Universal algebraic structures on polyvector fields

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Abstract

The theory of operads is a conceptual framework that has become a kind of universal language, relating branches of topology and algebra. This thesis uses the operadic framework to study the derived algebraic properties of polyvector fields on manifolds.

The thesis is divided into eight chapters. The first is an introduction to the thesis and the research field to which it belongs, while the second chapter surveys the basic mathematical results of the field.

The third chapter is devoted to a novel construction of differential graded operads, generalizing an earlier construction due to Thomas Willwacher. The construction highlights and explains several categorical properties of differential graded algebras (of some kind) that come equipped with an action by a differential graded Lie algebra. In particular, the construction clarifies the deformation theory of such algebras and explains how such algebras can be twisted by Maurer-Cartan elements.

The fourth chapter constructs an explicit strong homotopy deformation of polynomial polyvector fields on affine space, regarded as a two-colored noncommutative Gerstenhaber algebra. It also constructs an explicit strong homotopy quasi-isomorphism from this deformation to the canonical two-colored noncommmutative Gerstenhaber algebra of polydifferential operators on the affine space. This explicit construction generalizes Maxim Kontsevich's formality morphism.

The main result of the fifth chapter is that the deformation of polyvector fields constructed in the fourth chapter is (generically) nontrivial and, in a sense, the unique such deformation. The proof is based on some cohomology computations involving Kontsevich's graph complex and related complexes. The chapter ends with an application of the results to properties of a derived version of the Duflo isomorphism.

The sixth chapter develops a general mathematical framework for how and when an algebraic structure on the germs at the origin of a

sheaf on Cartesian space can be "globalized" to a corresponding algebraic structure on the global sections over an arbitrary smooth manifold. The results are applied to the construction of the fourth chapter, and it is shown that the construction globalizes to polyvector fields and polydifferential operators on an arbitrary smooth manifold.

The seventh chapter combines the relations to graph complexes, explained in chapter five, and the globalization theory of chapter six, to uncover a representation of the Grothendieck-Teichmüller group in terms of A_∞ morphisms between Poisson cohomology cochain complexes on a manifold.

Chapter eight gives a simplified version of a construction of a family of Drinfel'd associators due to Carlo Rossi and Thomas Willwacher. Our simplified construction makes the connections to multiple zeta values more transparent–in particular, one obtains a fairly explicit family of evaluations on the algebra of formal multiple zeta values, and the chapter proves certain basic properties of this family of evaluations.

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Contents

Introduction

We first give an informal introduction to the subject of the thesis. After that follows a more technical overview of the results and of the history of the field of research to which they belong.

1.1 Informal introduction.

> *"Well! I've often seen a cat without a grin," thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in all my life!"*

Lewis Carroll may have penned those lines as a humurous reference to the tendency of mathematicians to dissociate their craft from the natural world (according to the witty annotations of Carroll-authority and popular mathematics and science writer Martin Gardner, in [Carroll 1999]), but we shall take them as an explanatory metaphor for what an *operad* is. Operads are the main mathematical objects and tools in this thesis, so to explain our results we need to first explain what an operad is, and we will do this by a motivational example. Say that an *associative algebra* is a space[1]A equipped with a multiplication operation, mapping

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[1] The knowledgeable reader may want to insert the more specific term *vector* space. We shall cheat a lot during this informal introduction and deliberately minimize the use of such technical adjectives. Many of the things we discuss make sense for very general notions of *space* anyhow.

a pair of elements a, b in A to a product $a \cdot b$ (a third element) in A, and satisfying the associativity axiom that

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

for all elements a, b and c of A. Anyone that has gone through primary school has encountered the associative algebra $A = \mathbf{Q}$ of rational numbers. Now, instead of a grin without a cat, try to picture an associative algebra without a space. To answer this puzzle, imagine the multiplication as a machine or "black box" that has two inputs (where we insert a and b) and one output (where the final result, the product $a \cdot b$, appears), and draw this in the following form:

$$a \qquad b$$
$$a \cdot b$$

This is the smile of the multiplication operation on A. The multiplication should be associative, that is, it should satisfy $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, which translates into pictures as the statement that the following two trees are equal:

$$a \quad b \qquad\qquad b \quad c$$
$$c \qquad\qquad a$$
$$(a \cdot b) \cdot c \ = \ a \cdot (b \cdot c)$$
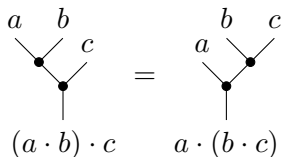
To get the smile *without* A we simply draw these two pictures of trees without any elements:

$$1 \quad 2 \qquad\qquad 1 \quad 2 \qquad\qquad 2 \quad 3$$
$$\qquad\qquad\qquad 3 \qquad 1$$
$$\text{and} \qquad =$$
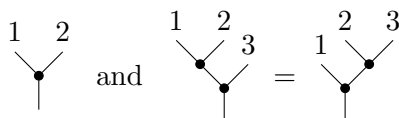
This is the grin of an associative algebra! The reason we number the inputs of the trees is because it matters in which order we multiply. If $a \cdot b = b \cdot a$ always holds, then the algebra is said to be *commutative*, and the grin of that commutativity relation would be drawn

$$1 \quad 2 \qquad 2 \quad 1$$
$$=$$
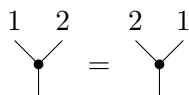
Thus, without numbering the inputs we would not be able to distinguish the smile of an associative algebra from the smile of a commutative algebra.

