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Combinatorial Methods in Complex Analysis

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Abstract

The theme of this thesis is combinatorics, complex analysis and algebraic geometry. The thesis consists of six articles divided into four parts.

Part A: *Spectral properties of the Schrödinger equation*

This part consists of Papers I-II, where we study a univariate Schrödinger equation with a complex polynomial potential. We prove that the set of polynomial potentials that admits solutions to the Schrödinger equation satisfying certain boundary conditions, is connected. We also study a similar problem for even polynomial potentials, where a similar result is obtained.

Part B: *Graph monomials and sums of squares*

In this part, consisting of Paper III, we study natural bases for the space of homogeneous, symmetric and translation-invariant polynomials in terms of multigraphs. We find all multigraphs with at most six edges that give rise to non-negative polynomials, and determine which of these that can be expressed as a sum of squares.

Part C: *Eigenvalue asymptotics of banded Toeplitz matrices*

This part consists of Papers IV-V. We give a new and generalized proof of a theorem by P. Schmidt and F. Spitzer concerning asymptotics of eigenvalues of Toeplitz matrices. We also generalize the notion of eigenvalues to rectangular matrices, and partially prove the multivariate analogue of the above theorem.

Part D: *Stretched Schur polynomials*

This part consists of Paper VI, where we give a combinatorial proof of the fact that certain sequences of skew Schur polynomials satisfy linear recurrences with polynomial coefficients.

List of papers

I P. Alexandersson, A. Gabrielov, *On eigenvalues of the Schrödinger operator with a complex-valued polynomial potential*, CMFT 12 No.1 (2012) 119–144.

II P. Alexandersson, *On eigenvalues of the Schrödinger operator with an even complex-valued polynomial potential*, CMFT 12 No. 2 (2012) 465–481.

III P. Alexandersson, B. Shapiro, *Discriminants, symmetrized graph monomials, and sums of squares*, Experimental Math. 21 No. 4 (2012) 353–361.

IV P. Alexandersson, *Schur polynomials, banded Toeplitz matrices and Widom's formula*, Electr. Jour. Comb. 19, No. 4 (2012).

V P. Alexandersson, B. Shapiro, *Around multivariate Schmidt-Spitzer theorem*, arXiv:1302.3716. Submitted, (2013).

VI P. Alexandersson, *Stretched skew Schur polynomials are recurrent*, arXiv:1210.0377. Submitted, (2012).

Contents

1. Introduction and summary of the papers

1.1 Introduction

The thesis consists of four different parts, corresponding to papers I-II, paper III, papers IV-V and finally paper VI.

The theme of the thesis is combinatorics, complex analysis and algebraic geometry. Most of the problems we consider are not a priori combinatorial in nature, but rather of analytic character. By using different methods, the problems are reduced to discrete combinatorial statements. For example, the main objects in papers I-III are graphs and multigraphs, while papers IV-VI mainly deals with Young tableaux. The latter is a well-studied combinatorial object with many applications.

The advantage of discrete problems is that these are easier to analyze with computer calculations, which is used extensively in the present thesis. A vast number of computer-generated examples where created for each article and these gave good indications on what exact statements to prove and the technique needed in the proofs. This is an aspect of the thesis which might not be obvious from the text.

1.2 Spectral properties of the Schrödinger equation

The first part was supervised by Andrei Gabrielov at Purdue University, USA, and concerns certain properties of solutions of the Schrödinger equation. The work relies on earlier results and techniques by A. Gabrielov and A. Eremenko, which are of a combinatorial nature.

In the first and second paper, we examine the Schrödinger-type equation $-y'' + P(z)y = 0$ for an arbitrary respectively arbitrary even polynomial potential P of degree n with complex coefficients. In short, only some polynomials P admits solutions to $-y'' + P(z)y = 0$ when we fix appropriate boundary conditions. In physics, one is for example interested in the boundary conditions given by $y(x) \to 0$ as $x \to \pm\infty$ for $x \in \mathbb{R}$.

We show that in the general case, the space of coefficients of P, which admits a solution satisfying the boundary conditions, is connected. By using previous results by A. Gabrielov and A. Eremenko, this problem is reduced to a discrete combinatorial problem on certain types of graphs with an extra structure. It is classically known that any solution y of the Schrödinger equation above can essentially be described by an ordered set of $n+2$ continuous parameters in $\mathbb{C}$, called *asymptotic values*, together with a discrete graph.

Thus, suppose we have two such solutions, y_1 and y_2 for two polynomials P_1 and P_2. We may continuously change the asymptotic values of y_1 so that they match the asymptotic values of y_2. The coefficients of P_1 depends continuously on the asymptotic values. Now, the only thing that remains is the discrete graph, which we can deform by interchanging the asymptotic values. This gives a braid group action on the graphs, and in order to prove connectedness, it suffices to show that each graph may be reached from every other using this action.

My contribution to Paper I was to analyze how the braid group acts on these graphs. By proving that the braid group acts transitively on the set of graphs, the result follows. The same technique was used in Paper II, where P is restricted to be an even polynomial. In this case, the space of parameters admitting a solution consists of two connected components, unless we pose some very restrictive boundary conditions.

1.3 Graph monomials and sums of squares

The second part concerns homogeneous, symmetric and translation-invariant polynomials in n variables. A polynomial P is called translation-invariant if $P(x_1+t, x_2+t, \ldots, x_n+t) = P(x_1, x_2, \ldots, x_n)$ for all $t \in \mathbb{R}$. We give a natural basis for the space of such polynomials in terms of multigraphs. When $\deg P = 2d$ is even, a second basis is constructed from multigraphs. This basis consists of squares of homogeneous, symmetric and translation-invariant polynomials of degree d. The proofs of linear independence in the different bases is done via a combinatorial argument.

As an application, motivated by an interesting example found by A. and P. Lax, see [2], we find all multigraphs with six or less edges, that give rise to a non-negative polynomial which is not a sum of squares. Most of this is done by computer-aided computations.

A good example of such polynomial is the discriminant of the kth derivative of a general polynomial $(t-x_1)\cdots(t-x_n)$. We conjecture that these discriminants are always sums of squares, and provide several examples indicating this. For the case $k = 1$, the representability as a sum of squares was earlier conjectured by F. Sottile and E. Mukhin, which is now settled, see [3].

My contribution to this paper consists of the computer calculations and the combinatorial proofs.

1.4 Eigenvalue asymptotics of banded Toeplitz matrices

The third part concerns a result in [4], from 1960 by P. Schmidt and F. Spitzer, which describes the asymptotic eigenvalue distribution for banded Toeplitz matrices. A banded $n \times n$ Toeplitz matrix has the form

$$(c_{j-i}),\, 1 \leq i < n, 1 \leq j < n \text{ with } s_l := 0 \text{ for } l > k, l < -h,$$

where $h, k > 0$ are fixed constants. The theorem by Palle Schmidt and Frank Spitzer states that the *limit set of eigenvalues* (with a suitable definition) coincides with a certain semi-algebraic curve depending on $c_{-h}, \dots, c_k$. Their proof relies on Widom's formula, see [5], which is used to compute determinants of banded Toeplitz matrices.

In Paper IV, we give a new (and generalized) proof of Schmidt and Spitzer's theorem using a new recurrence for skew Schur polynomials that we prove in the paper. We also show that Widom's formula is a special case of a known formula for Schur polynomials.

In Paper V, we generalize the notion of eigenvalues to any rectangular matrix and partially prove a multivariate version of the theorem by P. Schmidt and F. Spitzer. We also suggest a new way of how to view certain families of multivariate orthogonal polynomials.

My main contributions to Paper V consists of the proof of compactness of the conjectured limit set of generalized eigenvalues, as well as the proof of the inclusion of the limit set of eigenvalues in the conjectured limit set.

1.5 Stretched Schur polynomials

The final part consists of Paper VI and considerably generalizes the combinatorial part of the recurrence for Toeplitz determinants in Paper IV. There is a close connection between Toeplitz determinants and

Schur polynomials. Schur polynomials, and skew Schur polynomials are obtained from integer partitions, where a partition is a non-increasing sequence $(\lambda_1, \lambda_2, \dots, \lambda_p)$ of natural numbers. A sequence of stretched partitions is obtained by multiplying the entries of a partition by an integer factor, $\{(k\lambda_1, k\lambda_2, \dots, k\lambda_p)\}_{k=1}^{\infty}$ is thus such a sequence.

We show that any sequence of stretched skew partitions yields a sequence of corresponding skew Schur polynomials which satisfy a linear recurrence with polynomial coefficients.

To prove this result, a new ring structure on skew Young tableaux is introduced. In this ring, we give a combinatorial proof of a linear recurrence which is then mapped to a corresponding linear recurrence on Schur polynomials via a ring homomorphism. The characteristic polynomials of these recurrences may be used to determine the asymptotic root distribution of certain sequences of skew Schur polynomials. As in Paper IV, this may be used to give descriptions on the asymptotic eigenvalue distribution for certain matrices.

Sequences of stretched partitions and related combinatorial objects have been studied before, see for example the famous result of A. Knutson and T. Tao [1] regarding Littlewood-Richardson coefficients.

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2. Sammanfattning

Denna avhandling är uppdelad i fyra delar, med sammanlagt sex artiklar. Avhandlingen tillhör de matematiska områdena kombinatorik, komplex analys samt algebraisk geometri.

Den första delen består av artikel I-II och behandlar Schrödingerekvationen $-y'' + P(z)y = 0$, där potentialen P är ett polynom med komplexa koefficienter. Vi inför också ett antal randvillkor på denna ekvation. I den första artikeln visar vi att mängden av polynom som tillåter en lösning med randvillkoren uppfyllda, är sammanhängande. Detta visas genom att reducera problemet till ett kombinatoriskt problem på en viss slags grafer med extra struktur. I den andra artikeln används samma metodik för att visa ett motsvarande resultat, men där potentialen är ett jämnt polynom.

Den andra delen utgörs av artikel III, där vi studerar polynom som är homogena, symmetriska och translationsinvarianta. Sådana polynom dyker naturligt upp när man studerar diskriminanter. Vi studerar naturliga baser för detta rum av polynom med hjälp av multigrafer. Vi kartlägger sedan alla multigrafer med upp till sex kanter vars motsvarande polynom är icke-negativa men inte en summa av kvadrater. Detta motiveras av ett exempel som gavs av A. och P. Lax.

Den tredje delen, bestående av artikel IV-V, behandlar det asymptotiska beteendet hos egenvärden till vissa Toeplitzmatriser, med ökande storlek. Det finns ett klassiskt resultat om detta av P. Schmidt och F. Spitzer, som vi ger ett nytt och generaliserat bevis på. I den femte artikeln definierar vi egenvärden för rektangulära matriser. Därefter så formulerar vi en motsvarighet i flera variabler till Schmidt och Spitzers resultat, som vi delvis bevisar. Detta har en koppling till ortogonala polynom i flera variabler.

I den sista delen, bestående av artikel VI, studerar vi vissa serier av Schurpolynom. Vi visar att dessa polynom uppfyller linjära rekurrenser med polynomiella koefficienter. Detta är en stor generalisering av ett delresultat i artikel IV. Beviset är rent kombinatoriskt och bygger på studier av Youngtablåer.

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ON EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A COMPLEX-VALUED POLYNOMIAL POTENTIAL

PER ALEXANDERSSON AND ANDREI GABRIELOV

Abstract. We consider the eigenvalue problem with a complex-valued polynomial potential of arbitrary degree d and show that the spectral determinant of this problem is connected and irreducible. In other words, every eigenvalue can be reached from any other by analytic continuation.

We also prove connectedness of the parameter spaces of the potentials that admit eigenfunctions satisfying $k > 2$ boundary conditions, except for the case d is even and $k = d/2$. In the latter case, connected components of the parameter space are distinguished by the number of zeros of the eigenfunctions.

The first results can be derived from H. Habsch, while the case of a disconnected parameter space is new.

1. Introduction

In this paper we study analytic continuation of eigenvalues of the Schrödinger operator with a complex-valued polynomial potential. In other words, we are interested in the analytic continuation of eigenvalues $\lambda = \lambda(\alpha)$ of the boundary value problem for the differential equation

$$-y'' + P_\alpha(z)y = \lambda y, \tag{1}$$

where

$$P_\alpha(z) = z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_1 z \ \text{ with } \ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{d-1}),\ d \geq 2.$$

The boundary conditions are given by either (2) or (3) below. Namely, set $n = d + 2$ and divide the plane into n disjoint open sectors of the form

$$S_j = \left\{ z \in \mathbb{C} \setminus \{0\} : \left| \arg z - \frac{2\pi j}{n} \right| < \frac{\pi}{n} \right\}, \quad j = 0, 1, 2, \ldots, n-1.$$

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These sectors are called the *Stokes sectors* of the equation (1). It is well-known that any solution y of (1) in each open Stokes sector S_j satisfy $y(z) \to 0$ or $y(z) \to \infty$ as $z \to \infty$ along each ray from the origin in S_j, see [11]. In the first case, we say that y is *subdominant*, and in the second case, *dominant* in S_j. We impose the boundary conditions that for two *non-adjacent* sectors S_j and S_k, i.e. for $j \neq k \pm 1 \mod n$:

$$y \text{ is subdominant in } S_j \text{ and } S_k. \tag{2}$$

For example, $y(\infty) = y(-\infty) = 0$ on the real axis, the boundary conditions usually imposed in physics for even potentials, correspond to y being subdominant in S_0 and $S_{n/2}$.

It is well-known that analytic continuation of eigenvalues of (1) exists, see [11]. The eigenvalues tend to infinity, and depend analytically on the coefficients of P_α. Furthermore, there are no singularities in the whole space, except algebraic branch points, see [4].

The main theorems of this paper are:

Theorem 1. *For any eigenvalue $\lambda_k(\alpha)$ of equation* (1) *and boundary condition* (2)*, there is an analytic continuation in the α-plane to any other eigenvalue $\lambda_m(\alpha)$.*

A generalization of Theorem 1 to the case where y is subdominant in more than two sectors:

Theorem 2. *Given $k < n/2$ non-adjacent Stokes sectors $S_{j_1}, \dots, S_{j_k}$, the set of all $(\alpha, \lambda) \in \mathbb{C}^d$ for which the equation $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with*

$$y \text{ subdominant in } S_{j_1}, \dots, S_{j_k} \tag{3}$$

is connected.

Remark 3. *After this project was finished, the authors found out that Theorems 1 and 2 follows from a result in* [5, p. 36].

Theorem 4. *For n even and $k = n/2$, the set of all $(\alpha, \lambda) \in \mathbb{C}^d$ for which $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with*

$$y \text{ subdominant in } S_0, S_2, \dots, S_{n-2} \tag{4}$$

is disconnected. Additionally, the solutions to (1) *with conditions* (3)*, have finitely many zeros, and the set of α corresponding to a given number of zeros is a connected component of the former set.*

Nevanlinna parametrization in the study of linear differential equations was first used by Sibuya [11]. In [4] it was applied for the first time to this analytic continuation problem.

1.1. **Some previous results.** In the foundational paper [3], C. Bender and T. Wu studied analytic continuation of λ in the complex β-plane for the problem

$$-y'' + (\beta z^4 + z^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0.$$

Based on numerical computations, they conjectured for the first time the connectivity of the sets of odd and even eigenvalues. This paper generated considerable further research in both physics and mathematics literature. See e.g. [12] for early mathematically rigorous results in this direction.

In this paper, we reproduce the result in [5] of two reasons. First, it is now restated in modern language. Second, the results are needed to prove Theorem 4.

The intermediate results in this paper are also used in a forthcoming paper, [1], generalizing [4] to arbitrary even polynomial potentials.

2. Preliminaries

First, we recall some basic notions from Nevanlinna theory.

Lemma 5 (see [11]). *For any j, there is a solution y of* (1) *subdominant in the Stokes sector S_j. This solution is unique, up to multiplication by a non-zero constant. Each solution $y \neq 0$ is an entire function, and the ratio $f = y/y_1$ of any two linearly independent solutions of* (1) *is a meromorphic function, with the following properties:*

1. *For any Stokes sector S_j, we have $f(z) \to w \in \bar{\mathbb{C}}$ as $z \to \infty$ along any ray in S_j. This value w is called* the asymptotic value *of f in S_j.*
2. *For any j, the asymptotic values of f in S_j and S_{j+1} (index taken modulo n) are different. The function f has at least 3 distinct asymptotic values.*
3. *The asymptotic value of f is zero in S_j if and only if y is subdominant in S_j. It is convenient to call such sector* subdominant *as well. Note that the boundary conditions in* (2) *imply that the two sectors S_j and S_k are subdominant for f when y is an eigenfunction of* (1), (2).
4. *f does not have critical points, hence $f : \mathbf{C} \to \bar{\mathbb{C}}$ is unramified outside the asymptotic values.*
5. *The Schwarzian derivative S_f of f given by*

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

equals $-2(P_\alpha - \lambda)$. Therefore one can recover P_α and λ from f.

From now on, f always denotes the ratio of two linearly independent solutions of (1), with y being an eigenfunction of the boundary value problem (1), with conditions (2), (3) or (4).

2.1. **Cell decompositions.** Set $n = d + 2$, $d = \deg P$ where P is the polynomial potential and assume that all non-zero asymptotic values of f are distinct and finite. Let w_j be the asymptotic values of f, ordered arbitrarily with the only restriction that $w_j = 0$ if and only if S_j is subdominant. For example, one can denote by w_j the asymptotic value in the Stokes sector S_j. We will later need different orders of the non-zero asymptotic values, see Section 2.3.

Consider the cell decomposition Ψ_0 of $\bar{\mathbb{C}}_w$ shown in Figure 1(a). It consists of closed directed loops γ_j starting and ending at ∞, where the index is considered mod n, and γ_j is defined only if $w_j \neq 0$. The loops γ_j only intersect at ∞ and have no self-intersection other than ∞. Each loop γ_j contains a single non-zero asymptotic value w_j of f. For example, the boundary condition $y \to 0$ as $z \to \pm\infty$ for $z \in \mathbb{R}$ for even n implies that $w_0 = w_{n/2} = 0$, so there are no loops γ_0 and $\gamma_{n/2}$. We have a natural cyclic order of the asymptotic values, namely the order in which a small circle around ∞ counterclockwise intersects the associated loops γ_j, see Figure 1(a).

We use the same index for the asymptotic values and the loops, which motivates the following notation:

$$j_+ = j + k \text{ where } k \in \{1, 2\} \text{ is the smallest integer such that } w_{j+k} \neq 0.$$

Thus, γ_{j_+} is the loop around the next to w_j (in the cyclic order mod n) non-zero asymptotic value. Similarly, γ_{j_-} is the loop around the previous non-zero asymptotic value.

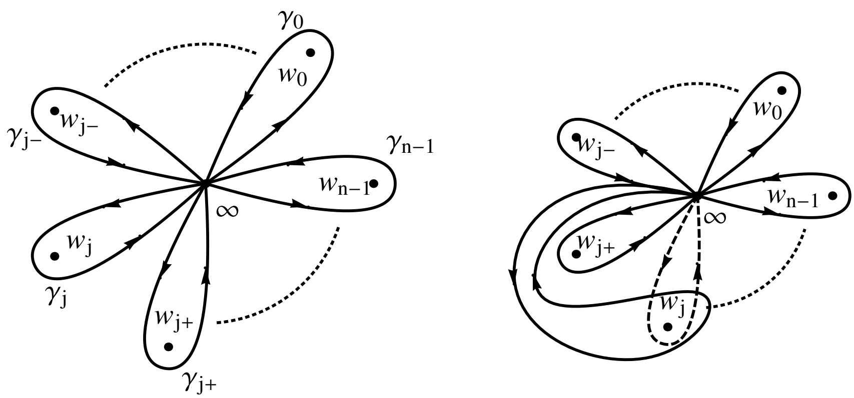
2.2. **From cell decompositions to graphs.** We may simplify our work with cell decompositions with the help of the following result.

Lemma 6 (See Section 3 [4]). *Given Ψ_0 as in Figure 1(a), one has the following properties:*

(a) The preimage $\Phi_0 = f^{-1}(\Psi_0)$ gives a cell decomposition of the plane $\mathbb{C}_z$. Its vertices are the poles of f, and the edges are preimages of the loops γ_j. These edges are labeled by j, and are called j-edges.

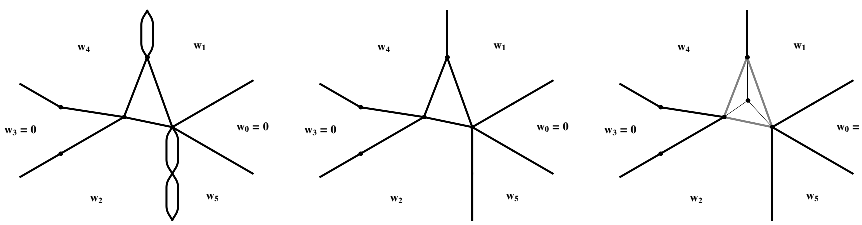
(b) The edges of Φ_0 are directed, their orientation is induced from the orientation of the loops γ_j. Removing all loops of Φ_0, we obtain an infinite, directed planar graph Γ, without loops.

(c) Vertices of Γ are poles of f, each bounded connected component of $\mathbb{C} \setminus \Gamma$ contains one simple zero of f, and each zero of f belongs to one such bounded connected component.



(a) Ψ_0 (b) $A_j(\Psi_0)$.

FIGURE 1. Permuting w_j and w_{j+} in Ψ_0.



(a) Γ (b) T_Γ (c) T_Γ^*

FIGURE 2. The correspondence between Γ, T_Γ and T_Γ^*.

(d) There are at most two edges of Γ *connecting any two of its vertices. Replacing each such pair of edges with a single undirected edge and making all other edges undirected, we obtain an undirected graph* T_Γ.

(e) T_Γ *has no loops or multiple edges, and the transformation from* Φ_0 *to* T_Γ *can be uniquely reversed.*

An example of the transformation from Γ to T_Γ is shown in Figure 2.

A *junction* is a vertex of Γ (and of T_Γ) at which the degree of T_Γ is at least 3. From now on, Γ refers to both the directed graph without loops and the associated cell decomposition Φ_0.

2.3. **Standard order.** For a potential of degree d, the graph Γ has $d+2=n$ infinite branches and n unbounded faces corresponding to the Stokes sectors. We defined earlier the ordering $w_0, w_1, \ldots, w_{n-1}$ of the asymptotic values of f.

If each w_j is the asymptotic value in the sector S_j, we say that the asymptotic values have *the standard order* and the corresponding cell decomposition Γ is a *standard graph.*

Lemma 7 (See Prop 6. [4]). *If a cell decomposition Γ is a standard graph, the corresponding undirected graph T_Γ is a tree.*

This property is essential in the present paper, and we classify cell decompositions of this type by describing the associated trees.

Below we define the action of the braid group that permute non-zero asymptotic values of Ψ_0. This induces the corresponding action on graphs. Each unbounded face of Γ (and T_Γ) will be labeled by the asymptotic value in the corresponding Stokes sector. For example, by labeling an unbounded face corresponding to S_k with w_j or just with the index j, we indicate that w_j is the asymptotic value in S_k.

From the definition of the loops γ_j, a face corresponding to a dominant sector has the same label as any edge bounding that face. The label in a face corresponding to a subdominant sector S_k is always k, since the actions defined below only permute non-zero asymptotic values. We say that an unbounded face of Γ is (sub)dominant if the corresponding Stokes sector is (sub)dominant.

For example, in Figure 2, the Stokes sectors S_0 and S_3 are subdominant, indicated by labeling the corresponding faces with 0. We do not have the standard order for Γ, since w_2 is the asymptotic value for S_4, and w_4 is the asymptotic value for S_2. The associated graph T_Γ is not a tree.

2.4. **Properties of graphs and their face labeling.**

Lemma 8 (see [4]). *The following holds:*

- *(I) Two bounded faces of Γ cannot have a common edge, since a j-edge is always at the boundary of an unbounded face labeled j.*
- *(II) The edges of a bounded face of a graph Γ are directed clockwise, and their labels increase in that order. Therefore, a bounded face of T_Γ can only appear if the order of w_j is non-standard.*

  (As an example, the bounded face in Figure2 has the labels $1, 2, 4$ (clockwise) of its boundary edges.)
- *(III) Each label appears at most once in the boundary of any bounded face of Γ.*
- *(IV) Unbounded faces of Γ adjacent to its junction u always have the labels cyclically increasing counterclockwise around u.*
- *(V) To each graph T_Γ, we associate a tree by inserting a new vertex inside each of its bounded faces, connecting it to the vertices of the bounded face and removing the boundrary edges of the original face. Thus we may associate a tree T^*_Γ with* any *cell decomposition,*

not necessarily with standard order, as in Figure 2(c). The order of w_j *above together with this tree uniquely determines* Γ. *This is done using the two properties above.*

(VI) The boundary of a dominant face labeled j *consists of infinitely many directed* j*-edges, oriented counterclockwise around the face.*

(VII) If $w_j = 0$ *there are no* j*-edges.*

(VIII) Each vertex of Γ *has even degree, since each vertex in* $\Phi_0 = f^{-1}(\Psi_0)$ *has even degree, and removing loops to obtain* Γ *preserves this property.*

Following the direction of the j-edges, the first vertex that is connected to an edge labeled j_+ is the vertex where the j-edges and the j_+-edges *meet*. The last such vertex is where they *separate*. These vertices, if they exist, must be junctions.

Definition 9. *Let* Γ *be a standard graph, and let* $j \in \Gamma$ *be a junction where the* j*-edges and* j_+*-edges separate. Such junction is called a* j-junction.

There can be at most one j-junction in Γ, the existence of two or more such junctions would violate property (III) of the face labeling. However, the same junction can be a j-junction for different values of j.

There are three different types of j-junctions, see Figure 3.

Case (a) only appears when $w_{j+1} \neq 0$. Cases (b) and (c) can only appear when $w_{j+1} = 0$. In (c), the j-edges and j_+-edges meet and separate at different junctions, while in (b), this happens at the same junction.

Definition 10. *Let* Γ *be a standard graph with a* j*-junction* u. *A* structure *at the* j*-junction is the subgraph* Ξ *of* Γ *consisting of the following elements:*

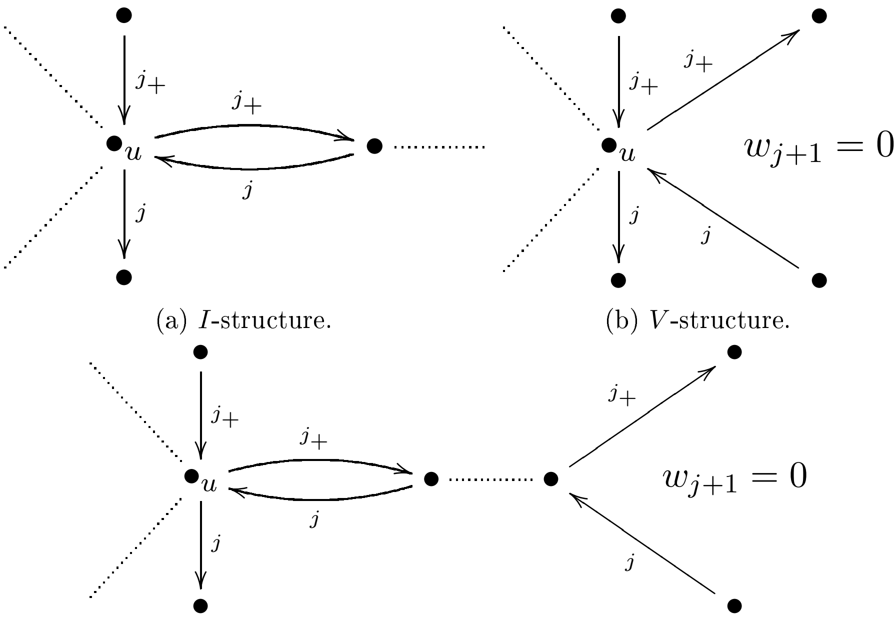
- *The edges labeled* j *that appear before* u *following the* j*-edges.*
- *The edges labeled* j_+ *that appear after* u *following the* j_+*-edges.*
- *All vertices the above edges are connected to.*

If u *is as in Figure 3(a),* Ξ *is called an* I-structure at the j-junction. *If* u *is as in Figure 3(b),* Ξ *is called a* V-structure at the j-junction. *If* u *is as in Figure 3(c),* Ξ *is called a* Y-structure at the j-junction.

Since there can be at most one j-junction, there can be at most one structure at the j-junction.

A graph Γ shown in Figure 4 has one (dotted) I-structure at the 1-junction v, one (dotted) I-structure at the 4-junction u, one (dashed) V-structure at the 2-junction v and one (dotdashed) Y-structure at the 5-junction u.

Note that the Y-structure is the only kind of structure that contains an additional junction. We refer to such junctions as Y*-junctions*. For example, the junction marked y in Figure 4 is a Y-junction.



(a) I-structure. (b) V-structure.

(c) Y-structure.

FIGURE 3. Different types of j-junctions.

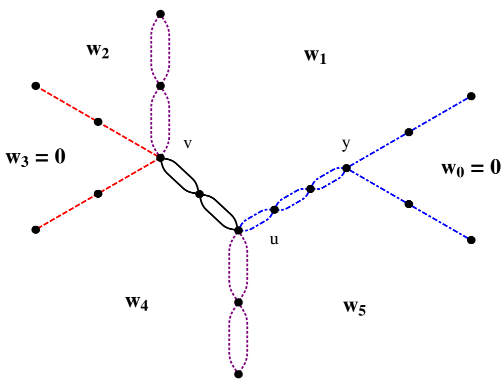


FIGURE 4. Graph Γ with (dotted) I-structures, a (dashed) Y-structure and a (dotdashed) Y-structure.

2.5. **Describing trees and junctions.** Let Γ be a graph with n branches, and Λ be the associated tree with all non-junction vertices removed. The dual graph $\hat{\Lambda}$ of Λ, is an n-gon where some non-intersecting chords are present. The junctions of Λ is in one-to-one correspondence with faces of $\hat{\Lambda}$ and vice versa. Two vertices are connected with an edge in $\hat{\Lambda}$ if and only if the corresponding faces are adjacent in Λ.

The extra condition that subdominant faces do not share an edge, implies that there are no chords connecting vertices in $\hat{\Lambda}$ corresponding to subdominant faces. For trees without this condition, we have the following lemma:

Lemma 11. *The number of $n+1$-gons with non-intersecting chords is equal to the number of bracketings of a string with n letters, such that each bracket pair contains at least two symbols.*

Proof. See [10, Thm. 1]. □

The sequence $s(n)$ of bracketings of a string with $n+1$ symbols are called the small Schröder numbers, see [10]. The first entries are $s(n)_{n\geq 0} = 1, 1, 3, 11, 45, 197, \dots$.

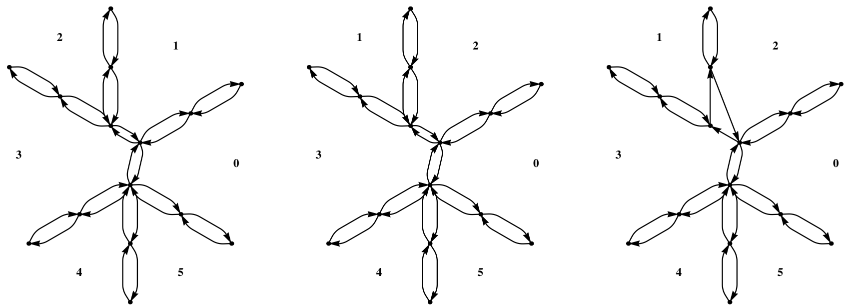
The condition that chords should not connect vertices corresponding to subdominant faces, translates into a condition on the first and last symbol in some bracket pair.

