# Waring-type problems for polynomials Algebra meets Geometry 

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Algebra meets Geometry
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## SAMMANFATNNING

I denna avhandling analyserar vi olika typer av additiva uppdelningar av homogena polynom. Dessa problem kallas vanligen vara av Waringtyp, och deras historia går tillbaka till mitten av 1800-talet. Nyligen har de uppmärksammats av en stor grupp matematiker och ingenjörer på grund av de många tillämningar de har. Samtidigt har de anknytning till grenar av Kommutativ algebra och Algebraisk geometri.

Det klassiska Waringproblemet handlar om uppdelning av homogena polynom som summor av potenser av linjärformer. Via s.k. Apolarity Theory, är studiet av uppdelningen för ett givet polynom $F$ relaterad till studiet av punkter "apolära" till $F$, nämligen konfigurationer av punkter vars ideal är innehållna i "perpidealet" till $F$. Speciellt analyserar vi vilka minimala punktmängder som kan vara apolära till ett givet polynom i fall av låg grad och litet antal variabler. Detta tillåter oss introducera begreppet Waring loci för homogena polynom.

Från en geometrisk synpunkt kan frågor om additiva uppdelningar av polynom beskrivas med hjälp av sekantvarieteter av projektiva varieteter. Speciellt är vi intresserade av dimensionen av dessa varieteter. Genom att använda ett gammalt resulatat av Terracini, kan vi bestämma dessa dimensioner genom beräkna Hilbertserier för homogena ideal.

Hilbertserier är viktiga invarianter för homogena ideal. För det klassiska Waringproblemet måste vi titta på s.k. power ideals, d.v.s. ideal genererade av potenser av linjärformer. Via Apolarity Theory är deras Hilbertserier relaterade till Hilbertserier för ideal till "fat points", d.v.s. ideal för konfigurationer av punkter med multiplicitet. I avhandlingen betraktar vi några speciella konfigurationer av feta punkter. Hilbertseries till ideal av feta punkter är ett mycket aktivt forskningsområde. Vi förklarar hur det är relaterat till Fröbergs kända förmodan om Hilbertseries till generiska ideal, vilken är öppen sedan 1985.

Dessutom använder vi Fröbergs förmodan till att härleda dimensionen av flera sekantvarieteter av projektiva varieteter, för att sedan härleda resultat om speciella problem av Waringtyp för polynom.

I avhandlingen arbetar vi för det mesta över komplexa talen. Emellertid analyserar vi också klassiska Waringdekompositioner över de reella
talen. Speciellt klassificerar vi för vilka monom den minimala längden av uppdelningen i summor av linjärformer är oberoende av om kroppen är den komplexa eller den reella.

## Preface

Classical Waring problem investigates additive decompositions of homogeneous polynomials as sums of powers of linear forms. This is the symmetric version of a problem about additive decompositions of tensors. Recently, these kinds of problems attracted the attention of a large community of mathematicians and engineers. They have several applications in different fields, such as: Algebraic Statistics (in statistical models with hidden variables), Phylogenetic (in reconstructions of evolutionary trees of DNA), Signal Processing (in decompositions of signals) and others.

In the present thesis we analyze different types of additive decompositions of homogeneous polynomials. We also see how these problems are related to questions in Commutative Algebra and Algebraic Geometry.

## Overview

Chapter 1: Introduction. First, we give a historical introduction of the topic. We also explain how questions about additive decompositions of polynomials can be described geometrically in terms of secant varieties.

ChAPter 2: APOLARITY THEORY AND POINT CONFIGURATIONS. In this chapter, we introduce the basic notions of apolarity theory. In particular, we see how the Waring decompositions of a homogeneous polynomial $F$ as sums of of powers of linear forms are related to set of points apolar to $F$, i.e., to configurations of points contained in the "perp" ideal of $F$ (Apolarity Lemma 2.1.13). In Section 2.3, we study which kinds of minimal set of points can be apolar to particular families of homoegeneous polynoimals (quadrics, monomials, binary forms and plane cubics). This is part of a joint work with Carlini and Catalisano [CCO] where we introduce the notion of Waring loci of homogeneous polynomials.

In the second part of the chapter, we describe a more general version of Apolarity Lemma which allows us to relate power ideals (i.e., ideals generated by powers of linear forms) and ideals of configurations of fat points
(i.e., configurations of points considered with certain multiplicity). In Section 2.7, we study Hilbert series and Betti numbers of special configurations of fat points. These computations have been already presented in [One14; BO15].

Chapter 3: Fröberg's conjecture and dimensions of secant varieties. The aim of this chapter is to view Fröberg's conjecture on the Hilbert series of generic ideals in relation to more recent problems. In Section 3.2 , we analyze the relation between power ideals that fail to have the Hilbert series prescribed by Fröberg's conjecture and linear systems of hypersurfaces in projective spaces which are linearly special with respect to the definition given by Brambilla, Dumitrescu and Postinghel [BDP15a].

In Section 3.3, we relate Fröberg's conjecture to the computation of the dimensions of secant varieties of varieties of powers, i.e., varieties whose points are $d$-th powers of homogeneous polynomials of degree $k$. In particular, we review computations already explained in [One14] by formulating a general conjecture on the dimensions of these secant varieties. Such a conjecture has been suggested in private communication by Ottaviani.

In Section 3.4, we study secant varieties of varieties of $\mu$-power, i.e., varieties whose points are $\mu$-powers of linear forms; namely, homogeneous polynomials of type $L_{1}^{m_{1}} \cdots L_{s}^{m_{s}}$, where $\mu=\left(m_{1}, \ldots, m_{s}\right)$ is a partition of a positive integer $d$. This is part of a joint work with Catalisano, Chiantini and Geramita [CCGO15]. This questions have been already considered in the literature for special partitions as $\mu=(1, \ldots, 1)$ and $\mu=(d-1,1)$.

Chapter 4: Decompositions of monomials. In this chapter, we study special additive decompositions of monomials.

In Section 4.1, we consider the classical Waring decompositions as sums of powers of linear forms over the fiels of real numbers instead of the field of complex numbers. In particular, we characterize for which monomials the smallest lenght of such a decomposition does not depend on the ground field. This is a joint work with Carlini, Kummer and Ventura [CKOV16].

In Section 4.2, we study decompositions of monomials as sums of powers of degree $k$ forms over the complex numbers. We get partial results in case of small degrees and small number of variables. This is a joint work with Carlini [CO15], already presented in [One14].

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under their supervision represent only a first step for future works and I will always be grateful to them when, hopefully, I will find more answers to their interesting questions.

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## CHAPTER 1

## INTRODUCTION

After a historical introduction, in this chapter we explain what kinds of $a d$ ditive decompositions of homogeneous polynomials we want to consider. In particular, we see how these explicit problems fit into several other interesting topics. We relate them to geometric questions about the secant varieties of projective varieties, on one side, and to algebraic questions about Hilbert functions of homogeneous ideals, on the other side.

## SECTION 1.1

## AdDitive decompositions of integers

In 1770, the English number theorist E. Waring (1736-1798), in his paper Meditationes Algebraicae, stated, with no proof, that
every natural number is a sum of at most 9 cubes;
every natural number is a sum of at most 19 fourth powers;
and so on...
Apparently, he believed that, for every natural number $d \geq 2$, there exists a number $N(d)$ such that every positive integer $n$ can be written as

$$
n=a_{1}^{d}+\ldots+a_{N(d)}^{d}, \quad a_{i} \in \mathbb{Z}_{>0}
$$

The smallest number with that property is denoted by $g(d)$. We had to wait more than a century for a proof of this statement.

Theorem 1.1.1 (Hilbert, 1909). For any $d \geq 2, g(d)$ exists.

In fact, from the famous Lagrange's four-square Theorem (1770), we know that $g(2)=4$; later, Wieferich and Kempner showed that $g(3)=9$ [Wie08; Kem12]. In 1986, Balasubramanian, Dress and Deshouillers established that $g(4)=19$ [BDD86]. In general, we have the conjectured formula ${ }^{1} g(d)=2^{d}+\left[(3 / 2)^{d}\right]-2$. As a consequence of the work of many mathematicians, it is known that this is true when $2^{d}\left\{(3 / 2)^{d}\right\}+\left[(3 / 2)^{d}\right] \leq 2^{d}$. In [Mah36], Mahler proved that there are only finitely many counterexamples to the inequality. Due to massive computer checking, it is believed that it is always true and that the value of $g(d)$ conjectured is correct.

In [Dav39], Davenport proved that any sufficiently large integer can be written as the sum of at most 16 fourth powers. Hence, for any $d \geq 2$, we define $G(d)$ as the least integer such that all sufficiently large integers can be expressed as sum of at most $G(d) d$-th powers of integers. Clearly, $G(d) \leq g(d)$. Gauss observed that every number congruent to $7 \bmod (8)$ is a sum of four squares, which proves that $G(2)=g(2)=4$; but the inequality can be strict. Indeed, it has been shown, for example, that $G(3) \leq 7$ and $G(4)=16$. However, only little is known about numbers $G(d)$ and this problem is currently a very active area of research in number theory.

## SECTION 1.2

## ADDITIVE DECOMPOSITIONS OF POLYNOMIALS

A standard result of linear algebra is that, given a quadratic homogeneous polynomial, or form, $Q$ in $n$ variables over a field $\mathbb{K}$ of characteristic different from two, we can find a symmetric matrix $M_{Q}$ such that $Q(x)=x M_{Q} x^{t}$, where $x \in \mathbb{K}^{n}$. Then, after a suitable change of coordinates, we can diagonalize the matrix $M_{Q}$ and we can represent $Q$ as a sum of $s$ squares of linear forms if and only if the matrix $M_{Q}$ has rank smaller than or equal to $s$. We consider similar decompositions for higher degree polynomials.

Since the 19-th century, geometers and algebraists seek for normal forms of (homogeneous) polynomials. For example, Hilbert, in his famous list of 23 problems posed in 1900, asked the following.

Hilbert's 17Th Problem. Is it true that any multivariate complex polynomial that takes only non-negative values over the reals can be written as a sum of squares of rational functions?

An affirmative answer was given by Artin in 1927 [Art27].
During the last decades, due to practical needs to decompose homogeneous polynomials and tensors, these types of questions again attracted the attention of many researchers, from pure mathematicians to engineers.
${ }^{1}$ If $x \in \mathbb{Q}$, we denote with $[x]$ and $\{x\}$ the integral and fractional part of $x$, respectively.

### 1.2.1 Classical Waring decompositions

Due to the nature of the question and its similarity with the problems for integers formulated by Waring, these problems about additive decompositions of polynomials are called Waring problems for polynomials.

We work over the field of complex numbers, except for Section 4.1, where we consider Waring decompositions over the field of real numbers.

Definition 1.2.1. Given a homogeneous polynomial $F$ of degree $d$ in $n+1$ variables, we define the Waring rank of $F$ as the smallest number of linear forms needed to decompose $F$ as a sum of their powers, i.e.,

$$
\operatorname{rk}(F):=\min \left\{s \mid \exists L_{1}, \ldots, L_{s} \text { linear forms s.t. } F=L_{1}^{d}+\ldots+L_{s}^{d}\right\}
$$

We refer to decompositions of $F$ as sums of powers of linear forms of length equal to the rank as Waring decompositions of $F$.

In the projective space $\mathbb{P}\left(S_{d}\right)$ of degree $d$ homogeneous polynomials ${ }^{2}$, we consider the set of polynomials of rank $r$, i.e.,

$$
U_{d, r}:=\left\{[F] \in \mathbb{P}\left(S_{d}\right) \mid \operatorname{rk}(F)=r\right\} .
$$

Observe that these sets are not necessarily closed as in the following.
Example 1.2.2. Consider $M=x y^{2} \in \mathbb{C}[x, y]_{3}$. It is a straightforward computation to check that $\operatorname{rk}(M)$ cannot be two. However, we can easily see that $M$ is contained in the closure of $U_{3,2}$. Indeed

$$
x y^{2}=\lim _{t \rightarrow 0} \frac{1}{3 t}\left(y^{3}+(t x-y)^{3}\right) .
$$

Definition 1.2.3. Fixing positive integers $n, d$ as above, we define:

1. the generic rank, i.e., the rank of the generic form in $S_{d}$, namely the rank that occurs in a Zariski open subset ${ }^{3}$ of $\mathbb{P}\left(S_{d}\right)$.
We denote it by $\mathrm{rk}^{\circ}(n+1, d)$.
2. the maximal rank, i.e., the maximal rank among all forms in $S_{d}$. We denote it by $\mathrm{rk}^{\max }(n+1, d)$.
Consideration of Waring problems for homogeneous polynomials was initiated by Sylvester who considered the case of two variables.

Theorem 1.2.4 (Sylvester [Syl51]). A generic binary form of degree d can be written as a sum of $\left\lceil\frac{d+1}{2}\right\rceil$ powers of linear forms, i.e., $\mathrm{rk}^{\circ}(2, d)=\left\lceil\frac{d+1}{2}\right\rceil$.

[^0]Decompositions of homogeneous polynomials as sums of powers of linear forms were the first natural kind of additive decompositions under consideration. Since the mid-19th century, they attracted a lot of attention especially due to their connections with the classic geometry of secant varieties. However, we had to wait until 1995 when a substantial generalization of Sylvester's result was proven. After a series of enlightening papers AH92a; AH92b; AH95], Alexander and Hirschowitz obtained the Waring rank of generic forms in any degree and any number of variables.

Theorem 1.2.5 (Alexander-Hirschowitz [AH95]). The Waring rank of the generic degree $d$ form in $n+1$ variables is $\mathrm{rk}^{\circ}(n+1, d)=\left\lceil\frac{\binom{n+d}{n}}{n+1}\right\rceil$, except for:

1. $d=2$, any $n$, where $\mathrm{rk}^{\circ}(n+1,2)=n+1$;
2. $n=2, d=4$, where $\operatorname{rk}^{\circ}(3,4)=6$ and not 5 as expected;
3. $n=3, d=4$, where $\operatorname{rk}^{\circ}(4,4)=10$ and not 9 as expected;
4. $n=4, d=3$, where $\operatorname{rk}^{\circ}(5,3)=8$ and not 7 as expected;
5. $n=4, d=4$, where $\operatorname{rk}^{\circ}(5,4)=15$ and not 14 as expected.

When the problem concerning generic forms was solved, pure mathematicians celebrated the result, but the community coming from the applied world was not yet satisfied. Indeed, the challenge is to compute the Waring rank of a given explicit polynomial and then to find a Waring decomposition. This question is much more difficult and, in full generality, is still open. Only a few special cases are solved. Comas and Seiguer in [CS11] describe a simple and beautiful algorithm to determine the rank of a given binary form; such algorithm, in some cases, can also provide the Waring decomposition, see Section 2.3.3. In 2012, Carlini, Catalisano and Geramita found a complete solution in the case of monomials in any number of variables and sums of pairwise coprime monomials, see [CCG12].

Theorem 1.2.6 (Carlini-Catalisano-Geramita). If $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}}$, with $d_{0} \leq \ldots \leq d_{n}$, then,

$$
\operatorname{rk}(M)=\frac{1}{\left(d_{0}+1\right)} \prod_{i=0}^{n}\left(d_{i}+1\right)
$$

Theorem 1.2.7 (Carlini-Catalisano-Geramita). If $F=\sum_{i=1}^{s} M_{i}$ where the $M_{i}$ 's are degree d monomials in pairwise different sets of variables, then

$$
\operatorname{rk}(F)=\sum_{i=1}^{s} \operatorname{rk}\left(M_{i}\right)
$$

The case of sums of forms in different sets of variables is of particular interest. A lot of work in this direction has been done in the last years. The so-called Strassen's conjecture claims that, under this assumption, the Waring rank is additive. We will return to this in Section 2.4 .

Conjecture 1.1 (Strassen's conjecture). If $F=\sum_{i=1}^{s} F_{i}$ where the $F_{i}$ 's are degree $d$ forms in different sets of variables, then

$$
\operatorname{rk}(F)=\sum_{i=1}^{s} \operatorname{rk}\left(F_{i}\right)
$$

Several other algorithms to compute Waring rank and Waring decompositions of specific polynomials have been proposed, see [0013; BGI11; BCMT09; CGLM08]. However, they often require additional assumptions on the input polynomial in order to work efficiently.

Concerning the maximal rank of homogeneous polynomials in $n+1$ variables and degree $d$, a brilliant and surprisingly elementary observation by Blekherman and Teitler in [BT14] states that the maximal rank is always at most twice the generic rank. Some better upper bound has been found in cases of low degree [BDP13; Jel14]. However, the precise answer to this problem is known in very few cases.

Theorem 1.2.8. The following maximal ranks are known:

1. (Sylvester) The maximal rank of degree $d$ binary forms is $d$;
2. (Classical result) The maximal rank of degree 2 and 3 ternary forms is 3 and 5, respectively;
3. (Kleppe, [Kle99]) The maximal rank of degree 4 ternary forms is 7;
4. (De Paris, [DP15]) The maximal rank of degree 5 ternary forms is 10 .

### 1.2.2 d-TH WARING DECOMPOSITIONS

In 2012 [FOS12], Fröberg, Ottaviani and Shapiro suggested an additive decomposition of forms that generalizes the original Waring problem.

Definition 1.2.9. Given a homogeneous polynomial $F$ of degree $k d$ in $n+1$ variables, we define the $d$-th Waring rank of $F$ as the smallest number of degree $k$ forms needed to decompose $F$ as sums of their $d$-th powers, i.e.,

$$
\operatorname{rk}_{d}(F):=\min \left\{s \mid \exists G_{1}, \ldots, G_{s} \text { degree } k \text { forms s.t. } F=G_{1}^{d}+\ldots+G_{s}^{d}\right\} \bigsqcup^{4}
$$

Clearly, if $k=1$, then $\mathrm{rk}_{d}(F)$ is the classical Waring rank. We denote with $\mathrm{rk}_{d}^{\circ}(n+1, k d)$ the $d$-th Waring rank of a generic degree $k d$ form in $n+1$ variables and we refer as $d$-th Waring decomposition of $F$ for decompositions as sums of $d$-th powers of minimal length. The main result is the following.

Theorem 1.2.10 (Fröberg-Ottaviani-Shapiro [FOS12]). For a given $d \geq 2$, the generic form of degree $k d$ in $n+1$ variables can be written as a sum of at most $d^{n} d$-th powers of forms of degree $k$.

[^1]Remark 1.2.11. This result is surprising for two remarkable reasons.
At first, this result is independent of $k$. Because of that, we have that such an upper bound on the rank is sharp for $k \gg 0$. Indeed, by a simple count of parameters, there is a natural lower bound

$$
\operatorname{rk}_{d}^{\circ}(n+1, k d) \geq\left\lceil\frac{\binom{n+k d}{n}}{\binom{n+k}{n}}\right\rceil
$$

which, when $k \rightarrow \infty$, tends to $d^{n}$.
The second surprising fact comes from the proof. The authors exhibit $d^{n}$ linear forms such that their linear combinations cover a Zariski open subset of forms of degree $k d$. In other words, even if our intuition might expect that considering $d$-th powers of degree $k$ forms would give us more flexibility, whenever $k$ is large enough, we cannot do better than by considering $k d$-th powers of linear forms.

In the present thesis, we conjecture the following solution to the problem, suggested by Ottaviani (private communication).

Conjecture 1.2. The $d$-th Waring rank of a generic degree $k d$ form in $n+1$ variables is

$$
\operatorname{rk}_{d}^{\circ}(n+1, k d)=\left\{\begin{array}{cc}
\min \left\{s \left\lvert\, \begin{array}{c}
\left.s\binom{n+k}{n}-\binom{s}{2} \geq\binom{ n+2 k}{n}\right\}
\end{array}\right.\right. & \text { for } d=2 \\
{\left[\frac{\binom{n+k d}{n}}{\binom{n+k}{n}}\right\rceil} & \text { for } d \geq 3
\end{array}\right.
$$

This conjecture holds in two variables as presented by Reznick, Rez13b]. In Section 3.3, we will see a different proof of this. A large number of other cases in low degrees and low number of variables has been checked numerically with the help of the Computer Algebra software Macaulay2 [GS].

Concerning the $d$-th Waring rank of specific polynomials, in [CO15] we started to analyze the case of monomials. So far, we have complete results only in case of low number of variables and/or low degree. Unfortunately, our approach cannot be generalized, but, as far as we know, there are no other results in the literature about this problem.

### 1.2.3 WARING-LIKE DECOMPOSITIONS

Other kinds of generalizations of Waring decompositions have been considered in the last decade. For example, in [AB11; Shi12; Abo14; CGG+15], the authors consider decompositions as sums of split forms or completely decomposable forms, namely decompositions of a degree $d$ homogeneous polynomial $F$ as $\sum_{i=1}^{s} L_{i, 1} \cdots L_{i, d}$, where the $L_{i, j}$ 's are linear forms. In [CGG02; Bal05; BCGI09], the authors consider decompositions of the type $\sum_{i=1}^{s} L_{i, 1}^{d-1} L_{i, 2}$. In the recent paper [CCGO15], together with Catalisano, Chiantini and Geramita, we collect all these decompositions under the more general term of Waring-like decompositions.

Definition 1.2.12. Given any partition $\mu=\left(m_{1} \geq \ldots \geq m_{r}\right) \vdash d$, we define the $\mu$-Waring rank of a homogeneous degree $d$ polynomial as the minimal number of what we call $\mu$-powers of linear forms ${ }^{5}$ needed to decompose $F$ as their sum, i.e.,
$\operatorname{rk}_{\mu}(F):=\min \left\{\begin{array}{l}\left.s \left\lvert\, \begin{array}{l}\exists L_{i, j} \text { linear forms, } \\ i=1, \ldots, s, j=1, \ldots, r\end{array}\right. \text { s.t. } F=L_{1,1}^{m_{1}} \cdots L_{1, r}^{m_{r}}+\ldots+L_{s, 1}^{m_{1}} \cdots L_{s, r}^{m_{r}}\right\} . ~ . ~ . ~ . ~\end{array}\right.$.
This problem generalizes both Waring problems mentioned above. Indeed, the classic Waring problem corresponds to the case of the trivial partition $\mu=(d)$. Instead, if we consider the partition $\mu=\left(k m_{1} \geq \ldots \geq k m_{r}\right) \vdash$ $k d$, then we have that, for any form $F$ of degree $k d, \operatorname{rk}_{d}(F) \leq \operatorname{rk}_{\mu}(F)$. We give the complete solution for any partition $\mu \vdash d$ for generic binary forms.

Theorem 1.2.13 (Catalisano-Chiantini-Geramita-Oneto [CCGO15]). For any partition $\mu=\left(m_{1} \geq \ldots \geq m_{r}\right)$ of $d$, the $\mu$-Waring rank of the generic binary form of degree $d$ is

$$
\mathrm{rk}_{\mu}^{\circ}(2, d)=\left\lceil\frac{d+1}{r+1}\right\rceil
$$

We will explain this in more details in Section 3.4.

### 1.2.4 REAL WARING DECOMPOSITIONS

Due to possible applications, it is also useful to consider the decompositions of forms over the real numbers. When it will not be clear from the context, in order to distinguish complex and real ranks, we add $\mathbb{C}$ or $\mathbb{R}$ as a subscript to the symbols defined above. Given a degree $d$ form $F \in \mathbb{R}\left[x_{0}, \ldots, x_{n}\right]$, we define a Waring decomposition of $F$ over $\mathbb{R}$ as an expression

$$
F=c_{1} L_{1}^{d}+\ldots+c_{s} L_{s}^{d}, \text { with }, c_{i} \in \mathbb{R} \text { and } L_{i} \text { 's are linear forms. }
$$

The Waring rank of $F$ over $\mathbb{R}$ is the smallest length $s$ of such a decomposition. We denote it by $\mathrm{rk}_{\mathbb{R}}(F)$ and clearly, $\mathrm{rk}_{\mathbb{C}}(F) \leq r k_{\mathbb{R}}(F)$.

If we consider complex coefficients, i.e., $S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$, and we fix the degree $d$, then the set $U_{d, r}=\left\{[F] \in \mathbb{P}\left(S_{d}\right) \mid \operatorname{rk}(F)=r\right\} \subset \mathbb{P}\left(S_{d}\right)$ has nonempty interior for a unique value of $r$, which is what we called the generic rank $\mathrm{rk}^{\circ}(n+1, d)$. This is true both in Euclidean and Zariski topology. If we consider real coefficients, i.e., $S=\mathbb{R}\left[x_{0}, \ldots, x_{n}\right]$, and we fix the degree $d$, then the set $U_{d, r}$ is semialgebraic and can have non-empty interior, with the Euclidean topology, for several values of $r$. Those are called the typical ranks. It is known that the minimal typical real rank coincides with generic complex rank [BT14].

The case of binary forms has been completely settled. Indeed, Comon and Ottaviani conjectured in [CO12] that typical ranks of degree $d$ binary forms take all integer values between $\left\lfloor\frac{d+2}{2}\right\rfloor$ and $d$; later, Blekherman

[^2]proved this conjecture in [Ble13]. For larger degrees the problem is open. Recent results are given in [BBO15; MMSV16].

In [BCG11], Boij, Carlini and Geramita considered the Waring rank of binary monomials over the real numbers.

Theorem 1.2.14 (Boij-Carlini-Geramita, [BCG11]). Let $M=x_{0}^{d_{0}} x_{1}^{d_{1}}$, then $\mathrm{rk}_{\mathbb{R}}(M)=d_{0}+d_{1}$.

In particular, we can observe that Theorem 1.2 .14 implies that in case of binary monomials the rank over the real and over the complex numbers coincide if and only if $d_{0}=1$. In a recent paper, we prove the following. We will come back to this in Section 4.1.

Theorem 1.2.15 (|CKOV16|). Given a monomial $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}}$, we have that $\mathrm{rk}_{\mathbb{R}}(M)=\mathrm{rk}_{\mathbb{C}}(M)$ if and only if $d_{i}=1$ for some $i=1, \ldots, n$.

## SECTION 1.3

## GEOMETRIC INTERPRETATION

We can interpret our Waring problems for polynomials in terms of geometry of secant varieties of projective varieties. Here is the general set-up.

### 1.3.1 SECANT VARIETIES

Definition 1.3.1. Let $X \subset \mathbb{P}^{N}$ be a projective variety ${ }^{6}$. We define the $s$-th secant variety of $X$ as the Zariski closure ${ }^{7}$ of the union of the linear spans of all $s$-tuples of points in $X$, i.e.,


Definition 1.3.2. Given any point $Q \in \mathbb{P}^{N}$, we define the $X$-rank of $Q$ as the minimal number of points in $X$ whose linear span contains $Q$, i.e.,

$$
\operatorname{rk}_{X}(Q)=\min \left\{s \mid \exists P_{1}, \ldots, P_{s} \in X, \quad Q \in\left\langle P_{1}, \ldots, P_{s}\right\rangle\right\} .
$$

Remark 1.3.3. It is important to notice that, since secant varieties are defined as Zariski closures, the fact that $Q \in \sigma_{s}(X)$ does not imply that

[^3]$\operatorname{rk}_{X}(Q)=s$. For example, let $X \subset \mathbb{P}^{3}$ be the twisted cubic parametrized by
\[

$$
\begin{aligned}
\nu_{1,3}: \mathbb{P}\left(S_{1}\right) & \rightarrow
\end{aligned}
$$ \mathbb{P}\left(S_{d}\right) ;
\]

If $x_{0}, \ldots, x_{3}$ are the homogeneous coordinates in $\mathbb{P}^{3}$, then $X$ is defined by the $2 \times 2$ minors of the matrix

$$
M=\left[\begin{array}{lll}
x_{0} & x_{1} & x_{2} \\
x_{1} & x_{2} & x_{3}
\end{array}\right]
$$

Set $Q=[0: 0: 1: 0]$. This point is not on $X$ and $\operatorname{rk}_{X}(Q) \leq 3$. Indeed, $Q \in\langle[1: 0: 0: 0],[1: 1: 1: 1],[1:-1: 1:-1]\rangle$. On the other hand, it is easy to check that $\mathrm{rk}_{X}(Q)$ cannot be equal to 2 since, if we assume that $Q$ is collinear with the two points $\nu_{1,3}([a: b])$ and $\nu_{1,3}([c: d])$, we get

$$
\operatorname{rk}\left[\begin{array}{cccc}
0 & 0 & 1 & 0 \\
a^{3} & a^{2} b & a b^{2} & b^{3} \\
c^{3} & c^{2} d & c d^{2} & d^{3}
\end{array}\right]=2
$$

which implies $a d-b c=0$. However, even if $\operatorname{rk}_{X}(Q)=3$, we have that $Q \in$ $\sigma_{2}(X)$. Indeed, if we consider the line between the points $[0: 0: 0: 1]$ and $\left[-t^{3}: t^{2}:-t: 1\right]$, described by the Cartesian equations $\left(x_{0}-t^{2} x_{2}, x_{1}-t x_{2}\right)$, then, the limit as $t \rightarrow 0$ is the tangent line to the rational normal curve at the point $[0: 0: 0: 1]$, described by $\left(x_{0}, x_{1}\right)$. Since the point $Q$ lies on this line, it is contained in the second secant variety of $X$; see Figure 1.1.

It is is an easy exercise to check that whenever $\sigma_{s}(X)=\sigma_{s+1}(X)$, then $\sigma_{s}(X)=\sigma_{s+h}(X)$, for any $h \geq 1$, and $\sigma_{s}(X)$ is a linear subspace of $\mathbb{P}^{N}$. It follows that, whenever we start with a non-degenerate projective variety $X$, we have a natural filtration of the ambient space, i.e.,

$$
X \subsetneq \sigma_{2}(X) \subsetneq \ldots \subsetneq \sigma_{s}(X)=\mathbb{P}^{N}
$$

It follows that the $X$-rank of a generic point in $\mathbb{P}^{N}$ can be described geometrically as the smallest $s$ such that $\sigma_{s}(X)$ fills the ambient space, i.e.,

$$
\mathrm{rk}_{X}^{\circ}:=\min \left\{s \mid \sigma_{s}(X)=\mathbb{P}^{N}\right\}
$$

Hence, in order to compute the $X$-rank of a generic point, it is enough to compute the dimensions of the secant varieties of $X$. Due to a simple count of parameters, we have an expected dimension for the $s$-th secant variety,

$$
\exp \cdot \operatorname{dim} \sigma_{s}(X)=\min \{s \operatorname{dim}(X)+s-1, N\}
$$

In terms of generic $X$-rank, we expect

$$
\exp \cdot \mathrm{rk}_{X}^{\circ}=\left\lceil\frac{N+1}{\operatorname{dim}(X)+1}\right\rceil
$$



Figure 1.1: The point $[0: 0: 1: 0]$ is contained in the second secant variety of the twisted cubic in $\mathbb{P}^{3}$.

Definition 1.3.4. We say that a projective variety $X \subset \mathbb{P}^{N}$ is s-defective whenever the actual dimension of its $s$-th secant variety is smaller than the expected one. If so, we call the difference

$$
\delta_{s}:=\exp \cdot \operatorname{dim} \sigma_{s}(X)-\operatorname{dim} \sigma_{s}(X)
$$

the $s$-defect of $X$.
Problems about additive decompositions of homogeneous polynomials can be described by using secant varieties of the projective varieties that parametrize the type of summands that we require.

### 1.3.2 CLASSICAL WARING PROBLEM: VERONESE VARIETIES

The varieties parametrizing powers of linear forms are the Veronese varieties which are defined as the images of the following embeddings.
Definition 1.3.5. The $d$-th Veronese embedding of $\mathbb{P}^{n}$ is

$$
\nu_{n, d}: \mathbb{P}^{n} \longrightarrow \mathbb{P}^{N_{n, d}}, \quad\left[a_{0}: \ldots: a_{n}\right] \mapsto\left[a_{0}^{d}: a_{0}^{d-1} a_{1}: \ldots: a_{n}^{d}\right]
$$

where $N_{n, d}:=\binom{n+d}{n}-1$. We call Veronese variety its image $V_{n, d}:=\nu_{n, d}\left(\mathbb{P}^{n}\right)$.
If we consider the basis of the vector space $S_{d}$ of forms of degree $d$ in $n+1$ variables given by the monomials

$$
X_{i_{0}, \ldots, i_{n}}:=\binom{d}{i_{0}, \ldots, i_{n}} x_{0}^{i_{0}} \cdots x_{n}^{i_{n}}, \quad \text { for any } i_{0}+\ldots+i_{n}=d, i_{j} \in \mathbb{N}
$$

we can describe the Veronese embeddings as

$$
\nu_{n, d}: \mathbb{P}\left(S_{1}\right) \longrightarrow \mathbb{P}\left(S_{d}\right), \quad[L] \mapsto\left[L^{d}\right] .
$$

In other words, the Veronese varieties parametrize powers of linear forms, namely

$$
V_{n, d}:=\left\{\left[L^{d}\right] \mid L \in S_{1}\right\} \subset \mathbb{P}\left(S_{d}\right) .
$$

It follows, that given a degree $d$ form $F$, the rank of the point $[F] \in \mathbb{P}\left(S_{d}\right)$ with respect to the Veronese variety $V_{n, d}$ is the Waring rank of $F$, i.e.,

$$
\mathrm{rk}_{d}(F)=\mathrm{rk}_{V_{n, d}}([F])
$$



Figure 1.2: The third secant variety of $V_{n, d}$.

Example 1.3.6. From this description, we have that the twisted cubic curve parametrizes the third powers of linear forms in two variables. Hence, the geometric description given in Example 1.3 .3 tells us that the monomial $x y^{2} \in S_{3}$, which corresponds to the point $[0: 0: 1: 0] \in \mathbb{P}\left(S_{3}\right)$ with respect to the standard monomial basis, can be written as the limit

$$
x y^{2}=\lim _{t \rightarrow 0} \frac{1}{3 t}\left(y^{3}+(t x-y)^{3}\right),
$$

as we have already observed in Example 1.2.2.
As explained above, if we want to find the rank of generic forms, we need to compute the dimensions of secant varieties and find the smallest secant variety which coincides with the ambient space. Unfortunately, Veronese varieties can be defective.

Example 1.3.7. Let $V_{2,2} \subset \mathbb{P}^{5}$ be the surface given by the degree two Veronese embedding of $\mathbb{P}^{2}$ in $\mathbb{P}^{5}$. Then, the expected dimension of $\sigma_{2}\left(V_{2,2}\right)$ is equal to $2 \cdot 2+2-1=5$, namely we expect that $\sigma_{s}\left(V_{2,2}\right)$ fills the ambient space. However, $V_{2,2}$ parametrizes the squares of linear forms in three variables which can be identified with the rank one symmetric $3 \times 3$ matrices. Consequently, its second secant variety corresponds to the rank two symmetric $3 \times 3$ matrices. It follows that $\sigma_{2}\left(V_{2,2}\right)$ is contained in, and actually equal to, the hypersurface described by the equation det $=0$. Hence, it is 4 -dimensional and, therefore, $V_{2,2}$ is 2-defective.

Secant varieties of Veronese varieties were studied by the classical geometers like Clebsch, Terracini, Bertini, Severi, Palatini since the end of the 19 -th century and by many other mathematicians during the last century. A list of defective Veronese varieties was classically well-known, however, we had to wait until 1995 to get a proof that it was the complete list of exceptions and a complete classification of defective Veronese varieties. Indeed, in AH95], Alexander and Hirschowitz obtained the following result which implies Theorem 1.2.5.

Theorem 1.3.8 (Alexander-Hirschowitz AH95]). The secant varieties of Veronese varieties $\sigma_{s}\left(V_{n, d}\right)$ have the expected dimension, i.e.,

$$
\operatorname{dim} \sigma_{s}\left(V_{n, d}\right)=\min \left\{s(n+1)-1, N_{n, d}\right\}
$$

except for

1. $d=2$, any $n$, with defect $\binom{s}{2}$;
2. $n=2, d=4$, with defect 1 ;
3. $n=3, d=4$, with defect 1 ;
4. $n=4, d=3$, with defect 1 ;
5. $n=4, d=4$, with defect 1 .

### 1.3.3 d-TH WARING PROBLEM: VARIETIES OF POWERS

We want to construct the variety inside the space of forms of degree $k d$ that parametrizes the $d$-th powers of degree $k$ forms.

Definition 1.3.9. We denote the variety of $d$-th powers of degree $k$ forms as

$$
V_{n, k, d}:=\left\{\left[G^{d}\right] \mid G \in S_{k}\right\} \subset \mathbb{P}\left(S_{k d}\right)
$$

These varieties have not been studied that much in the literature. In the recent paper [AC12], Abdesselam and Chipalkatti computed the equations of these varieties in connection with the invariant theory which goes back to Hilbert. We can describe them as linear projections of appropriate Veronese varieties. We start with an example.

Example 1.3.10. We want to describe $V_{1,2,2}$, i.e., the variety of squares of quadratic forms in $S=\mathbb{C}[x, y]$. We consider the monomial basis of $S_{2}$

$$
X_{20}:=x^{2}, \quad X_{11}:=x y, \quad X_{02}:=y^{2} .
$$

We can view any quadratic binary form $F=a_{0} x^{2}+a_{1} x y+a_{2} y^{2} \in S_{2}$ as a linear form $\operatorname{lin}(F):=a_{0} X_{20}+a_{1} X_{11}+a_{2} X_{02} \in \operatorname{Sym}^{1}\left(S_{2}\right)^{8}$. At this point, we can describe $V_{1,2,2}$ simply as the image of the map

$$
\nu_{1,2,2}: \mathbb{P}\left(S_{2}\right) \rightarrow \mathbb{P}\left(S_{4}\right), \quad[F] \mapsto\left[F^{2}\right]
$$

On the other hand, we can consider the degree two Veronese embedding

$$
\nu_{2,2}: \mathbb{P}\left(\operatorname{Sym}^{1}\left(S_{2}\right)\right) \rightarrow \mathbb{P}\left(\operatorname{Sym}^{2}\left(S_{2}\right)\right), \quad[\operatorname{lin}(F)] \mapsto\left[\operatorname{lin}(F)^{2}\right]
$$

We have a natural immersion of $S_{4}$ inside $S y m^{2}\left(S_{2}\right)$ and, then, we can construct the projection from $\mathbb{P}\left(S_{\text {mm }}{ }^{2}\left(S_{2}\right)\right)$ onto $\mathbb{P}\left(S_{4}\right)$ given by the multiplication map. In particular, we consider the basis of $\operatorname{Sym}^{2}\left(S_{2}\right)$ given by all the monomials of degree two in the $X$ 's, i.e.,

$$
Z_{20 ; 20}:=X_{20}^{2}, \quad X_{20 ; 11}:=X_{20} X_{11}, \ldots, Z_{02 ; 02}:=X_{02}^{2}
$$

Then, our projection is given by the substitutions $X_{i j} \mapsto x^{i} y^{j}$; namely,

$$
\begin{aligned}
& \pi: \mathbb{P}\left(\operatorname{Sym}^{2}\left(S_{2}\right)\right) \rightarrow \mathbb{P}\left(S_{4}\right), \\
& \pi\left(\left[a_{0} Z_{20 ; 20}+a_{1} Z_{20 ; 11}+\ldots+a_{5} Z_{02 ; 02}\right]\right)= \\
& \quad\left[a_{0} x^{4}+a_{1} x^{3} y+\left(a_{2}+a_{3}\right) x^{2} y^{2}+a_{4} x y^{3}+a_{5} y^{4}\right]
\end{aligned}
$$

in coordinates,

$$
\pi: \mathbb{P}^{5} \rightarrow \mathbb{P}^{4}, \quad\left[a_{0}: \ldots: a_{5}\right] \mapsto\left[a_{0}: a_{1}: a_{2}+a_{3}: a_{4}: a_{5}\right]
$$

Geometrically, $\pi$ is the projection from the point $[0: 0: 1:-1: 0: 0]$. Since the form $X_{11}^{2}-X_{20} X_{02}$ is not a square, the point $[0: 0: 1:-1: 0: 0]$ does not lie on the Veronese variety $V_{2,2}$. Therefore, we have that $\pi$ is a regular map on $V_{2,2}$ whose image is precisely $V_{1,2,2}$.

We can make this construction general, for any triple $n, k, d$. Consider the following basis:

1. monomial basis of $S_{k}$ :

$$
\left\{X_{I}:=x_{0}^{i_{0}} \cdots x_{n}^{i_{n}}\left|I=\left(i_{0}, \ldots, i_{n}\right) \in \mathbb{N}^{n+1},|I|:=i_{0}+\ldots+i_{n}=k\right\}\right.
$$

[^4]2. monomial basis of $\operatorname{Sym}^{d}\left(S_{k}\right)$ :
$$
\left\{Z_{I_{1} ; \ldots ; I_{d}}:=X_{I_{1}} \cdots X_{I_{d}}| | I_{j} \mid=k, \text { for all } j\right\} .
$$

Given multiindices $\mathbb{I}:=\left(I_{1}, \ldots, I_{d}\right)$, with $I_{h}=\left(i_{h, 0}, \ldots, i_{h, n}\right) \in \mathbb{N}^{n+1}$, denote

$$
\mathbb{I}_{l}:=i_{1, l}+\ldots+i_{d, l}, \quad \text { for any } l=1, \ldots, d
$$

and

$$
\mathbb{I}_{\bullet}:=\left(\mathbb{I}_{0}, \ldots, \mathbb{I}_{n}\right) \in \mathbb{N}^{n+1}
$$

We want to describe the variety $V_{n, k, d}$ of $d$-th powers given by the image of

$$
\nu_{n, k, d}: \mathbb{P}\left(S_{k}\right) \rightarrow \mathbb{P}\left(\operatorname{Sym}^{d}\left(S_{k}\right)\right), \quad[F] \mapsto\left[F^{d}\right] .
$$

We can view any element of $S_{k}$ as a linear form $\operatorname{lin}(F) \in \operatorname{Sym}^{1}\left(S_{k}\right)$ in the $X$ 's and, then, we consider the degree $d$ Veronese embedding

$$
\nu_{M, d}: \mathbb{P}\left(\operatorname{Sym}^{1}\left(S_{k}\right)\right) \rightarrow \mathbb{P}\left(\operatorname{Sym}^{d}\left(S_{k}\right)\right), \quad[\operatorname{lin}(F)] \mapsto\left[\operatorname{lin}(F)^{d}\right] .
$$

We have a natural embedding of $S_{k d}$ inside $\operatorname{Sym}^{d}\left(S_{k}\right)$. Thus, we can construct the linear projection from $\mathbb{P}\left(S y m^{d}\left(S_{k}\right)\right)$ onto $\mathbb{P}\left(S_{k d}\right)$ given by multiplication map. The projection is given by the substitutions $X_{I} \mapsto x_{0}^{i_{0}} \cdots x_{n}^{i_{n}}$; namely,

$$
\pi: \mathbb{P}\left(\operatorname{Sym}^{d}\left(S_{k}\right)\right) \rightarrow \mathbb{P}\left(S_{k d}\right), \quad P=\left[a_{I_{1} ; \ldots ; I_{d}}\right] \mapsto\left[L_{J}(P)\right],
$$

where,

Geometrically, we get that $\pi$ is the linear projection from the linear space $E$ defined by the ideal generated by the $L_{J}$ 's. Since the Veronese variety $V_{M, d}$ doesn't intersect $E$, we have that the projection $\pi$ is regular on the Veronese variety and, moreover, its image $\pi\left(V_{M, d}\right)$ coincides with the variety of powers $V_{n, k, d}$. Even if this is a very natural description of our varieties, these projections are not easy to study. Very little is known about these varieties and their secants. As far as we know, the only reference related to these varieties is [AC12] where the authors describe the equations of varieties $V_{1, k, d}$ of powers of binary forms in terms of Hilbert covariants. In this thesis, we conjecture a formula for the dimensions of all secant varieties of varieties of powers.

Conjecture 1.3. The secant varieties $\sigma_{s}\left(V_{n, k, d}\right)$ of varieties of powers are:

1. defective for $d=2$, with $s$-defect equal to $\binom{s}{2}$, if they do not fill the whole ambient space;
2. non-defective for $d \geq 3$.

In Section 3.3, we relate this conjecture to the computation of Hilbert series of some particular homogeneous ideals. We prove it in the case of two variables and up to four variables in the case of $d=2$. We present massive numerical support for Conjecture 1.3. Moreover, Conjecture 1.2 about the $d$-th Waring rank of a generic form follows from Conjecture 1.3 .

### 1.3.4 WARING-LIKE PROBLEMS: VARIETIES OF $\mu$-POWERS

Given any partition $\mu=\left(m_{1} \geq \ldots \geq m_{s}\right)$ of $d$, we consider the variety whose points are $\mu$-powers of linear forms.

Definition 1.3.11. We define the variety of $\mu$-powers as

$$
V_{n, \mu}:=\left\{\left[L_{1}^{m_{1}} \cdots L_{s}^{m_{s}}\right] \mid L_{i} \in S_{1}\right\} \subset \mathbb{P}\left(S_{d}\right)
$$

Also in this case we can see $V_{n, \mu}$ as the projection of a classical variety.
Definition 1.3.12. We define the Segre variety $\operatorname{Seg}_{n, s}$ as the image of

$$
\begin{aligned}
\underbrace{\mathbb{P}\left(S_{1}\right) \times \ldots \times \mathbb{P}\left(S_{1}\right)}_{r \text { times }} & \rightarrow \mathbb{P}\left(S_{1} \otimes \ldots \otimes S_{1}\right), \\
\left(\left[L_{1}\right], \ldots,\left[L_{s}\right]\right) & \mapsto\left[L_{1} \otimes \ldots \otimes L_{s}\right] .
\end{aligned}
$$

We define the Segre-Veronese variety $\operatorname{Seg}_{n, s}\left(m_{1}, \ldots, m_{s}\right)$ as the image of

$$
\begin{aligned}
\underbrace{\mathbb{P}\left(S_{1}\right) \times \ldots \times \mathbb{P}\left(S_{1}\right)}_{r \text { times }} & \rightarrow \mathbb{P}\left(S_{m_{1}} \otimes \ldots \otimes S_{m_{s}}\right), \\
\left(\left[L_{1}\right], \ldots,\left[L_{s}\right]\right) & \mapsto\left[L_{1}^{m_{1}} \otimes \ldots \otimes L_{s}^{m_{s}}\right]
\end{aligned}
$$

Then, given a partition $\mu=\left(m_{1} \geq \ldots \geq m_{s}\right)$, we can consider the Segre-Veronese variety $\operatorname{Seg}_{n, s}\left(m_{1}, \ldots, m_{s}\right)$ inside $\mathbb{P}\left(S_{m_{1}} \otimes \ldots \otimes S_{m_{s}}\right)$. We have a natural immersion of $S_{d}$ inside $S_{m_{1}} \otimes \ldots \otimes S_{m_{s}}$ and we consider the projection given by the multiplication map, namely

$$
\pi: \mathbb{P}\left(S_{m_{1}} \otimes \ldots \otimes S_{m_{s}}\right) \rightarrow \mathbb{P}\left(S_{d}\right), \pi\left(\left[L_{1}^{m_{1}} \otimes \ldots \otimes L_{s}^{m_{s}}\right]\right)=\left[L_{1}^{m_{1}} \cdots L_{s}^{m_{s}}\right]
$$

Hence, we get that our variety $V_{n, \mu}$ can be seen as the projection of the Segre variety $\operatorname{Seg}_{n, s}\left(m_{1}, \ldots, m_{s}\right)$.

Remark 1.3.13. The study of Segre varieties and their secant varieties is of particular interest in terms of Waring problems for tensors. Indeed, given $\mathbb{K}$-vector spaces $V_{1}, \ldots, V_{s}$, we can define for any tensor $T \in V_{1} \otimes \ldots \otimes V_{s}$ its tensor rank as the minimal number of decomposable tensors needed to decompose $T$ as their sum, i.e.,

$$
\operatorname{rk}(T):=\min \left\{r \mid T=\sum_{i=1}^{s} v_{1, i} \otimes \ldots \otimes v_{s, i}, \quad \text { with } v_{j, i} \in V_{j}\right\} .
$$

This set-up is of particular interest in applications. Waring decompositions of tensors appear in Algebraic Statistics (in statistical models with hidden variables), Phylogenetic (in reconstructions of evolutionary trees of DNA), Signal Processing (in decompositions of signals) and many others; for a more detailed exposition see for example [Lan12; DSS08]. In this case, the varieties parametrizing decomposable tensors are Segre varieties. Their secant varieties have been largely studied, see e.g. [CGG05; CGG08; AOP09], but in general the corresponding problem is still open.

