Cohomology of arrangements and moduli spaces

Olof Bergvall

Stockholm University

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Address: Matematiska Institutionen, Stockholms Universitet, 106 91 Stockholm
E-mail address: olofberg@math.su.se

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Abstract

This thesis mainly concerns the cohomology of the moduli spaces $\mathcal{M}_3[2]$ and $\mathcal{M}_{3,1}[2]$ of genus 3 curves with level 2 structure without respectively with a marked point and some of their natural subspaces. A genus 3 curve which is not hyperelliptic can be realized as a plane quartic and the moduli spaces $\mathcal{Q}[2]$ and $\mathcal{Q}_1[2]$ of plane quartics without respectively with a marked point are given special attention. The spaces considered come with a natural action of the symplectic group $\mathrm{Sp}(6,\mathbb{F}_2)$ and their cohomology groups thus become $\mathrm{Sp}(6,\mathbb{F}_2)$-representations. All computations are therefore $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant. We also study the mixed Hodge structures of these cohomology groups.

The computations for $\mathcal{M}_3[2]$ are mainly via point counts over finite fields while the computations for $\mathcal{M}_{3,1}[2]$ primarily use a description due to Looijenga [62] in terms of arrangements associated to root systems. This leads us to the computation of the cohomology of complements of toric arrangements associated to root systems. These varieties come with an action of the corresponding Weyl groups and the computations are equivariant with respect to this action.

Sammanfattning

Denna avhandling behandlar huvudsakligen kohomologin av modulirummen $\mathcal{M}_3[2]$ och $\mathcal{M}_{3,1}[2]$ av kurvor av genus 3 med nivå 2-struktur utan respektive med en markerad punkt samt några naturliga delrum av dessa rum. En icke hyperelliptisk kurva av genus 3 kan realiseras som en plan kvartisk kurva och vi ägnar extra uppmärksamhet åt modulirummen $\mathcal{Q}[2]$ och $\mathcal{Q}_1[2]$ av plana kvartiska kurvor utan respektive med en markerad punkt. Den symplektiska gruppen $\mathrm{Sp}(6,\mathbb{F}_2)$ verkar naturligt på de rum som betraktas i denna avhandling och kohomologigrupperna av dessa rum blir således representationer av $\mathrm{Sp}(6,\mathbb{F}_2)$. Våra beräkningar är därför ekvivarianta med avseende på gruppen $\mathrm{Sp}(6,\mathbb{F}_2)$. Vi behandlar också blandade Hodgestrukturer.

Beräkningarna för $\mathcal{M}_3[2]$ är huvudsakligen via punkträkningar över ändliga kroppar medan beräkningarna för $\mathcal{M}_{3,1}[2]$ främst använder en beskrivning av Looijenga [62] i termer av arrangemang associerade till rotsystem. Detta leder till beräkningar av kohomologin av komplement till toriska arrangemang associerade till rotsystem. Rotsystemens Weylgrupper verkar naturligt på dessa varieteter och våra beräkningar är ekvivarianta med avseende på denna verkan.

Preface

Since it is a bit unusual (at least in Sweden) to write up a thesis as a monograph rather than a collection of papers, it might be appropriate to explain this choice. First and foremost, the results presented here are all quite closely related and they are all building towards the same goal. Therefore, I felt that they were most naturally presented in one cohesive story along with the relevant theory. There are of course drawbacks of this choice - the most notable is probably that the resulting text became both longer to write and read. However, "shorter to read" does not necessarily mean "easier to understand" and this format also allows avoiding tedious repetitions that would occur in a collection where each paper itself would need to contain some extent of background theory.

The intended reader of this thesis is an algebraic geometer. This means that theory known to most algebraic geometers is used without much explanation, for theory that is known to exist (but perhaps not known) by most algebraic geometers I provide references while I discuss theory more specific to the topic in more detail. For example, the indended reader is expected to be familiar with sheaves and to at least have heard about the existence of the concept of mixed Hodge structures while Aronhold sets of theta characteristics are explained in more depth.

The exception to this rule is the introduction. Since at least parts of the topic at hand are in the rare but favorable position of being quite elementary we start at this point and the intended audience is therefore a bit broader here than in the rest of the thesis. At times this choice leads to simplifications, vagueness and imprecision but I hope this only occurs at points which are either well-known to experts or covered in more detail later in the thesis.

Acknowledgements

First and foremost I would like to thank my advisors Carel Faber and Jonas Bergström. A special thanks also goes to Dan Petersen for interesting discussions, answering many questions and providing useful comments on early versions of parts contained in this thesis.

I am very thankful for the many comments and suggestions on various parts I have received from Christian Espindola, Madeleine Leander, Ivan Martino, Alessandro Oneto, Travis Scrimshaw and Felix Wierstra. I would also like to express my gratitude towards Emanuele Delucchi for an invitation to Fribourg and for pointing out many interesting references regarding arrangements. Tomas Ericsson has always promptly and patiently solved any problem or request regarding computers that has arisen during this project and for this I am very grateful. Finally, I would like to thank Jörgen Backelin, Daniel Bergh, Alexander Berglund, Jan-Erik Björk, Mats Boij, Stefano Marseglia, David Rydh, Karl Rökaeus and Gustav Sædén Ståhl for interesting and helpful discussions.

Contents

1. Introduction

> *In any branch of mathematics, there are usually guiding problems, which are so difficult that one never expects to solve them completely, yet which provide stimulus for a great amount of work, and which serve as yardsticks for measuring progress in the field. In algebraic geometry such a problem is the classification problem. In its strongest form, the problem is to classify all algebraic varieties up to isomorphism.*

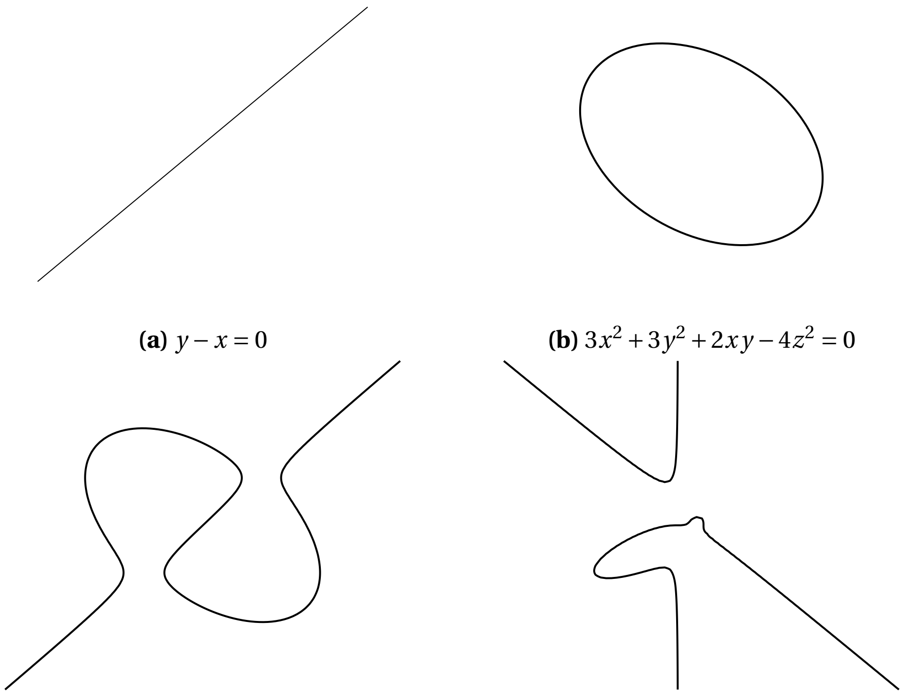
Robin Hartshorne, *Algebraic Geometry*, [53]

Let us begin by explaining a few words in the quote above. A *variety* is the set of solutions of a finite set of polynomial equations in an affine or projective space. Algebraic geometry views a variety as a geometric object and studies its geometrical properties. Two varieties with the same geometrical properties are said to be isomorphic. A variety of dimension 1 is called a curve, a variety of dimension 2 is called a surface and so on. This thesis mainly concerns the classification problem in the special case of a type of curves called plane quartics.

1.1 Moduli of curves

Let k be a field. Define a *smooth, plane, projective curve* C over k as a variety given as the zero locus in $\mathbb{P}^2(k)$ of a nonsingular, homogeneous polynomial $f(x, y, z) \in k[x, y, z]$. The *degree* of the curve C is the degree of f. If the degree is 1 we say that C is a *line*, if the degree is 2 we say that C is a *conic*, if the degree is 3 we say that C is a *cubic*, if the degree is 4 we say that C is a *quartic* and so on, see Figure 1.1.

By varying the coefficients of the polynomial f we get new polynomials and thus new curves. Intuitively, “small” variations in the coefficients should give curves that are “similar” or “close”, see Figure 1.2. This heuristic leads to the idea of a “space of curves” whose points are, not necessarily



(a) $y - x = 0$

(b) $3x^2 + 3y^2 + 2xy - 4z^2 = 0$

(c) $5y^3 - 5yz^2 - 6x^3 + 6xz^2 = 0$

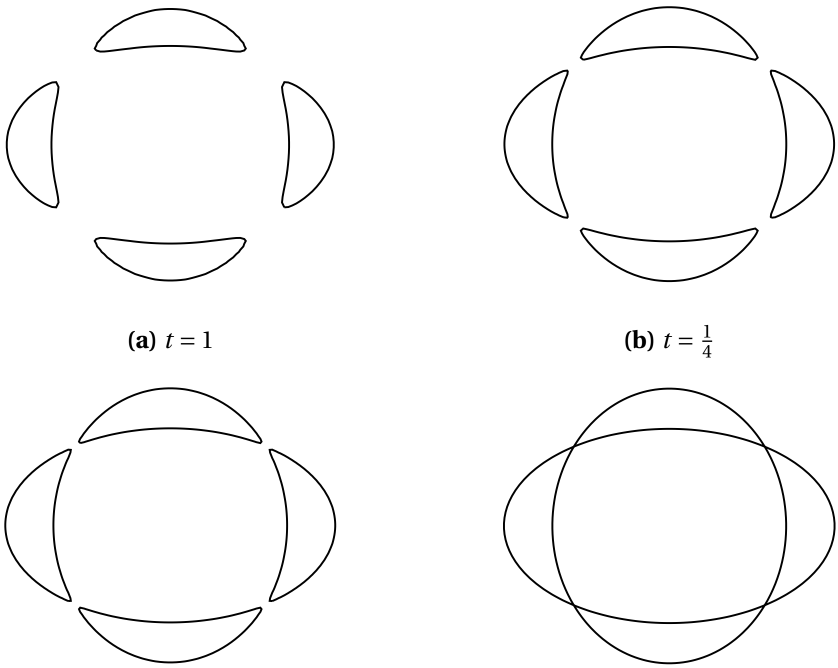
(d) $x^3(z-x) - y^3x - (z-x)^3y = 0$

Figure 1.1: The affine part, given by $z = 1$, of (A) a line (B) a conic (C) a cubic and (D) a quartic.

planar, curves and whose topology agrees with the above notion of closeness. This idea is surprisingly hard to make precise and the space itself is even more complicated to construct. Nevertheless, there is a *moduli space of curves* whose points are in bijection with isomorphism classes of curves and whose geometry reflects how curves vary in families. As a sidenote for experts we point out that, contrary to what the above might suggest, the moduli spaces occurring in this thesis will be considered as coarse spaces and not as stacks.

The moduli space of curves is not connected but has one component $\mathcal{M}_g$ for each nonnegative integer g (although we need to be extra careful in the cases where g is 0 or 1). The number g is the *genus* of a curve C whose isomorphism class lies in $\mathcal{M}_g$, i.e. $g = \dim_k H^0(C, \omega_C)$ where ω_C is the canonical sheaf of C. Over $\mathbb{C}$, each curve C is a Riemann surface and then the genus is simply the number of holes in C, see Figure 1.3.

Remark 1.1.1. The word "moduli" stems from the Latin word "modus" and means "modes" or "measures". In this context, the terminology was first



(a) $t = 1$ **(b)** $t = \frac{1}{4}$

(c) $t = \frac{1}{20}$ **(d)** $t = 0$

Figure 1.2: The affine part, given by $z = 1$, of the plane quartic $(x^2 + 2y^2 - z^2)(2x^2 + y^2 - z^2) + tz^4 = 0$ for four different values of t. Note that the curve in Figure 1.2d is singular.

used by Riemann [75] but it is likely that he was inspired by material science which at the time already had an established terminology of various "elastic moduli" describing how easily an object is reversibly deformed.

We also remark that classification first in terms of genus and then in terms of modulus is not unique for mathematics - for instance, in marine biology there are molluscs (sea snails) whose genera are further subdivided into moduli.

A smooth plane quartic has genus 3 but not all genus 3 curves are plane quartics. The set of genus 3 curves which are not plane quartics consists of the so-called hyperelliptic curves of genus 3. However, most genus 3 curves are plane quartics and the hyperelliptic curves will only play a minor role in this thesis. More precisely, the subspace $\mathcal{Q}$ of $\mathcal{M}_3$ consisting of isomorphism classes of plane quartics is dense and open so if we were able to pick a point of $\mathcal{M}_3$ at random we would get a plane quartic with probability 1.

Many of the complications in the construction of $\mathcal{M}_g$ stem from the ex-

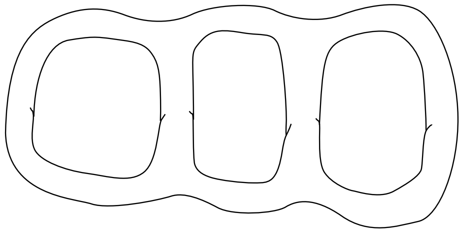


Figure 1.3: A curve of genus 3.

istence of curves with automorphisms. One approach towards a solution to this problem is to add extra structure to the curves. The simplest thing one can do is to label a number of points of the curves and require that isomorphisms take the i'th marked point of one curve to the i'th marked point of the other, see Figure 1.4. We thus obtain moduli spaces $\mathcal{M}_{g,n}$ of genus g curves with n ordered marked points. Adding more points yields more restrictions on the automorphisms and taking $n > 2g+2$ suffices to ensure that there is no nontrivial automorphisms if $g \geq 2$.

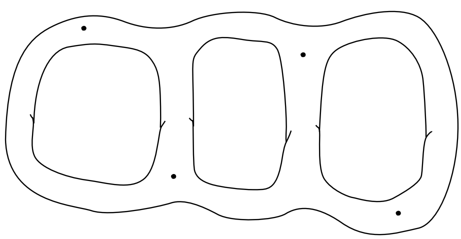


Figure 1.4: A curve of genus 3 with 4 marked points.

The downside of this path is that many aspects of the geometry of $\mathcal{M}_{g,n}$ are different from those of $\mathcal{M}_g$. Most notably we have $\dim(\mathcal{M}_g) = 3g-3$ for $g \geq 2$ while $\dim(\mathcal{M}_{g,n}) = 3g-3+n$, so each marked point adds one extra dimension. A method that avoids changing the dimension is to add something called a *level N structure*. We will discuss level structures in more detail in Chapter 3. Here we only mention that the most important part of a level N structure is a finite abelian group. A curve only has finitely many level N structures so the moduli space $\mathcal{M}_g[N]$ of genus g curves with level N structure has the same dimension as $\mathcal{M}_g$ and a curve with level N structure does not have any automorphisms if $N \geq 3$, see Section 2.A of [52]. The spaces $\mathcal{M}_g[N]$ were first studied by Mumford in [70].

In this thesis we take the perspective that curves with level structure are interesting in their own right and our main objects of study will be the moduli spaces $\mathcal{Q}[2]$ of smooth plane quartics with level 2 structure and $\mathcal{Q}_1[2]$

of smooth plane quartics with level 2 structure and one marked point. In particular, these spaces come with natural actions of the group $\mathrm{Sp}(6,\mathbb{F}_2)$ and this action will play an important role throughout the thesis.

1.2 Cohomology

One intuitive approach towards understanding a space is to try to understand its subspaces. Depending on what sort of questions we are interested in we might want to introduce an equivalence relation between the subspaces. For instance, if we are interested in the topology of a space we might want to consider two subspaces as equivalent if one can be continuously deformed to the other. This is the idea behind the *homology groups* of a space. However, it is often useful to study a space through the functions on the space. Taking this perspective one arrives at the *cohomology* of a space. Homology and cohomology share many properties but, since functions can be pointwise multiplied, cohomology has the advantage of having a ring structure.

Both the homology and the cohomology groups of a complex space are equipped with additional structures called *mixed Hodge structures* which are both interesting and often useful in computations. Mixed Hodge structures have many nice properties such as functoriality, compatibility with products of spaces (in the sense of the Künneth theorem) and they are also compatible with products in cohomology.

We are now in a position where we can state the aim of the thesis: we want to compute the cohomology groups of $\mathcal{Q}$ and related spaces. Since the group $\mathrm{Sp}(6,\mathbb{F}_2)$ acts on the spaces in question, their cohomology groups will be $\mathrm{Sp}(6,\mathbb{F}_2)$-representations and we want to determine these representations. The cohomology groups also carry mixed Hodge structures and we also want to determine these. We will not reach this goal in all cases, but at least in some, and we will also see some interesting byproducts along the way.

Since the spaces we are investigating are defined over the integers we have the flexibility of working over the complex numbers as well as over finite fields (although the spaces are not very nice in characteristic 2). Despite the fact that the complex and finite worlds are rather different, both perspectives will prove useful and in both cases there are notions of "purity" which will be essential. Another recurring theme will be the use of inclusion-exclusion arguments.

More precisely, when we are investigating a complex space X, the mixed Hodge structures allow the construction of a weighted Euler characteristic $E^m(X)$ which remembers the Hodge weights but forgets the cohomological grading. This Euler characteristic is additive and can therefore be computed by realizing X as a subspace of some larger space Y and then computing $E^m(Y)$ and $E^m(Y \setminus X)$. Finally, purity will allow us to recover the cohomological grading from the Hodge weights.

A space X over a finite field $\mathbb{F}_q$ with q elements has a finite number of points over $\mathbb{F}_q$. If X is a subspace of a larger space Y we have that the number of $\mathbb{F}_q$-points of Y is the number of $\mathbb{F}_q$-points of X plus the number of $\mathbb{F}_q$-points of $Y \setminus X$ so inclusion-exclusion arguments are applicable also in this setting. Such point counts, which are rather number theoretic in nature give certain Euler characteristics, which are of a topological nature, via the Lefschetz trace formula. Again, purity will allow us to go from Euler characteristics to actual cohomology.

We now give a more precise outline of the thesis.

1.3 Outline of thesis

Chapter 2 In this chapter we give much of the classical geometric background used later in the thesis. In particular, we discuss plane quartics, their relation to Del Pezzo surfaces of degree 2 as well as how they are determined by seven points in $\mathbb{P}^2$. Sections 2.2–2.4 are in large parts reworked material from my licentiate thesis [12].

Chapter 3 In this chapter we present some theory regarding lattices and quadratic forms over $\mathbb{F}_2$. This theory is then used to discuss level structures on curves and geometric markings on Del Pezzo surfaces. Also most of this chapter is reworked material from [12]. Section 3.6 is entirely new.

Chapter 4 Here we discuss some constructions and results of Looijenga from his paper [62], on which much of this thesis builds. Most importantly, we will see descriptions of certain subspaces of $\mathcal{Q}_1[2]$ in terms of arrangements of hyperplanes and arrangements of hypertori.

Chapter 5 In the first part of this chapter we review some of the theory of cohomology of general arrangements. In particular, we discuss the theory of purity as given by Dimca and Lehrer in [36] and we also give a theorem of Macmeikan [63] which will be important to us. In the second part of the chapter we give the results of the paper [13]. More precisely, we apply the theory from the first part of the chapter to the case of toric arrangements associated to root systems. In particular, we develop Algorithms 5.8.1 and

5.9.2 computing the cohomology of the complement of a general root system (equivariantly with respect to the corresponding Weyl group) and in Section 5.11 use this algorithm to compute the cohomology groups of the complements of toric arrangements associated to the exceptional root systems G_2, F_4, E_6 and E_7. We also compute the total cohomology of the complement of the toric arrangement associated to the root system A_n as well as its Poincaré polynomial, see Theorems 5.10.8 and 5.10.10.

Chapter 6 This chapter contains most of the results of [14]. More precisely, we apply the results from Chapter 5 to Looijenga's descriptions. We thereby obtain the $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant cohomology of the moduli space $\mathcal{Q}_{\mathrm{ord}}[2]$ of plane quartic curves with level 2 structure marked with one ordinary point, the moduli space $\mathcal{Q}_{\mathrm{btg}}[2]$ of plane quartic curves with level 2 structure marked with one bitangent point, the moduli space $\mathcal{Q}_{\mathrm{flx}}[2]$ of plane quartic curves with level 2 structure marked with one flex point, the moduli space $\mathcal{Q}_{\mathrm{hfl}}[2]$ of plane quartic curves with level 2 structure marked with one hyperflex point as well as some related spaces. The results are given in Tables 6.1–6.6. We also get a partial description of the cohomology of $\mathcal{Q}_1[2]$ and we relate the cohomology of $\mathcal{Q}[2]$ to that of $\mathcal{Q}_{\mathrm{flx}}[2]$, see Proposition 6.1.3 and 6.2.3.

Chapter 7 In this chapter we give most of the results of [15], which are improved versions of results of [12]. Specifically, through point counts over finite fields we are able to determine each cohomology group of $\mathcal{Q}[2]$ as a representation of S_7. The results are presented in Tables 7.1 and 7.2.

Chapter 8 Chapter 8 is where we discuss hyperelliptic curves. Specifically, we review a description of $\mathcal{H}_g[2]$ (due to work of Dolgachev and Ortland [38], Tsuyumine [84] and Runge [77]) and use this description to compute the cohomology of $\mathcal{H}_3[2]$ and $\mathcal{H}_{3,1}[2]$ equivariantly with respect to $\mathrm{Sp}(6,\mathbb{F}_2)$, see Tables 8.2 and 8.4. The results of this section can be found in [12], [14] and [15].

Chapter 9 In this chapter we give a partial description of the cohomology of $\mathcal{M}_{3,1}[2]$. We also determine $H^k(\mathcal{Q}[2])$ as a $\mathrm{Sp}(6,\mathbb{F}_2)$-representation for $0 \leq k \leq 3$. Both these results are from [14]. Finally, we discuss some possible directions for future work.

Appendices In the appendices we give some of the code used in this thesis.

2. Geometric background

In this chapter we discuss some classical geometry that we will use throughout the thesis. We start by investigating plane quartics - a subject with a fascinating history, starting with the 1486 inquisitional burning at the stake of the Spanish mathematician Valmes for solving the quartic equation, see [8].[1] Our focus will however be on the less dramatic but perhaps mathematically more interesting theory of the 19'th century, developed by Aronhold, Cayley, Hesse, Klein, Plücker, Schottky, Steiner and Weber among many others. In particular we shall see how the tangents to plane quartic curves behave. We shall then relate plane quartics to Del Pezzo surfaces, another rich and interesting topic. In particular, we will see how Del Pezzo surfaces can be realized as blowups of the projective plane in a set of points. This will give us the flexibility of three different perspectives, each with its own benifits and uses.

Before starting we establish some terminology. A *variety* shall be a reduced and separated scheme of finite type over an algebraically closed field. Thus, we do not require a variety to be irreducible. By the word *curve* shall mean an equidimensional variety of dimension 1 and a *surface* is an equidimensional variety of dimension 2. Thus, we also allow curves and surfaces to be reducible.

2.1 Plane quartics

Let K be an algebraically closed field of characteristic zero. A *smooth plane quartic curve*, or a plane quartic for short, is a nonsingular variety $C \subset \mathbb{P}^2(K)$ given by an equation

$$f(x, y, x) = 0,$$

[1] Tomás de Torquemada, the first inquisitor general, allegedly said that it was the will of God that the solution of the quartic should be beyond human understanding. However, this story is not very well documented and its truth is debated, see [8]. The first published solution of the quartic equation appears in Cardano's "Ars Magna" [20] from 1545 and is due to the Italian mathematician Ferrari who appears to have found his solution in 1540.

where $f(x, y, z) \in K[x, y, z]$ is a nonsingular homogeneous polynomial of degree 4. By the genus-degree formula we find that such a curve has genus $\frac{1}{2}(4-1)(4-2) = 3$. If we let $K[x, y, z]_4$ denote the set of homogeneous polynomials of degree 4 and let Δ_4 denote the set of singular homogeneous polynomials of degree 4, we can identify the space of plane quartics with the space $\mathbb{P}\left(K[x, y, z]_4 \setminus \Delta_4\right)$.

Let $\mathcal{H}_g$ denote the moduli space of hyperelliptic curves of genus g. Any smooth, nonhyperelliptic curve of genus $g \geq 2$ is embedded into $\mathbb{P}^{g-1}(K)$ as a curve of degree $2g-2$ via its canonical linear system. If C is a nonhyperelliptic curve of genus 3, a choice of basis of its space of global sections (up to a nonzero scalar multiple) gives an embedding of C into $\mathbb{P}^2(K)$ as a curve of degree 4. The projective general linear group $\mathrm{PGL}(3, K)$ acts on $\mathbb{P}\left(K[x, y, z]_4 \setminus \Delta_4\right)$ by changing bases and we may realize $\mathcal{M}_3 \setminus \mathcal{H}_3$ as the affine quotient $\mathbb{P}\left(K[x, y, z]_4 \setminus \Delta_4\right)/\mathrm{PGL}(3, K)$. We denote the space $\mathcal{M}_3 \setminus \mathcal{H}_3$ by $\mathcal{Q}$ and call it the *moduli space of plane quartics*. It is a dense open subset of $\mathcal{M}_3$ and its dimension can be computed as $\binom{4+3-1}{3-1} - 1 - 8 = 6$ (which of course agrees with the usual dimension formula $\dim(\mathcal{M}_g) = 3g-3$). When we view $\mathcal{Q}$ and $\mathcal{H}_3$ as subsets of $\mathcal{M}_3$ we will sometimes refer to them as the *quartic* and *hyperelliptic locus*, respectively.

We fix a plane quartic C and a point P on C and we let $T_P \subset \mathbb{P}^2(K)$ denote the tangent line of C at P. Since C is of degree 4, Bézout's theorem tells us that the intersection product $C \cdot T_P$ will consist of 4 points. There are four possibilities:

(*a*) $T_P \cdot C = 2P + Q + R$ where Q and R are two distinct points on C, both different from P. In this case, T_P is called an *ordinary tangent line* of C and P is called an *ordinary point* of C, see Figure 2.1a.

(*b*) $T_P \cdot C = 2P + 2Q$ where $Q \neq P$ is a point on C. In this case, T_P is called a *bitangent* of C and P is called a *bitangent point* of C, see Figure 2.1b.

(*c*) $T_P \cdot C = 3P + Q$ where $Q \neq P$ is a point on C. In this case, T_P is called a *flex line* of C and P is called a *flex point* of C, see Figure 2.1c.

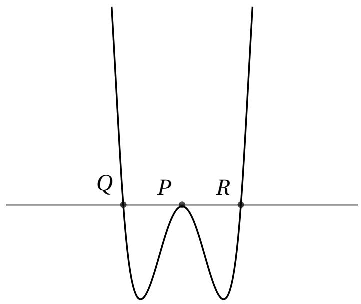
(*d*) $T_P \cdot C = 4P$. In this case, T_P is called a *hyperflex line* of C and P is called a *hyperflex point* of C, see Figure 2.1d.

Let (C, P) be an ordered pair where C is a plane quartic curve and P is a point of C. We say that two such pairs (C, P) and (C', P') are isomorphic if there is an isomorphism of curves $\phi : C \to C'$ such that $P' = \phi(P)$. We call the moduli space of such pairs the *moduli space of pointed plane quartics* and denote it by $\mathcal{Q}_1$. There is a natural forgetful morphism $\mathcal{Q}_1 \to \mathcal{Q}$ sending the isomorphism class of a pair (C,P) to the isomorphism class of the curve C and in particular we see that $\dim(\mathcal{Q}_1) = \dim(\mathcal{Q}) + 1 = 7$.

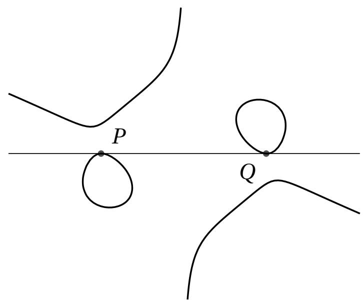
By considering the intersection product $C \cdot T_P$ at the marked point P we decompose $\mathcal{Q}_1$ as a disjoint union

$$\mathcal{Q}_1 = \mathcal{Q}_{\mathrm{ord}} \sqcup \mathcal{Q}_{\mathrm{btg}} \sqcup \mathcal{Q}_{\mathrm{flx}} \sqcup \mathcal{Q}_{\mathrm{hfl}},$$

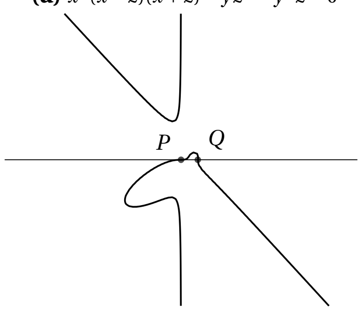
where $\mathcal{Q}_{\mathrm{ord}}$ consists of the isomorphism classes of pairs (C,P) where P is ordinary, $\mathcal{Q}_{\mathrm{btg}}$ consists of the isomorphism classes where P is a bitangent point, $\mathcal{Q}_{\mathrm{flx}}$ consists of the isomorphism classes where P is a flex point and $\mathcal{Q}_{\mathrm{hfl}}$ consists of the isomorphism classes where P is a hyperflex point.



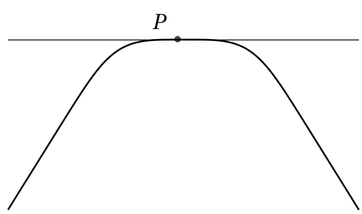
(a) $x^2(x-z)(x+z) - yz^3 - y^3z = 0$



(b) $(2z^2 - x^2 - xy)^2 - xy(x-y)(x+3y) = 0$



(c) $x^3(z-x) - y^3x - (z-x)^3y = 0$



(d) $x^4 + y^3z + yz^3 = 0$

Figure 2.1: The affine part, given by $z = 1$, of a plane quartic with: (A) an ordinary point P, (B) a bitangent point P, (C) a flex point P or (D) a hyperflex point P.

Remark 2.1.1. The curve in Figure 2.1b has been investigated in [57] and the equation for the curve in Figure 2.1d appears in [80]. We would also like to point out the amusing resemblance between Figure 2.1c and some of the photographs claimed to depict the Loch Ness monster.

As indicated by the terminology, most points of a plane quartic are ordinary points. Any plane quartic has a number of bitangents as well as a number of flex lines, but most plane quartics do not have hyperflex lines. More precisely, the locus $\mathcal{Q}_{\mathrm{ord}}$ is a dense open subset of $\mathcal{Q}_1$, the loci $\mathcal{Q}_{\mathrm{btg}}$ and $\mathcal{Q}_{\mathrm{flx}}$ both have codimension 1 in $\mathcal{Q}_1$ and the are forgetful morphisms $\mathcal{Q}_{\mathrm{btg}} \to \mathcal{Q}$ and $\mathcal{Q}_{\mathrm{flx}} \to \mathcal{Q}$ are surjective whereas $\mathcal{Q}_{\mathrm{hfl}}$ has codimension 2 in $\mathcal{Q}_1$ and the forgetful morphism $\mathcal{Q}_{\mathrm{hfl}} \to \mathcal{Q}$ is neither surjective nor injective.

In order to investigate the situation more closely, let $H_C \subset \mathbb{P}^2(K)$ be the Hessian of C, i.e. the curve defined by the equation

$$\det\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = 0,$$

where $f(x,y,z) = 0$ is the equation defining C. Thus, H_C is a curve of degree 6 and, for a general curve C, the intersection between H_C and C will consist of the flex points of C. By Bézout's theorem, we see that there are $6 \cdot 4 = 24$ flex points.

Let $C^\vee$ be the dual curve of C. The dual of a curve of degree d has degree $d \cdot (d-1)$ so $C^\vee$ has degree 12, arithmetic genus 55 and geometric genus 3. If C is general, then $C^\vee$ will only have double points as singularities and each singularity will either correspond to a flex line or a bitangent. By the genus-degree formula we conclude that the number of bitangents is

$$55 - 3 - 24 = 28.$$

If C is not general, then C still has 24 flex lines and 28 bitangents as long as we count a hyperflex line both as a flex line and as a bitangent line in the sense of intersection theory. For many purposes it makes much sense to adopt this broader definition of flex lines and bitangents and most authors indeed do so. However, the distinction between flex points, bitangent points and hyperflex points will at times be important to us. At these times we shall talk about *genuine* flex lines and bitangent lines in order to stress that we do not allow hyperflex lines. For a picture, see Figure 2.2.

At any rate, we have partially proven and hopefully convinced the reader of the following classical result from 1834. For a more complete account, see for instance [53], Chapter IV.2.

Theorem 2.1.2 (Plücker [74]). *Let C be a plane quartic and let N_{flx}, N_{btg} and N_{hfl} denote the number of genuine flex lines, genuine bitangent lines and hyperflex lines, respectively. Then*

$$N_{\mathrm{flx}} + 2N_{\mathrm{hfl}} = 24, \quad \textit{and}, \quad N_{\mathrm{btg}} + N_{\mathrm{hfl}} = 28.$$

Thus, if we let $\mathcal{Q}_{\overline{\mathrm{btg}}}$ and $\mathcal{Q}_{\overline{\mathrm{flx}}}$ be the closures of $\mathcal{Q}_{\mathrm{btg}}$ and $\mathcal{Q}_{\mathrm{flx}}$ in $\mathcal{Q}_1$, then

$$\mathcal{Q}_{\overline{\mathrm{btg}}} = \mathcal{Q}_{\mathrm{btg}} \sqcup \mathcal{Q}_{\mathrm{hfl}}, \quad \text{and} \quad \mathcal{Q}_{\overline{\mathrm{flx}}} = \mathcal{Q}_{\mathrm{flx}} \sqcup \mathcal{Q}_{\mathrm{hfl}},$$

and we see that the forgetful morphisms

$$\mathcal{Q}_{\overline{\mathrm{btg}}} \to \mathcal{Q}, \quad \text{and} \quad \mathcal{Q}_{\overline{\mathrm{flx}}} \to \mathcal{Q},$$

are finite of degrees 28 and 24, respectively.

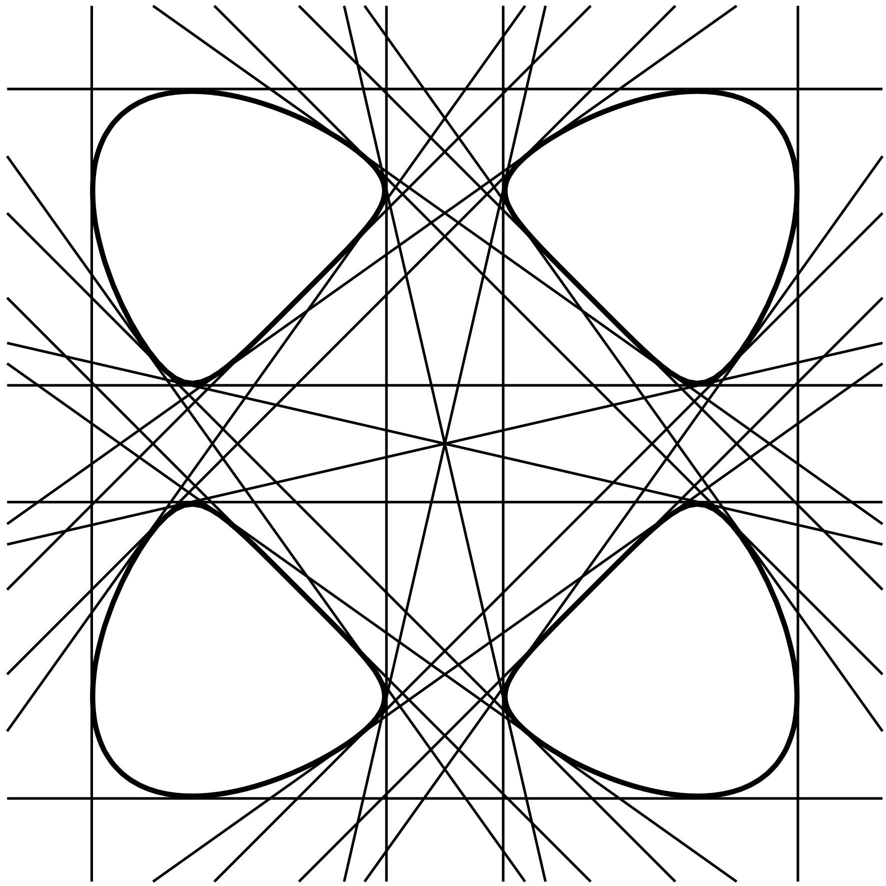


Figure 2.2: The affine part, given by $z = 1$, of the quartic $x^4 + y^4 - 6(x^2 + y^2)z^2 + 10z^4 = 0$ and its 28 bitangents of which 24 are genuine bitangents and 4 are hyperflex lines (namely the lines of slope ± 1 intersecting only one component). The above curve was discovered by Edge [41].

Similarly to the unpointed case, the moduli space $\mathcal{Q}_1$ is a dense open subset of the moduli space of pointed genus 3 curves, $\mathcal{M}_{3,1}$, and its complement in $\mathcal{M}_{3,1}$ is the moduli space of pointed hyperelliptic curves of genus 3, $\mathcal{H}_{3,1}$.

If we let $K = \mathbb{C}$, the moduli spaces introduced in this section all have de Rham cohomology groups and by Deligne [31], each of these groups is equipped with a mixed Hodge structure. In [62], Looijenga computed some of these cohomology groups together with their mixed Hodge structure. He found that the groups all had remarkably simple mixed Hodge structures - they are all pure of Tate type (k,k) for various values of k. This allowed him to present his result in an exceptionally compact way which he dubbed *Poincaré-Serre polynomials.*

Definition 2.1.3 (Looijenga [62])**.** Let X be a variety with de Rham cohomology groups $H^k(X)$ and let $d_{k,l}$ be the dimension of the subqoutient of $H^k(X)$ of Hodge weight l. The Poincaré-Serre polynomial of X is the polynomial

$$PS_X(t,u) = \sum_{k,l} d_{k,l} t^k u^l.$$

In particular, we note that $PS_X(t,1)$ is equal to the usual Poincaré polynomial $P_X(t)$ of X. We are now ready to state the results of Looijenga.

Theorem 2.1.4 (Looijenga [62])**.** *The de Rham cohomology groups of the moduli spaces $\mathcal{M}_3$, $\mathcal{Q}$ and $\mathcal{H}_3$ are all pure of Tate type (k,k) for various k and their Poincaré-Serre polynomials are given by*

$$PS_{\mathcal{M}_3}(t,u) = 1 + t^2u^2 + t^6u^{12}, \quad PS_{\mathcal{Q}}(t,u) = 1 + t^6u^{12}, \quad PS_{\mathcal{H}_3}(t,u) = 1.$$

Theorem 2.1.5 (Looijenga [62])**.** *The de Rham cohomology groups of the moduli spaces $\mathcal{Q}_{\mathrm{btg}}$, $\mathcal{Q}_{\mathrm{flx}}$, $\mathcal{Q}_{\mathrm{hfl}}$, $\mathcal{H}_{3,1}$, $\mathcal{Q}_{\overline{\mathrm{btg}}}$ and $\mathcal{Q}_{\overline{\mathrm{flx}}}$ are all pure of Tate type (k,k) for various k and their Poincaré-Serre polynomials are given by*

$$\begin{aligned}
&PS_{\mathcal{Q}_{\mathrm{btg}}}(t,u) = 1 + tu^2 + t^5u^{10} + 2t^6u^{12}, && PS_{\mathcal{Q}_{\mathrm{flx}}}(t,u) = 1 + t^6u^{12},\\
&PS_{\mathcal{Q}_{\mathrm{hfl}}}(t,u) = 1, && PS_{\mathcal{H}_{3,1}}(t,u) = 1 + t^2u^2,\\
&PS_{\mathcal{Q}_{\overline{\mathrm{btg}}}}(t,u) = 1 + t^5u^{10} + 2t^6u^{12}, && PS_{\mathcal{Q}_{\overline{\mathrm{flx}}}}(t,u) = 1 + t^2u^2 + t^6u^{12}.
\end{aligned}$$

In [62], Looijenga also made computations for the cohomology of $\mathcal{Q}_{\mathrm{ord}}$, $\mathcal{Q}_{\overline{\mathrm{ord}}} = \mathcal{Q}_1 \setminus \mathcal{Q}_{\overline{\mathrm{btg}}}$, $\mathcal{Q}_1$ and $\mathcal{M}_{3,1}$. Unfortunately, there was a small error in the calculation of the cohomology of $\mathcal{Q}_{\mathrm{ord}}$ which then propagated to the computations for the other two spaces. This was noticed and partially remedied in [47]. Using our results of Chapter 6, we can reprove the above results. Also, by reading off the first columns of Table 6.1 and Table 6.5 we can add the following two spaces to the list.

Theorem 2.1.6. *The de Rham cohomology groups of the moduli spaces $\mathcal{Q}_{\mathrm{ord}}$ and $\mathcal{Q}_{\overline{\mathrm{ord}}}$ both have Tate type (k,k) for various k and their Poincaré-Serre polynomials are given by*

$$\begin{aligned} PS_{\mathcal{Q}_{\mathrm{ord}}}(t,u) &= 1 + tu^2 + 2t^6u^{12} + 2t^7u^{14}, \\ PS_{\mathcal{Q}_{\overline{\mathrm{ord}}}}(t,u) &= 1 + 2t^6u^{12} + t^7u^{14} \end{aligned}$$

We mention that the cohomology of $\mathcal{Q}_1$ fits into a Gysin exact sequence

$$\cdots \to H^k(\mathcal{Q}_1) \to H^k\big(\mathcal{Q}_{\overline{\mathrm{ord}}}\big) \to H^{k-1}\Big(\mathcal{Q}_{\overline{\mathrm{btg}}}\Big) \to H^{k+1}(\mathcal{Q}_1) \to \cdots.$$

By considering the Hodge weights, this sequence splits into four term sequences

$$0 \to W_kH^k(\mathcal{Q}_1) \to H^k\big(\mathcal{Q}_{\overline{\mathrm{ord}}}\big) \to H^{k-1}\Big(\mathcal{Q}_{\overline{\mathrm{btg}}}\Big)(-1) \to W_kH^{k+1}(\mathcal{Q}_1) \to 0,$$

where $W_kH^i(\mathcal{Q}_1)$ denotes the weight k part of $H^i(\mathcal{Q}_1)$. Using Theorems 2.1.5 and 2.1.6 we now see that

$$PS_{\mathcal{Q}_1}(t,u) = 1 + t^2u^2 + t^6u^{12} + t^8u^{14} + a(t^6u^{12} + t^7u^{12}) + b(t^7u^{14} + t^8u^{14}),$$

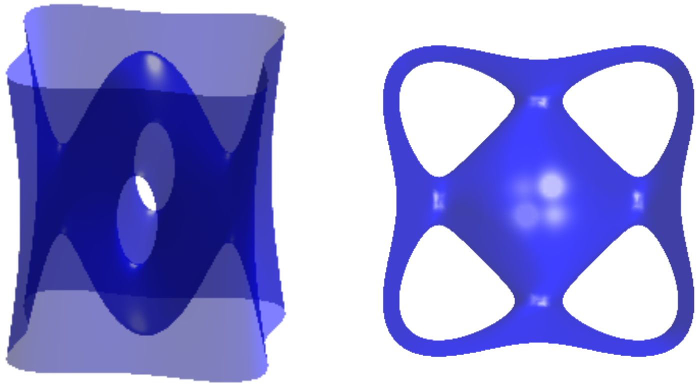
where a and b are either 0 or 1. The above formula also appears in [47] where it is obtained in a slightly different fashion.

Remark 2.1.7. Although the undetermined coefficients a and b are admittedly unsatisfactory, their appearance should not be unexpected. On the contrary, it should be seen as surprising that similar coefficients do not occur elsewhere, for instance in the Poincaré-Serre polynomial of $\mathcal{M}_3$. The simple reason is that the cohomology groups of $\mathcal{Q}$ and $\mathcal{H}_3$ match up in such a way that the corresponding Gysin sequence splits into short exact sequences whereby the cohomology of $\mathcal{M}_3$ is determined. For $\mathcal{Q}_1$ this is not the case and therefore one needs to actually understand either the Gysin map or the restriction map in order to determine a and b by this method.

We conclude this section by pointing out that Bergström and Tommasi [11] have used a different method to show that a and b are both zero.

Theorem 2.1.8 (Bergström and Tommasi [11])**.** *The Poincaré-Serre polynomials of $\mathcal{Q}_1$ and $\mathcal{M}_{3,1}$ are given by*

$$\begin{aligned} PS_{\mathcal{Q}_1}(t,u) &= 1 + t^2u^2 + t^6u^{12} + t^8u^{14}, \\ PS_{\mathcal{M}_{3,1}}(t,u) &= 1 + 2t^2u^2 + t^4u^4 + t^6u^{12} + t^8u^{14}. \end{aligned}$$



(a) **(b)**

Figure 2.3: Two pictures of the Del Pezzo surface of degree 2 given by the equation $t^2 = x^4 + y^4 - 6(x^2 + y^2)z^2 + 10z^2$.

2.2 Del Pezzo Surfaces and Plane Quartics

Once again, let K be an algebraically closed field of characteristic zero. A *Del Pezzo surface* is a smooth, rational surface S such that the *anticanonical class* $-K_S$ is ample. The number K_S^2 is called the *degree* of S. In other words, there are positive integers m and n and a closed embedding

$$i : S \hookrightarrow \mathbb{P}^m(K),$$

such that the *anticanonical sheaf* ω_S^{-1} satisfies $(\omega_S^{-1})^n = i^*\mathcal{O}_{\mathbb{P}^m}(1)$. We denote the moduli space of Del Pezzo surfaces of degree d by $\mathcal{DP}_d$.

In the upcoming sections we describe the classical relationship between Del Pezzo surfaces and plane quartics. For more complete treatments of the topic of Del Pezzo surfaces we recommend [65] and [56].

Let C be a quartic curve given by the equation $f(x,y,z) = 0$ and let t be another variable of degree 2. The equation

$$t^2 = f(x,y,z), \tag{2.2.1}$$

describes a surface S in the weighted projective space $\mathbb{P}(1,1,1,2)$, see Figure 2.3 and compare with Figure 2.2 (for an introduction to weighted projective spaces, see [56]). We shall show that S is a Del Pezzo surface of degree 2.

Proposition 2.2.1. *Let $f(x,y,z) \in K[x,y,z]$ be a smooth, homogeneous polynomial of degree 4. Then the surface $S \subset \mathbb{P}(1,1,1,2)$ defined by the equation $t^2 - f(x,y,z) = 0$ is a Del Pezzo surface of degree 2.*

Proof. The space $\mathbb{P}(1,1,1,2)$ has a singularity at the point $P = [0:0:0:1]$, which clearly is not a point of S, and is smooth elsewhere. We may thus verify that S is smooth by checking the partial derivatives of $g(x,y,z,t) = t^2 - f(x,y,z)$:

$$\frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x}, \quad \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y},$$
$$\frac{\partial g}{\partial z} = -\frac{\partial f}{\partial z}, \quad \frac{\partial g}{\partial x} = 2t.$$

Since C is smooth we have that the partial derivatives of f do not vanish simultaneously unless $x = y = z = 0$ and since P is not contained in S we conclude that S is smooth.

To simplify notation, let $\mathcal{O}(m)$ denote the sheaf $\mathcal{O}_{\mathbb{P}(1,1,1,2)}(m)$. The dualizing sheaf of $\mathbb{P}(1,1,1,2)$ is given by

$$\omega_{\mathbb{P}(1,1,1,2)} = \mathcal{O}(-1-1-1-2) = \mathcal{O}(-5).$$

Since the weighted degree of S is 4, the canonical sheaf of S is

$$\omega_S = \omega_{\mathbb{P}(1,1,1,2)} \otimes \mathcal{O}(4) \otimes \mathcal{O}_S = \mathcal{O}(-5+4)|_S = \mathcal{O}(-1)|_S.$$

In particular, see that $\omega_S^{-1} = \mathcal{O}(1)|_S$ is ample. It now follows that $h^0(S, mK_S) = 0$ for all $m \geq 1$. Furthermore, the Riemann-Roch theorem for surfaces, see for example [53], Theorem V.1.6, states that if D is a divisor on a smooth surface X, then

$$h^0(X,D) - h^1(X,D) + h^0(K_X - D) = \frac{1}{2}D(D-K_X) + \chi(\mathcal{O}_X).$$

We now take $X = S$ and let D be the zero divisor to get $\chi(\mathcal{O}_S) = 1$. It now follows from Castelnuovo's rationality criterion that S is rational, see [53], Theorem V.6.2. We conclude that S is a Del Pezzo surface.

From Equation 2.2.1 we see that the map

$$\iota : S \to S,$$

given by

$$[x:y:z:t] \mapsto [x:y:z:-t],$$

is an involution. The quotient $S/\langle\iota\rangle$ is isomorphic to the projective plane and we have a degree 2 covering map $p : S \to \mathbb{P}^2$ given by

$$[x:y:z:t] \mapsto [x:y:z].$$

The ramification divisor R of p is exactly the fixed point locus of ι and since C is given by the equation $f(x,y,z) = 0$, we see that the fixed point locus of ι is a curve isomorphic to C. By the Riemann-Hurwitz formula we have

$$K_S \sim p^* K_{\mathbb{P}^2} + R.$$

Let L be the class of a line in $\mathbb{P}^2$. We have $K_{\mathbb{P}^2} \sim -3L$ so $p^*K_{\mathbb{P}^2} \sim -3p^*L$. The curve C is the branch divisor of p so

$$p^*C \sim 2R.$$

Since $C \sim 4L$ we have $p^*C = 4p^*L$ and we thus see that $R \sim 2p^*L$. It now follows that

$$K_S \sim -3p^*L + 2p^*L = -p^*L.$$

We concde that

$$\begin{aligned} K_S^2 &= (-p^*L)^2 = \\ &= p^*(L^2) = \\ &= p^*(1) = \\ &= 2, \end{aligned}$$

since p has degree 2. We have thus shown that S is a Del Pezzo surface of degree 2. □

We shall now prove that the converse is also true. To see this, it is natural to study sections of powers of the anticanonical sheaf. Therefore, let S be a Del Pezzo surface and let the *anticanonical ring* of S be the ring

$$R(S) = \bigoplus_{m \geq 0} H^0\big(S, (\omega_S^{-1})^m\big).$$

Lemma 2.2.2. *Let S be a Del Pezzo surface of degree 2. Then the anticanonical ring of S is generated by $H^0\big(S, \omega_S^{-1}\big)$ and $H^0\big(S, (\omega_S^{-1})^2\big)$. Moreover, the cohomology group $H^0\big(S, (\omega_S^{-1})^m\big)$ has dimension m^2+m+1 as a K-vector space.*

For a proof, see [56], Corollary III.3.2.5 and Proposition III.3.4.

Proposition 2.2.3. *Let S be a Del Pezzo surface of degree 2. Then S is isomorphic to a surface in $\mathbb{P}(1,1,1,2)$ given by an equation $t^2 - f(x,y,z) = 0$ of degree 4 where $f(x,y,z)$ is smooth.*

Proof. By Lemma 2.2.2 we have that the dimension of $H^0\big(S, \omega_S^{-1}\big)$ is 3. Let $\{x, y, z\}$ be a basis. Since $|-K_S|$ is ample and base point free, see Proposition III.3.4 of Kollar [56] , so $H^0\big(S, \omega_S^{-1}\big)$ generates a subspace of $H^0\big(S, (\omega_S^{-1})^2\big)$ of dimension 6. By Lemma 2.2.2 we have that $H^0\big(S, (\omega_S^{-1})^2\big)$ has dimension 7 and we let $\{x^2, xy, xz, y^2, yz, z^2, t\}$ be a basis. Applying Lemma 2.2.2 once more we have that $H^0\big(S, (\omega_S^{-1})^3\big)$ has dimension 13 and that $H^0\big(S, (\omega_S^{-1})^4\big)$ has dimension 21. There are 13 monomials in x, y, z and t of degree 3 and 22 of degree 4 so there must be a relation

$$g(x,y,z,t) = t^2 + tf_2(x,y,z) + f_4(x,y,z) = 0,$$

in degree 4. In the expression above, $f_2(x, y, z)$ and $f_4(x, y, z)$ are polynomials in x, y and z of degrees 2 and 4 respectively.

Since the characteristic of K is not 2, we may complete the square and make a change of variables so that the relation becomes of the form

$$g(x, y, z, t) = t^2 - f(x, y, z),$$

where $f(x, y, z)$ has degree 4. Thus

$$S \cong \mathrm{Proj}(R(S)) \cong \mathrm{Proj}(K[x, y, z, t]/(g)),$$

expresses S in the desired form. One may easily verify the smoothness of $f(x, y, z)$ by using the smoothness of S and checking partial derivatives. $\square$

From Proposition 2.2.3 we see that a Del Pezzo surface of degree 2 has an involution ι which we call the *anticanonical involution*, the fixed point locus of ι is isomorphic to a plane quartic and that we have a morphism $p : S \to S/\langle\iota\rangle \cong \mathbb{P}^2$ of degree 2. We have thus seen that we can get a plane quartic from a Del Pezzo surface of degree 2 by taking the fixed point locus and, conversely, that we can get a Del Pezzo surface of degree 2 from a plane quartic C by taking the double cover of $\mathbb{P}^2$ ramified along C. In fact, this sets up an isomorphism between the corresponding moduli spaces, see Chapter IX of [38].

Theorem 2.2.4. *Sending a Del Pezzo surface of degree* 2 *to the fixed point locus of its anticanonical involution yields an isomorphism*

$$\mathcal{DP}_2 \cong \mathcal{Q}.$$

Let C be a plane quartic and let $p : S \to \mathbb{P}^2$ be the double cover of $\mathbb{P}^2$ ramified along C. If $L \subset \mathbb{P}^2$ is a bitangent to C, then $p^{-1}(L)$ will consist of two irreducible rational curves E_1 and E_2 of self intersection -1. Since C has 28 bitangents, we obtain 56 curves on S in this way. We will discuss this in more detail in the next section.

2.3 Del Pezzo Surfaces as Blowups

In this section we discuss the classical description of Del Pezzo surfaces in terms of blowups of $\mathbb{P}^2$.

Recall that a (-1)-curve on a smooth surface S is a rational curve E with self intersection equal to -1. Castelnuovo's contraction theorem states that if E is a (-1)-curve on a surface S then there is a nonsingular surface X, a point P on X and a morphism $\pi : S \to X$ such that $S \cong \mathrm{Bl}_P X$ via π and such that E is the exceptional curve of the blowup. In other words, (-1)-curves are exceptional curves, see [53], Theorem V.5.7. If S is a Del Pezzo surface, we can weaken this condition further.

Lemma 2.3.1. *Let S be a Del Pezzo surface. If C is an irreducible curve on S with negative self intersection, then C is exceptional.*

Proof. Let C be an irreducible curve of genus g on S. Since $-K_S$ is ample, we have $-K_S.C > 0$. If $C^2 < 0$, the adjunction formula gives

$$2g-2 = C.(C+K_S) = C^2 + K_S.C < 0.$$

We conclude that $g = 0$. This implies that $2g-2 = -2$ so $C^2 + K_S.C = -2$. We now have two negative integers C^2 and $K_S.C$ which add to -2. The only possibility is that $C^2 = -1$ and $K_S.C = -1$. We have thus proven that C is a rational curve of self intersection -1 and thus, by Castelnuovo's contraction theorem, that C is exceptional. $\square$

Lemma 2.3.2. *Let S be a Del Pezzo surface of degree d where $2 \le d \le 7$. Then S is isomorphic to the blowup of $\mathbb{P}^2$ in $9-d$ points.*

Proof. Since S is rational, there is a birational morphism $f : S \to X$ where X is a minimal rational surface. The minimal rational surfaces are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and the ruled surfaces. If X is a nontrivial ruled surface, then it contains a curve of self intersection less than -1, see [53], Proposition V.2.9. This contradicts Lemma 2.3.1 so X is either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. We have $K^2_{\mathbb{P}^2} = 9$ and $K^2_{\mathbb{P}^1\times\mathbb{P}^1} = 8$ so f must be a nontrivial birational morphism, i.e. S itself cannot be minimal.

Now suppose that $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $P \in X$ be point where f is not defined. Let $Y = \mathrm{Bl}_P X$. The morphism f now factors as

$$S \xrightarrow{g} Y \xrightarrow{\pi} X,$$

where π is a monoidal transformation centered at P. Let $p_i : X \to \mathbb{P}^1$ be the projection to the first respectively second factor and let $E_i = \pi^{-1}(p_i^{-1}(p_i(P)))$, $i = 1,2$. Then E_1 and E_2 are exceptional and may be blown down to get a birational morphism $h : Y \to \mathbb{P}^2$. The composition $h \circ g : S \to \mathbb{P}^2$ is now a birational morphism to $\mathbb{P}^2$.

We may thus assume that $X = \mathbb{P}^2$. We factor $f : S \to X$ as a finite sequence of monoidal transformations

$$S = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = \mathbb{P}^2,$$

where $X_i = \mathrm{Bl}_{P_i} X_{i-1}$, $i = 1,\ldots,n$, and $P_i \in X_{i-1}$ is a point. If P_i lies on an exceptional curve, then X_i contains a curve of self intersection less than -1

which is impossible by Lemma 2.3.1. Thus, for each of the points P_i there is a unique point in $\mathbb{P}^2$. We have $d = K_S^2 = K_{\mathbb{P}^2}^2 - n = 9 - n$ so $n = 9 - d$. $\square$

Since the canonical sheaf of $\mathbb{P}^2$ can be identified with $\mathcal{O}_{\mathbb{P}^2}(-3)$ we see that

$$K_S = -3L + E_1 + \cdots + E_n,$$

where L is the total transform of a line in $\mathbb{P}^2$ and E_i is the exceptional divisor corresponding to P_i.

Suppose that P_1, P_2 and P_3 are points in $\mathbb{P}^2$ which lie on a line L. Let $S = \mathrm{Bl}_{P_1,P_2,P_3}\mathbb{P}^2$ and let $\pi : S \to \mathbb{P}^2$ be the corresponding birational morphism. Let $\tilde{L}$ be the strict transform of L. We then have that $\tilde{L}$ is irreducible and

$$\begin{aligned}\tilde{L}^2 &= (\pi^* L - E_1 - E_2 - E_3)^2 = \\ &= \pi^* L^2 + E_1^2 + E_2^2 + E_3^2 = \\ &= 1 - 1 - 1 - 1 = \\ &= -2.\end{aligned}$$

Thus, by Lemma 2.3.1, S cannot be a Del Pezzo surface. Similarly, if 6 of the n points of a blowup $S = \mathrm{Bl}_{P_1,\ldots,P_n}$ lie on a conic, then S contains an irreducible curve of self intersection -2. This motivates the following definition.

Definition 2.3.3. Let $P_1, \ldots, P_n$ be n points in $\mathbb{P}^2$ where $1 \leq n \leq 7$. The points are in *general position* if no three of the points lie on a line and no six of the points lie on a conic.

Intuitively, a set of points is in general position if there is no unexpected curve passing through them. Thus, there is a more general definition which works for any n but this is enough for our purposes.

A small calculation shows that the blow up of $2 \leq n \leq 7$ points in $\mathbb{P}^2$ in general position gives a Del Pezzo surface of degree $9 - n$. We thus have the following theorem.

Theorem 2.3.4. *A surface S is a Del Pezzo surface of degree $2 \leq d \leq 7$ if and only if S is isomorphic to the blowup of $\mathbb{P}^2$ in $9 - d$ points in general position.*

We now specialize to the case of a Del Pezzo surface S of degree 2. We then have that S is isomorphic to the blowup of $\mathbb{P}^2$ in seven points $P_1, \ldots, P_7$ in general position. Let E_i denote the exceptional curve corresponding to P_i and let $L \subset S$ denote the total transform of a line in $\mathbb{P}^2$. We then have that the Picard group $\mathrm{Pic}(S)$ of S is

$$\mathrm{Pic}(S) = \mathbb{Z}L \oplus \mathbb{Z}E_1 \oplus \cdots \oplus \mathbb{Z}E_7. \tag{2.3.1}$$

The intersection theory is given by

$$L^2 = 1, \quad E_i^2 = -1, \quad L.E_i = 0, \quad E_i.E_j = 0,\, i \neq j. \tag{2.3.2}$$

We may now describe the exceptional curves of S explicitly.

Lemma 2.3.5. *Let $S = \mathrm{Bl}_{P_1,\dots,P_7}\mathbb{P}^2$ be a Del Pezzo surface of degree 2. Let L be the total transform of a line in $\mathbb{P}^2$ and let E_i be the exceptional curve corresponding to the point P_i. If E is an exceptional curve on S, then either*

- (i) *$E = E_i$, $i = 1\dots,7$, or,*
- (ii) *$E = L - E_i - E_j$, $1 \leq i < j \leq 7$, i.e. E is the strict transform of the line passing through P_i and P_j, or,*
- (iii) *$E = 2L - \sum_{k=1}^{7} E_k + E_i + E_j$, $1 \leq i < j \leq 7$, i.e. E is the strict transform of the conic passing through all of the points except P_i and P_j, or,*
- (iv) *$E = 3L - \sum_{k=1}^{7} E_k - E_i$, $i = 1,\dots,7$, i.e. E is the strict transform of the cubic passing through the seven points with a double point in P_i.*

In particular, S has exactly 56 exceptional curves.