3. Actions on graphs

3.1. **Definitions.** Let us now return to the cell decomposition Ψ_0 in Figure 1(a). Let w_j be a non-zero asymptotic value of f. Choose non-intersecting paths $\beta_j(t)$ and $\beta_{j+}(t)$ in $\bar{\mathbb{C}}_w$ with $\beta_j(0) = w_j$, $\beta_j(1) = w_{j+}$ and $\beta_{j+}(0) = w_{j+}$, $\beta_{j+}(1) = w_j$ so that they do not intersect γ_k for $k \neq j, j_+$ and such that the union of these paths is a simple contractible loop oriented counterclockwise. These paths define a continuous deformation of the loops γ_j and γ_{j+} such that the two deformed loops contain $\beta_j(t)$ and $\beta_{j+}(t)$, respectively, and do not intersect any other loops during the deformation (except at ∞). We denote the action on Ψ_0 given by $\beta_j(t)$ and $\beta_{j+}(t)$ by A_j. Basic properties of the fundamental group of a punctured plane, allows one to express the new loops in terms of the old ones:

$$A_j(\gamma_k) = \begin{cases} \gamma_j \gamma_{j+} \gamma_j^{-1} \text{ if } k = j, \\ \gamma_j \text{ if } k = j_+, \\ \gamma_k \text{ otherwise }, \end{cases} \qquad A_j^{-1}(\gamma_k) = \begin{cases} \gamma_{j+} \text{ if } k = j, \\ \gamma_{j+}^{-1} \gamma_j \gamma_{j+} \text{ if } k = j_+, \\ \gamma_k \text{ otherwise.} \end{cases}$$

Let f_t be a deformation of f. Since a continuous deformation does not change the graph, the deformed graph corresponding to $f_1^{-1}(A_j(\Psi_0))$ is the same as Γ. Let Γ' be this deformed graph with labels j and j_+ exchanged. Then the j-edges of Γ' are $f_1^{-1}(A_j(\gamma_{j+})) = f_1^{-1}(\gamma_j)$, hence they are the same as the j-edges of $A_j(\Gamma)$. The j_+-edges of Γ' are $f_1^{-1}(A_j(\gamma_j))$. Since $\gamma_{j+} = \gamma_j^{-1} A_j(\gamma_j)\gamma_j$, (reading left to right) this



(a) Γ (b) Γ' (c) $A_1(\Gamma)$

FIGURE 5. The action A_1. All sectors are dominant.

means that a j_+-edge of $A_j(\Gamma)$ is obtained by moving backwards along a j-edge of Γ', then along a j_+-edge of Γ', followed by a j-edge of Γ'.

These actions, together with their inverses, generate the Hurwitz (or sphere) braid group $\mathcal{H}_m$, where m is the number of non-zero asymptotic values. For a definition of this group, see [7]. The action A_j on the loops in Ψ_0 is presented in Figure 1(b).

The property (4) of the eigenfunctions implies that each A_j induces a monodromy transformation of the cell decomposition Φ_0, and of the associated directed graph Γ.

Reading the action *right to left* gives the new edges in terms of the old ones, as follows:

Applying A_j to Γ can be realized by first interchanging the labels j and j_+. This gives an intermediate graph Γ'. A j-edge of $A_j(\Gamma)$ starting at the vertex v ends at a vertex obtained by moving from v following first the j-edge of Γ' backwards, then the j_+-edge of Γ', and finally the j-edge of Γ'. If any of these edges does not exist, we just do not move. If we end up at the same vertex v, there is no j-edge of $A_j(\Gamma)$ starting at v. All k-edges of $A_j(\Gamma)$ for $k \neq j$ are the same as k-edges of Γ'.

An example of the action A_1 is presented in Figure 5. Note that A_j^2 preserves the standard cyclic order.

3.2. **Properties of the actions.**

Lemma 12. *Let Γ be a standard graph with no j-junction. Then $A_j^2(\Gamma) = \Gamma$.*

Proof. Since we assume $d > 2$, Lemma 8 implies that the boundaries of the faces of Γ labeled j and j_+ do not have a common vertex. From the definition of the actions in subsection 3.1, the graphs Γ and $A_j(\Gamma)$ are the same, except that the labels j and j_+ are permuted. Applying the same argument again gives $A_j^2(\Gamma) = \Gamma$. □

Theorem 13. *Let* Γ *be a standard graph with a* j*-junction* u*. Then* $A_j^2(\Gamma) \neq \Gamma$*, and the structure at the* j*-junction is moved one step in the direction of the* j*-edges under* A_j^2*. The inverse of* A_j^2 *moves the structure at the* j*-junction one step backwards along the* j_+*-edges.*

Proof. There are three cases to consider, namely I-structures, V-structures and Y-structures respectively.
Case 1: The structure at the j-junction is an I-structure and Γ is as in Figure 6(a). The action A_j first permutes the asymptotic values w_j and w_{j_+}, then transforms the new j- and j_+-edges, as defined in subsection 3.1. The resulting graph $A_j(\Gamma)$ is shown in Figure 6(b). Applying A_j to $A_j(\Gamma)$ yields the graph shown in Figure 6(c).
Case 2: The structure at the j-junction is a V-structure and Γ is as in Figure 7(a). The graphs $A_j(\Gamma)$ and $A_j^2(\Gamma)$ are as in Figure7(b) and in Figure 7(c) respectively.
Case 3: The structure at the j-junction is a Y-structure and Γ is as in Figure 8(a). The graphs $A_j(\Gamma)$ and $A_j^2(\Gamma)$ are as in Figure 8(b) and in Figure 8(c) respectively. The statement for A_j^{-2} is proved similarly. □

Examples of the actions are given in Appendix, Figures 16, 17 and 18.

3.3. **Contraction theorems.**

Definition 14. *Let* Γ *be a standard graph and let* u_0 *be a junction of* Γ*. The* u_0-metric *of* Γ*, denoted* $|\Gamma|_{u_0}$ *is defined as*

$$|\Gamma|_{u_0} = \sum_v (\deg(v) - 2)\, |v - u_0|$$

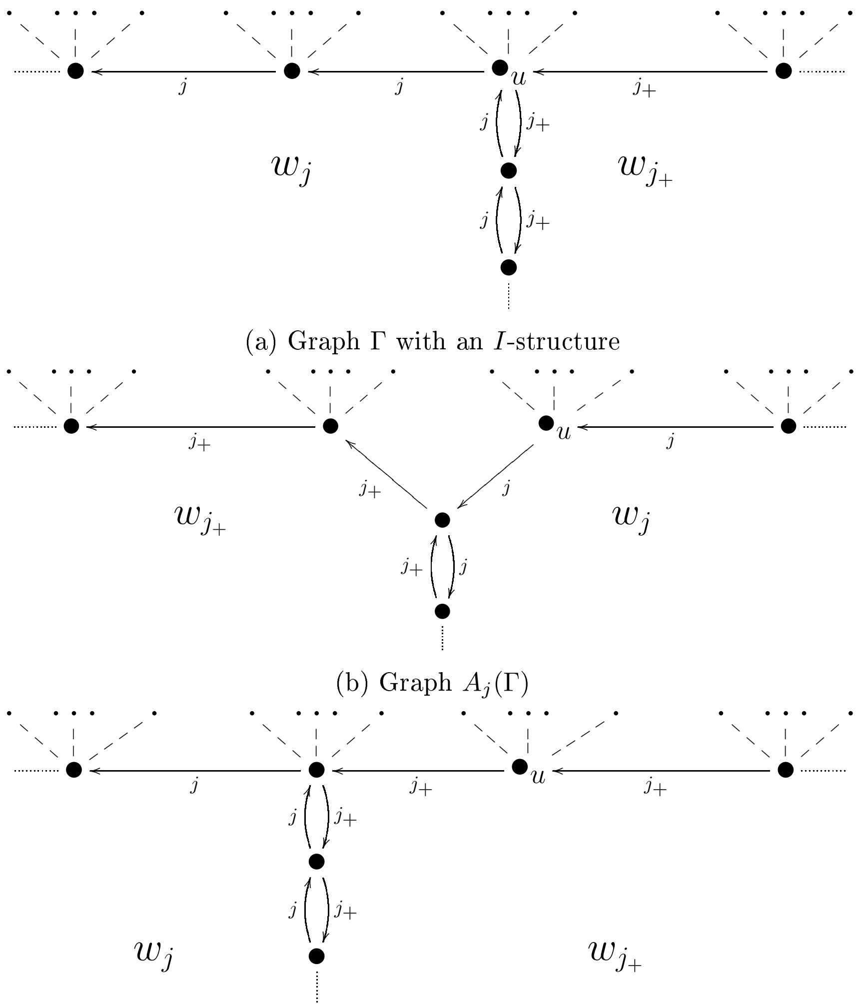
where the sum is taken over all vertices v *of* T_Γ*. Here* $deg(v)$ *is the total degree of the vertex* v *in* T_Γ *and* $|v - u_0|$ *is the length of the shortest path from* v *to* u_0 *in* T_Γ*. (Note that the sum in the right hand side is finite, since only junctions make non-zero contributions.)*

Definition 15. *A standard graph* Γ *is in* ivy form *if* Γ *is the union of the structures connected to a junction* u*. Such junction is called a* root junction*.*

Lemma 16. *The graph* Γ *is in ivy form if and only if all but one of its junctions are* Y*-junctions.*

Proof. This follows from the definitions of the structures. □

Theorem 17. *Let* Γ *be a standard graph. Then there is a sequence of actions* $A^* = A_{j_1}^{\pm 2} A_{j_2}^{\pm 2} \cdots$ *such that* $A^*(\Gamma)$ *is in* ivy form.



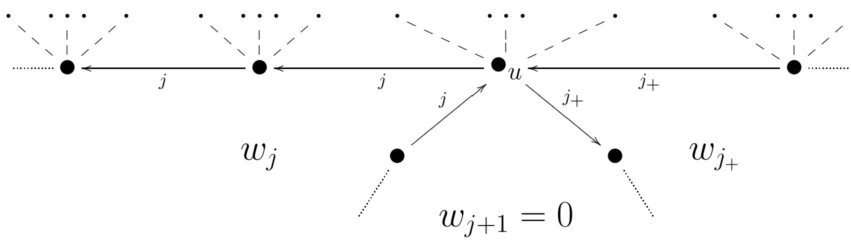
(a) Graph Γ with an I-structure

(b) Graph $A_j(\Gamma)$

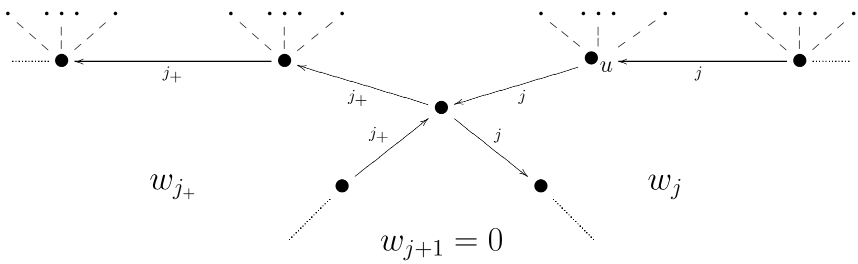
(c) Graph $A_j^2(\Gamma)$

FIGURE 6. Case 1, moving an I-structure.

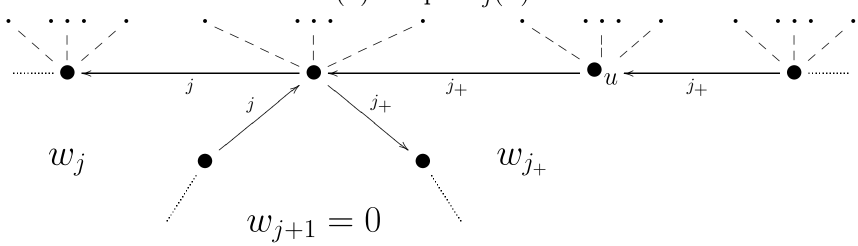
Proof. Assume that Γ is not in ivy form. Let U be the set of junctions in Γ that are not Y-junctions. Since Γ is not in ivy form, $|U| \geq 2$. Let $u_0 \neq u_1$ be two junctions in U such that $|u_0 - u_1|$ is maximal. Let p be the path from u_0 to u_1 in T_Γ. It is unique since T_Γ is a tree. Let v be the vertex immediately preceeding u_1 on the path p. The edge from v to u_1 in T_Γ is adjacent to at least one dominant face with label j such that $w_j \neq 0$. Therefore, there exists a j-edge between v and u_1 in Γ. Suppose first that this j-edge is directed from u_1 to v. Let us show that



(a) Graph Γ with a V-structure



(b) Graph $A_j(\Gamma)$



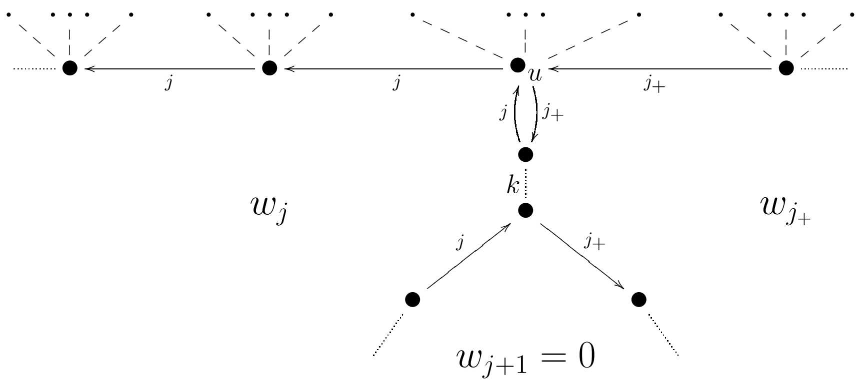
(c) Graph $A_j^2(\Gamma)$

FIGURE 7. Case 2, moving a V-structure.

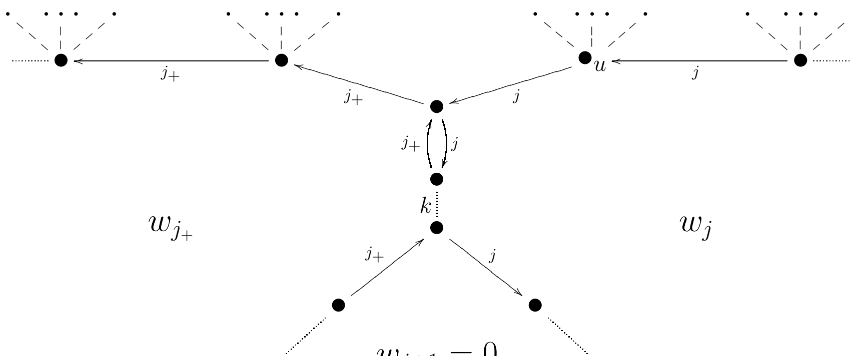
in this case u_1 must be a j-junction, i.e., the dominant face labeled j_+ is adjacent to u_1.

Since u_1 is not a Y-junction, there is a dominant face adjacent to u_1 with a label $k \neq j, j_+$. Hence no vertices of p, except possibly u_1 may be adjacent to j_+-edges. If u_1 is not a j-junction, there are no j_+-edges adjacent to u_1. This implies that any vertex of Γ adjacent to a j_+-edge is further away from u_0 that u_1.

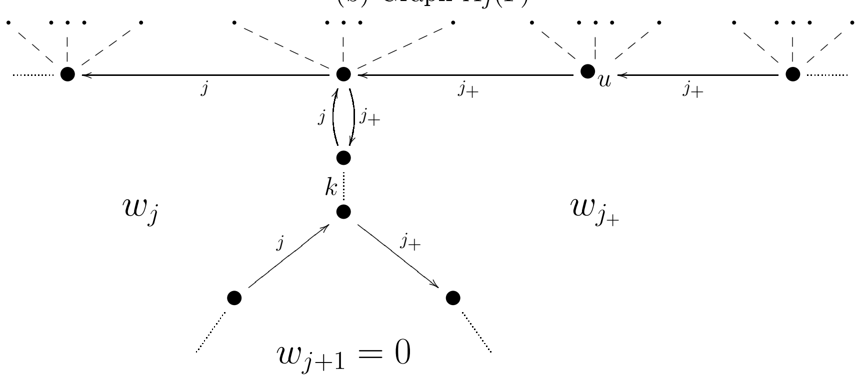
Let u_2 be the closest to u_1 vertex of Γ adjacent to a j_+-edge. Then u_2 should be a junction of T_Γ, since there are two j_+-edges adjacent to u_2 in Γ and at least one more vertex (on the path from u_1 to u_2) which is connected to u_2 by edges with labels other than j_+. Since u_2 is further away from u_0 than u_1 and the path p is maximal, u_2 must be a Y-junction. If the j-edges and j_+-edges would meet at u_2, u_1 would be a j-junction. Otherwise, a subdominant face labeled $j+1$ would



(a) Graph Γ with a Y-structure



(b) Graph $A_j(\Gamma)$



(c) Graph $A_j^2(\Gamma)$

FIGURE 8. Case 3, moving a Y-structure.

be adjacent to both u_1 and u_2, while a subdominant face adjacent to a Y-junction cannot be adjacent to any other junctions.

Hence u_1 must be a j-junction. By Theorem 13, the action A_j^2 moves the structure at the j-junction u_1 one step closer to u_0 along the path p, decreasing $|\Gamma|_{u_0}$ at least by 1.

The case when the j-edge is directed from v to u_1 is treated similarly. In that case, u_1 must be a j_--junction, and the action $A_{j_-}^{-2}$ moves the structure at the j_--junction u_1 one step closer to u_0 along the path p.

We have proved that if $|U| > 1$ then $|\Gamma|_{u_0}$ can be reduced. Since it is a non-negative integer, after finitely many steps we must reach a stage where $|U| = 1$, hence the graph is in ivy form. □

Remark 18. *The outcome of the algorithm is in general non-unique, and might yield different final values of* $|A^*(\Gamma)|_{u_0}$.

Lemma 19. *Let* Γ *be a standard graph with a junction* u_0 *such that* u_0 *is both a* j_-*-junction and a* j*-junction. Assume that the corresponding structures are of types* Y *and* V*, in any order. Then there is a sequence of actions from the set* $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ *that interchanges the* Y*-structure and the* V*-structure.*

Proof. We may assume that the Y- and V-structures are attached to u_0 counterclockwise around u_0, as in Figure 9, otherwise we reverse the actions. By Theorem 13, the action A_j^{2k} moves the V-structure k steps in the direction of the j-edges. Choose k so that the V-structure is moved all the way to u_1, as in Figure 10. Then u_1 becomes both a j_--junction and j-junction, with two V-structures attached. Proceed by applying $A_{j_-}^{2k}$ to move the V-structure at the j_--junction u_1 up to u_0, as in Figure 11. □

Lemma 20. *Let* Γ *be a standard graph with a junction* u_0*, such that* u_0 *is both a* j_-*-junction and a* j*-junction, with the corresponding structures of type* I *and* Y*, in any order. Then there is a sequence of actions from the set* $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ *converting the* Y*-structures to a* V*-structure.*

Proof. We may assume that the I- and Y-structures are attached to u_0 counterclockwise around u_0, as in Figure 12, otherwise, we just reverse the actions. By Theorem 13, we can apply $A_{j_-}^{-2}$ several times to move the I-structure down to u_1. *(For example, in Figure 12, we need to do this twice. This gives the configuration shown in Figure 13.)* Now u_1 becomes a j_--junction and a j-structure, with the I- and V-structures attached. Applying A_j^{2k}, we can move the V-structure at u_1 up to u_0. *(In our example, this final configuration is presented in Figure 14.)* Thus the Y-structure has been transformed to a V-structure. □

Theorem 21. *Let* Γ *be a standard graph with at least two adjacent dominant faces. Then there exists a sequence of actions* $A^* = A_{j_1}^{\pm 2} A_{j_2}^{\pm 2} \cdots$ *such that* $A^*(\Gamma)$ *have only* one *junction.*

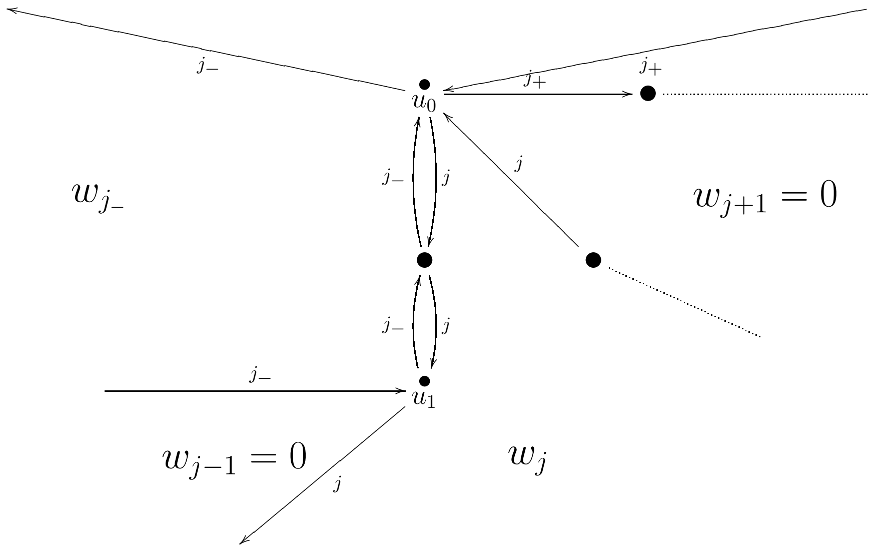


FIGURE 9. Adjacent Y- and V-structures.

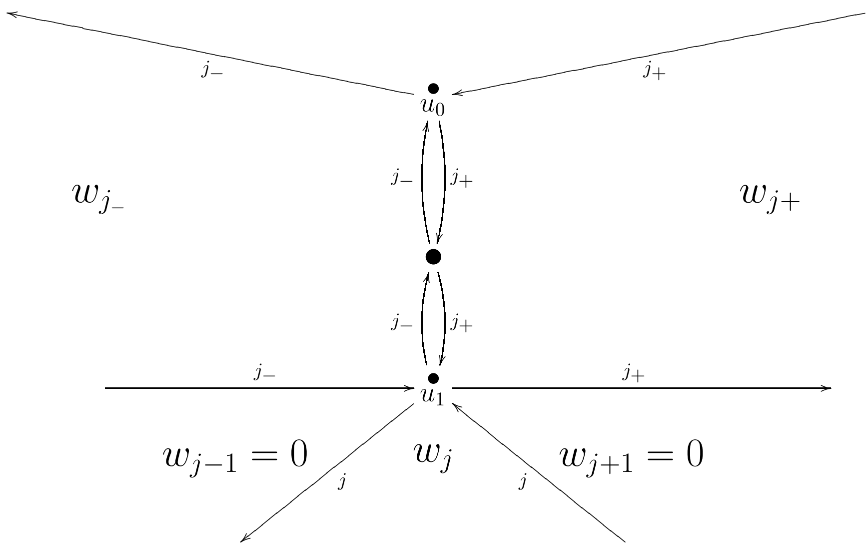


FIGURE 10. Intermediate configuration: two adjacent V-structures.

Proof. By Theorem 17 we may assume that Γ is a graph in ivy form with the root junction u_0. The existence of two adjacent dominant faces implies the existence of an I-structure. If there are only I-structures and V-structures, then u_0 is the only junction of Γ. Assume that there is at least one Y-structure. By Lemma 19, we may move a Y-structure

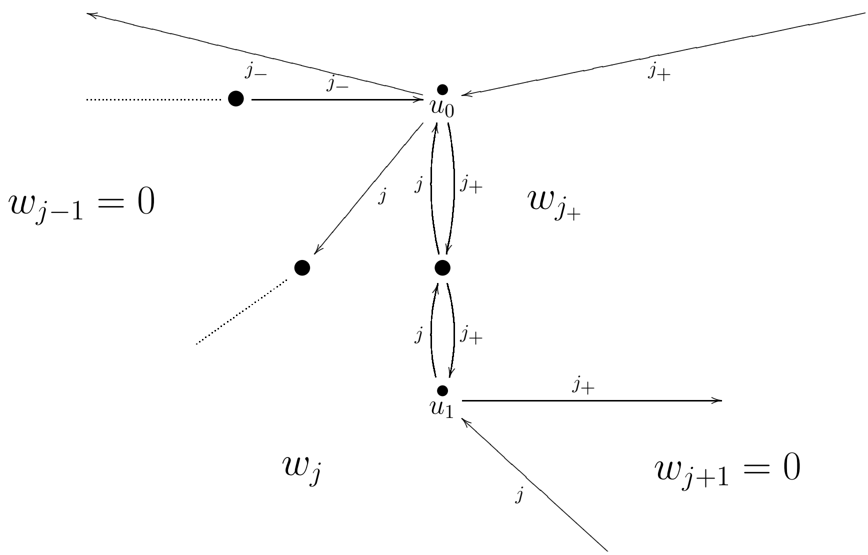


FIGURE 11. Y- and V-structures exchanged.

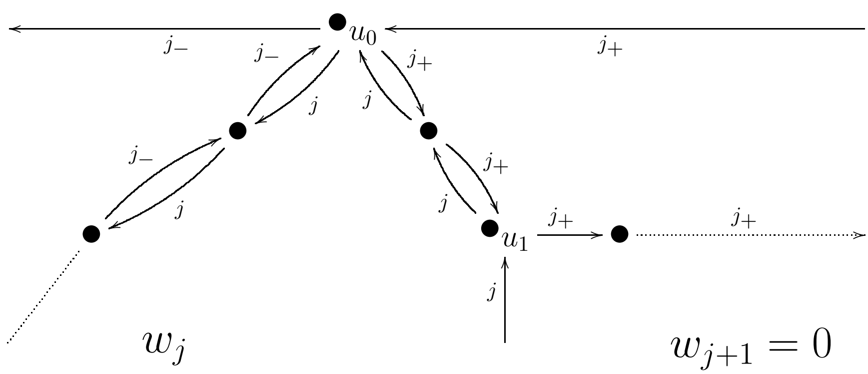


FIGURE 12. Adjacent I- and Y-structures

so that it is counterclockwise next to an I-structure. By Lemma 20, the Y-structure can be transformed to a V-structure, and the Y-junction removed. This can be repeated, eventually removing all junctions of Γ except u_0. □

Lemma 22. *Let Γ be a standard graph with a junction u_0, such that u_0 is both a j_--junction and a j-junction, with two adjacent Y-structures attached. Then there is a sequence of actions from the set $\{A_j^2, A_{j_-}^2, A_j^{-2}, A_{j_-}^{-2}\}$ converting one of the Y-structures to a V-structure.*

Proof. This can be proved by the arguments similar to those in the proof of Theorem 21. □

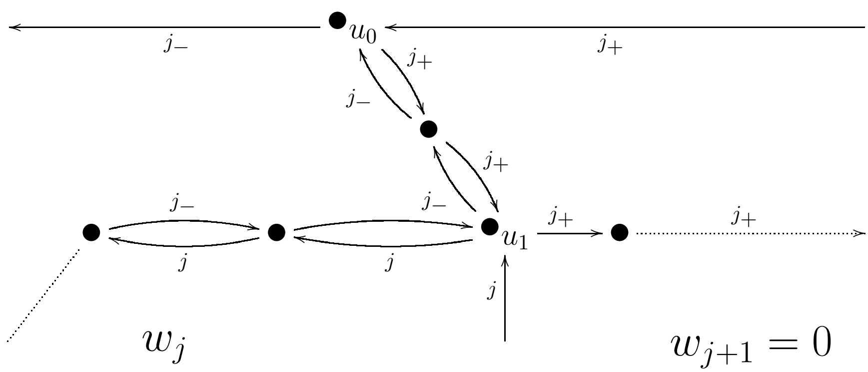


FIGURE 13. Moving the I-structure to u_1

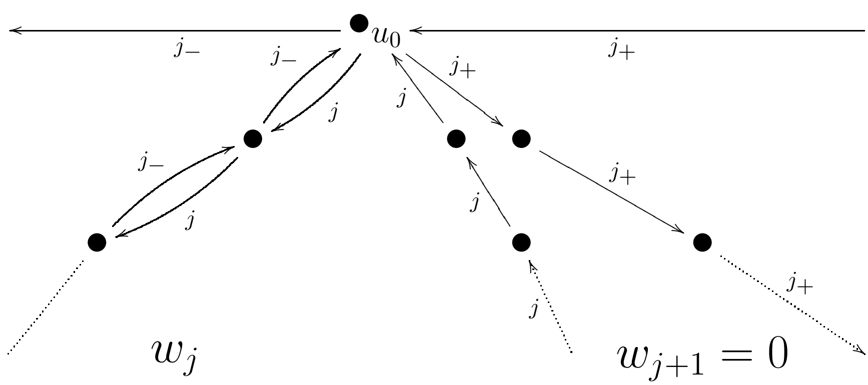


FIGURE 14. Moving the V-structure to u_0

Theorem 23. *Let Γ be a standard graph such that no two dominant faces are adjacent. Then there exists a sequence of actions $A^* = A_{j_1}^{\pm 2}, A_{j_2}^{\pm 2}, \dots,$ such that $A^*(\Gamma)$ is in ivy form, with at most one Y-structure.*

Proof. One may assume by Theorem 17 that Γ is in ivy form, with the root junction u_0. Since no two dominant faces are adjacent, there are only V- and Y-structures attached to u_0. If there are at least two Y-structures, we may assume, by Lemma 19, that two Y-structures are adjacent. By Lemma 22, two adjacent Y-structures can be converted to a V-structure and a Y-structure. This can be repeated until at most one Y-structure remains in Γ. □

Lemma 24. *Let Γ be a standard graph such that no two dominant faces are adjacent. Then the number of bounded faces of Γ is finite and does not change after any action A_j^2.*

Proof. The bounded faces of Γ correspond to the edges of T_Γ separating two dominant faces. Since no two dominant faces are adjacent, any two dominant faces have a finite common boundary in T_Γ. Hence the number of bounded faces of Γ is finite. Lemma 12 and Theorem 13 imply that this number does not change after any action A_j^2. □

4. Irreducibility and connectivity of the spectral locus

In this section, we obtain the main results stated in the introduction. We start with the following statements.

Lemma 25. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbb{C}^d$ such that equation (1) admits a solution subdominant in non-adjacent Stokes sectors $S_{j_1}, \dots, S_{j_k}$, $k \le (d+2)/2$. Then Σ is a smooth complex analytic submanifold of $\mathbb{C}^d$ of the codimension $k-1$.*

Proof. Let f be a ratio of two linearly independent solutions of (1), and let $w = (w_0, \dots, w_{d+1})$ be the set of asymptotic values of f in the Stokes sectors $S_0, \dots, S_{d+1}$. Then w belongs to the subset Z of $\bar{\mathbb{C}}^{d+2}$ where the values w_j in adjacent Stokes sectors are distinct and there are at least three distinct values among w_j. The group G of fractional-linear transformations of $\bar{\mathbb{C}}$ acts on Z diagonally, and the quotient Z/G is a $(d-1)$-dimensional complex manifold.

[2, Thm. 7.2] implies that the mapping $W : \mathbb{C}^d \to Z/G$ assigning to (α, λ) the equivalence class of w is submersive. More precisely, W is locally invertible on the subset $\{\alpha_{d-1} = 0\}$ of $\mathbb{C}^d$ and constant on the orbits of the group $\mathbb{C}$ acting on $\mathbb{C}^d$ by translations of the independent variable z. In particular, the preimage $W^{-1}(Y)$ of any smooth submanifold $Y \subset Z/G$ is a smooth submanifold of $\mathbb{C}^d$ of the same codimension as Y.

The set Σ is the preimage of the set $Y \subset Z/G$ defined by the $k-1$ conditions $w_{j_1} = \cdots = w_{j_k}$. Hence Σ is a smooth manifold of codimension $k-1$ in $\mathbb{C}^d$. □

Proposition 26. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbb{C}^d$ such that equation* (1) *admits a solution subdominant in the non-adjacent Stokes sectors $S_{j_1}, \dots, S_{j_k}$. If at least two remaining Stokes sectors are adjacent, then Σ is an irreducible complex analytic manifold.*

Proof. Let Σ_0 be the intersection of Σ with the subspace $\mathbb{C}^{d-1} = \{\alpha_{d-1} = 0\} \subset \mathbb{C}^d$. Then Σ has the structure of a product of Σ_0 and $\mathbb{C}$ induced by translation of the independent variable z. In particular, Σ is irreducible if and only if Σ_0 is irreducible.