The case of varieties of split or completely reducible forms, i.e., when $\mu=$ $(1, \ldots, 1)$, has been actively studied in the last decades. After the first work [AB11], Shin found in [Shi12] the dimension of the second secant variety in the ternary case and, in Abo14, Abo determined the dimensions of higher secant varieties. All these cases are non-defective. It is conjectured that varieties of split forms are never defective. Some new cases have been recently proved in [CGG ${ }^{+}$15].

The case of $(d-1,1)$-powers is known as the tangential variety $T_{n, d}$ of $V_{n, d}$, which is the union of all the tangent spaces to $V_{n, d}$. In [CGG02], Catalisano, Geramita and Gimigliano conjectured that secant varieties of tangential variety $\sigma_{s}\left(T_{n, d}\right)$ are never defective except for $d=2,2 \leq 2 s \leq n$ and $d=3, s=n=2,3,4$. Abo and Vannieuwenhoven proved it in [AV15].

In [CCGO15], we collect these particular cases in the more general framework of varieties of $\mu$-powers. We will come back to this work in Section 3.4 . Our main results are the following.

Theorem 1.3.14 (Catalisano-Chiantini-Geramita-Oneto [CCGO15]).

1. for any partition $\mu, \sigma_{s}\left(V_{1, \mu}\right)$ is non-defective;
2. the second secant variety $\sigma_{2}\left(V_{n, \mu}\right)$ is never defective, except for $n=2$ and $\mu=(2,1)$, where the defect is equal to one.

### 1.3.5 Terracini's Lemma

Whenever we want to compute the dimension of a variety, it is enough to look at the dimension of the tangent space at a generic (smooth) point.

In 1911, A. Terracini described the tangent space to secant varieties.
Lemma 1.3.15 (Terracini's Lemma [Ter11]). Let $X$ be a projective variety, $P$ be a generic point of $\sigma_{s}(X)$ such that $P \in\left\langle P_{1}, \ldots, P_{s}\right\rangle$, with $P_{i} \in X$; then,

$$
T_{P} \sigma_{s}(X)=\left\langle T_{P_{1}} X, \ldots, T_{P_{s}} X\right\rangle .
$$

Using Terracini's Lemma, we can first compute the tangent space of the varieties described above at $s$ generic points and then look at their span. For the varieties mentioned above, we additionally have that the tangent spaces to their secant varieties can be given as a homogeneous part of some particular families of ideals. In this way, we reduce the problem of computation of the dimension of the secant varieties to an algebraic question.

Veronese varieties and power ideals. As described above, we can see the $d$-th Veronese embedding of $\mathbb{P}^{n}$ as the image of the map

$$
\nu_{n, d}: \mathbb{P}\left(S_{1}\right) \longrightarrow \mathbb{P}\left(S_{d}\right), \quad[L] \mapsto\left[L^{d}\right] .
$$

Hence, the computation of the tangent space at the point $\left[L^{d}\right] \in V_{n, d}$ becomes an easy exercise in differential geometry. We consider all parametric
curves passing through $\left[L^{d}\right]$, obtained by mapping the lines in $\mathbb{P}\left(S_{1}\right)$ passing through $[L]$, and then we compute the span of their tangent vectors at $\left[L^{d}\right]$. Namely, considering a parametric line $r_{M}(t):=\{[L+t M] \mid t \in[-1,1]\}$, for any $M \in S_{1}$, we compute

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0} \nu_{n, d}\left(r_{M}(t)\right)=\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0}(L+t M)^{d}=d L^{d-1} M
$$

thus,

$$
T_{\left[L^{d}\right]} V_{n, d}=\left\{\left[L^{d-1} M\right] \mid M \in S_{1}\right\} \subset \mathbb{P}\left(S_{d}\right)
$$

By Terracini's Lemma, it follows that given a generic point $[F] \in \sigma_{s}\left(V_{n, d}\right)$ lying on the span of $s$ generic points of $V_{n, d}$, i.e., $[F] \in\left\langle\left[L_{1}^{d}\right], \ldots,\left[L_{s}^{d}\right]\right\rangle$, then

$$
\begin{aligned}
T_{[F]} \sigma_{s}\left(V_{n, d}\right) & =\left\langle T_{\left[L_{1}^{d}\right]} V_{n, d}, \ldots, T_{\left[L_{s}^{d}\right]} V_{n, d}\right\rangle= \\
& =\mathbb{P}\left(\left[\left(L_{1}^{d-1}, \ldots, L_{s}^{d-1}\right)\right]_{d}\right) \subset \mathbb{P}\left(S_{d}\right) .
\end{aligned}
$$

Now, we give a geometric explanation of the defectiveness of the Veronese surface in $\mathbb{P}^{5}$, see Example 1.3.7.

Example 1.3.16. Consider the tangent space to $\sigma_{2}\left(V_{2,2}\right) \subset \mathbb{P}^{5}$ at a generic point. By Terracini's Lemma, we look at the span of the tangent planes to the surface at two generic points $\left[L^{2}\right],\left[M^{2}\right] \in V_{2,2}$. Then, we would expect that these two planes do not intersect in $\mathbb{P}^{5}$ and then, by Grassmann's formula, we expect that the dimension of the tangent space to the secant variety is equal to five and it fills the ambient space. However, it is easy to check that the intersection of the two tangent planes is not empty; indeed,

$$
[L M] \in T_{\left[L^{2}\right]} V_{2,2} \cap T_{\left[M^{2}\right]} V_{2,2}
$$

Thus, the dimension is at most four.
We have reduced the problem of computing the dimension of secant varieties of Veronese varieties to the problem of computing the dimension of the $d$-th homogeneous part of an ideal generated by $(d-1)$-th powers of generic linear forms.

These ideals are known as power ideals and they have been intensively studied during the last decades, especially due to their relation with other branches of Algebraic Geometry, Commutative Algebra and Combinatorics; see [AP10] for a complete survey.

Varieties of powers and generalized power ideals. Given positive integers $n, k, d$, we parametrize the variety of powers $V_{n, k, d}$ as the image of the embedding

$$
\nu_{n, k, d}: \mathbb{P}\left(S_{k}\right) \longrightarrow \mathbb{P}\left(S_{k d}\right), \quad[G] \mapsto\left[G^{d}\right] .
$$

As before, we compute the tangent space at a generic point $\left[G^{d}\right] \in V_{n, k, d}$ by considering the parametric lines $r_{H}(t):=\{[G+t H] \mid t \in[-1,1]\}$, for any


Figure 1.3: The Veronese surface of degree 2 in $\mathbb{P}^{5}$ is 2-defective.
$H \in S_{k}$; the tangent vectors to the images are given by

$$
\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0} \nu_{n, k, d}\left(r_{H}(t)\right)=\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0}(G+t H)^{d}=d G^{d-1} H
$$

thus,

$$
T_{\left[G^{d}\right]} V_{n, d}=\left\{\left[G^{d-1} H\right] \mid H \in S_{k}\right\} \subset \mathbb{P}\left(S_{k d}\right)
$$

By using Terracini's Lemma, we have that

$$
T_{P} \sigma_{s}\left(V_{n, k, d}\right)=\mathbb{P}\left(\left[\left(G_{1}^{d-1}, \ldots, G_{s}^{d-1}\right)\right]_{k d}\right) \subset \mathbb{P}\left(S_{k d}\right)
$$

VARIETIES OF $\mu$-POWERS. Given any partition $\mu=\left(m_{1} \geq \ldots \geq m_{r}\right) \vdash d$ of a positive integer $d$, we parametrize the variety of $\mu$-powers $V_{n, \mu}$ as the image of the map

$$
\nu_{n, \mu}: \mathbb{P}\left(S_{1}\right) \times \cdots \times \mathbb{P}\left(S_{1}\right) \longrightarrow \mathbb{P}\left(S_{d}\right), \quad\left(\left[L_{1}\right], \ldots,\left[L_{r}\right]\right) \mapsto\left[L_{1}^{m_{1}} \cdots L_{r}^{m_{r}}\right]
$$

For short, we will use the notation $\mathbb{L}:=\left(L_{1}, \ldots, L_{r}\right) \in S_{1} \times \cdots \times S_{1}$ and $\mathbb{L}^{\mu}:=L_{1}^{m_{1}} \cdots L_{r}^{m_{r}} \in S_{d}$. Then, we consider the image of a line through a generic point $\left(\left[L_{1}\right], \ldots,\left[L_{r}\right]\right)$ in the direction of $\left(\left[M_{1}\right], \ldots,\left[M_{r}\right]\right)$ whose points are parametrized by

$$
\nu_{n, \mu}\left(r_{\left(\left[M_{1}\right], \ldots,\left[M_{r}\right]\right)}(t)\right)=\left[\left(L_{1}+t M_{1}\right)^{m_{1}} \cdots\left(L_{r}+t M_{r}\right)^{m_{r}}\right], \quad t \in[-1,1] .
$$

Hence, the tangent vector to this curve at $t=0$ equals

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0} \nu_{n, \mu}\left(r_{\left(\left[M_{1}\right], \ldots,\left[M_{r}\right]\right)}(t)\right) & =\left.\frac{\mathrm{d}}{\mathrm{~d} t}\right|_{t=0}\left(L_{1}+t M_{1}\right)^{m_{1}} \cdots\left(L_{r}+t M_{r}\right)^{m_{r}}= \\
& =\frac{\mathbb{L}^{\mu}}{L_{1}} M_{1}+\ldots+\frac{\mathbb{L}^{\mu}}{L_{r}} M_{r}
\end{aligned}
$$

In other words, we have that the tangent space to the variety of $\mu$-powers is given by

$$
T_{\left[\mathbb{L}^{\mu}\right]} V_{n, \mu}=\mathbb{P}\left(\left[\left(\frac{\mathbb{L}^{\mu}}{L_{1}}, \ldots, \frac{\mathbb{L}^{\mu}}{L_{r}}\right)\right]_{d}\right) \subset \mathbb{P}\left(S_{d}\right)
$$

In this case, we can use Terracini's Lemma as before, but in order to describe the tangent space of secant varieties we need more work that we postpone to Section 3.4 for the moment.

In conclusion, by using Terracini's Lemma we reduce the computations of dimensions of secant varieties to a purely algebraic question about the dimension of specific homogeneous parts of particular homogeneous ideals. In this way, we connect our problems of additive decompositions of polynomials to the computation of Hilbert functions and Hilbert series of particular families of homogeneous ideals. We refer to Appendix A for the basic notions about these algebraic invariants.

## CHAPTER 2

## ApOLARITY THEORY AND POINT CONFIGURATIONS

In this chapter, we focus on ideals of configurations of points. We first consider configurations of reduced points and, via Apolarity Theory, we see how these can be used to find Waring decompositions of a given homogeneous polynomial. Later, we look at ideals of fat points and we see how they are related to secant varieties of Veronese varieties and power ideals.

## SECTION 2.1

## APOLARITY THEORY

For the material in this section we refer mostly to [IK99; Ger96].
APOLARITY ACTION. Take two polynomial rings with the same number of variables over the complex numbers equipped with the standard grading, i.e.,

$$
S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]=\bigoplus_{i \in \mathbb{N}} S_{i} \text { and } \mathcal{S}=\mathbb{C}\left[X_{0}, \ldots, X_{n}\right]=\bigoplus_{i \in \mathbb{N}} \mathcal{S}_{i} ;
$$

where $S_{i}$ (resp. $\mathcal{S}_{i}$ ) is the $\mathbb{C}$-vector space of degree $i$ homogeneous polynomials in $S$ (resp. $\mathcal{S}$ ). We introduce an action of $\mathcal{S}$ over $S$ as partial derivatives ${ }^{1}$.
Definition 2.1.1. We define the apolarity action of $\mathcal{S}$ on $S$ as, for any $F \in S$ and $H \in \mathcal{S}$,

$$
H \circ F:=H\left(\frac{\partial}{\partial x_{0}}, \ldots, \frac{\partial}{\partial x_{n}}\right) F\left(x_{0}, \ldots, x_{n}\right) \in S .
$$

[^5]Example 2.1.2. Let $F=x_{0}^{4}+2 x_{0}^{3} x_{2}+x_{1}^{2} x_{2}^{2} \in S_{4}$ and $H=X_{0}^{2}+X_{1} X_{2} \in \mathcal{S}_{2}$. Then,

$$
H \circ F=\left(\frac{\partial^{2}}{\partial x_{0}^{2}}+\frac{\partial^{2}}{\partial x_{1} \partial x_{2}}\right) F=12 x_{0}^{2}+12 x_{0} x_{2}+4 x_{1} x_{2} \in S_{2} .
$$

Definition 2.1.3. Let $I$ be a homogeneous ideal in $\mathcal{S}$, then we define its inverse system $I^{-1} \subset S$ as the annihilator of $I$, i.e., as the $\mathcal{S}$-module

$$
I^{-1}:=\{F \in S \mid H \circ F=0, \text { for all } H \in I\}
$$

Remark 2.1.4. The inverse system of a homogeneous ideal is a graded $\mathcal{S}$-module, but, in general, $I^{-1}$ is not an ideal since it is not closed under multiplication. E.g. $X^{2} \circ x=0$, but $X^{2} \circ x^{2} \neq 0$.

For any $i \in \mathbb{N}$, the apolarity action induces a non-degenerate bilinear pairing ${ }^{2} \mathcal{S}_{i} \times S_{i} \rightarrow \mathbb{C}$. Hence, for any homogeneous ideal $I \subset \mathcal{S}$, we can consider the $\mathbb{C}$-vector space $I_{i}$ and construct the orthogonal space

$$
I_{i}^{\perp}=\left\{F \in S_{i} \mid H \circ F=0, \text { for any } H \in I_{i}\right\}
$$

By construction, $\left[I^{-1}\right]_{i} \subset I_{i}^{\perp}$, but actually equality holds.
Proposition 2.1.5. Given a homogeneous ideal I in $\mathcal{S}$, for any $i \in \mathbb{N}$,

$$
\left[I^{-1}\right]_{i}=I_{i}^{\perp}
$$

Proof. Let $F \in I_{i}^{\perp}$. We have to prove that $H \circ F=0$, for any $H \in I$.
If $\operatorname{deg} H>i, H \circ F=0$ because the degree of $H$ is bigger than the degree of $F$. Assume that deg $H<i$ and let $\mathrm{x}^{\alpha}:=x_{0}^{\alpha_{0}} \cdots x_{n}^{\alpha_{n}}$ be a monomial of degree $i-\operatorname{deg} H$. Hence, $\mathrm{x}^{\alpha} H \circ F=0$. By basic properties of differentials, we have that $\mathrm{x}^{\alpha} \circ(H \circ F)=0$, for any monomial of degree $i-\operatorname{deg} H$. Since the apolarity action gives a non-degenerate pairing, we have that $H \circ F=0$.

One motivation behind the study of inverse systems is that they can be helpful for computations of Hilbert functions of homogeneous ideals.

Given a homogeneous ideal $I$ in $S$, observe that both $I$ and the quotient ring $S / I$ inherit the graded structure. Namely, we denote with $I_{i}$ the $\mathbb{C}$ vector space of homogeneous polynomials of degree $i$ in $I$ and similarly, we denote with $[S / I]_{i}$ the quotients of $\mathbb{C}$-vector spaces $S_{i} / I_{i}$.

[^6]We say that $f$ is non-degenerate if $f_{m}:=f(m,-): N \longrightarrow L$ and $f_{n}:=f(-, n): M \longrightarrow L$ are isomorphisms.

Definition 2.1.6. Given a homogeneous ideal $I$, we define the Hilbert function of the quotient ring $S / I$ as

$$
\operatorname{HF}_{S / I}: \mathbb{N} \longrightarrow \mathbb{N}, \quad \operatorname{HF}_{S / I}(i):=\operatorname{dim}_{\mathbb{C}}[S / I]_{i}
$$

we define the Hilbert series of the quotient ring $S / I$ as the power series

$$
\operatorname{HS}_{S / I}(t):=\sum_{i \in \mathbb{N}} \operatorname{HF}_{S / I}(i) t^{i} \in \mathbb{C}[[t]] \underbrace{3}
$$

Due to Proposition 2.1.5, we have the following relation between the inverse system of $I$ and the Hilbert function of $S / I$ :

$$
\operatorname{dim}\left[I^{-1}\right]_{i}=\operatorname{dim} I_{i}^{\perp}=\operatorname{dim} S_{i}-\operatorname{dim} I_{i}=\mathrm{HF}_{S / I}(i), \quad \text { for any } i \in \mathbb{N}
$$

"Perp" ideals and Macaulay's Theorem. Another useful application of Apolarity Theory is a characterization of Artinian Gorenstein graded algebras.

Definition 2.1.7. Let $F \in S_{d}$ be a homogeneous polynomial. We define the perp ideal of $F$ as

$$
F^{\perp}:=\{G \in \mathcal{S} \mid G \circ F=0\} \subset \mathcal{S}
$$

Example 2.1.8. Let $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}} \in S_{d}$ with $d_{0} \leq \ldots \leq d_{n}$. It is obvious from the properties of differentials that the perp ideal of a monomial has to be a monomial ideal; moreover, it is easy to observe that a monomial is contained in $M^{\perp}$ if and only if it doesn't divide $M$. In other words,

$$
M^{\perp}=\left(X_{0}^{d_{0}+1}, \ldots, X_{n}^{d_{n}+1}\right) .
$$

We say that a quotient algebra $S / I$ is Artinian if it is a finitely dimensional $\mathbb{C}$-vector space; namely, there exists a degree $i \in \mathbb{N}$ where the ideal generates the whole ring $S$, i.e. $I_{h}=S_{h}$ for any $h \geq i$.

It is easy to observe that the quotient algebra $A_{F}:=\mathcal{S} / F^{\perp}$ is Artinian. Indeed, if $F \in S_{d}$, we have that $F$ vanishes under the action of any polynomial of higher degree; then we have that $\left[A_{F}\right]_{i}=0$, for any $i>\operatorname{deg}(F)$.

Definition 2.1.9. We say that an Artinian algebra $A=S / I$ has socle degree equal to $l$ if $A=A_{0} \oplus \ldots \oplus A_{l}$, with $A_{l} \neq 0$. The socle of $A$ is

$$
\operatorname{soc}(A):=(0: \mathfrak{m})=\{G \in A \mid G \mathfrak{m}=0\}
$$

where $\mathfrak{m}=\left(x_{0}, \ldots, x_{n}\right)$ is the irrelevant maximal ideal.
We say that an Artinian algebra $A=S / I$ of socle degree $l$ is Gorenstein if the dimension of its socle is equal to one.

In other words, since we have a trivial inclusion $A_{l} \subset \operatorname{soc}(A)$, we have that $A$ is Gorenstein if and only if $A_{l}=\operatorname{soc}(A)$ and $\operatorname{dim}_{\mathbb{C}} A_{l}=1$.
${ }^{3}$ See Appendix Afor more details and basic properties of Hilbert functions and Hilbert series.

Lemma 2.1.10. Let $F \in S_{d}$, then $A_{F}$ is Artinian Gorenstein algebra.
Proof. We have already observed that $A_{F}$ is Artinian.
Let $\langle F\rangle$ denote the one-dimensional vector space in $S_{d}$ spanned by $F$. We have that $\left[F^{\perp}\right]_{d}=\langle F\rangle^{\perp}$, where $\langle F\rangle^{\perp}$ denotes the orthogonal of the one dimensional vector space $\langle F\rangle$ with respect to the non-singular pairing $\mathcal{S}_{d} \times S_{d} \rightarrow \mathbb{C}$ induced by the apolarity action. Hence, $\operatorname{dim}\left[A_{F}\right]_{d}=1$. In order to conclude the proof, it is enough to show that the socle of $A_{F}$ is concentrated in degree $d$.

Take $G \in \mathcal{S}_{i}$, with $i<d$, such that $G \notin F^{\perp}$ and assume that $G X_{i} \in F^{\perp}$, for any $i=0, \ldots, n$. Then, $G \circ F \in S_{d-i}$ with $d-i>0$; hence, there is at least one $X_{i}$ such that $0 \neq X_{i} \circ(G \circ F)=G X_{i} \circ F$; contradiction.

For an Artinian algebra to be Gorenstein one needs strong restrictions in terms of the Hilbert function.

Lemma 2.1.11. The Hilbert function of an Artinian Gorenstein algebra $A$ is symmetric; namely, if $\operatorname{soc}(A)=l$, then

$$
\operatorname{HF}_{A}(i)=\operatorname{HF}_{A}(l-i), \quad \text { for all } t=0, \ldots, l .
$$

Idea of the proof. From the assumptions on $A$, it follows that the pairing induced by multiplication $A_{i} \times A_{l-i} \longrightarrow A_{l} \simeq \mathbb{C}$ is perfect.

In 1919, Macaulay gave a complete characterization of Artinian Gorenstein algebras. We don't give here a proof of this theorem, but refer to the lecture notes by Geramita Ger96] for a modern presentation.

Theorem 2.1.12 (Macaulay's Theorem [Mac19]). An Artinian algebra $A$ of socle degree $d$ is Gorenstein if and only if there exists a homogeneous polynomial $F$ of degree $d$ such that $A \simeq A_{F}$.

APOLARITY LEMMA. The study of perp ideals of homogeneous polynomials is also related to the study of their Waring decompositions, namely to the decompositions as sum of powers of linear forms. The key point is the following lemma, due to Iarrobino and Kanev.

Lemma 2.1.13 (Apolarity Lemma [IK99]). Let $F \in S_{d}$. Then, the following are equivalent:

1. $F=L_{1}^{d}+\ldots+L_{s}^{d}$, where $L_{i} \in S_{1}$;
2. $F^{\perp} \supset I_{\mathbb{X}}$, where $\mathbb{X}$ is the set of reduced points $\left\{\left[\check{L_{1}}\right], \ldots,\left[\check{L_{s}}\right]\right\} \subset \mathbb{P}\left(\mathcal{S}_{1}\right)$, where $L_{i}=L_{i}\left(X_{1}, \ldots, X_{n}\right) \in \mathcal{S}_{1}$.

Definition 2.1.14. We say that a set of reduced points $\mathbb{X} \subset \mathbb{P}\left(\mathcal{S}_{1}\right)$ is apolar to $F$ if $I_{\mathbb{X}} \subset F^{\perp}$.

Example 2.1.15. Consider the monomial $M=x y z \in \mathbb{C}[x, y, z]$.
We have seen that $M^{\perp}=\left(X^{2}, Y^{2}, Z^{2}\right)$; hence, we can find the ideal $I=\left(X^{2}-Y^{2}, X^{2}-Z^{2}\right) \subset M^{\perp}$ corresponding to the set of four reduced points $\mathbb{X}=\{[1: \pm 1: \pm 1]\}$. Thus,

$$
M=\frac{1}{24}\left[(x+y+z)^{3}-(x-y+z)^{3}-(x+y-z)^{3}+(x-y-z)^{3}\right]
$$

In other words, the study of Waring decompositions of a given homogeneous polynomial $F$ is related to the study of sets of reduced points apolar to $F$. Although Apolarity Theory has been largely developed, it is very difficult to describe efficiently the perp ideal of $F$ and to find ideals of reduced points contained in it. This is a major obstruction in the computation of the rank of a given homogeneous polynomial.

Algorithm 2.1.16. We give a Macaulay 2 function to compute the perp ideal of a homogeneous polynomial $F$. The idea is to construct the ideal degree by degree via simple linear algebra. Indeed, in each degree $d \leq$ $\operatorname{deg}(F)$, the perp ideal of $F$ is, by definition, the kernel of the linear map

$$
-\circ F: \mathcal{S}_{d} \longrightarrow S_{\operatorname{deg}(F)-d}, \quad G \mapsto G \circ F .
$$

```
perpId := method();
perpId (RingElement) := F -> (
    if (not isHomogeneous(F)) then
        error "expected homogeneous polynomial";
    R := ring(F);
    D := first degree F;
    L := for j from 0 to D list (
            basis(j,R) *
                mingens kernel
                            transpose diff(transpose basis(j,R),
                                    diff(basis(D-j,R),F))
        );
    return trim ideal L;
)
```


## SECTION 2.2

## Hilbert FUNCTIONS OF CONFIGURATIONS OF REDUCED POINTS

We recall some useful properties of the Hilbert functions of ideals of configurations of reduced points.

Lemma 2.2.1. The Hilbert function of a set of reduced points in $\mathbb{P}^{n}$ is strictly increasing until it becomes constant.

Proof. If $I_{\mathbb{X}}$ is the ideal defining a reduced set of points, we have that $S / I_{\mathbb{X}}$ is Cohen-Macaulay ${ }^{4}$ of dimension one. Then, by [BH98, Theorem 4.1.3], it follows that $\mathrm{HF}_{S / I_{\mathrm{X}}}$ is eventually constant.

Lemma 2.2.2. Let $\mathbb{X}$ be a set of $s$ reduced points in $\mathbb{P}^{n}$, then

$$
\mathrm{HF}_{S / I_{\mathrm{X}}}(i)=s, \quad \text { for all } i \gg 0
$$

Idea of the proof. Let $\mathbb{X}$ be the set of $s$ distinct reduced points. The Hilbert function of $S / I_{\mathbb{X}}$ in degree $i$ is the dimension of the vector space of hypersurfaces of degree $i$ in $\mathbb{P}^{n}$ passing through the points in $\mathbb{X}$; hence, each simple point impose at most one condition on the vector space of degree $i$ hypersurfaces. It follows that,

$$
\operatorname{dim}_{\mathbb{C}}\left[I_{\mathbb{X}}\right]_{i} \geq \operatorname{dim}_{\mathbb{C}} S_{i}-s
$$

i.e.,

$$
\operatorname{HF}_{S / I_{\mathbb{X}}}(i) \leq s
$$

Set $\mathbb{X}=P_{1}+\ldots+P_{s}$. For any $i=1, \ldots, s$, we can consider the hyperplanes $H_{i}$ such that $P_{i} \in H_{i}$ and $P_{j} \notin H_{i}$, for any $j \neq i$. Then, we define $F_{i}=\prod_{j \neq i} H_{j}$. It is not difficult to check that $F_{i}$ 's are linearly independent and $\mathrm{HF}_{S / I_{\mathrm{X}}}(s-1)=s$. Lemma 2.2.1 concludes the proof.

The Hilbert function of a configuration of reduced points can also tell us about the position of these points.

Example 2.2.3. Consider the three standard coordinate points

$$
\mathbb{X}=\{[1: 0: 0],[0: 1: 0],[0: 0: 1]\} \subset \mathbb{P}^{2}
$$

Then, $I_{\mathbb{X}}=(x y, x z, y x) \subset \mathbb{C}[x, y, z]$. It follows that $\left[S / I_{\mathbb{X}}\right]_{i}=\left\langle x^{i}, y^{i}, z^{i}\right\rangle$, for any $i \geq 1$. Hence,

$$
\mathrm{HF}_{S / I_{\mathrm{x}}}: 13333 \begin{array}{llll}
1 & 3
\end{array}
$$

On the other hand, if we consider a set of three distinct collinear points, we have one extra linear form in the ideal $I_{\mathbb{X}}$, namely the one corresponding to the line on which the points lie. Hence,

$$
\mathrm{HF}_{S / I_{\mathrm{X}}}: 12333 \cdots
$$

From the previous lemmas, we have that the set of the three coordinate points considered in the example have the maximal possible Hilbert function. In [GO81], Geramita and Orecchia introduced this as a definition of points in generic position.

[^7]Definition 2.2.4. We say that $s$ points are in generic position if the Hilbert function of the associated quotient ring $S / I_{\mathbb{X}}$ is the maximal possible, i.e.,

$$
\operatorname{HF}_{S / I_{\mathrm{X}}}=\min \left\{s,\binom{n+i}{n}\right\}, \text { for all } i \geq 0
$$

Remark 2.2.5. This definition of points in generic position implies the notion of general points in modern Algebraic Geometry. Indeed, in [GM84], Geramita and Maroscia proved the following.
Claim: the space of configurations of $s$ distinct reduced points in generic position accordingly to Definition 2.2.4 is a Zariski open subset of $\left(\mathbb{P}^{n}\right)^{s}$.

Proof of the claim. Let $\mathbb{X}=P_{1}+\ldots+P_{s}$. For any $i \in \mathbb{N}$, let

$$
\mathcal{B}_{i}=\left\{M_{1}, \ldots, M_{N_{n, i}}\right\}
$$

be the standard monomial basis of $S_{i}$, where $N_{n, i}=\operatorname{dim}_{\mathcal{C}} S_{i}=\binom{n+i}{n}$.
Let $F=\sum_{j=1}^{N_{n, i}} c_{j} M_{j}$ be a general form of degree $i$. Requiring that $F$ passes through a point $P_{l}$, we impose a linear equation on the vector space $S_{i}$ given by

$$
M_{1}\left(P_{l}\right) c_{1}+\ldots+M_{N_{n, i}}\left(P_{l}\right) c_{l}=0
$$

Hence, the dimension of $\left[I_{\mathbb{X}}\right]_{i}$ coincides with the dimension of the solution space of the linear system

$$
\mathcal{M}_{i ; P_{1}, \ldots, P_{s}} \cdot \mathbf{c}=\left[\begin{array}{ccc}
M_{1}\left(P_{1}\right) & \cdots & M_{N(n, i)}\left(P_{1}\right) \\
\vdots & \ddots & \vdots \\
M_{1}\left(P_{s}\right) & \cdots & M_{N(n, i)}\left(P_{s}\right)
\end{array}\right] \cdot\left[\begin{array}{c}
c_{1} \\
\vdots \\
c_{N_{n, i}}
\end{array}\right]=0
$$

namely,

$$
\operatorname{HF}_{S / I_{\mathrm{X}}}(i)=\operatorname{rk} \mathcal{M}_{i ; P_{1}, \ldots, P_{s}}
$$

It follows that the space of configurations of $s$ distinct reduced points which are in general position is the locus of configurations $\mathbb{X}=P_{1}+\ldots+P_{s}$ such that $\mathcal{M}_{i ; P_{1}, \ldots, P_{s}}$ have maximal rank for any $i \geq 0$. Since to require maximal rank for each $i$ is a Zariski closed condition in the space of configurations of $s$ points in $\mathbb{P}^{n}$. More precisely, it is defined by the maximal minors of $\mathcal{M}_{i ; P_{1}, \ldots, P_{s}}$, we can conclude the claim.

## SECTION 2.3

## WARING LOCI OF HOMOGENEOUS POLYNOMIALS

In this section, we introduce the concept of Waring locus and forbidden points for a given homogeneous polynomial. Of particular interest are the cases when the minimal Waring decomposition is unique, called in the literature the identifiable cases [CC06; BCO14; COV15]. Since finding a minimal Waring decomposition is often not doable in practice, we want to study
properties of all minimal Waring decompositions of a given form at once. This section is based on joint work with Carlini and Catalisano [CCO].

Definition 2.3.1. Given a form $F \in S_{d}$, we define the Waring locus of $F$ as the set of linear forms which appear in a Waring decomposition of $F$, namely

$$
\mathcal{W}_{F}:=\left\{[L] \in \mathbb{P}\left(S_{1}\right) \mid \exists L_{2}, \ldots, L_{r} \in S_{1}, F=L^{d}+L_{2}^{d}+\ldots+L_{r}^{d}, r=\operatorname{rk}(F)\right\}
$$

We call forbidden points the linear forms in the complement $\mathcal{W}_{F}$; namely, the set of linear forms that cannot appear in a Waring decomposition of $F$,

$$
\mathcal{F}_{F}:=\mathbb{P}\left(S_{1}\right) \backslash \mathcal{W}_{F} .
$$

In [BC13], using a different kind of approach, the authors started to point out the importance of understanding which linear forms are in the Waring locus of a given homogeneous polynomial. In particular, they show that $\mathcal{W}_{F}$ has no isolated points for any non identifiable form of degree $d$ and Waring rank strictly smaller than $\frac{3 d}{2}$.

As described by the Apolarity Lemma, if we want to find a Waring decomposition of a homogeneous polynomial $F$, we might look at minimal sets of points apolar to $F$. In other words, we can rephrase the definition of Waring locus of a homogeneous polynomial $F$ as the locus of the points that appear in a set of reduced points of cardinality equal to the rank of the polynomial and apolar to $F$; namely,

$$
\mathcal{W}_{F}:=\left\{P \in \mathbb{P}\left(\mathcal{S}_{1}\right) \mid P \in \mathbb{X}, I_{\mathbb{X}} \subset F^{\perp} \text { and }|\mathbb{X}|=\operatorname{rk}(F)\right\}
$$

Before getting into the results, we should be more careful with our definitions in order to avoid confusion. Indeed, it can be the case that a homogeneous polynomial in a certain number of variables, after a suitable change of coordinates, can actually be expressed in fewer variables. Hence, we should be more precise about where we are looking at the Waring loci.

Definition 2.3.2. We say that a homogeneous polynomial $F \in \mathbb{C}\left[x_{0}, \ldots, x_{m}\right]$ essentially involves $n+1$ variables if $\operatorname{dim}\left[F^{\perp}\right]_{1}=m-n$. In other words, if there exist linear forms $y_{0}, \ldots, y_{n} \in \mathbb{C}\left[x_{0}, \ldots, x_{m}\right]$ such that $F \in \mathbb{C}\left[y_{0}, \ldots, y_{n}\right]$.

The following result, also mentioned in [BL13] in the case of tensors, allows us to study Waring decompositions of a homogeneous polynomial $F$ in the ring of polynomials with its essential number of variables.

Proposition 2.3.3. Let $F \in \mathbb{C}\left[x_{0}, \ldots, x_{n}, x_{n+1}, \ldots, x_{m}\right]$ be a degree $d$ homogeneous polynomial such that $\left(F^{\perp}\right)_{1}=\left(X_{n+1}, \ldots, X_{m}\right)$. If

$$
F=L_{1}^{d}+\ldots+L_{r}^{d}
$$

where $r=\operatorname{rk}(F)$ and the $L_{i}$ are linear forms in $\mathbb{C}\left[x_{0}, \ldots, x_{n}, x_{n+1}, \ldots, x_{m}\right]$, then

$$
L_{i} \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right] \subset \mathbb{C}\left[x_{0}, \ldots, x_{n}, x_{n+1}, \ldots, x_{m}\right]
$$

for all $i, 1 \leq i \leq r$.

Proof. We proceed by contradiction. Assume that $L_{1}=x_{n+1}+\sum_{i \neq n+1} a_{i} x_{i}$, i.e. assume that $L_{1}$ actually involves the variable $x_{n+1}$. By assumption $\operatorname{rk}\left(F-L_{1}^{d}\right)<r=\operatorname{rk}(F)$. However, since $L_{1}$ is linearly independent with $x_{1}, \ldots, x_{n}$ we can apply the following fact (see [CCC15a, Proposition 3.1]):
if $y$ is a new variable, then

$$
\operatorname{rk}\left(F+y^{d}\right)=\operatorname{rk}(F)+1
$$

Hence, $\operatorname{rk}\left(F-L_{1}^{d}\right)=\operatorname{rk}(F)+1$ and this is a contradiction.
As a corollary, we have that if $F \in \mathbb{C}\left[x_{0}, \ldots, x_{m}\right]$ essentially involves $n+1$ variables, there exists an $n$-dimensional linear space $H \subset \mathbb{P}^{m}$, such that all the minimal sets of points apolar to $F$ are contained in $H$. In conclusion, we always consider the Waring locus of a form $F$ inside the space of linear forms in the essential number of variables of $F$.

Now, we compute the Waring loci of classes of homogeneous polynomials for which we have a very explicit description of their perp ideals.

### 2.3.1 QUADRICS

We first study homogeneous polynomials of degree two, i.e. quadrics. We recall that, given a quadric $Q$ in $m+1$ variables, we can associate a symmetric $(m+1) \times(m+1)$ matrix $M_{Q}$. Moreover, if $Q$ essentially involves $n+1$ variables, after diagonalizing the matrix $M_{Q}$, we may assume that $Q=x_{0}^{2}+\ldots+x_{n}^{2}$. Namely, in this case, the Waring rank of $Q$ equals the rank of $M_{Q}$ and the number of essential variables. We have that the forbidden points of a quadric are given by the quadric itself.

Proposition 2.3.4. If $Q \in S_{2}$ essentially involves $n+1$ variables, then

$$
\mathcal{F}_{Q}=Z(\check{Q}) \subset \mathbb{P}^{n}
$$

where $\check{Q}=Q\left(X_{0}, \ldots, X_{n}\right) \in \mathcal{S}_{2}$.
Proof. After a change of variables we may assume that $Q=x_{0}^{2}+\ldots+x_{n}^{2}$. A point $P=\left[a_{0}: \ldots: a_{n}\right]$ is a forbidden point for $Q$ if and only if $\operatorname{rk}\left(Q-L_{P}^{2}\right)=$ $n+1$ where $L_{P}=\sum_{i=0}^{n} a_{i} x_{i}$. Thus, $P$ is a forbidden point for $Q$ if and only if the symmetric matrix corresponding to the quadratic form $Q-L_{P}^{2}$ has non-zero determinant and therefore, $P$ is a forbidden point if and only if the symmetric matrix $M_{L_{P}^{2}}$ corresponding to $L_{P}^{2}$ only has zero eigenvalues. Since $M_{L_{P}^{2}}$ is a rank one matrix, $M_{L_{P}^{2}}$ has at most one non-zero eigenvalue. Note that

$$
\left(\begin{array}{lll}
a_{0} & \ldots & a_{n}
\end{array}\right) M_{L_{P}^{2}}=\left(a_{0}^{2}+\ldots+a_{n}^{2}\right)\left(\begin{array}{lll}
a_{0} & \ldots & a_{n}
\end{array}\right) .
$$

Also note that, if $\sum_{i=0}^{n} a_{i}^{2}=0$, then $M_{L_{P}^{2}}^{2}=0$ and thus zero is the only eigenvalue. Thus $\sum_{i=0}^{n} a_{i}^{2}$ is the only possible non-zero eigenvalue of $M_{L_{P}^{2}}$. Hence, $P$ is a forbidden point if and only if $\sum_{i=0}^{n} a_{i}^{2}=0$ and we conclude the proof.

### 2.3.2 Monomials

We consider now the case of monomials. In [CCG12], Carlini, Catalisano and Geramita gave an explicit formula for their Waring rank. Here is their result. We want to recall also the proof since we will use the same idea.

Theorem 2.3.5 (Carlini-Catalisano-Geramita, [CG12]). Consider $M=$ $x_{0}^{d_{0}} \cdots x_{n}^{d_{n}} \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$, with $0<d_{0} \leq \ldots \leq d_{n}$. Then,

$$
\operatorname{rk}(M)=\frac{1}{d_{0}+1} \prod_{i=0}^{n}\left(d_{i}+1\right)
$$

Proof. As observed in Example 2.1.8, we can easily compute its perp ideal

$$
M^{\perp}=\left(X_{0}^{d_{0}+1}, \ldots, X_{n}^{d_{n}+1}\right)
$$

Generalizing Example 2.1.15, we may observe that

$$
I_{\mathbb{X}}=\left(X_{0}^{d_{1}+1}-X_{1}^{d_{1}+1}, \ldots, X_{0}^{d_{n}+1}-X_{n}^{d_{n}+1}\right) \subset M^{\perp}
$$

with

$$
\begin{equation*}
\mathbb{X}=\left\{\left[1: \xi_{d_{1}}^{i_{1}}: \ldots: \xi_{d_{n}}^{i_{n}}\right] \mid i_{j}=0, \ldots, d_{j}-1, j=1, \ldots, n\right\} \tag{2.1}
\end{equation*}
$$

where $\xi_{d_{j}}$ represents a $d_{j}$-th primitive root of unity. By the Apolarity Lemma, it follows that

$$
\operatorname{rk}(M) \leq \frac{1}{d_{0}+1} \prod_{i=0}^{n}\left(d_{i}+1\right)
$$

On the other hand, let $\mathbb{X}$ be a set of $s$ reduced points apolar to $M$. In order to conclude our proof, it is enough to show that $s \geq \frac{1}{d_{0}+1} \prod_{i=0}^{n}\left(d_{i}+1\right)$.

Let $I_{\mathbb{X}^{\prime}}=I_{\mathbb{X}}:\left(X_{0}\right)$ be the ideal of the set of points $\mathbb{X}^{\prime} \subset \mathbb{X}$ which are outside the linear space $X_{0}=0$. Say $\left|\mathbb{X}^{\prime}\right|=s^{\prime} \leq s$. Hence, $X_{0}$ is a non-zero divisor for $S / I_{\mathbb{X}^{\prime}}$ and then,

$$
\operatorname{HS}_{\mathcal{S} /\left(I_{\mathbb{X}^{\prime}}+\left(X_{0}\right)\right)}(t)=(1-t) \operatorname{HS}_{\mathcal{S} / I_{\mathrm{x}^{\prime}}}(t) \square^{5}
$$

By Lemma 2.2.2, the Hilbert function of $S / I_{\mathbb{X}^{\prime}}$ is strictly increasing until it stabilizes at the value $s^{\prime}$; hence,

$$
s^{\prime} \geq \sum_{i \geq 0} \operatorname{HF}_{\mathcal{S} /\left(I_{\mathbb{x}^{\prime}}+\left(X_{0}\right)\right)}(i)
$$

Since

$$
I_{\mathbb{X}^{\prime}}+\left(X_{0}\right)=\left(\partial_{X_{0}} M\right)^{\perp}+\left(X_{0}\right)=\left(X_{0}, X_{1}^{d_{1}+1}, \ldots, X_{n}^{d_{n}+1}\right),
$$

we have that

$$
\sum_{i \geq 0} \operatorname{HF}_{\mathcal{S} /\left(I_{\mathrm{X}^{\prime}}+\left(X_{0}\right)\right)}(i)=\frac{1}{d_{0}+1} \prod_{i=0}^{n}\left(d_{i}+1\right)
$$

[^8]We may observe that the set of points (2.1) apolar to a general monomial $M$ is a complete intersection. This is not a lucky coincidence for this particular choice. In [BBT13], Buczyńska, Buczyńsky and Teitler proved that it is the case for any minimal set of points apolar to $M$.

Theorem 2.3.6 ([BBT13, Theorem 1]). Let $M=x_{0}^{d_{0}} x_{1}^{d_{1}} \ldots x_{n}^{d_{n}}$ with $0<$ $d_{0} \leq d_{1} \leq \ldots \leq d_{n}$. Then, any ideal $I_{\mathbb{X}}$ of a set of $\mathrm{rk}_{\mathbb{C}}(M)$ points apolar to $M$ is a complete intersection of the form

$$
\left(X_{1}^{d_{1}+1}-F_{1} X_{0}^{d_{0}+1}, \ldots, X_{n}^{d_{1}+1}-F_{n} X_{0}^{d_{0}+1}\right),
$$

where the forms $F_{i}$ 's have degrees $d_{i}-d_{0}$.
Using the idea behind the proof of Theorem 2.3.5, we get the following.
Proposition 2.3.7. Let $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}} \in S$ with $0<d_{0}=\ldots=d_{m}<$ $d_{m+1} \leq \ldots \leq d_{n}$, then

$$
Z\left(X_{0} \cdots X_{m}\right) \subset \mathcal{F}_{M} \subset \mathbb{P}^{n}
$$

Proof. Let $\mathbb{X}$ be a minimal set of apolar points for $M$. From the proof of the previous theorem, it follows that the cardinality of the set of points $\mathbb{X}^{\prime} \subset \mathbb{X}$ which lie outside the hyperplane $X_{0}=0$ equals

$$
s^{\prime}=\frac{1}{d_{0}+1} \prod_{i=0}^{n}\left(d_{i}+1\right)=|\mathbb{X}|
$$

In other words, there are no points of $\mathbb{X}$ lying on the hyperplane $X_{0}$.
The same computations work for any hyperplane $X_{i}=0$, for $i=0, \ldots, m$.

Remark 2.3.8. Apart from the algebraic technicalities required in the proof, we can explain the previous result with a easier argument in the case $d_{0} \geq 2$. Given a Waring decomposition $M=\sum_{j=1}^{r} L_{j}^{d}$, by differentiating both sides we get a Waring decomposition of the derivative,

$$
\begin{equation*}
\partial_{x_{i}} M=\sum_{j=1}^{r} c_{j}\left(\partial_{x_{i}} L_{j}\right)^{d-1}, \quad c_{j} \in \mathbb{C} . \tag{2.2}
\end{equation*}
$$

If $d_{0}=\ldots=d_{m} \geq 2$, from the formula given by Theorem 2.3.5, we have that $\operatorname{rk}(M)=\operatorname{rk}\left(\partial_{x_{i}} M\right)$, for any $i=0, \ldots, m$. Hence, every summand in (2.2) must be non-zero; equivalently, it has to be $\left[L_{j}\right] \notin Z\left(X_{0} \cdots X_{m}\right)$, for any $j=1, \ldots, r$.

Actually, we also prove the opposite inclusion by constructing, for any $P \notin Z\left(X_{0} \cdots X_{m}\right)$, a minimal set of apolar points for $M$ containing $P$.

Theorem 2.3.9. Take $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}} \in S$ with $0<d_{0}=\ldots=d_{m}<d_{m+1} \leq$ $\ldots \leq d_{n}$, then

$$
\mathcal{F}_{M}=Z\left(X_{0} \cdots X_{m}\right) \subset \mathbb{P}^{n}
$$

Proof. Consider any point $P=\left[p_{0}: \ldots: p_{n}\right] \notin Z\left(X_{0} \cdots X_{m}\right)$; with no loss of generality, we may assume $p_{0}=1$. For any $i=1, \ldots, n$, we construct the following hypersurfaces in $\mathbb{P}^{n}$, given by union of $d_{i}+1$ hyperplanes, respectively,

$$
H_{i}:= \begin{cases}X_{i}^{d_{i}+1}-p_{i}^{d_{i}+1} X_{0}^{d_{i}+1} & \text { if } p_{i} \neq 0 \\ X_{i}^{d_{i}+1}-X_{i} X_{0}^{d_{i}} & \text { if } p_{i}=0\end{cases}
$$

The ideal $I=\left(H_{1}, \ldots, H_{n}\right)$ is contained in $M^{\perp}$ and the zero locus of $I$ is the set of reduced points $\left[1: q_{1}: \ldots: q_{n}\right]$, where

$$
q_{i} \in \begin{cases}\left\{\xi_{i}^{j} p_{i} \mid j=0, \ldots, d_{i}\right\}, & \text { if } p_{i} \neq 0, \text { where } \xi_{i}^{d_{i}+1}=1 \\ \left\{\xi_{i}^{j} \mid j=0, \ldots, d_{i}-1\right\} \cup\{0\}, & \text { if } p_{i}=0, \text { where } \xi_{i}^{d_{i}}=1\end{cases}
$$

Thus, we have a set of $\operatorname{rk}(M)$ distinct points apolar to $M$ and containing the point $P$; hence, by definition, $P \in \mathcal{W}_{M}$.

### 2.3.3 BINARY FORMS

In this section we deal with the case $n=1$, that is the case of forms in two variables. In this case, the knowledge on the Waring rank of binary forms goes back to Sylvester [Syl51]. Comas and Seiguer [CS11] presented a detailed description in terms of secant varieties of rational normal curves. We refer to [Rez13a] for a complete survey about Waring decompositions of binary forms. The generic rank for binary forms is given by

$$
\mathrm{rk}^{\circ}(2, d)=\left\lceil\frac{d+1}{2}\right\rceil .
$$

Let's recall the well-known algorithm to compute the Waring rank of any binary form by studying its perp ideal.

Lemma 2.3.10. If $F \in \mathbb{C}[x, y]$ be a binary form of degree $d$, then,

$$
F^{\perp}=\left(G_{1}, G_{2}\right), \text { with } \operatorname{deg} G_{1}+\operatorname{deg} G_{2}=d+2
$$

Proof. In codimension two, we have that Gorenstein rings are also complete intersections ${ }^{6}$, hence, $F^{\perp}=\left(G_{1}, G_{2}\right)$. The sum of the degrees has to be equal to $d+2$ because the Hilbert function of $\mathcal{S} / F^{\perp}$ has to be symmetric by Lemma 2.1.11. In particular, it is given by

$$
\operatorname{HF}_{\mathcal{S} / F^{\perp}}(i)= \begin{cases}i+1 & \text { for } i=0, \ldots, \operatorname{deg} G_{1}-1 \\ \operatorname{deg} G_{1} & \text { for } i=\operatorname{deg} G_{1}, \ldots, \operatorname{deg} G_{2}-1 \\ \operatorname{deg} G_{1}-\left(i+1-\operatorname{deg} G_{2}\right) & \text { for } i=\operatorname{deg} G_{2}, \ldots, d+1\end{cases}
$$

[^9]Proposition 2.3.11 ([CS11]). Let $F \in \mathbb{C}[x, y]$ be a binary form with $F^{\perp}=$ $\left(G_{1}, G_{2}\right)$ and $\operatorname{deg} G_{i}=d_{i}$, respectively. Assume $d_{1} \leq d_{2}$. Then, we have that

$$
\operatorname{rk}(F)= \begin{cases}d_{1} & \text { if } G_{1} \text { is square free } \\ d_{2} & \text { otherwise }\end{cases}
$$

Proof. If $G_{1}$ is square-free, i.e. $G_{1}=L_{1} \cdots L_{d_{1}}$ and then

$$
\bigcap_{i=1}^{d_{1}}\left(L_{i}\right)=\left(L_{1} \cdots L_{d_{1}}\right) \subset F^{\perp}
$$

where the left hand side is the ideal of a set of $d_{1}$ reduced distinct points. Otherwise, it is easy to check that we have a square-free polynomial of degree exactly $d_{2}$, and not smaller, inside $F^{\perp}$.