Proof. Since the genus of E is 0 and $E^2 = -1$, the adjunction formula gives

$$-2 = E.(E + K_S) = -1 + K_S.E,$$

so $K_S.E = -1$.

Let $E = bL + a_1E_1 + \cdots + a_7E_7$. Then

$$E^2 = b^2 - a_1^2 - \cdots - a_7^2 = -1.$$

We have $K_S = -3L + E_1 + \cdots E_7$ so

$$-K_S.E = 3b + a_1 + \cdots + a_7 = 1.$$

We rewrite these equalities as $a_1^2 + \cdots + a_7^2 = b^2 + 1$ and $a_1 + \cdots + a_7 = 1 - 3b$.

Recall that the Schwartz inequality says that if x and y are column vectors $\mathbb{R}^n$, then $|x^T y|^2 \leq |x|^2 \cdot |y|^2$. We take $x = (a_1,\dots,a_7)$ and $y = (1,\dots,1)$ and get

$$(a_1 + \cdots + a_7)^2 \leq 7 \cdot (a_1^2 + \cdots + a_7^2).$$

This gives that $(1-3b)^2 \leq 7(b^2+1)$ which yields $0 \leq b \leq 3$. It is now an easy matter to check by hand that the only possible choices for $a_1,\dots,a_7$ are the ones in the above list. □

Different sets of points do not necessarily determine different Del Pezzo surfaces. On the other hand, a set $\{P_1,\ldots,P_7\}$ does not only determine a Del Pezzo surface S - it also determines a $\mathbb{Z}$-basis for the Picard group of S. Similarly, an ordered septuple $(P_1,\ldots,P_7)$ determines an ordered $\mathbb{Z}$-basis for S. We call an ordered basis of the Picard group of S a marking of S and a marking coming from a blowup of $\mathbb{P}^2$ is called a geometric marking. More precisely, we have the following.

Definition 2.3.6. Let S be a Del Pezzo surface of degree $2 \le d \le 7$ and let $n = 9-d$. A *geometric marking* of S is an isomorphism $\varphi : S \to \mathrm{Bl}_{P_1,\ldots,P_n}\mathbb{P}^2$ for some $P_1,\ldots,P_n \in \mathbb{P}^2$. Two geometric markings $\varphi : S \to \mathrm{Bl}_{P_1,\ldots,P_n}\mathbb{P}^2$ and $\psi : S \to \mathrm{Bl}_{P'_1,\ldots,P'_n}\mathbb{P}^2$ are equivalent if there is an element $\theta \in \mathrm{PGL}(3,K)$ such that the diagram

$$\begin{tikzcd}
\mathrm{Pic}\left(\mathrm{Bl}_{P'_1,\ldots,P'_n}\mathbb{P}^2\right) \arrow[r, "\psi^*"] \arrow[d, "\overline{\theta}^*"'] & \mathrm{Pic}(S) \\
\mathrm{Pic}\left(\mathrm{Bl}_{P_1,\ldots,P_n}\mathbb{P}^2\right) \arrow[ur, "\varphi^*"'] &
\end{tikzcd}$$

commutes, where $\overline{\theta}$ denotes the morphism $\mathrm{Bl}_{P_1,\ldots,P_n}\mathbb{P}^2 \to \mathrm{Bl}_{P'_1,\ldots,P'_n}\mathbb{P}^2$ induced by θ.

A pair (S,φ), where S is a Del Pezzo surface and φ is a geometric marking of S, is called a *geometrically marked Del Pezzo surface* and we denote the moduli space of geometrically marked Del Pezzo surfaces of degree d by $\mathcal{DP}_d^{\mathrm{gm}}$. If let $\mathcal{P}_n^2$ denote the moduli space of ordered n-tuples of points in the projective plane in general position up to projective equivalence we have

$$\mathcal{DP}_2^{\mathrm{gm}} \cong \mathcal{P}_7^2.$$

We now turn our attention back to the relation between Del Pezzo surfaces of degree 2 and plane quartics. In particular we shall investigate the relationship between exceptional curves and geometric markings of a Del Pezzo surface S of degree 2 and the bitangents of the corresponding plane quartic.

Let S be a Del Pezzo surface of degree 2. We define $K_S^\perp \subset \mathrm{Pic}(S)$ as

$$K_S^\perp = \{D \in \mathrm{Pic}(S) \,|\, K_S.D = 0\}. \tag{2.3.3}$$

It is a free $\mathbb{Z}$-module of rank 7.

Lemma 2.3.7. *Let S be a Del Pezzo surface of degree 2 with anticanonical involution ι. Then ι acts as -1 on $K_S^\perp$.*

Proof. Let $D \in \operatorname{Pic}(S)$. Note that $D+\iota(D)$ is fixed by ι. Thus, it is the pullback of some class under the morphism $|-K_S| : S \to \mathbb{P}^2$. In other words, $D+\iota(D) = mK_S$ for some integer m.

If $D \in K_S^{\perp}$, then $\iota(D.K_S) = \iota(D).\iota(K_S) = \iota(D).K_S = 0$ so $\iota(D) \in K_S^{\perp}$ and therefore $D+\iota(D) \in K_S^{\perp}$. We thus have

$$0 = (D+\iota(D)).K_S = mK_S.K_S = 2m,$$

which implies $m = 0$. Hence, $D+\iota(D) = 0$ so $\iota(D) = -D$. □

Proposition 2.3.8. *Let S be a Del Pezzo surface of degree 2, $p : S \to \mathbb{P}^2$ a double cover ramified along a plane quartic C with anticanonical involution ι and let E be an exceptional curve of S. Then*

(i) *$\iota(E) = -K_S - E$ is an exceptional curve, and*

(ii) *$p(E) = p(\iota(E))$ is a (not necessarily genuine) bitangent of C.*

Proof. (*i*) Define $E' = -K_S - E$. Then

$$E'^2 = K_S^2 + 2K_S.E + E^2 = -1,$$

and

$$-K_S.E' = K_S^2 + K_S.E = 1,$$

so E' is exceptional. Note that $\iota(E') = -K_S - \iota(E)$. Define

$$D = -K_S - E - \iota(E) = E' - \iota(E) = \iota(E') - E.$$

We have

$$K_S.D = -K_S^2 - K_S.E - K_S.\iota(E) = -2+1-\iota(K_S.E) = -1+1 = 0.$$

We conclude that $D \in K_S^{\perp}$. By Lemma 2.3.7 we now have $\iota(D) = -D$. Thus

$$\begin{aligned} -D = \iota(D) &= \\ &= -\iota(K_S) - \iota(E) - \iota(\iota(E)) = \\ &= -K_S - \iota(E) - E = \\ &= D. \end{aligned}$$

Thus, $2D = 0$ and since $\operatorname{Pic}(S)$ is a free $\mathbb{Z}$-module we have that $D = 0$. Hence, $\iota(E) = E'$ as desired.

(*ii*) In Section 2.2 we saw that $p^{-1}(C) \sim -2K_S$ so the intersection product $E \cdot p^{-1}(C) = P + Q$ for some points P and Q (which might be equal). Similarly, E' also intersects $p^{-1}(C)$ in two points. These points must be P and Q since ι fixes $p^{-1}(C)$ pointwise and interchanges E and E'. Thus, $p(E) = p(E')$ and $p(E) \cdot C = 2p(P) + 2p(Q)$ so $p(E)$ is a genuine bitangent if $P \neq Q$ and a hyperflex line if $P = Q$. □

Recall that if $S = \mathrm{Bl}_{P_1,\dots,P_7}\mathbb{P}^2$, then $K_S = -3L + E_1 + \cdots + E_7$, where L is the total transform of a line in $\mathbb{P}^2$. If we use this in Proposition 2.3.8 we obtain the following corollary.

Corollary 2.3.9. *Let $S = \mathrm{Bl}_{P_1,\dots,P_7}\mathbb{P}^2$ be a Del Pezzo surface of degree 2 with covering involution ι. Then the action of ι on the exceptional curves is described by*

$$E_i \overset{\iota}{\longleftrightarrow} 3L - E_1 - \cdots - E_7 - E_i,$$
$$L - E_i - E_j \overset{\iota}{\longleftrightarrow} 2L - E_1 - \cdots - E_7 + E_i + E_j.$$

Corollary 2.3.10. *If L is a bitangent of C, then $\pi^{-1}(L)$ consists of two exceptional curves which are conjugate under ι.*

Proof. By Lemma 2.3.5 we know that S has exactly 56 exceptional curves, by Theorem 2.1.2 we know that C has exactly 28 bitangents and by Proposition 2.3.8 we have that exceptional curves which are conjugate under ι map to the same bitangent. It is thus enough to show that two different pairs of conjugate exceptional curves do not map to the same bitangent.

Let D_1 and D_2 be two exceptional curves which are not conjugate under ι. By Corollary 2.3.9 we may assume that both D_1 and D_2 are of the form E_i or $L - E_i - E_j$. In particular, we may assume that D_1 and D_2 are two distinct lines. Thus, D_1 and D_2 intersect in at most 1 point and can thus not map to the same bitangent. □

Corollary 2.3.11. *Let S be a Del Pezzo surface of degree 2 and let C be the associated plane quartic. There is a natural two-to-one map from the set of exceptional curves on S to the set of bitangents of C.*

Let $p : S \to \mathbb{P}^2$ be a double cover ramified along a smooth quartic C. If we view a geometric marking of S as an ordered basis $(L, E_1, \dots, E_7)$ of $\mathrm{Pic}(S)$, then we obtain an ordered set $(p(E_1), \dots, p(E_7))$ of seven bitangents of C. However, not every set of seven bitangents of C arises in this way (just as not every set of seven exceptional curves on S constitutes a geometric marking or even a basis for $\mathrm{Pic}(S)$). Those which do arise in this way are called *Aronhold sets*. We shall discuss them in more detail in Section 3.2 and Section 3.5.

2.4 The Geiser involution

In the preceding sections we have seen that a Del Pezzo surface S of degree 2 can be realized both as the double cover $p : S \to \mathbb{P}^2$ ramified along a smooth quartic C and as the blowup $\pi : S \to \mathbb{P}^2$ in seven points $P_1, \dots, P_7$ in general position. This implies that seven points in general position determine a plane quartic, but at this point we need to go via a Del Pezzo surface. In this section we shall investigate how we can obtain a nonhyperelliptic genus 3 curve directly from the seven points.

Corollary 2.4.1. *Let $S = \mathrm{Bl}_{P_1,\dots,P_7}\mathbb{P}^2$ be a Del Pezzo surface of degree 2, let ι be the anticanonical involution of S and let L be the total transform of a line in $\mathbb{P}^2$. Then*

$$\iota(L) = 8L - 3E_1 - \cdots - 3E_7.$$

Thus, ι acts on $\mathbb{P}^2$ as the Cremona transformation given by the linear system of octics through $P_1,\dots,P_7$ with triple points in each of the points $P_1,\dots,P_7$.

Proof. We have $\iota(K_S) = K_S = -3L + E_1 + \cdot + E_7$ and $\iota(E_i) = 3L - E_1 - \cdots - E_7 - E_i$. Thus

$$\begin{aligned}
-3L + E_1 + \cdots + E_7 = \iota(K_S) = & \\
&= -3\iota(L) + \iota(E_1) + \cdots + \iota(E_7) = \\
&= -3\iota(L) + 21L - 8E_1 - \cdots - 8E_7.
\end{aligned}$$

We conclude that $\iota(L) = 8L - 3E_1 - \cdots - 3E_7$. □

A birational involution $\iota : \mathbb{P}^2 \to \mathbb{P}^2$ given by the linear system of octics through seven points $P_1,\dots,P_7$ with triple points in each of the points $P_1,\dots,P_7$ is classically known as the *Geiser involution*. An alternative description can be given as follows.

Let $\mathcal{N}$ be the net of cubics through $P_1,\dots,P_7$. A point Q in $\mathbb{P}^2 \setminus \{P_1,\dots,P_7\}$ defines a pencil $\mathcal{P}_Q \subset \mathcal{N}_Q$ of cubics passing through Q. By the Cayley-Bacharach theorem, the pencil $\mathcal{P}_Q$ has nine base points. Eight of these points are $P_1,\dots,P_7$ and Q so we obtain a ninth base point Q' and sending Q to Q' gives a birational involution of $\mathbb{P}^2$. This involution can be extended to the whole projective plane by sending P_i to itself. For a proof of the equivalence of the two descriptions, see Chapter VII.8 of [76].

From the first description of the Geiser involution ι it is clear that $\pi(p^{-1}(C))$ is the fixed point locus of ι. In the second description, we have that $Q \in \mathbb{P}^2$ is a fixed point if and only if all the members of $\mathcal{P}_Q$ share a tangent at Q. By a suitable choice of coordinates, we may make sure that $Q = [0:0:1]$. The pencil $\mathcal{P}_Q$ can then be given as

$$t_0 F_0^Q(x,y,z) + t_1 F_1^Q(x,y,z) = 0,$$

where

$$F_i^Q(x,y,z) = z^2(a_i x + b_i y) + z \cdot f_i(x,y) + g_i(x,y) = 0$$

are the defining equations of two general members C_0 and C_1 of $\mathcal{P}_Q$. Here, $f_i(x,y)$ and $g_i(x,y)$ are polynomials of degrees 2 and 3 and a_i and b_i are constants. A simple computation shows that the tangent of C_i at Q is given by

$$a_i x + b_i y = 0.$$

Thus, Q is a fixed point of the Geiser involution if and only if there is a nonzero λ such that

$$a_1 = \lambda a_0, \quad b_1 = \lambda b_0.$$

The curve C_Q defined by $\lambda F_0^Q(x,y,z) - F_1^Q(x,y,z) = 0$ will thus be the unique member of $\mathcal{P}_Q$ with a singularity at Q. On the other hand, if every member of $\mathcal{P}_Q$ is smooth at Q, then they all have distinct tangents at Q. Thus, Q is a fixed point if and only if $\mathcal{P}_Q$ has a member with a singularity at Q.

Let the net $\mathcal{N}$ be generated by the three curves given by the equations $F_i(x,y,z) = 0$, $i = 0,1,2$. Then, the fixed point locus B of the Geiser involution is given by the equation

$$\det\begin{pmatrix} \frac{\partial F_0}{\partial x} & \frac{\partial F_0}{\partial y} & \frac{\partial F_0}{\partial z} \\ \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix} = 0.$$

Thus B is a sextic curve with singularities at $P_1,\dots,P_7$. On the other hand, $B = \pi(p^{-1}(C))$ so B has geometric genus 3 so we conclude by the genus-degree formula that $P_1,\dots,P_7$ are the only singularities of B and that they are all double points.

We conclude by investigating the singularity type of B at P_i. Let C_i be a cubic passing through $P_1,\dots,P_7$ with a double point in P_i and let $\widetilde{C}_i$ be the strict transform of C_i in S. By Corollary 2.3.9 we have that $\iota(E_i) = \widetilde{C}_i$. If C_i is nodal, then E_i and $\widetilde{C}_i$ will intersect in two distinct points. These points are clearly fixed by ι and are thus points of $p^{-1}(C)$ and we conclude that B is nodal at P_i. Similarly, if C_i has a cusp at P_i, then E_i and $\widetilde{C}_i$ will intersect in a single point with multiplicity 2 and B will consequently be cuspidal at P_i. We have thus proven the following proposition.

Proposition 2.4.2. *Let $P_1,\dots,P_7 \in \mathbb{P}^2$ be seven points in general position. There is then a unique sextic curve passing through the points $P_1,\dots,P_7$ with double points precisely at $P_1,\dots,P_7$. Moreover, the singularity at P_i is a node if the unique cubic through $P_1,\dots,P_7$ with a singularity at P_i is nodal and otherwise it is a cusp.*

3. Algebraic background

In Chapter 2 we reviewed some classical algebraic geometry of curves and surfaces. However, we shall study objects equipped with extra algebraic structures. We have already seen geometrically marked Del Pezzo surfaces in Section 2.3, for instance. In order to be able to discuss these structures clearly and efficiently we shall review some theory of lattices and quadratic forms and see how these concepts are related to geometric notions such as bitangents, theta characteristics and geometric markings. For a more thorough treatment of lattices and bilinear forms, see [79], and for complete introductions to root systems and Weyl groups we refer the reader to [21] and [55].

3.1 Lattices

Let A be an integral domain. A *lattice* over A is a free A-module L equipped with a nondegenerate bilinear form $b : L \times L \to A$. Here, nondegenerate means that if $b(x, y) = 0$ for all $y \in L$, then $x = 0$. We will sometimes leave the bilinear form implicit and simply denote the lattice (L, b) by L. An *isometry* between two lattices (L, b) and (L', b') is an isomorphism $\phi : L \to L'$ of A-modules such that

$$b(x, y) = b'(\phi(x), \phi(y)),$$

for all $x, y \in L$. If there is an isometry between (L, b) and (L', b'), then (L, b) and (L', b') are said to be *isometric*.

The first example of a lattice that comes to mind is perhaps $\mathbb{R}^n$ with the standard Euclidean inner product, but there are of course many others. One example that will be of particular use to us is the following.

3.1.1. Hyperbolic lattices Let $A = \mathbb{Z}$ and let $L = H_r$ be a free $\mathbb{Z}$-module of rank $r + 1$ with generators l and $e_1 \dots, e_r$. We define a bilinear form b on H_r by setting

$$b(l, l) = 1, \quad b(e_i, e_i) = -1, \; i = 1, \dots, r, \quad b(l, e_i) = b(e_i, e_j) = 0, \; i \neq j.$$

The lattice (H_r, b) is called the *standard hyperbolic lattice* of rank $r + 1$. The group of isometries $H_r \to H_r$ is called *the orthogonal group* of H_r and will be denoted $O(H_r)$.

Recall Equations 2.3.1 and 2.3.2 describing the Picard group of a Del Pezzo surface S of degree 2. The Picard group $\mathrm{Pic}(S)$ is a lattice under the intersection pairing and the function $\phi : H_7 \to \mathrm{Pic}(S)$ defined by

$$\phi(l) = L, \quad \phi(e_i) = E_i, \quad i = 1, \ldots 7,$$

is an isometry. Thus, an alternative way to define a geometrically marked Del Pezzo surface of degree 2 is as pair (S, ϕ) where S is a Del Pezzo surface of degree 2 and $\phi : \mathrm{Pic}(S) \to H_7$ is an isometry defined via a blowdown structure on S. We denote the subgroup of $O(H_7)$ consisting of isometries $\psi : H_7 \to H_7$ such that $\psi \circ \phi$ is a geometric marking by $O_S(H_7)$.

3.1.2. The E_7-lattice Let $A = \mathbb{Z}$ and let $L_{E_7} \subset H_7$ be the free $\mathbb{Z}$-module generated by

$$\begin{aligned}
\beta_1 &= e_1 - e_2, \\
&\vdots \\
\beta_6 &= e_6 - e_7, \\
\beta_7 &= l - e_1 - e_2 - e_3.
\end{aligned}$$

The module L_{E_7} with the bilinear form inherited from H_7 is called the *E_7-lattice*. The reason is that the lattice is intimately related to the root system of type E_7. We refer the reader who is unfamiliar with root systems to Definition 5.6.1 and explain the connection right away.

The vectors $\beta_1, \ldots, \beta_7$ above are a choice of simple roots in a root system Φ_{E_7} in $H_7 \otimes \mathbb{R} \cong \mathbb{R}^8$ of type E_7. The positive roots with this choice of simple roots are

$$\begin{aligned}
\alpha_i &= 2l - e_1 - \ldots - e_7 + e_i, && 1 \le i \le 7, \\
\alpha_{i,j} &= e_i - e_j, && 1 \le i < j \le 7, \\
\alpha_{i,j,k} &= l - e_i - e_j - e_k, && 1 \le i < j < k \le 7.
\end{aligned}$$

Recall from Equation 2.3.3 that if S is a Del Pezzo surface of degree 2, then

$$K_S^{\perp} = \{D \in \mathrm{Pic}(S) \mid K_S \cdot D = 0\}.$$

If $S = \mathrm{Bl}_{P_1,\ldots,P_7}\mathbb{P}^2$, then $K_S^{\perp}$ is generated by the elements

$$\begin{aligned}
B_1 &= E_1 - E_2, \\
&\vdots \\
B_6 &= E_6 - E_7, \\
B_7 &= L - E_1 - E_2 - E_3,
\end{aligned}$$

and the map ϕ defined by $B_i \mapsto \beta_i$, $1 \le i \le 7$ is an isometry from $\operatorname{Pic}(S)$ to the E_7-lattice. We denote the image of the set of exceptional curves by $\mathcal{E}$. The set $\mathcal{E}$ is an orbit under the Weyl group W_{E_7} of Φ_{E_7}.

We have thus seen that Del Pezzo surfaces of degree 2 are naturally connected to both the hyperbolic lattice H_7 and the E_7-lattice. Plane quartics, on the other hand, are connected to symplectic vector spaces.

3.1.3. Symplectic vector spaces We now give a brief introduction to the theory of symplectic vector spaces over the field of two elements. For a more complete treatment, see [7].

Let $\mathbb{F}_2$ be the field with two elements and let V be a vector space over $\mathbb{F}_2$ of dimension $2g$. Fix a nondegenerate, alternating bilinear form

$$b : V \times V \to \mathbb{F}_2,$$

i.e. $b(v,v) = 0$ for all v and the map $v \mapsto b(v,-)$ is an isomorphism from V to $\operatorname{Hom}(V,\mathbb{F}_2)$. The lattice (V,b) is called a *symplectic space* over $\mathbb{F}_2$.

A subspace $X \subset V$ such that $b(x,x') = 0$ for all x and x' in X is called *isotropic*. Any maximal isotropic subspace has dimension g. To see this, define

$$X^\perp := \{v \in V \mid b(v,x) = 0 \text{ for all } x \in X\},$$

and note that X is a subspace of $X^\perp$ if X is isotropic. By the rank-nullity theorem we have $\dim X + \dim X^\perp = \dim V = 2g$ and it follows that $\dim X \le g$. If the inequality is strict we can find a vector v in $X^\perp \setminus X$ and the space $X \oplus \mathbb{F}_2 v$ is then isotropic of dimension $\dim(X)+1$.

Given a maximal isotropic subspace X, we may complete X into an *isotropic decomposition*

$$V = X \oplus Y,$$

i.e. a decomposition of V such that Y is also isotropic and such that X and Y are in mutual duality via b. Given a basis $x_1,\dots,x_g$ of X this duality provides a dual basis $y_1,\dots y_g$ of Y such that $b(x_i,y_j) = \delta_{i,j}$, where $\delta_{i,j}$ denotes the Kronecker delta function. The basis $x_1,\dots,x_g,y_1,\dots,y_g$ of V is called a *symplectic basis*. The vector space $\mathbb{F}_2^{2g}$ with symplectic basis

$$\begin{aligned} x_i &= \text{ the } i\text{'th coordinate vector}, \quad i = 1,\dots,g, \\ y_i &= \text{ the } (g+i)\text{'th coordinate vector}, \quad i = 1,\dots,g, \end{aligned}$$

is called the *standard symplectic space* of dimension $2g$ over $\mathbb{F}_2$.

The *symplectic group* of degree $2g$ is defined as

$$\operatorname{Sp}(2g,\mathbb{F}_2) := \{M \in \operatorname{GL}(2g,\mathbb{F}_2) \mid b(Mu,Mv) = b(u,v)\,, \text{ for all } u,v \in V\}.$$

It is clear that $\mathrm{Sp}(2g,\mathbb{F}_2)$ acts simply transitively on the set of symplectic bases of V, so by counting the symplectic bases one finds

$$|\mathrm{Sp}(2g,\mathbb{F}_2)| = 2^{g^2}(2^{2g}-1)(2^{2g-2}-1)\cdots(2^2-1).$$

Let u be a vector in V. The *transvection* T_u is the element in $\mathrm{Sp}(2g,\mathbb{F}_2)$ defined by

$$T_u(v) := v + b(u,v)\,u.$$

We have

$$\begin{aligned} T_u(T_u(v)) &= T_u(v + b(u,v)\,u) = \\ &= v + b(u,v)\,u + b(u, v + b(u,v)\,u)\,u = \\ &= v + 2b(u,v)\,u + b(u,v)\,b(u,u)\,u = \\ &= v, \end{aligned}$$

so the transvections are involutions. One may show that $\mathrm{Sp}(2g,\mathbb{F}_2)$ is generated by the transvections. The idea of the proof is to first show that the subgroup generated by the tranvections acts transitively on V. Then one shows that it also also acts transitively on pairs of vectors (u,v) such that $b(u,v)=1$ and then, finally, one shows that it acts transitively on the set of symplectic bases.

3.2 Quadratic forms on symplectic vector spaces

We shall now give some definitions and key properties of quadratic forms on quadratic vector spaces over $\mathbb{F}_2$ and some related topics. See also [49].

Let (V,b) be a symplectic vector space over $\mathbb{F}_2$ of dimension $2g$. A function $q : V \to \mathbb{F}_2$ is called a *quadratic form* relative to b if

$$q(u+v) + q(u) + q(v) = b(u,v), \qquad (3.2.1)$$

for all u and v in V. We shall denote the set of quadratic forms on V by $Q(V)$. Note that we can recover the bilinear form from a fixed quadratic form via the relation 3.2.1. We remark that the action of the symplectic group on V induces an action on $Q(V)$ by

$$M.q(v) = q(M^{-1}v),$$

for $M \in \mathrm{Sp}(2g,\mathbb{F}_2)$.

Specifying values for a quadratic form q on a basis of V determines q, so $|Q(V)| = 2^{2g}$. The vector space V acts on $Q(V)$ by

$$(v.q)(u) = q(u) + b(v,u).$$

Since b is nondegenerate, this action is free. Since $|V| = |Q(V)| = 2^{2g}$ and V acts freely on $Q(V)$ we conclude that action is also transitive so $Q(V)$ is a V-torsor. In particular, for each pair q and q' of quadratic forms, there is a unique vector $v \in V$ such that $v.q = q'$. We define

$$q + q' = v,$$

and

$$q + v = q'.$$

This turns the set $W = V \cup Q(V)$ into a $\mathbb{F}_2$-vector space. Since $|W| = 2^{2g} + 2^{2g} = 2^{2g+1}$ we have that the dimension of W is $2g+1$.

The actions of $\mathrm{Sp}(2g, \mathbb{F}_2)$ on V and $Q(V)$ induce an action of $\mathrm{Sp}(2g, \mathbb{F}_2)$ on the whole of W. This gives us a short exact $\mathrm{Sp}(2g, \mathbb{F}_2)$-equivariant sequence of vector spaces

$$0 \to V \to W \to \mathbb{F}_2 \to 0.$$

Given an isotropic decomposition $V = X \oplus Y$ we define a quadratic form $q_{X\oplus Y}$ in the following way. Let v be any vector in V. Then v can be uniquely expressed as $v = x + y$ with $x \in X$ and $y \in Y$. Now define

$$q_{X\oplus Y}(v) = b(x, y).$$

More explicitly, if $x_1, \dots, x_g$ is a basis for X and $y_1, \dots, y_g$ is the dual basis of Y we define

$$q_{X\oplus Y}\left(\sum_{i=1}^{g} \alpha_i x_i + \sum_{i=1}^{g} \beta_i y_i\right) = \sum_{i=1}^{g} \alpha_i \beta_i.$$

Definition 3.2.1. Let $x_1, \dots, x_g, y_1, \dots, y_g$ be a symplectic basis of V and let $q \in Q(V)$. The *Arf invariant* of q is defined as

$$\mathrm{Arf}(q) = \sum_{i=1}^{g} q(x_i) q(y_i).$$

A quadratic form is called *even* if $\mathrm{Arf}(q) = 0$ and *odd* if $\mathrm{Arf}(q) = 1$.

Although the definition seems to depend on the chosen symplectic basis, this is not the case.

Proposition 3.2.2. *The Arf invariant does not depend on the choice of symplectic basis.*

Proof. We first observe that since $\mathrm{Sp}(2g,\mathbb{F}_2)$ acts transitively on the set of symplectic bases, it is enough to show that $\mathrm{Arf}(A.q) = \mathrm{Arf}(q)$ for all $A \in \mathrm{Sp}(2g,\mathbb{F}_2)$. Further, since $\mathrm{Sp}(2g,\mathbb{F}_2)$ is generated by transvections it is enough to show that $\mathrm{Arf}(T_u.q) = \mathrm{Arf}(q)$ for any transvection T_u.

Let $u \in V$ and let T_u be the transvection defined by u. By Equation 3.2.1 we have

$$\begin{aligned} q\,(T_u v) &= q\,((v + b(u,v)u) = \\ &= q(v) + b(u,v)q(u) + b(u,v)^2 = \\ &= q(v) + \big(q(u)+1\big)\, b(u,v), \end{aligned}$$

where the last equality follows since $x^2 = x$ for all x in $\mathbb{F}_2$.

Let $x_1,\dots.x_g, y_1,\dots,y_g$ be a symplectic basis for V. Define $\mu = q(u)+1$ and write $u = \sum_{i=1}^g \alpha_i x_i + \sum_{i=1}^g \beta_i y_i$. We now have

$$\begin{aligned} \mathrm{Arf}(T_u.q) &= \sum_{i=1}^g \left[q(x_i)+\mu\beta_i\right]\cdot\left[q(y_i)+\mu\alpha_i\right] = \\ &= \sum_{i=1}^g q(x_i)q(y_i) + \sum_{i=1}^g \mu\alpha_i q(x_i) + \sum_{i=1}^g \mu\beta_i q(y_i) + \sum_{i=1}^g \mu^2\alpha_i\beta_i = \\ &= \mathrm{Arf}(q) + \mu\left(\sum_{i=1}^g \alpha_i q(x_i) + \beta_i q(y_i)\right) + \sum_{i=1}^g \mu\alpha_i\beta_i. \end{aligned} \tag{3.2.2}$$

Observe that $\lambda q(v) = q(\lambda v)$ for all $\lambda \in \mathbb{F}_2$ and all $v \in V$. Thus, $\alpha_i q(x_i) = q(\alpha_i x_i)$ and $\beta_i q(y_i) = q(\beta_i y_i)$. By using this observation and applying Equation 3.2.1 repeatedly we see that

$$\sum_{i=1}^g \alpha_i q(x_i) + \beta_i q(y_i) = q(u) + \sum_{i=1}^g \alpha_i\beta_i.$$

We now insert this expression into Equation 3.2.2 to get

$$\begin{aligned} \mathrm{Arf}(T_u.q) &= \mathrm{Arf}(q) + \mu\left(q(u) + \sum_{i=1}^g \alpha_i\beta_i\right) + \sum_{i=1}^g \mu\alpha_i\beta_i = \\ &= \mathrm{Arf}(q) + \mu q(u) + 2\mu\sum_{i=1}^g \alpha_i\beta_i = \\ &= \mathrm{Arf}(q) + \mu q(u). \end{aligned}$$

But $\mu = q(u)+1$ so either $q(u) = 0$ or $\mu = 0$ which gives that $\mu q(u) = 0$. Hence, $\mathrm{Arf}(T_u.q) = \mathrm{Arf}(q)$ as desired. □

Proposition 3.2.2 was first proven by Arf [5]. The argument we present above is essentially a simplified special case of an argument due to Dye [39].

We now list some properties of the Arf invariant. These properties are well known, see for instance [49], but since complete proofs seem hard to come by in the literature, we provide one here.

Proposition 3.2.3. *Let $V = X \oplus Y$ be an isotropic decomposition of V, let $v \in V$, $q \in Q(V)$ and let M be an element of* $\mathrm{Sp}(2g, \mathbb{F}_2)$. *Then*

(i) $\mathrm{Arf}(q_{X\oplus Y}) = 0$,

(ii) $\mathrm{Arf}(M.q) = \mathrm{Arf}(q)$,

(iii) $\mathrm{Arf}(v.q) = \mathrm{Arf}(q) + q(v)$, *and*

(iv) *the action of* $\mathrm{Sp}(2g, \mathbb{F}_2)$ *on* $Q(V)$ *has two orbits: the* $2^{g-1}(2^g+1)$ *even forms and the* $2^{g-1}(2^g-1)$ *odd forms.*

Proof. (*i*) Let $x_1, \dots, x_g$ be a basis for X and let $y_1, \dots, y_g$ be the corresponding dual basis of Y. We have that

$$\mathrm{Arf}(q_{X\oplus Y}) = \sum_{i=1}^{g} b(x_i, 0)\, b(0, y_i) = 0.$$

Since the Arf invariant does not depend on the choice of symplectic basis, this proves the claim.

(*ii*) This is just a reformulation of the fact that the Arf invariant is independent of the choice of symplectic basis.

(*iii*) Since $0.q = q$ and $q(0) = 0$ we have $\mathrm{Arf}(0.q) = \mathrm{Arf}(q) + q(0)$. We now assume that $v \neq 0$, set $x_1 = v$ and we extend x_1 to a symplectic basis $x_1, \dots, x_g, y_1, \dots, y_g$ of V. Then

$$\begin{aligned}
\mathrm{Arf}(v.q) &= \sum_{i=1}^{g} (q(x_i) + b(x_1, x_i))(q(y_i) + b(x_1, y_i)) = \\
&= \sum_{i=1}^{g} q(x_i)(q(y_i) + \delta_{i,1}) = \\
&= \sum_{i=1}^{g} q(x_i) q(y_i) + q(x_1) = \\
&= \mathrm{Arf}(q) + q(v),
\end{aligned}$$

as claimed.

(*iv*) Note that if $g = 1$, then a form $q_{X\oplus Y}$ corresponding to a isotropic decomposition $V = X \oplus Y$ has the three zeros $(1,0), (0,1)$ and $(0,0)$. Assume that $q_{X\oplus Y}$ has

$$2^{n-2}(2^{n-1} + 1)$$

zeros if $g = n-1$. Now let $g = n$ and let

$$v = \sum_{i=1}^{n} \alpha_i x_i + \sum_{i=1}^{n} \beta_i y_i$$

be a zero of $q_{X\oplus Y}$. We either have $\alpha_n = 0$ or $\alpha_n = 1$.

Suppose first that $\alpha_n = 0$. Then it does not matter if we choose $\beta_n = 0$ or $\beta_n = 1$; in either case the induction hypothesis gives that there are $2^{n-2}(2^{n-1}+1)$ choices for $\alpha_1,\ldots,\alpha_{n-1},\beta_1,\ldots,\beta_{n-1}$ such that $q_{X\oplus Y}(v) = 0$. Thus, $q_{X\oplus Y}$ has $2\cdot 2^{n-2}(2^{n-1}+1)$ zeros such that $\alpha_n = 0$.

If $\alpha_n = 1$ on the other hand, we may choose $\alpha_1,\ldots,\alpha_{n-1},\beta_1,\ldots,\beta_{n-1}$ however we want in order to obtain a zero of $q_{X\oplus Y}$ but then β_n is determined. Hence, $q_{X\oplus Y}$ has $2^{n-1}\cdot 2^{n-1}$ zeros with $\alpha_n = 1$. We now add the two cases together to see that $q_{X\oplus Y}$ has

$$2\cdot 2^{n-2}(2^{n-1}+1) + 2^{n-1}\cdot 2^{n-1} = 2^{n-1}(2^n+1)$$

zeros. By induction we thus have that $q_{X\oplus Y}$ has $2^{g-1}(2^g+1)$ zeros.

By (i) we have that $\operatorname{Arf}\big(q_{X\oplus Y}\big) = 0$ and by (iii) we have

$$\operatorname{Arf}\big(v.q_{X\oplus Y}\big) = \operatorname{Arf}\big(q_{X\oplus Y}\big) + q_{X\oplus Y}(v).$$

Since V acts simply transitively on $Q(V)$ we now see that there are $2^{g-1}(2^g+1)$ quadratic forms q such that $\operatorname{Arf}\big(q\big) = 0$ and $2^{g-1}(2^g-1)$ quadratic forms such that $\operatorname{Arf}\big(q\big) = 1$.

To complete the proof, fix q and consider the action of the transvection T_u. We have

$$\begin{aligned}(T_u.q)(v) &= q(T_u v) = \\ &= q(v + b(v,u)\,u) = \\ &= q(v) + q(b(v,u)\,u) + b(v, b(v,u)\,u) = \\ &= q(v) + b(v,u)\,q(u) + b(v,u)\,.\end{aligned}$$

Thus, if $q(u) = 1$, then $T_u.q = q$ and if $q(u) = 0$ we have $T_u.q = u.q$. Since V acts simply transitively on $Q(V)$ and $\operatorname{Sp}\big(2g,\mathbb{F}_2\big)$ preserves the Arf invariant we now see that $\operatorname{Sp}\big(2g,\mathbb{F}_2\big)$ acts transitively on the set of quadratic forms with even respectively odd Arf invariant. □

Corollary 3.2.4. *The following are equivalent*

(i) $\operatorname{Arf}\big(q\big) = 0$,

(ii) *q has $2^{g-1}(2^g+1)$ zeros on V,*

(iii) *there is an isotropic decomposition $V = X \oplus Y$ such that q restricts to zero on X and Y.*

Proof. We saw the equivalence of (*i*) and (*ii*) in the proof of Proposition 3.2.3. To see the equivalence of (*i*) and (*iii*), let $V = X \oplus Y$ be an isotropic decomposition and let $q_{X\oplus Y}$ be the corresponding quadratic form. If $\mathrm{Arf}(q) = 0$, then there is some $M \in \mathrm{Sp}(2g, \mathbb{F}_2)$ such that $M.q_{X\oplus Y} = q$. Then q is the quadratic form corresponding to the isotropic decomposition $MX \oplus MY$. □

Remark 3.2.5. One way to state Corollary 3.2.4 is that the Arf invariant of a quadratic form q on V is the value q takes on a majority of the elements in V. Therefore, the Arf invariant has sometimes been called the *democratic invariant* or *majority invariant.*

If one identifies $\mathbb{F}_2$ with $\{1, -1\}$ one can write the Arf invariant multiplicatively. If we add the value 0 and let the Arf invariant be zero if the "vote" is a tie, then the Arf invariant can be defined also for quadratic forms on odd dimensional vector spaces. It seems to have been this pursuit that led Browder to the parliamentary observation above, [17].

Corollary 3.2.6. *The stabilizer* $\mathrm{Stab}(q)$ *of* q *in* $\mathrm{Sp}(2g, \mathbb{F}_2)$ *has order*

$$2^{g^2-g+1}(2^{2g-2}-1)\cdot(2^{2g-4}-1)\cdots(2^2-1)\cdot(2^g-1),$$

if q is even and

$$2^{g^2-g+1}(2^{2g-2}-1)\cdot(2^{2g-4}-1)\cdots(2^2-1)\cdot(2^g+1)$$

if q is odd.

Proof. This follows from Proposition 3.2.3 and the orbit-stabilizer theorem. □

Corollary 3.2.7. *Let $u \in V$ be a nonzero vector. Then the transvection T_u lies in* $\mathrm{Stab}(q)$ *if and only if $q(u) = 1$.*

Proof. In the proof of Proposition 3.2.3 we saw that $q(T_u.v) = q(v) + (1 + q(u))b(u, v)$. Thus, if $q(u) = 1$ then $q(T_u.v) = q(v)$. Since b is nondegenerate, there is a vector v such that $b(u, v) \neq 0$. Thus, if $q(u) = 0$ then $q(T_u.v) = q(v) + b(u, v) \neq q(v)$. □

3.2.1. Aronhold sets Recall that $W = V \cup Q(V)$ is a $\mathbb{F}_2$-vector space of dimension $2g+1$ and that $Q(V)$ is a V-torsor. Suppose that $A = \{q_1, \dots, q_{2g+1}\}$ is a basis for W with all $q_i \in Q(V)$. An element $w \in W$ can then be expressed as

$$w = \sum_{i=1}^{2g+1} w_i q_i,$$

where $w_i \in \{0,1\}$. If we regard the w_i as integers we may define an integer $0 \le n_A(w) \le 2g+1$ as

$$n_A(w) = \sum_{i=1}^{2g+1} w_i.$$

Since $q_i + q_j \in V$ we see that if $q \in Q(V)$, then $n_A(q)$ is odd.

Definition 3.2.8. A set A is called an Aronhold set if $\mathrm{Arf}(q)$ only depends on $n_A(q) \mod 4$.

Note that $n_A(q_1) = \cdots = n_A(q_{2g+1}) = 1$ so if A is an Aronhold set we must have $\mathrm{Arf}(q_1) = \cdots = \mathrm{Arf}(q_{2g+1})$. We also point out that there is a unique form q_A such that $n_A(q_A) = 2g+1$, namely the form given by

$$q_A = \sum_{i=1}^{2g+1} q_i.$$

Proposition 3.2.9. *There are Aronhold sets. If $g \equiv 0,1 \mod 4$ then*

$$\mathrm{Arf}(q_1) = \cdots = \mathrm{Arf}(q_{2g+1}) = 0,$$

and if $g \equiv 2,3 \mod 4$ then

$$\mathrm{Arf}(q_1) = \cdots = \mathrm{Arf}(q_{2g+1}) = 1.$$

Furthermore, the group $\mathrm{Sp}(2g, \mathbb{F}_2)$ acts transitively on the collection of Aronhold sets and the stabilizer of an Aronhold set A is the symmetric group of A,

$$\mathrm{Sym}(A) \hookrightarrow \mathrm{Stab}(q_A) \hookrightarrow \mathrm{Sp}(2g, \mathbb{F}_2).$$

The proof is a quite long, but rather elementary computation. The interested reader is referred to [49] where one also finds the following fact.

Proposition 3.2.10. *The map sending an ordered Aronhold set $\{q_1, \dots, q_{2g+1}\}$ to the ordered symplectic basis*

$$\begin{aligned}
x_1 &= q_1 + q_2, & y_1 &= q_1 + q_{2g+1}, \\
x_2 &= q_3 + q_4 & y_2 &= q_1 + q_2 + q_3 + q_{2g+1}, \\
&\vdots & &\vdots \\
x_g &= q_{2g-1} + q_{2g}, & y_g &= q_1 + q_2 + \cdots q_{2g-1} + q_{2g+1},
\end{aligned}$$

is a bijection from the set of ordered Aronhold sets to the set of ordered symplectic bases.

3.3 Curves with symplectic level two structure

Let K be an algebraically closed of characteristic zero. Let C be a smooth, projective and irreducible curve of genus g over K and let $\mathrm{Jac}(C)$ be its Jacobian. Recall that the Picard group, $\mathrm{Pic}(C)$, of C is defined as the group of isomorphism classes of line bundles on C under tensor product and that $\mathrm{Pic}^n(C)$ is the subset of line bundles of degree n. Since C is a smooth, projective and irreducible scheme, $\mathrm{Pic}(C)$ is naturally isomorphic (as a graded $\mathbb{Z}$-module) to the group $\mathrm{Cl}(C)$ of divisor classes modulo linear equivalence, see [53], Corollary II.6.16. Since we shall only consider group theoretic properties of $\mathrm{Jac}(C)$ we can make the identifications

$$\mathrm{Jac}(C) = \mathrm{Pic}^0(C) = \mathrm{Cl}^0(C).$$

If $D \in \mathrm{Cl}(C)$, we shall denote the corresponding line bundle by $\mathcal{L}(D)$ and we shall use the notation $h^n(D)$ for the dimension of $H^n(C, \mathcal{L}(D))$.

The Jacobian $\mathrm{Jac}(C)$ has a 2-torsion subgroup

$$\mathrm{Jac}(C)[2] := \{v \in \mathrm{Jac}(C) \,|\, 2v = 0\}.$$

This group is evidently a vector space over the field of two elements, $\mathbb{F}_2$, and it is well known that its dimension is $2g$.

We may define a bilinear form on $\mathrm{Jac}(C)[2]$ in the following way. Let u and v be any two elements of $\mathrm{Jac}(C)[2]$ and think of them as linear equivalence classes of divisors. Pick a divisor $D \in u$ and a divisor $E \in v$ such that D and E have disjoint support. Since $2u = 2v = 0$ we have $2D = \mathrm{div}(f)$ and $2E = \mathrm{div}(g)$ for some functions f and g. We may now define the *Weil pairing*, b_C, by

$$(-1)^{b_C(u,v)} = \frac{f(E)}{g(D)},$$

where

$$f(E) := \prod_{P \in C} f(P)^{\mathrm{mult}_P(E)},$$

and

$$g(D) := \prod_{P \in C} g(P)^{\mathrm{mult}_P(D)}.$$

There are several things that should be checked here, for instance that the form does not depend on the choices of divisors D and E or the functions f and g and that the Weil pairing is nondegenerate and alternating so that the pair $(\mathrm{Jac}(C)[2], b_C)$ is a symplectic vector space of dimension $2g$ over $\mathbb{F}_2$. For these verifications, see for instance [48], [3] or [67].

Definition 3.3.1. A *symplectic level two structure* on a curve C of genus g is an isometry ϕ from the standard symplectic vector space of dimension $2g$ to $(\mathrm{Jac}(C)[2], b_C)$. Equivalently, a symplectic level two structure is a choice of an (ordered) symplectic basis $x_1, \ldots, x_g, y_1, \ldots, y_g$ of $(\mathrm{Jac}(C)[2], b_C)$.

Since we shall not talk about other level two structures we shall often just say "level two structure" instead of the more cumbersome "symplectic level two structure". Two curves with level two structures $(C, x_1, \ldots, y_g)$ and $(C', x'_1, \ldots, y'_g)$ are isomorphic if there is an isomorphism of curves $\varphi : C \to C'$ such that the induced morphism $\widetilde{\varphi} : \mathrm{Jac}(C')[2] \to \mathrm{Jac}(C)[2]$ takes one symplectic basis to the other in the sense that

$$\widetilde{\varphi}(x'_i) = x_i, \quad i = 1, \ldots g,$$
$$\widetilde{\varphi}(y'_i) = y_i, \quad i = 1, \ldots g.$$

We will write (C, φ) and $(C, x_1, \ldots, x_g, y_1, \ldots, y_g)$ interchangeably depending on what best suits our needs in different situations. We denote the moduli space of curves of genus g with symplectic level 2 structure by $\mathcal{M}_g[2]$. The morphism $\phi : \mathcal{M}_2[2] \to \mathcal{M}_g$ forgetting the level 2 structure has degree $|\mathrm{Sp}(2g, \mathbb{F}_2)| = 1\,451\,520$ and if S is a subset of $\mathcal{M}_g[2]$ we define $S[2] = \phi^{-1}(S)$.

3.4 Theta characteristics

Again, let C be a smooth curve of genus g over an algebraically closed field K of characteristic zero. We explained above that $\mathrm{Jac}(C)[2]$ equipped with the Weil pairing b is a symplectic space over $\mathbb{F}_2$ of dimension $2g$. We shall now explain how this symplectic space and its quadratic forms are connected to the so-called theta characteristics on C.

Definition 3.4.1. Let C be a smooth curve over K and let K_C be its canonical class. An element $\theta \in \mathrm{Pic}(C)$ such that $2\theta = K_C$ is called a theta characteristic.

If θ is a theta characteristic of C and v is an element of $\mathrm{Jac}(C)[2]$, then $2(\theta + v) = 2\theta + 2v = 2\theta$ so $\theta + v$ is again a theta characteristic and we see that $\mathrm{Jac}(C)[2]$ acts on the set of theta characteristics of C. This reminds us of the way a symplectic vector space V over $\mathbb{F}_2$ acts on its quadratic forms $Q(V)$. In fact, there is a natural identification between the set of theta characteristics of C and the quadratic forms of $\mathrm{Jac}(C)[2]$.

Theorem 3.4.2 (The Riemann-Mumford relation)**.** *We may identify the set of theta characteristics of C with the set of quadratic forms on* $\mathrm{Jac}(C)[2]$ *(with the Weil pairing) via*

$$q_\theta(v) := h^0(\theta + v) + h^0(\theta) \mod 2.$$

The Arf invariant is given by

$$\mathrm{Arf}(\theta) = h^0(\theta) \mod 2.$$

For a proof, see [51] or [3].

With the Riemann-Mumford relation at hand, we may translate the definitions and results of Section 3.2 to the language of theta characteristics. For instance, we may talk about Aronhold sets of theta characteristics. We also have the following corollary.

Corollary 3.4.3. *If C has genus g, then there are $2^{g-1}(2^g + 1)$ even theta characteristics of C and $2^{g-1}(2^g - 1)$ odd theta characteristics.*

3.5 The genus three case

We now specialize the theory above to the case $g = 3$.

3.5.1. Symplectic vector spaces of dimension 6

Lemma 3.5.1. *Let $S = \{q_1, \ldots, q_7\}$ be a set of distinct odd quadratic forms on a 6-dimensional symplectic space. Then S is an Aronhold set if and only if the forms*

$$q_i + q_j + q_k, \quad 1 \le i < j < k \le 7,$$

are all even.

Proof. Define $v_{i,j} = q_i + q_j$, $1 \le i < j \le 7$. If $v_{i,j} = v_{k,l}$ for $(i, j) \neq (k, l)$, then

$$v_{i,j} + v_{k,l} = q_i + q_j + q_k + q_l = 0,$$

so $q_l = q_i + q_j + q_k$. But q_l is odd while $q_i + q_j + q_k$ is even so this impossible. Thus, the 21 vectors $v_{i,j}$ are nonzero and distinct.

By Proposition 3.2.3 (*iii*) we have

$$\mathrm{Arf}\big(q + v\big) = \mathrm{Arf}\big(q\big) + q(v).$$

Since $q_j = q_i + v_{i,j}$ we get

$$\mathrm{Arf}\big(q_j\big) = \mathrm{Arf}\big(q_i\big) + q_i(v_{i,j}) = 1,$$

so $q_i(v_{i,j}) = 0$. Similarly, if i, j and k are distinct, then $q_i + q_j + q_k = q_k + v_{i,j}$ is even and we get

$$\mathrm{Arf}\big(q_k + v_{i,j}\big) = \mathrm{Arf}\big(q_k\big) + q_k(v_{i,j}) = 0,$$

so $q_k(v_{i,j}) = 1$. Suppose that

$$q = \sum_{i=1}^{7} a_i q_i = 0.$$

If i, j and k are distinct we have

$$q(v_{i,k}) + q(v_{j,k}) = a_i + a_j = 0.$$

Thus, $a_1 = a_2 = \ldots = a_7 = 0$ so the quadratic forms $q_1, \ldots, q_7$ are linearly independent and thus form a basis.

The $\binom{7}{3} = 35$ forms $q_i + q_j + q_k$, $1 \le i < j < k \le 7$, are all even. Proposition 3.2.3 (*iv*) says that there are $2^{3-1}(2^3+1) = 36$ even forms so only one remains. We want to show that the form $q_S = q_1 + \cdots + q_7$ is even. There are $\binom{7}{5} = 21$ quadratic forms of the form $q_{i_1} + \cdots + q_{i_5}$ where $|\{i_1, \ldots, i_5\}| = 5$. At most one of them is even so at least 20 of them are odd. Let $q = q_{i_1} + \cdots + q_{i_5}$ be odd and let $\{1, \ldots, 7\} \setminus \{i_1, \ldots, i_5\} = \{r, s\}$. We have $q = q_S + v_{r,s}$ and thus

$$\mathrm{Arf}\big(q\big) = \mathrm{Arf}\big(q_S\big) + q_S(v_{r,s}) = 1.$$

But since $q_r(v_{r,s}) = q_s(v_{r,s}) = 0$ and $q_i(v_{r,s}) = 1$ if i is not equal to r or s we see that $q_S(v_{r,s}) = 1$ and conclude that $\mathrm{Arf}\big(q_S\big) = 0$ as desired. □

Proposition 3.5.2. *Let (V, b) be a symplectic space of dimension 6. Then there is a bijection between the set of symplectic bases of V and the set of Aronhold sets of odd quadratic forms of $Q(V)$.*

Proof. We follow the proof given in [38].

Recall that we can recover b from a given quadratic form q via relation 3.2.1:

$$b(u, v) = q(u + v) + q(u) + q(v).$$

Let $q_1, \ldots, q_7$ be an ordered Aronhold set of odd quadratic forms. Define

$$v_i = q_i + q_7, \quad i = 1, \ldots, 6. \tag{3.5.1}$$

If we take $q = q_7$ in Equation 3.2.1 we get

$$b(v_i, v_j) = q_7(v_i + v_j) + q_7(v_i) + q_7(v_j).$$

By Proposition 3.2.3 (*iii*) we have

$$\operatorname{Arf}\big(q+v\big)=\operatorname{Arf}\big(q\big)+q(v).$$

Since $q_7+v_i=q_i$ we get that $q_7(v_i)=0$ for $i=1,\dots,6$. Further, since $q_i+q_j+q_7$ is even if $i\neq j$ we get that $q_7(v_i+v_j)=1$. Hence, $b(v_i,v_j)=1$ if $i\neq j$. Define

$$\begin{array}{lllllllll} x_1 & = & v_1, & x_2 & = & v_2+v_3, & x_3 & = & v_4+v_5 \\ y_1 & = & v_1+\cdots+v_6, & y_2 & = & v_3+v_4+v_5+v_6, & y_3 & = & v_5+v_6. \end{array}$$

It is easily verified that this is a symplectic basis.

Now suppose that we are given a symplectic basis x_1,x_2,x_3,y_1,y_2,y_3 of V. We may now solve the above equations to get six vectors $v_1,\dots,v_6$ such that $b(v_i,v_j)=1$ if $i\neq j$. We define a quadratic form q by

$$q\left(\sum_{i=1}^{6}a_iv_i\right)=\sum_{i<j}a_ia_j.$$

Let $u=\sum_{i=1}^{6}a_iv_i$ and $v=\sum_{i=1}^{6}b_iv_i$. Then

$$\begin{aligned} q(u+v)=q\left(\sum_{i=1}^{6}(a_i+b_i)v_i\right)&=\\ =\sum_{i<j}(a_i+b_i)(a_j+b_j)&=\\ =\sum_{i<j}a_ia_j+a_ib_j+a_jb_i+b_ib_j&=\\ =q(u)+\sum_{i\neq j}a_ib_j+q(v)&=\\ =q(u)+b(u,v)+q(v).& \end{aligned}$$

Thus, q is a quadratic form of (V,b). A tedious check shows that q has precisely 28 zeros in the vectors $v=\sum_{i=1}^{6}a_iv_i$ with 0, 1, 4 or 5 nonzero coefficients a_i. Thus, q is odd by Corollary 3.2.4, and by Proposition 3.2.3 we have that

$$q_i'=q+v_i,\quad i=1,\dots,6, \tag{3.5.2}$$

are all odd. Define $q_7'=q$. We now have seven odd quadratic forms $q_1',\dots,q_7'$. Note that $q_i'+q_j'+q_k'=q+v_i+v_j+v_k$. Further note that $\operatorname{Arf}\big(q\big)=1$, $q(v_i+v_j)=1$ if $i\neq j$ and $q(v_i+v_j+v_k)=1$ if i, j and k are distinct. It follows that if $1\leq i<j<k\leq 7$ are distinct and we define $v_7=0$, then

$$\begin{aligned} \operatorname{Arf}\Big(q_i'+q_j'+q_j'\Big)&=\operatorname{Arf}\big(q+v_i+v_j+v_k\big)=\\ &=\operatorname{Arf}\big(q\big)+q(v_i+v_j+v_k)=\\ &=0. \end{aligned}$$

Thus, $q'_1, \ldots, q'_7$ is an Aronhold set by Lemma 3.5.1.

With (3.5.1) and (3.5.2) in mind we see that to check that we have ended up with the Aronhold set we started with, it suffices to show that $q'_7 = q_7$. But using Proposition 3.2.3 (*iii*) it is easily verified that q_7 has exactly the same zeros as q'_7. Hence, $q_7 = q'_7$ as desired. $\square$

Let (V, b) be the standard symplectic space of dimension 6 and let q be the quadratic form defined by

$$q\left(\sum_{i=1}^{3} \alpha_i x_i + \beta_i y_i\right) = \sum_{i=1}^{3} \alpha_i \beta_i.$$

Then q has Arf invariant 0 so, by Corollary 3.2.6, its stabilizer $\mathrm{Stab}(q)$ has order

$$2^7(2^4-1)(2^2-1)(2^3-1) = 40320 = 8!.$$

This is a first indication that $\mathrm{Stab}(q)$ in fact is isomorphic to the symmetric group on eight elements, S_8.

By Corollary 3.2.4 we have that q has $2^{3-1}(2^3+1) = 36$ zeros so Corollary 3.2.7 gives that there are $2^6 - 36 = 28$ nontrivial transvections contained in $\mathrm{Stab}(q)$. This is the same as the number of transpositions in S_8 so this gives an idea to how one might cook up the suspected isomorphism.

Recall that S_8 has a presentation given by generators σ_i, $i = 1, \ldots, 7$ and relations

$$\begin{aligned} \sigma_i^2 &= \mathrm{id}, \\ \sigma_i\sigma_j &= \sigma_j\sigma_i, \qquad \text{if } j \neq i \pm 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}. \end{aligned}$$

The generator σ_i corresponds to the transposition of i and $i+1$. We define a map ϕ by

$$\begin{aligned} \sigma_1 &\mapsto T_{x_1+y_1}, \\ \sigma_2 &\mapsto T_{x_1+x_2+x_3+y_3}, \\ \sigma_3 &\mapsto T_{x_2+y_2}, \\ \sigma_4 &\mapsto T_{x_1+x_2+x_3+y_1}, \\ \sigma_5 &\mapsto T_{x_3+y_3}, \\ \sigma_6 &\mapsto T_{x_2+y_2+y_3}, \\ \sigma_7 &\mapsto T_{x_1+x_2+x_3+y_1+y_2+y_3}. \end{aligned}$$

It is easily checked that ϕ preserves the relations, so ϕ is a group homomorphism from S_8 to $\mathrm{Sp}(2g, \mathbb{F}_2)$. However, the only normal subgroups of S_8 are

S_8 itself, the alternating group A_8 and the trivial group. Since the image of ϕ clearly contains at least seven elements it follows that ϕ in fact is an isomorphism.

If q and q' are two even quadratic forms, then $\mathrm{Stab}(q)$ and $\mathrm{Stab}(q')$ are conjugate since $q' = M.q$ for some $M \in \mathrm{Sp}(2g, \mathbb{F}_2)$. We have

$$|\mathrm{Sp}(6, \mathbb{F}_2)| = 2^{3^2}(2^{2\cdot 3} - 1)(2^{2\cdot 3 - 2} - 1)(2^{2\cdot 3 - 4} - 1)(2^2 - 1) = 1\,451\,520,$$

and

$$\frac{|\mathrm{Sp}(6, \mathbb{F}_2)|}{|\mathrm{Stab}(q)|} = 36.$$

In particular we see that there are 36 copies of S_8 inside $\mathrm{Sp}(6, \mathbb{F}_2)$. It can also been shown that these are the only ways to embed S_8 into $\mathrm{Sp}(2g, \mathbb{F}_2)$, see for instance [22].

3.5.2. Curves of genus three with symplectic level two structure Let C be a curve of genus 3 over an algebraically closed field K of characteristic zero.

If C is a plane quartic, K_C is the restriction of a line L to C and since C has degree 4 we have $K_C = Q_1 + Q_2 + Q_3 + Q_4$ for some, not necessarily distinct, points $Q_1, \dots, Q_4 \in C$. Thus, an effective theta characteristic must be of the form $\theta = P_1 + P_2$ and such that $2\theta = K_C = 2P_1 + 2P_2$ is the restriction of a line to C. In other words, P_1 and P_2 must be the points of tangency of a bitangent. Thus, there is a natural identification between the set of bitangents of C and the set of effective theta characteristics of C. If θ is an effective theta characteristic, $\theta = Q_1 + Q_2$, then $h^0(K_C - \theta) = h^0(K_C - Q_1 - Q_2) = h^0(K_C) - 2 = 1$ since K_C is very ample. Thus, the effective theta characteristics of C are all odd and we may now apply Proposition 3.2.3 (*iv*) to see once more that a plane quartic has precisely $2^{3-1}(2^3 - 1) = 28$ bitangents.

In the hyperelliptic case, C will no longer have 28 bitangents but will still have 28 odd theta characteristics. The canonical class K_C can be expressed as the sum of two divisors in the unique g_2^1 of C, see [53] Chapter IV.5. Let $Q_1, \dots, Q_8$ be the ramification points of $g_2^1 : C \to \mathbb{P}^1$. We now find that the divisors $\theta_{i,j} = Q_i + Q_j$, $1 \le i < j \le 8$ are odd theta characteristics of C.

In Proposition 2.3.8 we saw that if $(S, E_1, \dots, E_7)$ is a geometrically marked Del Pezzo surface of degree 2 realized as a double cover $p : S \to \mathbb{P}^2$ ramified along a plane quartic C, then $p(E_1), \dots, p(E_7)$ are bitangents to C. Let $q_i = \frac{1}{2}p(E_i)$. By Proposition 3.2.10 we have that

$$x_1 = q_1 + q_2, \quad y_1 = q_1 + q_7,$$

$$x_2 = q_3 + q_4, \quad y_2 = q_1 + q_2 + q_3 + q_7,$$

$$x_3 = q_5 + q_6, \quad y_3 = q_1 + q_2 + q_3 + q_4 + q_5 + q_7,$$

is a symplectic basis of $\mathrm{Jac}(C)[2]$ (note that in the above equations we use addition of quadratic forms and not addition of divisors). This gives a morphism from the moduli space $\mathcal{DP}_2^{\mathrm{gm}}$ of geometrically marked Del Pezzo surfaces of degree 2 to the moduli space $\mathcal{Q}[2]$ of plane quartics with symplectic level 2 structure.