Let us choose a point $w = (w_0, \dots, w_{d+1})$ so that $w_{j_1} = \cdots = w_{j_k} = 0$, with all other values w_j distinct, non-zero and finite. Let Ψ_0 be a cell decomposition of $\bar{\mathbb{C}} \setminus \{0\}$ defined by the loops γ_j starting and ending at ∞ and containing non-zero values w_j, as in Section 2.1.

Nevanlinna theory (see [8, 9]), implies that, for each standard graph Γ with the properties listed in Lemma 8, there exists $(\alpha, \lambda) \in \mathbb{C}^d$ and a meromorphic function $f(z)$ such that f is the ratio of two linearly independent solutions of (1) with the asymptotic values w_j in the Stokes sectors S_j, and Γ is the graph corresponding to the cell decomposition $\Phi_0 = f^{-1}(\Psi_0)$. This function, and the corresponding point (α, λ) is defined uniquely up to translation of the variable z. We can choose f uniquely if we require that $\alpha_{d-1} = 0$ in (α, λ). Conditions on the asymptotic values w_j imply then that $(\alpha, \lambda) \in \Sigma'$. Let f_Γ be this uniquely selected function, and $(\alpha_\Gamma, \lambda_\Gamma)$ the corresponding point of Σ'.

Let $W : \Sigma' \to Y \subset Z/G$ be as in the proof of Lemma 25. Then Σ' is an unramified covering of Y. Its fiber over the equivalence class of w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ. Each action A_j^2 corresponds to a closed loop in Y starting and ending at w. Since for a given list of subdominant sectors a standard graph with one vertex is unique, Theorem 21 implies that the monodromy action is transitive. Hence Σ' is irreducible as a covering with a transitive monodromy group (see, e.g., [6, §5]). $\square$

This immediately implies Theorem 2, and we may also state the following corollary equivalent to Theorem 1:

Corollary 27. *For every potential P_α of even degree, with* $\deg P_\alpha \geq 4$ *and with the boundary conditions $y \to 0$ for $z \to \pm\infty$, $z \in \mathbb{R}$, there is an analytic continuation from any eigenvalue λ_m to any other eigenvalue λ_n in the α-plane.*

Proposition 28. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbb{C}^d$, for even d, such that equation* (1) *admits a solution subdominant in the* $(d+2)/2$ *Stokes sectors $S_0, S_2, \ldots, S_d$. Then irreducible components Σ_k, $k = 0, 1, \ldots$ of Σ, which are also its connected components, are in one-to-one correspondence with the sets of standard graphs with k bounded faces. The corresponding solution of* (1) *has k zeros and can be represented as $Q(z)e^{\phi(z)}$ where Q is a polynomial of degree k and ϕ a polynomial of degree* $(d+2)/2$.

Proof. Let us choose w and Ψ_0 as in the proof of Proposition 26. Repeating the arguments in the proof of Proposition 26, we obtain an unramified covering $W : \Sigma' \to Y$ such that its fiber over w consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ with the properties listed in Lemma 8. Since we have no adjacent dominant sectors, Theorem 23 implies that any standard graph Γ can be transformed by the monodromy action to a graph Γ_0 in ivy form with at most one Y-structure attached at its j-junction, where j is any index such that S_j is a dominant sector. Lemma 24 implies that Γ and Γ_0 have the same number k of bounded faces. If $k = 0$, the graph Γ_0 is unique. If $k > 0$, the graph Γ_0 is completely determined by k and j. Hence for each $k = 0, 1, \dots$ there is a unique orbit of the monodromy group action on the fiber of W over w consisting of all standard graphs Γ with k bounded faces. This implies that Σ' (and Σ) has one irreducible component for each k.

Since Σ is smooth by Lemma 25, its irreducible components are also its connected components.

Finally, let $f_\Gamma = y/y_1$ where y is a solution of (1) subdominant in the Stokes sectors $S_0, S_2, \dots, S_d$. Then the zeros of f and y are the same, each such zero belongs to a bounded domain of Γ, and each bounded domain of Γ contains a single zero. Hence y has exactly k simple zeros. Let Q be a polynomial of degree k with the same zeros as y. Then y/Q is an entire function of finite order without zeros, hence $y/Q = e^\phi$ where ϕ is a polynomial. Since y/Q is subdominant in $(d+2)/2$ sectors, $\deg \phi = (d+2)/2$. □

The above proposition immediately implies Theorem 4.

5. Alternative viewpoint

In this section, we provide an example of the correspondence between the actions on cell decompositions with some subdominant sectors and actions on cell decompositions with no subdominant sectors. This correspondence can be used to simplify calculations with cell decompositions. We will illustrate our results on a cell decomposition with 6 sectors, the general case follows immediately.

Let C_6 be the set of cell decompositions with 6 sectors, none of them subdominant. Let $C_6^{03} \subset C_6$ be the set of cell decompositions such that for any $\Gamma \in C_6^{03}$, the sectors S_0 and S_3 do not share a common edge in the associated undirected graph T_Γ. Define D_6^{03} to be the set of cell decompositions with 6 sectors where S_0 and S_3 are subdominant.

Lemma 29. *There is a natural bijection between C_6^{03} and D_6^{03}.*

Proof. Let $\Gamma \in C_6^{03}$ be a cell decomposition, and let T_Γ be the associated undirected graph, see section 2.2. Then consider T_Γ as the (unique) undirected graph associated with some cell decomposition $\Delta \in D_6^{03}$. This is possible since the condition that the sectors 0 and 3 do not share a common edge in Γ, ensures that the subdominant sectors in Δ do not share a common edge. Let us denote this map π. Conversely, every cell decomposition $\Delta \in D_6^{03}$ is associated with a cell decomposition $\Gamma \in C_6^{03}$ by the inverse procedure π^{-1}. □

We have previously established that $\mathcal{H}_6$ acts on C_6 and that $\mathcal{H}_4$ acts on D_6^{03}. Let $B_0, B_1, \dots, B_5$ be the actions generating $\mathcal{H}_6$, as described in subsection 3.1, and let A_1, A_2, A_4, A_5 generate $\mathcal{H}_4$. Let $\mathcal{H}_6^{03} \subset \mathcal{H}_6$ be the

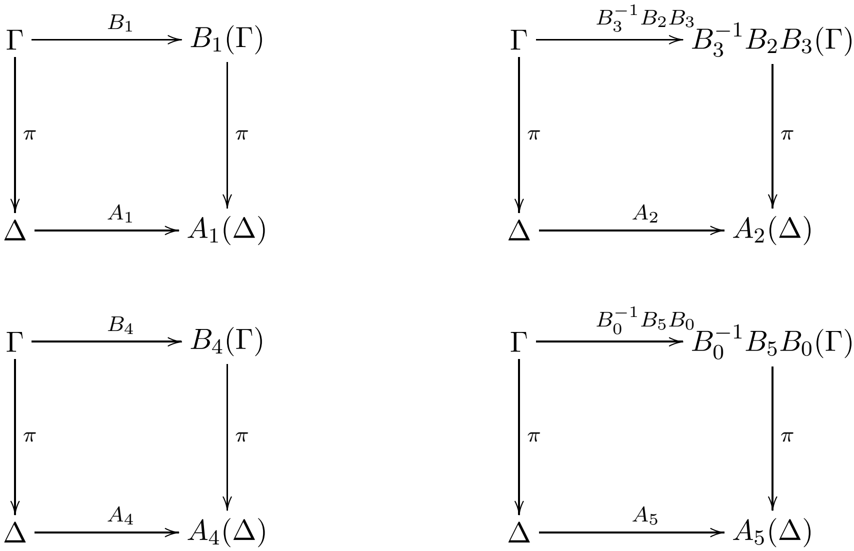


Figure 15. The commuting actions

subgroup generated by $B_1, B_2B_3B_2^{-1}, B_4, B_5B_0B_5^{-1}$, and their inverses. It is easy to see that $\mathcal{H}_6^{03}$ acts on elements in C_6^{03} and preserves this set.

Lemma 30. *The diagrams in Figure 15 commute.*

Proof. Let (a, b, c, d, e, f) be the 6 loops of a cell decomposition Ψ_0 as in Figure 1, looping around the asymptotic values $(w_0, \dots, w_5)$. Let Ψ_0' be the cell decomposition with the four loops (b, c, e, f), such that if $\Gamma \in C_6^{03}$ is the preimage of Ψ_0, then $\pi(\Gamma)$ is the preimage of Ψ_0'. That is, the preimages of the loops a and d in Ψ_0 are removed under π.

B_j acts on Ψ_0 and A_j acts on Ψ_0'. (See subsection 3.1 for the definition.) We have

$$A_1(b, c, e, f) = (bcb^{-1}, e, f),\ A_4(b, c, d, e) = (b, c, efe^{-1}, e). \tag{5}$$

and

$$\begin{aligned} B_1(a, b, c, d, e, f) &= (a, bcb^{-1}, d, e, f), \\ B_4(a, b, c, d, e, f) &= (a, b, c, efe^{-1}, e, f). \end{aligned} \tag{6}$$

Equation (5) and (6) shows that the left diagrams commute, since applying π to the result from (6) yields (5). We also have that

$$A_2(b, c, e, f) = (b, cec^{-1}, c, f),\ A_5(b, c, e, f) = (f, c, e, fbf^{-1}). \tag{7}$$

We now compute $B_3^{-1}B_2B_3(a,b,c,d,e,f)$. Observe that we must apply these actions *left to right*:

$$\begin{aligned} B_3^{-1}B_2B_3(a,b,c,d,e,f) &= B_2B_3(a,b,c,e,e^{-1}de,f) \\ &= B_3(a,b,cec^{-1},c,e^{-1}de,f) \\ &= (a,b,cec^{-1},c(e^{-1}de)c^{-1},c,f) \end{aligned} \tag{8}$$

A similar calculation gives

$$B_0^{-1}B_5B_0(a,b,c,d,e,f) = (f(b^{-1}ab)f^{-1},f,c,d,e,f,b,f^{-1}), \tag{9}$$

and applying π to the results (8) and (9) give (7). □

Remark 31. *Note that $B_j^{-1}B_{j-1}B_j(\Gamma) = B_{j-1}B_jB_{j-1}^{-1}(\Gamma)$ for all $\Gamma \in C_6$, which follows from basic properties of the braid group.*

The above result can be generalized as follows: Let C_n be the set of cell decompositions with n sectors such that all sectors are dominant. Let $C_n^{\mathbf{l}} \subset C_n$, $\mathbf{l} = \{l_1, l_2, \dots, l_k\}$ be the set of cell decompositions such that for any $\Gamma \in C_n^{\mathbf{l}}$, no two sectors in the set $S_{l_1}, S_{l_2}, \dots, S_{l_k}$ have a common edge in the associated undirected graph T_Γ. Let $D_n^{\mathbf{l}}$ be the set of cell decompositions with n sectors such that the sectors $S_{l_1}, S_{l_2}, \dots, S_{l_k}$ are subdominant. Let $\{A_j\}_{j\notin \mathbf{l}}$ be the $n-k$ actions acting on $C_n^{\mathbf{l}}$ indexed as in subsection 3.1. Let $\{B_j\}_{j=0}^{n-1}$ be the actions on C_n. Let $\pi : C_n^{\mathbf{s}} \to D_n^{\mathbf{s}}$ be the map similar to the bijection above, where one obtain a cell decomposition in $D_n^{\mathbf{s}}$ by removing edges with a label in $\mathbf{l}$ from a cell decomposition in $C_n^{\mathbf{s}}$. Then

$$\begin{cases} \pi(B_j(\Gamma)) = A_j(\pi(\Gamma)), & \text{if } j, j+1 \notin \mathbf{l}, \\ \pi(B_j^{-1}B_{j-1}B_j(\Gamma)) = A_j(\pi(\Gamma)), & \text{if } j \notin \mathbf{l}, j+1 \in \mathbf{l}. \end{cases} \tag{10}$$

Remark 32. *There are some advantages with cell decompositions with no subdominant sectors:*

- *An action A_j always interchanges the asymptotic values w_j and w_{j+1}.*
- *Lemma 8(II) implies T_Γ have no bounded faces if and only if order of the asymptotic values is a cyclic permutation of the standard order.*

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6. Appendix

6.1. **Examples of monodromy action.** Below are some specific examples on how the different actions act on trees and non-trees.

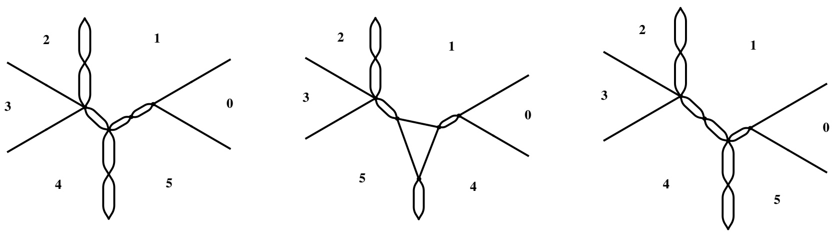

Figure 16. Example action of A_4^{-1} and A_4^{-2} in case 1.

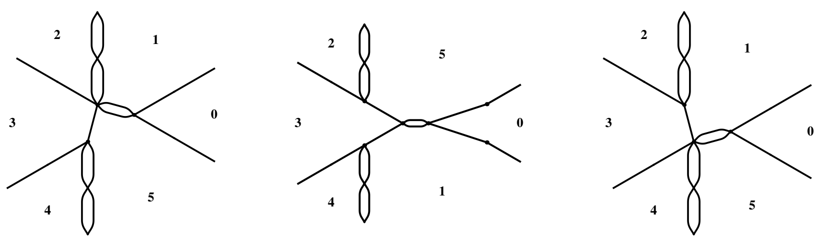

Figure 17. Example action of A_5 and A_5^2 in case 2.

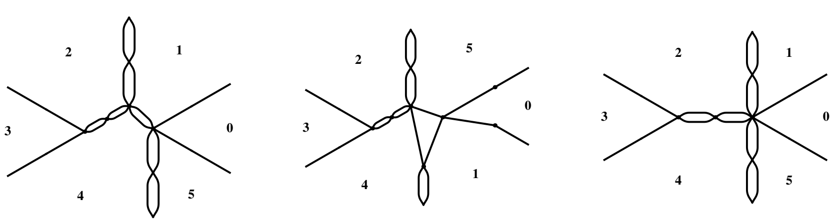

Figure 18. Example action of A_5^{-1} and A_5^{-2} in case 3.

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ON EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH AN EVEN COMPLEX-VALUED POLYNOMIAL POTENTIAL

PER ALEXANDERSSON

ABSTRACT. In this paper, we generalize several results in the article "Analytic continuation of eigenvalues of a quartic oscillator" of A. Eremenko and A. Gabrielov.

We consider a family of eigenvalue problems for a Schrödinger equation with even polynomial potentials of arbitrary degree d with complex coefficients, and $k < (d+2)/2$ boundary conditions. We show that the spectral determinant in this case consists of two components, containing even and odd eigenvalues respectively.

In the case with $k = (d+2)/2$ boundary conditions, we show that the corresponding parameter space consists of infinitely many connected components.

1. INTRODUCTION

We study the problem of analytic continuation of eigenvalues of the Schrödinger operator with an even complex-valued polynomial potential. In other words, analytic continuation of eigenvalues $\lambda = \lambda(\alpha)$ in the differential equation

$$-y'' + P_\alpha(z)y = \lambda y, \tag{1}$$

where $\alpha = (\alpha_2, \alpha_4, \dots, \alpha_{d-2})$ and $P_\alpha(z)$ is the even polynomial

$$P_\alpha(z) = z^d + \alpha_{d-2}z^{d-2} + \cdots + \alpha_2 z^2.$$

The boundary conditions are as follows: Set $n = d + 2$ and divide the plane into n disjoint open sectors

$$S_j = \left\{ z \in \mathbb{C} \setminus \{0\} : \left| \arg z - \frac{2\pi j}{n} \right| < \frac{\pi}{n} \right\}, \quad j = 0, 1, 2, \dots, n-1.$$

The index j should be considered mod n. These are the *Stokes sectors* of the equation (1). A solution y of (1) satisfies $y(z) \to 0$ or $y(z) \to \infty$

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as $z \to \infty$ along each ray from the origin in S_j, see [10]. The solution y is called *subdominant* in the first case, and *dominant* in the second case.

The main result of this paper is as follows:

Theorem 1. *Let $\nu = d/2 + 1$ and let $J = \{j_1, j_2, \dots, j_{2m}\}$ with $j_{k+m} = j_k + \nu$ and $|j_p - j_q| > 1$ for $p \neq q$. Let Σ be the set of all $(\alpha, \lambda) \in \mathbb{C}^\nu$ for which the equation $-y'' + (P_\alpha - \lambda)y = 0$ has a solution with with the boundary conditions*

$$y \text{ is subdominant in } S_j \text{ for all } j \in J, \tag{2}$$

where $P_\alpha(z)$ is an even polynomial of degree d. For $m < \nu/2$, Σ consists of two irreducible connected components. For $m = \nu/2$, (which can only happen when $d \equiv 2 \mod 4$), Σ consists of infinitely many connected components, distinguished by the number of zeros of the corresponding solution of (1)*.*

1.1. **Previous results.** The first study of analytic continuation of λ in the complex β-plane for the problem

$$-y'' + (\beta z^4 + z^2)y = \lambda y, \quad y(-\infty) = y(\infty) = 0$$

was done by Bender and Wu [3]. They discovered the connectivity of the sets of odd and even eigenvalues and rigorous results was later proved in [11].

In [4], the even quartic potential $P_a(z) = z^4 + az^2$ and the boundary value problem

$$-y'' + (z^4 + az^2)y = \lambda_a y, \quad y(\infty) = y(-\infty) = 0$$

was considered.

The problem has discrete real spectrum for real a, with $\lambda_1 < \lambda_2 < \dots \to +\infty$. There are two families of eigenvalues, those with even index and those with odd. If λ_j and λ_k are two eigenvalues in the same family, then λ_k can be obtained from λ_j by analytic continuation in the complex α-plane. Similar results have been found for other potentials, such as the PT-symmetric cubic, where $P_\alpha(z) = (iz^3 + i\alpha z)$, with $y(z) \to 0$, as $z \to \pm\infty$ on the real line. See for example [5].

2. Preliminaries on general theory of solutions to the Schrödinger equation

We will review some properties for the Schrödinger equation with a general polynomial potential. In particular, these properties hold for an even polynomial potential. These properties may also be found in [4, 1].

The general Schrödinger equation is given by

$$-y'' + P_\alpha(z)y = \lambda y, \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{d-1})$ and $P_\alpha(z)$ is the polynomial

$$P_\alpha(z) = z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_1 z.$$

We have the associated Stokes sectors

$$S_j = \left\{ z \in \mathbb{C} \setminus \{0\} : \left| \arg z - \frac{2\pi j}{n} \right| < \frac{\pi}{n} \right\}, \quad j = 0, 1, 2, \ldots, n-1,$$

where $n = d + 2$, and index considered mod n. The boundary conditions to (3) are of the form

$$y \text{ is subdominant in } S_{j_1}, S_{j_2}, \ldots, S_{j_k} \tag{4}$$

with $|j_p - j_q| > 1$ for all $p \neq q$.

Notice that any solution $y \neq 0$ of (3) is an entire function, and the ratio $f = y/y_1$ of any two linearly independent solutions of (3) is a meromorphic function with the following properties, (see [10]).

(i) For any j, there is a solution y of (3) subdominant in the Stokes sector S_j, where y is unique up to multiplication by a non-zero constant.

(ii) For any Stokes sector S_j, we have $f(z) \to w \in \bar{\mathbb{C}}$ as $z \to \infty$ along any ray in S_j. This value w is called *the asymptotic value* of f in S_j.

(iii) For any j, the asymptotic values of f in S_j and S_{j+1} (index still taken modulo n) are distinct. Furthermore, f has at least 3 distinct asymptotic values.

(iv) The asymptotic value of f in S_j is zero if and only if y is subdominant in S_j. We call such sector *subdominant* for f as well. Note that the boundary conditions given in (4) imply that sectors $S_{j_1}, \ldots, S_{j_k}$ are subdominant for f when y is an eigenfunction of (3), (4).

(v) f does not have critical points, hence $f : \mathbb{C} \to \bar{\mathbb{C}}$ is unramified outside the asymptotic values.

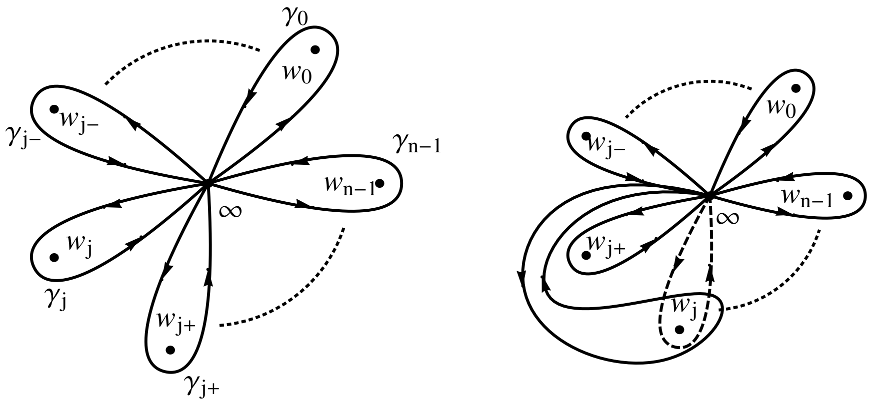
(vi) The Schwartzian derivative S_f of f given by

$$S_f = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

equals $-2(P_\alpha - \lambda)$. Therefore one can recover P_α and λ from f.

From now on, f denotes the ratio of two linearly independent solutions of (3) and (4).

2.1. **Cell decompositions.** As above, set $n = \deg P + 2$ where P is our polynomial potential and assume that all non-zero asymptotic values of f are distinct and finite. Let w_j be the asymptotic values of f with an arbitrary ordering satisfying the only restriction that if S_j is subdominant, then $w_j = 0$. One can denote by w_j the asymptotic value in the Stokes sector S_j, which will be called the *standard order*, see Section 2.3.



(a) Ψ_0 (b) $A_j(\Psi_0)$.

FIGURE 1. Permuting w_j and w_{j_+} in Ψ_0.

Consider the cell decomposition Ψ_0 of $\bar{\mathbb{C}}_w$ shown in Figure 1(a). It consists of closed directed loops γ_j starting and ending at ∞, where the index is considered mod n, and γ_j is defined only if $w_j \neq 0$. The loops γ_j only intersect at ∞ and have no self-intersection other than ∞. Each loop γ_j contains a single non-zero asymptotic value w_j of f. For example, for even n, the boundary condition $y \to 0$ as $z \to \pm\infty$ for $z \in \mathbb{R}$ implies that $w_0 = w_{n/2} = 0$, so there are no loops γ_0 and $\gamma_{n/2}$. We have a natural cyclic order of the asymptotic values, namely the order in which a small circle around ∞ traversed counterclockwise intersects the associated loops γ_j, see Figure 1(a).

We use the same index for the asymptotic values and the loops, so define

$$j_+ = j + k \quad \text{where } k \in \{1, 2\} \text{ is the smallest integer such that } w_{j+k} \neq 0.$$

Thus, γ_{j_+} is the loop around the next to w_j (in the cyclic order mod n) non-zero asymptotic value. Similarly, γ_{j_-} is the loop around the previous non-zero asymptotic value.

2.2. **From cell decompositions to graphs.** Proofs of all statements in this subsection can be found in [4].

Given f and Ψ_0 as above, consider the preimage $\Phi_0 = f^{-1}(\Psi_0)$. Then Φ_0 gives a cell decomposition of the plane $\mathbb{C}_z$. Its vertices are the poles of f and the edges are preimages of the loops γ_j. An edge that is a preimage of γ_j is labeled by j and called a j-edge. The edges are directed, their orientation is induced from the orientation of the loops γ_j. Removing all loops of Φ_0, we obtain an infinite, directed planar graph Γ, without loops. Vertices of Γ are poles of f, each bounded connected component of $\mathbb{C} \setminus \Gamma$ contains one simple zero of f, and each zero of f belongs to one such bounded connected component. There are at most two edges of Γ connecting any two of its vertices. Replacing each such pair of edges with a single undirected edge and making all other edges undirected, we obtain an undirected graph T_Γ. It has no loops or multiple edges, and the transformation from Φ_0 to T_Γ can be uniquely reversed.

A *junction* is a vertex of Γ (and of T_Γ) at which the degree of T_Γ is at least 3. From now on, Γ refers to both the directed graph without loops and the associated cell decomposition Φ_0.

2.3. **The standard order of asymptotic values.** For a potential P of degree d, the graph Γ has $n = d + 2$ infinite branches and n unbounded faces corresponding to the Stokes sectors of P. We fixed earlier the ordering $w_0, w_1, \dots, w_{n-1}$ of the asymptotic values of f.

If each w_j is the asymptotic value in the sector S_j, we say that the asymptotic values have *the standard order* and the corresponding cell decomposition Γ is a *standard graph.*

Lemma 2 (See [4], Proposition 6)**.** *If a cell decomposition Γ is a standard graph, then the corresponding undirected graph T_Γ is a tree.*

In the next section, we define some actions on Ψ_0 that permute non-zero asymptotic values. Each unbounded face of Γ (and T_Γ) will be labeled by the asymptotic value in the corresponding Stokes sector. For example, labeling an unbounded face corresponding to S_k with w_j or just with the index j, indicates that w_j is the asymptotic value in S_k.

From the definition of the loops γ_j, a face corresponding to a dominant sector has the same label as any edge bounding that face. The label in a face corresponding to a subdominant sector S_k is always k, since the actions defined below only permute non-zero asymptotic values.

An unbounded face of Γ is called (sub)dominant if the corresponding Stokes sector is (sub)dominant.

2.4. **Properties of graphs and their face labeling.**

Lemma 3 (See Section 3 in [4])**.** *Any graph Γ have the following properties:*

- *(i) Two bounded faces of Γ cannot have a common edge, (since a j-edge is always at the boundary of an unbounded face labeled j.)*
- *(ii) The edges of a bounded face of a graph Γ are directed clockwise, and their labels increase in that order. Therefore, a bounded face of T_Γ can only appear if the order of w_j is non-standard.*
- *(iii) Each label appears at most once in the boundary of any bounded face of Γ.*
- *(iv) The unbounded faces of Γ adjacent to a junction u, always have the labels cyclically increasing counterclockwise around u.*

(v) The boundary of a dominant face labeled j consists of infinitely many directed j-edges, oriented counterclockwise around the face.
(vi) If $w_j = 0$ there are no j-edges.
(vii) Each vertex of Γ has even degree, since each vertex in $\Phi_0 = f^{-1}(\Psi_0)$ has even degree, and removing loops to obtain Γ preserves this property.

Following the direction of the j-edges, the first vertex that is connected to an edge labeled j_+ is the vertex where the j-edges and the j_+-edges *meet.* The last such vertex is where they *separate.* These vertices, if they exist, must be junctions.

Definition 4. *Let Γ be a standard graph, and let $u \in \Gamma$ be a junction where the j-edges and j_+-edges separate. Such junction is called a* j-junction.

There can be at most one j-junction in Γ, the existence of two or more such junctions would violate property (iii) of the face labeling. However, the same junction can be a j-junction for different values of j.

There are three different types of j-junctions, see Figure 2.

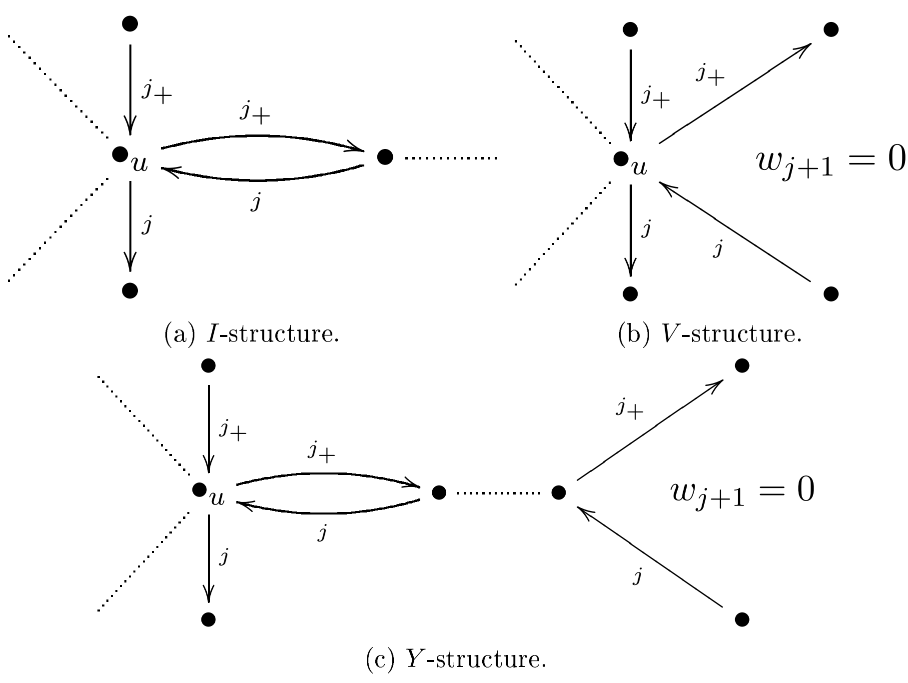


FIGURE 2. Different types of j-junctions.

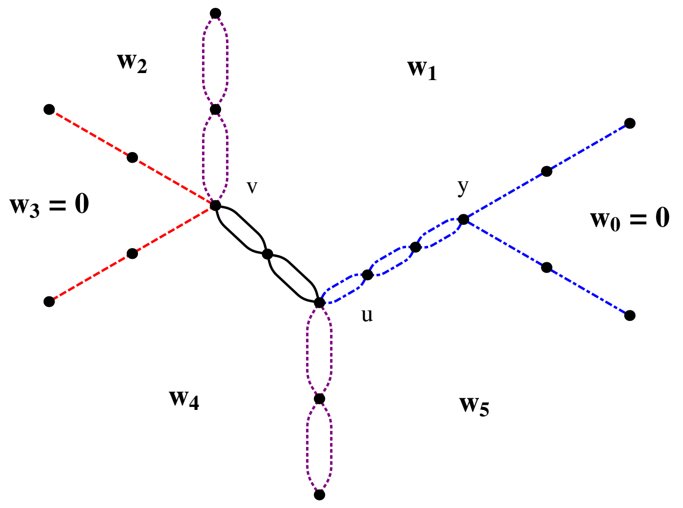


FIGURE 3. Graph Γ with (dotted) I-structures, a (dashed) Y-structure and a (dotdashed) Y-structure.

Case (a) only appears when $w_{j+1} \neq 0$. Cases (b) and (c) can only appear when $w_{j+1} = 0$. In (c), the j-edges and j_+-edges meet and separate at different junctions, while in (b), this happens at the same junction.

Definition 5. *Let Γ be a standard graph with a j-junction u. A* structure *at the j-junction is the subgraph Ξ of Γ consisting of the following elements:*

- *The edges labeled j that appear before u following the j-edges.*
- *The edges labeled j_+ that appear after u following the j_+-edges.*
- *All vertices the above edges are connected to.*

If u is as in Figure 2(a), Ξ is called an I-structure at the j-junction*. If u is as in Figure 2(b), Ξ is called a* V-structure at the j-junction*. If u is as in Figure 2(c), Ξ is called a* Y-structure at the j-junction.

Since there can be at most one j-junction, there can be at most one structure at the j-junction.

A graph Γ shown in Figure 3 has one (dotted) I-structure at the 1-junction v, one (dotted) I-structure at the 4-junction u, one (dashed) V-structure at the 2-junction v and one (dotdashed) Y-structure at the 5-junction u.