Having this explicit description of the perp ideal of binary forms, we can compute the Waring loci and the forbidden points of binary forms.

Theorem 2.3.12. Let $F$ be a degree $d$ binary form and let $G \in F^{\perp}$ be an element of minimal degree inside the perp ideal. Then,

1. if $\operatorname{rk}(F)<\left\lceil\frac{d+1}{2}\right\rceil$, then $\mathcal{W}_{F}=Z(G)$;
2. if $\operatorname{rk}(F)>\left\lceil\frac{d+1}{2}\right\rceil$, then $\mathcal{F}_{F}=Z(G)$;
3. if $\operatorname{rk}(F)=\left\lceil\frac{d+1}{2}\right\rceil$ and d is even, then $\mathcal{F}_{F}$ is finite and not empty; if $\operatorname{rk}(F)=\left\lceil\frac{d+1}{2}\right\rceil$ and $d$ is odd, then $\mathcal{W}_{F}=Z(G)$.
Proof. (1) In this numerical case, we have that $G$ is (up to scalar multiplication) the generator of $F^{\perp}$ of minimal degree and it is also square-free. Hence, the Waring decomposition of $F$ is unique and is precisely $Z(G)$.
(2) In this case we have $F^{\perp}=\left(G_{1}, G_{2}\right)$ with $d_{1}=\operatorname{deg}\left(G_{1}\right)<\operatorname{deg}\left(G_{2}\right)=$ $d_{2}, d_{1}+d_{2}=d+2, G_{1}$ is not square-free, and $\operatorname{rk}(F)=d_{2}$. In particular, $G_{1}$ is an element of minimal degree in the perp ideal. We first show that $\mathcal{F}_{F} \supseteq Z\left(G_{1}\right)$. Let $P=Z(L) \in Z\left(G_{1}\right)$ for some linear form $L$, i.e. $L$ divides $G_{1}$. We want to show that there is no set of points apolar to $F$ containing $P$. Thus, it is enough to show that there is no square-free element of $F^{\perp}$ divisible by $L$. Since $G_{1}$ and $G_{2}$ have no common factors, and $L$ divides $G_{1}$, it follows that the only elements of $F^{\perp}$ divisible by $L$ are multiple of $G_{1}$. Thus they are not square-free. Hence, $P \in \mathcal{F}_{F}$. We now prove that $\mathcal{F}_{F} \subseteq Z\left(G_{1}\right)$ by showing that, if $P=Z(L) \notin Z\left(G_{1}\right)$, then $P \in \mathcal{W}_{F}$. Note that $L$ does not divide $G_{1}$ and consider

$$
F^{\perp}:(L)=(L \circ F)^{\perp}=\left(H_{1}, H_{2}\right)
$$

where $c_{1}=\operatorname{deg}\left(H_{1}\right), c_{2}=\operatorname{deg}\left(H_{2}\right)$ and $c_{1}+c_{2}=d+1$. Since $G_{1}$ is a minimal degree element in $F^{\perp}$ and $L$ does not divide $G_{1}$, we have $H_{1}=G_{1}$ and $c_{2}=d_{2}-1$. Thus $\operatorname{rk}(F)=\operatorname{rk}(L \circ F)+1$. Let $H$ be a degree $d_{2}-1$ square
free element in $(L \circ F)^{\perp}=F^{\perp}:(L)$. Hence, $P \in Z(L H)$ and $Z(L H)$ is a set of $d_{2}$ points apolar to $F$.
(3) Let $F^{\perp}=\left(G_{1}, G_{2}\right), d_{1}=\operatorname{deg}\left(G_{1}\right)$, and $d_{2}=\operatorname{deg}\left(G_{2}\right)$. If $d$ is odd, then $d_{2}=d_{1}+1$ and $\operatorname{rk}(F)=d_{1}$; thus $G_{1}$ is a square-free element of minimal degree and $F$ has a unique apolar set of $d_{1}$ distinct points, namely $Z\left(G_{1}\right)$. This proves the case of odd $d$. If $d$ is even, then $d_{1}=d_{2}=\operatorname{rk}(F)$ and $F$ has infinitely many apolar sets of $\operatorname{rk}(F)$ distinct points. However, for each $P \in \mathbb{P}^{1}$ there is a unique set of $\mathrm{rk}(F)$ points (maybe not distinct) apolar to $F$ and containing $P$. That is, there is a unique (up to scalar) element $G \in\left(F^{\perp}\right)_{d_{1}}$ vanishing at $P$. Thus, $P \in \mathcal{F}_{F}$ if and only if $G$ is not squarefree. There are finitely many not square-free elements in $\left(F^{\perp}\right)_{d_{1}}$ since they correspond to the intersection of the line given by $\left(F^{\perp}\right)_{d_{1}}$ in $\mathbb{P}\left(\mathcal{S}_{d_{1}}\right)$ with the hypersurface given by the discriminant. Note that the line is not contained in the hypersurface since $\left[F^{\perp}\right]_{d_{1}}$ contains square-free elements.

We can improve Part (3) of Theorem 2.3.12, when $d$ is even by adding a genericity assumption.

Proposition 2.3.13. Let $d=2 h$. If $F \in S_{d}$ is a generic form of rank $h+1$, then $\mathcal{F}_{F}$ is a set of $2 h^{2}$ distinct points.

Proof. Let $\mathcal{D} \subset \mathbb{P}^{h+1}$ be the variety of degree $h+1$ binary forms having at least a factor of multiplicity two. Note that forms having higher degree factors, or more than one repeated factor, form a variety of codimension at least one in $\mathcal{D}$. In particular, a generic line $L$ will meet $\mathcal{D}$ in $\operatorname{deg} \mathcal{D}$ distinct points, each point corresponding to a form of the type $B_{1}^{2} B_{2} \ldots B_{h}$, where $B_{i}$ is not proportional to $B_{j}$ for $i \neq j$.

Note that $F^{\perp}=\left(G_{1}, G_{2}\right)$, where $\operatorname{deg}\left(G_{1}\right)=\operatorname{deg}\left(G_{2}\right)=h+1$. Since the Grassmannian of lines in $\mathbb{P}^{h}$ has dimension $2 h$, the form $F$ determines a generic line and viceversa. The non square-free elements of $\left(F^{\perp}\right)_{h+1}$ correspond to $L \cap \mathcal{D}$, where $L$ is the line given by $\left(F^{\perp}\right)_{h+1}$. By genericity, $L \cap \mathcal{D}$ consists of exactly $\operatorname{deg} \mathcal{D}$ points each corresponding to a degree $h+1$ form $F_{i}$ having exactly one repeated factor of multiplicity two. For each degree one form $l$, there exists exactly one element in $\left(F^{\perp}\right)_{h+1}$, thus $\operatorname{gcd}\left(F_{i}, F_{j}\right)=1$ if $i \neq j$. Hence,

$$
\mathcal{F}_{F}=\bigcup_{i} Z\left(F_{i}\right)
$$

is a set of $h \operatorname{deg} \mathcal{D}$ distinct points and the result is now proven.
We can iterate the use of Theorem 2.3 .12 to construct a Waring decomposition for a given binary form. Take $F \in S_{d}$ with rank $r$ larger than the generic one, so that the Waring decomposition of $F$ is not unique. The idea is to construct a Waring decomposition of $F$ step-by-step, choosing one summand at a time. From our result, we know that in this case the forbidden locus is a closed subset $\mathcal{F}_{F}=Z(G)$, where $G$ is an element in $F^{\perp}$ of minimal degree; hence, we can pick any point $\left[L_{1}\right]$ in the open set $\mathbb{P}^{1} \backslash Z(G)$ to start our Waring decomposition of $F$. Consider now $F_{1}=F-L_{1}^{d}$. If the rank of $F_{1}$ (which is simply one less than the rank of $F$ ) is still large
enough not to have a unique decomposition, we can proceed in the same way as before. We may observe that $\mathcal{F}_{F_{1}}=\mathcal{F}_{F} \cup\left[L_{1}\right]$. Indeed, by Theorem 2.3.12, $\mathcal{F}_{F_{1}}=Z\left(G_{1}\right)$, where $G_{1}$ is an element of minimal degree of $F_{1}^{\perp}$. Since $\operatorname{rk}\left(F_{1}\right)=\operatorname{rk}(F)-1$, we have that $\operatorname{deg}\left(G_{1}\right)=\operatorname{deg}(G)+1$. In particular, it has to be $G_{1}=G L_{1}^{\perp}$, where $L_{1}^{\perp}$ is the linear differential operator annihilating $L_{1}$. Hence, we can continue to construct our decomposition for $F$ by taking any point $\left[L_{2}\right] \in \mathbb{P}^{1} \backslash Z\left(G_{1}\right)$ and then looking at $F_{2}=F-L_{1}^{d}-L_{2}^{d}$. We can continue this procedure until we get a form $F_{i}$ with a unique Waring decomposition; namely, until $i=r-\left\lceil\frac{d+1}{2}\right\rceil$, if $d$ is odd, and $i=r-\left\lfloor\frac{d+1}{2}\right\rfloor$, if $d$ is even. In other words, we get the following result.

Proposition 2.3.14. Let $F$ be a degree d binary form of rank $r \geq\left\lceil\frac{d+1}{2}\right\rceil$. For any choice of distinct $L_{1}, \ldots, L_{s} \notin \mathcal{F}_{F}$, where $s=r-\left\lceil\frac{d+1}{2}\right\rceil$, if $d$ is odd, and $s=r-\left\lfloor\frac{d+1}{2}\right\rfloor$, if $d$ is even, there exists a unique minimal Waring decomposition of $F$ involving $L_{1}^{d}, \ldots, L_{s}^{d}$.

### 2.3.4 Plane cubics

In this section we describe Waring loci and forbidden points for $n=2$ and $F \in S_{3}$, that is for plane cubics. For simplicity, we denote $S=\mathbb{C}[x, y, z]$ and $\mathcal{S}=\mathbb{C}[X, Y, Z]$. In this case, we have that the generic rank is equal to four and the maximal rank is equal to five. In particular, we have the following characterization of plane cubics given in [LT10].

|  | Normal form | Waring rank | Computation of $\mathcal{W}_{F}$ |
| :---: | :---: | :---: | :---: |
| $(1)$ | $x^{3}$ | 1 | Theorem 2.3.9 |
| (2) | $x y(x+y)$ | 2 | Theorem 2.3.12 |
| (3) | $x^{2} y$ | 3 | Theorem 2.3.9 |
| (4) | $x^{3}+y^{3}+z^{3}$ | 3 | Proposition 2.4.5 |
| (5) | $x y z$ | 4 | Theorem 2.3.9 |
| (6) | $x\left(y z+x^{2}\right)$ | 4 | Theorem 2.3.18 |
| (7) | $x y z-(y+z)^{3}$ | 4 | Theorem 2.3.19 |
| (8) | $x^{3}-y^{2} z$ | 4 | Theorem 2.4.8 |
| (9) | $x^{3}+y^{3}+z^{3}+a x y z$, | 4 | Theorem 2.3.21 |
|  | with $a^{3} \neq-27,0,6^{3}$ |  |  |
| (10) | $x\left(x y+z^{2}\right)$ | 5 | Theorem 2.3.24 |

Note. In case (9), $a^{3} \neq 0,6^{3}$ so that the rank is actually three and $a^{3} \neq-27$ for smoothness of the Hessian canonical form, Dol12].

Remark 2.3.15. Some of these families can be treated in different classes of homogeneous polynomials:
(1),(3),(5): they are monomials, the results follow from Theorem 2.3 .9 ,
(2): they are in two variables, the result follows from Theorem 2.3.12,
(4),(8): smooth plane cubics and plane cuspidal cubic can be seen as examples of sums of pairwise coprime monomials. We treat them later in Section 2.4 in connection with Strassen's conjecture. The computations of their Waring loci are in Proposition 2.4.5 and Theorem 2.4.8, respectively.

We now study plane cubics of rank four.
Lemma 2.3.16. Let $F$ be a plane cubic of rank four and let $\mathbb{X}$ be a set of four distinct points apolar to $F$. If $\mathbb{X}$ has exactly three collinear points, then $F$ is a cusp, that is $F$ is of type (8).

Proof. After changing coordinates, we may assume that the three collinear points lie on the line defined by $X$ and that the point not on the line is [1:0:0]. Thus, $X Y, X Z \in F^{\perp}$ and $F=X^{3}+G(Y, Z)$. By [CCC15a, Proposition 3.1] we have that $\operatorname{rk}(F)=1+\operatorname{rk}(G)$ and thus $\operatorname{rk}(G)=3$. Since all degree three binary cubics of rank three are monomials, we get that, after a change of variables, $G$ can be written as $L_{1} L_{2}^{2}$, with $L_{i} \in \mathbb{C}[Y, Z]_{1}$. Hence, $F=X^{3}+L_{1} L_{2}^{2}$ and this completes the proof.

We focus on families (6),(7) and (9). Due to Lemma 2.3.16, we can study these families with the same strategy, explained in the following.

Remark 2.3.17. Let $F$ be a rank four plane cubic which is not cuspidal. Hence, first of all, we have that $F$ essentially involves three variables and cannot be written as a binary form; therefore, since $\mathcal{S} / F^{\perp}$ is Artin Gorenstein, we have that its Hilbert function is forced to be

$$
\mathrm{HF}_{\mathcal{S} / F^{\perp}}: 1331-
$$

It follows that $\mathcal{L}=\left[F^{\perp}\right]_{2}$ is a net of conics, say $\mathcal{L}=\left\langle\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}\right\rangle$. Since we assume that $F$ is not cuspidal, by Lemma 2.3.16, all sets of four points apolar to $F$ are complete intersections of two conics. Thus, when we look for minimal Waring decompositions of $F$, we only need to look at pencils of conics contained in $\mathcal{L}$ with four distinct base points. In particular, fixing a point $P \in \mathbb{P}^{2}$, we can consider the pencil $\mathcal{L}(-P)$ of plane conics in $\mathcal{L}$ passing through $P$. If $\mathcal{L}(-P)$ has four distinct base points, then $P \in \mathcal{W}_{F}$; otherwise, we have that the base locus of $\mathcal{L}(-P)$ is not reduced and $P \in \mathcal{F}_{F}$.

We recall that a pencil of conics $\mathcal{L}^{\prime}$ has four distinct base points, no three of them collinear, if and only if the pencil contains exactly three reducible conics; see Figure 2.1. In conclusion, given a point $P \in \mathbb{P}^{2}$, we consider the line $\mathbb{P}(\mathcal{L}(-P)) \subset \mathbb{P}(\mathcal{L})$ : if the line is a proper secant line of the degree three curve $\Delta \subset \mathbb{P}(\mathcal{L})$ of reducible conics in $\mathcal{L}$, then we have that $P \in \mathcal{W}_{F}$; otherwise, $\mathcal{F}_{F}$. Thus, we have to study the dual curve $\check{\Delta} \subset \check{\mathbb{P}}(\mathcal{L})$ of lines not intersecting $\Delta$ in three distinct points ${ }^{7}$.

An equation for $\check{\Delta}$ can be found with a careful use of elimination theory. Hence, to explicitly find a $\mathcal{F}_{F}$, we consider the map

$$
\varphi: \mathbb{P}\left(S_{1}\right) \longrightarrow \check{\mathbb{P}}(\mathcal{L}), \quad \varphi([a: b: c])=\left[C_{1}(a, b, c): C_{2}(a, b, c): C_{3}(a, b, c)\right] .
$$

Note that $\varphi$ is defined everywhere and it is generically 4-to-1. In particular,

$$
\mathcal{F}_{F}=\varphi^{-1}(\check{\Delta})
$$

[^10]

Figure 2.1: The three reducible conics in a pencil of plane conics with four distinct base points, with no three of them collinear.

Theorem 2.3.18. If $F=x\left(y z+x^{2}\right)$, then $\mathcal{F}_{F}=Z\left(X Y Z\left(X^{2}-12 Y Z\right)\right)$.
Proof. Let $\mathcal{L}=\left(F^{\perp}\right)_{2}$ and let $\mathcal{C}_{1}: X^{2}-6 Y Z=0, \mathcal{C}_{2}: Y^{2}=0$, and $\mathcal{C}_{3}: Z^{2}=0$ be the conics generating $\mathcal{L}$. In the plane $\mathbb{P}(\mathcal{L})$ with coordinate $\alpha, \beta$ and $\gamma$, let $\Delta$ be the cubic of reducible conics in $\mathcal{L}$. By computing we get the following equation for $\Delta$ :

$$
\operatorname{det}\left[\begin{array}{ccc}
\alpha & 0 & 0 \\
0 & \beta & -3 \alpha \\
0 & -3 \alpha & \gamma
\end{array}\right]=\alpha \beta \gamma-9 \alpha^{3}=0
$$

In this case, $\Delta$ is the union of the conic $\mathcal{C}: 9 \alpha^{2}-\beta \gamma=0$ and the secant line $r: \alpha=0$. The line $r$ corresponds to $\mathcal{L}(-[1: 0: 0])$ and then, by Remark 2.3.17, we have that $[1: 0: 0] \in \mathcal{F}_{F}$.

By Remark 2.3.17, in order to completely describe $\mathcal{F}_{F}$, we have to study two families of lines in $\mathbb{P}(\mathcal{L})$ : the tangents to the conic $\mathcal{C}$ and all the lines passing through the intersection points between the line $r$ and the conic $\mathcal{C}$, i.e., all the lines through the points $[0: 0: 1]$ and $[0: 1: 0]$. More precisely, the point $P=[X: Y: Z]$ belongs to $\mathcal{F}_{F}$ if and only if the line $l: \alpha \mathcal{C}_{1}(P)+\beta \mathcal{C}_{2}(P)+\gamma \mathcal{C}_{3}(P)=0 \subset \mathbb{P}(\mathcal{L})$, i.e., the line

$$
l: \alpha\left(X^{2}-6 Y Z\right)+\beta Y^{2}+\gamma Z^{2}=0
$$

falls in one of the following cases:
(1) $l$ is tangent to the conic $\mathcal{C}: \beta \gamma-9 \alpha^{2}=0$;
(2) $l$ passes through the point $[0: 1: 0]$;
(3) $l$ passes through the point $[0: 0: 1]$.

In the cases (2) and (3) we get that $Y^{2}=0$ and $Z^{2}=0$, respectively. So $V(Y Z) \subset \mathcal{F}_{F}$. Now, assume that $P \notin\{Y Z=0\}$. By direct computation, we get that the line $l$ is tangent to the conic $\mathcal{C}$ if $X^{2}\left(X^{2}-12 Y Z\right)=0$.

It follows that $\mathcal{F}_{F}=V\left(X Y Z\left(X^{2}-12 Y Z\right)\right)$; see Figure 2.2.


Figure 2.2: The forbidden points of $F=x\left(y z+x^{2}\right)$.

We now consider family (7), that is the family of nodal cubics.
Theorem 2.3.19. If $F=y^{2} z-x^{3}-x z^{2}$, then

$$
\mathcal{F}_{F}=Z\left(G_{1} G_{2}\right),
$$

where $G_{1}=X^{3}-6 Y^{2} Z+3 X Z^{2}$ and $G_{2}=9 X^{4} Y^{2}-4 Y^{6}-24 X Y^{4} Z-$ $30 X^{2} Y^{2} Z^{2}+4 X^{3} Z^{3}-3 Y^{2} Z^{4}-12 X Z^{5}$.

Proof. Take $\mathcal{L}=\left(F^{\perp}\right)_{2}=\left\langle\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}\right\rangle$ with

$$
\mathcal{C}_{1}=X Y, \mathcal{C}_{2}=X^{2}-3 Z^{2}, \mathcal{C}_{3}=Y^{2}+X Z
$$

In the plane $\mathbb{P}(\mathcal{L})$ with coordinates $\alpha, \beta$, and $\gamma$ let $\Delta$ be the cubic of reducible conics in $\mathcal{L}$. By computing we see that $\Delta$ is defined by

$$
\operatorname{det}\left[\begin{array}{ccc}
\beta & \frac{1}{2} \alpha & \frac{1}{2} \gamma \\
\frac{1}{2} \alpha & \gamma & 0 \\
\frac{1}{2} \gamma & 0 & -3 \beta
\end{array}\right]=\frac{3}{4} \alpha^{2} \beta-3 \beta^{2} \gamma-\frac{1}{4} \gamma^{3}=0
$$

In this case, we have that $\Delta$ is an irreducible smooth cubic; hence,

$$
\begin{equation*}
\mathcal{F}_{F}=\left\{P \in \mathbb{P}^{2}: \mathbb{P}(\mathcal{L}(-P)) \text { is a tangent line to } \Delta \subset \mathbb{P}(\mathcal{L})\right\} \tag{2.3}
\end{equation*}
$$

Thus, we are looking for points $P \in \mathbb{P}^{2}$ such that the line

$$
\begin{equation*}
l: C_{1}(P) \alpha+C_{2}(P) \beta+C_{3}(P) \gamma=0 \tag{2.4}
\end{equation*}
$$

is tangent to $\Delta$. We consider two distinct cases if $C_{1}(P)=0$ and $C_{1}(P) \neq 0$.

If $C_{1}(P) \neq 0$, we compute $\alpha$ from the equation (2.4) and we substitute in the equation of of the cubic $\Delta$. Then, it is enough to compute the discriminant of the following form in $\beta$ and $\gamma$

$$
3\left(C_{2} \beta+C_{3} \gamma\right)^{2} \beta-12 C_{1}^{2} \beta^{2} \gamma-C_{1}^{2} \gamma^{3}
$$

and we get $D=27 C_{1}^{4} G_{1}^{2} G_{2}$. Thus, if $C_{1}(P) \neq 0$, we have that $P \in \mathcal{F}_{F}$ if and only if $P \in Z\left(G_{1} G_{2}\right)$. If $C_{1}(P)=0$, by direct computation, we check that $\mathcal{F}_{F} \cap Z\left(C_{1}\right)=Z\left(G_{1} G_{2}\right) \cap Z\left(C_{1}\right)$. Hence, the proof is completed.

Remark 2.3.20. The description of the forbidden locus for a plane cubic given in (2.3) reminds one of an old observation of De Paolis. He gave an algorithm to construct a decomposition of a general plane cubic as a sum of four cubes of linear forms whenever starting with a linear form which defines a line intersecting the Hessian of the plane cubic in precisely three points. This algorithm has been recently recalled in [Ban14].

Now, we consider the case of cubics in family (9) and we use the map $\varphi$ defined in Remark 2.3.17.

Theorem 2.3.21. If $F=x^{3}+y^{3}+z^{3}+a x y z$ belongs to family (9), then

1. if $\left(\frac{a^{3}-54}{9 a}\right)^{3} \neq 27$, then $\mathcal{F}_{F}$ is the dual curve to the smooth plane cubic

$$
\alpha^{3}+\beta^{3}+\gamma^{3}-\frac{a^{3}-54}{9 a} \alpha \beta \gamma=0
$$

2. otherwise, $\mathcal{F}_{F}$ is the union of three lines pairwise intersecting in three distinct points.

Proof. Take $\mathcal{L}=\left(F^{\perp}\right)_{2}=\left\langle\mathcal{C}_{1}, \mathcal{C}_{2}, \mathcal{C}_{3}\right\rangle$ with

$$
\mathcal{C}_{1}=a X^{2}-6 Y Z, \mathcal{C}_{2}=a Y^{2}-6 X Z, \mathcal{C}_{3}=a Z^{2}-6 X Y
$$

In the plane $\mathbb{P}(\mathcal{L})$ with coordinates $\alpha, \beta$, and $\gamma$ let $\Delta$ be the cubic curve consisting of reducible conics. An equation for $\Delta$ is given by

$$
\operatorname{det}\left[\begin{array}{ccc}
a \alpha & -3 \gamma & -3 \beta \\
-3 \gamma & a \beta & -3 \alpha \\
-3 \beta & -3 \alpha & a \gamma
\end{array}\right]=\left(a^{3}-54\right) \alpha \beta \gamma-9 a \alpha^{3}-9 a \beta^{3}-9 a \gamma^{3}=0
$$

In the numerical case $\left(\frac{a^{3}-54}{9 a}\right)^{3} \neq 27, \Delta$ is a smooth cubic curve. Hence,

$$
\mathcal{F}_{F}=\left\{P \in \mathbb{P}^{2} \mid \mathbb{P}(\mathcal{L}(-P)) \text { is a tangent line to } \Delta \subset \mathbb{P}(\mathcal{L})\right\}
$$

Hence, we get $\mathcal{F}_{F}$ as described in Remark 2.3 .17 by using the map $\varphi$.
Otherwise, $\Delta$ is the union of three lines intersecting in three distinct points $Q_{1}, Q_{2}$ and $Q_{3}$. Hence,

$$
\mathcal{F}_{F}=\left\{P \in \mathbb{P}^{2} \mid Q_{i} \in \mathbb{P}(\mathcal{L}(-P)), \text { for some } i\right\} .
$$

Example 2.3.22. Consider $F=x^{3}+y^{3}+z^{3}-6 x y z$, thus we are in the case (1) of Theorem 2.3.21 for $a=-6$. We can compute $\mathcal{F}_{F}$ using the following.

Algorithm 2.3.23. This algorithm is written with the language of the Computer Algebra Software Macaulay2 [GS].

```
R = QQ[x..z,a..c];
S = R[A..C];
-- x..z : parameters of a point in P^2;
-- a..c : parameters of a point over Delta;
-- A..C : in place of alpha, beta and gamma;
-- input: generators of linear system L corresponding to
-- the degree 2 part of perp ideal of F;
-- (give them in the variables x,y,z)
    C1 = ;
    C2 = ;
    C3 = ;
-- web of conics in L and through the point P=[x:y:z];
    LP}=C1*A+C2*B+C3*C
-- equation for a point Q=[a:b:c] over Delta;
    D = det (1/2*
                                    diff(transpose matrix{{x,y,z}},
                                    diff(matrix{{x,y,z}},LP))
            );
    DD = sub(D,{A => a,B => b,C => c});
-- tangent to Delta in the point Q=[a:b:c];
    TT = diff(a,DD)*A+diff(b,DD)*B+diff(c,DD)*C;
-- we impose that LL define a line tangent to Delta by
-- imposingthat LP is equal to the line TT and that the
-- point Q is on Delta;
    I = ideal((coefficients(TT-LP))_1,DD);
-- elimination of the extra variables: the result is the
-- locus of points P=[x:y:z] such that the corresponding
-- net of LP gives a tangent line to Delta, hence the
-- complement of the Waring locus;
    II = sub(I,R);
    radical eliminate({a,b,c},II);
```

Hence, in this specific example, we get that $\mathcal{F}_{F}=Z\left(G_{1} G_{2}\right)$, where $G_{1}=$ $x^{3}+y^{3}-5 x y z+z^{3}$ and $G_{2}=27 X^{6}-58 X^{3} Y^{3}+27 Y^{6}-18 X^{4} Y Z-18 X Y^{4} Z-$ $109 X^{2} Y^{2} Z^{2}-58 X^{3} Z^{3}-58 Y^{3} Z^{3}-18 X Y Z^{4}+27 Z^{6}$.

We conclude with family (10), that is cubics of rank five.
Theorem 2.3.24. If $F=x\left(x y+z^{2}\right)$, then $\mathcal{F}_{F}=\{[1: 0: 0]\}$.
Proof. Let $L$ be a linear form. The following are equivalent:
(1) $[L] \in \mathcal{F}_{f}$;
(2) $\operatorname{rk}\left(F-\lambda L^{3}\right)=5$, for all $\lambda \in \mathbb{C}$;
(3) $F-\lambda L^{3}=0$ is the union of an irreducible conic and a tangent line, for all $\lambda \in \mathbb{C}$;
(4) $F$ and $L^{3}$ must have the common factor $L$, that is, the line $L=0$ is the line $x=0$.

It easy to show that (1) and (2) are equivalent. For the equivalence between (2) and (3), see the table at the beginning of this section on the classification of plane cubics.

If (3) holds, then all the elements in the linear system given by $F$ and $L^{3}$ are reducible. Thus, by the second Bertini's Theorem, the linear system has the fixed component $\{x=0\}$. On the other hand, to see that (4) implies (3), note that for all $\lambda \in \mathbb{C}$, the cubic $x\left(x y+z^{2}+\lambda x^{2}\right)=0$ is the union of an irreducible conic and a tangent line.

## SECTION 2.4

## Waring loci and the Strassen conjecture

The standard algorithm to calculate the product of two $2 \times 2$ matrices involves 8 multiplications. Let $A=\left(a_{i, j}\right)$ and $B=\left(b_{i, j}\right)$, then $C=A B=\left(c_{i, j}\right)$, where

$$
\begin{aligned}
& c_{1,1}=a_{1,1} b_{1,1}+a_{1,2} b_{2,1} ; \\
& c_{1,2}=a_{1,1} b_{1,2}+a_{1,2} b_{2,2} ; \\
& c_{2,1}=a_{2,1} b_{1,1}+a_{2,2} b_{2,1} ; \\
& c_{2,2}=a_{2,1} b_{1,2}+a_{2,2} b_{2,2} .
\end{aligned}
$$

In general, we have that the multiplication of two $n \times n$ matrices uses $n^{3}$ multiplications and $n^{3}-n^{2}$ additions, namely we have on the order of $n^{3}$ arithmetic operations that are needed by the standard algorithm. Indeed, it has been proven in [BCS13], that the exponent on the order of the required arithmetic operations equals the one about the number of multiplications needed. Hence, the challenge is to reduce the number of multiplications in the algorithm, see [BCS13; Lan12] for a complete description.

In 1969, Strassen managed to give an algorithm for the multiplication of two $2 \times 2$ matrices, using only seven multiplications. Namely,

$$
\begin{aligned}
\mathrm{I} & =\left(a_{1,1}+a_{2,2}\right)\left(b_{1,1}+b_{2,2}\right) ; \\
\mathrm{II} & =\left(a_{2,1}+a_{2,2}\right) b_{1,1} ; \\
\mathrm{III} & =a_{1,1}\left(b_{1,2}-b_{2,2}\right) ; \\
\mathrm{IV} & =a_{2,2}\left(-b_{1,1}+b_{2,1}\right) ; \\
\mathrm{V} & =\left(a_{1,1}+a_{1,2}\right) b_{2,2} ; \\
\mathrm{VI} & =\left(-a_{1,1}+a_{2,1}\right)\left(b_{1,1}+b_{1,2}\right) ; \\
\mathrm{VII} & =\left(a_{1,2}-a_{2,2}\right)\left(b_{2,1}+b_{2,2}\right) ;
\end{aligned}
$$

then,

$$
\begin{aligned}
& c_{1,1}=\mathrm{I}+\mathrm{IV}-\mathrm{V}+\mathrm{VII} \\
& c_{2,1}=\mathrm{II}+\mathrm{IV} \\
& c_{1,2}=\mathrm{III}+\mathrm{V} \\
& c_{2,2}=\mathrm{I}+\mathrm{III}-\mathrm{II}+\mathrm{VI}
\end{aligned}
$$

Since the multiplication of two $n \times n$ matrices can be made in blocks, Strassen's algorithm can be generalized to multiplication of bigger matrices. As a consequence, we have that matrix multiplication can be performed by using on the order of $n^{2.81}$ arithmetic operations. ${ }^{8}$

After Strassen's result, it became clear that even natural procedures might require fewer operations than the expected. In 1973, Strassen formulated the following conjecture for bilinear maps:
let $\phi, \psi$ be two bilinear maps and consider two pairs of matrices $A, B$ and $C, D$, then, the computational complexity of simultaneously computing $\phi(A, B)$ and $\psi(C, D)$ is the sum of the complexities of $\phi$ and $\psi$.

The conjecture is still open. In [Lan12], it is explained how to interpret the Strassen conjecture in terms of tensors and their tensor rank. We are interested in its symmetric version which is also still largely open.

Conjecture 2.1 (Strassen additive conjecture). If $F \in \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ and $G \in \mathbb{C}\left[y_{0}, \ldots, y_{m}\right]$ are degree $d$ forms, with $d \geq 2$, then

$$
\operatorname{rk}(F+G)=\operatorname{rk}(F)+\operatorname{rk}(G) .
$$

Throughout this section, we use the following notations. We denote

$$
\begin{aligned}
S & =\mathbb{C}\left[x_{1,0}, \ldots, x_{1, n_{1}}, \ldots, x_{m, 0}, \ldots, x_{m, n_{m}}\right] \\
S^{[i]} & =\mathbb{C}\left[x_{i, 0}, \ldots, x_{i, n_{i}}\right], \text { for any } i=1, \ldots, m .
\end{aligned}
$$

Steps towards a proof of the Strassen conjecture have been done recently in [ $\left.\mathrm{CCC}^{+} 15 \mathrm{~b}\right]$. The most complete result is the following.

[^11]Theorem 2.4.1 (Carlini-Catalisano-Chiantini-Geramita-Woo). Let $F=$ $F_{1}+\ldots+F_{m} \in S$ be a homogeneous polynomial of degree $d$ with $F_{i} \in S^{[i]}$. If for every $i=0, \ldots, m, F_{i}$ is one of the following types:

1. monomial;
2. form in one or two variables;
3. $F_{i}=x_{0}^{a}\left(x_{1}^{b}+\ldots+x_{n}^{b}\right)$, with $a+1 \geq b$;
4. $F_{i}=x_{0}^{a}\left(x_{1}^{b}+x_{2}^{b}\right)$;
5. $F_{i}=x_{0}^{a}\left(x_{0}^{b}+\ldots+x_{n}^{b}\right)$, with $a+1 \geq b$;
6. $F_{i}=x_{0}^{a}\left(x_{0}^{b}+x_{1}^{b}+x_{2}^{b}\right)$;
7. $F_{i}=x_{0}^{a} G\left(x_{1}, \ldots, x_{n}\right)$, where $G^{\perp}=\left(g_{1}, \ldots, g_{n}\right)$ is a complete intersection with $a<\operatorname{deg}\left(g_{i}\right)$, for any $i=1, \ldots, n$;
8. $F_{i}$ is a Vandermonde determinant;
then

$$
\operatorname{rk}(F)=\operatorname{rk}\left(F_{1}\right)+\ldots+\operatorname{rk}\left(F_{m}\right)
$$

A stronger version of the Strassen conjecture can be rephrased in terms of minimal Waring decompositions. Indeed, we might expect that Waring decompositions of $F$ are always constructed as sums of Waring decompositions of the $F_{i}$ 's. Unfortunately, this is not always the case and, as often happens, quadrics have unexpected behaviors.
Example 2.4.2. Take $F=x^{2}-2 y z$. By Theorem 2.4.1, the rank of $F$ is three. However, we can find a Waring decomposition of $F$ that doesn't involve $x^{2}$, i.e.,

$$
F=(x+y)^{2}+(x+z)^{2}-(x+y+z)^{2} .
$$

We believe that this is an exceptional case which happens only for quadrics and we conjecture the following.

Conjecture 2.2. Let $F=F_{1}+\ldots+F_{m} \in S$ be a homogeneous polynomial of degree $d$ with $F_{i} \in S^{[i]}$. If $d \geq 3$, then any minimal Waring decomposition of $F$ is a sum of minimal Waring decompositions of the $F_{i}$ 's; that is,

$$
F=\sum_{i=1}^{m} \sum_{j=1}^{\operatorname{rk}\left(F_{i}\right)} L_{i, j}^{d}
$$

where $F_{i}=\sum_{j=1}^{\mathrm{rk}\left(F_{i}\right)} L_{i, j}^{d}$ and $L_{i, j} \in S^{[i]}$.
From Conjecture 2.2, it is natural to interpret the Strassen conjecture in terms of the concept of Waring loci of homogeneous polynomials. More precisely, we want to describe the Waring locus of a sum of homogeneous polynomials, which are pairwise in different sets of variables, as the union of the Waring loci of the single summands.

Conjecture 2.3. Let $F=F_{1}+\ldots+F_{m} \in S$ be a homogeneous polynomial of degree $d$ with $F_{i} \in S^{[i]}$. If $d \geq 3$, then

$$
\mathcal{W}(F)=\bigcup_{i=1, \ldots, r} \mathcal{W}\left(F_{i}\right) \subset \mathbb{P}(S)
$$

We always assume that each $F_{i} \in S^{[i]}$ is written in the essential number of variables and then, as explained in Proposition 2.3.3, we look at each $\mathcal{W}\left(F_{i}\right)$ inside the linear subspace $\mathbb{P}\left(S^{[i]}\right) \subset \mathbb{P}(S)$.

Example 2.4.3. We can view Example 2.4 .2 in terms of Waring loci. Let $F=F_{1}+F_{2}$ with $F_{1}=x^{2}$ and $F_{2}=2 y z$. The union of the two Waring loci is

$$
\mathcal{W}\left(F_{1}\right) \cup \mathcal{W}\left(F_{2}\right)=\{[1: 0: 0]\} \cup\left(\mathbb{P}_{y, z}^{1} \backslash\{[0: 1: 0],[0: 0: 1]\}\right) \subset \mathbb{P}_{x, y, z}^{2}
$$

Then, since we have rank equal to three, we look for ideals of three reduced points inside $F^{\perp}=\left(x y, x z, y^{2}, z^{2}, x^{2}+y z\right)$. We can find the ideal

$$
I_{\mathbb{X}}=\left(y(x-y), z(x-z), x^{2}-y^{2}+y z-z^{2}\right) \subset F^{\perp}
$$

which cut out the set of points $\mathbb{X}=\{[1: 1: 0],[1: 0: 1],[1: 1: 1]\}$ which is not contained in $\mathcal{W}\left(F_{1}\right) \cup \mathcal{W}\left(F_{2}\right)$.

Lemma 2.4.4. Conjecture 2.2 and Conjecture 2.3 are equivalent and they imply the Strassen additive conjecture (Conjecture 2.1) for $d \geq 3$.

Proof. The fact that Conjecture 2.2 implies Conjecture 2.3 and the Strassen conjecture is obvious. Assume that Conjecture 2.3 is true. If $\mathbb{X}$ is a minimal set of points apolar to $F$, then we have that $\mathbb{X} \subset \mathcal{W}(F)$. Let $\mathbb{X}_{i}:=$ $\mathbb{X} \cap \mathcal{W}\left(F_{i}\right) \subset \mathbb{P}\left(S^{[i]}\right)$, for any $i=1, \ldots, m$, say $\mathbb{X}_{i}=\left\{P_{i, 1}, \ldots, P_{i, r_{i}}\right\}$. Then,

$$
F=\sum_{i=1}^{m} \sum_{j=1}^{r_{i}} L_{i, j}^{d}
$$

where $L_{i, j}$ is the linear form corresponding to the point $P_{i, j}$, for any $i, j$, respectively. Restricting this identity to the linear space $\mathbb{P}\left(S^{[i]}\right)$, we get a Waring decomposition $F_{i}=\sum_{j=1}^{r_{i}} L_{i, j}^{d}$. Hence, $r_{i} \geq \operatorname{rk}\left(F_{i}\right)$ and then $\operatorname{rk}(F)=$ $|\mathbb{X}| \geq \sum_{i=1, \ldots, m} \operatorname{rk}\left(F_{i}\right)$. Since the opposite inequality is holds, it follows that $\operatorname{rk}(F)=\sum_{i=1, \ldots, m} \operatorname{rk}\left(F_{i}\right)$ and $r_{i}=\operatorname{rk}\left(F_{i}\right)$. Thus, Conjecture 2.2 holds.

The following preliminary result is our main tool to prove that Conjecture 2.2 and Conjecture 2.3 hold for some specific families of polynomials.

Proposition 2.4.5. Let $F=\sum_{i=1}^{s} F_{i} \in S$ be a form such that $F_{i} \in S^{[i]}$ for all $i=1, \ldots$, s. If the following conditions hold

1. for each $1 \leq i \leq s$, there exists a linear derivation $\partial_{i} \in T_{1}^{[i]}$ such that

$$
\operatorname{rk}\left(\partial_{i} F_{i}\right)=\operatorname{rk}\left(F_{i}\right)
$$

2. the Strassen conjecture holds for $F_{1}+\ldots+F_{s}$,
3. the Strassen conjecture holds for $\partial_{1} F_{1}+\ldots+\partial_{s} F_{s}$,
then F satisfies Conjecture 2.2 and Conjecture 2.3
Proof. Consider the linear form $t=\alpha_{1} \partial_{1}+\ldots+\alpha_{s} \partial_{s} \in T$, with $\alpha_{i} \neq 0$, for all $i=1, \ldots, s$. With no loss of generality, we may assume that $\partial_{i}=X_{i, 0}$.

Let $I_{\mathbb{X}} \subset F^{\perp}$ be the ideal of a set of points giving a Waring decomposition of $F$, i.e., the cardinality of $\mathbb{X}$ is equal to $\operatorname{rk}(F)$. Thus, $I_{\mathbb{X}}:(t)$ is the ideal of the points of $\mathbb{X}$ lying outside the linear space $t=0$. Then,

$$
I_{\mathbb{X}}:(t) \subset F^{\perp}:(t)=\left(\frac{\partial F}{\partial t}\right)^{\perp}=: \widetilde{F}^{\perp}
$$

By our assumptions, we get that $\operatorname{rk}(F)=\operatorname{rk}(\widetilde{F})$. Therefore, the set of points corresponding to $I_{\mathbb{X}}:(t)$ has cardinality equal to $\operatorname{rk}(F)$; it follows that $\mathbb{X}$ does not have points on the hyperplane $t=0$.

Claim. If $P=\left[a_{1,0}: \ldots: a_{1, n_{1}}: \ldots: a_{s, 0}: \ldots: a_{s, n_{s}}\right]$ belongs to $\mathcal{W}(F)$, then in the set $\left\{a_{1,0}, \ldots, a_{s, 0}\right\}$ there is exactly one non-zero coefficient.

Proof of the Claim. The claim follows from the first part, since if we have either no or at least two non-zero coefficients in the set $\left\{a_{1,0}, \ldots, a_{s, 0}\right\}$ it is easy to find a linear space $\{t=0\}$ containing the point $P$ and contradicting the assumption that it belongs to the Waring locus of $F$.

Let's consider $\mathbb{X}_{i}:=\mathbb{X} \backslash\left\{X_{i, 0}=0\right\}$, for all $i=1, \ldots, s$. As above, from

$$
I_{\mathbb{X}_{i}}=I_{\mathbb{X}}:\left(X_{i, 0}\right) \subset F^{\perp}:\left(X_{i, 0}\right)=\left(\frac{\partial F_{i}}{\partial x_{i, 0}}\right)^{\perp}
$$

we can conclude that the cardinality of each $\mathbb{X}_{i}$ is at least $\operatorname{rk}\left(F_{i}\right)$. Moreover, by the claim, we have that the $\mathbb{X}_{i}$ 's are all distinct. By additivity of the rank, we conclude that

$$
\begin{gathered}
\mathbb{X}=\bigcup_{i=1, \ldots, r} \mathbb{X}_{i}, \quad \text { with } \mathbb{X}_{i} \cap \mathbb{X}_{j}=\emptyset, \text { for all } i \neq j, \quad \text { and } \\
\left|\mathbb{X}_{i}\right|=\operatorname{rk}\left(F_{i}\right), \text { for all } i=1, \ldots, s
\end{gathered}
$$

Hence, we have that the sets $\mathbb{X}_{i}$ give minimal Waring decompositions of the forms $\partial_{i} F_{i}$ 's and, by Proposition 2.3 .3 , they lie in $\mathbb{P}_{X_{i, 0}, \ldots, X_{i, n_{i}}}^{n_{i}}$, respectively. Since $\mathbb{X}$ gives a minimal Waring decomposition of $F$, specializing to zero the variables not in $S^{[i]}$ we see that $\mathbb{X}_{i}$ gives a minimal Waring decomposition of $F_{i}$. Hence, $\mathcal{W}(F) \subset \bigcup_{i=1, \ldots, s} \mathcal{W}\left(F_{i}\right)$. The other inclusion is trivial.

Now, we can prove that Conjecture 2.2 and Conjecture 2.3 hold for some family of polynomials for which the assumptions of Proposition 2.4.5 hold.

Lemma 2.4.6. If $F \in S_{d}$ is one of the following degree $d$ forms

1. a monomial $x_{0}^{d_{0}} \ldots x_{n}^{d_{n}}$ with $d_{i} \geq 2$ for $0 \leq i \leq n$;
2. a binary form $F \neq L M^{d-1}$;
3. $x_{0}^{a}\left(x_{1}^{b}+\ldots+x_{n}^{b}\right)$ with $b, n \geq 2$ and $a+1 \geq b$;
4. $x_{0}^{a}\left(x_{0}^{b}+x_{1}^{b}+\ldots+x_{n}^{b}\right)$ with $b, n \geq 2$ and $a+1 \geq b$;
5. $x_{0}^{a} G\left(x_{1}, \ldots, x_{n}\right)$ such that $G^{\perp}=\left(g_{1}, \ldots, g_{n}\right), a \geq 2$, and $\operatorname{deg} g_{i} \geq a+1$.
then there exists a linear derivation $\partial \in T_{1}$ such that

$$
\operatorname{rk}(\partial F)=\operatorname{rk}(F)
$$

Proof. (1.) Take $F=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}}$ with $d_{0} \leq \ldots \leq d_{n}$. Then, we know by [CCG12] that $\operatorname{rk}(F)=\left(d_{1}+1\right) \cdots\left(d_{n}+1\right)$. If we let $\partial=X_{0}$, then $\operatorname{rk}(F)=$ $\operatorname{rk}(\partial F)$.
(2.) We know that $F^{\perp}=\left(G_{1}, G_{2}\right)$ with $\operatorname{deg}\left(G_{i}\right)=d_{i}, d_{1} \leq d_{2}$ and $d_{1}+d_{2}=$ $d+2$. We have to consider different cases.
a) If $d_{1}<d_{2}$ and $G_{1}$ is square-free, then $\operatorname{rk}(F)=\operatorname{deg}\left(G_{1}\right)$. Consider any linear form $L \in T_{1}$ which is not a factor of $G_{1}$. Fix $\partial:=L \in T_{1}$ to be the corresponding linear derivation. Then, $(\partial F)^{\perp}=F^{\perp}:(L)=$ $\left(H_{1}, H_{2}\right)$, with $\operatorname{deg}\left(H_{1}\right)+\operatorname{deg}\left(H_{2}\right)=d+1$. Since $L$ is not a factor of $G_{1}$, then we have that $G_{1}=H_{1}$ and, since it is square-free, we have that $\operatorname{rk}(\partial F)=\operatorname{rk}(F)$.
b) If $d_{1}<d_{2}$ and $G_{1}$ is not square-free, say $G_{1}=L_{1}^{m_{1}} \cdots L_{s}^{m_{s}}$, with $m_{1} \leq$ $\ldots \leq m_{s}$, then we have $\operatorname{rk}(F)=\operatorname{deg}\left(G_{2}\right)$. Fix $\partial=L_{1} \in T_{1}$. Then, we have that $(\partial F)^{\perp}=F^{\perp}:\left(L_{1}\right)=\left(H_{1}, H_{2}\right)$ with $\operatorname{deg}\left(H_{1}\right)+\operatorname{deg}\left(H_{2}\right)=$ $d+1$. Since $L_{1}^{m_{1}-1} \cdots L_{s}^{m_{s}} \in(\partial F)^{\perp}$, but not in $F^{\perp}$, it has to be $H_{1}=$ $L_{1}^{m_{1}-1} \cdots L_{s}^{m_{s}}$. In particular, $\operatorname{rk}(\partial F)=\operatorname{deg}\left(H_{2}\right)=\operatorname{deg}\left(G_{2}\right)=\operatorname{rk}(F)$.
c) If $d_{1}=d_{2}$, we can always consider a non square-free element $G \in$ $\left[F^{\perp}\right]_{d_{1}}$. Indeed, if both $G_{1}$ and $G_{2}$ are square-free, then it is enough to consider one element lying on the intersection between the hypersurface in $\mathbb{P}\left(S_{d_{1}}\right)$ defined by the vanishing of the discriminant of polynomials of degree $d_{1}$ with the line passing through $\left[G_{1}\right]$ and $\left[G_{2}\right]$. Say $G=L_{1}^{m_{1}} \cdots L_{s}^{m_{s}}$, with $m_{1} \leq \ldots \leq m_{s}$. Fix $\partial=L_{1} \in T_{1}$. Hence, we conclude the proof as in b).
(3.) If $F=x_{0}^{a}\left(x_{1}^{b}+\ldots+x_{n}^{b}\right)$ with $b, n \geq 2$ and $a+1 \geq b$, then we have that $\operatorname{rk}(F)=(a+1) n$, by [CCC15a]. If we set $\partial:=X_{1}+\ldots+X_{n}$, then $\partial F=x_{0}^{a}\left(x_{1}^{b-1}+\ldots+x_{n}^{b-1}\right)$ and the rank is preserved.
(4.) If $F=x_{0}^{a}\left(x_{0}^{b}+\ldots+x_{n}^{b}\right)$ with $b, n \geq 2$ and $a+1 \geq b$, then we have that $\operatorname{rk}(F)=(a+1) n$, by [CCC15a]. If we set $\partial:=X_{1}+\ldots+X_{n}$, then $\partial F=x_{0}^{a}\left(x_{1}^{b-1}+\ldots+x_{n}^{b-1}\right)$ and the rank is preserved.
(5.) If $F=x_{0}^{a} G\left(x_{1}, \ldots, x_{n}\right)$ with $G^{\perp}=\left(g_{1}, \ldots, g_{n}\right), a \geq 2$, and $\operatorname{deg} g_{i} \geq$ $a+1$, we know that $\operatorname{rk}(F)=d_{1} \cdots d_{n}$, by [CCC15a]. If we consider $\partial:=X_{0}$, then we have that $\partial F=x_{0}^{a-1} G\left(x_{1}, \ldots, x_{n}\right)$ and the rank is preserved.