The group $\mathrm{Sp}(6,\mathbb{F}_2)$ acts on $\mathcal{Q}[2]$ by changing symplectic structures. In a similar fashion we have that the group $W_{E_7}^+$ acts naturally on $\mathcal{DP}_2^{\mathrm{gm}}$ by changing geometric markings. However, acting on the geometrically marked Del Pezzo surface with the element $-1 \in W_{E_7}^+$ gives a geometrically marked Del Pezzo surface only differing from the original one by the anticanonical involution ι. These two geometrically marked Del Pezzo surfaces are thus isomorphic and we conclude that -1 acts as the identity on $\mathcal{DP}_2^{\mathrm{gm}}$. Thus, the quotient $W_+(E_7) = W(E_7)/\{\pm 1\}$ acts on $\mathcal{DP}_2^{\mathrm{gm}}$. The group $W_{E_7}^+$ is isomorphic to the direct product $\mathrm{Sp}(6,\mathbb{F}_2) \times \{\pm 1\}$, see [23], so $W_+(E_7)$ is isomorphic to $\mathrm{Sp}(6,\mathbb{F}_2)$.

Theorem 3.5.3 (van Geemen [38], Chapter IX, Theorem 1)**.** *There is a* $\mathrm{Sp}(6,\mathbb{F}_2)$*-equivariant isomorphism*

$$\mathcal{DP}_2^{\mathrm{gm}} \to \mathcal{Q}[2].$$

3.6 The group of the 28 bitangents

We shall now sketch a more geometric description of the symplectic group $\mathrm{Sp}(6,\mathbb{F}_2)$. For a more complete account, see [25], Chapter II.4.4.

Let C be a plane quartic and let L_1 and L_2 be two bitangents to C. Let $C \cdot L_1 = 2P_1 + 2Q_1$ and $C \cdot L_2 = 2P_2 + 2Q_2$. There is a pencil $\mathcal{P}$ of conics through P_1, P_2, Q_1 and Q_2 and a small computation, see [54], shows that there are exactly five conics H_i, $i = 1, \ldots, 5$, in $\mathcal{P}$ such that their intersection with C consists entirely of bitangent points. More precisely, each of the conics H_i intersects C in eight bitangent points belonging to four bitangents.

A set T of four bitangents of C is called a *syzygetic tetrad* if its eight points of intersection with C lie on a conic, see Figure 3.1. If T is not a syzygetic tetrad it is called an *asyzygetic tetrad*.

Cayley denoted the bitangents of C by the set of unordered pairs of 8 objects $\{a,b,c,d,e,f,g,h\}$ or, in other words, by the edges of a complete graph K_8 with 8 vertices. In this notation, the syzygetic tetrads correspond to the edges of quadrilaterals or four disjoint edges, i.e. subgraphs isomorphic either to a cycle graph with 4 vertices or four disjoint edges with two vertices each, see Figure 3.2. There are $3 \cdot \binom{8}{4} = 210$ quadrilaterals and $7 \cdot 5 \cdot 3 \cdot 1 = 105$ quadruples of disjoint edges. Thus, there are 315 syzygetic tetrads.

Let S denote the set of syzygetic tetrads on C. Clearly, the symmetric group S_8 on the eight vertices of K_8 preserves S but there are more elements

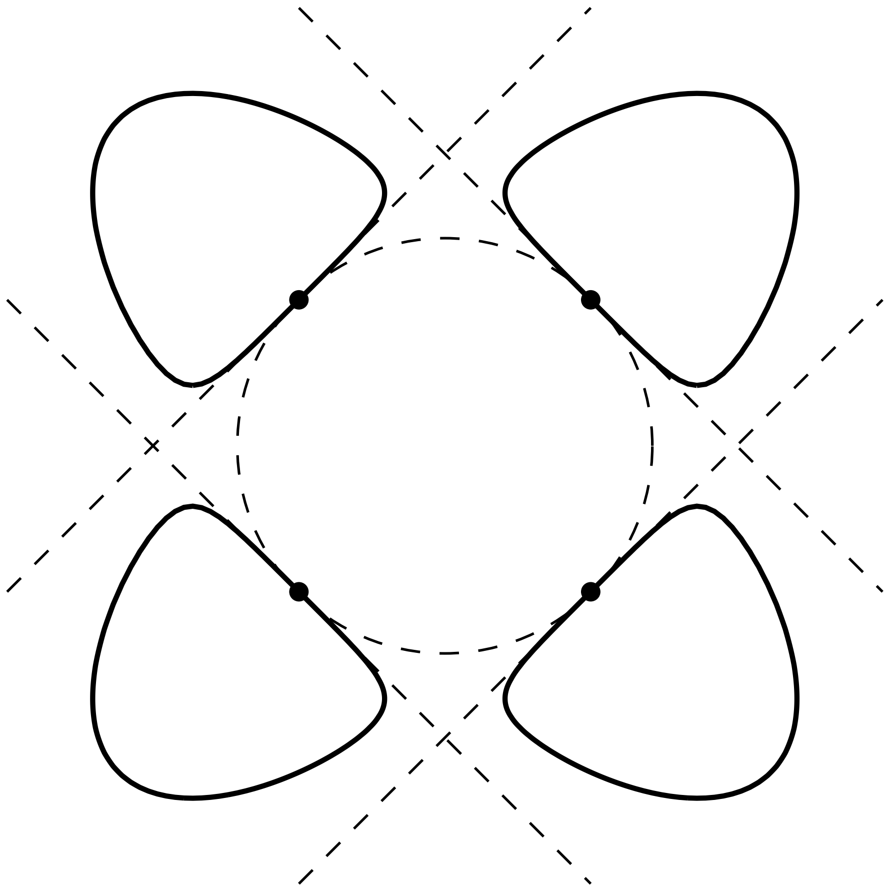
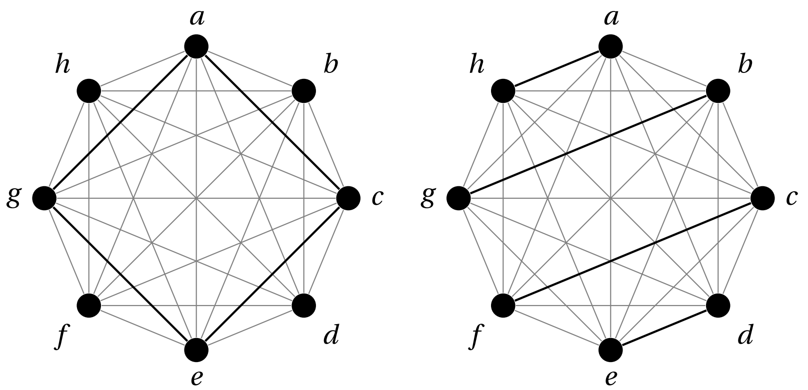


Figure 3.1: A syzygetic tetrad on a plane quartic.

in the symmetric group on the edges with this property. Hesse's *bifid maps* (bifid means "cleft into two equal parts") are defined via partitions of the vertices into two equal parts. The map fixes an edge whose vertices lie in different parts of the partition and sends an edge between two vertices in the same part to the edge between the other two vertices in that part. For example, in cycle notation we have

$$\frac{abcd}{efgh} = (\{a,b\}\,\{c,d\})(\{a,c\}\,\{b,d\})(\{a,d\}\,\{b,c\})(\{e,f\}\,\{g,h\})(\{e,g\}\,\{f,h\})(\{e,h\}\,\{f,g\}).$$

The symmetric group S_8 has two orbits in S: the quadrilaterals and the quadruples of disjoint lines. The action of the group G generated by S_8 and the bifid maps is however transitive on S. The identity element and the $\binom{8}{4}/2 = 35$ bifid maps constitute a set of coset representatives for S_8 in G so $|G| = 36 \cdot 8! = 1\,451\,520$. In fact we have that G is isomorphic to $\mathrm{Sp}(6, \mathbb{F}_2)$. The group G is generated by the 28 transpositions in S_8 and the 35 bifid



(a) A subgraph isomorphic to C_4. **(b)** Four disjoint edges.

Figure 3.2: The two types of syzygetic tetrads.

maps so the total number of generators in this description is 63. The group $\mathrm{Sp}(6,\mathbb{F}_2)$, on the other hand, is generated by the transvections. Since there are $2^6 - 1 = 63$ nonzero vectors in the standard symplectic space of dimension 6 over $\mathbb{F}_2$, there are also 63 transvections. One may now construct an isomorphism by mapping the transpositions and bifid maps to the transvections in a suitable manner.

4. Looijenga's results

In this chapter we discuss a description of the moduli space $\mathcal{Q}_1[2]$ of plane quartics with level 2 structure and one marked point given by Looijenga in his paper [62]. The construction uses the classification of tangents that we discussed in Section 2.1 and the relation between plane quartics and Del Pezzo surfaces that we discussed in Section 2.2 in order to decompose the moduli space of geometrically marked Del Pezzo surfaces of degree 2 into four subspaces. We shall then see how to use the description of the Picard group of a Del Pezzo surface from Section 2.3 in order to describe these four subspaces in terms of some of the lattices we discussed in Chapter 3.

4.1 Anticanonical curves

Let S be a Del Pezzo surface of degree 2 over $\mathbb{C}$. Recall from Chapter 2 that we can realize S both as a double cover $p : S \to \mathbb{P}^2$ ramified over a plane quartic C and as the blowup $\pi : S \to \mathbb{P}^2$ in seven points $P_1, \ldots, P_7$ in general position. We denote the exceptional curve corresponding to P_i by E_i, let L denote the total transform of a line in $\mathbb{P}^2$ and we denote the anticanonical involution of S by ι. In Section 2.2 we identified the fixed points of ι in S with $p^{-1}(C)$. We shall now give another characterization of the fixed points of ι.

A curve $A \subset S$ in the anticanonical linear system $|-K_S|$ is called an *anticanonical curve*. The anticanonical class $-K_S = 3L - E_1 - \cdots - E_7$ corresponds to the linear system $\mathcal{C}$ of cubics in $\mathbb{P}^2$ passing through $P_1, \ldots, P_7$. In Section 2.4 we saw that the curve $B = \pi(p^{-1}(C))$ consists of all the singular points of members of $\mathcal{C}$. We thus see that a point $Q \in S$ is a point of $p^{-1}(C)$ if and only if there is a unique anticanonical curve A with a singularity at Q, see Section 2.4. Note that since A is isomorphic to a singular plane cubic, its irreducible components will be rational.

By the above construction we have that if (C, P) is a plane quartic with a marked point, the double cover $p : S \to \mathbb{P}^2$ ramified along C naturally becomes equipped with an anticanonical curve A with a singularity at P. Thus, if we introduce the moduli space $\mathcal{DP}_{2,\mathrm{a}}$ of Del Pezzo surfaces of degree 2 marked with a singular point of an anticanonical curve we have an isomorphism $\mathcal{Q}_1 \cong \mathcal{DP}_{2,\mathrm{a}}$. Similarly, we introduce the moduli space $\mathcal{DP}_{2,\mathrm{a}}^{\mathrm{gm}}$

of geometrically marked Del Pezzo surfaces marked with a singular point of an anticanonical curve and get an isomorphism between the moduli space of plane quartics with level two structure and one marked point $\mathcal{Q}_1[2]$ and $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{a}}$.

In Section 2.2 we saw that $p^{-1}(C) \sim -2K_S$. Thus

$$A.p^{-1}(C) = (-K_S).(-2K_S) = 4.$$

We have that A intersects $p^{-1}(C)$ with multiplicity at least 2 so $p(A)$ is a tangent to C. The anticanonical curve A can be of the following types.

(*i*) The anticanonical curve A can be an irreducible curve with a node. Then $p(A)$ intersects C with multiplicity 2 at P so $p(A)$ is either an ordinary tangent line or a bitangent. But we have shown that the inverse image of a bitangent under p consists of two exceptional curves which are conjugate under ι and we conclude that $p(A)$ is an ordinary tangent line. We may thus identify the locus $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}} \subset \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{a}}$ consisting of surfaces such that the anticanonical curve through the marked point is irreducible and nodal with the locus $\mathcal{Q}_{\mathrm{ord}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is ordinary.

(*ii*) The anticanonical curve A can be an irreducible curve with a cusp. Then $p(A)$ intersects C with multiplicity 3 at P so $p(A)$ must be a flex line. We may thus identify the locus $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{c}} \subset \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{a}}$ consisting of surfaces such that the anticanonical curve through the marked point is irreducible and cuspidal with the locus $\mathcal{Q}_{\mathrm{flx}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is a genuine flex point.

(*iii*) The anticanonical curve A can consist of two rational curves intersecting with multiplicity one at P. Thus, the cubic $\pi(A)$ must be the product of a conic through five of the points $P_1, \ldots, P_7$ with a line through the remaining two. Hence, A consists of two conjugate exceptional curves and $p(A)$ is a genuine bitangent. We may thus identify the locus $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{tr}} \subset \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{a}}$ consisting of surfaces such that the anticanonical curve through the marked point consists of two rational curves intersecting transversally in two distinct points with the locus $\mathcal{Q}_{\mathrm{btg}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is a genuine bitangent point.

(*iv*) The anticanonical curve A can consist of two rational curves intersecting with multiplicity two at P. An analysis similar to the one above shows that $p(A)$ is then a hyperflex line. We may thus identify the locus $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{d}} \subset \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{a}}$ consisting of surfaces such that the anticanonical curve through the marked point consists of two rational curves with a double intersection with the locus $\mathcal{Q}_{\text{hfl}}[2] \subset \mathcal{Q}_1[2]$ consisting of curves whose marked point is a hyperflex point.

We investigate the four cases separately.

4.2 The irreducible nodal case

Let S be a Del Pezzo surface of degree 2, let $\phi : \operatorname{Pic}(S) \to H_7$ be a geometric marking and let P be a point of S such that there is a unique rational anticanonical curve A on S which is nodal at P. The Jacobian $\operatorname{Jac}(A)$ is isomorphic to $\mathbb{C}^*$ as a group, see Chapter II.6 of [53], and the restriction homomorphism

$$\operatorname{Pic}(S) \to \operatorname{Pic}(A),$$

induces a homomorphism

$$r : K_S^\perp \to \operatorname{Jac}(A).$$

Recall that $K_S^\perp$ is a lattice isometric to the E_7-lattice L_{E_7}. We thus see that r is an element of the 7-dimensional algebraic torus $T_{S,P} = \operatorname{Hom}\big(K_S^\perp, \operatorname{Jac}(A)\big) \cong (\mathbb{C}^*)^7$ and we have a natural action of W_{E_7} on $T_{S,P}$ via its action on $K_S^\perp$.

Let T be the torus $\operatorname{Hom}\big(L_{E_7}, \mathbb{C}^*\big)$. The action of W_{E_7} on L_{E_7} induces an action on T. If $\varphi : \operatorname{Jac}(A) \to \mathbb{C}^*$ is one isomorphism, then the function defined by $\tilde{\varphi}(D) = \varphi(D)^{-1}$ is another one and these are the only isomorphisms between $\operatorname{Jac}(A)$ and $\mathbb{C}^*$, i.e. $\operatorname{Hom}(\operatorname{Jac}(A), \mathbb{C}^*) = \{\phi, \frac{1}{\phi}\}$. We have that W_{E_7} contains -1, so there is a unique isomorphism

$$W_{E_7} \setminus T_{S,P} \to W_{E_7} \setminus T.$$

Thus, r determines an element $\chi_{S,P}$ in $W_{E_7} \setminus T$. The geometric marking ϕ in turn determines a lift $\chi_{S,P}^\phi$ in $\{\pm 1\} \setminus T$ and, since the anticanonical involution ι acts as -1 on $K_S^\perp$, the lift $\chi_{S,P}^\phi$ is unchanged by composition with ι. This gives a $W_{E_7}^+$-equivariant morphism

$$\begin{aligned} X_{E_7} : \mathcal{DP}_{2,\mathrm{n}}^{\mathrm{gm}} &\to \{\pm 1\} \setminus T, \\ (S, P, \phi) &\mapsto \chi_{S,P}^\phi. \end{aligned}$$

Every root α in $\Phi(E_7)$ determines a multiplicative character on T by evaluation, i.e. by sending an element $\chi \in T$ to $\chi(\alpha) \in \mathbb{C}^*$. Recall that α determines a reflection $r_\alpha : L_{E_7} \to L_{E_7}$

$$r_\alpha(v) = v - 2\frac{b(\alpha, v)}{b(\alpha, \alpha)} \cdot \alpha.$$

Note that

$$\chi(r_\alpha(v)) = \frac{\chi(v)}{\chi(\alpha)^{2b(\alpha,v)/b(\alpha,\alpha)}},$$

is equal to $\chi(v)$ for all v if and only if $\chi(\alpha) = 1$. In other words, the kernel of the character determined by α is exactly the fixed point locus in T of the reflection r_α. This locus is a *hypertorus*, i.e. a subtorus of codimension 1, which we denote by T_α. Define

$$D_{E_7} = \bigcup_{\alpha \in \Phi(E_7)} T_\alpha,$$

and let T_{E_7} be the complement $T \setminus D_{E_7}$.

Proposition 4.2.1 (Looijenga [62])**.** *The morphism X_{E_7} gives a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}} \to \{\pm 1\} \setminus T_{E_7}.$$

Proof. We shall construct an inverse.

Let A be a nodal rational curve. We have a natural map $A_{\mathrm{reg}} \to \mathrm{Pic}^1(A)$ from the regular locus of A to the first Picard group of A defined by sending a point P to the linear system $|P|$. By the Riemann-Roch theorem for singular curves we have that the dimension of any linear system of degree 1 is zero, so the map is bijective. Since $\mathrm{Pic}^1(A)$ is a $\mathrm{Jac}(A)$-torsor, we may use this map to define a $\mathrm{Jac}(A)$-torsor structure on A_{reg}.

Let $\chi \in T_{E_7}$, i.e. let χ be a character $L_{E_7} \to \mathbb{C}^*$ which does not take the value 1 on any root, and choose an isomorphism $\varphi : \mathbb{C}^* \to \mathrm{Jac}(A)$ and a point P_1 in A_{reg}. Using the $\mathrm{Jac}(A)$-torsor structure of A_{reg} we define points P_{i+1} by

$$P_{i+1} = P_i + \varphi(\chi(e_{i+1} - e_i)) = P_i + \varphi(\chi(-\alpha_i)),$$

for $i = 1, \dots, 6$. We also define the linear system

$$\begin{aligned} |D| = |P_1 + P_2 + P_3 + \varphi(\chi(l - e_1 - e_2 - e_3))| = \\ = |P_1 + P_2 + P_3 + \varphi(\chi(\alpha_7))|. \end{aligned}$$

The linear system $|D|$ has degree 3 and by the Riemann-Roch theorem it also has dimension 3. We therefore have that $|D|$ determines a map $f : A \to \mathbb{P}^{3-1} = \mathbb{P}^2$, and since $3 \geq 2p_a(A) + 1$, this map is a closed immersion by Corollary IV.3.2 of [53].

We now want to check that the points $f(P_1), \dots, f(P_7)$ lie in general position. We begin by showing that the points are distinct. Since $P_j = P_i + \varphi(\chi(e_j - e_i))$ we have that $f(P_i) = f(P_j)$ if and only if $\varphi(\chi(e_j - e_i)) = id$, i.e. if and only if $\chi(e_j - e_i) = 1$. But, if $i < j$ then $e_j - e_i$ is a root so this does not happen.

We continue by showing that no three points lie on a line. This condition is equivalent to the condition

$$\chi(l-e_i-e_j-e_k)\neq 1.$$

Since $l-e_i-e_j-e_k$ is a root if $i<j<k$ and $\chi\in T_{E_7}$, this condition is satisfied.

Finally, we check that no six of the points lie on a conic. This condition is equivalent to the condition

$$\chi(2l-e_1-\cdots-e_7+e_i)\neq 1.$$

Since $2l-e_1-\cdots-e_7+e_i$ is a root and $\chi\in T_{E_7}$, this condition is satisfied.

We have thus proven that the points $f(P_1),\ldots,f(P_7)$ are in general position. We obtain the desired inverse

$$\{\pm 1\}\setminus T_{E_7}\to \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}}.$$

by sending χ to the blowup of $\mathbb{P}^2$ in $f(P_1),\ldots,f(P_7)$. □

4.3 The irreducible cuspidal case

The cuspidal and nodal cases are rather similar and we follow the exposition of Section 4.2 closely.

Let S be a Del Pezzo surface of degree 2, let $\phi:\mathrm{Pic}(S)\to H_7$ be a geometric marking and let P be a point of S such that there is a unique rational anticanonical curve A on S which is cuspidal at P. The Jacobian $\mathrm{Jac}(A)$ is isomorphic to $\mathbb{C}$ as a group, see Chapter II.6 of [53]. The restriction homomorphism $\mathrm{Pic}(S)\to\mathrm{Pic}(A)$ induces a homomorphism r in the 7-dimensional $\mathbb{C}$-vector space $V_{S,P}=\mathrm{Hom}\big(K_S^{\perp},\mathrm{Jac}(A)\big)\cong(\mathbb{C}^*)^7$ and we have a natural action of W_{E_7} on $V_{S,P}$ via its action on $K_S^{\perp}$.

We define $V=\mathrm{Hom}\big(L_{E_7},\mathbb{C}\big)$. The action of W_{E_7} on L_{E_7} induces an action on V. If $\varphi:\mathrm{Jac}(A)\to\mathbb{C}$ is one isomorphism and $\lambda\in\mathbb{C}^*$, then the function defined by $\tilde{\varphi}(D)=\lambda\varphi(D)$ is another one and all isomorphisms between $\mathrm{Jac}(A)$ and $\mathbb{C}$ are of this form. We thus get a unique isomorphism

$$W_{E_7}\setminus\mathbb{P}\big(V_{S,P}\big)\to W_{E_7}\setminus\mathbb{P}(V),$$

so r determines an element $\rho_{S,P}$ in $W_{E_7}\setminus\mathbb{P}(V)$. The geometric marking ϕ determines a lift $\rho^{\phi}_{S,P}$ in $\mathbb{P}(V)$. This construction gives a $W^+_{E_7}$-equivariant morphism

$$\begin{aligned}R_{E_7}:\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}}&\to\mathbb{P}(V),\\ (S,P,\phi)&\mapsto\rho^{\phi}_{S,P}.\end{aligned}$$

Each root $\alpha \in \Phi_{E_7}$ determines a reflection hyperplane $V_\alpha \subset V$ and we define

$$H_{E_7} = \bigcup_{\alpha \in \Phi_{E_7}} V_\alpha .$$

We now introduce $V_{E_7} = V \setminus H_{E_7}$ and its projectivization $\mathbb{P}\left(V_{E_7}\right)$.

Proposition 4.3.1 (Looijenga [62]). *The morphism R_{E_7} gives a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{c}} \to \mathbb{P}\left(V_{E_7}\right).$$

The proof is completely analogous to the proof of Proposition 4.2.1 and is therefore omitted.

4.4 The case of two rational curves with transversal intersections

Let S be a Del Pezzo surface of degree 2, let $\phi : \mathrm{Pic}(S) \to H_7$ be a geometric marking and let P be a point of S such that there is a unique rational anticanonical curve A on S which consists of two rational curves F_1 and F_2 intersecting transversally at P and at another point Q. As we have already mentioned, the curves F_1 and F_2 are exceptional and conjugate under the anticanonical involution ι. The curves F_1 and F_2 span a sublattice $\langle F_1, F_2\rangle$ of $\mathrm{Pic}(S)$ of rank 2 and we denote its orthogonal complement by $\langle F_1, F_2\rangle^\perp$. The restriction homomorphism $\mathrm{Pic}(S) \to \mathrm{Pic}(A)$ induces a homomorphism

$$r : \langle F_1, F_2\rangle^\perp \to \mathrm{Jac}(A).$$

We have that $\mathrm{Jac}(A)$ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ as a group, see Chapter II.6 of [53], and if we require that the first factor comes from F_1 and the second from F_2 there are precisely four such homomorphisms.

Let $\varphi : \mathrm{Jac}(A) \to \mathbb{C}^* \times \mathbb{C}^*$ be an isomorphism. As in the proof of Proposition 4.2.1, we have that A_{reg} is a $\mathrm{Jac}(A)$-torsor. Let $P_1 \in F_1 \cap A_{\mathrm{reg}}$ and $P_2 \in F_2 \cap A_{\mathrm{reg}}$. If we require

$$\lim_{t_i \to 0} \left(P_i + \varphi^{-1}(t_1, t_2)\right) \longrightarrow P,$$

for $i = 1, 2$ then φ is uniquely determined and replacing P by Q corresponds to replacing $\varphi = (\varphi_1, \varphi_2)$ by the isomorphism given by

$$\tilde{\varphi}(D) = \left(\varphi_1(D)^{-1}, \varphi_2(D)^{-1}\right).$$

Let $D \in \langle F_1, F_2\rangle^{\perp}$. Since $F_1 + \iota(F_1) = F_1 + F_2 = K_S$ we have that $\langle F_1, F_2\rangle^{\perp} \subset K_S^{\perp}$ so ι acts as -1 on $\langle F_1, F_2\rangle^{\perp}$. We thus have

$$r(D)^{-1} = r(-D) = r(\iota.D) = \iota.r(D).$$

In other words, r is anti-invariant under ι. Denote the subgroup of $\mathrm{Jac}(A)$ consisting of elements which are anti-invariant under ι by $\mathrm{Jac}(A)^{-}$. Under the isomorphism $\varphi : \mathrm{Jac}(A) \to \mathbb{C}^* \times \mathbb{C}^*$, the group $\mathrm{Jac}(A)^{-1}$ is identified with the subgroup parametrized by $s \mapsto (s, s^{-1})$.

The marking ϕ gives two elements $f_1 = \phi(F_1)$ and $f_2 = \phi(F_2)$. Let $\langle f_1, f_2\rangle$ denote the span of these elements in L_{E_7} and let $\langle f_1, f_2\rangle^{\perp}$ be the orthogonal complement. Let $\Phi(f_1, f_2) = \langle f_1, f_2\rangle^{\perp} \cap \Phi_{E_7}$. We claim that $\Phi(f_1, f_2)$ is a sub-root system of type E_6. To see this, recall that W_{E_7} acts transitively on $\mathcal{E}$, the image of the set of exceptional curves under ϕ. Thus, there is no loss in generality in assuming that $f_1 = e_7$. Then it is easily checked that $\{\alpha_1, \ldots, \alpha_5, \alpha_7\}$ is a set of simple roots for $\langle f_1, f_2\rangle^{\perp}$.

Thus, r and F_1 give an element in the algebraic torus

$$T(f_1) = \mathrm{Hom}\Big(\langle f_1, f_2\rangle^{\perp}, \mathbb{C}^*\Big),$$

whereas if we choose the other irreducible component, F_2, we get an element in the torus

$$T(f_2) = \mathrm{Hom}\Big(\langle f_2, f_1\rangle^{\perp}, \mathbb{C}^*\Big)$$

The two tori are interchanged by ι so in the end we get an element

$$\chi^{\phi}_{S,P} \in \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} T(e),$$

and sending (S, P, ϕ) to $\chi^{\phi}_{S,P}$ gives a morphism

$$X_{E_6} : \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{tr}} \to \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} T(e).$$

Similarly to the previous cases, for each $\alpha \in \Phi(e, \iota.e)$ we define

$$T_\alpha = \{\chi \in T(e) | \chi(\alpha) = 1\},$$

and

$$D_{e,\iota.e} = \bigcup_{\alpha \in \Phi(e,\iota.e)} T_\alpha,$$

as well as the complement $T_{E_6}(e) = T(e) \setminus D_{e,\iota.e}$.

Proposition 4.4.1 (Looijenga [62])**.** *The morphism X_{E_6} gives a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{tr}} \to \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} T_{E_6}(e).$$

For the proof, which is rather similar to the proof of Proposition 4.2.1, see [62].

4.5 The case of two rational curves intersecting in one point with multiplicity two

Let S be a Del Pezzo surface of degree 2, let $\phi : \mathrm{Pic}(S) \to H_7$ be a geometric marking and let P be a point of S such that there is a unique rational anticanonical curve A on S which consists of two rational curves F_1 and F_2 intersecting at P with multiplicity two and nowhere else. As in the case above, the curves F_1 and F_2 are exceptional and conjugate under the anticanonical involution ι. The curves F_1 and F_2 span a sublattice $\langle F_1, F_2\rangle$ of $\mathrm{Pic}(S)$ of rank 2 and we denote its orthogonal complement by $\langle F_1, F_2\rangle^{\perp}$. The restriction homomorphism $\mathrm{Pic}(S) \to \mathrm{Pic}(A)$ induces a homomorphism

$$r : \langle F_1, F_2\rangle^{\perp} \to \mathrm{Jac}(A).$$

We have that $\mathrm{Jac}(A)$ is isomorphic to $\mathbb{C}\times\mathbb{C}$ as a group, see Chapter II.6 of [53].

From here on, the construction is very similar to that in Section 4.4. The geometric marking ϕ, the point P and the choice of the component F_1 gives an element in $V(f_1) = \mathrm{Hom}\left(\left\langle f_1, f_2\right\rangle^{\perp}, \mathbb{C}\right)$. Choosing F_2 instead of F_1 gives an element in $V(f_2) = \mathrm{Hom}\left(\left\langle f_2, f_1\right\rangle^{\perp}, \mathbb{C}\right)$ and we use the involution ι, which acts as -1, to identify the two vector spaces $V(f_1)$ and $V(f_2)$. We thus obtain an element

$$\rho^{\phi}_{S,P} \in \{\pm 1\} \setminus \coprod_{e\in\mathcal{E}} \mathbb{P}(V(e)),$$

and sending (S, P, ϕ) to $\rho^{\phi}_{S,P}$ gives a morphism

$$R_{E_6} : \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{d}} \to \{\pm 1\} \setminus \coprod_{e\in\mathcal{E}} \mathbb{P}(V(e)).$$

For each $\alpha \in \Phi(e, \iota.e)$ we define

$$V_\alpha = \{\rho \in V(e) | \rho(\alpha) = 0\}.$$

Finally, we define

$$H_{e,\iota.e} = \bigcup_{\alpha\in\Phi(e,\iota.e)} V_\alpha,$$

as well as the complement $V_{E_6}(e) = V(e) \setminus H_{e,\iota.e}$.

Proposition 4.5.1 (Looijenga [62])**.** *The morphism R_{E_6} gives a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{d}} \to \{\pm 1\} \setminus \coprod_{e\in\mathcal{E}} \mathbb{P}\left(V_{E_6}(e)\right).$$

4.6 Putting the pieces together

The four loci $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}}$, $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{c}}$, $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{tr}}$ and $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{d}}$ are naturally grouped in two pairs. Firstly, we have the two spaces $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}}$ and $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{c}}$ which correspond to surfaces marked with irreducible anticanonical curves and which are described in terms of the root system E_7 and, secondly, we have the two spaces $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{tr}}$ and $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{d}}$ which correspond to surfaces marked with reducible anticanonical curves and which are described in terms of the root system E_6. We therefore introduce

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{irr}} = \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{n}} \cup \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{c}},$$

and

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{red}} = \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{tr}} \cup \mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{d}}.$$

We begin by investigating $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{irr}}$. Consider the algebraic torus T defined in Section 4.2 and identify the tangent space at the origin of T with the vector space V defined in Section 4.3. If we let $\tilde{T}$ be the blowup of T in the origin, then the exceptional divisor is identified with the projectivization $\mathbb{P}(V)$. Furthermore, if we let $\tilde{D}_{E_7}$ be the strict transform of D_{E_7} then $\mathbb{P}(V) \setminus \tilde{D}_{E_7} = \mathbb{P}(V_{E_7})$. Thus, if we let $\tilde{T}_{E_7} = \tilde{T} \setminus \tilde{D}_{E_7}$, then we may identify both T_{E_7} and $\mathbb{P}(V_{E_7})$ as subvarieties and $\tilde{T}_{E_7}$ is their disjoint union.

Proposition 4.6.1 (Looijenga [62])**.** *The maps X_{E_7} and P_{E_7} define a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{irr}} \to \{\pm 1\} \setminus \tilde{T}_{E_7}.$$

For a proof, we again refer to [62].

The case $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{red}}$ is completely analogous. We let $e \in \mathcal{E}$, define $\tilde{T}(e)$ to be the blowup of $T(e)$ in the origin, let $\tilde{D}_{e,\iota.e}$ be the strict transform of $D_{e,\iota.e}$ and finally define $\tilde{T}_{E_6}(e) = \tilde{T}(e) \setminus \tilde{D}_{e,\iota.e}$. We then have the following.

Proposition 4.6.2 (Looijenga [62])**.** *The maps X_{E_6} and P_{E_6} define a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{red}} \to \{\pm 1\} \setminus \coprod_{e \in \mathcal{E}} \tilde{T}_{E_6}(e).$$

For completeness, we also give a brief description of Looijenga's gluing of $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{irr}}$ and $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{red}}$. To this end, we remind the reader of some standard definitions in toric geometry.

4.6.1. Interlude on toric geometry Recall that to the standard affine torus $T = (\mathbb{C}^*)^n$ we may associate two groups:

(*i*) The character group of morphisms of group schemes

$$M = \{\chi : T \to \mathbb{C}^*\},$$

under multiplication, and,

(*ii*) The group of one-parameter subgroups consisting of morphisms of group schemes

$$N = \{\lambda : \mathbb{C}^* \to T\},$$

under multiplication.

The group M is isomorphic to $\mathbb{Z}^n$ where the isomorphism is given by mapping $m = (m_1, \ldots, m_n)$ to the map $\chi^m : T \to \mathbb{C}^*$ defined by

$$(t_1, \ldots, t_n) \mapsto t_1^{m_1} \cdots t_n^{m_n}.$$

Also N is isomorphic to $\mathbb{Z}^n$. We now map $u = (u_1, \ldots, u_n)$ to $\lambda^u : \mathbb{C}^* \to T$ given by

$$t \mapsto (t^{u_1}, \ldots, t^{u_n}).$$

The composition $\chi \circ \lambda$ is a map $\mathbb{C}^* \to \mathbb{C}^*$ of the form $t \mapsto t^k$ for some integer k. We may thus define $\langle \chi, \lambda \rangle$ by

$$\chi \circ \lambda(t) = t^{\langle \chi, \lambda \rangle}.$$

This gives a perfect pairing $M \times N \to \mathbb{Z}$. In the notation of our previous isomorphisms we have

$$\langle \chi^m, \lambda^u \rangle = m_1 u_1 + \cdots + m_n u_n.$$

We shall write $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

A *rational polyhedral cone* $\sigma \subset N_{\mathbb{R}}$ is a cone generated by finitely many elements of N, i.e.

$$\sigma = \{c_1 u_1 + \cdots + c_s u_s | c_1, \ldots, c_s \geq 0\},$$

where $u_1, \ldots, u_s \in N$. We say that σ is strongly convex if $\sigma \cap (-\sigma) = \{0\}$. A *face* of σ is an intersection $\{l = 0\} \cap \sigma$ where l is a linear form which is nonnegative on σ. A one dimensional face is called an *edge* and a face of codimension 1 is called a *facet*.

Given a strongly convex polyhedral cone $\sigma \subset N_{\mathbb{R}}$ we define the *dual cone* $\sigma^\vee \subset M_{\mathbb{R}}$ by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle \chi^m, \lambda^u \rangle \geq 0 \text{ for all } u \in \sigma\}.$$

This is a rational polyhedral cone of dimension n (regardless of the dimension of σ). If we define

$$R_\sigma = \mathbb{C}[\sigma^\vee \cap M],$$

with multiplication $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$, then

$$X_\sigma = \mathrm{Spec}(R)$$

is called an affine toric variety. Since the trivial cone $\{0\}$ is a subset of any cone σ we will have $R_\sigma \subset \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ for any cone σ and this inclusion corresponds to an inclusion

$$(\mathbb{C}^*)^n \subset X_\sigma.$$

In other words, any n-dimensional affine toric variety contains the standard n-dimensional torus as a Zariski open subset.

A *fan*, Σ, is a finite collection of strongly polyhedral cones satisfying:

(*i*) if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$, and

(*ii*) if $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is both a face of σ and of τ.

Every cone $\sigma \in \Sigma$ defines an affine toric variety X_σ. Further, if σ and τ are two cones in Σ then $X_{\sigma\cap\tau}$ is a Zariski open subset of both X_σ and X_τ. Finally, recall that a *toric variety* is a variety obtained from a fan Σ by gluing together X_σ and X_τ along $X_{\sigma\cap\tau}$ for all pairs of cones σ and τ in Σ. This variety is denoted X_Σ.

4.6.2. The final gluing We now return to the gluing of $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{irr}}$ and $\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{red}}$. Let $e \in \mathcal{E}$ and let $b(-,-)$ be the bilinear form of L_{E_7}. We define a one-parameter subgroup p_e by $t \mapsto t^{b(e,-)}$ and a strongly convex cone $\sigma_e = \langle p_e \rangle$. The cone σ_e determines an affine torus T_e and a torus embedding $\varphi_e : T \hookrightarrow T_e$. Further, we have a fan $\Sigma = \{\sigma_e | e \in \mathcal{E}\}$ which determines a toric variety $T_{\mathcal{E}}$ which is the variety obtained from the T_e's by gluing along T.

For each $e \in \mathcal{E}$ we have $T_e \cong \mathbb{C} \times (\mathbb{C}^*)^6$ so $T_e \setminus T = \{0\} \times (\mathbb{C}^*)^6$ is abstractly isomorphic to $T(e)$. More explicitly, we have that

$$p_e(t)|_{\langle e,\iota.e\rangle^\perp} = 1 \quad \text{for all } t \in \mathbb{C}^*.$$

Let $\chi_1 = p_e^\vee \in M_{\mathbb{R}}$ and extend to a basis for $M_{\mathbb{R}}$ with $\chi_2, \ldots, \chi_7$. Then the inclusion $T \subset T_e$ corresponds to the inclusion of rings

$$R_{T_e} = \mathbb{C}[t_1, t_2^{\pm 1}, \ldots, t_7^{\pm 1}] \subset \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_7^{\pm 1}] = R_T,$$

and the torus $T(e)$ is embedded into T_e via

$$R_{T_e} = \mathbb{C}[t_1, t_2^{\pm 1} \dots, t_7^{\pm 1}] \to \frac{\mathbb{C}[t_1, t_2^{\pm 1} \dots, t_7^{\pm 1}]}{(t_1)} = \mathbb{C}[t_2^{\pm 1} \dots, t_7^{\pm 1}] = R_{T(e)}.$$

Here t_i is the variable corresponding to the character χ_i.

For each $e \in \mathcal{E}$, the closure of $\varphi_e(p_e)$ is a projective line $\mathbb{P}^1_e$ and the $\mathbb{P}^1_e$'s intersect in the origin of $T_{\mathcal{E}}$. Let $\tilde{T}_{\mathcal{E}}$ be the blowup of $T_{\mathcal{E}}$ in the origin and let $\tilde{\mathbb{P}}^1_e$ be the strict transform of $\mathbb{P}^1_e$. Then, the $\tilde{\mathbb{P}}^1_e$'s are disjoint. Let

$$L = \bigcup_{e \in \mathcal{E}} \tilde{\mathbb{P}}^1_e,$$

and let $\overline{T}_{\mathcal{E}}$ be the blowup of $\tilde{T}_{\mathcal{E}}$ in L. The exceptional divisor corresponding to $\tilde{\mathbb{P}}^1_e$ is of the form $\mathbb{P}^1_e \times \mathbb{P}(V(e))$. We now contract the $\mathbb{P}^1_e$-factors in order to obtain a variety $\hat{T}$. The construction allows us to identify the spaces D_{E_7} and H_{E_7}, the $D_{e,\iota.e}$'s and the $H_{e,\iota.e}$'s in $\hat{T}$ and we let $\hat{T}_{\mathcal{E}}$ be the complement of their union. We then have the following.

Theorem 4.6.3 (Looijenga [62])**.** *The maps X_{E_7}, R_{E_7}, X_{E_6} and R_{E_6} together give a $W^+_{E_7}$-equivariant isomorphism*

$$\mathcal{DP}^{\mathrm{gm}}_{2,\mathrm{a}} \to \{\pm 1\} \setminus \hat{T}_{\mathcal{E}}.$$

5. Cohomology of complements of arrangements

In Chapter 4 we saw that the spaces $\mathcal{Q}_{\mathrm{ord}}[2]$, $\mathcal{Q}_{\mathrm{btg}}[2]$, $\mathcal{Q}_{\mathrm{flx}}[2]$ and $\mathcal{Q}_{\mathrm{hfl}}[2]$ are all isomorphic to quotients of complements of unions of closed subvarieties in varieties of larger dimension. These unions are all examples of so-called *arrangements* and in this chapter we shall see how to compute the cohomology of the complement of arrangements of the types occurring in Chapter 4.

Arrangements of hyperplanes have been studied for a long time and their cohomology has been investigated by others in sufficient detail for our purposes. In particular, Fleischmann and Janiszczak [45] have computed the cohomology of complements of arrangements of hyperplanes associated to root systems of exceptional type equivariantly with respect to the corresponding Weyl groups.

Also arrangements of hypertori have been studied quite extensively but not as much as arrangements of hyperplanes. In particular, equivariant computations of complements of arrangements of hypertori associated to root systems have not been performed before. The main purpose of this chapter is therefore to fill this gap, at least enough for the computation of cohomology of moduli of plane quartics. We do this by developing an algorithm for the computation of the cohomology of the complement of an arrangement of hypertori associated to a root system. The results are presented in Section 5.11.

As a side track we also compute the total cohomology of the complement of the arrangement of hypertori associated to the root system A_n by quite different methods. The result is given in Theorem 5.10.8. We also compute the Poincaré polynomial of the complement of the arrangement of hypertori associated to the root system A_n. The result is given in Theorem 5.10.10.

5.1 Weights

Let X be a variety over $\mathbb{C}$ of dimension n and let G be a finite group of automorphisms of X. Let $H^k_c(X)$ denote the k'th compactly supported de Rham cohomology group of X with complex coefficients. Deligne, see [31] and [32], has shown that $H_c^k(X)$ carries a mixed Hodge structure. In particular, there is a filtration on $H_c^k(X)$ called the *weight filtration*

$$0 = W_{-1}H_c^k(X) \subset W_0 H_c^k(X) \subset \cdots \subset W_{2k} H_c^k(X) = H_c^k(X),$$

with many nice properties. For instance, an automorphism of X respects the weight filtration, so G also respects the filtration. Hence, the *graded components*

$$\mathrm{Gr}_m^W H_c^k(X) = W_m H_c^k(X) / W_{m-1} H_c^k(X),$$

are G-modules. In what follows we shall identify the character of a representation of G with the vector space affording the representation. Let $R(G)$ denote the complex representation ring of G.

Definition 5.1.1. The *weight m (compactly supported) equivariant Euler characteristic* of X is the element of $R(G)$ given by

$$E_c^{G,m}(X) = \sum_{k \geq 0} (-1)^k \mathrm{Gr}_m^W H_c^k(X).$$

The *(compactly supported) equivariant Euler weight polynomial* of X is the element of $R(G)[t]$ given by

$$W_c^G(X,t) = \sum_{m \geq 0} E_c^{G,m}(X)\, t^m.$$

The *(compactly supported) equivariant Poincaré polynomial* of X is the element of $R(G)[t]$ given by

$$P_c^G(X,t) = \sum_{k \geq 0} H_c^k(X)\, t^k.$$

In less precise terms we have that the cohomology of X has two gradings - the cohomological grading and the weight grading. The Poincaré polynomial remembers the cohomological grading but forgets about the weights while the weight polynomial remembers the weights but forgets the cohomological grading.

The weight polynomial has the following nice additivity property.

Lemma 5.1.2. *Let X be a variety, G a finite group of automorphisms of X and let $Z \subset X$ be a G-invariant closed subset. Then*

$$W_c^G(X,t) = W_c^G(X \setminus Z, t) + W_c^G(Z,t).$$

Proof. There is a G-equivariant long exact sequence of mixed Hodge structures

$$\cdots \to H_c^k(X \setminus Z) \to H_c^k(X) \to H_c^k(Z) \to H_c^{k+1}(X \setminus Z) \to \cdots. \tag{5.1.1}$$

From this the result follows via a routine computation. □

5.2 Purity

This section is based on work of Dimca and Lehrer [36], and all the results herein are theirs.

Again, let X be a variety over $\mathbb{C}$ of dimension n and let G be a finite group of automorphisms of X. If X is smooth, then $W_i H_c^k(X)$ can only be nonzero when $2k-2n \le i \le k$, see Section 4.5.2 and Appendix B of [73] (recall that since X has complex dimension n we have $0 \le k \le 2n$). Moreover, $W_{2k-2n} H_c^k(X)$ has Tate type $(k-n, k-n)$.

If $W_{2k-2n} H_c^k(X) = H_c^k(X)$ for all k, then $\mathrm{Gr}_{2k-2n}^W H_c^k(X) = H_c^k(X)$ and for $i \neq 2k-2n$ we have $\mathrm{Gr}_i^W H_c^k(X) = 0$. Then

$$\begin{aligned}
W_c^G(X,t) &= \sum_{m\ge 0} E_c^{G,m}(X)\, t^m = \\
&= \sum_{m\ge 0}\sum_{k\ge 0} (-1)^k \mathrm{Gr}_m^W H_c^k(X)\, t^m = \\
&= \sum_{k\ge 0} (-1)^k \mathrm{Gr}_{2k-2n}^W H_c^k(X)\, t^{2k-2n} = \\
&= \sum_{k\ge 0} (-1)^k H_c^k(X)\, t^{2k-2n} = \\
&= t^{-2n} P_c^G(X, -t^2).
\end{aligned} \tag{5.2.1}$$

In particular, we see that the equivariant weight polynomial determines the Poincaré polynomial. This is the motivation for the following definition.

Definition 5.2.1 (Dimca and Lehrer [36])**.** Let X be a variety over $\mathbb{C}$ of dimension n.

(i) The cohomology group $H_c^k(X)$ is called *pure of weight w* if $\mathrm{Gr}_i^W H_c^k(X) = 0$ for all $i \neq w$.

(ii) If X is an irreducible variety, then X is called *minimally pure* if each cohomology group $H_c^k(X)$ is pure of weight $2k-2n$.

(iii) A general variety X is called minimally pure if it is equidimensional and, for each collection of irreducible components $X_1, \ldots, X_s$ of X, the irreducible variety

$$X_1 \setminus (X_2 \cup \cdots \cup X_s),$$

is minimally pure.

In particular, if X is a minimally pure irreducible variety, then $H_c^i(X)$ can only be nonzero in the range $n \le i \le 2n$. Recall that if X is smooth and compact, then $H^{2n-k}(X)$ has weight $2n-k$, see [27], Section 3.3. By Poincaré duality we have $\left(H^{2n-k}(X)\right)^\vee \cong H_c^k(X)\,(\text{-}n)$ so $H_c^k(X)$ has weight k. We conclude that if X is smooth, compact and minimally pure then $k = 2k - 2n$ for $k = 0, \ldots, 2n$. Thus, X is a collection of points. Hence, minimal purity has its main use for noncompact varieties.

The following lemma is a direct consequence of Definition 5.2.1.

Lemma 5.2.2. *Let X be a minimally pure variety, let $\{X_i\}_{i\in I}$ be its set of irreducible components and let $J_1 \subset I$ and $J_2 \subset I$. Then*

$$\bigcup_{i\in J_1} X_i \setminus \bigcup_{j\in J_2} X_j,$$

is minimally pure.

Lemma 5.2.3. *Let X be a minimally pure variety of pure dimension n. Then $H_c^k(X)$ is pure of weight $2k - 2n$.*

Proof. Let $\{X_1, \ldots, X_r\}$ be the set of irreducible components of X. If $r = 1$, then the result is part of Definition 5.2.1. We thus assume that $r > 1$ and proceed by induction. Let $Z = X_2 \cup \cdots \cup X_r$ and use the exact sequence 5.1.1. By the induction hypothesis we have that $H_c^k(Z)$ is pure of weight $2k - 2n$ and $X \setminus Z = X_1 \setminus Z$ is an irreducible variety, since it is open in X_1, and thus also $H_c^k(X \setminus Z)$ is pure of weight $2k - 2n$. It now follows that $H_c^k(X)$ is pure of weight $2k - 2n$. □

Lemma 5.2.4. *Let X be an irreducible and minimally pure variety and let $D \subset X$ be a minimally pure divisor. Then $X \setminus D$ is minimally pure.*

Proof. We again use the exact sequence 5.1.1:

$$\cdots \to H_c^{k-1}(D) \to H_c^k(X \setminus D) \to H_c^k(X) \to \cdots.$$

By Lemma 5.2.3 we have that $H_c^{k-1}(D)$ is pure of weight $2(k-1) - 2(n-1) = 2k - 2n$ and $H_c^k(X)$ is also pure of weight $2k - 2n$. It follows that also $H_c^k(X \setminus D)$ is pure of weight $2k - 2n$. □

5.3 Arrangements

Definition 5.3.1. Let X be a variety. An *arrangement* $\mathcal{A}$ in X is a finite set $\{A_i\}_{i\in I}$ of closed subvarieties of X.

Given an arrangement $\mathcal{A}$ in a variety X one may define its *cycle*

$$D_{\mathcal{A}} = \bigcup_{i\in I} A_i \subset X,$$

and its *open complement*

$$X_{\mathcal{A}} = X \setminus D_{\mathcal{A}}.$$

The variety $X_{\mathcal{A}}$ will be our main object of study.

In certain situations, of which we will see numerous examples later, many interesting properties of the variety $X_{\mathcal{A}}$ can be deduced from properties of $D_{\mathcal{A}}$ via inclusion-exclusion arguments. The object that governs the principle of inclusion and exclusion in this setting is the *intersection poset* of $\mathcal{A}$.

Definition 5.3.2. Let $\mathcal{A}$ be an arrangement in a variety X. The intersection poset of $\mathcal{A}$ is the set

$$\mathscr{P}(\mathcal{A}) = \left\{ \cap_{j \in J} A_j \,\middle|\, J \subseteq I \right\}.$$

of intersections of elements of $\mathcal{A}$, ordered by inclusion. We include X as an element of $\mathscr{P}(\mathcal{A})$ corresponding to the empty intersection.

The definition of the poset $\mathscr{P}(\mathcal{A})$ is deceivingly similar to a poset $\mathscr{L}(\mathcal{A})$ used in many combinatorial texts whose elements are *irreducible components* of intersections of elements in $\mathcal{A}$ and where the order is given by *reverse* inclusion. The combinatorialists have good reasons for their definition - it turns intervals into geometric lattices and the poset is graded by codimension. For us, $\mathscr{P}(\mathcal{A})$ will be the most important object but at times we will also consider the poset $\mathscr{L}(\mathcal{A})$.

Since $\mathscr{L}(\mathcal{A})$ is a poset, it has a Möbius function $\mu : \mathscr{L}(\mathcal{A}) \times \mathscr{L}(\mathcal{A}) \to \mathbb{Z}$ defined inductively by setting $\mu(Z, Z) = 1$ and

$$\sum_{Y \le Z' \le Z} \mu(Z', Z) = 0, \quad \text{if } Y \ne Z,$$

where the sum is over all $Z' \in \mathscr{L}(\mathcal{A})$ between Y and Z. Since we shall exclusively be interested in the values of the Möbius function at the maximal element X, we shall use the simplified notation $\mu(Z) = \mu(Z, X)$.

The two types of arrangements that will be of importance to us are *hyperplane arrangements* and *toric arrangements*, i.e. arrangements of hyperplanes in an affine or projective space respectively arrangements of codimension 1 tori inside a higher dimensional torus. As it happens, these are also the types of arrangements that have been given most attention in the literature.

Example 5.3.3. Let X be the affine plane $\mathbb{A}^2$ and let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ consist of the varieties given by the equations

$$A_1 : 2x = 0, \quad A_2 : 2y = 0, \quad A_3 : 2x + 2y = 0, \quad A_4 : 2x - 2y = 0.$$

The resulting hyperplane arrangement is illustrated in Figure 5.1. The intersection poset is given in Figure 5.2 as well as the values of the Möbius function.

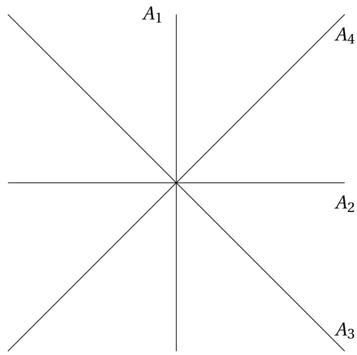


Figure 5.1: The hyperplane arrangement of Example 5.3.3.

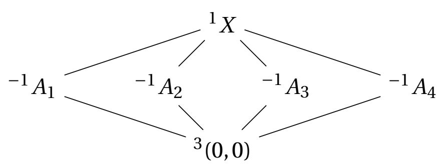


Figure 5.2: The intersection poset of the hyperplane arrangement of Example 5.3.3. The numbers in the upper left corners are the values of the Möbius function.

Example 5.3.4. In this example we consider the "exponentiation" of Example 5.3.3. Let X be the 2-torus $(\mathbb{C}^*)^2$ and let $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ consist of the varieties given by the equations

$$A_1 : x^2 = 1, \quad A_2 : y^2 = 1, \quad A_3 : x^2 y^2 = 1, \quad A_4 : x^2 y^{-2} = 1.$$

The resulting toric arrangement is illustrated in Figure 5.3. This time, the four varieties intersect in the four points $(\pm 1, \pm 1)$. The intersection poset is given in Figure 5.4 as well as the values of the Möbius function.

Example 5.3.5. We shall now consider an example with a slightly more general behaviour. Let X be the 2-torus $(\mathbb{C}^*)^2$ and let $\mathcal{A} = \{A_1, A_2, A_3\}$ consist of the varieties given by the equations

$$A_1 : x = 1, \quad A_2 : y = 1, \quad A_3 : x^3 y = 1.$$

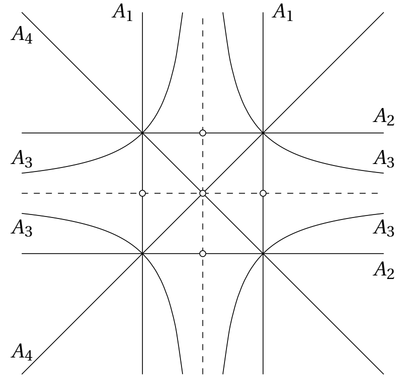


Figure 5.3: The toric arrangement of Example 5.3.4.

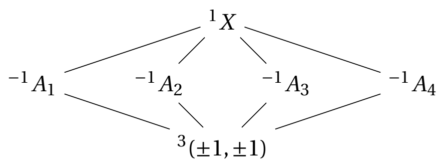


Figure 5.4: The intersection poset of the toric arrangement of Example 5.3.4. The numbers in the upper left corners are the values of the Möbius function.

In this example, the real picture is a bit too misleading to be of use. Nevertheless, we can still compute the intersection poset. Let ζ be a primitive third root of unity. We then have

$$A_1 \cap A_2 = A_1 \cap A_3 = \{(1,1)\} \quad \text{and} \quad A_2 \cap A_3 = \{(1,1), (\zeta, 1), (\zeta^2, 1)\}.$$

The intersection poset is given in Figure 5.5 as well as the values of the Möbius function.

Definition 5.3.6. Let X be a complex variety, let $\mathcal{A} = \{A_i\}_{i \in I}$ be an arrangement in X and let $Z \subset X$ be a closed subset. The arrangement

$$\mathcal{A}_Z = \{A_i \cap Z | A_i \in \mathcal{A}, A_i \cap Z \neq Z\},$$

in Z is called the *restriction* of $\mathcal{A}$ to Z.

Definition 5.3.7 (Dimca and Lehrer [36])**.** Let X be a minimally pure variety over $\mathbb{C}$ and let $\mathcal{A} = \{A_i\}_{i \in I}$ be an arrangement in X. The arrangement $\mathcal{A}$ is called *minimally pure* if

(i) Each A_i is either a union of irreducible components of X or the codimension of $A_i \cap X_j$ in X_j is 1 for all irreducible components X_j of X such that $A_i \cap X_j \neq \emptyset$.

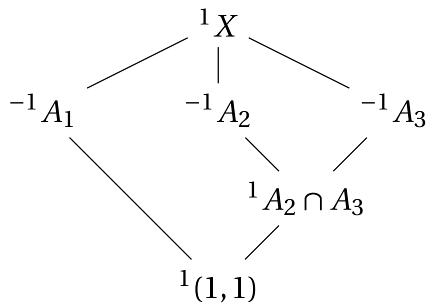


Figure 5.5: The intersection poset of the toric arrangement of Example 5.3.5. The numbers in the upper left corners are the values of the Möbius function.

(*ii*) For each $Z \in \mathscr{P}(\mathcal{A})$ we have

(a) Z is minimally pure,

(b) if $Z \cap X_j \neq \emptyset$, then $\dim Z \cap X_j = \dim Z$,

(*iii*) For each $Z \in \mathscr{P}(\mathcal{A})$, the restriction arrangement $\mathcal{A}_Z$ in Z satisfies (*i*) and (*ii*) above.

In particular we see that if X is irreducible and minimally pure and $\mathcal{A}$ is a minimally pure arrangement in X, then either $D_{\mathcal{A}}$ is a divisor or $X_{\mathcal{A}} = \emptyset$.

Theorem 5.3.8 (Dimca and Lehrer [36])**.** *Let X be a minimally pure variety and let $\mathcal{A} = \{A_i\}_{i=1}^r$ be a minimally pure arrangement in X. Then the open complement $X_{\mathcal{A}}$ is minimally pure.*

Proof. We start by proving the theorem for an irreducible X of dimension n. There is nothing to prove if $A_i = X$ for some i. We proceed by double induction on r and n. The cases where either $r = 0$ or $n = 0$ are trivial.

Let $n > 0$ and $|I| > 0$. Then there is an element $A_1 \in \mathcal{A}$ and there is no loss in generality in assuming that the dimension of A_1 is $n-1$. Define

$$Y = X \setminus \bigcup_{i=2}^{r} A_i, \quad \text{and} \quad Z = A_1 \setminus \bigcup_{i=2}^{r} A_i.$$

We then have that $X_{\mathcal{A}} = Y \setminus Z$. By induction on r we have that Y is minimally pure and by induction on n we have that Z is minimally pure of dimension $n-1$. Hence, $H_c^k(Y)$ is a pure Hodge structure of weight $2k-2n$ while $H_c^k(Z)$ is a pure Hodge structure of weight $2k+2-2n$. Thus, the long exact sequence 5.1.1

$$\cdots \to H_c^{k-1}(Y) \to H_c^{k-1}(Z) \to H_c^k(Y \setminus Z) \to H_c^k(Y) \to H_c^k(Z) \to \cdots,$$

splits into short exact sequences

$$0 \to H_c^{k-1}(Z) \to H_c^k(Y \setminus Z) \to H_c^k(Y) \to 0.$$

Thus, $H_c^k(Y \setminus Z)$ is a pure Hodge structure of weight $2k-2n$.

Now let X be general and let $\{X_1, \dots, X_r\}$ be the irreducible components of X. The set $\tilde{X}_j = X_{\mathcal{A}} \cap X_j$ is either irreducible or empty and the set $\{\tilde{X}_1, \dots, \tilde{X}_r\}$ is the set of irreducible components of $X_{\mathcal{A}}$ (possibly plus the empty set).

Pick an irreducible component X_j and consider the restricted arrangement $\mathcal{A}_{X_j}$. We then have the following.

(*i*) By Definition 5.3.7 we have that for each $A_i \in \mathcal{A}$, either $A_i \cap X_j$ is equal to X_j or $\emptyset$ or we have that the codimension in X_j is equal to 1. Thus, each element of $\mathcal{A}_{X_j}$ satisfies Definition 5.3.7 (*i*).

(*ii*) Let $Z \in \mathscr{P}\left(\mathcal{A}_{X_j}\right)$. Then $Z \in \mathscr{P}(\mathcal{A})$ so Z is minimally pure and we either have $Z \cap X_j = \emptyset$ or $\dim Z \cap X_j = \dim Z$, since $\mathcal{A}$ is a minimally pure arrangement.

(*iii*) Let $Z \in \mathscr{P}\left(\mathcal{A}_{X_j}\right)$. The restriction of $\mathcal{A}_{X_i}$ to Z is the same as the restriction of $\mathcal{A}$ to Z so the arrangement $\mathcal{A}_{X_i}$ satisfies Definition 5.3.7 (*iii*).

We conclude that $\mathcal{A}_{X_i}$ is minimally pure.

Now let $X_{i_1}, \dots, X_{i_s}$ be a collection of irreducible components of X and let $Y = X_{i_1} \setminus \bigcup_{k=2}^s X_{i_k}$. Replacing X_j by Y in the above argument and using Lemma 5.2.2 we get that Y is irreducible and minimally pure and that $\mathcal{A}_Y$ is a minimally pure arrangement. Since we have seen that the complement of a minimally pure arrangement in an irreducible minimally pure variety is minimally pure, the result now follows, see Definition 5.2.1 (*iii*). □

It is not very hard to see that both hyperplane arrangements and toric arrangements are minimally pure. Thus, the open complement of an arrangement of this type is minimally pure. These facts were known before [36] - the hyperplane case was first proven by Brieskorn [16] while the toric case was proven by Looijenga [62] (although they did not use the language of minimal purity).

Before moving on, we state a result of Looijenga which will be of use to us later.