Note that the Y-structure is the only kind of structure that contains an additional junction. We refer to such additional junctions as *Y-junctions.* For example, the junction marked y in Figure 3 is a Y-junction.

2.5. **Braid actions on graphs.** As in [1], we define continuous deformations A_j of the loops in Figure 1(a), such that the new loops are given in terms of the old ones by

$$A_j(\gamma_k) = \begin{cases} \gamma_j \gamma_{j_+} \gamma_j^{-1} \text{ if } k = j, \\ \gamma_j \text{ if } k = j_+, \\ \gamma_k \text{ otherwise,} \end{cases}, \quad A_j^{-1}(\gamma_k) = \begin{cases} \gamma_{j_+} \text{ if } k = j, \\ \gamma_{j_+}^{-1} \gamma_j \gamma_{j_+} \text{ if } k = j_+ \\ \gamma_k \text{ otherwise.} \end{cases}$$

These actions, together with their inverses, generate the Hurwitz (or sphere) braid group $\mathcal{H}_m$, where m is the number of non-zero asymptotic values. (For a definition of this group, see [7].) The action of the generators A_j and A_k commute if $|j-k| \geq 2$.

The property (v) of the eigenfunctions implies that each A_j induces a monodromy transformation of the cell decomposition Φ_0, and of the associated directed graph Γ.

3. Properties of even actions on centrally symmetric graphs

3.1. **Additional properties for even potential.** In addition to the previous properties for general polynomials, these additional properties holds for even polynomial potentials P (see [4]). From now until the end of the article, $\nu = (\deg(P)+2)/2$.

Each solution y of (1) is either even or odd and we may choose y and y_1 such that $f = y/y_1$ is odd.

If the asymptotic values $w_0, w_1, \dots, w_{2\nu-1}$ are ordered in the standard order, we have that $w_j = -w_{j+\nu}$.

We may choose the loops centrally symmetric in Figure 1(a) which implies that Φ_0 and Γ are centrally symmetric.

3.2. **Even braid actions.** Define the *even actions* E_j as $E_j = A_j \circ A_{j+\nu}$.

Assume that Γ is a graph with the property that if w_j is the asymptotic value in S_k, then w_{j+n} is the asymptotic value in $S_{k+\nu}$. (For example, all standard graphs have this property, with $j = k$.) It follows from the symmetric property of E_j that E_j preserves this property. To illustrate, we have that $E_j(\Psi_0)$ is given in Figure 4.

Lemma 6. *If Γ is centrally symmetric, then $E_j(\Gamma)$ and $E_j^{-1}(\Gamma)$ are centrally symmetric graphs.*

Proof. We may choose the deformations of the paths γ_j and $\gamma_{j+\nu}$ being centrally symmetric, which implies that the composition $A_j \circ A_{j+\nu}$ preserves the property of Γ being centrally symmetric, see details in [4]. $\square$

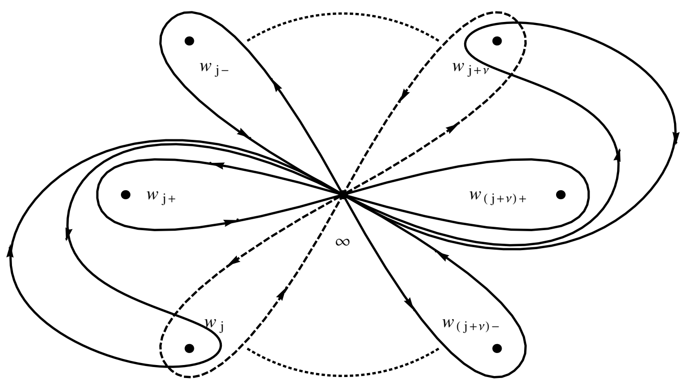


FIGURE 4. $E_j(\Psi_0)$

Lemma 7. *Let* Γ *be a centrally symmetric standard graph without a* j*-junction. Then* $E_j^2(\Gamma) = \Gamma$.

Proof. Since A_j and A_{j+n} commute, we have that $E_j^2 = A_j^2 A_{j+\nu}^2$, and the statement then follows from [1, Lemma 12]. □

Theorem 8. *Let* Γ *be a centrally symmetric standard graph with a* j*-junction* u*. Then* $E_j^2(\Gamma) \neq \Gamma$*, and the structure at the* j*-junction is moved one step in the direction of the* j*-edges under* E_j^2*. The inverse of* E_j^2 *moves the structure at the* j*-junction one step backwards along the* j_+*-edges.*

Since Γ *is centrally symmetric, it also has a* $j+\nu$*-junction, and the structure at the* $j+\nu$*-junction is moved one step in the direction of the* $j+\nu$*-edges under* E_j^2*. The inverse of* E_j^2 *moves the structure at the* $j+\nu$*-junction one step backwards along the* $(j+\nu)_+$*-edges.*

Proof. Since $E_j^2 = A_j^2 A_{j+\nu}^2$, the result follows from [1, Theorem 13]. □

4. Proving Main Theorem 1

Notice that each centrally symmetric standard graph Γ has either a vertex in its center, or a double edge, connecting two vertices. This property follows from the fact that Γ_T is a centrally symmetric tree.

Lemma 9. *Let* Γ *be a centrally symmetric graph. Then for every action* E_j*,* Γ *has a vertex at the center if and only if* $E_j(\Gamma)$ *has a vertex at the center.*

Proof. This is evident from the definition of the actions, since an action only changes edges, and preserves the vertices. □

Corollary 10. *The spectral determinant has at least two connected components.*

Each centrally symmetric standard graph Γ is of one of two types:

1. Γ has a central double edge. The vertices of the central double edge are called *root junctions.*
2. Γ has a junction at its center. This junction is called the *root junction* u_r.

Definition 11. *A centrally symmetric standard graph* Γ *is in* ivy form *if* Γ *consists of structures connected to one or two root junctions.*

Definition 12. *Let* Γ *be a centrally symmetric standard graph.*
The root metric *of* Γ*, denoted* $|\Gamma|_r$ *is defined as*

$$|\Gamma|_r = \sum_{v \in \Gamma} (\deg(v) - 2)\, |v - u_r|$$

where the sum is taken over all vertices v *of* Γ_1*. Here* $deg(v)$ *is the total degree of the vertex* v *in* T_Γ *and* $|v - u_r|$ *is the length of the shortest path from* v *to the closest root junction* u_r *in* T_Γ*.*

Lemma 13. *The graph* Γ *is in ivy form if and only if all but its root junctions are* Y*-junctions.*

Proof. This follows from the definitions of the structures. □

Theorem 14. *Let* Γ *be a centrally symmetric standard graph. Then there is a sequence of even actions* $E^* = E_{j_1}^{\pm 2}, E_{j_2}^{\pm 2}, \dots,$ *such that* $E^*(\Gamma)$ *is in* ivy form*.*

Proof. Assume that Γ is not in ivy form.

Let U be the set of junctions in Γ that are not Y-junctions. Since Γ is not in ivy form we have that $|U| \geq 3$. Let $u_r \neq u_1$ be two junctions in U such that $|u_r - u_1|$ is maximal, and u_r is the root junction closest to u_1. Let p be the path from u_r to u_1 in T_Γ. It is unique since T_Γ is a tree. Let v be the vertex preceding u_1 on the path p. The edge from v to u_1 in T_Γ is adjacent to at least one dominant face with label j such that $w_j \neq 0$. Therefore, there exists a j-edge between v and u_1 in Γ. Suppose first that this j-edge is directed from u_1 to v. Let us show that in this case u_1 must be a j-junction, i.e., the dominant face labeled j_+ is adjacent to u_1.

Since u_1 is not a Y-junction, there is a dominant face adjacent to u_1 with a label $k \neq j, j_+$. Hence no vertices of p, except possibly u_1 can be adjacent to j_+-edges. If u_1 is not a j-junction, there are no j_+-edges adjacent to u_1. This implies that any vertex of Γ adjacent to a j_+-edge is further away from u_r than u_1.

Let u_2 be the closest to u_1 vertex of Γ adjacent to a j_+-edge. Then u_2 should be a junction of T_Γ, since there are two j_+-edges adjacent to u_2 in Γ and at least one more vertex (on the path from u_1 to u_2) which is connected to u_2 by edges with labels other than j_+. Since u_2 is further away from u_r that u_1 and the path p is maximal, u_2 must be a Y-junction. If the j-edges and j_+-edges would meet at u_2, u_1 would be a j-junction. Otherwise, a subdominant face labeled $j+1$ would be adjacent to both u_1 and u_2, while a subdominant face adjacent to a Y-junction cannot be adjacent to any other junctions.

Hence u_1 must be a j-junction. By Theorem 8, the action E_j^2 moves the structure at the j-junction u_1 one step closer to u_r along the path p, and similarly happens on the opposite side of Γ, decreasing $|\Gamma|_r$ by at least 2.

The case when the j-edge is directed from v to u_1 is treated similarly. In that case, u_1 must be a j_--junction, and the action $A_{j_-}^{-2}$ moves the structure at the j_--junction u_1 one step closer to u_r along the path p.

We have proved that if $|U| > 1$ then $|\Gamma|_r$ can be reduced. Since it is a non-negative integer, after finitely many steps we must reach a stage where U consists only of the root junctions. Hence there exists E^* such that $E^*(\Gamma)$ is in ivy form. □

The above theorem shows that for every centrally symmetric standard graph Γ, there is a sequence of actions that turns Γ into ivy form. A graph in ivy form consists of one or two root junctions, with attached structures. These structures can be ordered counterclockwise around each root junction. These observations motivates the following lemmas:

Lemma 15. *Let Γ be a centrally symmetric standard graph, and let $u_r \in \Gamma$ be a root junction of type j_- and of type j. Let S_1 and S_2 be the corresponding structures attached to u_r.*

(1) If S_1 and S_2 are of type Y resp. V, then there is a sequence of even actions that interchange these structures.
(2) If S_1 and S_2 are of type I resp. Y, then there is a sequence of even actions that converts the type Y structure to a type V structure.
(3) If S_1 and S_2 are both of type Y, then there is a sequence of even actions that converts one of the Y-structures to a V-structure.

Proof. By symmetry, there are identical structures in Γ attached to a root junction of type $\nu + j_-$ and $\nu + j$, with attached structures S_1' and S_2' of the same type as S_1 resp. S_2.

Lemma 19, 20 and 22 in [1], gives the existence of a non-even sequence of actions, that only acts on S_1 and S_2 in the desired way.

In all these cases, the sequence is of the form

$$A^* = A_{k_1}^{\pm 2} A_{k_2}^{\pm 2} \cdots A_{k_m}^{\pm 2}$$

where $k_1, k_2, \dots, k_m \in \{j, j_-\}$. It follows that the action

$$B^* = A^{\pm 2}_{k_1+\nu} A^{\pm 2}_{k_2+\nu} \cdots A^{\pm 2}_{k_m+\nu}$$

do the same as A^* but on S'_1 and S'_2.

Now, $E^* = A^* \circ B^*$ is even, since by commutativity[1], it is equal to

$$(A^{\pm 2}_{k_1} A^{\pm 2}_{k_1+\nu})(A^{\pm 2}_{k_2} A^{\pm 2}_{k_2+\nu}) \cdots (A^{\pm 2}_{k_m} A^{\pm 2}_{k_m+\nu})$$

which easily may be written in terms of our even actions as

$$E^{\pm 2}_{k_1} E^{\pm 2}_{k_2} \cdots E^{\pm 2}_{k_m}.$$

This sequence of actions has the desired property. □

Corollary 16. *Let Γ be a centrally symmetric graph, with two adjacent dominant faces. Then there is a sequence of even actions E^* such that $E^*(\Gamma)$ has either one or two junctions.*

Proof. We may apply even actions to make Γ into a standard graph, and then convert it to ivy form. The condition that we have two dominant faces, is equivalent to existence of I-structures. If there are no Y-structures, then the only junctions of Γ are the root junctions, and we are done. Otherwise, we may interchange the Y- and V-structures, so that a Y-structure appears next to the I-structure. By using the second part of the above lemma, we decrease the number of Y-structures of Γ by two. After a finite number of actions, we arrive at a graph in ivy form without Y-structures. □

Lemma 17. *Let Γ be a centrally symmetric graph, with no adjacent dominant faces. Then there is a sequence of even actions E^* such that $E^*(\Gamma)$ is in ivy form, with at most two Y-structures.*

Proof. By Theorem 14, we may assume that Γ is in ivy form. Since there are no adjacent dominant sectors, the only structures of Γ are of Y and V type. These are attached to the one or two root junctions.

Assume that there are more than two Y-structures present. Two of these must be attached to the same root junction, u_r. By repeatedly applying part one of Lemma 15, we may interchange the Y- and V-structures attached to u_r such that the two Y-structures are adjacent. Applying part three of Lemma 15, we may then convert one of the two Y-structures to a V-structure.

By symmetry, the same change is done on the opposite side of Γ and total number of Y-structures of Γ have therefore been reduced by two. We may repeat this procedure a finite number of times, until the number of Y-structures is less than three. This implies the lemma. □

[1]We have at least 4 structures, 2 of them are Y or V structures. Hence $\nu \geq 3$ and we have commutativity.

Lemma 18 (See [1]). *Let Γ be a standard graph such that no two dominant faces are adjacent. Then the number of bounded faces of Γ is finite and does not change after any action A_j^2.*

Corollary 19. *The number of bounded faces of Γ does not change under any* even *action $E_j^2 = A_j^2 A_{j+\nu}^2$.*

Lemma 20. *Let $\nu = n/2 = d/2 + 1$ and let Σ be the space of all $(\alpha, \lambda) \in \mathbb{C}^{\nu-1}$ such that equation* (1) *admits a solution subdominant in non-adjacent Stokes sectors*

$$S_{j_1}, S_{j_2}, \ldots, S_{2m} \tag{5}$$

with $j_{k+m} = j_k + \nu$ and $1 \leq m \leq \nu/2$. Then Σ is a smooth complex analytic submanifold of $\mathbb{C}^{\nu-1}$ of the codimension m.

Proof. We consider the space $\mathbb{C}^{\nu-1}$ as a subspace of the space $\mathbb{C}^{n-2}$ of all (α, λ) corresponding to the general polynomial potentials in (3), with $\alpha = (\alpha_1, \ldots, \alpha_{d-1})$. Let f be a ratio of two linearly independent solutions of (3), and let $w = (w_0, \ldots, w_{n-1})$ be the set of the asymptotic values of f in the Stokes sectors $S_0, \ldots, S_{n-1}$.

Then w belongs to the subset Z of $\bar{\mathbb{C}}^{n-1}$ where the values w_j in adjacent Stokes sectors are distinct and there are at least three distinct values among w_j. The group G of fractional-linear transformations of $\bar{\mathbb{C}}$ acts on Z diagonally, and the quotient Z/G is a $(n-3)$-dimensional complex manifold.

Theorem 7.2, [2] implies that the mapping $W : \mathbb{C}^{n-2} \to Z/G$ assigning to (α, λ) the equivalence class of w is submersive. More precisely, W is locally invertible on the subset $\{\alpha_{d-1} = 0\}$ of $\mathbb{C}^{n-2}$

For an even potential, there exists an odd function f. The corresponding set of asymptotic values satisfies ν linear conditions $w_{j+\nu} = -w_j$ for $j = 0, \ldots, \nu - 1$. For $(\alpha, \lambda) \in \Sigma$, we can assume that $S_{j_1}, \ldots, S_{j_m}$ are subdominant sectors for f. This adds m linearly independent conditions $w_{j_1} = \cdots = w_{j_m} = 0$. Let Z_0 be the corresponding subset of Z. Its codimension in Z is $\nu + m$. The one-dimensional subgroup $\mathbb{C}^*$ of G consisting of multiplications by non-zero complex numbers preserves Z_0, and $gZ_0 \cap Z_0 = \emptyset$ for each $g \in G \setminus \mathbb{C}^*$. The explanation is as follows:

Since we have at least two subdominant sectors, only fractional linear transforms that preserves 0 are allowed. Furthermore, there exists a sector S_k with the value w_k different from 0 and ∞ (otherwise we would have only two asymptotic values). There is a unique transformation, multiplication by w_k^{-1}, preserving 0 and sending $\pm w_k$ to ± 1. This implies that the only transformation preserving 0 and sending $\pm w_k$ to another pair of opposite numbers is multiplication by a non-zero constant.

Hence GZ_0 is a G-invariant submanifold of Z of codimension $\nu + m - 2$, and its image $Y_0 \subset Y$ is a smooth submanifold of codimension $\nu + m -$

2. Due to Bakken's theorem, $W^{-1}(Y_0)$ intersected with the $(n-3)$-dimensional space of (α, λ) with $\alpha_{d-1} = 0$ is a smooth submanifold of codimension $\nu + m - 2$, dimension $\nu - m - 1$. Accordingly, it is a smooth submanifold of codimension m of the space $\mathbb{C}^{\nu-1}$. □

Proposition 21. *Let Σ be as in Lemma 20. If at least two adjacent Stokes sectors are missing in* (5)*, then Σ consists of two irreducible complex analytic manifolds.*

Proof. Nevanlinna theory (see [8, 9]), implies that, for each symmetric standard graph Γ with the properties listed in Lemma 3, there exists $(\alpha, \lambda) \in \mathbb{C}^{n-1}$ and an *odd* meromorphic function $f(z)$ such that f is the ratio of two linearly independent solutions of (1) with the asymptotic values w_j in the Stokes sectors S_j, and Γ is the graph corresponding to the cell decomposition $\Phi_0 = f^{-1}(\Psi_0)$. This function, and the corresponding point (α, λ) is defined uniquely.

Let $W : \Sigma \to Y_0$ be as in the proof of Lemma 20. Then Σ is an unramified covering of Y_0. Its fiber over the equivalence class of $w \in Y_0$ consists of the points $(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ. Each action A_j^2 corresponds to a closed loop in Y_0 starting and ending at w.

It should be noted that Y_0 is a connected manifold: Since for a given list of subdominant sectors the standard graph with one junction is unique. Lemma 9 then implies that the monodromy group has two orbits; odd and even eigenfunctions cannot be exchanged by any path in Y_0, while any odd (even) can be transferred into any other odd (even) eigenfunction by a sequence of $E_k^{\pm 2}$, by Theorem 15.

Hence Σ consists of two irreducible connected components (see, e.g., [6]). □

This immediately implies Theorem 1, for $m < \nu/2$. The following proposition implies the case where $m = \nu/2$.

Proposition 22. *Let Σ be the space of all $(\alpha, \lambda) \in \mathbb{C}^{\nu-1}$, for even ν, such that equation* (1) *admits a solution subdominant in every other Stokes sector, that is, in $S_0, S_2, \dots, S_{n-2}$.*

Then irreducible components Σ_k, $k = 0, 1, \dots$ of Σ, which are also its connected components, are in one-to-one correspondence with the sets of centrally symmetric standard graphs with k bounded faces. The corresponding solution of (1) *has k zeros and can be represented as $Q(z)e^{\phi(z)}$ where Q is a polynomial of degree k and ϕ a polynomial of degree $(d+2)/2$.*

Proof. Let us choose w and Ψ_0 as in the proof of Proposition 21. Repeating the arguments in the proof of Proposition 21, we obtain an unramified covering $W : \Sigma \to Y_0$ such that its fiber over w consists of the points

$(\alpha_\Gamma, \lambda_\Gamma)$ for all standard graphs Γ with the properties listed in Lemma 3.

Since we have no adjacent dominant sectors, Lemma 17 implies that any standard graph Γ can be transformed by the monodromy action to a graph Γ_0 in ivy form with at most two Y-structures attached at the root junction(s) of type j and $j + \nu$.

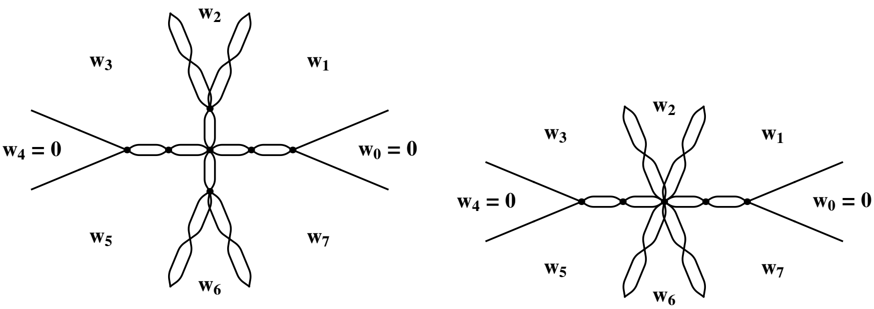
Lemma 18 implies that Γ and Γ_0 have the same number k of bounded faces. If $k = 0$, the graph Γ_0 is unique. If $k > 0$, the graph Γ_0 is completely determined by k. Hence for each $k = 0, 1, \dots$ there is a unique orbit of the monodromy group action on the fiber of W over w consisting of all standard graphs Γ with k bounded faces. This implies that Σ has one irreducible component for each k.

Since Σ is smooth by Lemma 20, its irreducible components are also its connected components.

Finally, let $f_\Gamma = y/y_1$ where y is an odd solution of (1) subdominant in the Stokes sectors $S_0, S_2, \dots, S_{n-2}$. Then the zeros of f and y are the same, each such zero belongs to a bounded domain of Γ, and each bounded domain of Γ contains a single zero. Hence y has exactly k simple zeros. Let Q be a polynomial of degree k with the same zeros as y. Then y/Q is an entire function of finite order without zeros, hence $y/Q = e^\phi$ where ϕ is a polynomial. Since y/Q is subdominant in $(d+2)/2$ sectors, $\deg \phi = (d+2)/2$. □

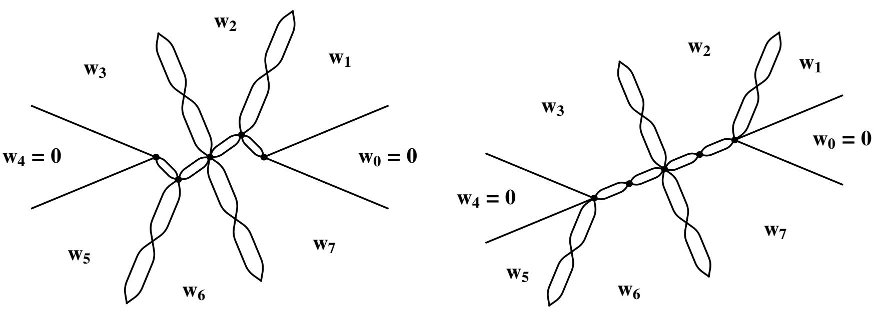
5. Illustrating example

We will now give a small example on how to apply the method given in Theorem 14 and Lemma 15. Let Γ be as in Figure 5(a). From subsection 2.4, we have that a dominant face with label j have j-edges as boundaries. Hence the faces 0 and 4 are subdominant. Also, the direction of the edges are directed counterclockwise in each of the dominant faces. Applying E_1^2, moves the I-structure at the 1-junction one step to the right, following the 1-edges. Similarly, the I-structure at the 5-junction moves one step to the left. Therefore, $E_1^2(\Gamma)$ is given in Figure 5(b). The graph $E_1^2(\Gamma)$ is now in ivy form, it consists of a center junction connected to four I-structures and two Y-structures. We proceed by using the algorithm in Lemma 15, and apply E_1^2 two times more. These steps are given in Figure 6. The next step in the lemma is to move the newly created V-structures to the center junction. We therefore apply E_3^2 two times. These final steps are presented in Figure 7, and we have reached the unique graph with only one junction.



(a) (b)

FIGURE 5. The graphs Γ and $E_1^2(\Gamma)$



(a) (b)

FIGURE 6. The graphs $E_1^4(\Gamma)$ and $E_1^6(\Gamma)$

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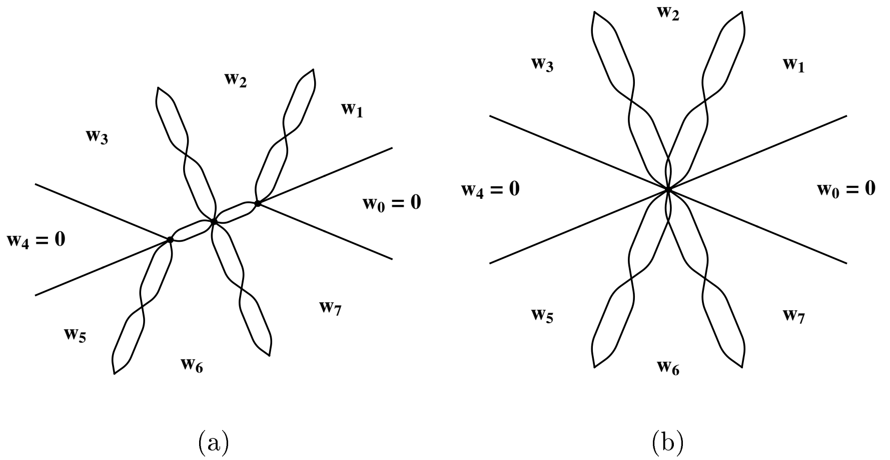

FIGURE 7. The graphs $E_3^2 E_1^6(\Gamma)$ and $E_3^4 E_1^6(\Gamma)$

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DISCRIMINANTS, SYMMETRIZED GRAPH MONOMIALS, AND SUMS OF SQUARES

PER ALEXANDERSSON AND BORIS SHAPIRO

Abstract. Motivated by the necessities of the invariant theory of binary forms, J. J. Sylvester constructed in 1878 for every graph with possible multiple edges but without loops, its symmetrized graph monomial, which is a polynomial in the vertex labels of the original graph. We pose the question for which graphs this polynomial is a nonnegative or a sum of squares. This problem is motivated by a recent conjecture of F. Sottile and E. Mukhin on discriminant of the discriminant of the derivative of a univariate polynomial and by an interesting example of P. and A. Lax of a graph with four edges whose symmetrized graph monomial is non-negative but not a sum of squares. We present detailed information about symmetrized graph monomials for graphs with four and six edges, obtained by computer calculations.

1. Introduction

In what follows, by a *graph* we will always mean a (directed or undirected) graph with (possibly) multiple edges but no loops. The classical construction of [13, 8] associates to an arbitrary directed loopless graph a symmetric polynomial as follows.

Definition 1. *Let g be a directed graph with vertices $x_1, \dots, x_n$ and adjacency matrix (a_{ij}), where a_{ij} is the number of directed edges connecting x_i and x_j. Define its graph monomial P_g as*

$$P_g(x_1, \dots, x_n) := \prod_{1 \le i,j \le n} (x_i - x_j)^{a_{ij}}.$$

The symmetrized graph monomial of g is defined as

$$\tilde{g}(\mathbf{x}) = \sum_{\sigma \in S_n} P_g(\sigma \mathbf{x}), \quad \mathbf{x} = x_1, \dots, x_n.$$

Observe that if the original g is undirected, one can still define $\tilde{g}$ up to a sign by choosing an arbitrary orientation of its edges. Symmetrized graph monomials are closely related to SL_2-invariants and covariants

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and were introduced in 1870s in an attempt to find new tools in invariant theory. Namely, to obtain an SL_2-coinvariant from a given $\tilde{g}(\mathbf{x})$, we have to perform two standard operations. First, we express the symmetric polynomial $\tilde{g}(\mathbf{x})$ in n variables in terms of the elementary symmetric functions $e_1, \dots, e_n$ and obtain the polynomial $\hat{g}(e_1, \dots, e_n)$. Second, we perform the standard homogenization of a polynomial of a given degree d,

$$Q_g(a_0, a_1, \dots, a_n) := a_0^d \hat{g}\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right).$$

The following fundamental proposition apparently goes back to A. Cayley; see [9, Theorem 2.4].

Theorem 1.

(i) If g is a d-regular graph with n vertices, then $Q_g(a_0, \dots, a_n)$ is either an SL_2-invariant of degree d in n variables, or it is identically zero.

(ii) Conversely, if $Q(a_0, \dots, a_n)$ is an SL_2-invariant of degree d and order n, then there exist d-regular graphs $g_1, \dots, g_r$ with n vertices and integers $\lambda_1, \dots, \lambda_r$ such that

$$Q = \lambda_1 Q_{g_1} + \cdots + \lambda_r Q_{g_r}.$$

Remark 1. *Recall that a graph is called* d-regular *if each of its vertices has valency d. Observe that if g is an arbitrary graph, then it is natural to interpret its polynomial $Q_g(a_0, \dots, a_n)$ as the SL_2-coinvariant.*

The question about the kernel of the map sending g to $\tilde{g}(\mathbf{x})$ (or to Q_g) was discussed already by J. Petersen, who claimed that he has found a necessary and sufficient condition when g belongs to the kernel; see [9]. This claim turned out to be false. (An interesting correspondence among J. J. Sylvester, D. Hilbert and F. Klein related to this topic can be found in [10].) The kernel of this map seems to be related to several open problems such as the Alon-Tarsi conjecture [3] and the Rota basis conjecture [14]. (We want to thank Professor A. Abdesselam for this valuable information; see [1].)

In the present paper, we are interested in examples of graphs with a symmetrized graph monomial that is nonnegative or a sum of squares. Our interest in this matter has two sources.

The first one is a recent conjecture by F. Sottile and E. Mukhin formulated on the AIM meeting 'Algebraic systems with Only Real Solutions' in October 2010. This conjecture is now settled; see [11, Corollary 14].

Theorem 2. *The discriminant $\mathcal{D}_n$ of the derivative of a polynomial p of degree n is the sum of squares of polynomials in the differences of the roots of p.*

Based on our calculations and computer experiments, we propose the following extension and strengthening Theorem 2. We call an arbitrary graph with all edges of even multiplicity a *square graph.* Observe that the symmetrized graph monomial of a square graph is obviously is a sum of squares.

Conjecture 1. *For every nonnegative integer* $0 \le k \le n-2$, *the discriminant* $\mathcal{D}_{n,k}$ *of the kth derivative of a polynomial p of degree n is a finite positive linear combination of the symmetrized graph monomials, where all underlying graphs are square graphs with n vertices. The vertices* $x_1, \dots, x_n$ *are the roots of p. In other words,* $\mathcal{D}_{n,k}$ *lies in the convex cone spanned by the symmetrized graph monomials of the square graphs with n vertices and* $\binom{n-k}{2}$ *edges.*

Observe that $\deg \mathcal{D}_{n,k} = (n-k)(n-k-1)$ and is therefore even. The following examples support the above conjectures. Below we use the following convention. If a displayed graph has fewer than n vertices, then we always assume that it is appended by the required number of isolated vertices so that there are n vertices altogether.

Example 1. *If* $k = 0$, *then* $\mathcal{D}_{n,0}$ *is proportional to* $\tilde{g}$, *where g is the complete graph on n vertices with all edges of multiplicity* 2.

Example 2. *For* $k \ge 0$, *the discriminant* $\mathcal{D}_{k+2,k}$ *equals*

$$k!(k+1)! \sum_{1 \le i < j \le k+2} (x_i - x_j)^2.$$

In other words, $\mathcal{D}_{k+2,k} = \frac{(k+2)!}{2}\tilde{g}$ where the graph g is given in Figure 1 (appended with k isolated vertices).

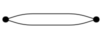

FIGURE 1. The graph g for the case $\mathcal{D}_{k+2,k}$

Example 3. *For* $k \ge 0$, *we conjecture that the discriminant* $\mathcal{D}_{k+3,k}$ *equals*

$$(k!)^3 \left[\frac{(k+1)^3(k+2)(k+6)}{72}\tilde{g_1} + \frac{(k+1)^3 k(k+2)}{12}\tilde{g_2} + \frac{(k-1)k(k+1)^2(k+2)(k-2)}{96}\tilde{g_3} \right],$$

where the graphs g_1, g_2 *and* g_3 *are given in Figure 2. (This claim is verified for* $k = 1, \dots, 12$.*)*

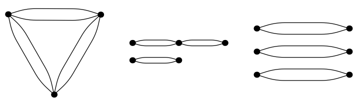

FIGURE 2. The graphs g_1, g_2 and g_3 for the case $\mathcal{D}_{k+3,k}$.