Theorem 2.4.7. Let $F=\sum_{i=1}^{s} F_{i} \in S$ be a form such that $F_{i} \in S^{[i]}$, for all $i=1, \ldots, s$. If each $F_{i}$ is contained in one of the following families:

1. a monomial $x_{0}^{d_{0}} \cdot \ldots \cdot x_{n}^{d_{n}}$ with $d_{i} \geq 2$ for $0 \leq i \leq n$;
2. a binary form $F \neq L M^{d-1}$;
3. $x_{0}^{a}\left(x_{1}^{b}+\ldots+x_{n}^{b}\right)$ with $b, n \geq 2$ and $a+1 \geq b$;
4. $x_{0}^{a}\left(x_{0}^{b}+x_{1}^{b}+\ldots+x_{n}^{b}\right)$ with $b, n \geq 2$ and $a+1 \geq b$;
5. $x_{0}^{a} G\left(x_{1}, \ldots, x_{n}\right)$ such that $G^{\perp}=\left(g_{1}, \ldots, g_{n}\right), a \geq 2$, and $\operatorname{deg} g_{i} \geq a+1$.
then Conjecture 2.2 holds for $F$.
In [CCO], we prove Conjecture 2.2 in a few more cases without using Proposition 2.4.5. Note, for example, that Proposition 2.4.5 cannot be applied if one of the summands is a monomial with lowest exponent equal to one, since Lemma 2.4.6 does not hold.

Theorem 2.4.8. Conjecture 2.2 is true for a form $F$ of degree $d \geq 3$

$$
F=x_{0} x_{1}^{a_{1}} \ldots x_{n}^{a_{n}}+y_{0}^{b_{0}} y_{1}^{b_{1}} \ldots y_{m}^{b_{m}} \in \mathbb{C}\left[x_{0}, x_{1}, \ldots x_{n}, y_{0}, y_{1}, \ldots y_{m}\right]
$$

with $d=1+\sum_{i=0}^{n} a_{i}=\sum_{i=0}^{m} b_{i}$ and $b_{0} \leq b_{i}(i=1, \ldots, m)$.

## SECTION 2.5

## Apolarity Lemma:

## POWER IDEALS AND FAT POINTS

In the previous sections, we have seen how to relate Waring decompositions of homogeneous polynomials to ideals of configurations of reduced points. The key point was the Apolarity Lemma (Lemma 2.1.13). In this section, we describe a more general version of the Apolarity Lemma that allows us to relate configurations of non-reduced points with power ideals, namely ideals generated by powers of linear forms.

### 2.5.1 IDEALS OF FAT POINTS

Using our geometric intuition, we already have a naïve concept of nonreduced points. For example, consider a line intersecting a plane conic in two distinct points and let's imagine that we move the secant line to a tangent position as in Figure 2.3. Since high school, we are used to say that the tangent meets the conic twice since the associated system of equations have one double root. Algebraic geometers describe this more rigorously by using the notion of a scheme. Roughly speaking, the idea is to generalize the notion of algebraic variety to include also nilpotent elements, namely to
consider also non-radical ideals. Let's go back to the example of the intersection between a conic and a tangent line in order to describe it in more detail.


Figure 2.3: Example of non-reduced point: the intersection of a conic and a tangent line.

Example 2.5.1. Let $\mathcal{C}: y-x^{2}=0$ be a conic in the affine plane $\mathbb{A}_{x, y}^{2}$ and $r$ : $y=0$ be the tangent line at the origin $O$. Their intersection is described by the ideal $I=\left(x^{2}-y, y\right)=\left(x^{2}, y\right)$. As a variety, we have that the intersection is simply one point, the origin $O$, given by the radical of $I, \sqrt{I}=(x, y)$. The coordinate ring is a one-dimensional vector space $\mathbb{C}[x, y] /(x, y) \simeq \mathbb{C}$.

On the other hand, if we consider the ideal $I$, we have that the quotient ring is given by $\mathbb{C}[x, y] /\left(x^{2}, y\right) \simeq \mathbb{C} \oplus \mathbb{C} x$ which is a vector space of dimension two. Heuristically, we imagine the scheme as two points infinitesimally close and supported at the origin along the direction of the $x$-axis. Indeed, consider the lines $r(t): y=t$, with $t \in(0,1]$, intersecting the conic at the two distinct points $( \pm \sqrt{t}, t)$. For $t \rightarrow 0$, both points tends to the origin $O$.

Another way of thinking is as follows. The ideal $\sqrt{I}=(x, y) \subset \mathbb{C}[x, y]$ is the ideal of homogeneous polynomials vanishing at the point $O=(0,0)$. Instead, the ideal $I=\left(x^{2}, y\right)$ is given by the homogeneous polynomials $F \in \mathbb{C}[x, y]$ that vanish at the origin $O$ and such that also their partial derivative $\partial_{x} F$ vanishes at $O$. For this reason, we can imagine the scheme given by the intersection of the conic and the tangent line as a simple point together with a tangent vector.

Since in this thesis we only need the concept of multiple points, we don't
give details about scheme theory. For that, see [EH00, Har77]. We give an ad hoc definition for the 0-dimensional ${ }^{9}$ schemes we are interested in.

Definition 2.5.2. A fat point of multiplicity $d$ in $\mathbb{P}^{n}$ is a 0 -dimensional scheme associated to the $d$-th power $\wp^{d}$ of a prime ideal $\wp$ of height $n$. It is denoted by $d P$. The reduced point $P$, defined by $\wp$, is called its support. The quotient $S_{\wp} / \wp^{d} S_{\wp}$, where $S_{\wp}$ is the graded localization of $S$ at $\wp$, is a finite dimensional vector space whose dimension is called the degree of $d P$.

In general, we define a scheme of fat points, or a configuration of fat points, as $\mathbb{X}=d_{1} P_{1}+\ldots+d_{s} P_{s} \subset \mathbb{P}^{n}$, where $P_{1}, \ldots, P_{s}$ are distinct points, with corresponding ideals $\wp_{1}, \ldots, \wp_{s}$, respectively and $d_{1}, \ldots, d_{s}$ are positive integers; namely, $\mathbb{X}$ is the 0-dimensional scheme defined by the ideal $I_{\mathbb{X}}=\wp_{1}^{d_{1}} \cap \ldots \cap \wp_{s}^{d_{s}}$. The degree of $\mathbb{X}$ is the sum of the degrees of the $d_{i} P_{i}$ 's.

Remark 2.5.3. In the case $d_{1}=\ldots=d_{s}=d$, let $\mathbb{X}_{d}=d P_{1}+\ldots+d P_{s}$, for any $d \geq 1$. Then, we have that $I_{\mathbb{X}_{d}}=\wp_{1}^{d} \cap \ldots \cap \wp_{s}^{d}$ is the $d$-th symbolic power ${ }^{10}$ of the ideal $I_{\mathbb{X}_{1}}=\wp_{1} \cap \ldots \cap \wp_{s}$. We denote it by $I_{\mathbb{X}_{1}}^{(d)}$.

The study of the behavior between symbolic and regular powers of homogeneous ideals involved many mathematicians and different areas. By definition, we always have the inclusion $I^{d} \subset I^{(d)}$, but the equality is not always true. Consequently, people started to study containment problems, see e.g. [ELS01; HH02]. In [BH10], the authors showed that for any $c<n$, there exists an ideal of points in $\mathbb{P}^{n}$ such that $I^{(d)} \not \subset I^{r}$ for some $d>c r$. In [BCH14], there is a list of open conjectures about containment problems.

### 2.5.2 InVERSE SYSTEMS OF IDEALS FAT POINTS.

In particular, we are interested in studying the Hilbert series of ideals of fat points. This is a sort of interpolation problem. Indeed, if $\wp$ is the prime ideal defining a point $P \in \mathbb{P}^{n}$, we have that

$$
F \in\left[\wp^{d}\right]_{m} \quad \text { if and only if } \quad Z(F) \text { is singular of multiplicity } d \text { at } P .
$$

Therefore, $\mathrm{HF}_{S / \wp^{d}}(m)$ equals the codimension of the vector space of hypersurfaces of degree $m$ in $\mathbb{P}^{n}$ which are singular at $P$ with multiplicity $d$. As we have explained previously, one way to study Hilbert functions of homogeneous ideals is to look at their inverse systems. In EI95], Emsalem and Iarrobino started to study inverse systems of ideals of fat points and it continued in the famous book by Iarrobino and Kanev [IK99]. However, our exposition refers to the lecture notes by Geramita [Ger96].

[^12][^13]Example 2.5.4. Let $P_{0}=[1: 0: \ldots: 0] \in \mathbb{P}^{n}$ be the coordinate point associated to $\wp_{0}=\left(x_{1}, \ldots, x_{n}\right)$. We consider the fat point $d P_{0}$ defined by the ideal $\wp_{0}^{d}$. Since $\wp_{0}^{d}$ is the monomial ideal generated by all degree $d$ monomials in the variables $x_{1}, \ldots, x_{n}$, we have that

$$
\left[\left(\wp_{0}^{d}\right)^{-1}\right]_{i}=\left\{\begin{array}{cl}
\mathcal{S}_{i} & \text { for all } i<d \\
X_{0}^{i-d+1} \mathcal{S}_{d-1} & \text { for all } i \geq d
\end{array}\right.
$$

In general, given any point $P=\left[p_{0}: \ldots: p_{n}\right] \in \mathbb{P}^{n}$ and its defining ideal $\wp$, with no loss of generality, we may assume that $p_{0} \neq 0$. Hence, after a linear change of coordinates, we may assume $P=P_{0}$. Using the previous case and after that performing the inverse change of coordinates, we get

$$
\left[\left(\wp^{d}\right)^{-1}\right]_{i}=\left\{\begin{array}{cc}
\mathcal{S}_{i} & \text { for all } i<d  \tag{2.5}\\
L^{i-d+1} \mathcal{S}_{d-1} & \text { for all } i \geq d
\end{array}\right.
$$

where $L=p_{0} X_{0}+\ldots+p_{n} X_{n}$.
Theorem 2.5.5 (Apolarity Lemma [IK99; Ger96]). Let $P_{1}, \ldots, P_{s}$ be distinct points in $\mathbb{P}^{n}$ with coordinates $P_{i}=\left[p_{0}^{(i)}: \ldots: p_{n}^{(i)}\right]$, respectively. Consider the corresponding linear forms, $L_{i}=p_{0}^{(i)} X_{0}+\ldots+p_{n}^{(i)} X_{n}$, for any $i=1, \ldots, s$. Given positive integers $d_{1}, \ldots, d_{s}$, we consider the scheme of fat points $\mathbb{X}=d_{1} P_{1}+\ldots+d_{s} P_{s}$ defined by the ideal $I=\wp_{1}^{d_{1}} \cap \ldots \cap \wp_{s}^{d_{s}}$, where the $\wp_{i}$ 's are the defining ideals of the $P_{i}$ 's, respectively. Then,

$$
\left[I^{-1}\right]_{i}= \begin{cases}\mathcal{S}_{i} & \text { for all } i \leq \max \left\{d_{i}-1\right\} \\ L_{1}^{i-d_{1}+1} \mathcal{S}_{d_{1}-1}+\ldots+L_{s}^{i-d_{s}+1} \mathcal{S}_{d_{s}-1} & \text { for all } i \geq \max \left\{d_{i}\right\}\end{cases}
$$

Idea of the proof. It follows directly from (2.5) and the following fact.
Claim. Let $I, J$ be two homogeneous ideals $I$ and $J$,

$$
(I \cap J)^{-1}=I^{-1}+J^{-1}
$$

Remark 2.5.6. The weaker version of the Apolarity Lemma used in the previous section (Lemma 2.1.13) is a special case of Theorem 2.5.5. Indeed, in the same notation as above, let $i=d_{1}=\ldots=d_{s}=d$. By Theorem 2.5.5, we have that a homogeneous polynomial $F$ of degree $d$ can be expressed as the sum of $L_{1}^{d}, \ldots, L_{s}^{d}$ if and only if $F$ is contained in the inverse system of $I=\wp_{1} \cap \ldots \cap \wp_{s}$. Since $\left[I^{-1}\right]_{i}^{\perp}=I_{i}$, it is equivalent to say that the ideal $I$ is contained in the perp ideal of $F$.

As explained in Section 2.1, the dimensions of the homogeneous parts of the inverse system $I^{-1}$ are related to the Hilbert functions of $S / I$.

Corollary 2.5.7. Let $\mathbb{X}=d_{1} P_{1}+\ldots+d_{s} P_{s}$ be a scheme of fat points in $\mathbb{P}^{n}$ with defining ideal $I=\wp_{1}^{d_{1}} \cap \ldots \cap \wp_{s}^{d_{s}}$. Let $L_{1}, \ldots, L_{s}$ be the linear forms defined as in Theorem 2.5.5. Then,

$$
\operatorname{HF}_{S / I}(i)= \begin{cases}\operatorname{dim}_{\mathbb{C}} \mathcal{S}_{i}, & \text { for all } i \leq \max \left\{d_{i}-1\right\} \\ \operatorname{dim}_{\mathbb{C}}\left[\left(L_{1}^{i-d_{1}+1}, \ldots, L_{s}^{i-d_{s}+1}\right)\right]_{i}, & \text { for all } i \geq \max \left\{d_{i}\right\}\end{cases}
$$

Moreover, in Section 1.3.1, we saw that, given a generic point $P \in \sigma_{s}\left(V_{n, d}\right)$ on a secant variety of a Veronese variety lying on the linear span of $s$ points $\left[L_{1}^{d}\right], \ldots,\left[L_{s}^{d}\right] \in V_{n, d}$, we have that the tangent space of $\sigma_{s}\left(V_{n, d}\right)$ at $P$ is

$$
T_{P} \sigma_{s}\left(V_{n, d}\right)=\mathbb{P}\left(\left[L_{1}^{d-1}, \ldots, L_{s}^{d-1}\right]_{d}\right) \subset \mathbb{P}\left(S_{d}\right)
$$

Combining this description with Corollary 2.5.7, we get a relation between Hilbert functions of ideals of double points and the dimensions of secant varieties of Veronese varieties.

Corollary 2.5.8. Let $I_{\mathbb{X}}=\wp_{1}^{2} \cap \ldots \cap \wp_{s}^{2}$ be the ideal of a scheme $\mathbb{X}$ of $s$ fat points of multiplicity d in $\mathbb{P}^{n}$, then

$$
\mathrm{HF}_{S / I_{\mathrm{X}}}(d)=\operatorname{dim} \sigma_{s}\left(V_{n, d}\right)
$$

## SECTION 2.6

## Hilbert Functions of CONFIGURATIONS OF FAT POINTS

Similarly as in Section 2.2, we have the following result on the structure of the Hilbert functions of configurations of points.

Lemma 2.6.1. Let $\mathbb{X}$ be a 0 -dimensional scheme of degree $D$ in $\mathbb{P}^{n}$, then the Hilbert function of the associated quotient ring $S / I_{\mathbb{X}}$ is strictly increasing until it becomes constant; moreover, $\mathrm{HF}_{S / I_{\mathrm{X}}}(i)=D$, for $i \gg 0$.

Example 2.6.2. Consider again the point $P=[1: 0: \ldots: 0]$ defined by the prime ideal $\wp=\left(x_{1}, \ldots, x_{n}\right)$. From Corollary 2.5.7, we have that

$$
\mathrm{HF}_{S / \wp^{d}}(i)= \begin{cases}\operatorname{dim}_{\mathbb{C}} \mathcal{S}_{i}=\binom{n+i}{n}, & \text { for all } i \leq d-1 \\ \operatorname{dim}_{\mathbb{C}}\left[\left(X_{0}^{i-d+1}\right)\right]_{i}=\binom{n+d-1}{n}, & \text { for all } i \geq d\end{cases}
$$

This coincides with the Hilbert function of $\binom{n+d-1}{n}$ simple points in generic position, see Lemma 2.2.1 and Lemma 2.2.2. Note that $\binom{n+d-1}{n}$ coincides with the degree of the fat point $d P$. Therefore, we can visualize a fat point of multiplicity $d$ in $\mathbb{P}^{n}$ as $\binom{n+d-1}{n}$ simple points collapsed at the point $P$.

Now, let $\mathbb{X}=d_{1} P_{1}+\ldots+d_{s} P_{s}$. The degree of $\mathbb{X}$ is $D=\sum_{i=1}^{s}\binom{n+d_{i}-1}{n}$. Then, similarly to the case of reduced points, if the $P_{i}$ 's are in generic position, we expect that the Hilbert function of $S / I_{\mathbb{X}}$ is the largest possible, i.e., that $\mathbb{X}$ behaves as $D$ distinct points in generic position,

$$
\exp \cdot \operatorname{HF}_{S / I_{\mathbb{X}}}(i)=\min \left\{\binom{n+i}{n}, D\right\}, \text { for all } i \geq 0
$$

This is equivalent to expect that, in each degree $i \geq 0$, the vanishing of all derivatives of order $\leq d_{j}-1$ at each point $P_{j}$, respectively, impose independent conditions on the linear system $\Gamma\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(j)\right)$ of hypersurfaces of degree $j$ in $\mathbb{P}^{n}$. If the latter assumption holds, we say that $\mathbb{X}$ imposes independent conditions on $\mathcal{O}_{\mathbb{P}^{n}}(j)$. Unfortunately, this is not always the case and we can easily find examples when the Hilbert function of a scheme $\mathbb{X}$ of fat points in generic position is not the expected one.

Example 2.6.3. Consider two distinct double points $\mathbb{X}=2 P_{1}+2 P_{2}$ in the projective plane $\mathbb{P}^{2}$ defined by the ideal $I_{\mathbb{X}}=\wp_{1}^{2} \cap \wp_{2}^{2}$. The space of conics in $\mathbb{P}^{2}$ has (affine) dimension 6 and each double point imposes exactly 3 condition. Hence, we expect to have no conic with two distinct singular points and therefore the Hilbert function of the quotient ring $S / I_{\mathbb{X}}$ in degree 2 should be equal to 6 , i.e.,

$$
\exp . \mathrm{HF}_{S / \wp_{1}^{2} \cap \wp_{2}^{2}}: 13 \begin{array}{lllll} 
& 6 & 6 & \cdots
\end{array}
$$

However, the double line passing through the two points gives a reducible conic with two singular points. By Bezout's Theorem, we can see that the double line is also the unique conic in $I_{\mathbb{X}}$ and, in fact,

$$
\mathrm{HF}_{S / \wp_{1}^{2} \cap \wp_{2}^{2}}: 1 \begin{array}{lllll} 
& 3 & 5 & 6 & 6
\end{array} \cdots
$$

Example 2.6.4. Consider five general double points $\mathbb{X}=2 P_{1}+\ldots+2 P_{5}$ in $\mathbb{P}^{2}$ defined by the ideal $I_{\mathbb{X}}=\wp_{1}^{2} \cap \ldots \cap \wp_{5}^{2}$. The space of quartics in $\mathbb{P}^{2}$ has (affine) dimension $\binom{2+4}{2}=15$ and each double point imposes 3 conditions. Hence, we expect to have no quartics with five singular points and therefore

$$
\mathrm{HF}_{S / \not \wp_{1}^{2} \cap \ldots \cap \wp_{5}^{2}}: 1 \begin{array}{lllllll}
1 & 6 & 10 & \mathbf{1 5} & 15 & 15 & \cdots
\end{array}
$$

However, there exists a unique conic passing through five points in generic position; thus, the double conic is a quartic inside the ideal $\wp_{1}^{2} \cap \ldots \cap \wp_{5}^{2}$. By Bezout's theorem, it is also unique and we have that

$$
\mathrm{HF}_{S / \wp_{1}^{2} \cap \ldots \wp_{5}^{2}}: 1 \quad 3 \quad 6 \quad 10141515 \cdots
$$

From these two examples, it is clear that the study of the Hilbert functions of ideals of configurations of fat points is far from being obvious. Apart from the case of double points in generic position and some special configurations, this problem is still largely open.

### 2.6.1 Double points: the Alexander-Hirschowitz Theorem

In 1995, after a series of brilliant articles AH92a; AH92b], Alexander and Hirschowitz gave the solution of the problem regarding the Hilbert function of double points in generic position. This is the precise formulation of Alexander-Hirschowitz Theorem. From this, due to Corollary 2.5.8, we get the dimension of all secant varieties of Veronese varieties (Theorem 1.3.8) and then, from the geometric description of the classical Waring problem, we also get the Waring rank of generic forms in any number of variables and any degree (Theorem 1.2.5).

Theorem 2.6.5 (Alexander-Hirschowitz Theorem, [AH95]). Consider $P_{1}, \ldots, P_{s}$ distinct general points of $\mathbb{P}^{n}$, with defining ideals $\wp_{1}, \ldots, \wp_{s}$, respectively. Then, the scheme of double points $\mathbb{X}=2 P_{1}+\ldots+2 P_{s}$ defined by the ideal $\wp_{1}^{2} \cap \ldots \cap \wp_{s}^{2}$ has the expected Hilbert function, i.e.

$$
\operatorname{HF}_{S / \wp_{1}^{2} \cap \ldots \cap \wp_{s}^{2}}(d)=\min \left\{s(n+1),\binom{n+d}{n}\right\},
$$

except for the following cases,

1. $d=2 ; 2 \leq s \leq n$; where it must be reduced by $\binom{s}{2}$;
2. $n=2 ; d=4$; $s=5$; where it must be reduced by 1 ;
3. $n=3 ; d=4$; $s=9$; where it must be reduced by 1 ;
4. $n=4 ; d=4 ; s=14$; where it must be reduced by 1 ;
5. $n=4 ; d=3$; $s=7$; where it must be reduced by 1 .

For ideals of fat points with higher multiplicity not much is known. Even the case of triple points in generic position is solved only up to three variables, see [Dum13]. We come back to this problem about the Hilbert function of ideals of fat points with higher multiplicity in Section 3.2 where we relate it with the Fröberg Conjecture.

## SECTION 2.7

## SPECIAL CONFIGURATIONS OF FAT POINTS

In this section we compute Hilbert functions and Betti numbers of some special configurations of fat points. As we said above, given a scheme of general fat points $\mathbb{X}=d_{1} P_{1}+\ldots+d_{s} P_{s}$ in $\mathbb{P}^{n}$, the expected Hilbert function is given by

$$
\exp \cdot \operatorname{HF}_{S / I_{\mathrm{X}}}(i)=\min \left\{\binom{n+i}{n}, \sum_{j=1}^{s}\binom{n+d_{j}-1}{n}\right\}, \text { for all } i \geq 0
$$

If we specialize the points $P_{1}, \ldots, P_{s}$, the number of conditions imposed by $\mathbb{X}$ can only decrease, hence the Hilbert function of $S / I_{\mathbb{X}}$ can only increase. Moreover, the property of having the smallest expected Hilbert function is an open condition, then it follows that it is enough to find one explicit specialization of $\mathbb{X}$ with the expected Hilbert series in order to prove that the generic configuration have the smallest Hilbert function.

### 2.7.1 8 GENERIC FAT POINTS IN $\mathbb{P}^{3}$

Here, we describe a special configuration of fat points studied in [One14]. We consider eight fat points in generic position with the same multiplicity in $\mathbb{P}^{3}$, say $\mathbb{X}_{d}=d P_{1}+\ldots+d P_{8} \in \mathbb{P}^{3}$. Then, the defining ideal $I_{\mathbb{X}_{d}}$ is the $d$-th symbolic power $I_{\mathbb{X}_{1}}^{(d)}$ of the radical ideal $I_{\mathbb{X}_{1}}=\wp_{1} \cap \ldots \cap \wp_{8}$. Thus, we have

$$
\exp \cdot \operatorname{HF}_{S / I_{\mathbb{X}_{d}}}(i)=\min \left\{\binom{i+3}{3}, 8\binom{d+2}{3}\right\}
$$

In order to prove that this is the correct Hilbert function, we specialize the $P_{i}$ 's to the following configuration:

$$
\begin{aligned}
& P_{1}=[1: 0: 0: 0], P_{2}=[0: 1: 0: 0], P_{3}=[0: 0: 1: 0], P_{4}=[0: 0: 0: 1], \\
& P_{5}=[1: 1: 1: 0], P_{6}=[1: 1: 0: 1], P_{7}=[1: 0: 1: 1], P_{8}=[0: 1: 1: 1] .
\end{aligned}
$$

Let $I=\wp_{1} \cap \ldots \cap \wp_{8}$ be the ideal defining such a configuration.
Lemma 2.7.1. The $d$-symbolic power $I^{(d)}$ is:

1. empty in degree $<2 d$;
2. of dimension $d+1$ in degre $2 d$.

Proof. For any possible pairs of our variables, we define the hyperplanes:

$$
H_{01}=\left\{x_{0}-x_{1}=0\right\}, \ldots, H_{23}=\left\{x_{2}-x_{3}=0\right\} .
$$

Among such hyperplanes, we choose an independent 4-tuple, say

$$
H_{01}, H_{02}, H_{13}, H_{23}
$$

We can easily check the following two properties:

- every hyperplane passes through exactly 4 of our points, e.g.,

$$
\left\{P_{1}, P_{2}, P_{7}, P_{8}\right\} \in H_{23}
$$

- there are two pairs of such a 4-tuple which do not intersect at the $P_{i}$ 's.

Considering these two pairs, i.e. $\left\{H_{01}, H_{23}\right\}$ and $\left\{H_{02}, H_{13}\right\}$, we can construct the following two quadrics

$$
\mathcal{C}_{1}=H_{01} H_{23}, \mathcal{C}_{2}=H_{02} H_{13} .
$$

From the above mentioned properties of our hyperplanes, we may observe that each of the two quadrics passes through all the $P_{i}^{\prime} s$ exactly once and then, each power $\mathcal{C}_{i}^{m}$ has singularities of degree $m$ at each point; in other words, $\mathcal{C}_{i}^{m}$ belongs to $I^{(m)}$ for any positive integer $m$. Hence, the following polynomials are inside the degree $2 d$ part of the ideal $I^{(d)}$,

$$
G_{i, d-i}:=\mathcal{C}_{1}^{i} \cdot \mathcal{C}_{2}^{d-i}, \quad \text { for } i=0, \ldots, d
$$

We want to show that these are the generators of the ideal in degree $2 d$. Firstly, we may observe that such polynomials are linearly independent. We proceed by induction on the degree $d$. Base of induction: consider a linear combination $\alpha \mathcal{C}_{1}+\beta \mathcal{C}_{2}=0$. By intersecting with the hyperplane $H_{01}=\left\{x_{0}-x_{1}=0\right\}$, we get $\beta=0$ and, consequently, $\alpha=0$. Similarly, given a linear combination $\alpha \mathcal{C}_{1}^{2}+\beta \mathcal{C}_{1} \mathcal{C}_{2}+\gamma \mathcal{C}_{2}^{2}=0$, we get $\alpha=0$, by intersecting with the hyperplane $H_{01}$, and $\gamma=0$, by intersecting with $H_{02}$; consequently, $\beta=0$. With these two base steps, we can do the step of induction. Consider a linear combination $\sum_{i=0, \ldots, d} \alpha_{i} G_{d-i, i}=0$. By intersecting with the planes $H_{01}$ and $H_{02}$, we get $\alpha_{0}=0$ and $\alpha_{d}=0$, respectively. Consequently, we get a linear combination,

$$
\mathcal{C}_{1} \mathcal{C}_{2} \cdot \sum_{j=0}^{d-2} \alpha_{j+1} \mathcal{C}_{1}^{d-2-j} \mathcal{C}_{2}^{j}=0
$$

by induction, we are done.
After showing that the above polynomials are linearly independents, we now have to prove that they are enough to generate the whole degree $2 d$ part of the ideal, namely that the dimension of the linear system $\mathcal{L}_{2 d}\left(\mathbb{X}_{d}\right)$ is equal to $d+1$ and not bigger. We proceed by induction on the degree $d$.

The case $d=1$ is trivial. For the general case, we look at the linear system $\mathcal{L}_{2 d}\left(\mathbb{X}_{d}+P\right)$ of hypersurfaces of degree $2 d$ through our scheme $\mathbb{X}_{d}$ plus one extra simple point $P$.

Claim: the dimension of $\mathcal{L}_{2 d}\left(\mathbb{X}_{d}+P\right)$ is equal to $d$.
Proof of Claim. The aim is to show that, choosing a point $P$ in a clever way, the quadric given by the two plane $H_{01}$ and $H_{23}$ is a fixed component for all the surfaces of degree $2 d$ through the scheme $\mathbb{X}_{d}+Q$; thus, by induction,

$$
\operatorname{dim}_{\mathbb{C}} \mathcal{L}_{2 d}\left(\mathbb{X}_{d}+Q\right)=\operatorname{dim}_{\mathbb{C}} \mathcal{L}_{2 d-2}\left(\mathbb{X}_{d-1}\right)=d
$$

The symmetry of the problem suggests that we choose the point $Q$ on the intersection line $l=\left\{H_{01}=H_{23}=0\right\}$ between the two planes. Let's take, for example, a point on the line with coordinates $Q_{m}:=[2 m+1: 2 m+1$ : $2 m+3: 2 m+3]$ for some $m \geq 0$. On the plane $H_{01}$ we consider the coordinates $\left\{x_{0}, x_{2}, x_{3}\right\}$ and we consider the general conic given by the equation

$$
\alpha_{1} x_{0}^{2}+\alpha_{2} x_{0} x_{2}+\alpha_{3} x_{0} x_{3}+\alpha_{4} x_{2}^{2}+\alpha_{5} x_{2} x_{3}+\alpha_{6} x+3^{2}=0
$$

imposing the passage through the points $P_{3}, P_{4}, P_{5}, P_{6}$ on the plane, we get the following conditions

$$
\alpha_{4}=\alpha_{6}=0, \alpha_{1}+\alpha_{2}=\alpha_{1}+\alpha_{3}=0,
$$

thus, we get the pencil of conics

$$
\alpha_{1} x_{0}^{2}-\alpha_{1} x_{0} x_{2}-\alpha_{1} x_{2} x_{3}-\alpha_{5} x_{2} x_{3}=0 .
$$

Imposing the passage through the point $Q_{m}$, we get the condition

$$
\alpha_{5}=\frac{2(2 m+1)(2 m+3)-(2 m+1)^{2}}{(2 m+3)^{2}} \alpha_{1}=\frac{(2 m+1)(2 m+5)}{(2 m+3)^{2}} \alpha_{1},
$$

and then the unique conic
$\mathcal{Q}=(2 m+3)^{2} x_{0}^{2}-(2 m+3)^{2} x_{0} x_{2}-(2 m+3)^{2} x_{0} x_{3}+(2 m+1)(2 m+5) x_{3}^{2}=0$.
Intersecting $\mathcal{Q}$ with the intersection line $l$, i.e., imposing $x_{2}-x_{3}=0$, we get the equation

$$
\begin{aligned}
& (2 m+3)^{2} x_{0}^{2}-2(2 m+3)^{2} x_{0} x_{2}+(2 m+1)(2 m+5) x_{3}^{2}=0 \\
\Longrightarrow & {\left[(2 m+3) x_{0}-(2 m+1) x_{3}\right]\left[(2 m+3) x_{0}-(2 m+5) x_{3}\right]=0 . }
\end{aligned}
$$

Thus, the conic $\mathcal{Q}$ intersects the line $l$ also at two distinct points: the point $Q_{m}$ and also $R_{m}=[2 m+5: 2 m+5: 2 m+3: 2 m+3]$.

We play the same game on the plane $H_{23}$ considering the point $R_{m}$. Imposing the passage through the points $P_{1}, P_{2}, P_{7}, P_{8}$ to the conics in the variables $\left\{x_{0}, x_{1}, x_{2}\right\}$, we get the pencil of conics

$$
\beta_{2} x_{0} x_{1}-\beta_{6} x_{0} x_{2}-\beta_{6} x_{1} x_{2}+\beta_{6} x_{2}^{2}=0 .
$$

Imposing the passage through the point $R_{m}$, we get the condition

$$
\beta_{2}=\frac{2(2 m+5)(2 m+3)-(2 m+3)^{2}}{(2 m+5)^{2}} \beta_{6}=\frac{(2 m+7)(2 m+3)}{(2 m+5)^{2}} \beta_{6} .
$$

Thus, we get a unique conic intersecting the line $l$ at the two distinct points $R_{m}$ and $Q_{m+2}$. Hence, we come back to the plane $H_{01}$ and we continue the same game starting from the point $Q_{m+2}$, and so on.

In this way, on the two planes we can find infinitely many conics which have to be fixed component for our linear system; thus, the two planes are fixed components and the Claim is proven.

From the Claim, we have that the linear system $\mathcal{L}_{2 d}(\mathbb{X})$ has the right dimension; indeed,

$$
d^{\text {Claim }}=\operatorname{dim}_{\mathbb{C}} \mathcal{L}_{2 d}\left(\mathbb{X}_{d}+P\right) \geq \operatorname{dim}_{\mathbb{C}} \mathcal{L}_{2 d}\left(\mathbb{X}_{d}\right)-1 \geq(d+1)-1
$$

To complete the proof of the theorem, we just need to prove that there are no elements in the degree $2 d-1$ part of the ideal $I^{(d)}$. Indeed, assume

$F \in I^{(d)}$ of degree $2 d-1$; thus, $x_{0} F \in I^{(d)}$. Consequently, there exists a linear combination between $x_{0} F$ and the $G_{i}$ 's which is identically zero. By specializing to the hyperplane $x_{0}=0$ and using the linear independence of the $G_{i}$ 's, we are done.

Theorem 2.7.2. Let $P_{1}, \ldots, P_{8}$ be general points in $\mathbb{P}^{3}$ with defining ideals $\wp_{1}, \ldots, \wp_{8}$, respectively. For any $d \geq 1$, let $I^{(d)}=\wp_{1}^{d} \cap \ldots \cap \wp_{8}^{d}$ be the defining ideal of the scheme of fat points $\mathbb{X}_{d}=d P_{1}+\ldots+d P_{8}$. Then,

$$
\mathrm{HS}_{S / I^{d}}(t)=\sum_{0 \leq i \leq 2 m-1}\binom{i+3}{3} t^{i}+\sum_{i \geq 2 m} 8\binom{m+2}{3} t^{i}
$$

Proof. From the first part of Lemma 2.7.1, we have that there are no generators in degree $\leq 2 d-1$. Hence $\operatorname{HF}_{S / I^{(d)}}(i)=\operatorname{dim}_{\mathbb{C}} S_{i}$, for $i \leq 2 d-1$.

Moreover, we know that the Hilbert function $S / I^{(d)}$ is weakly increasing and eventually equals to $8\binom{d+2}{3}$. Hence, in order to conclude, it is enough to observe that

$$
\operatorname{HF}_{S / I^{(d)}}(2 d)=\binom{2 d+3}{3}-(d+1)=8\binom{d+2}{3}
$$

where the first inequality follows from the second part of Lemma 2.7.1.

Betti numbers. Set $R_{d}=S / I^{(d)}$. We have proved in Theorem 2.7.2 that

$$
\operatorname{HS}_{R_{d}}(t)=\sum_{j=0}^{2 d-1}\binom{3+j}{j} t^{j}+8\binom{d+2}{3} \frac{t^{2 d}}{1-t}
$$

Since $R_{d}$ is a Cohen-Macaulay ring of dimension one, we can take a nonzero divisor $L$ of degree 1 . Consequently, from the exact sequence given by
the multiplication by $L$ (see Example A.1.4), we have that

$$
\begin{aligned}
\mathrm{HS}_{R_{d} /(L)}(t) & =(1-t) \operatorname{HS}_{R_{d}}(t)= \\
& =(1-t) \sum_{j=0}^{2 d-1}\binom{3+j}{j} t^{j}+8\binom{d+2}{3} t^{2 d}
\end{aligned}
$$

In particular, $\left[R_{d} /(L)\right]_{2 d+1}=0$ and, since $L$ is a non-zero divisor of $R_{d}$,

$$
\beta_{i, j}=\operatorname{dim} \operatorname{Tor}_{i, j}^{S}\left(R^{(d)}, \mathbb{C}\right)=\operatorname{dim} \operatorname{Tor}_{i, j}^{S}\left(R^{(d)} /(L), \mathbb{C}\right)
$$

In order to compute the last Tor, one can start from the resolution of $\mathbb{C} \simeq S /\left(x_{0}, \ldots, x_{n}\right)$ given by the Koszul complex $\mathbb{K}_{\bullet}$, and then tensoring by $R_{d} /(L)$. We denote it by

$$
\mathcal{K}_{\bullet}=\mathbb{K} \bullet \otimes_{S} R_{d} /(L)
$$

By definition of the Koszul complex, which is given by the exterior algebra $\wedge \mathbb{C}^{4}$, we have that $\mathbb{K}_{i}=0$, for all $i \geq 5$. Otherwise, we have that the elements of the graded part $\mathcal{K}_{i, j}$ of $\mathcal{K}_{i}$ consist of linear combinations of elements of type $f \cdot t_{j_{1}} \cdots t_{j_{i}}$, where $\operatorname{deg}(f)=j-i$ and where $\left\{t_{1}, \ldots, t_{4}\right\}$ is a basis of $\mathbb{C}^{4}$. Thus, since $\left[R_{d} /(L)\right]_{2 d+1}=0$, by looking at the homology,

$$
\operatorname{Tor}_{i, j}^{S}\left(R_{d} /(L), \mathbb{C}\right)=0, \text { for all } j \geq i+2 d+1
$$

Thus, we can only have a few (possibly) non-zero Betti numbers, namely

|  | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | - | - | - | - | - |
| $2 d-1$ | - | $\beta_{1,2 d}$ | $\beta_{2,2 d+1}$ | $\beta_{3,2 d+2}$ | $\beta_{4,2 d+3}$ | - |
| $2 d$ | - | $\beta_{1,2 d+1}$ | $\beta_{2,2 d+2}$ | $\beta_{3,2 d+3}$ | $\beta_{4,2 d+4}$ | - |
| $2 d+1$ | - | - | - | - | - | - |

Lemma 2.7.3. There are no linear syzygies among the generators of our ideal $I^{(d)}$ in degree $2 d$, i.e. $\beta_{2,2 d+1}=0$.
Proof. Since the vanishing of a Betti number is equivalent to the vanishing of the corresponding Tor, it is given by the surjectivity or the injectivity of some linear map, i.e. by the maximality of the rank of some matrix. Hence, it is an open condition and, then, it is enough to show that such condition holds for a specific example in order to get the result for the generic case.

Let's consider the same arrangement of eight fat points as in Theorem 2.7.2. We have proved that the corresponding ideal has $d+1$ generators in degree $2 d$, i.e.

$$
G_{i, d-i}:=\mathcal{C}_{1}^{i} \mathcal{C}_{2}^{d-i}, \text { for } i=0 \ldots, d
$$

where $\mathcal{C}_{1}=\left(x_{0}-x_{1}\right)\left(x_{2}-x_{3}\right)$ and $\mathcal{C}_{2}=\left(x_{0}-x_{2}\right)\left(x_{1}-x_{3}\right)$. We proceed by induction over $d$. First, assume that there is a linear syzygy, i.e. there exists linear forms $L_{0}, L_{1}$ such that

$$
L_{0} \mathcal{C}_{1}+L_{1} \mathcal{C}_{2}=L_{0}\left(x_{0}-x_{1}\right)\left(x_{2}-x_{3}\right)+L_{1}\left(x_{0}-x_{2}\right)\left(x_{1}-x_{3}\right)=0
$$

Thus, all the third order derivatives of such polynomial should be equal to zero. Since the monomials $x_{0} x_{2}$ and $x_{1} x_{3}$ appear only in $\mathcal{C}_{1}$ and not in $\mathcal{C}_{2}$, the derivatives $\partial_{x_{0}^{2} x_{2}}, \partial_{x_{1}^{2} x_{3}}, \partial_{x_{0} x_{2}^{2}}, \partial_{x_{1} x_{3}^{2}}$ give us $L_{0}=0$. Similarly, by using the monomials $x_{0} x_{1}$ and $x_{2} x_{3}$, we get $L_{1}=0$.

Let's now assume that there is a linear syzygy in the case $d=2$, namely there exist linear forms $L_{0}, L_{1}$ and $L_{2}$ such that

$$
L_{0} G_{2,0}+L_{1} G_{1,1}+L_{2} G_{0,2}=0
$$

Again, monomials $x_{0}^{2} x_{2}^{2}$ and $x_{1}^{2} x_{3}^{2}$ appear only in $G_{2,0}$, thus, imposing that the degree 5 derivatives $\partial_{x_{0}^{3} x_{2}^{2}}, \partial_{x_{1}^{3} x_{3}^{2}}, \partial_{x_{0}^{2} x_{2}^{3}}, \partial_{x_{1}^{2} x_{3}^{3}}$ vanish, we get $L_{0}=0$. Similarly, by considering monomials $x_{0}^{2} x_{1}^{2}$ and $x_{2}^{2} x_{3}^{2}$, which appear only in $G_{0,2}$, and derivatives $\partial_{x_{0}^{3} x_{1}^{2}}, \partial_{x_{0}^{2} x_{1}^{3}}, \partial_{x_{2}^{3} x_{3}^{2}}, \partial_{x_{2}^{2} x_{3}^{3}}$, we get $L_{2}=0$. Consequently, $L_{1}=0$. In general, assuming that we have a linear syzygy

$$
\sum_{i=0}^{d} L_{i} G_{d-i, i}=\sum_{i=0}^{d} L_{i} \mathcal{C}_{1}^{d-i} \mathcal{C}^{i}=0
$$

we can consider the monomials $x^{d} z^{d}$ and $y^{d} t^{d}$ which appear only in $G_{d, 0}$. By imposing the vanishing of all the derivatives $\partial_{x_{0}^{d+1} x_{2}^{d}}, \partial_{x_{1}^{d+1} x_{3}^{d}}, \partial_{x_{0}^{d} x_{2}^{d+1}}, \partial_{x_{1}^{d} x_{3}^{d+1}}$, we get $L_{0}=0$. Similarly, we can use the monomials $x_{0}^{d} x_{1}^{d}$ and $x_{2}^{d} x_{3}^{d}$ to conclude $L_{d}=0$. Thus, we can rewrite the linear syzygy as

$$
\mathcal{C}_{1} \mathcal{C}_{2} \cdot \sum_{j=0}^{d-2} l_{j+1} \mathcal{C}_{1}^{d-2-j} \mathcal{C}_{2}^{j}=0
$$

and then use induction.
Now, since there are no linear syzygies among the generators of degree $2 d$, we have that $\beta_{2,2 d+1}=0$ and consequently, by a basic property of Betti numbers, also $\beta_{3,2 d+2}=\beta_{4,2 d+3}=0$. Thus, we can read the remaining Betti numbers directly from the Hilbert series of $R_{d}$, which is of the form

$$
\frac{1-\beta_{1,2 d} t^{2 d}-\beta_{1,2 d+1} t^{2 d+1}+\beta_{2,2 d+2} t^{2 d+2}-\beta_{3,2 d+3} t^{2 d+3}+\beta_{4.2 d+4} t^{2 d+4}}{(1-t)^{4}}
$$

From Theorem 2.7.2, we get

$$
\begin{equation*}
(1-t)^{4} \operatorname{HS}_{R_{d}}(t)=(1-t)^{4} \sum_{j=0}^{2 d-1}\binom{3+j}{j} t^{j}+(1-t)^{3} 8\binom{d+2}{3} t^{2 d} \tag{2.6}
\end{equation*}
$$

From that, we already can see that $\beta_{4,2 d+4}=0$. Now, one can show that

$$
(1-t)^{4} \sum_{j=0}^{2 d-1}\binom{3+j}{j} t^{j}=1+\sum_{j=0}^{3}(-1)^{j+1}\binom{3}{j}\binom{3+2 d}{2 d} \frac{2 d}{2 d+j} t^{2 d+j}
$$

The idea to prove such an equality is to show that both sides have the same derivatives, see [FL02, Proposition 5.2] for a similar computation. Thus, we can continue from equation (2.6), getting

$$
(1-t)^{4} \operatorname{HS}_{R_{d}}(t)=1+\sum_{j=0}^{3}(-1)^{j+1}\binom{3}{j}\left[\binom{3+2 d}{2 d}-8\binom{d+2}{3}\right] t^{2 d+j}
$$

By direct computations, we finally get

$$
(1-t)^{4} \mathrm{HS}_{R_{d}}(t)=1-(d+1) t^{2 d}-4\binom{d+1}{2} t^{2 d+1}+d(4 d+5) t^{2 d+2}-4\binom{d+1}{2}
$$

Hence, the Betti diagram of $R_{d}$ is as follows

|  | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | - | - | - | - |
| $2 \mathrm{~d}-1$ | - | $d+1$ | - | - | - |
| 2 d | - | $4\binom{d+1}{2}$ | $d(4 d+5)$ | $4\binom{d+1}{2}$ | - |
| $2 \mathrm{~d}+1$ | - | - | - | - | - |

### 2.7.2 Configurations of $\xi$-POINTS

In 2012, Fröberg, Ottaviani and Shapiro started the study of $d$-th Waring decompositions. Namely, decompositions of homogeneous polynomials of degree $k d$ as sums of $d$-th powers of forms of degree $k$. As explained in Section 1.3.3, these decompositions can be interpreted geometrically in terms of secant varieties of varieties of powers. In particular, the $d$-th Waring rank of generic forms of degree $k d$ corresponds to the first of such secant varieties that fills the whole ambient space, i.e.,

$$
\operatorname{rk}_{d}^{\circ}(n+1, k d)=\min \left\{s \mid \sigma_{s}\left(V_{n, k, d}\right)=\mathbb{P}\left(S_{k d}\right)\right\}
$$

Via Terracini's Lemma, we can describe the tangent space at a generic point $P \in \sigma_{s}\left(V_{n, k, d}\right)$ as

$$
T_{P}\left(\sigma_{s}\left(V_{n, k, d}\right)\right)=\mathbb{P}\left(\left[G_{1}^{d-1}, \ldots, G_{s}^{d-1}\right]_{k d}\right)
$$

where the $G_{i}$ 's are generic forms of degree $k$. In [FOS12], the authors proved that, for $s=d^{n}$, there exists a choice of the $G_{i}$ 's such that the ideal generated by their $d$-th powers coincides with the whole polynomial ring in degree $k d$. Their idea was to fix a primitive $d$-th root of unity and then consider the power ideal $I_{n, k, d}$ generated by all the $d^{n}$ forms

$$
\left(x_{0}+\xi^{g_{1}} x_{1}+\ldots+\xi^{g_{n}} x_{n}\right)^{k(d-1)}, \quad \text { where } 0 \leq g_{i} \leq d-1, \text { for all } i=1, \ldots, n
$$

Theorem 2.7.4 (Fröberg-Ottaviani-Shapiro, [FOS12]). Given positive integers $n, k, d$, in the same notation as above,

$$
\left[I_{n, k, d}\right]_{k d}=S_{k d}
$$

As a consequence, they get the following.
Theorem 2.7.5 (Fröberg-Ottaviani-Shapiro, [FOS12]). Fixed positive integers $n, k, d$, we have that

$$
\operatorname{rk}_{d}^{\circ}(n+1, k d) \leq d^{n}
$$

In this section, based on [BO15], we compute the Hilbert function of these power ideals and the corresponding ideals of fat points; namely, the schemes of fat points supported on the configuration of $\xi$-points

$$
\mathbb{X}_{n, k, d}=\left\{\left[1: \xi^{g_{1}}: \ldots: \xi^{g_{n}}\right] \in \mathbb{P}^{n} \mid 0 \leq g_{i} \leq d-1, \text { for all } i=1, \ldots, n\right\} .
$$

The key idea is to refine the standard grading.

