^{\mathbf{a}_j}\binom{z_j t^{-1}}{\mathbf{a}_j} = \frac{z_j(z_j - t)\cdots(z_j - (\mathbf{a}_j - 1)t)}{\mathbf{a}_j!}.$$

Notice that if $\gamma = (\gamma_1, \dots, \gamma_n)$, with $|\gamma| = dk$, then $\gamma/k = (\gamma_1/k, \dots, \gamma_n/k)$ is contained in $\mathcal{N}$, and

$$k^d \varphi\left(\frac{\gamma}{k}; \frac{1}{k}\right) = \sum_{\mathbf{a} \in A} x_{\mathbf{a}} \binom{\gamma}{\mathbf{a}}.$$

We have that $\varphi(z,t) \to F(z)$ as $t \to 0$. Hence, writing $\varphi(z;0) = F(z)$, the function $\varphi(z;t)$ is continuous in the region given by $z \in \mathcal{N}$ and $0 \leq t \leq 1$. Therefor, there is an ε such that $\varphi(z;t)$ is strictly positive for each $z \in \mathcal{N}$ and $0 \leq t < \varepsilon$.

Consider the product

$$\frac{G_{k-1}(z)F(z)}{(d(k-1))!} = \sum_{\beta \in L^+_{d(k-1)}} \sum_{\mathbf{a} \in A} x_{\mathbf{a}} \frac{z^{\mathbf{a}+\beta}}{\mathbf{a}!\,\beta!}.$$

Write $\gamma = \mathbf{a} + \beta$, so that $\gamma \in L^+_{kd}$. We find that

$$\frac{G_{k-1}(z)F(z)}{(d(k-1))!} = \sum_{\gamma \in L^+_{kd}} \sum_{\mathbf{a} \in A(\gamma)} x_{\mathbf{a}} \frac{z^\gamma}{(\gamma - \mathbf{a})!\mathbf{a}!} = \sum_{\gamma \in L^+_{kd}} \frac{z^\gamma}{\gamma!} \sum_{\mathbf{a} \in A(\gamma)} x_{\mathbf{a}} \binom{\gamma}{\mathbf{a}}. \tag{3.14}$$

The index set $A(\gamma)$ consist of all $\mathbf{a} \in A$ such that $\gamma - \mathbf{a} \in L^+_{d(k-1)}$. Since L contains L_A, we have that $\gamma - \mathbf{a} \in L_{d(k-1)}$. Hence, we will only have that $\gamma - \mathbf{a} \notin L^+_{d(k-1)}$ if one component of $\gamma - \mathbf{a}$ is negative. If so, then the binomial coefficient $\binom{\gamma}{\mathbf{a}}$ vanishes. Hence, the above expression is unaltered if the second sum is taken over A instead of $A(\gamma)$. We conclude that

$$\frac{G_{k-1}(z)F(z)}{(d(k-1))!} = \sum_{\gamma \in L^+_{kd}} \frac{z^\gamma}{\gamma!} \sum_{\mathbf{a} \in A} x_{\mathbf{a}} \binom{\gamma}{\mathbf{a}} = \sum_{\gamma \in L^+_{kd}} \frac{z^\gamma}{\gamma!} \varphi\left(\frac{\gamma}{k}; \frac{1}{k}\right),$$

where the coefficients of the polynomial in the right hand side are positive if $1/k < \varepsilon$. In particular, we can choose K as any integer greater than $1/\varepsilon$. □

Proof of Theorem 3.3.1. Let us first assume that $\mathcal{N}_A$ is a dilation of the standard n-simplex in $\mathbb{R}^n$. Let $f \in \mathbb{C}^A_*$, and let $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{C}}_f$. By (2.10) we find that f is nonvanishing on the closure of $\operatorname{Arg}^{-1}(\theta)$. Consider the polynomial $f_\theta(z) = f(e^{i\theta}z)$, which is nonvanishing on the closed positive orthant. Let $\bar{f}_\theta(z)$ denote the polynomial whose coefficients are the conjugates of those of $f_\theta(z)$. It follows that the real polynomial

$$\hat{f}_\theta(z) = \bar{f}_\theta(z) f_\theta(z)$$

is positive on the closed positive orthant. Notice that $\hat{f}_\theta(z)$ has support in L_A. Let $F_\theta(z)$ denote the homogenization of $\hat{f}_\theta(z)$, that is

$$F_\theta(z) = z_0^{2d} \hat{f}_\theta\left(\frac{z}{z_0}\right).$$

Then $F_\theta(z)$ fulfills the requirement of Theorem 3.3.2. In particular, we find a polynomial $G_k(z)$ such that $G_k(z)F_\theta(z)$ has positive coefficients. Let $g_k(z)$ be the dehomogenization of $G_k(z)$, obtained by setting $z_0 = 1$. It follows that all non-vanishing coefficients of $g_k(z)\hat{f}_\theta(z)$ are positive. We conclude that

$$g_k(e^{-i\theta}z)\bar{f}_\theta(e^{-i\theta}z)f(z) \in \langle f \rangle$$

is colopsided at θ, proving the theorem in this case. Notice that we can choose L so that $g_k(z)$ has support in L_A.

Now consider the case when $\mathcal{N}$ is an arbitrary simplex. Let $f \in \mathbb{C}^A_*$ and $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{C}}_f$. Let T be an integer affine transformation mapping A to a point configuration A', the latter being a dilation of the standard n-simplex in $\mathbb{R}^n$. Let $f' = T(f)$ denote the image of f in $\mathbb{C}_*^{A'}$, and let $\theta' \in \mathbf{T}^n \setminus \overline{\mathcal{C}}_{f'}$ denote the corresponding image of θ. By the previous case we find a polynomial $g'(z)$, with support in $L_{A'}$, such that $g'(z)f'(z)$ is colopsided at θ'. Since $g'(z)$ has support in $L_{A'}$ there is a polynomial $g(z)$ with support in L_A such that $g'(z) = T(g(z))$. It follows that the polynomial $g(z)f(z)$ is colopsided at θ. □

3.4 Hyperfields

Multigroups and hyperfields have appeared on several occasions throughout history, but seem to be easily forgotten. In the context of tropical geometry, they were introduced by Viro in [57], in an effort to put tropical geometry on firm algebra-geometric foundations. We will not here give an introduction to this rather unknown field, but we refer instead to Viro's survey [57] and the references therein. We will relate the (co)lopsidedness criterion to the theory of hyperfields, for two reasons. Firstly, we claim that (co)lopsidedness is tropical geometry. Secondly, this is the setting in which lopsidedness can be generalized to arbitrary fields.

Definition 3.4.1. A pair $(X, \oplus)$ of a set X and a multivalued binary operation $\oplus$ on X is said to be a commutative *multigroup* if

i) the operation $\oplus$ is associative and commutative,

ii) X contains an element 0 such that $0 \oplus x = x \oplus 0 = x$ for all $x \in X$, and

iii) for each $x \in X$ there is a unique element $-x \in X$ such that $0 \in x \oplus (-x)$.

Definition 3.4.2. A map $\varphi\colon X_1 \to X_2$ between multigroups is said to be a homomorphism if $\varphi(0) = 0$ and $\varphi(x_1 \oplus x_2) \subset \varphi(x_1) \oplus \varphi(x_2)$.

The following three remarks are meant to emphasize some difficulties encountered when working with multigroups. We refer the reader to [57] for details.

Remark 3.4.3. Let $(X, \oplus_X)$ be a multigroup, and let Y be a subset of X such that, firstly, $y \in Y$ if and only if $-y \in Y$ and, secondly, $y_1 \oplus_X y_2$ meets Y for all y_1 and y_2. Consider the multivalued binary operation on Y defined by

$$y_1 \oplus_Y y_2 = (y_1 \oplus_X y_2) \cap Y.$$

The multigroup $(Y, \oplus_Y)$ is said to be a submultigroup of $(X, \oplus_X)$. It is important to note that, in difference to groups, the sum $y_1 \oplus y_2$ depend on whether we consider the operation to be $\oplus_Y$ or $\oplus_X$. In particular, that we require in Definition 3.4.2 inclusion instead of equality, is motivated by that we wish the inclusion map of Y into X to be a multigroup homomorphism. ○

Remark 3.4.4. Even though, in a multigroup, we have the notion of inverse, we cannot cancel terms of an identity by adding to an element its inverse. In particular, if Y is a subset of a multigroup X, then the condition that $x \oplus Y = Y \oplus x$ for all $x \in X$ is not equivalent to that $x \oplus Y \oplus (-x) = Y$ for all $x \in X$. Thus, we get two distinct notions of normal subgroups. ⎔

Remark 3.4.5. A multigroup homomorphism φ can have trivial kernel even if φ is not injective. For example, the sign function $\operatorname{sgn}\colon \mathbb{R} \to \{-1, 0, 1\}$ is a multigroup homomorphism from the reals to the *sign multigroup.* ⎔

Definition 3.4.6. A triple $(X, \oplus, \otimes)$ of a set X, a multivalued binary operation $\oplus$ on X, and a binary operation $\otimes$ on X, is said to be a *hyperfield* if

i) $(X, \oplus)$ is a commutative multigroup with identity 0,

ii) $(X_*, \otimes)$ is a commutative group with identity 1,

iii) $0 \otimes x = 0$ for all $x \in X$,

iv) $\otimes$ distributes over $\oplus$ in the sense that $x \otimes (y \oplus z) \subset (x \otimes y) \oplus (x \otimes z)$ for all $x, y, z \in X$.

Example 3.4.7. The real tropical hyperfield is the set $\mathbb{R} \cup \{-\infty\}$ with the multiplication $x_1 \otimes x_2 = x_1 + x_2$ and the addition

$$x_1 \oplus x_2 = \begin{cases} \max(x_1, x_2) & \text{if } x_1 \neq x_2 \\ [-\infty, x_1] & \text{if } x_1 = x_2. \end{cases}$$

We will now describe how hyperfields naturally arise from group homomorphisms of the multiplicative subgroup of a field, e.g., from projections of the complex torus $\mathbb{C}_*$.

Theorem 3.4.8 (Krasner[1])**.** *Let $\Bbbk$ be a field, and let $\varphi\colon \Bbbk_* \to H$ be an epimorphism of multiplicative groups. Extend φ to a surjection $\widehat{\varphi}\colon \Bbbk \to \widehat{H}$, where $\widehat{H} = H \sqcup \{0\}$. Extend the multiplication in H to $\widehat{H}$ by $0 \otimes h = h \otimes 0 = 0$ for all h. Then $\widehat{H}$ is a hyperfield with the addition*

$$h_1 \oplus h_2 = \widehat{\varphi}\left(\widehat{\varphi}^{-1}(h_1) + \widehat{\varphi}^{-1}(h_2)\right).$$

Proof. Since $\widehat{\varphi}$ is surjective, the operation $\oplus$ is well-defined. Associativity and commutativity follows from the corresponding properties of the field $\Bbbk$. The additive unit is 0. Furthermore, if $h = \widehat{\varphi}(x)$ then $-h = \widehat{\varphi}(-x)$ fulfills that $0 \in h \oplus (-h)$, and this inverse is clearly unique. Note also that $\widehat{\varphi}$ is a multigroup homomorphism from $\Bbbk$ to $\widehat{H}$.

[1] According to O. Viro, this construction is due to M. Krasner.

It remains to prove only that $\otimes$ distributes over $\oplus$. Let $x_k \in \widehat{\varphi}^{-1}(h_k)$ for $k = 1, 2, 3$. It suffices to show that

$$\widehat{\varphi}(x_1) \otimes \widehat{\varphi}(x_2 + x_3) \subset \left(\widehat{\varphi}(x_1) \otimes \widehat{\varphi}(x_2)\right) \oplus \left(\widehat{\varphi}(x_1) \otimes \widehat{\varphi}(x_3)\right), \tag{3.15}$$

for any choices of x_k. If $x_k = 0$ for some k, then equality holds in (3.15). If $x_2 = -x_3$, then the left hand side of (3.15) equals 0, while the right hand side evaluates to

$$\varphi(x_1 x_2) \oplus \varphi(x_1 x_3) = \varphi(x_1 x_2) \oplus \varphi(-x_1 x_2) \ni 0.$$

Finally, if also $x_2 + x_3 \neq 0$, then

$$\widehat{\varphi}(x_1) \otimes \widehat{\varphi}(x_2 + x_3) = \varphi(x_1(x_2 + x_3)) \subset \varphi(x_1 x_2) \oplus \varphi(x_1 x_3),$$

since $\widehat{\varphi}$ is a multigroup homomorphism. As the right hand side in the last display is equal to the right hand side of (3.15), the proof is finished. $\square$

Example 3.4.9. The *ultratriangle hyperfield* is the set $\mathbb{R} \cup \{-\infty\}$ with the multiplication $\otimes$ given by usual addition, and the addition $\oplus$ given by

$$h_1 \oplus h_2 = \left[\log|e^{h_1} - e^{h_2}|, \log|e^{h_1} + e^{h_2}|\right].$$

It is the hyperfield obtained from $\mathbb{C}_*$ by the epimorphism $\log\colon \mathbb{C}_* \to \mathbb{R}$. $\bigcirc$

Example 3.4.10. The *phase hyperfield* is the set $S^1 \sqcup \{0\} \subset \mathbb{C}$ with multiplication $\otimes$ being usual complex multiplication, and the addition $\oplus$ being

$$h_1 \oplus h_2 = \begin{cases} h_1 & \text{if } h_1 = h_2 \text{ or } h_2 = 0, \\ \{h_1, 0, -h_1\} & \text{if } h_1 = -h_2, \\ \text{the smallest open arc from } h_1 \text{ to } h_2 & \text{otherwise.} \end{cases}.$$

It is the hyperfield obtained from $\mathbb{C}_*$ by the epimorphism $\arg\colon \mathbb{C}_* \to \mathbf{T}$.

Let H be a hyperfield, and let H_*^A be a torus of polynomial functions on H^n with support A. Further, let $\varphi\colon H \to X$ be a hyperfield homomorphism. We denote by φ also the induced map of H-modules (in the hyperfield sense) $\varphi\colon H^n \to X^n$ and $\varphi\colon H^A \to X^A$ acting componentwise by φ. The following theorem follows immediately from the definitions.

Theorem 3.4.11. *The lopsided amoeba $\mathcal{L}$ is the tropical variety of* $\mathrm{Log}(f)$ *in the ultratriangle hyperfield. The lopsided coamoeba $\mathcal{D}$ is the tropical variety of* $\mathrm{Arg}(f)$ *in the phase hyperfield.*

Let $\varphi\colon H \to X$ be a hyperfield morphism. Furthermore, let $f \in H^A$ and let $z \in H^n$. Then,

$$\varphi\big(f(z)\big) \subset \varphi(f)\big(\varphi(z)\big),$$

where $\varphi(f) \in X_A$. In particular,

$$\varphi\big(Z_H(f)\big) \subset Z_X(\varphi(f)).$$

We conclude that $\mathcal{A} \subset \mathcal{L}$ and that $\mathcal{C} \subset \mathcal{D}$.

Remark 3.4.12. There is no hyperfield morphism between the complex numbers and the real tropical hyperfield. However, the identity map on $\mathbb{R} \cup \{-\infty\}$ is a hyperfield morphism from the real tropical hyperfield to the ultratriangle hyperfield. We will investigate further this link between classical tropical geometry and lopsidedness in Chapter 6.

Let us end with a construction that allows us to recover the tropical semifield from the real tropical hyperfield.

Definition 3.4.13. Let x denote a finite sum $x_1 \oplus \cdots \oplus x_m$. The set

$$[x] := \bigcup_{n=1}^{\infty} n(x \oplus (-x)), \tag{3.16}$$

where

$$ny = \underbrace{y \oplus \cdots \oplus y}_{n \text{ times}},$$

is called the *inverter* of x. The set of all inverters, as x runs over all finite sums of elements in X, is called the inverter semifield of X, denoted by $\mathrm{Inv}(X)$. An element $x \in X$ such that $[x] = 0$ is known as a *scalar*, see [41], and the set of all scalars in X will be denoted by $\mathrm{Scal}(X)$.

By abuse of notation, we will use $[x]$ to denote both the subset of X defined by (3.16) and the corresponding element of $\mathrm{Inv}(X)$. Inverters are similar to commutators of non-commutative algebra; a hyperfield is a field if and only if $\mathrm{Scal}(X) = X$.

Lemma 3.4.14. *We have that* $[x] = [x \oplus x] = [x \oplus (-x)]$.

Proof. As X is commutative, $[x \oplus x] = [x \oplus (-x)]$. Both of these sets consists of the union (3.16) taken only over even numbers n. Thus the result follows from that $n(x \oplus (-x)) \subset (n+1)(x \oplus (-x))$. □

In many cases, however not for the ultratriangle hyperfield, it holds that $x \oplus x \oplus (-x) \oplus (-x) = x \oplus (-x)$, which is equivalent to that $[x] = x \oplus (-x)$. Let us now motivate the name "inverter semifield" used above.

Theorem 3.4.15. *The addition* $[x]+[y]=[x\oplus y]$ *and the multiplication* $[x]\times[y]=[x\otimes y]$ *defines a semifield structure on* $\mathrm{Inv}(X)$.

Proof. Let us first prove that the addition is well-defined. That is, that $[y]=[z]$ implies that $[x\oplus y]=[x\oplus z]$, as subsets of X. Let $a\in[x\oplus y]$. Then $a\in n(x\oplus y\oplus(-x)\oplus(-y))=n(x\oplus(-x))\oplus n(y\oplus(-y))$. That is, there exists $a_x\in n(x\oplus(-x))$ and $a_y\in n(y\oplus(-y))$ such that $a\in a_x\oplus a_y$. As $[y]=[z]$, and $a_y\in[y]$, we find that there exist an $m\in\mathbb{N}_+$ such that $a_y\in m(z\oplus\bar{z})$. Hence,

$$a\in a_x\oplus a_y\subset n(x\oplus(-x))\oplus m(z\oplus\bar{z})\subset\max(n,m)(x\oplus(-x)\oplus z\oplus\bar{z})\subset[x\oplus z].$$

Thus, $[x\oplus y]\subset[x\oplus z]$. The other inclusion follows by symmetry. The identity element is for addition is $[0]$.

To see that the multiplication is well-defined, we note that if $[y]=[z]$, then, as subsets of X,

$$[x]\times[y]=[x\otimes y]=x\otimes[y]=x\otimes[z]=[x\otimes z]=[x]\times[z].$$

The multiplicative identity is $[1]$ and the multiplicative inverse of $[x]$ is $[x^{-1}]$. The distributive law $[x]\times([y]+[z])=[x]\times[y]+[x]\times[z]$ is straightforward to check from the definitions. □

It is clear that $\mathrm{Inv}(X)$ is not a group; as $0\in[y]$ for each y, we find that $[x]\subset[x\oplus y]$ for every x and y. Hence, no nonzero element of $\mathrm{Inv}(X)$ has an additive inverse. The set $\mathrm{Inv}(X)$ has a natural partial order, defined by that $[x]\le[y]$ if and only if $[x\oplus y]\subset[y]$, which is equivalent to that $[x\oplus y]=[y]$. With this ordering, the inverter $[0]$ is a global minimum.

Theorem 3.4.16. *Assume that* $x\in[x]$ *for each* $x\in X$*, then each hyperfield homomorphism* $\varphi\colon X\to Y$ *induces a homomorphism* $\mathrm{Inv}(\varphi)\colon\mathrm{Inv}(X)\to\mathrm{Inv}(Y)$ *of semifields, defined as*

$$\mathrm{Inv}(\varphi)([x])=[\varphi(x)].$$

Proof. We note first that $\varphi([x])\subset[\varphi(x)]$ as subsets of Y. To see that the given map is well-defined, note that if $[x]=[z]$, then $x\in[x]=[z]$ by assumption. Hence, $\varphi(x)\subset\varphi([z])\subset[\varphi(z)]$. As each inverter is a strong submultigroup, we find that $[\varphi(x)]\subset[\varphi(z)]$. (A strong submultigroup is a submultigroup such that, in the notation of Remark 3.4.3, $\oplus_Y=\oplus_X$, see [57].) The reverse inclusion follows by symmetry.

To see that $\mathrm{Inv}(\varphi)$ is a homomorphism of semifields, note first that

$$\mathrm{Inv}(\varphi)([x]+[y])=[\varphi(x\oplus y)]\subset[\varphi(x)\oplus\varphi(y)]=\mathrm{Inv}(\varphi)([x])+\mathrm{Inv}(\varphi)([y]).\quad(3.17)$$

On the other hand, as $x\in[x]\subset[x\oplus y]$, we find that $\varphi(x)\in\varphi([x\oplus y])\subset[\varphi(x\oplus y)]$. Using once more that each inverter is a strong submultigroup, we find that $[\varphi(x) \oplus \varphi(y)] \subset [\varphi(x \oplus y)]$. Thus the inclusion in (3.17) is an equality. It remains to show that $\mathrm{Inv}(\varphi)$ is compatible with the multiplication. We find that

$$\begin{aligned}\mathrm{Inv}(\varphi)([x] \times [y]) &= \mathrm{Inv}(\varphi)([x \otimes y]) = [\varphi(x \otimes y)] \\ &= [\varphi(x) \otimes \varphi(y)] = \mathrm{Inv}(\varphi)([x]) \times \mathrm{Inv}(\varphi)([y]).\end{aligned} \qquad \square$$

4. Coamoebas of polynomials supported on a circuit

In this chapter we study coamoebas of polynomials supported on a circuit A. Our main tool is the order map $\mathrm{ord}_{\mathcal{D}}$ from the previous chapter. As seen in Theorem 3.2.16, it is particularly suitable when A is a circuit. We will index the points of A by $k = 0, \dots, n+1$. Furthermore, we partition the set of circuits into two classes; *simplex circuits*, for which $\mathcal{N}$ is a simplex, and *vertex circuits*, for which $A = \mathrm{vert}(\mathcal{N})$.

4.1 Real polynomials and the A-discriminant

We will say that f is *real at* θ if there is a real subvector space $\ell \subset \mathbb{C}$ such that $\hat{f}_{\mathbf{a}}(\theta) \in \ell$ for all $\mathbf{a} \in A$. (Here, $\hat{f}$ is the function from Section 2.2.) If such a θ exists, then f is real. That is, after a change of variables and multiplication with a Laurent monomial, $f \in \mathbb{R}^A_*$. Our first lemma is valid for any polynomial f. However, we will only use it in the case when A is a circuit.

Lemma 4.1.1. *Assume that the polynomial f is real at $\theta_0 \in \mathbb{R}^n$. Then, f is real at $\theta \in \mathbb{R}^n$ if and only if $\theta \in \theta_0 + \pi L_A^*$, where L_A^* is the dual lattice of L_A.*

Proof. Applying an integer affine transformation, we can assume that $\mathbf{0} \in A$. Multiplying f with a constant we can assume that $\ell_0 = \mathbb{R}$, and that the constant monomial has coefficient equal to 1. Furthermore, by a translation in θ we can assume that $\theta_0 = 0$. That is, all coefficients of f are real, in particular proving *if*-part of the statement. To show the *only if*-part, notice first that $\hat{f}(\theta) \subset \ell$ implies that ℓ contains both the origin and 1. That is, $\ell = \mathbb{R}$. Furthermore, $\hat{f}(\theta) \subset \mathbb{R}$ only if for each $\mathbf{a} \in A$ there is a $k \in \mathbb{Z}$ such that $\langle \mathbf{a}, \theta \rangle = \pi k$, which concludes the proof. □

The following characterization of the coamoeba of the A-discriminant of a circuit is an immediate consequence of the Horn–Kapranov parametrization. Our proof is instead based on properties of the function $\hat{f}(\theta)$. As it is more cumbersome, we remark that the lemmas of this section are of equal importance.

Proposition 4.1.2. *Let A be a nondegenerate circuit, and let δ_k be as in* (2.9)*. Then,* $\operatorname{Arg}(f) \in \mathcal{C}_\Delta$ *if and only if, after possibly multiplying f with a constant, there is a point $\theta \in \mathbb{R}^n$ such that $\hat{f}_k(\theta) = \delta_k$ for all k.*

The A-discriminant Δ of a nondegenerate circuit A has been described in [19, Proposition 9.1.8], where they obtained the formula

$$\Delta(f) = \prod_{\delta_k=1} \mathbf{b}_k^{\mathbf{b}_k} \prod_{\delta_k=-1} x_k^{-\mathbf{b}_k} - \prod_{\delta_k=-1} \mathbf{b}_k^{-\mathbf{b}_k} \prod_{\delta_k=1} x_k^{\mathbf{b}_k}. \tag{4.1}$$

In particular, Δ is a binomial. We see from the proof of Theorem 3.2.16 that if the complement $\mathbf{T}^n \setminus \overline{\mathcal{C}}$ does not have $\operatorname{Vol}(A)$-many connected components then

$$\operatorname{Arg}_\pi(f)B \equiv 2\pi \operatorname{Vol}(A) \mod 2\pi. \tag{4.2}$$

Lemma 4.1.3. *Let A_κ be as in Example 2.1.5. For each $\kappa = 0, 1, \ldots, n+1$, there are exactly* $\operatorname{Vol}(A_\kappa)$*-many points $\theta \in \mathbf{T}$ such that*

$$\hat{f}_k(\theta) = \delta_k, \quad \forall\, k \neq \kappa. \tag{4.3}$$

Proof. After applying an integer affine transformation we reduce to the case when A_κ consists of the vertices of the standard simplex, a case in which the statement is obvious. □

Lemma 4.1.4. *Fix $\kappa \in \{0, \ldots, n+1\}$. For each θ fulfilling* (4.3)*, let $\varphi_\theta \in \mathbf{T}$ be defined by the criterion that if* $\arg_\pi(x_\kappa) = \varphi_\theta$ *then*

$$\hat{f}_\kappa(\theta) = \delta_\kappa. \tag{4.4}$$

Assume that $L_A = \mathbb{Z}^n$. Then, the numbers φ_θ are distinct.

Proof. We can assume that $\mathbf{a}_0 = \mathbf{0}$ and that $x_0 = 1$. Assume that $\varphi_{\theta_1} = \varphi_{\theta_2}$. Then,

$$\langle \mathbf{a}, \theta_2 \rangle = \langle \mathbf{a}, \theta_1 \rangle + 2\pi r, \quad \forall \mathbf{a} \in A.$$

After a translation in θ, we can assume that $\theta_1 = 0$. Hence, since 1 is a monomial of f, all coefficients are real. Consider the lattice L consisting of all points $\theta \in \mathbb{R}^n$ such that f is real at θ. Since $L_A = \mathbb{Z}^n$, Lemma 4.1.1 implies that $L = \pi\mathbb{Z}^n$. However, we find that

$$\left\langle \mathbf{a}, \frac{\theta_2}{2} \right\rangle = \pi r,$$

and hence $\frac{\theta_2}{2} \in L$. This implies that $\theta_2 \in 2\pi\mathbb{Z}^n$, and hence $\theta_2 = 0$ in $\mathbf{T}^n$. □

Proof of Proposition 4.1.2. Assume first that there is a θ as in the statement of the proposition, where we can assume that $\theta = 0$. Then, $\arg(x_k) = \arg(\delta_k)$. It follows that the monomials

$$\prod_{\delta_k=1} \mathbf{b}_k^{\mathbf{b}_k} \prod_{\delta_k=-1} x_k^{-\mathbf{b}_k} \quad \text{and} \quad \prod_{\delta_k=-1} \mathbf{b}_k^{\mathbf{b}_k} \prod_{\delta_k=1} x_k^{-\mathbf{b}_k}$$

are equal in sign. We find that Δ vanishes for $x_k = \delta_k|\mathbf{b}_k|$, implying that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$.

Conversely, fix an index κ, and reduce f to a univariate polynomial by setting $x_k = \delta_k|\mathbf{b}_k| = \mathbf{b}_k$ for $k \neq \kappa$. Let $\mathcal{I}$ denote the set of points $\theta \in \mathbf{T}^n$ such that $\hat{f}_k(\theta) = \delta_k$ for $k \neq \kappa$, which by Lemma 4.1.3 has cardinality V_κ. By Lemma 4.1.4, the set $\mathcal{I}$ is in a bijective correspondence with values of $\arg(x_\kappa)$ such that $\hat{f}_\kappa(\theta) = \delta_\kappa$. Therefor, we find that Δ vanishes at $x_\kappa = V_\kappa e^{i\varphi}$ for each $\varphi \in \mathcal{I}$. However, the discriminant Δ specializes, up to a constant, to the binomial

$$\Delta_\kappa(x_\kappa) = x_\kappa^{|\mathbf{b}_\kappa|} - \mathbf{b}_\kappa^{|\mathbf{b}_\kappa|} = x_\kappa^{V_\kappa} - \mathbf{b}_\kappa^{V_\kappa},$$

which has exactly V_κ-many solutions in $\mathbb{C}_*$ of distinct arguments. Hence, since $\Delta(f) = 0$ by assumption, and comparing the number of solutions, it holds that $\hat{f}_\kappa(\theta) = \delta_\kappa$ for one of the points $\theta \in \mathcal{I}$. □

4.2 The space of coamoebas

Let $U_k \subset \mathbb{C}_*^A$ denote the set of all f such that the number of connected components of the complement of $\overline{\mathcal{C}}$ is $\mathrm{Vol}(A) - k$. Describing the sets U_k is known as the problem of describing the *space of coamoebas* of $\mathbb{C}_*^A$. In this section, we will give explicit descriptions of the sets U_k in the case when A is a circuit. As a first observation we note that the cardinality of the image of the map $\mathrm{ord}_\mathcal{D}$ is at least $\mathrm{Vol}(A) - 1$, implying that

$$\mathbb{C}_*^A = U_0 \cup U_1,$$

and in particular $U_k = \emptyset$ for $k \geq 2$. Hence, it suffices for us to give an explicit description of the set U_1. Our main results are the following two theorems, which elucidate the difference between vertex circuits and simplex circuits.

Theorem 4.2.1. *Assume that A is a nondegenerate simplex circuit, with $\mathbf{a}_{n+1}$ as an interior point. Choose B such that $\delta_{n+1} = -1$, and let Δ be as in* (4.1)*. Then, $f \in U_1$ if and only if $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$ and*

$$(-1)^{\mathrm{Vol}(A)} \Delta\big(\delta_0|x_0|, \ldots, \delta_{n+1}|x_{n+1}|\big) \leq 0. \tag{4.5}$$

Theorem 4.2.2. *Assume that A is a vertex circuit. Then, $f \in U_1$ if and only if* $\operatorname{Arg}(f) \in \mathcal{C}_\Delta$.

The article [54] describes the *space of amoebas* in the case when A is a simplex circuit in dimension at least two. In this case, the number of connected components of the amoeba complement is either equal to the number of vertices of $\mathcal{N}$ or one greater. One implication of [54, Theorems 4.4 and 5.4] is that, if the amoeba complement has the minimal number of connected components, then

$$(-1)^{\operatorname{Vol}(A)}\,\Delta\big(\delta_0|x_0|,\ldots,\delta_{n+1}|x_{n+1}|\big) \geq 0.$$

Furthermore, this set intersect U_1 only in the discriminant locus $\Delta(f) = 0$. The space of amoebas in the case when A is a simplex circuit in dimension $n = 1$ has been studied in [55], and is a more delicate problem. On the other hand, if A is a vertex circuit then the fact that maximally sparse polynomials have solid amoebas implies that the number of connected components of the amoeba complement does not depend on f. From Theorems 4.2.1 and 4.2.2 we see that a similar discrepancy between simplex circuits and vertex circuits occur for coamoebas.

Example 4.2.3. The reduced family

$$f(z) = 1 + z_1^3 + z_2^3 + \xi\, z_1 z_2.$$

was considered in [50, Example I.6], where the study of the space of amoebas was initiated. We have drawn the space of amoebas and coamoebas jointly in the left picture in Figure 4.1. The light gray region, whose boundary is a hypocycloid, marks values of ξ for which the amoeba complement has no bounded component. The set U_1 is seen in dark gray. The black dots is the discriminant locus $\Delta(\xi) = 0$, which is contained in the circle $|\xi| = 3$ corresponding an equality in (4.5). ⬡

Example 4.2.4. The reduced family

$$f(z) = 1 + z_1 + z_2^3 + \xi\, z_1^3 z_2$$

is a vertex circuit. In this case, the topology of the amoeba does not depend on the coefficient ξ. The space of coamoebas is drawn in the right picture in Figure 4.1. The set U_1 comprises the three dark gray lines emerging from the origin. The black dots is the discriminant locus $\Delta(\xi) = 0$. It might seem like the set U_0 is disconnected, however this a consequence of that we consider f in reduced form. In $\mathbb{C}_*^A$ the set U_0 is connected, though not simply connected. ⬡

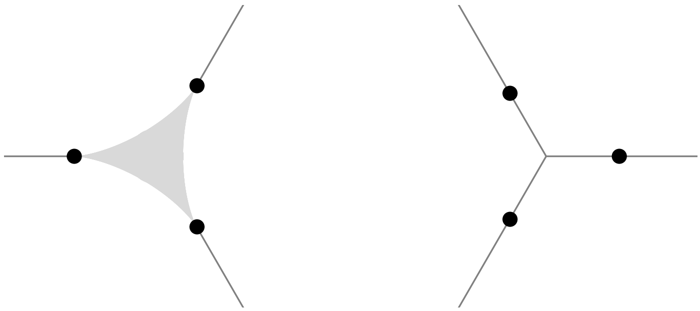

Figure 4.1: The amoeba and coamoeba spaces of Examples 4.2.3 and 4.2.4.