Example 4. *The discriminant* $\mathcal{D}_{5,1}$ *is given by*

$$\mathcal{D}_{5,1} = \frac{19}{6}\tilde{g}_1 + 14\tilde{g}_2 + 2\tilde{g}_3$$

where g_1, g_2, g_3 *are given in Figure 3.*

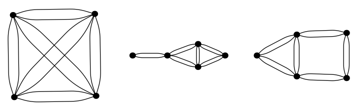

FIGURE 3. The graphs g_1, g_2 and g_3 for the case $\mathcal{D}_{5,1}$.

Example 5. *Finally,*

$$\mathcal{D}_{6,2} = 19200\tilde{g}_1 + 960\tilde{g}_2 + 3480\tilde{g}_3 + 3240\tilde{g}_4 + \frac{3440}{3}\tilde{g}_5 + 2440\tilde{g}_6,$$

where $g_1, \dots, g_6$ *are given in Figure 4. (Note that this representation as sum of graphs is not unique.)*

It is classically known that for any given number n of vertices and d edges, the linear span of the symmetrized graph monomials coming from all graphs with n vertices and d edges coincides with the linear space $\mathbf{PST}_{n,d}$ of all symmetric translation-invariant polynomials of degree d in n variables.

We say that a pair (n, d) is *stable* if $n \geq 2d$. For stable (n, d), we suggests a natural basis in $\mathbf{PST}_{n,d}$ of symmetrized graph monomials that seems to be new; see Proposition 7 and Corollary 4. In the case of even degree, there is a second basis in $\mathbf{PST}_{n,d}$ of symmetrized graph monomials consisting of only square graphs; see Proposition 9 and Corollary 5.

The second motivation of the present study is an interesting example of a graph whose symmetrized graph monomial is nonnegative but not a sum of squares. Namely, the main result of [6] shows that $\tilde{g}$ for the graph given in Figure 5 has this property.

Finally, let us present our main computer-aided results regarding the case of graphs with four and six edges. Observe that there exist 23 graphs

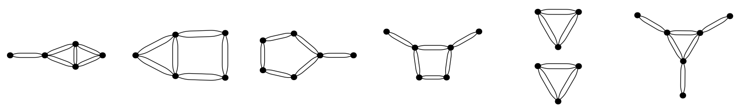

FIGURE 4. The graphs $g_1, \ldots, g_6$ for the case $\mathcal{D}_{6,2}$.

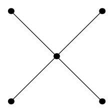

FIGURE 5. The Lax graph, i.e., the only four-edged graph that yields a nonnegative polynomial that is not SOS.

with four edges and 212 graphs with six edges. We say that two graphs are *equivalent* if their symmetrized graph monomials are nonvanishing identically and proportional. Note that two graphs do not need to be isomorphic to be equivalent; see, for example, the equivalence classes in Figure 6.

Proposition 3.

(i) Ten graphs with four edges have identically vanishing symmetrized graph monomial.

(ii) The remaining 13 *graphs are divided into four equivalence classes presented in Figure 6.*

(iii) The first two classes contain square graphs, and thus their symmetrized monomials are nonnegative.

(iv) The third graph is nonnegative (as a positive linear combination of the Lax graph and a polynomial obtained from a square graph). Since it effectively depends only on three variables, it is SOS, see [5].

(v) The last graph is the Lax graph, which is thus the only nonnegative graph with four edges not being a SOS.

Proposition 4.

(i) 102 *graphs with* 6 *edges have identically vanishing symmetrized graph monomial.*

(ii) The remaining 110 *graphs are divided into 27 equivalence classes.*

(iii) 12 *of these classes can be expressed as nonnegative linear combinations of square graphs, i.e., they lie in the convex cone spanned by the square graphs.*

(iv) Of the remaining 15 classes, the symmetrized graph monomial of 7 *of them change sign.*

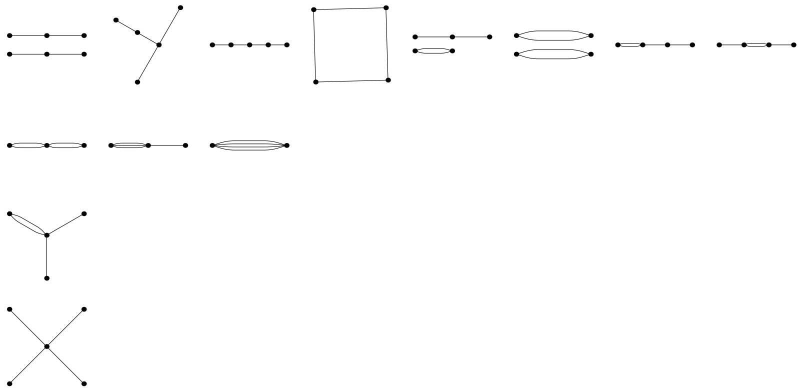

FIGURE 6. Four equivalence classes of the 13 graphs with four edges, whose symmetrized graph monomials do not vanish identically.

(v) *Of the remaining eight classes (which are presented on Figure 7) the first five are sums of squares. (Observe, however, that these symmetrized graph monomials do not lie in the convex cone spanned by the square graphs.)*

(vi) *The last three classes contain all nonnegative graphs with six edges, which are not SOS and therefore, give new examples of graphs á la Lax.*

Proving Proposition 3 is simply a matter of straightforward computation. Cases (i)-(iv) in Proposition 4 also follows from a longer calculation, by examining each of the 212 graphs. Proofs of case (v) requires the notion of certificates.

It is well known that a polynomial is a sum of squares if and only if it can be represented as vQv^T, where Q is positive semidefinite and v is a monomial vector. Such a representation is called a *certificate.* Certificates for the eight classes in Proposition 4, case (v), in the form of positive semidefinite matrices and corresponding monomial vectors can be found in [2].

The simplest certificate, for the third class, is given by the vector

$$v_3 = \{x_3x_4^2, x_3^2x_4, x_2x_4^2, x_2x_3x_4, x_2x_3^2, x_2^2x_4, x_2^2x_3, x_1x_4^2, x_1x_3x_4, x_1x_3^2, x_1x_2x_4, x_1x_2x_3, x_1x_2^2, x_1^2x_4, x_1^2x_3, x_1^2x_2\}$$

together with the positive semidefinite matrix Q_3, shown as Figure 8.

Case (vi) was analyzed with the Yalmip software, which provides a second kind of certificate that shows that the last three classes are not SOS.

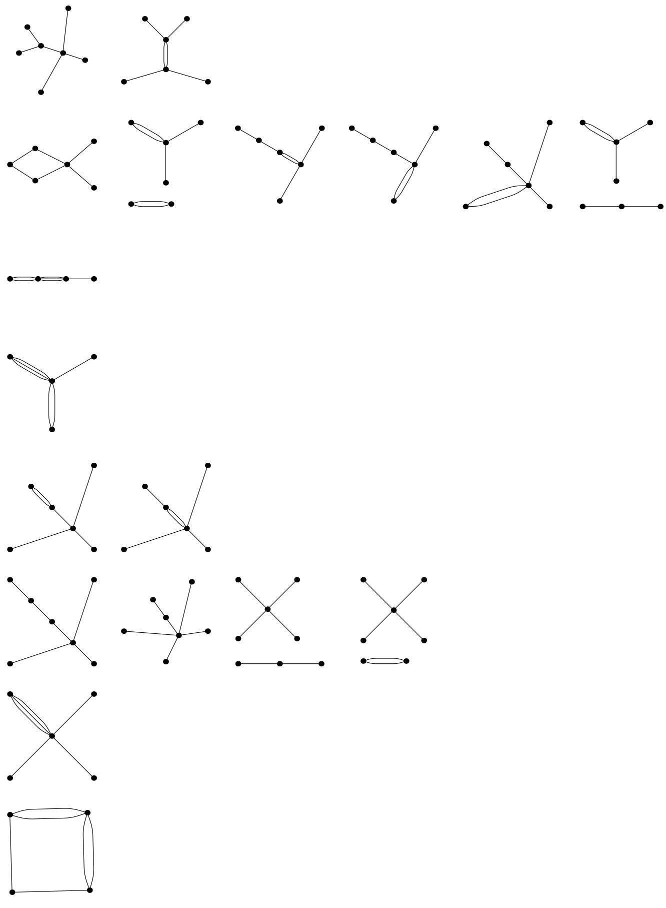

FIGURE 7. Eight equivalence classes of all nonnegative graphs with six edges.

Finally, notice that translation-invariant symmetric polynomials appeared also in the early 1970s in the study of integrable N-body problems in mathematical physics, particularly in the famous paper [4]. A few much more recent publications related to the ring of such polynomials in connection with the investigation of multiparticle interactions and the quantum Hall effect have been published since then; see e.g., [12], [7]. In particular, the ring structure and the dimensions of the homogeneous components of this ring were calculated. It was shown [12, Section IV] and [7] that the ring of translation invariant symmetric polynomials (with integer coefficients) in $x_1, \dots, x_n$ is isomorphic, as a graded ring to the polynomial ring $\mathbb{Z}[e_2, \dots, e_n]$ where e_i stands for the ith elementary symmetric function in $x_1 - x_{avg}, \dots, x_n - x_{avg}$ with $x_{avg} = \frac{1}{n}(x_1 + \cdots + x_n)$.

$$\begin{pmatrix}
10 & -6 & -5 & -4 & 3 & 3 & -1 & -5 & -4 & 3 & 8 & 0 & -2 & 3 & -1 & -2 \\
-6 & 10 & 3 & -4 & -5 & -1 & 3 & 3 & -4 & -5 & 0 & 8 & -2 & -1 & 3 & -2 \\
-5 & 3 & 10 & -4 & -1 & -6 & 3 & -5 & 8 & -2 & -4 & 0 & 3 & 3 & -2 & -1 \\
-4 & -4 & -4 & 24 & -4 & -4 & -4 & 8 & -8 & 8 & -8 & -8 & 8 & 0 & 0 & 0 \\
3 & -5 & -1 & -4 & 10 & 3 & -6 & -2 & 8 & -5 & 0 & -4 & 3 & -2 & 3 & -1 \\
3 & -1 & -6 & -4 & 3 & 10 & -5 & 3 & 0 & -2 & -4 & 8 & -5 & -1 & -2 & 3 \\
-1 & 3 & 3 & -4 & -6 & -5 & 10 & -2 & 0 & 3 & 8 & -4 & -5 & -2 & -1 & 3 \\
-5 & 3 & -5 & 8 & -2 & 3 & -2 & 10 & -4 & -1 & -4 & 0 & -1 & -6 & 3 & 3 \\
-4 & -4 & 8 & -8 & 8 & 0 & 0 & -4 & 24 & -4 & -8 & -8 & 0 & -4 & -4 & 8 \\
3 & -5 & -2 & 8 & -5 & -2 & 3 & -1 & -4 & 10 & 0 & -4 & -1 & 3 & -6 & 3 \\
8 & 0 & -4 & -8 & 0 & -4 & 8 & -4 & -8 & 0 & 24 & -8 & -4 & -4 & 8 & -4 \\
0 & 8 & 0 & -8 & -4 & 8 & -4 & 0 & -8 & -4 & -8 & 24 & -4 & 8 & -4 & -4 \\
-2 & -2 & 3 & 8 & 3 & -5 & -5 & -1 & 0 & -1 & -4 & -4 & 10 & 3 & 3 & -6 \\
3 & -1 & 3 & 0 & -2 & -1 & -2 & -6 & -4 & 3 & -4 & 8 & 3 & 10 & -5 & -5 \\
-1 & 3 & -2 & 0 & 3 & -2 & -1 & 3 & -4 & -6 & 8 & -4 & 3 & -5 & 10 & -5 \\
-2 & -2 & -1 & 0 & -1 & 3 & 3 & 3 & 8 & 3 & -4 & -4 & -6 & -5 & -5 & 10
\end{pmatrix}$$

FIGURE 8. The positive semidefinite matrix Q_3.

From this fact one can easily show that the dimension of its dth homogeneous component equals the number of distinct partitions of d in which each part is strictly bigger than 1 and the number of parts is at most n. Several natural linear bases have also been suggested for each such homogeneous component, see [12, (29)] and [7]. It seems that the authors of the latter papers were unaware of the mathematical developments in this field related to graphs.

2. Some generalities on symmetrized graph monomials

We begin with a few definitions.

Definition 2. *An* integer partition *of d is a d-tuple $(\alpha_1, \dots, \alpha_d)$ such that $\sum_i \alpha_i = d$ and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_d \geq 0$.*

Definition 3. *Let g be a directed graph with d edges and n vertices $v_1, \dots, v_n$. Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be an integer partition of d. A* partition-coloring *of g with α is an assignment of colors to the edges and vertices of g satisfying the following conditions:*

- *For each color i, $1 \leq i \leq d$, we paint with the color i some vertex v_j and exactly α_i edges connected to v_j.*
- *Each vertex is painted exactly once.*

An edge is called odd-colored *if it is directed toward a vertex with the same color. The coloring is said to be* negative *if there is an odd number of odd-colored edges in g, and* positive *otherwise.*

Definition 4. *Given a polynomial $P(\mathbf{x})$ and a multi-index α, we use the notation $Coeff_\alpha(P(\mathbf{x}))$ to denote the coefficient of $\mathbf{x}^\alpha$ in $P(\mathbf{x})$.*

Note that we may view α as a partition of the sum of the exponents.

Lemma 5. *Let g be a directed graph with d edges and vertices $v_1, \dots, v_n$. Then $Coeff_\alpha(\tilde{g})$ is given by the difference of positive and negative partition-colorings of g with α.*

Proof. See [9, Lemma 2.3]. □

2.1. **Bases for $\mathbf{PST}_{n,d}$.** It is known that the dimension of $\mathbf{PST}_{n,d}$ with $n \geq 2d$ is given by the number of integer partitions of d in which each nonzero part is of at least size 2; see [7]. Such integer partition will be called a 2-*partition*.

To each 2-partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$, $\alpha_i \neq 1$, we associate the graph b_α consisting of a root vertex, connected to α_i other vertices, with the edges directed away from the root vertex. Since α is an integer partition of d, it follows that b_α has exactly d edges. This type of graph will be called a *partition graph*.

The dimension of $\mathbf{PST}_{n,d}$ is independent of n (as long as $n \geq 2d$), and we deal only with homogeneous symmetric polynomials of degree d. Thus, each monomial is essentially determined only by the way the powers of the variables are partitioned. The variables themselves become unimportant, since every permutation of the variables is present. For example, the monomials x^3zw and xy^3w are always present simultaneously with the same coefficient, while x^3z^3 is different from the previous two.

2.2. **Partition graphs.**

Definition 5. *Let $P(\mathbf{x})$ be a polynomial in $|\mathbf{x}|$ variables. We use the following notation*

$$Sym_{(\mathbf{x},\mathbf{y})}P = \sum_{(\tau_1,\tau_2,\dots,\tau_n)\subseteq \mathbf{x}\cup\mathbf{y}} P(\tau_1, \tau_2, \dots, \tau_n),$$

where we sum over all possible permutations *and choices of n variables among the $|\mathbf{x}| + |\mathbf{y}|$ variables.*

The following is obviously true.

Lemma 6. *Let $P(\mathbf{x})$ be a polynomial. Then*

$$Sym_{(\mathbf{x},\mathbf{y})}P = \sum_{i=0}^{|\mathbf{y}|} \sum_{\substack{\sigma\subseteq\mathbf{y}\\ |\sigma|=i}} \sum_{\substack{\tau\subseteq\mathbf{x}\\ |\tau|=|\mathbf{x}|-i}} Sym_{(\tau\cup\sigma)}P.$$

Here, the two inner sums denote choices of all subsets *of a certain size.*

Corollary 1. *If $Sym_{\mathbf{x}}P$ is non-negative, then $Sym_{(\mathbf{x},\mathbf{y})}P$ is nonnegative.*

Corollary 2. *If $Sym_{\mathbf{x}}P$ is a sum of squares, then $Sym_{(\mathbf{x},\mathbf{y})}P$ is a sum of squares.*

Corollary 3. *If $\sum_i \lambda_i Sym_{\mathbf{x}} P_i = 0$ then we have $\sum_i \lambda_i Sym_{(\mathbf{x},\mathbf{y})} P_i = 0$.*

We will use the notation that every symmetric polynomial $\tilde{g}$ associated with a graph on d edges is symmetrized over $2d$ variables. Corollary 3 says that if a relation holds for symmetrizations in $2d$ variables, it will also hold for $2d+k$ variables ($k \geq 0$). Therefore, each relation derived in this section also holds for $2d+k$ variables.

Proposition 7. *Let b_α be a partition graph with d edges, $\alpha = (\alpha_1, \dots, \alpha_d)$, and let $\beta = (\beta_1, \beta_2, \dots, \beta_d)$ be a 2-partition.*

Then

$$Coeff_\beta(\tilde{b_\alpha}) = \begin{cases} 0 & \text{if } \beta \neq \alpha \\ \prod_{j=2}^{d} \#\{i|\alpha_i = j\}! & \text{if } \beta = \alpha. \end{cases}$$

Proof. We will try to color the graph b_α with β. Since $\beta_i \neq 1$, we may only color the roots of b_α. Hence, all edges in each component of b_α must have the same color as the corresponding root. It is clear that such coloring is impossible if $\alpha \neq \beta$. If $\alpha = \beta$, we see that each coloring has positive sign, since only roots are colored and all connected edges are directed outward.

The only difference between two colorings must be the assignment of the colors to the roots. Hence, components with the same size can permute colors, which yields

$$\prod_{j=2}^{d} \#\{i|\alpha_i = j\}!$$

ways to color g with the partition $(\alpha_1, \dots, \alpha_d)$. □

Corollary 4. *All partition graphs yield linearly independent polynomials, since each partition graph b_α contributes with the unique monomial $\mathbf{x}^\alpha$. The number of partition graphs on d edges equals the dimension of $\mathbf{PST}_{d,n}$, and therefore, when $n \geq 2d$, they must span the entire vector space.*

2.3. **Square graphs.** We will use the notation

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k | \alpha_{k+1}, \alpha_{k+1}, \dots, \alpha_d)$$

to denote a partition where $\alpha_1, \dots, \alpha_k$ are the odd parts in nonincreasing order, and $\alpha_{k+1}, \dots, \alpha_d$ are the even parts in nonincreasing order. (Note that this convention differs from the standard one for partitions.) As before, parts are allowed to be equal to zero, so that α can be used as a multi-index over d variables.

Now we define a second type of graphs, which we associate with 2-paritions of even integers: Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k | \alpha_{k+1}, \dots, \alpha_d)$, $\alpha_i \neq 1$, be a 2-partition of d. Since this is a partition of an even integer, k must be even.

For each even $\alpha_i \geq 2$, we have a connected component of h_α consisting of a root, connected to $\alpha_i/2$ other vertices, with the edges directed away from the root, and with multiplicity 2.

For each pair $\alpha_{2j-1}, \alpha_{2j}$ of odd parts, $j = 1, 2, \dots, k/2$, we have a connected component consisting of two roots v_{2j-1} and v_{2j}, such that v_i is connected to $\lfloor \alpha_i/2 \rfloor$ other vertices for $i = 2j-1, 2j$, with edges of multiplicity 2 and the roots are connected with a double edge. This type of component will be called a *glued component.*

Thus, each edge in h_α has multiplicity 2, and the number of edges, counting multiplicity, is d. This type of graph will be called a *partition square graph.* Note that all edged have even multiplicity, so $\tilde{h}_\alpha(\mathbf{x})$ is a sum of squares.

Lemma 8. *Let h_α be a partition square graph such that $\alpha = (\alpha_1, \dots, \alpha_d)$. Then*

$$Coeff_\alpha(\tilde{h}) = (-1)^{\frac{1}{2}\#\{i|\alpha_i \equiv_2 1\}} 2^{\#\{i|\alpha_i = 2\}} \prod_{j=0}^{n} \#\{i|\alpha_i = j\}!.$$

Proof. Similarly to Proposition 7, it is clear that a coloring of h with d colors require that each root must be colored.

The root of a component with only two vertices is not uniquely determined, so we have $2^{\#\{i|\alpha_i=2\}}$ choices of the root.

It is clear that each glued component contributes exactly one odd edge for every coloring, and therefore the sign sign is the same for each coloring. The number of glued components is precisely $\frac{1}{2}\#\{i|\alpha_i \equiv_2 1\}$.

Lastly, we may permute the colors corresponding to roots with the same degree. These observations together yields the formula

$$(-1)^{\frac{1}{2}\#\{i|\alpha_i \equiv_2 1\}} 2^{\#\{i|\alpha_i = 2\}} \prod_{j=0}^{n} \#\{i|\alpha_i = j\}!,$$

which completes the proof. ∎

Define a total order on 2-partitions as follows:

Definition 6. *Let*

$$\alpha = (\alpha_1, \dots, \alpha_k | \alpha_{k+1}, \dots, \alpha_d) \text{ and } \alpha' = (\alpha'_1, \dots, \alpha'_{k'} | \alpha'_{k'+1}, \dots, \alpha_{k'})$$

be 2-partitions. We say that $\alpha \prec \alpha'$ if $\alpha_i = \alpha'_i$ for $i = 1, \dots, j-1$, $j \geq 1$ and one of the following holds:

- $\alpha_j > \alpha'_j$ *and* $\alpha_j = \alpha'_j \mod 2$*;*
- α_j *is odd and* α'_j *is even.*

Proposition 9. *Let* h_α *be a square graph. Then we may write*

$$\tilde{h_\alpha} = \sum_\beta \lambda_\beta \tilde{b}_\beta, \quad b_\beta \text{ is a partition graph,} \tag{1}$$

where $\lambda_\beta = 0$ *if* $\beta \prec \alpha$*.*

Proof. Let α be the partition $(\alpha_1, \dots, \alpha_q | \alpha_{q+1} \dots, \alpha_d)$ and let $\beta = (\beta_1, \dots, \beta_r | \beta_{r+1}, \dots, \beta_d)$, with $\beta \prec \alpha$. Consider equation (1) and apply $Coeff_\beta$ to both sides. Proposition 7 implies

$$Coeff_\beta(\tilde{h}_\alpha) = \lambda_\beta \cdot C_\beta, \text{ where } C_\beta > 0.$$

It suffices to show that there is no partition-coloring of h_α with β if $\beta \prec \alpha$, since this implies $\lambda_\beta = 0$.

We now have three cases to consider:

Cases 1 and 2: $\alpha_i = \beta_i$ for $i = 1, \dots, j-1$ and $\beta_j > \alpha_j$, where α_j and β_j are either both odd or both even. We must color a root and β_j connected edges, since $\beta_j > \alpha_j \geq 2$. There is no vacant root in g_α with degree at least β_j, all such roots having already been colored with the colors $1, \dots, j-1$. Hence a coloring is impossible in this case.

Case 3: $\alpha_i = \beta_i$ for $i = 1, \dots, j-1$, β_j is odd and α_j is even. This condition implies that $q < r$.

Every component of h_α has an even number of edges, and only vertices with degree at least three can be colored with an odd color. Therefore, glued components must be colored with exactly zero or two odd colors, and nonglued component must have an even number of edges of each present color. This implies that a coloring is possible only if $r \leq q$, a contradiction.

Hence, there is no coloring of h_k with the colors given by β, and therefore, $Coeff_\beta(\tilde{h}_\alpha) = 0$, implying $\lambda_\beta = 0$. □

Corollary 5. *The polynomials obtained from the partition square graphs with* d *edges is a basis for* $\mathbf{PST}_{d,n}$*, for even* d*.*

Proof. Let $\alpha_1 \prec \dots \prec \alpha_k$ be the 2-partitions of d. Since $\tilde{b}_{\alpha_1}, \dots, \tilde{b}_{\alpha_k}$ is a basis, there is a uniquely determined matrix M such that

$$(\tilde{h}_{\alpha_1}, \dots, \tilde{h}_{\alpha_k})^T = M(\tilde{b}_{\alpha_1}, \dots, \tilde{b}_{\alpha_k})^T.$$

Proposition 9 implies that M is lower-triangular. Proposition 7 and Lemma 8 imply that the entry at (α_i, α_i) in M is given by

$$(-1)^{\frac{1}{2}\#\{j|\alpha_{ij} \equiv_2 1\}} 2^{\#\{j|\alpha_{ij}=2\}},$$

which is nonzero. Hence M has an inverse, and the partition square graphs form a basis. See Figure 9 for an example of the two sets of bases for the case $d = 6$. □

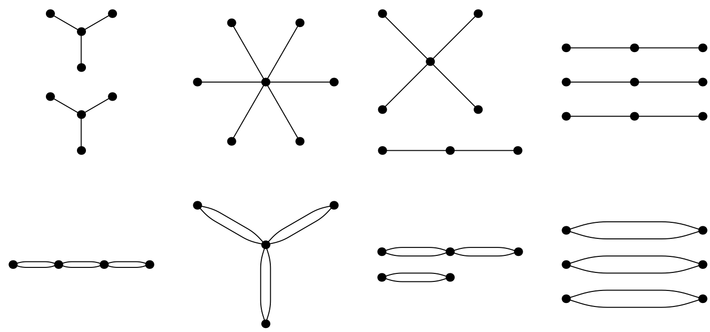

FIGURE 9. A base of partition graphs and a base of partition square graphs in the stable case with six edges.

3. Final remarks

Some obvious challenges related to this project are as follows.

(1) Prove Conjecture 1.
(2) Describe the boundary of the convex cone spanned by all square graphs with a given number of (double) edges and vertices.
(3) Find more examples of graphs á la Lax.

4. Acknowledgements

We are sincerely grateful to Professors A. Abdesselam, F. Sottile and B. Sturmfels for discussions and important references and to Dr. P. Rostalski for his assistance with our computer-aided proof of the fact that certain symmetrized graph monomials are SOS.

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SCHUR POLYNOMIALS, BANDED TOEPLITZ MATRICES AND WIDOM'S FORMULA

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ABSTRACT. We prove that for arbitrary partitions $\boldsymbol{\lambda} \subseteq \boldsymbol{\kappa}$, and integers $0 \leq c < r \leq n$, the sequence of Schur polynomials

$$S_{(\boldsymbol{\kappa}+k\cdot\mathbf{1}^c)/(\boldsymbol{\lambda}+k\cdot\mathbf{1}^r)}(x_1,\dots,x_n)$$

for k sufficiently large, satisfy a linear recurrence. The roots of the characteristic equation are given explicitly. These recurrences are also valid for certain sequences of minors of banded Toeplitz matrices.

In addition, we show that Widom's determinant formula from 1958 is a special case of a well-known identity for Schur polynomials.

1. INTRODUCTION

1.1. Minors of banded Toeplitz matrices. Fix a positive integer n and a finite sequence $s_0, s_1, \dots, s_n$ of complex numbers. Define an infinite banded Toeplitz matrix A by the formula

$$(1)\qquad A := (s_{j-i}),\ 1 \leq i < \infty, 1 \leq j < \infty \text{ with } s_i := 0 \text{ for } i > n, i < 0.$$

Given an increasing $r-$tuple $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r)$ and an increasing $c-$tuple $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_c)$ of positive integers with $r \leq c \leq n$, define $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ as the $k \times k-$matrix obtained by first removing rows indexed by $\{\alpha_i\}_{i=1}^r$ and columns indexed by $\{\beta_i\}_{i=1}^c$ from A and then selecting the leading $k \times k-$sub-matrix. In particular, we let D^k_c to be $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for $\boldsymbol{\alpha} = \emptyset, \boldsymbol{\beta} = (1, 2, \dots, c)$. We will also require $s_0 = 1$ which is a natural assumption[1].

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[1]If $s_0 = 0$, the first column of $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ will consist of zeros, unless $\beta_1 = 1$. In the first case, $\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is therefore 0 for every $k > 0$ and uninteresting. In the latter case, we may just as well use the sequence $s_1, s_2, \dots, s_n$ and decrease all entries in $\boldsymbol{\beta}$ by 1 and obtain the exact same sequence. Thus, there is no loss of generality if we assume $s_0 \neq 0$. Furthermore, we are interested on the determinants of $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, so assuming $s_0 := 1$ is not a big restriction and the general case can easily be recovered.

A great deal of research has been focused on the asymptotic eigenvalue distribution of D_c^k as $k \to \infty$, the most important are the Szegö limit theorem from 1915, and the strong Szegö limit theorem from 1952.

There are many ways to generalize the strong Szegö limit theorem, for example, the Fisher-Hartwig conjecture from 1968. Some cases of the conjecture have been promoted to a theorem, based on the works of many people the last 20 years, including Widom, Basor, Silberman, Böttcher and Tracy. A possible refinement of the conjecture is the Basor-Tracy conjecture, [7, 2], which recently has been proved in the general case, see [9].

Asymptotics of Toeplitz determinants arises naturally in many areas; Szegö himself considered the two-dimensional Ising model. For a more recent application in combinatorics, see [1], where the length of the longest increasing subsequence in a random permutation is studied.

A classic result in the theory of banded Toeplitz matrices was obtained by H. Widom [16]. We use $[n]$ to denote the set $\{1, 2, \dots, n\}$ and the symbol $\binom{[n]}{c}$ as the set of subsets of $[n]$ with cardinality c. In a modern setting, Widom's formula may then be formulated as follows:

Theorem 1. *(Widom's determinant formula,* [6]*) Let $\psi(t) := \sum_{i=0}^{n} s_i t^i$. If the zeros $t_1, t_2, \dots, t_n$ of $\psi(t) = 0$ are distinct then, for every $k \geq 1$,*

$$\det D_c^k = \sum_{\sigma} C_\sigma w_\sigma^k, \quad \sigma \in \binom{[n]}{n-c} \tag{2}$$

where

$$w_\sigma := (-1)^{n-c} s_n \prod_{i \in \sigma} t_i \text{ and } C_\sigma := \prod_{i \in \sigma} t_i^c \prod_{\substack{j \in \sigma \\ i \notin \sigma}} (t_j - t_i)^{-1}.$$

In 1960, by using Widom's formula, P. Schmidt and F. Spitzer gave a description of the limit set of the eigenvalues of D_c^k as $k \to \infty$. In the above notation, part of their theorem can be stated as follows:

Theorem 2. *(P. Schmidt, F. Spitzer,* [15]*)*

Let I_k denote the $k \times k$-identity matrix and define

$$B = \left\{ v \middle| v = \lim_{k \to \infty} v_k, \det(D_c^k - v_k I_k) = 0 \right\},$$

that is, B is the set of limit points of eigenvalues of $\{D_c^k\}_{k=0}^{\infty}$. Let

$$f(z) = \sum_{i=0}^{n} s_i z^{i-c} \text{ and } Q(v, z) = z^c (f(z) - v).$$

Order the moduli of the zeros, $\rho_i(v)$, of $Q(v, z)$ in increasing order,

$$0 < \rho_1(v) \leq \rho_2(v) \leq \dots \leq \rho_n(v),$$

with possible duplicates counted several times, according to multiplicity. Let $C = \{v|\rho_c(v) = \rho_{c+1}(v)\}$ *. Then,* $B = C$*.*

The Laurent polynomial $f(z)$ is called the *symbol* associated with the Toeplitz matrix D_c^k, and it is an important tool[2] for studying asymptotics.

More recently, a newer approach using the theory of Schur polynomials has been successfully used to further investigate the series $\{\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}\}_{k=1}^{\infty}$, e.g. [8]. For a recent application of Schur functions in the non-banded case, see [5].

There is also a connection between multivariate orthogonal polynomials and certain determinants of $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, considered as functions of $(s_0, s_1, \dots, s_n)$. The solution set to a system of polynomial equations obtained from some $\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ converges to the measure of orthogonality as $k \to \infty$. For example, in 1980, a bivariate generalization of Chebyshev polynomials was constructed by K. B. Dunn and R. Lidl. Some more recent applications of the theory of symmetric functions are [3, 11], where use of Schur polynomials and representation theory gives multivariate Chebyshev polynomials. These multivariate Chebyshev polynomials are also minors of certain Toeplitz matrices.