Theorem 5.3.9 (Looijenga [62])**.** *Let X be a minimally pure variety and let $\mathcal{A}$ be a minimally pure arrangement in X. Then there is a spectral sequence of* *mixed Hodge structures*

$$E_1^{-p,q} = \bigoplus_{\substack{Z\in\mathscr{L}(\mathcal{A})\\ \mathrm{cd}(Z)=p}} H^{q-2p}(Z)\otimes_{\mathbb{Z}} \mathbb{Z}^{|\mu_{\mathscr{L}}(Z)|}(-p),$$

that converges to $H^{q-p}(X_{\mathcal{A}})$ and degenerates at the first page.

We refer to [62] for the precise definition of the differentials.

5.4 Macmeikan's Theorem

In this section we shall present a theorem due to Macmeikan [63], which will be one of our main tools in this thesis.

Let X be a variety over $\mathbb{C}$ of dimension n and let G be a finite group of automorphisms of X. Let $\mathcal{A}$ be an arrangement in X. We then have two posets naturally associated to $\mathcal{A}$, the poset $\mathscr{P}(\mathcal{A})$ of intersections of elements in $\mathcal{A}$ and the poset $\mathscr{L}(\mathcal{A})$ of irreducible components of intersections of elements in $\mathcal{A}$ where $\mathscr{P}(\mathcal{A})$ is ordered by inclusion and $\mathscr{L}(\mathcal{A})$ is ordered by reverse inclusion. For each element $g\in G$, let $\mathscr{P}^g(\mathcal{A})$ and $\mathscr{L}^g(\mathcal{A})$ denote the subposets consisting of elements which are setwise fixed by g and denote their Möbius functions by $\mu^g_{\mathscr{P}}$ and $\mu^g_{\mathscr{L}}$, respectively. We also introduce the notation $\langle g\rangle$ for the cyclic subgroup of G generated by g and $\mathrm{cd}_X(Z)$ for the codimension of a subvariety Z of X.

We are now in a position to state Macmeikan's theorem. Recall that $P_c^G(X_{\mathcal{A}},t)$ is a sum of characters of G with polynomials in t as coefficients. As such, we can evaluate $P_c^G(X_{\mathcal{A}},t)$ at an element g of G.

Theorem 5.4.1 (Macmeikan [63])**.** *Let X be a minimally pure variety over $\mathbb{C}$ and let $\mathcal{A}$ be a minimally pure arrangement in X. Then the value of the G-equivariant Poincaré polynomial of the open complement $X_{\mathcal{A}}$ at $g\in G$ is given by*

$$P_c^G(X_{\mathcal{A}},t)(g) = \sum_{Z\in\mathscr{P}^g(\mathcal{A})} \mu^g_{\mathscr{P}}(Z)(-t)^{\mathrm{cd}_X(Z)} P_c^{\langle g\rangle}(Z,t)(g).$$

Even though this is the version we want, we shall prove the analogous "combinatorial" version simply because it does not seem to exist in the literature. The proof of Theorem 5.4.1 can be found in [63] and the proof below is rather similar.

Theorem 5.4.2. *Let X be a minimally pure variety over $\mathbb{C}$ and let $\mathcal{A}$ be a minimally pure arrangement in X. Then the value of the G-equivariant Poincaré polynomial of the open complement $X_{\mathcal{A}}$ at $g\in G$ is given by*

$$P_c^G(X_{\mathcal{A}},t)(g) = \sum_{Z\in\mathscr{L}^g(\mathcal{A})} \mu^g_{\mathscr{L}}(Z)(-t)^{\mathrm{cd}_X(Z)} P_c^{\langle g\rangle}(Z,t)(g).$$

Proof. Let n be the dimension of X. For an element $Z \in \mathscr{L}(\mathcal{A})$ we define

$$Z^* = Z \setminus \bigcup W, \quad \text{where} \quad W \in \mathscr{L}(\mathcal{A}), W \subsetneq Z.$$

Note that $X^* = X_{\mathcal{A}}$. We also define sets U_i by

$$U_i = \bigcup_{\substack{Z \in \mathscr{L}(\mathcal{A}) \\ \mathrm{cd}_X(Z) \geq i}} Z^* = \bigcup_{\substack{Z \in \mathscr{L}(\mathcal{A}) \\ \mathrm{cd}_X(Z) \geq i}} Z.$$

The sets U_i are closed, since they are finite unions of closed sets, and they form a decreasing filtration

$$U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 = X.$$

We extend the above filtration by setting $U_{n+1} = \emptyset$.

Define $V_i = U_i \setminus U_{i+1}$. Note that the sets V_i are disjoint unions of varieties of the same dimension. Each of these varieties is the complement of a minimally pure arrangement in a minimally pure variety and thus itself minimally pure. Hence, the sets V_i are all minimally pure. The group G stabilizes the sets V_i and, by Lemma 5.1.2, we have

$$W_c^G(V_i, t) = W_c^G(U_i, t) - W_c^G(U_{i+1}, t).$$

We thus get a telescoping sum

$$\begin{aligned}\sum_{i=0}^{n} W_c^G(V_i, t) &= \sum_{i=0}^{n} \left(W_c^G(U_i, t) - W_c^G(U_{i+1}, t) \right) = \\ &= W_c^G(U_0, t) - W_c^G(U_{n+1}, t) = \\ &= W_c^G(X, t) - 0.\end{aligned}$$

Hence,

$$W_c^G(X, t) = \sum_{i=0}^{n} W_c^G(V_i, t).$$

We also have the following.

Claim: For any $Z \in \mathscr{L}(\mathcal{A})$, the relation

$$W_c^G(V_i, t)(g) = \sum_{\substack{Z \in \mathscr{L}^g(\mathcal{A}) \\ \mathrm{cd}_X(Z) = i}} W_c^{\langle g \rangle}(Z^*, t)(g)$$

holds.

Proof of claim: Let $Z \in \mathscr{L}(\mathcal{A})$ be of codimension i in X. Let Y be an intersection of elements in $\mathcal{A}$ such that Z is an irreducible component of Y. Consider the restriction of $\mathcal{A}$ to Y, $\mathcal{A}_Y$. Since Y is minimally pure and $\mathcal{A}_Y$ is a minimally pure arrangement in Y we have that $Y_{\mathcal{A}_Y}$ is minimally pure. But Z^* is clearly an irreducible component of $Y_{\mathcal{A}_Y}$ and thus minimally pure. In particular, Z^* is equidimensional and closed in V_i. On the other hand we have

$$V_i = \bigsqcup_{\substack{Z\in\mathscr{L}(\mathcal{A})\\ \mathrm{cd}_X(Z)=i}} Z^*,$$

so each Z^* is also open. The Mayer-Vietoris sequence thus gives that

$$H_c^k(V_i) \cong \bigoplus_{\substack{Z\in\mathscr{L}(\mathcal{A})\\ \mathrm{cd}_X(Z)=i}} H_c^k\big(Z^*\big).$$

The Mayer-Vietoris sequence is G-equivariant. An element $g \in G$ stabilizes V_i but permutes the Z^*. Clearly, only elements $Z \in \mathscr{L}(\mathcal{A})$ which are fixed by g can contribute to the evaluation at g. We thus have that

$$P_c^G(V_i,t)(g) = \sum_{\substack{Z\in\mathscr{L}(\mathcal{A})\\ \mathrm{cd}_X(Z)=i}} P_c^{\langle g\rangle}(Z^*,t)(g) = \sum_{\substack{Z\in\mathscr{L}^g(\mathcal{A})\\ \mathrm{cd}_X(Z)=i}} P_c^{\langle g\rangle}(Z^*,t)(g).$$

By Equation 5.2.1, this is equivalent to

$$W_c^G(V_i,t)(g) = \sum_{\substack{Z\in\mathscr{L}^g(\mathcal{A})\\ \mathrm{cd}_X(Z)=i}} W_c^{\langle g\rangle}(Z^*,t)(g).$$

This proves the claim.

□

By the claim we now have

$$W_c^G(X,t)(g) = \sum_{i=0}^{n} W_c^G(V_i,t)(g) = \qquad (5.4.1)$$
$$= \sum_{Z\in\mathscr{L}^g(\mathcal{A})} W_c^{\langle g\rangle}(Z^*,t)(g).$$

Define $f : \mathscr{L}^g(\mathcal{A}) \to \mathbb{C}[t]$ by

$$f(Z) = W_c^{\langle g\rangle}(Z,t)(g),$$

and $g : \mathscr{L}^g(\mathcal{A}) \to \mathbb{C}[t]$ by

$$h(Z) = W_c^{\langle g\rangle}(Z^*,t)(g).$$

By Equation 5.4.1 we have

$$f(Z) = \sum_{Y \subseteq Z} h(Y).$$

We now apply the Möbius inversion formula, see Chapter 3.7 of [82], to get

$$h(Z) = \sum_{Y \subseteq Z} \mu^g_{\mathscr{L}}(Y) f(Y).$$

Recall that $X^* = X_{\mathcal{A}}$. Deciphering all our definitions we now have

$$\begin{aligned} h(X) &= W_c^{\langle g \rangle}(X^*, t)(g) = \\ &= W_c^{\langle g \rangle}(X_{\mathcal{A}}, t)(g) = \\ &= \sum_{Z \subseteq X} \mu^g_{\mathscr{L}}(Z) f(Z) = \\ &= \sum_{Z \subseteq X} \mu^g_{\mathscr{L}}(Z) W_c^{\langle g \rangle}(Z, t)(g), \end{aligned}$$

from which we conclude that

$$W_c^{\langle g \rangle}(X_{\mathcal{A}}, t)(g) = \sum_{Z \subseteq X} \mu^g_{\mathscr{L}}(Z) W_c^{\langle g \rangle}(Z, t)(g).$$

The action of g does not depend on whether it is considered as an element in G or $\langle g \rangle$ so $W_c^{\langle g \rangle}(X_{\mathcal{A}}, t)(g) = W_c^G(X_{\mathcal{A}}, t)(g)$. The theorem now follows by an application of Equation 5.2.1. □

We now have the following corollary of Theorem 5.4.1.

Corollary 5.4.3. *Let $\mathcal{A}$ be a minimally pure arrangement in a minimally pure variety X and let G be a finite group of automorphisms of X that stabilizes $\mathcal{A}$ as a set. Suppose also that both $X_{\mathcal{A}}$ and each element of $\mathscr{P}(\mathcal{A})$ satisfy Poincaré duality. Then, for each $g \in G$*

$$P^G(X_{\mathcal{A}}, t)(g) = \sum_{Z \in \mathscr{P}^g(\mathcal{A})} \mu^g_{\mathscr{P}}(Z)(-t)^{\mathrm{cd}_X(Z)} P^{\langle g \rangle}(Z, t)(g),$$

where $P^G(Z, t)$ denotes the ordinary G-equivariant Poincaré polynomial of Z.

Proof. Poincaré duality tells us that if M is a smooth manifold of complex dimension m, then (see Chapter 13 of [64])

$$P_c^G(M, t) = t^{2m} \cdot P^G(M, t^{-1}).$$

Let the complex dimension of X be n. We apply Poincaré duality to Theorem 5.4.1 and get

$$t^{2n}P(X_{\mathcal{A}},t^{-1})(g)=\sum_{Z\in\mathscr{P}^g(\mathcal{A})}\mu^g_{\mathscr{P}}(Z)(-t)^{\mathrm{cd}_X(Z)}\cdot t^{2\dim(Z)}\cdot P(Z,t^{-1})(g).$$

We thus have that

$$\begin{aligned}P(X_{\mathcal{A}},t^{-1})(g)&=\sum_{Z\in\mathscr{P}^g(\mathcal{A})}\mu^g_{\mathscr{P}}(Z)(-t)^{\mathrm{cd}_X(Z))}\cdot t^{2\dim(Z)-2n}\cdot P(Z,t^{-1})(g)=\\&=\sum_{Z\in\mathscr{P}^g(\mathcal{A})}\mu^g_{\mathscr{P}}(Z)(-t)^{\mathrm{cd}_X(Z)}\cdot t^{-2\mathrm{cd}_X(Z)}\cdot P(Z,t^{-1})(g)=\\&=\sum_{Z\in\mathscr{P}^g(\mathcal{A})}\mu^g_{\mathscr{P}}(Z)(-t^{-1})^{\mathrm{cd}_X(Z)}\cdot P(Z,t^{-1})(g).\end{aligned}$$

We now arrive at the desired formula by substituting t^{-1} for t. □

By replacing $\mathcal{P}$ with $\mathcal{L}$ in the above proof, we get an analogous corollary of Theorem 5.4.2.

Corollary 5.4.4. *Let $\mathcal{A}$ be a minimally pure arrangement in a minimally pure variety X and let G be a finite group of automorphisms of X that stabilizes $\mathcal{A}$ as a set. Suppose also that both $X_{\mathcal{A}}$ and each element of $\mathscr{L}(\mathcal{A})$ satisfy Poincaré duality. Then, for each $g\in G$*

$$P^G(X_{\mathcal{A}},t)(g)=\sum_{Z\in\mathscr{L}^g(\mathcal{A})}\mu^g_{\mathscr{L}}(Z)(-t)^{\mathrm{cd}_X(Z)}P^{\langle g\rangle}(Z,t)(g).$$

If we apply Corollary 5.4.3 to a hyperplane arrangement $\mathcal{A}$ and observe that $P^{\langle g\rangle}(\mathbb{A}^n,t)(g)=1$ we arrive at the formula

$$P^G(X_{\mathcal{A}},t)(g)=\sum_{Z\in\mathscr{P}^g(\mathcal{A})}\mu^g_{\mathscr{P}}(Z)(-t)^{\mathrm{cd}_X(Z)},$$

known as the *Orlik-Solomon formula*, see [72]. Also Dimca and Lehrer proved a generalization of this formula, see Corollary 3.11 of [36], and Macmeikan's theorem is a further generalization of their result.

Example 5.4.5. Let $X=(\mathbb{C}^*)^2$ and let the arrangement $\mathcal{A}$ in X consist of the four subtori given by the equations

$$A_1: z_1=1,\quad A_2: z_1^2z_2=1,\quad A_3: z_1z_2^2=1,\quad A_4: z_2=1.$$

Let ξ be a primitive third root of unity. We then have

$$\begin{aligned}&A_1\cap A_2=A_1\cap A_4=A_3\cap A_4=\{(1,1)\},\\&A_1\cap A_3=\{(1,1),\,(1,-1)\},\\&A_2\cap A_3=\{(1,1),\,(\xi,\xi),\,(\xi^2,\xi^2)\},\\&A_2\cap A_4=\{(1,1),\,(-1,1)\},\end{aligned}$$

and all further intersections are equal to $\{(1,1)\}$. We thus have the poset $\mathscr{P}(\mathcal{A})$ given in Figure 5.6.

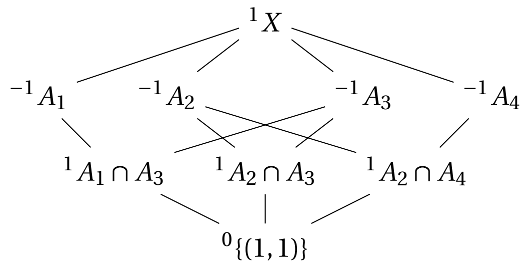


Figure 5.6: The intersection poset of the toric arrangement of Example 5.4.5. The numbers in the upper left corners are the values of the Möbius function

The Poincaré polynomial of X is $(1+t)^2$ and the Poincaré polynomial of A_i is $1+t$. By Corollary 5.4.3 we now get that the Poincaré polynomial of $X_{\mathcal{A}}$ is

$$\begin{aligned} P(X_{\mathcal{A}},t) &= (1+t)^2 + 4\cdot(-1)\cdot(-t)^1\cdot(1+t) + (-t)^2\cdot(2+3+2) = \\ &= 1+6\,t+12\,t^2. \end{aligned}$$

We invite the reader to verify that we indeed get the same result by using Corollary 5.4.4. We remark that the above arrangement is the toric arrangement associated to the root system A_2.

5.5 The total cohomology of an arrangement

Even though Theorem 5.4.1 is a useful tool, it is often hard to apply if the poset $\mathscr{P}(\mathcal{A})$ is too complicated. Sometimes it is easier to say something about the action of G on the cohomology as a whole (of course, at the expense of getting weaker results). This is the point of view in the following discussion, which is a direct generalization of that of Felder and Veselov in [44].

Let $\mathcal{A}$ be an arrangement in a variety X and let G be a finite group of automorphisms of X that fixes $\mathcal{A}$ as a set. The group G will then act on the individual cohomology groups of $X_{\mathcal{A}}$ and thus on the total cohomology

$$H^*(X_{\mathcal{A}}) = \bigoplus_{i\geq 0} H^i(X_{\mathcal{A}}).$$

The value of the *total character* at $g \in G$ is defined as

$$P(X_{\mathcal{A}})(g) = P(X_{\mathcal{A}},1)(g) = \sum_{i\geq 0} \mathrm{Tr}\Big(g, H^i(X_{\mathcal{A}})\Big),$$

and the *Lefschetz number* of $g \in G$ is defined as

$$L(X_{\mathcal{A}})(g) = P(X_{\mathcal{A}}, -1)(g) = \sum_{i \geq 0} (-1)^i \cdot \mathrm{Tr}\Big(g, H^i(X_{\mathcal{A}})\Big).$$

Let $X_{\mathcal{A}}^g$ denote the fixed point locus of $g \in G$. The Lefschetz fixed point theorem, see [18], then states that the Euler characteristic $E\big(X_{\mathcal{A}}^g\big)$ of $X_{\mathcal{A}}^g$ equals the Lefschetz number of g:

$$E\big(X_{\mathcal{A}}^g\big) = L(X_{\mathcal{A}})(g).$$

We now specialize to the case when each cohomology group $H^i(X_{\mathcal{A}})$ is pure of Tate type (i,i) and $\mathcal{A}$ is fixed under complex conjugation. This is, for instance, the case when $\mathcal{A}$ is minimally pure. We define an action of $G \times \mathbb{Z}_2$ on X by letting $(g,0) \in G \times \mathbb{Z}_2$ act as $g \in G$ and $(0,1) \in G \times \mathbb{Z}_2$ act by complex conjugation. Since $\mathcal{A}$ is fixed under conjugation, this gives an action on $X_{\mathcal{A}}$. We write $\bar{g}$ to denote the element $(g,1) \in G \times \mathbb{Z}_2$.

Remark 5.5.1. This action is somewhat different from the action described in [44]. However, it seems that this is the action actually used. The difference is rather small and only affects some minor results.

Since $H^i(X_{\mathcal{A}})$ has Tate type (i,i), complex conjugation acts as $(-1)^i$ on $H^i(X_{\mathcal{A}})$. We thus have

$$\begin{aligned} L(X_{\mathcal{A}})(\bar{g}) &= \sum_{i \geq 0} (-1)^i \cdot \mathrm{Tr}\Big(\bar{g}, H^i(X_{\mathcal{A}})\Big) = \\ &= \sum_{i \geq 0} (-1)^i \cdot (-1)^i \cdot \mathrm{Tr}\Big(g, H^i(X_{\mathcal{A}})\Big) = \\ &= P(X_{\mathcal{A}})(g). \end{aligned}$$

Since $L(X_{\mathcal{A}})(\bar{g}) = E\Big(X_{\mathcal{A}}^{\bar{g}}\Big)$ we have proved the following lemma.

Lemma 5.5.2. *Let X be a smooth variety and let $\mathcal{A}$ be an arrangement in X which is fixed by complex conjugation and such that $H^i(X_{\mathcal{A}})$ is of pure Tate type (i,i). Let G be a finite group which acts on X as automorphisms and which fixes $\mathcal{A}$ as a set. Then*

$$P(X_{\mathcal{A}})(g) = E\Big(X_{\mathcal{A}}^{\bar{g}}\Big).$$

5.6 Toric arrangements associated to root systems

Recall that if α is a nonzero vector in a real Euclidean space V, then α defines a reflection r_α through the hyperplane perpendicular to it. More explicitly,

$$r_\alpha(v) = v - 2 \cdot \frac{\alpha \cdot v}{\alpha \cdot \alpha} \cdot \alpha.$$

Since the vectors α and $-\alpha$ define the same reflection hyperplane, we have $r_\alpha = r_{-\alpha}$.

Definition 5.6.1. A finite subset Φ of a real Euclidean space V is called a *root system* if

(*i*) Φ spans V,

(*ii*) if $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$, then $\lambda = \pm 1$,

(*iii*) if $\alpha, \alpha' \in \Phi$, then $r_\alpha(\alpha') \in \Phi$,

(*iv*) for any two vectors α and α' in Φ, the number $2 \cdot \frac{\alpha \cdot \alpha'}{\alpha \cdot \alpha}$ is an integer.

The elements of Φ are called *roots*.

If Φ is a root system, then the reflections defined by elements in Φ generate a finite group W called the *Weyl group* of Φ. A set Φ^+ with the property that for each $\alpha \in \Phi$, precisely one of the vectors α and $-\alpha$ lies in Φ^+ is called a set of *positive roots*. If Φ^+ is a set of positive roots and $\beta \in \Phi^+$ is an element which cannot be written as a sum of two other positive roots in Φ^+, then β is called a *simple root* with respect to Φ^+. For every root system, there are several choices of sets of positive roots and for each set of positive roots there is precisely one basis of V consisting of simple roots with respect to Φ^+.

A root system Φ is called *reducible* if it can be partitioned into a disjoint union $\Phi = \Phi_1 \sqcup \Phi_2$ such that $\alpha_1 \cdot \alpha_2 = 0$ if $\alpha_1 \in \Phi_1$ and $\alpha_2 \in \Phi_2$. A root system which is not reducible is called *irreducible*. There are several types of irreducible root systems. Firstly, there are the four infinite *classical series* A_n, B_n, C_n and D_n. Secondly, there are the five *exceptional root systems* G_2, F_4, E_6, E_7 and E_8.

Let Φ be a root system, let Φ^+ be a set of positive roots of Φ and let $\Delta = \{\beta_1, \ldots, \beta_n\}$ be the set of simple roots with respect to Φ^+. Let M be the $\mathbb{Z}$-linear span of Φ. Thus, M is a free $\mathbb{Z}$-module of finite rank n.

Define $T = \mathrm{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. The Weyl group W acts on T from the right by precomposition, i.e.

$$(\chi.g)(v) = \chi(g.v), \quad \chi \in T, g \in W.$$

For each $g \in W$ we define

$$T^g = \{\chi \in T | \chi.g = \chi\},$$

and for each $\alpha \in \Phi$ we define

$$T_\alpha = \{\chi \in T | \chi(\alpha) = 1\}.$$

We thus obtain two arrangements of hypertori in T

$$\mathcal{T}^{r_\Phi} = \left\{T^{r_\alpha}\right\}_{\alpha\in\Phi}, \quad \text{and} \quad \mathcal{T}_\Phi = \{T_\alpha\}_{\alpha\in\Phi}.$$

Observe that in the definition of $\mathcal{T}^{r_\Phi}$, we only use reflections and not general group elements. To avoid cluttered notation we shall write T^{r_Φ} instead of the more cumbersome $T_{\mathcal{T}_{r^\Phi}}$. Similarly, we write T_Φ to mean $T_{\mathcal{T}_\Phi}$.

Lemma 5.6.2. *Let α be an element of Φ. The two subtori T^{r_α} and T_α of T coincide if and only if the expression*

$$2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha}$$

takes the value 1 *for some $v \in M$.*

Proof. By definition we have

$$r_\alpha(v) = v - 2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha}\alpha.$$

Hence

$$\chi(r_\alpha(v)) = \frac{\chi(v)}{\chi(\alpha)^{2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha}}},$$

and we thus see that $\chi(r_\alpha(v)) = \chi(v)$ for all $v \in M$ if and only if

$$\chi(\alpha)^{2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha}} = 1,$$

for all $v \in M$. Hence, $T_\alpha \subset T^{r_\alpha}$ always holds. Also, if v is such that $2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha} = 1$ then $\chi(\alpha)$ must be 1 and it then follows that $T^{r_\alpha} = T_\alpha$.

On the other hand, if $2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha} \neq 1$ for all $v \in M$, then $2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha} \in n\mathbb{Z}$ for some integer $n > 1$. To see this, assume the contrary, namely that $2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha} \neq 1$ for all v but there is no $n > 1$ which divides $2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha}$ for all v. Then there are elements v_1 and v_2 of M such that

$$n_1 = 2\cdot\frac{\alpha\cdot v_1}{\alpha\cdot\alpha}, \quad \text{and} \quad n_2 = 2\cdot\frac{\alpha\cdot v_2}{\alpha\cdot\alpha},$$

are coprime. Let a and b be integers such that $an_1 + bn_2 = 1$. Then $av_1 + bv_2 \in M$ and

$$2\cdot\frac{\alpha\cdot(av_1 + bv_2)}{\alpha\cdot\alpha} = 1.$$

Hence, $2\cdot\frac{\alpha\cdot v}{\alpha\cdot\alpha} \in n\mathbb{Z}$ for some integer $n > 1$. Thus, the character χ which takes the value ζ, a primitive n'th root of unity, on α is an element of T_{r_α} but clearly not an element of T_α. □

Example 5.6.3. The root system A_n can be defined as the set of vectors in $\mathbb{R}^{n+1}$ of the form $e_i - e_j$, $i \neq j$, where e_i is the ith coordinate vector. Since $(e_i - e_j)\cdot(e_i - e_j) = 2$ and $(e_i - e_j)\cdot(e_i - e_k) = 1$ if $j \neq k$, we see that

$$2 \cdot \frac{(e_i - e_j)\cdot(e_i - e_k)}{(e_i - e_j)\cdot(e_i - e_j)} = 1.$$

Thus, if $n > 1$ then every root in A_n fulfills Lemma 5.6.2.

We will study the root systems A_n in more detail in Section 5.10. In particular, we will compute cohomology of the varieties T_{A_n}.

Example 5.6.4. The root system B_n can be defined as the set of vectors in $\mathbb{R}^n$ of the form

$$\begin{array}{ll} \pm e_i, & i = 1, \ldots, n, \\ e_i - e_j, & i \neq j, \\ \pm(e_i + e_j), & i \neq j, \end{array}$$

where e_i is the ith coordinate vector. Since $e_i \cdot e_i = 1$ we have

$$2\frac{e_i \cdot v}{e_i \cdot e_i} = 2(e_i \cdot v) \in 2\mathbb{Z},$$

for all $v \in M$. Thus, by Lemma 5.6.2 we have that $T^{r_{e_i}} \neq T_{e_i}$.

Let $\chi \in T$. We introduce the notation $\chi(\beta_i) = z_i$ for the simple roots β_i, $i = 1, \ldots, n$. The coordinate ring of T is then

$$\mathbb{C}[T] = \mathbb{C}[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}].$$

If α is a root, there are integers $m_1, \ldots, m_n$ such that

$$\alpha = m_1 \cdot \beta_1 + \cdots + m_n \cdot \beta_n.$$

With this notation we have that $\chi(\alpha) = 1$ if and only if

$$z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} = 1.$$

We denote the Laurent polynomial $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n} - 1$ by f_α. Thus, χ is an element of T_α if and only if $f_\alpha(\chi) = 0$. If we differentiate f_α with respect to z_i we get

$$\frac{\partial f_\alpha}{\partial z_i} = m_i \cdot z_1^{m_1} \cdots z_i^{m_i - 1} \cdots z_n^{m_n},$$

which clearly is nonzero everywhere. Thus, each T_α is smooth.

Describing the cohomology of T_Φ as a W-representation is a nontrivial task. However, in low cohomological degrees we can say something in general. To begin, $H^0(T_\Phi)$ is of course always the trivial representation. We can also describe $H^1(T_\Phi)$ but to do so we need some notation.

Definition 5.6.5. Let Φ be a root system of rank n, realized in a vector space V of dimension n.

(i) The representation given by the action of W on $V = M \otimes_{\mathbb{Z}} \mathbb{C}$ is called the *standard representation* and is denoted χ_{std}.

(ii) The group W permutes the lines generated by elements of Φ. These lines are in bijective correspondence with the positive roots Φ^+. We call the resulting permutation representation the *positive representation* and denote it by χ_{pos}.

Remark 5.6.6. The positive representation is positive in two senses. Firstly, it is a permutation representation so it only takes non-negative values. Secondly, it is defined in terms of positive roots.

Lemma 5.6.7. *Let Φ be a root system. Then*

$$H^1(T_\Phi) = \chi_{\text{std}} + \chi_{\text{pos}}.$$

Proof. Define

$$P = \bigoplus_{\alpha \in \Phi} H^0(T_\alpha)$$

By Theorem 5.3.9 we have

$$H^1(T_\Phi) = \bigoplus_{q-p=1} E_1^{-p,q},$$

where

$$E_1^{0,1} = H^1(T), \quad E_1^{-1,2} = \bigoplus_{\substack{Z \in \mathscr{L}(\mathcal{A}) \\ \text{cd}(Z)=1}} H^0(Z)(-1),$$

and $E_1^{-p,q} = 0$ for $p > 1$. Since $T = \text{Hom}(M, \mathbb{C}^*)$ it follows that $H^1(T, \mathbb{Z}) = M = \chi_{\text{std}}$. An element in $\mathscr{L}(\mathcal{A})$ of codimension 1 corresponds to a root $\alpha \in \Phi$. But α and $-\alpha$ define the same torus T_α so

$$E_1^{-1,2} = \bigoplus_{\alpha \in \Phi^+} H^0(T_\alpha)(-1),$$

which equals χ_{pos} as a representation. □

5.7 Binomial ideals

We have seen that the condition $\chi(\alpha) = 1$ can be translated into an equation of the form

$$x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} = 1.$$

Further conditions $\chi(\alpha') = 1$ lead to further equations of the above form. Thus, the varieties $Z \in \mathscr{P}(\mathcal{T}_\Phi)$ will all be defined by ideals generated by binomials. Such ideals are called *binomial ideals* or, if we want to emphasize that we allow negative exponents, *Laurent binomial ideals*.

We identify M with $\mathbb{Z}^n$ via the choice of simple roots and we define

$$\mathbb{C}[M] = \mathbb{C}[x^\pm] = \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}].$$

A binomial in $\mathbb{C}[x^\pm]$ is an element with at most two terms, $ax^{m_\alpha} + bx^{m_\beta}$. A Laurent binomial ideal is an ideal generated by binomials. By multiplying each binomial $ax^{m_\alpha} + bx^{m_\beta}$ by $-a^{-1}x^{-m_\beta}$ we may assume that a Laurent binomial ideal is generated by elements of the form $x^m - c_m$ where $m \in \mathbb{Z}^n$ and $c_m \in \mathbb{C}^*$.

We regard $\mathbb{C}[x^\pm]$ as the coordinate ring of $T = \mathrm{Hom}(M, \mathbb{C}^*)$. A *partial character* on $\mathbb{Z}^n$ is a homomorphism

$$\rho : L_\rho \to \mathbb{C}^*,$$

where L_ρ is a sublattice of $\mathbb{Z}^n$. A partial character defines a Laurent binomial ideal

$$I(\rho) = (x^m - \rho(m) : m \in L_\rho).$$

We now state a few results regarding Laurent binomial ideals due to Eisenbud and Sturmfels, [43].

Lemma 5.7.1 (Eisenbud and Sturmfels [43])**.** *Let $I \subset \mathbb{C}[x^\pm]$ be a proper Laurent binomial ideal. Then there is a unique partial character ρ on $\mathbb{Z}^n$ such that $I = I(\rho)$.*

Proof. Let S be the set

$$\{m \in \mathbb{Z}^n | \text{there exists a binomial of the form } x^m - c \in I,\ c \in \mathbb{C}^*\}$$

Suppose that both $x^m - c$ and $x^m - c'$ lie in I. Then $x^m - c - (x^m - c') = c' - c$ lies in I. Since I is proper we must have that $c = c'$ so the constant is determined by m.

Note that

$$x^{m+m'} - cd = (x^m - c)x^{m'} + c(x^{m'} - d).$$

From this we conclude that if $x^m - c_m$ and $x^{m'} - c_{m'}$ lie in I, then so does $x^{m+m'} - c_m c_{m'}$. Also, if $x^{m+m'} - c_m c_{m'}$ and $x^m - c_m$ lie in I, then so does $x^{m'} - c_{m'}$. Thus, S is a sublattice of $\mathbb{Z}^n$ and the map $\rho : S \to \mathbb{C}^*$ taking m to c_m is a character.

For the uniqueness part, see [43]. $\square$

For a sublattice $L \subseteq \mathbb{Z}^n$, define the *saturation*

$$\mathrm{Sat}(L) = \mathbb{Q}L \cap \mathbb{Z}^n.$$

If $L = \mathrm{Sat}(L)$ we say that L is *saturated.*

Lemma 5.7.2 (Eisenbud and Sturmfels [43])**.** *Let ρ be a partial character and let r be the order of $\mathrm{Sat}(L_\rho)/L_\rho$. There are r distinct characters $\rho_1,\dots,\rho_r$ on $\mathrm{Sat}(L_\rho)$ extending ρ.*

(a) The ideal $I(\rho)$ is radical.

(b) The associated primes of $I(\rho)$ are $I(\rho_j)$, $j = 1,\dots,r$.

(c) The minimal primary decomposition of $I(\rho)$ is

$$I(\rho) = \bigcap_{j=1}^{r} I(\rho_j).$$

Lemma 5.7.3 (Eisenbud and Sturmfels [43])**.** *A Laurent binomial ideal $I(\rho)$ is prime if and only if L_ρ is a direct summand of $\mathbb{Z}^n$.*

Lemma 5.7.4 (Eisenbud and Sturmfels [43])**.** *The algebraic set $V(I(\rho)) \subset T$ is the set of characters that are extensions of ρ. If it is non-empty, it is the translate of the subgroup*

$$\mathrm{Hom}\left(\mathbb{Z}^n/L_\rho, \mathbb{C}^*\right) \subset T,$$

by any extension of ρ.

5.8 Equivariant cohomology of intersections of hypertori

Let f be a Laurent polynomial and let $V(f)$ denote the subvariety of T defined by f. A variety $Z \in \mathscr{P}(\mathcal{T}_\Phi)$ is an intersection

$$Z = \bigcap_{\alpha\in S} T_\alpha = \bigcap_{\alpha\in S} V\left(f_\alpha\right),$$

where S is a subset of Φ. We define the ideal

$$I_S = \left(f_\alpha\right)_{\alpha\in S} \subseteq \mathbb{C}[T].$$

Then $Z = V(I_S)$. The ideal I_S is entirely determined by the exponents occurring in the various Laurent polynomials f_α generating it or, in other words, the coefficients occurring in the elements of S when expressed in terms of the simple roots Δ. Thus, if we define the *module of exponents*

$$N_S = \mathbb{Z}\langle S\rangle \subseteq M,$$

then the module N_S determines I_S and

$$V(I_S) = \mathrm{Hom}(M/N_S, \mathbb{C}^*) \subseteq \mathrm{Hom}(M, \mathbb{C}^*) = T.$$

We now recall some facts about the cohomology of tori. For these statements, see Chapter 9 of [24]. Let L be a free $\mathbb{Z}$-module. The torus $T_L = \mathrm{Hom}(L, \mathbb{C}^*)$ has cohomology given by

$$H^i(T_L) = \bigwedge^i H^1(T_L) = \bigwedge^i L.$$

Suppose L' is another free $\mathbb{Z}$-module and let $T_{L'} = \mathrm{Hom}\big(L', \mathbb{C}^*\big)$. A morphism $L \to L'$ of free $\mathbb{Z}$-modules gives rise to a morphism $T_{L'} \to T_L$ and the induced map $H^i(T_L) \to H^i(T_{L'})$ is the map

$$\bigwedge^i L \to \bigwedge^i L'.$$

The modules M/N_S will not always be free but do still determine the cohomology of $V(I_S)$ in a sense very similar to the above. Let L be a free $\mathbb{Z}$-module, $N \subset L$ a submodule and let $Q = L/N$. The module Q will split as a direct sum $Q = Q^T \oplus Q^F$, where Q^T is the torsion part and Q^F is the free part of Q. The variety $T_Q = \mathrm{Hom}(Q, \mathbb{C}^*)$ consists of $\left|Q^T\right|$ connected components, each isomorphic to $\mathrm{Hom}\big(Q^F, \mathbb{C}^*\big)$. The ith cohomology group of T_Q is given by

$$H^i\big(T_Q\big) = \bigoplus_{v \in Q^T} \bigwedge^i Q^F.$$

Let $\varphi : L \to L'$ be a homomorphism and define $Q' = L'/\varphi(N)$. The morphism φ induces a morphism $Q \to Q'$ which in turn gives rise to a morphism $T_{Q'} \to T_Q$ and the induced map $H^i\big(T_Q\big) \to H^i\big(T_{Q'}\big)$ is the map

$$\bigoplus_{v \in Q^T} \bigwedge^i Q^F \to \bigoplus_{v' \in Q'^T} \bigwedge^i Q'^F.$$

The situation is perhaps clarified by the following. Consider the module $Q_S = M/N_S$. The module Q_S is determined by the echelon basis matrix of N_S. Torsion elements of Q_S stem from rows in the echelon basis matrix whose entries have greatest common divisor greater than 1. The module Q_S^F is the module $M/\mathrm{Sat}(N_S)$, where $\mathrm{Sat}(N_S) = N_S \otimes_{\mathbb{Z}} \mathbb{Q} \cap M$ is the saturation of

N_S, i.e. the module obtained from N_S by dividing each row in the echelon basis matrix of N_S by the greatest common divisor of its entries.

A row $(m_1,\dots,m_n)$ in the echelon basis matrix of N_S corresponds to the equation

$$z_1^{m_1}\cdots z_n^{m_n} = 1.$$

If $\gcd(m_1,\dots,m_n) = d$ we may write $m_i = d\cdot m_i'$ and

$$(z_1^{m_1'}\cdots z_n^{m_n'})^d = 1,$$

i.e. an equation for d non-intersecting hypertori, namely the hypertorus given by

$$z_1^{m_1'}\cdots z_n^{m_n'} = 1,$$

translated by multiplication by powers of a primitive dth root of unity.

A linear map $g: M\to M$ which fixes N_S can be analyzed in two steps. Firstly, we can investigate how it "permutes different roots of unity", more precisely, how it acts on Q_S^T. The elements of Q_S^T correspond to connected components of $V(I_S)$ and a component is fixed by g if and only if the corresponding element of Q_S^T is fixed. Of course, only fixed components can contribute to the trace of g on $H^i(V(I_S))$. Once we have determined which components are fixed it suffices to compute the trace of g on the cohomology on one of those components, e.g. the component corresponding to the zero element of Q_S^T.

We may now write down an algorithm for computing the equivariant Poincaré polynomial of an element $Z\in\mathscr{P}(\mathcal{T}_\phi)$.

Algorithm 5.8.1. *Let Φ be a root system of rank n with Weyl group W and let $Z\in\mathscr{P}(\mathcal{T}_\phi)$ correspond to the subset S of Φ, i.e.*

$$Z = \bigcap_{\alpha\in S} T_\alpha.$$

Let g be an element of W stabilizing Z. Then $P(Z,t)(g)$ can be computed via the following steps.

(1) Compute the number m of elements in Q_S^T which are fixed by g (for instance by lifting each element of Q_S^T to M, acting on the lifted element by g and pushing the result down to Q_S^T).

(2) Compute $\operatorname{Tr}(g,Q_S^F)$ (for instance as $\operatorname{Tr}(g,M)-\operatorname{Tr}(g,N_S)$).

(3) Using the knowledge of $\operatorname{Tr}(g,Q_S^F)$, compute $\operatorname{Tr}(g,\wedge^i Q_S^F)$ for $i=1,\dots,n$ (for instance via the Newton-Girard method).

The polynomial $P(Z,t)(g)$ is now given by

$$P(Z,t)(g) = m \cdot \sum_{i=0}^{n} \mathrm{Tr}(g, \wedge^i Q_S^F)\, t^i.$$

5.9 Posets of toric arrangements associated to root systems

Section 5.8 tells us how to compute $\mathrm{Tr}(g, H^i(Z))$ for any $Z \in \mathscr{P}(\mathcal{T}_\phi)$ and thus, via Poincaré duality, how to compute $\mathrm{Tr}(g, H_c^i(Z))$. However, in order to use Corollary 5.4.3 to compute $P(T_\Phi, t)(g)$ we also need to compute the poset $\mathscr{P}^g(\mathcal{T}_\phi)$. The following discussion takes its inspiration from Fleischmann and Janiszczak [45].

We have seen that each element $Z = \cap_{\alpha \in S} T_\alpha$ is given by its module of exponents $N_S = \mathbb{Z}\langle S \rangle \subset M$. We may therefore equally well investigate the modules N_S. However, an inclusion $N_S \subset N_{S'}$ gives a surjection $M/N_S \twoheadrightarrow M/N_{S'}$ and thus an inclusion $\mathrm{Hom}(M/N_{S'}, \mathbb{C}^*) \hookrightarrow \mathrm{Hom}(M/N_S, \mathbb{C}^*)$. Hence, the poset structure should be given by reverse inclusion.

Definition 5.9.1. Let Φ be a root system. The poset of modules of exponents is the set

$$\mathscr{N}(\Phi) = \{\mathbb{Z}\langle S \rangle | S \subseteq \Phi\},$$

ordered by reverse inclusion. If g is an element of the Weyl group of Φ, we write $\mathscr{N}^g(\Phi)$ to denote the subposet of $\mathscr{N}(\Phi)$ of modules fixed by g.

By construction we have that the poset $\mathscr{N}^g(\Phi)$ is isomorphic to the poset $\mathscr{P}^g(\mathcal{T}_\Phi)$. The benefit of considering the posets $\mathscr{N}^g(\Phi)$ instead is that they are more easily computed.

Let g be an element of the Weyl group of Φ. If N is an element of $\mathscr{N}^g(\Phi)$, then $N \cap \Phi$ is a union of g-orbits of Φ. Since $N = \mathbb{Z}\langle N \cap \Phi \rangle$, we may compute $\mathscr{N}^g(\Phi)$ via the following steps.

Algorithm 5.9.2. *Let Φ be a root system and let W be its Weyl group. Let g be an element of g. Then the following algorithm computes the poset $\mathscr{N}^g(\Phi)$:*

(1) Compute the g-orbits of Φ.

(2) Compute the set $\mathscr{N}^g(\Phi)_{\mathrm{set}}$ of all (distinct) $\mathbb{Z}$-spans of unions of g-orbits.

(3) Investigate the inclusion relations of the elements of $\mathscr{N}^g(\Phi)_{\mathrm{set}}$.

Using Algorithms 5.8.1 and 5.9.2 as well as Corollary 5.4.3 we have constructed a `SageMath` [35] program computing the equivariant Poincaré polynomials of the complement of a toric arrangement associated to a root system Φ. The program is presented in Appendix A. This program has been used to compute these polynomials for many root systems of small rank. In Section 5.11 we give the results for all exceptional root systems except for E_8. At the moment the E_8-case seems beyond reach for an ordinary computer but we hope to complete the computation also for the E_8 case, using high performance computing.

In practice, we represent elements of $\mathcal{N}^g(\Phi)_{\text{set}}$ by their echelon basis matrices (which need to be computed with some care since we are working over the integers), so step (2) consists of performing Gaussian elimination on matrices and making sure we only save each matrix once. Step (3) is by far the most computationally demanding since, in principle, one needs to make $|\mathcal{N}^g(\Phi)|^2$ comparisons. Fortunately, it is very parallelizable.

In order to use Corollary 5.4.3, we need to compute certain values of the Möbius function of $\mathcal{N}^g(\Phi)$. At least for computational purposes, the inclusion data of $\mathcal{N}^g(\Phi)$ is most conveniently encoded as a matrix A of zeros and ones whose columns and rows are indexed by the elements of $\mathcal{N}^g(\Phi)$ and where the element at place (i, j) is one if and only if the ith element of $\mathcal{N}^g(\Phi)$ is less than the jth element of $\mathcal{N}^g(\Phi)$. In this setting, the computation of the Möbius function of $\mathcal{N}^g(\Phi)$ amounts to the inversion of A and this is indeed what most computer algebra systems do. However, inverting a matrix using Gauss-Jordan elimination requires up to $|\mathcal{N}^g(\Phi)|^3$ operations so if we were to compute the entire Möbius function, this would be the most time consuming step (there are better ways to invert matrices but they all have complexity worse than $O(n^2)$ where n is the dimension of the matrix). Fortunately, we only need one row of the inverted matrix and this computation only requires at worst $|\mathcal{N}^g(\Phi)|^2$ operations.

Remark 5.9.3. Both the computation of the inclusion data for $\mathcal{N}^g(\Phi)$ and the computations for the Möbius function require about $|\mathcal{N}^g(\Phi)|^2$ operations. However, in the former case an operation typically consists of solving n linear equations, where n is the rank of Φ whereas the operations in the latter case are additions, multiplications and subtractions of integers. Therefore, the computation of the inclusion data for $\mathcal{N}^g(\Phi)$ is the most time consuming step.

5.9.1. Hyperplane arrangements associated to root systems Given a root system Φ we can also define a corresponding hyperplane arrangement. Let

$V = \mathrm{Hom}(M,\mathbb{C}) \cong \mathbb{C}^n$ and for each $\alpha \in \Phi$ define

$$V_\alpha = \{\phi \in V | \phi(\alpha) = 0\}.$$

We thus obtain a hyperplane arrangement in V

$$\mathcal{V}_\Phi = \{V_\alpha\}_{\alpha\in\Phi}.$$

As usual, we denote the corresponding poset by $\mathscr{P}(\mathcal{V}_\Phi)$.

We can analyze $\mathscr{P}(\mathcal{V}_\Phi)$ in a way analogous to how we analyzed $\mathscr{P}(\mathcal{T}_\Phi)$. For instance, if we take $\mathbb{C}$-spans instead of taking $\mathbb{Z}$-spans in step (2) of Algorithm 5.9.2 we obtain a smaller poset $\mathscr{H}^g(\Phi)$ which is isomorphic to $\mathscr{P}(\mathcal{V}_\Phi)$. We have an order preserving surjection

$$\tau : \mathscr{N}^g(\Phi) \to \mathscr{H}^g(\Phi), \tag{5.9.1}$$

sending a module N to $N \otimes_{\mathbb{Z}} \mathbb{C}$. Note that τ sends a module of rank r to a vector space of dimension r. In terms of echelon basis matrices, τ sends a echelon basis matrix to its saturation. Thus, the map τ is an isomorphism of posets if and only if each module of $\mathscr{N}^g(\Phi)$ is saturated.

In past few sections we have discussed arrangements of tori from relatively algebraic perspective but there is also a parallel, more enumerative story. For instance, the work of Ehrenborg, Readdy and Slone [42] and that of Lawrence [58] has a flavour quite similar to what we have presented in the previous two sections, but in a different context. We also mention that although we have worked exclusively with cohomology with field coefficients, there has been great progress also for cohomology with coefficients in $\mathbb{Z}$. In particular we would like to point out the works [28] and [29] of d'Antonio and Delucchi showing that the integer cohomology groups of complements of toric arrangements are torsion free. Finally, we point the work by Ardila, Castillo and Henley [4] computing so-called arithmetic Tutte polynomials of toric arrangements associated to classical root systems in a slightly different setup. Using the work of Moci [69], such Tutte polynomials can be translated into Poincaré polynomials.

5.10 The toric arrangement associated to A_n

The simplest irreducible root systems are the root systems of type A_n. The root system A_n is most naturally viewed as a subset of an n-dimensional subspace of $\mathbb{R}^{n+1}$. Denote the ith coordinate vector of $\mathbb{R}^{n+1}$ by e_i. The roots Φ can then be chosen to be

$$\alpha_{i,j} = e_i - e_j, \quad i \neq j.$$

A choice of positive roots is

$$\alpha_{i,j} = e_i - e_j, \quad i < j,$$

and the simple roots with respect to this choice of positive roots are

$$\beta_i = e_i - e_{i+1}, \quad i = 1, \ldots, n.$$

The Weyl group W of A_n is isomorphic to the symmetric group S_{n+1} and an element of S_{n+1} acts on an element in $M = \mathbb{Z}\langle\Phi\rangle$ by permuting the indices of the coordinate vectors in $\mathbb{R}^{n+1}$.

5.10.1. The total character Let W be the Weyl group of A_n. In this section we shall compute the total character evaluated at any element $g \in W$. This will determine the total cohomology $H^*(T_\Phi)$ as an W-representation. Although we have not pursued this, similar methods should allow the computation of $H^*(T_\Phi)$ also in the case of root systems of type B_n, C_n and D_n.

Lemma 5.10.1. *Let W be the Weyl group of A_n and suppose that $g \in W$ has a cycle of length greater than two. Then $T_\Phi^{\bar{g}}$ is empty.*

Proof. The statement only depends on the conjugacy class of g so suppose that g contains the cycle $(12\ldots s)$, where $s \geq 3$. We then have

$$\begin{aligned}
g.\beta_1 &= e_2 - e_3 = \beta_2,\\
g.\beta_2 &= e_3 - e_4 = \beta_3,\\
&\vdots\\
g.\beta_{s-2} &= e_{s-1} - e_s = \beta_{s-1},\\
g.\beta_{s-1} &= e_s - e_1 = -(\beta_1 + \ldots + \beta_{s-1}).
\end{aligned}$$

If $\overline{g}.\chi = \chi$ we must have

$$z_i = \overline{z}_{i+1} \quad \text{for} \quad i = 1, \ldots, s-2, \tag{1}$$
$$z_{s-1} = \overline{z}_1^{-1}\overline{z}_2^{-1}\cdots\overline{z}_{s-1}^{-1}. \tag{2}$$

We insert (1) into (2) and take absolute values to obtain $|z_1|^s = 1$. We thus see that $|z_1| = 1$. Since we have $z_2 = \overline{z}_1$ it follows that

$$\chi(\alpha_{1,3}) = \chi(\beta_1 + \beta_2) = z_1 \cdot z_2 = z_1 \cdot \overline{z}_1 = |z_1|^2 = 1.$$

Thus, χ lies in $T_{\alpha_{13}}$ so $T_\Phi^{\overline{g}}$ is empty. $\square$

Corollary 5.10.2. *Let Φ be a root system of rank n containing A_n as a subroot system so that the Weyl group of Φ contains the Weyl group of A_n as a subgroup. If g is an element of the Weyl group of A_n of order greater than 2, then $T_\Phi^{\bar{g}}$ is empty.*

Proof. Since A_n is a subroot system of Φ we have $\mathcal{T}_{A_n} \subseteq \mathcal{T}_\Phi$. Hence, $T_\Phi \subseteq T_{A_n}$ so the Corollary follows since $T_{A_n}^{\bar{g}} = \emptyset$. □

We point out that Corollary 5.10.2 together with Lemma 5.5.2 provides a test for the program from Section 5.9 for quite a wide range of root systems. In particular, if we apply Lemma 5.5.2 to Lemma 5.10.1 we obtain the following corollary.

Corollary 5.10.3. *If g is an element of the Weyl group of A_n such that $g^2 \neq \mathrm{id}$, then*

$$P(T_\Phi)(g) = 0.$$

We thus know the total character of all elements in the Weyl group of A_n of order greater than 2. We shall therefore turn our attention to the involutions.

Lemma 5.10.4. *If g is an element of the Weyl group of A_n of order 2 which is not a reflection, then*

$$E(T_\Phi^{\bar{g}}) = P(T_\Phi)(g) = 0.$$

Proof. Let $k > 1$ and consider the element $g = (12)(34)\cdots(2k-1\,2k)$. We define a new basis for M:

$$\begin{aligned} \gamma_i &= \beta_i + \beta_{i+1}, & i &= 1,\dots,2k-2, \\ \gamma_j &= \beta_j & j &= 2k-1,\dots,n \end{aligned}$$

Then

$$g.\gamma_{2i-1} = \gamma_{2i}, \quad \text{and} \quad g.\gamma_{2i} = \gamma_{2i-1},$$

for $i = 1,\dots k-1$,

$$g.\gamma_{2k-1} = -\gamma_{2k-1}, \quad g.\gamma_{2k} = \gamma_{2k-1} + \gamma_{2k},$$

and $g.\gamma_i = \gamma_i$ for $i > 2k$. If we put $\chi(\gamma_i) = t_i$, then $T_\Phi^{\bar{g}} \subseteq T_\Phi$ is given by the equations

$$\begin{array}{lcll} t_{2i-1} & = & \bar{t}_{2i} & i = 1,\dots,k-1, \\ t_{2k-1} & = & \bar{t}_{2k-1}^{-1}, & \\ t_{2k} & = & \bar{t}_{2k-1} \cdot \bar{t}_{2k}, & \\ t_i & = & \bar{t}_i, & i = 2k+1,\dots,n. \end{array}$$

Thus, the points of $T_\Phi^{\bar{g}}$ have the form

$$(t_1, \bar{t}_1, t_2, \bar{t}_2, \ldots, t_{k-1}, \bar{t}_{k-1}, s, s^{-1/2} \cdot r, t_{2k}, \ldots, t_n),$$

where $s \in S^1 \setminus \{1\} \subset \mathbb{C}$, $r \in \mathbb{R}^+$ and $t_i \in \mathbb{R}$ for $i = 2k, \ldots, n$.

We can now see that each connected component of $T_\Phi^{\bar{g}}$ is homeomorphic to $\mathcal{C}_{k-1}(\mathbb{C} \setminus \{0,1\}) \times (0,1) \times \mathbb{R}^{n-2k+1}$, where $\mathcal{C}_{k-1}(\mathbb{C} \setminus \{0,1\})$ is the configuration space of $k-1$ points in the twice punctured complex plane. This space is in turn homotopic to the configuration space $\mathcal{C}_{k+1}(\mathbb{C})$ of $k+1$ points in the complex plane. The space $\mathcal{C}_{k+1}(\mathbb{C})$ is known to have Euler characteristic zero for $k \geq 1$. □

Remark 5.10.5. Note that it is essential that we use ordinary cohomology in the above proof, since compactly supported cohomology is not homotopy invariant. However, since T_Φ satisfies Poincaré duality, the corresponding result follows also in the compactly supported case.

We now turn to the reflections.

Lemma 5.10.6. *If g is a reflection in the Weyl group of A_n, then*

$$E(T_\Phi^{\bar{g}}) = P(T_\Phi)(g) = n!.$$

Proof. Let $g = (12)$. We then have

$$\begin{aligned} g.\beta_1 &= -\beta_1, \\ g.\beta_2 &= \beta_1 + \beta_2, \\ g.\beta_i &= \beta_i, && i > 2. \end{aligned}$$

This gives the equations

$$\begin{aligned} z_1 &= \overline{z}_1^{-1}, \\ z_2 &= \overline{z}_1 \cdot \overline{z}_2, \\ z_i &= \overline{z}_i, && i > 2. \end{aligned}$$

Thus $z_1 \in S^1 \setminus \{1\}$, z_2 is not real and satisfies $z_2 = \overline{z}_1 \cdot \overline{z}_2$ so we choose z_2 from a space isomorphic to $\mathbb{R}^*$. Hence, $T_\Phi^{\bar{g}} \cong [0,1] \times \mathbb{R}^* \times M$ where M is the space where the last $n-2$ coordinates $z_3, \ldots, z_n$ take their values.

These coordinates satisfy $z_i = \overline{z}_i$, i.e. they are real. We begin by choosing z_3. Since $\chi(e_i - e_j) \neq 0,1$ we need $z_3 \neq 0,1$. We thus choose z_3 from $\mathbb{R} \setminus \{0,1\}$. We then choose z_4 in $\mathbb{R} \setminus \{0, 1, \frac{1}{z_3}\}$, z_5 in $\mathbb{R} \setminus \{0, 1, \frac{1}{z_4}, \frac{1}{z_3 \cdot z_4}\}$ and so on. In the ith step we have i components to choose from. Thus, M consists of

$$3 \cdot 4 \cdots n = \frac{n!}{2},$$

components, each isomorphic to $\mathbb{R}^{n-2}$. Hence, $E(M) = \frac{n!}{2}$ and it follows that

$$E(T_\Phi^{\bar{g}}) = E([0,1]) \cdot E(\mathbb{R}^*) \cdot E(M) = n!.$$

□

It remains to compute the value of the total character at the identity element.

Lemma 5.10.7. $E(T_\Phi^{\bar{\mathrm{id}}}) = P(T_\Phi)(\mathrm{id}) = \frac{(n+2)!}{2}$.

Proof. The proof is a calculation similar to that in the proof of Lemma 5.10.6. We note that the equations for $T_\Phi^{\bar{\mathrm{id}}}$ are $z_i = \bar{z}_i$, $i = 1, \ldots, n$ so the computation of $E(T_\Phi^{\bar{\mathrm{id}}})$ is essentially the same as that for $E(M)$ above. The difference is that we have n steps and in the i'th step we have $i+1$ choices. This gives the result. □

Let W denote the Weyl group of A_n. Lemmas 5.10.1, 5.10.4, 5.10.6 and 5.10.7 together determine the character of W on $H^*(T_\Phi)$. The corresponding calculation for the affine hyperplane case was first carried out by Lehrer in [59] and later by Felder and Veselov in [44]. In the hyperplane case, the total cohomology turned out to be $2\,\mathrm{Ind}_{\langle s\rangle}^W(\mathrm{Triv}_{\langle s\rangle})$, where s is a transposition, i.e. the cohomology is twice the representation induced up from the trivial representation of the subgroup generated by a transposition. It turns out that the representation $\mathrm{Ind}_{\langle s\rangle}^W(\mathrm{Triv}_{\langle s\rangle})$ accounts for most of the cohomology also in the toric case.

The representation $\mathrm{Ind}_{\langle s\rangle}^W(\mathrm{Triv}_{\langle s\rangle})$ takes value $(n+1)!$ on the identity, $2(n-1)!$ on transpositions and is zero elsewhere. Since the character of $H^*(T_\Phi)$ takes the value $(n+2)!/2$ on the identity, $n!$ on transpositions and is zero elsewhere we can now see the following.

Theorem 5.10.8. *Let Φ be the root system A_n and let W be the Weyl group of A_n. Then the total cohomology of T_Φ is the W-representation*

$$H^*(T_\Phi) = \mathrm{Reg}_W + n \cdot \mathrm{Ind}_{\langle s\rangle}^W(\mathrm{Triv}_{\langle s\rangle}),$$

where Reg_W is the regular representation of W and $\mathrm{Ind}_{\langle s\rangle}^W(\mathrm{Triv}_{\langle s\rangle})$ denotes the representation of W induced up from the trivial representation of the subgroup generated by the simple reflection $s = (12)$.

5.10.2. The Poincaré polynomial In [6] Arnold proved that the Poincaré polynomial of the complement of the affine hyperplane arrangement associated to A_n is

$$\prod_{i=1}^{n}(1+i\cdot t). \tag{5.10.1}$$

This result was later reproved and generalized by Orlik and Solomon in [72]. In this section we shall see that the Poincaré polynomial of the complement of the toric arrangement associated to A_n satisfies a similar formula.

Recall the map $\tau : \mathscr{N}(\Phi) \to \mathscr{H}(\Phi)$ defined in Equation 5.9.1. The map τ is an isomorphism if and only if each module of $\mathscr{N}(\Phi)$ is saturated. For $\Phi = A_n$ this is indeed the case. The pivotal observation for proving this is the following lemma.

Lemma 5.10.9. *Let M be a binary matrix of full rank such that the 1's in each row of M are consecutive. Then each pivot element in the row reduced echelon matrix (over $\mathbb{Z}$) obtained from M is 1.*

Proof. The proof goes by induction on the number of rows in M. If M only has one row, there is nothing to prove. Assume that the result is true for matrices with at most n rows and assume that M has $n+1$ rows.

There is no loss in generality in assuming that the rows of M are ordered lexicographically. Suppose that the first row has its first one in column i. We begin the row reduction by reducing downward, subtracting the first row from all rows containing a one in column i. Let M_1 be the resulting matrix. Since the rows of M were ordered lexicographically, we have only shortened the sequences of ones in some rows, i.e. M_1 is of the form in the statement of the lemma. If we delete the first row of M_1 we obtain a matrix M_2 with n rows which is still of this form and all the elements of M_2 in columns up to i are zero. Applying the induction hypothesis, we have that the row reduced echelon matrix M_3 obtained from M_2 only has ones as pivot elements. The only thing remaining to obtain the row reduced echelon matrix of M is to possibly subtract rows from M_3 from the first row of M. But this row has a pivot element 1 in the ith column and M_3 only has nonzero entries to the right of i. We thus see that the row reduced echelon form of M will have the required form. □

If one expresses the roots of A_n in terms of the simple roots β_i, each root is a vector of zeros and ones with all ones consecutive. It thus follows from Lemma 5.10.9 that the modules in $\mathscr{N}(\Phi)$ are saturated. Hence, the map $\tau : \mathscr{N}(\Phi) \to \mathscr{H}(\Phi)$ is an isomorphism of posets. This is a key ingredient in proving the following formula.