If A is a simplex circuit, with $\mathbf{a}_{n+1}$ as an interior point, then

$$\mathbf{b}_0 + \cdots + \mathbf{b}_n = -\mathbf{b}_{n+1} = \mathrm{Vol}(A),$$

and in particular $V_{n+1} = \mathrm{Vol}(A)$. By Lemma 4.1.3 there is a set $\mathcal{I}$ of cardinality $\mathrm{Vol}(A)$ consisting of all points θ such that $\hat{f}_0(\theta) = \cdots = \hat{f}_n(\theta) = \delta_k = 1$ In particular, f is colopsided at $\theta \in \mathcal{I}$ unless $\hat{f}_{n+1}(\theta) = -1$. We claim here that $\mathcal{I}$ is an *index set* of the complement of $\overline{\mathcal{C}}$. That is, that each connected component of $\mathbf{T}^n \setminus \overline{\mathcal{C}}$ contains exactly one point of $\mathcal{I}$.

Proposition 4.2.5. *Let A be a simplex circuit. Assume that* $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$*, i.e., that there exists a point $\theta \in \mathcal{I}$ such that $\hat{f}_{n+1}(\theta) = \delta_{n+1}$. Then, the complement of $\overline{\mathcal{C}}$ has* $\mathrm{Vol}(A)$*-many connected components if and only if it contains θ.*

Proof. We can assume that $\theta = 0$. To prove the *if*-part, assume that $0 \in \Theta$ for some connected component Θ of the complement of $\overline{\mathcal{C}}$. We wish to show that f is never colopsided in Θ, for this implies that the complement of $\overline{\mathcal{C}}$ has $\mathrm{Vol}(A)$-many connected components. Assume, to the contrary that, there exists a point $\hat{\theta} \in \Theta$ such that f is colopsided at $\hat{\theta}$. Then, $\mathrm{ord}_{\mathcal{D}}(\hat{\theta}) = m\pi$ for some integer m with $|m| < \mathrm{Vol}(A)$. Let $f^\varepsilon = (x_0, \ldots, x_n, x_{n+1}e^{i\varepsilon})$, and let $\mathcal{C}_\varepsilon$ and $\mathcal{D}_\varepsilon$ denote the coamoeba and lopsided coamoeba of f^ε respectively. We have that f^ε is colopsided at 0 for $\varepsilon \notin 2\pi\mathbb{Z}$. By continuity of roots, for $\varepsilon > 0$ sufficiently small, the points 0 and $\hat{\theta}$ are contained in the same connected component of the complement of $\overline{\mathcal{C}}_\varepsilon$. Hence, by Proposition 3.1.2, they are contained in the same connected component of the complement of $\overline{\mathcal{D}}_\varepsilon$. However, if ord_ε denotes the order map for $\overline{\mathcal{D}}_\varepsilon$, then

$$\mathrm{ord}_\varepsilon(0) = \mathrm{Vol}(A)(\pi - \varepsilon) \neq m(\pi - \varepsilon) = \mathrm{ord}_\varepsilon(\hat{\theta}),$$

contradicting that ord_ε is constant on connected components of the complement of $\overline{\mathcal{D}}_\varepsilon$.

To prove the *only if*-part, assume that there exists a connected component Θ of the complement of $\overline{\mathcal{C}}$ in which f is never colopsided. We wish to prove that $0 \in \Theta$. As f^ε is colopsided at 0 for $\varepsilon > 0$ sufficiently small, we find that $0 \in \overline{\Theta}$. Indeed, if this was not the case, then the complement of $\overline{\mathcal{C}}_\varepsilon$ has $(\mathrm{Vol}(A)+1)$-many connected components, a contradiction. As $0 \notin \mathcal{H}_f$, it follows from Lemma 2.2.3 that there exists a disc D_0 centered at 0 such that

$$D_0 \cap (\mathbf{T}^n \setminus \overline{\mathcal{C}}) = D_0 \cap \Theta.$$

Furthermore, $D_0 \cap \Theta \neq \emptyset$, since $0 \in \overline{\Theta}$. Let $\theta \in D_0 \cap \Theta$. Since f is a real polynomial, conjugation yields that $-\theta \in D_0 \cap \Theta$. However, $\Theta \subset \mathbb{R}^n$ is convex, implying that $0 \in \Theta$. $\square$

Proof of Theorem 4.2.1. If $\mathrm{Arg}(f) \notin \mathcal{C}_\Delta$ then the image of $\mathrm{ord}_{\mathcal{D}}$ is of cardinality $\mathrm{Vol}(A)$, and hence $f \in U_0$. Thus, we only need to consider a polynomial $f \in \mathbb{C}_*^A$ such that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$, for which can assume that $\hat{f}(0) = \delta_k$ for all k. In particular, f is a real polynomial. By Proposition 4.2.5, it holds that the complement of $\overline{\mathcal{C}}$ has $\mathrm{Vol}(A)$-many connected components if and only if it contains 0. Keeping $x_0, \ldots, x_n$ and $\arg(x_{n+1})$ fixed, let us consider the dependence of f on $|x_{n+1}|$. As f is a real polynomial, it restricts to a map $f\colon \mathbb{R}^n_{\geq 0} \to \mathbb{R}$, whose image depends nontrivially on $|x_{n+1}|$. Notice that $0 \in \overline{\mathcal{C}}$ if and only if $f(\mathbb{R}^n_{\geq 0})$ contains the origin. Since $\hat{f}_k(0) = \delta_k = 1$ for $k \neq n+1$, and since $\mathbf{a}_{n+1}$ is an interior point of A, the map f takes the boundary of $\mathbb{R}^n_{\geq 0}$ to $[1, \infty)$. In particular, if $0 \in f(\mathbb{R}^n_{\geq 0})$, then $0 \in f(\mathbb{R}^n_+)$. The boundary of the set of all $|x_{n+1}|$ for which $0 \in f(\mathbb{R}^n_{\geq 0})$ is the set of all values of $|x_{n+1}|$ for which $f(\mathbb{R}^n_+) = [0, \infty)$. Furthermore, $f(\mathbb{R}^n_+) = [0, \infty)$ holds if and only if there exists an $r \in \mathbb{R}^n_+$ such that $f(r) = 0$ and $f(r) \geq 0$ in a neighborhood of r, implying that r is a critical point of f. That is,

$$\Delta\big(\delta_0|x_0|, \ldots, \delta_{n+1}|x_{n+1}|\big) = 0.$$

Since Δ is a binomial, there is exactly one such value of $|x_{n+1}|$. Finally, we note that $0 \in \overline{\mathcal{C}}$ if $|x_{n+1}| \to \infty$, which finishes the proof. $\square$

Proof of Theorem 4.2.2. If $\mathrm{Arg}(f) \notin \mathcal{C}_\Delta$, then the image of $\mathrm{ord}_{\mathcal{D}}$ is of cardinality $\mathrm{Vol}(A)$ and hence $f \in U_0$. Assume that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$, and that $\hat{f}_k(0) = \delta_k$ for all k. It holds that $0 \in \mathcal{H}_f$ since there exists two adjacent vertices $\mathbf{a}_0$ and $\mathbf{a}_1$ of A such that $\delta_0 = 1$ and $\delta_1 = -1$. Let

$$H = \{\theta \mid \langle \mathbf{a}_0 - \mathbf{a}_1, \theta \rangle = 0\}$$

be the hyperplane of $\mathcal{H}_f$ containing 0. Assume that exists connected component Θ of the complement of $\overline{\mathcal{C}}$ in which f is nowhere colopsided. As in the proof of Proposition 4.2.5, we conclude that $0 \in \overline{\Theta}$, for otherwise we can construct a coamoeba with $(\mathrm{Vol}(A)+1)$-many connected components of its complement. As $H \subset \overline{\mathcal{C}}$, we find that Θ is contained in one of the half-spaces

$$H_\pm = \{\theta \mid \pm \langle \mathbf{a}_0 - \mathbf{a}_1, \theta\rangle > 0\},$$

say that $\Theta \subset H_+$. Let $f^\varepsilon = (x_0 e^{i\varepsilon}, x_1, \dots, x_{n+1})$, and let H^ε denote the corresponding hyperplane

$$H^\varepsilon = \{\theta \mid \langle \mathbf{a}_0 - \mathbf{a}_1, \theta\rangle = -\varepsilon\}.$$

For $|\varepsilon|$ sufficiently small, continuity of roots implies that there is a connected component $\Theta^\varepsilon \subset H^\varepsilon_+$ in which f^ε is never colopsided. However, by choosing the sign of ε, we can force $0 \in H^\varepsilon_-$. This implies that the coamoeba $\overline{\mathcal{C}}_\varepsilon$ has $(\mathrm{Vol}(A)+1)$-many connected components of its complement, a contradiction. □

4.3 The maximal area of planar circuit coamoebas

In this section, we will prove the following upper bound on the area of a planar circuit coamoeba.

Theorem 4.3.1. *Let A be a planar circuit, and let $f \in \mathbb{C}^A_*$. Then* $\mathrm{Area}(\mathcal{C}) \leq 2\pi^2$.

Note that we calculate area without multiplicities, in contrast to [31]. However, the relation between (co)amoebas of maximal area and Harnack curves is made visible also in this setting.

Theorem 4.3.2. *Let A be a planar circuit. Then, there exists a polynomial $f \in \mathbb{C}^A_*$ such that* $\mathrm{Area}(\mathcal{C}) = 2\pi^2$ *if and only if A admits an equimodular triangulation.*

Example 4.3.3. Let $f(z) = 1 + z_1 + z_2 - r z_1 z_2$ for $r \in \mathbb{R}_+$. Notice that A admits a unimodular triangulation. The shell $\mathcal{H}$ consists of the families $\theta_1 = k_1\pi$ and $\theta_2 = k_2\pi$ for $k_1, k_2 \in \mathbb{Z}$. Hence, $\mathcal{H}$ divides $\mathbf{T}^2$ into four regions of equal area. At least two of these regions are contained in the coamoeba, which implies that $\mathrm{Area}(\mathcal{C}) = 2\pi^2$. See the left picture of Figure 4.2. ○

Example 4.3.4. Let $f(z) = 1 + zw^2 + z^2 w - rzw$ for $r \in \mathbb{R}_+$. As in the previous example, A admits a unimodular triangulation. Notice that $\mathrm{Arg}(f) \in \mathcal{C}_\Delta$. The complement of the coamoeba of the trinomial $g(z) = 1 + zw^2 + z^2 w$ has three connected components, of which f is colopsided in two. We have that $\mathcal{H}_f = \mathcal{H}_g$. Thus, if $\mathbf{T}^2 \setminus \overline{\mathcal{C}}_f$ has two connected components, i.e., if $r \geq 3$, then one of the three connected components of $\mathbf{T}^2 \setminus \overline{\mathcal{C}}_g$ is contained in $\overline{\mathcal{C}}_f$. This implies that $\mathrm{Area}(\mathcal{C}_f) = 2\pi^2$, see the right picture of Figure 4.2. ⬡

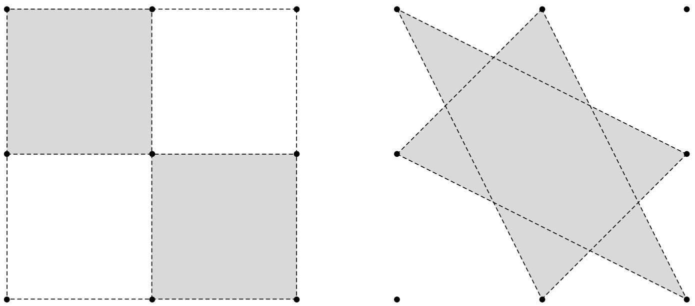

Figure 4.2: The coamoebas of Examples 4.3.3 and 4.3.4.

Let us compare to the case of a planar circuit amoeba. It was shown in [50, Theorem I.12] that the sharp upper bound on the number of connected components of the complement of the amoeba is $\#A$. In [34], a bound on the area of the amoeba was given as $\pi^2 \mathrm{Vol}(A)$, and it was shown that maximal area was obtained for Harnack curves. For coamoebas, to roles of the integers $\mathrm{Vol}(A)$ and $\#A$ are reversed. The upper bound on the number of connected components of the coamoeba complement is given by $\mathrm{Vol}(A)$. While, at least for codimension $m \leq 1$, the maximal area of the coamoeba is $\pi^2(m+1) = \pi^2(\#A - n)$. Note also that the coamoebas of Examples 4.3.3 and 4.3.4 are coamoebas of Harnack curves.

Consider a bivariate trinomial f, with one marked monomial. Let $\Sigma = \Sigma(f)$ denote the quadruple of polynomials obtained by flipping signs of the unmarked monomials. Furthermore, let

$$\mathcal{H}_\Sigma = \bigcup_{g \in \Sigma} \mathcal{H}_g,$$

which is a hyperplane arrangement in $\mathbb{R}^2$ (or $\mathbf{T}^2$). Let $\mathcal{P}_\Sigma$ denote the set of all intersection points of distinct hyperplanes in $\mathcal{H}_\Sigma$.

Proposition 4.3.5. *Let $f(z)$ be a bivariate trinomial. Then, the union*

$$\overline{\mathcal{C}}_\Sigma = \bigcup_{g \in \Sigma} \overline{\mathcal{C}}_g,$$

covers $\mathbb{R}^2$. To be specific, $\mathcal{P}_\Sigma$ is covered thrice, $\mathcal{H}_\Sigma \setminus \mathcal{P}_\Sigma$ is covered twice, and $\mathbb{R}^2 \setminus \mathcal{H}_\Sigma$ is covered once.

Proof. After applying an integer affine transformations, we reduce to the case when A consist of the vertices of the standard simplex. This case follows, e.g., from the description in [40]. See also Figure 2.1. $\square$

Corollary 4.3.6. *If $f(z)$ is a bivariate trinomial, then* $\mathrm{Area}(\mathcal{C}) = \pi^2$.

Proof. The coamoebas appearing in the union $\overline{\mathcal{C}}_\Sigma$, when considered in $\mathbb{R}^2$, are merely translations of each other. Hence, they have equal area. As they cover the torus once a.e., and $\mathrm{Area}(\mathbf{T}^2) = 4\pi^2$, the result follows. $\square$

Let $f_{\hat{k}}$ denote the image of f under the projection $\mathrm{pr}_k\colon \mathbb{C}_*^A \to \mathbb{C}_*^{A_k}$.

Lemma 4.3.7. *Let A be a planar circuit, and let $f \in \mathbb{C}_*^A$. Assume that $\theta \in \mathbf{T}$ is generic in the sense that no two components of $\hat{f}(\theta)$ are antipodal, and assume further that f is* not *colopsided at θ. Then, exactly two of the trinomials $f_{\hat{1}}, \dots, f_{\hat{4}}$ are colopsided at θ.*

Proof. Fix an arbitrary point $\mathbf{a}_1 \in A$, and let $\ell \subset \mathbb{C}$ denote the real subvector space containing $\hat{f}_1(\theta)$. As f is not colopsided at θ, both half spaces relative ℓ contains at least one component of $\hat{f}(\theta)$. There is no restriction to assume that the upper half space contains the two components $\hat{f}_2(\theta)$ and $\hat{f}_3(\theta)$, and that the latter is of greatest angular distance from $\hat{f}_1(\theta)$. Then, $f_{\hat{4}}$ is colopsided at θ. Furthermore, we find that $f_{\hat{2}}$ is not colopsided at θ, for if it where then so would f. As $\mathbf{a}_1 \in A_4$ and $\mathbf{a}_1 \in A_2$, there is at least one trinomial obtained from f containing $\mathbf{a}_1$ which is not colopsided at θ, and at least one which is colopsided at θ. As $\mathbf{a}_1$ was arbitrary, it follows that exactly two of the trinomials $f_{\hat{1}}, \dots, f_{\hat{4}}$ are colopsided at θ, and exactly two are not. $\square$

Proof of Theorem 4.3.1. By containment, it holds that $\mathrm{Area}(\mathcal{C}_f) \le \mathrm{Area}(\mathcal{D}_f)$, and thus it suffices to calculate the area of $\mathcal{D}_f$. By Proposition 3.1.10, we have that

$$\mathcal{D}_f = \bigcup_{k=1}^{4} \mathcal{C}_{f_{\hat{k}}}. \tag{4.6}$$

For a generic point $\theta \in \mathcal{D}_f$, Lemma 4.3.7 gives that θ (and, in fact, a small neighborhood of θ) is contained in the interior of exactly two out of the four coamoebas in the right hand side of (4.6). Hence,

$$\mathrm{Area}(\mathcal{D}_f) = \frac{1}{2}\big(\mathrm{Area}(\mathcal{C}_{f_{\hat{1}}}) + \cdots + \mathrm{Area}(\mathcal{C}_{f_{\hat{4}}})\big) = 2\pi^2. \qquad \square$$

Proof of Theorem 4.3.2. To prove the *if* part, we will prove that A admits an equimodular triangulation only if, after applying an integer affine transformation, it is equal to the point configuration of either Example 4.3.3 or Example 4.3.4. Assume that $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ are vertices of $\mathcal{N}_A$. After applying an integer affine transformation, we can assume that $\mathbf{a}_1 = k_1\mathbf{e}_1$, that $\mathbf{a}_2 = k_2\mathbf{e}_2$ with $k_1 \geq k_2$, and that $\mathbf{a}_3 = \mathbf{0}$. Notice that such a transformation rescales A, though it does not affect the area of the coamoeba $\mathcal{C}$. Let $\mathbf{a}_4 = m_1\mathbf{e}_1 + m_2\mathbf{e}_2$.

If A is a vertex circuit, then each triangulation of A consist of two simplices, which are equal in area by assumption. Comparing the areas of the subsimplices of A, we obtain the relations

$$|k_1k_2 - k_1m_2 - k_2m_1| = k_1k_2 \quad \text{and} \quad k_1m_2 = k_2m_1.$$

In m, this system has (k_1, k_2) as the only nontrivial solution, and we conclude that A is the unit square, up to integer affine transformations.

If A is a simplex circuit, then A has one triangulation with three simplices of equal area. Comparing areas, we obtain the relations

$$3k_1m_2 = 3k_2m_1 = k_1k_2.$$

Thus, $3m_1 = k_1$ and $3m_2 = k_2$, and we conclude that A is the simplex from Example 4.3.4, up to integer affine transformations.

To prove the *only if*-statement, consider $f \in \mathbb{C}_*^A$. Let $S = \{\mathbf{a}_1, \mathbf{a}_2\} \subset A$ be such that the line segment $[\mathbf{a}_1, \mathbf{a}_2]$ is interior to $\mathcal{N}$. Denote by $\mathcal{C}_S$ the coamoeba of the truncated polynomial f_S. Applying an integer affine transformation, we can assume that $[\mathbf{a}_1, \mathbf{a}_2] \subset \mathbb{R}\mathbf{e}_1$, and that $\mathbf{a}_3$ and $\mathbf{a}_4$ lies in the upper and lower half space respectively. Then, the hyperplane arrangement $\mathcal{C}_S \subset \mathbf{T}$ consists of Length$[\mathbf{a}_1, \mathbf{a}_2]$-many lines, each parallel to the θ_2-axis. If $\mathbf{a}_3 = m_{31}\mathbf{e}_1 + m_{32}\mathbf{e}_2$ and $\mathbf{a}_4 = m_{41}\mathbf{e}_1 + m_{42}\mathbf{e}_2$, then $\hat{f}_3(\theta)$ and $\hat{f}_4(\theta)$ takes m_{32} respectively m_{42} turns around the origin when θ traverses once a line of $\mathcal{C}_S$. Notice that $\mathcal{C}_S \subset \overline{\mathcal{D}}$, as $\hat{f}_1(\theta)$ and $\hat{f}_2(\theta)$ are antipodal for $\theta \in \mathcal{C}_S$. That is, for such θ, there is a real subvector space $\ell_\theta \subset \mathbb{C}$ containing all components of $\hat{f}_S(\theta)$.

Assume that f is colopsided for some $\theta \in \mathcal{C}_S$, so that in particular $\theta \not\in \mathcal{C}$. If $\theta \in \mathcal{H}_f$, then at exactly one of the points $\hat{f}_3(\theta)$ and $\hat{f}_4(\theta)$ is contained in ℓ_θ, for otherwise f would not be colopsided at θ. By wiggling θ in $\mathcal{C}_S$ we can assume that $\theta \not\in \mathcal{H}_f$. Under this assumption, we find that $\theta \notin \overline{\mathcal{C}}$. Thus, there is a neighborhood N_θ which is separated from $\overline{\mathcal{C}}$. As $\theta \in \overline{\mathcal{D}}$, the intersection $N_\theta \cap \overline{\mathcal{D}}$ has positive area, implying that Area$(\overline{\mathcal{C}}) <$ Area$(\overline{\mathcal{D}})$.

Thus, if f is such that Area$(\overline{\mathcal{C}}) = 2\pi^2$, then f can never be colopsided in $\mathcal{C}_S$. In particular, for $\theta \in \mathcal{C}_S$ such that $\hat{f}_3(\theta) \in \ell$, it must hold $\hat{f}_4(\theta) \in \ell$, and vice versa. As there are $2m_{32}$ points of the first kind, and $2m_{42}$ points of the second kind, it holds that $m_{32} = m_{42}$. Hence, the simplices with vertices $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ have equal area.

If A is a vertex circuit, this suffices to conclude that A admits an equimodular triangulation. If A is a simplex circuit, then we can assume that $\mathbf{a}_1$ is an interior point of $\mathcal{N}_A$. Repeating the argument for either $S = \{\mathbf{a}_1, \mathbf{a}_3\}$ or $S = \{\mathbf{a}_1, \mathbf{a}_4\}$ yields that A has a triangulation with three triangles of equal area. That is, it admits an equimodular triangulation. □

4.4 The set of critical arguments

Let $C(f)$ denote the set of critical points of f, that is, the variety defined by (2.3). Let $\mathcal{I} = \operatorname{Arg}(C(f))$ denote the coamoeba of $C(f)$. We will say that $\mathcal{I}$ is the set of *critical arguments* of f. In this section we will prove that, under certain assumptions on A, the set $\mathcal{I}$ is an index set of the coamoeba complement. That is, each connected component of the complement $\mathbf{T}^n \setminus \overline{\mathcal{C}}$ contains exactly one point of $\mathcal{I}$. This settles a conjecture used in in [27] when computing monodromy in the context of dimer models and mirror symmetry. That it is necessary to impose assumptions on A is related to the fact that an integer affine transformation acts nontrivially on the set of critical points.

Let A be a circuit, with the elements $\mathbf{a} \in A$ ordered so that it has a Gale dual $B = (B_1, B_2)^t$ such that $B_1 \in \mathbb{R}_+^{m_1+1}$ and that $B_2 \in \mathbb{R}_-^{m_2+1}$. That is, B_1 has only positive entries, while B_2 has only negative entries. We have that $m_1 + m_2 = n$. Let $A = (A_1, A_2)$ denote the corresponding decomposition of the matrix A. We will say that A is in *orthogonal form* if

$$A = \begin{pmatrix} 1 & 1 \\ \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}, \tag{4.7}$$

where $\tilde{A}_1$ is an $m_1 \times (m_1+1)$-matrix and $\tilde{A}_2$ is an $m_2 \times (m_2+1)$-matrix. In particular, the Newton polytopes $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ has $\mathbf{0}$ as a relatively interior point, and as their only intersection point.

With A in the form (4.7), we can act by integer affine transformations affecting $\tilde{A}_1$ and $\tilde{A}_2$ separately. Therefor, if A is in orthogonal form, then we can assume that

$$\tilde{A}_k = (-p_1\mathbf{e}_1, \ldots, -p_{m_k}\mathbf{e}_{m_k}, \mathbf{a}_{m_k+1}), \tag{4.8}$$

where $p_1, \ldots, p_{m_k}$ are positive integers and $\mathbf{a}_{m_k+1}$ has positive coordinates. We will say that A is in *special orthogonal form* if (4.8) holds. The main results of this section are the following lemma and theorem.

Lemma 4.4.1. *Each circuit A can be put in (special) orthogonal form by applying and integer affine transformation.*

Theorem 4.4.2. *Let A be in special orthogonal form. Then, for each $f \in \mathbb{C}_*^A$, the set of critical arguments is an index set of the complement of $\overline{\mathcal{C}}$.*

Remark 4.4.3. The conditions of Theorem 4.4.2 can be relaxed in small dimensions. When $n = 1$, it is enough to require that $\mathbf{0}$ is an interior point of $\mathcal{N}$. When $n = 2$, for generic f, it is enough to require that each quadrant Q fulfills that $\overline{Q} \setminus \{\mathbf{0}\}$ has nonempty intersection with A. ⬡

Proof of Lemma 4.4.1. Let $\mathbf{u}_1 \ldots, \mathbf{u}_{m_2}$ be a basis for the left kernel of A_1, and let $\mathbf{v}_1, \ldots, \mathbf{v}_{m_1}$ be a basis for the left kernel of A_2. Multiplying A from the left by

$$T = \left(\mathbf{e}_1, \mathbf{v}_1, \ldots, \mathbf{v}_{m_1}, \mathbf{u}_1, \ldots, \mathbf{u}_{m_2}\right)^t,$$

it takes the desired form. We need only to show that $\det(T) \neq 0$.

Notice that $\ker(A_1) \cap \ker(A_2) = 0$, since A is assumed to be of full dimension. Assume that there is a linear combination

$$\lambda_0 \mathbf{e}_1 + \sum_{i=1}^{m_1} \lambda_i \mathbf{v}_i + \sum_{j=1}^{m_2} \lambda_j \mathbf{u}_j = 0.$$

Then, since B is a Gale dual of A,

$$0 = \left(\sum_{j=1}^{m_2} \lambda_j \mathbf{u}_j\right) AB = (0, \ldots, 0, -\lambda_0, \ldots, -\lambda_0) B = -\lambda_0 \sum_{\mathbf{a} \in A_2} \mathbf{b}_{\mathbf{a}} = \lambda_0 \operatorname{Vol}(A),$$

which implies that $\lambda_0 = 0$. Therefor, $\sum_{i=1}^{m_1} \lambda_i \mathbf{v}_i = -\sum_{j=1}^{m_2} \lambda_j \mathbf{u}_j \in \ker(A_2)$. As $\ker(A_1) \cap \ker(A_2) = 0$, we can conclude that $\sum_{i=1}^{m_1} \lambda_i \mathbf{v}_i = \mathbf{0}$. Thus, $\lambda_i = 0$ for all i by linear independence of the vectors $\mathbf{v}$. Then, linear independence of the vectors $\mathbf{u}_j$ imply that $\lambda_j = 0$ fo all j. □

Proof of Theorem 4.4.2. We find that

$$\begin{aligned} z_i \partial_i f(z) &= -p_i x_i z^{\mathbf{a}_i} + \langle \mathbf{a}_{m_k}, \mathbf{e}_i \rangle x_{m_1} z^{\mathbf{a}_{m_1}}, \quad i = 0, \ldots, m_1 \\ z_j \partial_j f(z) &= -p_j x_j z^{\mathbf{a}_j} + \langle \mathbf{a}_{n+1}, \mathbf{e}_j \rangle x_{n+1} z^{\mathbf{a}_{n+1}}, \quad j = m_1 + 1, \ldots, n. \end{aligned}$$

Hence, for each $\theta \in \mathcal{I}$, it holds that

$$\hat{f}_0(\theta) = \cdots = \hat{f}_{m_1}(\theta) \quad \text{and} \quad \hat{f}_{m_1+1}(\theta) = \cdots = \hat{f}_{n+1}(\theta). \tag{4.9}$$

In particular, f is colopsided at θ unless, after a rotation, $\hat{f}_k(\theta) = \delta_k$ for all k. In the latter case, we refer to Theorems 4.2.1 and 4.2.2.

To see that the points $\theta \in \mathcal{I}$ for which f is colopsided at θ are contained in distinct connected components of the complement of $\overline{\mathcal{D}}$, consider a line segment ℓ in $\mathbb{R}^n$ with endpoints in $\mathcal{I}$. Not all identities of (4.9) can hold identically along ℓ. Since the argument of each monomial is linear in θ, this implies that for a pair such that the identity in (4.9) does not hold identically along ℓ, there is an intermediate point $\theta \in \ell$ for which the corresponding monomials are antipodal, and hence $\theta \in \overline{\mathcal{D}}$. □

4.5 On systems supported on a circuit

In this section we will consider a system

$$F_1(z) = F_2(z) = 0 \tag{4.10}$$

of two bivariate polynomials. We will write $f(z) = 0$ for the system (4.10). The system is said to be generic if it has finitely many roots in $\mathbb{C}_*^2$, and it is said to be supported on a circuit A if the supports of F_1 and F_2 are contained in, but not necessarily equal to, A. That is, we allow coefficients in $\mathbb{C}$ rather than $\mathbb{C}_*$. By the Bernstein–Kushnirenko theorem, a generic system $f(z) = 0$ has at most $\mathrm{Vol}(A)$-many roots in $\mathbb{C}_*^2$. However, if f is real, then fewnomial theory states that a generic system $f(z)$ has at most three roots in $\mathbb{R}_+^2 = \mathrm{Arg}^{-1}(0)$. We will solve the complexified fewnomial problem, i.e., for $f(z)$ with complex coefficients we will bound the number of roots in each sector $\mathrm{Arg}^{-1}(\theta)$. Our intention is to offer a new approach to fewnomial theory. We will restrict to the case of simplex circuits, for the following two reasons. Firstly, it allows for a simpler exposition. Secondly, for vertex circuits our method recovers the known (sharp) bound, while for simplex circuits we obtain a sharpening of the fewnomial bound.

Theorem 4.5.1. *Let f be supported on a planar simplex circuit $A \subset \mathbb{Z}^2$. Then, each sector $\mathrm{Arg}^{-1}(\theta)$ contains at most two solutions of $f(z) = 0$.*

A generic system $f(z)$ is, after taking appropriate linear combinations, equivalent to a system of two trinomials whose support intersect in a dupleton. That is, we can assume that $f(z)$ is in the form

$$\begin{cases} F_1(z) = & x_1 z^{\mathbf{a}_0} & + & z^{\mathbf{a}_2} & + & x_2 z^{\mathbf{a}_3} & = 0 \\ F_2(z) = & x_3 z^{\mathbf{a}_1} & + & z^{\mathbf{a}_2} & + & x_4 z^{\mathbf{a}_3} & = 0. \end{cases} \tag{4.11}$$

From now on, we will assume that $f(z)$ is in the form (4.11). We will use the notation

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \mathbf{a}_0 & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix} \quad \text{and} \quad \hat{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ A_1 & A_2 \end{pmatrix},$$

where A_k denotes the support of F_k (notice that this differs from the notation used in previous sections). We can identify a system $f(z)$ in the form (4.11) with its coefficient vector in $\mathbb{C}_*^{\hat{A}}$.

When reducing $f(z)$ to the form (4.11) by taking linear combinations, there is a choice of which monomials to eliminate in F_1 and F_2 respectively. In order for the arguments of the roots of $f(z) = 0$ to depend continuously on the coefficients, we need to be careful with which choice to make.

Lemma 4.5.2. *Let ℓ denote the line containing $\mathbf{a}_2$ and $\mathbf{a}_3$, and let γ be a compact path in $\mathbb{C}_*^{\hat{A}}$. If ℓ intersects the interior of $\mathcal{N}_A$, then the arguments of the solutions to $f(z) = 0$ vary continuously along γ.*

Proof. It is enough to show that along a compact path γ, the set

$$\bigcup_{f\in\gamma} \mathcal{A}_f = \bigcup_{f\in\gamma} \operatorname{Log}(Z(f)) \tag{4.12}$$

is bounded, for it implies that for $f \in \gamma$, the roots of f are uniformly separated from the boundary of $\mathbb{C}_*^2$.

We first claim that our assumptions imply that the normal fans of $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ have no common one-dimensional cone. Indeed, these fans have a common one-dimensional cone if and only if the Newton polytopes $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ have facets Γ_1 and Γ_2 respectively with a common outward normal vector $\mathbf{n}$. As A is a circuit, it holds that $\Gamma_1 = \Gamma_2 = [\mathbf{a}_2, \mathbf{a}_3] \subset \ell$. Since the normal vector $\mathbf{n}$ is common for $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$, we find that Γ_1 (and Γ_2) is a facet of $\mathcal{N}_A$. But then ℓ contains a facet of $\mathcal{N}_A$, and hence it cannot intersect the interior of $\mathcal{N}_A$, a contradiction.

Consider a point $f \in \mathbb{C}_*^{\hat{A}}$. Since the normal fans of $\mathcal{N}_{A_1}$ and $\mathcal{N}_{A_2}$ has no common one-dimensional cones, the intersection of the amoebas $\mathcal{A}_{F_1}$ and $\mathcal{A}_{F_2}$ is bounded (this follows, e.g., from the fact the amoeba has finite Hausdorff distance from the Archimedean tropical variety, see [4]). Thus, the amoeba $\mathcal{A}_f$ is bounded, say that $\mathcal{A}_f \subset D(R_f)$, where $D(R_f)$ denotes the disk of radii R_f centered around the origin. By continuity of roots, $\mathcal{A}_{\tilde{f}} \subset D(R_f)$ for all $\tilde{f}$ in some small neighborhood of f. Finally, as γ is compact, we need only to consider finitely many of the sets $D(R_f)$. Then, we can take R to be the maximum of R_f. □

In order for the assumptions of Lemma 4.5.2 to be fulfilled, for a simplex circuit A, we need that $\mathbf{a}_0$ and $\mathbf{a}_1$ are vertices of $\mathcal{N}_A$, see Figure 4.3.

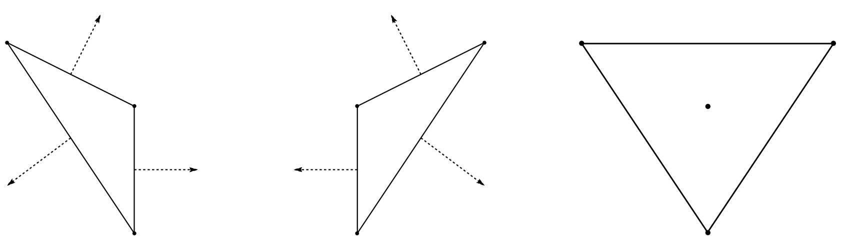

Figure 4.3: The Newton polytopes $\mathcal{N}_A$, $\mathcal{N}_{A_1}$, and $\mathcal{N}_{A_2}$.