For example, if $n = 2$ and $P_j(s_1, s_2) := \det D_1^j$, we have that

$$T_j(x) = P_j(x - \sqrt{x^2-1}, x + \sqrt{x^2-1}) = S_{(j)}(x - \sqrt{x^2-1}, x + \sqrt{x^2-1}),$$

where $T_j(x)$ is the jth Chebyshev polynomial of the second kind, and $S_{(j)}$ is the Schur polynomial for the partition with one part of size j, in two variables.

However, the close connection between multivariate Chebyshev polynomials and Schur polynomials (and thus minors of banded Toeplitz matrices) has not yet been sufficiently investigated.

1.2. **Main results.** We start with giving a Schur polynomial interpretation of $\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

Set $s_i := s_i(x_1, x_2, \dots, x_n)$ where s_i is the i :th elementary symmetric polynomial. We impose a natural[3] restriction on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, namely $\alpha_i \geq \beta_i$ for $i = 1, 2, \dots, r$.

Proposition 3. *In the above notation, for k sufficiently large, we have*

$$\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} = S_{(\boldsymbol{\lambda}+k\boldsymbol{\mu})/(\boldsymbol{\kappa}+k\boldsymbol{\nu})}(x_1, x_2, \dots, x_n), \tag{3}$$

[2]Note that f has a close resemblance with ψ in Widom's formula.

[3]This ensures that no leading matrix of $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is upper-triangular with a zero on the main diagonal, which would force $\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to vanish.

where $S_{(\boldsymbol{\lambda}+k\boldsymbol{\mu})/(\boldsymbol{\kappa}+k\boldsymbol{\nu})}$ *is a skew Schur polynomial defined below. Here* $\boldsymbol{\lambda}, \boldsymbol{\kappa}, \boldsymbol{\mu}, \boldsymbol{\nu}$ *are partitions given by*

$$\boldsymbol{\lambda} = (1-\beta_1, 2-\beta_2, \dots, c-\beta_c), \quad \boldsymbol{\kappa} = (1-\alpha_1, 2-\alpha_2, \dots, r-\alpha_r)$$
$$\boldsymbol{\mu} = (\underbrace{1,1,\dots,1}_{c}), \quad \boldsymbol{\nu} = (\underbrace{1,1,\dots,1}_{r}).$$

The conditions on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ ensure that $S_{(\boldsymbol{\lambda}+k\boldsymbol{\mu})/(\boldsymbol{\kappa}+k\boldsymbol{\nu})}$ is well-defined for $k \geq \max(\alpha_r - r, \beta_c - c)$. (Identity (3) is proven below in Proposition 10, a similar identity is proven in [8].)

To state our main first result, we need to define the following. Set $b := \binom{n}{c-r}$ and define the finite sequence of polynomials $\{Q_i(x_1,\dots,x_n)\}_{i=0}^{b}$ by the identity

$$\sum_{k=0}^{b} Q_{b-k} t^k = \prod_{\substack{\sigma \subseteq [n] \\ |\sigma| = c-r}} (t - x_{\sigma_1} x_{\sigma_2} \cdots x_{\sigma_{c-r}}). \tag{4}$$

Theorem 4. *Given strictly increasing sequences* $\boldsymbol{\alpha}, \boldsymbol{\beta}$ *of positive integers of length* r *resp.* c *with* $c \leq r$*, satisfying* $\alpha_i \geq \beta_i$ *for* $i = 1, 2, \dots, r$*, we have*

$$\sum_{k=0}^{b} Q_{b-k} \det(D^{k+j}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) = 0 \quad \textit{for all } j \geq \max(\alpha_c - c, \beta_r - r). \tag{5}$$

(Here, we use the convention that the determinant of an empty matrix is 1.)

Remark 5. For the case D^k_c, the existence of recurrence (5) was previously shown in [14, Theorem 2], but its length and coefficients were not given explicitly. Also, Theorem 4 has close resemblance to a result given in [12, Theorem 5.1]. It is however unclear whether [12] implies Theorem 4. Additionally, in contrast to [12], our proof of Theorem 4 is short and purely combinatorial.

To formulate the second result, define

$$\chi(t) = \prod_{i=1}^{n}(t - x_i) = (-1)^n \sum_{i=0}^{n} (-t)^{n-i} s_i(x_1, x_2, \dots, x_n). \tag{6}$$

We then have the following theorem, which is equivalent to Widom's formula:

Theorem 6. *(Modified Widom's formula)*

If the zeros $x_1, x_2, \dots, x_n$ *of* $\chi(t) = 0$ *are distinct then, for every* $k \geq 1$,

$$\det D^k_c = \sum_{\tau} \prod_{i \in \tau} x_i^k \prod_{\substack{i \in \tau \\ j \notin \tau}} \frac{x_i}{x_i - x_j}, \quad \tau \in \binom{[n]}{c}$$

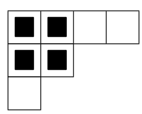
Remark 7. Below, we show that this (and therefore Widom's original formula) follows immediately from a known identity for the Hall polynomials.

Note that Theorem 4 can be verified easily using Widom's original formula. I was informed that there is an unpublished result by S. Delvaux and A. L. García which uses a Widom-type formula for block Toeplitz matrices to give recurrences similar to (5).

2. Preliminaries

Given two integer partitions $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$, we say that $\boldsymbol{\lambda} \supseteq \boldsymbol{\mu}$ if $\lambda_j \geq \mu_j$ for $j = 1, \ldots, n$. Given two such partitions, one constructs the associated *skew Young diagram*[4] by having n left-adjusted rows of boxes, where row j contains λ_j boxes, and then removing the first μ_j boxes from row j. The removed boxes is called the *skew part* of the tableau.

Example 8. The following diagram is obtained from the partitions $(4, 2, 1)$ and $(2, 2)$, and it is said to be of the *shape* $(4, 2, 1)/(2, 2)$:



(We will omit/add trailing zeros in partitions when the intended length is known from the context.)

The *conjugate* of a partition is the partition obtained by transposing the corresponding tableau. For example, the conjugate of $(4, 2, 1)/(2, 2)$ is $(3, 2, 1, 1)/(2, 2)$.

Given such a diagram, a *(skew) semi-standard Young tableau* (we will sometimes use just the word tableau from now on) is an assignment of positive integers to the boxes, such that each row is weakly increasing, and each column is strictly increasing.

We define the *(skew) Schur polynomial* $S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, x_2, \ldots, x_n)$ as

$$S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, x_2, \ldots, x_n) = \sum x_1^{h_1} \cdots x_n^{h_n} \tag{7}$$

where the sum is taken over all tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$, and h_j counts the number of boxes containing j for each particular tableau. No box may contain an integer greater than n. When $\boldsymbol{\mu} = (0, 0, \ldots, 0)$ we just write $S_{\boldsymbol{\lambda}}$. To clarify, each Schur polynomial is associated with a Young diagram, and each monomial in such polynomial corresponds to a set

[4] In the case $\boldsymbol{\mu} = (0, 0, \ldots, 0)$, the word *skew* is to be omitted.

of tableaux. We use this correspondence extensively. For example, the tableau above yields the Schur polynomial

$$x_1^3 + x_2^3 + x_3^3 + 2(x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2) + 3x_1x_2x_3.$$

The following formula express the (skew) Schur polynomials in a determinant form:

Proposition 9. *(Jacobi-Trudi identity* [13]*)*
Let $\boldsymbol{\lambda} \supseteq \boldsymbol{\mu}$ be partitions with at most n parts and let $\boldsymbol{\lambda}', \boldsymbol{\mu}'$ be their conjugate partitions (with at most k parts). Then the (skew) Schur polynomial $S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ is given by

$$S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, x_2, \dots, x_n) = \begin{vmatrix} s_{\lambda'_1-\mu'_1} & s_{\lambda'_1-\mu'_1+1} & \cdots & s_{\lambda'_1-\mu'_1+k-1} \\ s_{\lambda'_2-\mu'_2-1} & s_{\lambda'_2-\mu'_2} & \cdots & s_{\lambda'_2-\mu'_2+k-2} \\ \vdots & & \ddots & \vdots \\ s_{\lambda'_k-\mu'_k-k+1} & \cdots & & s_{\lambda'_k-\mu'_k} \end{vmatrix}$$

where $s_j := s_j(x_1, \dots, x_n)$, the elementary symmetric functions in the variables $x_1, \dots, x_n$. Here, $s_j \equiv 0$ for $j < 0$.

It is clear that every (skew) Schur polynomial $S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \dots, x_n)$ is symmetric in $x_1, \dots, x_n$.

3. Proofs

The following proposition shows that certain minors of banded Toeplitz matrices may be interpreted as Schur polynomials.

Proposition 10. *Let $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be defined as above. Then,*

$$\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} = S_{(\boldsymbol{\lambda}+k\boldsymbol{\mu})/(\boldsymbol{\kappa}+k\boldsymbol{\nu})}(x_1, x_2, \dots, x_n)$$

where

$$\boldsymbol{\lambda} = (1-\beta_1, 2-\beta_2, \dots, c-\beta_c), \quad \boldsymbol{\kappa} = (1-\alpha_1, 2-\alpha_2, \dots, r-\alpha_r)$$

and

$$\boldsymbol{\mu} = (\underbrace{1, 1, \dots, 1}_{c}), \quad \boldsymbol{\nu} = (\underbrace{1, 1, \dots, 1}_{r}).$$

Proof. Consider the matrix A defined in (1), where the indices (of s) on the main diagonal are all 0. Now, removing the rows $\boldsymbol{\alpha}$ will *decrease* the index on row i by $\#\{j|\alpha_j - j + 1 \leq i\}$. Similarly, removing the columns $\boldsymbol{\beta}$ will *increase* the index in column i by $\#\{j|\beta_j - j + 1 \leq i\}$. *After* removing rows and columns, the diagonal of the resulting matrix, $\tilde{A}$, is given by

$$\left(\#\{j|\beta_j - j + 1 \leq i\} - \#\{j|\alpha_j - j + 1 \leq i\}\right)_{i=1}^{\infty}.$$

Now, the leading $k \times k$ minor of $\tilde{A}$ is $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ and its anti-diagonal transpose has the same determinant as $D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$. The main diagonal in the anti-diagonal transposed matrix equals

(8)
$$(\#\{j|\beta_j - j + 1 \leq k - i + 1\} - \#\{j|\alpha_j - j + 1 \leq k - i + 1\})_{i=1}^k =$$
$$(\#\{j|\beta_j \leq k + j - i\} - \#\{j|\alpha_j \leq k + j - i\})_{i=1}^k$$

Now, well-known properties of partition conjugation imply that the partition $(\boldsymbol{\lambda} + k\boldsymbol{\mu})'$ equals

$$(\#\{j|k + j - \beta_j \geq 1\}, \#\{j|k + j - \beta_j \geq 2\}, \ldots, \#\{j|k + j - \beta_j \geq k\}),$$

and $(\boldsymbol{\kappa} + k\boldsymbol{\nu})'$ is given by

$$(\#\{j|k + j - \alpha_j \geq 1\}, \#\{j|k + j - \alpha_j \geq 2\}, \ldots, \#\{j|k + j - \alpha_j \geq k\}).$$

Rewriting this we obtain

$$(\boldsymbol{\lambda}+k\boldsymbol{\mu})' = (\#\{j|\beta_j \leq k+j-i\})_{i=1}^k, \ (\boldsymbol{\kappa}+k\boldsymbol{\nu})' = (\#\{j|\alpha_j \leq k+j-i\})_{i=1}^k.$$

Finally, using $(\boldsymbol{\kappa}+k\boldsymbol{\nu})/(\boldsymbol{\lambda}+k\boldsymbol{\mu})$ in the Jacobi-Trudy identity, Proposition 9, yields a $k \times k-$matrix with diagonal entries

$$(\boldsymbol{\lambda} + k\boldsymbol{\mu})' - (\boldsymbol{\kappa} + k\boldsymbol{\nu})' = (\#\{j|\beta_j \leq k + j - i\} - \#\{j|\alpha_j \leq k + j - i\})_{i=1}^k.$$

This expression coincides with the expression for $\det D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in (8), and now it is straightforward to see that all other matrix entries coincides as well. □

3.1. **Young tableaux and sequence insertion.** To prove Theorem 4, we need to define a new combinatorial operation on semi-standard skew Young tableaux. Namely, given a tableau T with n rows, we define an insertion of a sequence $\mathbf{t} = t_1 < t_2 < \cdots < t_c$ into T as follows. Each t_i is inserted into row i, such that the resulting row is still weakly increasing. (Clearly, there is a unique way to do this.) If there is no row i, we create a new left-adjusted row consisting of one box which contains t_i. We call this operation *sequence insertion* of $\mathbf{t}$ into T.

Lemma 11. *The result of sequence insertion is a semi-standard Young tableau.*

Proof. It is clear that it suffices to check that the resulting columns are strictly increasing. Furthermore, it suffices to show that any two boxes in adjacent rows are strictly increasing. Let us consider rows i and $i+1$ after inserting t_i and t_{i+1}, $t_i < t_{i+1}$. There are three cases to consider:

Case 1: The numbers t_i and t_{i+1} are in the same column:

$$\begin{bmatrix} \cdots & a_1 & t_i & a_2 & \cdots & a_m & \cdots \\ \cdots & b_1 & t_{i+1} & b_2 & \cdots & b_m & \cdots \end{bmatrix}$$

Since $t_i < t_{i+1}$, and all the other columns are unchanged, the columns are strictly increasing.

Case 2: The number t_i is to the right of t_{i+1}:

$$\begin{bmatrix} \cdots & t_i & a_1 & a_2 & \cdots & a_{m-1} & a_m & \cdots \\ \cdots & b_1 & b_2 & b_3 & \cdots & b_m & t_{i+1} & \cdots \end{bmatrix}$$

The columns where strictly increasing before the insertion. Therefore, $t_i \leq a_1 < b_1$, $a_m < b_m \leq t_{i+1}$ and $a_j < b_j \leq b_{j+1}$. It follows that all the columns are strictly increasing.

Case 3: The number t_i to the left of t_{i+1}:

$$\begin{bmatrix} \cdots & a_1 & a_2 & \cdots & a_{m-1} & a_m & t_i & \cdots \\ \cdots & t_{i+1} & b_1 & b_2 & b_3 & \cdots & b_m & \cdots \end{bmatrix}$$

We have that $a_j \leq t_i < t_{i+1} \leq b_k$ for $1 \leq j, k \leq m$, since the rows are increasing. Thus, it is clear that all the columns are strictly increasing. It is easy to see that the result is a tableau even if $c \neq n$. □

Notice that different sequence insertions commute, i.e., inserting sequence $\mathbf{s}$ into T followed by $\mathbf{t}$, yields the same result as the reverse order of insertion.

We may extend the notion of sequence insertion to skew tableaux as follows: First put negative integers in the skew part, such that the negative integers in each particular row have the same value, and the columns are strictly increasing. The result is a regular tableau, (but with some negative entries), so we may perform sequence insertion. The negative entries still form a skew part of the tableau, and we may remove these to obtain a skew tableau.

Note that we may also allow negative entries in a sequence, which after insertion, are removed. The result is a skew tableau. The following example illustrates this:

Example 12. Here, we insert the sequence $(-1, 2, 3)$ into a skew tableau of shape $(4, 3, 3, 2)/(2, 1, 1)$:

$$\begin{array}{cccc} \blacksquare & \blacksquare & 1 & 1 \\ \blacksquare & 1 & 2 & \\ \blacksquare & 3 & 4 & \\ 1 & 4 & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccccc} \blacksquare & \blacksquare & \blacksquare & 1 & 1 \\ \blacksquare & 1 & 2 & 2 & \\ \blacksquare & 3 & 3 & 4 & \\ 1 & 4 & & & \end{array}$$

Lemma 13. *Let $S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(x_1, \ldots, x_n)$ be a (skew) Schur polynomial. Then, for any $k \geq 0$, the coefficient of $x_1^{h_1} \cdots x_n^{h_n}$ in $x_{t_1} \cdots x_{t_c} S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ with $0 < t_1 < t_2 < \cdots < t_c$ counts the number of (skew) tableau of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ that results in a (skew) tableau that has exactly h_i boxes with value i, after insertion of the sequence $(-k, \ldots, -2, -1, t_1, t_2, \ldots, t_c)$.*

Proof. This follows immediately from the definition of sequence insertion and the definition of the skew Schur polynomials. □

Expressing Schur polynomials and products of the form $x_{t_1} \cdots x_{t_c} S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ as a sum of *monic* monomials, we have a 1-1-correspondence between a monic monomials and tableaux. Thus, in what follows, we may sloppily identify these two objects when proving Theorem 4:

Proof of Theorem 4. We may assume that $\alpha_i \geq \beta_i$ for $i = 1, \dots, r$. Otherwise, all determinants vanish, and the identity is trivially true. With these assumptions we may use the Schur polynomial interpretation.

Let $b := \binom{n}{c-r}$ and let $j \geq \max(r - \alpha_r, c - \beta_c)$. Rewriting (4) using identity (3) yields

$$S_{(\boldsymbol{\lambda}+(b+j)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j)\boldsymbol{\nu})} = \sum_{k=0}^{b-1} Q_{b-k} S_{(\boldsymbol{\lambda}+(k+j)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(k+j)\boldsymbol{\nu})}. \tag{9}$$

Now, notice that the difference between tableaux of shape $(\boldsymbol{\lambda} + k\boldsymbol{\mu})/(\boldsymbol{\kappa} + k\boldsymbol{\nu})$ and tableaux of shape $(\boldsymbol{\lambda} + (k-1)\boldsymbol{\mu})/(\boldsymbol{\kappa} + (k-1)\boldsymbol{\nu})$ is that the former contains an extra column of the form

$$\underbrace{\blacksquare, \dots, \blacksquare}_{r} \underbrace{\square, \dots, \square}_{c-r}.$$

Therefore, each tableau of shape $(\boldsymbol{\lambda}+k\boldsymbol{\mu})/(\boldsymbol{\kappa}+k\boldsymbol{\nu})$, $(k > \max(r-\alpha_r, c-\beta_c))$ may be obtained from some tableau of shape $(\boldsymbol{\lambda} + (k-1)\boldsymbol{\mu})/(\boldsymbol{\kappa} + (k-1)\boldsymbol{\nu})$ by inserting a sequence of the form

$$(-r, \dots, -1, t_1, t_2, \dots, t_{c-r}).$$

Together with Lemma 13, this implies that all Young tableaux[5] in $S_{(\boldsymbol{\lambda}+(b+j)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j)\boldsymbol{\nu})}$ are also tableaux[5] in

$$Q_1 S_{(\boldsymbol{\lambda}+(b+j-1)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j-1)\boldsymbol{\nu})}. \tag{10}$$

Hence, there is almost an equality between $S_{(\boldsymbol{\lambda}+(b+j)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j)\boldsymbol{\nu})}$ and (10), but some tableaux in $S_{(\boldsymbol{\lambda}+(b+j)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j)\boldsymbol{\nu})}$ may be obtained by using different sequence insertions. Those tableaux are exactly the tableaux that may be obtained by using

$$S_{(\boldsymbol{\lambda}+(b+j-2)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j-2)\boldsymbol{\nu})}$$

using two *different* sequence insertions.

Thus, $S_{(\boldsymbol{\lambda}+(b+j)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j)\boldsymbol{\nu})}$ is almost given by

$$Q_1 S_{(\boldsymbol{\lambda}+(b+j-1)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j-1)\boldsymbol{\nu})} + Q_2 S_{(\boldsymbol{\lambda}+(b+j-2)\boldsymbol{\mu})/(\boldsymbol{\kappa}+(b+j-2)\boldsymbol{\nu})}.$$

(Multiplying with Q_2 can be viewed as performing all possible pairs of two different sequence insertions, and then there is a sign.)

Repeating this reasoning using inclusion/exclusion yields (9). □

[5] monic monomials

Remark 14. Note that the technical condition $j \geq \max(\alpha_r - r, \beta_c - c)$ in (5) is indeed necessary. For example, with $n = 2$, $\{\det D^k_{(),(2)}\}_{k=0}^2$ do not satisfy the recurrence but $\{\det D^k_{(),(2)}\}_{k=1}^3$ do:

$$x_1x_2 \cdot 1 - (x_1 + x_2)\left|1\right| + 1\begin{vmatrix}1 & x_1x_2\\0 & 1\end{vmatrix} \neq 0$$

but

$$x_1x_2 \cdot \left|1\right| - (x_1 + x_2)\begin{vmatrix}1 & x_1x_2\\0 & 1\end{vmatrix} + 1\begin{vmatrix}1 & x_1x_2 & 0\\0 & x_1+x_2 & x_1x_2\\0 & 1 & x_1+x_2\end{vmatrix} = 0.$$

This circumstance is a clear distinction of our result to the result in [12], where the corresponding recurrence (for a slightly different type of objects) does not need such additional restriction.

3.2. **Widom's formula.** We will now show that Theorem 6 is equivalent to Widom's formula.

Lemma 15. *Theorem 6 is equivalent to Widom's formula* (2)*.*

Proof. It is clear from (6) that $(-t)^n\psi(-1/t) = \chi(t)$, so the roots of these polynomials are related by $t_i = -1/x_i$. Substituting $t_i \mapsto -1/x_i$ in (2) and canceling signs yields

$$\det D^k_c = \sum_\sigma \left(\frac{s_n}{\prod_{i\in\sigma} x_i}\right)^k \left(\prod_{i\in\sigma} x_i^{-c}\right) \prod_{\substack{j\in\sigma\\ i\notin\sigma}} \left(\frac{1}{x_j} - \frac{1}{x_i}\right)^{-1}.$$

Using that $s_n = x_1x_2\cdots x_n$ and rewriting the last product, we get

$$\det D^k_c = \sum_\sigma \prod_{i\notin\sigma} x_i^k \left(\prod_{i\in\sigma} x_i^{-c}\right) \prod_{\substack{j\in\sigma\\ i\notin\sigma}} x_j \left(\frac{x_i}{x_i - x_j}\right).$$

Now notice that the last product produces x_j^c, since $|[n]\setminus\sigma| = c$. Thus, we may cancel these with the middle product. Finally, putting $\tau = [n]\setminus\sigma$ yields the desired identity. □

Thus, to prove Widom's formula, it suffices to prove Theorem 6. However, it is a direct consequence of the following identity:

Proposition 16. *(Identity for Hall polynomials,* [13, p. 104, eqn. (2.2)]*) The Schur polynomial $S_\lambda(x_1,\dots,x_n)$ satisfy*

$$S_\lambda(x_1,\dots,x_n) = \sum_{w\in\mathfrak{S}_n/\mathfrak{S}_n^\lambda} w\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{\lambda_i>\lambda_j} \frac{x_i}{x_i - x_j}\right)$$

where $\mathfrak{S}_n^\lambda$ is the subgroup of permutations with the property that $\lambda_{w(j)} = \lambda_j$ for $j = 1,\dots,n$, and w acts on the indices of the variables.

Proof of Theorem 6. Let $\boldsymbol{\lambda} = (k, \ldots, k, 0, \ldots, 0)$ with c entries equal to k. Then $\mathfrak{S}_n^\lambda$ is the subgroup consisting of permutations, permuting the first c variables, and the last $n-c$ variables independently. The condition $\lambda_i > \lambda_j$ will only be satisfied if $\lambda_i = k$ and $\lambda_j = 0$. Therefore Proposition 16 immediately implies Theorem 6. □

3.3. **Applications.** Theorem 4 can be used to give a shorter proof a result of Schmidt and Spitzer in [15], by using the main result in [4], which reads as follows:

Let $\{P_n(z)\}$ be a sequence of polynomials satisfying

$$P_{n+b} = -\sum_{j=1}^{b} q_j(z) P_{n+b-j}(z), \tag{11}$$

where the q_j are polynomials. The number $x \in \mathbb{C}$ is a limit of zeros of $\{P_n\}$ if there is a sequence of z_n s.t. $P_n(z_n) = 0$ and $\lim_{n\to\infty} z_n = x$.

For fixed z, we have roots $v_i, 1 \leq i \leq b$ of the characteristic equation

$$v^b + \sum_{j=1}^{b} q_j(z) v^{b-j} = 0.$$

For any z such that the $v_i(z)$ are distinct, we may express $P_n(z)$ as follows:

$$P_n(z) = \sum_{j=1}^{b} r_j(z) v_i(z)^n. \tag{12}$$

Under the non-degeneracy conditions that $\{P_n\}$ do not satisfy a recurrence of length less than b, and that there is no w with $|w| = 1$ such that $v_i(z) = w v_j(z)$ for some $i \neq j$, the following holds:

Theorem 17. *(See* [4]*).*

Suppose $\{P_n\}$ satisfy (11)*. Then x is a limit of zeros if and only if the roots v_i of the characteristic equation can be numbered so that one of the following is satisfied:*

(1) $|v_1(x)| > |v_j(x)|, 2 \leq j \leq b$ *and* $r_1(z) = 0$.
(2) $|v_1(x)| = |v_2(x)| = \cdots = |v_l(x)| > |v_j(x)|, l+1 \leq j \leq b$ *for some* $l \geq 2$.

We are now ready to prove a generalization of Theorem 2:

Theorem 18. *Fix natural numbers n and $0 < c < n$. Let $\gamma_1, \gamma_2, \ldots, \gamma_d$ be a sequence of d integers such that $c < \gamma_1 < \gamma_2 < \cdots < \gamma_d$. Set $\boldsymbol{\alpha} = (\gamma_1, \ldots, \gamma_d)$ and set $\boldsymbol{\beta} = (1, 2, \ldots, c, \gamma_1, \ldots, \gamma_d)$. Define*

$$B = \left\{ v | v = \lim_{k\to\infty} v_k, \det(D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} - v_k I_k) = 0 \right\}.$$

Let

$$f(z) = \sum_{i=0}^{n} s_i z^{i-c}, \quad Q(v,z) = z^c(f(z) - v).$$

Order the moduli of the zeros, $\rho_i(v)$, of $Q(v,z)$ in increasing order, with possible duplicates counted several times, according to multiplicity:

$$0 < \rho_1(v) \leq \rho_2(v) \leq \cdots \leq \rho_n(v).$$

Let $C = \{v | \rho_c(v) = \rho_{c+1}(v)\}$. Then, $B = C \cup W$ where $W \subset \mathbb{C}$ is a finite set of points.

Proof. Consider the sequence of matrices $\{D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} - vI_k\}_{k=K}^{\infty}$, $K = \gamma_d - d$. It is easy to see that the main diagonal of all these matrices will be of the form $s_c - v$, and no other entries involve either s_c or v. Now, define $s'_i(v) = s_i - \delta_{ic} v$, where δ_{ij} is the Dirac delta. Let us modify (6) and define

$$\chi(v,t) = \prod_{i=1}^{n}(t - x_i(v)) = (-1)^n \sum_{i=0}^{n} (-t)^{n-i} s'_i(v). \tag{13}$$

Notice that $\chi(v,t) = (-1)^n Q(v, -1/t)$. If we enumerate the roots of $\chi(v,t)$ according to their magnitude,

$$0 < |x_1(v)| \leq |x_2(v)| \leq \cdots \leq |x_n(v)|,$$

we have that $|x_i(v)| = 1/\rho_i(v)$ for $1 \leq i \leq n$.

From Theorem 4 it follows that the series $\{D^k_{\boldsymbol{\alpha},\boldsymbol{\beta}} - vI_k\}_{k=K}^{\infty}$ satisfy the characteristic equation

$$\prod_{\substack{\sigma \subseteq [n] \\ |\sigma| = c}} (t - x_{\sigma_1}(v) x_{\sigma_2}(v) \cdots x_{\sigma_c}(v)) = 0. \tag{14}$$

It is evident that for this characteristic equation the non-degeneracy conditions hold. All roots are different, and we require all of them for the equation to be symmetric under permutation of the x_i, hence, the recurrence is minimal. The second condition holds since the left-hand side of the characteristic equation is irreducible, see [4] for details.

From Theorem 17, it follows that the zeros of $\det(D^{k_m}_{\boldsymbol{\alpha},\boldsymbol{\beta}} - v_m I_{k_m}) = 0$ accumulate exactly where two or more of the largest zeros of (14) coincide in magnitude, or when the corresponding $r_j(z) = 0$ in (12). The latter case can only hold for only a finite number of points; (alternative 1 cannot be satisfied if $d = 0$, equation (12) is then Widom's formula, and all coefficients $r_i(z)$ are non-zero since all roots $x_j(v)$ are nonzero). The first case is satisfied exactly when

$$|x_{n-c-1}(v) x_{n-c+1}(v) x_{n-c+2}(v) \cdots x_n(v)| =$$
$$|x_{n-c}(v) x_{n-c+1}(v) x_{n-c+2}(v) \cdots x_n(v)|$$

This condition is equivalent to $|x_{n-c-1}(v)| = |x_{n-c}(v)|$, which is exactly $\rho_c(v) = \rho_{c+1}(v)$. This concludes the proof. □

The same strategy as above may be used to find limits of *generalized eigenvalues*, as defined in [10].

It is also possible to generalize Theorem 4 to more general sequences of skew Schur polynomials, $\{S_{(\boldsymbol{\kappa}+k\boldsymbol{\nu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}\}_{k=0}^{\infty}$ for $\boldsymbol{\nu} \subseteq \boldsymbol{\mu}$. This may be used to find asymptotics for the number of skew tableaux of certain shapes, and asymptotics for the set of zeros of the Schur polynomials.

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AROUND MULTIVARIATE SCHMIDT-SPITZER THEOREM

PER ALEXANDERSSON AND BORIS SHAPIRO

ABSTRACT. Given an arbitrary complex-valued infinite matrix $\mathcal{A} = (a_{ij})$, $i = 1, \dots, \infty$; $j = 1, \dots, \infty$ and a positive integer n we introduce a naturally associated polynomial basis $\mathfrak{B}_\mathcal{A}$ of $\mathbb{C}[x_0, \dots, x_n]$. We discuss some properties of the locus of common zeros of all polynomials in $\mathfrak{B}_A$ having a given degree m; the latter locus can be interpreted as the spectrum of the $m \times (m+n)$-submatrix of $\mathcal{A}$ formed by its m first rows and $(m+n)$ first columns. We initiate the study of the asymptotics of these spectra when $m \to \infty$ in the case when $\mathcal{A}$ is a banded Toeplitz matrix. In particular, we present and partially prove a conjectural multivariate analog of the well-known Schmidt-Spitzer theorem which describes the spectral asymptotics for the sequence of principal minors of an arbitrary banded Toeplitz matrix. Finally, we discuss relations between polynomial bases $\mathfrak{B}_\mathcal{A}$ and multivariate orthogonal polynomials.

1. INTRODUCTION

The approach of this paper is motivated by the modern interpretation of the Heine-Stieltjes multiparameter spectral problem as presented in [9] and [10]. Let us recall some relevant results in the matrix set-up.

Given integers $m > 0$ and $n \geq 0$ consider the space $Mat(m, m+n)$ of complex-valued $m \times (m+n)$-matrices. For $s = 0, \dots, n$ define the s-th unit matrix

$$\mathcal{I}_s := (\delta_{s+i-j}) \in Mat(m, m+n).$$

(In what follows the sizes of matrices can be infinite.)

Definition 1 (see [10]). *Given a matrix $A \in Mat(m, m+n)$ define its* eigenvalue locus $\mathcal{E}_A$ *as*

$$\mathcal{E}_A := \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1} \colon \operatorname{rank} \left(A - \sum_{s=0}^{n} x_s \mathcal{I}_s \right) < m \right\}.$$

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For $n = 0$, $\mathcal{E}_A$ coincides with the usual set of eigenvalues of a square matrix A.