## MULTICYCLE GRADING

Let $\mathbb{Z}_{k}=\left\{[0]_{d},[1]_{d}, \ldots,[d-1]_{d}\right\}$ be the cyclic group of integers modulo $d$. Let $\xi$ be a primitive $d^{t h}$-root of unity and observe that, for any $\nu \in \mathbb{Z}_{d}$, the complex number $\xi^{\nu}$ is well-defined. We will usually use a small abuse of notation denoting a class of integer modulo $d$ simply with its representative between 0 and $d-1$; e.g. when we will consider the scalar product between two vectors $\mathbf{g}, \mathbf{h} \in \mathbb{Z}_{d}^{n+1}$, denoted by $\langle\mathbf{g}, \mathbf{h}\rangle$, we will mean the usual scalar product considering each entry of the two vectors as the smallest positive representative of the corresponding class.

Consider, for each $\mathbf{g}=\left(g_{0}, \ldots, g_{n}\right) \in \mathbb{Z}_{d}^{n+1}$, the polynomial

$$
\phi_{\mathbf{g}}:=\left(\sum_{i=0}^{n} \xi^{g_{i}} x_{i}\right)^{D}, \text { where } D:=k(d-1)
$$

Hence, $I_{n, k, d}$ is by definition the ideal generated by all $\phi_{\mathbf{g}}$, with $\mathbf{g} \in 0 \times$ $\mathbb{Z}_{d}^{n}$. It is homogeneous with respect to the standard grading, but it is also homogeneous with respect to the $\mathbb{Z}_{d}^{n+1}$-grading we are going to define.

Consider the projection $\pi_{d}: \mathbb{N} \longrightarrow \mathbb{Z}_{d}$ given by $\pi_{d}(n)=[n]_{d}$. For any vector $\mathbf{a}=\left(a_{0}, \ldots, a_{n}\right) \in \mathbb{N}^{n+1}$, we define the multicyclic degree as follows.

Given a monomial $\mathbf{x}^{\mathbf{a}}:=x_{0}^{a_{0}} \ldots x_{n}^{a_{n}}$, we set

$$
\operatorname{mcdeg}\left(\mathbf{x}^{\mathbf{a}}\right):=\pi_{d}^{n+1}(\mathbf{a})=\left(\left[a_{0}\right]_{d}, \ldots,\left[a_{n}\right]_{d}\right) .
$$

Thus, combining this multicyclic degree with the standard grading, we get the multigrading on the polynomial ring $S$ given by

$$
S=\bigoplus_{i \in \mathbb{N}} S_{i}=\bigoplus_{i \in \mathbb{N}} \bigoplus_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} S_{i, \mathbf{g}}, \text { where } S_{i, \mathbf{g}}:=S_{i} \cap S_{\mathbf{g}}
$$

where, for any $i_{1}, i_{2} \in \mathbb{N}$ and $\mathbf{g}_{1}, \mathbf{g}_{2} \in \mathbb{Z}_{d}^{n+1}$, we have that

$$
S_{i_{1}, \mathbf{g}_{1}} \cdot S_{i_{2}, \mathbf{g}_{2}}=S_{i_{1}+i_{2}, \mathbf{g}_{1}+\mathbf{g}_{2}}
$$

Remark 2.7.6. For $0:=(0, \ldots, 0)$, we get obviously that $S_{\mathbf{0}}=\mathbb{C}\left[x_{0}^{d}, \ldots, x_{n}^{d}\right]$, and then, for any $i \in \mathbb{N}$,

$$
S_{i, \mathbf{0}} \neq 0 \text { if and only if } i=j d \text { for some } j \in \mathbb{N},
$$

in which case

$$
\operatorname{dim}_{\mathbb{C}} S_{j d, \mathbf{0}}=\binom{n+j}{n}
$$

For any multicycle $\mathbf{g}=\left(g_{0}, \ldots, g_{n}\right) \in \mathbb{Z}_{d}^{n+1}$, we define:
the partition vector of $g$ to be

$$
\operatorname{part}(\mathbf{g}):=\left(\#\left\{g_{i}=0\right\}, \ldots, \#\left\{g_{i}=d-1\right\}\right)
$$

and the weight of g as

$$
\mathrm{wt}(\mathbf{g}):=\sum_{j=0}^{n} g_{j}
$$

Clearly, the weight is non-negative and

$$
\mathrm{wt}(\mathbf{g})=0 \text { if and only if } \mathbf{g}=\mathbf{0}
$$

Lemma 2.7.7. Let $i \in \mathbb{N}$ and $\mathrm{g} \in \mathbb{Z}_{d}^{n+1}$. Then,

$$
S_{i, \mathbf{g}} \neq 0 \text { if and only if } i-\mathrm{wt}(\mathbf{g})=j d \text {, for some } j \in \mathbb{N} .
$$

In that case,

$$
\operatorname{dim}_{\mathbb{C}} S_{i, \mathbf{g}}=\binom{n+j}{n}
$$

Proof. Given a monomial $\mathbf{x}^{\mathbf{a}}$ with $i=\operatorname{deg}\left(\mathbf{x}^{\mathbf{a}}\right)$, consider $\mathbf{g}=\pi_{d}^{n+1}(\mathbf{a})$. Hence, we have that $\mathrm{x}^{\mathrm{a}-\mathrm{g}} \in S_{i-\mathrm{wt}(\mathrm{g}), \mathbf{0}}$. Furthermore,

$$
\operatorname{dim}_{\mathbb{C}} S_{i, \mathbf{g}}=\operatorname{dim}_{\mathbb{C}} S_{i-\mathrm{wt}(\mathbf{g}), \mathbf{0}}=\binom{n+j}{n}
$$

Now, we denote with $\mathcal{G}_{d, n, i}$ the set set of all multicycles satisfying the two equivalent conditions of Lemma 2.7.7, i.e.,

$$
\mathcal{G}_{d, n, i}:=\left\{\mathbf{h} \in \mathbb{Z}_{d}^{n+1} \mid i-\mathrm{wt}(\mathbf{h}) \in d \mathbb{N}\right\}=\left\{\mathbf{h} \in \mathbb{Z}_{d}^{n+1} \mid S_{i, \mathbf{h}} \neq 0\right\} .
$$

Coming back to our ideal, since we can write $S_{D}=\bigoplus_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} S_{D, \mathbf{g}}$, we can represent the generator $\phi_{\mathbf{0}}=\left(x_{0}+\ldots+x_{n}\right)^{D}$ of $I_{n, k, d}$ as

$$
\phi_{\mathbf{0}}=\sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} \psi_{\mathbf{g}}, \text { where } \psi_{\mathbf{g}} \in S_{D, \mathbf{g}}
$$

Clearly, if $\psi_{\mathbf{g}} \neq 0$ then $\mathbf{g} \in \mathcal{G}_{d, n, D}$, but one can also check that actually

$$
\psi_{\mathbf{g}} \neq 0 \Longleftrightarrow \mathbf{g} \in \mathcal{G}_{d, n, D}
$$

In particular, under the equivalent latter conditions, we have that,

$$
\psi_{\mathbf{g}}=\sum_{\substack{a_{0}+\ldots+a_{n}=D \\ \pi_{d}^{n+1}\left(a_{0}, \ldots, a_{n}\right)=\mathbf{g}}}\binom{D}{a_{0}, \ldots, a_{n}} \mathbf{x}^{\mathbf{a}} .
$$

This example should make the reader more familiar with these notation.
Example 2.7.8. Let $k=4, n=2, d=2$ and $\phi_{\mathbf{0}}=\left(x_{0}+x_{1}+x_{2}\right)^{4}$. We have

$$
\begin{aligned}
& \psi_{(0,0,0)}=x_{0}^{4}+6 x_{0}^{2} x_{1}^{2}+6 x_{0}^{2} x_{2}^{2}+x_{1}^{4}+6 x_{1}^{2} x_{2}^{2}+x_{2}^{4} \\
& \psi_{(1,0,0)}=\psi_{(0,1,0)}=\psi_{(0,0,1)}=\psi_{(1,1,1)}=0 \\
& \psi_{(1,1,0)}=4 x_{0}^{3} x_{1}+12 x_{0} x_{1} x_{2}^{2}+4 x_{0} x_{1}^{3} \\
& \psi_{(1,0,1)}=4 x_{0}^{3} x_{2}+12 x_{0} x_{1}^{2} x_{2}+4 x_{0} x_{2}^{3} \\
& \psi_{(0,1,1)}=4 x_{1} x_{2}^{3}+12 x_{0}^{2} x_{1} x_{2}+4 x_{1} x_{2}^{3} .
\end{aligned}
$$

We can notice that, since $(1,0,0) \notin \mathcal{G}_{2,2,4}$, we already expected $\psi_{(1,0,0)}=0$, and similarly for $(0,1,0),(0,0,1)$ and $(1,1,1)$.
Lemma 2.7.9. For any $\mathrm{g} \in \mathbb{Z}_{d}^{n+1}$, one has

$$
\phi_{\mathbf{g}}=\sum_{\mathbf{h} \in \mathcal{G}_{d, n, D}} \xi^{\langle\mathbf{g}, \mathbf{h}\rangle} \psi_{\mathbf{h}}
$$

and, conversely,

$$
\psi_{\mathbf{g}}=d^{-n-1} \sum_{\mathbf{h} \in \mathbb{Z}_{d}^{n+1}} \xi^{-\langle\mathbf{g}, \mathbf{h}\rangle} \phi_{\mathbf{h}}
$$

Proof. From the definition, we can write

$$
\begin{aligned}
\phi_{\mathbf{g}} & =\left(\sum_{i=0}^{n} \xi^{g_{i}} x_{i}\right)^{D}=\sum_{a_{0}+\ldots+a_{n}=D}\binom{D}{a_{0}, \ldots, a_{n}} \prod_{l=0}^{n} \xi^{g_{l} a_{l}} x_{l}^{a_{l}}= \\
& =\sum_{a_{0}+\ldots+a_{n}=D}\binom{D}{a_{0}, \ldots, a_{n}} \xi^{\langle\mathbf{g}, \mathbf{d}\rangle} \mathbf{x}^{\mathbf{a}} .
\end{aligned}
$$

Now, for each $\mathbf{a}=\left(a_{0}, \ldots, a_{n}\right)$, we can consider the vector $\pi_{d}^{n+1}(\mathbf{a})=\mathbf{h} \in$ $\mathbb{Z}_{d}^{n+1}$. Since $\xi$ is a $d^{t h}$ root of unity, we have $\xi^{\langle\mathbf{g}, \mathbf{a}\rangle}=\xi^{\langle\mathbf{g}, \mathbf{h}\rangle}$. Thus,

$$
\phi_{\mathbf{g}}=\sum_{\mathbf{h} \in \mathcal{G}_{d, n, D}} \xi^{\langle\mathbf{g}, \mathbf{h}\rangle} \sum_{\substack{a_{0}+\ldots+a_{n}=D \\ \pi_{d}^{n+1}(\mathbf{a})=\mathbf{h}}}\binom{D}{a_{0}, \ldots, a_{n}} \mathbf{x}^{\mathbf{a}}=\sum_{\mathbf{h} \in \mathcal{G}_{d, n, D}} \xi^{\langle g, h\rangle} \psi_{\mathbf{h}} .
$$

For the second part of the statement, we consider the following equality which follows from the first part already proven. For any $\mathbf{m} \in \mathbb{Z}_{d}^{n+1}$,

$$
\sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} \xi^{-\langle\mathbf{g}, \mathbf{m}\rangle} \phi_{\mathbf{g}}=\sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} \sum_{\mathbf{h} \in \mathcal{G}_{d, n, D}} \xi^{\langle\mathbf{g}, \mathbf{h}\rangle-\langle\mathbf{g}, \mathbf{m}\rangle} \psi_{\mathbf{h}} .
$$

In the right hand-side, we have

$$
\begin{cases}\text { if } \mathbf{m}=\mathbf{h}: & \sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} \psi_{\mathbf{h}}=d^{n+1} \psi_{\mathbf{h}} ; \\ \text { if } \mathbf{m} \neq \mathbf{h}: & \sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} \xi^{(\mathbf{g}, \mathbf{h}-\mathbf{m})} \psi_{\mathbf{h}}=\sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} \xi_{0}^{g_{0}} \ldots \xi_{n}^{g_{n}} \psi_{\mathbf{h}}=0 .\end{cases}
$$

Hence, the set $\left\{\psi_{\mathbf{g}}\right\}_{\mathbf{g} \in \mathcal{G}_{d, n, D}}$ of nonzero polynomials has distinct multicyclic degree and, consequently, is linearly independent. In other words, we have proved the following proposition.

Proposition 2.7.10. $I_{n, k, d}$ is minimally generated by $\left\{\psi_{\mathbf{g}}\right\}_{\mathbf{g} \in \mathcal{G}_{d, n, D}}$.
Theorem 2.7.11. The cardinality of $\mathcal{G}_{d, n, D}$ is given by:

$$
\begin{gathered}
\left|\mathcal{G}_{d, n, D}\right|= \\
=\sum_{i \geq 0} \sum_{\nu_{2}, \ldots, \nu_{d-1} \geq 0}\binom{n+1}{D-d i-\sum_{j=1}^{d-1}(j-1) \nu_{j}}\binom{D-d i-\sum_{j=1}^{d-1}(j-1) \nu_{j}}{\nu_{2}, \ldots, \nu_{d-1}, D-\sum_{j=2}^{d-1} j v_{j}}= \\
=\sum_{i, \nu_{2}, \ldots, \nu_{d-1} \geq 0}\binom{n+1}{\nu_{2}, \ldots, \nu_{d-1}, D-d i-\sum_{j=2}^{d-1} j \nu_{j}, n+1-D+d i+\sum_{j=2}^{d-1}(j-1) \nu_{j}} .
\end{gathered}
$$

In particular, if $d=2$, then this number of generators equals $\sum_{i \geq 0}\binom{n+1}{k-2 i}$.
Proof. For any $\mathbf{g} \in \mathcal{G}_{d, n, D}$, we can write $\psi_{\mathbf{g}}=f\left(x_{0}^{d}, \ldots, x_{n}^{d}\right) \mathbf{x}^{\mathbf{g}}$ where $f$ is a homogeneous polynomial of degree $i$ and $\operatorname{part}(\mathbf{g})=\left(0, \nu_{1}, \ldots, \nu_{d-1}\right)$. To count the number of elements of $\mathcal{G}_{d, n, D}$, there are $\left(\begin{array}{c} \\ D-d i-\sum_{j=1}^{n+1}(j-1) \nu_{j}\end{array}\right)$ ways to choose the variables with nonzero exponent modulo $d$ and, for each such choice, there are $\binom{D-d i-\sum_{j=1}^{d-1}(j-1) \nu_{j}}{\nu_{2}, \ldots, \nu_{d-1}, D-\sum_{j=2}^{d-1} j v_{j}}$ ways to distribute the exponents.

Example 2.7.12. For $k=2, d=3, n=4$, the number of minimal generators is $\binom{3}{0,3,0,0}+\binom{3}{0,1,2,0}+\binom{3}{1,1,0,1}+\binom{3}{2,0,1,0}+\binom{3}{0,0,1,2}=16$. This means that the original generators $\phi_{\mathbf{g}}$ are linearly independents.

Theorem 2.7.13. If $d=2$, the generators $\left\{\phi_{\mathbf{g}}\right\}_{\mathbf{g} \in 0 \times \mathbb{Z}_{2}^{n}}$ are linearly independent if and only if $n+1 \leq k$.

Proof. $\left\{\psi_{\mathbf{g}}\right\}$ is linearly independent, and they are $\sum_{i \geq 0}\binom{n+1}{d-2 i}$ many. This sum equals $2^{n}$ if and only if $n+1 \leq k$.

Hilbert function of the power ideals $I_{n, k, d}$
In order to simplify the notation, when there will be no ambiguity, we will denote $I:=I_{n, k, d}$ and $R:=R_{n, k, d}=S / I$ with the multicycling grading described in the previous section $R=\bigoplus_{i \in \mathbb{N}} \bigoplus_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} R_{i, \mathbf{g}}$.

Definition 2.7.14. For $0 \leq i \leq k$ and given a vector $\mathbf{h} \in \mathbb{Z}_{d}^{n+1}$, we define the map

$$
\begin{aligned}
\mu_{i, \mathbf{h}}: \quad D_{i, \mathbf{h}}:=\bigoplus_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} S_{i, \mathbf{h}-\mathbf{g}} & \longrightarrow S_{i+D, \mathbf{h}} \\
\left(\ldots, f_{\mathbf{g}}, \ldots\right) & \longmapsto \sum_{\mathbf{g} \in \mathbb{Z}_{d}^{n+1}} f_{\mathbf{g}} \psi_{\mathbf{g}}
\end{aligned}
$$

given by the multiplication by each $\psi_{\mathbf{g}} \in S_{D, \mathbf{g}}$.
Remark 2.7.15. In order to have $S_{i+D, \mathbf{h}} \neq 0$, we always assume $i+D-$ $\mathrm{wt}(\mathbf{h}) \in d \mathbb{Z}$. Observe that, under such assumption, we have the following equivalence

$$
i-\mathrm{wt}(\mathbf{h}-\mathbf{g}) \in d \mathbb{Z} \Longleftrightarrow D-\mathrm{wt}(\mathbf{g}) \in d \mathbb{Z}
$$

In other words, using the properties of the multicyclic grading explained in the previous section, we have

$$
S_{i, \mathbf{h}-\mathbf{g}} \neq 0 \Longleftrightarrow \psi_{\mathbf{g}} \neq 0
$$

Thus, it makes sense to study the injectivity of the $\mu_{i, \mathrm{~h}}$ 's which will be the crucial step for our computations.
Lemma 2.7.16. Given $0 \leq i \leq d$ and $\mathbf{h} \in \mathbb{Z}_{k}^{n+1}$, if $i+D-\mathrm{wt}(\mathbf{h}) \in k \mathbb{N}$ and $\mathrm{wt}(\mathbf{h}) \leq(k-1)(d-i)$,

$$
\operatorname{dim}\left(D_{i, \mathbf{h}}\right) \leq \operatorname{dim}\left(S_{i+D, \mathbf{h}}\right)
$$

with equality if $\mathrm{wt}(\mathbf{h})=(k-1)(d-i)$.
Proof. Under such numerical assumptions, we have that $D_{i, \mathrm{~h}}$ coincides with $S_{i}$; thus,

$$
\operatorname{dim}_{\mathbb{C}} D_{i, \mathbf{h}}=\binom{n+i}{n}
$$

Moreover, we may observe that, for some integer $m \geq 0$,

$$
d m=i+D-\mathrm{wt}(\mathbf{h}) \geq i+D-(d-1)(k-i)=d i
$$

Hence, $i+D-\mathrm{wt}(\mathbf{h})=d(i+j)$ for some $j \geq 0$ and

$$
\operatorname{dim}\left(S_{i+D, \mathbf{h}}\right)=\binom{n+i+j}{n}
$$

For any $0 \leq i \leq k$ and $\mathbf{h} \in \mathbb{Z}_{d}^{n+1}$, the image of the map $\mu_{i, \mathbf{h}}$ is simply the part of multicycling degree $(i, \mathbf{h})$ of our ideal $I$. These maps will be the main tool in our computations regarding the Hilbert function of $I$ and its quotient ring $R$. By Remark 2.7.23 and Lemma 2.7.16, it makes sense to ask if $\mu_{i, \mathbf{h}}$ is injective whenever $\mathrm{wt}(\mathbf{h}) \leq(d-1)(k-i)$ and $i+D-\mathrm{wt}(\mathbf{h}) \in d \mathbb{Z}$. In these cases, the dimension of $I_{i+D, \mathrm{~h}}$ in degree $i$ will be simply the dimension of $D_{i, \mathrm{~h}}=S_{i}$. On the other hand, again by Lemma 2.7.16, one could hope that $\mu_{i, \mathbf{h}}$ is surjective in all the other cases implying, consequently, $R_{i+D, \mathbf{h}}=0$. This is true for $d=2$ as we are going to prove in the next section.

CASE $d=2$. In this case, $D=k(d-1)=k$. Moreover, as we said in Remark 2.7.23, we consider only the maps $\mu_{i, \mathbf{h}}$ with $i+k-\mathrm{wt}(\mathbf{h})$ is even.

Lemma 2.7.17. In the same notation as above, we have:

1. $\mu_{k, 0}$ is bijective;
2. $\mu_{i, \mathbf{h}}$ is injective if $\mathrm{wt}(\mathbf{h}) \leq k-i$;
3. $\mu_{i, \mathbf{h}}$ is surjective if $\mathrm{wt}(\mathbf{h}) \geq k-i$.

Proof. (1) The map $\mu_{k, 0}$ is surjective, due to Theorem 2.7.4. The map $\mu_{k, 0}$ is also injective because we are in the limit case of Lemma 2.7.16, i.e., where the dimensions of the source and the target are equal.
(2) Given a monomial $M$ with $M \in S_{k+i, \mathbf{h}}$, there exists a monomial $M^{\prime}$ such that $M M^{\prime} \in S_{2 k, \mathbf{0}}$. Indeed, it is enough to consider the monomial $\mathbf{x}_{\mathbf{h}}$ to get $\operatorname{mcdeg}\left(\mathbf{x}_{\mathbf{h}} M\right)=\mathbf{0}$ and then we can multiply by any monomial of the right degree to get total degree equal to $2 d$ and multicyclic degree equal to $\mathbf{0}$. Hence, the injectivity of $\mu_{i, \mathbf{h}}$ follows from (1).
(3) If $\mathrm{wt}(\mathbf{h})=(k-i)$, we are in the limit case of Lemma 2.7.16 and then, from injectivity of $\mu_{i, \mathbf{h}}$, also the surjectivity follows. Instead, the case $\mathrm{wt}(\mathbf{h})>k-i$ follows from the previous one because, given any monomial $M$ with $M \in S_{n, \mathbf{h}}$ and $n-\mathrm{wt}(\mathbf{h})=2 m$, then $M$ is a product of a monomial $M^{\prime}$ with $M^{\prime} \in S_{n-2 m, \mathbf{h}}$.

We are now able to study the Hilbert function of our class of power ideals.
Lemma 2.7.18. In the same notation as above, we have:

1. if $i<k, I_{i}=0$;
2. if $i=j+k$ with $j \geq 0, R_{i, \mathbf{h}} \neq 0$ if and only if

$$
\mathbf{h} \in \mathcal{H}_{j}:=\left\{\mathbf{h}^{\prime} \mid i-\mathrm{wt}\left(\mathbf{h}^{\prime}\right) \in 2 \mathbb{N}, \mathrm{wt}\left(\mathbf{h}^{\prime}\right)<k-j, \mathrm{wt}\left(\mathbf{h}^{\prime}\right) \leq n+1\right\}
$$

moreover, if $\mathbf{h} \in \mathcal{H}_{j}$, then

$$
\operatorname{dim}_{\mathbb{C}} R_{i, \mathbf{h}}=\operatorname{dim}_{\mathbb{C}} S_{i, \mathbf{h}}-\binom{n+j}{n}
$$

Proof. Since $I$ has generators in degree $k$, then $I_{i}=0$ for all $i<k$. Consider now $i=k+j$ for some $j \geq 0$. Since $R_{i}=\bigoplus_{\mathbf{h} \in \mathbb{Z}_{d}^{n+1}} R_{i, \mathbf{h}}$, we will focus on the dimension of each summand $R_{i, \mathbf{h}}$. Given $\mathbf{h} \in \mathbb{Z}_{2}^{n+1}$, we have seen that $I=\left(\psi_{\mathbf{g}} \mid \mathbf{g} \in \mathcal{G}_{2, n, D}\right)$; hence, $I_{i, \mathbf{h}}=\operatorname{Im}\left(\mu_{j, \mathbf{h}}\right)$. By Lemma 2.7.17, for $\mathrm{wt}(\mathbf{h}) \geq$ $k-j$, we know that $\mu_{j, \mathbf{h}}$ is surjective and then $I_{i, \mathbf{h}}=S_{i, \mathbf{h}}$. Consequently, $R_{i, \mathbf{h}}=0$. Moreover, by Lemma 2.7.7, we need to consider only $\mathbf{h} \in \mathbb{Z}_{2}^{n+1}$ such that $i-\mathrm{wt}(\mathbf{h}) \in 2 \mathbb{N}$. Otherwise $S_{i, \mathbf{h}}=0$ and consequently, $R_{i, \mathbf{h}}=0$. Thus, we just need to consider $h$ from the set $\mathcal{H}_{j}$ defined in the statement. By Lemma 2.7.17, under these numerical assumptions, $\mu_{j, \mathbf{h}}$ is injective and

$$
\operatorname{dim}_{\mathbb{C}} I_{i, \mathbf{h}}=\sum_{\mathbf{g} \in \mathbb{Z}_{2}^{n+1}} \operatorname{dim}_{\mathbb{C}} S_{j, \mathbf{h}-\mathbf{g}}=\operatorname{dim}_{\mathbb{C}} S_{j}=\binom{n+j}{n}
$$

or, equivalently,

$$
\operatorname{dim}_{\mathbb{C}} R_{i, \mathbf{h}}=\operatorname{dim}_{\mathbb{C}} S_{i, \mathbf{h}}-\binom{n+j}{n}
$$

Theorem 2.7.19. The Hilbert function of the quotient ring $R$ is given by:

1. if $i<k, \operatorname{HF}_{R}(i)=\binom{n+i}{n}$;
2. if $i=j+k$ with $j \geq 0$,

$$
\operatorname{HF}_{R}(i)=\sum_{\mathbf{h} \in \mathcal{H}_{j}} \operatorname{dim}_{\mathbb{C}} R_{i, \mathbf{h}}=\sum_{\substack{h<k-j \\ i-h \in 2 \mathbb{N}}}\binom{n+1}{h}\left(\binom{n+\frac{i-h}{2}}{n}-\binom{n+j}{n}\right) .
$$

Proof. The case $i<k$ is trivial. Consider $i=j+k$ with $j \geq 0$. First, we may observe that, by Lemma 2.7.18, whenever $\mathbf{h} \in \mathcal{H}_{j}$, the dimension of $R_{i, \mathbf{h}}$ depends only on the weight of $h$. Indeed, considering $h \in \mathcal{H}_{j}$ and denoting $h:=\mathrm{wt}(\mathbf{h})$, we get, by Lemma 2.7.7,

$$
\operatorname{dim}_{\mathbb{C}} R_{i, \mathbf{h}}=\operatorname{dim}_{\mathbb{C}} S_{i-h, \mathbf{0}}-\binom{n+j}{n}=\binom{n+\frac{i-h}{2}}{n}-\binom{n+j}{n}
$$

To conclude our proof, we just need to observe that, fixing a weight $h$, we have exactly $\binom{n+1}{h}$ vectors $\mathbf{h} \in \mathbb{Z}_{2}^{n+1}$ with such a weight.

Corollary 2.7.20. $R_{2 k-1}=0$.
Proof. In order to get $R_{2 k-1, \mathbf{h}} \neq 0$, we should have that $\mathrm{wt}(\mathbf{h})$ is odd and $\mathrm{wt}(\mathbf{h})<1$, which is impossible.

In the following example, we explicit our algorithm in a particular case in order to help the reader in the comprehension of the theorem.

Example 2.7.21. Fix $n+1=4$, i.e. $S=\mathbb{C}\left[x_{0}, \ldots, x_{3}\right]$ and $k=5$. We compute the Hilbert function of the quotient $R=S / I_{3,5,2}$ where

$$
I_{3,5,2}=\left(\left(x_{0} \pm x_{1} \pm x_{2} \pm x_{3}\right)^{5}\right)
$$

For $i<5$, we have

$$
\operatorname{HF}_{R}(i)=\binom{3+i}{3}
$$

For $i=5(j=0)$, we have that $\mathcal{H}_{0}=\{\mathbf{h} \mid \operatorname{wt}(\mathbf{h})=1,3\}$. Hence,

$$
\begin{aligned}
\mathrm{HF}_{R}(5) & =\sum_{\mathrm{wt}(\mathbf{h})=1} \operatorname{dim}_{\mathbb{C}} R_{5, \mathbf{h}}+\sum_{\mathrm{wt}(\mathbf{h})=3} \operatorname{dim}_{\mathbb{C}} R_{5, \mathbf{h}}= \\
& =\binom{4}{1}\left(\operatorname{dim}_{\mathbb{C}}\left(S_{4, \mathbf{0}}\right)-1\right)+\binom{4}{3}\left(\operatorname{dim}_{\mathbb{C}}\left(S_{2, \mathbf{0}}\right)-1\right)= \\
& =4(10-1)+4(4-1)=36+12=48
\end{aligned}
$$

For $i=6(j=1)$, we have that $\mathcal{H}_{1}=\{\mathbf{h} \mid \operatorname{wt}(\mathbf{h})=0,2\}$. Hence,

$$
\begin{aligned}
\mathrm{HF}_{R}(6) & =\operatorname{dim}_{\mathbb{C}} R_{6, \mathbf{0}}+\sum_{\mathrm{wt}(\mathbf{h})=2} \operatorname{dim}_{\mathbb{C}} R_{6, \mathbf{h}}= \\
& =\left(\operatorname{dim}_{\mathbb{C}}\left(S_{6, \mathbf{0}}\right)-4\right)+\binom{4}{2}\left(\operatorname{dim}_{\mathbb{C}}\left(S_{4, \mathbf{0}}\right)-4\right)= \\
& =(20-4)+6(10-4)=16+36=52 .
\end{aligned}
$$

For $i=7(j=2)$, we have that $\mathcal{H}_{2}=\{\mathbf{h} \mid \operatorname{wt}(\mathbf{h})=1\}$. Hence,

$$
\operatorname{HF}_{R}(7)=\sum_{\mathrm{wt}(\mathbf{h})=1} \operatorname{dim}_{\mathbb{C}} R_{7, \mathbf{h}}=\binom{4}{1}\left(\operatorname{dim}_{\mathbb{C}}\left(S_{6, \mathbf{0}}\right)-10\right)=4(20-10)=40
$$

For $i=8(j=3)$, we have that $\mathcal{H}_{3}=\{\mathbf{0}\}$. Hence,

$$
\operatorname{HF}_{R}(8)=\operatorname{dim}_{\mathbb{C}} R_{8, \mathbf{0}}=\operatorname{dim}_{\mathbb{C}}\left(S_{8, \mathbf{0}}\right)-20=35-20=15
$$

For $i \geq 9(j \geq 4)$, we can easily see that $\mathcal{H}_{j}=\emptyset$. Thus,

$$
\begin{array}{ccccccccccc}
\mathrm{HF}_{R}: & 1 & 4 & 10 & 20 & 35 & 48 & 52 & 40 & 15 & -
\end{array}
$$

Below we give a more explicit formula for the Hilbert series in cases with small number of variables.

Theorem 2.7.22. The Hilbert series of $R_{1, k, 2}$ is given by

$$
\operatorname{HS}_{R_{1, k, 2}}(t)=\frac{1-2 t^{k}+t^{2 k}}{(1-t)^{2}}
$$

The Hilbert series of $R_{2, k, 2}$, for $k \geq 2$ is given by

$$
\begin{aligned}
\mathrm{HS}_{R_{2, k, 2}}(t) & =\frac{\left(1-4 t^{k}+d t^{2 k-1}+3 t^{2 k}-k t^{2 k+1}\right)}{(1-t)^{3}}= \\
& =\sum_{i=0}^{k-1}\binom{i+2}{2} t^{i}+\sum_{i=0}^{k-2}\left(\binom{k+i+2}{2}-4\binom{i+2}{2}\right) t^{k+i}
\end{aligned}
$$

The Hilbert series of $R_{3, k, 2}$, for $k \geq 3$ is given by

$$
\begin{aligned}
& \operatorname{HS}_{R_{3, k, 2}}(t)= \\
& =\frac{\left(1-8 t^{k}+\binom{k}{2} t^{2 k-2}+4 k t^{2 k-1}-\left(k^{2}-7\right) t^{2 k}-4 k t^{2 k+1}+\binom{k+1}{2} t^{2 k+2}\right)}{(1-t)^{4}}= \\
& =\sum_{i=0}^{k-1}\binom{i+3}{3} t^{i}+\sum_{i=0}^{k-3}\left(\binom{k+i+3}{3}-8\binom{i+3}{3}\right) t^{k+i}+\binom{k+1}{2} t^{2 k-2} .
\end{aligned}
$$

Proof. Case $n+1=2$. The result follows from the fact that in this case we have a complete intersection, see Example A.1.4.

Case $n+1=3$. From Lemma 2.7.17, for any $0 \leq j \leq k-3$, we have that $\left[I_{2, k, 2}\right]_{k+j}=S_{j}\left[I_{2, k, 2}\right]_{k}$, since $\mathrm{wt}(\mathbf{h}) \leq k-3$, for all possible h. Since $2 k-2$ is even, we get that $\mathrm{wt}(\mathbf{h})$ should be even and therefore, $\mathrm{wt}(\mathbf{h}) \leq 2=$ $k-(k-2)$. Thus, we get injectivity in this degree also. Now, from Theorem 2.7.19, we get that $\operatorname{dim}_{\mathbb{C}}\left(\left[R_{2, k, 2}\right]_{k+j}\right)=\operatorname{dim}_{\mathbb{C}}\left(S_{k+j}\right)-\#\left(\mathcal{H}_{j}\right) \cdot\binom{n+j}{n}$. Under our numerical assumption, it is clear that, for $0 \leq i \leq k-3, \mathcal{H}_{i}$ is exactly the half of all possible vectors in $\mathbb{Z}_{2}^{n+1}$, i.e., $\#\left(\mathcal{H}_{i}\right)^{-}=2^{n}$. Hence,

$$
\operatorname{HS}_{R_{2, k, 2}}(t)=\sum_{i=0}^{k-1}\binom{i+2}{2} t^{i}+\sum_{i=0}^{k-2}\left(\binom{k+i+2}{2}-4\binom{i+2}{2}\right) t^{k+i}
$$

A simple calculation shows that

$$
(1-t)^{3} \operatorname{HS}_{R_{2, k, 2}}(t)=\left(1-4 t^{k}+k t^{2 k-1}+3 t^{2 k}-k t^{2 k+1}\right)
$$

Case $n+1=4$. Since $\operatorname{wt}(\mathbf{h}) \leq 4$, for all possible $\mathbf{h}$, we get from Lemma 2.7.17 that $\left[I_{3, k, 2}\right]_{k+i}=S_{i}\left[I_{3, k, 2}\right]_{k}$, for all $0 \leq i \leq k-4$. Moreover, since $2 k-3$ is odd, we get that $\mathrm{wt}(\mathbf{h})$ should be odd and consequently $\mathrm{wt}(\mathbf{h}) \leq$ $3=k-(k-3)$. Hence, we have injectivity also in this degree. Moreover, for all $0 \leq i \leq k-3$, we get that $\mathcal{H}_{i}$ is half of all possible vectors in $\mathbb{Z}_{2}^{n+1}$, i.e. $\mathcal{H}_{i}$ has cardinality equal to $2^{n}$. Now, we just need to compute the dimension of $\left[R_{3, k, 2}\right]_{2 k-2}$. By definition, the vectors $\mathbf{h} \in \mathcal{H}_{k-2}$ have to be odd, since $2 k-2$ is odd, and they have to satisfy the condition $\mathrm{wt}(\mathbf{h})<2$. Thus, we get only the case $\mathbf{h}=\mathbf{0}$ and $\#\left(\mathcal{H}_{k-2}\right)=1$. Thus, by Theorem 2.7.19,

$$
\begin{aligned}
\operatorname{dim}_{\mathbb{C}}\left(\left[R_{3, k, 2}\right]_{2 k-2}\right) & =\operatorname{dim}_{\mathbb{C}}\left(\left[R_{3, k, 2}\right]_{2 k-2, \mathbf{0}}\right)=\operatorname{dim}_{\mathbb{C}}\left(S_{2 k-2, \mathbf{0}}\right)-\binom{3+k-2}{3}= \\
& =\binom{k+2}{3}-\binom{k+1}{3}=\binom{k+1}{2}
\end{aligned}
$$

Putting together our last observations, we get

$$
\begin{aligned}
& \operatorname{HS}_{R_{3, k, 2}}(t)= \\
& =\sum_{i=0}^{k-1}\binom{i+3}{3} t^{i}+\sum_{i=0}^{k-3}\left(\binom{k+i+3}{3}-8\binom{i+3}{3}\right) t^{k+i}+\binom{k+1}{2} t^{2 k-2} .
\end{aligned}
$$

A simple calculation shows that

$$
\begin{aligned}
& (1-t)^{4} \mathrm{HS}_{R_{3, k, 2}}(t)= \\
& =1-8 t^{k}+\binom{k}{2} t^{2 k-2}+4 k t^{2 k-1}-\left(k^{2}-7\right) t^{2 k}-4 k t^{2 k+1}+\binom{k+1}{2} t^{2 k+2}
\end{aligned}
$$

Remark 2.7.23. From the proof of Theorem 2.7 .22 , we can say something more about the Hilbert series of $R_{n, k, 2}$, even for more variables. Assuming $k \geq n$, by using the same ideas as in Theorem 2.7.22, we get that, for all $0 \leq j \leq k-n$, the $(k+j)^{t h}$-coefficient of our Hilbert series is equal to

$$
\operatorname{HF}_{R_{n, k, 2}}(k+j)=\binom{n+k+j}{n}-2^{n}\binom{n+j}{n}
$$

Moreover, we get that, for any $k \geq 2, \mathcal{H}_{k-2}=\{0\}$ and consequently,

$$
\begin{aligned}
& \operatorname{HF}_{R_{n, k, 2}}(2 k-2)=\operatorname{dim}_{\mathbb{C}}\left(\left[R_{n, k, 2}\right]_{2 k-2}\right)=\operatorname{dim}_{\mathbb{C}}\left(\left[R_{n, k, 2}\right]_{2 k-2, \mathbf{0}}\right)= \\
& =\operatorname{dim}_{\mathbb{C}}\left(S_{2 k-2, \mathbf{0}}\right)-\binom{n+k-2}{n}=\binom{n+k-1}{n}-\binom{n+k-2}{n}= \\
& =\binom{n+k-2}{n-1}
\end{aligned}
$$

Similarly, we have that, for any $k \geq 3, \mathcal{H}_{k-3}=\left\{h \in \mathbb{Z}_{2}^{n+1} \mid \operatorname{wt}(h)=1\right\}$. Thus,

$$
\begin{aligned}
\operatorname{HF}_{R_{n, k, 2}}(2 k-3) & =\operatorname{dim}_{\mathbb{C}}\left(\left[R_{n, k, 2}\right]_{2 k-3}\right)=\sum_{\mathrm{wt}(h)=1} \operatorname{dim}_{\mathbb{C}}\left(\left[R_{n, k, 2}\right]_{2 k-3, h}\right)= \\
& =(n+1)\left[\operatorname{dim}_{\mathbb{C}}\left(S_{2 k-2, \mathbf{0}}\right)-\binom{n+k-2}{n}\right]= \\
& =(n+1)\binom{n+k-2}{n-1}
\end{aligned}
$$

Conjecture 2.4. $R_{n, k, 2}$ is a level algebra, i.e. $\operatorname{Soc}\left(R_{n, k, 2}\right)=\left[R_{n, k, 2}\right]_{2 k-2}$. If so, from Remark 2.7.23. we get that $\operatorname{Soc}\left(R_{n, k, 2}\right)$ has dimension $\binom{n+k-2}{n-1}$.

CASE $d>2$. We want to generalize our results for $d>2$. Inspired by Lemma 2.7.16, we conjecture the following behavior of the maps $\mu_{i, \mathbf{h}}$.

Conjecture 2.5. In the same notation as in Definition 2.7.14, we have

1. $\mu_{i, \mathbf{h}}$ is injective if $\mathrm{wt}(\mathbf{h}) \leq(d-1)(k-i)$;
2. $\mu_{i, \mathbf{h}}$ is surjective if $\mathrm{wt}(\mathbf{h}) \geq(d-1)(k-i)$.

By following ideas similar to the one used in Lemma 2.7.18, we have that Conjecture 2.5 would imply the following.

Conjecture 2.6. In the above notation, if $i=j+D$ with $j \geq 0$, then $R_{i, \mathbf{h}} \neq 0$ if and only if

$$
\mathbf{h} \in \mathcal{H}_{j}:=\left\{\mathbf{h}^{\prime} \mid i-\mathrm{wt}\left(\mathbf{h}^{\prime}\right) \in d \mathbb{N}, \mathrm{wt}\left(\mathbf{h}^{\prime}\right)<k-j, \mathrm{wt}\left(\mathbf{h}^{\prime}\right) \leq(d-1)(n+1)\right\}
$$

moreover, if $\mathbf{h} \in \mathcal{H}_{j}$, then

$$
\operatorname{dim}_{\mathbb{C}} R_{i, \mathbf{h}}=\operatorname{dim}_{\mathbb{C}}\left(S_{i, \mathbf{h}}\right)-\binom{n+j}{n}
$$

## Proposition 2.7.24. Conjecture 2.5 implies Conjecture 2.6

Proof. Follow the proof of Theorem 2.7.19.
Remark 2.7.25. The above conjectures imply a direct generalization of the algorithm to compute the Hilbert function of the quotient rings $R$, described in Example 2.7.21. Trivially, we already know that, for $i<D$, since the ideal $I$ has generators only in degree $D$,

$$
\operatorname{HF}_{R}(i)=\binom{n+i}{n}
$$

For the cases $i=D+j$ with $j \geq 0$, from Conjecture 2.6, we would have

$$
\operatorname{HF}_{R}(i)=\sum_{\substack{h<(d-1)(k-j) \\ i-h \in d \mathbb{N}}} N_{h}\left(\binom{n+\frac{i-h}{d}}{n}-\binom{n+j}{n}\right)
$$

where $N_{h}$ is simply the number of vectors $\mathbf{h} \in \mathbb{Z}_{d}^{n+1}$ of weight wt $(\mathbf{h})=h$. In order to compute the numbers $N_{h}$, we may look at the following formula,:

$$
\sum_{h=0}^{(d-1)(n+1)} N_{h} x^{h}=\left(1+x+\ldots+x^{d-1}\right)^{n+1}=\left(\frac{1-x^{d}}{1-x}\right)^{n+1}
$$

By expanding the right hand side, we get, for all $h=0, \ldots,(d-1)(n+1)$,

$$
N_{h}=\sum_{s=0}^{\left\lfloor\frac{h}{d}\right\rfloor}(-1)^{s}\binom{n+1}{s}\binom{n+h-d s}{n}
$$

Remark 2.7.26. From the conjectures, we can extend Corollary 2.7.20 to the case $d>2$, i.e.,

$$
\left[R_{n, k, d}\right]_{k d-1}=0
$$

Indeed, with the same notation as above, let's take $j=k-1$. Thus, to compute the Hilbert function of the quotient in position $k d-1$ we should compute the set $\mathcal{H}_{k-1}$, i.e. the set of $\mathbf{h} \in \mathbb{Z}_{d}^{n+1}$ satisfying the following conditions:

$$
k d-1-\mathrm{wt}(\mathbf{h}) \in d \mathbb{Z}, \quad \mathrm{wt}(\mathbf{h})<(d-1)(k-k+1)=d-1
$$

From the first condition, we get that $\mathrm{wt}(\mathbf{h}) \in(d-1)+d \mathbb{Z}_{\geq 0}$ which is clearly in contradiction with the second condition above. Thus, $\mathcal{H}_{k-1}$ is empty and $\mathrm{HF}_{R}(k d-1)=0$.

Example 2.7.27. Let's give one explicit example of the computations in order to clarify the algorithm. Consider the parameters: $n=2, k=8, d=$ 4. Thus we have $D=24$. Let's compute, for example, the Hilbert function of the corresponding quotient ring in degree $i=28$, i.e., $j=4$. Using a

Computer Algebra software, as CoCoA5 by [CoC] or Macaulay2 by [GS] and the implemented functions involving Gröbner basis, one can see that

$$
\operatorname{HF}_{R}(28)=195 .
$$

Let's apply our algorithm to compute the same number. First, we need to write down the vector $N$ where, for $l=0 \ldots(d-1)(n+1), N_{l}:=\#\{\mathbf{h} \in$ $\left.\mathbb{Z}_{d}^{n+1} \mid \operatorname{wt}(\mathbf{h})=l\right\}$. Under our numerical assumptions, we have

$$
N=\left(N_{0}, \ldots, N_{9}\right)=(1,3,6,10,12,12,10,6,3,1)
$$

Now, we need to compute the vector $H$, where we store all the possible weights for the vectors $\mathbf{h} \in \mathcal{H}_{4}$, i.e. all the number $0 \leq h \leq 9$ such that the numerical conditions

$$
28-h \in 4 \mathbb{Z}, h<(d-1)(k-j)=12
$$

holds. Thus, $H=\left(H_{0}, H_{1}, H_{2}\right)=(0,4,8)$. Finally, we can compute $H F_{R}(28, \mathbf{h})$ for each $h \in \mathcal{H}_{4}$. From our formula, it is clear that such numbers depend only on the weight of $h$; thus, we just need to consider each single element in the vector $H$. Assuming wt $(\mathbf{h})=0$, we get,

$$
R_{0}:=\operatorname{HF}_{R}(28, \mathbf{0})=\operatorname{dim}_{\mathbb{C}} S_{28, \mathbf{0}}-\binom{n+j}{n}=36-15=21
$$

Similarly, we get: for $\mathrm{wt}(\mathbf{h})=4$,

$$
R_{4}:=\operatorname{HF}_{R}(28, \mathbf{h})=\operatorname{dim}_{\mathbb{C}} S_{24,0}-\binom{n+j}{n}=28-15=13
$$

and, for $\mathrm{wt}(\mathbf{h})=8$,

$$
R_{8}:=\operatorname{HF}_{R}(28, \mathbf{h})=\operatorname{dim}_{\mathbb{C}} S_{20, \mathbf{0}}-\binom{n+j}{n}=21-15=6
$$

Now, we are able to compute the Hilbert function in degree 28.

$$
\begin{aligned}
\mathrm{HF}_{R}(28) & =N_{H_{0}} R_{H_{0}}+N_{H_{1}} R_{H_{1}}+N_{H_{2}} R_{H_{2}}= \\
& =21+12 \cdot 13+3 \cdot 6=21+156+18=195 .
\end{aligned}
$$

Algorithm 2.7.28. In this section we want to show our algorithm implemented in CoCoA5 programming language, see [CoC]. As we have seen in the previous section, in the case $d>2$, it is just conjectured that the algorithm give the correct solution. However, as we will see in Section 2.7.38, we made several computer experiments supporting our conjectures and, then, our algorithm. Here is the CoCoA5 script of our algorithm based on Theorem 2.7.19 and Remark 2.7.25.

```
-- 1) Input parameters N, K, D;
    N := ;
    K := ;
    D := ;
    DD :=K* (D-1);
-- HF will be the vector representing the Hilbert
-- function of the quotient ring;
    HF := [];
-- 2) Input vector NN where NN[I] counts the number
-- of vectors in ZZ^{n+1} modulo K of weight I;
    Foreach H In 0..((N+1)*(D-1)) Do
        M := 0;
        Foreach S In O..(Div(H,D)) Do
            M := M+ (-1)^S*Bin (N+1,S) *Bin (N+H-D*S,N);
        EndForeach;
        Append(Ref NN,M);
    EndForeach;
-- 3) Compute the Hilbert Function:
-- in degree <DD:
    Foreach L In 0..(DD-1) Do
        Append(Ref HF,Bin(N+L,N));
    EndForeach;
-- in degree =DD,..,K*D-1:
    Foreach J In O..(K-2) Do
    I:=DD+J;
    H:=[];
    M:=0;
    Foreach S In O..I Do
        If Mod(I-S,D)=0 Then
                If S<(D-1)*(K-J) Then
                    If S< (D-1)* (N+1)+1 Then
                    Append(Ref H,S);
                    M:=M+1;
                    EndIf;
                EndIf;
        EndIf;
    EndForeach;
    HH:=0;
    If M>0 Then
        Foreach S In 1..M Do
```

```
                HH:=HH+NN[H[S]+1]*
                        (Bin(N+Div(I-H[S],D),N)-Bin(N+J,N));
        EndForeach;
    EndIf;
    Append(Ref HF,HH);
EndForeach;
-- 4) Print the Hilbert function:
HF;
```

Remark 2.7.29. In the case $d=2$, our algorithm (which gives the correct answer by Theorem 2.7.19) works very fast even for large values of $n$ and $k$, e.g. for $n, k \sim 300$; when the Computer Algebra softwares, involving the computation of Gröbner basis, cannot do in a reasonable amount of time and memory. Concerning the case $d>2$, with the support of Computer Algebra software Macaulay 2 and its implemented function to compute Hilbert series of quotient rings, we have checked, for low $k$ and $d$, that our numerical algorithm produces the right Hilbert function for two and three variables. Moreover, in Section 2.7.2, we study the schemes of fat points related to our power ideals and our results about their Hilbert series, will support Conjecture 2.6 in many more cases. More exactly, using combinatorics, we check the case $n+1=2$ and, using the Computer Algebra software CoCoA 5 , we have checked that the conjectured algorithm gives the correct Hilbert function for all $n, d \leq 20$ and $k \leq 150$, see Remark 2.7.40.