Proposition 4.5.3. *If f is nonreal at θ, then there is at most one zero of $f(z) = 0$ contained in the sector $\operatorname{Arg}^{-1}(\theta)$.*

Proof. If F_k is nonreal, then the fiber in $Z(F_k)$ over a point $\theta \in \mathcal{C}_{F_k}$ is a singleton. Hence, if the number of roots of $f(z) = 0$ in $\mathrm{Arg}^{-1}(\theta)$ is greater than one, then both F_1 and F_2 are real at θ. $\square$

One implication of Proposition 4.5.3 is that the complexified fewnomial problem reduces to the real fewnomial problem. However, our approach is dependent on allowing coefficients to be nonreal. In fact, we will consider a partially complexified problem, allowing $x_1, x_3 \in \mathbb{C}_*$ but requiring that $x_2, x_4 \in \mathbb{R}_*$.

We define the lopsided coamoeba of the system $f(z)$ by

$$\mathcal{D}_f = \mathcal{D}_{F_1} \cap \mathcal{D}_{F_2} = \mathcal{C}_{F_1} \cap \mathcal{C}_{F_2},$$

where the last equality follows from that F_1 and F_2 are simple polynomials. That is, f is said to be colopsided at θ if either F_1 or F_2 is colopsided at θ. We will say that f is real at θ if both F_1 and F_2 are real at θ.

The lopsided coamoeba $\mathcal{D}_f$ consists of a number of polygons in $\mathbf{T}^2$, possibly degenerated to singletons. The following two lemmas will allow us to count the number of such polygons.

Lemma 4.5.4. *Assume that f is nonreal. Let g be a binomial constructed by choosing two monomials from* (4.11)*, possibly alternating signs. If x_2 and x_4 are of opposite signs, then $\mathcal{C}_g \subset \mathbf{T}^2 \setminus \mathcal{D}_f$. If x_2 and x_4 are equal in sign, then $\mathcal{C}_g \subset \mathbf{T}^2 \setminus \mathcal{D}_f$* except *for $g(z) = \pm(x_1 z^{\mathbf{a}_0} - x_3 z^{\mathbf{a}_1})$.*

Proof. If, for $\theta \in \mathbf{T}^2$, two components of $\hat{F}_1(\theta)$ is contained in a real subvector space $\ell \subset \mathbb{C}$, then either F_1 is colopsided at θ or $\hat{F}_1(\theta) \subset \ell$. However, the latter implies that two components of $\hat{F}_2(\theta)$ are contained in ℓ. Repeating the argument yields that f is either real or colopsided at θ.

Thus, the only binomials we need to consider is $g_\pm(z) = x_1 z^{\mathbf{a}_0} \pm x_3 z^{\mathbf{a}_1}$. For each $\theta \in \mathcal{C}_{g_+}$ the vectors $\hat{F}_1(\theta)$ and $\hat{F}_2(\theta)$ differ in sign in their first component, and hence at least one is colopsided at θ unless f is real. For each $\theta \in \mathcal{C}_{g_-}$, the vectors $\hat{F}_1(\theta)$ and $\hat{F}_2(\theta)$ differ in sign in the the last component only if x_2 and x_4 differ in sign. If this is the case, then at least one is colopsided at θ unless f is real. $\square$

Lemma 4.5.5. *Let $\theta \in \mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$ for truncated binomials g_1 and g_2 of F_1 and F_2 respectively. If the Newton polytopes (i.e., line segments) of g_1 and g_2 are nonparallel, then $\theta \in \overline{\mathcal{D}_f}$.*

Proof. If F_1 and F_2 are both real at θ, then $\theta \in \mathcal{D}_f$. If F_1 is nonreal at θ, then for a sufficiently small neighborhood $N_\theta \subset \mathbb{R}^2$, it holds that

$$\mathcal{C}_{F_1} \cap N_\theta = \{\varphi \,|\, \langle \varphi, \mathbf{n} \rangle > \langle \theta, \mathbf{n} \rangle\} \cap N_\theta,$$

where $\mathbf{n}$ is a normal vector of $\mathcal{N}(g_1)$. Since connected components of the complement of $\mathcal{C}_{F_2}$ are convex, either $\mathcal{C}_{F_2}$ intersect $\mathcal{C}_{F_1}$ in N_θ, or the boundary of $\mathcal{C}_{F_2}$ is contained in the line $\ell = \{\varphi \,|\, \langle \varphi, \mathbf{n}\rangle = \langle \theta, \mathbf{n}\rangle\}$. As the boundary of $\mathcal{C}_{F_2}$ contains $\mathcal{C}_{g_2}$ it holds, in the latter case, that $\mathcal{C}_{g_2} \subset \ell$. Hence, in this case, $\mathbf{n}$ is a normal vector of $\mathcal{N}(g_2)$, a contradiction to the assumptions of the lemma. We conclude that $\mathcal{C}_{F_2} \cap \mathcal{C}_{F_1} \cap N_\theta \neq \emptyset$. Since this holds for any sufficiently small neighborhood N_θ, the result follows. □

Example 4.5.6. Consider the system

$$f(z) = \begin{cases} x_1 z_1 z_2^2 + 1 + x_2 z_1 z_2 \\ x_3 z_1^2 z_2 + 1 + x_4 z_1 z_2. \end{cases}$$

We have that $\mathrm{Vol}(A) = 3$. Hence H divides $\mathbf{T}^2$ into three cells. The lopsided coamoeba $\mathcal{D}_f$, and the hyperplane arrangement H, can be seen in Figure 4.4. In the upper two pictures is the generic respectively real situation when x_2 and x_4 differ in sign. In lower two pictures is the generic respectively real situation when x_2 and x_4 are equal in sign. In the generic case, the lopsided coamoeba $\mathcal{D}_f$ consists of three polygons. When passing from the generic to the real case, we observe the following behavior. Some polygons of $\mathcal{D}_f$ deform into single points. These points are necessarily the center of mass of the cells of H. Some pairs of polytopes of $\mathcal{D}_f$ deform to nonconvex polygons, typically with a single intersection point. Our proof of Theorem 4.5.1 is based on the observation that, when passing from a generic to a real system, at most two polytopes of $\mathcal{D}_F$ deform to a nonconvex polygon intersecting H. ⬡

Proof of Theorem 4.5.1. Let us consider the auxiliary binomials

$$g_1(z) = x_1 z^{\mathbf{a}_0} - z^{\mathbf{a}_2}, \qquad g_2(z) = x_3 z^{\mathbf{a}_1} - z^{\mathbf{a}_2},$$
$$h_1(z) = x_1 z^{\mathbf{a}_0} + z^{\mathbf{a}_2}, \qquad \text{and} \quad h_2(z) = x_3 z^{\mathbf{a}_1} + z^{\mathbf{a}_2}.$$

The vectors $\mathbf{a}_2 - \mathbf{a}_0$ and $\mathbf{a}_2 - \mathbf{a}_1$ span the simplex $\mathcal{N}_A$, hence the hyperplane arrangement $H = \mathcal{C}_{g_1} \cup \mathcal{C}_{g_2}$ divides $\mathbf{T}^2$ into $\mathrm{Vol}(A)$-many parallelograms with the points $P = \mathcal{C}_{h_1} \cap \mathcal{C}_{h_2}$ as their centers of mass.

If f is nonreal, then we conclude from Lemma 4.5.4 that $H \subset \mathbf{T}^2 \setminus \mathcal{D}_f$, and from Lemma 4.5.5 that $P \subset \overline{\mathcal{D}_f}$. By Lemma 4.5.2 we find that $\mathcal{D}_f$ has at most $\mathrm{Vol}(A)$-many connected components. Hence, $\mathcal{D}_f$ has at exactly one connected component in each of the cells of H, and the number of roots of $f(z) = 0$ projected by the argument map into each such component is exactly one.

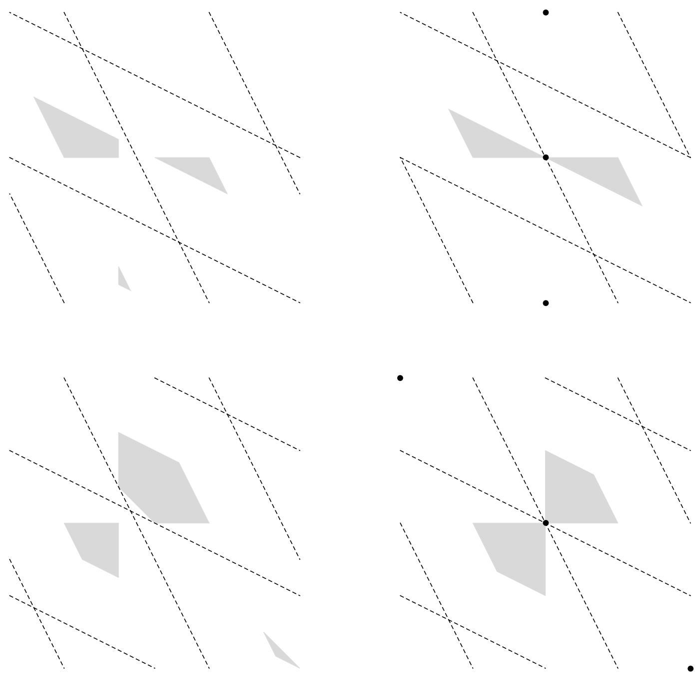

Figure 4.4: The lopsided coamoebas from Example 4.5.6.

Consider now the real case when x_2 and x_4 differ in sign. Then, at least one of F_1 and F_2 are colopsided at each point in the intersection $\mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$. Thus, if $\operatorname{Arg}^{-1}(\theta)$ contains a root of $f(z) = 0$, then each sufficiently small neighborhood N_θ of θ intersect at most two of the cells of the hyperplane arrangement H. Hence, using Lemma 4.5.2, and wiggling the arguments of coefficients of f by ε, we find that N_θ intersect at most two of the polygons of $\mathcal{D}_{f^\varepsilon}$. Hence, there can be at most two roots contained in $\operatorname{Arg}^{-1}(\theta)$.

Consider now the case when f real with x_2 and x_4 equal in sign. In this case, a point $\theta \in \mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$ can be contained in $\mathcal{D}_f$. See the left picture of Figure 4.5, where the hyperplane arrangement H is given in black, and the shells $\mathcal{H}_{F_1}$ and $\mathcal{H}_{F_2}$ are drawn in dotted respectively dashes lines, with indicated orientation. (The dash-dot line is contained in both shells.) Wiggling the arguments of coefficient x_1 and/or x_3 by ε, we claim the we obtain a situation as in the right picture of Figure 4.5. That is, at most two polygons of $\mathcal{D}_{f^\varepsilon}$ intersect a small neighborhood N_θ of θ. Let us prove this last claim.

Let f be generic, with x_2 and x_4 real and equal in sign. The hyperplanes $\mathcal{C}_{g_1}$ and $\mathcal{C}_{g_2}$ locally divide the plane into four regions. We can assume that $\mathbf{a}_2 = \mathbf{0}$. Then, $\mathcal{C}_{g_1}$ consists of all θ such that $\hat{f}_1(\theta) = 1$, and $\mathcal{C}_{g_1}$ consists of all θ such that $\hat{f}_3(\theta) = 1$. Thus, locally, the cells of H can be indexed by the signs

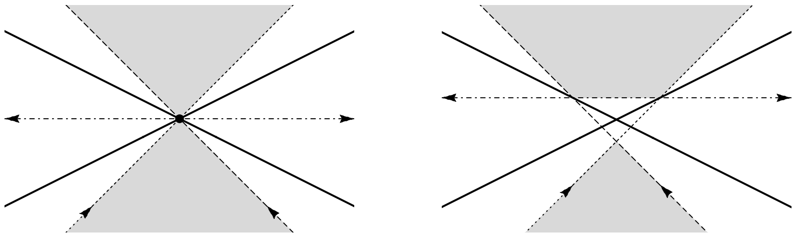

Figure 4.5: To the left: the coamoeba $\mathcal{D}_f$ close to a point of $\mathcal{C}_{g_1} \cap \mathcal{C}_{g_2}$ when x_2 and x_4 have equal in signs and x_1 and x_3 are real. To the right: the same picture after wiggling the argument of x_1 or x_3.

of the imaginary parts of $\hat{f}_1(\theta)$ and $\hat{f}_3(\theta)$. Assume that $\widetilde{\theta} \in \mathcal{D}_f \cap N_\theta$. Then neither F_1 nor F_2 is colopsided at $\widetilde{\theta}$. Observe that $\hat{f}_2(\widetilde{\theta}) = \hat{f}_4(\widetilde{\theta})$, since x_2 and x_4 are equal in sign. We find that

$$\operatorname{sgn}(\Im(\hat{f}_1(\widetilde{\theta}))) = -\operatorname{sgn}(\Im(\hat{f}_2(\widetilde{\theta}))) = -\operatorname{sgn}(\Im(\hat{f}_4(\widetilde{\theta}))) = \operatorname{sgn}(\Im(\hat{f}_4(\widetilde{\theta}))),$$

where the first and the last equality hold since neither F_1 nor F_2 is colopsided at $\widetilde{\theta}$. This implies that each polygon of $\mathcal{D}_f$ that intersects a small neighborhood of θ is necessarily contained in one of the cells of H which correspond to that the imaginary parts of $\hat{f}_1(\widetilde{\theta})$ and $\hat{f}_4(\widetilde{\theta})$ are equal in sign. As there are two such cells, we find that there are at most two polygons of $\mathcal{D}_f$ that intersect a small neighborhood of θ. $\square$

5. Hypergeometric functions

In this chapter, we study solutions of the A-hypergeometric system $H_A(\beta)$ from Definition 2.1.7. For convenience, we assume that $\iota_A = 1$. We will focus on the dependence of A-hypergeometric functions on the parameter β. In particular, a function $\Phi(\beta; x)$ that is analytic in β in a domain $V \subset \mathbb{C}_*^{q+n}$ is said to be A-hypergeometric if $\Phi(\beta; x) \in \mathrm{Sol}_x(H_A(\beta))$ for all $\beta \in V$. We take in this chapter the more general approach that A is a Cayley configuration of q sets $A_1, \ldots, A_q$. Our primary tool will be Euler type hypergeometric integrals;

$$M_C(\beta; x) := \int_C \frac{f(z)^{\beta_\mathbf{1}}}{z^{\beta_\mathbf{2}}} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n}, \tag{5.1}$$

where $f \in \mathbb{C}_*^A$ and C is some n-cycle. Notice that we employ multi-index notation both in the numerator and in the denominator. We collect the two vectors $\beta_\mathbf{1} = (\beta_1, \ldots, \beta_q)$ and $\beta_\mathbf{2} = (\beta_{q+1}, \ldots, \beta_{q+n})$ in the parameter vector $\beta = (\beta_\mathbf{1}, \beta_\mathbf{2})$.

Theorem 5.0.7 (Gelf'and, Kapranov, and Zelevinsky, [18])**.** *If the integral* (5.1) *converges and defines a germ of an analytic function in the variables x, then it represents a solution to the A-hypergeometric system $H_A(\beta)$.* □

If C is compact then the convergence of (5.1) is immediate. The main theorem of [18] states that if β is a non-resonant parameter then one can find compact cycles $C_i \subset \mathbb{C}_*^n \setminus Z(f)$, for $i = 1, \ldots, \mathrm{Vol}(A)$, such that the corresponding integrals (5.1) is a basis of $\mathrm{Sol}_x(H_A(\beta))$.

The integral (5.1) is multivalued in two senses. Firstly, analytic continuation along a loop in $\mathbb{C}_*^A$ need not preserve the cycle C. Secondly, we need to choose branches of the exponential functions $f(z)^{\beta_\mathbf{1}}$ and $z^{\beta_\mathbf{2}}$. Note that the values of the integral $M_C(\beta; x)$ for different branches of the exponential functions differ only by multiplication with an exponential function in the parameters. In particular, for a fixed cycle C, the $\mathbb{C}$-vector space spanned by the integral in question is uniquely determined.

While (5.1) defines an analytic function of the parameter β, understanding the behavior of the solution space $\mathrm{Sol}_x(H_A(\beta))$ is a more subtle question. In general, the nth singular homology group $H_n(\mathbb{C}_*^n \setminus Z(f), \mathbb{C})$ has rank greater than $\mathrm{Vol}(A)$, and it is per se not clear that the compact cycles generating a basis of $\mathrm{Sol}_x(H_A(\beta))$ can be chosen independently of β. A similar problem occurs when using series to construct a basis of $\mathrm{Sol}_x(H_A(\beta))$; the cardinality of the set of possible initial monomials of such series is greater than expected.

An additional difficulty occurs since we wish also to include resonant parameters in our study. In particular, at a rank-jumping parameter $\beta \in \mathscr{E}_A$ one cannot span the full solution space using integrals of the form (5.1). In fact, it follows from results in Section 5.2 that at rank-jumping parameters not all solutions are analytic in the parameters.

To overcome the latter difficulty we introduce the following stratification of the parameter space $\mathbb{C}^{q+n}$. The resonant arrangement $\mathscr{R}_A$ defines a filtration

$$S_0 \subset S_1 \subset \cdots \subset S_{q+n} = \mathbb{C}^{q+n}$$

whose component S_d in degree d consist of the union of all intersections of $q+n-d$ resonant hyperplanes. In our stratification, a d-dimensional stratum s is a connected component of $S_d \setminus S_{d-1}$, where $S_{-1} = \emptyset$ by convention. Furthermore, the set of all strata (including the empty set) has the structure of a poset lattice by the partial order defined by that $s_2 \prec s_1$ if $s_2 \subset \overline{s}_1$. In a poset lattice an element s_1 is said to *cover* s_2 if $s_2 \prec s_1$ and for any $s_2 \prec s_3 \prec s_1$ we have that exactly one of $s_3 = s_1$ and $s_3 = s_2$ holds. Let $\hat{s}$ denote the set of all stratum that covers s.

The computations performed in [5] implies that the rank of the solution space $\mathrm{Sol}_x(H_A(\beta))$ is constant along each stratum s. Let us denote this constant by $\mathrm{rk}(s)$. The present work suggests the following conjecture.

Conjecture 5.0.8. Let $U \subset \mathbb{C}^A_* \setminus Z(D_A)$ be a simply connected domain. In the following statements, each function $\Phi(\beta; x)$ is analytic for $x \in U$.

i) For each stratum s there are functions $\Phi_i(\beta; x)$, $i = 1, \ldots, \mathrm{rk}(s)$, analytic in s and meromorphic on the closure $\overline{s}$, which generate the solution space $\mathrm{Sol}_U(H_A(\beta))$ for each $\beta \in s$.

ii) Let s' cover s. Then, there are functions $\Phi_i(\beta; x)$, $i = 1, \ldots, \mathrm{rk}(s')$, analytic in $s' \cup s$, which for each $\beta \in s' \cup s$ generate a subspace $\mathrm{Sol}_U^{s'}(H_A(\beta)) \subset \mathrm{Sol}_U(H_A(\beta))$ of rank $\mathrm{rk}(s')$.

iii) For each s with $\dim(s) < q+n$, and for each $\beta \in s$, it holds that

$$\mathrm{Sol}_U(H_A(\beta)) = \sum_{s' \in \hat{s}} \mathrm{Sol}_U^{s'}(H_A(\beta)), \qquad (5.2)$$

where, for each β, the summation is understood as a sum of vector subspaces of $\mathrm{Sol}_U(H_A(\beta))$. ◯

This chapter is organized as follows. In Section 5.1 we develop the theory of *Euler–Mellin integrals*, a specific Euler type integral related to the coamoeba $\mathcal{C}$. In Section 5.2 we prove some general properties of solutions of differential equations depending on parameters, in particular implying that Conjecture 5.0.8 is true in the case that $\mathscr{E}_A = \emptyset$. In Section 5.3 we study Euler–Mellin integrals for resonant parameters in detail, culminating in the proof of Conjecture 5.0.8 in the case when A represent a monomial curve (i.e., when $n = 1$).

5.1 Euler–Mellin integrals

By an *Euler–Mellin integral* we mean the integral (5.1) taken over a cycle $C = \mathrm{Arg}^{-1}(\theta)$. In order for such an integral to converge, restrictions must a priori be placed on both the exponent vector β and the polynomial f. After considering the conditions for convergence, we will discuss the relation to coamoebas and the question of whether it is possible to construct a basis of the A-hypergeometric system using Euler–Mellin integrals.

Let $\mathcal{N}_i$ denote the convex hull of A_i, and let $\mathcal{N}$ denote the Minkowski sum $\mathcal{N}_1 + \cdots + \mathcal{N}_q$.

5.1.1. Convergence. Let us initially consider the case $C = \mathbb{R}^n_+$. We will now provide a domain of convergence for the Euler–Mellin integral (5.1) as a function of the parameter β, generalizing [37, Theorem 1].

Definition 5.1.1. The polynomial f is said to be *completely nonvanishing* on a set X if for each face $\Gamma \prec \mathcal{N}$ (including $\mathcal{N}$ itself), the truncated polynomial f_Γ has no zeros on X. ◯

For a vector $b_\mathbf{1} \in \mathbb{R}^q$, we denote by $b_\mathbf{1}\mathcal{N}$ the weighted Minkowski sum $\sum_{i=1}^q b_{\mathbf{1}i}\mathcal{N}_i$ of the Newton polytopes $\mathcal{N}_i$ with respect to $b_\mathbf{1}$.

Theorem 5.1.2. *If each of the polynomials $f_1, \ldots, f_q$ is completely nonvanishing on the positive orthant $\mathbb{R}^n_+$, then the Euler–Mellin integral $M_0(\beta; x)$ from* (5.1) *converges and defines an analytic function in the tube domain*

$$\left\{\beta \in \mathbb{C}^{n+q} \mid b_\mathbf{1} = -\Re(\beta_\mathbf{1}) \in \mathbb{R}^q_+,\ b_\mathbf{2} = -\Re(\beta_\mathbf{2}) \in \mathrm{int}(b_\mathbf{1}\mathcal{N})\right\}. \qquad (5.3)$$

Proof. It suffices to prove that for any β with all components of $b_\mathbf{1}$ positive and $b_\mathbf{2} \in \mathrm{int}(b_\mathbf{1}\mathcal{N})$, there exist positive constants c and K such that

$$\left|f(e^\tau)^{-\beta_1} e^{\langle \beta_2, \tau\rangle}\right| = \left|f(e^\tau)^{-\beta_1}\right| e^{-\langle b_2, \tau\rangle} \geq c e^{K|\tau|} \qquad \text{for all} \quad \tau \in \mathbb{R}^n.$$

In fact, it is enough to show that this inequality holds outside of some ball $B(0)$ in $\mathbb{R}^n$.

Since $b_\mathbf{2} \in \mathrm{int}(b_\mathbf{1}\mathcal{N})$, we can expand $b_\mathbf{2}$ as a sum $b_\mathbf{2} = b_{\mathbf{2}1} + \cdots + b_{\mathbf{2}q}$ of q vectors such that $b_{\mathbf{2}i}/b_{\mathbf{1}i} \in \mathrm{int}(\mathcal{N}_i)$. It is shown in the proof of [37, Theorem 1] that for each $b_{\mathbf{2}i} \in \mathrm{int}(\mathcal{N}_i)$ there are positive constants τ_i and K_i such that for τ outside of some ball $B_i(0)$,

$$|f_i(e^\tau)|e^{-\langle b_{\mathbf{1}i},\tau\rangle} \geq \tau_i e^{K_i|\tau|}.$$

Note that it is essential in [37, Theorem 1] that f_i is completely nonvanishing on the positive orthant. Thus for τ outside of $B(0) = \bigcup_{i=0}^q B_i(0)$, we have

$$|f(e^\tau)^{-\beta_1}|e^{-\langle b_\mathbf{2},\tau\rangle} = \prod_{i=1}^q \left(|f_i(e^\tau)|e^{-\left\langle \frac{b_{\mathbf{2}i}}{b_{\mathbf{2}i}},\tau\right\rangle}\right)^{b_{\mathbf{1}i}} \geq \prod_{i=1}^q \tau_i^{b_{\mathbf{1}i}} e^{b_{\mathbf{1}i}K_i|\tau|} = ce^{K|\tau|},$$

where $c = \tau_1^{b_{11}} \cdots \tau_q^{b_{\mathbf{1}q}}$ and $K = b_{\mathbf{1}1}K_1 + \cdots + b_{\mathbf{1}q}K_q$ are the desired positive constants. □

Example 5.1.3. By a classical integral representation of the Gauss hypergeometric function ${}_2F_1$,

$$\int_0^\infty \frac{(1+z)^{\beta_1}(\xi+z)^{\beta_2}}{z^{\beta_3}} \frac{dz}{z} = G(\beta)\, {}_2F_1\left(\begin{matrix} -\beta_2, \beta_1-\beta_2-\beta_3 \\ -\beta_1-\beta_2 \end{matrix} ;1-\xi\right), \qquad (5.4)$$

where

$$G(\beta) = \frac{\Gamma(\beta_3-\beta_1-\beta_2)\Gamma(-\beta_3)}{\Gamma(-\beta_1-\beta_2)}.$$

To ensure convergence, we need that $\Re(\beta_1+\beta_2) < \Re(\beta_1+\beta_2-\beta_3) < 0$ and that $|\arg_\pi(\xi)| < \pi$. The latter condition is equivalent to that $f(z) = (1+z)(\xi+z)$ is completely nonvanishing on $\mathbb{R}_+$. Since $\mathcal{N}_1 = \mathcal{N}_2 = [0,1]$, the condition that $b_\mathbf{2} \in \mathrm{int}(b_\mathbf{1}\mathcal{N})$ is equivalent to $0 > \Re(\beta_3) > \Re(\beta_1+\beta_2)$. We note further that the right hand side of (5.4) is analytic in this domain. ○

As the right hand side of (5.4) is a meromorphic function in β, it provides a meromorphic extension of the corresponding Euler–Mellin integral. On this right side, we have the regularized ${}_2F_1$ as one factor, thus the polar locus of the meromorphic extension is contained in two families of hyperplanes given by the polar loci of the Gamma functions. That this kind of meromorphic continuation is possible for all Euler–Mellin integrals is the essence of Hadamard's *partie finie*, as understood by Riesz [49]. We will perform this meromorphic extension explicitly, as it makes use of combinatorial data that is crucial for our description of the behavior of Euler–Mellin integrals at resonant parameters.

To obtain the strongest form of this result, we choose a specific presentation for $b_{\mathbf{1}}\mathcal{N}$. To begin, each Newton polytope $\mathcal{N}_i$ can be written uniquely as the intersection of a finite number of halfspaces

$$\mathcal{N}_i = \bigcap_{j=1}^{M_i} \left\{ b_{\mathbf{2}} \in \mathbb{R}^n \mid \langle \mu_{ij}, b_{\mathbf{2}} \rangle \geq \nu_{ij} \right\}, \tag{5.5}$$

where the μ_{ij} and the ν_{ij} are primitive integer vectors.

Fixing an order, let $\{\mu_1, \ldots, \mu_M\}$ be equal to the set $\{\mu_{ij} \mid 1 \leq i \leq q, \leq j \leq M_i\}$, where we assume that $\mu_i \neq \mu_j$ for all $i \neq j$. We now extend the definitions of μ_{ij} from (5.5) to each μ_k; namely, for each k, let $\nu_k = (\nu_{1k}, \ldots, \nu_{qk})$ with

$$\nu_{ik} = \min \left\{ \langle \mu_k, \mathbf{a} \rangle \mid \mathbf{a} \in \mathcal{N}_i \right\},$$

and set $|\nu_k| = \nu_{1k} + \cdots + \nu_{qk}$. By definition of the ν_k, we have $\operatorname{int}(b_{\mathbf{1}}\mathcal{N}) = \sum_{i=1}^{q} b_{\mathbf{1}i} \operatorname{int}(\mathcal{N}_i)$ and

$$b_{\mathbf{1}}\mathcal{N} = \bigcap_{k=1}^{M} \left\{ b_{\mathbf{2}} \in \mathbb{R}^n \mid \langle \mu_k, b_{\mathbf{2}} \rangle \geq \langle \nu_k, b_{\mathbf{1}} \rangle \right\}. \tag{5.6}$$

We are now prepared to state the main result of this section, which provides a meromorphic continuation of (5.1), generalizing [37, Theorem 2].

Theorem 5.1.4. *If the polynomials $f_1, \ldots, f_q$ are completely nonvanishing on the positive orthant $\mathbb{R}^n_+$ (as in Definition 5.1.1) and the Newton polytope $\mathcal{N}$ is of full dimension n, then the Euler–Mellin integral $M(\beta; x)$ admits a meromorphic continuation of the form*

$$M(\beta; x) = \Phi(\beta; x) \prod_{k=1}^{M} \Gamma(\langle \nu_k, \beta_{\mathbf{1}} \rangle - \langle \mu_k, \beta_{\mathbf{2}} \rangle), \tag{5.7}$$

where $\Phi(\beta; x)$ is an entire function in β and μ_k, ν_k are given by (5.6).

Proof. By Theorem 5.1.2, the Euler–Mellin integral $M(\beta; x)$ converges for β in the domain

$$\left\{ \beta \in \mathbb{C}^{n+q} \,\middle|\, b_{\mathbf{1}} \in \mathbb{R}^q_+ \text{ and } \langle \mu_k, b_{\mathbf{2}} \rangle > \langle \nu_k, b_{\mathbf{1}} \rangle,\ k = 1, \ldots, M \right\},$$

which is a domain since $\mathcal{N}$ is of full dimension. Our goal is to expand the convergence domain of the integral (5.1), at the cost of multiplication by factors corresponding to the poles of the Gamma functions appearing in (5.7). We do this iteratively, integrating by parts in the direction of a vector μ_k at each step. This expands the domain of convergence in the opposite direction of μ_k by a distance d_k, which we determine explicitly.

To begin, we set notation for the first iteration in one direction. Fix k between 1 and M, and let Γ be the face of $\mathcal{N}_i$ corresponding to μ_k and ν_k. For $\mathbf{a} \in A$, consider the integers

$$d_k^{\mathbf{a}} = \langle \mu_k, \mathbf{a} \rangle - |\nu_k|.$$

Since $\mathbf{a} \in \mathcal{N}$, it follows that $d_k^{\mathbf{a}} \geq 0$. In particular, since there is a decomposition $\mathbf{a} = \sum_i \mathbf{a}_i$ with $\mathbf{a}_i \in \mathcal{N}_i$, we see that $d_k^{\mathbf{a}} = 0$ if and only if $\langle \mu_k, \mathbf{a}_i \rangle = \nu_{ik}$ for all i.

Let i be fixed. The polynomial $(f_i)_\Gamma$ has the homogeneity $(f_i)_\Gamma(\lambda^{\mu_k} z) = \lambda^{\nu_{ik}} (f_i)_\Gamma(z)$, where $\lambda \in \mathbb{C}_*$ and $\lambda^{\mu_k} z = (\lambda^{\mu_{1k}} z_1, \lambda^{\mu_{2k}} z_2, \ldots, \lambda^{\mu_{nk}} z_n)$. Hence the coefficients of the scaled polynomial $\lambda^{-\nu_{ik}} (f_i)_\Gamma(\lambda^{\mu_k} z)$ are independent of k and λ. In particular, we have that the Newton polytope of

$$f_i'(z) = \frac{d}{d\lambda} \Big(\lambda^{-\nu_{ik}} f_i(\lambda^{\mu_k} z) \Big)\Big|_{\lambda=1}$$

is disjoint from Γ. This fact allows us to extend the domain of convergence of (5.1) over the hyperplane defined by $\langle \mu_k, b_\mathbf{2} \rangle = \langle \nu_k, b_\mathbf{1} \rangle$ as follows. Since $M(\beta; x)$ is independent of λ, we have

$$0 = \frac{d}{d\lambda} \int_{\mathbb{R}_+^n} \frac{f(\lambda^{\mu_k} z)^{\beta_\mathbf{1}}}{(\lambda^{\mu_k} z)^{\beta_\mathbf{2}}} \frac{dz}{z} = \frac{d}{d\lambda} \left(\lambda^{\langle \nu_k, \beta_\mathbf{1} \rangle - \langle \mu_k, \beta_\mathbf{2} \rangle} \int_{\mathbb{R}_+^n} \frac{\lambda^{\langle \nu_k, \beta_\mathbf{1} \rangle} f(\lambda^{\mu_k} z)^{\beta_\mathbf{1}}}{z^{\beta_\mathbf{2}}} \frac{dz}{z} \right).$$

Differentiating with respect to λ and setting $\lambda = 1$ yields the identity

$$M(\beta; x) = \frac{1}{\langle \nu_k, \beta_\mathbf{1} \rangle - \langle \mu_k, \beta_\mathbf{2} \rangle} \int_{\mathbb{R}_+^n} \frac{g_k(z) f(z)^{\beta_\mathbf{1} - 1}}{z^{\beta_\mathbf{2}}} \frac{dz}{z}, \tag{5.8}$$

where g_k is the polynomial

$$g_k = \sum_{i=1}^{q} -\beta_{\mathbf{1}i} \cdot f_1 \cdots f_i' \cdots f_q$$

and $\beta_\mathbf{1} - 1 = (\beta_1 - 1, \ldots, \beta_q - 1)$. Note that $\operatorname{supp}(g_k)$ is contained in A. Moreover, since Γ is the face of $\mathcal{N}$ corresponding to μ_k and $\operatorname{supp}(f_i')$ is disjoint from $\mathcal{N}_i \cap \Gamma$, we see that $\operatorname{supp}(g_k)$ is disjoint from Γ. In other words, $d_k^{\mathbf{a}} > 0$ for each $\mathbf{a} \in \operatorname{supp}(g_k)$.