Proposition 2 (Lemma 1 of [10]). *For arbitrary $A \in Mat(m, m+n)$ the eigenvalue locus $\mathcal{E}_A$ consists of $\binom{m+n}{n+1}$ points counting multiplicities. In other words, counting multiplicities there exist $\binom{m+n}{n+1}$ eigenvalue tuples $(x_0, x_1, \ldots, x_n)$ such that $A - \sum_{s=0}^{n} x_s \mathcal{I}_s$ has rank smaller than m.*

Remark 3. *Notice that for $n > 0$, the locus $\mathcal{E}_A$ is not a complete intersection since it is given by the vanishing of all maximal minors of A. (A similar phenomenon can be observed for common zeros of multivariate Schur polynomials, since Schur polynomials are given by determinant formulas.)*

Notation 4. *Given an infinite matrix $\mathcal{A} = (a_{ij})$, $i = 1, \ldots, \infty; j = 1, \ldots, \infty$, an integer $n \geq 0$, and an m-tuple of positive integers $I = (i_1, i_2, \ldots, i_m)$ satisfying $1 \leq i_1 < i_2 < \cdots < i_m \leq m + n$, consider the submatrix A_I of $\mathcal{A} - \sum_{s=0}^{n} x_s \mathcal{I}_s$ formed by the first m rows and m columns indexed by I. Define*

$$P^I_{\mathcal{A}}(x_0, x_1, \ldots, x_n) := \det A_I. \tag{1}$$

Evidently, $P^I_{\mathcal{A}}(x_0, \ldots, x_n)$ is a maximal minor of the principal $m \times (m+n)$ submatrix of $\mathcal{A} - \sum_{s=0}^{n} x_s \mathcal{I}_s$ formed by its m first rows and $m + n$ first columns. Therefore $P^I_{\mathcal{A}}(x_0, \ldots, x_n)$ is a polynomial in $x_0, \ldots, x_n$.

Proposition 5. *In the above notation the following holds:*

(i) for any multiindex I with $|I| = m$, $\deg P^I_{\mathcal{A}}(x_0, \ldots, x_n) = m$;
(ii) all $\binom{m+n}{m}$ polynomials $P^I_{\mathcal{A}}(x_0, ..., x_n) \in \mathbb{C}[x_0, \ldots, x_n]$ with $|I| = m$ are linearly independent which implies that the totality of all polynomials $P^I_{\mathcal{A}}(x_0, ..., x_n)$ is a linear basis of $\mathbb{C}[x_0, \ldots, x_n]$;
(iii) the set $\mathcal{E}^{(m)}_{\mathcal{A}}$ of common zeros of all $P^I_{\mathcal{A}}(x_0, ..., x_n)$ with $|I| = m$ is a finite subset of $\mathbb{C}^{n+1}$ of cardinality $\binom{m+n}{n+1}$ counting multiplicities.

Remark 6. *Notice that for $\binom{m+n}{m}$ randomly chosen polynomials in $\mathbb{C}[x_0, x_1 \ldots, x_n]$ of degree m, the set of their common zeros is typically empty.*

Proposition 5 together with our numerical experiments motivate the following question.

Given an arbitrary infinite matrix $\mathcal{A}$ as above, associate to each $\mathcal{E}^{(m)}_{\mathcal{A}}$ its "root-counting" measure $\mu^{(m)}_{\mathcal{A}}$ supported on $\mathcal{E}^{(m)}_{\mathcal{A}} \subset \mathbb{C}^{n+1}$ by assigning to every point $p \in \mathcal{E}^{(m)}_{\mathcal{A}}$ the point mass $\kappa(p)/\binom{m+n}{n+1}$ where $\kappa(p)$ is the multiplicity of p. (Obviously, $\mu^{(m)}_{\mathcal{A}}$ is a discrete probability measure.)

Main Problem. Under which assumptions on $\mathcal{A}$ does the weak limit $\mu_{\mathcal{A}} = \lim_{m\to\infty} \mu_{\mathcal{A}}^{(m)}$ exists? In case when $\mu_{\mathcal{A}}$ exists, is it possible to describe the support and density of the measure?

In the classical case $n = 0$, the above problem was intensively studied by many authors. The main focus has been when $\mathcal{A}$ is either a Jacobi or a Toeplitz matrix (or their generalizations such as block-Toeplitz matrices etc.), see e.g. [4, 3, 11, 12].

The main goal of this note is to present a multivariate analogue of the well-known theorem by P. Schmid and F. Spitzer [8], where they describe $\mu_{\mathcal{A}}$ for an arbitrary banded Toeplitz matrix $\mathcal{A}$ in the case $n = 0$.

Namely, let $\mathcal{A} = (c_{i-j})$, with $i, j = 1, 2, \dots$ be an infinite, banded Toeplitz matrix, where $c_i = 0$ if $i < -k$ or $i > h$. Fixing $n \geq 0$ as above, we obtain for each positive integer m the eigenvalue locus $\mathcal{E}_{\mathcal{A}}^{(m)}$ of the principal $m \times (m+n)$ submatrix $A^{(m)}$ of $\mathcal{A}$.

Define the *limit set $B_{\mathcal{A}}$ of eigenvalue loci* as

$$(2) \quad B_{\mathcal{A}} = \left\{\mathbf{x} \in \mathbb{C}^{n+1} \colon \mathbf{x} = \lim_{m\to\infty} \mathbf{x}_m, \mathbf{x}_m \in \mathcal{E}_{\mathcal{A}}^{(m)}\right\}, \ \mathbf{x} = (x_0, \dots, x_n).$$

In other words, $B_{\mathcal{A}}$ is the set of limit points of the sequence $\{\mathcal{E}_{\mathcal{A}}^{(m)}\}$. Thus $B_{\mathcal{A}}$ is the support of the limiting measure $\mu_{\mathcal{A}}$ if it exists. (For a general infinite matrix $\mathcal{A}$ as above, its limit set $B_{\mathcal{A}}$ might be empty.)

Set

$$(3) \qquad Q(t, \mathbf{x}) = t^k \left(\sum_{j=-k}^{h} c_j t^j - \sum_{j=0}^{n} x_j t^j \right),$$

and let $\alpha_1(\mathbf{x}), \alpha_2(\mathbf{x}), \dots, \alpha_{k+h}(\mathbf{x})$ be the roots of $Q(t, \mathbf{x}) = 0$, ordered according to their absolute values, i.e. $|\alpha_i(\mathbf{x})| \leq |\alpha_{i+1}(\mathbf{x})|$ for all $0 < i < k + h$. Let $C_{\mathcal{A}}$ be the real semi-algebraic set given by the condition:

$$(4) \qquad C_{\mathcal{A}} = \{\mathbf{x} \in \mathbb{C}^{n+1} \colon |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|\}.$$

Our main conjecture is as follows.

Conjecture 7. *For any banded Toeplitz matrix $\mathcal{A}$, if $B_{\mathcal{A}}$ is defined by* (2) *and $C_{\mathcal{A}}$ defined by* (4) *then $B_{\mathcal{A}} = C_{\mathcal{A}}$.*

By Conjecture 7 the set $B_{\mathcal{A}}$ is a real semi-algebraic $(n+1)$-dimensional subset of $\mathbb{C}^{n+1}$. In the classical case $n = 0$, Conjecture 7 is settled by P. Schmidt and F. Spitzer in [8]. Another important case when Conjecture 7 has been proved follows from some known results on multivariate Chebyshev polynomials, which is is presented in Example 8 below. Namely, when $k = 1$ and $h = n + 1$ with c_{-1} and c_{n+1} non-zero, we may do a affine change of the variables and a scaling of $\mathcal{A}$. This reduces to the latter case to $c_{-1} = c_{n+1} = 1$ and all other $c_i = 0$.

For these particular values, the family $\{P^I_{\mathcal{A}}(\mathbf{x})\}$ becomes the multivariate Chebyshev polynomials of the second kind, see e.g. [5, 7, 2, 13]. These polynomials also have a close connection to another well-known family of polynomials, namely the Schur polynomials.

Example 8. For the above matrices corresponding to the multivariate Chebyshev polynomials the eigenlocus $\mathcal{E}^{(m)}_{\mathcal{A}}$ can be described explicitly, see for example [6].

More precisely, the points in $\mathcal{E}^{(m)}_{\mathcal{A}}$ lie on a real, n-dimensional surface $C_{\mathcal{A}} \subset \mathbb{C}^{n+1}$ which is naturally parametrized by an $(n+1)$-dimensional torus T^{n+1}. This parametrization is given by

(5)
$$C_{\mathcal{A}} = \left\{\mathbf{x} \in \mathbb{C}^{n+1} | x_j = -e_{j+1}(\exp(i\theta_1), \dots, \exp(i\theta_{n+1}), \exp(i\theta_{n+2}))\right\}$$

where $(\theta_1, \dots, \theta_{n+1})$ lie on T^{n+1}, $\sum_{j=0}^{n+2} \theta_j = 0$, and e_j is the j-th elementary symmetric function in $n+2$ variables.

Notice that for $\mathbf{x} \in C_{\mathcal{A}}$,

$$\begin{aligned} Q(t, \mathbf{x}) &= 1 + x_0 t + x_1 t^2 + \cdots + x_n t^{n+1} + t^{n+2} \\ &= \prod_j (t + e^{i\theta_j}) \end{aligned}$$

by the Vieta formulas. Thus, for $\mathbf{x} \in C_{\mathcal{A}}$, *all* roots of Q, (as a polynomial in t) have absolute value equal to 1 when the x_j are parametrized as in (5).

Furthermore, the points in $\mathcal{E}^{(m)}_{\mathcal{A}}$ are also expressed by (5), with the parameters $(\theta_1, \dots, \theta_{n+2})$ being certain rational multiples of π, distributed in a regular lattice. The mapping from the 2-torus to the eigenlocus is illustrated in Figure 1.

Another interesting aspect of Example 8 is that all the points $\mathbf{x} = (x_0, \dots, x_n)$ in the sets $\mathcal{E}^{(m)}_{\mathcal{A}}$ satisfy the conditions $x_j = \overline{x_{n-j}}$, $j = 0, 1, \dots, n$, which explains why we can draw $C_{\mathcal{A}} \subset \mathbb{C}^2$ in Figure 1a as a 2-dimensional set. For larger n, $C_{\mathcal{A}}$ is a $(n+1)$-dimensional analogue of the two-dimensional deltoid, shown in Figure 1a.

For general $\mathcal{A}$, we do not have the inclusion $\mathcal{E}^{(m)}_{\mathcal{A}} \subseteq C_{\mathcal{A}}$ for arbitrary finite m. However, if $\mathcal{A}$ has an additional extra symmetry, this seems to be case.

Definition 9. *A banded Toeplitz matrix such that its $Q(t, \mathbf{x})$ in (3) satisfies*

$$\overline{Q(t, x_0, x_1, \dots, x_n)} = \bar{t}^{h+k-1} Q(1/\bar{t}, \overline{x}_n, \overline{x}_{n-1}, \dots, \overline{x}_0)$$

is called multihermitian *of order n.*

Conjecture 10. *If $\mathcal{A}$ is multihermitian of order n, then each point* $\mathbf{x} = (x_0, x_1, \ldots, x_n) \in \mathcal{E}_{\mathcal{A}}^{(m)}$ *satisfies* $x_j = \overline{x_{n-j}}$ *for* $j = 0, 1, \ldots, n$.

Conjecture 10 obviously holds for the case $n = 0$, as it reduces to the fact that hermitian matrices have real eigenvalues. It is also straightforward to check that Conjecture 10 is true for the Chebyshev case of Example 8 above.

We have extensive numerical evidence for this conjecture. Another strong indication supporting Conjecture 10 is that if $\mathcal{A}$ is multi-hermitian, then every point $\mathbf{x} \in C_{\mathcal{A}}$ (which by Conjecture 7 is in the limit eigenlocus) satisfies the required symmetry $x_j = \overline{x_{n-j}}$ for $j = 0, 1, \ldots, n$.

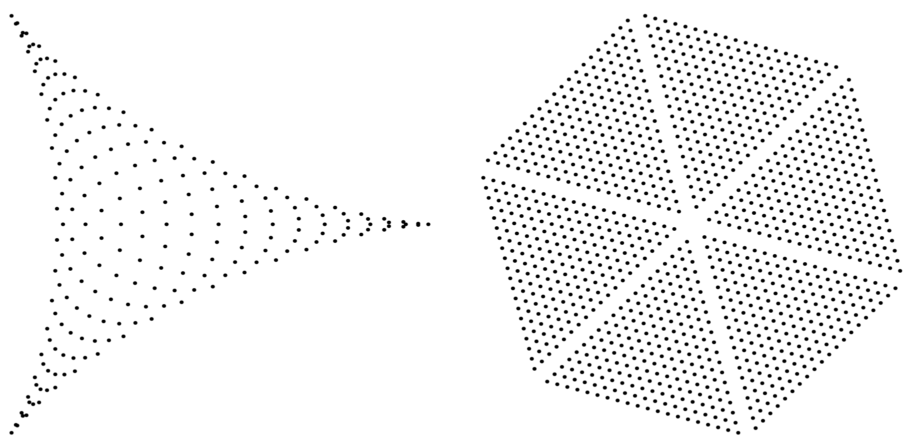

FIGURE 1. The eigenvalue locus $\mathcal{E}_2^{(20)}$ and its pull-back to T^2. The torus T^2 is covered with a hexagon, where each triangle is mapped to the eigenlocus. The 6-fold symmetry is due to the S_3-action by permutation of the arguments $\theta_1, \theta_2, \theta_3$ in (5). (Notice $\theta_1 + \theta_2 + \theta_3 = 0$ and this is the subspace which is illustrated in the figure to the right.)

The next group of examples are bivariate analogues of special univariate cases originally studied in [8], and later in [3], where they are referred to as "star-shaped curves":

Example 11. The bivariate case when $Q(t, \mathbf{x}) = 1 + t^d x_0 + t^{d+1} x_1 + t^{2d+1}$, $d \geq 1$ gives sets in $\mathbb{C}^2$ where $x_0 = \overline{x_1}$, by Conjecture 10. They correspond

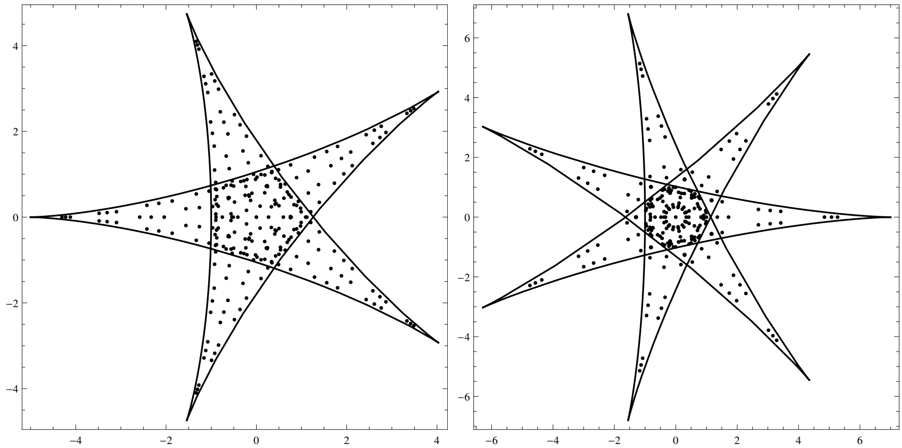

FIGURE 2. 5-edged star, when $d = 2$ and 7-edged star, when $d = 3$

to Toeplitz matrices of the form

$$\begin{pmatrix} x_0 & x_1 & 1 & 0 & 0 & \cdots \\ 1 & x_0 & x_1 & 1 & 0 & \cdots \\ 0 & 1 & x_0 & x_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \begin{pmatrix} x_0 & x_1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & x_0 & x_1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & x_0 & x_1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix}, \dots$$

The above two matrices represent $d = 1$ and $d = 2$.

Figures 2 and 3 present the distributions of $x_0 \in \mathbb{C}$, for $d = 2, 3, 4$. (Recall that $x_1 = \bar{x}_0$.) The points shown on these figures belong to $\mathcal{E}_{\mathcal{A}}^{(m)}$ for $m = 13, 14, 15$, and the curves are certain hypocycloids, parametrizing the boundary of $C_{\mathcal{A}}$. More explicitly, for a given integer $d \geq 1$ the hypocycloid boundary for $x_0 \in \mathbb{C}$ is given by

$$x_0 = (-1)^d e^{-i(d+2)\theta} \left((d+2)e^{i(2d+3)\theta} + d + 1 \right) \text{ where } \theta \in [0, 2\pi],$$

which is one of the implications of Conjecture 7.

Finally, the main result of this note is as follows;

Theorem 12. *For any banded Toeplitz matrix $\mathcal{A}$, where $B_{\mathcal{A}}$ is defined by* (2) *and $C_{\mathcal{A}}$ is defined by* (4)*, one has $B_{\mathcal{A}} \subseteq C_{\mathcal{A}}$.*

2. PROOFS

Proof of Proposition 5. We shall prove items (i) and (ii) simultaneously by calculating the leading homogeneous part of $P_{\mathcal{A}}^{I}(x_0, ..., x_n)$. Let us order the set of all admissible indices $I = (1 \leq i_1 < \ldots < i_m \leq m + n)$

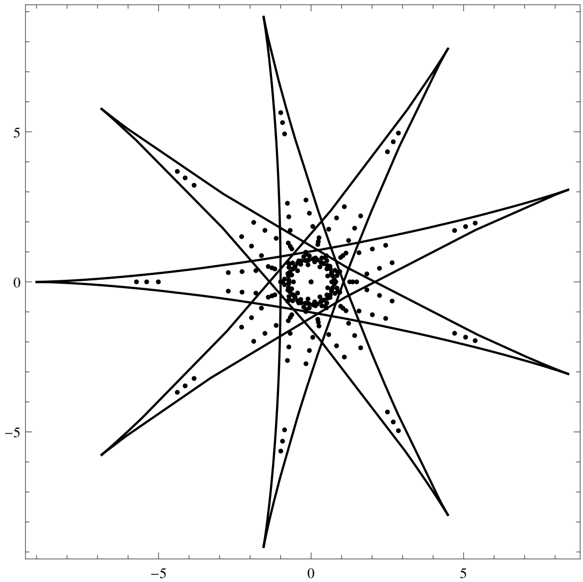

FIGURE 3. 9-edged star, when $d = 4$.

lexicographically. We can also order lexicographically all monomials of degree m in $x_0, \ldots, x_n$. By equation (1) $P^I_{\mathcal{A}}(x_0, ..., x_n) = \det A_I$ where the columns of A_I are indexed by I. Let $\widetilde{P}^I_{\mathcal{A}}(x_0, ..., x_n)$ be the homogeneous part of $P^I_{\mathcal{A}}(x_0, ..., x_n)$ of degree m. One can easily see that the product of all entries on the main diagonal of A_I contains the monomial $\mathfrak{m}_I$ of degree m given by $\mathfrak{m}_I = \prod_{j=1}^m x_{i_j - j+1}$. Moreover it is straightforward that $\widetilde{P}^I_{\mathcal{A}}(x_0, ..., x_n) = \mathfrak{m}_I + \ldots$ where $\ldots$ stands for the linear combination of monomials $\mathfrak{m}_{I'}$ of degree m coming other I' which are lexicographically smaller than I. In other words, the matrix formed by $\widetilde{P}^I_{\mathcal{A}}(x_0, ..., x_n)$ versus monomials is triangular in the lexicographic ordering with unitary main diagonal which proves items (i) and (ii).

Item (iii) is just a reformulation of Proposition 2 above. □

Throughout the rest of the paper, we put $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{h+k})$. We will also assume that $c_h = 1$, which corresponds to a rescaling of the original matrix $\mathcal{A}$. This is equivalent to the assumption that $Q(t, \mathbf{x})$ is monic. By shifting the variables in $\mathbf{x}$, we may also assume, without loss of generality, that $c_0 = c_1 \cdots = c_n = 0$ in $\mathcal{A}$.

In the above notation, it is convenient to work with the roots of $Q(t, \mathbf{x})$. This motivates the following definitions. Let $\Gamma_j \subset \mathbb{C}^{h+k}$, $j = k, \ldots, k+n$ denote the real semi-algebraic hypersurface consisting of all $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{h+k})$ such that when the α_j are ordered with increasing modulus, $|\alpha_j| = |\alpha_{j+1}|$. Similarly, let G_j be defined as the real semi-algebraic set

$$\{\mathbf{x} \in \mathbb{C}^{n+1} \colon Q(t, \mathbf{x}) = (t - \alpha_1) \cdots (t - \alpha_{h+k}) \text{ where } \boldsymbol{\alpha} \in \Gamma_j\}.$$

Then, by definition, $C_{\mathcal{A}} = \bigcap_{j=k}^{k+n} G_j$.

Proposition 13. *For any banded Toeplitz matrix $\mathcal{A}$ and any non-negative $n < h$, the set $C_{\mathcal{A}}$ defined by* (3)-(4) *is compact.*

Proof. As discussed above, we may without loss of generality assume that $c_h = 1$ and $c_0 = c_1 = \cdots = c_n = 0$. Since Q may be assumed to be monic, we have $c_j = e_{h-j}(-\boldsymbol{\alpha})$ for $-k \leq j < 0$ and $n < j \leq h$, and $x_j = -e_{h-j}(-\boldsymbol{\alpha})$ when $0 \leq j \leq n$. Thus, it suffices to show that the set of $\boldsymbol{\alpha} \in \mathbb{C}^{h+k}$ that satisfies the conditions (3)-(4), is compact. It is also evident that the set $C_{\mathcal{A}}$ is closed, so we only need to show that it is bounded. We show this fact by contradiction.

Assume we have a sequence of roots $\{\boldsymbol{\alpha}^m\}_{m=1}^{\infty}$ of (3) such that $\|\boldsymbol{\alpha}^m\| \to \infty$ where (4) is satisfied for each $\boldsymbol{\alpha}^m$. We assume that the modulus of the roots are always ordered increasingly. There are two cases to consider.

Case 1: Assume that for some $0 \leq b < k$, a sequence of individual roots satisfies the condition $|\alpha_{b+1}^m| \to \infty$ but $|\alpha_j^m|$ are bounded for all m and $j \leq b$. Then consider $e_{h+k-b}(\boldsymbol{\alpha})$. Since $b < k$, in our notation $e_{h+k-b}(\boldsymbol{\alpha})$ equals the coefficient c_{b-k}. Notice that e_{h+k-b} contains the term $\alpha_{b+1}\alpha_{b+2} \cdots \alpha_{h+k}$ which grows quicker than all other terms in the expansion of $e_{h+k-b}(\boldsymbol{\alpha})$. This contradicts the assumption $e_{h+k-b}(\boldsymbol{\alpha}) = c_{b-k}$.

Case 2: Assume that for some b with $k + n \leq b < h + k$, we have a sequence of individual roots $|\alpha_{b+1}^m| \to \infty$ but $|\alpha_j^m|$ are bounded for all m and $j \leq b$. Consider

$$e_b(\boldsymbol{\alpha}) = e_b(\alpha_1, \ldots, \alpha_{h+k}) = \sum_{\sigma \in \binom{[h+k]}{b}} \frac{e_0}{\alpha_{\sigma_1}\alpha_{\sigma_2} \cdots \alpha_{\sigma_b}}.$$

This contains an expression with the denominator $\alpha_1\alpha_2 \cdots \alpha_b$, i.e. the product of all bounded roots. Now, since $h+k-b$ roots among all $h+k$ roots grow in absolute value, and the product $\alpha_1 \ldots \alpha_{h+k}$ equals c_h, it follows that $|\alpha_1\alpha_2 \cdots \alpha_b| \to 0$ as $m \to \infty$, and this term converges to 0 quicker than any other product $\alpha_{\sigma_1}\alpha_{\sigma_2} \cdots \alpha_{\sigma_b}$. Thus, $|e_b|$ should grow. This contradicts the assumption $e_b(\boldsymbol{\alpha}) = c_{h-b}$.

Notice that under our assumptions, the above cases cover all possible ways for a sequence of roots to diverge. Since both cases yield a contradiction, it follows that any sequence of roots of (3) satisfying (4) must be bounded. Thus, $C_{\mathcal{A}}$ is compact. □

The following result is multivariate analog of a known fact in the case $n = 0$, see [4, Prop. 11.18 and 11.19].

Proposition 14. *In the notation of* (3)–(4)*, for any* $\mathbf{x}$ *belonging to the boundary* $\partial C_{\mathcal{A}}$ *of* $C_{\mathcal{A}}$*, at least one of the following three conditions is satisfied:*

(i) the discriminant of $Q(t, \mathbf{x})$ *with respect to* t *vanishes, i.e.* $Q(t, \mathbf{x})$ *has (at least) a double root in* t*.*
(ii) $|\alpha_{k-1}(\mathbf{x})| = |\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|$.
(iii) $|\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})| = |\alpha_{k+n+2}(\mathbf{x})|$.

Proof. We need the following two simple statements.

Lemma 15. *Let* Pol_d *be the set of all monic polynomials of degree* d *with complex coefficients. Let* $\Sigma_{p,q} \subset Pol_d$ *be the subset of polynomials satisfying*

$$|\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q|, \tag{6}$$

where $1 \le p < q \le d$ *and* $\alpha_1, \alpha_2, \dots, \alpha_d$ *being the roots of polynomials ordered according to their increasing absolute values. Then* $\Sigma_{p,q}$ *is a real semi-algebraic set of codimension* $q - p$ *whose boundary is the union of three pieces:* $\Sigma_{p-1,q}$*,* $\Sigma_{p,q+1}$ *and the intersection of* $\Sigma_{p,q}$ *with the standard discriminant in* Pol_d*, i.e. the set of polynomials having multiple roots. (Notice that if* $p = 1$ *then* $\Sigma_{p-1,q}$ *is empty, and if* $q = d$ *then* $\Sigma_{p,q+1}$ *is empty by definition.)*

Proof. $\Sigma_{p,q}$ is obtained as the image under the Vieta map of an obvious semi-algebraic set $|\alpha_1| \le |\alpha_2| \le \cdots \le |\alpha_p| = |\alpha_{p+1}| = \cdots = |\alpha_q| \le |\alpha_{q+1}| \le \cdots \le |\alpha_d|$. Notice that the Vieta map is a local diffeomorphism outside the preimage of the standard discriminant. Therefore the boundary of $\Sigma_{p,q}$ must either belong to the standard discriminant or to one of $\Sigma_{p-1,q}$ or $\Sigma_{p,q+1}$. The former is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p-1,q-1}$ and the latter is the common boundary between $\Sigma_{p,q}$ and $\Sigma_{p+1,q+1}$. □

Given a closed Whitney stratified set X (for example, semi-analytic) we say that X is *a k-dimensional stratified set without boundary* if

(i) the top-dimensional strata of X have dimension k;
(ii) for any point x lying in any stratum of dimension $k - 1$, one can choose orientation of the (germs of) k-dimensional strata of a sufficiently small neighborhood of x in X so that $\partial X = 0$.

Lemma 16. *The boundary of the intersection of any closed semi-algebraic set* Γ *with any closed algebraic set* Θ *is included in the intersection of the boundary* $\partial\Gamma$ *with* Θ*.*

Proof. Observe that any real algebraic variety X of dimension k is a stratifiable set without boundary. Indeed, the fact we are proving is local, and it suffices to prove it for generic x on $(k-1)$-dimensional strata.

Consider an embedding of X in some high-dimensional linear space, take the Whitney stratification with x on its stratum $Y \subset B$ of dimension $k-1$, and a transversal to Y of codimension $k-1$ at x.

Therefore, we may now assume that the germ of X near x is topologically a product of a germ of algebraic curve and a germ of a smooth manifold of dimension $k-1$. Furthermore, a germ of any real algebraic curve Γ can be always oriented so that $\partial\Gamma = 0$ which follows from the existence of Puiseux series for an arbitrary branch of algebraic curve. This argument shows that any point in the intersection $\Gamma \cap \Theta$ which does not belong to the boundary of Γ can not lie on the boundary of this intersection which settles Lemma 16. □

Lemmas 15 and 16 immediately imply Proposition 14 since every $C_{\mathcal{A}}$ is the intersection of an appropriate $\Sigma_{p,q}$ with an appropriate affine subspace in Pol_{k+h}. □

Proof of Theorem 12. In our notation, let $D_j^m(\mathbf{x})$ be the determinant of the $m \times m$-matrix A_I with $I = \{j+1, j+2, \dots, j+m\}$ for $0 \leq j \leq n$. It is evident that $\mathcal{E}_{\mathcal{A}}^{(m)}$ is a subset of the set $\widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ of solutions to the system of polynomial equations

$$D_0^m(\mathbf{x}) = D_1^m(\mathbf{x}) = \cdots = D_n^m(\mathbf{x}) = 0. \tag{7}$$

We will show a stronger statement that, in notation of Theorem 12,

$$\lim_{m\to\infty} \widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)} \subseteq C_{\mathcal{A}}.$$

Although each individual $\widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ (considered as a points set with multiplicities) is strictly bigger than $\mathcal{E}_{\mathcal{A}}^{(m)}$ the limits $B_{\mathcal{A}} = \lim_{m\to\infty} \mathcal{E}_{\mathcal{A}}^{(m)}$ and $\lim_{m\to\infty} \widetilde{\mathcal{E}}_{\mathcal{A}}^{(m)}$ seem to coincide as infinite sets.

The next proposition accomplishes the proof of Theorem 12. □

In Theorem 4 of [1] it was shown that each sequence of determinants $\{D_j^m(\mathbf{x})\}_{m=1}^{\infty}$ as above satisfies a linear recurrence relation with coefficients depending on $\mathbf{x}$. The characteristic polynomial $\chi_j(t)$ of the j-th recurrence can be factorized as

$$\chi_j(t, \mathbf{x}) = \prod_{\sigma} (t - r_{j\sigma}), \text{ where } r_{j\sigma} = (-1)^{k+j} (\alpha_{\sigma_1} \cdots \alpha_{\sigma_{k+j}})^{-1}, \tag{8}$$

and σ is a $k+j$-subset of $[k+h]$.

Proposition 17. *Suppose that $\{\mathbf{x}_m\}_1^\infty$, is a sequence of points in $\mathbb{C}^{n+1}$ satisfying the system of equations:*

$$D_j^m(\mathbf{x}_m) = 0 \text{ for } j = 0, 1, \dots, n \text{ and } m = 1, 2, \dots \tag{9}$$

and such that the limit $\lim_{m\to\infty} \mathbf{x}_m =: \mathbf{x}^*$ exists. Then for all $j = 0, \dots, n$ $|\alpha_{k+j}(\mathbf{x}^*)| = |\alpha_{k+j+1}(\mathbf{x}^*)|$ when the α_i are indexed with increasing order of their modulus.

Proof. Provided that all the roots of $\chi_j(t, \mathbf{x})$ are distinct, by using a version of Widom's formula, (see [1, 4]) we have

$$D_j^m(\mathbf{x}) = \sum_\sigma \prod_{l\in\sigma, i\notin\sigma} \left(1 - \frac{\alpha_l(\mathbf{x})}{\alpha_i(\mathbf{x})}\right)^{-1} \cdot r_{j\sigma}(\mathbf{x})^m. \tag{10}$$

We may assume that for $\mathbf{x}^*$ and fixed j, the $r_{j\sigma}(\mathbf{x}^*)$ are ordered decreasingly with respect to their modulus (for some ordering $\sigma = 1, 2, \dots$). The goal is to prove that $|r_{j1}(\mathbf{x}^*)| = |r_{j2}(\mathbf{x}^*)|$ since this implies $|\alpha_{k+j}(\mathbf{x}^*)| = |\alpha_{k+j+1}(\mathbf{x}^*)|$. We show this fact by contradiction.

Assume that $|r_{j1}(\mathbf{x}^*)| > |r_{j2}(\mathbf{x}^*)| \geq \cdots \geq |r_{jb}(\mathbf{x}^*)|$, i.e. that the largest root is simple and has modulus strictly larger than any other root of the characteristic equation (8). By examining (10), it is evident that $r_{j1}(\mathbf{x}_m)^m$ is the dominating term for sufficiently large m, that is, $D_j^m(\mathbf{x}_m)/r_{j1}(\mathbf{x}_m)^m \to L \neq 0$ as $m \to \infty$.

By standard properties of linear recurrences, this holds even when there are multiple zeros among the smaller roots; remember that our assumption was that $r_{j1}(x_m)$ is a simple zero of (8) when m is large enough.