## Hilbert functions of $\xi$-POints in $\mathbb{P}^{n}$

As we mentioned in Section 2.5, there is a relation between power ideals and schemes of fat points; in particular, between their Hilbert functions, see Corollary 2.5.7. Here, we want to compute the Hilbert function of the schemes of fat points associated to the power ideals $I_{n, k, d}$. More exactly, we consider schemes of fat points with support over the $d^{n}$ points

$$
\mathbb{X}_{n, k, d}=\left\{\left[1: \xi^{g_{1}}: \ldots: \xi^{g_{n}}\right] \in \mathbb{P}^{n} \mid 0 \leq g_{j} \leq k-1\right\} .
$$

We call them $\xi$-points for short. In the case $d=2$, from our computations of the Hilbert functions of power ideals $I_{n, k, 2}$, we obtain the Hilbert function for fat points with support on the ( $\pm 1$ )-points. Consequently, after investigating the results for the case $d=2$, we have been able to compute the Hilbert functions for any $d>2$ as well.

CAE $d=2$. We begin by considering our class of power ideals for $d=2$, namely the ideal $I_{k}$ generated by the $k$-th powers

$$
\left(x_{0} \pm x_{1} \pm \ldots \pm x_{n}\right)^{k}
$$

We have described an easy numerical algorithm to compute the Hilbert function of the quotient rings $R_{k}=S / I_{k}$. Thus, by the Apolarity Lemma
(see Corollary 2.5.7), we can compute the Hilbert function of schemes of fat points supported at the $( \pm 1)$-points of $\mathbb{P}^{n}$.
Proposition 2.7.30. Let $I^{(m)}$ be the ideal associated to the scheme of fat points of multiplicity $m$ supported on the $( \pm 1)$-points of $\mathbb{P}^{n}$. Then,

$$
\operatorname{HF}_{S / I^{(m)}}(i)=\left\{\begin{array}{ll}
\binom{n+i}{n} & \text { for } i \leq 2 m-1 \\
n+2 m \\
n
\end{array}\right)-\binom{m+n-1}{n-1} \quad l \begin{array}{ll}
\text { for } i=2 m \\
\left(\begin{array}{c}
n+2 m+1
\end{array}\right)-(n+1)\binom{m+n-1}{n-1} & \text { for } i=2 m+1 \\
\left.2^{n} \begin{array}{c}
n+m-1 \\
n
\end{array}\right) & \text { for } i \geq 2 m+n-2
\end{array}
$$

Proof. By Corollary 2.7.20, we know that, for all $i$ satisfying the inequality $i \geq 2(i-m+1)-1$, i.e., $i \leq 2 m-1$,

$$
\operatorname{HF}_{R_{i-m+1}}(i)=0
$$

Moreover, by Remark 2.7.23, we have that

$$
\operatorname{HF}_{R_{m+1}}(2 m)=\binom{n+m-1}{n-1}, \quad \operatorname{HF}_{R_{m+2}}(2 m+1)=(n+1)\binom{n+m-1}{n-1}
$$

and

$$
\operatorname{HF}_{R_{i-m+1}}(i)=\binom{n+i}{n}-2^{n}\binom{n+m-1}{n}
$$

for $i \leq 2(i-m+1)-n$, i.e., $i \geq 2 m+n-2$. By apolarity, we are done.
Remark 2.7.31. Proposition 2.7 .30 tells us that the ideal $I^{(m)}$ is generated in degrees $\geq 2 m$ and, in particular, has $\binom{m+n-1}{n-1}$ generators in degree $2 m$. Thanks to the geometrical meaning of the symbolic power $I^{(m)}$, we can easily find such generators.

We may observe that we have exactly $n$ pairs of hyperplanes which split our $2^{n}$ points. Namely, for any variable except $x_{n}$, we can consider the hyperplanes

$$
H_{i}^{+}=\left\{x_{i}+x_{n}=0\right\} \text { and } H_{i}^{-}=\left\{x_{i}-x_{n}=0\right\}, \text { for all } i=0, \ldots, n-1
$$

It is clear that, for all $i$, half of our $( \pm 1)$-points lie on $H_{i}^{+}$and half on $H_{i}^{-}$. Consequently, we have $n$ quadrics passing through our points exactly once, i.e., $\mathcal{Q}_{i}=H_{i}^{+} H_{i}^{-}=x_{i}^{2}-x_{n}^{2}$, for all $i=0, \ldots, n-1$.

Now, we want to find the generators of $I^{(m)}$, hence we want to find hypersurfaces passing through our points with multiplicity $m$. We can consider, for example, all the monomials of degree $m$ evaluated at these quadrics, i.e., the degree $2 m$ forms

$$
\mathcal{G}_{1}:=\mathcal{Q}_{0}^{m}, \mathcal{G}_{2}:=\mathcal{Q}_{0}^{m-1} \mathcal{Q}_{1}, \mathcal{G}_{3}:=\mathcal{Q}_{0}^{m-1} \mathcal{Q}_{2}, \ldots, \mathcal{G}_{N}:=\mathcal{Q}_{n-1}^{m}
$$

where $N=\binom{n+m-1}{n-1}$. We can actually prove that they generate the part of degree $2 m$ of $I^{(m)}$ as a $\mathbb{C}$-vector space. Since the number of $\mathcal{G}_{i}$ 's is equal to
the dimension of $\left[I^{(m)}\right]_{2 m}$ computed in Proposition 2.7.30, it is enough to prove the following.

Claim. These $\mathcal{G}_{i}$ 's are linearly independent over $\mathbb{C}$.
Proof of the Claim. We use double induction over the number of variables $n$ and the degree $m$. For two variables, i.e. $n=1$, we have that the dimension of $\left[I^{(m)}\right]_{2 m}$ is equal to 1 for all $m$ and then, $\mathcal{G}_{1}=\mathcal{Q}_{0}^{m}$ is the unique generator. For $n>1$, we consider first the case $m=1$. Assume to have

$$
\alpha_{0} \mathcal{Q}_{0}+\ldots+\alpha_{n-1} \mathcal{Q}_{n-1}=\alpha_{0}\left(x_{0}^{2}-x_{n}^{2}\right)+\ldots+\alpha_{n-1}\left(x_{n-1}^{2}-x_{n}^{2}\right)=0
$$

Specializing on the hyperplane $H_{0}^{-}=\left\{x_{0}=x_{n}\right\}$, we reduce the linear combination in one variable less and, by induction, we get that $\alpha_{i}=0$ for all $i=1, \ldots, n-1$; consequently, also $\alpha_{0}=0$.

Assume that we have a linear combination for $m \geq 2$, i.e.,

$$
\begin{aligned}
& \alpha_{1} \mathcal{G}_{1}+\alpha_{2} \mathcal{G}_{2}+\ldots+\alpha_{N} \mathcal{G}_{N}= \\
& \quad=\alpha_{1}\left(x_{0}^{2}-x_{n}^{2}\right)^{m}+\alpha_{2}\left(x_{0}^{2}-x_{n}^{2}\right)^{m-1}\left(x_{1}^{2}-x_{n}^{2}\right)+\ldots+\alpha_{N}\left(x_{n-1}^{2}-x_{n}^{2}\right)^{m}=0 .
\end{aligned}
$$

Again, by specializing to the hyperplane $H_{0}^{-}=\left\{x_{0}=x_{n}\right\}$, we get a linear combination in the same degree, but with one variable less and, by induction over $n$, we have that $\alpha_{i}=0$, for all $i$, where the definition $\mathcal{G}_{i}$ doesn't involve $\left(x_{0}^{2}-x_{n}^{2}\right)^{m}$. Thus, the remaining linear combination is of type

$$
\left(x_{0}^{2}-x_{n}^{2}\right)\left[\alpha_{0} \mathcal{Q}_{0}^{m-1}+\alpha_{1} \mathcal{Q}_{0}^{m-2} \mathcal{Q}_{1}+\ldots+\alpha_{m} \mathcal{Q}_{n-1}^{m-1}\right]=0
$$

We are done by induction over $m$.
Hence, we can consider the ideal $J_{m}=\left(x_{0}^{2}-x_{n}^{2}, \ldots, x_{n-1}^{2}-x_{n}^{2}\right)^{m}$. It is clearly contained in $I^{(m)}$ but, a priori, it could be smaller. In order to show that the equality holds and that $I^{(m)}$ is minimally generated by the $\mathcal{G}_{i}$ 's, we calculate the Hilbert series of the ideal $J_{m}$.

Lemma 2.7.32. the Hilbert series of $T_{m}=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right] / J_{m}$ is given by

$$
\operatorname{HS}_{T_{m}}(t)=\frac{1+\sum_{i=1}^{n}(-1)^{i} \beta_{i} t^{2 m+2(i-1)}}{(1-t)^{n+1}}
$$

where $\beta_{i}:=\beta_{i, 2 m+2(i-1)}=\binom{m+i-2}{i-1}\binom{m+n-1}{n-i}$, for all $i=1, \ldots, n$, and the multiplicity is $e\left(T_{m}\right)=2^{n}\binom{m+n-1}{n}$.

Proof. The quotient $T_{m}$ is a 1-dimensional Cohen-Macaulay ring and $x_{n}$ is a non-zero divisor. Thus, we have that $T_{m}$ and the quotient $T_{m} /\left(x_{n}\right)$ have the same Betti numbers. Moreover,

$$
T_{m} /\left(x_{n}\right)=\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] /\left(x_{0}^{2}, \ldots, x_{n-1}^{2}\right)^{m}
$$

and the resolution of those quotients is well-known. Namely, the quotient ring $\mathbb{C}\left[x_{0}, \ldots, x_{n}\right] /\left(x_{0}, \ldots, x_{n-1}\right)^{m}$ has a pure resolution of type $(m, m+$
$1, \ldots, m+n-1)$ and its Betti numbers and multiplicity are expressed by an explicit formula, see [BH98, Theorem 4.1.5]. Thus, $T_{m} /\left(x_{n}\right)$ has a pure resolution of type $(2 m, 2 m+2,2 m+4, \ldots, 2 m+2(n-1))$, i.e.

$$
\cdots \longrightarrow S(-2 m-4)^{\beta_{3,2 m+4}} \longrightarrow S(-2 m-2)^{\beta_{2,2 m+2}} \longrightarrow S(-2 m)^{\beta_{1,2 m}} \longrightarrow 0
$$

where $S$ is the graded polynomial ring $\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right]$ and $S(-i)$ is its $i^{t_{-}}$ shifting, i.e. $[S(-i)]_{j}:=S_{j-i}$. Moreover, the Betti numbers and the multiplicity of the quotient are given by the following formulas:

$$
\begin{aligned}
\beta_{i}:=\beta_{i, 2 m+2(i-1)} & =(-1)^{i+1} \prod_{j \neq i} \frac{m+j-1}{j-i}= \\
& =(-1)^{i+1} \frac{m(m+1) \cdots(m+i-2)}{(-1)^{i-1}(i-1)!} \cdot \frac{(m+i) \cdots(m+n-1)}{(n-i)!}= \\
& =\binom{m+i-2}{i-1}\binom{m+n-1}{n-i} ; \\
e\left(T_{m}\right) & =\frac{1}{n!} \prod_{i=1}^{n}(2 m+2(i-1))=2^{n}\binom{m+n-1}{n} .
\end{aligned}
$$

Therefore, we easily get the following Hilbert series of $T_{m}=S / J_{m}$,

$$
\mathrm{HS}_{T_{m}}(t)=\frac{1+\sum_{i=1}^{n}(-1)^{i} \beta_{i} t^{2 m+2(i-1)}}{(1-t)^{n+1}}
$$

Corollary 2.7.33. Let $T_{m}=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right] /\left(x_{0}^{2}-x_{n}^{2}, \ldots, x_{n-1}^{2}-x_{n}^{2}\right)^{m}$. Then,

$$
\operatorname{HF}_{T_{m}}(i)= \begin{cases}\binom{n+i}{n} & \text { for } i \leq 2 m-1 \\
\binom{n+2 m}{n}-\binom{m+n-1}{n-1} & \text { for } i=2 m \\
\left(\begin{array}{c}
n+2 m+1
\end{array}\right)-(n+1)\binom{m+n-1}{n-1} & \text { for } i=2 m+1 \\
2^{n}\binom{n+m-1}{n} & \text { for } i \gg 0\end{cases}
$$

Proof. The values of the Hilbert function for $i \leq 2 m+1$ follow directly by extending the Hilbert series computed in Lemma 2.7.32, by recalling that $\frac{1}{(1-t)^{n+1}}=\sum_{j \geq 0}\binom{n+j}{n} t^{j}$. Since $T_{m}$ is a 1-dimensional CM ring, we have that its Hilbert function is eventually constant and equal to the multiplicity.

Now, we complete our study of the ideal of fat ( $\pm 1$ )-points in $\mathbb{P}^{n}$.
Theorem 2.7.34. Let $I^{(m)}$ be the ideal associated to the scheme of fat points of multiplicity $m$ and supported on the $2^{n}$ points $[1: \pm 1: \ldots: \pm 1] \in \mathbb{P}^{n}$. The generators are given by the monomials of degree $m$ evaluated at the $n$ quadrics $\mathcal{Q}_{i}=x_{i}^{2}-x_{n}^{2}$, for all $i=0, \ldots, n-1$, and the Hilbert series is

$$
\operatorname{HS}_{S / I^{(m)}}(t)=\frac{1+\sum_{i=1}^{n}(-1)^{i} \beta_{i} t^{2 m+2(i-1)}}{(1-t)^{n+1}}
$$

where the Betti numbers are given by

$$
\beta_{i}:=\beta_{i, 2 m+2(i-1)}=\binom{m+i-2}{i-1}\binom{m+n-1}{n-i}, \text { for } i=1, \ldots, n
$$

Proof. Let's write $I^{(m)}=J_{m}+J$, where $J_{m}=\left(\mathcal{Q}_{0}, \ldots, \mathcal{Q}_{n-1}\right)^{m}$. By Lemma 2.7.32, it is enough to show that $J=0$. We consider the quotient $T_{m}=$ $S /\left(I^{(m)}+\left(x_{n}\right)\right)=\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] /\left(\left(x_{0}^{2}, \ldots, x_{n}\right)^{m}+\bar{J}\right)$ and the exact sequence

$$
0 \longrightarrow \operatorname{Ann}\left(x_{n}\right) \longrightarrow S / I^{(m)} \xrightarrow{\cdot x_{n}} S / I^{(m)} \longrightarrow T_{m} \longrightarrow 0
$$

Consequently, we get

$$
\mathrm{HS}_{T_{m}}(t)=(1-t) \mathrm{HS}_{S / I^{(m)}}(t)+\mathrm{HS}_{\operatorname{Ann}\left(x_{n}\right)}(t) .
$$

Since $S / I^{(m)}$ is 1-dimensional ring, we have that $\mathrm{HS}_{S / I^{(m)}}(t)=\frac{h(t)}{(1-t)}$ and the multiplicity is $e\left(S / I^{(m)}\right)=h(1)$. Thus, the multiplicity of $T_{m}$ is

$$
\begin{equation*}
e\left(T_{m}\right)=h(1)+\operatorname{HS}_{\operatorname{Ann}\left(x_{n}\right)}(1) \geq e\left(S / I^{(m)}\right)=2^{n}\binom{m+n-1}{n} \tag{2.7}
\end{equation*}
$$

Moreover, the equality holds if and only if $x_{n}$ is a non-zero divisor of $T_{m}$. On the other hand, we have that $T_{m}=\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] /\left(x_{0}^{2}, \ldots, x_{n-1}\right)^{m}+\bar{J}$ and consequently, by Lemma 2.7.32, we have

$$
\begin{equation*}
e\left(T_{m}\right) \leq e\left(\mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right] /\left(x_{0}^{2}, \ldots, x_{n-1}\right)^{m}\right)=2^{n}\binom{m+n-1}{n} \tag{2.8}
\end{equation*}
$$

where the equality holds if and only if $\bar{J}=0$. From (2.7) and (2.8), we can conclude that

- $x_{n}$ is a non-zero divisor for $T_{m}=S / I^{(m)}$;
- $\bar{J}=0$.

Now, assume that $J \neq 0$ and take a non-zero element $f \in J$ of minimal degree. Then, since $\bar{J}=0$, we get that $f=x_{n} \cdot g$, for some $g$, thus we have $x_{n} \cdot g=0$ in $T_{m}$. This contradicts the fact that $x_{n}$ is a non-zero divisor in $T_{m}$, since $g \notin J$ because of minimality of $f$ in $J$ and $g \notin J_{m}$ because $f$ is not.

Corollary 2.7.35. Let I be the ideal defining the set of reduced ( $\pm 1$ )-points of $\mathbb{P}^{n}$ and let $I^{(m)}$ be its $m$-th symbolic power, i.e. $I^{(m)}$ is the ideal of fat points of multiplicity $m$ supported on the $( \pm 1)$-points of $\mathbb{P}^{n}$. Then,

$$
I^{(m)}=I^{m}
$$

Moreover, for any monomial ordering such that $x_{n} \succ x_{i}$ for all $i=0, \ldots, n-$ 1 , the set of generators given in Theorem 2.7 .34 is a Gröbner basis for $I^{(m)}$.

Proof. The first part follows directly from Theorem 2.7.34, since

$$
I=\left(x_{0}^{2}-x_{k}^{2}, \ldots, x_{n-1}^{2}-x_{n}^{2}\right)
$$

Let $\left\{\mathcal{G}_{1}, \ldots, \mathcal{G}_{N}\right\}$ be the generators of $I^{m}$, namely the degree $m$ forms obtained by evaluating all the possible monomial of degree $d$ at the quadrics $x_{i}-x_{n}$, for all $i=0, \ldots, n-1$. We have that their leading terms generate the initial ideal, i.e. they are a Gröbner basis. Indeed, we clearly have the inclusion

$$
\left(\operatorname{in}\left\{\mathcal{G}_{1}, \ldots, \mathcal{G}_{N}\right\}\right) \subset \operatorname{in}(I) ;
$$

but, we also have that the left hand side is exactly

$$
\left(\operatorname{in}\left\{\mathcal{G}_{1}, \ldots, \mathcal{G}_{N}\right\}\right)=\left(x_{0}^{2}, \ldots, x_{n-1}^{2}\right)^{d}
$$

which has the same Hilbert function as $I$, see Theorem 2.7.34, Consequently the same Hilbert function of in $(I)$. Hence, equality holds.

CASE $d>2$. Let $\xi$ be a $d^{t h}$-root of unity and consider the ideal $I_{d}^{(m)}$ corresponding to the scheme of fat points of multiplicity $m$ and supported on the $d^{n} \xi$-points of type $\left[1: \xi^{g_{1}}: \ldots: \xi^{g_{n}}\right] \in \mathbb{P}^{n}$ with $0 \leq g_{i} \leq d-1$, for all $i=1, \ldots, n$. Previously, we have considered the power ideals $I_{n, k, d}$ generated by all possible

$$
\left(x_{0}+\xi^{g_{1}} x_{1}+\ldots+\xi^{g_{n}} x_{n}\right)^{k(d-1)}, \quad \text { with } 0 \leq g_{i} \leq d-1, \text { for all } i=1, \ldots, n
$$

Since the considered powers were only multiples of $(d-1)$, we cannot hope to completely get the Hilbert series of our scheme of fat points of multiplicity $m$ directly from the Hilbert series of $R_{n, k, d}=S / I_{n, k, d}$. However, we can easily observe that

$$
\operatorname{HF}_{I^{(k)}}(k d-1)=\operatorname{HF}_{R_{n, k, d}}(k d-1)
$$

by Remark 2.7.26, we conjecture that the ideal $I_{d}^{(k)}$ should be generated at least in degree $k d$. Thus, inspired by the case $d=2$, we can actually claim that $I_{d}^{(k)}$ is nonzero in degree $k d$. Indeed, we have that, for any variable $x_{0}, \ldots, x_{n-1}$, we can consider the $d$ hyperplanes

$$
H_{i}^{0}=\left\{x_{i}-x_{n}=0\right\}, H_{i}^{1}=\left\{x_{i}-\xi x_{n}=0\right\}, \ldots, H_{i}^{d-1}=\left\{x_{i}-\xi^{d-1} x_{n}=0\right\} ;
$$

such hyperplanes divide the $d^{n}$ points in $d$ distinct groups of $d^{n-1}$ points; thus, their products give a set of degree $d$ forms which vanish with multiplicity 1 at each point, i.e.

$$
\mathcal{Q}_{i}=H_{i}^{0} \cdot H_{i}^{1} \cdots H_{i}^{d-1}=x_{i}^{d}-x_{n}^{d}, \text { for all } i=0, \ldots, n-1
$$

Consequently, we get

$$
J_{k, d}=\left(\mathcal{Q}_{0}, \mathcal{Q}_{1}, \ldots, \mathcal{Q}_{n-1}\right)^{d} \subset I_{d}^{(k)}
$$

Now, by using the same ideas as for the case $d=2$, we can get the analogs of Lemma 2.7 .32 and Theorem 2.7.34, for all $d \geq 2$. Therefore, we get the following general result.

Theorem 2.7.36. Let $I_{d}^{(m)}$ be the ideal associated to the scheme of fat points of multiplicity $m$ and supported on the $d^{n} \xi$-points $\left[1: \xi^{g_{1}}: \ldots: \xi^{g_{n}}\right] \in \mathbb{P}^{n}$ for $0 \leq g_{i} \leq d-1$. The generators are given by the monomials of degree $m$ evaluated at the $n$ forms of degree $d \mathcal{Q}_{i}=x_{i}^{d}-x_{n}^{d}$, for all $i=0, \ldots, n-1$ and the Hilbert series is

$$
\mathrm{HS}_{S / I_{d}^{(m)}}(t)=\frac{1+\sum_{i=1}^{n}(-1)^{i} \beta_{i} t^{d m+d(i-1)}}{(1-t)^{n+1}}
$$

where the Betti numbers are given by

$$
\beta_{i}:=\beta_{i, d m+d(i-1)}=\binom{m+i-2}{i-1}\binom{m+n-1}{n-i}, \text { for } i=1, \ldots, n
$$

Remark 2.7.37. Moreover, similarly to Corollary 2.7.35, we have that

- $I_{d}^{(m)}=I_{d}^{m}$;
- the set of generators given in Theorem 2.7.36, is a Gröbner basis.

Remark 2.7.38. Since we have explicitly computed the Hilbert series of $\xi$-points in $\mathbb{P}^{n}$, we can go back to the Hilbert series of the power ideals $I_{n, k, d}$ by using apolarity. In particular, we can check that our Conjecture 2.6 holds in several special cases. We have seen that, since $I_{n, k, d}$ is generated in degree $k(d-1)$ and generate the whole polynomial ring in degree $k d-1$, the Hilbert function of $R_{n, k, d}$ has to be computed only in degrees $i=(k-$ $1) d+j$, with $j=0, \ldots, d-2$. In these degrees, by apolarity, we get

$$
\operatorname{HF}_{R_{n, k, d}}(i)=\operatorname{HF}_{I_{d}^{(j+1)}}(i)
$$

By Theorem 2.7.36, we can explicitly compute such Hilbert function, for all $j=0, \ldots, d-2$, i.e.,

$$
\begin{align*}
& \operatorname{HF}_{R_{n, k, d}}(i)=  \tag{2.9}\\
& =\sum_{\substack{s \in \mathbb{N} \\
s \leq \frac{d-1}{d}(k-j)}}(-1)^{s+1}\binom{n+(d-1)(k-j)-d s}{n}\binom{j+s-1}{s-1}\binom{j+n}{n-s}
\end{align*}
$$

We also conjectured the following formula: for all $j=0, \ldots, k-2$,

$$
\begin{equation*}
\operatorname{HF}_{R_{n, k, d}}(i)=\sum_{\substack{h<(d-1)(k-j) \\ i-h \in d \mathbb{N}}} N_{h}\left(\binom{n+\frac{i-h}{d}}{n}-\binom{n+j}{n}\right), \tag{2.10}
\end{equation*}
$$

where $N_{h}$ is the number of vectors $\mathbf{h} \in \mathbb{Z}_{d}^{n+1}$ of weight $\mathrm{wt}(\mathbf{h})=h$, see Remark 2.7.25. In order to show that formula (2.10) is correct and therefore to prove Conjecture 2.6 , we should show that the right-hand side of such formula is equal to the right-hand side of formula (2.9).

Proposition 2.7.39. Assuming $n=1$, i.e. in the binary case, formulas (2.9) and (2.10) are equal and Conjecture 2.6 is true.

Proof. For any $k$ and $d$, the unique non-zero summand is the one for $s=1$; thus,

$$
2.9)=1+(d-1)(k-j)-d .
$$

Now, we look at formula (2.10). At first, we may observe that, for $n=1$, the number of vectors in $\mathbb{Z}_{d}^{2}$ with fixed weight $h$ can be computed very easily, indeed

$$
N_{h}= \begin{cases}h+1 & \text { for } 0 \leq h \leq d-1 \\ 2 d-(h+1) & \text { for } d \leq h \leq 2(d-1) .\end{cases}
$$

Thus, any $i=(d-1) k+j$ can be written as $c d+r$ for some positive integers $c, r$ with $0 \leq r \leq d-1$ and then, we get

$$
\begin{aligned}
\mathbf{2 . 1 0} & =N_{r}(1+c-(j+1))+N_{r+d}(1+(c-1)-(j+1))= \\
& =(r+1)(1+c-(j+1))+(d-r-1)(1+(c-1)-(j+1))= \\
& =(r+1) c+r+1-(r+1)(j+1)+d c-(r+1) c-d j-d+(r+1)(c+1) .
\end{aligned}
$$

Moreover, recalling that $i=c d+r=(d-1) k+j$, we finally get

$$
2.10)=1+(d-1) k+j-d j-d=1+(d-1)(k-j)-d .
$$

Remark 2.7.40. With similar, but longer and more intricate arguments as for Proposition 2.7.2, we have also been able to check the case $n+1=3$. Unfortunately, we have not been able to prove that the two expressions given in (2.9) and (2.10) give the same answer for all possible triples of parameters $(k, n, d)$. Implementing such formulas in the CoCoA5 language, we have been able to check all cases $n, d \leq 20, k \leq 150$. Here is the implementation of formula (2.9) in CoCoA5 language. Concerning formula (2.10, we have used the Algorithm 2.7.28.

```
-- 1) Input of the parameters K, N, D;
    K := ;
    N := ;
    D := ;
    DD := (K-1)*D;
-- HF will be the vector containing the relevant part of
-- the Hilbert function, i.e. from (K-1)D to KD-2;
    HF := [];
-- 2) Compute the Hilbert function;
    Foreach J In 0..(K-2) Do
    B := 0;
```

```
    KK := (D-1)* (K-J)/D;
    Foreach S In 1..N Do
        If S <= KK Then
            B := B+ (-1)^ (S+1)*Bin(N+(D-1)* (K-J) -D*S,N)*
                                    Bin(J+S-1,S-1)*Bin(J+N,N-S);
        EndIf;
    EndForeach;
    Append(Ref HF , B );
EndForeach;
    3) Print the Hilbert function;
HF;
```


## CHAPTER 3

## FRÖBERG'S CONJECTURE AND DIMENSIONS OF SECANT VARIETIES

In this chapter, we focus on Fröberg's conjecture about the Hilbert series of generic ideals. In particular, we relate that to the study of secant varieties of the projective varieties introduced in the previous sections.

## SECTION 3.1

## Fröberg's conjecture

In 1985, Fröberg was studying the Hilbert series of generic ideals $I \subset S=$ $\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$, i.e., of ideals generated by generic forms. More precisely, an ideal $I=\left(F_{1}, \ldots, F_{g}\right)$ can be viewed as a point in $\mathbb{A}^{N_{1}} \times \cdots \times \mathbb{A}^{N_{g}}$, where $N_{i}=\operatorname{dim} S_{d_{i}}=\binom{n+d_{i}}{n}$. Fröberg was interested in the Hilbert series that occurs in some Zariski open dense subset of this affine space. Whenever we fix the number of the generators to be less than the number of the variables, we have that generic ideals are complete intersection. Therefore, if $g \leq n+1$, then the Hilbert series of a generic ideal $I=\left(F_{1}, \ldots, F_{g}\right)$, with $\operatorname{deg} F_{i}=d_{i}$, is given by

$$
\begin{equation*}
\operatorname{HS}_{S / I}(t)=\frac{\prod_{i=1}^{g}\left(1-t^{d_{i}}\right)}{(1-t)^{n+1}} \tag{3.1}
\end{equation*}
$$

see Example A.1.4. When the number of generators is higher than the number of the variables, it is easy to check that the power series in (3.1) has negative coefficients and, then, it cannot be the Hilbert series of some homogeneous ideal. However, Fröberg conjectured the following.

Conjecture 3.1 (Fröberg's conjecture, Frö85]). Let $I=\left(F_{1}, \ldots, F_{g}\right)$ be a generic ideal with $\operatorname{deg} F_{i}=d_{i}$, for any $i=1, \ldots, g$. Then,

$$
\begin{equation*}
\operatorname{HS}_{S / I}(t)=\left[\frac{\prod_{i=1}^{g}\left(1-t^{d_{i}}\right)}{(1-t)^{n+1}}\right] \tag{3.2}
\end{equation*}
$$

where [.] denotes the truncation of the power series at the first non positive coefficient; namely, $\left[\sum_{i} a_{i} t^{i}\right]:=\sum_{j} b_{j} t^{j}$ with $b_{j}=a_{i}$, if $a_{i}>0$ for any $i \leq j$, and $b_{j}=0$, otherwise.

As we said, the conjecture is true for $g \leq n+1$, since generic ideals are complete intersections. Moreover, it has been proven for:

1. $g=n+2$, by Stanley, see Frö85, Example 3.2];
2. $n=1$, by Fröberg [Frö85];
3. $n=2$, by Anick Ani86].

In [FL91], Fröberg and Löfwall proved that, fixing the number of generators and their degrees, there exists a Zariski dense open subset of the affine space $\mathbb{A}^{N_{1}} \times \cdots \times \mathbb{A}^{N_{g}}$ where the coefficient-wise smallest Hilbert series is attained. Moreover, in [Frö85], Fröberg proved the following.

Theorem 3.1.1 (|Frö85]). Let $I=\left(F_{1}, \ldots, F_{g}\right)$ be a generic ideal with $\operatorname{deg} F_{i}=d_{i}$, for any $i=1, \ldots, g$. Then,

$$
\operatorname{HS}_{S / I}(t) \succeq_{\mathrm{Lex}}\left[\frac{\prod_{i=1}^{g}\left(1-t^{d_{i}}\right)}{(1-t)^{n+1}}\right]
$$

where the inequality is in the lexicographic sense.
From these remarks, we have that, fixing the parameters $\left(n ; d_{1}, \ldots, d_{g}\right)$, it is enough to exhibit one ideal with the Hilbert series as in the formula (3.2) in order to prove Fröberg's conjecture. An ideal with Hilbert series as in (3.2) is called Hilbert generic. For example, Stanley's proof of Fröberg's conjecture was by showing that the ideal $\left(x_{0}^{d_{0}}, \ldots, x_{n}^{d_{n}}, L^{d_{n+1}}\right)$ is Hilbert generic, for any generic choice of a linear form $L$. In particular, this tells us the generic power ideals with $g \leq n+2$ generators are Hilbert generic. Unfortunately, generic power ideals are not always Hilbert generic, as we see in the next section.

## SECTION 3.2

## POWER IDEALS AND FRÖBERG-IARROBINO'S CONJECTURE

Since the 1990s, the interest around the Hilbert series of power ideals increased a lot, especially due to their connections with several different branches of Commutative Algebra, Algebraic Geometry and Combinatorics. See [AP10] for a recent survey on the topic.

### 3.2.1 Fröberg-Hollman list of exceptions

In [FH94], Fröberg and Hollman produced a list of cases in which generic power ideals fail to be Hilbert generic. Here, we consider generic power ideals with generators of same degrees, i.e., of type $\left(L_{1}^{d}, \ldots, L_{g}^{d}\right) \subset \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$.

| (d,n,g) | Computed Hilbert series (Fröberg's conjecture formula (3.2)) |
| :---: | :---: |
| $(2,4,7)$ | $\begin{array}{r} 1+5 t+8 t^{2}+\mathbf{t}^{3} \\ \left(\ldots+0 \cdot t^{3}\right) \end{array}$ |
| $(2,6,9)$ | $\begin{aligned} 1+7 t+19 t^{2}+ & 21^{3}+\mathbf{t}^{4} \\ & \left(\ldots+0 \cdot t^{4}\right) \end{aligned}$ |
| $(2,7,10)$ | $\begin{array}{r} 1+8 t+26 t^{2}+40 t^{3}+\mathbf{1 6 t}^{4} \\ \left(\ldots+15 t^{4}\right) \end{array}$ |
| $(2,8,11)$ | $\begin{array}{r} 1+9 t+34 t^{2}+66 t^{3}+55 t^{4}+\mathbf{t}^{5} \\ \left(\ldots+0 \cdot t^{5}\right) \end{array}$ |
| $(2,9,12)$ | $\begin{array}{r} 1+10 t+43 t^{2}+100 t^{3}+121 t^{4}+\mathbf{3 2} \mathbf{t}^{5} \\ \left(\ldots+22 t^{5}\right) \end{array}$ |
| $(2,10,13)$ | $\begin{array}{r} 1+11 t+53 t^{2}+143 t^{3}+221 t^{4}+144 \mathbf{1 t}^{\mathbf{5}}+\mathbf{t}^{\mathbf{6}} \\ \left(\ldots+143 t^{5}+0 \cdot t^{6}\right) \end{array}$ |
| $(2,10,14)$ | $\begin{array}{r} 1+11 t+52 t^{2}+132 t^{3}+168 t^{4}+\mathbf{1 4 t}^{5} \\ \left(\ldots+0 \cdot t^{5}\right) \end{array}$ |
| $(3,2,5)$ | $\begin{aligned} & 1+3 t+6 t^{2}+5 t^{3}+\mathbf{t}^{4} \\ &\left(\ldots+0 \cdot t^{4}\right) \end{aligned}$ |
| $(3,3,9)$ | $\begin{array}{r} 1+4 t+10 t^{2}+11 t^{3}+\mathbf{t}^{4} \\ \left(\ldots+0 \cdot t^{4}\right) \end{array}$ |
| $(3,4,7)$ | $\begin{array}{r} 1+5 t+15 t^{2}+28 t^{3}+35 t^{4}+21 t^{5}+\mathbf{t}^{6} \\ \left(\ldots+0 \cdot t^{6}\right) \end{array}$ |
| $(3,4,14)$ | $\begin{aligned} & 1+5 t+15 t^{2}+ 21 t^{3}+\mathbf{t}^{4} \\ &\left(\ldots+0 \cdot t^{4}\right) \end{aligned}$ |
| $(3,5,8)$ | $\begin{array}{r} 1+6 t+21 t^{2}+48 t^{3}+78 t^{4}+84 t^{5}+43 \mathrm{t}^{6} \\ \left(\ldots+42 t^{6}\right) \end{array}$ |
| $(3,5,9)$ | $\begin{array}{r} 1+6 t+21 t^{2}+47 t^{3}+72 t^{4}+63 t^{5}+3 \mathbf{3 t}^{6} \\ \left(\ldots+0 \cdot t^{6}\right) \end{array}$ |
| $(3,6,9)$ | $\begin{array}{r} 1+7 t+28 t^{2}+75 t^{3}+147 t^{4}+210 t^{5}+204 t^{6}+\mathbf{8 5 t}^{\mathbf{7}}+\mathbf{t}^{8} \\ \left(\ldots+78 t^{7}+0 \cdot t^{8}\right) \end{array}$ |
| $(3,7,10)$ | $\begin{array}{r} 1+8 t+36 t^{2}+110 t^{3}+250 t^{4}+432 t^{5}+561 t^{6}+492 t^{7}+\mathbf{1 7 1 7}^{8} \\ \left(\ldots+135 t^{8}\right) \end{array}$ |

We can easily explain some of these exceptions by using the relation between power ideals and ideals of fat points given by Apolarity Lemma (Lemma 2.5.5. Some of them are very classical geometrical constructions.

Alexander-Hirschowitz cases. We consider the cases

$$
(d, n, g)=(2,4,7),(3,3,5),(3,4,9),(3,5,14)
$$

From the Fröberg-Hollman list, we have that they fail to be Hilbert generic in one specific degree. Hence, by using the Apolarity Lemma, we are interested in the following cases:

1. for $(2,4,7)$ : we want to compute $\operatorname{HF}_{S / I_{\mathbb{X}}}(3)$, where $\mathbb{X}$ is the scheme of seven double points in $\mathbb{P}^{3}$ in general position;
2. for $(3,3,5)$ : we want to compute $\mathrm{HF}_{S / I_{\mathbb{X}}}(4)$, where $\mathbb{X}$ is the scheme of five double points in $\mathbb{P}^{2}$ in general position;
3. for $(3,4,9)$ : we want to compute $\mathrm{HF}_{S / I_{\mathbb{X}}}(4)$, where $\mathbb{X}$ is the scheme of nine double points in $\mathbb{P}^{3}$ in general position;
4. for $(3,5,14)$ : we want to compute $\mathrm{HF}_{S / I_{\mathrm{X}}}(4)$, where $\mathbb{X}$ is the scheme of fourteen double points in $\mathbb{P}^{4}$ in general position.

These are exceptional cases in the Alexander-Hirschowitz Theorem about the Hilbert series of double points in general position (Theorem 2.6.5). In particular, we have already considered the case 2 in Example 2.6.4.
Example 3.2.1. Case 3: the scheme of nine double points in $\mathbb{P}^{3}$ has multiplicity $9 \cdot 4=36$, while the dimension of the space of quartics in $\mathbb{P}^{3}$ is 35 . Hence, we expect to have no quartics with nine singular points in general position. On the other hand, there is always a quadric passing through nine generic points in $\mathbb{P}^{3}$ and, then, the double quadric gives us the unexpected quartic singular at the nine points.

Case 4: the scheme of fourteen double points in $\mathbb{P}^{4}$ has multiplicity 14. $5=70$, which equals the dimension of the space of quartics in $\mathbb{P}^{4}$. Hence, we expect to have no quartics with fourteen singular points in general position. On the other hand, there is always a quadric passing through fourteen generic points in $\mathbb{P}^{4}$ and, then, the double quadric gives us the unexpected quartic singular at the fourteen points.

The case 1 is different.
Example 3.2.2. In this case, we have that the multiplicity of seven double points in $\mathbb{P}^{4}$ is $7 \cdot 5=35$ which is equal to the dimension of cubics in $\mathbb{P}^{4}$. Hence, we expect to have no cubics with seven singular points in general position. However, we have the following classical result; e.g. see [Har92].

Theorem 3.2.3. For any $n+3$ points in general position in $\mathbb{P}^{n}$ there exists a unique Rational Normal Curve $\mathcal{C}_{n}$ of degree $n$ passing through these points.

Rational Normal Curves are very classical and well studied geometrical objects. In particular, they are determinantal varieites, namely they can be set-theoretically described as the zero locus of the $2 \times 2$ minors of the matrix

$$
M=\left[\begin{array}{ccccc}
x_{0} & x_{1} & x_{2} & \ldots & x_{h} \\
x_{1} & x_{2} & . & \ldots & x_{h+1} \\
x_{2} & . & . & \ldots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
x_{n-h} & . & . & \ldots & x_{n}
\end{array}\right]
$$

Moreover, also their secant varieties are determinantal.
Proposition 3.2.4. The s-th secant variety $\sigma_{s}\left(\mathcal{C}_{n}\right)$ is the zero locus of the $(s+1) \times(s+1)$ minors of the matrix $M$.

Coming back to our example, we have a unique Rational Normal Curve $\mathcal{C}_{4}$ of degree 4 passing through the seven general points. Hence, we have that the 2 -nd secant variety $\sigma_{2}\left(\mathcal{C}_{4}\right)$ is the cubic hypersurface of $\mathbb{P}^{4}$ given by

$$
\operatorname{det}\left[\begin{array}{lll}
x_{0} & x_{1} & x_{2} \\
x_{1} & x_{2} & x_{3} \\
x_{2} & x_{3} & x_{4}
\end{array}\right]=0
$$

Since, this determinant can be defined as a combination of the $2 \times 2$ minors, it is easy to see that it also singular along all the Rational Normal Curve and, in particular, at the seven points. Thus, $\sigma_{2}\left(\mathcal{C}_{4}\right)$ is an unexpected cubic hypersurface singular at the seven points in general position. One can prove that this is the unique cubic surface singular at the 7 given points.

OTHER CASES WITH $n+3$ POINTS. The construction explained in the Example 3.2.2 can be used to describe several other exceptions.

1. $(2,6,9)$ : we want to compute $\operatorname{HF}_{S / I_{\mathbb{X}}}(4)$, where $\mathbb{X}$ is the scheme of nine triple points in $\mathbb{P}^{6}$ in general position;
2. ( $2,8,11$ ): we want to compute $\mathrm{HF}_{S / I_{\mathbb{X}}}(5)$, where $\mathbb{X}$ is the scheme of eleven points of multiplicity four in $\mathbb{P}^{8}$ in general position;
3. $(2,10,13)$ : we want to compute $\mathrm{HF}_{S / I_{\mathbb{X}}}(6)$, where $\mathbb{X}$ is the scheme of thirteen points of multiplicity five in $\mathbb{P}^{1} 0$ in general position;
4. $(3,4,7)$ : we want to compute $\operatorname{HF}_{S / I_{\mathbb{X}}}(6)$, where $\mathbb{X}$ is the scheme of seven points of multiplicity four in $\mathbb{P}^{4}$ in general position;
5. $(3,6,9)$ : we want to compute $\operatorname{HF}_{S / I_{\mathbb{X}}}(8)$, where $\mathbb{X}$ is the scheme of nine points of multiplicity six in $\mathbb{P}^{6}$ in general position;

In all these cases we have that an unexpected hypersurface singular at the prescribed points is given by a secant variety of the unique Rational Normal Curve $\mathcal{C}_{n}$ of degree $n$ passing through the given $n+3$ points in $\mathbb{P}^{n}$. In particular, we have:

1. $\sigma_{3}\left(\mathcal{C}_{6}\right)$ is a quartic hypersurface of $\mathbb{P}^{6}$ with nine generic points of multiplicity three;
2. $\sigma_{4}\left(\mathcal{C}_{8}\right)$ is a quintic hypersurface of $\mathbb{P}^{8}$ with eleven generic points of multiplicity four;
3. $\sigma_{5}\left(\mathcal{C}_{10}\right)$ is a sextic hypersurface of $\mathbb{P}^{10}$ with thirteen generic points of multiplicity five;
4. the double $\sigma_{2}\left(\mathcal{C}_{4}\right)$ is a sextic hypersurface of $\mathbb{P}^{4}$ with seven generic points of multiplicity four;
5. the double $\sigma_{3}\left(\mathcal{C}_{6}\right)$ is an octic hypersurface of $\mathbb{P}^{6}$ with nine generic points of multiplicity six;

Other exceptions: $(2,2 m, 2 m+3)$. Consider the cases

$$
(d, n, g)=(2,2 m, 2 m+3), \text { for any } m \geq 0 .
$$

From some numerical properties of the Fröberg's conjecture formula 3.2, we have that the first 0 term of the series is in degree $m+1$. Thus, we need to look at $\mathrm{HF}_{S / I_{\mathbb{X}}}(m+1)$ where $\mathbb{X}$ is the scheme of $2 m+3$ points of multiplicity $m$ in $\mathbb{P}^{2 m}$. Thus, we expect to have no hypersurfaces of degree $m$ in $\mathbb{P}^{2 m}$ passing through $2 m+3$ points in general position and singular at them with multiplicity $m$. On the other hand, consider the unique Rational Normal Curve $\mathbb{C}_{2 m}$ of degree $2 m$ inside $\mathbb{P}^{2 m}$ passing trough the given $2 m+3$ points. The $m^{t h}$-secant variety is the hypersurface given by the equation

$$
\operatorname{det}\left[\begin{array}{cccc}
x_{0} & x_{1} & \cdots & x_{m} \\
x_{1} & x_{2} & \cdots & x_{m+1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m} & x_{m+1} & \cdots & x_{2 m}
\end{array}\right]=0 .
$$

Thus, $\sigma_{m}\left(\mathcal{C}_{2 m}\right)$ is an unexpected hypersurface showing that generic power ideals with $(d, n, g)=(2,2 m, 2 m+3)$ are not Hilbert generic.

Unfortunately, we can not analyze all the exceptions listed by Fröberg and Hollman in this way. In the cases just described, we were expecting to have no hypersurfaces with some prescribed singulartities and, then, it was enough to exhibit one of such unexpected hypersurfaces. In all the other cases, we already expect to have several hypersurfaces with prescribed singularities and we should exhibit more than the expected ones. This is much more difficult. For example, it is not easy to write down 171, instead of the expected 135 , octics in $\mathbb{P}^{7}$ with ten points of multiplicity six.

### 3.2.2 LINEAR SPECIALITY

In BDP15a], Brambilla, Dumitrescu and Postinghel suggested a new direction in order to study dimensions of linear systems. Let $\mathcal{L}_{n}\left(m, d^{g}\right)$ be the linear system of hypersurfaces of degree $m$ in $\mathbb{P}^{n}$ with $g$ base points of multiplicity $d$ in general position ${ }^{11}$. Considering the notation used previously, if $\mathbb{X}$ is the scheme of $g$ points in $\mathbb{P}^{n}$, of multiplicity $d$ and in general position, we have that $\mathcal{L}_{n}\left(m, d^{g}\right)$ is the homogeneous part $\left[I_{\mathbb{X}}\right]_{m}$. Naïvely, we have that, since a fat point of multiplicity $d$ in $\mathbb{P}^{n}$ has degree $\binom{n+d-1}{n}$, we expect

$$
\begin{equation*}
\exp \cdot \operatorname{dim} \mathcal{L}_{n}\left(m, d^{g}\right)=\max \left\{0,\binom{n+m}{n}-g\binom{n+d-1}{n}\right\} . \tag{3.3}
\end{equation*}
$$

[^14]However, this expected dimension takes in consideration only the multiple base points. In [BDP15a], Brambilla, Dumitrescu and Postinghel started to consider also linear spaces contained with multiplicity in the base locus. In particular, they observe that a linear space of dimension $r$, contained in the base locus of $\mathcal{L}_{n}\left(m, d^{g}\right)$ with multiplicity $d$, gives an obstruction of

$$
\begin{equation*}
(-1)^{r+1}\binom{n+[(r+1) d-r m]-r-1}{n} \tag{3.4}
\end{equation*}
$$

where we set $\binom{a}{b}=0$, whenever $a<b$. Thus, they refine the expected dimension given with the formula (3.3).