Let us rewrite (5.8) as the sum

$$M(\beta; x) = \sum_{\mathbf{a} \in \operatorname{supp}(g_k)} \frac{h_\mathbf{a}(\beta_\mathbf{1})}{\langle \nu_k, \beta_\mathbf{1} \rangle - \langle \mu_k, \beta_\mathbf{2} \rangle} \int_{\mathbb{R}_+^n} \frac{f(z)^{\beta_\mathbf{1} - 1}}{z^{\beta_\mathbf{2} - \mathbf{a}}} \frac{dz}{z}, \tag{5.9}$$

for some linear polynomials $h_\mathbf{a}(\beta_\mathbf{1})$, and note that each term of (5.9) is a translation in β of the original Euler–Mellin integral. By Theorem 5.1.2, the term corresponding to $\mathbf{a}$ converges on the domain given by $b_\mathbf{1} + 1 > 0$ and

$$\langle \mu_j, b_\mathbf{2} + \mathbf{a} \rangle > \langle \nu_j, b_\mathbf{1} + 1 \rangle, \quad \text{for } j = 1, \ldots, M,$$

where the latter is equivalent to

$$\langle \mu_j, b_{\mathbf{2}} \rangle > \langle \nu_j, b_{\mathbf{1}} + 1 \rangle - \langle \mu_j, \mathbf{a} \rangle = \langle \nu_j, b_{\mathbf{1}} \rangle - d_j^{\mathbf{a}}, \quad \text{for } j = 1, \ldots, M.$$

The sum (5.9) converges on the intersection of these domains, given by

$$b_{\mathbf{1}} + 1 > 0, \quad \langle \mu_j, b_{\mathbf{2}} \rangle > \langle \nu_j, b_{\mathbf{1}} \rangle \quad \text{if } j \neq k, \quad \text{and} \quad \langle \mu_k, b_{\mathbf{2}} \rangle > \langle \nu_k, b_{\mathbf{1}} \rangle - d_k,$$

where $d_k = \min\{d_k^{\mathbf{a}} \mid \mathbf{a} \in \operatorname{supp}(g_k)\}$. Since d_k is positive, (5.9) has a strictly larger domain of convergence than (5.1); we say that it has been extended by the "distance" d_k in the direction μ_k.

Before iterating this procedure, we set some notation. Let G_k be the semigroup generated by the integers $\{d_k^{\mathbf{a}} \mid \mathbf{a} \in \operatorname{supp}(f) \text{ and } 1 \leq k \leq M\} \subseteq \mathbb{N}$. Let $\alpha = (\mathbf{a}_1, \ldots, \mathbf{a}_r)$ be an ordered r-tuple with $\mathbf{a}_i \in \operatorname{supp}(f)$ for each i. We sometimes write α as an exponent of z, viewing $\alpha = \mathbf{a}_1 + \cdots + \mathbf{a}_r$. Similarly, set $d_k^{\alpha} = d_k^{\mathbf{a}_1} + \cdots + d_k^{\mathbf{a}_r} \in G_k$.

After r iterations, let $\mu_{j(i)}$ denote the direction of the extension in the ith iteration. Let $d_{j(i)}^{\alpha_i} = d_{j(i)}^{\mathbf{a}_1} + \cdots + d_{j(i)}^{\mathbf{a}_{i-1}} \in G_{j(i)}$ be the sum of the distances of the first $i-1$ components of α in the direction $\mu_{j(i)}$. Then there is a rational function of the type

$$L_\alpha(\beta) = \prod_{i=1}^{r} \frac{h_{\alpha_i}(\beta_{\mathbf{1}})}{\langle \nu_{j(i)}, \beta_{\mathbf{1}} \rangle - \langle \mu_{j(i)}, \beta_{\mathbf{2}} \rangle + d_{j(i)}^{\alpha_i}}, \tag{5.10}$$

where $h_\alpha(\beta_{\mathbf{1}}) = (h_{\alpha_1}(\beta_{\mathbf{1}}), \ldots, h_{\alpha_r}(\beta_{\mathbf{1}}))$ is an ordered r-tuple of linear polynomials such that $M(\beta; x)$ can be expressed as a finite sum of translations of the original Euler–Mellin integral:

$$M(\beta; x) = \sum_{\alpha} L_\alpha(\beta) \int_{\mathbb{R}_+^n} \frac{f(z)^{\beta_1 - r}}{z^{\beta_2 - \alpha}} \frac{dz}{z}. \tag{5.11}$$

Fixing k, we next expand the domain of convergence of (5.11) in the direction determined by μ_k. This is achieved through simultaneous expansion of the domains of convergence of all terms, arguing as above. This yields the expression

$$\begin{aligned} M(\beta; x) &= \sum_{\alpha} L_\alpha(\beta) \sum_{\mathbf{a} \in \operatorname{supp}(g_k)} \frac{h_{(\alpha, \mathbf{a})_{r+1}}(\beta_{\mathbf{1}})}{\langle \nu_k, \beta_{\mathbf{1}} \rangle - \langle \mu_k, \beta_{\mathbf{2}} \rangle + d_k^{\alpha}} \int_{\mathbb{R}_+^n} \frac{f(z)^{\beta_1 - r - 1}}{z^{\beta_2 - \alpha - \mathbf{a}}} \frac{dz}{z} \\ &= \sum_{\alpha'} L_{\alpha'}(\beta) \int_{\mathbb{R}_+^n} \frac{f(z)^{\beta_1 - r'}}{z^{\beta_2 - \alpha'}} \frac{dz}{z}, \end{aligned} \tag{5.12}$$

where $\beta' = (\beta, \mathbf{a})$, $r' = r + 1$, and the resulting rational function $L_{\alpha'}(\beta)$ is given by

$$L_{\alpha'}(\beta) = L_\alpha(\beta) \frac{h_{\alpha'_{r'}}(\beta_{\mathbf{1}})}{\langle \nu_k, \beta_{\mathbf{1}} \rangle - \langle \mu_k, \beta_{\mathbf{2}} \rangle + d_k^{\alpha}}.$$

Since the convergence domain of each term in (5.11) is extended by the distance d_k in the direction determined by μ_k, the convergence domain of the sum is similarly extended. In addition, since $d_k^{\mathbf{a}} > 0$, we have that $d_k^{\alpha+\mathbf{a}} > d_k^{\alpha}$; therefore, the products $L_\alpha(\beta)$ will never repeat factors in their denominators. As (5.12) is in the same form as (5.11), we may iterate this procedure to extend the domain of convergence.

Finally, note that after r iterations that have extended the domain of convergence of $M(\beta;x)$ in the direction determined by μ_j for r_j of the r steps, we obtain a meromorphic function on the tube domain given by $\beta \in \mathbb{C}^{n+q}$ such that $b_{\mathbf{1}} + \sum_{j=1}^{M} r_j = b_{\mathbf{1}} + r > 0$ and

$$\langle \mu_j, b_{\mathbf{2}}\rangle > \langle \nu_j, b_{\mathbf{1}}\rangle - q_j d_j, \quad \text{for } j = 1, \ldots, M.$$

Continuing this process, $M(\beta;x)$ can be extended to a meromorphic function in β on $\mathbb{C}^{n+q}$ as in (5.7). We note that because the denominator of the products of the rational functions $L_\alpha(\beta)$ never has repeated terms, all poles of the extended Euler–Mellin integral are simple. It now follows from the removable singularities theorem that $\Phi(\beta;x)$ in (5.7) is an entire function in β, as desired. □

Remark 5.1.5. In the proof of Theorem 5.1.4, we see that the linear form $\langle \mu_k, b_{\mathbf{2}}\rangle - \langle \nu_k, b_{\mathbf{1}}\rangle - d$ appears in the denominator of some rational function L_α if and only if $d \in G_k$. Hence if $G_k \neq \mathbb{N}$, then our meromorphic continuation has introduced unnecessary zeros into the entire function $\Phi(\beta;x)$. ○

Remark 5.1.6. If $q = 1$, then $h_{\alpha_i}(\beta_{\mathbf{1}}) = k_{\alpha_i}(\beta_{\mathbf{1}} - i)$ for some constant k_{α_i}, where h_{α_i} is as in (5.10). Therefore each L_α is divisible by the Pochammer symbol $(-\beta_{\mathbf{1}})_{i+1}$, which can thus be factored outside of the sum (5.11). In particular, $\tilde{\Phi}(\beta;x) = \Gamma(-\beta_{\mathbf{1}})\Phi(\beta;x)$ is an entire function. ○

Example 5.1.7. Consider the case of $q+1$ linear functions of one variable,

$$M(\beta;\xi) = \int_0^\infty \frac{(1+z)^{\beta_0}(\xi_1+z)^{\beta_1}\cdots(\xi_q+z)^{\beta_q}}{z^{\beta_{q+1}}}\frac{dz}{z}. \tag{5.13}$$

Note that we have reindexed β for this example. When $q = 0$, (5.13) is the Beta-function. That is, $\Phi(\beta) = 1/\Gamma(-\beta_1)$, or with the notation of Remark 5.1.6, $\tilde{\Phi}(\beta) = 1$. When $q = 1$, we showed in Example 5.1.3 that

$$\Phi(\beta;\xi) = \frac{1}{\Gamma(-\beta_0-\beta_1)}\,{}_2F_1\left(\begin{matrix} -\beta_1, \beta_2-\beta_0-\beta_1 \\ -\beta_0-\beta_1 \end{matrix}; 1-\xi\right).$$

This equality is obtained by the change of variables $w = z/(1+z)$ and application of the generalized binomial theorem. By similar calculations in the case $q = 2$,

$$\Phi(\beta;\xi) = \frac{1}{\Gamma(-\beta_0-\beta_1-\beta_2)}\, F_1\left(\begin{array}{c}\beta_3-\beta_0-\beta_1-\beta_2, -\beta_1, -\beta_2 \\ -\beta_0-\beta_1-\beta_2\end{array}; 1-\xi_1, 1-\xi_2\right),$$

where F_1 denotes the first Appell series. For arbitrary q and $|\xi_i| < 1$,

$$\Phi(\beta;\xi) = \frac{1}{\Gamma(-|\beta_{\mathbf{1}}|)} \sum_{k\in\mathbb{N}^q} \frac{(\beta_{q+1}-|\beta_{\mathbf{1}}|)_{|k|}}{(-|\beta_{\mathbf{1}}|)_{|k|}} \frac{(-\hat{\beta}_{\mathbf{1}})_k}{k!} (1-\xi)^k,$$

where $|\beta_{\mathbf{1}}| = \beta_0 + \cdots + \beta_q$, and $(-\hat{\beta}_{\mathbf{1}})_k = (-\beta_1)_{k_1}\cdots(-\beta_q)_{k_q}$. ⎔

Example 5.1.8. Consider the case of one linear function of n variables,

$$M(\beta) = \int_{\mathbb{R}_+^n} \frac{(1+z_1+\cdots+z_n)^{\beta_0}}{z_1^{\beta_1}\cdots z_n^{\beta_n}} \frac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_n},$$

where we again have reindexed β. We claim that

$$M(\beta) = \frac{\Gamma(-\beta_1)\cdots\Gamma(-\beta_n)\Gamma(\beta_1+\cdots+\beta_n-\beta_0)}{\Gamma(-\beta_0)},$$

and hence $\Phi(\beta) = 1/\Gamma(-\beta_0)$. This is clear when $n = 1$ because we again have the Beta-function. For $n > 1$ one can argue by induction, making the change of variables given by $w_n = z_n$ and $w_i = z_i/(1+z_n)$ for $i \neq n$. To generalize this example to an arbitrary simplex, consider the Euler–Mellin integral

$$M(\beta) = \int_{\mathbb{R}_+^n} \frac{(1+z^{T_1}+\cdots+z^{T_n})^{\beta_0}}{z_1^{\beta_1}\cdots z_n^{\beta_n}} \frac{dz_1\wedge\cdots\wedge dz_n}{z_1\cdots z_n},$$

where the exponent vectors T_i are the columns of an integer affine transformation T. We find that

$$M(\beta) = \frac{\Gamma(-(T^{-1}\beta)_1)\cdots\Gamma(-(T^{-1}\beta)_n)\Gamma(\beta_0+|T^{-1}\beta|)}{|\det(T)|\,\Gamma(-\beta_0)},$$

after employing the change of variables $w_i = z^{T_i}$. ⎔

For Theorems 5.1.2 and 5.1.4 to hold, each $f_i(z)$ must be completely nonvanishing on the positive orthant. This is a strong restriction that many polynomials will not fulfill. We will now relax this condition by considering the coamoeba of $f(z)$.

Corollary 5.1.9. *For $\theta \in \mathbf{T}^n$, a polynomial $f(z)$ is completely nonvanishing on the set* $\operatorname{Arg}^{-1}(\theta)$ *if and only if $\theta \notin \overline{\mathcal{C}}$.*

Proof. The claim is equivalent to (2.10). □

By Corollary 5.1.9, when polynomials $f_1, \ldots, f_q$ are such that the closure of the coamoeba of $f(z) = \prod_{i=1}^{q} f_i(z)$ is a proper subset of $\mathbf{T}^n$, there is a $\theta \notin \overline{\mathcal{C}}$ for which the Euler–Mellin integral with respect to θ is well-defined:

$$M_\theta(\beta; x) = \int_{\mathrm{Arg}^{-1}(\theta)} \frac{f(z)^{\beta_1}}{z^{\beta_2}} \frac{dz}{z}. \tag{5.14}$$

It is immediate that θ-analogues of Theorems 5.1.2 and 5.1.4 hold as $\mathrm{Arg}^{-1}(\theta)$ is a rotation of $\mathbb{R}_+^n$. In addition, a slight perturbation of θ does not impact the value of (5.14).

Theorem 5.1.10. *The Euler–Mellin integral M_θ of* (5.14) *is a locally constant function in θ. Thus it depends only on the choice of connected component Θ of the complement of $\overline{\mathcal{C}}$. We write $M_\Theta = M_\theta$ and accordingly $\Phi_\Theta = \Phi_\theta$.*

Proof. First consider the case $n = 1$, and suppose that θ_1 and θ_2 lie in the same connected component of the complement of $\overline{\mathcal{C}}$; in fact, assume that the interval $[\theta_1, \theta_2] \subseteq \mathbb{R}^n \setminus \overline{\mathcal{C}}$. In other words, $f(z)$ has no zeros with arguments in this interval, and hence $f(z)^{\beta_1} z^{-\beta_2 - 1}$ is analytic in the corresponding domain. Connecting the two rays $\mathrm{Arg}^{-1}(\theta_1)$ and $\mathrm{Arg}^{-1}(\theta_2)$ with the circle section of radius r yields a closed curve, and the integral of $f(z)^{\beta_1} z^{-\beta_2 - 1}$ over this (oriented) curve is zero by residue calculus. By the proof of Theorem 5.1.2, the integral over the circle section tends to 0 as $r \to \infty$, so the two Euler–Mellin integrals M_{θ_1} and M_{θ_2} are equal.

In arbitrary dimensions, we obtain the desired equality by considering one variable at a time while the remaining variables are fixed. □

5.1.2. Linear independence I: Mellin–Barnes integrals. In this section, we will investigate linear independence of extended Euler–Mellin integrals. Our main tool is so called *Mellin–Barnes integrals*, see [6; 35]. The main result of this section is Theorem 5.1.13, which identifies the set of Mellin–Barnes integral solutions of $H_A(\beta)$ with Euler–Mellin integral solutions from components of the complement of the lopsided coamoeba.

We say that $\beta \in \mathbb{C}^{q+n}$ is *totally non-resonant* for A if the shifted lattice $\beta + \mathbb{Z}^{q+n}$ has empty intersection with any hyperplane spanned by any $q+n-1$ linearly independent columns of A.

Definition 5.1.11. Fix a Gale dual B of A, and let γ be such that $A\gamma = \beta$. Then for $x \in \mathbb{C}^A$, the *Mellin–Barnes integral* has the form

$$L(x) = \int_{(i\mathbb{R})^m} \prod_{i=1}^{N} \Gamma(-\gamma_i - \langle \mathbf{b}_i, w \rangle) x_i^{\gamma_i + \langle \mathbf{b}_i, w \rangle} \, dw_1 \wedge \cdots \wedge dw_m. \tag{5.15}$$

Given $\theta \in \mathbf{T}^n$ and $x \in \mathbb{C}_*^A$, we write

$$L^\theta(x) = L(x_1 e^{i\langle \mathbf{a}_1, \theta\rangle}, \ldots, x_N e^{i\langle \mathbf{a}_N, \theta\rangle}),$$

viewed as the germ of an analytic function at x. ○

The following result on Mellin–Barnes integrals summarizes Corollary 4.2, Theorem 3.1, and Proposition 4.3 of [6].

Theorem 5.1.12 (Beukers, [6])**.** *Consider $x, x_1, \ldots, x_K \in \mathbb{C}_*^A$.*

i) If $\operatorname{Arg}(x)B \in \operatorname{int}(\mathcal{Z}_B)$, then the integral $L(x)$ converges absolutely.

ii) If $\operatorname{Arg}(x)B \in \operatorname{int}(\mathcal{Z}_B)$ and $\gamma_i < 0$ for each i, then $L(x) \in \operatorname{Sol}_x(H_A(\beta))$.

iii) Assume that the m-tuples $\operatorname{Arg}(x_1)B, \ldots, \operatorname{Arg}(x_K)B$ are distinct elements of the set $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(x)B + 2\pi L_B)$. If β is a totally non-resonant homogeneity parameter then the Mellin–Barnes integrals $L(x_1), \ldots, L(x_K)$ are linearly independent.

By choosing $x_1, \ldots, x_K$ as in Theorem 5.1.12, one obtains a set of linearly independent solutions to $H_A(\beta)$ that are in bijective correspondence with the set $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(x)B + 2\pi L_B)$, provided that β is sufficiently generic.

We now show that the order map for the set of components of $\mathbf{T}^n \setminus \overline{\mathcal{D}}$ lifts to a bijection between the set of Mellin–Barnes integrals corresponding to points in $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(x)B + 2\pi L_B)$ and the set of Euler–Mellin integrals arising from the connected components of $\mathbf{T}^n \setminus \mathcal{D}$.

Theorem 5.1.13. *For each $\theta \in \mathbf{T}^n \setminus \overline{\mathcal{D}}$ and β the tubular domain* (5.3)*, the Mellin–Barnes integral $L^\theta(x)$ and Euler–Mellin integral $M_\theta(x)$ satisfy the relation*

$$\iota_B\, L^\theta(x) = 2\pi i\, e^{i\langle \beta_1, \theta\rangle}\, \Gamma(-\beta_1)\, \iota_A\, M_\theta(x),$$

where ι_A and ι_B are the indices of the lattices L_A and L_B respectively.

Proof. As the order map $\operatorname{ord}_{\mathcal{D}}$ from Theorem 3.2.1 maps a point in $\mathbf{T}^n \setminus \overline{\mathcal{D}}$ to a point in the set $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(x)B + 2\pi L_B)$, the Mellin–Barnes integral $L^\theta(x)$ is convergent by Theorem 5.1.12.

By meromorphic extension, it is enough to give the proof in the case when the A-hypergeometric homogeneity parameter β is such that the integral expression in (5.1) converges. We may also assume that A is of the form (3.10) namely

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & A_1 & A_2 \end{pmatrix},$$

where $A_{\mathbf{1}}$ is a nonsingular $n\times n$-matrix; we will use the same decomposition for $x=(x_0,x_{\mathbf{1}},x_{\mathbf{2}})$. For convenience we will take β of the form $\beta=(\beta_{\mathbf{1}},A_{\mathbf{1}}\beta_{\mathbf{2}})$. Let B denote a dual matrix of A of the form

$$B=\begin{pmatrix} * \\ A_{\mathbf{1}}^{-1}A_{\mathbf{2}} \\ -I_m \end{pmatrix} D,$$

where the first row is chosen so that each column of B sums to zero and D is an integer diagonal matrix such that B is an integer matrix. It will later be useful that

$$\frac{\iota_B}{\iota_A}=\frac{|\det(D)|}{|\det(A_{\mathbf{1}})|}. \tag{5.16}$$

To see this, assume that $\iota_A=1$. Following [35, Proposition 4.2], this implies that A can be extended to a $N\times N$ unimodular matrix

$$\widetilde{A}=\begin{pmatrix} 1 & 1 & 1 \\ 0 & A_{\mathbf{1}} & A_{\mathbf{2}} \\ * & * & * \end{pmatrix} \text{ with inverse } \widetilde{A}^{-1}=\begin{pmatrix} * & \widetilde{B} \end{pmatrix}=\begin{pmatrix} * & \tilde{b}_0 \\ * & \widetilde{B}_1 \\ * & \widetilde{B}_2 \end{pmatrix}.$$

It follows that $\widetilde{B}$ is a Gale dual of A, and by the Schur complement formula, $|A_{\mathbf{1}}|=|\widetilde{B}_2|$. As $B=\widetilde{B}T$ for some affine transformation T the equality (5.16) holds.

It is enough to prove the statement with this particular Gale dual B. We have that

$$\begin{aligned} M_\theta(\beta;x) &= \int \frac{f(z)^{\beta_1}}{z^{A_1\beta_2}}\frac{dz}{z} \qquad (5.17)\\ &= \frac{x_{\mathbf{1}}{}^{\beta_2}}{x_0^{|\beta_2|-\beta_1}}\int \frac{\left(1+z^{\mathbf{a}_1}+\cdots+z^{\mathbf{a}_n}+\xi_1^{1/d_1}z^{\mathbf{a}_{1+n}}+\cdots+\xi_m^{1/d_m}z^{\mathbf{a}_{m+n}}\right)^{\beta_1}}{z^{A_1\beta_2}}\frac{dz}{z}, \end{aligned}$$

where the integration takes place over $\mathrm{Arg}^{-1}(\theta)$ in the first integral and over $\mathrm{Arg}^{-1}(\theta+\mathrm{Arg}(x_{\mathbf{1}})A_{\mathbf{1}}^{-1})$ in the second. Let us denote by $M_\theta(\beta;\xi)$ the function given by the integral in (5.17). By Lemma 3.1.9, that $\theta\in\mathbf{T}^n\setminus\overline{\mathcal{D}}$ is equivalent to the convergence of the integral

$$\int \xi^w M_\theta(\beta;\xi)\,\frac{d\xi}{\xi},$$

where the integration takes place over the domain given by the fiber of Arg over the point $\mathrm{Arg}(\xi)=\mathrm{Arg}(x)B$ and w is chosen to fulfill the requirements of Theorem 5.1.2. However, this integral is precisely the Mellin transform with respect to ξ of $M_\theta(\beta;\xi)$ with variables w. Consequently, after making the change of variables $\xi_i^{1/d_i} z^{\mathbf{a}_{i+n}} \mapsto \xi_i$, we find that

$$\mathcal{M}\, M_\theta(\beta;\xi)(w) = \frac{|\det(D)|}{|\det(A_\mathbf{1})|} \int \frac{(1+z_1+\cdots+z_n+\xi_1+\cdots+\xi_m)^{\beta_1}}{z^{\beta_2 - A_\mathbf{1}^{-1}A_2 D w}\xi^{Dw}} \frac{dz\wedge d\xi}{z\xi}.$$

For γ in (5.15), write $\gamma = (\gamma_0, \gamma_\mathbf{1}, \gamma_\mathbf{2})$. Assuming that $\beta_{\mathbf{2}j} > 0$ for all j, that $\beta_1 < |\beta_\mathbf{2}|$ (note that this is in accordance with our previous assumptions on β), and that $-1 \gg \gamma_\mathbf{2} > 0$. We set $\gamma_\mathbf{1} = \beta_\mathbf{2} - A_\mathbf{1}^{-1} A_\mathbf{2}\gamma_\mathbf{2}$ and $\gamma_0 = -|\beta_2| - \beta_\mathbf{1} + \langle \mathbf{b}_0, \gamma_\mathbf{2}\rangle$. It follows that $\gamma_k < 0$ for all k. With this notation, evaluating the above integral as in Example 5.1.8, we find that

$$\mathcal{M}\, M_\theta(\beta;\xi)(w) = \frac{|\det(D)|}{|\det(A_\mathbf{1})|} \frac{\prod_{i=1}^N \Gamma(-\gamma_i - \langle \mathbf{b}_i, w-\gamma_\mathbf{2}\rangle)}{\Gamma(-\beta_\mathbf{1})},$$

Furthermore,

$$\sum_{i=0}^{r-1} \gamma_i \mathbf{a}_i = A\gamma = \begin{pmatrix} \beta_\mathbf{1} \\ A_\mathbf{1}\beta_\mathbf{2} \end{pmatrix}.$$

Turning to the Mellin–Barnes integral, we find that

$$\begin{aligned} L^\theta(x) &= \int_{(i\mathbb{R})^m} \prod_{i=1}^N \Gamma(-\gamma_i - \langle \mathbf{b}_i, w\rangle) x_i^{\gamma_i + \langle \mathbf{b}_i, w\rangle}\, dw \\ &= \int_{\gamma_\mathbf{2}+(i\mathbb{R})^m} \prod_{i=1}^N \Gamma(-\gamma_i - \langle \mathbf{b}_i, w-\gamma_\mathbf{2}\rangle) x_i^{\gamma_i + \langle \mathbf{b}_i, w-\gamma_2\rangle}\, dw \\ &= \frac{e^{i\langle A_\mathbf{1}\beta_\mathbf{2}, \theta\rangle} x_\mathbf{1}^{-\beta_\mathbf{2}}}{x_0^{|\beta_2|-\beta_\mathbf{1}}} \int_{\gamma_\mathbf{2}+(i\mathbb{R})^m} \left(\prod_{i=1}^N \Gamma(-\gamma_i - \langle \mathbf{b}_i, w-\gamma_\mathbf{2}\rangle)\right) \frac{dw}{x^w}. \end{aligned}$$

The bounds in the proof of Theorem 5.1.2 imply that we can apply the Mellin inversion formula, which yields the equality

$$|\det(D)|\, L^\theta(\xi) = 2\pi i\, e^{i\langle A_\mathbf{1}\beta_\mathbf{2}, \theta\rangle}\Gamma(-\beta_\mathbf{1})\, |\det(A_\mathbf{1})|\, M_\theta(\xi).$$

Applying (5.16) thus completes the proof. □

Corollary 5.1.14. *If β is totally non-resonant, then when viewed as analytic germs at some $x \in \mathbb{C}_*^A \setminus Z(D_A)$, the extended Euler–Mellin integrals $\Phi_\Theta(\beta;x)$, where Θ ranges over the components of $\mathbf{T}^n \setminus \overline{\mathcal{D}}$, are linearly independent solutions of the A-hypergeometric system $H_A(\beta)$.*

Proof. Let $\theta_1,\ldots,\theta_K$ be representatives for the components of $\mathbf{T}^n \setminus \overline{\mathcal{D}}$. If the indicated set of extended Euler–Mellin integrals is linearly dependent, then there exist constants $\ell_1, \ldots, \ell_K$ providing a vanishing linear combination of $M_{\theta_1}(\beta; x), \ldots, M_{\theta_K}(\beta; x)$, so that

$$\iota_B \sum_{j=1}^{K} \ell_j e^{i\langle \beta_2, \theta_j \rangle} L^{\theta_j}(x) = 2\pi i\, \Gamma(-\beta_1)\, \iota_A \sum_{j=1}^{K} \ell_j M_{\theta_j}(\beta; x) = 0.$$

It then follows from Theorem 5.1.12 that $\ell_1 = \cdots = \ell_K = 0$. □

5.1.3. Linear independence II: residue integrals. This section contains a brief study of residue integrals in the case $n = 1$. Assume that $\mathbf{0} \in A$, so that $d = \mathrm{Vol}(A)$ is the degree of each polynomial $f \in \mathbb{C}_*^A$. The integrand in (5.1) has, for very general β (including all non-resonant parameters), singularities at $0, \infty$, and at each $\rho \in Z(f)$. This yields $\mathrm{Vol}(A) + 2$ residue integrals

$$\mathrm{Res}_\rho(\beta; x) = \int_{C(\rho)} \frac{f(z)^{\beta_1}}{z^{\beta_2}} \frac{dz}{z},$$

where $C(\rho)$ is a small counterclockwise loop around ρ if $\rho = 0$ or $\rho \in Z(f)$, or it is a large clockwise oriented loop if $\rho = \infty$.

For generic parameters, $H_A(\beta)$ admits only a solution space of dimension $\mathrm{Vol}(A)$. Thus, there are two nontrivial linear relations among these $\mathrm{Vol}(A) + 2$ residue integrals.

Example 5.1.15. Let us consider the case

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Here, $f(z) = x_1 + x_2 z$, so $\rho = -x_1/x_2$ is the unique solution to $f(z) = 0$. Assume that $\arg(\rho) \neq 0$. For any $\theta \neq \mathrm{Arg}(\rho)$, after choosing branches, a integral formula for the Beta-function yields that

$$\int_{\mathrm{Arg}^{-1}(\theta)} \frac{f(z)^{\beta_1}}{z^{\beta_2}} \frac{dz}{z} = \frac{\Gamma(-\beta_2)\Gamma(\beta_2 - \beta_1)}{\Gamma(-\beta_1)} x_1^{\beta_1 - \beta_2} x_2^{\beta_2}.$$

That is, the extended Euler–Mellin integral provides the unique (up to multiplication by scalars) nonzero A-hypergeometric function of parameter β.

The fact that the above Euler–Mellin integral is independent of θ might cause the misconception that the residue integral at ρ vanishes. In fact, to write the residue integral as a difference of Euler–Mellin integrals, we cannot choose branches of the latter so that they coincide. Actually,

$$\mathrm{Res}_\rho(\beta; x) = (1 - e^{2\pi i \beta_1}) M_\theta(\beta; x),$$

where M_θ denotes some branch of the Euler–Mellin integral. As the coefficient is nonvanishing for generic β_1, we find that residue integral at ρ span the solution space for such parameters.

The present example is special in the sense that a change of variables takes the residue at the origin to the residue at ρ. Thus, also the residue at the origin span the full solution space for generic β. ◯

We use the following convention. Let $\rho_1, \ldots, \rho_d$ denote the zeros of f with indices considered modulo d, ordered by their arguments. Similarly, label the components of the complement of $\mathcal{C}$ by Θ_i, where the indexing is chosen so that the sector $\operatorname{Arg}^{-1}(\Theta_i)$ contains ρ_{i-1} and ρ_i in its boundary.

Theorem 5.1.16. *Let β be a non-resonant parameter. Then, the residue integrals $\operatorname{Res}_\rho(\beta; x)$, for $\rho \in Z(f)$, span the solution space $\operatorname{Sol}_x(H_A(\beta))$.*

Proof. Let $U \subset \mathbb{C}_*^A$ be a domain such that

$$\max(|x_2|, \ldots, |x_{N-1}|) \ll \min(|x_1|, |x_N|)$$

for all $x \in U$. It is enough to prove the proposition under the assumption that $x \in U$. Let ℓ denote a loop in $\mathbb{C}_*^A$ such that its induced permutation of the roots $\rho_1, \ldots, \rho_d$ is the d-cycle $(1\,2 \cdots d)$. Since $x \in U$, such a loop can be explicitly constructed as follows. The loop $\{(x_1, e^{2\pi i t} x_N) \mid t \in [0,1]\}$ permutes the roots of the polynomial $x_1 + x_N z^d$; hence, by continuity of roots, we may choose ℓ to be

$$\ell = \Big\{(x_1, x_2, \ldots, x_{N-1}, e^{2\pi i t} x_N) \mid t \in [0,1]\Big\}.$$

Let $\eta^d = e^{2\pi i \beta_2}$. We can choose branches of the residue integrals so that analytic continuation along the loop ℓ takes $\operatorname{Res}_j(\beta; x)$ to $\eta \operatorname{Res}_{j+1}(\beta; x)$, where $\operatorname{Res}_j(\beta; x)$ denotes the residue integral at the root ρ_j. In particular, traversing d times the loop ℓ is equivalent to multiplying each residue integral by $\eta^d \neq 1$. The benefit of this choice of branches lies in the following fact. If $\hat{x}$ is obtained by letting $x_1, \ldots, x_{N-1} \to 0$, which is possible in U, then

$$\eta^j \operatorname{Res}_j(\beta, \hat{x}) = \operatorname{Res}_1(\beta, \hat{x}).$$

Assume that there is a linear relation

$$\sum_{j=1}^{d} c_j \eta^j \operatorname{Res}_j(\beta; x) = 0. \tag{5.18}$$

By performing analytic continuation along the loop ℓ we obtain $\mathrm{Vol}(A)$-many relations, captured by the coefficient matrix

$$C = \begin{pmatrix} c_1\eta & c_2\eta^2 & c_3\eta^3 & \cdots & c_d\eta^d \\ c_d\eta^{d+1} & c_1\eta^2 & c_2\eta^3 & \cdots & c_{d-1}\eta^d \\ c_{d-1}\eta^{d+1} & c_d\eta^{d+2} & c_1\eta^3 & \cdots & c_{d-2}\eta^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2\eta^{d+1} & c_3\eta^{d+2} & c_4\eta^{d+3} & \cdots & c_1\eta^d \end{pmatrix}.$$

Since the number non-trivial relations among the $\mathrm{Vol}(A)+2$ residue integrals is 2, the rowspan of C is of rank at most 2. Assuming that the rank is positive, we will obtain a contradiction.

Case 1: Assume first the the rowspan of C has rank two. Taking linear combinations we can eliminate one coefficient, and then perform analytic continuation along ℓ once more. Thus, we can assume that $c_1 = 0$. Note that, if two consecutive coefficients vanish, then the fact that all 3×3-minors of C vanishes implies that all coefficients vanish.

If c_{d-1} does not vanish, then we can eliminate the first coefficient of the second row using the third row. In the resulting difference, either all coefficients vanish, or we obtain two consecutive coefficients that vanish. Assuming the former, we find that c_2 vanishes, that is, two consecutive coefficients vanish.