Theorem 5.10.10. *Let Φ be the root system A_n. The Poincaré polynomial of T_Φ is given by*

$$P(T_\Phi, t) = \prod_{i=1}^{n} (1 + (i+1) \cdot t).$$

Proof. Let $\mu_{\mathscr{H}}$ denote the Möbius function of $\mathscr{H}(\Phi)$. Theorem 5.4.1 and Equation 5.10.1 tell us that

$$\sum_{V \in \mathscr{H}(\Phi)} \mu_{\mathscr{H}}(V) \cdot (-t)^{\dim(V)} = \prod_{i=1}^{n} (1 + i \cdot t).$$

By equating the coefficients of t^r we get

$$\begin{aligned} \sum_{\substack{V \in \mathscr{H}(\Phi) \\ \dim(V)=r}} \mu_{\mathscr{H}}(V)(-1)^r &= \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=r}} \prod_{i \in I} i = \\ &= e_r(1, \ldots, n), \end{aligned}$$

where e_r denotes the rth elementary symmetric polynomial. We saw above that the map $\tau : \mathscr{N}(\Phi) \to \mathscr{H}(\Phi)$ is an isomorphism of posets. Hence, if $\mu_{\mathscr{N}}$ denotes the Möbius function of $\mathscr{N}(\Phi)$ we have that

$$\mu_{\mathscr{N}}(N) = \mu_{\mathscr{H}}(\tau(N)).$$

It thus follows that

$$\sum_{\substack{N \in \mathscr{N}(\Phi) \\ \mathrm{rk}(N)=r}} \mu_{\mathscr{N}}(N)(-1)^r = e_r(1, \ldots, n). \tag{5.10.2}$$

If we apply Theorem 5.4.1 to T_Φ, we obtain

$$\begin{aligned} P(T_\Phi, t) &= \sum_{N \in \mathscr{N}(\Phi)} \mu_{\mathscr{N}}(N)(-t)^{\mathrm{rk}(N)} \cdot (1+t)^{n-\mathrm{rk}(N)} \\ &= \sum_{r=0}^{n} t^r \cdot (1+t)^{n-r} \sum_{\mathrm{rk}(N)=r} \mu_{\mathscr{N}}(N) \cdot (-1)^r. \end{aligned}$$

If we use Equation 5.10.2 we now see that the coefficient of t^k in $P(T_\Phi, t)$ is

$$\sum_{j=0}^{k} \binom{n-j}{k-j} \cdot e_j(1, \ldots, n).$$

The coefficient of t^k in $\prod_{i=1}^n (1+(i+1)\cdot t)$ is

$$\begin{aligned}\sum_{\substack{I\subseteq\{1,\ldots,n\}\\|I|=k}} (i_1+1)\cdots(i_k+1) &= \sum_{\substack{I\subseteq\{1,\ldots,n\}\\|I|=k}} \sum_{j=0}^{k} e_j(i_1,\ldots,i_k) = \\ &= \sum_{j=0}^{k} \sum_{\substack{I\subseteq\{1,\ldots,n\}\\|I|=k}} e_j(i_1,\ldots,i_k) = \\ &= \sum_{j=0}^{k} \binom{n-j}{k-j} \cdot e_j(1,\ldots,n).\end{aligned}$$

This proves the claim. □

Note that setting $t=1$ in Theorem 5.10.10 gives another proof of Lemma 5.10.7. One can also prove Theorem 5.10.10 by applying Theorem 5.11 of [69] to Theorem 5.1 of [4] once one has carefully checked that their setup coincides with ours in the case of A_n.

5.11 Toric arrangements associated to root systems of exceptional type

As mentioned before, we have used Algorithms 5.8.1 and 5.9.2 to construct a `Sage` program computing the equivariant cohomology of the complement of a toric arrangement associated to a root system. One application of the results in Section 5.10 has been to verify the validity of this program. Below, we also compare our results with earlier work. In this section we present the results produced by this program for the exceptional root systems G_2, F_4, E_6 and E_7. This only leaves the exceptional root system E_8. We will complete the list with this case in future work.

The irreducible representations of Weyl groups of exceptional root systems can be described in terms of two integers, d and e. The integer d is the degree of the representation. The integer e can be defined as follows, see [21], p. 411.

Definition 5.11.1. Let Φ be an exceptional root system and let W be its Weyl group. Then each irreducible representation of W occurs in some symmetric power $\mathrm{Sym}^i \chi_{\mathrm{std}}$ of χ_{std}. Given an irreducible representation χ, let e be the integer such that χ occurs as a direct summand in $\mathrm{Sym}^e \chi_{\mathrm{std}}$ but not in $\mathrm{Sym}^i \chi_{\mathrm{std}}$ for all $i<e$.

For root systems of type E, the integers d and e uniquely determine the irreducible representations. However, for the root systems F_4 and G_2 this is not the case. We denote the irreducible character corresponding to d and e by ϕ_d^e. If there are two characters corresponding to the same d and e we add a second subscript to distinguish between them.

We refer to [21] for character tables of the Weyl groups of type G_2 and F_4. It should be noted that Carter denotes the characters ϕ_d^e, $\phi_{d,1}^e$ and $\phi_{d,2}^e$ by $\phi_{d,e}$, $\phi'_{d,e}$ and $\phi''_{d,e}$. We have chosen to denote the characters differently because we are in need of notational compactness.

5.11.1. The root system G_2 Let Φ be the root system of type G_2. Then the Poincaré polynomial of T_Φ is

$$P(T_\Phi, t) = 1 + 8\,t + 19\,t^2.$$

The cohomology of T_Φ as a representation of the Weyl group of G_2 is given in Table 5.1.

	ϕ_1^0	ϕ_1^6	$\phi_{1,1}^3$	$\phi_{1,2}^3$	ϕ_2^1	ϕ_2^2
$H^0(T_\Phi)$	1	0	0	0	0	0
$H^1(T_\Phi)$	2	0	0	0	1	2
$H^2(T_\Phi)$	2	1	1	1	3	4

Table 5.1: The cohomology groups of the complement of the toric arrangement associated to G_2 as representations of the Weyl group.

5.11.2. The root system F_4 Let Φ be the root system of type F_4. Then the Poincaré polynomial of T_Φ is

$$P(T_\Phi, t) = 1 + 28\,t + 286\,t^2 + 1260\,t^3 + 2153\,t^4.$$

The cohomology of T_Φ as a representation of the Weyl group of F_4 is given in Table 5.2.

Remark 5.11.2. The polynomial $P(T_\Phi, t)$ was computed by Moci in [68] as the Poincaré polynomial of the complement of the toric arrangement associated to the *coroot* system of F_4.

5.11.3. The root system E_6 Let Φ be the root system of type E_6. Then the Poincaré polynomial of T_Φ is

$$P(T_\Phi, t) = 1 + 42\,t + 705\,t^2 + 6020\,t^3 + 27459\,t^4 + 63378\,t^5 + 58555\,t^6.$$

The cohomology of T_Φ as a representation of the Weyl group of E_6 is given in Table 5.3.

	ϕ_1^0	ϕ_1^{24}	$\phi_{1,1}^{12}$	$\phi_{1,2}^{12}$	$\phi_{2,1}^{16}$	$\phi_{2,2}^{4}$	$\phi_{2,2}^{16}$	$\phi_{2,1}^{4}$	ϕ_4^1	$\phi_{4,2}^7$
$H^0(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	2	0	0	0	0	1	0	1	1	0
$H^2(T_\Phi)$	2	0	0	0	0	3	0	3	2	0
$H^3(T_\Phi)$	3	0	1	2	1	6	2	5	3	3
$H^4(T_\Phi)$	3	1	2	3	3	6	4	5	6	7

	$\phi_{4,1}^7$	ϕ_4^{13}	ϕ_4^8	$\phi_{6,2}^6$	$\phi_{6,1}^6$	$\phi_{8,1}^3$	$\phi_{8,2}^9$	$\phi_{8,2}^3$	$\phi_{8,1}^9$	ϕ_9^{10}
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	2	1	1	2	0	2	0	0
$H^3(T_\Phi)$	2	1	8	7	9	7	5	8	4	7
$H^4(T_\Phi)$	6	5	10	12	14	13	13	14	12	16

	$\phi_{9,1}^6$	$\phi_{9,2}^6$	$\phi_{9,}^2$	ϕ_{12}^4	ϕ_{16}^5
$H^0(T_\Phi)$	0	0	0	0	0
$H^1(T_\Phi)$	0	0	2	0	0
$H^2(T_\Phi)$	3	4	9	3	2
$H^3(T_\Phi)$	12	15	20	16	12
$H^4(T_\Phi)$	18	20	22	25	26

Table 5.2: The cohomology groups of the complement of the toric arrangement associated to F_4 as representations of the Weyl group.

Remark 5.11.3. The column ϕ_1^0 gives the cohomology of T_Φ/W. It was first computed by Looijenga in [62].

5.11.4. The root system E_7

Let Φ be the root system of type E_7. Then the Poincaré polynomial of T_Φ is

$$\begin{aligned} P(T_\Phi, t) = \quad & 1 + 70\,t + 2016\,t^2 + 30800\,t^3 + 268289\,t^4 + \\ & + 1328670\,t^5 + 3479734\,t^6 + 3842020\,t^7. \end{aligned}$$

The cohomology of T_Φ as a representation of the Weyl group of E_7 is given in Table 5.4.

Remark 5.11.4. The column ϕ_1^0 gives the cohomology of T_Φ/W. It was first computed by Looijenga in [62] and later corrected by Getzler and Looijenga in [47].

	ϕ_1^0	ϕ_1^{36}	ϕ_6^{25}	ϕ_6^1	ϕ_{10}^9	ϕ_{15}^{17}	ϕ_{15}^{16}	ϕ_{15}^5	ϕ_{15}^4	ϕ_{20}^{20}
$H^0(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	1	0	0	0	0	1	0
$H^2(T_\Phi)$	0	0	0	1	0	0	0	1	2	0
$H^3(T_\Phi)$	0	0	0	1	1	0	0	5	4	0
$H^4(T_\Phi)$	0	0	0	5	7	4	3	12	10	2
$H^5(T_\Phi)$	1	0	4	10	12	15	14	20	19	17
$H^6(T_\Phi)$	2	1	6	8	12	16	18	17	19	21

	ϕ_{20}^2	ϕ_{20}^{10}	ϕ_{24}^{12}	ϕ_{24}^6	ϕ_{30}^{15}	ϕ_{30}^3	ϕ_{60}^{11}	ϕ_{60}^5	ϕ_{60}^8	ϕ_{64}^{13}
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	3	0	0	0	0	2	0	2	1	0
$H^3(T_\Phi)$	6	2	1	4	0	9	3	11	6	1
$H^4(T_\Phi)$	15	11	10	16	9	24	23	38	30	21
$H^5(T_\Phi)$	30	24	26	31	30	41	68	80	74	69
$H^6(T_\Phi)$	25	23	27	29	33	36	66	69	69	69

	ϕ_{64}^4	ϕ_{80}^7	ϕ_{81}^6	ϕ_{81}^{10}	ϕ_{90}^8
$H^0(T_\Phi)$	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0
$H^2(T_\Phi)$	3	0	2	0	0
$H^3(T_\Phi)$	14	9	14	5	10
$H^4(T_\Phi)$	45	45	50	36	50
$H^5(T_\Phi)$	88	99	103	94	111
$H^6(T_\Phi)$	74	91	92	90	101

Table 5.3: The cohomology groups of the complement of the toric arrangement associated to E_6 as representations of the Weyl group.

	ϕ_1^0	ϕ_1^{63}	ϕ_7^{46}	ϕ_7^1	ϕ_{15}^7	ϕ_{15}^{28}	ϕ_{21}^{36}	ϕ_{21}^3	ϕ_{21}^{33}	ϕ_{21}^6
$H^0(T_\Phi)$	1	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	1	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	0	1	0	0	0	1	0	1
$H^3(T_\Phi)$	0	0	0	0	1	0	0	2	0	3
$H^4(T_\Phi)$	0	0	0	1	4	0	0	3	0	7
$H^5(T_\Phi)$	0	0	0	3	7	3	2	11	2	17
$H^6(T_\Phi)$	2	0	4	9	14	18	19	26	16	34
$H^7(T_\Phi)$	2	1	8	10	21	19	25	30	23	34

	ϕ_{27}^2	ϕ_{27}^{37}	ϕ_{35}^{31}	ϕ_{35}^4	ϕ_{35}^{22}	ϕ_{35}^{13}	ϕ_{56}^{30}	ϕ_{56}^3	ϕ_{70}^{18}	ϕ_{70}^9
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	1	0	0	1	0	0	0	0	0	0
$H^2(T_\Phi)$	2	0	0	2	0	0	0	1	0	0
$H^3(T_\Phi)$	3	0	0	3	0	1	0	3	0	1
$H^4(T_\Phi)$	8	0	0	9	2	4	0	9	5	6
$H^5(T_\Phi)$	25	1	3	30	16	14	11	30	30	30
$H^6(T_\Phi)$	50	16	23	63	45	36	53	70	86	80
$H^7(T_\Phi)$	43	30	43	52	47	44	74	71	101	85

	ϕ_{84}^{12}	ϕ_{84}^{15}	ϕ_{105}^5	ϕ_{105}^{26}	ϕ_{105}^{12}	ϕ_{105}^6	ϕ_{105}^{15}	ϕ_{105}^{21}	ϕ_{120}^4	ϕ_{120}^{25}
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	1	0	0	1	0	0	2	0
$H^3(T_\Phi)$	1	0	4	0	2	7	0	0	9	0
$H^4(T_\Phi)$	9	5	15	1	14	27	5	1	33	0
$H^5(T_\Phi)$	50	27	53	29	63	78	34	20	99	19
$H^6(T_\Phi)$	127	78	122	113	154	160	101	94	194	99
$H^7(T_\Phi)$	117	108	134	137	147	159	133	124	185	136

	ϕ_{168}^6	ϕ_{168}^{21}	ϕ_{189}^{22}	ϕ_{189}^{20}	ϕ_{189}^5	ϕ_{189}^7	ϕ_{189}^{17}	ϕ_{189}^{10}	ϕ_{210}^{10}	ϕ_{210}^6
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	1	0	0	0	1	0	0	0	0	2
$H^3(T_\Phi)$	7	0	0	0	6	4	0	5	4	13
$H^4(T_\Phi)$	36	2	5	7	25	23	6	33	32	51
$H^5(T_\Phi)$	128	35	61	73	90	86	55	125	128	157
$H^6(T_\Phi)$	267	145	215	233	216	205	182	277	295	326
$H^7(T_\Phi)$	249	200	255	251	239	243	226	276	307	313

	ϕ_{210}^{13}	ϕ_{210}^{21}	ϕ_{216}^{16}	ϕ_{216}^9	ϕ_{280}^8	ϕ_{280}^{17}	ϕ_{280}^{18}	ϕ_{280}^9	ϕ_{315}^{16}	ϕ_{315}^7
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	0	0	2	0	0	0	0	0
$H^3(T_\Phi)$	0	0	1	3	9	0	0	4	0	4
$H^4(T_\Phi)$	9	2	13	21	47	7	19	28	21	31
$H^5(T_\Phi)$	68	45	99	87	191	73	126	121	141	136
$H^6(T_\Phi)$	214	185	287	224	427	257	351	306	393	347
$H^7(T_\Phi)$	253	248	296	275	404	343	388	345	441	385

	ϕ_{336}^{14}	ϕ_{336}^{11}	ϕ_{378}^{14}	ϕ_{378}^9	ϕ_{405}^{15}	ϕ_{405}^8	ϕ_{420}^{10}	ϕ_{420}^{13}	ϕ_{512}^{12}	ϕ_{512}^{11}
$H^0(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^1(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^2(T_\Phi)$	0	0	0	0	0	0	0	0	0	0
$H^3(T_\Phi)$	2	2	2	2	0	11	7	0	6	2
$H^4(T_\Phi)$	33	26	33	28	12	73	61	20	61	29
$H^5(T_\Phi)$	179	128	188	150	118	268	258	139	290	180
$H^6(T_\Phi)$	456	344	498	405	397	588	598	417	710	524
$H^7(T_\Phi)$	468	416	533	467	480	598	602	510	731	627

Table 5.4: The cohomology groups of the complement of the toric arrangement associated to E_7 as representations of the Weyl group.

6. Quartics with marked points

In this chapter we shall apply the results of Chapter 5 to the descriptions of Chapter 4. More precisely, we compute the cohomology groups of $\mathcal{Q}_{\mathrm{ord}}[2]$, $\mathcal{Q}_{\mathrm{btg}}[2]$, $\mathcal{Q}_{\mathrm{flx}}[2]$ and $\mathcal{Q}_{\mathrm{hfl}}[2]$ as representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ together with their mixed Hodge structures. The results are presented in Tables 6.1–6.4. We also compute the cohomology groups of $\mathcal{Q}_{\overline{\mathrm{ord}}}[2] = \mathcal{Q}_{\mathrm{ord}}[2] \sqcup \mathcal{Q}_{\mathrm{flx}}[2]$ and $\mathcal{Q}_{\overline{\mathrm{btg}}}[2] = \mathcal{Q}_{\mathrm{btg}}[2] \sqcup \mathcal{Q}_{\mathrm{hfl}}[2]$ as representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ together with their mixed Hodge structures and we obtain a partial result about the cohomology of $\mathcal{Q}_1[2]$. The results are given in Table 6.5, Table 6.6 and Proposition 6.1.3. Finally, we prove Proposition 6.2.3 which states that the cohomology of $\mathcal{Q}[2]$ is a $\mathrm{Sp}(6,\mathbb{F}_2)$-subrepresentation of the cohomology of $\mathcal{Q}_{\mathrm{flx}}[2]$.

Throughout the chapter we shall work over the field of complex numbers.

6.1 Consequences of Looijenga's results

Recall from Chapter 4 that there are $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant isomorphisms

$$\begin{aligned} \mathcal{Q}_{\mathrm{ord}}[2] &\cong T_{E_7}/\{\pm 1\}, & \mathcal{Q}_{\mathrm{btg}}[2] &\cong \coprod_{e\in\mathcal{E}} T_{E_6}/\{\pm 1\}, \\ \mathcal{Q}_{\mathrm{flx}}[2] &\cong \mathbb{P}\left(V_{E_7}\right), & \mathcal{Q}_{\mathrm{hfl}}[2] &\cong \coprod_{e\in\mathcal{E}} \mathbb{P}\left(V_{E_6}\right)/\{\pm 1\}. \end{aligned}$$

It follows from the results of Chapter 5 that the cohomology groups $H^k(\mathcal{Q}_{\mathrm{ord}}[2])$ and $H^k\left(\mathcal{Q}_{\mathrm{btg}}[2]\right)$ are pure Hodge structures of Tate type (k,k). Also, since the Weyl group of E_7 is isomorphic to $\mathrm{Sp}(6,\mathbb{F}_2)\times\{\pm 1\}$, we can read off the $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant cohomology of $\mathcal{Q}_{\mathrm{ord}}[2]$ from Table 5.4 by restriction to $\mathrm{Sp}(6,\mathbb{F}_2)$. The result is given in Table 6.1. For $\mathcal{Q}_{\mathrm{btg}}[2]$ we may first induce up the cohomology of T_{E_6} as a representation from the Weyl group of E_6 to a representation of the Weyl group of E_7 and then restrict the result down to $\mathrm{Sp}(6,\mathbb{F}_2)$. The result is given in Table 6.2.

If Φ is a root system with Weyl group W and V_Φ is the complement of the associated affine hyperplane arrangement, then V_Φ is a trivial $\mathbb{C}^*$-bundle over $\mathbb{P}(V_\Phi)$. We have $H^0(\mathbb{C}^*) = H^1(\mathbb{C}^*) = \mathbb{C}$ and $H^k(\mathbb{C}^*) = 0$ for $k \geq 2$. Thus, using the Leray-Serre spectral sequence, see [81] and [78], we get W-

equivariant isomorphisms of mixed Hodge structures

$$H^k(V_\Phi) \cong H^k(\mathbb{P}(V_\Phi)) \oplus H^{k-1}(\mathbb{P}(V_\Phi))(-1).$$

We conclude that $H^k(\mathbb{P}(V_\Phi))$ is a pure Hodge structure of Tate type (k,k) and that the W-equivariant Poincaré polynomial of $\mathbb{P}(V_\Phi)$ can be deduced from the W-equivariant Poincaré polynomial of V_Φ via the formula

$$P^W(\mathbb{P}(V_\Phi), t) = \frac{P^W(V_\Phi, t)}{1+t}. \tag{6.1.1}$$

Fleischmann and Janiszczak [45], have computed the equivariant cohomology of the complements of affine hyperplane arrangements associated to root systems of exceptional type. Using their results it is an easy matter to compute the equivariant cohomology of $\mathbb{P}(V_\Phi)$ using Equation 6.1.1. Using the same induction and restriction procedure as in the toric case above, we can then deduce the $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant cohomology also of $\mathcal{Q}_{\mathrm{flx}}[2]$ and $\mathcal{Q}_{\mathrm{hfl}}[2]$. The results are given in Tables 6.3 and 6.4.

Now recall that

$$\mathcal{Q}_{\overline{\mathrm{ord}}}[2] \cong \tilde{T}_{E_7}/\{\pm 1\}, \quad \mathcal{Q}_{\overline{\mathrm{btg}}}[2] \cong \coprod_{e\in\mathcal{E}} \tilde{T}_{E_6}/\{\pm 1\}.$$

Let Φ be a root system, M be the $\mathbb{Z}$-linear span of Φ and let $T = \mathrm{Hom}(M, \mathbb{C}^*)$. Let $\mathcal{T}_\Phi = \{T_\alpha\}_{\alpha\in\Phi}$ be the arrangement in T associated to Φ and let $D_\Phi = \cup_\alpha T_\alpha$ be its divisor. Define $\tilde{T}$ as the blowup of T in the origin and let $\tilde{D}_\Phi$ be the strict transform of D_Φ. We now explain how to compute the cohomology of the complement $\tilde{T}_\Phi = \tilde{T} \setminus \tilde{D}_\Phi$.

Lemma 6.1.1 (Looijenga [62])**.** *Let Φ be a root system with Weyl group W, let T_Φ be the open complement of the toric arrangement associated to Φ and let $\mathbb{P}(V_\Phi)$ be the projectivization of the open complement of the hyperplane arrangement associated to Φ. There is then a W-equivariant short exact sequence of mixed Hodge structures*

$$0 \to H^\bullet\left(\tilde{T}_\Phi\right) \to H^\bullet(T_\Phi) \to H^\bullet(\mathbb{P}(V_\Phi))(-1)[1] \to 0.$$

In particular, $H^k\left(\tilde{T}_\Phi\right)$ is a pure Hodge structure of Tate type (k,k).

Proof. We only sketch a proof. The complement of the exceptional divisor $E \setminus \tilde{D}_\Phi$ can naturally be identified with $\mathbb{P}(V_\Phi)$. One then applies the Thom isomorphism, see [73] Theorem B.29, to the long exact sequence coming from the pair $(\tilde{T}_\Phi, T_\Phi)$. We refer to [62] for the details. □

We conclude that the de Rham cohomology groups $H^k(\mathcal{Q}_{\overline{\mathrm{ord}}}[2])$ and $H^k\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)$ are pure Hodge structures of Tate type (k,k) and that they are easily computed from the cohomology groups of the spaces $\mathcal{Q}_{\mathrm{ord}}[2]$, $\mathcal{Q}_{\mathrm{btg}}[2]$, $\mathcal{Q}_{\mathrm{flx}}[2]$ and $\mathcal{Q}_{\mathrm{hfl}}[2]$. The results are given in Table 6.5 and Table 6.6.

In order to compute the cohomology of $\mathcal{Q}_1[2]$ we need to patch together the cohomology of $\mathcal{Q}_{\overline{\mathrm{ord}}}[2]$ and $\mathcal{Q}_{\overline{\mathrm{btg}}}[2]$. Unfortunately, we have not been able to achieve this goal but the following lemma takes us a bit on the way.

Lemma 6.1.2 (Looijenga [62])**.** *Let X be a variety of pure dimension and let $Y \subset X$ be a hypersurface. Then there is a Gysin exact sequence of mixed Hodge structures*

$$\cdots \to H^{k-2}(Y)(\text{-}1) \to H^k(X) \to H^k(X \setminus Y) \to H^{k-1}(Y)(\text{-}1) \to \cdots$$

We now take $X = \mathcal{Q}_1[2]$ and $Y = \mathcal{Q}_{\overline{\mathrm{btg}}}[2]$. Then $X \setminus Y = \mathcal{Q}_{\overline{\mathrm{ord}}}[2]$ and we get a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant exact sequence of mixed Hodge structures

$$\cdots \to H^{k-2}\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)(\text{-}1) \to H^k(\mathcal{Q}_1[2]) \to H^k\left(\mathcal{Q}_{\overline{\mathrm{ord}}}[2]\right) \to \cdots.$$

Since $H^k(\mathcal{Q}_{\overline{\mathrm{ord}}}[2])$ and $H^k\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)$ are both pure Hodge structures of Tate type (k,k), the sequence splits into four term exact sequences

$$0 \to W_{2k}H^k(\mathcal{Q}_1[2]) \to H^k\left(\mathcal{Q}_{\overline{\mathrm{ord}}}[2]\right) \to H^{k-1}\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)(\text{-}1) \to W_{2k}H^{k+1}(\mathcal{Q}_1[2]) \to 0.$$

We have thus obtained the following result.

Proposition 6.1.3. *The above construction gives a* $\mathrm{Sp}(6,\mathbb{F}_2)$*-equivariant exact sequence of mixed Hodge structures*

$$0 \to W_{2k}H^k(\mathcal{Q}_1[2]) \to H^k\left(\mathcal{Q}_{\overline{\mathrm{ord}}}[2]\right) \to H^{k-1}\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)(\text{-}1) \to W_{2k}H^{k+1}(\mathcal{Q}_1[2]) \to 0.$$

The Hodge structure of $H^k(\mathcal{Q}_1[2])$ consists of one part of Tate type $(k-1,k-1)$ and one part of Tate type (k,k).

In [62], Looijenga shows a similar result for $\mathcal{Q}_1$, the moduli space of plane quartics with one marked point. In this case, it happens that the (k,k)-part is always zero when the $(k-1,k-1)$-part is nonzero and vice versa, so the cohomology of $\mathcal{Q}_1$ is pure in this sense. This can of course also be seen from the result of Bergström and Tommasi [11] that the cohomology of $\mathcal{Q}_1$ is the tensor product of the cohomology of $\mathcal{Q}$ and the cohomology of $\mathbb{P}^1$. In our case, the spaces involved have much more cohomology and we will not see such purity in the case of $\mathcal{Q}_1[2]$.

6.2 Quartics without marked points

In this section we shall relate the cohomology of $\mathcal{Q}[2]$ to the cohomology of various subspaces of $\mathcal{Q}_1[2]$.

The forgetful morphism

$$\pi : \mathcal{Q}_{\overline{\mathrm{btg}}}[2] \to \mathcal{Q}[2],$$

is finite so the map

$$\pi_* \circ \pi^* : H^k(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]) \to H^k(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]),$$

is multiplication with $\deg(\pi)$ (and since each quartic has 28 bitangents which, counted with multiplicity, have 2 points of tangency each we have that $\deg(\pi) = 56$). Thus, since we are using cohomology with coefficients in a field whose characteristic does not divide 56, the map

$$\pi^* : H^k(\mathcal{Q}[2]) \to H^k\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)$$

is injective. We now see the following.

Lemma 6.2.1. *The cohomology $H^k(\mathcal{Q}[2])$ is pure of Tate type (k,k) and is a subrepresentation of the $\mathrm{Sp}(6,\mathbb{F}_2)$-representation $H^k\left(\mathcal{Q}_{\overline{\mathrm{btg}}}[2]\right)$.*

We define $\mathcal{Q}_{\overline{\mathrm{flx}}}[2] = \mathcal{Q}_{\mathrm{flx}}[2] \sqcup \mathcal{Q}_{\mathrm{hfl}}[2] \subset \mathcal{Q}_1[2]$. We apply Lemma 6.1.2 with $X = \mathcal{Q}_{\overline{\mathrm{flx}}}[2]$ and $Y = \mathcal{Q}_{\mathrm{hfl}}[2]$ so that $X \setminus Y = \mathcal{Q}_{\mathrm{flx}}[2]$. We thus get a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant exact sequence of mixed Hodge structures

$$\cdots \to H^{k-2}(\mathcal{Q}_{\mathrm{hfl}}[2])\,(\text{-}1) \to H^k\big(\mathcal{Q}_{\overline{\mathrm{flx}}}[2]\big) \to H^k(\mathcal{Q}_{\mathrm{flx}}[2]) \to \cdots.$$

Since $H^k(\mathcal{Q}_{\mathrm{flx}})$ and $H^k(\mathcal{Q}_{\mathrm{hfl}})$ are both pure Hodge structures of Tate type (k,k), the long exact sequence splits into four term exact sequences

$$0 \to W_{2k}H^k\big(\mathcal{Q}_{\overline{\mathrm{flx}}}[2]\big) \to H^k(\mathcal{Q}_{\mathrm{flx}}[2]) \to H^{k-1}(\mathcal{Q}_{\mathrm{hfl}}[2])\,(\text{-}1) \to W_{2k}H^{k+1}\big(\mathcal{Q}_{\overline{\mathrm{flx}}}[2]\big) \to 0.$$

We conclude the following.

Lemma 6.2.2. *There is a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant exact sequence of mixed Hodge structures*

$$0 \to W_{2k}H^k\big(\mathcal{Q}_{\overline{\mathrm{flx}}}[2]\big) \to H^k(\mathcal{Q}_{\mathrm{flx}}[2]) \to H^{k-1}(\mathcal{Q}_{\mathrm{hfl}}[2])\,(\text{-}1) \to W_{2k}H^{k+1}\big(\mathcal{Q}_{\overline{\mathrm{flx}}}[2]\big) \to 0.$$

The Hodge structure of $H^k\big(\mathcal{Q}_{\overline{\mathrm{flx}}}[2]\big)$ consists of one part of Tate type $(k-1,k-1)$ and one part of Tate type (k,k).

The forgetful morphism

$$\pi : \mathcal{Q}_{\overline{\text{flx}}}[2] \to \mathcal{Q}[2],$$

is finite so the map

$$\pi_* \circ \pi^* : H^k(\mathcal{Q}_{\overline{\text{flx}}}[2]) \to H^k(\mathcal{Q}_{\overline{\text{flx}}}[2]),$$

is multiplication with $\deg(\pi) = 24$ (since each plane quartic has 24 flex points). Since we are using cohomology with coefficients in a field whose characteristic does not divide 24, we conclude that the map

$$\pi^* : H^k(\mathcal{Q}[2]) \to H^k\big(\mathcal{Q}_{\overline{\text{flx}}}[2]\big),$$

is injective. But by Lemma 6.2.1 we have that $H^k(\mathcal{Q}[2])$ is pure of Tate type (k,k). Hence, $H^k(\mathcal{Q}[2])$ must sit inside the (k,k)-part of $H^k\big(\mathcal{Q}_{\overline{\text{flx}}}[2]\big)$ which in turn sits inside $H^k(\mathcal{Q}_{\text{flx}})$.

Proposition 6.2.3. *The cohomology $H^k(\mathcal{Q}[2])$ is pure of Tate type (k,k) and is a subrepresentation of the* $\mathrm{Sp}(6,\mathbb{F}_2)$*-representation $H^k(\mathcal{Q}_{\text{flx}}[2])$.*

Both Proposition 6.2.3 and Lemma 6.2.1 relates the cohomology of $\mathcal{Q}[2]$ to the cohomology of spaces which we know as representations of $\mathrm{Sp}(6,\mathbb{F}_2)$-representations. The main advantage of Proposition 6.2.3 over Lemma 6.2.1 is that the cohomology of $\mathcal{Q}_{\text{flx}}[2]$ is smaller than that of $\mathcal{Q}_{\overline{\text{btg}}}[2]$ so Proposition 6.2.3 provides a sharper bound. It is of course also true that the multiplicity of an irreducible representation ϕ occurring in $H^k(\mathcal{Q}[2])$ is at most the minimum of the multiplicity of ϕ in $H^k(\mathcal{Q}_{\text{flx}})$ and the multiplicity of ϕ in $H^k\Big(\mathcal{Q}_{\overline{\text{btg}}}[2]\Big)$. However, as one can see in Table 6.3 and Table 6.6, the multiplicity of ϕ in $H^k(\mathcal{Q}_{\text{flx}})$ is always smaller so this insight does unfortunately not provide new information about the cohomology of $\mathcal{Q}[2]$.

6.3 Tables

Tables 6.1 - 6.6 give $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant cohomology for various spaces. The rows correspond to cohomology groups and the columns correspond to the irreducible representations of $\mathrm{Sp}(6,\mathbb{F}_2)$, see Table 6.7. A number n in row H^k and column ϕ means that the irreducible representation ϕ occurs in H^k with multiplicity n.

For convenience, we start by giving the Poincaré-Serre polynomials of the spaces occurring in the tables below.

Proposition 6.3.1. *Let $v = tu^2$. The Poincaré-Serre polynomials of $\mathcal{Q}_{\mathrm{ord}}[2]$, $\mathcal{Q}_{\mathrm{btg}}[2]$, $\mathcal{Q}_{\mathrm{flx}}[2]$, $\mathcal{Q}_{\mathrm{hfl}}[2]$, $\mathcal{Q}_{\overline{\mathrm{ord}}}[2]$ and $\mathcal{Q}_{\overline{\mathrm{btg}}}[2]$ are*

$$
\begin{aligned}
PS_{\mathcal{Q}_{\mathrm{ord}}[2]}(t,u) &= 1+63v+1638v^2+22680v^3+10089v^4+\\
&\quad +820827v^5+2004512v^6+2068430v^7,\\
PS_{\mathcal{Q}_{\mathrm{btg}}[2]}(t,u) &= 28+1176v+19740v^2+168560v^3+768852v^4+\\
&\quad +1774584v^5+1639540v^6,\\
PS_{\mathcal{Q}_{\mathrm{flx}}[2]}(t,u) &= 1+62v+1555v^2+20180v^3+142739v^4+\\
&\quad +521198v^5+765765v^6,\\
PS_{\mathcal{Q}_{\mathrm{hfl}}[2]}(t,u) &= 28+980v+13300v^2+85400v^3+278992v^4+\\
&\quad +344960v^5,\\
PS_{\mathcal{Q}_{\overline{\mathrm{ord}}}[2]}(t,u) &= 1+62v+1576v^2+21125v^3+159909v^4+\\
&\quad +678068v^5+1483314v^6+1302665v^7,\\
PS_{\mathcal{Q}_{\overline{\mathrm{btg}}}[2]}(t,u) &= 28+1148v+18760v^2+155260v^3+681352v^4+\\
&\quad +1495592v^5+1294580v^6.
\end{aligned}
$$

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	1	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	2	0	2	0	0
H^3	0	0	0	3	0	3	0	3	0	0
H^4	0	0	0	7	0	8	2	9	0	5
H^5	0	0	3	17	2	25	16	30	11	30
H^6	2	4	18	34	19	50	45	63	53	86
H^7	2	8	19	34	25	43	47	52	74	101

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	1	0	2	1	0	0	0	0
H^3	1	0	7	2	9	7	5	0	0	4
H^4	9	1	27	14	33	36	33	5	7	32
H^5	50	29	78	63	99	128	125	61	73	128
H^6	127	113	160	154	194	267	277	215	233	295
H^7	117	137	159	147	185	249	276	255	251	307

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	2	0	0	2	0	0	0	0	0	0
H^3	13	1	0	9	0	2	2	11	7	6
H^4	51	13	19	47	21	33	33	73	61	61
H^5	157	99	126	191	141	179	188	268	258	290
H^6	326	287	351	427	393	456	498	588	598	710
H^7	313	296	388	404	441	468	533	598	602	731

Table 6.1: The cohomology of $\mathcal{Q}_{\mathrm{ord}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	1	0	0	1	0	3	0	2	0	0
H^2	0	0	0	2	0	4	0	5	0	0
H^3	0	0	0	6	0	7	2	10	0	4
H^4	0	0	3	17	2	20	15	25	11	30
H^5	1	4	14	30	17	41	39	49	51	80
H^6	2	7	18	25	22	35	39	44	60	78

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	2	1	0	0	0	1
H^2	1	0	6	2	9	8	4	0	0	11
H^3	10	1	24	15	30	35	31	4	7	48
H^4	46	27	72	60	89	114	118	58	69	146
H^5	105	105	140	129	169	229	243	198	206	282
H^6	98	112	124	119	143	197	215	207	207	244

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	3	1	0	10	0	2	1	9	5	5
H^3	27	14	16	49	18	32	30	67	56	58
H^4	120	92	120	173	134	168	179	252	242	272
H^5	265	250	319	364	360	401	447	522	526	629
H^6	241	243	309	323	349	375	423	463	477	578

Table 6.2: The cohomology of $\mathcal{Q}_{\mathrm{btg}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	0	0	1	0	1	0	0
H^3	0	0	0	2	0	2	0	2	0	0
H^4	0	0	0	5	0	6	1	6	0	5
H^5	0	0	3	10	2	15	11	18	9	20
H^6	1	2	7	13	8	17	18	22	23	35

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	1	0	2	1	0	0	0	2
H^3	1	0	6	2	6	6	4	0	0	10
H^4	8	1	18	11	24	27	25	5	7	35
H^5	32	22	45	39	59	76	77	45	51	93
H^6	47	47	60	58	69	98	104	88	92	120

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	0	2	0	0	0	0	0	0
H^3	4	1	0	7	0	2	2	10	7	6
H^4	25	12	17	36	19	27	29	55	47	50
H^5	80	68	83	118	96	116	124	164	160	184
H^6	111	111	140	157	155	175	193	221	226	272

Table 6.3: The cohomology of $\mathcal{Q}_{\mathrm{flx}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	0	0	0	0	0	1	0	2	0	0
H^2	0	0	0	0	0	1	0	2	0	0
H^3	0	0	0	4	0	3	1	3	0	3
H^4	0	0	1	6	1	7	5	7	6	13
H^5	0	1	4	5	4	8	9	11	10	14

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	1	1	0	0	0	1
H^2	1	0	3	2	4	4	2	0	0	6
H^3	5	1	10	7	13	15	16	3	5	21
H^4	15	12	25	19	31	39	41	26	26	49
H^5	23	22	26	27	31	45	46	40	44	53

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	2	1	0	7	0	2	1	6	4	4
H^3	5	8	10	21	12	17	19	33	29	31
H^4	44	34	48	57	54	60	68	90	85	99
H^5	49	53	62	74	69	81	86	96	102	122

Table 6.4: The cohomology of $\mathcal{Q}_{\mathrm{hfl}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	1	0	1	0	0
H^3	0	0	0	3	0	2	0	2	0	0
H^4	0	0	0	5	0	6	2	7	0	5
H^5	0	0	3	12	2	19	15	24	11	25
H^6	2	4	15	24	17	35	34	45	44	66
H^7	1	6	12	21	17	26	29	30	51	66
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	0	0	1	0	2	1	0	0	0	0
H^3	1	0	6	2	7	6	5	0	0	2
H^4	8	1	21	12	27	30	29	5	7	22
H^5	42	28	60	52	75	101	100	56	66	93
H^6	95	91	115	115	135	191	200	170	182	202
H^7	70	90	99	89	116	151	172	167	159	187
	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	0	0	0	0	0
H^2	2	0	0	2	0	0	0	0	0	0
H^3	13	1	0	7	0	2	2	11	7	6
H^4	47	12	19	40	21	31	31	63	54	55
H^5	132	87	109	155	122	152	159	213	211	240
H^6	246	219	268	309	297	340	374	424	438	526
H^7	202	185	248	247	286	293	340	377	376	459

Table 6.5: The cohomology of $\mathcal{Q}_{\overline{\mathrm{ord}}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	1	0	0	0	0
H^1	0	0	0	1	0	2	0	2	0	0
H^2	0	0	0	2	0	3	0	3	0	0
H^3	0	0	0	6	0	6	2	8	0	4
H^4	0	0	3	13	2	17	14	22	11	27
H^5	1	4	13	24	16	34	34	42	45	67
H^6	2	6	14	20	18	27	30	33	50	64
	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	1	0	2	1	0	0	0	1
H^2	1	0	5	2	8	7	4	0	0	10
H^3	9	1	21	13	26	31	29	4	7	42
H^4	41	26	62	53	76	99	102	55	64	125
H^5	90	93	115	110	138	190	202	172	180	233
H^6	75	90	98	92	112	152	169	167	163	191
	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	3	1	0	9	0	2	1	9	5	5
H^3	25	13	16	42	18	30	29	61	52	54
H^4	105	84	110	152	122	151	160	219	213	241
H^5	221	216	271	307	306	341	379	432	441	530
H^6	192	190	247	249	280	294	337	367	375	456

Table 6.6: The cohomology of $\mathcal{Q}_{\overline{\mathrm{btg}}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$.

cent.	1451520	23040	4608	1536	384	2160	648	108	384	192	192	128	32	30	144	144	72	48	36	36	12	7	16	16	9	10	24	24	12	15
p pow.	–	A	A	A	A	A	A	A	B	C	C	B	C	A	AA	AB	BB	AC	CA	CB	CD	A	A	D	B	AA	DB	DC	CA	AA
conj. cl.	$1A$	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	7	-5	-1	3	-1	4	-2	1	3	1	-3	-1	1	2	-2	2	2	0	1	-1	-1	0	-1	1	1	0	-2	0	0	-1
χ_3	15	-5	7	3	-1	0	-3	3	-1	-3	1	3	1	0	-2	-2	1	0	1	1	-1	1	1	-1	0	0	0	-2	-1	0
χ_4	21	9	-3	1	-3	6	3	0	5	-1	3	1	-1	1	0	0	3	-2	0	0	0	0	-1	1	0	-1	2	0	-1	1
χ_5	21	-11	5	5	-3	6	3	0	1	-3	-3	1	1	1	-2	2	-1	2	-2	2	0	0	-1	-1	0	-1	0	0	1	1
χ_6	27	15	3	7	3	9	0	0	3	1	5	-1	1	2	3	3	0	1	0	0	0	-1	1	-1	0	0	1	-1	0	-1
χ_7	35	-5	3	-5	3	5	-1	2	7	-1	-1	-1	-1	0	1	-3	3	1	-2	0	0	0	1	1	-1	0	-1	-1	1	0
χ_8	35	15	11	7	3	5	-1	2	-1	5	1	3	1	0	3	-1	-1	1	0	2	0	0	-1	1	-1	0	-1	1	-1	0
χ_9	56	-24	-8	8	0	11	2	2	0	4	-4	0	0	1	-3	1	-2	-1	0	-2	0	0	0	0	-1	1	1	-1	0	1
χ_{10}	70	-10	-10	6	-2	-5	7	1	2	2	2	2	-2	0	-1	-1	-1	3	-1	-1	1	0	0	0	1	0	-1	-1	-1	0
χ_{11}	84	4	20	4	4	-6	3	3	4	0	0	4	0	-1	-2	2	-1	-2	1	-1	1	0	0	0	0	-1	0	0	1	-1
χ_{12}	105	-35	1	5	1	15	-3	-3	5	-1	-5	1	-1	0	1	1	1	-1	1	1	1	0	1	-1	0	0	-1	1	-1	0
χ_{13}	105	25	-7	9	1	0	6	3	-3	-3	-3	-3	1	0	4	-4	2	0	1	-1	1	0	-1	-1	0	0	0	0	0	0
χ_{14}	105	5	17	-3	-7	0	6	3	-3	3	-1	1	-1	0	2	2	2	0	-1	-1	-1	0	1	-1	0	0	0	2	0	0
χ_{15}	120	40	-8	8	0	15	-6	0	0	-4	4	0	0	0	1	1	-2	-1	-2	-2	0	1	0	0	0	0	-1	1	0	0
χ_{16}	168	40	8	8	8	6	6	-3	0	0	0	0	0	-2	-2	2	2	2	1	-1	-1	0	0	0	0	0	0	0	0	1
χ_{17}	189	21	-3	-11	-3	9	0	0	9	1	1	1	1	-1	-3	-3	0	1	0	0	0	0	-1	-1	0	1	1	1	0	-1
χ_{18}	189	-51	-3	13	-3	9	0	0	-3	1	1	-3	1	-1	-3	-3	0	1	0	0	0	0	1	1	0	-1	1	1	0	-1
χ_{19}	189	-39	21	1	-3	9	0	0	-3	-5	-1	1	-1	-1	3	3	0	1	0	0	0	0	-1	1	0	1	1	-1	0	-1
χ_{20}	210	50	2	2	-6	15	3	0	-2	2	2	-2	-2	0	-1	-1	-1	-1	2	2	0	0	0	0	0	0	-1	-1	1	0
χ_{21}	210	10	-14	10	2	-15	-6	3	6	-2	-2	-2	-2	0	1	1	-2	1	1	1	-1	0	0	0	0	0	1	1	0	0
χ_{22}	216	-24	24	8	0	-9	0	0	0	-4	4	0	0	1	-3	-3	0	-1	0	0	0	-1	0	0	0	1	-1	1	0	1
χ_{23}	280	-40	-8	-8	8	10	10	1	0	0	0	0	0	0	2	-2	-2	-2	-1	1	-1	0	0	0	1	0	0	0	0	0
χ_{24}	280	40	24	8	0	-5	-8	-2	0	4	-4	0	0	0	1	-3	0	-1	-2	0	0	0	0	0	1	0	1	-1	0	0
χ_{25}	315	-45	-21	3	3	0	-9	0	-5	3	3	3	-1	0	0	0	3	0	0	0	0	0	-1	-1	0	0	0	0	1	0
χ_{26}	336	-16	16	-16	0	6	-6	0	0	0	0	0	0	1	2	-2	-2	2	2	-2	0	0	0	0	0	-1	0	0	0	1
χ_{27}	378	-30	-6	2	-6	-9	0	0	6	2	2	-2	2	-2	3	3	0	-1	0	0	0	0	0	0	0	0	-1	-1	0	1
χ_{28}	405	45	-27	-3	-3	0	0	0	-3	-3	-3	5	1	0	0	0	0	0	0	0	0	-1	1	1	0	0	0	0	0	0
χ_{29}	420	20	4	-12	4	0	-3	3	-4	0	0	-4	0	0	-4	4	1	0	-1	1	1	0	0	0	0	0	0	0	-1	0
χ_{30}	512	0	0	0	0	-16	8	-4	0	0	0	0	0	2	0	0	0	0	0	0	0	1	0	0	-1	0	0	0	0	-1

Table 6.7: The character table of $\mathrm{Sp}(\mathbb{F}_2^6)$ as found in [23]. The irreducible characters $\chi_1, \ldots, \chi_{30}$ are listed by increasing dimension. The conjugacy classes are denoted by their order followed by a letter to be able to distinguish between conjugacy classes of the same order, e.g. $2C$ denotes the third class of order 2. The top row contains the order of the centralizers. The row "p pow." tells which conjugacy class the class ends up in after raising it to a prime power dividing its order where the prime divisors are listed in increasing order. For instance, the AC above $6D$ denotes the fact that an element of $6D$ raised to the power 2 lies in $3A$ and an element raised to the power 3 lies in $2C$.

7. S_7-equivariant cohomology of the moduli space of plane quartics

In Chapter 6 we found a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant inclusion of mixed Hodge structures

$$H^k(\mathcal{Q}[2]) \hookrightarrow H^k(\mathcal{Q}_{\mathrm{flx}}[2]),$$

between the cohomology of the moduli space of plane quartics with level 2 structure and the cohomology of the moduli space of plane quartics with level 2 structure and one marked flex point. This gives a lot of restrictions on the cohomology of $\mathcal{Q}[2]$, but it does not determine it completely.

Recall from Section 2.3 that we have a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant isomorphism

$$\mathcal{Q}[2] \cong \mathcal{P}^2_7,$$

to the moduli space of seven ordered points in general position in $\mathbb{P}^2$. However, the action of $\mathrm{Sp}(6,\mathbb{F}_2)$ on $\mathcal{P}^2_7$ is rather subtle and has this far eluded direct analysis. Nevertheless, the subgroup $S_7 \subset \mathrm{Sp}(6,\mathbb{F}_2)$ acts by permuting the seven points and is therefore more easily analyzed.

In this chapter we shall use this description to determine the cohomology of $\mathcal{Q}[2]$ as a S_7-representation and we thus get further restrictions on the cohomology of $\mathcal{Q}[2]$ as a $\mathrm{Sp}(6,\mathbb{F}_2)$-representation. Previously, we have mainly worked over $\mathbb{C}$ but $\mathcal{Q}[2]$ is a variety over $\mathbb{Z}$ and we may thus also consider its incarnation over a finite field. In this setting, we have the Lefschetz trace formula.

7.1 The Lefschetz trace formula

Let p be a prime number, let $n \geq 1$ be an integer and let $q = p^n$. Let $\mathbb{F}_q$ be a finite field with q elements, let $\mathbb{F}_{q^m}$ a degree m extension and let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. If X is a scheme defined over $\overline{\mathbb{F}}_q$, we shall denote its geometric Frobenius endomorphism induced from $\mathbb{F}_q$ by F. Furthermore, let l be another prime number, different from p, and let $H^k_{\mathrm{et,c}}(X,\mathbb{Q}_l)$ denote the i'th compactly supported l-adic cohomology group of X.

Let Γ be a finite group of rational automorphisms of X. Then each cohomology group $H^k_{\mathrm{et,c}}(X,\mathbb{Q}_l)$ is a Γ-representation. The Lefschetz trace formula allows us to obtain information about these representations by counting the number of fixed points of $F\sigma$ for different $\sigma \in \Gamma$.

Theorem 7.1.1 (Lefschetz trace formula)**.** *Let X be a separated scheme of finite type over $\overline{\mathbb{F}}_q$ with Frobenius endomorphism F and let σ be a rational automorphism of X of finite order. Then*

$$|X^{F\sigma}| = \sum_{k\geq 0} (-1)^k \cdot \mathrm{Tr}\left(F\sigma, H^k_{\mathrm{et,c}}(X,\mathbb{Q}_l)\right),$$

where $X^{F\sigma}$ denotes the fixed point set of $F\sigma$.

Remark 7.1.2. For a proof, see [33], Rapport - Théorème 3.2. This theorem is usually only stated in terms of F. To get the above version one simply applies the "usual" theorem to the twist of X by σ.

Remark 7.1.3. If Γ is a finite group of rational automorphisms of X and $\sigma \in \Gamma$, then $|X^{F\sigma}|$ will only depend on the conjugacy class of σ.

Recall that the Γ-equivariant Euler characteristic of X is the element of the representation ring $R(\Gamma)$ defined as the virtual representation

$$\mathrm{Eul}^{\Gamma}_X = \sum_{k\geq 0} (-1)^k \cdot H^k_{\mathrm{et,c}}(X,\mathbb{Q}_l) \in R(\Gamma).$$

By Theorem 7.1.1 we may determine Eul^{Γ}_X by computing $|X^{F\sigma}|$ for each $\sigma \in \Gamma$ and by Remark 7.1.3 it is enough to do so for one representative of each conjugacy class. This motivates the following definition.

Definition 7.1.4. Let X be a separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism F and let Γ be a finite group of rational automorphisms of X. The determination of $|X^{F\sigma}|$ for all $\sigma \in \Gamma$ is then called a Γ*-equivariant point count* of X over $\mathbb{F}_q$.

7.2 Minimal purity

We have seen that equivariant point counts give equivariant Euler characteristics. In the complex setting there is a notion of minimal purity which allows us to translate weighted Euler characteristics into Poincaré polynomials. Now we are working in positive characteristic so there are no Hodge decompositions, but there is a similar notion of purity which allows us to draw similar conclusions. This phenomenon has earlier mainly been used for compact spaces, see [10], [9], [85] and [11].

Definition 7.2.1 ([36])**.** Let X be an irreducible and separated scheme of finite type over $\mathbb{F}_q$ with Frobenius endomorphism F and let l be a prime not dividing q. The scheme X is called *minimally pure* if F acts on $H^k_{\text{et,c}}(X,\mathbb{Q}_l)$ with all eigenvalues equal to $q^{k-\dim(X)}$.

A pure dimensional and separated scheme X of finite type over $\mathbb{F}_q$ is minimally pure if for any collection $\{X_1,\dots,X_r\}$ of irreducible components of X, the irreducible scheme $X_1 \setminus (X_2 \cup \cdots \cup X_r)$ is minimally pure.

There is also an analogue of Definition 5.3.7 of minimally pure arrangements.

Thus, if X is minimally pure, then a term $q^{k-\dim(X)}$ in $|X^F|$ can only come from $H^k_{\text{et,c}}(X,\mathbb{Q}_l)$ and we can determine the Γ-equivariant Poincaré polynomial of X

$$P^\Gamma_X(t) = \sum_{k\geq 0} H^k_{\text{et,c}}(X,\mathbb{Q}_l)\cdot t^k \in R(\Gamma)[t].$$

via the relation

$$\mathrm{Eul}^\Gamma_X(\sigma) = q^{-2\dim(X)}\cdot P^\Gamma_X(\sigma)(-q^2).$$

Proposition 7.2.2. *The moduli spaces $\mathcal{Q}[2] \cong \mathcal{P}^2_7$ are minimally pure.*

Proof. Recall that the forgetful morphism $\pi : \mathcal{Q}_{\overline{\text{btg}}}[2] \to \mathcal{Q}[2]$ is finite. Using the same argument as in the proof of Lemma 6.2.1 and the comparison theorem, see Appendix C of [53], we get a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant injection

$$H^k_{\text{et,c}}(\mathcal{Q}[2],\mathbb{Q}_l) \hookrightarrow H^k_{\text{et,c}}\Big(\mathcal{Q}_{\overline{\text{btg}}}[2],\mathbb{Q}_l\Big).$$

We conclude that the group $H^k_{\text{et,c}}(\mathcal{Q}[2],\mathbb{Q}_l)$ is an F-invariant subspace of $H^k_{\text{et,c}}\Big(\mathcal{Q}_{\overline{\text{btg}}}[2],\mathbb{Q}_l\Big)$. Repeating the argument of the proof of Lemma 6.1.1, we have that $H^k_{\text{et,c}}\Big(\mathcal{Q}_{\overline{\text{btg}}}[2],\mathbb{Q}_l\Big)$ is a F-invariant subspace of $H^k_{\text{et,c}}\big(\mathcal{Q}_{\text{btg}}[2],\mathbb{Q}_l\big)$ and $\mathcal{Q}_{\text{btg}}[2]$ is isomorphic to the complement of a toric arrangement by Proposition 4.4.1.

Dimca and Lehrer [36] have shown that complements of toric arrangements are minimally pure in the sense of Definition 7.2.1 so F acts on $H^k_{\text{et,c}}\big(\mathcal{Q}_{\text{btg}}[2],\mathbb{Q}_l\big)$ with all eigenvalues equal to q^{k-6}. Since $H^k_{\text{et,c}}(\mathcal{Q}[2],\mathbb{Q}_l)$ is an F-invariant subspace of $H^k_{\text{et,c}}\big(\mathcal{Q}_{\text{btg}}[2],\mathbb{Q}_l\big)$, it follows that F acts on $H^k_{\text{et,c}}(\mathcal{Q}[2],\mathbb{Q}_l)$ with all eigenvalues equal to q^{k-6}. We conclude that $\mathcal{Q}[2]$ is minimally pure. □

We can therefore determine the structure of $H^k_{\text{et,c}}(\mathcal{Q}[2],\mathbb{Q}_l)$ as a S_7-representation by making S_7-equivariant point counts of $\mathcal{P}^2_7$.

7.3 Equivariant point counts

In this section we shall perform a S_7-equivariant point count of $\mathcal{P}_7^2$. This amounts to the computation of $\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right|$ for one representative σ of each of the fifteen conjugacy classes of S_7. The computations will be rather different in the various cases but at least the underlying idea will be the same. Throughout this section we shall work over a finite field $\mathbb{F}_q$ where q is odd.

Let U be a subset of $\left(\mathbb{P}^2\right)^7$ and interpret each point of U as an ordered septuple of points in $\mathbb{P}^2$. Define the *discriminant locus* $\Delta \subset U$ as the subset consisting of septuples which are not in general position. If U contains the subset of $\left(\mathbb{P}^2\right)^7$ consisting of all septuples which are in general position, then

$$\mathcal{P}_7^2 = (U \setminus \Delta)\,/\,\mathrm{PGL}(3).$$

An element of PGL(3) is completely specified by where it takes four points in general position. Therefore, the points of $\mathcal{P}_7^2$ do not have any automorphisms and we have the simple relation

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = \frac{\left|U^{F\sigma}\right| - \left|\Delta^{F\sigma}\right|}{|\mathrm{PGL}(3)|}. \tag{7.3.1}$$

We will choose the set U in such a way that counting fixed points of $F\sigma$ in U is easy. We shall therefore focus on the discriminant locus.

The discriminant locus can be decomposed as

$$\Delta = \Delta_l \cup \Delta_c,$$

where Δ_l consists of septuples where at least three points lie on a line and Δ_c consists of septuples where at least six points lie on a conic. The computation of $|\Delta^{F\sigma}|$ will consist of the following three steps:

- the computation of $|\Delta_l^{F\sigma}|$,
- the computation of $|\Delta_c^{F\sigma}|$,
- the computation of $|(\Delta_l \cap \Delta_c)^{F\sigma}|$.

We can then easily determine $|\Delta^{F\sigma}|$ via the principle of inclusion and exclusion.

In the analysis of $\Delta_l \cap \Delta_c$ the following definition will sometimes be useful, see Figure 7.1.

Definition 7.3.1. Let C be a smooth conic over $\mathbb{F}_q$ and let $P \in \mathbb{P}^2$ be a $\mathbb{F}_q$-point. We then say

- that P is on the $\mathbb{F}_q$-*inside* of C if there is no $\mathbb{F}_q$-tangent to C passing through P,
- that P is on C if there is precisely one $\mathbb{F}_q$-tangent to C passing through P,
- that P is on the $\mathbb{F}_q$-*outside* of C if there are two $\mathbb{F}_q$-tangents to C passing through P.

For a motivation of the terminology, see Figure 7.1.

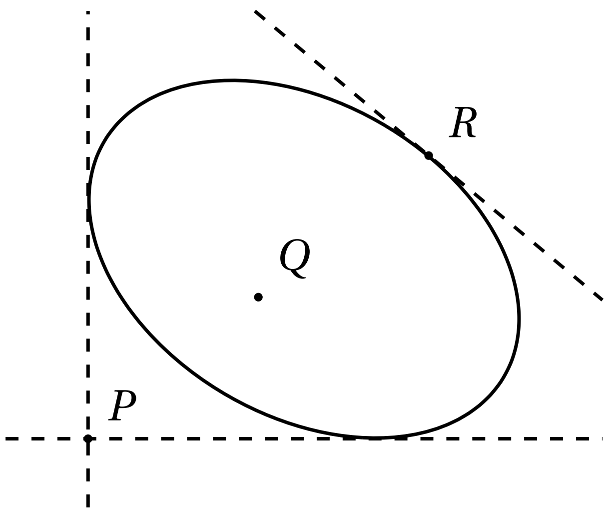


Figure 7.1: A conic C with a point P on the outside of C, a point Q on the inside of C and a point R on C.

It is not entirely clear that the above definition makes sense. To see that it in fact does, we need the following lemma.

Lemma 7.3.2. *Let $C \subset \mathbb{P}^2$ be a smooth conic over a field k. If there is a point P such that three tangents of C pass through P, then the characteristic of k is* 2.

Proof. Since C is smooth, the three points of tangency will be in general position, so by a projective change of coordinates they can be transformed to $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ and C will then be given by a polynomial $F = XY + \alpha XZ + \beta YZ$, where $\alpha, \beta \in k^*$. The tangent lines thus become $Y + \alpha Z$, $X + \beta Z$ and $\alpha X + \beta Y$.

Let the coordinates of P be $[a : b : c]$. Since these lines all pass through P, the first tangent equation gives that $b = -\alpha c$ and the second gives $a = -\beta c$. Inserting these expressions into the third tangent equation gives $-2\alpha\beta c = 0$. If $c = 0$, then also $a = b = 0$ which is impossible. Since also α and β are nonzero we see that the only possibility is that the characteristic of k is 2. $\square$

Let $\sigma^{-1} = (i_1 \ldots i_r)$ be a cycle in S_7. An ordered septuple $(P_1, \ldots, P_7)$ of points in $\mathbb{P}^2$ will be fixed by $F\sigma$ if and only if $FP_{i_j} = P_{i_{j+1}}$ for $i = 1, \ldots, r-1$ and $FP_{i_r} = P_{i_1}$. This is the motivation for the following definition.

Definition 7.3.3. Let X be a $\mathbb{F}_q$-scheme with Frobenius endomorphism F and let $Z \subset X_{\overline{\mathbb{F}}_q}$ be a subscheme. If

$$\left|\left\{F^i Z\right\}_{i \geq 0}\right| = m,$$

we say that Z is a *strict* $\mathbb{F}_{q^m}$-subscheme.

If Z is a strict $\mathbb{F}_{q^m}$-subscheme, the m-tuple $(Z, \ldots, F^{m-1}Z)$ is called a *conjugate m-tuple*. Let r be a positive integer and let $\lambda = [1^{\lambda_1}, \ldots, r^{\lambda_r}]$ be a partition of r. An r-tuple $(Z_1, \ldots, Z_r)$ of closed subschemes of X is called a *conjugate λ-tuple* if it consists of λ_1 conjugate 1-tuples, λ_2-conjugate 2-tuples and so on. We denote the set of conjugate λ-tuples of closed points of X by $X(\lambda)$.

Since the conjugacy class of an element in S_7 is given by its cycle type, we want to count the number of conjugate λ-tuples in both U and Δ for each partition of seven. In this pursuit, the following formula is helpful. Its proof is a simple application of the principle of inclusion and exclusion.

Lemma 7.3.4. *Let X be a $\mathbb{F}_q$-scheme and let $\lambda = [1^{\lambda_1}, \ldots, n^{\lambda_n}]$ be a partition. Then*

$$|X(\lambda)| = \prod_{i=1}^{\nu} \prod_{j=0}^{\lambda_i - 1} \left[\left(\sum_{d|i} \mu\left(\frac{i}{d}\right) \cdot |X(d)| \right) - i \cdot j \right],$$

where μ is the Möbius function.