Definition 3.2.5. In the same notation as above, the linear virtual dimension of $\mathcal{L}_{n}\left(m, d^{g}\right)$ is

$$
\operatorname{vdim} \mathcal{L}_{n}\left(m, d^{g}\right)=\sum_{r=-1}^{g-1}(-1)^{r+1}\binom{g}{r+1}\binom{n+[(r+1) d-r m]-r-1}{n}
$$

Hence, the expected linear dimension is

$$
\exp \cdot \operatorname{ldim} \mathcal{L}_{n}\left(m, d^{g}\right)=\max \left\{0, \operatorname{vdim} \mathcal{L}_{n}\left(m, d^{g}\right)\right\}
$$

A linear system with dimension different than the linear expected dimension is called linearly special.

A priori, we don't know if the actual dimension of a linear system is always at least its expected dimension. However, it is conjectured that this is always the case.

Conjecture 3.2 (Weak Fröberg-Iarrobino's conjecture, [Cha05]). With the same notation as above,

$$
\operatorname{dim} \mathcal{L}_{n}\left(m, d^{g}\right) \geq \exp . \operatorname{ldim} \mathcal{L}_{n}\left(m, d^{g}\right)
$$

This expected linear dimension is precisely the geometric version of the formula (3.2) prescribed by Fröberg's conjecture. Before motivating this claim in detail, we start by explain it with an easy example.

Example 3.2.6. Consider once again the case of two double points in $\mathbb{P}^{2}$ from Example 2.6.3. In the previous chapter, we described that as a defective case. Indeed, two double points give six conditions and, then, we were expecting to have no conics with two distinct singular points; although, we could easily find one by considering the double line through the two points. On the other hand, the expected linear dimension takes in consideration the fact that the double line is contained in the base locus of $\mathcal{L}_{2}\left(2,2^{2}\right)$ :

$$
\exp \cdot \operatorname{ldim} \mathcal{L}_{2}\left(2,2^{2}\right)=\max \left\{0, \operatorname{vdim} \mathcal{L}_{2}\left(2,2^{2}\right)\right\}=\max \{0,6-2 \cdot 3+1\}=1
$$

Thus, we can now say that this example was not really an exception. From the point of view of power ideals, by Apolarity Lemma (Theorem 2.5.5), we
know that the dimension of $\mathcal{L}_{2}\left(2,2^{2}\right)$ coincides with the Hilbert function in degree 2 of the quotient ring with respect to the ideal $I=\left(L_{1}, L_{2}\right) \subset S=$ $\mathbb{C}\left[x_{0}, x_{1}, x_{2}\right]$, where $L_{1}, L_{2}$ are generic linear forms. This is a trivial case, since it is a complete intersection. In particular, the formula 3.1 holds:

$$
\mathrm{HS}_{S / I}(t)=\frac{1}{1-t} \quad \Longrightarrow \quad \mathrm{HF}_{S / I}(2)=1
$$

Hence, also from the perspective of the Fröberg-Hollman list, this case was not an exceptional case. In general, the claim is that:
power ideals which fail to be Hilbert generic correspond to linear systems that are linearly special.

Indeed, consider the power series given by the formula (3.2) in Fröberg's conjecture:

$$
H_{n, d, g}(t)=\frac{\left(1-t^{d}\right)^{g}}{(1-t)^{n+1}}
$$

Now, it is an easy exercise to check that

$$
\begin{equation*}
\operatorname{coeff}_{t^{m}} H_{n, d, g}(t)=\sum_{s=0}^{g}(-1)^{s}\binom{g}{s}\binom{n+(m-s d)}{n} \tag{3.5}
\end{equation*}
$$

The linear system $\mathcal{L}_{n}\left(m, d^{g}\right)$, by Apolarity Lemma (Theorem 2.5.5), corresponds to the degree $m$ homogeneous part of a power ideal with $g$ generators of degree $m-d+1$. Hence, the formula 3.5 becomes:

$$
\begin{equation*}
\operatorname{coeff}_{t^{m}} H_{n, m-d+1, g}(t)=\sum_{s=0}^{g}(-1)^{s}\binom{g}{s}\binom{n+m-s(m-d+1)}{n} \tag{3.6}
\end{equation*}
$$

By substituting $r=s-1$,

$$
\operatorname{coeff}_{t^{m}} H_{n, m-d+1, g}(t)=\sum_{r=-1}^{g-1}(-1)^{r+1}\binom{g}{r+1}\binom{n+[(r+1) d-r m]-r-1}{n}
$$

which is precisely the virtual linear dimension of $\mathcal{L}_{n}\left(m, d^{g}\right)$. In other words, the expected obstructions obtained by looking at linear spaces contained with certain multiplicity in the base locus of a linear system $\mathcal{L}_{n}\left(m, d^{g}\right)$ corresponds to the expected Koszul syzygies that we always have in the power ideal. In [BDP15b], Brambilla, Dumitrescu and Postinghel continued their analysis of base loci of linear systems with prescribed multiple base points. In particular, they conjecture that a linear system with $n+3$ points are linearly special only if it contains in the base locus, with certain multiplicity, secant varieties to the Rational Normal Curve of degree $n$ and their joins with linear subspaces spanned by some of the base points. Moreover, they give a new definition of expected dimension, called secant linear dimension, which extends the expected linear dimension given in Definition 3.2.5. In particular, they claim that the obstructions given by one of
such rational varieties equal the obstructions given by a linear space of the same dimension and multiplicity, see formula (3.4). This would give a geometrical justification of all the cases with $n+3$ and $n+4$ points of the Fröberg-Hollman list.

| Linear system | linear virtual dimension | computed dimension |
| :--- | :---: | :---: |
| $\mathcal{L}_{4}\left(3,2^{7}\right)$ | 0 | $1($ gap $=\mathbf{1})$ |
| $\mathcal{L}_{6}\left(4,3^{9}\right)$ | -6 | $1(\mathbf{7})$ |
| $\mathcal{L}_{7}\left(4,3^{10}\right)$ | 15 | $16(\mathbf{1})$ |
| $\mathcal{L}_{8}\left(5,4^{11}\right)$ | -33 | $1(\mathbf{3 4 )}$ |
| $\mathcal{L}_{9}\left(5,4^{12}\right)$ | 22 | $32(\mathbf{1 0})$ |
| $\mathcal{L}_{10}\left(5,4^{13}\right)$ | 143 | $144(\mathbf{1})$ |
| $\mathcal{L}_{10}\left(5,4^{14}\right)$ | 0 | $14(\mathbf{1 4})$ |
| $\mathcal{L}_{2}\left(4,2^{5}\right)$ | 0 | $1(\mathbf{1})$ |
| $\mathcal{L}_{4}\left(6,4^{7}\right)$ | -14 | $1(\mathbf{1 5})$ |
| $\mathcal{L}_{5}\left(6,4^{8}\right)$ | 42 | $43(\mathbf{1})$ |
| $\mathcal{L}_{5}\left(6,4^{9}\right)$ | -6 | $3(\mathbf{9})$ |
| $\mathcal{L}_{6}\left(8,6^{9}\right)$ | -147 | $1(\mathbf{1 4 8})$ |
| $\mathcal{L}_{7}\left(8,6^{10}\right)$ | 135 | $171(\mathbf{3 6})$ |

Here is the heuristic justification, based on the analysis on secant linear dimension suggested by [BDP15b], of the difference between the computed dimension and the linear virtual dimension.
$-\mathcal{L}_{4}\left(3,2^{7}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $2(=2 \cdot 7-3 \cdot 4)$ and, then, gives an obstruction of

$$
\binom{4+2-2}{4}=1 .
$$

- $\mathcal{L}_{6}\left(4,3^{9}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $3(=3 \cdot 9-4 \cdot 6)$ and, then, gives an obstruction of

$$
\binom{6+3-2}{6}=7 .
$$

- $\mathcal{L}_{7}\left(4,3^{10}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $2(=3 \cdot 10-4 \cdot 7)$ and, then, gives an obstruction of

$$
\binom{7+2-2}{7}=1
$$

$-\mathcal{L}_{8}\left(5,4^{11}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $4(=4 \cdot 11-4 \cdot 8)$ and, then, gives an obstruction

$$
\binom{8+4-2}{8}=45
$$

Moreover, any line connecting one of the base points with any other point of the Rational Normal Curve is contained in the base locus with multiplicity
$3(=4+4-5)$. Hence, any of the 11 cones, with one of the base points as vertex and with the Rational Normal Curve as base, is contained in the base locus of the linear system with multiplicity 3 and, then, gives an obstruction of

$$
-\binom{8+3-3}{8}=-1
$$

thus, in total, we have

$$
45-11 \cdot 1=34
$$

- $\mathcal{L}_{9}\left(5,4^{12}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $3(=4 \cdot 12-5 \cdot 9)$ and, then, gives an obstruction of

$$
\binom{9+3-2}{9}=10
$$

$-\mathcal{L}_{10}\left(5,4^{13}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $2(=4 \cdot 13-5 \cdot 10)$ and, then, gives an obstruction of

$$
\binom{10+2-2}{10}=1
$$

$-\mathcal{L}_{10}\left(5,4^{14}\right)$ : each one of the 14 Rational Normal Curves defined by a choice of 13 of the base points is contained inside the base locus with multiplicity $2(=4 \cdot 13-5 \cdot 10)$ and, then, all together, they give an obstruction of

$$
14 \cdot\binom{9+2-2}{9}=14
$$

- $\mathcal{L}_{2}\left(4,2^{5}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $2(=2 \cdot 5-2 \cdot 4)$ and, then, gives an obstruction of

$$
\binom{2+2-2}{2}=1
$$

- $\mathcal{L}_{4}\left(6,4^{7}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $4(=4 \cdot 7-4 \cdot 6)$ and, then, gives an obstruction of

$$
\binom{4+4-2}{4}=15 .
$$

$-\mathcal{L}_{5}\left(6,4^{8}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $2(=4 \cdot 8-5 \cdot 6)$ and, then, gives an obstruction of

$$
\binom{5+2-2}{5}=1
$$

$-\mathcal{L}_{5}\left(6,4^{9}\right)$ : each one of the 9 Rational Normal Curves defined by a choice of 8 of the base points is contained inside the base locus with multiplicity $2(=4 \cdot 8-5 \cdot 6)$ and, then, gives an obstruction of

$$
9 \cdot\binom{5+2-2}{5}=9
$$

$-\mathcal{L}_{6}\left(8,6^{9}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $6(=6 \cdot 9-6 \cdot 8)$ and, then, gives an obstruction of

$$
\binom{6+6-2}{6}=210
$$

Moreover, any line connecting one of the base points with any other point of the Rational Normal Curve is contained in the base locus with multiplicity $4(=6+6-8)$. Hence, any of the 9 cones, with one of the base points as vertex and with the Rational Normal Curve as base, is contained in the base locus of the linear system with multiplicity 4 and, then, gives an obstruction of

$$
-\binom{6+4-3}{6}=-7
$$

Finally, any secant line to the Rational Normal Curve is contained in the base locus with multiplicity $4(6+6=8)$. Hence, the 2-nd secant variety is also contained in the base locus of the linear system with multiplicity 4. Since it is a threefold, it gives an obstruction of

$$
\binom{6+4-4}{6}=1
$$

Therefore, in total, we have

$$
210-9 \cdot 7+1=148
$$

$-\mathcal{L}_{7}\left(8,6^{10}\right)$ : the Rational Normal Curve is contained inside the base locus with multiplicity $4(=6 \cdot 10-7 \cdot 8)$ and, then, gives an obstruction of

$$
\binom{7+4-2}{7}=36
$$

This work about linear speciality pursued by Brambilla, Dumitrescu and Postinghel goes in the direction of trying to give a complete answer to the general question about the Hilbert series of ideals of fat points in general position. In [Cha05], the following conjecture is suggested. Observe that, as explained above, this conjecture can rephrased in terms of powers ideals that might not be Hilbert generic.

Conjecture 3.3 (Strong Fröberg-Iarrobino's conjecture, [Cha05]). In the same notation as above, the linear system $\mathcal{L}_{n}\left(m, d^{g}\right)$ has dimension equal to the linear expected dimension except perhaps when one of the following conditions hold:

1. $g=n+3$;
2. $g=n+4$;
3. $n=2$ and $g=7,8$;
4. $n=3, g=9$ and $m \geq 2 d$;
5. $n=4, g=14$ and $m=2 d, d=2,3$.

## SECTION 3.3

## SECANT VARIETIES OF VARIETIES OF POWERS

In this section, we use Fröberg's conjecture to conjecture the dimension of all secant varieties of varieties of powers, see Conjecture 1.3. As explained in Section 1.3.3, this conjecture would imply Conjecture 1.2 about the $d$-th Waring rank of generic forms. We defined the variety of powers $V_{n, k, d}$ as

$$
V_{n, k, d}=\left\{\left[G^{k}\right] \mid G \in S_{d}\right\} \subset \mathbb{P}\left(S_{k d}\right)
$$

By using Terracini's Lemma, we have seen that the generic tangent space to the $s$-th secant variety to the variety of powers is given by

$$
T_{P} \sigma_{s}\left(V_{n, k, d}\right)=\mathbb{P}\left(\left[\left(G_{1}^{d-1}, \ldots, G_{s}^{d-1}\right)\right]_{k d}\right),
$$

where $G_{1}, \ldots, G_{s}$ are generic forms of degree $k$. Thus, in order to compute the dimension of all secant varieties of varieties of powers, we have to compute the Hilbert function in degree $k d$ of the ideal $\left(G_{1}^{d-1}, \ldots, G_{s}^{d-1}\right)$. We call these ideals generalized power ideals.

### 3.3.1 BINARY CASE

First, we consider the case of binary forms ( $n=1$ ).
Lemma 3.3.1. Let $I=\left(G_{1}^{d}, \ldots, G_{g}^{d}\right)$ be a generalized power ideals where $F_{i}$ is a generic binary forms. Then, I is Hilbert generic.

Proof. The idea is to specialize the $G_{i}$ 's to be linear forms. Then, we are reduced to the case of power ideals in two variables. They are Hilbert generic by [GS98, Corollary 2.3].

Proposition 3.3.2. The secant varieties $\sigma_{s}\left(V_{1, k, d}\right)$ are never defective.
Proof. For $d=2$, it is easy to see that $\sigma_{2}\left(V_{1, k, 2}\right)$ fills the whole ambient space. Indeed, if $F \in S_{2 d}$, we can decompose it as $F=X Y$, where $X, Y$ are homogeneous polynomials of degree $d$; hence,

$$
F=X Y=\frac{1}{4}\left[(X+Y)^{2}-(X-Y)^{2}\right] .
$$

Assume now $d \geq 3$. By Lemma 3.3.1, we know that the Hilbert function of a generalized power ideal, whose generators are powers of generic forms, equals the formula (3.2) prescribed by Fröberg's conjecture. Hence,

$$
\operatorname{codim} \sigma_{s}\left(V_{1, k, d}\right)=\operatorname{coeff} k d\left[\frac{\left(1-t^{k(d-1)}\right)^{s}}{(1-t)^{2}}\right]=\max \{0,(k d+1)-s(k+1)\}
$$

As a corollary, we get the $d$-th Waring rank of generic binary forms. This result has been proved by Reznick in [Rez13b] with a different approach.

Corollary 3.3.3. The generic $d$-th Waring rank of binary forms is

$$
\mathrm{rk}_{d}^{\circ}(2, d k)=\left\lceil\frac{k d+1}{k+1}\right\rceil .
$$

### 3.3.2 SUM OF SQUARES

In this case ( $d=2$ ), we get rid of the powers in the ideal and, then, we are precisely under the assumptions of Fröberg's conjecture. Thus, we have that, assuming that Fröberg's conjecture holds,

$$
\begin{aligned}
\operatorname{codim} \sigma_{s}\left(V_{n, k, 2}\right) & =\max \left\{0, \operatorname{coeff}_{2 k} \frac{\left(1-t^{k}\right)^{s}}{(1-t)^{n+1}}\right\}= \\
& =\max \left\{0,\binom{n+2 k}{n}-s\binom{n+k}{n}+\binom{s}{2}\right\} .
\end{aligned}
$$

Theorem 3.3.4. If Fröberg's conjecture holds, then the secant varieties $\sigma_{s}\left(V_{n, k, 2}\right)$ are defective, with defect $\binom{s}{2}$, until they fill the ambient space.

Remark 3.3.5. The fact that these cases are defective is not unexpected. Indeed, similarly as for the case of degree 2 Veronese embeddings, we can explain the defectiveness by Terracini's Lemma. Indeed, the tangent spaces to the variety of powers at two generic points, $\left[F^{2}\right]\left[G^{2}\right] \in V_{n, k, 2}$, has a non-empty intersection

$$
[F G]=T_{\left[F^{2}\right]} V_{n, k, 2} \cap T_{\left[G^{2}\right]} V_{n, k, 2}
$$

Hence, the dimension of the span of $s$ generic tangent spaces to the variety of powers $V_{n, k, 2}$ is at most $s\binom{n+k}{n}-\binom{s}{2}$.

Corollary 3.3.6. If Fröberg's conjecture holds, then the 2-nd Waring rank of a generic form of degree $2 k$ in $n+1$ variables is

$$
\operatorname{rk}_{2}^{\circ}(n+1, k d)=\min \left\{s \left\lvert\, s\binom{n+k}{n}-\binom{s}{2} \geq\binom{ n+2 k}{n}\right.\right\} .
$$

In the case of three variables, we know that Fröberg's conjecture holds by the result of Anick [Ani86]. Then, we get the following.

Theorem 3.3.7. The 2-nd Waring rank of a generic ternary form is

$$
\mathrm{rk}_{2}^{\circ}(3,2 k)=\left\lceil\frac{\binom{2 k+2}{2}}{\binom{k+2}{2}}\right\rceil \text {, }
$$

except for

$$
d=1,3,4 \quad \text { where } \quad \mathrm{rk}_{d}^{\circ}(2,2 k)=\left\lceil\frac{\binom{2 k+2}{2}}{\binom{k+2}{2}}\right\rceil+1
$$

In the case of four variables, we have that the generic 2-nd Waring rank is less or equal than 8 , by Theorem 2.7.5. Moreover, we have that the lower bound, given by a simple parameter count, is equal to 8 for $d \geq 21$. Thus, in order to have a complete description of the 2-nd Waring rank also in the case of four variables, it is enough to check with the support of the Computer Algebra software Macaulay2 [GS] that Fröberg's conjecture holds for $d \leq 21$; see [One14] for details. Therefore, we have the following.

Theorem 3.3.8. The 2-nd Waring rank of a generic quaternary form is

$$
\mathrm{rk}_{2}^{\circ}(4,2 k)=\left\lceil\frac{\binom{2 k+3}{3}}{\binom{k+3}{3}}\right\rceil,
$$

except for

$$
d=1,2 \quad \text { where } \quad \mathrm{rk}_{2}^{\circ}(4,2 k)=\left\lceil\frac{\binom{2 k+3}{3}}{\binom{k+3}{3}}\right\rceil+1
$$

### 3.3.3 General case

Assume now $d \geq 3$. Supported by massive computer experiments with Macaulay 2 [GS], we have the following conjecture; see [Nic15].

Conjecture 3.4. Generalized power ideals generated by powers of generic forms of degree at least 2 are Hilbert generic.

Thus, similarly as above we get the following.
Theorem 3.3.9. If Conjecture 3.4 holds, then, for $d \geq 3$, the secant varieties $\sigma_{s}\left(V_{n, k, d}\right)$ are never defective.

Proof. If Conjecture 3.4 holds, we can use the formula given by Fröberg's conjecture. Therefore,

$$
\begin{aligned}
\operatorname{codim} \sigma_{s}\left(V_{n, k, d}\right) & =\max \left\{0, \operatorname{coeff}_{k d} \frac{\left(1-t^{(k(d-1)}\right)^{s}}{(1-t)^{n+1}}\right\}= \\
& =\max \left\{0,\binom{n+k d}{n}-s\binom{n+k}{n}\right\}
\end{aligned}
$$

Corollary 3.3.10. If Conjecture 3.4 holds, then the $d$-th Waring rank of a generic form of degree $k d$ in $n+1$ variables is

$$
\mathrm{rk}_{d}^{\circ}(n+1, k d)=\left\lceil\frac{\binom{n+k d}{n}}{\binom{n+k}{n}}\right\rceil .
$$

## SECTION 3.4

## SECANT VARIETIES OF DECOMPOSABLE FORMS

Generalizing classical ideas coming from Segre, Spampinato, Bordiga and Mammana, in [CGG+15], the authors considered, for any partition $\mu=$ $\left(m_{1}, \ldots, m_{r}\right) \vdash d$, the variety of $\mu$-reducible forms; namely,

$$
\left\{[F] \in \mathbb{P}\left(S_{d}\right) \mid F=F_{1} \cdots F_{r}, \text { for some } 0 \neq F_{i} \in S_{m_{i}}\right\}
$$

In particular, they are interested in computing the dimensions of all their secant varieties. The particular case $\mu=(1, \ldots, 1)$ was first considered by Arrondo and Bernardi [AB11]; followed by the work of Shin [Shi12], who computed the dimension of all the 2-nd secant varieties in the three variable case. More recently, Abo [Abo14] completed the three variable case for all higher secant varieties. In all these cases, secant varieties are never defective and it is conjectured that this is always the case.

In [CGG $\left.{ }^{+} 15\right]$, the authors give a conjectural formula for the dimensions of all secant varieties, for any partition $\mu$. They also observe that, in the case of partitions $\mu=\left(m_{1}, m_{2}\right)$, such a formula is implied by Fröberg's conjecture.

Now, we want to report on recent results obtained in [CCGO15], where we started to study a special case of varieties of decomposable forms. Fixed a partition $\mu$ as above, we define the variety of $\mu$-powers $\sum^{2}$ as

$$
V_{n, \mu}=\left\{[F] \in \mathbb{P}\left(S_{d}\right) \mid F=L_{1}^{\mu_{1}} \cdots L_{s}^{\mu_{s}}, \quad L_{i} \in S_{1}\right\}
$$

This is related to the computation of the $\mu$-Waring rank of generic forms of degree $d$, introduced in Section 1.2.3. Apart from the case $\mu=(1, \ldots, 1)$, also the special partition $\mu=(d-1,1)$ has been considered in the literature. Indeed, in that case $V_{n, \mu}$ is the tangential variety of the Veronese variety $V_{n, d}$, namely the union of all tangent spaces to $V_{n, d}$. Catalisano, Geramita and Gimigliano conjectured in [CGG02] that secant varieties to the $V_{n,(d-1,1)}$ are never defective, except for $d=2,2 \leq 2 s \leq n$ and $d=3, s=n=2,3,4$. Abo and Vannieuwenhoven proved it in [AV15].

In our work, we look at the problem for any arbitrary partition $\mu$. Moreover, since the cases $r=1$ and $d=1,2$ have been already considered in previous works, we assume $r \geq 2$ and $d \geq 3$.

[^15]We have already seen in Section 1.3.5, that the generic tangent space to the variety of $\mu$-powers at the point $\left[\mathbb{L}^{\mu}\right]=\left[L_{1}^{m_{1}} \cdots L_{r}^{m_{r}}\right] \in V_{n, \mu}$ is given by

$$
T_{\left[\mathbb{L}^{\mu}\right]} V_{n, \mu}=\mathbb{P}\left(\left[\left(\frac{\mathbb{L}^{\mu}}{L_{1}}, \ldots, \frac{\mathbb{L}^{\mu}}{L_{r}}\right)\right]\right) \subset \mathbb{P}\left(S_{d}\right)
$$

This can also be written as

$$
\begin{equation*}
\left(\frac{\mathbb{L}^{\mu}}{L_{1}}, \ldots, \frac{\mathbb{L}^{\mu}}{L_{r}}\right)=\left(L_{1}^{m_{1}} \cdots L_{r}^{m_{r}}\right) \cdot\left(\frac{\mathbb{L}}{L_{1}}, \ldots, \frac{\mathbb{L}}{L_{r}}\right) \tag{3.7}
\end{equation*}
$$

where $\mathbb{L}=L_{1} \cdots L_{r}$. We give a more practical description of these ideals.
Proposition 3.4.1. (for binary forms) Let $S=\mathbb{C}[x, y]$ and $\mu=\left(m_{1}, \ldots, m_{r}\right) \vdash$ d. If $L_{1}, L_{2}, \ldots, L_{r}$ are general linear forms, $P=\left[L_{1}^{m_{1}} \cdots L_{r}^{m_{r}}\right] \in V_{1, \mu}$ and we set $I^{\prime}$ to be the principal ideal

$$
I^{\prime}:=\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right)
$$

then we have

$$
T_{P}\left(V_{1, \mu}\right)=\mathbb{P}\left(I_{d}^{\prime}\right)
$$

Proof. In view of equation (3.7) above, we first consider the ideal

$$
J:=\left(L_{2} L_{3} \cdots L_{r}, L_{1} L_{3} \cdots L_{r}, \ldots, L_{1} L_{1} \cdots L_{r-1}\right)
$$

Claim. $J=(x, y)^{r-1}$.
Proof of the Claim. We proceed by induction on $r$. The claim is obvious for $r=2$, so let $r>2$. Since

$$
J=\left(L_{r} \cdot\left(L_{2} L_{3} \cdots L_{r-1}, L_{1} L_{3} \cdots L_{r-1}, \ldots, L_{1} L_{2} \cdots L_{r-2}\right), L_{1} L_{2} \cdots L_{r-1}\right)
$$

we have, by the induction hypothesis, that

$$
J=\left(L_{r} \cdot(x, y)^{r-2}, L_{1} L_{2} \cdots L_{r-1}\right)
$$

The form $L_{1} L_{2} \cdots L_{r-1}$ is a general form in $S_{r-1}$, hence not in the space $L_{r}(x, y)^{r-2}$. This last implies that $\operatorname{dim} J_{r-1}=r$. Since the ideal $J$ begins in degree $r-1$, we are done with the proof of the claim.

Now, using the Claim and equation (3.7), we have that

$$
I=\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right)(x, y)^{r-1} \subseteq\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right) .
$$

Since $(x, y)^{r-1}=\bigoplus_{j \geq r-1} S_{j}$, we have that

$$
I_{d}=\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right)_{d} .
$$

Proposition 3.4.2. Let $S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ and $\mu=\left(m_{1}, \ldots, m_{r}\right) \vdash d$. If $L_{1}, L_{2}, \ldots, L_{r}$ are general linear forms and $\mathbb{L}^{\mu}=L_{1}^{m_{1}} \cdots L_{r}^{m_{r}}$, then

$$
\left(\frac{\mathbb{L}^{\mu}}{L_{1}}, \ldots, \frac{\mathbb{L}^{\mu}}{L_{r}}\right)=\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right) \cap\left(\bigcap_{1 \leq i<j \leq r}\left(L_{i}, L_{j}\right)^{m_{i}+m_{j}-1}\right)
$$

Proof. By equation (3.7), we need to prove that the two ideals

$$
I=\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right) \cdot\left(L_{2} L_{3} \cdots L_{r}, \ldots, L_{1} \cdots L_{r-1}\right)
$$

and

$$
J=\left(L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}\right) \cap\left(\bigcap_{1 \leq i<j \leq r}\left(L_{i}, L_{j}\right)^{m_{i}+m_{j}-1}\right)
$$

are equal. Since each generator of $I$ is in $J$, we have $I \subseteq J$.
Now, let $H=L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1}$, and suppose that $F=H G \in J$, for some homogeneous polynomial $G$. Since the $L_{i}$ are general linear forms, we have

$$
H \in\left(L_{i}, L_{j}\right)^{m_{i}+m_{j}-2} \quad \text { for every } 1 \leq i<j \leq r
$$

but

$$
H \notin\left(L_{i}, L_{j}\right)^{m_{i}+m_{j}-1} \quad \text { for every } 1 \leq i<j \leq r,
$$

so $G \in \sqrt{\left(L_{i}, L_{j}\right)^{m_{i}+m_{j}-1}}=\left(L_{i}, L_{j}\right)$, for all $1 \leq i<j \leq r$.
Since $\bigcap_{1 \leq i<j \leq r}\left(L_{i}, L_{j}\right)=\left(L_{2} \cdots L_{r}, \ldots, L_{1} \cdots L_{r-1}\right)$, we are done.

### 3.4.1 BINARY CASE

For binary forms, we can cover all cases. First, we observe the following.
Lemma 3.4.3. Let $S=\mathbb{C}[x, y]$ and $\mu=\left(m_{1}, \ldots, m_{r}\right) \vdash d$. For $i=1, \ldots, s$, let $\mathbb{L}_{i}^{\mu}=L_{i, 1}^{m_{1}} \cdots L_{i, r}^{m_{r}}$, where $L_{i, j}$ 's are generic linear forms. Then, the ideal $\left(\mathbb{L}_{1}^{\mu}, \ldots, \mathbb{L}_{s}^{\mu}\right)$ is Hilbert generic.

Proof. For any $i=1, \ldots, s$, we can specialize all the $L_{i, j}$ 's to be equal. In that case, we are reduced to the case of power ideals in two variables. They are Hilbert generic by [GS98, Corollary 2.3].

Theorem 3.4.4. Let $S=\mathbb{C}[x, y]$ and $\mu=\left(m_{1}, \ldots, m_{r}\right) \vdash d$. Then, $\sigma_{s}\left(V_{1, \mu}\right)$ is never defective, for any s; i.e.,

$$
\operatorname{dim} \sigma_{s}\left(V_{1, \mu}\right)=\min \{s r+s-1, d\}
$$

Proof. Since every form of degree $d$ in $S$ splits as a product of linear forms and the general form of degree $d$ is square-free, we conclude that, for $r=d$, i.e., $\mu=(1, \ldots, 1)$, we have $V_{1, \mu}=\mathbb{P}\left(S_{d}\right)$. Then, assume $r<d$.

By Proposition 3.4.1 and Terracini's Lemma 1.3.15, we get the following description of the generic tangent space to secant varieties of $V_{1, \mu}$. Let $P_{i}=$ $\left[\mathbb{L}_{i}^{\mu}\right]=\left[L_{i, 1}^{m_{1}} \cdots L_{i, r}^{m_{r}}\right] \in V_{1, \mu}$ and $P \in\left\langle P_{1}, \ldots, P_{s}\right\rangle$ be generic points. Then,

$$
T_{P} \sigma_{s}\left(V_{1, \mu}\right)=\mathbb{P}\left(\left[\left(\mathbb{L}_{1}^{\mu-1}, \ldots, \mathbb{L}_{s}^{\mu-1}\right)\right]_{d}\right)
$$

where $\mu-1=\left(m_{1}-1, \ldots, m_{r}-1\right)$. By Lemma 3.4.3, we can compute the dimension of such a tangent space by using the formula (3.2) given by Fröberg's conjecture and we conclude the proof.

Corollary 3.4.5. For any partition $\mu=\left(m_{1}, \ldots, m_{r}\right) \vdash d$, the $\mu$-Waring rank of a generic binary form is

$$
\mathrm{rk}_{\mu}^{\circ}(2, d)=\left\lceil\frac{d+1}{r+1}\right\rceil .
$$

Remark 3.4.6. In general, we call $\mu$-power ideals the ideals generated by $\mu$-powers $\left(\mathbb{L}_{1}^{\mu}, \ldots, \mathbb{L}_{s}^{\mu}\right)$, with $\mathbb{L}_{i}^{\mu}=L_{i, 1}^{m_{1}} \cdots L_{i, r}^{m_{r}}$, where $L_{i, j}$ 's are linear forms, for any $i=1, \ldots, s$. These ideals clearly generalize the definition of power ideals, which are the $\mu$-power ideals for $\mu=(d)$. Supported by massive computer experiments, we conjecture the following.

Conjecture 3.5. Let $S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$. For any $\mu \neq(d)$, generic $\mu$-power ideals are Hilbert generic.

### 3.4.2 2-ND SECANT VARIETIES

In this section, we find all the dimensions of 2-nd secant varieties, for any partition $\mu$ and for any number of variables.

Theorem 3.4.7. Let $S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ and $\mu=\left(m_{1}, \ldots, m_{r}\right)$. Assume $n \geq$ $2, r \geq 2, d \geq 3$. Then, $\sigma_{s}\left(V_{n, \mu}\right)$ is not defective, except for

$$
\mu=(2,1), n=2, \text { where the defect is equal to } 1 .
$$

Proof. We always have $r \leq d$. The case $r=d$ is covered by [Shi12, Theorem 4.4], so we may assume that $r<d$. By Terracini's Lemma, we need to find the vector space dimension of

$$
\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]}+I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d}
$$

where the $\left[I_{\left[\mathbb{L}_{i}^{\mu}\right]}\right]$ 's define the tangent spaces at two generic points $\mathbb{L}_{1}^{\mu}=$ [ $\left.L_{1}^{m_{1}} \cdots L_{r}^{m_{r}}\right]$ and $\mathbb{L}_{2}^{\mu}=\left[N_{1}^{m_{1}} \cdots N_{r}^{m_{r}}\right]$ of $V_{1, \mu}$, where ${ }_{i}$ 's and $N_{i}$ 's are generic linear forms. Namely, by Proposition 3.4.2,

$$
I_{\left[\mathbb{L}_{1}^{\mu}\right]}=\left(L_{1}^{d_{1}-1} \cdots L_{r}^{d_{r}-1}\right) \cap\left(\bigcap_{1 \leq i<j \leq r}\left(L_{i}, L_{j}\right)^{d_{i}+d_{j}-1}\right),
$$

$$
I_{\left[\mathbb{L}_{2}^{\mu}\right]}=\left(N_{1}^{d_{1}-1} \cdots N_{r}^{d_{r}-1}\right) \cap\left(\bigcap_{1 \leq i<j \leq r}\left(N_{i}, N_{j}\right)^{d_{i}+d_{j}-1}\right) .
$$

By the exact sequence

$$
\begin{equation*}
0 \longrightarrow\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d} \longrightarrow\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \oplus I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d} \longrightarrow\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]}+I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d} \longrightarrow 0 \tag{3.8}
\end{equation*}
$$

and the fact that $\operatorname{dim}\left[I_{\left[\mathbb{L}_{i}^{\mu}\right]}\right]_{d}=r n+1$, for $i=1,2$, it is enough to find $\operatorname{dim}\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d}$. Recall that the expected dimension of $\sigma_{2}\left(V_{n, \mu}\right)$ is

$$
\exp \cdot \operatorname{dim} \sigma_{2}\left(V_{n, \mu}\right)=\min \left\{2 r n+1,\binom{d+n-1}{n-1}-1\right\}
$$

Note that, if $\operatorname{dim}\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d}=0$, then, by (3.8), the dimension of $\sigma_{2}\left(V_{n, \mu}\right)$ is as expected.

Let $V$ be the subscheme of $\mathbb{P}^{n}$ defined by $I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}$ and let $F$ be a form of degree $d I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}$. Clearly

$$
F=L_{1}^{m_{1}-1} \cdots L_{r}^{m_{r}-1} \cdot N_{1}^{m_{1}-1} \cdots N_{r}^{m_{r}-1} \cdot G
$$

where $G$ is a form of degree $d-2(d-r)=2 r-d$.
If $2 r-d<0$, there are no forms of this degree, hence $\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d}=0$ and we are done. So assume $2 r-d \geq 0$.

The form $G$ vanishes on the residual scheme $W$ of $V$ with respect to the $2 r$ multiple hyperplanes $\left\{L_{i}=0\right\}$ and $\left\{N_{i}=0\right\}(1 \leq i \leq r)$. It is easy to see that $W$ is defined by the ideal

$$
\left(\cap_{1 \leq i<j \leq r}\left(L_{i}, L_{j}\right)\right) \cap\left(\cap_{1 \leq i<j \leq r}\left(N_{i}, N_{j}\right)\right) .
$$

The form $G$ cannot be divisible by all the $L_{i}$ 's and $N_{j}$ 's. Then, it has degree at least $2 r$.

Without loss of generality assume that $G$ is not divisible by $L_{1}$, and let $H$ be the hyperplane defined by $L_{1}$. The form $G$ cuts out on $H$ a hypersurface $\mathcal{Q}$ of $H$ having degree $2 r-d$, and containing the $r-1$ hyperplanes of $H$ cut out by $L_{2}, \ldots, L_{r}$. Hence, in order for $G$ to exist, $2 r-d$ has to be at least $r-1$. That is, $d \leq r+1$. Since we are assuming $d \geq r+1$, we get that $d=r+1$.

It follows that $\mathcal{Q}$ has degree $r-1$, contains the $r-1$ hyperplanes of $H$ cut out by $L_{2}, \ldots, L_{r}$, and contains the trace on $H$ of the schemes defined by the ideals $\left(N_{i}, N_{j}\right)$. Since the $N_{i}$ are generic with respect to $H$ and to the $L_{i}$ 's, the only possibility is that the schemes $Y_{i, j}$ defined by the ideals $\left(N_{i}, N_{j}\right)$ do not intersect $H$. Since $H \simeq \mathbb{P}^{n-1} \subseteq \mathbb{P}^{n}$ and $Y_{i, j} \simeq \mathbb{P}^{n-2} \subseteq \mathbb{P}^{n}$, then $H \cap Y_{i, j}=\emptyset$ only for $n \leq 2$.

Recalling that $n>2,2 \leq r \leq n$ and $d=r+1$, we are left with the following cases:
(1) $n=2, r=3, d=4$;
(2) $n=2, r=2, d=3$.
(1). In this case, $\mu=(2,1,1)$, the form $G$ has degree $2 r-d=2$ and the scheme $W$ is the union of 6 general points of $\mathbb{P}^{2}$. Hence, $G$ does not exist, and so $\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{d}=0$.
(2). In this case, $\mu=(2,1)$, the form $G$ has degree $2 r-d=1$ and the scheme $W$ is the union of 2 points of $\mathbb{P}^{2}$. Hence, $G$ exists and describes the line through the two points. It follows that $\operatorname{dim}\left(I_{P_{1}} \cap I_{P_{2}}\right)_{3}=1$. So, we get

$$
\operatorname{dim}\left[I_{\left[\mathbb{L}_{1}^{\mu}\right]} \cap I_{\left[\mathbb{L}_{2}^{\mu}\right]}\right]_{3}=9
$$

that is, $\operatorname{dim} \sigma_{2}\left(V_{2,(2,1)}\right)=8$, while $\exp \cdot \operatorname{dim} \sigma_{2}\left(V_{2,(2,1)}\right)=9$. Thus, we have that $V_{2,(2,1)}$ has 2-defect equal to 1 .

## CHAPTER 4

## DECOMPOSITIONS OF MONOMIALS

In this chapter we focus on additive decompositions of monomials. In Section 2.3.2, we discussed classical Waring decompositions over the complex numbers. In Section 4.1, we describe recent results about Waring ranks of monomials over the real numbers. In particular, we classify the monomials whose real Waring rank coincides with the complex rank. In Section 4.2, we begin to investigate $d$-th Waring decompositions of monomials over the complex numbers. As far as we know, this is the state of the art of these decompositions of monomials.

## SECTION 4.1

## REAL WARING RANK OF MONOMIALS

We consider Waring decompositions of monomials over the real numbers. We introduced them in Section 1.2.4. Here, we want to explain recent results from [CKOV16], where, in particular, we classify the monomials whose real rank coincides with the complex rank; see Theorem 4.1.8.

### 4.1.1 Binary case

In [BCG11], Boij, Carlini and Geramita computed real rank of binary monomials. Their result surprisingly rely on the following straightforward application of the well-known Descartes' rule of signs.

Lemma 4.1.1 ([BCG11, Lemma 4.1-4.2]). Consider a real polynomial

$$
F=x^{d}+c_{1} x^{d-1}+\ldots+c_{d-1} x+c_{d} \in \mathbb{R}[x]
$$

Then,

1. for any $i<d$, there is a choice of $c_{j}$ 's such that $F$ has distinct real roots and $c_{i}=0$;
2. if $c_{i}=c_{i+1}=0$ for some $0<i<d$, then $F$ has a non real root.

We recall the proof of the result about the real rank of binary monomials since we will generalize the same idea in the case of more variables.

Proposition 4.1.2 ([BCG11, Proposition 4.4]). Consider $M=x_{0}^{d_{0}} x_{1}^{d_{1}}$, then $\mathrm{rk}_{\mathbb{R}}(M)=d_{0}+d_{1}$.
Proof. We use Apolarity Lemma 2.1.13. We have that $M^{\perp}=\left(X_{0}^{d_{0}+1}, X_{1}^{d_{1}+1}\right)$. In this case, ideals of reduced points in two variables are principal ideals whose generator is square-free. Hence, we seek for square-free homogeneous polynomials inside $M^{\perp}$. Let $F=X_{0}^{d_{0}+1} G_{0}+X_{1}^{d_{1}+1} G_{1} \in M^{\perp}$ a homogeneous polynomial of degree $D$, namely $\operatorname{deg} G_{i}=D-d_{i}-1$.

If $D=d_{0}+d_{1}$, then the only missing monomial in $F$ is $X_{0}^{d_{0}} X_{1}^{d_{1}}$. Therefore, by Lemma 4.1.1(1), there exists a choice of the $G_{i}$ 's such that $F$ has only distinct real solutions. In particular, $\mathrm{rk}_{\mathbb{R}}(M) \leq d_{0}+d_{1}$. On the other hand, if $D<d_{0}+d_{1}$, we have several consecutive zero coefficients in $F$. Therefore, by Lemma 4.1.1(2), for any choice of the $G_{i}$ 's we have a non-real solution. In particular, $\mathrm{rk}_{\mathbb{R}} \geq d_{0}+d_{1}$ and we are done.

As a corollary of Theorem 2.3.5 and Proposition 4.1.2, we get the following.
Corollary 4.1.3. Consider $M=x_{0}^{d_{0}} x_{1}^{d_{1}}$, with $0<d_{0} \leq d_{1}$. Then, we have that $\mathrm{rk}_{\mathbb{C}}(M)=\mathrm{rk}_{\mathbb{R}}(M)$ if and only if $d_{0}=1$.

### 4.1.2 More variables

We give an upper bound for the real rank of monomials in more variables.
Theorem 4.1.4. If $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}}$ with $0<d_{0} \leq \ldots \leq d_{n}$, then

$$
\mathrm{rk}_{\mathbb{R}}(M) \leq \frac{1}{2 d_{0}} \prod_{i=0}^{n}\left(d_{i}+d_{0}\right)
$$

Proof. We know that $M^{\perp}=\left(X_{0}^{d_{0}+1}, \ldots, X_{n}^{d_{n}+1}\right)$. Let us consider

$$
F_{i}=X_{0}^{a_{0}+1} G_{i}\left(X_{0}, X_{i}\right)+X_{i}^{a_{i}+1} H_{i}\left(X_{0}, X_{i}\right),
$$

where $\operatorname{deg} G_{i}=a_{i}-1$ and $\operatorname{deg} H_{i}=a_{0}-1$, for every $i=1, \ldots, n$. Each $F_{i}$ is a binary form of degree $d_{0}+d_{i}$ where the monomial $X_{0}^{d_{0}} X_{i}^{d_{i}}$ does not appear. Thus, by Lemma 4.1.1(1), there exists a choice of $G_{i}$ 's and $H_{i}$ such that $F_{i}$ has $d_{0}+d_{i}$ distinct real roots; say $p_{i, j}$, with $j=1, \ldots, a_{0}+a_{i}$. Therefore, the ideal $\left(F_{1}, \ldots, F_{n}\right) \subset M^{\perp}$ is the ideal of the following set of distinct real points:

$$
\mathbb{X}=\left\{\left[1: p_{1, j_{1}}: \ldots: p_{n, j_{n}}\right] \mid 1 \leq j_{i} \leq d_{0}+d_{i}, \text { for } i=1, \ldots, n\right\}
$$

By the Apolarity Lemma 2.1.13, we are done.

However, we have to observe that such upper bound is not always sharp.
Proposition 4.1.5. Let $M=x_{0}^{2} \cdots x_{n}^{2}$. Then $\operatorname{rk}_{\mathbb{R}}(M) \leq\left(3^{n+1}-1\right) / 2$.
Proof. We explicitly give an apolar set of points for $M$ as follows. For any $i=0, \ldots, n$, let us consider the set

$$
\mathbb{X}_{i}=\left\{\left[p_{0}: \ldots: p_{i-1}: 1: p_{i+1}: \ldots: p_{n}\right] \in \mathbb{P}^{n} \mid p_{i} \in\{0, \pm 1\}\right\}
$$

We can easily determine the cardinality of $\mathbb{X}=\bigcup_{i=0}^{n} \mathbb{X}_{i}$. From all $(n+1)$ tuples $\left(p_{0}, \ldots, p_{n}\right)$ with $p_{i}=0, \pm 1$, we need to discard $(0, \ldots, 0)$, since it does not correspond to any point in the projective space. We are double counting, since $\left(p_{0}, \ldots, p_{n}\right)$ and $\left(-p_{0}, \ldots,-p_{n}\right)$ define the same point in the projective space. Thus, $|\mathbb{X}|=\left(3^{n+1}-1\right) / 2$. For each $P \in \mathbb{X}$, let $L_{P}$ denote the corresponding linear form $p_{0} x_{0}+\ldots+p_{n} x_{n}$ and $n(P)$ the number of entries different from zero. For each $i=1, \ldots, n+1$, we set

$$
R_{i}=\sum_{\substack{P \in \mathbb{X} \\ n(P)=i}} L_{P}^{2 n+2} .
$$

By direct computation, we obtain

$$
\frac{(2 n+2)!}{2} x_{0}^{2} \cdots x_{n}^{2}=\sum_{i=1}^{n+1}(-2)^{n+1-i} R_{i}
$$

Thus, $\mathbb{X}$ is apolar to $M$ and this concludes the proof.

Example 4.1.6. We have the following real decomposition of $M=x_{0}^{2} x_{1}^{2}$ :

$$
12 x_{0}^{2} x_{1}^{2}=R_{2}-2 R_{1}=\left(x_{0}+x_{1}\right)^{4}+\left(x_{0}-x_{1}\right)^{4}-2\left(x_{0}^{4}+x_{1}^{4}\right)
$$

For $n=2$, we have $\operatorname{rk}_{\mathbb{C}}\left(x_{0}^{2} x_{1}^{2} x_{2}^{2}\right)=9$ and $\operatorname{rk}_{\mathbb{R}}\left(x_{0}^{2} x_{1}^{2} x_{2}^{2}\right) \leq 13:$

$$
\begin{gathered}
360 x_{0}^{2} x_{1}^{2} x_{2}^{2}=R_{3}-2 R_{2}+4 R_{1}= \\
=\left(x_{0}+x_{1}+x_{2}\right)^{6}+\left(x_{0}+x_{1}-x_{2}\right)^{6}+\left(x_{0}-x_{1}+x_{2}\right)^{6}+\left(x_{0}-x_{1}-x_{2}\right)^{6}+ \\
-2\left[\left(x_{0}+x_{1}\right)^{6}+\left(x_{0}-x_{1}\right)^{6}+\left(x_{0}+x_{2}\right)^{6}+\left(x_{0}-x_{2}\right)^{6}+\left(x_{1}+x_{2}\right)^{6}+\left(x_{1}-x_{2}\right)^{6}\right]+ \\
+4\left(x_{0}^{6}+x_{1}^{6}+x_{2}^{6}\right)
\end{gathered}
$$

Since the complex rank is always a lower bound for the real rank, by Theorem 2.3 .5 and Theorem 4.1.4, we get the following result.

Corollary 4.1.7. If $M=x_{0} x_{1}^{d_{1}} \ldots x_{n}^{d_{n}}$, then $\mathrm{rk}_{\mathbb{C}}(M)=\operatorname{rk}_{\mathbb{R}}(M)$.

### 4.1.3 Complex rank vs. Real rank

We characterize monomials whose real rank coincides with the complex rank by generalizing Corollary 4.1.3 in more variables.

Theorem 4.1.8. Let $M=x_{0}^{d_{0}} \cdots x_{n}^{d_{n}}$ be a degree d monomial with $0<d_{0} \leq$ $\ldots \leq d_{n}$. Then,

$$
\mathrm{rk}_{\mathbb{R}}(M)=\operatorname{rk}_{\mathbb{C}}(M) \text { if and only if } d_{0}=1
$$

We present two proofs. The first one relies on elimination theory. This makes it more explicit, but it works only in three variables. Instead, the second one will prove the theorem in full generality.