Hence, we can assume that c_{d-1} vanishes. Repeating the argument, we find that every second coefficient vanishes. In particular, we can assume that d is even. Under this assumption, that the row span of C is of rank two implies that every second row of C defines the same linear relation. In particular, we have a well-defined quotient

$$\varepsilon = \frac{c_2}{c_d\eta^d} = \frac{c_4}{c_2} = \cdots = \frac{c_d}{c_{d-2}}.$$

Then,

$$\varepsilon = \frac{c_2}{c_d\eta^d} = \frac{1}{\eta^d}\frac{c_2}{c_4}\cdots\frac{c_{d-2}}{c_d} = \frac{1}{\eta^d}\varepsilon^{1-\frac{d}{2}},$$

and hence $\varepsilon = \eta^{-2}$. Normalizing, we can assuming that $c_2 = \eta^{-2}$, implying that $c_{2j} = \eta^{-2j}$. In particular, also taking the limit as $|x_1|, \ldots |x_{N-1}| \to 0$, we find that

$$0 = \mathrm{Res}_1(\beta; \hat{x}) \sum_{j=1}^{\frac{d}{2}} \eta^{2j}.$$

In the right hand side, the first factor is not identically vanishing (it is a change of variables away from the residue integral of Example 5.1.15). Hence, we can conclude that η^2 is a $d/2$th root of unity. In particular, η is a dth root of unity, implying that $e^{2\pi i\beta_2} = 1$, a contradiction.

Case 2: Assume now that the rows of C span a linear space of rank 1. Introducing the quotient

$$\varepsilon = \frac{c_1}{c_d\eta^d} = \frac{c_2}{c_1} = \cdots = \frac{c_d}{c_{d-1}},$$

we proceed as in Case 1 only to conclude that η is a dth root of unity, contradicting that β_2 is generic. $\square$

Theorem 5.1.17. *Let $U \subset \mathbb{C}_*^A$ be a domain such that the roots of f are separated in argument. Then, the Euler–Mellin integrals $M_\Theta(\beta; x)$, where Θ ranges over the set of connected components of $\mathbf{T} \setminus \overline{\mathcal{B}}$ provide a basis for the solution space $\mathrm{Sol}_U(H_A(\beta))$ for each non-resonant parameter β.*

Proof. Considering first a domain V for which the original Euler–Mellin integrals converges. It is clear that the set of Euler–Mellin integrals span the space of residue integrals, and hence they span the full solution space for each non-resonant parameter. $\square$

If A is a circuit, then for each totally non-resonant parameter there exists a Mellin–Barnes, and hence also an Euler–Mellin, basis of solutions. However, we saw in Section 3.2.2 that it is in general not possible to construct neither a Mellin–Barnes nor an Euler–Mellin basis of solutions, as one cannot always find a coamoeba with sufficiently many connected components of its complement. In general $\mathbf{T}^n \setminus \overline{\mathcal{C}}$ has more connected components than $\mathbf{T}^n \setminus \overline{\mathcal{D}}$. In many cases, it is possible to construct a basis of Euler–Mellin integral solutions even though Mellin–Barnes integrals do not suffice.

Example 5.1.18. Consider the point configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 6 \end{pmatrix}.$$

By [36], since the coamoeba of the A-discriminant covers $\mathbf{T}^4$, the maximal number of points in the set $\mathrm{int}(\mathcal{Z}_B) \cap (\mathrm{Arg}(x)B + 2\pi L_B)$ is five. Hence there is not a basis of solutions of $H_A(\beta)$ represented by Mellin–Barnes integrals. However, for a generic polynomial $f \in \mathbb{C}_*^A$ the coamoeba $\mathcal{C}$ has six components in its complement. Thus by Theorem 5.1.17, at each non-resonant β, this set of components provide a basis of solutions of $H_A(\beta)$ represented by extended Euler–Mellin integrals. ◯

5.1.4. An A-hypergeometric rank-jumping example. We conclude this section with an example first studied in [9], where it was shown that some parameters β admit a higher-dimensional solution space for $H_A(\beta)$ than the expected dimension of $\mathrm{Vol}(A)$. The point configuration that we consider is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{pmatrix},$$

for which there is a unique parameter $\beta = (1,2)$ for which the dimension of the solution space of $H_A(\beta)$ is one larger than expected.

Let $\theta \in \Theta$ for a fixed connected component of $\mathbf{T}^2 \setminus \overline{\mathcal{C}}$. In order to calculate the corresponding function Φ_Θ at the rank-jumping parameter $\beta = (1,2)$, we first expand the Euler–Mellin integral M_Θ five times in different directions to obtain convergence. Upon expansion,

$$\begin{aligned} M_\Theta(\beta;x) = & \frac{(-\beta_1)_2}{-\beta_2}\int \frac{z^{-\beta_2}h_1(z)}{f(z)^{2-\beta_1}}\frac{dz}{z} + \frac{(-\beta_1)_3}{-\beta_2}\int \frac{z^{-\beta_2}h_2(z)}{f(z)^{3-\beta_1}}\frac{dz}{z} \\ & + \frac{(-\beta_1)_4}{-\beta_2}\int \frac{z^{-\beta_2}h_3(z)}{f(z)^{4-\beta_1}}\frac{dz}{z} + \frac{(-\beta_1)_5}{-\beta_2}\int \frac{z^{-\beta_2}h_4(z)}{f(z)^{5-\beta_1}}\frac{dz}{z}, \end{aligned} \tag{5.19}$$

where all integrals are taken over $\mathrm{Arg}^{-1}(\theta)$ and $(-\beta_1)_n$ denotes the Pochhammer symbol. This shows that when $\beta = (1,2)$, the entire function Φ_Θ falls into the situation noted in Remark 5.1.6, and we thus ignore the factor (β_1+1) in (5.19). To be explicit,

$$\begin{aligned} h_1(z) = & \frac{3x_2x_3z^4}{1-\beta_2} + \frac{3x_2x_3z^4}{3-\beta_2} + \frac{4x_2x_4z^5}{1-\beta_2} + \frac{4x_2x_4z^5}{4-\beta_2}, \\ h_2(z) = & \frac{36x_1x_3^2z^6}{(3-\beta_2)(\beta_2-4\beta_1+2)} + \frac{48x_1x_3x_4z^7}{(3-\beta_2)(\beta_2-4\beta_1+1)} + \frac{48x_1x_3x_4z^7}{(4-\beta_2)(\beta_2-4\beta_1+1)} \\ & + \frac{64x_1x_4^2z^8}{(4-\beta_2)(\beta_2-4\beta_1)} + \frac{x_2^3z^3}{(1-\beta_2)(2-\beta_2)} + \frac{3x_2^2x_3z^5}{(1-\beta_2)(2-\beta_2)} \\ & + \frac{4x_2^2x_4z^6}{(1-\beta_2)(2-\beta_2)} + \frac{27x_2x_3^2z^7}{(3-\beta_2)(\beta_2-4\beta_1+2)} + \frac{36x_2x_3x_4z^8}{(3-\beta_2)(\beta_2-4\beta_1+1)} \\ & + \frac{36x_2x_3x_4z^8}{(4-\beta_2)(\beta_2-4\beta_1+1)} + \frac{48x_2x_4^2z^9}{(4-\beta_2)(\beta_2-4\beta_1)} + \frac{9x_3^3z^9}{(3-\beta_2)(\beta_2-4\beta_1+2)}, \\ h_3(z) = & \frac{48x_1x_3^2x_4z^{10}}{(3-\beta_2)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)} + \frac{48x_1x_3^2x_4z^{10}}{(4-\beta_2)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+12)} \\ & + \frac{64x_1x_3x_4^2z^{11}}{(4-\beta_2)(\beta_2-4\beta_1+1)^2} + \frac{36x_2x_3^2x_4z^{11}}{(3-\beta_2)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)} \\ & + \frac{36x_2x_3^2x_4z^{11}}{(4-\beta_2)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)} + \frac{48x_2x_3x_4^2z^{12}}{(4-\beta_2)(\beta_2-4\beta_1)(\beta_2-4\beta_1+1)} \\ & + \frac{12x_3^3x_4z^{13}}{(3-\beta_2)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)} + \frac{12x_3^3x_4z^{13}}{(4-\beta_2)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)}, \end{aligned}$$

and

$$
\begin{aligned}
h_4(z) = {} & \frac{64 x_1 x_3^2 x_4^2 z^{14}}{(4-\beta_2)(\beta_2-4\beta_1)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)} \\
& + \frac{48 x_2 x_3^2 x_4^2 z^{15}}{(4-\beta_2)(\beta_2-4\beta_1)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)} \\
& - \frac{16 x_3^3 x_4^2 z^{17}}{\beta_2(4-\beta_2)(\beta_2-4\beta_1)(\beta_2-4\beta_1+1)(\beta_2-4\beta_1+2)}.
\end{aligned}
$$

Each term in (5.19) is a translation in β of the original Euler–Mellin integral which converges at $\beta = (1,2)$. In addition, the lack of a degree 2 term in f is manifested in that no term of any $h_i(z)$ has both $(2-\beta_2)$ and $(\beta_2-4\beta_1+2)$ as factors in its denominator. Thus, there are entire functions Φ_1, Φ_2, and Φ_3 in β such that

$$\Phi_\Theta = (\beta_2-4\beta_1+2)\Phi_1 + (2-\beta_2)\Phi_2 + (2-\beta_2)(\beta_2-4\beta_1+2)\Phi_3.$$

From this expression we see that while $\Phi_\Theta(1,2;x) = 0$ independently of x and Θ. We also obtain two functions Φ_1 and Φ_2 that are solutions of $H_A(\beta)$ at $\beta = (1,2)$. Explicit calculation reveals that

$$\Phi_1(1,2;x) = 2\,\frac{x_2^2}{x_1} \qquad \text{and} \qquad \Phi_2(1,2;x) = 2\,\frac{x_3^2}{x_4},$$

for any choice of Θ. These span the Laurent series solutions of the system $H_A(1,2)$, which has dimension two only at this parameter [9].

5.2 Constructing solutions at resonant parameters

In this section we discuss how one can construct solutions at resonant parameters from linearly independent solutions at non-resonant parameters. We will use classical techniques, applicable in a wider setting than the A-hypergeometric environment. The crucial property of the system $H_A(\beta)$ is the fact that the resonant arrangement is a (locally finite) hyperplane arrangement.

Theorem 5.2.1. *Let $\Phi_1,\dots,\Phi_M$ be germs of analytic functions in x depending continuously on a parameter β. If, for a fixed $\hat\beta$, they are linearly independent (over $\mathbb{C}$), then there is a neighborhood $N(\hat\beta)$ such that they are linearly independent for any $\beta \in N(\hat\beta)$.*

Proof. Assume the contrary. Then, there is a sequence of parameters $\{\beta_k\}_{k=0}^{\infty}$ such that $\beta_k \to \hat{\beta}$ as $k \to \infty$, and, furthermore, for all β_k there exists a linear combination

$$\sum_{j=1}^{M} c_{kj} \Phi_j(\beta_k; x) = 0.$$

Let $c_k = (c_{k1}, \ldots, c_{kM})$. We can consider $\{c_k\}_{k=0}^{\infty}$ as a sequence in the compact space $\mathbb{P}^{M-1}$. In particular we we can choose a convergent subsequence with limit $c \in \mathbb{P}^{M-1}$, which we identify with a representative $c \in \mathbb{C}^M$. By continuity, we find that

$$\sum_{j=1}^{M} c_j \Phi_j(\hat{\beta}; x) = \lim_{k\to\infty} \sum_{j=1}^{M} c_{kj} \Phi_j(\beta_k; x) = 0,$$

contradicting that the function Φ_j are linearly independent for $\hat{\beta}$. □

Let $P_1(x,\partial), \ldots, P_K(x,\partial)$ be linear partial differential operators in $D[\beta]$, the Weyl algebra on $\mathbb{C}^A$ with additional commuting variables β. Let $\varphi_i(\beta)$, for $i = 1, \ldots K$, be analytic functions of $\beta \in \mathbb{C}^{q+n}$. Consider the system $H(\beta)$ of linear partial differential equations, depending on the parameter β,

$$P_i(x,\partial) \bullet \Phi(\beta; x) = \varphi_i(\beta) \Phi(\beta; x), \qquad i = 1, \ldots, K. \tag{5.20}$$

We view $H(\beta)$ as a left ideal in $D[\beta]$ defined by the operators $P_i(x,\partial) - \varphi_i(\beta)$.

Assume further that $\Phi(\beta; x)$ is analytic for x in a domain $U \subseteq \mathbb{C}_*^A$ and β in a domain V. For $\gamma \in \mathbb{C}^{q+n}$, denote by ∇_γ the differentiation operator with respect to β in the direction of γ. Applying ∇_γ to both sides of (5.20) yields

$$P_i(x,\partial) \bullet \nabla_\gamma \Phi(\beta; x) = \nabla_\gamma \varphi_i(\beta) \cdot \Phi(\beta; x) + \varphi_i(\beta) \cdot \nabla_\gamma \Phi(\beta; x), \qquad i = 1, \ldots, K.$$

Thus, if $\beta \in V$ is such that $\Phi(\beta; x) \equiv 0$ for $x \in U$, then it follows that $\nabla_\gamma \Phi(\beta; x)$ solves (5.20) at β. [1] We say that $\nabla_\gamma \Phi(\beta; x)$ has been *constructed from* $\Phi(\beta; x)$ *by taking parametric derivatives.*

If the function $\nabla_\gamma \Phi(\beta; x)$ happens to vanish identically at β, then the parametric derivative procedure can be iterated. We now state a sufficient condition for this algorithm to terminate after a finite number of steps.

Proposition 5.2.2. *Let* $\Phi(\beta; x)$ *be a nontrivial solution to* (5.20) *which is analytic for* x *in a domain* $U \subset \mathbb{C}_*^n$ *and* β *in a domain* V*. Assume further that* $\Phi(\beta; x)$ *vanishes identically when restricted to a hyperplane* $L \subset \mathbb{C}^{q+n}$ *in the parameters. Then, for each* $\beta \in L$ *and* $\gamma \notin T_L(\beta)$ *the above process terminates after a finite number of steps. That is, for each* $\beta \in L$*, there is an integer*

[1] This argument was shown to the author by P. Kurasov.

$p = p(\beta,\gamma)$ such that $\nabla_\gamma^{(p)}\Phi(\beta;x) \not\equiv 0$ for $\beta \in L \cap V$. Furthermore, the number p does not depend on γ, and thus it defines a function $\beta \mapsto p(\beta)$, which in turn is upper semicontinuous in the analytic topology of $\mathbb{C}^{q+n}$.

Proof. Let $v_1,\dots,v_{q+n-1}$ be a basis of $T_L(\beta)$, which we extend to a basis of $\mathbb{C}^{q+n}$ by adding $v_0 = \gamma$. Then

$$\nabla_{v_i}^{(p)}\Phi(\beta;x) \equiv 0 \quad \text{for} \quad i \in \{1,\dots,q+n-1\}, \quad p \geq 0, \quad \text{and} \quad \beta \in L \cap V.$$

If, in addition, $\nabla_\gamma^{(p)}\Phi(\beta;x) = 0$ for all p and $\beta \in L \cap V$, then all mixed derivatives $\nabla_{v_i}^{(p_i)}\nabla_\gamma^{(p)}\Phi(\beta;x)$ vanish. Hence, Taylor's formula implies that $\Phi(\beta;x) = 0$ in a neighborhood of β, a contradiction. That p does not depend on γ follows by a change of variables fixing $v_1,\dots,v_{q+n-1}$. To see that the map $\beta \mapsto q(\beta)$ is upper semicontinuous, it suffices to note that $\nabla_\gamma^{(p)}\Phi(\beta;x) \not\equiv 0$ for $\beta \in L \cap V$ implies that $\nabla_\gamma^{(p)}\Phi(\beta;x) \not\equiv 0$ for β in some (analytic) open neighborhood of β. □

Proposition 5.2.3. *With the hypotheses of Proposition 5.2.2, assume further that the function $\Phi(\beta;x)$ is not identically vanishing in x for any fixed $\beta \in V \setminus L$. Then $\nabla_\gamma^{(p)}\Phi(\beta;x)$ is not identically vanishing in x at β.*

Proof. We can assume that γ is the normal vector for the hyperplane L, so that L is defined by a linear equation $\gamma\cdot\beta = \kappa$. By l'Hôpital's rule, for $\beta \in V \cap L$,

$$\nabla_\gamma^{(p)}\Phi(\beta;x) = \Phi(\beta;x)((\gamma\cdot\beta)-\kappa)^{-p}.$$

Hence for $\beta \in V \cap L$, the function $\nabla_\gamma^{(p)}\Phi(\beta;x)$ can be replaced by the analytic extension to L of the quotient

$$\Phi(\beta;x)((\gamma\cdot\beta)-\kappa)^{-p},$$

which gives a solution of $H_A(\beta)$ which is analytic in V.

The vanishing locus of $\Phi(\beta;x)((\gamma\cdot\beta)-\kappa)^{-p}$ is an analytic subvariety of L of codimension at least one. Thus, by assumption, this vanishing locus is a subvariety of $\mathbb{C}_*^{q+n}$ of codimension at least two, which implies that it is empty. □

A spanning tree T of the stratification of the parameter space $\mathbb{C}^{q+d}$ (considered as a poset lattice), is a function associating to each stratum s of positive codimension an element $T(s)$ in its set of covering strata $\hat{s}$.

Theorem 5.2.4. *Let $U \subset \mathbb{C}_*^A \setminus Z(D_A)$ be a simply connected domain, and let T be a spanning tree of the stratification poset lattice. Then, for each stratum s, there are $\mathrm{Vol}(A)$-many functions $\Phi_i(\beta;x)$, for $i = 1,\dots,\mathrm{Vol}(A)$, constructed from the solutions along $T(s)$, which span a subspace of $\mathrm{Sol}_U(H_A(\beta))$ of rank $\mathrm{Vol}(A)$ for each $\beta \in s$.*

Proof. We use induction over the codimension of the stratum s. The basis of the induction is the full-dimensional stratum $\mathbb{C}^{q+n} \setminus \mathscr{R}_A$, for which we can use the residue integrals of [18] to define the functions $\Phi_i(\beta; x)$.

Assume now that we have proven the existence of the functions $\Phi_i(\beta; x)$ for each stratum of codimension m, and let s be a stratum of codimension $m+1$. In particular, the statement is proven for the stratum $T(s)$. Using Proposition 5.2.2 repeatedly, we construct $\mathrm{Vol}(A)$-many linearly independent solutions along s from the solutions along $T(s)$. By Proposition 5.2.3, the constructed solutions can, considered as meromorphic functions on $\overline{s}$, only be linearly dependent at strata of higher codimension. A similar argument yields that they can only have singularities at strata of higher codimension. In particular, they are analytic in s and span a subspace of $\mathrm{Sol}(H_A(\beta))$ of rank $\mathrm{Vol}(A)$ for each $\beta \in s$. $\square$

Corollary 5.2.5. *If $\mathscr{E}_A = \emptyset$, then all claims of Conjecture 5.0.8 holds.*

Proof. The first and the third claim follows from Theorem 5.2.4. To see that the second claim is true, choose a spanning tree of the stratification poset lattice such that $s' = T(s)$. $\square$

5.3 Projective toric curves

We now aim to prove Conjecture 5.0.8 in the case when A represent a projective toric curve. Without loss of generality, the A-hypergeometric system associated to a projective monomial curve is determined by a matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & d_2 & d_3 & \cdots & d_{N-1} & d \end{pmatrix} \tag{5.21}$$

where $0 = d_1 < d_2 < \cdots < d_{N-1} < d_N = d$ and such that $\iota_A = 1$. The curve alluded to above is the one obtained by taking the Zariski closure in $\mathbb{P}^{N-1}_{\mathbb{C}}$ of the image of the map

$$\mathbb{C}^2_* \to \mathbb{C}^A_* \quad \text{given by} \quad (s,t) \mapsto (s, st^{d_2}, \ldots, st^{d_{N-1}}, st^d).$$

In the case of curves, the article [9] carried out a series of computations to show that

$$\mathscr{E}_A = [(\mathbb{N}A + \mathbb{Z}\Gamma_0) \cap (\mathbb{N}A + \mathbb{Z}\Gamma_d)] \setminus \mathbb{N}A, \tag{5.22}$$

and that $\mathrm{rk}(H_A(\beta)) = \mathrm{Vol}(A) + 1$ for $\beta \in \mathscr{E}_A$. The solutions computed in [9] are specific to integer parameters. An alternative proof of these facts can be found in [51, Section 4.2].

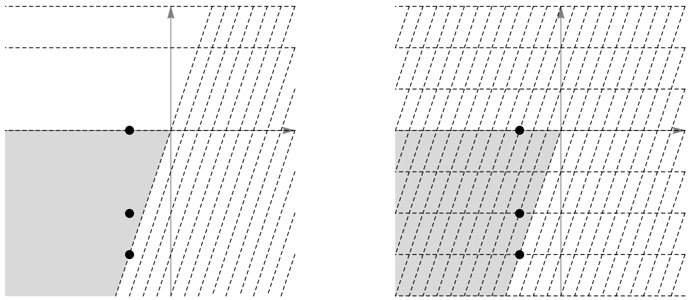

Figure 5.1: The polar lines (left) and resonant lines (right) when The opposites of the points of A are highlighted. The shaded region is the domain of convergence of $M_\Theta(\beta;x)$.

Recall the monoids

$$G_0 = \mathbb{N}\{d-d_{N-1},\dots,d-d_2,d\} \quad \text{and} \quad G_d = \mathbb{N}\{d_2,\dots,d_{N-1},d\} \tag{5.23}$$

which appeared in the meromorphic extension of the Euler–Mellin integral $M_\theta(\beta;x)$.

Definition 5.3.1. We define the *polar locus* of A, denoted by $\mathscr{P}_A$, as the union in $\mathbb{C}^2$ of the zero loci of the linear polynomials $(\gamma_0\cdot\beta)+\kappa$, for $\kappa\in G_0$ and $(\gamma_d\cdot\beta)+\kappa$, for $\kappa\in G_d$. ◯

In other words,

$$\mathscr{P}_A = \bigcup_{\kappa\in G_0} Z((\gamma_0\cdot\beta)+\kappa) \;\cup\; \bigcup_{\kappa\in G_d} Z((\gamma_d\cdot\beta)+\kappa). \tag{5.24}$$

Note that $\mathscr{P}_A$ is strictly contained in the set of resonant parameters $\mathscr{R}_A$, see Figure 5.1. It contains the poles of all Euler–Mellin integrals $M_\Theta(\beta;x)$ by Theorem 5.1.2.

The meromorphic extension (5.12) is in the case of curves given by the expressions

$$M_\Theta(\beta;x) = \frac{-\beta_2}{\gamma_0\cdot\beta}\sum_{i=1}^{N} d_i\, x_i\, M_\Theta(x,\beta-\mathbf{a}_i) \tag{5.25}$$

$$\text{and} \quad M_\Theta(\beta;x) = \frac{-\beta_2}{\gamma_d\cdot\beta}\sum_{i=1}^{N} (d-d_i)\, x_i\, M_\Theta(x,\beta-\mathbf{a}_i). \tag{5.26}$$

When extending the convergence domain of $M_\theta(\beta;x)$, choices must be made as to the order in which the two types of extensions are performed.

We focus on the two orderings given by either first extending over lines resonant with respect to the face Γ_0, and then extending over lines resonant with respect to the face Γ_d, or vice versa.

If β is contained in a polar line L with defining equation $(\gamma_j \cdot \beta) + \kappa = 0$, then restricting the entire function $\Phi_\Theta(\beta; x)$ in β to L, the only terms in the iterated expansion of $M_\Theta(\beta; x)$ that do not vanish are those for which the linear form $(\gamma_j \cdot \beta) + \kappa$ appears in the denominators of their coefficients. Thus, in order to explain the behavior of Φ_Θ at such resonant parameters, it is necessary to carefully track the combinatorics of the coefficients in the expansion process of $M_\Theta(\beta; x)$. This is of particular importance when the parameter β is contained in the intersection of two polar lines.

5.3.1. Polar resonant lines. Let us now study the behavior of Euler–Mellin integrals at parameters that are resonant with respect to precisely one facet of A. We obtain an explicit understanding of how these solutions behave along a line in $\mathscr{P}_A$, while the behavior along nonpolar lines seems to be fundamentally different.

We first consider the case that β is polar, where the behavior is illustrated by the following example. A series solution $\sum_{v \in \mathbb{C}^n} c_v x^v$ of $H_A(\beta)$ is said to have *finite support* if the set $\{v \in \mathbb{C}^n \mid c_v \neq 0\}$ is finite.

Example 5.3.2. For any A as in (5.21), consider the supporting line of the cone (5.3) given by $\beta_2 = 0$, which is contained in $\mathscr{P}_A$. In order to evaluate Φ_Θ, we must expand $M_\Theta(\beta; x)$ once in the direction of μ_0, which yields

$$M_\Theta(\beta; x) = \frac{\beta_1}{\beta_2} \int_{\operatorname{Arg}^{-1}(\theta)} \frac{f'(z) f(z)^{\beta_1 - 1}}{z^{\beta_2 - 1}} \frac{dz}{z},$$

for $\theta \in \Theta$. Hence, along the line $\beta_2 = 0$, the Euler–Mellin integral for Θ evaluates, after applying an extension formula once, as

$$\left. \frac{M_\Theta(\beta; x)}{\Gamma(\beta_2)} \right|_{\beta_2 = 0} = \beta_1 \int_{\operatorname{Arg}^{-1}(\theta)} f'(z) f(z)^{\beta_1 - 1}\, dz = \beta_1 f(0)^{\beta_1} = \beta_1 x_1^{\beta_1}.$$

Most notably, $\Phi_\Theta(\beta_1, 0; x)$ is independent of Θ and equal to a series solution of $H_A(\beta)$ with finite support. ⬡

Theorem 5.3.3. *Let L be a line contained in $\mathscr{P}_A$. Then all extended Euler–Mellin integrals $\Phi_\Theta(\beta; x)$ coincide and evaluate to a finitely supported series solution of $H_A(\beta)$. After possibly removing a factor which is constant with respect to x, this series is nonvanishing in x for each β and defines an analytic function on each domain U in $\mathbb{C}_*^A \setminus Z(D_A)$.*

Proof. It is enough to consider a domain U as in Theorem 5.1.17. We first consider the case $\beta_2 = M$, for some integer M, so that $L = L_0(M)$ is an integer translate of the span of the face Γ_0. From (5.23), since $L \subseteq \mathscr{P}_A$, there exists a partition of M using only $d_2, \ldots, d_N$. Let $P(A, M)$ denote the set of all ordered partitions p of M with parts $d_2, \ldots, d_N$. Let $m_i(p)$ denote the number of times d_i appears in p, and set $m(p) = (m_1(p), \ldots, m_N(p))$, so that $|m(p)|$ is the length of p. Furthermore, for each $i \in \{1, \ldots, |m(p)|\}$, let $s_i = s_i(p)$ denote the ith partial sum of p. Now parameterize the line $L_0(M) = \{\beta \in \mathbb{C}^2 \mid \beta_2 = M\}$ by

$$\lambda \mapsto (\lambda, M) \quad \text{for} \quad \lambda \in \mathbb{C}.$$

We claim that, for each connected component Θ of the complement of $\mathcal{C}$, the restriction of the Euler–Mellin integral $M_\Theta(\beta; x)/\Gamma(\beta_2)\big|_{L_0(M)}$ equals

$$x_1^\lambda \sum_{p \in P(A,M)} \left(\prod_{i=1}^{|m(p)|-1} \frac{\lambda - i}{M + \beta_{\mathbf{1}i}(p)} \right) \left(\prod_{i=2}^{N} d_i^{m_i(p)} \left(\frac{x_i}{x_1} \right)^{m_i(p)} \right). \tag{5.27}$$

Indeed, the only terms in the expansion of $M_\Theta(\beta; x)$ that are relevant at $\beta_2 = M$ are those containing the factor $\beta_2 - M$ in the denominators of their coefficients. This is the case only for terms corresponding to ordered partitions of M, and the term corresponding to $p \in P(A, M)$, when restricted to $L_0(M)$, is the one given in (5.27), including the monomial $x^{m(p)}$. The integral of this term evaluates, in the manner of Example 5.3.2, to $x_1^{\beta_1 - |m(p)|}$.

By symmetry, a similar formula can be given when $d\beta_{\mathbf{1}} - \beta_2 = M$, so that $L = L_d(M)$ is a translate of the span of the face Γ_d. In this case, consider the parameterization of $L_d(M)$ given by

$$\lambda \mapsto (\lambda, d\lambda + M) \quad \text{for} \quad \lambda \in \mathbb{C}.$$

The analog of (5.27), for this line is, is that $M_\Theta(\beta; x)/\Gamma(d\beta_{\mathbf{1}} - \beta_2)\big|_{L_d(M)}$ equals

$$x_N^\lambda \sum_{p \in P(\bar{A},M)} \left(\prod_{i=1}^{|m(p)|-1} \frac{\lambda - i}{M + \beta_{\mathbf{1}i}(p)} \right) \left(\prod_{i=1}^{M-1} (d - d_i)^{m_i(p)} \left(\frac{x_i}{x_N} \right)^{m_i(p)} \right). \tag{5.28}$$

The coefficient of a monomial in (5.27) or (5.28) is a sum of positive multiples of the product

$$\prod_{i=1}^{|m(p)|-1} (\lambda - i), \tag{5.29}$$

and hence it is a positive multiple of (5.29); in particular, the coefficient of a monomial vanishes only if the product (5.29) vanishes. Thus a function of the form (5.27) or (5.28) is nontrivial unless λ is such that all the coefficients vanish, which is equivalent to λ being a positive integer such that every $p \in P(A, M)$ has more than λ terms. However, this vanishing can be removed, as it is caused by a factor $(\lambda - i)$ appearing in all coefficients of (5.27) or (5.28). Removing such factors, (5.27) and (5.28) provide everywhere non-trivial solutions that are analytic in λ along $L_0(M)$ or $L_d(M)$, respectively. □

Corollary 5.3.4. *If $\beta \in \mathscr{P}_A$, then $\operatorname{Res}_\rho(\beta; x) = 0$ for each $\rho \in Z(f)$.*

Proof. This follows Theorem 5.3.3 and the fact that each residue integral can be written as a difference of Euler–Mellin integrals. Notice that there is no ambiguity of branches for resonant parameters. □

The behavior of extended Euler–Mellin integrals along nonpolar resonant lines is more difficult to track. Focusing instead on the residue integrals, we observe a behavior that is different than that along polar lines.

Proposition 5.3.5. *If $\beta \in \mathscr{R}_A \setminus \mathscr{P}_A$ is resonant with respect to Γ_0, then the residue $\operatorname{Res}_0(\beta; x)$ vanishes identically in x. If β is resonant with respect to Γ_d, then $\operatorname{Res}_\infty(\beta; x)$ vanishes identically in x.*

Proof. Consider the case $\beta_2 = M$ for some $M \in \mathbb{Z}$, so that β is resonant with respect to the face Γ_0. If M is a negative integer, then the integrand of (5.1) is analytic in a neighborhood of $z = 0$. Thus the only case to consider is when M is positive, which corresponds to a gap in the set $\mathscr{P}_A$ of polar lines.

The residue integrals, being evaluated over a compact cycle, converge for every β. However, by uniqueness of meromorphic extension, the identities (5.25) and (5.26) hold. Hence

$$\operatorname{Res}_0(\beta; x) = \frac{\beta_1}{\beta_2} \int_{C(0)} \frac{f'(z) f(z)^{\beta_1 - 1}}{z^{\beta_2 - 1}} \frac{dz}{z},$$

where $f'(z)$ denotes the derivative of $f(z)$ with respect to z. By applying this formula repeatedly, we conclude that it is enough to show that each term of $f'(z)$ which corresponds to $d_i > \beta$ vanishes. Indeed, this term equals

$$\frac{\beta_1}{\beta_2} \int_{C(0)} \frac{d_i z_i^{d_i - 1} f(z)^{\beta_1 - 1}}{z^{\beta_2 - 1}} \frac{dz}{z},$$

whose integrand has a zero at the origin of order $d_i - \beta_1 - 1 \geq 0$. The case that β is resonant with respect to the other facet of A follows similarly. The vanishing $\operatorname{Res}_\infty(\beta; x)$ of for non-polar resonant lines with respect to the other facet Γ_d follows from a change of variables. □

Remark 5.3.6. It is not clear if any further dependencies of the residue integrals exist at nonpolar resonant parameters. Thus, the implications of

Proposition 5.3.5 for extended Euler–Mellin integrals also remain unclear. Computational evidence suggests that at nonpolar resonant parameters, the analytic continuations of the extended Euler–Mellin integrals remain linearly independent. If such a statement was proven, the the stratification induced by the resonant arrangement $\mathscr{R}_A$ could be exchanged for the stratification induced by the polar arrangement $\mathscr{P}_A$. ◯

5.3.2. Intersections of polar resonant lines. In this section, we explain the behavior of extended Euler–Mellin integrals at parameters β that lie at the intersection of two resonant lines, so that β is resonant with respect to both faces of A, see Definition 2.1.8. The subdivision of the resonant arrangement into polar and nonpolar lines gives a natural subdivision of these parameters into three cases.

At intersections of two nonpolar lines, and also at intersections of one polar and one nonpolar line, the rank of $H_A(\beta)$ is $\mathrm{Vol}(A) = d$. As discussed in Remark 5.3.6, it not clear how (or if) the extended Euler–Mellin integrals form linear dependencies at such parameters.

Thus for now, we consider only the third case, when a parameter β is contained in the intersection of two polar lines, which, in particular, includes all rank-jumping parameters. Associated to these two lines, respectively, are the two finitely supported series as described in Theorem 5.3.3. Before we (possibly) remove unnecessary (vanishing) coefficients at the end of the proof of Theorem 5.3.3, these two series are evaluations of the same integral and therefore coincide. However, after removing such coefficients, the resulting two series may or may not coincide. For $\beta \in \mathbb{Z}^2$, this is decided by whether or not β is rank-jumping.