We now recall a number of basic results regarding point counts. First, note that if we apply Lemma 7.3.4 to $X = (\mathbb{P}^n)^\vee$, the dual projective space, we see that the number of conjugate λ-tuples of hyperplanes is equal to the number of conjugate λ-tuples of points in $\mathbb{P}^n$. We also recall that

$$\left|\mathbb{P}^n_{\mathbb{F}_q}\right| = \sum_{i=0}^{n} q^i,$$

and that

$$|\mathrm{PGL}(3)| = q^3 \cdot (q^3 - 1) \cdot (q^2 - 1).$$

A slightly less elementary result is that the number of smooth conics defined over $\mathbb{F}_q$ is

$$q^5 - q^2.$$

To see this, note that there is a $\mathbb{P}^5$ of conics. Of these there are q^2+q+1 double $\mathbb{F}_q$-lines, $\frac{1}{2}\cdot(q^2+q+1)\cdot(q^2+q)$ intersecting pairs of $\mathbb{F}_q$-lines and $\frac{1}{2}\cdot(q^4-q)$ conjugate pairs of $\mathbb{F}_{q^2}$-lines.

We are now ready for the task of counting the number of conjugate λ-tuples for each element of S_7.

Remark 7.3.5. Since $\mathcal{P}^2_7$ is minimally pure, Equation 7.3.1 gives that $\left|\left(\mathcal{P}^2_7\right)^{F\sigma}\right|$ is a monic polynomial in q of degree six so it is in fact enough to make counts for six different finite fields and interpolate. This is however hard to carry out in practice, even with a computer, as soon as λ contains parts of large enough size (where "large enough" means 3 or 4). However, one can always obtain partial information which provides important checks for our computations.

7.3.1. The case $\lambda = [7]$ Let $\lambda = [7]$. Since we only need to make the computation for one permutation σ of cycle type λ, we may as well assume that $\sigma^{-1} = (1234567)$ so that F acts as $FP_i = P_{i+1}$ for $i = 1,\dots,6$ and $FP_7 = P_1$. In this case, we simply take $\left(\mathbb{P}^2\right)^7$ as our set U.

The main observation is the following.

Lemma 7.3.6. *If $(P_1,\dots,P_7)$ is a λ-tuple with three of its points on a line, then all seven points lie on a line defined over $\mathbb{F}_q$.*

Proof. Suppose that the set $S = \{P_i, P_j, P_k\}$ is contained in the line L. Then L is either defined over $\mathbb{F}_q$ or $\mathbb{F}_{q^7}$. One easily checks that for each of the $\binom{7}{3} = 35$ possible choices of S there is an integer $1 \le i \le 6$ such that $|F^i S \cap S| = 2$. Since a line is defined by any two points on it we have that $L = F^i L$. Hence, we have that L is defined over $\mathbb{F}_q$ and that $\{P_i, FP_i, \dots, F^6 P_i\} = \{P_1,\dots,P_7\} \subset L$. □

Lemma 7.3.7. *If $(P_1,\dots,P_7)$ is a λ-tuple with six of its points on a smooth conic, then all seven points lie on a smooth conic defined over $\mathbb{F}_q$.*

Proof. Suppose that the set $S = \{P_{i_1},\dots,P_{i_6}\}$ lies on a smooth conic C. We have $|FS \cap S| = 5$ and since a conic is defined by any five points on it we have $FC = C$. Hence, we have that C is defined over $\mathbb{F}_q$ and that all seven points lie on C. □

We conclude that Δ_l and Δ_c are disjoint. We obtain $|\Delta_l|$ by first choosing a $\mathbb{F}_q$-line L and then picking a λ-tuple on L. We thus have

$$|\Delta_l| = (q^2+q+1)\cdot(q^7-q).$$

To obtain $|\Delta_c|$ we first choose a smooth conic C and then a conjugate λ-tuple on C. We thus have

$$|\Delta_c| = (q^5 - q^2) \cdot (q^7 - q).$$

Equation 7.3.1 now gives

$$\left|\left(\mathcal{P}^2_7\right)^{F\sigma}\right| = q^6 + q^3.$$

7.3.2. The case $\lambda = [1,6]$ Let $\lambda = [1,6]$. Since we only need to make the computation for one permutation σ of cycle type λ, we may as well assume that $\sigma^{-1} = (123456)(7)$ so that F acts as $FP_i = P_{i+1}$ for $i = 1,\ldots,5$, $FP_6 = P_1$ and $FP_7 = P_7$. Also in this case we take $\left(\mathbb{P}^2\right)^7$ as our set U.

The main observation is the following.

Lemma 7.3.8. *If a λ-tuple has three points on a line, then either*

(1) the first six points of the λ-tuple lie on a $\mathbb{F}_q$-line or,

(2) the first six points lie on two conjugate $\mathbb{F}_{q^2}$-lines, the $\mathbb{F}_{q^2}$-lines contain three $\mathbb{F}_{q^6}$-points each and these triples are interchanged by F, or,

(3) the first six points lie pairwise on three conjugate $\mathbb{F}_{q^3}$-lines which intersect in P_7.

Proof. Suppose that $S = \{P_i, P_j, P_k\}$ lie on a line L. Then L is either defined over $\mathbb{F}_q$, $\mathbb{F}_{q^2}$, $\mathbb{F}_{q^3}$ or $\mathbb{F}_{q^6}$. One easily checks that for each of the $\binom{7}{3} = 35$ possible choices of S there is an integer $1 \le i \le 3$ such that $|F^i S \cap S| = 2$ so L is defined over $\mathbb{F}_q$, $\mathbb{F}_{q^2}$ or $\mathbb{F}_{q^3}$, i.e. we are in one of the three cases above. $\square$

Let $\Delta_{l,i}$ be the subset of Δ_l corresponding to case (i) in Lemma 7.3.8. The set $\Delta_{l,1}$ is clearly disjoint from the other two.

Lemma 7.3.9. *If six of the points of a λ-tuple lie on a smooth conic, then all of the first six points of the tuple lie on the conic and the conic is defined over $\mathbb{F}_q$.*

Proof. Suppose that $S = \{P_{i_1}, \ldots, P_{i_6}\}$ lie on a smooth conic C. Then $|FS \cap S| \ge 5$ so $FC = C$. Let $P \in S$ be a $\mathbb{F}_{q^6}$-point. Then $\{P, FP, \ldots, F^5\} = \{P_1, \ldots, P_6\} \subset C$. $\square$

Since a smooth conic does not contain a line, we have that Δ_c only intersects $\Delta_{l,3}$.

We compute $|\Delta_{l,1}|$ by first choosing a $\mathbb{F}_q$-line L and then a $\mathbb{F}_{q^6}$ point on L. Finally we choose a $\mathbb{F}_q$-point P_7 anywhere. We thus have

$$|\Delta_{l,1}| = (q^2 + q + 1) \cdot (q^6 - q^3 - q^2 + q) \cdot (q^2 + q + 1).$$

To obtain $|\Delta_{l,2}|$ we first choose a $\mathbb{F}_{q^2}$-line, L. By Lemma 7.3.4 there are $q^4 - q$ such lines. The other $\mathbb{F}_{q^2}$-line must then be FL. We then choose a $\mathbb{F}_{q^6}$-point P_1 on L. The points $P_2 = FP_1, \dots, P_6 = F^5 P_1$ will then be the rest of our conjugate sextuple. By Lemma 7.3.4 (with $\mathbb{F}_{q^2}$ as the ground field) there are $q^6 - q^2$ choices. We now have two $\mathbb{F}_{q^2}$-lines with three of our six $\mathbb{F}_{q^6}$-points on each so all that remains is to choose a $\mathbb{F}_q$-point anywhere we want in $q^2 + q + 1$ ways. Hence,

$$|\Delta_{l,2}| = (q^4 - q)(q^6 - q^2)(q^2 + q + 1).$$

To count $|\Delta_{l,3}|$ we first choose a $\mathbb{F}_q$-point P_7 in $q^2 + q + 1$ ways. There is a $\mathbb{P}^1$ of lines through P_7 and we want to choose a $\mathbb{F}_{q^3}$-line L through P. By Lemma 7.3.4 there are $q^3 - q$ choices. Finally, we choose a $\mathbb{F}_{q^6}$-point P_1 on L. By Lemma 7.3.4 there are $q^6 - q^3$ possible choices. We thus have

$$|\Delta_{l,3}| = (q^2 + q + 1)(q^3 - q)(q^6 - q^3).$$

In order to finish the computation of Δ_l, we need to compute $|\Delta_{l,2} \cap \Delta_{l,3}|$. We first choose a pair of conjugate $\mathbb{F}_{q^2}$-lines in $\frac{1}{2}(q^4 - q)$ ways. These intersect in a $\mathbb{F}_q$-point and we choose P_7 away from this point in $q^2 + q$ ways. We then choose a $\mathbb{F}_{q^3}$-line through P_7 in $q^3 - q$ ways. This line intersects the two $\mathbb{F}_{q^2}$-lines in 2 distinct points which clearly must be defined over $\mathbb{F}_{q^6}$. We choose one of them to become P_1 in 2 ways. Thus, in total we have

$$|\Delta_{l,2} \cap \Delta_{l,3}| = (q^4 - q) \cdot (q^2 + q) \cdot (q^3 - q).$$

To compute $|\Delta_c|$ we first choose a smooth conic C in $q^5 - q^2$ ways and then use Lemma 7.3.4 to see that we have $q^6 - q^3 - q^2 + q$ ways of choosing a conjugate sextuple on C. Finally, we choose P_7 anywhere we want in $q^2 + q + 1$ ways. We thus see that

$$|\Delta_c| = (q^5 - q^2)(q^6 - q^3 - q^2 + q)(q^2 + q + 1).$$

It remains to compute the size of the intersection between Δ_l and Δ_c. To compute do this, we begin by choosing a smooth conic C in $q^5 - q^2$ ways and then a $\mathbb{F}_q$-point P_7 not on C in $q^2 + q + 1 - (q + 1) = q^2$ ways. By Lemma 7.3.4 there are $q^3 - q$ strict $\mathbb{F}_{q^3}$-lines passing through P. All of these intersect C in two $\mathbb{F}_{q^3}$-points since, by Lemma 7.3.2, these lines cannot be tangent to C since the characteristic of $\mathbb{F}_q$ is odd. More precisely, choosing any of the $q^3 - q$ strict $\mathbb{F}_{q^3}$-points of C gives a strict $\mathbb{F}_{q^3}$-line, and since every such line cuts C in exactly two points we conclude that there are precisely $\frac{1}{2}(q^3 - q)$ strict $\mathbb{F}_{q^3}$-lines through P intersecting C in two $\mathbb{F}_{q^3}$-points. Thus, the remaining

$$q^3 - q - \frac{1}{2}(q^3 - q) = \frac{1}{2}(q^3 - q)$$

$\mathbb{F}_{q^3}$-lines through P_7 will intersect C in two $\mathbb{F}_{q^6}$-points. If we pick one of them and label it P_1 we obtain an element in $\Delta_l \cap \Delta_c$. Hence,

$$|\Delta_l \cap \Delta_c| = (q^5 - q^2)q^2(q^3 - q).$$

We now conclude that

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6 - 2q^3 + 1.$$

7.3.3. The case $\lambda = [2,5]$ Throughout this section, λ will denote the partition $[2,5]$. We take $U = \left(\mathbb{P}^2\right)^7$.

Lemma 7.3.10. *If $(P_1,\ldots,P_7)$ is a λ-tuple with three of its points on a line, then all five $\mathbb{F}_{q^5}$-points lie on a line defined over $\mathbb{F}_q$. If six of the points lie on a smooth conic C, then all seven points lie on C and C is defined over $\mathbb{F}_q$.*

Proof. The proof is very similar to the proofs of Lemmas 7.3.6 and 7.3.7 and is therefore omitted. □

By Lemma 7.3.4, there are $q^{10} + q^5 - q^2 - q$ conjugate quintuples whereof $(q^2 + q + 1)(q^5 - q)$ lie on a line. We may thus choose a conjugate quintuple whose points do not lie on a line in $q^{10} - q^7 - q^6 + q^3$ ways. This quintuple defines a smooth conic C. By Lemma 7.3.10, it is enough to choose a conjugate pair outside C in order to obtain an element of $\left(\mathbb{P}^2\right)^7 \setminus \Delta$ of the desired type. Since there are $q^4 - q$ conjugate pairs of which $q^2 - q$ lie on C there are $q^4 - q^2$ remaining choices. We thus obtain

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6 - q^2.$$

7.3.4. The case $\lambda = [1^2,5]$ The computation in this case is very similar to that of the case $\lambda = [2,5]$ and we therefore simply state the result:

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6 - q^2.$$

7.3.5. The case $\lambda = [3^1,4^1]$ Throughout this section, λ shall mean the partition $[3^1,4^1]$. Since we only need to make the computation for one permutation, we shall assume that the Frobenius permutes points $P_1, P_2, P_{3,4}$ according to (1234) and the three points P_5, P_6, P_7 according to (567). We take $U = \left(\mathbb{P}^2\right)^7$.

Lemma 7.3.11. *If a conjugate λ-tuple has three points on a line, then either*

(1) the four $\mathbb{F}_{q^4}$-points lie on a $\mathbb{F}_q$-line, or

(2) *the three* $\mathbb{F}_{q^3}$*-points lie on a* $\mathbb{F}_q$*-line.*

Proof. It is easy to see that if three $\mathbb{F}_{q^4}$-points lie on a line, then all four $\mathbb{F}_{q^4}$-points lie on that line and even easier to see the corresponding result for three $\mathbb{F}_{q^3}$-points.

Suppose that two $\mathbb{F}_{q^4}$-points P_i and P_j and a $\mathbb{F}_{q^3}$-point P lie on a line L. Since $F^4 P_i = P_i$ and $F^4 P_j = P_j$ we see that $FF^4 L = L$. However, $F^4 P = FP \neq P$. Repeating this argument again, with FP in the place of P, shows that also $F^2 P$ lies on L. We are thus in case (1).

If we assume that two $\mathbb{F}_{q^3}$-points and a $\mathbb{F}_{q^4}$-point lie on a line, then a completely analogous argument shows that all four $\mathbb{F}_{q^4}$-points lie on that line. □

We decompose Δ_l as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of tuples with the four $\mathbb{F}_{q^4}$-points on a line and $\Delta_{l,2}$ consists of tuples with the three $\mathbb{F}_{q^3}$-points on a line. The computations of $|\Delta_{l,1}|$, $\Delta_{l,2}$ and $|\Delta_{l,1} \cap \Delta_{l,2}|$ are completely straightforward and we get

$$|\Delta_l| = q^{13} + 2q^{12} - 3q^{10} - 2q^9 + q^8 + q^7 - q^6 - q^5 + q^4 + q^3.$$

To compute $|\Delta_c|$ we start by noting that if six of the points of a λ-tuple lie on a smooth conic C, then all seven points lie on C and C is defined over $\mathbb{F}_q$. Thus, the problem consists of choosing a smooth conic C over $\mathbb{F}_q$ and then picking a λ-tuple on C. We thus have

$$|\Delta_c| = (q^5 - q^2)(q^4 - q^2)(q^3 - q).$$

Since no three points on a smooth conic lie on a line we conclude that the intersection $\Delta_l \cap \Delta_c$ is empty. We now obtain

$$\left| \left(\mathcal{P}_7^2\right)^{F\sigma} \right| = q^6 - q^5 - 2q^4 + q^3 + q^2 :$$

7.3.6. The case $\lambda = [1,2,4]$ Throughout this section, λ shall mean the partition $[1^1, 2^1, 4^1]$. Since we only need to make the computation for one permutation, we shall assume that the Frobenius permutes points P_1, P_2, P_3, P_4 according to (1234), switches the two points P_5, P_6 and fixes the point P_7. The computation will turn out to be quite a bit more complicated in this case than in the previous cases, mainly because both 1 and 2 divide 4. We take $U = \left(\mathbb{P}^2\right)^7$.

We have the following trivial decomposition of Δ_l

$$\Delta_l = \bigcup_{i=1}^{6} \Delta_{l,i},$$

where

- $\Delta_{l,1}$ consists of λ-tuples with three $\mathbb{F}_{q^4}$-points lying on a line,
- $\Delta_{l,2}$ consists of λ-tuples with two $\mathbb{F}_{q^4}$-points and a $\mathbb{F}_{q^2}$-point lying on a line,
- $\Delta_{l,3}$ consists of λ-tuples with two $\mathbb{F}_{q^4}$-points and the $\mathbb{F}_q$-point lying on a line,
- $\Delta_{l,4}$ consists of λ-tuples with a $\mathbb{F}_{q^4}$-point and two $\mathbb{F}_{q^2}$-points lying on a line,
- $\Delta_{l,5}$ consists of λ-tuples with a $\mathbb{F}_{q^4}$-point, a $\mathbb{F}_{q^2}$-point and a $\mathbb{F}_q$-point lying on a line, and,
- $\Delta_{l,6}$ consists of λ-tuples with two $\mathbb{F}_{q^2}$-points and the $\mathbb{F}_q$-point lying on a line.

This decomposition is of course naive and is not very nice to work with since none of the possible intersections are empty. The reader can surely think of many other decompositions which a priori look more promising. However, the more "clever" approaches we have tried have turned out to be quite hard to work with in practice. The positive thing about the above decomposition is that most intersections are rather easily handled and that quadruple intersections (and higher) all consist of tuples where all seven points lie on a $\mathbb{F}_q$-line.

The two slightly more complicated sets in the above list are $\Delta_{l,2}$ and $\Delta_{l,3}$. We shall therefore comment a bit about the computations involving them.

The set $\Delta_{l,2}$ splits into three disjoint subsets

$$\Delta_{l,2} = \Delta_{l,2}^1 \cup \Delta_{l,2}^2 \cup \Delta_{l,2}^3.$$

where

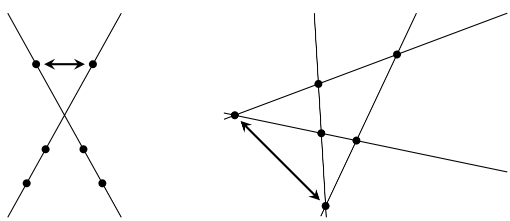
- $\Delta_{l,2}^1$ consists of λ-tuples such that the four $\mathbb{F}_{q^4}$-points and the two $\mathbb{F}_{q^2}$-points lie on a $\mathbb{F}_q$-line, or,
- $\Delta_{l,2}^2$ consists of λ-tuples such that the two $\mathbb{F}_{q^4}$-points and the $\mathbb{F}_{q^2}$-point lie on a $\mathbb{F}_{q^2}$-line L (and the other two $\mathbb{F}_{q^4}$-points and the second $\mathbb{F}_{q^2}$-point lie on FL), or,
- $\Delta_{l,2}^3$ consists of λ-tuples such that the four $\mathbb{F}_{q^4}$-points and the two $\mathbb{F}_{q^2}$-points are intersection points of four conjugate $\mathbb{F}_{q^4}$-lines.

The sets $\Delta_{l,2}^2$ and $\Delta_{l,2}^3$ are illustrated in Figure 7.2 below. The cardinality of $\Delta_{l,2}^1$ is easily computed to be $(q^2+q+1)^2(q^4-q^2)(q^2-q)$. To get the cardinality of $\Delta_{l,2}^2$, we first choose a $\mathbb{F}_{q^2}$-line L in q^4-q ways and then a $\mathbb{F}_{q^4}$-point P_1 on L in q^4-q^2 ways. This determines all the four $\mathbb{F}_{q^4}$ points since they must be $P_2 = FP_1$, $P_3 = F^2P_1$ and $P_4 = F^3P_1$. We must now decide if P_5 should lie on L or FL. We then choose a $\mathbb{F}_{q^2}$-point on the chosen line. The lines L and FL both contain q^2+1 points defined over $\mathbb{F}_{q^2}$ of which precisely one is defined over $\mathbb{F}_q$ (namely the point $L \cap FL$). Hence, there are q^2 choices for P_5. It now only remains to choose P_7 in one of q^2+q+1 ways. We thus have

$$|\Delta_{l,2}^2| = 2(q^4-q)(q^4-q^2)q^2(q^2+q+1).$$

It remains to compute $|\Delta_{l,2}^3|$. We first choose a $\mathbb{F}_{q^2}$-point P_5 not defined over $\mathbb{F}_q$ in one of q^4-q ways. There are q^4-q^2 lines L strictly defined over $\mathbb{F}_{q^4}$ through P_5 and we choose one. We thus get four $\mathbb{F}_{q^4}$-lines which intersect in the two $\mathbb{F}_{q^2}$-points P_5 and P_6 as well as in four $\mathbb{F}_{q^4}$-points. We choose one of these to become P_1 and the labels of the other three points are then given. However, we could as well have chosen the line F^2L and ended up with the same four $\mathbb{F}_{q^4}$-points. We therefore must divide by 2. Finally, we choose any of the q^2+q+1 $\mathbb{F}_q$-points to become P_7. We thus have

$$|\Delta_{l,2}^3| = 2(q^4-q)(q^4-q^2)(q^2+q+1).$$



$\Delta_{l,2}^2$ $\Delta_{l,2}^3$

Figure 7.2: Illustration of elements of the sets $\Delta_{l,2}^2$ and $\Delta_{l,2}^3$.

The set $\Delta_{l,3}$ splits into two disjoint subsets

$$\Delta_{l,3} = \Delta_{l,3}^1 \cup \Delta_{l,3}^2,$$

where

- $\Delta_{l,3}^1$ consists of λ-tuples such that the four $\mathbb{F}_{q^4}$-points and the $\mathbb{F}_q$-point lie on a $\mathbb{F}_q$-line, or,

- $\Delta^2_{l,3}$ consists of λ-tuples such that there are two conjugate $\mathbb{F}_{q^2}$-lines intersecting in the $\mathbb{F}_q$-point, each $\mathbb{F}_{q^2}$-line containing two of the $\mathbb{F}_{q^4}$-points.

To compute $|\Delta^1_{l,3}|$ we first choose a $\mathbb{F}_q$-line L, then a conjugate quadruple and a $\mathbb{F}_q$-point on L and finally a conjugate pair of $\mathbb{F}_{q^2}$-points anywhere. Hence

$$|\Delta^1_{l,3}| = (q^2+q+1)(q^4-q^2)(q+1)(q^4-q).$$

To compute $|\Delta^2_{l,3}|$ we first choose a $\mathbb{F}_{q^2}$-line L not defined over $\mathbb{F}_q$, then a $\mathbb{F}_{q^4}$-point P_4 not defined over $\mathbb{F}_{q^2}$ on L and finally a pair of conjugate $\mathbb{F}_{q^2}$-points anywhere. We thus have

$$|\Delta^2_{l,3}| = (q^4-q)^2(q^4-q^2).$$

We now consider the intersection $\Delta_{l,2} \cap \Delta_{l,3}$. The decompositions above yield a decomposition

$$\Delta_{l,2} \cap \Delta_{l,3} = \bigcup_{i,j} \Delta^i_{l,2} \cap \Delta^j_{l,3}.$$

The intersection $\Delta^1_{l,2} \cap \Delta^1_{l,3}$ consists of configurations where all seven points lie on a $\mathbb{F}_q$-line. There are

$$(q^2+q+1)(q^4-q^2)(q^2-q)(q+1)$$

such λ-tuples. Both the intersections $\Delta^1_{l,2} \cap \Delta^2_{l,3}$ and $\Delta^2_{l,2} \cap \Delta^1_{l,3}$ are empty.

To compute the cardinality of $\Delta^2_{l,2} \cap \Delta^2_{l,3}$ we first choose a $\mathbb{F}_{q^2}$-line L in q^4-q ways and then a strict $\mathbb{F}_{q^4}$-point P_1 on L in q^4-q^2 ways. We must now decide if P_5 should lie on L or FL. We then choose a strict $\mathbb{F}_{q^2}$-point on the chosen line in q^2 ways. We are now sure to have a tuple in $\Delta^2_{l,2}$. To make sure that the tuple also lies in $\Delta^2_{l,3}$ we have no choice but to put P_7 at the intersection of L and FL. There are thus

$$2(q^4-q)(q^4-q^2)q^2$$

elements in the intersection $\Delta^2_{l,2} \cap \Delta^2_{l,3}$.

The intersection $\Delta^3_{l,2} \cap \Delta^1_{l,3}$ is empty so there is only one intersection remaining. As explained in the computation of $|\Delta^3_{l,2}|$, there are $2(q^4-q)(q^4-q^2)$ ways to obtain four strict $\mathbb{F}_{q^4}$-points and two strict $\mathbb{F}_{q^2}$-points which are the intersection points of four conjugate $\mathbb{F}_{q^4}$-lines. We now note that there are precisely six lines through pairs of points among the four $\mathbb{F}_{q^4}$-points. Four of these lines are of course the four $\mathbb{F}_{q^4}$-lines. The remaining two lines are defined over $\mathbb{F}_{q^2}$ and therefore intersect in a $\mathbb{F}_q$-point. To obtain a tuple in $\Delta^2_{l,3}$ we have no choice but to choose P_7 as this intersection point. Therefore, there are

$$2(q^4-q)(q^4-q^2),$$

elements in the intersection $\Delta^3_{l,2}\cap\Delta^2_{l,3}$.

The remaining computations are rather straightforward. One obtains

$$|\Delta_l| = q^{13}+5q^{12}-4q^{10}-5q^9-3q^8+2q^7+q^6+3q^5+q^4-q^3.$$

We now turn to Δ_c. We have that if six points of a conjugate λ-tuple lie on a smooth conic C, then the four $\mathbb{F}_{q^4}$-points and the two $\mathbb{F}_{q^2}$-points lie on C and C is defined over $\mathbb{F}_q$. Thus, the computation of $|\Delta_c|$ consists of choosing a smooth conic C defined over $\mathbb{F}_q$, choose a $\mathbb{F}_{q^4}$-point and a $\mathbb{F}_{q^2}$-point on C and finally choose a $\mathbb{F}_q$-point anywhere. We thus have

$$|\Delta_c| = (q^5-q^2)(q^4-q^2)(q^2-q)(q^2+q+1).$$

It remains to investigate the intersection $\Delta_l\cap\Delta_c$. Since a smooth conic cannot contain three collinear points, we only have nonempty intersection between Δ and the sets $\delta^2_{l,3}$ and $\Delta_{l,6}$.

To compute $|\Delta^2_{l,3}\cap\Delta_c|$ we first choose a smooth conic C, then a conjugate quadruple on C and finally a pair of conjugate $\mathbb{F}_{q^2}$-points on C. The $\mathbb{F}_q$-point is then uniquely defined as the intersection point of the two $\mathbb{F}_{q^2}$-lines through pairs of the four $\mathbb{F}_{q^4}$-points. We thus have

$$|\Delta^2_{l,3}\cap\Delta_c| = (q^5-q^2)(q^4-q^2)(q^2-q).$$

The same construction as above works for the intersection $\Delta_{l,6}\cap\Delta_c$ if we remember that we now have some choice for the $\mathbb{F}_q$-point since it can lie anywhere on the line through the two $\mathbb{F}_{q^2}$-points. We thus have

$$|\Delta_{l,6}\cap\Delta_c| = (q^5-q^2)(q^4-q^2)(q^2-q)(q+1).$$

The only thing that remains to compute is the cardinality of the triple intersection $\Delta^2_{l,3}\cap\Delta_{l,6}\cap\Delta_c$. We first assume that the $\mathbb{F}_q$-point is on the outside of a smooth conic C containing the other six points. We first choose C in q^5-q^2 ways. There are $\frac{1}{2}(q+1)q$ ways to choose two $\mathbb{F}_q$-points P and Q on C. Intersecting the tangents T_PC and T_QC gives a $\mathbb{F}_q$-point P_7 which will clearly lie on the outside of C. Hence, there are precisely $\frac{1}{2}(q+1)q$ ways to choose a $\mathbb{F}_q$-point on the outside of C.

We now want to choose a $\mathbb{F}_q$-line through P_7 intersecting C in two $\mathbb{F}_{q^2}$-points. There are $q+1$ $\mathbb{F}_q$-lines through P_7 of which two are tangents to C.

These tangent lines contain a $\mathbb{F}_q$-point of C each so there are $q-1$ remaining $\mathbb{F}_q$-points on C. Picking such a point gives a line through this point, P_7 and one further point on C. We thus see that exactly $\frac{1}{2}(q-1)$ of the $\mathbb{F}_q$-lines through P_7 intersect C in two $\mathbb{F}_q$-points. Hence, there are precisely

$$q+1-2-\frac{1}{2}(q-1)=\frac{1}{2}(q-1)$$

$\mathbb{F}_q$-lines through P_7 which intersect C in two $\mathbb{F}_{q^2}$-points. These points are clearly conjugate under F. We label one of them as P_5.

We shall now choose a conjugate pair of $\mathbb{F}_{q^2}$-lines through P_7 intersecting C in four $\mathbb{F}_{q^4}$-points. There are q^2-q conjugate pairs of $\mathbb{F}_{q^2}$-lines through P_7. No $\mathbb{F}_{q^2}$-line through P_7 is tangent to C so each $\mathbb{F}_{q^2}$-line through P_7 will intersect C in two points. The conic C contains q^2-q points which are defined over $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$. Picking such a point gives a line through this point and P_7 as well as one further $\mathbb{F}_{q^2}$-point not defined over $\mathbb{F}_q$. Thus, there are $\frac{1}{2}(q^2-q)$ lines obtained in this way. Typically, such a line will be defined over $\mathbb{F}_{q^2}$ but not $\mathbb{F}_q$. We saw above that the number of such lines which are defined over $\mathbb{F}_q$ is precisely $\frac{1}{2}(q-1)$. Thus, there are precisely

$$\frac{1}{2}(q^2-q)-\frac{1}{2}(q-1)=\frac{1}{2}(q^2-2q+1)$$

$\mathbb{F}_{q^2}$-lines, not defined over $\mathbb{F}_q$, which intersect C in two $\mathbb{F}_{q^4}$-points. Thus, the remaining

$$q^2-q-\frac{1}{2}(q^2-2q+1)=\frac{1}{2}(q^2-1) \tag{7.3.2}$$

$\mathbb{F}_{q^2}$-lines must intersect C in two $\mathbb{F}_{q^4}$-points. Picking such a line and labeling one of the points P_1 gives a configuration belonging to $\Delta_{l,3}^2\cap\Delta_{l,6}\cap\Delta_c$ and we thus see that there are

$$\frac{1}{2}(q^5-q^2)q(q+1)(q-1)(q^2-1)$$

such configurations with P_7 on the outside of C.

We now assume that P_7 is on the inside of C. We first choose C in q^5-q^2 ways. Since the number of $\mathbb{F}_q$-points is q^2+q+1 and $q+1$ of these lie on C the number of $\mathbb{F}_q$-points not on C is precisely q^2. We just saw that $\frac{1}{2}(q+1)q$ of these lie on the outside of C so there must be

$$q^2-\frac{1}{2}(q+1)q=\frac{1}{2}(q^2-q)$$

$\mathbb{F}_q$-points which lie on the inside of C.

Since P_7 now lies on the inside of C, every $\mathbb{F}_q$-line through P_7 will intersect C in two points. Exactly $\frac{1}{2}(q+1)$ will intersect C in two $\mathbb{F}_q$-points so the remaining $\frac{1}{2}(q+1)$ will intersect C in two conjugate $\mathbb{F}_{q^2}$-points. We pick such a pair of points and label one of them P_5.

We now choose a conjugate pair of $\mathbb{F}_{q^2}$-lines through P_7 intersecting C in a conjugate quadruple of $\mathbb{F}_{q^4}$-points. The number of $\mathbb{F}_{q^2}$-lines, not defined over $\mathbb{F}_q$, through P_7 is q^2-q. Two of these are tangent to C so, using ideas similar to those above, we see that

$$\frac{1}{2}(q^2-q-2)-\frac{1}{2}(q+1)+2=\frac{1}{2}(q^2-2q+1),$$

of these lines intersect C in points defined over $\mathbb{F}_{q^2}$. Hence, the remaining

$$q^2-q-\frac{1}{2}(q^2-2q+1)=\frac{1}{2}(q^2-1) \tag{7.3.3}$$

lines intersect C in two $\mathbb{F}_{q^4}$-points which are not defined over $\mathbb{F}_{q^2}$. If we pick one of these points to become P_1 we end up with a configuration in $\Delta_{l,3}^2\cap\Delta_{l,6}\cap\Delta_c$. We thus have

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q^2-1)$$

such configurations with P_7 on the inside of C. One may note that the expression above actually is the same as the expression when P_7 was on the outside, but this will not always be the case.

We thus have

$$|\Delta_l\cap\Delta_c|=q^{12}+q^{11}-4q^{10}-2q^9+3q^8+4q^7-4q^5+q^3,$$

and finally also

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right|=q^6-q^5-2q^4+q^3-2q^2+3.$$

7.3.7. The case $\lambda=[1^3,4]$ Throughout this section, λ shall mean the partition $[1^3,4]$. We shall assume that the Frobenius automorphism permutes the points P_1,P_2,P_3,P_4 according to (1234) and fixes the $\mathbb{F}_q$-points P_5, P_6 and P_7. Let U be the open subset of $(\mathbb{P}^2)^7$ consisting of septuples such that the last three points of the tuple are not collinear. In other words, we choose any conjugate quadruple and three $\mathbb{F}_q$-points which do not lie on a line.

We can decompose Δ_l into a disjoint as

$$\Delta_l=\Delta_{l,1}\cup\Delta_{l,2},$$

where $\Delta_{l,1}$ consists of septuples such that all four $\mathbb{F}_{q^4}$-points lie on a $\mathbb{F}_q$-line and $\Delta_{l,2}$ consists of septuples such that the $\mathbb{F}_{q^2}$-line through P_1 and P_3 intersects the $\mathbb{F}_{q^2}$-line through P_2 and P_4 in P_5, P_6 or P_7.

To compute $|\Delta_{l,1}|$ we simply choose a $\mathbb{F}_q$-line L, a conjugate quadruple on L and finally place the three $\mathbb{F}_q$-points in such a way that they do not lie on a line. This can be done in

$$(q^2+q+1)^2(q^4-q^2)(q^2+q)q^2$$

ways. To compute $|\Delta_{l,2}|$ we first choose P_5, P_6 or P_7 then a $\mathbb{F}_{q^2}$-line L not defined over $\mathbb{F}_q$ through this point. Finally, we choose a $\mathbb{F}_{q^4}$-point P_1 which is not defined over $\mathbb{F}_{q^2}$ on L and make sure that the final two $\mathbb{F}_q$-points are not collinear with the first. This can be done in

$$3(q^2+q+1)(q^2-q)(q^4-q^2)(q^2+q)q^2$$

ways. This gives that

$$|\Delta_l| = 4q^{12}+6q^{11}+q^{10}-4q^9-5q^8-q^7-q^5.$$

We now turn to investigate Δ_c. It is not hard to see that if six of the points lie on a smooth conic C, then the four $\mathbb{F}_{q^4}$-points must lie on that conic and C must be defined over $\mathbb{F}_q$. We thus choose a smooth conic C over $\mathbb{F}_q$ and a conjugate quadruple on C. Then we choose P_5, P_6 or P_7 to possibly not lie on C and place the other two on C. Finally, we place the final $\mathbb{F}_q$-point anywhere except on the line through the other two $\mathbb{F}_q$-points. This gives the number

$$3(q^5-q^2)(q^4-q^2)(q+1)q\cdot q^2.$$

However, we have now counted the configurations where all seven points lie on a smooth conic three times. There are

$$(q^5-q^2)(q^4-q^2)(q+1)q(q-1)$$

such configurations and it thus follows that

$$|\Delta_c| = 3q^{13}+q^{12}-3q^{11}-2q^{10}-q^9+q^8-q^7+2q^5.$$

We now turn to the intersection $\Delta_l \cap \Delta_c = \Delta_{l,1} \cap \Delta_c$. We begin by choosing one of the $\mathbb{F}_q$-points P_5, P_6 and P_7 to not lie on the conic C. We call the chosen point P and the remaining two points P_i and P_j where $i<j$. We now have three disjoint possibilities:

(*i*) the point P may lie on the outside of C with one of the tangents through P also passing through P_i,

(*ii*) the point P may lie on the outside of C with none of the tangents through P passing through P_i, (*iii*) the point P may lie on the inside of C.

We consider the three cases (*i*)-(*iii*) separately.

(i). We begin by choosing a smooth conic C in q^5-q^2 ways and a $\mathbb{F}_q$-point P on the outside of C in $\frac{1}{2}(q+1)q$ ways. By Equation 7.3.2, there are now q^2-1 ways to choose P_1. Since we require P_i to lie on a tangent to C which passes through P, we only have 2 choices for P_i. Finally, we may choose P_j as any of the q remaining points on C. We thus have

$$3(q^5-q^2)(q+1)q(q^2-1)q$$

possibilities in this case.

(ii). Again, we begin by choosing a smooth conic C in q^5-q^2 ways and a $\mathbb{F}_q$-point P on the outside of C in $\frac{1}{2}(q+1)q$ ways and choose P_1 in one of q^2-1 ways. The point P_i should now not lie on a tangent to C which passes through P so we have $q-1$ choices. Since the line between P and P_i is not a tangent to C, there is one further intersection point between this line and C. We must choose P_j away from this point and P_i and thus have $q-1$ possible choices. Hence, we have

$$\frac{3}{2}(q^5-q^2)(q+1)q(q^2-1)(q-1)^2$$

possibilities in this case.

(iii). We start by choosing a smooth conic C in q^5-q^2 ways and then a point P on the inside of C in $\frac{1}{2}(q^2-q)$ ways. We continue by using Equation 7.3.3 to see that we can choose P_1 in q^2-1 ways. We now choose P_i as any of the $\mathbb{F}_q$-points on C and thus have $q+1$ choices. Finally, we may choose P_j as any $\mathbb{F}_q$-point on C, except in the intersection of C with the line through P_i and P. Hence, we have $q-1$ choices. We thus have

$$\frac{3}{2}(q^5-q^2)(q^2-q)(q^2-1)(q+1)(q-1)$$

possibilities in this case.

We now conclude that

$$|\Delta_l\cap\Delta_c| = 3q^{11}-3q^9-3q^5+3q^3,$$

and, finally,

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6-q^5-2q^4+q^3-2q^2+3.$$

7.3.8. The case $\lambda = [1,3^2]$ Throughout this section, λ shall mean the partition $[1^1,3^2]$. We shall use the notation P_1, P_2, P_3 for the first conjugate triple and Q_1, Q_2, Q_3 for the second. The $\mathbb{F}_q$-point will be denoted by R. We take $U = \left(\mathbb{P}^2\right)^7$.

We decompose Δ_l as

$$\Delta_l = \bigcup_{i=1}^{5} \Delta_{l,i},$$

where

- $\Delta_{l,1}$ consists of λ-tuples such that the points P_1, P_2 and P_3 lie on a $\mathbb{F}_q$-line,
- $\Delta_{l,2}$ consists of λ-tuples such that the points P_1, P_2 and P_3 are the intersection points of a conjugate triple of $\mathbb{F}_{q^3}$-lines with each of the lines containing one of the points Q_1, Q_2 and Q_3,
- $\Delta_{l,3}$ consists of λ-tuples such that the points Q_1, Q_2 and Q_3 are the intersection points of a conjugate triple of $\mathbb{F}_{q^3}$-lines with each of the lines containing one of the points P_1, P_2 and P_3,
- $\Delta_{l,4}$ consists of λ-tuples such that the points Q_1, Q_2 and Q_3 lie on a $\mathbb{F}_q$-line, and
- $\Delta_{l,5}$ consists of λ-tuples such that the point R is the intersection of three conjugate $\mathbb{F}_{q^3}$-lines with each of the $\mathbb{F}_{q^3}$-lines containing one of the points P_1, P_2 and P_3 and one of the points Q_1, Q_2 and Q_3.

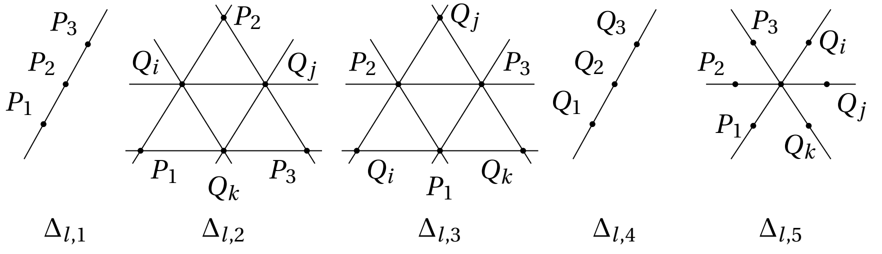


Figure 7.3: Illustration of the decomposition of Δ_l.

$\Delta_{l,1}$ and $\Delta_{l,4}$

The cardinalities of $\Delta_{l,1}$ and $\Delta_{l,4}$ are obviously the same so we only make the computation for $\Delta_{l,1}$. We thus choose a $\mathbb{F}_q$-line L, a conjugate $\mathbb{F}_{q^3}$-tuple P_1,

P_2, P_3 on L, a conjugate $\mathbb{F}_{q^3}$-tuple Q_1, Q_2, Q_3 anywhere except equal to the other $\mathbb{F}_{q^3}$-tuple and, finally, a $\mathbb{F}_q$-point anywhere. We thus get

$$|\Delta_{l,1}| = |\Delta_{l,4}| = (q^2+q+1)^2(q^3-q)(q^6+q^3-q^2-q-3).$$

$\Delta_{l,2}$ and $\Delta_{l,3}$

The cardinalities of $\Delta_{l,2}$ and $\Delta_{l,3}$ are of course also the same. To compute $|\Delta_{l,2}|$ we first choose a conjugate triple of lines, L, FL, F^2L, which do not intersect in a point. There are $q^6+q^3-q^2-q$ conjugate triples of lines of which $(q^2+q+1)(q^3-q)$ intersect in a point. There are thus $q^6-q^5-q^4+q^3$ possible triples. We give the label P_1 to the point $L\cap FL$ which determines the labels of the other two intersection points. We must now choose if Q_1 should lie on L, FL or F^2L and then place Q_1 on the chosen line. There are $3(q^3-1)$ ways to do this. Finally, we choose any $\mathbb{F}_q$-point. We thus have

$$|\Delta_{l,2}| = |\Delta_{l,3}| = 3(q^6-q^5-q^4+q^3)(q^3-1)(q^2+q+1).$$

$\Delta_{l,5}$

We first choose a $\mathbb{F}_q$-point R anywhere and then a conjugate triple of lines, L, FL, F^2L through R. We then choose a point P_1 somewhere on L in q^3 ways. We must now decide if Q_1 should lie on L, FL or F^2L and then pick a point on the chosen line in one of q^3-1 ways. We thus see that

$$|\Delta_{l,5}| = 3(q^2+q+1)(q^3-q)q^3(q^3-1).$$

We must now compute the cardinalities of the different intersections. Firstly, note that the intersection between $\Delta_{l,1}$ and $\Delta_{l,2}$ is empty. Secondly, the size of the intersection of $\Delta_{l,1}$ and $\Delta_{l,3}$ is equal to that of the intersection of $\Delta_{l,2}$ and $\Delta_{l,4}$. To obtain $|\Delta_{l,1}\cap\Delta_{l,3}|$ we first choose a conjugate triple of lines, L, FL, F^2L, which do not intersect in a point and label the intersection $L\cap FL$ by Q_1. We then choose a $\mathbb{F}_q$-line L' and thus get three $\mathbb{F}_{q^3}$-points $L'\cap L$, $L'\cap FL$ and $L'\cap F^2L$. We label one of these by P_1. We may now choose any $\mathbb{F}_q$-point to become R. We thus see that that

$$|\Delta_{l,1}\cap\Delta_{l,3}| = |\Delta_{l,2}\cap\Delta_{l,4}| = 3(q^6-q^5-q^4+q^3)(q^2+q+1)^2.$$

When we consider the intersection between $\Delta_{l,1}$ and $\Delta_{l,4}$ we must distinguish between the cases where the two triples lie on the same line and when they do not. A simple computation then gives

$$|\Delta_{l,1}\cap\Delta_{l,4}| = (q^2+q+1)^2(q^3-q)(q^3-q-3)+(q^2+q+1)^2(q^2+q)(q^3-q)^2.$$

We continue by observing that $|\Delta_{l,1} \cap \Delta_{l,5}| = |\Delta_{l,4} \cap \Delta_{l,5}|$. To compute $|\Delta_{l,1} \cap \Delta_{l,5}|$ we first choose a $\mathbb{F}_q$-point R and then a conjugate $\mathbb{F}_{q^3}$-tuple of lines L, FL, F^2L through R. We continue by choosing a $\mathbb{F}_q$-line L' not through R in one of q^2 ways and then label $L' \cap L$, $L' \cap FL$ or $L' \cap F^2L$ by Q_1. Finally, we choose one of the remaining $q^3 - 1$ points of L to become P_1. Hence,

$$|\Delta_{l,1} \cap \Delta_{l,5}| = |\Delta_{l,4} \cap \Delta_{l,5}| = 3(q^2+q+1)(q^3-q)q^2(q^3-1).$$

The sets $\Delta_{l,2}$ and $\Delta_{l,3}$ do not intersect and neither do the sets $\Delta_{l,3}$ and $\Delta_{l,4}$. Hence, there are only two intersections left, namely the one between $\Delta_{l,2}$ and $\Delta_{l,5}$ and the one between $\Delta_{l,3}$ and $\Delta_{l,5}$. These have equal cardinalities. To compute $|\Delta_{l,2} \cap \Delta_{l,5}|$ we first choose a conjugate triple of lines, L, FL, F^2L, which do not intersect in a point and label the intersection $L \cap FL$ by Q_1. We then choose a $\mathbb{F}_q$-point R. The lines between R and the points Q_1, Q_2 and Q_3 intersect the lines L, FL and F^2L in three points and we label one of them by P_1. We thus have

$$|\Delta_{l,2} \cap \Delta_{l,5}| = |\Delta_{l,3} \cap \Delta_{l,5}| = 3(q^6-q^5-q^4+q^3)(q^2+q+1).$$

There is only one triple intersection, namely between $\Delta_{l,1}$, $\Delta_{l,4}$ and $\Delta_{l,5}$. To compute the size of this intersection we first choose a $\mathbb{F}_q$-point R and then a conjugate triple of lines, L, FL, F^2L through R. We then choose a $\mathbb{F}_q$-line L' not through R and label the intersection $L \cap L'$ by P_1. Finally, we choose another $\mathbb{F}_q$-line L'' and label one of the intersections $L'' \cap L$, $L'' \cap FL$ and $L'' \cap F^2L$ by Q_1. This shows that

$$|\Delta_{l,1} \cap \Delta_{l,4} \cap \Delta_{l,5}| = 3(q^2+q+1)(q^3-q)q^2(q^2-1).$$

We now turn to the computation of $|\Delta_c|$. If six points of a λ-tuple lie on a smooth conic C, then both of the conjugate triples must lie on C and C must be defined over $\mathbb{F}_q$. Hence, to obtain $|\Delta_c|$ we only have to choose a smooth $\mathbb{F}_q$-conic C, two conjugate triples on C and a $\mathbb{F}_q$-point anywhere. We thus have that,

$$|\Delta_c| = (q^5-q^2)(q^3-q)(q^3-q-3)(q^2+q+1).$$

Since the sets $\Delta_{l,1}$, $\Delta_{l,2}$, $\Delta_{l,3}$ and $\Delta_{l,4}$ all require three of the $\mathbb{F}_{q^3}$-points to lie on a line, they will have empty intersection with Δ_c. This is however not true for the set $\Delta_{l,5}$. To obtain such a configuration we first choose a smooth conic C and a $\mathbb{F}_q$-point R. Now choose a $\mathbb{F}_{q^3}$-point P_1 on C which is not defined over $\mathbb{F}_q$ in $q^3 - q$ ways. Since both C and R are defined over $\mathbb{F}_q$ we know that any tangent to C which passes through R must either be defined over $\mathbb{F}_{q^2}$ (or $\mathbb{F}_q$). Hence, the line through R and P_1 will intersect C in P_1 and one point more. We label this point with Q_1, Q_2 or Q_3. We thus have

$$|\Delta_l \cap \Delta_c| = 3(q^5-q^2)q^2(q^3-q).$$

We now obtain

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6 - 2q^5 - 2q^4 - 8q^3 + 16q^2 + 10q + 21.$$

7.3.9. The case $\lambda = [2^2, 3]$ Throughout this section, λ shall mean the partition $[2^2, 3^1]$. We shall use the notation P_1, P_2, P_3 for the conjugate triple and Q_1, Q_2 and R_1, R_2 for the two conjugate pairs. We let $U = \left(\mathbb{P}^2\right)^7$.

We can decompose Δ_l as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of septuples such that the three $\mathbb{F}_{q^3}$-points lie on a line and $\Delta_{l,2}$ consists of septuples such that the four $\mathbb{F}_{q^2}$-points lie on a line.

We have

$$|\Delta_{l,1}| = (q^2 + q + 1)(q^3 - q)(q^4 - q)(q^4 - q - 2),$$

and

$$|\Delta_{l,2}| = (q^2 + q + 1)(q^2 - q)(q^2 - q - 2)(q^6 + q^3 - q^2 - q).$$

The cardinality of the intersection is easily computed to be

$$|\Delta_{l,1} \cap \Delta_{l,2}| = (q^2 + q + 1)^2 (q^3 - q)(q^2 - q)(q^2 - q - 2).$$

This allows us to compute $|\Delta_l|$.

We have that if six of the points of a λ-tuple lie on a smooth conic C, then all seven points lie on C and C is defined over $\mathbb{F}_q$. We thus have that Δ_c is disjoint from Δ_l. We also see that

$$|\Delta_c| = (q^5 - q^2)(q^3 - q)(q^2 - q)(q^2 - q - 2),$$

so,

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6 - q^5 - 2q^4 + 3q^3 + q^2 - 2q.$$

7.3.10. The case $\lambda = [1^2, 2, 3]$ Throughout this section, λ shall mean the partition $[1^2, 2^1, 3^1]$. We shall use the notation P_1, P_2, P_3 for the conjugate triple, Q_1, Q_2 for the conjugate pair and use R_6 and R_7 to denote the two $\mathbb{F}_q$-points. Let $U = \left(\mathbb{P}^2\right)^7$.

We decompose Δ_l as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of λ-tuples such that the three $\mathbb{F}_{q^3}$-points lie on a $\mathbb{F}_q$-line and $\Delta_{l,2}$ consists of λ-tuples such that the two $\mathbb{F}_{q^2}$-points and one of the $\mathbb{F}_q$-points lie on a $\mathbb{F}_q$-line.

We have

$$|\Delta_{l,1}| = (q^2+q+1)^2(q^3-q)(q^4-q)(q^2+q).$$

We decompose $\Delta_{l,2}$ as

$$\Delta_{l,2} = \Delta_{l,2}^6 \cup \Delta_{l,2}^7,$$

where $\Delta_{l,2}^i$ consists of tuples such that the line through the two $\mathbb{F}_{q^2}$-points passes through R_i. We have

$$|\Delta_{l,2}^6| = |\Delta_{l,2}^7| = (q^2+q+1)(q^2-q)(q+1)(q^6+q^3-q^2-q)(q^2+q).$$

We now turn to the double intersections. We have

$$|\Delta_{l,1} \cap \Delta_{l,2}^6| = |\Delta_{l,1} \cap \Delta_{l,2}^7| = (q^2+q+1)^2(q^3-q)(q^2-q)(q+1)(q^2+q),$$

and

$$|\Delta_{l,2}^6 \cap \Delta_{l,2}^7| = (q^2+q+1)(q^2-q)(q+1)q(q^6+q^3-q^2-q).$$

Finally, we compute the cardinality of the intersection of all three sets

$$|\Delta_{l,1} \cap \Delta_{l,2}^6 \cap \Delta_{l,2}^7| = (q^2+q+1)^2(q^3-q)(q^2-q)(q+1)q.$$

This now allows us to compute Δ_l.

If six points of a λ-tuple lie on a smooth conic C, then the three $\mathbb{F}_{q^3}$-points and the two $\mathbb{F}_{q^2}$-points lie on C and C is defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_c|$ we begin by choosing a smooth conic C over $\mathbb{F}_q$. We then choose a conjugate triple and a conjugate pair of $\mathbb{F}_{q^2}$-points on C. Then, we choose either R_6 or R_7 and place the chosen point on C. Finally, we place the remaining point anywhere we want. We thus obtain the number

$$2(q^5-q^2)(q^3-q)(q^2-q)(q+1)(q^2+q).$$

However, in the above we have counted the configurations where all seven points lie on the conic twice. We thus have to take away

$$(q^5-q^2)(q^3-q)(q^2-q)(q+1)q,$$

in order to obtain $|\Delta_c|$.

It only remains to compute the cardinality of the intersection $\Delta_l \cap \Delta_c$. We only have nonempty intersection between the set Δ_c and the set $\Delta_{l,2}$. To compute the cardinality of this intersection, we only have to make sure to choose the point R_6 (resp. R_7) on the line through the two $\mathbb{F}_{q^2}$-points. Hence, we have

$$|\Delta_{l,2}^6 \cap \Delta_c| = |\Delta_{l,2}^7 \cap \Delta_c| = (q^5-q^2)(q^3-q)(q^2-q)(q+1)^2,$$

and, therefore,

$$|\Delta_l \cap \Delta_c| = 2(q^5 - q^2)(q^3 - q)(q^2 - q)(q+1)^2.$$

This gives us

$$\left|\left(\mathcal{P}^2_7\right)^{F\sigma}\right| = q^6 - 3q^5 - q^4 + 5q^3 - 2q.$$

7.3.11. The case $\lambda = [1^4, 3^1]$ Throughout this section, λ shall mean the partition $[1^4, 3^1]$. We shall denote the four $\mathbb{F}_q$-points by P_1, P_2, P_3 and P_4 and denote the conjugate triple by Q_1, Q_2, Q_3. Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of points with the first four in general position.

A septuple in Δ_l will have the three $\mathbb{F}_{q^3}$-points on a $\mathbb{F}_q$-line. Thus, to compute the size of Δ_l, we only need to place the four $\mathbb{F}_q$-points in general position, choose $\mathbb{F}_q$-line L and place the conjugate $\mathbb{F}_{q^3}$-tuple on L. We thus have,

$$|\Delta_l| = (q^2 + q + 1)(q^2 + q)q^2(q^2 - 2q + 1)(q^2 + q + 1)(q^3 - q).$$

A septuple in Δ_c will have the three $\mathbb{F}_{q^3}$-points on a smooth conic C defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_c|$, we first choose a smooth conic C defined over $\mathbb{F}_q$ and then a conjugate triple on C. We then choose one of the points P_1, P_2, P_3 and P_4 to possibly not lie on C. Call this point P. We then place the other three points on C. These three points define three lines which, in total, contain $(q+1) + q + (q-1) = 3q$ points. As long as we choose P away from these points, the four $\mathbb{F}_q$-points will be in general position. We thus obtain

$$4(q^5 - q^2)(q^3 - q)(q+1)q(q-1)(q^2 - 2q + 1).$$

However, we have counted the septuples with all seven points on a smooth conic four times. We thus need to take away

$$3(q^5 - q^2)(q^3 - q)(q+1)q(q-1)(q-2).$$

Since Δ_l and Δ_c are disjoint we are done and conclude that

$$\left|\left(\mathcal{P}^2_7\right)^{F\sigma}\right| = q^6 - 5q^5 + 10q^4 - 5q^3 - 11q^2 + 10q.$$

7.3.12. The case $\lambda = [1, 2^3]$ Throughout this section, λ shall mean the partition $[1^1, 2^3]$. We shall denote the three conjugate pairs of $\mathbb{F}_{q^2}$-points by P_1, P_2, Q_1, Q_2 and R_1, R_2 and the $\mathbb{F}_q$-point by O. Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of points such that the first six points have no three on a line.

We decompose Δ_l as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2},$$

where $\Delta_{l,1}$ consists of those septuples where two conjugate $\mathbb{F}_{q^2}$-points and the $\mathbb{F}_q$-point lie on a $\mathbb{F}_q$-line and $\Delta_{l,2}$ consists of those septuples where two conjugate $\mathbb{F}_{q^2}$-lines, containing two $\mathbb{F}_{q^2}$-points each, intersect in the point defined over $\mathbb{F}_q$.

The set $\Delta_{l,1}$ naturally decomposes into three equally large, but not disjoint, subsets:

- the set $\Delta^a_{l,1}$ where P_1, P_2 and O lie on a $\mathbb{F}_q$-line,
- the set $\Delta^b_{l,1}$ where Q_1, Q_2 and O lie on a $\mathbb{F}_q$-line, and
- the set $\Delta^c_{l,1}$ where R_1, R_2 and O lie on a $\mathbb{F}_q$-line.

Similarly, the set $\Delta_{l,2}$ decomposes into six disjoint and equally large subsets:

- the two sets $\Delta^{P_1,Q_i}_{l,2}$ where the line through the points P_1 and Q_i also passes through the point O,
- the two sets $\Delta^{P_1,R_i}_{l,2}$ where the line through the points P_1 and R_i also passes through the point O, and
- the two sets $\Delta^{Q_1,R_i}_{l,2}$ where the line through the points Q_1 and R_i also passes through the point O.

The cardinalities of these sets are easily computed to be

$$|\Delta^a_{l,1}| = |\Delta^b_{l,1}| = |\Delta^c_{l,1}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8)(q + 1),$$

and

$$|\Delta^{P_1,Q_i}_{l,2}| = |\Delta^{P_1,R_i}_{l,2}| = |\Delta^{Q_1,R_i}_{l,2}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).$$

To compute $|\Delta^a_{l,1} \cap \Delta^b_{l,1}|$ we note that if we place the three pairs of $\mathbb{F}_{q^2}$-points such that no three lie on a line, then the line through P_1 and P_2 and the line through Q_1 and Q_2 will intersect in a $\mathbb{F}_q$-point. By choosing this point as O we obtain an element of $\Delta^a_{l,1} \cap \Delta^b_{l,1}$. We now see that

$$|\Delta^a_{l,1} \cap \Delta^b_{l,1}| = |\Delta^a_{l,1} \cap \Delta^c_{l,1}| = |\Delta^b_{l,1} \cap \Delta^c_{l,1}| = (q^4 - q)(q^4 - q^2)(q^4 - 6q^2 + q + 8).$$

To compute $|\Delta^a_{l,1} \cap \Delta^{Q_1,R_1}_{l,2}|$ we first choose a conjugate pair Q_1, Q_2 and then a conjugate pair of $\mathbb{F}_{q^2}$-points R_1, R_2 which do not lie on the line through Q_1 and Q_2. We now only have one choice for O. We choose a $\mathbb{F}_q$-line L

through O. There are two possibilities: either L will pass through the intersection point P of the line through Q_1 and R_2 and the line through Q_2 and R_1 or it will not. If L passes through P, then we have q^2-q possible choices for P_1 and P_2 on L. Otherwise, we only have q^2-q-2 choices. Hence

$$\begin{aligned}&|\Delta_{l,1}^{a}\cap\Delta_{l,2}^{Q_1,R_i}|=|\Delta_{l,1}^{b}\cap\Delta_{l,2}^{P_1,R_i}|=|\Delta_{l,1}^{c}\cap\Delta_{l,2}^{P_1,Q_i}|=\\&=(q^4-q)(q^4-q^2)(q^2-q)+(q^4-q)(q^4-q^2)q(q^2-q-2).\end{aligned}$$

The only nonempty triple intersection is $\Delta_{l,1}^{a}\cap\Delta_{l,1}^{b}\cap\Delta_{l,1}^{c}$. A computation very similar to the one for $|\Delta_{l,1}^{a}\cap\Delta_{l,2}^{Q_1,R_1}|$ gives

$$\begin{aligned}|\Delta_{l,1}^{a}\cap\Delta_{l,1}^{b}\cap\Delta_{l,1}^{c}|=2(q^4-q)(q^4-q^2)(q^2-q-2)+\\+(q^4-q)(q^4-q^2)(q-3)(q^2-q-4).\end{aligned}$$

This finishes the investigation of Δ_l.

We now turn to Δ_c. If six points of a λ-tuple lie on a smooth conic C, then the six $\mathbb{F}_{q^2}$-points lie on C and C is defined over $\mathbb{F}_q$. Since no three points of a smooth conic can lie on a line, we shall obtain an element of Δ_c simply by choosing a smooth conic C, three conjugate pairs on C and, finally, a $\mathbb{F}_q$-point anywhere. We thus have

$$|\Delta_c|=(q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4)(q^2+q+1).$$

We shall now compute the cardinality of the intersection between Δ_l and Δ_c. The intersections with the cases $\Delta_{l,1}^{a}$, $\Delta_{l,1}^{b}$, and $\Delta_{l,1}^{c}$ are easily handled: we simply choose a smooth conic with three conjugate pairs on it and then place O on the line through the right conjugate pair. We thus get

$$\begin{aligned}|\Delta_{l,1}^{a}\cap\Delta_c|=|\Delta_{l,1}^{b}\cap\Delta_c|=|\Delta_{l,1}^{c}\cap\Delta_c|=\\=(q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4)(q+1).\end{aligned}$$

The intersections with the sets $\Delta_{l,2}^{a_1,Q_i}$, $\Delta_{l,2}^{P_1,R_i}$ and $\Delta_{l,2}^{Q_1,R_i}$ are perhaps even simpler: once we have chosen our conic C and our conjugate pairs we have only one choice for O. Hence,

$$\begin{aligned}&|\Delta_{l,2}^{P_1,Q_i}\cap\Delta_c|=|\Delta_{l,2}^{P_1,R_i}\cap\Delta_c|=|\Delta_{l,2}^{Q_1,R_i}\cap\Delta_c|=\\&=(q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4).\end{aligned}$$

An analogous argument shows that

$$\begin{aligned}&|\Delta_{l,1}^{a}\cap\Delta_{l,1}^{b}\cap\Delta_c|=|\Delta_{l,1}^{a}\cap\Delta_{l,1}^{c}\cap\Delta_c|=|\Delta_{l,1}^{b}\cap\Delta_{l,1}^{c}\cap\Delta_c|=\\&=(q^5-q^2)(q^2-q)(q^2-q-2)(q^2-q-4).\end{aligned}$$

The remaining intersections are quite a bit harder than the previous ones. We consider $\Delta^{a}_{l,1} \cap \Delta^{Q_1,R_1}_{l,2} \cap \Delta_c$, but the other intersections of this type are completely analogous and have the same size.