Three variables case: via elimination theory. The idea is to use elimination theory in order to reduce our problem to the case of two variables and use again Lemma 4.1.1. Hence, we recall some facts about elimination theory; for more details we refer to [CLO92, Chapter 3].

Consider two homogeneous polynomials in $n+1$ variables written as

$$
F=\sum_{i=0}^{s} A_{i} x_{n}^{i} \quad \text { and } \quad G=\sum_{i=0}^{t} B_{i} x_{n}^{i}, \quad \text { where } A_{i}, B_{i} \in \mathbb{C}\left[x_{0}, \ldots, x_{n-1}\right]
$$

We define the resultant $\operatorname{Res}\left(F, G, x_{n}\right)$ of $F$ ad $G$ with respect to $x_{n}$ as the determinant of the $(s+t) \times(s+t)$ Sylvester matrix

$$
\operatorname{Syl}\left(F, G, x_{n}\right)=\left(\begin{array}{ccccccc}
A_{0} & 0 & 0 & 0 & B_{0} & 0 & 0  \tag{4.1}\\
A_{1} & A_{0} & 0 & 0 & B_{1} & 0 & 0 \\
\vdots & \vdots & \ddots & 0 & \vdots & \ddots & 0 \\
A_{s} & A_{s-1} & & A_{0} & B_{t-1} & & B_{0} \\
0 & A_{s} & & A_{1} & B_{t} & & B_{1} \\
0 & 0 & \ddots & \vdots & 0 & \ddots & \vdots \\
0 & 0 & 0 & A_{s} & 0 & 0 & B_{t}
\end{array}\right) .
$$

Resultants are fundamental tools in elimination theory. As explained in [CLO92, Section 3.6], in the monic case $A_{0}=1$, it is straightforward to show that the roots of resultants lift to solutions of the system of polynomial equations $F=G=0$.

Proposition 4.1.9. Let $\mathbf{c}=\left(c_{0}, \ldots, c_{n-1}\right)$. If $A_{0}=1$, then there exists $c_{n} \in \mathbb{C}$ such that $F\left(\mathbf{c}, c_{n}\right)=G\left(\mathbf{c}, c_{n}\right)=0$ if and only if $H(\mathbf{c})=0$, where $H\left(x_{0}, \ldots, x_{n-1}\right)=\operatorname{Res}\left(F, G, x_{n}\right)$.

We are now ready to prove our theorem in the case of three variables.
Proof of Theorem 4.1.8 in three variables. If $d_{0}=1$, then the real rank is equal to the complex rank by Corollary 4.1.7. If $d_{0} \geq 2$, we give a proof by contradiction. Let $\mathbb{X}$ be an apolar set of real points to $M$. If the cardinality
of $\mathbb{X}$ is $\mathrm{rk}_{\mathbb{C}}(M)$, then its defining ideal $I_{\mathbb{X}} \subset M^{\perp}=\left(X_{0}^{d_{0}+1}, X_{1}^{d_{1}+1}, X_{2}^{d_{2}+1}\right)$ is a complete intersection by Proposition 2.3.6. Hence, $I_{\mathbb{X}}=(F, G)$ where

$$
F=X_{1}^{d_{1}+1}+X_{0}^{d_{0}+1} \cdot F^{\prime} \text { and } G=X_{2}^{d_{2}+1}+X_{0}^{d_{0}+1} \cdot G^{\prime} .
$$

Here $F^{\prime}, G^{\prime} \in \mathbb{C}\left[X_{0}, X_{1}, X_{2}\right]$ are forms of degrees $d_{1}-d_{0}$ and $d_{2}-d_{0}$ respectively. Let us consider the Sylvester matrix of $F$ and $G$ with respect to $X_{2}$ as in (4.1). Thus, the monomial $X_{1}^{d_{1}+1}$ appears in $A_{0}$ and $B_{d_{2}+1}=1$. The monomial $X_{1}^{\left(d_{1}+1\right)\left(d_{2}+1\right)}$ appears in the resultant. On the other hand, since $X_{0}$ has exponent at least $d_{0}+1$ in all the entries of the Sylvester matrix (4.1), we conclude that

$$
H\left(X_{0}, X_{1}\right)=\sum_{i=0}^{\left(d_{1}+1\right)\left(d_{2}+1\right)} c_{i} X_{0}^{i} X_{1}^{\left(d_{1}+1\right)\left(d_{2}+1\right)-i}
$$

where $c_{1}=\ldots=c_{d_{0}}=0$. Therefore, since $d_{0} \geq 2$, by Lemma 4.1.1, $H\left(1, X_{1}\right)$ has some non real root. Thus, there exists $c_{1} \in \mathbb{C} \backslash \mathbb{R}$ such that $H\left(1, c_{1}\right)=0$. By Proposition 4.1.9, there exists $c_{2}$ such that $F\left(1, c_{1}, c_{2}\right)=G\left(1, c_{1}, c_{2}\right)=0$. Hence, $\left[1: c_{1}: c_{2}\right] \in \mathbb{X}$ and it is not real. This is a contradiction.

GENERAL CASE: VIA LINEAR ALGEBRA. We need to recall some fact about trace bilinear form of finite $\mathbb{K}$-algebras; we refer to [PRS93] for details.

Let $A=\mathbb{K}\left[X_{0}, \ldots, X_{n}\right] / I$ be a finite $\mathbb{K}$-algebra. For any element $F \in A$, we define the endomorphism $m_{F} \in \operatorname{End}(A)$ to be the multiplication by $F$. Since $A$ is a finite dimensional $\mathbb{K}$-vector space, we have a trace map $\operatorname{Tr}_{A / \mathbb{K}}: \operatorname{End}(A) \rightarrow \mathbb{K}$, which is the trace of the corresponding matrix. We define a symmetric bilinear form

$$
B(F, G): A \otimes A \rightarrow \mathbb{K},
$$

by

$$
B(F, G)=\operatorname{Tr}\left(m_{F} \cdot m_{G}\right)=\operatorname{Tr}_{A / \mathbb{K}}\left(m_{F \cdot G}\right) .
$$

The following result is featured in [PRS93, Theorem 2.1]; we give an elementary proof for the sake of completeness.

Proposition 4.1.10. Let $A$ be a reduced finite $\mathbb{R}$-algebra of dimension $N$. If $\operatorname{Spec} A$ consists only of $\mathbb{R}$-points, then the bilinear form

$$
B: A \otimes A \rightarrow \mathbb{R},(F, G) \mapsto \operatorname{Tr}_{A / \mathbb{R}}\left(m_{F \cdot G}\right)
$$

is positive definite.
Proof. The $\mathbb{R}$-algebra $A$ is isomorphic to $\mathbb{R} \times \cdots \times \mathbb{R}$ because $A$ is reduced. The representing matrix of the $\mathbb{R}$-linear map $A \rightarrow A$ given by multiplication by $F=\left(F_{1}, \ldots, F_{N}\right) \in A$ is the diagonal matrix with diagonal entries $F_{1}, \ldots, F_{N}$. Thus, we have $B(F, F)=\operatorname{Tr}_{A / \mathbb{R}}\left(m_{F^{2}}\right)=F_{1}^{2}+\ldots+F_{N}^{2} \geq 0$ and $B(F, F)=0$ if and only if $F=0$. Hence $B$ is positive definite.

Now, we can give a result on the number of real solutions of some family of complete intersections, which has a similar flavour of the result given in Lemma 4.1.1, based on Descartes' rule of signs.

Theorem 4.1.11. Let $2 \leq d_{0} \leq \ldots \leq d_{n}$. Let $F_{1}, \ldots, F_{n} \in \mathbb{R}\left[X_{1}, \ldots, X_{n}\right]$ be polynomials of degree at most $d_{i}-d_{0}$, respectively. Then the system of polynomial equations defined by

$$
\begin{align*}
G_{1}=X_{1}^{d_{1}+1}+F_{1} & =0 \\
& \vdots  \tag{4.2}\\
G_{n}=X_{n}^{d_{n}+1}+F_{n} & =0
\end{align*}
$$

does not have $\prod_{i=1}^{n}\left(d_{i}+1\right)$ real distinct solutions.
Proof. We proceed by contradiction. Assume that the number of real distinct solutions is $\prod_{i=1}^{n}\left(a_{i}+1\right)$. Let $I=\left(G_{1}, \ldots, G_{n}\right) \subseteq \mathbb{R}\left[X_{1}, \ldots, X_{n}\right]$ and consider the $\mathbb{R}$-algebra $A=\mathbb{R}\left[x_{1}, \ldots, x_{n}\right] / I$. We define the bilinear form

$$
B: A \otimes A \rightarrow \mathbb{R},(H, K) \mapsto \operatorname{Tr}_{A / \mathbb{R}}\left(m_{H \cdot K}\right)
$$

Since the system has $\prod_{i=1}^{n}\left(a_{i}+1\right)$ real distinct solutions, $B$ is positive definite by Proposition 4.1.10. The residue classes of the monomials $X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}$ with $0 \leq \alpha_{i} \leq d_{i}$ form a basis of $A$ as a vector space over $\mathbb{R}$. We want to show that the representing matrix $M$ of the $\mathbb{R}$-linear map

$$
\varphi: A \rightarrow A, H \mapsto X_{1}^{2} \cdot H
$$

with respect to this basis has only zeros on the diagonal. This would imply $B\left(X_{1}, X_{1}\right)=\operatorname{Tr}_{A / \mathbb{R}}\left(m_{X_{1}^{2}}\right)=0$, which, in turn, would imply that $B$ is not positive definite. For $0 \leq \alpha_{i} \leq d_{i}$, we have

$$
\varphi\left(X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}\right)=X_{1}^{\alpha_{1}+2} \cdot X_{2}^{\alpha_{2}} \cdots X_{n}^{\alpha_{n}}
$$

If $\alpha_{1}+2 \leq d_{1}$, then the column of $M$ corresponding to the basis element $X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}$ has its only nonzero entry at the row corresponding to the basis element $X_{1}^{\alpha_{1}+2} \cdot X_{2}^{\alpha_{2}} \cdots X_{n}^{\alpha_{n}}$. If $\alpha_{1}+2>d_{1}$, then

$$
\varphi\left(X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}\right)=-F_{1} \cdot X_{1}^{\alpha_{1}+1-a_{1}} \cdot X_{2}^{\alpha_{2}} \cdots X_{n}^{\alpha_{n}}
$$

It follows from our assumptions on the degrees of the $F_{i}$ 's, that the element $\varphi\left(X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}\right)$ is in the span of all basis elements corresponding to monomials of degree smaller than $\sum_{i=1}^{n} \alpha_{i}$. In both cases, the corresponding diagonal entry of $M$ is zero and we conclude the proof.

With this result, we can give a proof of Theorem 4.1.8.
Proof of Theorem 4.1.8. If $d_{0}=1$, Corollary 4.1.7 proves the statement. Viceversa, suppose that $\mathrm{rk}_{\mathbb{R}}(M)=\operatorname{rk}_{\mathbb{C}}(M)$ and let $\mathbb{X}$ be a minimal set of real points apolar to $M$. Assume by contradiction that $d_{0} \geq 2$. By Theorem[2.3.6, we know that $\mathbb{X}$ is a complete intersection and we may dehomogenize by $X_{0}=1$. The set $\mathbb{X}$ gives the solutions to a system of polynomial equations of the form (4.2) in Theorem 4.1.11. This is a contradiction and we conclude the proof.

## SECTION 4.2

## MONOMIALS AS SUMS OF $d$-TH POWERS

In this section, we consider the $d$-th Waring decompositions introduced by Fröberg, Ottaviani and Shapiro in [FOS12]. Here, we work again over complex numbers. Given a form $F \in S_{k d}$, we define its $d$-th Waring rank as

$$
\mathrm{rk}_{d}(F):=\min \left\{s \mid \exists G_{1}, \ldots, G_{s} \in S_{k}, F=G_{1}^{d}+\ldots+G_{s}^{d}\right\} .
$$

In Section 3.3, we analyzed the $d$-th Waring rank of generic forms. Here, we explain the recent results obtained in [CO15] where we considered $d$-th Waring decompositions of monomials. As far as we know, this is the only work in the literature about these type of decompositions for specific forms.

### 4.2.1 BASIC TOOLS

We introduce the elementary tools that we are going to use.
Remark 4.2.1. Consider a monomial $M$ of degree $k d$ in the variables $\left\{x_{0}, \ldots, x_{n}\right\}$. We say that a monomial $M^{\prime}$ of degree $k^{\prime} d$ in the variables $\left\{X_{0}, \ldots, X_{m}\right\}$ is a grouping of $M$ if there exists a positive integer $l$ such that $k=l k^{\prime}$ and $M$ can be obtained from $M^{\prime}$ by substituting each variable $X_{i}$ with a monomial of degree $l$ in the $x$ 's, i.e. $X_{i}=N_{i}\left(x_{0}, \ldots, x_{n}\right)$ for each $i=1, \ldots, m$ with $\operatorname{deg}\left(N_{i}\right)=l$. Then,

$$
\mathrm{rk}_{d}\left(M^{\prime}\right) \geq \mathrm{rk}_{d}(M)
$$

Indeed, given a decomposition of $M^{\prime}$ as sum of $d$-th powers

$$
M^{\prime}=\sum_{i=1}^{r} G_{i}\left(X_{0}, \ldots, X_{n}\right)^{d}, \text { with } \operatorname{deg}\left(F_{i}\right)=d^{\prime}
$$

we can write a decomposition for $M$ by using the substitution given above, i.e.,

$$
M=\sum_{i=1}^{r} G_{i}\left(N_{0}\left(x_{0}, \ldots, x_{n}\right), \ldots, N_{m}\left(x_{0}, \ldots, x_{n}\right)\right)^{d}
$$

Remark 4.2.2. Consider a monomial $M$ of degree $k d$ in the variables $\left\{x_{0}, \ldots, x_{n}\right\}$. We say that a monomial $M^{\prime}$ of the same degree is a specialization of $M$ if $M^{\prime}$ can be found from $M$ after a certain number of identifications of the type $x_{i}=x_{j}$. Then,

$$
\operatorname{rk}_{d}(M) \geq \operatorname{rk}_{d}\left(M^{\prime}\right)
$$

Indeed, given a decomposition of $M$ as sum of $r d$-th powers, we can write a decomposition for $M^{\prime}$ with the same number of summands applying the identifications between variables to each addend.

Remark 4.2.3. Consider a monomial $M$ of degree $k d$ and $N$ a monomial of degree $k^{\prime}$. We can look at the monomial $M^{\prime}=M N^{d}$. Clearly the degree of $M^{\prime}$ is also divisible by $d$; again, it makes sense to compare the $d$-th Waring rank of $M$ and $M^{\prime}$. The relation is

$$
\mathrm{rk}_{d}(M) \geq \mathrm{rk}_{d}\left(M^{\prime}\right)
$$

Indeed, given a decomposition as $M=\sum_{i=1}^{r} F_{i}^{d}$, with $F_{i} \in S_{k}$, we can find a decomposition for $M^{\prime}$ with the same number of summands, i.e.,

$$
M^{\prime}=M N^{d}=\sum_{i=1}^{r}\left(F_{i} N\right)^{d}
$$

The inequality on the $d$-th Waring rank in Remark 4.2.3 can be strict.
Example 4.2.4. Consider $d=3$ and the monomials

$$
M=x_{1} x_{2} x_{3} \quad \text { and } \quad M^{\prime}=\left(x_{0}^{2}\right)^{3} \cdot M=x_{0}^{6} x_{1} x_{2} x_{3}
$$

By Theorem 2.3.5, we know that $\mathrm{rk}_{3}(M)=\operatorname{rk}_{3}\left(x_{1} x_{2} x_{3}\right)=4$. On the other hand, we can consider the grouping $M^{\prime}=\left(x_{0}^{3}\right)^{2}\left(x_{1} x_{2} x_{3}\right)=X_{0}^{2} X_{1}$. Thus, by Remark 4.2.1 and Theorem 2.3.5, we have $\operatorname{rk}_{3}\left(M^{\prime}\right) \leq \operatorname{rk}_{3}\left(X_{0}^{2} X_{1}\right)=3$.

As a straightforward application of these remarks, we get the following.
Lemma 4.2.5. Given a monomial $M=x_{0}^{d_{0}} x_{1}^{d_{1}} \cdots x_{n}^{d_{n}}$ of degree $k d$, then

$$
\operatorname{rk}_{d}(M) \leq \operatorname{rk}_{d}([M])
$$

with $[M]:=x_{0}^{\left[a_{0}\right]_{d}} \cdots x_{n}^{\left[a_{n}\right]_{d}}$, where $\left[d_{i}\right]_{d}$ is the smallest positive representative of the equivalence class of $d_{i}$ modulo $d$, for any $i$.
Proof. We can write $d_{i}=d \alpha_{i}+\left[d_{i}\right]_{d}$ for each $i=0, \ldots, n$. Hence, we get that $M=N^{d}[M]$, where $N=x_{0}^{\alpha_{0}} \cdots x_{n}^{\alpha_{n}}$. Obviously, $d$ divides $\operatorname{deg}([M])$ and, by Remark 4.2.3, we are done.

Remark 4.2.6. With the above notation, we have that $\left[d_{0}\right]_{d}+\cdots+\left[d_{n}\right]_{d}$ is a multiple of $d$ and also, it has to be at most $(d-1)(n+1)=d n-n+d-1$. This observation will be useful to restrict the cases to be considered when we will seek for upper bounds for the $d$-th rank.

### 4.2.2 GENERAL CASE

First, we get some general result on the $d$-th Waring rank of monomials.
Remark 4.2.7. Using the idea of grouping variables, we get a complete description of the $d=2$ case. Given a monomial $M$ of degree $2 k$ which is not a square, we have $\mathrm{rk}_{d}(M)=2$. Indeed,

$$
M=X Y=\left[\frac{1}{2}(X+Y)\right]^{2}+\left[\frac{i}{2}(X-Y)\right]^{2}
$$

where $X$ and $Y$ are two monomials of degree $k$.

In the following result, we give a general upper bound on the $d$-th rank of monomials. Moreover, we see that the case $d=2$ is the only one in which the $d$-th Waring rank of a monomial can be equal to two.

Theorem 4.2.8. If $M$ is a monomial of degree $k d$, then $\operatorname{rk}_{d}(M) \leq 2^{d-1}$. Moreover, $\mathrm{rk}_{d}(M)=2$ if and only if $d=2$ and $M$ is not a square.

Proof. Any monomial $M \in S_{k d}$ is a specialization of $x_{1} \cdots x_{k d}$. Now, we can consider the grouping given by

$$
X_{1}=x_{1} \cdots x_{d}, \ldots, X_{k}=x_{(k-1) d+1} \cdots x_{k d}
$$

Thus, by Remark 4.2.2, Remark 4.2.1 and Theorem 2.3.5, we get the bound

$$
\mathrm{rk}_{d}(M) \leq \mathrm{rk}_{d}\left(X_{1} \cdots X_{d}\right)=2^{d-1}
$$

Now, suppose that $d>2$ and $\operatorname{rk}_{d}(M)=2$. Hence, we can write $M=A^{d}-B^{d}$, for suitable $A, B \in S_{k}$. Factoring we get

$$
M=\prod_{i=1}^{d}\left(A-\xi_{i} B\right)
$$

where the $\xi_{i}$ are the $d$-th roots of 1 . In particular, the forms $A-\xi_{i} B$ are monomials. If $M$ is not a $d$-th power, from $A-\xi_{1} B$ and $A-\xi_{2} B$, we get that $A$ and $B$ are not trivial binomials. Hence, a contradiction since $A-\xi_{3} B$ cannot be a monomial. Remark 4.2.7 concludes the proof.

Remark 4.2.9. For $n \geq 2$ and $d$ small enough, we may observe that our result gives an upper bound for the $d$-th Waring rank of monomials of degree $k d$ smaller than the general result of [FOS12]. Indeed, if we look for which $d$ the inequality $2^{d-1} \leq d^{n}$ holds, we get: for $n=2, d \leq 6$ and, for $n=3$, $d \leq 9$. Increasing $n$, we go even further, e.g. for $n=10$, we get $d \leq 59$.

### 4.2.3 TWO VARIABLES

For binary monomials ( $n=1$ ), we can improve the upper bound in Theorem 4.2.8.

Proposition 4.2.10. Let $M=x_{0}^{d_{0}} x_{1}^{d_{1}}$ be a binary monomial of degree $k d$. Then,

$$
\operatorname{rk}_{d}(M) \leq \max \left\{\left[d_{0}\right]_{d},\left[d_{1}\right]_{d}\right\}+1
$$

Proof. By Lemma 4.2.5, we know that $\mathrm{rk}_{d}(M) \leq \operatorname{rk}_{d}([M])$; hence, we consider the monomial $[M]=x_{0}^{\left[d_{0}\right]_{d}} x_{1}^{\left[d_{1}\right]_{d}}$. Now, we observe that, as we said in Remark 4.2.6, the degree of $[M]$ is a multiple of $k$ and also $\leq 2 d-2$; hence, $\operatorname{deg}([M])$ is either equal to 0 , i.e. $[M]=1$, or $d$. In the first case, it means that $M$ was a pure $d$-th power, and the $d$-th Waring rank is

$$
\operatorname{rk}_{d}(M)=1=\max \left\{\left[d_{0}\right]_{d},\left[d_{1}\right]_{d}\right\}+1 .
$$

If $\operatorname{deg}([M])=d$, we can apply Theorem 2.3 .5 to $[M]$ and we get

$$
\operatorname{rk}_{d}(M) \leq \operatorname{rk}_{d}([M])=\max \left\{\left[d_{0}\right]_{d},\left[d_{1}\right]_{d}\right\}+1
$$

Remark 4.2.11. As a consequence of Proposition 4.2.10, for binary monomials we have that $\operatorname{rk}_{d}(M) \leq d$. We observe that this upperbound is sharp by considering $\mathrm{rk}_{d}\left(x_{0} x_{1}^{d-1}\right)=d$.

We may observe that, in view of Theorem 1.2 .14 , with the same proof of Proposition 4.2.10, we can give the following upper bound on the $d$-th Waring rank of binary monomials over the real numbers.

Proposition 4.2.12. Let $M=x_{0}^{d_{0}} x_{1}^{d_{1}}$ be a binary monomial of degree $k d$. Then,

$$
\operatorname{rk}_{d, \mathbb{R}}(M) \leq\left[d_{0}\right]_{d}+\left[d_{1}\right]_{d} .
$$

As a corollary of Theorem 4.2.8, we solve the $d=3$ case.
Corollary 4.2.13. Given a binary monomial $M$ of degree $3 k$, we have

1. $\operatorname{rk}_{3}(M)=1$ if $M$ is a pure cube;
2. $\mathrm{rk}_{3}(M)=3$ otherwise.

Proof. By Remark 4.2.11, we have that the 3-rd Waring rank can be at most three; on the other hand, by Theorem 4.2.8, we have that, if $M$ is not a pure cube, then the rank has to be at least three.

For $d \geq 4$, we can get only partial results by performing a case by case analysis as the following one for the $d=4$ case.

Remark 4.2.14. We consider the $d=4$ case for binary monomials. From our previous results, we can only have 4 -th Waring ranks equal to 1,3 or 4 .

Let $M=x_{0}^{d_{0}} x_{1}^{d_{1}}$ be a binary monomial of degree $4 k$. By Remark 4.2.5, we can consider the monomial $[M]$ obtained by considering the exponents modulo 4 . Since $[M]$ has degree divisible by 4 and less or equal to 6 , we have to consider only three cases with respect the remainders of the exponents modulo 4, i.e.

$$
\left(\left[d_{0}\right]_{4},\left[d_{1}\right]_{4}\right) \in\{(0,0),(1,3),(2,2)\}
$$

The $(0,0)$ case corresponds to pure fourth powers, i.e. monomials with 4 -th Waring rank equal to 1 . In the $(2,2)$ case we have

$$
\operatorname{rk}_{4}(M) \leq \operatorname{rk}_{4}\left(x_{0}^{2} x_{1}^{2}\right)=3
$$

since the 4 -th Waring rank cannot be two, we have that binary monomials in the $(2,2)$ class have $4^{\text {th }}$-rank equal to three. Unfortunately, we can not conclude in the same way the $(1,3)$ case. Since $\mathrm{rk}_{4}\left(x_{0} x_{1}^{3}\right)=4$, a monomial in the $(1,3)$ class may have rank equal to 4 . For example, by using the Computer Algebra software CoCoA , we have computed $\operatorname{rk}_{4}\left(x_{0} x_{1}^{7}\right)=\operatorname{rk}_{4}\left(x_{0}^{3} x_{1}^{5}\right)=$ 4.

### 4.2.4 $d=3$ CASE IN THREE AND MORE VARIABLES

In this section we consider the case $d=3$ with more than two variables. By Theorem 4.2.8, 3-rd Waring ranks can only be equal to 1,3 or 4 . This lack of space allows us to give a complete solution for ternary monomials.

Proposition 4.2.15. Given a monomial $M=x_{0}^{d_{0}} x_{1}^{d_{1}} x_{2}^{d_{2}}$ of degree $3 k$, we have that

1. $\operatorname{rk}_{3}(M)=1$ if $M$ is a pure cube;
2. $\operatorname{rk}_{3}(M)=4$ if $M=x_{0} x_{1} x_{2}$;
3. $\mathrm{rk}_{3}(M)=3$ otherwise.

Proof. By Lemma 4.2.5, we consider the monomials [ $M$ ] with degree divisible by 3 and less or equal than 6 . Hence, we have only four possible cases, i.e.

$$
\left(\left[d_{0}\right]_{3},\left[d_{1}\right]_{3},\left[d_{2}\right]_{3}\right) \in\{(0,0,0),(0,1,2),(1,1,1),(2,2,2)\}
$$

The $(0,0,0)$ case corresponds to pure cubes and then to monomials with 3 -rd Waring rank equal to one. In the $(0,1,2)$ case we have, by Theorem 2.3.5,

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{1} x_{2}^{2}\right)=3 ;
$$

since, by Theorem 4.2.8, the rank of monomials which are not pure cubes is at least 3 , we get the equality. Similarly, we conclude that we have rank three for monomials in the $(2,2,2)$ class. Indeed, by using grouping and Theorem 2.3.5, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{0}^{2} x_{1}^{2} x_{2}^{2}\right) \leq \operatorname{rk}_{3}\left(X Y^{2}\right)=3
$$

Now, we just need to consider the $(1,1,1)$ class. By Theorem 2.3.5, we have $\operatorname{rk}_{3}\left(x_{0} x_{1} x_{2}\right)=4$. Hence, we can consider monomials $M=x_{0}^{d_{0}} x_{1}^{d_{1}} x_{2}^{d_{2}}$ with $d_{0}=3 \alpha+1, d_{1}=3 \beta+1, d_{2}=3 \gamma+1$ and where at least one of $\alpha, \beta, \gamma$ is at least one, say $\alpha>0$. By Remark 4.2.3, we have

$$
\operatorname{rk}_{3}(M)=\mathrm{rk}_{3}\left(\left(x_{0}^{\alpha-1} x_{1}^{\beta} x_{2}^{\gamma}\right)^{3} \cdot x_{0}^{4} x_{1} x_{2}\right) \leq \operatorname{rk}_{3}\left(x_{0}^{4} x_{1} x_{2}\right)
$$

Now, to conclude the proof, it is enough to show that $\mathrm{rk}_{3}\left(x_{0}^{4} x_{1} x_{2}\right)=3$. In fact, we can write

$$
x_{0}^{4} x_{1} x_{2}=\left[\sqrt{\frac{1}{6}} x_{0}^{2}+x_{1} x_{2}\right]^{3}+\left[-\frac{1}{6} x_{0}^{2}+x_{1} x_{2}\right]^{3}+\left[\sqrt[3]{-2} x_{1} x_{2}\right]^{3}
$$

and therefore, we are done.
With similar analysis, we produce partial results in four and five variables.

Remark 4.2.16. Given a monomial $M=x_{0}^{d_{0}} x_{1}^{d_{1}} x_{2}^{d_{2}} x_{3}^{d_{3}}$ with degree $3 d k$, we consider the monomial $[M]$ which has degree divisible by 4 and less or equal than 8 . Hence, we need to consider only the following classes with respect to the remainders of the exponents modulo 3
$\left(\left[a_{0}\right]_{3},\left[a_{1}\right]_{3},\left[a_{2}\right]_{3},\left[a_{3}\right]_{3}\right) \in\{(0,0,0,0),(0,0,1,2),(0,1,1,1),(0,2,2,2),(1,1,2,2)\}$.
The $(0,0,0,0)$ case corresponds to pure cubes and we have rank equal to one. Now, we use again Lemma 4.2.5, grouping and Theorem 2.3.5.

In the $(0,0,1,2)$ class, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{2} x_{3}^{2}\right)=3 ;
$$

in the $(0,2,2,2)$ class, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{1}^{2} x_{2}^{2} x_{3}^{2}\right) \leq \operatorname{rk}_{3}\left(X Y^{2}\right)=3
$$

in the $(1,1,2,2)$ class, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{0} x_{1} x_{2}^{2} x_{3}^{2}\right) \leq \operatorname{rk}_{3}\left(\left(x_{0} x_{1}\right)\left(x_{2} x_{3}\right)^{2}\right)=\operatorname{rk}_{3}\left(X Y^{2}\right)=3
$$

Again, since the 3-rd Waring rank has to be at least three by Theorem 4.2.8, we conclude that in these classes the 3 -rd Waring rank is equal to three. The $(0,1,1,1)$ class is a unique missing case because the upper bound with $\operatorname{rk}_{3}\left(x_{1} x_{2} x_{3}\right)=4$ is not helpful to give a complete classification.
Remark 4.2.17. Given a monomial $M=x_{0}^{d_{0}} x_{1}^{d_{1}} x_{2}^{d_{2}} x_{3}^{d_{3}} x_{4}^{d_{4}}$ with degree $3 k$, we consider the monomial $[M]$ which has degree divisible by 4 and less or equal to 10 . Hence, we need to consider only the following classes with respect to the remainders of the exponents of $M$ modulo 3 .

The $(0,0,0,0,0)$ class corresponds to pure cubes and 3-rd Waring rank equal to one. By using Lemma 4.2.5, grouping, previous results in three or four variables and Theorem 2.3.5, we get the following results.

In the $(0,0,0,1,2)$ case, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{3} x_{4}^{2}\right)=3
$$

in the $(0,0,2,2,2)$ case, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{1}^{2} x_{2}^{2} x_{3}^{2}\right)=3
$$

in the $(0,1,1,2,2)$ case, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{0} x_{1} x_{2}^{2} x_{3}^{2}\right)=3
$$

in the $(1,2,2,2,2)$ case, we have

$$
\operatorname{rk}_{3}(M) \leq \operatorname{rk}_{3}\left(x_{0} x_{1}^{2} x_{2}^{2} x_{3}^{2} x_{4}^{2}\right)=\operatorname{rk}_{3}\left(\left(x_{0} x_{1}^{2}\right) \cdot\left(x_{2} x_{3} x_{4}\right)^{2}\right) \leq \operatorname{rk}_{3}\left(X Y^{2}\right)=3
$$

Hence, by Theorem 4.2.8, in these cases we have 3-rd Waring rank is equal to three. There are only two missing cases: the $(0,0,1,1,1)$ case, which can be reduced to the missing case in four variables, and the ( $1,1,1,1,2$ ) case.

## Appendix A

## ALGEBRAIC INVARIANTS

Let $S=\mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ be the polynomial ring in $n+1$ variables with complex coefficients. We consider $S$ equipped with the standard grading, i.e., we present $S=\bigoplus_{i \in \mathbb{N}} S_{i}$, where $S_{i}$ denotes the $\mathcal{C}$-vector space of homogeneous polynomials of degree $i$. Any homogeneous ideal $I \subset S$ inherits such grading and the associated quotient algebra $R=S / I$ is graded as $R=\bigoplus_{i \in \mathbb{N}} R_{i}$ where $R_{i}=S_{i} / I_{i}$.

Given a homogeneous ideal $I$, we introduce the corresponding projective algebraic set defined as the zero locus of the forms in $I$. In order to study such an algebraic set, we consider invariants associated to $I$.

## SECtion A. 1

## Hilbert functions

Definition A.1.1. Given a graded $S$-module $M=\bigoplus_{i \in \mathbb{N}} M_{i}$, we define its Hilbert function as

$$
\operatorname{HF}_{M}(i):=\operatorname{dim}_{\mathcal{C}} M_{i}, \quad \text { for all } i \in \mathbb{N} .
$$

Using this function, we also define the Hilbert series of $M$ as

$$
\operatorname{HS}_{M}(t):=\sum_{i \in \mathbb{N}} \operatorname{HF}(M ; i) t^{i} \in \mathbb{Z}[[t]] .
$$

Example A.1.2. Let $I=\left(x_{0}^{3}, x_{1}^{2}, x_{2}^{2}\right) \subset \mathbb{C}\left[x_{0}, x_{1}, x_{2}\right]$. Then, the quotient ring $R=\mathbb{C}\left[x_{0}, x_{1}, x_{2}\right] / I$ is generated as a $\mathbb{C}$-vector space by

$$
\begin{array}{cccc}
\operatorname{deg}=0: & 1, & & \\
\operatorname{deg}=1: & x_{0}, & x_{1}, & x_{2}, \\
\operatorname{deg}=2: & x_{0}^{2}, & x_{0} x_{1}, & x_{0} x_{2}, \quad x_{1} x_{2} \\
\operatorname{deg}=3: 3: & x_{0}^{2} x_{1}, & x_{0}^{2} x_{2} & x_{0} x_{1} x_{2},
\end{array} \quad \begin{aligned}
& \operatorname{HF}_{R}(0)=1 ; \\
& \operatorname{HF}_{R}(1)=3 ; \\
& \operatorname{HF}_{R}(2)=4 ; \\
& \operatorname{HF}_{R}(3)=3 ; \\
& H_{R}(i)=0, \quad \text { for all } i \geq 4 .
\end{aligned}
$$

and the Hilbert series equals $\operatorname{HS}_{R}(t)=1+3 t+4 t^{2}+3 t^{3}$.

Before going further, let's recall that a grading can be defined over any monoid, for example over all integers. In particular, we can consider the standard grading as defined over $\mathbb{Z}$ by setting $S_{i}=0$, for any $i<0$. In this way, it makes sense to define the shifting of an $S$-module.

Definition A.1.3. Let $M=\bigoplus_{i \in \mathbb{Z}} M_{i}$ be a graded $S$-module. For any integer $j \in \mathbb{Z}$, we define the shifted module $M(j)$ as the $S$-module $M$ with the graded structure given by

$$
[M(j)]_{i}:=M_{i-j}, \text { for all } i \in \mathbb{Z}
$$

In the category of graded modules, the maps are morphisms of $S$-modules preserving the grading. In other words, given two graded $S$-modules $M$ and $N$, we say that a morphism of $S$-modules $f: M \longrightarrow N$ is graded of degree $j$ if,

$$
f\left(M_{i}\right) \subset N_{i+j}, \quad \text { for all } i \in \mathbb{Z}
$$

One of the reasons for introducing shifting of modules is that, among the graded morphisms, the ones which we like most are of course the ones of degree 0 . For example, given a short exact sequence of graded $S$-modules with degree 0 maps

$$
0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0
$$

we obtain induced short exact sequence of $\mathcal{C}$-vector spaces for each degree $i \in \mathbb{Z}$. Therefore,

$$
\operatorname{HF}_{N}(i)=\operatorname{HF}_{M}(i)+\operatorname{HF}_{P}(i), \quad \text { for any } i \in \mathbb{Z}
$$

Example A.1.4. (Complete intersections) Given an $S$-module $M$, we say that a non-zero element $f \in S$ is $M$-regular if $f$ is a non-zero divisor on $M$ (for short NZD), i.e. if $f g=0$ for $g \in M$ then $g=0$. In general, a sequence $\left(f_{1}, \ldots, f_{g}\right)$ of elements in $S$ is called an $M$-regular sequence, or simply an $M$-sequence, if the following conditions are satisfied:

1. $f_{i}$ is $M /\left(f_{1}, \ldots, f_{i-1}\right) M$-regular element for all $i=1, \ldots, g$;
2. $M /\left(f_{1}, \ldots, f_{g}\right) M$ is non zero.

A quotient ring $R=S / I$ is called a complete intersection if $I$ is a homogeneous ideal generated by an $S$-sequence $\left(f_{1}, \ldots, f_{g}\right)$. Under these assumptions, the computation of the Hilbert series of $R$ is an easy exercise. We proceed by induction on the number of generators. Assuming $g=1$ and $\operatorname{deg} f_{1}=d_{1}$, we get the following exact sequence with degree 0 maps:

$$
0 \longrightarrow S\left(-d_{1}\right) \xrightarrow{\cdot f_{1}} S \longrightarrow S / I \longrightarrow 0
$$

Hence, we get that $\operatorname{HF}(S / I ; i)=\operatorname{HF}(S ; i)-\operatorname{HF}\left(S ; i+d_{1}\right)$, for all $i \in \mathbb{Z}$. Thus, looking at their Hilbert series, we get

$$
\mathrm{HS}_{S / I}(t)=\left(1-t^{d_{1}}\right) \operatorname{HS}_{S}(t)=\left(1-t^{d_{1}}\right) \sum_{i \in \mathbb{N}}\binom{n+i}{n} t^{i}=\frac{1-t^{d_{1}}}{(1-t)^{n+1}}
$$

Now, let $I=\left(f_{1}, \ldots, f_{g}\right)$ with $g>1$. By induction, from the exact sequence of degree 0 maps

$$
0 \longrightarrow S /\left(f_{1}, \ldots, f_{g-1}\right)\left(-d_{g}\right) \xrightarrow{\cdot f_{g}} S /\left(f_{1}, \ldots, f_{g-1}\right) \longrightarrow S / I \longrightarrow 0
$$

Hence, similarly to the case $g=1$, we get

$$
\begin{equation*}
\operatorname{HS}_{S / I}(t)=\left(1-t^{d_{g}}\right) \operatorname{HS}_{S /\left(f_{1}, \ldots, f_{g-1}\right)}(t)=\frac{\prod_{i=1}^{g}\left(1-t^{d_{i}}\right)}{(1-t)^{n+1}} \tag{A.1}
\end{equation*}
$$

## SECTION A. 2

## Resolutions and Betti numbers

Given a finitely generated $S$-module $M$, there exists a finitely generated free $S$-module $F_{0}$ and a surjective homomorphism $\varphi_{0}$ mapping the generators of $F_{0}$ to a set of generators of $M$. The kernel of such homomorphism is again a finitely generated $S$-module and we have an exact sequence of $S$-modules

$$
0 \longrightarrow \operatorname{ker} \varphi_{0} \longrightarrow F_{0} \xrightarrow{\varphi_{0}} M \longrightarrow 0
$$

Now, we can find another surjective homomorphism $\varphi_{1}$ mapping the generators of a free finitely generated $S$-module $F_{1}$ to the generators of $\operatorname{ker} \varphi_{0}$, i.e., the relations among the generators of $F_{0}$. Such relations are called the syzygies of the module. We continue this procedure by considering the kernel of $\varphi_{1}$ and so on. Hence, we get a free resolution of our $S$-module $M$, i.e a long exact sequence

$$
\mathbb{F}_{\bullet}: \ldots \longrightarrow F_{2} \longrightarrow F_{1} \longrightarrow F_{0} \longrightarrow M \longrightarrow 0
$$

By Hilbert's syzygy theorem, we also know that, for any finitely generated module $M$ over a polynomial ring $S$, there is a finite free resolution, i.e., there exist finitely generated free $S$-modules $F_{1}, \ldots, F_{n}$ such that

$$
\mathbb{F}_{\bullet}: 0 \longrightarrow F_{n} \longrightarrow \ldots \longrightarrow F_{1} \longrightarrow F_{0} \longrightarrow M \longrightarrow 0
$$

Of course, if we want to optimize this construction, we can start by considering a minimal set of generators of our module $M$ and then, at each step, a minimal set of generators for the module of syzygies. In that way, we get a minimal free resolution $\mathbb{F}$ • for our module $M$. The length of the minimal free resolution $n$ is called the projective dimension of $M$ and denoted by $\operatorname{pdim}(M)$. Moreover, if we also have a graded structure on $M$, we can additionally assume to have graded maps of degree 0 in such a minimal resolution. In this way, we get a set of invariants for the module $M$. Indeed, considering a minimal free resolution with degree 0 maps, we can write, for all $i=1, \ldots, n$,

$$
F_{i}=\bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i, j}}
$$

Integers $\beta_{i, j}$ are called the graded Betti numbers of $M$. Such numbers can be seen as homological quantities since one can prove that

$$
\beta_{i, j}=\operatorname{dim}\left[\operatorname{Tor}_{i}^{S}(M, \mathbb{C})\right]_{j}
$$

this fact shows that such numbers depends only on the module $M$ and not on the minimal resolution.

Often, the Betti numbers are represented as Betti table as

$\beta(M)=$|  | $\ldots$ | i | $\ldots$ |
| :---: | :---: | :---: | :---: |
| $\vdots$ |  | $\vdots$ |  |
| $j$ | $\ldots$ | $\beta_{i, i+j}$ | $\ldots$ |
| $\vdots$ |  | $\vdots$ |  |

Remark A.2.1. The entry $(i, j)$ of the table is the Betti number $\beta_{i, i+j}$. Moreover, the minimality of the resolution gives us another piece of useful information about the Betti numbers and the shape of the Betti table, namely

$$
\min \left\{j \mid \beta_{i+1, j} \neq 0\right\}>\min \left\{j \mid \beta_{i, j} \neq 0\right\} .
$$

Moreover, because of the relation between the exact sequences of $S$-modules with degree 0 maps and the Hilbert function of the modules in the sequence, the Betti numbers allow us to write the Hilbert series of a module $M$ in a more explicit way. Hence, given an $S$-module $M$ and its Betti numbers $\left(\beta_{i, j}\right)$, we obtain

$$
\operatorname{HS}_{M}(t)=\frac{\sum_{i=0}^{\operatorname{pdim}(M)} \sum_{j \in \mathbb{Z}}(-1)^{i} \beta_{i, j} t^{j}}{(1-t)^{n+1}}
$$

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[^0]:    ${ }^{2}$ Given any $\mathbb{K}$-vector space $V$ of dimension $N+1$, we define the associated $N$-dimensional projective space, denoted $\mathbb{P}(V)$, as the space of lines in $V$; more precisely, as the quotient of $V \backslash 0$ by the equivalence relation given by $v \simeq w$ if and only if there exists $\lambda \in \mathbb{K} \backslash 0$ such that $v=\lambda w$. A vector basis for $V$, is called a set of coordinates of $V$ and we denote it by $\mathbb{P}^{N}$ when there is no confusion about the coordinates we are using. In the case of the space of homogeneous forms of degree $d$, we consider the monomial basis.
    ${ }^{3}$ We consider $\mathbb{P}^{N}$ equipped with the Zariski topology, namely the closed sets are defined as the zero loci of homogeneous polynomials with respect to the coordinates used. Closed sets are also called varieties and when non-empty, their complements are always dense.

[^1]:    ${ }^{4}$ In the original paper (FOS12] the roles of $d$ and $k$ are inverted. We use this notation here to conform with the notation used in the classical Waring problem.

[^2]:    ${ }^{5}$ In the original paper, we use a different terminology. In this thesis, we want to use this notation because $\mu$-powers of linear forms will appear again from a different perspective.

[^3]:    ${ }^{6}$ We define a variety in the affine space $\mathbb{A}_{\mathbb{C}}^{n+1}$ as the zero locus of polynomials in a radical ideal $I \subset \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$. If the ideal is homogeneous, we view the zero locus up to scalar multiplication as being inside the projective space $\mathbb{P}^{n}$ and we call it a projective variety. We denote it by $Z(I)$. Given a variety, we call the quotient $\mathbb{C}\left[x_{0}, \ldots, x_{n}\right] / I$ by an ideal (homogeneous ideal, resp.) the coordinate ring (homogeneous coordinate ring, resp.).
    ${ }^{7}$ The Zariski topology of the affine space (projective space, resp.) is defined as the topology where the closed set are varieties (projective varieties, resp.).

[^4]:    ${ }^{8}$ Given a $\mathbb{K}$-vector space $V$, we can construct the tensor algebra associated to $V$ as the graded algebra $T(V)=\bigoplus_{i \in \mathbb{N}} V^{\otimes i}$. Then, we can define the associated symmetric algebra, $\operatorname{Sym}(V)=\bigoplus_{i \in \mathbb{N}} \operatorname{Sym}^{i}(V)$ as the quotient of the tensor algebra by the symmetrizing ideal generated by all elements of type $v \otimes w-$ $w \otimes v, v, w \in V$. If $V$ is $N$-dimensional and we fix a basis, we can see the symmetric algebra as the polynomial ring on $N$ variables with coefficients in $\mathbb{K}$.

[^5]:    ${ }^{1}$ Over fields of positive characteristic, we can develop apolarity thory, but we need to be more careful. We should consider divided powers rings instead of polynomial rings and we should consider the action given by contraction; see [IK99].

[^6]:    ${ }^{2}$ Given vector spaces $M, N, L$ over a field $\mathbb{K}$, a bilinear pairing is a map $f: M \times N \longrightarrow L$ such that,

    1. $f(k m, n)=f(m, k n)=k f(m, n)$, for any $m \in M, n \in N, k \in \mathbb{K}$;
    2. $f\left(m_{1}+m_{2}, n\right)=f\left(m_{1}, n\right)+f\left(m_{2}, n\right)$, for any $m_{1}, m_{2} \in M, n \in N$;
    3. $f\left(m, n_{1}+n_{2}\right)=f\left(m, n_{1}\right)+f\left(m, n_{2}\right), \quad$ for any $m \in M, n_{1}, n_{2} \in N$.
[^7]:    ${ }^{4}$ A finite $S$-module $M$ is Cohen-Macaulay if $\operatorname{depth} M=\operatorname{dim} M$, where the depth is defined as the length of the longest regular sequence in $M$. Then, it directly follows from this definition that any reduced $S$-module of dimension 1 is Cohen-Macaulay.

[^8]:    ${ }^{5}$ Basic property of Hilbert series; see Example A.1.4

[^9]:    ${ }^{6}$ In general, complete intersections are Gorenstein, but the opposite implication is not always true. There are counterexamples also of ideals of 0 -dimensional schemes.

[^10]:    ${ }^{7}$ Given a projective space $\mathbb{P}^{n}=\mathbb{P}(V)$ associated to a vector space $V$ of dimension $n+1$, we can consider the dual projective space as $\check{\mathbb{P}}^{n}$ as the projective space $\mathbb{P}\left(V^{*}\right)$ associated to the dual vector space. This can be identified with the space of hyperplanes of $\mathbb{P}^{n}$.

[^11]:    ${ }^{8}$ More recent results have brought the number of operations required closer to $n^{2}$.

[^12]:    ${ }^{9}$ The dimension of a scheme associated to an ideal $I \subset \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]$ is given by $n-\operatorname{ht}(I)$, where the height of an ideal $I$ is defined as $\operatorname{ht}(I):=\max \left\{s \mid \exists \wp_{1}, \ldots, \wp_{s}\right.$ prime ideals s.t. $\left.I=\wp_{1} \subset \ldots \subset \wp_{s} \subset \mathbb{C}\left[x_{0}, \ldots, x_{n}\right]\right\}$.

[^13]:    ${ }^{10}$ For any prime ideal $\wp \subset S$ in a Noetherian ring, we define $\wp^{(d)}$ as $\wp^{d} S_{\wp} \cap S$, namely the contraction of the extension of the $d$-th power power via the localization map $S \rightarrow S_{\wp}$. Given an ideal $I \subset S$, we define the $d$-th symbolic power $I^{(d)}$ as the contraction of the extension via the localization map $S \rightarrow S_{\Lambda}$, where $\Lambda=S \backslash \bigcup_{\wp \in \operatorname{Ass}(I)} \wp$. In case of a radical ideal with minimal primes $\wp_{1}, \ldots, \wp_{s}$, we have that $I^{(d)}=\wp_{1}^{(d)} \cap \ldots \cap \wp_{s}^{(d)}$. If $\wp$ is the ideal of a simple point, then we additionally have $\wp^{(d)}=\wp^{d}$.

[^14]:    ${ }^{1}$ similarly as for the general version of Fröberg's conjecture, in BDP15a], the authors do not require that the points have the same multiplicity. We are assuming that for the sake of simplicity.

[^15]:    ${ }^{2}$ In the original paper, we use a different terminology. In this thesis, we want to use this notation because $\mu$-powers of linear forms appear from several different perspective.