Theorem 5.3.7. *Suppose that β is at the intersection of two resonant lines, both of which are contained in $\mathscr{P}_A$.*

i) If $\beta \notin \mathbb{Z}^2$ or $\beta \in \mathscr{E}_A$, then the two solutions obtained in Theorem 5.3.3 at β are finitely supported and linearly independent.

ii) If $\beta \in \mathbb{Z}^2 \setminus \mathscr{E}_A$, then for $x \in (\mathbb{C}^n \setminus Z(D_A))$, then the two solutions obtained in Theorem 5.3.3 at β coincide and equal a single finitely supported series.

If $\beta \in \mathbb{N}A$, [51, Lemma 3.4.10] (or [9, Proposition 1.1]) show that $H_A(\beta)$ has a unique (up to multiplication by a constant) polynomial solution. By Theorem 5.3.7, this solution is given by an extended Euler–Mellin integral.

Proof of Theorem 5.3.7. If $\beta \notin \mathbb{Z}^2$, then the series solutions (5.27) and (5.28) of $H_A(\beta)$ have disjoint supports. In particular, they are linearly independent.

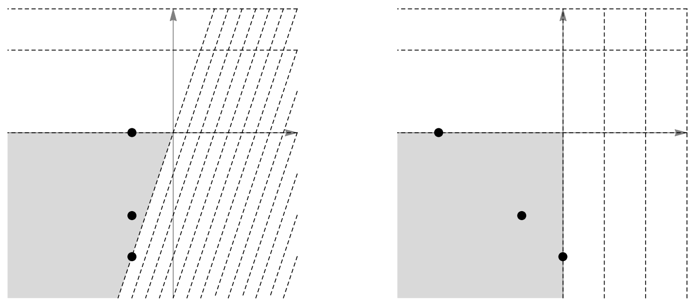

Figure 5.2: The left diagram shows $\mathscr{P}_A$, while the right shows $\mathscr{P}_\Delta$, mimicking Figure 5.1.

To handle the case $\beta \in \mathbb{Z}^2$, consider the matrix $\Delta = TA$, where

$$T = \begin{pmatrix} d & -1 \\ 0 & 1 \end{pmatrix}.$$

Since $T \in \mathrm{GL}_2(\mathbb{Q})$, it induces an isomorphism of A-hypergeometric systems for any $\beta \in \mathbb{C}^2$:

$$H_A(\beta) \cong H_\Delta(\gamma) \quad \text{where } \gamma = T\beta.$$

Figure 5.2 illustrates the change of coordinates induced by T on the parameter space. Let $\mathbb{N}\Delta$ denote the monoid spanned by the columns of Δ.

With this notation, when $\beta \in \mathbb{Z}^2$ is contained in two lines of $\mathscr{P}_A$, consider all quadruples (if any) $(\delta^{i,j} \in \mathbb{N}\Delta \mid 1 \le i, j \le 2)$ such that, with $\gamma = T\beta$, all of the following hold:

$$\delta_1^{1,1} = \gamma_1, \quad \delta_2^{1,1} + \delta_2^{1,2} = \gamma_2, \quad \delta^{1,2} \neq 0, \tag{5.30}$$

$$\delta_2^{2,1} = \gamma_2, \quad \delta_1^{2,1} + \delta_1^{2,2} = \gamma_1, \quad \delta^{2,2} \neq 0. \tag{5.31}$$

The conditions (5.30) and (5.31) are necessary for the equations of the polar lines containing β to appear in the coefficient of some term after the Euler–Mellin integrals have been expanded either first over the face Γ_0, or first over the face Γ_d respectively. From this, we immediately conclude that if there is no quadruple $(\delta^{i,j} \in \mathbb{N}\Delta \mid 1 \le i, j \le 2)$ such that (5.30) and (5.31) hold, then all extended Euler–Mellin integrals vanish.

Let us now show that when $\beta \in \mathbb{Z}^2$, all extended Euler–Mellin integrals vanish at β if and only if β is rank-jumping for A. We first claim that, as $\beta \in \mathbb{Z}^2$ is contained in the intersection of two lines in $\mathscr{P}_A$, there exists a pair $(\delta^{1,1}, \delta^{1,2})$ such that (5.30) holds if and only if there exists a pair $(\delta^{2,1}, \delta^{2,2})$ such that (5.31) holds. To see this, recall that $\Delta = TA$ and $\gamma = T\beta$, for T as above, so $\beta \in \mathbb{N}A$ if and only if $\gamma \in \mathbb{N}\Delta$. Further, by symmetry, it is enough to show that (5.30) is equivalent to $\gamma \in \mathbb{N}\Delta$. Notice that each $\delta \in \mathbb{N}\Delta$ fulfills that $\delta_1 + \delta_2 \in dl$, for some $l \in \mathbb{N}$. This implies that if δ and $\tilde{\delta}$ are such that $\delta_1 = \tilde{\delta}_1$, then $\delta_1 = \delta_2 + h(0,d)^t$ for some $h \in \mathbb{Z}$. If $(\delta^{1,1}, \delta^{1,2})$ satisfies (5.30), then $\delta_1^{1,1} = \gamma_1$ and $\delta_2^{1,1} \leq \gamma_2$. Hence, $\gamma = \delta_1 + h(0,d)^t$ for some $h \geq 0$. Since $(0,d)^t \in \Delta$, $\gamma \in \mathbb{N}\Delta$. Conversely, if $\gamma \in \mathbb{N}\Delta$, then because $(d,0)^t \in \Delta$, $\delta_1 = \gamma$ and $\delta_2 = (d,0)^t$ satisfies (5.30). This establishes the proof of the claim.

From the above calculation, it also follows that the failure of (5.30) is equivalent to β belonging to $(\mathbb{Z}\Gamma_0 + \mathbb{N}A) \setminus \mathbb{N}A$. Similarly, the failure of (5.31) is equivalent to β belonging to $(\mathbb{Z}\Gamma_d + \mathbb{N}A) \setminus \mathbb{N}A$. Thus the desired conclusion for $\beta \in \mathbb{Z}^2 \setminus \mathscr{E}_A$ follows from (5.22).

Finally, if $\beta \in \mathscr{E}_A$, we must show that the two finitely supported solutions recovered from (5.27) and (5.28) are linearly independent. Expressions for the two Laurent polynomial solutions at rank-jumping parameters are known from [9], and they are identified by the negative supports of their monomials. One solution, which by necessity is obtained from (5.27), contains negative powers only of x_1. The other, by necessity is obtained from (5.28), contains negative powers only of x_N, establishing the result. □

5.3.3. The solution space of a projective toric curve. In this section, we aim to prove Conjecture 5.0.8 in the case when A represents a projective toric curve. To complete the proof, we need the following result, which we include without proof.

Theorem 5.3.8 (Matusevich)**.** *Let $\beta \in \mathbb{Z}^2$ be an integer parameter, and let V be a small domain containing β. Let U be as in Theorem 5.1.16. Then, there are series $\Phi_i(\beta; x)$, for $i = 1, \ldots, \mathrm{Vol}(A) - 1$, that converge uniformly in U, such that*

i) each series is analytic in the variables β in V,

ii) for each $\beta \in V$ they span a subspace of the solution space $\mathrm{Sol}(H_A(\beta))$ of rank $\mathrm{Vol}(A) - 1$, and

iii) no series solution of $H_A(\beta)$ in their span has finite support.

Proof of Conjecture 5.0.8 in the case of a curve. Applying the methods of Section 5.2, it only remains to prove that the third claim holds at rank-jumping parameters. By Theorem 5.3.7 we have two finitely supported solutions in $H_A(\beta)$, each of which can be viewed as the restriction of a function that is analytic along one of the resonant lines containing β. We need only to show that each of these two finitely supported solutions is not contained in the linear span of the $\mathrm{Vol}(A)$-many solutions obtained along the other resonant line. However, this follows from Theorem 5.3.8. Indeed, we find that along each of the resonant lines there is a unique solution which is finitely supported at β. $\square$

6. Discriminant amoebas

In [43], the amoeba of the principal A-determinant was related to domains of convergence of hypergeometric series. The main result of that article is that the amoeba of the principal A-determinant is solid. That is, that the connected components of its complement is in a bijective relation with the vertices of its Newton polytope (which equals the secondary polytope Σ_A).

In [36], the coamoeba of the A-discriminant, in the case $m = 2$, was studied in relation to the zonotope $\mathcal{Z}_B$. On the one hand, by Theorem 5.1.12, this relates the coamoeba of the A-discriminant to domains of convergence of Mellin–Barnes integrals. On the other hand, by the order map $\mathrm{ord}_{\mathcal{D}}$, this relates the A-discriminant from to colopsidedness.

In this chapter, we will study the amoeba and coamoeba of the principal A-determinant D_A in relation to lopsidedness and colopsidedness respectively, completing the picture of [43].

6.1 The amoeba of D_A

Let $f \in \mathbb{C}_*^A$. If, for each connected component $E_{\mathbf{a}}$ of the complement of $\mathcal{A}$ we are given a point $\tau_{\mathbf{a}} \in E_{\mathbf{a}}$, then the coefficients $s_{\mathbf{a}}$ of the tropical polynomial defining the spine of $\mathcal{A}$ can be computed. However, to find such a set of points is a highly nontrivial matter, and often includes ad hoc knowledge of the spine. Let us consider instead the tropicalization of f, which in this text denotes the tropical polynomial

$$\mathrm{trop}(f)(\tau) = \max_{\mathbf{a}\in A}\left(\log|f_{\mathbf{a}}| + \langle \mathbf{a}, \tau\rangle\right).$$

Let us denote the tropical variety $Z_{\mathrm{trop}}(\mathrm{trop}(f))$ by $\mathcal{S} = \mathcal{S}_f$. If we consider $\mathcal{S}$ as a polyhedral cell complex in $\mathbb{R}^n$, then we can index each cell σ by the set of elements $\mathbf{a}$ of A such that for each $\tau \in \sigma$ the maximum of $\mathrm{trop}(f)$ is attained by the monomial of index $\mathbf{a}$. In this notation, a cell σ is full-dimensional if and only if $\sigma = \sigma_{\mathbf{a}}$ for some singleton $\mathbf{a} \in A$. The dual triangulation of $\mathcal{S}$ is the triangulation whose simplices are precisely the (convex hulls of the) index sets of $\mathcal{S}$.

In accordance with the previous notation we will denote by $\widehat{U}_{\mathbf{a}}$ the set of all $f \in \mathbb{C}_*^A$ such that the complement of $\mathcal{L}$ has a component of order $\mathbf{a}$. The inclusion $\widehat{E}_{\mathbf{a}} \subset E_{\mathbf{a}}$ implies that $\widehat{U}_{\mathbf{a}} \subset U_{\mathbf{a}}$.

Proposition 6.1.1. *The set $\widehat{E}_{\mathbf{a}}$ is contained in the interior of the cell $\sigma_{\mathbf{a}}$ of $\mathcal{S}$.*

Proof. If $|f_{\mathbf{a}}|e^{\langle \mathbf{a},\tau\rangle} > \sum_{\beta\neq\mathbf{a}}|f_\beta|e^{\langle\beta,\tau\rangle}$ then $\log|f_{\mathbf{a}}| + \langle\mathbf{a},\tau\rangle > \log|f_\beta| + \langle\beta,\tau\rangle$ for each $\beta\neq\mathbf{a}$. □

We will say that $\mathcal{S}$ is a *pseudo-spine* of the lopsided amoeba $\mathcal{L}$, the prefix being added as, in general, not all full dimensional cells of $\mathcal{S}$ contain a component of the complement of $\mathcal{L}$. The main theorem of this section is the following.

Theorem 6.1.2. *Let* $\mathrm{Log}(f) \in \mathbf{T}^A \setminus \mathcal{A}_D$*, with* $\mathrm{Log}(f) \in E = E(T)$ *for a coherent triangulation T of $\mathcal{N}$. Then,* $\mathrm{Im}(\mathrm{ord}_{\mathcal{L}}) = \mathrm{vert}(T)$.

The contour $\mathcal{B} = \mathcal{B}_f$ of the amoeba $\mathcal{A}$ is defined as the image of the set of critical points of Log, when restricted to the hypersurface $Z(f)$. See [32] and [46] a more detailed description of the contour and, e.g., [43] for the relation to A-hypergeometric series.

Proposition 6.1.3. *The set-valued map $f \mapsto$* $\mathrm{Im}(\mathrm{ord}_{\mathcal{L}})$ *is locally constant outside of* $\mathrm{Log}^{-1}(\mathcal{B}_D)$.

Proof. It suffices to prove that for each $\mathbf{a}\in A$, the characteristic map of the set $\widehat{E}_{\mathbf{a}}$ is locally constant outside of $\mathrm{Log}^{-1}(\mathcal{B}_D)$.

If $\mathbf{a}$ is a vertex of A, then $\mathbf{a}\in\mathrm{Im}(\mathrm{ord}_{\mathcal{L}})$ for all f, hence there is nothing to prove in this case. Assume now that $\mathbf{a}$ is an interior point of some face Γ of $\mathcal{N}$. We first claim that (3.1) holds at some point τ if and only if there exists an $\tau_\Gamma\in\mathbb{R}^n$ such that

$$|f_{\mathbf{a}}|e^{\langle\mathbf{a},\tau_\Gamma\rangle} > \sum_{\beta\in\Gamma\setminus\mathbf{a}} |f_\beta|e^{\langle\beta,\tau_\Gamma\rangle}.$$

The "only if" direction follows from choosing $\tau_\Gamma = \tau$. For the "if" direction, we need only to consider the case when Γ is a proper face of $\mathcal{N}$, in which case there is a nonempty set of linearly independent vectors $\mu_1,\ldots,\mu_k\in\mathbb{Z}^n$, such that $\beta\in\Gamma$ if and only if β minimizes $\langle\mu_j,\beta\rangle$ over A for each $j=1,\ldots,k$. Let $d_j(\beta) = \langle\mu_j,\beta-\mathbf{a}\rangle$, so that there exists a j with $d_j(\beta)>0$ if and only if $\beta\notin\Gamma$. For any real number t, we find that

$$\frac{|f_\beta|e^{\langle\tau_\Gamma+\sum_{j=1}^k t\mu_j,\beta\rangle}}{e^{\langle\sum_{j=1}^k t\mu_j,\mathbf{a}\rangle}} = |f_\beta|e^{\langle\tau_\Gamma,\beta\rangle}e^{(t\sum_{j=1}^k d_j(\beta)},$$

which tends to 0 as $t\to-\infty$ unless $\beta\in\Gamma$. Hence, as A is finite, by choosing t sufficiently large and negative, we find that $\mathbf{a}$ fulfills (3.1) at $\tau = \tau_\Gamma + t\sum_{j=1}^k \mu_j$.

The fact that the logarithmic Gauss map is a birational isomorphism [28], the Horn–Kapranov parametrization, and the description of the contour of the amoeba of $\Delta_{A\cap\Gamma}$ given in [32, Lemma 4.3] and [46], implies that the contour of the amoeba of $\Delta_{A\cap\Gamma}$ contains the image of the locus of all real coefficients $f \in \mathbb{C}_*^{A\cap\Gamma}$ such that f has a singular point in $\mathbb{R}^{\dim(\Gamma)}$. By reduction of variables, we only need to consider the case when $A\cap\Gamma$ is of full dimension. In this case, that $\mathbf{a}$ is an interior point of Γ implies that the given map is locally constant outside of the set of all f such that there exists a $\tau \in \mathbb{R}^n$ with

$$|f_{\mathbf{a}}|e^{\langle\tau,\mathbf{a}\rangle} = \sum_{\beta\in\Gamma\setminus\mathbf{a}} |f_\beta| e^{\langle\tau,\beta\rangle}$$
$$\text{and} \quad |f_{\mathbf{a}}|e^{\langle\tilde{\tau}y,\mathbf{a}\rangle} \leq \sum_{\beta\in\Gamma\setminus\mathbf{a}} |f_\beta| e^{\langle\tilde{\tau},\beta\rangle},$$

for each $\tilde{\tau}$ in some small neighborhood of τ. Set $\tilde{f}_{\mathbf{a}} = -|f_{\mathbf{a}}|$ and $\tilde{f}_\beta = |f_\beta|$ for $\beta \in \Gamma\setminus\mathbf{a}$. It follows that $\tilde{f}(z)$ is a real polynomial which has a singular point in $\mathbb{R}^n$, and hence

$$\mathrm{Log}(f) = \mathrm{Log}(\tilde{f}) \in \mathcal{B}_{\Delta_{A\cap\Gamma}}.$$ □

Proposition 6.1.4. *Let E be a component of the complement of $\mathcal{A}_D$, with recession cone $C = C(E)$. If $\mathrm{Log}(f) \in \mathrm{int}(E\cap C)$, then $\mathcal{S}$ is a spine of $\mathcal{L}$, i.e., $\mathcal{S}$ is a strong deformation retract of $\mathcal{L}$.*

Proof. We only have to show that if $\sigma = \sigma_{\mathbf{a}}$ is a full dimensional face of $\mathcal{S}$, then $\mathbf{a} \in \mathrm{Im}(\mathrm{ord}_{\mathcal{L}})$. If $\mathrm{Log}(f) \in \mathrm{int}(E\cap C)$, then so does $r\,\mathrm{Log}(f)$ for each $r \geq 1$. By Proposition 6.1.3, the image of the order map of the lopsided amoeba is constant on each connected component of the complement of $\mathcal{L}_D$. Hence, it suffices to show the statement for some r. If there exists $\tau \in \mathbb{R}^n$ such that

$$\mathrm{Log}|f_{\mathbf{a}}| + \langle\mathbf{a},\tau\rangle > \max_{\beta\neq\mathbf{a}}\big(\mathrm{Log}|f_\beta| + \langle\beta,\tau\rangle\big),$$

then for r big enough we find that

$$|f_{\mathbf{a}}^r|e^{\langle\mathbf{a},r\tau\rangle} > \sum_{\beta\neq\mathbf{a}} |f_\beta^r| e^{\langle\beta,r\tau\rangle},$$

which proves the proposition. □

It follows from (2.4) that the Newton polytope of Δ_A is a Minkowski summand of the secondary polytope Σ_A. As noted in the introduction, the secondary polytope Σ_A is closely related to the set of coherent triangulations of

A. Let us describe this relation in some detail. Consider a triangulation T of $\mathcal{N}$, with vertices in A. The characteristic function $\varphi_T\colon A \to \mathbb{R}$ is the function

$$\varphi_T(\mathbf{a}) = \sum_{\mathbf{a}\in\mathrm{vert}(\tau)} \mathrm{Vol}(\tau),$$

where the sum is taken of all simplices $\tau \subset T$ such that $\mathbf{a}$ is a vertex of τ. Then, Σ_A is the convex hull in $\mathbb{R}^A$ of the vectors $\varphi_T(A) = (\varphi_T(\mathbf{a}_1),\ldots,\varphi_T(\mathbf{a}_N))$ as T runs over all triangulations of $\mathcal{N}$ with vertices in A. The vertices of Σ_A are precisely the points $\varphi_T(A)$ given by coherent triangulations T [19, Theorem 7.1.7b]. The normal cone of Σ_A at $\varphi_T(A)$ consists of all linear forms ψ on $\mathbb{R}^A$ such that $\psi(\varphi_T(A)) = \max_{\varphi\in\Sigma(A)} \psi(\varphi)$. This cone coincides with the cone $C(T)$ consisting of all functions $\varphi: A \to \mathbb{R}$ such that the dual triangulation of the tropical variety

$$\mathrm{trop}(\varphi)(\tau) = \max_{\mathbf{a}\in A}(\varphi(\mathbf{a}) + \langle \tau, \mathbf{a}\rangle)$$

is T [19, Theorem 7.1.7c].

Proposition 6.1.5. *Let E and $C = C(E)$ be as in the previous proposition, and let T be the (coherent) triangulation of $\mathcal{N}$ corresponding to E. If* $\mathrm{Log}(f) \in \mathrm{int}(E\cap C)$*, then T is the dual triangulation of $\mathcal{S}$.*

Proof. If we denote the dual triangulation of $\mathcal{S}$ by T', then the function $\psi\colon A \to \mathbb{R}^A$ given by $\mathbf{a} \mapsto \mathrm{Log}(f_{\mathbf{a}})$ belongs to the cone $C(T')$. Therefor, the function $\overline{\psi}\colon \mathbb{R}^A \to \mathbb{R}$ given by $\varphi \mapsto \langle\varphi, \mathrm{Log}(f)\rangle$ attains its maximum at the vertex $\varphi_{T'}(A)$. Hence $\mathrm{Log}(f)$ is contained in the normal cone $N_{\varphi_{T'}}\Sigma(A)$. However, $\mathrm{Log}(f)$ is contained in the interior of the normal cone $N_{\varphi_T}\Sigma(A)$ by assumption, which implies that $T = T'$. □

Example 6.1.6. Both previous propositions fail if we replace the criterion that $\mathrm{Log}(f) \in \mathrm{int}(E\cap C)$ by the criterion that $\mathrm{Log}(f) \in E$. For example, if $f(z) = 1 + \xi z + z^2$, so that $\Delta_B(\xi) = \xi^2 - 4$, then the complement of $\mathcal{A}_\Delta$ has two connected components, namely $E_0 = (-\infty, \log(2))$ and $E_2 = (\log(2), \infty)$. The former of these intervals is not contained in the corresponding recession cone, given by $(-\infty, 0)$. Thus, if $\log(\xi) \in (0, \mathrm{Log}(2))$, then $\mathcal{S}$ will have three full-dimensional faces, while the complement of $\mathcal{L}$ has only two connected components. ⬡

Proof of Theorem 6.1.2. Note first that $E\cap C$ is nonempty. If $\mathrm{Log}(f) \in E\cap C$, where $C = C(E)$ is the recession cone of E, then Propositions 6.1.4 and 6.1.5 implies that $\mathrm{Im}(\mathrm{ord}_{\mathcal{L}}) = \mathrm{vert}(T)$. The general case now follows from Proposition 6.1.3. □

Example 6.1.7. Consider the point configuration

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix},$$

which is associated to the family polynomials of the (reduced) form

$$f(z_1, z_2) = 1 + z_1 + z_2 + \xi_1 z_1^2 + \xi_2 z_2^2.$$

The principal A-determinant is given by

$$\Delta_B(\xi) = (4\xi_1\xi_2 - \xi_1 - \xi_2)(1 - 4\xi_1)(1 - 4\xi_2).$$

Its amoeba can be seen in Figure 6.1, where the full-dimensional set is the amoeba of the A-discriminant. In each connected component of the complement $\mathbb{R}^2 \setminus \mathcal{A}_D$ is the corresponding coherent triangulation of A. Note that the set of connected components of $\mathbb{R}^2 \setminus \mathcal{A}_\Delta$ is neither in a bijective correspondence with triangulations of A, nor in a bijective correspondence with vertex sets of such triangulations. ⬡

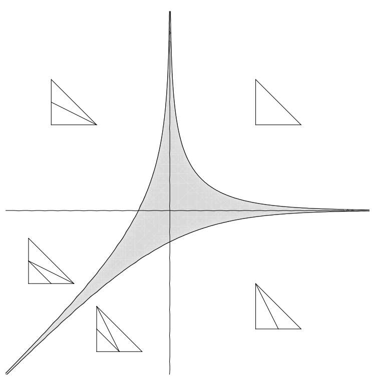

Figure 6.1: The amoeba of the principal A-determinant from Example 6.1.7, with corresponding triangulations of A.

6.2 The coamoeba of D_A

In [36], L. Nilsson and M. Passare proved the following statement. Let A be of codimension two, and let $f \in \mathbb{C}_*^A$. Reduce f with respect to a Gale dual B, and let ξ be the coordinates of $\mathbb{C}_*^m$, so that $\operatorname{Arg}(\xi) = \operatorname{Arg}(f)B$. Then, counting on the one hand the number of points of the affine lattice $\operatorname{Arg}(\xi) +$

$2\pi L_B$ contained in the interior of the zonotope $\mathcal{Z}_B$, and on the other hand the (finite) number of preimages over $\mathrm{Arg}(\xi)$ in the discriminantal variety $Z(\Delta_B)$, these two integers sums up to $\mathrm{Vol}(A)$. However, it should be noted that they prove this *granted some generically satisfied conditions*, i.e., under the assumption that A is not dual defect. Removing this assumption, the statement is not true.

Example 6.2.1. If A is a pyramid, i.e., if there is a strict face $\Gamma \subset \mathcal{N}$ containing all but one element $\mathbf{a}$ of A, then $\Delta_A = 1$. In this case, $\mathbf{b_a} = 0$. Hence, the zonotope $\mathcal{Z}_B$ does not depend on $\mathbf{b_a}$. Deleting the row $\mathbf{b_a}$ from B we obtain a Gale dual of the point configuration $A \cap \Gamma$. Notice that $A \cap \Gamma$ is of codimension 2 as a subset of $\mathbb{Z}^{n-1}$. In particular, the Nilsson–Passare theorem is applicable to $A \cap \Gamma$. Hence, $\mathcal{Z}_B$ need not cover $\mathbf{T}^2$. Furthermore, the fiber in the variety $Z(D_A)$ over a point $\theta \in \Delta_{A\cap\Gamma}$ is not finite, since the Newton polytope of $A \cap \Gamma$ is not of full dimension in $\mathbb{Z}^n$. Hence, in general, we cannot count multiplicities as in [36]. ⬡

In [44], the Nilsson–Passare theorem was reproved with a different approach. Starting from the matrix B, they defined the variety $Z(\Delta_B)$ in question through the Horn–Kapranov parametrization (2.8). Our suggestion is that one should instead focus on the principal A-determinant. If, for example, $\Delta_A = 1$, then the relevant information is contained in the factors of D_A that correspond to strict faces of A. Notice that the criterion that $\Delta_A = 1$ is not equivalent to the criterion, appearing in *loc. sit.*, that B has parallel rows. See, for example, any Gale dual of the point configuration A from Example 6.1.7. The following two theorems provide a weaker version of the Nilsson–Passare theorem. For the latter statement we have only obtained a complete proof in the case $n = 1$.

Theorem 6.2.2. *Assume that B is a Gale dual of A, and that f is such that the set* $\mathrm{int}(\mathcal{Z}_B) \cap (\mathrm{Arg}(f)B + 2\pi L_B)$ *is of cardinality* $\mathrm{Vol}(A)$. *Then* $\mathrm{Arg}(f) \in \mathbf{T}^A \setminus \overline{\mathcal{C}}_D$.

Theorem 6.2.3. *Assume that $n = 1$. Let $f \in \mathbb{C}_*^A$, and assume that* $\mathrm{Arg}(f) \in \mathbf{T}^A \setminus \overline{\mathcal{C}}_D$. *Let B be a Gale dual of A. Then, the set* $\mathrm{int}(\mathcal{Z}_B) \cap (\mathrm{Arg}(f)B + 2\pi L_B)$ *is of cardinality* $\mathrm{Vol}(A)$.

Before we turn to the proofs of Theorems 6.2.2 and 6.2.3, let us give the Nilsson–Passare theorem for principal A-determinants.

Lemma 6.2.4. *We have that* $\mathcal{H}(D_B) = \mathrm{pr}_m(\partial \mathcal{Z}_B)$, *where* $\mathrm{pr}_m \colon \mathbb{R}^m \to \mathbf{T}^m$ *denotes the natural projection.*

Proof. Assume that θ belongs to some facet Γ of the zonotope $\mathcal{Z}_B$. Then there is a rational linear transformation T, and a rational $(n+1) \times (n+1)$-

matrix B_1, such that

$$B = MT = \begin{pmatrix} B_1 \\ I_m \end{pmatrix} T,$$

where I_m is the $m \times m$ identity matrix, and where Γ has the normal vector $T^{-1}e_m$, see (3.11). Thus, θ belongs to the boundary of the zonotope $\mathcal{Z}_{B'}$, where B' is the Gale dual of the point configuration defined by removing all elements $\mathbf{a} \in A$ such that the row of M corresponding to $\mathbf{a}$ has vanishing mth coordinate. As B' is a column vector, it is the Gale dual of a subcircuit of A (of dimension two less than the number of rows of M with nonvanishing last coordinate). However, the edges of Σ_A correspond to subconfigurations, of some dimension, that has exactly two triangulations. Hence, they are precisely the set of subcircuits of A. □

Theorem 6.2.5 (Nilsson–Passare theorem for D)**.** *If A is of codimension two, then the set* $\operatorname{int}(\mathcal{Z}_B) \cap (\operatorname{Arg}(f)B + 2\pi\mathbb{Z}^2)$ *is of cardinality* $\operatorname{Vol}(A)$ *if and only if* $\operatorname{Arg}(f)$ *is contained in the complement of* $\overline{\mathcal{C}}_D$.

Proof. If $\Delta_A \neq 1$, then A is not a pyramid, and hence each strict face of A is at most a circuit. Hence, the theorem follows from the usual Nilsson–Passare theorem and Lemma 6.2.4 in this case.

Assume now that $\Delta_A = 1$. It was shown in [11] that, if $\iota_B = 1$, then B can be subdivided into sets of parallel rows such that each family sums to zero individually. This implies that all vertices of the zonotope $\mathcal{Z}_B$ are contained in $2\pi\mathbb{Z}^2$. Furthermore, a classical formula for the area of a zonotope, [53], yields that

$$\operatorname{Area}(\mathcal{Z}_B) = 4\pi^2 \sum_{i<j} |\det(\mathbf{b}_{\mathbf{a}_i}, \mathbf{b}_{\mathbf{a}_j})| = 4\pi^2 \operatorname{Vol}(A),$$

where the last equality is contained in [11] (see also [36, Theorem 3]). These two facts implies that the zonotope $\mathcal{Z}_B$ covers the torus $\mathbf{T}^2$ precisely $\operatorname{Vol}(A)$-many times. □

Proof of Theorem 6.2.2. By assumption, there is a Mellin–Barnes basis of solutions for totally non-resonant parameters. In particular, for totally non-resonant parameters, there is no A-hypergeometric function with singularities in $\operatorname{Arg}^{-1}(f)$, as the Mellin–Barnes integrals converges and defines analytic functions in this domain. However, the singular locus $Z(D_A)$ of $H_A(\beta)$ is is independent of β [20; 21]. Hence, $\operatorname{Arg}(f) \in \mathbf{T}^A \setminus \mathcal{C}_D$. Since the number of points of the affine lattice $\operatorname{Arg}(f)B + 2\pi\mathbb{Z}^m$ contained in the zonotope $\mathcal{Z}_B$ is a lower semi-continuous function of f, we conclude that $\operatorname{Arg}(f) \in \mathbf{T}^A \setminus \overline{\mathcal{C}}_D$. □

To prove Theorem 6.2.3 we will introduce further notation. Let us consider the affine situation, i.e., we consider any phase to be an element of $\mathbb{R}$.

For a pair of distinct elements $\{\mathbf{a}_i, \mathbf{a}_j\} \subset A$, choose $\zeta_0^{i,j} \in \mathbb{R}$ as some argument such that $\hat{f}_i(\theta)$ and $\hat{f}_j(\theta)$ are antipodal for $\theta = \zeta_0^{i,j}$. Define

$$\zeta_k^{i,j} = \zeta_0^{i,j} + \frac{2\pi k}{|\mathbf{a}_i - \mathbf{a}_j|},$$

so that $\{\zeta_k^{i,j}\}_{k\in\mathbb{Z}}$ is the set of all points such that $\hat{f}_i(\theta)$ and $\hat{f}_j(\theta)$ are antipodal, with the ordering induced by the index set $\mathbb{Z}$ in agreement with the ordering inherited from $\mathbb{R}$. For any fix $\theta \in \mathbb{R}$, we define

$$\begin{aligned} \zeta_+^{i,j} &= \zeta_+^{i,j}(\theta) = \min\left\{\zeta_k^{i,j} \,|\, \zeta_0^{i,j} \geq \theta\right\}, \\ \text{and} \quad \zeta_-^{i,j} &= \zeta_-^{i,j}(\theta) = \max\left\{\zeta_k^{i,j} \,|\, \zeta_0^{i,j} \leq \theta\right\}. \end{aligned}$$

Lemma 6.2.6. *Let $A = \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$, with the coordinates of $\hat{f}(\theta)$ counter-clockwise ordered as $\hat{f}_0(\theta), \hat{f}_1(\theta), \hat{f}_2(\theta)$ in $S^1 \subset \mathbb{C}$. Assume that f is not colopsided at θ.*

i) If $\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2$, then f is not colopsided in $\left[\zeta_-^{0,2}, \theta\right]$.

ii) If $\mathbf{a}_0 < \mathbf{a}_2 < \mathbf{a}_1$, then f is not colopsided in $\left[\theta, \zeta_+^{0,1}\right]$.

Proof. Let $\theta(t) = \theta + t$, and let $\theta_{jk}(t)$ denote the (smallest) intermediate angle of the points $\hat{f}_j(\theta(t))$ and $\hat{f}_k(\theta(t))$. Then, $|\theta'_{jk}(t)| = |\mathbf{a}_k - \mathbf{a}_j|$. The two parts of the lemma follows from that non-colopsidedness ensures that $\pi - \theta_{02} < \min(\theta_{01}, \theta_{12})$ and that $\pi - \theta_{01} < \min(\theta_{02}, \theta_{12})$ respectively. □

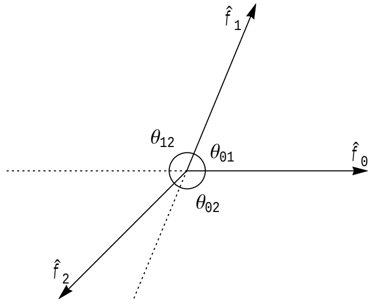

Figure 6.2: Lemma 6.2.6.i: decreasing θ, the intermediate angle θ_{02} is the only one increasing. Non-colopsidedness, i.e. that $\hat{f}_2(\theta)$ lies between the dotted half-lines, ensures that θ_{02} will reach π before θ_{01} decrease to 0.