We first consider the case when O is on the outside of C. There are $q+1$ lines through O. Of these, precisely two are tangents and $\frac{1}{2}(q-1)$ intersect C in $\mathbb{F}_q$-points. Thus, the remaining $\frac{1}{2}(q-1)$ lines will intersect C in two conjugate $\mathbb{F}_{q^2}$-points. We thus pick one of these lines and label one of the intersection points by P_1.

Picking a $\mathbb{F}_{q^2}$-point not defined over $\mathbb{F}_q$ on C will typically define a $\mathbb{F}_{q^2}$-line through O which is not defined over $\mathbb{F}_q$. However, some of these choices will give $\mathbb{F}_q$-lines and we saw above that the number of such $\mathbb{F}_q$-lines is $\frac{1}{2}(q-1)$. Thus, the number of $\mathbb{F}_{q^2}$-lines, not defined over $\mathbb{F}_q$, intersecting C in two $\mathbb{F}_{q^2}$-points is

$$\frac{1}{2}(q^2-q) - \frac{1}{2}(q-1) = \frac{1}{2}(q^2-2q+1).$$

We pick one such line, label one of the intersection points Q_1 and the other intersection point R_1. This gives us a configuration of the desired type. Hence, the number of tuples in $\Delta^{a}_{l,1} \cap \Delta^{Q_1,R_1}_{l,2} \cap \Delta_c$ with O on the outside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q^2-2q+1).$$

We now turn to the case when O is on the inside of C. Of the $q+1$ lines defined over $\mathbb{F}_q$ which pass through O, $\frac{1}{2}(q+1)$ will now intersect C in $\mathbb{F}_q$-points and equally many in conjugate $\mathbb{F}_{q^2}$-points. We thus pick a line that intersects C in two conjugate $\mathbb{F}_{q^2}$-points and label one of them by P_1.

We now want to pick a $\mathbb{F}_{q^2}$-line through O which is not defined over $\mathbb{F}_q$ and which intersects C in two $\mathbb{F}_{q^2}$-points that are not defined over $\mathbb{F}_q$. To obtain such a line we pick a $\mathbb{F}_{q^2}$-point which is not defined over $\mathbb{F}_q$ on C. However, two such points define tangents to C which pass through O and $\frac{1}{2}(q+1)$ of the lines obtained in this way are actually defined over $\mathbb{F}_q$. We thus have

$$\frac{1}{2}(q^2-q-2) - \frac{1}{2}(q+1) = \frac{1}{2}(q^2-2q-3)$$

choices. We pick such a line and label the intersection points by Q_1 and R_1. Hence, the number of tuples in $\Delta^{a}_{l,1} \cap \Delta^{Q_1,R_1}_{l,2} \cap \Delta_c$ with O on the inside of C is

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q^2-2q-3).$$

This finishes the computation of $|\Delta^{a}_{l,1} \cap \Delta^{Q_1,R_i}_{l,2} \cap \Delta_c|$, $|\Delta^{b}_{l,1} \cap \Delta^{P_1,R_i}_{l,2} \cap \Delta_c|$ and $|\Delta^{c}_{l,1} \cap \Delta^{P_1,Q_i}_{l,2} \cap \Delta_c|$.

The only remaining intersection is $\Delta^a_{l,1} \cap \Delta^b_{l,1} \cap \Delta^c_{l,1} \cap \Delta_c$ which we shall handle in a way similar to that above. Fortunately, much of the work has already been done. To start, if O is on the outside of C, then there are $\frac{1}{2}(q-1)$ lines though O which are defined over $\mathbb{F}_q$ and intersect C in conjugate pairs of $\mathbb{F}_{q^2}$-points. Thus, there are

$$(q-1)(q-3)(q-5)$$

ways to pick three lines and label the intersection points with P_1 and P_2, Q_1 and Q_2 and R_1 and R_2. Hence, the number of λ-tuples in $\Delta^a_{l,1} \cap \Delta^b_{l,1} \cap \Delta^c_{l,1} \cap \Delta_c$ with O on the outside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q-3)(q-5).$$

Similarly, if O lies on the inside of C we have seen that there are $\frac{1}{2}(q+1)$ lines through O which are defined over $\mathbb{F}_q$ and which intersect C in a pair of conjugate $\mathbb{F}_{q^2}$-points. Thus, there are

$$(q+1)(q-1)(q-3)$$

ways to pick three lines and label the intersection points with P_1 and P_2, Q_1 and Q_2 and R_1 and R_2. Hence, the number of λ-tuples in $\Delta^a_{l,1} \cap \Delta^b_{l,1} \cap \Delta^c_{l,1} \cap \Delta_c$ with O on the inside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q+1)(q-1)(q-3).$$

We finally obtain

$$\left|\left(\mathcal{P}^2_7\right)^{F\sigma}\right| = q^6-3q^5-6q^4+19q^3+6q^2-24q+7.$$

7.3.13. The case $\lambda = [1^3, 2^2]$ Throughout this section, λ shall mean the partition $[1^3, 2^2]$. We shall denote the $\mathbb{F}_q$-points by P_1, P_2 and P_3 and the two conjugate pairs of $\mathbb{F}_{q^2}$-points by Q_1, Q_2 and R_1, R_2. Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of points such that the first five points lie in general position.

The set Δ_l can be decomposed as

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2} \cup \Delta_{l,3},$$

where

- $\Delta_{l,1}$ consists of tuples such that the line through R_1 and R_2 also passes through P_1, P_2 or P_3,

- $\Delta_{l,2}$ consists of tuples such that the points R_1 and R_2 lie on the line through Q_1 and Q_2, and

- $\Delta_{l,3}$ consists of tuples such that a line through Q_1 and one of the points P_1, P_2 and P_3 also contains R_1 or R_2.

The set $\Delta_{l,1}$ decomposes as a union of the sets $\Delta^1_{l,1}$, $\Delta^2_{l,1}$ and $\Delta^3_{l,1}$ consisting of tuples with the line through R_1 and R_2 passing through P_1, P_2 and P_3, respectively. Similarly, the set $\Delta_{l,3}$ is the union of the six sets $\Delta^{i,j}_{l,3}$, $i = 1,2$, $j = 1,2,3$, where $\Delta^{i,j}_{l,3}$ contains all tuples such that Q_1, R_i and P_j lie on a line.

The cardinalities of the above sets are easily computed to be

$$|\Delta^i_{l,1}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q+1)(q^2-q),$$
$$|\Delta_{l,2}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^2-q-2),$$

and

$$|\Delta^{i,j}_{l,3}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^2-1).$$

The cardinality of $\Delta^i_{l,1} \cap \Delta^j_{l,1}$, $i \neq j$, is also easily computed:

$$|\Delta^i_{l,1} \cap \Delta^j_{l,1}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)(q^2-q).$$

There is only nonempty intersection between the set $\Delta^i_{l,1}$ and the set $\Delta^{j,k}_{l,3}$ if $k \neq i$. We then place the first five points in general position and choose a $\mathbb{F}_q$-line through P_k which does not pass through P_i in q ways. This gives a tuple of the desired form. We thus see that

$$|\Delta^i_{l,1} \cap \Delta^{j,k}_{l,3}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q)q.$$

We also have nonempty intersection between the sets $\Delta^{1,i}_{l,3}$ and the set $\Delta^{2,j}_{l,3}$ where $i \neq j$. Such a configuration is actually given by specifying the first five points in general position since we must then take R_1 as the intersection point of the line between Q_1 and P_i and the line between Q_2 and P_j and similarly for R_2. Hence,

$$|\Delta^{1,i}_{l,3} \cap \Delta^{2,j}_{l,3}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q).$$

Since the set $\Delta_{l,2}$ cannot intersect any of the other sets, because this would require Q_1 and Q_2 to lie on a line through one of the $\mathbb{F}_q$-points, it is now time to consider the triple intersections.

Since P_1, P_2 and P_3 do not lie on a line we have that the intersection of $\Delta^1_{l,1}$, $\Delta^2_{l,1}$ and $\Delta^3_{l,1}$ is empty. We thus only have two types of triple intersections, namely $\Delta^i_{l,1} \cap \Delta^j_{l,1} \cap \Delta^{r,s}_{l,3}$ and $\Delta^i_{l,1} \cap \Delta^{1,j}_{l,3} \cap \Delta^{2,k}_{l,3}$ where, of course, i, j and k are assumed to be distinct.

An element of $\Delta_{l,1}^{i} \cap \Delta_{l,1}^{j} \cap \Delta_{l,3}^{r,s}$ is specified by choosing the first five points in general position. The point R_r must then be chosen as the intersection point of the line between P_i and P_j and the line between Q_1 and P_s and similarly for FR_r. We thus have

$$|\Delta_{l,1}^{i} \cap \Delta_{l,1}^{j} \cap \Delta_{l,3}^{r,s}| = (q^2+q+1)(q^2+q)q^2(q^4-3q^3+3q^2-q).$$

To compute the cardinality of the intersection $\Delta_{l,1}^{i} \cap \Delta_{l,3}^{1,j} \cap \Delta_{l,3}^{2,k}$ we first choose two $\mathbb{F}_q$-points P_j and P_k. We then choose a conjugate pair of $\mathbb{F}_{q^2}$-lines through each of these points. The intersections of these lines give four $\mathbb{F}_{q^2}$-points which we only have one way to label with Q_1, Q_2, R_1 and R_2. We must now place the point P_i somewhere on the line L through R_1 and R_2. The line through P_i and P_k intersects L in one $\mathbb{F}_q$-point and the line through Q_1 and Q_2 intersects L in another. Thus, we have $q-1$ choices for P_i. We thus see that

$$|\Delta_{l,1}^{i} \cap \Delta_{l,3}^{1,j} \cap \Delta_{l,3}^{2,s}| = (q^2+q+1)(q^2+q)(q^2-q)^2(q-1).$$

This completes the investigation of Δ_l.

If a smooth conic C contains six of the points, then C contains both the conjugate $\mathbb{F}_{q^2}$-pairs and C is defined over $\mathbb{F}_q$. Thus, to compute $|\Delta_c|$ we first choose a smooth conic C over $\mathbb{F}_q$ and then pick one of the points P_1, P_2 and P_3 to possibly lie outside C. We call the chosen point P. We then place the other two points and the two $\mathbb{F}_{q^2}$-pairs on C. Finally, we must place P somewhere to make P_1, P_2, P_3, Q_1 and Q_2 lie in general position. Hence, we must choose P away from the line through the two other $\mathbb{F}_q$-points and away from the line through Q_1 and Q_2. This gives us

$$(q^5-q^2)(q+1)q(q^2-q)(q^2-q-2)(q^2-q).$$

However, in the above we have counted the configurations where all seven points lie on C three times. We must therefore take away

$$2\cdot(q^5-q^2)(q+1)q(q-1)(q^2-q)(q^2-q-2)$$

in order to obtain $|\Delta_c|$.

The intersection $\Delta_{l,2} \cap \Delta_c$ is empty but the intersections of Δ_c with the other sets in the decomposition of Δ_l are not. To compute $|\Delta_{l,1}^{i} \cap \Delta_c|$ we shall first assume that P_i lies on the outside of C. Of the $q+1$ lines through P_i which are defined over $\mathbb{F}_q$ we have that 2 are tangent to C and $\frac{1}{2}(q-1)$ intersect C in two $\mathbb{F}_q$-points. Thus, there are $\frac{1}{2}(q-1)$ lines left which must intersect C in a pair of conjugate $\mathbb{F}_{q^2}$-points. We pick such a line and label the intersection points by R_1 and R_2 in one of two ways. We shall now place the other two $\mathbb{F}_q$-points on C. There are $\frac{1}{2}(q+1)q$ ways to choose two $\mathbb{F}_q$-points on C of which $\frac{1}{2}(q-1)$ pairs lie on a $\mathbb{F}_q$-line through P_i. There are thus $\frac{1}{2}(q^2+1)$ pairs which do not lie on a line through P_i and, since there are two ways to label each pair, we thus have q^2+1 choices for the two $\mathbb{F}_q$-points. Finally, we shall place Q_1 and Q_2 somewhere on C but we have to make sure that the points P_1, P_2, P_3, Q_1 and Q_2 are in general position. Since the lines between P_i and the other two $\mathbb{F}_q$-points intersect C only in $\mathbb{F}_q$-points, the only thing that might go wrong when choosing Q_1 and Q_2 is that the line through Q_1 and Q_2 might also go through P_i. As seen above, there are exactly $q-1$ choices for Q_1 and Q_2 for which this happens, so the remaining $q^2-q-(q-1)=q^2-2q+1$ choices will give a configuration of the desired type. We thus have that the number of elements in $\Delta^i_{l,1}\cap\Delta_c$ such that P_i lies on the outside of C is

$$\frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q^2+1)(q^2-2q+1).$$

We now assume that P_i lies on the inside of C. We proceed similarly to the above. First we observe that the number of $\mathbb{F}_q$-lines through P_i is $q+1$ of which half intersect C in two $\mathbb{F}_q$-points and half intersect C in conjugate pairs of $\mathbb{F}_{q^2}$-points. We choose a line which intersects C in two conjugate $\mathbb{F}_{q^2}$-points and label the intersection points by R_1 and R_2. We now choose a $\mathbb{F}_q$-point P_j on C in one of $q+1$ ways. The line through P_i and P_j intersects C in another $\mathbb{F}_q$-point and we choose the final $\mathbb{F}_q$-point away from this intersection point and P_j. Finally, we shall place the points Q_1 and Q_2 on C in a way so that the points P_1, P_2, P_3, Q_1 and Q_2 are in general position. As above, the only thing that might go wrong is that the line through Q_1 and Q_2 might go through P_i and there are precisely $q+1$ choices for Q_1 and Q_2 for which this happens. Thus, there are $q^2-q-(q+1)=q^2-2q-1$ valid choices for Q_1 and Q_2. Hence, there are

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q+1)(q-1)(q^2-2q-1)$$

elements in $\Delta^i_{l,1}\cap\Delta_c$ such that P_i lies on the inside of C.

To compute the intersection $\Delta^{i,j}_{l,3}\cap\Delta_c$ we note that if we place P_j outside of C and then choose two $\mathbb{F}_q$-points on C and two conjugate $\mathbb{F}_{q^2}$-points Q_1 and Q_2 on C such that P_1, P_2, P_3, Q_1 and Q_2 are in general position, then we must choose R_i as the other intersection point of C with the line through Q_1 and P_j. We may thus use constructions analogous to those above to see that there are

$$\frac{1}{2}(q^5-q^2)(q+1)q(q^2+1)(q^2-2q+1)$$

elements in $\Delta_{l,1}^{i,j} \cap \Delta_c$ with P_j on the outside of C and

$$\frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q-1)(q^2-2q-3)$$

elements with P_j on the inside of C.

We may now put all the pieces together to obtain

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6-7q^5+10q^4+15q^3-26q^2-8q+15.$$

7.3.14. The case $\lambda = [1^5,2]$ Throughout this section, λ shall mean the partition $[1^5,2]$. We shall denote the $\mathbb{F}_q$-points by P_1, P_2, P_3, P_4 and P_5 and the points of the conjugate pair of $\mathbb{F}_{q^2}$-points by Q_1 and Q_2. Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of points such that the first five points lie in general position.

If three points of a conjugate λ-tuple in $U(\lambda)$ lie on a line, then Q_1 and Q_2 lie on a line passing through one of the $\mathbb{F}_q$-points. There are

$$(q+1)+q+(q-1)+(q-2)+(q-3) = 5q-5,$$

$\mathbb{F}_q$-lines passing through P_1, P_2, P_3, P_4 or P_5 (or possibly two of them). Each of these lines contains q^2-q conjugate pairs and no conjugate pair lies on two such lines. We thus have

$$|\Delta_l| = (q^2+q+1)(q^2+q)q^2(q^2-2q+1)(q^2-5q+6)(5q-5)(q^2-q).$$

If six of the points of a conjugate λ-tuple lie on a smooth conic C, then C is defined over $\mathbb{F}_q$ and contains Q_1 and Q_2. Therefore, to compute the cardinality of Δ_c we first choose a smooth conic C defined over $\mathbb{F}_q$ and one of the points P_1, P_2, P_3, P_4 or P_5 to possibly lie outside C. We call the chosen point P. Then, we choose four $\mathbb{F}_q$-points and a conjugate pair on C. Finally, we choose P away from the six lines through pairs of the other four $\mathbb{F}_q$-points. We thus get

$$5(q^5-q^2)(q+1)q(q-1)(q-2)(q^2-q)(q^2-5q+6).$$

In the above we have counted the λ-tuples with all seven points on a conic five times. We therefore must take away

$$4(q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q^2-q),$$

in order to obtain $|\Delta_c|$.

To compute the size of the intersection $\Delta_l \cap \Delta_c$ we shall decompose this set into a disjoint disjoint union of five subsets A_i, $i=1,\dots,5$, where A_i consists of those tuples where P_i does not lie on the conic C through the other six points. Each of the sets A_i is then decomposed further into a union of the sets A_i^{out} and A_i^{in} where A_i^{out} consists of those tuples with P_i on the outside of C and A_i^{in} consists of those with P_i on the inside of C. Finally, we shall decompose A_i^{out} into a union of the three disjoint subsets:

- the set $A_{i,0}^{\text{out}}$ consisting of λ-tuples such that the tangent lines to C passing through P_i do not pass through any of the other points of the λ-tuple,
- the set $A_{i,1}^{\text{out}}$ consisting of λ-tuples such that exactly one of the tangent lines to C passing through P_i pass through one of the other points of the λ-tuple,
- the set $A_{i,2}^{\text{out}}$ consisting of λ-tuples such that both the tangent lines to C passing through P_i passes through another point of the λ-tuple.

To compute $|A_i^{\text{out}}|$, we first choose a smooth conic C defined over $\mathbb{F}_q$ in $q^5 - q^2$ ways and then a point P_i outside C in $\frac{1}{2}(q+1)q$ ways. As seen many times before, there are exactly $\frac{1}{2}(q-1)$ lines through P_i which are defined over $\mathbb{F}_q$ and which intersect C in a conjugate pair of points. We pick such a line and label the points Q_1 and Q_2 in one of two ways. From this point on, the computations are a little bit different for the three subsets of A_i^{out}.

The subset $A_{i,0}^{\text{out}}$. We shall now pick the other four $\mathbb{F}_q$-points of the λ-tuple. Since we should not pick points whose tangents pass through P_i, we have $q-1$ choices for the first point. For the second point, we should stay away from the tangent points, the first point and the other intersection point of C and the line through P_i and the first point. Hence, we have $q-3$ choices. In a similar way, we see that we have $q-5$ choices for the third point and $q-7$ for the fourth. Hence,

$$|A_{i,0}^{\text{out}}| = \frac{1}{2}(q^5 - q^2)(q+1)q(q-1)(q-1)(q-3)(q-5)(q-7).$$

The subset $A_{i,1}^{\text{out}}$. We begin by choosing one of the four $\mathbb{F}_q$-points to lie on a tangent to C passing through P_i and then we pick the tangent it should lie on. For the first of the remaining three points we now have $q-1$ choices and, similarly to the above case, we have $q-3$ choices for the second and $q-5$ for the third. Thus,

$$|A_{i,1}^{\text{out}}| = 4 \cdot 2 \cdot \frac{1}{2}(q^5 - q^2)(q+1)q(q-1)(q-1)(q-3)(q-5).$$

The subset $A_{i,2}^{\text{out}}$. We begin by choosing two of the four $\mathbb{F}_q$-points to lie on tangents to C passing through P_i and then we pick which point should lie on which tangent. For the first of the remaining two points we now have $q-1$ choices and we then have $q-3$ choices for the second. Thus,

$$|A_{i,2}^{\text{out}}| = \binom{4}{2} \cdot 2 \cdot \frac{1}{2}(q^5-q^2)(q+1)q(q-1)(q-1)(q-3).$$

It remains to compute $|A_i^{\text{in}}|$. We first choose a smooth conic C defined over $\mathbb{F}_q$ in q^5-q^2 ways and then a point P_i on the inside of C in $\frac{1}{2}(q^2-q)$ ways. We have already seen that there now are $\frac{1}{2}(q+1)$ lines passing through P_i which are defined over $\mathbb{F}_q$ and which intersect C in a conjugate pair of points. We thus pick such a line and label the intersection points by Q_1 and Q_2. Since any $\mathbb{F}_q$-line through P_i will intersect C in precisely two points, we have $(q+1)(q-1)(q-3)(q-5)$ choices for the remaining four $\mathbb{F}_q$-points of the λ-tuple. We thus see that

$$|A_i^{\text{in}}| = \frac{1}{2}(q^5-q^2)(q^2-q)(q+1)(q-1)(q-3)(q-5).$$

We now conclude that

$$\left|\left(\mathcal{P}_7^2\right)^{F\sigma}\right| = q^6 - 15q^5 + 90q^4 - 265q^3 + 374q^2 - 200q + 15.$$

7.3.15. The case $\lambda = [1^7]$ Throughout this section, λ shall mean the partition $[1^7]$. Since we shall almost exclusively be interested in objects defined over $\mathbb{F}_q$, we shall often omit the decoration "$\mathbb{F}_q$". For instance, we shall simply write "point" to mean "$\mathbb{F}_q$-point". Let $U \subset (\mathbb{P}^2)^7$ be the subset consisting of septuples of points such that the first four points lie in general position. We thus have

$$|U(\lambda)| = (q^2+q+1)(q^2+q)q^2(q^2-2q+1)(q^2+q-3)(q^2+q-4)(q^2+q-5).$$

The following notation will be quite convenient.

Definition 7.3.12. If P and Q are two points in $\mathbb{P}^2$, then the line through P and Q shall be denoted PQ.

Since we shall often want to stay away from lines through two of the first four points we define

$$\mathscr{S} = \bigcup_{1\le i<j\le 4} P_iP_j.$$

We note that $\mathscr{S}$ contains

$$6(q-2)+4+3 = 6q-5$$

points.

The set Δ_l

The set Δ_l decomposes into a disjoint union of three sets

$$\Delta_l = \Delta_{l,1} \cup \Delta_{l,2} \cup \Delta_{l,3},$$

where

- the points of $\Delta_{l,1}$ are such that at least one of the points P_5, P_6 or P_7 lies in $\mathscr{S}$,
- the points of $\Delta_{l,2}$ are such that one of the lines P_iP_j, $5 \le i < j \le 7$, contains one of the points P_1, P_2, P_3 and P_4, but $\{P_5, P_6, P_7\} \cap \mathscr{S} = \emptyset$, and
- the points of $\Delta_{l,3}$ are such that the three points P_5, P_6 and P_7 lie on a line which does not pass through P_1, P_2, P_3 or P_4.

We shall consider the three subsets separately.

The set $\Delta_{l,1}$. For each subset $I \subset \{5,6,7\}$, let $\Delta_{l,1}(I)$ denote the set of points in $\Delta_{l,1}$ such that $P_i \in \mathscr{S}$ for all $i \in I$. We can then decompose $\Delta_{l,1}$ further as

$$\Delta_{l,1} = \Delta_{l,1}(\{5\}) \cup \Delta_{l,1}(\{6\}) \cup \Delta_{l,1}(\{7\}).$$

Clearly, $\Delta_{l,1}(\{i\}) \cap \Delta_{l,1}(\{j\}) = \Delta_{l,1}(\{i,j\})$.

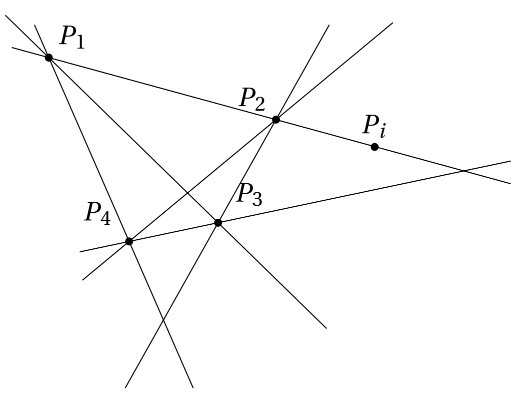


Figure 7.4: A typical element of $\Delta_{l,1}(\{i\})$.

A typical element of $\Delta_{l,1}(\{i\})$ is illustrated in Figure 7.4 above. To compute $|\Delta_{l,1}(\{i\})|$ we first place the first four points in general position, then choose P_i as any point in $\mathscr{S}$ and finally place the remaining two points anywhere. Hence

$$|\Delta_{l,1}(\{i\})| = \underbrace{(q^2+q+1)(q^2+q)q^2(q^2-2q+1)}_{|\mathrm{PGL}(3)|}(6q-9)(q^2+q-4)(q^2+q-5).$$

Similarly, we have

$$|\Delta_{l,1}(\{i,j\})| = |\mathrm{PGL}(3)| \cdot (6q-9)(6q-10)(q^2+q-5),$$

and

$$|\Delta_{l,1}(\{5,6,7\})| = |\mathrm{PGL}(3)| \cdot (6q-9)(6q-10)(6q-11).$$

This allows us to compute $|\Delta_{l,1}|$ as

$$|\Delta_{l,1}| = |\mathrm{PGL}(3)| \cdot (18q^5 - 99q^4 + 252q^3 - 414q^2 + 417q - 180).$$

The set $\Delta_{l,2}$. Let $\{i,j\} \in \{5,6,7\}$, $r \in \{1,2,3,4\}$ and let $\Delta^r_{l,2}(\{i,j\})$ be the subset of points in $\Delta_{l,2}$ such that $P_iP_j \cap \{P_1,P_2,P_3,P_4\} = \{P_r\}$. We also define

$$\Delta_{l,2}(\{i,j\}) = \bigcup_{r=1}^{4} \Delta^r_{l,2}(\{i,j\}).$$

A typical element of $\Delta^r_{l,2}(\{i,j\})$ is illustrated in Figure 7.5. To obtain an element of $\Delta^r_{l,2}(\{i,j\})$ we first place P_1, P_2, P_3 or P_4 in general position. There are $q+1$ lines through P_r of which 3 are contained in $\mathscr{S}$. We choose P_iP_j as one of the remaining $q-2$ lines. Note that P_iP_j will not pass through any of the points

$$Q_1 = P_1P_4 \cap P_2P_3, \quad Q_2 = P_2P_4 \cap P_1P_3, \quad Q_3 = P_3P_4 \cap P_1P_2. \tag{7.3.4}$$

Hence, P_iP_j will intersect $\mathscr{S}$ in P_r and three further points. There are thus $q-3$ ways to choose P_i and then $q-4$ ways to choose P_j. Finally, there are

$$|\mathbb{P}^2 \setminus \mathscr{S}| - 2 = q^2 + q + 1 - (6q-5) - 2 = q^2 - 5q + 4,$$

choices for the seventh point. We thus have

$$|\Delta^r_{l,2}(\{i,j\})| = |\mathrm{PGL}(3)| \cdot (q-2)(q-3)(q-4)(q^2-5q+4).$$

We have counted some tuples several times. To begin with, the points of

$$\Delta^r_{l,2}(\{5,6\}) \cap \Delta^r_{l,2}(\{5,7\}) \cap \Delta^r_{l,2}(\{6,7\}),$$

have been counted three times. There are

$$|\mathrm{PGL}(3)| \cdot (q-2)(q-3)(q-4)(q-5),$$

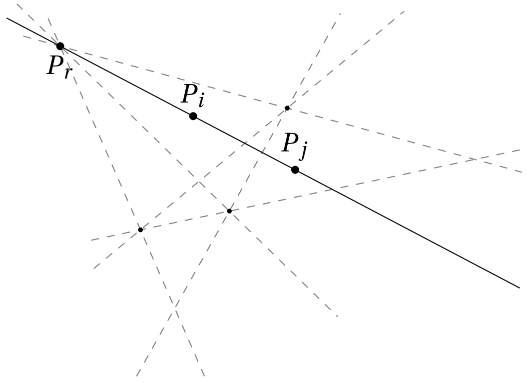


Figure 7.5: A typical element of $\Delta^r_{l,2}(\{i,j\}$.

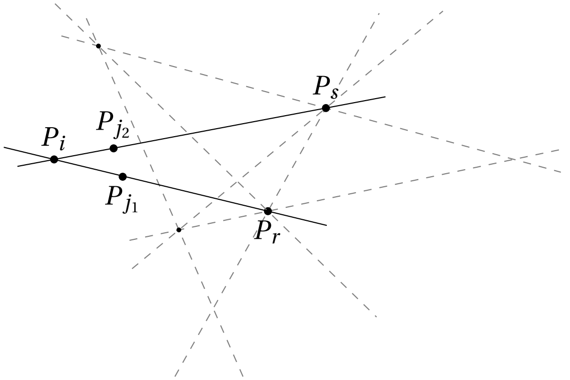


Figure 7.6: A typical element of $\Delta^r_{l,2}(\{i,j\}) \cap \Delta^s_{l,2}(\{i,k\})$.

of these.

Further, the sets $\Delta^r_{l,2}(\{i,j\})$ and $\Delta^s_{l,2}(\{i,k\})$ will intersect if $r \neq s$ and $j \neq k$. A typical element is illustrated in Figure 7.6.

To compute $|\Delta^r_{l,2}(\{i,j\}) \cap \Delta^s_{l,2}(\{i,k\})|$ we begin by choosing P_1, P_2, P_3 and P_4 in general position and continue by choosing P_i outside $\mathscr{S}$ in q^2-5q+6 ways. This gives us two lines P_iP_r and P_iP_s which intersect $\mathscr{S}$ in four points each. We choose P_j on P_iP_r away from P_i and $\mathscr{S}$ in $q-4$ ways and similarly for P_k. This gives

$$|\Delta^r_{l,2}(\{i,j\}) \cap \Delta^s_{l,2}(\{i,k\})| = |\mathrm{PGL}(3)| \cdot (q^2-5q+6)(q-4)^2.$$

Finally, we must compute the cardinality of the triple intersection

$$\Delta^r_{l,2}(\{5,6\}) \cap \Delta^s_{l,2}(\{5,7\}) \cap \Delta^t_{l,2}(\{6,7\}),$$

where r, s and t are distinct. A typical element of the intersection is illustrated in Figure 7.7.

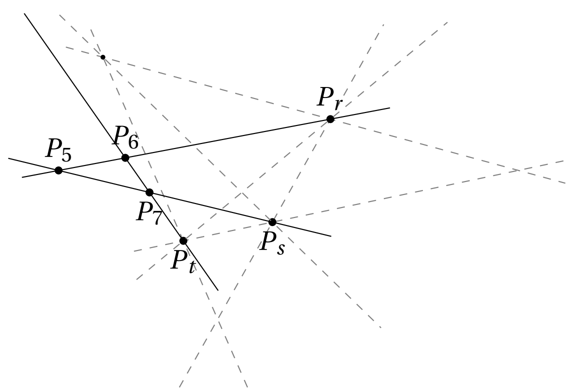


Figure 7.7: A typical element of $\Delta^r_{l,2}(\{5,6\}) \cap \Delta^s_{l,2}(\{5,7\}) \cap \Delta^t_{l,2}(\{6,7\})$.

This is where we have to pay for the awkward requirement that P_5, P_6 and P_7 should not be in $\mathscr{S}$. We shall view $\Delta^r_{l,2}(\{5,6\}) \cap \Delta^s_{l,2}(\{5,7\}) \cap \Delta^t_{l,2}(\{6,7\})$ as an open subset of the set $T^{r,s,t}$ consisting of tuples such that

- the line P_5P_6 passes through P_r, P_5P_7 passes through P_s and P_6P_7 passes through P_t but,
- we allow P_5, P_6 and P_7 to lie in $\mathscr{S}$, but,
- we do not allow the lines P_iP_j, $5 \leq i < j \leq 7$ to be contained in $\mathscr{S}$.

The complement of $\Delta^r_{l,2}(\{5,6\}) \cap \Delta^s_{l,2}(\{5,7\}) \cap \Delta^t_{l,2}(\{6,7\})$ in $T^{r,s,t}$ can be decomposed into a union of three subsets $T_i^{r,s,t}$, $i = 5,6,7$, consisting of those tuples with P_i in $\mathscr{S}$.

We begin with the computation of $|T^{r,s,t}|$. To obtain such a tuple, we begin by choosing a line L_r through P_r in $q-2$ ways. We shall then choose a line L_s through P_s. There are however two cases that may occur. Typically, the intersection point $P_5 = L_r \cap L_s$ will lie outside $\mathscr{S}$ but for one choice of L_s it will lie in $\mathscr{S}$. The situation is illustrated in Figure 7.8.

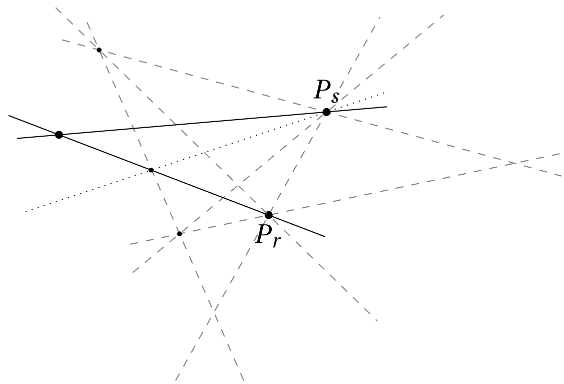


Figure 7.8

There are $q-3$ ways to choose L_s so that $L_r \cap L_s$ lies outside $\mathscr{S}$. When we choose the line L_t through P_t we must make sure that L_t is not contained in $\mathscr{S}$ and that L_t does not pass through $L_r \cap L_s$, since we want to end up with three distinct intersection points. We thus have $q-3$ choices. On the other hand, if we choose L_s as the one line making the intersection point $L_r \cap L_s$ lie in $\mathscr{S}$ we only need to make sure that L_t is not contained in $\mathscr{S}$ and we thus have $q-2$ choices. Hence, we see that

$$|T^{r,s,t}| = |\mathrm{PGL}(3)| \cdot \left((q-2)(q-3)^2 + (q-2)^2\right).$$

We now turn to the computation of $|T_i^{r,s,t}|$, $i = 5,6,7$. We then begin by choosing a line L_r through P_r in $q-2$ ways. The line L_s through P_s is then completely determined since we must have $P_i \in \mathscr{S}$. This gives us $q-2$ choices for the final line L_t through P_t. Hence,

$$|T_i^{r,s,t}| = |\mathrm{PGL}(3)| \cdot (q-2)^2.$$

We now turn to the computation of $|T_i^{r,s,t} \cap T_j^{r,s,t}|$, $5 \le i < j \le 7$. As above, we begin by choosing a line L_r through P_r in $q-2$ ways. Since P_i must lie in $\mathscr{S}$ we have only one choice for L_s. Since $P_j = L_s \cap L_t$ we see that we now have precisely one choice for L_t also. Hence,

$$|T_i^{r,s,t} \cap T_j^{r,s,t}| = |\mathrm{PGL}(3)| \cdot (q-2).$$

We now consider $T_5^{r,s,t} \cap T_6^{r,s,t} \cap T_7^{r,s,t}$. It turns out that once the four points P_1, P_2, P_3 and P_4 have been placed in general position, there is precisely one such tuple. The situation is illustrated in Figure 7.9.

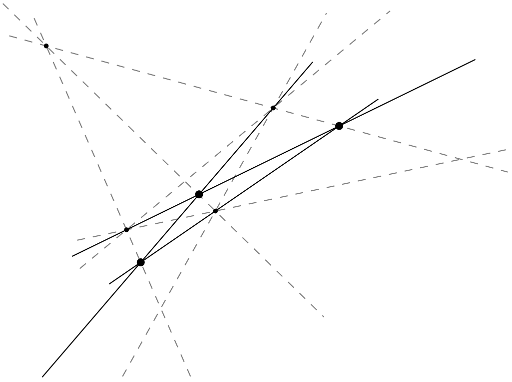


Figure 7.9: The only element in $T_5^{r,s,t} \cap T_6^{r,s,t} \cap T_7^{r,s,t}$.

This finally allows us to compute

$$\Delta_{l,2} = |\mathrm{PGL}(3)| \cdot (12q^5 - 212q^4 + 1504q^3 - 5320q^2 + 9296q - 6360).$$

The set $\Delta_{l,3}$. Recall the definition of the three points Q_1, Q_2 and Q_3 from Equation 7.3.4. Using these three points we may decompose $\Delta_{l,3}$ into a disjoint union of the following subsets:

- $\Delta_{l,3}(\{Q_r, Q_s\})$ consisting of those tuples of $\Delta_{l,3}$ where P_5, P_6 and P_7 lie on the line Q_rQ_s, $1 \le r < s \le 3$, and,
- $\Delta_{l,3}(\{Q_r\})$ consisting of those tuples of $\Delta_{l,3}$ with P_5, P_6 and P_7 on a line through Q_r, $1 \le r \le 3$, which does not pass through any of the other Q_i, and
- $\Delta_{l,3}(\emptyset)$ consisting of those tuples of $\Delta_{l,3}$ with P_5, P_6 and P_7 on a line which does not pass through Q_1, Q_2 or Q_3.

We begin by considering $\Delta_{l,3}(\{Q_r, Q_s\})$. The line Q_rQ_s contains $q+1$ points of which four lie in $\mathscr{S}$. There are thus $q-3$ choices for P_5, $q-4$ choices for P_6 and $q-5$ choices for P_7. Hence,

$$|\Delta_{l,3}(\{Q_r, Q_s\})| = |\mathrm{PGL}(3)| \cdot (q-3)(q-4)(q-5).$$

We continue with $|\Delta_{l,3}(\{Q_r\})|$. There are $q+1$ lines through Q_r of which two are contained in $\mathscr{S}$ and two are the lines through the other two Q_i. Hence, there are $q-3$ choices for a line L though Q_r. The line L intersects $\mathscr{S}$ in five points so we have $q-4$ choices for P_5, $q-5$ choices for P_6 and $q-6$ choices for P_7. We conclude that

$$|\Delta_{l,3}(\{Q_r\})| = |\mathrm{PGL}(3)| \cdot (q-3)(q-4)(q-5)(q-6).$$

To compute $|\Delta_{l,3}(\emptyset)|$ we begin by choosing a line L which does not pass through any of the points P_1, P_2, P_3, P_4, Q_1, Q_2 and Q_3. There are q^2+q+1 lines in $\mathbb{P}^2$, of which $q+1$ passes through P_i, $i = 1,2,3,4$. There is exactly one line through each pair of these points so there are

$$q^2+q+1-4(q+1)+6 = q^2-3q+3$$

lines which do not pass through P_1, P_2, P_3 and P_4. Of the $q+1$ lines through Q_i, $i = 1,2,3$, precisely two have been removed above and the line Q_iQ_j passes through both Q_i and Q_j. Hence, we have

$$q^2-3q+3-3(q-1)+3 = q^2-6q+9,$$

choices for L.

The line L intersects $\mathscr{S}$ in six points. We therefore have $q-5$ choices for P_5, $q-6$ choices for P_6 and $q-7$ choices for P_7. Hence,

$$|\Delta_{l,3}(\emptyset)| = |\mathrm{PGL}(3)| \cdot (q^2-6q+9)(q-5)(q-6)(q-7).$$

We now add everything together to obtain

$$|\Delta_{l,3}| = |\mathrm{PGL}(3)| \cdot (q^5-21q^4+173q^3-693q^2+1338q-990)$$

and, finally,

$$|\Delta_l| = |\mathrm{PGL}(3)| \cdot (31q^5-332q^4+1929q^3-6427q^2+11051q-7530).$$

The set Δ_c

We decompose Δ_c as

$$\Delta_c = \Delta_{c,1} \cup \Delta_{c,2},$$

where $\Delta_{c,1}$ consists of tuples where six points lie on a smooth conic C with one of the points P_1, P_2, P_3 or P_4 possibly outside C and $\Delta_{c,2}$ consists of tuples where six points lie on a smooth conic C with one of the points P_5, P_6 or P_7 possibly outside C.

To obtain an element of $\Delta_{c,1}$ we first choose one of the points P_1, P_2, P_3 and P_4 and call it P. Then we choose a smooth conic C in q^5-q^2 ways and place all of the seven points except P on C in

$$(q+1)q(q-1)(q-2)(q-3)(q-4),$$

ways. There are three lines through pairs of points in $\{P_1,P_2,P_3,P_4\}\setminus\{P\}$ which together contain $3q$ points. These lines do not contain P_5, P_6 and P_7 so we have

$$q^2+q+1-3q-3=q^2-2q-2,$$

choices for P. Multiplying everything together we obtain

$$N_1 := 4(q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)(q^2-2q-2),$$

which is almost $|\Delta_{c,1}|$ except that we have counted the tuples where all seven points lie on C four times.

To obtain an element of $\Delta_{c,2}$ we first choose P_5, P_6 and P_7 and call the chosen point P. We then choose a smooth conic C and place all but the chosen points on C. Finally, we place P anywhere in $\mathbb{P}^2$ except at the six chosen points. In this way we obtain the number

$$N_2 := 3(q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)(q^2+q-5),$$

which is almost equal to $|\Delta_{c,2}|$ except that we have counted the tuples with all seven points on C three times.

We now want to compute the number of tuples with all seven points on a smooth conic C. We thus choose a smooth conic C and place all seven points on it in

$$N_7 := (q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)(q-5),$$

ways. We thus have

$$|\Delta_c| = |\mathrm{PGL}(3)|\cdot(7q^5-74q^4+288q^3-517q^2+446q-168)$$

The set $\Delta_l\cap\Delta_c$

We introduce the filtration $\mathscr{F}_3\subset\mathscr{F}_2\subset\mathscr{F}_1=\Delta_l\cap\Delta_c$ where

- the set $\mathscr{F}_1$ consists of tuples such that at least one line contains three points of the tuple,
- the set $\mathscr{F}_2$ consists of tuples such that at least two lines contain three points of the tuple,

- the set $\mathscr{F}_3$ consists of tuples such that at least three lines contain three points of the tuple.

The strategy will be to compute the numbers:

$$\begin{aligned}
N_1 &= |\mathscr{F}_1| + |\mathscr{F}_2| + |\mathscr{F}_3|,\\
N_2 &= |\mathscr{F}_2| + 2|\mathscr{F}_3|,\\
N_3 &= |\mathscr{F}_3|,
\end{aligned}$$

and thereby obtain the desired cardinality.

Since the points P_1, P_2, P_3 and P_4 are assumed to constitute a frame, we must do things a little bit differently depending on whether the point not on the conic is one of these four or not. We therefore make further subdivisions.

The subsets with P_5, P_6 or P_7 not on the conic. We shall denote the subsets in question by $\mathscr{F}_i^{5,6,7}$ and, similarly

$$\begin{aligned}
N_1^{5,6,7} &= |\mathscr{F}_1^{5,6,7}| + |\mathscr{F}_2^{5,6,7}| + |\mathscr{F}_3^{5,6,7}|,\\
N_2^{5,6,7} &= |\mathscr{F}_2^{5,6,7}| + 2|\mathscr{F}_3^{5,6,7}|,\\
N_3^{5,6,7} &= |\mathscr{F}_3^{5,6,7}|.
\end{aligned}$$

To compute $N_1^{5,6,7}$, we first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C and call the remaining two points P_i and P_j. We then choose C in $q^5 - q^2$ ways and choose two points among $\{P_1, P_2, P_3, P_4, P_i, P_j\}$ and call them R_1 and R_2. There are $(q+1)q$ ways to place R_1 and R_2 on C and there are then $q-1$ ways to place P on the line R_1R_2. Finally, we place the remaining four points on C in $(q-1)(q-2)(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$N_1^{5,6,7} := 3 \cdot \binom{6}{2} \cdot (q^5 - q^2)(q+1)q(q-1)^2(q-2)(q-3)(q-4).$$

In order to compute $N_2^{5,6,7}$, we first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C and call the remaining two points P_i and P_j. We then choose C in $q^5 - q^2$ ways and choose two unordered pairs of unordered points among $\{P_1, P_2, P_3, P_4, P_i, P_j\}$. This can be done in $\frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2}$ ways. We call the points of the first pair R_1 and R_2 and those of the second O_1 and O_2. There are $(q+1)q(q-1)(q-2)(q-3)(q-4)$ ways to place $\{P_1, P_2, P_3, P_4, P_i, P_j\}$ on C and the point P is then completely determined as $P = R_1R_2 \cap O_1O_2$. Thus

$$N_2^{5,6,7} = 3 \cdot \frac{1}{2} \cdot \binom{6}{4} \cdot \binom{4}{2} \cdot (q^5 - q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).$$

The computation of $N_3^{5,6,7}$ is slightly more complicated since we need to subdivide into two subcases depending on if P is on the outside or on the inside of C. We call the two corresponding numbers $N_{3,\mathrm{out}}^{5,6,7}$ and $N_{3,\mathrm{in}}^{5,6,7}$.

To compute $N_{3,\mathrm{out}}^{5,6,7}$ we first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C. We proceed by choosing the smooth conic C in q^5-q^2 ways and then the point P on the outside of C in $\frac{1}{2}(q+1)q$ ways. We now place P_1 at one of the $q-1$ points of C whose tangent does not pass through P and choose one of the remaining 5 points as the other intersection point in $C \cap P_1P$. There are now four remaining points P_i, P_j, P_k and P_l to place on C. We place P_i at one of the $q-3$ remaining points of C whose tangent does not pass through P and choose one of the remaining three points as the other intersection point in $C \cap P_iP$. There are now two points P_r and P_s to place on C. We place P_r at one of the $q-5$ possible points and the point P_s is then determined. We thus have

$$N_{3,\mathrm{out}}^{5,6,7} = 3 \cdot (q^5-q^2) \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 5 \cdot (q-3) \cdot 3 \cdot (q-5).$$

We proceed by computing $N_{3,\mathrm{in}}^{5,6,7}$. We first choose one of the points P_5, P_6 or P_7 to be the point P not on the smooth conic C. We proceed by choosing the smooth conic C in q^5-q^2 ways and then the point P on the outside of C in $\frac{1}{2}(q-1)q$ ways.

We now place P_1 at one of the $q+1$ points of C whose tangent does not pass through P and choose one of the remaining 5 points as the other intersection point in $C \cap P_1P$. There are now four remaining points P_i, P_j, P_k and P_l to place on C. We place P_i at one of the $q-1$ remaining points of C and choose one of the remaining three points as the other intersection point in $C \cap P_iP$. There are now two points P_r and P_s to place on C. We place P_r at one of the $q-3$ possible points and the point P_s is then determined. We now see that

$$N_{3,\mathrm{in}}^{5,6,7} = 3 \cdot (q^5-q^2) \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 5 \cdot (q-1) \cdot 3 \cdot (q-3).$$

The subsets with P_1, P_2, P_3 or P_4 not on the conic. We shall denote the subsets in question by $\mathscr{F}_i^{1,2,3,4}$ and, similarly

$$\begin{aligned}
N_1^{1,2,3,4} &= |\mathscr{F}_1^{1,2,3,4}| + |\mathscr{F}_2^{1,2,3,4}| + |\mathscr{F}_3^{1,2,3,4}|,\\
N_2^{1,2,3,4} &= |\mathscr{F}_2^{1,2,3,4}| + 2|\mathscr{F}_3^{1,2,3,4}|,\\
N_3^{1,2,3,4} &= |\mathscr{F}_3^{1,2,3,4}|.
\end{aligned}$$

In order to compute $N_1^{1,2,3,4}$, we first choose one of the points P_1, P_2, P_3 or P_4 to be the point P not on the smooth conic C and call the remaining three points P_r, P_s and P_t. We continue by choosing a smooth conic C in q^5-q^2 ways.

We first assume that P lies on a line R_1R_2 where $\{R_1,R_2\}\subset\{P_5,P_6,P_7\}$. We therefore choose the two points in 3 ways and call the remaining point P_i. We then place R_1 and R_2 on C in $(q+1)q$ ways. We continue by choosing the three points P_r, P_s and P_t on C in $(q-1)(q-2)(q-3)$ ways. The lines P_rP_s, P_rP_t and P_sP_t intersect the line R_1R_2 in three distinct points so there are $q-4$ ways to choose the point P on R_1R_2 but away from these three points and R_1 and R_2. Finally, we place P_i at one of the $q-4$ remaining points of C. Multiplying everything together we get

$$4\cdot 3\cdot(q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q-4)^2.$$

We now assume that P lies on a line ab with $a\in\{P_1,P_2,P_3,P_4\}$ and $b\in\{P_5,P_6,P_7\}$. We thus first choose a as one of the points in $\{P_r,P_s,P_t\}$ and the point b as one of the points $\{P_5,P_6,P_7\}$ and place a and b on C in one of $(q+1)q$ ways. We then place the remaining two points, c and d, of $\{P_1,P_2,P_3,P_4\}$ on C in $(q-1)(q-2)$ ways. The line cd intersects ab in a point outside of C so there are $q-2$ ways to choose P on ab but away from this intersection point and a and b. Finally, we place the remaining two points of $\{P_5,P_6,P_7\}$ on C in one of $(q-3)(q-4)$ ways. Multiplying everything together we obtain

$$4\cdot 3\cdot 3\cdot(q^5-q^2)(q+1)q(q-1)(q-2)^2(q-3)(q-4).$$

We now add the two answers above together to get

$$N_1^{1,2,3,4}=24q^3(q-2)(q-3)(q-4)(2q-5)(q+1)(q^2+q+1)(q-1)^2.$$

To compute $N_2^{1,2,3,4}$, we first choose one of the points P_1, P_2, P_3 or P_4 to be the point P not on the smooth conic C and call the remaining three points P_r, P_s and P_t. We continue by choosing a smooth conic C in q^5-q^2 ways.

We first assume that P lies on two lines R_1R_2 and O_1O_2 where $\{R_1,O_1\}\subset\{P_5,P_6,P_7\}$ and $\{R_2,O_2\}\subset\{P_r,P_s,P_t\}$. We now choose two points among $\{P_5,P_6,P_7\}$ in three ways and choose two points among $\{P_r,P_s,P_t\}$ in three ways and rename the remaining two points to P_u and P_v. There are now two possible ways to label the four chosen points R_1,R_2,O_1 and O_2 in such a way that $\{R_1,O_1\}\subset\{P_5,P_6,P_7\}$ and $\{R_2,O_2\}\subset\{P_r,P_s,P_t\}$ and we choose one of them. We then place the four points R_1, R_2, O_1 and O_2 on the conic C in $(q+1)q(q-1)(q-2)$ ways. The point P is now given as $P=R_1R_2\cap O_1O_2$ and no matter how we place P_u and P_v, the three lines P_rP_s, P_rP_t and P_sP_t will not go through P. We can now multiply everything together to obtain

$$4\cdot 3\cdot 3\cdot 2\cdot(q^5-q^2)(q+1)q(q-1)(q-2)(q-3)(q-4).$$

The other possibility is that P lies on two lines R_1R_2 and R_3b where $\{R_1, R_2, R_3\}$ is the set $\{P_5, P_6, P_7\}$ and $b \in \{P_r, P_s, P_t\}$. We thus choose b in three ways and rename the remaining two points in $\{P_r, P_s, P_t\}$ to P_u and P_v. From now on, we must differentiate between when P is on the outside and on the inside of C.

First, we choose P on the outside of C in $\frac{1}{2}(q+1)q$ ways. We then choose b as a point on C whose tangent does not pass through P in $q-1$ ways. We then choose one of the points P_5, P_6 and P_7 to become the second intersection point in $C \cap bP$. Then, we place the remaining two points among $\{P_5, P_6, P_7\}$ on C such that the line through them passes through P in $q-3$ ways. There are now $q-5$ ways to choose P_u and P_v such that the line P_uP_v will pass through P. Thus, the remaining $(q-3)(q-4)-(q-5) = q^2-8q+17$ choices must give P_u and P_v such that none of the lines P_rP_s, P_rP_t and P_sP_t will contain P. We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 3 \cdot (q-3) \cdot (q^2 - 8q + 17).$$

Now we choose P on the inside of C in one of $\frac{1}{2}(q-1)q$ ways. We then choose b as a point on C whose tangent does not pass through P in $q+1$ ways. We then choose one of the points P_5, P_6 and P_7 to become the second intersection point in $C \cap bP$. Then, we place the remaining two points among $\{P_5, P_6, P_7\}$ on C such that the line through them passes through P in $q-1$ ways. There are now $q-3$ ways to choose P_u and P_v such that the line P_uP_v will pass through P. Thus, the remaining $(q-3)(q-4)-(q-3) = (q-3)(q-5)$ choices must give P_u and P_v such that none of the lines P_rP_s, P_rP_t and P_sP_t will contain P. We may now multiply everything together to obtain

$$4 \cdot (q^5 - q^2) \cdot 3 \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 3 \cdot (q-1) \cdot (q-3)(q-5).$$

We may now add everything together to get

$$N_2^{1,2,3,4} = 36q^3(q+1)(q^2+q+1)(5q^3-37q^2+82q-60)(q-1)^2.$$

Finally, we need to compute $N_3^{1,2,3,4}$. We begin by choosing one of the points P_1, P_2, P_3 or P_4 to be the point P not on the smooth conic C and call the remaining three points P_r, P_s and P_t. We continue by choosing a smooth conic C in $q^5 - q^2$ ways.

Here, we only have the possibility that P lies on three lines R_1O_1, R_2O_2 and R_3O_3 where $\{R_1, R_2, R_3\} = \{P_5, P_6, P_7\}$ and $\{O_1, O_2, O_3\} = \{P_r, P_s, P_t\}$. However, we must take care of the case that P is on the outside of C and the case that P is on the inside of C separately. We call the corresponding numbers $N^{1,2,3,4}_{3,\mathrm{out}}$ and $N^{1,2,3,4}_{3,\mathrm{in}}$.

We begin by computing $N^{1,2,3,4}_{3,\mathrm{out}}$. We thus choose the point P as a point on the outside of C in $\frac{1}{2}(q+1)q$ ways. We begin by placing P_5 at one of the points of C whose tangent does not pass through P in $q-1$ ways. We label the second intersection point of $C \cap P_5P$ with P_r, P_s or P_t and call the remaining two points P_u and P_v. We then place P_6 at one of the $q-3$ remaining points of C whose tangent does not pass through P and then choose one of the points P_u and P_v to become the other intersection point of $C \cap P_6P$. Finally, we place P_7 at one of the remaining $q-5$ points and label the other point of $C \cap P_7P$ in the only possible way. We thus have

$$N^{1,2,3,4}_{3,\mathrm{out}} = 4 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q+1)q \cdot (q-1) \cdot 3 \cdot (q-3) \cdot 2 \cdot (q-5).$$

We now turn to computing $N^{1,2,3,4}_{3,in}$. We thus choose the point P as a point on the inside of C in $\frac{1}{2}(q-1)q$ ways. We begin by placing P_5 at one of the points of C whose tangent does not pass through P in $q+1$ ways. We label the second intersection point of $C \cap P_5P$ with P_r, P_s or P_t and call the remaining two points P_u and P_v. We then place P_6 at one of the $q-1$ remaining points of C and then choose one of the points P_u and P_v to become the other intersection point of $C \cap P_6P$. Finally, we place P_7 at one of the remaining $q-3$ points and label the other point of $C \cap P_7P$ in the only possible way. We now see that

$$N^{1,2,3,4}_{3,\mathrm{in}} = 4 \cdot (q^5 - q^2) \cdot \frac{1}{2}(q-1)q \cdot (q+1) \cdot 3 \cdot (q-1) \cdot 2 \cdot (q-3),$$

and we get

$$N^{1,2,3,4}_{3} = 192q^3(q+1)(q^2+q+1)(q^2-3q+3)(q-1)^2.$$

We now obtain

$$|\Delta_l \cap \Delta_c| = |\mathrm{PGL}(3)| \cdot (93q^4 - 1245q^3 + 6195q^2 - 13470q + 10737),$$

and, finally,

$$\left|\left(\mathcal{P}^2_7\right)^{F\sigma}\right| = q^6 - 35q^5 + 490q^4 - 3485q^3 + 13174q^2 - 24920q + 18375.$$

7.3.16. Summary of computations We summarize the computations in Table 7.1 and in Proposition 7.3.13 we give the Poincaré-Serre polynomial of $\mathcal{Q}[2]$. In Table 7.2 we give the cohomology of $\mathcal{Q}[2]$ as a representation of

S_7. The rows correspond to the cohomology groups and the columns correspond to the irreducible representations of S_7. The symbol s_λ denotes the irreducible representation of S_7 corresponding to the partition λ and a number n in row H^k and column s_λ means that s_λ occurs in H^k with multiplicity n.

Proposition 7.3.13. *Let $v = tu^2$. The Poincaré-Serre polynomial of $\mathcal{Q}[2]$ is*

$$PS_{\mathcal{Q}[2]}(t,u) = 1 + 35v + 490v^2 + 3485v^3 + 13174v^4 + 24920v^5 + 18375v^6.$$

λ	$\lvert\mathcal{Q}[2]^{F\cdot\sigma_\lambda}\rvert$
$[7]$	q^6+q^3
$[6,1]$	q^6-2q^3+1
$[5,2]$	q^6-q^2
$[5,1^2]$	q^6-q^2
$[4,3]$	$q^6-q^5-2q^4+q^3+q^2$
$[4,2,1]$	$q^6-q^5-2q^4+q^3-2q^2+3$
$[4,1^3]$	$q^6-q^5-2q^4+q^3-2q^2+3$
$[3^2,1]$	$q^6-2q^5-2q^4-8q^3+16q^2+10q+21$
$[3,2^2]$	$q^6-q^5-2q^4+3q^3+q^2-2q$
$[3,2,1^2]$	$q^6-3q^5+5q^3-q^2-2q$
$[3,1^4]$	$q^6-5q^5+10q^4-5q^3-11q^2+10q$
$[2^3,1]$	$q^6-3q^5-6q^4+19q^3+6q^2-24q+7$
$[2^2,1^3]$	$q^6-7q^5+10q^4+15q^3-26q^2-8q+15$
$[2,1^5]$	$q^6-15q^5+90q^4-265q^3+374q^2-200q+15$
$[1^7]$	$q^6-35q^5+490q^4-3485q^3+13174q^2-24920q+18375$

Table 7.1: The S_7-equivariant point count of $\mathcal{Q}[2]$. We use σ_λ to denote any permutation in S_7 of cycle type λ.

	s_7	$s_{6,1}$	$s_{5,2}$	$s_{5,1^2}$	$s_{4,3}$	$s_{4,2,1}$	$s_{4,1^3}$	$s_{3^2,1}$	$s_{3,2^2}$	$s_{3,2,1^2}$
H^0	1	0	0	0	0	0	0	0	0	0
H^1	1	1	1	0	1	0	0	0	0	0
H^2	0	3	4	4	3	5	1	3	1	1
H^3	1	8	14	18	14	30	16	16	12	18
H^4	4	20	44	47	44	99	56	56	54	83
H^5	6	33	76	76	72	178	97	104	105	169
H^6	6	23	51	54	54	127	74	76	77	126

	$s_{3,1^4}$	$s_{2^3,1}$	$s_{2^2,1^3}$	$s_{2,1^5}$	s_{1^7}
H^0	0	0	0	0	0
H^1	0	0	0	0	0
H^2	0	0	0	0	0
H^3	4	6	3	0	0
H^4	32	31	25	6	1
H^5	71	65	64	26	3
H^6	54	54	50	22	5

Table 7.2: The cohomology of $\mathcal{Q}[2]$ as a representation of S_7.

8. Hyperelliptic curves

Up to this point we have almost exclusively discussed plane quartics. We shall now briefly turn our attention to the other type of genus 3 curves - the hyperelliptic curves. Most of the chapter is devoted to the moduli space $\mathcal{H}_3[2]$ of hyperelliptic curves of genus 3 without a marked point. We determine the cohomology of $\mathcal{H}_3[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$ together with its mixed Hodge structure. The results are presented in Table 8.4. For comparison with Chapter 7 we also present the cohomology of $\mathcal{H}_3[2]$ as a representation of the symmetric group S_7 in Table 8.5. Once these computations have been completed, we compute the cohomology of $\mathcal{H}_{3,1}[2]$ via a short and simple spectral sequence argument. The result is given in Proposition 8.2.1.

8.1 Hyperelliptic curves without marked points

There are many possible ways to approach the computation of the cohomology of $\mathcal{H}_3[2]$. Our choice is by means of equivariant point counts as in Chapter 7.

Recall that a hyperelliptic curve C of genus g is determined, up to isomorphism, by $2g+2$ distinct points on $\mathbb{P}^1$, up to projective equivalence and that any such collection S of $2g+2$ points determines a double cover $\pi : C \to \mathbb{P}^1$ branched precisely over S (and C is thus a hyperelliptic curve). Moreover, if we pick $2g+2$ ordered points $P_1,\ldots,P_{2g+2}$ on $\mathbb{P}^1$, the curve C also attains a level 2-structure. In the genus 3 case, we get 8 points $Q_i = \pi^{-1}(P_i)$ which determine $\binom{8}{2} = 28$ odd theta characteristics $Q_i + Q_j$, $i < j$ and $\{Q_1 + Q_8,\ldots,Q_7 + Q_8\}$ is an ordered Aronhold set, by Proposition 3.2.9, Theorem 3.4.2 and [3], Appendix B.32-33. By Proposition 3.2.10 we have that an ordered Aronhold set determines a level 2-structure. See also [49].