Hence, for sufficiently large m, $D_j^m(\mathbf{x}_m) \approx L r_{j1}(x_m)^m$, which is nonzero for sufficiently large m. This contradicts the condition that $\mathbf{x}_m$ satisfies (9). Consequently, $|r_{j1}(\mathbf{x}^*)| = |r_{j2}(\mathbf{x}^*)|$ for $j = 0, 1, \dots, n$ and this implies Proposition 17. □

Proposition 17 implies that $\mathbf{x}$ lies in $B_\mathcal{A}$ only if $\mathbf{x}$ is a limit of solutions to (9), but such limit $\mathbf{x}$ must satisfy that $|\alpha_k(\mathbf{x})| = |\alpha_{k+1}(\mathbf{x})| = \cdots = |\alpha_{k+n+1}(\mathbf{x})|$. Therefore, $B_\mathcal{A} \subseteq C_\mathcal{A}$.

3. Further directions

1. It seems relatively easy to describe the stratified structure of $C_\mathcal{A}$ at least in case of generic $\mathcal{A}$. In particular, in the Chebyshev case of Example 8 the set $C_\mathcal{A}$ has the same stratification as a simplex of corresponding dimension. One can also understand the stratified structure of the sets $\Sigma_{p,q}$ introduced in Lemma 15. Since each $C_\mathcal{A}$ is obtained from a corresponding $\Sigma_{p,q}$ by intersecting it with an affine subspace the stratified structure of the former for generic $\mathcal{A}$ is also describable. On the other hand, our Example 11 seems to show more complicated stratified structure due to the presence of additional symmetry.

2. We say that an (infinite) complex-valued matrix $\mathcal{A}$ has a *weak univariate orthogonality property* if the sequence of characteristic polynomials of its principal minors obeys the standard 3-term recurrence relation with complex coefficients. There is a straightforward version of this notion for finite square matrices. Obviously, any Jacobi matrix has this property. However, it seems that for any $m \geq 3$ the set $WO_m \subset Mat(m,m)$ of all $m \times m$-matrices with the latter property has a bigger dimension than the set $Jac_m \subset Mat(m,m)$ of all Jacobi $m \times m$-matrices.

Problem 18. *Find the dimension of* WO_m.

3. Analogously, given a non-negative integer n, we say that an (infinite) complex-valued matrix $\mathcal{A}$ has a *weak n-variate orthogonality property* if the above family $\{P^I_{\mathcal{A}}(x_0, x_1, \ldots, x_n)\}$ (see Definition 4) satisfies the 3-term recurrence relation (2.2) of Theorem 2.1 of [13] with complex coefficients.

There are many similarities between families $\{P^I_{\mathcal{A}}(x_0, x_1, \ldots, x_n)\}$ and families of multivariate orthogonal polynomials which by one of the standard definitions of such polynomials also satisfy (2.2) of Theorem 2.1 of [13] with real coefficients.

Our computer experiments show that in this aspect the case $n > 0$ is quite different from the classical case $n = 0$. In particular, we believe that the following conjecture holds.

Conjecture 19. *Given $n > 0$, a banded matrix $\mathcal{A}$ has a weak n-variate orthogonality property if it is of the form*

$$\mathcal{A} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n+1} & 0 & 0 & 0 & \dots \\ d_{-1} & d_0 & d_1 & \dots & d_n & d_{n+1} & 0 & 0 & \dots \\ 0 & d_{-1} & d_0 & \dots & d_{n-1} & d_n & d_{n+1} & 0 & \dots \\ 0 & 0 & d_{-1} & \dots & d_{n-2} & d_{n-1} & d_n & d_{n+1} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_0, \ldots, a_{n+1}, d_{-1}, \ldots, d_{n+1} \in \mathbb{C}$.

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STRETCHED SKEW SCHUR POLYNOMIALS ARE RECURRENT

PER ALEXANDERSSON

Abstract. We show that sequences of skew Schur polynomials obtained from stretched semi-standard Young tableaux satisfy a linear recurrence, which we give explicitly. We apply this to find certain asymptotic behavior of these Schur polynomials and present conjectures on minimal recurrences for stretched skew Schur polynomials.

1. Introduction

Stretched skew tableaux, i.e. skew semi-standard Young tableaux (SSYTs) of shape $k\boldsymbol{\mu}/k\boldsymbol{\nu}$ for positive integers k, $\boldsymbol{\mu}, \boldsymbol{\nu}$ partitions, appear in many areas. For example, they appear naturally when studying Toeplitz matrix minors, see e.g. [4]. In an earlier paper [1], we found asymptotics of certain families of minors of banded Toeplitz matrices by examining stretched skew tableaux. In this paper we generalize the technique used in [1] to explicitly give linear recurrences that skew Schur polynomials obtained from stretched semi-standard Young tableaux satisfy.

Our results appears to have close connection to systems of linear recurrences described in [5], and this paper suggests that a generalization of some results in [5] is possible.

As an easy consequence of this paper, it follows that the number of SSYTs of shape $k\boldsymbol{\mu}/k\boldsymbol{\nu}$ is a polynomial in k. This is a well-known fact which can be proved by elementary methods. However, it might be possible to apply the methods in this paper to prove polynomiality of stretched Kostka numbers. This is also known, but currently requires application of non-trivial tools of different areas, see [7, 6].

As a second consequence, we prove a certain asymptotic behavior of roots of stretched skew Schur polynomials, and conjecture the asymptotic behavior of a general system of stretched skew Schur polynomials. Asymptotics and root location of Schur polynomials seems to be a rather unexplored topic, except in areas where the Schur polynomials have an additional meaning, for example, as minors of Toeplitz matrices.

Key words and phrases. Schur polynomials, tableau concatenation, Young tableaux, recurrence, asymptotics.

We use multi-index notation, i.e. $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The length of a vector is considered to be n unless stated otherwise. Let us now formulate the main theorem of the paper:

Theorem 1. *Let n be a positive integer and let $\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$ be partitions of length at most n such that $\boldsymbol{\mu} \supseteq \boldsymbol{\nu}$ and $k(\boldsymbol{\mu} - \boldsymbol{\nu}) \supseteq \boldsymbol{\lambda} - \boldsymbol{\kappa}$ for some positive integer k. Then, for sufficiently large r, the sequence $\{S_{(\boldsymbol{\kappa}+k\boldsymbol{\mu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}(\mathbf{x})\}_{k=r}^{\infty}$ satisfy a linear recurrence with coefficients polynomial in $x_1, x_2, \dots, x_n$. A characteristic polynomial for the recurrence is given by*

$$\chi(t) = \prod_{T \in \mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}} (t - \mathbf{x}^{w(T)}) \tag{1}$$

where $\mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}$ is the set of semi-standard Young tableaux of shape $\boldsymbol{\mu}/\boldsymbol{\nu}$ with entries in $1, 2, \dots, n$ and $w(T)$ is the weight of the tableau T. In particular, if $\boldsymbol{\lambda} = \boldsymbol{\kappa} = \emptyset$ we may take $r = 0$.

Remark 2. *Notice that* (1) *above does not necessary give the shortest possible recurrence in general. In Corollary 15 below, we give a description of the minimal recurrence. In Corollary 16, we use* (1) *for finding certain asymptotics of the Schur polynomials in the sequence $\{S_{(\boldsymbol{\kappa}+k\boldsymbol{\mu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}(\mathbf{x})\}_{k=r}^{\infty}$.*

2. Preliminaries

For the sake of completeness we define the basic notions in the theory of Young tableaux and Schur polynomials. This material can be found in standard reference literature such as [8].

Definition 3. *A* partition $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ *is a finite weakly decreasing sequence of non-negative integers;*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

The parts *of a partition are the positive entries and the number of positive parts is the* length *of the partition, denoted $l(\boldsymbol{\lambda})$. The* weight, $|\boldsymbol{\lambda}|$ *is the sum of the parts.*

The empty partition $\emptyset$ is the partition with no parts. The partition $(1, 1, \dots, 1)$ with k entries equal to 1 is denoted $\mathbf{1}^k$. We use the standard convention that $\lambda_i = 0$ if $i > l(\boldsymbol{\lambda})$. Addition and multiplication with a scalar on partitions is performed elementwise.

Definition 4. *For partitions $\boldsymbol{\lambda}, \boldsymbol{\mu}$ we say that $\boldsymbol{\lambda} \supseteq \boldsymbol{\mu}$ if $\lambda_i \geq \mu_i$ for all i. This is the* inclusion order. *We also define $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if $|\boldsymbol{\lambda}| = |\boldsymbol{\mu}|$ and $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for all k. This is the* domination order.

2.1. **Young diagrams and Young tableaux.**

Definition 5. *Let* $\boldsymbol{\lambda} \supseteq \boldsymbol{\mu}$ *be partitions. A* skew Young diagram *of* shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ *is an arrangement of "boxes" in the plane with coordinates given by*

$$\{(i,j) \in \mathbb{Z}^2 | \mu_i < j \leq \lambda_i\}.$$

Here, i is the row coordinate, j is the column coordinate. If $\boldsymbol{\mu} = \emptyset$ *we will just refer to the shape as* $\boldsymbol{\lambda}$ *and the diagram is a regular Young diagram.*

There are at least two other ways to draw these diagrams. In this text, the English convention is used. Notice that the diagram of shape $\boldsymbol{\lambda}'/\boldsymbol{\mu}'$ is the transpose of the diagram with shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$.

In this context, it will be convenient to define the *skew part* of a skew diagram as special boxes with coordinates $\{(i,j) \in \mathbb{Z}^2 | 1 \leq j \leq \mu_i \leq \lambda_i\}$. We will call these boxes *skew*. (In Figure 1(b) there are for example seven skew boxes and six ordinary boxes.)

(a) Diagram of shape $(5,4,2,2)$ (b) Diagram of shape $(5,4,3,1)/(3,2,2)$

FIGURE 1

Definition 6. *A* semi-standard Young tableau[1] *(or SSYT) is a Young diagram with natural numbers in the boxes, such that each row is weakly increasing and each column is strictly increasing.*

We denote by $\mathbf{T}^n_{\boldsymbol{\lambda}/\boldsymbol{\mu}}$ the set of SSYTs of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ with entries in $1, 2, \ldots, n$. For an example of an SSYT, see Figure 2.

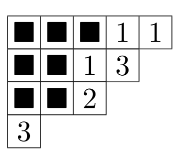

FIGURE 2. SSYT of shape $(5,4,3,1)/(3,2,2)$

[1]Also called column-strict tableau, or reverse plane partition

2.2. **Schur polynomials.**

Definition 7. *Given an SSYT T, with entries in $1, 2 \dots, n$, we define the* weight $w(T)$ *of T as a vector $\mathbf{t} = (t_1, t_2, \dots, t_n)$ given by $t_k = \#\{b_{ij} \in T | b_{ij} = k\}$. Thus, t_k counts the number of boxes containing the number k.*

Definition 8. *The* skew Schur polynomial *is defined as*

$$S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \sum_{T \in \mathbf{T}^n_{\boldsymbol{\lambda}/\boldsymbol{\mu}}} \mathbf{x}^{w(T)}$$

where $\mathbf{x} = (x_1, \dots, x_n)$. It can be shown that these polynomials are symmetric in $x_1, x_2, \dots, x_n$.

3. Proofs

3.1. **Tableau concatenation.**

We now define an operation on pairs of SSYTs:

Definition 9. *Given $T_1 \in \mathbf{T}^n_{\boldsymbol{\kappa}/\boldsymbol{\lambda}}$ and $T_2 \in \mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}$ we define the* tableau concatenation *$T_1 \boxtimes T_2$ as the SSYT obtained by concatenating the boxes row-wise and then sorting each row in increasing order, with respect to their content. The skew boxes are treated as being less than the ordinary boxes.*

We also use the same notation for the corresponding operation on diagram shapes.

From this definition, it is clear that the product $\boxtimes$ is commutative and associative. It is however not obvious that the result of this operation is an SSYT, so we prove this in the following proposition:

Proposition 10. *If $T_1 \in \mathbf{T}^n_{\boldsymbol{\kappa}/\boldsymbol{\lambda}}$ and $T_2 \in \mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}$ then $T_1 \boxtimes T_2 \in \mathbf{T}^n_{(\boldsymbol{\lambda}+\boldsymbol{\mu})/(\boldsymbol{\kappa}+\boldsymbol{\nu})}$ and $w(T_1 \boxtimes T_2) = w(T_1) + w(T_2)$.*

Proof. From the definition, it is evident that the shape of $T_1 \boxtimes T_2$ is $(\boldsymbol{\kappa} + \boldsymbol{\mu})/(\boldsymbol{\lambda} + \boldsymbol{\nu})$. It is also clear that the rows are weakly increasing, by construction. It suffices to show that the columns in $T_1 \boxtimes T_2$ are strictly increasing.

Given an SSYT T_1, we may view its columns $C_1, C_2, \dots, C_k$ as individual SSYTs. Since the rows are already ordered, it is evident that $C_1 \boxtimes C_2 \boxtimes \cdots \boxtimes C_k = T_1$. Therefore, $T_1 \boxtimes T_2 = C_1 \boxtimes C_2 \boxtimes \cdots \boxtimes C_k \boxtimes T_2$ and it suffices to show that $C \boxtimes T_2$ is an SSYT for a general column C.

Let C be a column with row entries $t_1, t_2, \dots, t_k$ where we treat skew boxes in row i as having the value $n - i$. This ensures that $t_1 < t_2 < \cdots < t_k$. We use the same treatment for the skew boxes in T_2.

It suffices to show that any two boxes in a column in adjacent rows are strictly increasing in $C \boxtimes T_2$. Let us consider rows i and $i+1$ in $C \boxtimes T_2$. There are three cases to consider:

Case 1: The numbers t_i and t_{i+1} are in the same column:

$$\begin{bmatrix} \cdots & a_1 & t_i & a_2 & \cdots & a_m & \cdots \\ \cdots & b_1 & t_{i+1} & b_2 & \cdots & b_m & \cdots \end{bmatrix}$$

Since $t_i < t_{i+1}$, and all the other columns are unchanged, the columns are strictly increasing.

Case 2: The number t_i is to the right of t_{i+1}:

$$\begin{bmatrix} \cdots & t_i & a_1 & a_2 & \cdots & a_{m-1} & a_m & \cdots \\ \cdots & b_1 & b_2 & b_3 & \cdots & b_m & t_{i+1} & \cdots \end{bmatrix}$$

The columns where strictly increasing before the concatenation. Therefore, $t_i \leq a_1 < b_1$, $a_m < b_m \leq t_{i+1}$ and $a_j < b_j \leq b_{j+1}$. It follows that all the columns are strictly increasing.

Case 3: The number t_i to the left of t_{i+1}:

$$\begin{bmatrix} \cdots & a_1 & a_2 & \cdots & a_{m-1} & a_m & t_i & \cdots \\ \cdots & t_{i+1} & b_1 & b_2 & b_3 & \cdots & b_m & \cdots \end{bmatrix}$$

We have that $a_j \leq t_i < t_{i+1} \leq b_k$ for $1 \leq j,k \leq m$, since the rows are increasing. Thus, it is clear that all the columns are strictly increasing. It is easy to see that the result is an SSYT even if $k \neq n$. □

Remark 11. *We observe that $\boxtimes$ gives a monoid*[2] *structure on the set of SSYTs. It is natural to construct the corresponding commutative* ring $\mathbf{T}_R^n$ *by considering formal sums of Young tableaux with entries in $1, 2, \ldots, n$, and the number of parts at most n. The operation $\boxtimes$ serves as multiplication, and the empty tableau $\emptyset$ acts as multiplicative identity.*

For tableaux T define the map $\phi(T) = \mathbf{x}^{w(T)}$ and extend it linearly to formal sums. It is evident that $\phi(T_1 \boxtimes T_2) = \phi(T_1)\phi(T_2)$ so ϕ acts as a ring homomorphism from $\mathbf{T}_R^n$ to $\mathbb{Z}[x_1, \ldots, x_n]$. It is therefore natural to consider $|w(\cdot)|$ as a grading on $\mathbf{T}_R^n$. Notice that the ring $\mathbf{T}_R^n$ is finitely generated for each n, a possible set of generators being all tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ with $\lambda_i \leq 1$ and $l(\boldsymbol{\lambda}) \leq n$. In other words, any tableau can be "factored as a product of columns". The cancellation property also hold in $\mathbf{T}_R^n$, namely if $T_1 \boxtimes T = T_2 \boxtimes T$ then $T_1 = T_2$.

The following definition and lemmas are needed for proving the existence and to determine the constant r in Theorem 1:

Definition 12. *Given two skew shapes $\boldsymbol{\mu}/\boldsymbol{\nu}, \boldsymbol{\kappa}/\boldsymbol{\lambda}$, we say that $\boldsymbol{\mu}/\boldsymbol{\nu}$* sits inside *$\boldsymbol{\kappa}/\boldsymbol{\lambda}$ if every column in the diagram of shape $\boldsymbol{\mu}/\boldsymbol{\nu}$ can be found in the diagram of shape $\boldsymbol{\kappa}/\boldsymbol{\lambda}$, counting multiplicities.*

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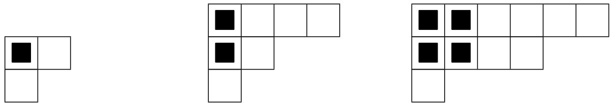
[2]Notice: this is *not* the plactic monoid which is a different type of monoid structure.



(a) (b) (c)

FIGURE 3. Diagram (a) do not sit inside any of the other two diagram, but (b) sits inside (c).

Lemma 13. *For every pair of skew shapes $\boldsymbol{\kappa}/\boldsymbol{\lambda}$ and $\boldsymbol{\mu}/\boldsymbol{\nu}$ there exists an integer $r \geq 0$ such that $\boldsymbol{\mu}/\boldsymbol{\nu}$ sits inside $(\boldsymbol{\kappa}+r\boldsymbol{\mu})/(\boldsymbol{\lambda}+r\boldsymbol{\nu})$.*

Proof. Notice that we can equivalently prove that for some $r \geq 0$ $\boldsymbol{\mu}/\boldsymbol{\nu}$ sits inside $\boldsymbol{\kappa}/\boldsymbol{\lambda} \boxtimes r\boldsymbol{\mu}/r\boldsymbol{\nu}$. The boxes in $\boldsymbol{\kappa}/\boldsymbol{\lambda}$ "push" the boxes in $r\boldsymbol{\mu}/r\boldsymbol{\nu}$ at most κ_1 places to the right when performing the tableau concatenation. If we choose $r > \kappa_1$, then we will have $r > \kappa_1$ copies of each column in $\boldsymbol{\mu}/\boldsymbol{\nu}$, and therefore a tableau concatenation with $\boldsymbol{\kappa}/\boldsymbol{\lambda}$ cannot deform all of them. This concludes the proof. □

Lemma 14. *Let $T \in \mathbf{T}^n_{\boldsymbol{\kappa}/\boldsymbol{\lambda}}$. If $\boldsymbol{\mu}/\boldsymbol{\nu}$ sits inside $\boldsymbol{\kappa}/\boldsymbol{\lambda}$ then there exists $T' \in \mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}$ and $T'' \in \mathbf{T}^n_{(\boldsymbol{\kappa}-\boldsymbol{\mu})/(\boldsymbol{\lambda}-\boldsymbol{\nu})}$ such that $T = T' \boxtimes T''$.*

Proof. Since $\boldsymbol{\mu}/\boldsymbol{\nu}$ sits inside $\boldsymbol{\kappa}/\boldsymbol{\lambda}$, we may find columns $C_1, \dots, C_k$ in $\boldsymbol{\kappa}/\boldsymbol{\lambda}$ such that $T' = C_1 \boxtimes C_2 \boxtimes \dots \boxtimes C_k$ has shape $\boldsymbol{\mu}/\boldsymbol{\nu}$. The tableau T' is of the correct shape, and deleting corresponding columns in T yields a tableau $T'' \in \mathbf{T}^n_{(\boldsymbol{\kappa}-\boldsymbol{\mu})/(\boldsymbol{\lambda}-\boldsymbol{\nu})}$. □

We are now ready to give a proof of Theorem 1:

Proof. We may assume that $\boldsymbol{\kappa} \supseteq \boldsymbol{\lambda}$ since otherwise, choose k such that $k(\boldsymbol{\mu}-\boldsymbol{\nu}) \supseteq \boldsymbol{\lambda}-\boldsymbol{\kappa}$ and take $\boldsymbol{\kappa}' := \boldsymbol{\kappa}+k\boldsymbol{\nu}$ and $\boldsymbol{\lambda}' := \boldsymbol{\lambda}+k\boldsymbol{\mu}$. Then $\boldsymbol{\kappa}' \supseteq \boldsymbol{\lambda}'$ and the sequence $\{S_{(\boldsymbol{\kappa}'+j\boldsymbol{\mu})/(\boldsymbol{\lambda}'+j\boldsymbol{\nu})}\}_{j=0}^{\infty}$ is the same as $\{S_{(\boldsymbol{\kappa}+j\boldsymbol{\mu})/(\boldsymbol{\lambda}+j\boldsymbol{\nu})}\}_{j=k}^{\infty}$.

Set $d := |\mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}|$, which is the degree of the characteristic polynomial $\chi(t)$. By Lemma 13, we may choose r_0 such that $\boldsymbol{\mu}/\boldsymbol{\nu}$ sits inside $\boldsymbol{\kappa} + r_0\boldsymbol{\mu}/\boldsymbol{\lambda} + r_0\boldsymbol{\nu}$. Let $r \geq r_0$ be arbitrary. It then suffices to prove that the sequence $\{S_{(\boldsymbol{\kappa}+k\boldsymbol{\mu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}\}_{k=r}^{r+d}$ satisfy the recurrence given by $\chi(t)$.

Let $\mathbf{T}_j$ be the set/formal sum of the elements in $\mathbf{T}^n_{(\boldsymbol{\kappa}+(j+r)\boldsymbol{\mu})/(\boldsymbol{\lambda}+(j+r)\boldsymbol{\nu})}$. By Lemma 14, it is clear that

$$\mathbf{T}_j \subset \sum_{T \in \mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}} \mathbf{T}_{j-1} \boxtimes T \tag{2}$$

as multisets, (and equality as sets) for $j = 1, 2, \dots, d$. Some tableaux appear multiple times on the right-hand side, and these are exactly the tableaux that may be decomposed as concatenations in (at least) two different ways, namely

$$\sum_{T_1,T_2\in\mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}} \mathbf{T}_{j-2}\boxtimes T_1\boxtimes T_2, \quad T_1\neq T_2.$$

Define the multisets/formal sums

$$Q_j := \sum_{\substack{T_1,T_2,\dots,T_j\in\mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}\\ a\neq b\Rightarrow T_a\neq T_b}} (-1)^{d-j} T_1\boxtimes T_2\boxtimes\cdots\boxtimes T_j, \quad Q_0 := \emptyset.$$

Hence, Q_j is, as a set, the tableaux in $\mathbf{T}_d$ that can be obtained from $\mathbf{T}_{d-j}$ by inserting j different tableaux from $\mathbf{T}^n_{\boldsymbol{\mu}/\boldsymbol{\nu}}$.

By using the principle of inclusion/exclusion, we obtain

$$Q_0\boxtimes\mathbf{T}_d + Q_1\boxtimes\mathbf{T}_{d-1} + Q_2\boxtimes\mathbf{T}_{d-2}+\cdots+Q_d\boxtimes\mathbf{T}_0 = 0. \tag{3}$$

Application of the ring homomorphism ϕ to this expression followed by factoring yields the desired identity. □

4. Applications and further development

4.1. **Asymptotics.** The following results are corollaries of Theorem 1:

Corollary 15. *The sequence* $\{S_{(\boldsymbol{\kappa}+k\boldsymbol{\mu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}(\mathbf{x})\}_{k=r}^{\infty}$ *satisfy a linear recurrence, with a minimal characteristic polynomial of the form* $\chi_m(t) = \prod_{\mathbf{w}\in W}(t-\mathbf{x}^{\mathbf{w}})$, *where* $W\subset\mathbb{N}^n$ *is invariant under permutations, i.e.* $\mathbf{w}\in W \Rightarrow (w_{\sigma_1}\cdots w_{\sigma_n})\in W$ *for every* $\sigma\in\mathfrak{S}_n$.

Proof. Clearly, the roots of $\chi_m(t)$ must be a subset of the roots of (1). The roots of $\chi_m(t)$ are invariant under permutation of variables, since this holds for the Schur polynomials. This implies the invariance on W. □

Corollary 16. *Let* $\boldsymbol{\kappa},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}$ *be partitions satisfying the conditions in Theorem 1, with the additional condition that* $\boldsymbol{\mu}\neq\boldsymbol{\nu}$. *Set*

$$P_k(z) = S_{(\boldsymbol{\kappa}+k\boldsymbol{\mu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}(z,\xi_2,\dots,\xi_n),\ \xi_i\in\mathbb{C}, |\xi_i| = R \text{ for } i=2,\dots,n.$$

Define the limit set of roots $A = \{z\in\mathbb{C} | z = \lim_{k\to\infty} z_k, P_k(z_k) = 0\}$. *Then* A *is a circle with radius* R, *possibly together with the point at the origin.*

Proof. This follows from Theorem 1, Corollary 15 together with the main theorem in [3]. □

Example 17. *If* $\boldsymbol{\lambda}_n = (n, n-1, n-2, \dots, 0)$, *then* all *zeros of* $S_{k\boldsymbol{\lambda}_n}(t, \mathbf{1}^n)$ *lie on the unit circle, for every* n, k.

The following conjecture is a generalization of Corollary 16:

Conjecture 18. *Let $1 \leq j \leq n$ and $\boldsymbol{\kappa}_i, \boldsymbol{\lambda}_i, \boldsymbol{\mu}_i, \boldsymbol{\nu}_i$, $1 \leq i \leq j$, be partitions satisfying the assumptions in Theorem 1.*

Let $x_i \in \mathbb{C}$, $|\xi_i| = R$ for $i = j+1, \dots, n$ and define

$$P_k^i(z_1, \dots, z_j) = S_{(\boldsymbol{\kappa}_i + k\boldsymbol{\mu}_i)/(\boldsymbol{\lambda}_i + k\boldsymbol{\nu}_i)}(z_1, z_2, \dots, z_j, \xi_{j+1}, \dots, \xi_n), 1 \leq i \leq j. \tag{4}$$

Set

$$A = \{\mathbf{z} \in \mathbb{C}^j | \mathbf{z} = \lim_{k\to\infty} \mathbf{z}_k, P_k^1(\mathbf{z}_k) = P_k^2(\mathbf{z}_k) = \dots = P_k^j(\mathbf{z}_k) = 0\}.$$

Then, under some mild non-degeneracy conditions on the partitions,

$$A = Z \cup \begin{cases} \{R(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_j}) | \theta_1, \theta_2, \dots, \theta_j \in \mathbb{R}\} \text{ if } j < n, \\ \{R(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_j}) | R, \theta_1, \theta_2, \dots, \theta_j \in \mathbb{R}\} \text{ if } j = n. \end{cases} \tag{5}$$

where Z is either $\emptyset$ or the set consisting of the origin.

Remark 19. *This is true in a slightly modified special case. Multivariate Chebyshev polynomials may be defined as certain polynomials P_k^i as above, with an appropriate change of variables. The support of the orthogonality measure is the image of A, under the same mapping as the change of variables. See [2] for the connection between multivariate Chebyshev polynomials of the second kind, and Schur polynomials.*

4.2. **Kostka coefficients.** The recurrence (1) is in some cases not the shortest possible. For some applications, this is not a problem but it is not completely satisfying. Below we conjecture the shortest possible recurrence. We need the following definitions:

Definition 20. *Given a partition $\boldsymbol{\lambda}$, we define the* monomial symmetric polynomial $m_{\boldsymbol{\lambda}}$ *as*

$$m_{\boldsymbol{\lambda}} = \sum_{\mathbf{w}} \mathbf{x}^{\mathbf{w}}, \tag{6}$$

where the sum is taken over distinct *permutations of $\boldsymbol{\lambda}$.*

Note that $m_{k\boldsymbol{\lambda}}(x_1, x_2, \dots, x_n) = m_{\boldsymbol{\lambda}}(x_1^k, x_2^k, \dots, x_n^k)$ by definition.

Definition 21. *The Kostka coefficient $K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}}$ is the number of tableaux of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ with weight $\mathbf{w}$.*

It is well-known that $K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} = K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\overline{\mathbf{w}}}$ where $\overline{\mathbf{w}}$ is the vector obtained from $\mathbf{w}$ by rearranging the elements as a partition, in decreasing order. It is evident that $K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} = 0$ if $|\mathbf{w}| \neq |\boldsymbol{\lambda}| - |\boldsymbol{\mu}|$. The Kostka numbers and the monomial symmetric polynomials are related by:

$$S_{\boldsymbol{\lambda}/\boldsymbol{\mu}}(\mathbf{x}) = \sum_{\mathbf{w}} K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} m_{\mathbf{w}}(\mathbf{x}), \tag{7}$$

where the sum is taken over all partitions $\mathbf{w}$.

We now give a hands-on application of tableau concatenation and Kostka coefficients:

Proposition 22. *If* $K_{\boldsymbol{\lambda}/\boldsymbol{\mu},\mathbf{w}} > 0$ *then* $K_{k\boldsymbol{\lambda}/k\boldsymbol{\mu},k\mathbf{w}} > 0$ *for any integer* $k > 0$.

Proof. Let T be a tableau of shape $\boldsymbol{\lambda}/\boldsymbol{\mu}$ with weight $\mathbf{w}$. Then the kth power $T \boxtimes \cdots \boxtimes T$ is a tableau with shape $k\boldsymbol{\lambda}/k\boldsymbol{\mu}$ and weight $k\mathbf{w}$, and hence $K_{k\boldsymbol{\lambda}/k\boldsymbol{\mu},k\mathbf{w}} > 0$. □

Remark 23. *In fact,* $K_{\boldsymbol{\lambda},\mathbf{w}} > 0 \Leftrightarrow K_{k\boldsymbol{\lambda},k\mathbf{w}} > 0$ *and this is known as Fulton's K-saturation conjecture. Its proof is given in* [6]*, which uses the K-hive model machinery.*

We now give a conjectural sharper version of Theorem 1:

Conjecture 24. *Let* $\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}$ *be partitions of length at most* n*, such that* $\boldsymbol{\mu} \supseteq \boldsymbol{\nu}$ *and* $k(\boldsymbol{\mu} - \boldsymbol{\nu}) \supseteq \boldsymbol{\kappa} - \boldsymbol{\lambda}$ *for some positive integer* k*. Set*

$$W = \{\mathbf{w} \in \mathbb{N}^n | K_{\boldsymbol{\mu}/\boldsymbol{\nu},\mathbf{w}} > 0 \text{ and } \overline{\mathbf{w}} \trianglerighteq \overline{\boldsymbol{\mu} - \boldsymbol{\nu}}\}.$$

Then, for sufficiently large r*, the sequence* $\{S_{(\boldsymbol{\kappa}+k\boldsymbol{\mu})/(\boldsymbol{\lambda}+k\boldsymbol{\nu})}(\mathbf{x})\}_{k=r}^{\infty}$ *satisfy a linear recurrence with the minimal characteristic polynomial*

$$\chi(t) = \prod_{\mathbf{w} \in W} (t - x^{\mathbf{w}}). \tag{8}$$

In Theorem 1, it is obvious how to interpret the coefficients in the linear recurrence as certain tableaux concatenations, mapped under the ring homomorphism ϕ. However, in the conjecture above, it is not even clear if such interpretation exists.

Remark 25. *The motivation for the conjecture is that in the case* $\boldsymbol{\nu}$ *is the empty partition, then the conjecture reduces to a formula in* [5]*, where a similar recurrence is considered. Another reason is that the recurrence is free from multiple roots, which indicates that it is minimal in some sense. The exact form of the recurrence is supported by computer experiments.*

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