Proof of Theorem 6.2.3. Assume that $\operatorname{Arg}(f) \in \mathbf{T}^A \setminus \overline{\mathcal{C}}_\Delta$. We will give an elementary proof by induction over m. To spare space we note that the cases

$m = 1$ and $m = 2$ are solved by Lemma 6.2.4 and by the Nilsson–Passare theorem respectively. We can assume that

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \mathbf{a}_1 & \dots & \mathbf{a}_{m+1} \end{pmatrix},$$

where $0 = \min(\mathbf{a}_\kappa)$ and $\mathbf{a}_{m+1} = \max(\mathbf{a}_\kappa)$. Define

$$I_k^{\kappa_1,\kappa_2} = \left[\zeta_k^{\kappa_1,\kappa_2}, \zeta_{k+1}^{\kappa_1,\kappa_2}\right], \quad k \in \mathbb{Z}. \tag{6.1}$$

If the coefficients $f_1, \dots, f_m$ are sufficiently small in moduli, then the zeros of $f(z)$ will have arguments close to the numbers $\zeta_k^{0,m+1}$. Thus, by continuity of roots, we find that $\mathbf{T}^A \setminus \overline{\mathcal{D}}$ has at most one connected component in the interior of each interval $I_k^{0,m+1}$, and we need to show that it has exactly one component in each such interval.

That $\mathrm{Arg}(f) \in \mathbf{T}^A \setminus \overline{\mathcal{C}}_\Delta$ implies that $\mathrm{Arg}(f_{\hat\kappa}) \in \mathbf{T}^{A_{\hat\kappa}} \setminus \overline{\mathcal{C}}_{\Delta_{\hat\kappa}}$ for each $\kappa = 0, \dots, m+1$. In particular, with k fixed, it follows from the induction hypothesis that for each $\kappa = 1, \dots, m$ there is an open subinterval $I_k^\kappa \subset I_k^{0,m+1}$ in which $f_{\hat\kappa}$ is colopsided, and furthermore there is a subinterval $I_k^{m+1} \subset I_k^{0,m}$ in which $f_{\widehat{m+1}}$ is colopsided.

Fix an integer k, and consider the interval $I_k^{0,m+1}$. Assume that the endpoint θ of I_k^1 is minimal among the set of endpoints of the intervals I_k^κ for $\kappa = 1, \dots, m$. Notice that $\theta = \zeta_k^{\kappa_2,\kappa_3}$ for some $\kappa_2, \kappa_3 \neq 1$. The index $\kappa = 1$ need not be unique with this property. We can assume that $\arg_\pi(f_{\kappa_2}\langle\theta\rangle) = 0$, that $\arg_\pi(f_{\kappa_3}\langle\theta\rangle) = \pi$, and that $\Im(\hat f_\kappa(\theta)) \geq 0$ fore each $\kappa \neq 1$. There cannot be a third component of $\hat f(\theta)$ contained in $\mathbb{R}$, for any choice of three elements of A defines a circuit, whence it would follow from Lemma 6.2.4 that $\mathrm{Arg}(f) \in \overline{\mathcal{C}}(\Delta_A)$. Hence, it suffices to show that $\Im(\hat f_1(\theta)) \geq 0$, as this implies that $\Im(\hat f_1(\theta)) > 0$, which in turn implies that f is colopsided at $\theta - \varepsilon$ for ε sufficiently small and positive.

If there is some $\kappa \notin \{0, 1, \kappa_2, \kappa_3, m+1\}$, then by minimality, θ is also the endpoint for $I_{\hat\kappa}$, which in turn implies that $\Im(\hat f_1(\theta)) > 0$. The only case in which this does not occur, is when $m = 3$ (so that $m + 1 = 4$) and $\kappa_2, \kappa_3 \notin \{0, 4\}$. We will consider this case separately, still under the assumptions of the induction hypothesis, writing $\kappa_2 = 2$ and $\kappa_3 = 3$.

Let us initially impose the further assumption that

$$0 = \arg_\pi(\hat f_2(\theta)) < \arg_\pi(\hat f_0(\theta)) \leq \arg_\pi(\hat f_4(\theta)) < \arg_\pi(\hat f_3(\theta)) = \pi.$$

It follows that $\mathbf{a}_3 > \mathbf{a}_2$, as θ is the endpoint of $I_{\hat 1}$. Assuming that $\Im(\hat f_1(\theta)) < 0$, there are three cases.

<u>Case 1</u>: Consider the case that

$$\arg_\pi(-\hat f_1(\theta)) \in \left[\arg_\pi(\hat f_2(\theta)), \arg_\pi(\hat f_0(\theta))\right] = \left[0, \arg_\pi(\hat f_0(\theta))\right].$$

Then $f_{\hat{3}}$ is not colopsided at θ. Thus, by minimality of θ, we find that $f_{\hat{3}}$ is colopsided for some $\tilde{\theta} > \theta$ in the interval $I_k^{0,4}$.

If $\mathbf{a}_1 > \mathbf{a}_2$, then Lemma 6.2.6.ii applied to $\{0, \mathbf{a}_1, \mathbf{a}_2\}$ implies that $f_{\hat{3}}$ is not colopsided in the interval $[\theta, \zeta_+^{0,1}]$. If $\zeta_+^{0,1} \geq \zeta_+^{0,4}$, this implies that $f_{\hat{3}}$ is not colopsided in the interval $[\theta, \zeta_+^{0,4}]$. If $\zeta_+^{0,1} < \zeta_+^{0,4}$, then Lemma 6.2.6.ii applied to $\{0, \mathbf{a}_1, \mathbf{a}_4\}$ at $\zeta_+^{0,1}$ implies that $f_{\hat{3}}$ is not colopsided in the interval $[\zeta_+^{0,1}, \zeta_+^{0,4}]$, again implying that $f_{\hat{3}}$ is not colopsided in the interval $[\theta, \zeta_+^{0,4}]$. However, this contradicts the existence of $\tilde{\theta}$.

Thus, we must have that $\mathbf{a}_2 > \mathbf{a}_1$. Since $\mathbf{a}_3 > \mathbf{a}_2$, Lemma 6.2.6.ii applied to $\{0, \mathbf{a}_2, \mathbf{a}_3\}$ implies that $f_{\hat{4}}$ is not colopsided in the interval $[\theta, \zeta_+^{0,3}]$. Furthermore, Lemma 6.2.6.i applied to $\{0, \mathbf{a}_1, \mathbf{a}_2\}$ yields that $f_{\hat{4}}$ is not colopsided in $[\zeta_-^{0,2}, \theta]$. Since $\mathbf{a}_3 > \mathbf{a}_2$, we have that

$$\arg_\pi(\hat{f}_3(\zeta_-^{0,2})) < \arg_\pi(\hat{f}_0(\zeta_-^{0,2})).$$

Hence, applying Lemma 6.2.6.ii to $\{0, \mathbf{a}_2, \mathbf{a}_3\}$ at $\zeta_-^{0,2}$ shows that $f_{\hat{4}}$ is not colopsided in $[\zeta_-^{0,3}, \zeta_-^{0,2}]$. All in all, we conclude that $f_{\hat{4}}$ is never colopsided in the interval $[\zeta_-^{0,3}, \zeta_+^{0,3}] = I_k^{0,3}$, a contradiction.

Case 2: Consider now when

$$\arg_\pi(-\hat{f}_1(\theta)) \in \left[\arg_\pi(\hat{f}_0(\theta)), \arg_\pi(\hat{f}_4(\theta))\right].$$

Then $f_{\hat{3}}$ is not colopsided at θ. Hence, since θ was minimal, it has to be colopsided for some larger $\tilde{\theta} > \theta$ in $I_k^{0,4}$. However, Lemma 6.2.6.i applied to $\{0, \mathbf{a}_1, \mathbf{a}_4\}$ shows that $f_{\hat{3}}$ is not colopsided in the interval $[\theta, \zeta_+^{0,4}]$, a contradiction since $\zeta_+^{0,4}$ is the endpoint of $I_k^{0,4}$.

Case 3: The case

$$\arg_\pi(-\hat{f}_1(\theta)) \in \left[\arg_\pi(\hat{f}_4(\theta)), \arg_\pi(\hat{f}_3(\theta))\right]$$

is similar to Case 1.

Finally, the situation

$$0 = \arg_\pi(\hat{f}_2(\theta)) < \arg_\pi(\hat{f}_4(\theta)) \leq \arg_\pi(\hat{f}_0(\theta)) < \arg_\pi(\hat{f}_3(\theta)) = \pi$$

is shown analogously, with the only difference that one should consider $f_{\hat{0}}$ instead of $f_{\hat{4}}$ in cases 1 and 3. All in all, we conclude that $\Im(\hat{f}_1(\theta)) \geq 0$, which finishes the induction step in this special case. □

7. Descartes pairs

In what follows we consider real univariate polynomials with non-vanishing coefficients. The famous Descartes' rule of signs claims that the number of positive roots of such a polynomial does not exceed the number of sign changes in its sequence of coefficients. An arbitrary ordered sequence $\sigma = (\sigma_0, \sigma_1, ..., \sigma_d) \in \{\pm 1\}^{d+1}$ is called a *sign pattern*. Given a sign pattern σ, we call by its *Descartes pair* (p_σ, n_σ) the pair of non-negative integers counting sign changes and sign preservations of σ. We have that $p_\sigma + n_\sigma = d$. The Descartes pair of σ gives the respective upper bounds on the number of positive and negative roots of any polynomial of degree d whose signs of coefficients are given by σ.

To any polynomial $f(z)$ we associate the pair (p_f, n_f) giving the numbers of its positive and negative roots, counted with multiplicities. Then, (p_f, n_f) satisfies the restrictions

$$p_f \le p_\sigma, \quad p_f \equiv p_\sigma \mod 2, \quad n_f \le n_\sigma, \quad \text{and} \quad n_f \equiv n_\sigma \mod 2. \tag{7.1}$$

We call pairs (p, n) satisfying (7.1) *admissible* for σ. In general, given a sign pattern σ, not all of its admissible pairs are realizable by polynomials f with sign pattern $\mathrm{sgn}(f) = \sigma$.

Problem 7.0.7. For a given sign pattern σ, which admissible pairs (p, n) are realizable by polynomials f such that $\mathrm{sgn}(f) = \sigma$?

In this section we will denote by Pol_d the (affine) space of all real monic univariate polynomials of degree d. We define the *standard real discriminant* $\Delta_d \subset \mathrm{Pol}_d$ as the subset of all polynomials having a real multiple root. Detailed information about a natural stratification of Δ_d can be found in, e.g., [52]. It is a well-known and simple fact that $\mathrm{Pol}_d \setminus \Delta_d$ consists of $\lfloor d/2 \rfloor + 1$ components distinguished by the number of real simple roots. Moreover, each such component is contractible. Strangely enough analogous statements in the case when one imposes additional restrictions on the signs of coefficients seems to be unknown.

To formulate our results we need to introduce some notation. For any pair (d, k) of non-negative integers with

$$d \ge k \quad \text{and} \quad d - k \equiv 0 \mod 2, \tag{7.2}$$

denote by $\mathrm{Pol}_{d,k}$, the set of all real polynomials of degree d with k real simple roots. Denote by $\mathrm{Pol}_d^{\sigma} \subset \mathrm{Pol}_d$ the set (orthant) of all polynomials $f \in \mathrm{Pol}_d$ with sign pattern $\mathrm{sgn}(f) = \sigma$. Finally, set $\mathrm{Pol}_{d,k}^{\sigma} = \mathrm{Pol}_{d,k} \cap \mathrm{Pol}_d^{\sigma}$. We have the natural action of $\mathbb{Z}_2^3$ on the space of polynomials and on the set of all sign patterns. It is generated by

$$f(z) \mapsto -f(z), \quad f(z) \mapsto f(-z), \quad \text{and} \quad f(z) \mapsto z^d f(1/z).$$

We will refer to this action as the *standard action.* The properties we will study below are invariant under the standard action.

We start with the following simple result.

Theorem 7.0.8. *We have the following characterization of when the set* $\mathrm{Pol}_{d,k}^{\sigma}$ *is nonempty.*

i) If d is even, then $\mathrm{Pol}_{d,0}^{\sigma}$ is nonempty if and only if $\sigma_0 = \sigma_d$.

ii) For any pair of positive integers (d,k) fulfilling (7.2) and any sign pattern σ, the set $\mathrm{Pol}_{d,k}^{\sigma}$ is nonempty.

Observe that in general, the intersection $\mathrm{Pol}_{d,k}^{\sigma}$ is not necessarily connected. The total number k of real zeros can be distributed between p positive and n negative in different ways satisfying (7.1). See examples below. On the other hand, some concrete intersections have to be connected. In particular, the following holds.

Theorem 7.0.9. *We have the following results concerning when the set* $\mathrm{Pol}_{d,k}^{\sigma}$ *is contractible.*

i) For any d and σ, the sets $\mathrm{Pol}_{d,d}^{\sigma}$ and $\mathrm{Pol}_{d,0}^{\sigma}$ are contractible. (The latter is empty for d odd.)

ii) For the sign pattern $\hat{+} = (+,+,\dots,+)$ consisting of all pluses, the intersection $\mathrm{Pol}_{d,k}^{\hat{+}}$ is contractible for any (d,k) fulfilling (7.2).

In addition, we provide the following result concerning the connectedness of the sets $\mathrm{Pol}_{d,k}^{\sigma}$.

Theorem 7.0.10. *For any sign pattern σ and integer k, the union*

$$\mathrm{Pol}_{d,\geq k}^{\sigma} = \bigcup_{m \geq k} \overline{\mathrm{Pol}}_{d,m}^{\sigma} \tag{7.3}$$

is simply connected.

While working on the project, we noticed a recent paper [2] dealing with the same problem and giving complete description of non-realizable patterns and pairs (p,n) for polynomials up to degree 6. This paper contains interesting historical material as well as references [3; 22] to the earlier research in this topic. The main result of [2] is as follows.

Theorem 7.0.11. *In degree $d \le 6$, the only nonrealizable pairs of a sign pattern σ and a Descartes pair (p,n) are those appearing in Table 7.1. In particular, in degree $d \le 3$, all admissible pairs are realizable.*

Trying to extend Theorem 7.0.11, we obtained a computer-aided classification of all non-realizable sign patterns and pairs for $d = 7$ and almost all for $d = 8$, see below.

Theorem 7.0.12. *For $d = 7$, among the 1472 possible combinations of a sign pattern and a pair (up to the standard $\mathbb{Z}_2^3$-action), there exist exactly 6 which are non-realizable. They are listed in Table 7.1.*

Theorem 7.0.13. *For $d = 8$, among the 3648 possible combinations of a sign pattern and a pair (up to the standard $\mathbb{Z}_2^3$-action), there exist 13 which are known to be non-realizable. They are listed in Table 7.1.*

Remark 7.0.14. For $d = 8$, there exist 7 (up to the standard $\mathbb{Z}_2^3$-action) combinations of a sign pattern and a pair for which it is unknown whether they are realizable or not. They are listed in Table 7.2. ○

Based on the above results, we formulate the following conjecture.

Conjecture 7.0.15. For an arbitrary sign pattern σ, the only type of pairs (p,n) which can be non-realizable has either p or n vanishing.

Rephrasing the above conjecture, we say that the only phenomenon implying non-realizability is that "real roots on one half-axis force real roots on the other half-axis". At the moment this conjecture is verified by computer-aided methods up to $d = 10$.

7.1 Realizability

In this subsection we will prove a number of results concerning realizability of Descartes pairs (p,n), including Theorems 7.0.12 and 7.0.13. We will begin with a class of pairs whose realizability is implied by the lopsidedness criterion.

Degree	Sign pattern	Descartes pairs
4	+ - - - +	(0,2)
5	+ - - - + +	(0,3)
6	+ - - - - - +	(0,2) and (0,4)
	+ - - - - + +	(0,4)
	+ - - - + - +	(0,2)
7	+ - - - - - - +	(0,3) and (0,5)
	+ - - - - - + +	(0,5)
	+ + - - - - + +	(0,5)
	+ - - - - + + +	(0,5)
	+ - - - - + - +	(0,3)
8	+ - - - - - - - +	(0,2), (0,4), and (0,6)
	+ - - - - - - + +	(0,6)
	+ + - - - - - + +	(0,6)
	+ - - - - - + + +	(0,6)
	+ - - - - + + + +	(0,6)
	+ - - - - - + - +	(0,2) and (0,4)
	+ - - - + - - - +	(0,2) and (0,4)
	+ - - - + - + - +	(0,2)
	+ - + - - - + - +	(0,2)

Table 7.1: Nonrealizable Descartes pairs in degree $d \leq 8$.

Degree	Sign pattern									Descartes pairs
8	+	-	-	+	-	-	-	-	+	(0,4)
	+	+	+	-	-	-	-	+	+	(0,6)
	+	+	-	-	-	-	-	-	+	(0,4)
	+	-	-	-	-	+	+	-	+	(0,4)
	+	+	-	-	-	-	+	-	+	(0,4)
	+	-	-	-	-	+	-	+	+	(0,4) and (4,0)

Table 7.2: Descartes pairs in degree $d = 8$ for which it is unknown whether they are realizable or not.

Lemma 7.1.1. *For a given sign pattern σ consider all possible sign patterns $\widetilde{\sigma}$ obtained from σ by removing an arbitrary subset of its entries except for the first and the last. (On the level of polynomials this corresponds to requiring that the corresponding coefficient vanishes.) For any such $\widetilde{\sigma}$, let $(\widetilde{p}, \widetilde{n})$ be its Descartes' pair, i.e., the number of its sign changes and the number of sign changes of the coefficients of $f(-z)$. Then $(\widetilde{p}, \widetilde{n})$ is realizable for σ.*

Proof. A *sign-independently real-rooted polynomial* is a real univariate polynomial such that it has only real roots and the same holds for an arbitrary sign change of its coefficients, see [42]. A polynomial is sign-independently real rooted if and only if for each monomial k, the lopsidedness criterion (3.1) is fulfilled for k and at $z_k \in \mathbb{R}_+$. Let $f(z)$ be a sign-independently real-rooted polynomial with the given sign pattern σ. For each $\widetilde{\sigma}$, let $g(z)$ denote the polynomial obtained by deleting those monomials from $f(z)$ which correspond to components of σ deleted when constructing $\widetilde{\sigma}$. Clearly the inequality (3.1) holds also for $g(z)$, since we are removing monomials from its right-hand side. Therefore, the sign of $g(z)$ at $z = z_k$ equals that of $x_k z_k^k$. Since $z_0 < z_1 < \cdots < z_d$, we conclude that $g(z)$ has at least $\widetilde{p}$ sign changes in $\mathbb{R}_+$. Similarly, we find that $g(z)$ has at least $\widetilde{n}$ sign changes in $\mathbb{R}_-$. Furthermore, by Descartes' rule of signs, this is the maximal number of positive and negative roots respectively of $g(z)$. Hence, this is the number of positive and negative roots of $g(z)$, and in particular each such root is of order one. Therefore, a perturbation of the coefficients does not change the number of real roots. Finally, by arbitrary small perturbations of the vanishing coefficients of $g(z)$, we can construct a polynomial with the sign pattern σ. □

Next, we consider how realizable pairs for sign patterns σ and $\widetilde{\sigma}$ yields realizable pairs for a *concatenation* of σ and $\widetilde{\sigma}$.

Lemma 7.1.2 (Concatenation Lemma). *Take polynomials*

$$f(z) = \sum_{k=0}^{d_1} x_k z^k \quad \textit{and} \quad g(z) = \sum_{k=0}^{d_2} y_k z^k$$

of degrees d_1 *and* d_2 *respectively and with non-vanishing coefficients. Assume that their sign patterns* σ *and* $\widetilde{\sigma}$ *are such that* $\widetilde{\sigma}_0 = \sigma_{d_1}$*, and that they realize the pairs* (p, n) *and* $(\widetilde{p}, \widetilde{n})$*. Then, for* $\varepsilon > 0$ *small enough, either the polynomial*

$$h(z) = \left(\frac{1}{x_{d_1}} \sum_{k=0}^{d_1-1} x_k z^k \right) + z^{d_1} + \frac{z^{d_1}}{y_0} \left(\sum_{k=1}^{d_2} y_k (\epsilon z)^k \right)$$

or the polynomial $-h(z)$ *realizes the pair* $(p + \widetilde{p}, n + \widetilde{n})$ *for the sign pattern*

$$(\sigma_0, \ldots, \sigma_{d_1}, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_2). \tag{7.4}$$

Proof. We can assume that $\sigma_{d_1} = \widetilde{\sigma}_0 = 1$, in which case the polynomial $h(z)$ has the sign sequence (7.4) for all $\varepsilon > 0$. Notice that, pointwise (and uniformly on compact subsets),

$$h(z) \to \frac{f(z)}{x_{d_1}}, \quad \epsilon \to 0, \quad \text{and} \quad \epsilon^{d_1} h(z/\epsilon) \to \frac{g(z)}{y_0}, \quad \epsilon \to 0.$$

Therefore, for ϵ sufficiently small, h has at least $p + \widetilde{p}$ positive roots, and at least $n + \widetilde{n}$ negative roots.

It remains to show that for ϵ small enough, the number of non-real roots of $h(z)$ is equal to the sum of the number of non-real roots of f and g. By continuity of roots, for each neighborhood N_w of a non-real root w of f of multiplicity m_w, there is a $t = t(w) > 0$ such that $h(z)$ has m_w roots in N_w if $\epsilon < t(w)$. Similarly, for each neighborhood $N_{\widetilde{w}}$ of a non-real root $\widetilde{w}$ of g of multiplicity $m_{\widetilde{w}}$ there is a $t = t(\widetilde{w}) > 0$ such that $h(z/\epsilon)$ has $m_{\widetilde{w}}$ roots in $N_{\widetilde{w}}$. This implies that $h(z)$ has $m_{\widetilde{w}}$ roots in the dilated set $\epsilon N_{\widetilde{w}}$, for $\epsilon < t(\widetilde{w})$. For each non-real root w of f, choose its neighborhood N_w such that all N_w's are pairwise disjoint and do not intersect the real axis. Choose the neighborhoods $N_{\widetilde{w}}$ of the non-real roots $\widetilde{w}$ of g similarly. If f and g has a common non-real root, then we cannot choose the neighborhoods N_w's and $N_{\widetilde{w}}$'s as above so that N_w is disjoint from $N_{\widetilde{w}}$ for every pair w and $\widetilde{w}$. However, for ϵ sufficiently small, the dilated sets $\epsilon N_{\widetilde{w}}$ are disjoint from N_w for any pair w and $\widetilde{w}$. Indeed, since the open sets N_w do not meet $\mathbb{R}$, there is a neighborhood N_0 of the origin disjoint from each N_w; and for ϵ small enough we have that $\epsilon N_{\widetilde{w}} \subset N_0$, implying the latter claim.

The fact that $N_{\widetilde{w}} \cap \mathbb{R} = \emptyset$, implies that $\epsilon N_{\widetilde{w}} \cap \mathbb{R} = \emptyset$ as well. Therefore, we can conclude that, for ϵ small enough, all roots of $h(z)$ contained in any of the sets $\epsilon N_{\widetilde{w}}$ or N_w, are non-real, which finishes the proof. $\square$

Let us, as a first application of the above lemmas, give a proposition with a flavor of Conjecture 7.0.15.

Proposition 7.1.3. *Given an arbitrary sign pattern σ, any of its admissible pair (p,n) satisfying the criterion that*

$$\min(p,n) > \left\lfloor \frac{d-4}{3} \right\rfloor$$

is realizable.

Proof. Notice first that, if $d \leq 3$, then $\lfloor (d-4)/3 \rfloor < 0$. Thus we claim that any admissible pair is realizable in this case, which follows from Theorem 7.0.11.

For arbitrary d, let us decompose σ in the following manner. Let

$$\tau_k = (\sigma_{3k+1}, \ldots, \sigma_{3k+4}), \quad k = 0, \ldots, \left\lfloor \frac{d-4}{3} \right\rfloor,$$

where we use slight abuse of notation as the last pattern is not necessarily of length four. Then, for each τ_k, the admissible pairs are among the pairs

$$(1,0), (1,2), (3,0), (0,1), (2,1), \text{ and, } (0,3),$$

and for each τ_k all admissible pairs are realizable as τ_k is of length at most four, and hence correspond to the case $d \leq 3$.

For each τ_k, associate initially an admissible pair $u_k = (1,0)$ or $u_k = (0,1)$ depending on whether τ_k admits an odd number of positive roots and an even number of negative roots, or vice versa. By assumption,

$$\sum_k u_k \leq (p,n)$$

(where the inequality should be understood componentwise). If this is not an equality, then the difference is of the form $(2a, 2b)$, where $a + b \leq \lfloor (d-4)/3 \rfloor$, since the original pair (p,n) is admissible. Define

$$\begin{aligned} v_k &= u_k + (2,0), && k = 0, \ldots, a-1, \\ v_k &= u_k + (0,2), && k = a, \ldots, a+b-1, \\ v_k &= u_k, && k = a+b, \ldots, \lfloor (d-4)/3 \rfloor. \end{aligned}$$

Then, v_k is an admissible pair for τ_k, and in addition

$$\sum_k v_k = (p,n).$$

Applying Lemma 7.1.2 repeatedly to the patterns τ_k, we can conclude Proposition 7.1.3. □

Proposition 7.1.4 (Kostov's Lemma)**.** *Consider a sign pattern σ consisting of a consecutive pluses followed by b consecutive minuses and then by c consecutive pluses, where $a+b+c=d+1$. Then*

i) for the pair $(0,d-2)$, this sign pattern is not realizable if

$$\kappa = \frac{d-a-1}{a}\cdot\frac{d-c-1}{c} \geq 4; \tag{7.5}$$

ii) any admissible pair of the form $(2,n)$, with $n>0$, is realizable.

iii) if d is even, then $(2,0)$ is realizable if and only if $\sigma_k=-1$ for some odd k.

Remark 7.1.5. Inequality (7.5) provides only sufficient conditions for non-realizability of the pattern σ with the pair $(0,d-2)$. One can ask how sharp this condition is. At the moment we do not have examples with (7.5) violated when the pair $(0,d-2)$ is not realizable. ○

Proof of Proposition 7.1.4. To prove the first claim, we show that the three-part sign pattern σ satisfying the assumptions of the proposition is not realizable by a polynomial $f(z)$ having $d-2$ negative and a double positive root. By a linear change of z the latter can be assumed to be equal to 1, that is $f(z)=(z^2-2z+1)g(z)$, where

$$g(z)=z^{d-2}+x_1z^{d-3}+\cdots+x_{d-2}.$$

Here, $x_j>0$ since the factor $g(z)$ has $d-2$ negative roots. The coefficients of $f(z)$ are equal to

$$\begin{aligned}1,\quad x_1-2,\quad x_2-2x_1+1,\quad \ldots,\\ x_{d-2}-2x_{d-3}+x_{d-4},\quad -2x_{d-2}+x_{d-3},\quad x_{d-2}.\end{aligned}$$

We want to show that it is impossible to have both inequalities:

$$x_a-2x_{a-1}+x_{a-2}<0 \tag{7.6}$$

$$\text{and}\quad x_{a+b-1}-2x_{a+b-2}+x_{a+b-3}<0 \tag{7.7}$$

satisfied.

Consider a polynomial having $d-2$ negative roots and a complex conjugate pair. If the polynomial has at least one negative coefficient, then its factor having complex roots must be of the form $z^2-2\beta z+\beta^2+\gamma$, where $\beta>0$ and $\gamma>0$. A linear change of variables brings the polynomial to the form

$$h(z)=(z^2-2z+1+\delta)g(z),$$

for some $\delta > 0$. The coefficients of $h(z)$ are obtained from that of $f(z)$ by adding the ones of the polynomial $\delta g(z)$. If inequality (7.7) fails, then the coefficient of $z^{d-a-b+1}$ in $Q(z)$ is positive (it equals $x_{a+b-1} - 2x_{a+b-2} + x_{a+b-3} + \delta x_{a+b+1} > 0$). So the sign pattern of $h(z)$ is different from σ. If inequality (7.7) holds, then inequality (7.6) fails and the coefficient of z^{d-a} in $h(z)$ is non-negative, so that $h(z)$ does not have the sign pattern σ.

The polynomial $g(z)$, having all roots negative and real, its coefficients satisfy the Newton inequalities:

$$\frac{x_k^2}{\binom{d-2}{k}^2} \geq \frac{x_{k-1}x_{k+1}}{\binom{d-2}{k+1}\binom{d-2}{k-1}}, \qquad k = 1, \ldots, d-3.$$

Here,

$$\kappa = \frac{\binom{d-2}{a}\binom{d-2}{a+b-3}}{\binom{d-2}{a-1}\binom{d-2}{a+b-2}} = \frac{d-a-1}{a} \cdot \frac{d-c-1}{c}.$$

I.e., $x_a x_{a+b-3} \geq \kappa x_{a-1} x_{a+b-2}$. Inequalities (7.6) and (7.7) imply respectively

$$x_a < 2x_{a-1} \quad \text{and} \quad x_{a+b-3} < 2x_{a+b-2}.$$

Thus $x_a x_{a+b-3} \geq \kappa x_{a-1} x_{a+b-2} > \kappa x_a x_{a+b-3}/4$, which is a contradiction since $\kappa \geq 4$ by assumption.

To prove the second and third claim, we use Lemma 7.1.1. Firstly, consider the sign pattern $\tilde{\sigma}$ obtained by keeping only the constant and leading terms and, in addition, one terms with negative coefficient. By assumption, we can choose the latter term such that the pair $(\tilde{p}, \tilde{n})$ related to $\tilde{\sigma}$ (in the notation of Lemma 7.1.1) is of the form $(2, \tilde{n})$, where n and $\tilde{n}$ are of equal parity and $\tilde{n} \leq n$. Adding any further terms to $\tilde{\sigma}$ does not alter the number of positive roots $\tilde{p}$. However, adding a single term to sigma, either $\tilde{n}$ is unaltered, or it is increased by two. Since $\tilde{n} \leq n$, we can add terms until eventually we have equality, implying realizability by Lemma 7.1.1. □

Turning towards Theorem 7.0.12, we have the following lemma concerning odd degrees.

Lemma 7.1.6. *Let d be odd. Consider a sign pattern σ such that*

i) $\sigma_0 = \sigma_d = +1$,

ii) all other entries at even positions are -1,

iii) there is at most one sign change in the group of signs at odd positions.

Then, of all admissible pairs $(0, s)$, *only* $(0, 1)$ *is realizable.*

Proof. Let us decompose a polynomial $f(z)$ with the sign pattern σ as the sum of the polynomials $f_o(z)$ and $f_e(z)$, containing only odd respectively even monomials. Then, each of f_e, f_o and f_o' have exactly one positive root, which we denote by z_e, z_o, respectively z_o'. We first claim that $z_o' < z_o < z_e$.

To prove the claim, assume initially that $z_e \le z_o$. Then both f_o and f_e are non-positive on the interval $[z_e, z_o]$. Therefore, also $f(z)$ is non-positive on the same interval, which contradicts to the assumption that $f(z)$ is positive for all positive z.

We now prove that $z_o' < z_o$. Present $f_o(z)$ as a sum $f_o^+(z) - f_o^-(z)$, where $f_o^+(z)$ is the sum of all odd degree monomials with positive coefficients and $f_o^-(z)$ is the negative of the sum of all odd degree monomials with negative coefficients. Observe that the degree of the smallest monomial in $f_o^+(z)$ is larger than $\delta = \deg f_o^-$ by assumption. If $f_o(z) \ge 0$, i.e., $z \ge z_o$ then

$$(f_o^+)'(z) > \delta f_o^+(z) \ge \delta f_{odd}^-(z) > (f_o^-)'(z),$$

which implies that $f_o'(z) > 0$, and hence $z_o > z_o'$.

Finally, we show that $f(z)$ has at most one negative root. Consider the interval $[0, z_o]$. Since $z_e > z_o$,, we have that $f_e(z) > 0$ in $[0, z_o]$. Additionally, $f_o(z)$ is non-positive in this interval, implying that $f(-z) = f_e(z) - f_o(z)$ is positive in the interval $[0, z_o]$. In the interval $[z_o', +\infty)$, the polynomial $f_o'(z)$ is positive which together with the fact that $f_e'(z)$ is negative implies that $f'(-z) = f_e'(z) - f_o'(z)$ is negative. Thus, being positive in $[0, z_o]$ and monotone decreasing to $-\infty$ in $[z_o', +\infty)$, we conclude that $f(-z)$ necessarily has exactly one positive root. □

Lemma 7.1.7. *Let σ be a sign sequence such that $\sigma_0 = \sigma_d$. Assume that it is possible to delete components of σ, other than the constant and leading term, so that the obtained sequence $\widetilde{\sigma}$ (in the notation of Lemma 7.1.1) has length four and fulfills that*

i) $\widetilde{\sigma}$ has two sign changes

ii) the flip of $\widetilde{\sigma}$ (i.e., the sign sequence obtained by $f(z) \mapsto f(-z)$) has three sign changes.

(Notice that d is necessarily odd.) Then, the pair $(0,3)$ is realizable for sigma.

Proof. The assumptions is, after applying the standard action, equal to that we can delete terms of a polynomial $f(z)$ with sign sequence $\operatorname{sgn}(f) = \sigma$ to a polynomial

$$\widetilde{f}(z) = x_0 - x_a z^a + x_b z^b + x_d z^d,$$

where a and b are even and each x_i is positive. Choose x_0, x_a, and x_b so that the auxiliary polynomial $x_0 - x_a z^a + x_b z^b$ has a positive root of multiplicity two. Then, with x_d sufficiently small, we obtain a polynomial with no positive roots and three negative roots. By Descartes' rule of sign, there are at most three negative roots. Hence, all real roots are simple, implying that small perturbations of the vanishing coefficients does not alter the number of real roots. □

Proof of Theorem 7.0.12. The fact that the patterns given in degree 7 in Table 7.1 are non-realizable follows from Proposition 7.1.4 and Lemma 7.1.6. It remains to show that all other admissible patterns and pairs are realizable.