However, not all level 2-structures on the hyperelliptic curve C arise from different orderings of the points. Nevertheless, there is an intimate relationship between the moduli space $\mathcal{H}_g[2]$ of hyperelliptic curves with level 2-structure and the moduli space $\mathcal{M}_{0,2g+2}$ of $2g+2$ ordered points on $\mathbb{P}^1$ given by the following theorem which can be found in [38], Theorem VIII.1.

Theorem 8.1.1. *Each irreducible component of $\mathcal{H}_g[2]$ is isomorphic to the moduli space $\mathcal{M}_{0,2g+2}$ of $2g+2$ ordered points on the projective line.*

Dolgachev and Ortland [38], pose the question whether the irreducible components of $\mathcal{H}_g[2]$ also are the connected components or, in other words, if $\mathcal{H}_g[2]$ is smooth. In the complex case, the question was answered positively by Tsuyumine in [84] and later, by a shorter argument, by Runge in [77]. Using the results of [2], the argument of Runge carries over word for word to an algebraically closed field of positive characteristic different from 2.

Theorem 8.1.2. *If $g \geq 2$, then each irreducible component of $\mathcal{H}_g[2]$ is also a connected component.*

We have a natural action of S_{2g+2} on the space $\mathcal{M}_{0,2g+2}$. Since different orderings of the points correspond to different symplectic level 2 structures, S_{2g+2} sits naturally inside $\mathrm{Sp}(2g,\mathbb{F}_2)$ and, in fact, for $g=3$ and for even g it is a maximal subgroup, see [40]. With Theorems 8.1.1 and 8.1.2 at hand, the following slight generalization of a corollary in [38] (p.145) is clear.

Corollary 8.1.3. *Let $g \geq 2$ and let $X_{[\tau]} = \mathcal{M}_{0,2g+2}$ for each left coset $[\tau] \in \mathscr{T} := \mathrm{Sp}(2g,\mathbb{F}_2)/S_{2g+2}$. Then*

$$\mathcal{H}_g[2] \cong \coprod_{[\tau]\in\mathscr{T}} X_{[\tau]},$$

and the group $\mathrm{Sp}(2g,\mathbb{F}_2)$ acts transitively on the set of connected components $X_{[\tau]}$ of $\mathcal{H}_g[2]$. In particular, there are

$$\frac{\left|\mathrm{Sp}(2g,\mathbb{F}_2)\right|}{\left|S_{2g+2}\right|} = \frac{2^{g^2}\left(2^{2g}-1\right)\left(2^{2g-2}-1\right)\cdots\left(2^2-1\right)}{(2g+2)!},$$

connected components of $\mathcal{H}_g[2]$.

Remark 8.1.4. As pointed out in [77], the argument of the corollary stated in [38] is not quite correct in full generality as it is given there. However, it is enough to prove the result for $g=3$ and for even g, and in [77] it is explained how to obtain the full result.

Let us now, once and for all, choose a set T of representatives of $\mathrm{Sp}(2g,\mathbb{F}_2)/S_{2g+2}$. If we denote the elements of $X_{[\mathrm{id}]}$ by x, then any element in $X_{[\tau]}$ can be written as τx for some $x \in X_{[\mathrm{id}]}$. Let α be any element of $\mathrm{Sp}(2g,\mathbb{F}_2)$. Then

$$\alpha\tau = \tau'\sigma,$$

for some $\sigma \in S_{2g+2}$ and some $\tau' \in T$. Since the Frobenius commutes with the action of $\mathrm{Sp}(2g,\mathbb{F}_2)$ we have that

$$F\alpha(\tau x) = \tau x,$$

if and only if

$$F(\tau'\sigma x) = \tau'(F\sigma x) = \tau x.$$

But the Frobenius acts on each of the components of $\mathcal{H}_g[2]$ so we see that $F\alpha(\tau x) = \tau x$ if and only if $\tau' = \tau$ and $F\sigma x = x$.

We now translate the above observation into more standard representation theoretic vocabulary. Define a class function ψ on S_{2g+2} by

$$\psi(\sigma) = |X_{\mathrm{id}}^{F\sigma}|,$$

and define a class function $\hat{\psi}$ on $\mathrm{Sp}\big(2g, \mathbb{F}_2\big)$ by setting

$$\hat{\psi}(\alpha) = |\mathcal{H}_g[2]^{F\alpha}|,$$

for any $\alpha \in \mathrm{Sp}(6, \mathbb{F}_2)$. By the above observation we have that

$$\hat{\psi}(\alpha) = \sum_{\tau \in T} \widetilde{\psi}(\tau^{-1}\alpha\tau),$$

where

$$\widetilde{\psi}(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \in S_{2g+2}, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, $\hat{\psi}$ is the class function ψ induced from S_{2g+2} up to $\mathrm{Sp}\big(2g, \mathbb{F}_2\big)$. Thus, to make an S_{2g+2}-equivariant point count of $\mathcal{H}_g[2]$ we can make an S_{2g+2}-equivariant point count of $\mathcal{M}_{0,2g+2}$ and then use the representation theory of S_{2g+2} and $\mathrm{Sp}\big(2g, \mathbb{F}_2\big)$ in order to first induce the class function up to $\mathrm{Sp}\big(2g, \mathbb{F}_2\big)$ and then restrict it down again to S_{2g+2}. Once this is done, we can obtain the S_{2g+1}-equivariant point count by restricting from S_{2g+2} to S_{2g+1}.

Using Lemma 7.3.4, the S_8-equivariant point count of $\mathcal{H}_3[2]$ is very easy. We first compute the number of λ-tuples of $\mathbb{P}^1$ for each partition of λ of 8 and then divide by $|\mathrm{PGL}(2)|$ in order to obtain $|\mathcal{M}_{0,8}^{F\sigma}|$, where σ is a permutation in S_8 of cycle type λ. The result is given in Table 8.1. Once this is done, we induce up to $\mathrm{Sp}(6, \mathbb{F}_2)$ in order to obtain the $\mathrm{Sp}(6, \mathbb{F}_2)$-equivariant cohomology of $\mathcal{H}_3[2]$. The results are given in Table 8.2 and 8.4. Finally, we restrict to S_7 to get the results of Table 8.3 and 8.5. The computations present no difficulties whatsoever. We also mention that the equivariant Poincaré polynomials of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$ have been computed for all $n \geq 3$ in [46].

It is not very hard to see that $M_{0,2g+2}$ is isomorphic to the complement of a hyperplane arrangement. One way to see this is to start by placing the first three points at 0, 1 and ∞. Then $\mathcal{M}_{0,2g+2}$ is isomorphic to $\big(\mathbb{A}^1 \setminus \{0,1\}\big)^{2g-1} \setminus \Delta$, where $\Delta \subset \big(\mathbb{A}^1 \setminus \{0,1\}\big)^{2g-1}$ is the subset of points where at least two coordinates are equal. Thus, by the results of Section 7.2 we can deduce the cohomology of $\mathcal{H}_3[2]$ from the equivariant point counts.

8.2 Hyperelliptic curves with marked points

Once we know the cohomology of $\mathcal{H}_3[2]$, the computation of the cohomology of the moduli space $\mathcal{H}_{3,1}[2]$ of hyperelliptic curves of genus 3 with level 2 structure and one marked point is easy. The moduli space $\mathcal{H}_{3,1}[2]$ is a $\mathbb{P}^1$-fibration over $\mathcal{H}_3[2]$ via the morphism $\pi : \mathcal{H}_{3,1}[2] \to \mathcal{H}_3[2]$ forgetting the marked point. The Leray-Serre spectral sequence of this fibration degenerates at the second page and allows us to compute the cohomology of $\mathcal{H}_{3,1}[2]$ as given in the following proposition.

Proposition 8.2.1. *The cohomology of $\mathcal{H}_{3,1}[2]$ is given by*

$$H^k\big(\mathcal{H}_{3,1}[2]\big) = H^{k-2}(\mathcal{H}_3[2])\,(\text{-}1) \oplus H^k(\mathcal{H}_3[2])\,.$$

8.3 Tables

Tables 8.1, 8.2 and 8.3 give equivariant point counts for various spaces and groups. In Table 8.4 we give the cohomology of $\mathcal{H}_3[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$. The rows correspond to the cohomology groups and the columns correspond to the irreducible representations of $\mathrm{Sp}(6,\mathbb{F}_2)$, see Table 6.7. Similarly, Table 8.5 gives the cohomology of $\mathcal{H}_3[2]$ as a representation of S_7. The rows correspond to the cohomology groups and the columns correspond to the irreducible representations of S_7. The symbol s_λ denotes the irreducible representation of S_7 corresponding to the partition λ and a number n in row H^k and column s_λ means that s_λ occurs in H^k with multiplicity n.

For convenience of we also give the Poincaré-Serre polynomial of $\mathcal{H}_3[2]$.

Proposition 8.3.1. *Let $v = tu^2$. The Poincaré-Serre polynomial of $\mathcal{H}_3[2]$ is*

$$PS_{\mathcal{H}_3[2]}(t,u) = 36 + 720v + 5580v^2 + 20880v^3 + 37584v^4 + 25920v^5.$$

λ	$\lvert\mathcal{M}_{0,8}^{F\cdot\sigma_\lambda}\rvert$
$[8]$	$\left(q^2+1\right)q^3$
$[7,1]$	$\left(q+1\right)\left(q^2+q+1\right)\left(q^2-q+1\right)$
$[6,2]$	$q\left(q-1\right)\left(q^3+q-1\right)$
$\left[6,1^2\right]$	$q\left(q+1\right)\left(q^3+q-1\right)$
$[5,3]$	$q\left(q-1\right)\left(q+1\right)\left(q^2+1\right)$
$[5,2,1]$	$q\left(q-1\right)\left(q+1\right)\left(q^2+1\right)$
$\left[5,1^3\right]$	$q\left(q-1\right)\left(q+1\right)\left(q^2+1\right)$
$\left[4^2\right]$	$q\left(q^4-q^2-4\right)$
$[4,3,1]$	$\left(q-1\right)q^2\left(q+1\right)^2$
$\left[4,2^2\right]$	$\left(q-1\right)\left(q-2\right)\left(q+1\right)q^2$
$\left[4,2,1^2\right]$	$\left(q-1\right)\left(q+1\right)q^3$
$\left[4,1^4\right]$	$\left(q-1\right)\left(q-2\right)\left(q+1\right)q^2$
$\left[3^2,2\right]$	$q\left(q-1\right)\left(q^3-q-3\right)$
$\left[3^2,1^2\right]$	$q\left(q+1\right)\left(q^3-q-3\right)$
$\left[3,2^2,1\right]$	$q\left(q-1\right)\left(q-2\right)\left(q+1\right)^2$
$\left[3,2,1^3\right]$	$\left(q+1\right)q^2\left(q-1\right)^2$
$\left[3,1^5\right]$	$q\left(q-1\right)\left(q-2\right)\left(q-3\right)\left(q+1\right)$
$\left[2^4\right]$	$\left(q-2\right)\left(q-3\right)\left(q+2\right)\left(q^2-q-4\right)$
$\left[2^3,1^2\right]$	$q\left(q-2\right)\left(q+1\right)\left(q^2-q-4\right)$
$\left[2^2,1^4\right]$	$q\left(q-1\right)\left(q+1\right)\left(q-2\right)^2$
$\left[2,1^6\right]$	$q\left(q-1\right)\left(q-2\right)\left(q-3\right)\left(q-4\right)$
$\left[1^8\right]$	$\left(q-2\right)\left(q-3\right)\left(q-4\right)\left(q-5\right)\left(q-6\right)$

Table 8.1: The S_8-equivariant point count of $\mathcal{M}_{0,8}$. We use σ_λ to denote any permutation in S_8 of cycle type λ.

λ	$\lvert\mathcal{H}_3[2]^{F\cdot\sigma_\lambda}\rvert$
$[8]$	$2\,q^5+2\,q^3$
$[7,1]$	$q^5+q^4+q^3+q^2+q+1$
$[6,2]$	$3\,q^5+3\,q^3-6\,q^2-3\,q^4+3\,q$
$[6,1^2]$	$q^5+q^4+q^3-q$
$[5,3]$	q^5-q
$[5,2,1]$	q^5-q
$[5,1^3]$	q^5-q
$[4^2]$	$4\,q^5-16\,q-4\,q^3$
$[4,3,1]$	$2\,q^5+2\,q^4-2\,q^3-2\,q^2$
$[4,2^2]$	$6\,q^5+12\,q^2-12\,q^4-6\,q^3$
$[4,2,1^2]$	$2\,q^5-2\,q^3$
$[4,1^4]$	$2\,q^5-4\,q^4-2\,q^3+4\,q^2$
$[3^2,2]$	$q^5-q^4-q^3-2\,q^2+3\,q$
$[3^2,1^2]$	$3\,q^5+3\,q^4-3\,q^3-12\,q^2-9\,q$
$[3,2^2,1]$	$2\,q^5-2\,q^4-6\,q^3+2\,q^2+4\,q$
$[3,2,1^3]$	$4\,q^5-4\,q^4-4\,q^3+4\,q^2$
$[3,1^5]$	$6\,q^5-30\,q^4+30\,q^3+30\,q^2-36\,q$
$[2^4]$	$12\,q^5+48\,q-60\,q^3+336\,q^2-48\,q^4-576$
$[2^3,1^2]$	$4\,q^5-8\,q^4-20\,q^3+24\,q^2+32\,q$
$[2^2,1^4]$	$8\,q^5-32\,q^4+24\,q^3+32\,q^2-32\,q$
$[2,1^6]$	$16\,q^5-160\,q^4+560\,q^3-800\,q^2+384\,q$
$[1^8]$	$36\,q^5-720\,q^4+5580\,q^3-20880\,q^2+37584\,q-25920$

Table 8.2: The S_8-equivariant point count of $\mathcal{H}_3[2]$. We use σ_λ to denote any permutation in S_8 of cycle type λ.

λ	$\lvert\mathcal{H}_3[2]^{F\cdot\sigma_\lambda}\rvert$
$[7]$	$q^5+q^4+q^3+q^2+q+1$
$[6,1]$	$q^5+q^4+q^3-q$
$[5,2]$	q^5-q
$[5,1^2]$	q^5-q
$[4,3]$	$2q^5+2q^4-2q^3-2q^2$
$[4,2,1]$	$2q^5-2q^3$
$[4,1^3]$	$2q^5-4q^4-2q^3+4q^2$
$[3^2,1]$	$3q^5+3q^4-3q^3-12q^2-9q$
$[3,2^2]$	$2q^5-2q^4-6q^3+2q^2+4q$
$[3,2,1^2]$	$4q^5-4q^4-4q^3+4q^2$
$[3,1^4]$	$6q^5-30q^4+30q^3+30q^2-36q$
$[2^3,1]$	$4q^5-8q^4-20q^3+24q^2+32q$
$[2^2,1^3]$	$8q^5-32q^4+24q^3+32q^2-32q$
$[2,1^5]$	$16q^5-160q^4+560q^3-800q^2+384q$
$[1^7]$	$36q^5-720q^4+5580q^3-20880q^2+37584q-25920$

Table 8.3: The S_7-equivariant point count of $\mathcal{H}_3[2]$. We use σ_λ to denote any permutation in S_7 of cycle type λ.

	ϕ_{1a}	ϕ_{7a}	ϕ_{15a}	ϕ_{21a}	ϕ_{21b}	ϕ_{27a}	ϕ_{35a}	ϕ_{35b}	ϕ_{56a}	ϕ_{70a}
H^0	1	0	0	0	0	0	0	1	0	0
H^1	0	0	0	0	0	1	0	1	0	0
H^2	0	0	0	1	0	0	0	0	0	0
H^3	0	0	0	1	0	0	0	0	0	1
H^4	0	0	0	0	0	1	1	1	0	1
H^5	0	0	1	0	1	1	1	2	0	0

	ϕ_{84a}	ϕ_{105a}	ϕ_{105b}	ϕ_{105c}	ϕ_{120a}	ϕ_{168a}	ϕ_{189a}	ϕ_{189b}	ϕ_{189c}	ϕ_{210a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	0	0	1	0	0	0	1
H^2	0	0	1	0	2	1	1	0	0	3
H^3	0	0	3	1	3	2	4	1	0	5
H^4	2	2	3	2	3	6	5	4	4	6
H^5	4	2	1	4	1	4	3	3	6	4

	ϕ_{210b}	ϕ_{216a}	ϕ_{280a}	ϕ_{280b}	ϕ_{315a}	ϕ_{336a}	ϕ_{378a}	ϕ_{405a}	ϕ_{420a}	ϕ_{512a}
H^0	0	0	0	0	0	0	0	0	0	0
H^1	0	0	0	1	0	0	0	0	0	0
H^2	1	0	0	2	0	0	1	3	2	2
H^3	5	1	3	3	4	3	5	10	7	7
H^4	6	5	7	8	7	9	9	10	12	14
H^5	2	7	4	8	3	8	6	4	8	9

Table 8.4: The cohomology of $\mathcal{H}_3[2]$ as a representation of $\mathrm{Sp}(6, \mathbb{F}_2)$.

	s_7	$s_{6,1}$	$s_{5,2}$	$s_{5,1^2}$	$s_{4,3}$	$s_{4,2,1}$	$s_{4,1^3}$	$s_{3^2,1}$	$s_{3,2^2}$	$s_{3,2,1^2}$
H^0	2	1	1	0	1	0	0	0	0	0
H^1	2	7	9	5	5	7	1	3	2	1
H^2	3	18	30	31	25	50	20	26	19	26
H^3	6	35	74	80	72	162	86	92	83	129
H^4	8	48	114	117	109	271	150	157	158	254
H^5	5	31	72	77	72	180	103	108	108	180

	$s_{3,1^4}$	$s_{2^3,1}$	$s_{2^2,1^3}$	$s_{2,1^5}$	s_{1^7}
H^0	0	0	0	0	0
H^1	0	0	0	0	0
H^2	5	7	4	0	0
H^3	43	45	36	10	1
H^4	105	96	92	35	4
H^5	77	72	72	31	5

Table 8.5: The cohomology of $\mathcal{H}_3[2]$ as a representation of S_7.

9. Consequences and concluding remarks

In this chapter we derive some partial results regarding the cohomologies of $\mathcal{Q}[2]$ and $\mathcal{M}_{3,1}[2]$ as representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ as well as the cohomology of $\mathcal{M}_3[2]$ as a representation of S_7. We also discuss some possible directions for future work.

9.1 The moduli space of marked genus three curves

Recall that the moduli space $\mathcal{M}_{3,1}[2]$ of genus 3 curves with level 2 structure and one marked point naturally decomposes into a disjoint union

$$\mathcal{M}_{3,1}[2] = \mathcal{Q}_1[2] \sqcup \mathcal{H}_{3,1}[2],$$

where $\mathcal{Q}_1[2]$ is the locus of plane quartics with level 2 structure and one marked point and $\mathcal{H}_{3,1}[2]$ is the locus of hyperelliptic curves of genus 3 with level 2 structure and one marked point. Applying Lemma 6.1.2 to $\mathcal{H}_{3,1}[2] \subset \mathcal{M}_{3,1}[2]$ gives the long $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant exact sequence of mixed Hodge structures

$$\cdots \to H^{k-2}\big(\mathcal{H}_{3,1}[2]\big)(\text{-}1) \to H^k\big(\mathcal{M}_{3,1}[2]\big) \to H^k(\mathcal{Q}_1[2]) \to H^{k-1}\big(\mathcal{H}_{3,1}[2]\big)(\text{-}1) \to \cdots$$

Recall also Proposition 6.1.3 and Proposition 8.2.1 as well as Table 6.5, Table 6.6 and Table 8.4.

9.2 The S_7-equivariant cohomology of $\mathcal{M}_3[2]$

Also the moduli space $\mathcal{M}_3[2]$ of genus 3 curves with level 2 structure decomposes into a disjoint union

$$\mathcal{M}_3[2] = \mathcal{Q}[2] \sqcup \mathcal{H}_3[2],$$

where $\mathcal{Q}[2]$ is the locus of plane quartics with level 2 structure and $\mathcal{H}_3[2]$ is the locus of hyperelliptic curves of genus 3 with level 2 structure. We apply

Lemma 6.1.2 to $\mathcal{H}_3[2] \subset \mathcal{M}_3[2]$ and get the long $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant exact sequence of mixed Hodge structures

$$\cdots \to H^{k-2}(\mathcal{H}_3[2])\,(\text{-}1) \to H^k(\mathcal{M}_3[2]) \to H^k(\mathcal{Q}[2]) \to H^{k-1}(\mathcal{H}_3[2])\,(\text{-}1) \to \cdots$$

The reader might want to recall Table 6.3 and Table 8.4. By restricting to S_7 we get an S_7-equivariant long exact sequence. In this context it might be interesting to recall Table 7.2 and Table 8.4.

9.3 The $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant cohomology of $\mathcal{Q}[2]$

In Chapter 7 we computed the cohomology of $\mathcal{Q}[2]$ as a S_7-representation and the results are found in Table 7.2. There are 30 irreducible representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ while there are 15 irreducible representations of S_7. Restricting from $\mathrm{Sp}(6,\mathbb{F}_2)$ to S_7 we thus get a set of 15 linear equations in 30 unknowns for each cohomological degree. However, the unknowns are the multiplicities of the irreducible representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ in the cohomology of $\mathcal{Q}[2]$. As such, they are nonnegative integers. Let R be the matrix describing the restriction of the irreducible representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ to S_7, let x^k be a vector of the 30 unknowns and let h^k be a vector describing the k'th cohomology group of $\mathcal{Q}[2]$ as a S_7-representation. We may now formulate the above as

$$Rx^k = h^k, \quad x^k \geq 0, \quad x^k \in \mathbb{Z}^{30}, \quad k = 0,\dots,6. \tag{9.3.1}$$

By Proposition 6.2.3 we know that there is a $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant injection of mixed Hodge structures

$$H^k(\mathcal{Q}[2]) \hookrightarrow H^k(\mathcal{Q}_{\mathrm{flx}}[2]),$$

and the cohomology of $\mathcal{Q}_{\mathrm{flx}}[2]$ is given as a $\mathrm{Sp}(6,\mathbb{F}_2)$-representation in Table 6.3. We thus get an upper bound for the multiplicity of each irreducible representation of $\mathrm{Sp}(6,\mathbb{F}_2)$ in each cohomology group of $\mathcal{Q}[2]$. Let f^k denote a vector describing the k'th cohomology group of $\mathcal{Q}_{\mathrm{flx}}[2]$ as a representation of $\mathrm{Sp}(6,\mathbb{F}_2)$. We may now formulate the above as

$$x^k \leq f^k, \quad k = 0,\dots,6. \tag{9.3.2}$$

Equations 9.3.1 and 9.3.2 together constitute the constraints of a so-called *integer programming problem*, see for instance [71]. We may thus respectively maximize and minimize each variable x_i^k subject to 9.3.1 and 9.3.2 and get values $x_{i,\max}^k$ and $x_{i,\min}^k$. If $x_{i,\max}^k = x_{i,\min}^k$ we conclude that the multiplicity of the irreducible representation of $\mathrm{Sp}(6,\mathbb{F}_2)$ corresponding to i in $H^k(\mathcal{Q}[2])$ is $x^k_{i,\max} = x^k_{i,\min}$.

The method described above is enough to determine the cohomology of $\mathcal{Q}[2]$ in degrees 0, 1, 2 and 3. Using the notation of Table 6.7 we have

$$
\begin{aligned}
H^0(\mathcal{Q}[2]) &= \phi_{1a}, && 1-\text{dimensional},\\
H^1(\mathcal{Q}[2]) &= \phi_{35b}, && 35-\text{dimensional},\\
H^2(\mathcal{Q}[2]) &= \phi_{210a}+\phi_{280b}, && 490-\text{dimensional},\\
H^3(\mathcal{Q}[2]) &= \phi_{21a}+\phi_{105b}+\phi_{189a}+2\phi_{210a}+\phi_{210b}+ && 3485-\text{dimensional}.\\
&\quad +\phi_{378a}+2\phi_{405a}+2\phi_{420a}+\phi_{512a}
\end{aligned}
$$

In degrees 4, 5 and 6 the above method does not suffice. For instance, for $H^4(\mathcal{Q}[2])$ the set of feasible solutions contains 1039 elements.

9.4 Directions for future work

Although we have come quite a long way, many natural questions remain unanswered. In particular, we have not computed the cohomology of $\mathcal{M}_3[2]$, $\mathcal{Q}_1[2]$ and $\mathcal{M}_{3,1}[2]$ and we have not computed the cohomology groups of $\mathcal{Q}[2]$ as a $\mathrm{Sp}(6,\mathbb{F}_2)$-representations. Here we indicate possible routes to go further towards these goals. We also discuss some areas which can be explored using the methods of this thesis as well as some further problems which can be of interest.

9.4.1. Cohomology of $\hat{T}_{\mathscr{E}}$ One way to understand the cohomology of $\mathcal{Q}_1[2]$ would be to trace through the construction of Section 4.6.2 in order to obtain the $\mathrm{Sp}(6,\mathbb{F}_2)$-equivariant cohomology of $\hat{T}_{\mathscr{E}}$. Since the construction of Section 4.6.2 is rather involved, the computations are likely to become rather complicated. However, others have considered similar situations. In particular, there are general methods developed by Danilov [26], which have been applied by Dolgachev and Lunts [37], and Stembridge [83], to varieties rather similar to $\hat{T}_{\mathscr{E}}$. There are also results of Lehrer [60]. Even though this is a very nice starting point, it should be mentioned that their results are of a more general but less precise nature than what we are after (e.g. tables similar to Tables 6.1–6.6) and it is not clear how far one could go in this direction.

9.4.2. Gysin morphisms One of our main unsolved problems is the following: given a variety X and a closed subvariety $Y \subset X$ as well as the cohomology of Y and $X \setminus Y$ we want to patch together the cohomology of X. We do have the Gysin sequence

$$\cdots \to H^{k-2}(Y)\,(\text{-}1) \xrightarrow{\gamma} H^k(X) \to H^k(X \setminus Y) \xrightarrow{\text{res}} H^{k-1}(Y)\,(\text{-}1) \to \cdots$$

where γ is the Gysin map and res is the restriction map. Thus, understanding the Gysin or restriction morphisms would solve our problem.

The Gysin and restriction morphisms are probably best studied at the level of a complex computing the cohomology of X. One natural candidate is of course the de Rham complex, see [50], but we also have the Salvetti complex, see [28]. Some related work has been done, see for instance [34], but a lot of ground work needs to be done. In particular, we need to make sure that our complex works well equivariantly. We also recall that the cohomology groups involved are typically rather large and that we therefore will deal with large complexes. It is therefore essential that the computations involved in the construction of the complex are not too time consuming.

9.4.3. Self-associated point sets In Section 9.3 we were unable to compute the cohomology groups $H^4(\mathcal{Q}[2])$, $H^5(\mathcal{Q}[2])$ and $H^6(\mathcal{Q}[2])$ as $\mathrm{Sp}(6,\mathbb{F}_2)$-representations. The reason was that simply that we had too few equations in too many unknowns. One way to obtain a few extra equations is to consider the action of the larger subgroup $S_8 \subset \mathrm{Sp}(6,\mathbb{F}_2)$ on $\mathcal{Q}[2]$.

Let $P_1,\ldots,P_{2n+2}$ be a collection of points in $\mathbb{P}^n$ and let M be a matrix whose columns are representatives of $P_1,\ldots,P_n$. We say that the collection $P_1,\ldots,P_{2n+2}$ is *self-associated* if there is a diagonal $(2n+2)\times(2n+2)$-matrix Λ with non-zero entries along the diagonal such that

$$M\cdot\Lambda\cdot M^T = 0.$$

We say that 8 points $P_1,\ldots,P_8 \in \mathbb{P}^3$ are in *typical position* if no 4 points lie on a plane and no seven lie on a twisted cubic and we consider two such ordered collections to be equivalent if there is a projective transformation taking one to the other. Let $\mathcal{S}_8^3$ denote the moduli space of ordered octuples of self-associated points in $\mathbb{P}^3$ in typical position. We then have that $\mathcal{S}_8^3$ is isomorphic to $\mathcal{Q}[2]$, see see [38] and [49].

The nice feature of the above model of $\mathcal{Q}[2]$ is of course that the action of S_8 is in plain sight. However, while $\mathcal{P}_7^2$ is a open subset of the moduli space of seven ordered points in $\mathbb{P}^2$ the space $\mathcal{S}_8^3$ is a subvariety of codimension 3 in the moduli space of eight ordered points in $\mathbb{P}^3$ and this makes computations harder.

From the viewpoint of equivariant point counts with a computer we need to iterate through a set of (at least) dimension 9 instead of dimension 6. However, we know that the value of the equivariant Poincaré polynomial at a group element $\sigma \in S_8$ is a polynomial in q of degree 6 and since we have determined $H^0(\mathcal{Q}[2])$, $H^1(\mathcal{Q}[2])$, $H^2(\mathcal{Q}[2])$ and $H^3(\mathcal{Q}[2])$ we know four of its coefficients. Therefore we only need to make computations for 3 finite fields in order to determine $P^{S_8}(\mathcal{Q}[2],t)(\sigma)$. We have been able to carry out these computations for 3 of the conjugacy classes of S_8 which are not coming from S_7. In the notation of Table 6.7, the classes are $4B = [2^2,4]$, $6E = [2,3^2]$ and $15A = [3,5]$ and the point counts are as given in Table 9.1. Using the results

Class	$\lvert \mathcal{Q}[2]^{F\sigma} \rvert$
$4B$	$q^6 - 5q^5 + 6q^4 + 5q^3 - 10q^2 + 3$
$6E$	$q^6 - 4q^3 + 2q^2 - 2q + 3$
$15A$	$q^6 - q^2$

Table 9.1: Equivariant point counts for the classes $4B$, $6E$ and $15A$ of $\mathrm{Sp}(6,\mathbb{F}_2)$ (where σ denotes any element of the corresponding conjugacy class).

of Table 9.1 the possibilities for $H^4(\mathcal{Q}[2])$, $H^5(\mathcal{Q}[2])$ and $H^6(\mathcal{Q}[2])$ are reduced further. For instance, the set of feasible decompositions of $H^4(\mathcal{Q}[2])$ into irreducible representations of $\mathrm{Sp}(6,\mathbb{F}_2)$ is reduced from 1039 to 6.

9.4.4. Moduli of Del Pezzo surfaces Since our focus has been quartic curves we have almost exclusively considered Del Pezzo surfaces of degree 2. Nevertheless, there are of course Del Pezzo surfaces of degrees $1 \leq d \leq 9$. There are two types of Del Pezzo surfaces of degree 8, the blowup of $\mathbb{P}^2$ in one point and $\mathbb{P}^1 \times \mathbb{P}^1$, but for all other degrees d every Del Pezzo surface is isomorphic to the blowup of $\mathbb{P}^2$ in $9-d$ points in general position.

If $n \leq 4$, then all collections of n points are projectively equivalent. Thus, there are no moduli for Del Pezzo surfaces of degree $5 \leq d \leq 9$. On the other hand, the moduli space of Del Pezzo surfaces of degree $1 \leq d \leq 4$ has dimension $2 \cdot (5-d)$ and is isomorphic to $\mathcal{P}^2_{9-d}$. We may thus compute the S_{9-d}-equivariant cohomology of the moduli space of geometrically marked Del Pezzo surfaces of degree d in a manner similar to what we did in Chapter 7.

We may also mimic the constructions of Looijenga described in Chapter 4 in order to obtain models of moduli spaces of geometrically marked Del Pezzo surfaces described by various arrangements associated to root systems. More precisely, if X is a Del Pezzo surface of degree $1 \leq d \leq 6$, then the elements $D \in K_X^{\perp}$ such that $D^2 = -2$ form a root system Φ. From [65] we have

We may then compute the cohomology of these varieties equivariantly with respect to the various Weyl groups using the methods of Chapter 5.

d	1	2	3	4	5	6
$9-d$	8	7	6	5	4	3
Φ	E_8	E_7	E_6	D_5	A_4	$A_1 \times A_2$

Table 9.2: The root systems on $K_X^{\perp}$.

9.4.5. Cohomology of toric arrangements In Section 5.10.1 we determined the total cohomology of the complement of the toric arrangement associated to A_n as a representation of the Weyl group of A_n. A natural question is then if similar arguments will also work for the other classical series B_n, C_n and D_n.

In another direction, we have not explored the order preserving morphism from Equation 5.9.1

$$\tau : \mathscr{N}(\Phi) \to \mathscr{H}(\Phi),$$

fully. In particular, recall that when $\Phi = A_n$ we have that τ is an isomorphism. Lehrer and Solomon [61] and Lehrer [59] have results describing each cohomology group of the complement of the hyperplane arrangement associated to A_n as a representation of the Weyl group of A_n. It would be very interesting to see if these results can be translated to the toric world via τ.

9.4.6. Ring structures In this thesis we have only studied the additive structure on the various cohomology groups occurring. But the total cohomology is a graded ring and it is therefore natural to ask how the multiplicative structure looks like. A first small step in this direction is the following.

Proposition 9.4.1. *Let X be the complement of a toric arrangement associated to a root system Φ which either is of type A_n or is an irreducible root system of exceptional type. Then $H^*(X)$ is generated in degree* 1.

Proof. By the results of Deshpande and Sutar [34], we only need to check that the arrangement associated to Φ is of what they call "deletion-restriction type". For a root system of exceptional type, this is easily verified using our program and for a root system of type A_n this is a consequence of the corresponding arrangement being unimodular. □

It of course natural to ask if the above result holds for a general toric arrangement associated to a root system and Deshpande and Sutar [34] seem to have made good progress in this direction.

Perhaps more interesting are the relations in $H^*(X)$. Callegaro and Delucchi [19], have given a method for computing the cohomology ring of the complement of a toric arrangement but it is rather involved. Nevertheless, it would of course be of interest to give a computer implementation of their method. Thus, there is some hope to determine the ring structure of the cohomology, at least of the moduli spaces which can be described in terms of arrangements. It would of course also be very interesting, but even more challenging, to compute the cohomology of the moduli spaces which are not directly associated to root systems and the root system cases seem to be a natural first step towards this more ambitious goal.

9.4.7. Degenerations and compactifications Looijenga's description of $\mathcal{Q}_1[2]$ is very nice and it is of course desirable to obtain a similar description for a larger class of curves, for instance $\mathcal{M}_{3,1}[2]$ or some compactification of $\mathcal{M}_{3,1}[2]$. In the description of $\mathcal{Q}_1[2]$ in terms of self-associated point sets one can let the eight points degenerate so that they lie on a twisted cubic, see [38]. This determines a hyperelliptic curve of genus 3 so this is one type of degeneration that would be interesting to investigate further from this perspective.

In a slightly different direction, one could take an existing compactification of $\mathcal{M}_3[2]$ as starting point. It is quite hard to give a modular compactification of $\mathcal{M}_3[2]$ but at least one such compactification has been constructed by Abramovich, Corti and Vistoli [1].

9.4.8. Curves with more marked points If we consider the moduli spaces $\mathcal{M}_{3,n}[2]$ for $n \geq 2$ we do no longer have descriptions in terms of arrangements. However, there are still ways to study the cohomology of these spaces. For instance, we still have the Lefschetz trace formula and Proposition 2.4.2 provides a possibility to make point counts over finite fields for $\mathcal{Q}_n[2]$. In particular, it would be interesting to see if the counts would show a behaviour similar to that for $\mathcal{Q}_n$ observed in [9].

Appendices

A. A program for computing cohomology of complements of toric arrangements associated to root systems

Algorithms 5.8.1 and 5.9.2 provide a method for computing the cohomology of the complement of a toric arrangement associated to a root system. In this section we give a `SageMath` program implementing these algorithms. We remark that only minor modifications would be required in order to make a program that works for more general arrangements. We have not pursued this because it convenient to use the root system structure and since there is not always a group acting naturally on a general toric arrangement (in fact, most of the code deals with the group action so a more general program would be much shorter).

Let us point out that in the code below we are at times using parallel computations and at these times we use all available CPU's. If one wants to use the computer while running the computations or if one is using a shared computer it is probably advisable to only use a portion of the CPU's. One then changes the code `sage.parallel.ncpus.ncpus()` to a suitable integer wherever it occurs.

A.1 Generating initial data

Let Φ be a root system and let g be an element of the corresponding Weyl group. Here, we give the code of the subprogram for computing the $\langle g\rangle$-orbits of Φ, i.e. Step (1) in Algorithm 5.9.2.

```
# *******************************************************
#
#
# Functions for convenience.
#
#
# *******************************************************

```

```
# Takes a root system R and returns a set of
# representatives for the conjugacy classes of the
# corresponding Weyl group.
def classreps(R):
    G=R.ambient_space().weyl_group()
    return G.conjugacy_classes_representatives()

# ******************************************************
#
# Functions for generating the set of modules.
#
# ******************************************************

# Checks if u=+/-v.
def pmeq(u,v):
    if u == v:
        return True
    elif u == -v:
        return True
    else:
        return False

# Returns the roots of R as vectors.
def vecroots(R):
    PR=R.ambient_space().roots()
    SR=R.ambient_space().simple_roots()
    out=[]
    M=[]
    for s in SR:
        M.append(vector(s))
    M=matrix(M)
    for i in range(len(PR)):
        v=M.solve_left(vector(PR[i]))
        out.append(v)
    return out

# Takes a group element g (in matrix form) and a
# set of roots vR (a list of vectors) and returns the
# orbits of vR under the action of g (more precisely, the
# cyclic group generated by g) as a list of lists.
def orbs(g,vR):
    out=[]
    S=vR[:]
    while len(S)>0:
        s=g*S[0]
        orb=[S[0],-S[0]]
        while pmeq(S[0],s)==False:
            orb.append(s)
            orb.append(-s)
            S.remove(s)
            S.remove(-s)
            s=g*s
        out.append(orb)
        S.remove(-S[0])
        S.remove(S[0])
```

```
    return out

# Takes an element g of the Weyl group (in matrix form)
# and the root system R and returns a set of the minimal
# proper g-stable submodules of  the module corresponding
# to the root system.
def stabsubmods(g,R):
    out=[]
    vR=vecroots(R)
    orb=orbs(g,vR)
    for i in range(len(orb)):
        V=matrix(ZZ,orb[i]).echelon_form(
            include_zero_rows=False)
        out.append(V)
    return set(out)

# Takes a list vecs of integer vectors and returns an
# echelon matrix representing the Z-module generated by
# vecs.
def giveModule(vecs):
    M=matrix(vecs)
    return M.echelon_form(include_zero_rows=False)
```

A.2 Generating the set of modules

Let Φ be a root system and let g be an element of the corresponding Weyl group. Once the $\langle g\rangle$-orbits of Φ have been computed we may compute the set of all $\mathbb{Z}$-modules generated by unions of such orbits, i.e. Step (2) of Algorithm 5.9.2. Below we give the code of the subprogram performing these computations.

```
# Converts the input of newmodules to something passable
# to parnewmodules.
def argtopar(S1,S2,S3):
    arg=[]
    for s in S1:
        arg.append((s,S2,S3))
    return arg

# Takes a module M (from the initial set of modules) and
# a set of previously generated modules (prev) and a
# total set of modules (old) and generates all sums of a
# module in prev and startM.
@parallel(sage.parallel.ncpus.ncpus())
def parnewmodules(M,prev,old):
    out=[]
    for P in prev:
        temp=block_matrix(2,1,[P,M]).echelon_form
            (include_zero_rows=False)#Compute P+M
        if temp not in old:
            out.append(temp)
    return set(out)
```

```

# Converts the output of parnewmodules to a set of
# modules.
def tomodlist(gen,pgend):
    out=[]
    for G in gen:
        for g in G[1]:
            #if isinstance(g,basestring):
            #    print(g)
            out.append(g)
    return pgend.union(set(out))

# Splits a list L into a bunch of lists with n elements.
# Since not all lists have a length divisible by L, the
# final list may be shorter than len(L)/n.
def split_list(L,n):
    copL=L.copy()
    l=(len(L)/n).floor()
    r=len(L)-n*l
    out=[]
    for i in range(0,l):
        out.append(set([copL.pop() for j in
            range(i*n,(i+1)*n)]))
    if r>0:
        out.append(set([copL.pop() for j in
            range(l*n,l*n+r)]))
    return out

# Takes a module an initial set of modules (start), a set
# of previously generated modules (prev) and a total set
# of modules (old) and generates all sums of a module in
# "prev" and a module in "start" which do not occur in
# "old".
def newmodules_par(start,prev,old,ct):
    ncpus=Integer(sage.parallel.ncpus.ncpus())
    lists=split_list(start,ncpus)
    nmds=set([])
    i=1
    for l in lists:
        print((i/len(lists)).n())
        f=open(str(ct)+"-"+str(i)+".sage","w")
        f.write(str((i/len(lists)).n()))
        f.close()
        i=i+1
        arg=argtopar(l,prev,old)
        gen=parnewmodules(arg)
        nmds=tomodlist(gen,nmds)
    return nmds

# Takes a set of previously (prev) generated modules, an
# initial set of modules (start) and an total set of
# modules (old) and generates the next set of modules.
def newmodules(start,prev,old):
    out=set([])
    for i in range(len(start)):
```

```
        for j in range(len(prev)):
            n=i*len(prev)+j
            if mod(n,1000)==0:
                print(n)
            temp=start[i]+prev[j]
            if temp not in old:
                out=out.union(set([temp]))
    return out

# Generates the set of g-stable submodules without poset
# structure.
def setofmodules(startlist):#, Integer n):
    t0=walltime()
    tcurr=walltime()
    start=startlist
    old=startlist
    prev=startlist
    ct=1
    while len(prev)>0:
        print(str(len(prev))+" sums of "+str(ct)+"
            modules were generated in
            "+str(walltime()-tcurr)+" seconds.")
        f=open("Modules"+str(ct)+".sage","w")
        f.write(str(len(prev))+" sums of "+str(ct)+"
            modules were generated in
            "+str(walltime()-tcurr)+" seconds.")
        f.close()
        tcurr=walltime()
        prev=newmodules_par(start,prev,old,ct)
        old=old.union(prev)
        ct=ct+1
    ZM=matrix([0]*1).echelon_form(
        include_zero_rows=False)
    old=old.union(set([ZM]))
    print("A set containing "+str(len(old))+" elements
        was generated in "+str(walltime()-t0)
        +" seconds.")
    return old

def convert_module_list_out(L):
    out=[]
    for l in L:
        temp=[]
        if l.rank()==0:
            v=vector([0]*l.degree())
            temp.append(v)
        else:
            for v in l.echelonized_basis():
                temp.append(v)
        out.append(temp)
    return out

def convert_module_list_in(L):
    out=[]
    for l in L:
```

```
        m=matrix(l)
        out.append(span(m,ZZ))
    return out

def save_modules(L,listname,filename):
    print("Saving list to \""+listname+"\" in file
        \""+filename+".sage\".")
    f=open(str(filename)+".sage","w")
    out=convert_module_list_out(L)
    f.write(listname+"="+str(out))
    f.close()
    print("List saved.")
```

A.3 Generating the poset

Let Φ be a root system and let g be an element of the corresponding Weyl group. Once the underlying set of $\mathscr{P}^g(\Phi)$ has been computed we need to, for each module in $\mathscr{P}^g(\Phi)$ check which other modules of $\mathscr{P}^g(\Phi)$ it is included in, i.e. Step (3) of Algorithm 5.9.2. Below we give the code of the subprogram performing these computations.

```
# Converts the input of gen_rels to something passable
# to par_gen_rels.
def argtopar_gen_rels(L,S):
    arg=[]
    for l in L:
        arg.append((l,S))
    return arg

@parallel(sage.parallel.ncpus.ncpus())
def par_gen_rels(M,S):
    m=matrix(1,len(S),sparse=True)
    #m=[0 for s in S]
    for i in range(len(S)):
        MS=block_matrix(2,1,[M,S[i]]).echelon_form
            (include_zero_rows=False) #Generates M+S[i]
        if S[i]==MS: #Checks if M is a submodule of S[i]
            #m[i]=1
            m[0,i]=1
    return m

# Converts the output of par_gen_rels to something
# useful.
def tolists(gen,pgendrel,pgendel):
    outrel=[]
    outel=[]
    for G in gen:
        outrel.append(G[1])
        outel.append(G[0][0][0])
    outrel=block_matrix(len(outrel),1,outrel)
    if pgendrel!=[]:
        outrel=block_matrix(2,1,[pgendrel,outrel])
```

```
    outel=pgendel+outel
    return [outrel,outel]

# Splits a list L into a bunch of lists with n elements.
# Since not all lists have a length divisible by L, the
# final list may be shorter than len(L)/n.
def split_list(L,n):
    l=(len(L)/n).floor()
    r=len(L)-n*l
    out=[]
    for i in range(0,l):
        out.append([L[j] for j in range(i*n,(i+1)*n)])
    if r>0:
        out.append([L[j] for j in range(l*n,l*n+r)])
    return out

# Takes two lists L and S with the same entries but in
# different order and returns a list of integers such
# that the i'th entry is the index of L[i] in S.
def give_inds(L,S):
    out=[]
    for l in L:
        b=l==S[0]
        i=0
        while b==False:
           i=i+1
           b=l==S[i]
        out.append(i)
    return out

# Takes a list L and a list of integers inds and returns
# a list with the same entries as L but with the order
# specified by inds.
def perm_list(L,inds):
    out=[0 for i in inds]
    for i in range(len(inds)):
        out[inds[i]]=L[i]
    return matrix(out,sparse=True)

# Computes the inclusion relations of the set of
# modules S.
def gen_rels(S):
    t0=walltime()
    rels=[]
    els=[]
    L=split_list(S,Integer(
        10*sage.parallel.ncpus.ncpus()))
    ct=0
    for l in L:
        #if Mod(ct,8)==0:
        print("The relations of the poset has been "
            +str((100*ct/len(L)).n(digits=3))+"%"
            +" completed in "+str(walltime()-t0)
            +" seconds.")
        arg=argtopar_gen_rels(l,S)
```

```
        gen=par_gen_rels(arg)
        temp=tolists(gen,rels,els)
        rels=temp[0]
        els=temp[1]
        ct=ct+1
    inds=give_inds(els,S)
    rels=perm_list(rels,inds)
    print("The relations of the poset were computed in
        "+str(walltime()-t0)+" seconds.")
    return rels


def add_zero(n):
    if n<10:
        return str(0)+str(n)
    else:
        return str(n)

def give_date():
    from datetime import datetime
    now=datetime.now()
    return str(now.year)+"-"+add_zero(now.month)+"-
        "+add_zero(now.day)+", at "+add_zero(now.hour)+
        ":"+add_zero(now.minute)+":"+add_zero(now.second)


# Generates a subset of the inclusion relations.
def part_gen_rels(S,stind,endind,list_name,file_name):
    Spart=[S[i] for i in range(stind,endind+1)]
    t0=walltime()
    rels=[]
    els=[]
    L=split_list(Spart,Integer(
        10*sage.parallel.ncpus.ncpus()))
    ct=0
    for l in L:
        print(give_date())
        print("The relations of the poset have been "
            +str((100*ct/len(L)).n(digits=3))+"%"
            +" completed in "+str(walltime()-t0)
            +" seconds.")
        arg=argtopar_gen_rels(l,S)
        gen=par_gen_rels(arg)
        temp=tolists(gen,rels,els)
        rels=temp[0]
        els=temp[1]
        ct=ct+1
    inds=give_inds(els,Spart)
    rels=perm_list(rels,inds)
    rels=sparse_zero_one_out(rels)
    print("A list of "+str(endind-stind+1)
        +" relations were computed in "
        +str(walltime()-t0)+" seconds.")
    save_rels(rels,list_name,file_name)
    return rels
```

```
# Generates N subsets of the inclusion relations.
def many_gen_rels(S,stind,endind,N,Rstr):
    n=(endind-stind+1)/N
    for i in range(1,N+1):
        si=stind+(i-1)*n
        ei=stind+i*n-1
        endstr=str(si)+"_"+str(ei)
        ln="R_"+endstr
        fn=Rstr+"_rels_"+endstr
        rels=part_gen_rels(S,si,ei,ln,fn)
    print("Done!")

# Computes the Mobius function of the poset given by
# rel_list with respect to the maximal element. The
# maximal element is assumed to correspond to the first
# entry.
def mob(rel_list):
    M=matrix(rel_list)
    v=matrix(1,len(rel_list))
    v[0,0]=1
    return M.solve_left(v).list()

def save_rels(rels,list_name,file_name):
    print("Saving relations to \""+list_name+
        "\" in the file \""+file_name+".sage\".")
    f=open(file_name+".sage","w")
    f.write(list_name+"="+str(rels))
    f.close()
    print("Saving complete.")

# Converts a matrix of zeros and ones to the form
# [nrows,ncols,
# [list of column indicies where row 1 has a 1],...,
# [list of column indicies where row nrows has a 1]]
def sparse_zero_one_out(M):
    m=M.dimensions()[0]
    n=M.dimensions()[1]
    out=[m,n]
    for i in range(m):
        temp=[]
        for j in range(n):
            if M[i,j]==1:
                temp.append(j)
        out.append(temp)
    return out

def sparse_zero_one_in(L):
    m=L[0]
    n=L[1]
    out=matrix(m,n)
    for i in range(n):
        r=L[i+2]
        for j in range(len(r)):
            out[i,r[j]]=1
```

```
    return out
```

A.4 Computing the Poincaré polynomial

Let Φ be a root system and let g be an element of the corresponding Weyl group. Once $\mathscr{P}^g(\Phi)$ with all inclusion relations has been computed we may compute the $\langle g\rangle$-equivariant Poincaré polynomial for each element of $\mathscr{P}^g(\Phi)$ using Algorithm 5.8.1 and then use Corollary 5.4.3 to compute the $\langle g\rangle$-equivariant Poincaré polynomial of the complement of the toric arrangement associated to Φ.

```
# Takes a matrix g and a positive integer n and returns
# a list of powers of g, [g,g^2,...,g^n].
def gpows(g,n):
    out=[g]
    temp=g
    for i in range(n-1):
        temp=temp*g
        out.append(temp)
    return out

# Takes a sequence of k numbers (the traces of g acting
# on V) and returns the result of performing
# Newton-Girard on these numbers (the traces of g acting
# on wedge 0,1,...,k of V).
def newtongirard(traces):
    trcs=[1]+traces
    k=len(trcs)
    trw=[0]*k
    trw[0]=trcs[0]
    for i in range(k-1):
        temp=0
        for j in range(i+1):
            temp=temp+(-1)**j*trcs[j+1]*trw[i-j]
        temp=temp/(i+1)
        trw[i+1]=temp
    return trw

# Computes the trace of g on the module M.
def M_trace(g,M):
    out=0
    for i in range(M.matrix().dimensions()[0]):
        out=out+M.matrix().solve_left(g*M.matrix()[i])[i]
    return out

# Computes the number of tori in Hom(Amb/M,C^*) which
# are setwise fixed by g.
def M_mult(g,M,Amb):
    Q=Amb/M
    invs=Q.invariants()
    i=0
    vecs=[vector(M.zero())]
```

```
    while i<len(invs) and invs[i]>1:
        v=Q.gen(i).lift()
        newvecs=[]
        for j in range(1,invs[i]):
            for k in range(len(vecs)):
                newvecs.append(vecs[k]+j*v)
        vecs=vecs+newvecs
        i=i+1
    out=0
    for v in vecs:
        if Q(v-g*v).is_zero():
            out=out+1
    return out

# Computes the <g>-equivariant Poincare polynomial
# of the torus defined by M.
def M_polynomial(gpws,Ambtraces,M):
    k=len(Ambtraces)-M.rank()
    Qtraces=[]
    for i in range(k):
       Qtraces.append(Ambtraces[i]-M_trace(gpws[i],M))
    wedges=newtongirard(Qtraces)
    t=var("t")
    pol=0
    for i in range(len(wedges)):
        pol=pol+wedges[i]*t^i
    return pol

# Copmutes the <g>-equivariant Poincare polynomial of
# the toric arrangement associated to r (whose
# g-invariant subposet is S with Mobius funciton mu).
def torpol(g,r,S,mu):
    pol=0
    simp=r.ambient_space().simple_roots()
    Amb=[]
    for s in simp:
        Amb.append(vector(s))
    Amb=span(Amb,ZZ)
    d=Amb.rank()
    gpws=gpows(g,d)
    Ambtraces=[]
    t=var("t")
    for i in range(len(gpws)):
        Ambtraces.append(M_trace(gpws[i],Amb))
    for i in range(len(S)):
        if mu[i]!=0:
            mult=M_mult(g,S[i],Amb)
            pol=pol+mult*mu[i]*M_polynomial(
                gpws,Ambtraces,S[i])*(-t)**(S[i].rank())
    return expand(pol)
```

B. A program for equivariant counts of seven points in general position

In this appendix we give an example of a `Maple`™[1] [66] program for counting seven points in general position over a finite field equivariantly with respect to an element in S_7. In the following example, the element is the permutation (12345)(67).

One remark is in order. The moduli space $\mathcal{P}_7^2$ has dimension 6. Thus, counting the points over the finite field $\mathbb{F}_q$ with q elements has complexity $\mathcal{O}(q^6)$. However, if we consider a group element σ, we need to make computations over $\mathbb{F}_{q^d}$ where d is the least common multiple of the lengths of the cycles in σ. Also, depending on σ it might be difficult to get rid of the action of PGL(3) whereby we can lose up to 8 dimensions. Thus, we may end up with complexity $\mathcal{O}(q^{14d})$. Of course, the variety is still of dimension 6 and there should therefore be ways to cut the complexity back down to $\mathcal{O}(q^6)$ although this is not always easy. However, some easy tricks are useful. For instance, one can always place up to four $\mathbb{F}_q$-rational points at four specified points in general position and the simple observation that two conjugate $\mathbb{F}_{q^2}$-points define a line defined over $\mathbb{F}_q$ also helps.

```
# Takes a field F=GF(p,n), a prime p and a vector
# v=[x,a,b] where x is assumed to be 0 or 1.
# Returns Frob(v), i.e. [x^p,a^p,b^p].
frob:=proc(F,p,v)
    return [v[1],F:-`^`(v[2],p),F:-`^`(v[3],p)]:
end proc:

# Checks if two points v and w with entries in
# the field F are equal.
vequals := proc(F,v,w)
    local i,tf:
    for i from 1 to 3 do
        tf := F:-`=`(v[i],w[i]):
        if evalb(tf=false) then
            return tf:
```

[1] Maple™ is a trademark of Waterloo Maple Inc.

```
            end if:
        end do:
        return tf:
end proc:

# Takes two points a and b with entries in the field F
# and produces the coefficients of the equation
# for the line through a and b.
giveline:=proc(F,a,b)
        local x,y,z:
        x:=F:-`-`(F:-`*`(a[2],b[3]),F:-`*`(a[3],b[2])):
        y:=F:-`-`(F:-`*`(a[3],b[1]),F:-`*`(a[1],b[3])):
        z:=F:-`-`(F:-`*`(a[1],b[2]),F:-`*`(a[2],b[1])):
        return [x,y,z]:
end proc:

# Takes the coefficients of the equations of two lines
# and computes the point of intersection.
# By projective duality, this is computed by giveline
# (so this procedure is mainly for purposes of code
# readability).
intlines:=proc(F,l1,l2)
        return giveline(F,l1,l2):
end proc:

# Checks if a point pt lies on a line L.
# L is given by the coefficients of its defining
# equation. L and pt both have coefficients
# in the field F.
online:=proc(F,L,pt)
        return F:-`=`(F:-`+`(F:-`*`(L[1],pt[1]),F:-`+`
            (F:-`*`(L[2],pt[2]),F:-`*`
            ((L[3],pt[3])))),F:-ConvertIn(0)):
end proc:

# Takes six points a1, a2, a3, b1, b2 and b3
# with coefficients in the field F and checks
# if they lie on a conic.
# The check is by means of Pascal's theorem
# which states that the new intersection points
# of the six lines
# <a1,b2>, <a1,b3>, <a2,b1>, <a2,b3>, <a3,b1> and
# <a3,b2> lie on a line iff the six points lie
# on a conic.
onconic:=proc(F,a1,a2,a3,b1,b2,b3)
        local l1,l2,i12,i13,i23,L:
        l1:=giveline(F,a1,b2):
        l2:=giveline(F,a2,b1):
        i12:=intlines(F,l1,l2):
        l1:=giveline(F,a1,b3):
        l2:=giveline(F,a3,b1):
        i13:=intlines(F,l1,l2):
        l1:=giveline(F,a2,b3):
        l2:=giveline(F,a3,b2):
        i23:=intlines(F,l1,l2):
```

```
    L:=giveline(F,i12,i13):
    return online(F,L,i23):
end proc:

# Checks if the point pt lies on any of the lines
# in Ls.
linestest:=proc(F,Ls,pt)
    local i:
    for i from 1 to numelems(Ls) do
        if evalb(online(F,Ls[i],pt)=true) then
            return false:
        end if:
    end do:
    return true:
end proc:

# Checks if at least six of the points
# p1, p2, p3, p4, p5, p6 and p7 lie on
# a conic.
contest:=proc(F,p1,p2,p3,p4,p5,p6,p7)
    if onconic(F,p2,p3,p4,p5,p6,p7) then
        return false:
    elif onconic(F,p1,p3,p4,p5,p6,p7) then
        return false:
    elif onconic(F,p1,p2,p4,p5,p6,p7) then
        return false:
    elif onconic(F,p1,p2,p3,p5,p6,p7) then
        return false:
    elif onconic(F,p1,p2,p3,p4,p6,p7) then
        return false:
    elif onconic(F,p1,p2,p3,p4,p5,p7) then
        return false:
    elif onconic(F,p1,p2,p3,p4,p5,p6) then
        return false:
    end if:
    return true:
end proc:

# Produces a conjugate 5-tuple in general position.
fivepoints:=proc(F,p,e,y,i,j)
    local a,b:
    local p1,p2,p3,p4,p5:
    local l12,l13,l23,l14,l24,l34,l15,l25,l35,l45:
    a:=F:-`^`(y,i):
    b:=F:-`^`(y,j):
    p1:=[e,a,b]:
    p2:=frob(F,p,p1):
    l12:=giveline(F,p1,p2):
    p3:=frob(F,p,p2):
    if linestest(F,[l12],p3) then
        l13:=giveline(F,p1,p3):
        l23:=giveline(F,p2,p3):
        p4:=frob(F,p,p3):
        if linestest(F,[l12,l13,l23],p4) then
            p5:=frob(F,p,p4):
```

```
                l14:=giveline(F,p1,p4):
                l24:=giveline(F,p2,p4):
                l34:=giveline(F,p3,p4):
                if linestest(F,[l12,l13,l23,
                             l14,l24,l34],p5)  then
                    l15:=giveline(F,p1,p5):
                    l25:=giveline(F,p2,p5):
                    l35:=giveline(F,p3,p5):
                    l45:=giveline(F,p4,p5):
                    return [p1,p2,p3,p4,p5,
                           [l12,l13,l23,l14,l24,
                           l34,l15,l25,l35,l45],true]:
                end if:
            end if:
        end if:
        return [0,0,0,0,0,0,false]:
end proc:

# Takes a conjugate 5-tuple and iterates through all
# conjugate pairs.
twopoints:=proc(F,p,e,o,z,p1,p2,p3,p4,p5,L)
    local a,b,i,j:
    local p6,p7:
    local out:=0:
    for i from 1 to p^2-1 do
        a:=F:-`^`(z,i):
        p6:=[o,e,a]:
        p7:=frob(F,p,p6):
        if evalb(vequals(F,p6,p7)=false) and
           linestest(F,L,p6) and
           evalb(onconic(F,p1,p2,p3,p4,p5,p6)=false)
           then
            out:=out+1:
        end if:
        p6:=[e,o,a]:
        p7:=frob(F,p,p6):
        if evalb(vequals(F,p6,p7)=false) and
           linestest(F,L,p6) and
           evalb(onconic(F,p1,p2,p3,p4,p5,p6)=false)
           then
            out:=out+1:
        end if:
        p6:=[e,a,o]:
        p7:=frob(F,p,p6):
        if evalb(vequals(F,p6,p7)=false) and
           linestest(F,L,p6) and
           evalb(onconic(F,p1,p2,p3,p4,p5,p6)=false)
           then
            out:=out+1:
        end if:
        for j from 1 to p^2-1 do
            b:=F:-`^`(z,j):
            p6:=[e,a,b]:
            p7:=frob(F,p,p6):
            if evalb(vequals(F,p6,p7)=false) and
```

```
                linestest(F,L,p6) and
                evalb(onconic(F,p1,p2,p3,p4,p5,p6)
                    =false)
                then
                 out:=out+1:
            end if:
        end do:
    end do:
    return out:
end proc:

# Counts the number of conjugate 7-tuples for a
# permutation of cycle type [2,5].
# Note that the result needs to be divided by
# |PGL(3)|.
ftcount:=proc(F,p)
    local o:=F:-ConvertIn(0), e:=F:-ConvertIn(1):
    local x:=F:-PrimitiveElement(F):
    local y:=F:-`^`(x,(p^10-1)/(p^5-1)):
    local z:=F:-`^`(x,(p^10-1)/(p^2-1)):
    local i,j,P,out:=0:
    for i from 1 to p^5-1 do
        print(i):
        for j from 1 to p^5-1 do
            P:=fivepoints(F,p,e,y,i,j):
            if P[7] then
                out:=out+twopoints(F,p,e,o,z,
                    P[1],P[2],P[3],P[4],P[5],P[6]):
            end if:
        end do:
    end do:
    return out:
end proc:
```

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Matematiska Institutionen, Stockholms Universitet, 106 91 Stockholm
E-mail address: olofberg@math.su.se

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