Using Lemma 7.1.1 and a Mathematica script, [1] the question is reduced to checking the cases:

Sign pattern								Pair
+	-	-	-	+	+	+	-	(1,4)
+	+	+	-	-	-	+	+	(0,5)
+	+	+	+	-	-	-	+	(0,5)
+	-	-	+	-	-	-	+	(0,3)
+	-	+	-	-	-	+	+	(0,3)
+	-	-	-	-	-	+	+	(0,3)
+	+	-	+	-	-	-	+	(0,3)
+	-	+	+	-	-	-	+	(0,3)

The realizability of the first four cases follows from Lemma 7.1.2, by concatenation of the following realizable patterns and pairs;

$$\begin{array}{llcll}
(+,-,-), & (1,1) & \text{and} & (-,-,+,+,+,-), & (0,3),\\
(+,+), & (0,1) & \text{and} & (+,+,-,-,-,+,+), & (0,4),\\
(+,+), & (0,1) & \text{and} & (+,+,+,-,-,-,+), & (0,4),\\
(+,-,-,+), & (0,3) & \text{and} & (+,-,-,-,+), & (0,0).
\end{array}$$

The realizability of the remaining four cases follows from Lemma 7.1.7, deleting all monomials except those with indices $\{0,4,6,7\}$ in the first two cases and $\{0,3,5,7\}$ in the last two cases. □

We have the following version of Lemma 7.1.6 for even degrees.

Proposition 7.1.8. *For d even, consider a sign pattern σ such that*

i) $\sigma_0 = \sigma_d = +1$,

ii) the signs of all odd monomials are $+1$,

[1] Which, as of this writing, is not available on the author's website.

iii) among the remaining signs of even monomials there are $a \geq 1$ minuses (at arbitrary positions).

Then, the pairs $(p,0)$ with $p > 0$, and only they, are non-realizable.

Proof. Firstly, to see that no pairs $(p,0)$ with $p \geq 0$ is realizable for σ, note that $f(-z) < f(z)$, that $f(0) > 0$, and that $f(z) \to \infty$ as $z \to \infty$. Hence, if $f(z)$ has a positive root, then it has a negative root.

To prove realizability for any other pair (p,n), let us first prove that of $\widetilde{\sigma}$ is a sign pattern for even degree such that minuses appear only for odd monomials, then all admissible pairs are realizable. Indeed, this follows by induction on the degree using Lemma 7.1.2; we view $\widetilde{\sigma}$ as the concatenation of the sequence τ_1 consisting of its first three entries, and τ_2 consisting of all but the first two entries.

Now, let (p,n) be an admissible pair for σ such that $n \geq 2$. Then, the pair $(p, n-2)$ is realizable for the sign pattern $\widetilde{\sigma}$ obtained from σ by deleting the first and the last entry. Using Lemma 7.1.2, concatenating once from the left and once from the right with the sign pattern $(+,+)$, we conclude that (p,n) is realizable. □

Proof of Theorem 7.0.13. The fact that the patterns given in degree 8 in Table 7.1 are non-realizable follows from Proposition 7.1.4 and Lemma 7.1.8. It remains to show that all other admissible patterns, except those appearing in Table 7.2, are realizable.

Using Lemma 7.1.1 and the above mentioned Mathematica script this reduces to checking 22 distinct pairs of a sign sequence and a Descartes pair. We refer from writing them all explicitly. As in the proof of Theorem 7.0.12, we now apply Lemma 7.1.2, which reduces seven cases consisting of the six cases in Table 7.2 and the additional case

$$(+,-,-,-,+,+,-,-,+) \quad \text{and} \quad (0,4).$$

To see that this pair is realizable, consider a polynomial

$$f(z) = x_0 - x_2 z^2 + x_4 z^4 - x_6 z^6 + x_8 z^6,$$

with each x_i positive and such that $f(z)$ has two distinct positive roots of order two. Adding the monomial z^5 with a positive coefficient that is sufficiently small, we obtain a polynomial that has no positive roots but four negative real simple roots. Now perturb the vanishing coefficients. □

7.2 Contractibility

We now turn to the statements concerning connectedness and contractibility of the sets $\mathrm{Pol}^{\sigma}_{d,k}$.

Proof of Theorem 7.0.8. To prove the first claim, note first that Descartes' rule of signs implies necessity. Conversely, of $\sigma_0 = \sigma_d$, let $\widetilde{\sigma}$ be the sign pattern obtained by deleting all other entries of σ, and apply Lemms 7.1.1.

To prove the second claim, let us begin with the case $k = 1$. Then, d is necessarily odd, and the same construction as in the case $k = 0$, using Lemma 7.1.1, gives realizability of either $(0,1)$ or $(1,0)$.

To prove the general case, let us use induction over d, with the case $k = 1$ as the basis of the induction. That is, let $k > 1$. Let $\widetilde{\sigma}$ be the sign pattern obtained by deleting the last entry of σ, for which there is a polynomial realizing $\widetilde{\sigma}$ with $k-1$ real roots by the induction hypothesis. Let $\hat{\sigma}$ be the sign patter consisting of the last two entries of σ. The, Lemma 7.1.2 implies that we can construct a polynomial with sign pattern σ and k real roots by concatenation of $\widetilde{\sigma}$ and $\hat{\sigma}$. □

To prove Proposition 7.0.9, we need the following lemma having an independent interest.

Lemma 7.2.1. *For any $\bar{\sigma}$, the intersection* $\mathrm{Pol}^{\sigma}_{d,d}$ *is path-connected.*

Proof. Recall that a real polynomial $f(z)$ is called sign-independently real-rooted if every polynomial obtained from $f(z)$ by an arbitrary sign change of its coefficients is real-rooted. It is shown in [42] that the logarithmic image of the set of all sign-independently real-rooted polynomials is convex. Hence the set of all sign-independently real-rooted polynomials itself is logarithmically convex and in particular, it is path-connected. As noted earlier, sign independent real-rootedness is equivalent to that for all k there exists a real point z_k such that (3.1) holds.

Using induction on the degree d, we will now prove that, for any polynomial $f \in \mathrm{Pol}^{\sigma}_{d,d}$, there is a path $t \mapsto f_t$ such that

i) $f_0 = f$,

ii) f_1 is sign-independently real-rooted, and

iii) $f_t \in \mathrm{Pol}^{\sigma}_{d,d}$ for all $t = [0,1]$.

Since the set consisting of all sign-independently real-rooted polynomials is path-connected, this claim settles Lemma 7.2.1. The case $d = 1$ is trivial, as any linear polynomial is sign-independently real-rooted.

Let f be a real-rooted polynomial of degree d. Then, $g = f'$ is a real-rooted polynomial of degree $d-1$. Hence, by the induction hypothesis, there is a path $t \mapsto g_t$ as above. Furthermore, since f is real-rooted, so is its polar derivative $f'_\alpha(z) := f(z) + \frac{z}{\alpha} f'(z)$ for all $\alpha \in \mathbb{R}^+$.

For each $t \in [0,1]$, let $\alpha_t > 0$ be such that $h_{t,\alpha}(z) := f(z) + \frac{z}{\alpha} g_t(z)$ is real-rooted for any $0 < \alpha < \alpha_t$. By continuity of roots, $h_{\hat{t},\alpha_t}$ is real-rooted for $\hat{t}$ in a small neighborhood of t. Since $[0,1]$ is compact, we can find a finite set $\alpha_{t_1}, \dots, \alpha_{t_N}$ such that $h_{t,\alpha}(z)$ is real-rooted for all $t \in [0,1]$ if $\alpha < \min(\alpha_{t_1}, \dots, \alpha_{t_N})$.

Since $zg_1(z)$ is sign-independently real-rooted, for all k and all monomials $x_k z^k$ of $zg_1(z)$, there exists a point z_k such that (3.1) holds. Since the signs of $f(z)$ are equal to the signs of $zg_1(z)$, there exists an $\alpha_k > 0$ such that (3.1) holds for $h_{1,\alpha}(z)$ for k at z_k. However, since (3.1) always holds for the constant term with z_0 sufficiently small, we conclude that $h_{1,\alpha}$ is sign-independently real-rooted when $\alpha < \min_{k=1,\dots,d-1} \alpha_k$.

Now fix a positive number $\alpha^* < \min(\alpha_{t_1}, \dots, \alpha_{t_N}, \alpha_1, \dots \alpha_{d-1})$ and consider the path composed of the two paths

$$\alpha \mapsto f'_\alpha, \quad \alpha \in [\infty, \alpha^*] \qquad \text{and} \qquad t \mapsto h_{t,\alpha^*}, \quad t \in [0,1].$$

By construction, this path is contained in $\mathrm{Pol}^\sigma_{d,d}$. Its starting point is $f(z)$ and its endpoint h_{1,α^*} is sign-independently real-rooted. This concludes the induction step. □

Proof of Proposition 7.0.9. To prove the first claim, notice firstly that the set $\mathrm{Pol}_{d,0}$ of all positive monic polynomials is a convex cone. Therefore its intersection with any orthant is convex and contractible (if nonempty).

Secondly, consider the set $\mathrm{Pol}^\sigma_{d,d}$. Take a real-rooted polynomial $f(z)$ realizing a given pattern. Consider the family $f(z)+\lambda z^a$, $a = 0,1,\dots,n-1$. Polynomials in this family are real-rooted and with the given sign pattern until either there is a confluence of roots of the polynomial, or its a-th derivative vanishes at the origin. In both cases further increase or decrease of the parameter λ never brings us back to the set of real-rooted polynomials.

Thus the set $\mathrm{Pol}^\sigma_{d,d}$ has what we call *Property A*: each of its connected components intersected with a line parallel to any coordinate axis in the space of coefficients is either empty, or a point, or, finally, an interval whose endpoints are continuous functions of other coefficients. (Indeed, they are values of the polynomial or of its derivatives at roots of the polynomial or its derivatives; therefore these roots are algebraic functions depending continuously on the coefficients.)

Maxima and minima of such functions are also continuous. Therefore the projection of each connected component of $\mathrm{Pol}^\sigma_{d,d}$ on each coordinate hyperplane in the space of the coefficients also enjoys Property A. (It suffices to fix the values of all coefficients but one and study the endpoints of the segments as functions of that coefficient).

Now replace $\mathrm{Pol}^{\sigma}_{d,d}$ by a smaller set obtained as follows. Choose some coefficient and, for fixed values of all other coefficients, substitute every nonempty intersection of $\mathrm{Pol}^{\sigma}_{d,d}$ with lines parallel to the axis corresponding to the chosen coefficient by the half-sum of the endpoints, i.e., substitute the intersection segment by its middle point. This operation produces the graph of a continuous function depending on the other coefficients. The projection of this graph to the coordinate hyperplane of other coefficients is a domain having Property A, but belonging to a space of dimension $n-1$. Continuing this process one contracts each connected component of the set $\mathrm{Pol}^{\sigma}_{d,d}$ to a point. Using Lemma 7.2.1 we conclude that $\mathrm{Pol}^{\sigma}_{d,d}$ is path-connected and therefore contractible.

To prove the second claim, let us show that any compact subsubset of $\mathrm{Pol}^{\hat{+}}_{d,k}$ can be contracted to a point inside $\mathrm{Pol}^{\hat{+}}_{d,k}$. Observe that for any polynomial $f(z)$ with positive coefficients the family of polynomials $f(z+t)$ for a positive parameter t consists of polynomials with all positive coefficients and the same number of real roots all being negative. Given a compact set $K \subset \mathrm{Pol}^{\hat{+}}_{d,k}$, consider its shift K_t obtained by applying the above shift to the left on the distance t, for t sufficiently large. Then all real roots of all polynomials in the compact set K_t will be very large negative numbers and all complex conjugate pairs will have very large negative real part. Therefore one can choose any specific polynomial $\widetilde{p}$ in K_t and contract the whole K_t to $\widetilde{f}$ along the straight segments, i.e., $\tau\widetilde{f}+(1-\tau)f$ for any $f \in K_t$. Obviously such contraction takes place inside $\mathrm{Pol}^{\hat{+}}_{d,k}$. □

Proof of Theorem 7.0.10. We will follow the steps of the proof of Lemma 7.2.1. For any polynomial f, the set K_f consisting of all exponents k such that there exists a $z_k \in \mathbb{R}_+$ for which (3.1) holds, provides a lower bound on the number of real roots of f. This lower bound is called the number of *lopsided induced zeros* of f. Fixing an arbitrary set of exponents K, let us denote by S_K the set of all polynomials such that $K \subseteq K_f$. We saw in Section 6.1 that S_K is logarithmically convex. Consider the family F_m consisting of all exponent sets K such that the number of lopsided induced zeros of polynomials in S_K is at least m. The set $S_m = \cup_{K\in F_m} S_K$ is a union of logarithmically convex sets, whose intersection contains the set of all sign-independently real-rooted polynomials. In particular, S_m is path-connected.

As in the proof of Lemma 7.2.1, for any polynomial f which has at least m real roots, all polynomials in the path

$$\alpha \mapsto f(z)+\frac{z}{\alpha}f'(z), \quad \alpha \in [\infty,\alpha^*]$$

have at least m real roots. The same argument as in the proof of Lemma 7.2.1 gives path-connectedness of the set (7.3) of polynomials with at least m real roots.

Let us now prove simply connectedness of the set (7.3) by induction on the degree d. Consider a closed loop in $\mathrm{Pol}^{\sigma}_{d,\geq m}$, i.e., a path ℓ given by $\theta \mapsto f_\theta(z)$, $\theta \in [0,1]$, such that $f_0(z) = f_1(z)$, and such that $f_\theta(z)$ has at least m real roots for all θ.

Consider the induced loop ℓ' given by $\theta \mapsto f'_\theta(z)$, where we use the notation $f'_\theta(z) = \frac{d}{dz} f_\theta(z)$. It is contained in the set $\mathrm{Pol}^{\hat{\sigma}}_{d-1,\geq m-1}$, where $\hat{\sigma}$ is obtained from σ by deleting its first entry. By the induction hypothesis, the loop ℓ' can be contracted to a point within the set of all polynomials of degree $d-1$ with at least $m-1$ real roots. In other words, we have a map $(\theta,\varphi) \mapsto f'_{(\theta,\varphi)}$, for $(\theta,\varphi) \in [0,1]^2$, satisfying the conditions:

i) $f'_{(\theta,0)}(z) = f'_\theta(z)$,

ii) $f'_{(\theta,1)}$ is independent of θ, and

iii) $f'_{(\theta,\varphi)}$ has at least $m-1$ real roots for all θ and φ.

The last property implies that $zf'_{(\theta,\varphi)}$ has at least m real roots for all θ and φ. Define $g_{(\theta,\varphi)}$ by the conditions that $\frac{d}{dz} g_{(\theta,\varphi)} = f'_{(\theta,\varphi)}$ and that the constant term of $g_{(\theta,\varphi)}$ is independent of φ.

Since the loop ℓ' is compact, we can find an $\alpha^* \in \mathbb{R}_+$ such that the polar derivative

$$f'_{(\theta,\varphi,\alpha)}(z) := g_{(\theta,\varphi)}(z) + \frac{z}{\alpha} f'_{(\theta,\varphi)}(z)$$

has at least m roots for each $\alpha < \alpha^*$ and all $(\theta,\varphi) \in [0,1]^2$. Thus, similarly to the proof of Lemma 7.2.1, the composition of the maps

$$\alpha \mapsto f'_{(\theta,0,\alpha)}, \quad \alpha \in [\infty, \alpha^*] \qquad \text{and} \qquad \varphi \mapsto f'_{(\theta,\varphi,\alpha^*)}, \quad \varphi \in [0,1]$$

provides a contraction of the loop ℓ in the set $\mathrm{Pol}^{\sigma}_{d,\geq m}$. $\square$

Sammanfattning

Denna avhandling behandlar tropiska aspekter av polynom över de reella talen och av hypergeometriska funktioner. Avhandlingen omfattar sju kapitel. Efter en kort inledning, behandlar vi i det andra kapitlet grundläggande begrepp som återkommer genom hela avhandlingen.

I de efterföljande två kapitlen studerar vi koamöban $\mathcal{C}_f$ av ett Laurentpolynom f i flera variabler. Denna utgörs, per definition, av bilden av den algebraiska nollställemängden $Z(f) \subset \mathbb{C}_*^n$ under den projektion som ges av den komponentvisa argumentavbildningen. Koamöbor introducerades av M. Passare och A. Tsikh som duala objekt till amöbor. Vi introducerar i denna avhandling en förenklad version av koamöban, kallad den *sidotunga koamöban*, och betecknad $\mathcal{D}_f$. Den sidotunga koamöban har en associerad avbildning

$$\mathrm{ord}_{\mathcal{D}} \colon \mathbf{T}^n \setminus \overline{\mathcal{D}}_f \to 2\pi\mathbb{Z}^m,$$

där $\mathbf{T}$ betecknar kvoten $\mathbb{R}/2\pi\mathbb{Z}$. Avbildningern $\mathrm{ord}_{\mathcal{D}}$ är lokalt konstant, och definierar därför en avbildning, också betecknad $\mathrm{ord}_{\mathcal{D}}$, från mängden bestående av alla sammanhängande komponenter av komplementet till den sidotunga koamöbans tillslutning. Vi kommer här bevisa, för det första, att denna avbildning är injektiv och, för det andra, att dess bild består av de gitterpunkter vilka befinner sig i det inre av en viss zonotop. På grundval av detta kallar vi $\mathrm{ord}_{\mathcal{D}}$ för *den sidotunga koamöbans ordningsavbildning*. Med hjälp av ordningsavbildningen studerar vi sedan koamöbans topologi. Särskilt så besvarar vi positivt en förmodan av M. Passare gällande antalet sammanhängande komponenter av komplementet till koamöbans tillslutning, under antagandet att polynomets Newton-polytop $\mathcal{N}_f$ har få hörn. Vi fokuserar sedan på koamöbor av polynom vars stöd består av en krets. För sådana familjer erhåller vi en komplett beskrivning av koamöba-rummet. Vidare så besvarar vi positivt en förmodan gällande argumenten av de kritiska punkterna till ett polynom.

I det fjärde kapitlet studerar vi hypergeometriska funktioner i Gel'fands, Kapranovs och Zelevinskys bemärkelse. Vi definierar Euler–Mellin integraler, en familj av Eulerska hypergeometriska integraler vilken är nära associerad till koamöban $\mathcal{C}_f$. Vi visar sedan att den sidotunga koamöbans ordningsavbildning ger en identitet mellan Euler–Mellin integraler och så kallade

Mellin–Barnes integraler. Till skillnad från tidigare utförda studier av hypergeometriska integraler så gör den explicita formen av Euler–Mellin integraler att dessa kan nyttjas för att i detalj studera hur A-hypergeometriska funktioner beror på homogenitetsparametern β. Vårt huvudsakliga resultat i detta kapitel är en komplett beskrivning av detta beroende i fallet då A representerar en projektiv torisk kurva.

I femte kapitlet så studerar vi amöban och koamöban av den principala A-determinanten. Där tidigare studier har relaterat dessa objekt till konvergensområden för serie- repsektive integralrepresentationer av hypergeometriska funktioner, kommer vi istället att fokusera på deras respektive relation till sidotunghet.

I det avslutande kapitlet applicerar vi resultaten från tidigare kapitel på teorin kring envariabelpolynom över de reella talen. Descartes teckenregel ger nödvändiga, men inte tillräckliga, villkor för att ett par (p, n) av heltal ska motsvara antalet positiva och negative nollställen till ett sådant polynom. Vi ger här ett antal tillräckliga villkor, och erhåller som en följd en komplett karakterisering av vilka par uppfyllande Descartes teckenregel som är realiserbara för polynom upp till grad sju.

Acknowledgements

I would like to express my sincere gratitude to my supervisor Boris Shapiro for his patience, advice, and never-ending stream of mathematical problems. I would also like to thank Ralf Fröberg who guided me into grad school. The work on this thesis was interrupted by the tragic death of Mikael Passare, who sparked my interest in tropical geometry. To him, I will be forever grateful. To everyone who gave their support during that time, I might not know all of you by name, I am forever in debt. Special thanks go to all people involved in this work. To Petter Johansson and Johannes Lundqvist, for the plenty discussions about amoebas and coamoebas. To Christine Berkesch Zamaere and Laura F. Matusevich, for the reception I received in the U.S., and all hours spent contemplating "hypergeometric stuff". To Vladimir P. Kostov for thoughtful remarks. I wish to thank everyone at the Department of Mathematics at Stockholm University, those still here and those that have left. It has been a pleasure! Finally, I thank all my friends and family, whose support has been invaluable.

References

[1] Adolphson, A., *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290. 19

[2] Albouy, A and Fu, Y., *Some remarks about Descartes' rule of signs*, Elem. Math. **69** (2014), no. 4, 186–194. 111

[3] Anderson, B., Jackson, J., and Sitharam, M., *Descartes' rule of signs revisited*, Amer. Math. Monthly **105** (1998), no. 5, 447–451. 111

[4] Avendaño, M., Kogan, R., Nisse, M., and Rojas, J. M., *Metric estimates and membership comcomplex for archimedean and tropical hypersurfaces*, arXiv:1307.3681, 2013. 64

[5] Berkesch, C., *The rank of a hypergeometric system*, Compos. Math. **147** (2011), no. 1, 284–318. 19, 70

[6] Beukers, F., *Monodromy of A-hypergeometric functions*, http://www.staff.science.uu.nl/~beuke106/Amonodromy.pdf, arXiv:1101.0493, 2013. 19, 39, 78, 79

[7] Birkeland, R., *Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen*, Math. Z. **26** (1927), no. 1, 566–578. 10

[8] Bochner, S., *A theorem on analytic continuation of functions in several variables*, Ann. of Math. (2) **39** (1938), no. 4, 14–19. 21

[9] Cattani, E., D'Andrea, C., and Dickenstein, A., *The A-hypergeometric system associated with a monomial curve*, Duke Math. J. **99** (1999), no. 2, 179–207. 19, 86, 87, 90, 95, 97

[10] Dickenstein, A., Feichtner, E. M., and Sturmfels, B., *Tropical discriminants*, J. Amer. Math. Soc. **20** (2007), no. 4, 1111–1133. 18

[11] Dickenstein, A. and Sturmfels, B., *Elimination theory in codimension 2*, J. Symbolic Comput. **34** (2002), no. 2, 119–135. 105

[12] Esterov, A, *Newton polyhedra of discriminants of projections*, Dicrete Comput. Geom. **44** (2010), no. 1, 96–148. 18

[13] Feng, B., He, Y-H., Kennaway, K. D., and Vafa, C., *Dimer models from mirror symmetry and quivering amoebae*, Adv. Theor. Math. Phys. **12** (2008), no. 3, 489–545. 32

[14] Forsberg, M., Passare, M., and Tsikh, A., *Laurent determinants and arrangements of hyperplane amoebas*, Adv. Math. **151** (2000), no. 1, 45–70. 11, 21

[15] Forsgård, J., *On hypersurface coamoebas and integral representations of A-hypergeometric functions*, Licentiate thesis, Stockholms universitet, Stockholm, 2012. 40

[16] Forsgård, J. and Johansson, P., *Coamoebas and line arrangements in dimension two*, Math. Z. **278** (2014), no. 1–2, 25–38. 23

[17] Gel'fand, I. M., Graev, M. I., and Zelevinsky, A. V., *Holonomic systems of equations and series of hypergeometric type*, (Russian) Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19, translation in Soviet Math. Dokl. **36** (1988), no. 1, 5–10. 18, 19

[18] Gel'fand, I. M., Kapranov, M. M., and Zelevinsky, A. V., *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math. **84** (1990), no. 2, 255–271. 15, 18, 19, 69, 90

[19] ______, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994. 17, 18, 52, 102

[20] Gel'fand, I. M., Zelevinsky, A. V., and Kapranov, M. M., *Equations of hypergeometric type and Newton polyhedra*, (Russian) Dokl. Akad. Nauk SSSR **300** (1988), no. 3, 529–534, translation in Soviet Math. Dokl. **37** (1988), no. 3, 678–682. 18, 19, 105

[21] ______, *Hypergeometric functions and toric varieties*, (Russian) Funktsional. Anal. i Prilozhen **23** (1989), no. 2, 12–26, translation in Funct. Anal. Appl. **23** (1989), no. 2, 94–106. 18, 19, 105

[22] Grabiner, D. J., *Descartes' rule of signs: another construction*, Amer. Math. Monthly **106** (1999), no. 9, 854–856. 111

[23] Hardy, G. H., Littlewood, J. E., and Pólya, G., *Inequalities*, Cambridge, at the University Press, 1952, 2d ed. 42

[24] Hayman, W. K., *The local growth of power series: a survey of the Wiman–Valiron method*, Canad. Math. Bull. **17** (1974), no. 3, 317–358. 28

[25] Johansson, P., *Coamoebas*, Licentiate thesis, Stockholms universitet, Stockholm, 2010. 22, 23, 24, 30

[26] ———, *The argument cycle and the coamoeba*, Complex Var. Elliptic Equ. **58** (2013), no. 3, 373–384. 21

[27] Jung, K. and van Straten, D., *Arctic Computation of Monodromy*, Report No. 40/2006, Oberwolfach, 2006. 61

[28] Kapranov, M. M., *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), no. 2, 277–285. 101

[29] Kiselman, C. O., *Questions inspired by Mikael Passare's mathematics*, Afr. Mat. **25** (2014), no. 2, 271–288. 40

[30] Marden, M., *Geometry of polynomials*, Second edition ed., Mathematical Surveys, no. 3, Amer. Math. Soc., Providence, RI, 1966. 28

[31] Mikhalkin, G., *Amoebas of half-dimensional varieties*, arXiv:1412.4658. 57

[32] ———, *Real algebraic curves, the moment map and amoebas*, Ann. of Math. (2) **151** (2000), no. 1, 309–326. 100, 101

[33] ———, *Amoebas of algebraic varieties and tropical geometry*, Different faces of geometry (Donaldson, S., Eliashberg, Y., and Gromov, M., eds.), Int. Math. Ser. (N. Y.), vol. 3, Kluwer/Plenum, New York, 2004, pp. 257–300. 21

[34] Mikhalkin, G. and Rullgård, H., *Amoebas of maximal area*, Internat. Math. Res. Notices (2001), no. 9, 441–451. 58

[35] Nilsson, L., *Amoebas, discriminants, and hypergeometric functions*, Doctoral thesis, Stockholms universitet, Stockholm, 2009. 19, 78, 80

[36] Nilsson, L. and Passare, M., *Discriminant coamoebas in dimension two*, J. Commut. Algebra **2** (2010), no. 4, 447–471. 85, 99, 103, 104, 105

[37] ———, *Mellin transforms of multivariate rational functions*, J. Geom. Anal. **23** (2013), no. 1, 24–46. 71, 72, 73

[38] Nisse, M., *Geometric and combinatorial structure of hypersurface coamoebas*, arXiv:0906.2729, 2009. 22, 23

[39] Nisse, M. and Sottile, F., *Non-Archimedean coamoebae*, Tropical and non-Archimedean geometry (Amini, O., Baker, M., and Faber, X., eds.), Contemp. Math., vol. 605, Amer. Math. Soc., Providence, RI, 2013, pp. 73–91. 20

[40] ______, *The phase limit set of a variety*, Algebra Number Theory **7** (2013), no. 2, 339–352. 21, 22, 59

[41] Oystein, O., *Theory of multigroups*, Amer. J. Math. **60** (1938), no. 3, 705–733. 47

[42] Passare, M., Rojas, J. M., and Shapiro, B., *New multiplier sequences via discriminant amoebae*, mosc. Math. J. **11** (2011), no. 3, 547–560, 631. 113, 121

[43] Passare, M., Sadykov, T., and Tsikh, A., *Singularities of hypergeometric functions in several variables*, Compos. Math. **141** (2005), no. 3, 787–810. 11, 99, 100

[44] Passare, M. and Sottile, F., *Discriminant coamoebas through homology*, J. Commut. Algebra **5** (2013), no. 3, 413–440. 17, 104

[45] Passare, M. and Tsikh, A., *Algebraic equations and hypergeometric series*, The legacy of Niels Henrik Abel (Laudal, O. A. and Piene, R., eds.), Springer, Berlin, 2004, pp. 653–672. 11

[46] ______, *Amoebas: their spines and their contours*, Idempotent mathematics and mathematical physics (Litvinov, G. L. and Maslov, V. P., eds.), Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 275–288. 21, 100, 101

[47] Pellet, M. A., *Sur un mode de séparation des racines des équations et la fomule de Lagrange*, Bull. Sci. Math. **5** (1881), 393–395. 27

[48] Purbhoo, K, *A Nullstellensatz for amoebas*, Duke Math. J. **141** (2008), no. 3, 407–445. 27, 41

[49] Riesz, M., *Intégrales de Riemann–Liouville et potentiels*, (French) Acta Sci. Math. (Szeged) **9** (1938), 1–42, Rectification au travail Intégrales de Riemann–Liouville et potentiels, (French) Acta Sci. Math. (Szeged), **9** (1938), 116–118. 72

[50] Rullgård, H., *Topics in geometry, analysis and inverse problems*, Doctoral thesis, Stockholms universitet, Stockholm, 2003. 21, 27, 54, 58

[51] Saito, M., Sturmfels, B., and Takayama, N., *Gröbner deformations of hypergeometric differential equations*, Algorithms and Computation in Mathematics, vol. 6, Springer-Verlag, Berlin, 2000. 12, 19, 90, 95

[52] Shapiro, B. Z. and Khesin, B. A., *Swallowtails and Whitney umbrellas are homeomorphic*, J. Algebraic Geom. **1** (1992), no. 4, 549–560. 109

[53] Shephard, G. C., *Combinatorial properties of associated zonotopes*, Canad. J. Math. **26** (1974), 302–321. 105

[54] Theobald, T. and de Wolff, T., *Amoebas of genus at most one*, Adv. Math. **239** (2013), 190–213. 54

[55] ———, *Norms of roots of trinomials*, arXiv:1411.6552, 2014. 54

[56] Ueda, K. and Yamazaki, M., *A note on dimer models and McKay quivers*, Comm. Math. Phys. **301** (2011), no. 3, 723–747. 32

[57] Viro, O., *Hyperfields for Tropical Geometry I. Hyperfields and dequantization*, arXiv:1006.3034, 2010. 44, 48

STOCKHOLM DISSERTATIONS IN MATHEMATICS

Dissertations at the Department of Mathematics Stockholm University

1-22. 2005-2008.
23. Strohmayer, Henrik: *Prop profiles of compatible Poisson and Nijenhuis structures.* 2009.
24. Rökaeus, Karl: *Grothendieck rings and motivic integration.* 2009.
25. Nilsson, Lisa: *Amoebas, discriminants, and hypergeometric functions.* 2009.
26. Lundqvist, Samuel: *Computational algorithms for algebras.* 2009.
27. Lindell, Andreas: *Theoretical and practical applications of probability.* 2009.
28. Geli Rolfhamre, Patricia: *From penicillin binding proteins to community interventions: mathematical and statistical models related to antibiotic resistance.* 2009.
29. Zusmanovich, Pasha: *Low-dimensional cohomology of current Lie algebras.* 2010.
30. Bojarova, Jelena: *Toward sequential data assimilation for numerical weather prediction (NWP) models using Kalman filter tools.* 2010.
31. Granåker, Johan: *Wheeled operads in algebra, geometry, and quantization.* 2010.
32. Xantcha, Qimh: *The theory of polynomial functors.* 2010.
33. Emtander, Eric: *Chordal and complete structures in combinatorics and commutative algebra.* 2011.
34. Grünewald, Maria: *Design and analysis of response selective samples in observational studies.* 2011.
35. Samieinia, Shiva: *Digital geometry, combinatorics, and discrete optimization.* 2011.
36. Björkwall, Susanna: *Stochastic claims reserving in non-life insurance: Bootstrap and smoothing models.* 2011.
37. Waliullah, Shoyeb: *Topics in nonlinear elliptic differential equations.* 2011.
38. Andersson, Patrik: *Four applications of stochastic processes: Contagious disease, credit risk, gambling and bond portfolios.* 2011.
39. Tadesse, Yohannes: *Tangential derivations, Hilbert series and modules over Lie algebroids.* 2011.
40. Al Abdulaali, Ahmad Khalid: *On the extension and wedge product of positive currents.* 2012.
41. Lindenstrand, David: *Stochastic epidemic models in heterogeneous communities.* 2012.
42. Lundqvist, Johannes: *On amoebas and multidimensional residues.* 2013.
43. Jost, Christine: *Topics in computational algebraic geometry and deformation quantization.* 2013.
44. Alexandersson, Per: *Combinatorial methods in complex analysis.* 2013.
45. Höhna, Sebastian: *Bayesian phylogenetic inference: Estimating diversification rates from reconstructed phylogenies.* 2013.
46. Aermark, Lior: *Hardy and spectral inequalities for a class of partial differential operators.* 2014.
47. Alm, Johan: *Universal algebraic structures on polyvector fields.* 2014.
48. Jafari-Mamaghani, Mehrdad: *A treatise on measuring Wiener–Granger causality.* 2014.
49. Martino, Ivan: *Ekedahl invariants, Veronese modules and linear recurrence varieties.* 2014.
50. Ekheden, Erland: *Catastrophe, ruin and death: Some perspectives on insurance mathematics.* 2014.
51. Johansson, Petter: *On the topology of the coamoeba.* 2014.
52. Lopes, Fabio: *Spatial marriage problems and epidemics.* 2014.
53. Bergh, Daniel: *Destackification and motivic classes of stacks.* 2014.
54. Olsson, Fredrik: *Inbreeding, effective population sizes and genetic differentiation: A mathematical analysis of structured populations.* 2015.
55. Forsgård, Jens: *Tropical aspects of real polynomials and hypergeometric functions.* 2015.

Distributor
Department of Mathematics
Stockholm University
SE-106 91 Stockholm, Sweden