The pictures we have drawn are, secretely, examples of operads, namely, the operads Ass and Com, governing associative and commutative algebras. In slightly more detail, an operad is a collection of spaces $O(n)$ (one for each natural number $n \geq 1$) together with a collection of so-called "partial composition" functions

$$\circ_i : O(n) \times O(k) \to O(n-1+k),$$

mapping a pair of elements φ (in $O(n)$) and ψ (in $O(k)$), to some element $\varphi \circ_i \psi$ in the space $O(n-1+k)$. These functions have to satisfy certain axioms, but they are not important to us during this informal treatment. To define the operad of associative algebras, Ass, we let Ass(n) be the space of all trees that, first of all, have exactly n input edges and, secondly, have all vertices attached to exactly three edges, considered modulu the smile of the associativity relation. For example, the tree

$$
\begin{array}{c}
1 \quad 2 \\
\searrow \quad 3 \\
\bullet \quad \swarrow \quad 4 \\
\bullet \swarrow \\
|
\end{array}
$$
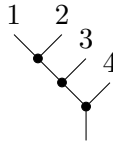
represents an element in Ass(4), but since everything has to be taken modulu the smile of associativity, the tree

$$
\begin{array}{c}
2 \quad 3 \\
1 \quad \searrow \\
\searrow \quad \bullet \quad 4 \\
\bullet \swarrow \\
|
\end{array}
$$
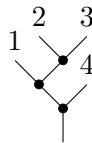
will represent the same element. The functions $\circ_i$ are given by grafting trees together (at the input labelled i) and suitably renumbering the inputs. For example,

$$
\begin{array}{ccc}
\begin{array}{c} 1 \quad 2 \\ \searrow 3 \\ \bullet \swarrow \\ \bullet \\ | \end{array}
& \circ_2 \quad
\begin{array}{c} 1 \quad 2 \\ \bullet \\ | \end{array}
\quad = \quad
\begin{array}{c} 2 \quad 3 \\ 1 \quad \searrow \\ \searrow \bullet \quad 4 \\ \bullet \swarrow \\ | \end{array}
\end{array}
$$
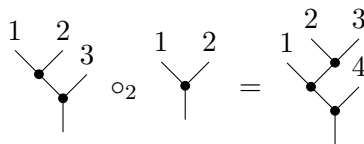
The operad Com is defined in the completely analogous manner, except we now regard the trees modulu also the commutativity relation.

Let V be a space. We can then form an operad $\mathsf{End}\langle V\rangle$, traditionally called the *endomorphism operad of* V, with $\mathsf{End}\langle V\rangle(n)$ defined as the collection of all functions

$$f : \underbrace{V \times \cdots \times V}_{n} \to V$$

from n copies of V to V. The partial compositions are defined by substitution of inputs, i.e., if f is as displayed above and $g \in \mathsf{End}\langle V\rangle(k)$, then $f \circ_i g$ is the function with $n - 1 + k$ inputs given by the formula

$$(f \circ_i g)(x_1, \ldots, x_{n-1+k})$$
$$= f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+k-1}), x_{i+k}, \ldots, x_{n-1+k}).$$

In words, we insert the output of g into the i-th input of f.

Let us say that a *morphism* of operads $F : \mathsf{O} \to \mathsf{P}$ is

- a function F_n from $\mathsf{O}(n)$ to $\mathsf{P}(n)$, for each $n \geq 1$,

- such that the collection $\{F_1, F_2, F_3, \ldots\}$ respects all the partial compositions.

Explicitly, the second point means that $F_{n-1+k}(\varphi \circ_i \psi) = F_n(\varphi) \circ_i F_k(\psi)$, for all φ in $\mathsf{O}(n)$ and all ψ in $\mathsf{O}(k)$. The reader may now like to try to prove the following claim:

Claim. Specifying the structure of an associative algebra on a space A (that is, equipping A with an associative multiplication operation) is the same thing as specifying a morphism of operads

$$\mathsf{Ass} \to \mathsf{End}\langle V\rangle.$$

Giving A the structure of a commutative algebra is, analogously, the same thing as a morphism $\mathsf{Com} \to \mathsf{End}\langle V\rangle$. Moreover, since all commutative algebras are, in particular, associative algebras, there is a morphism $\mathsf{Ass} \to \mathsf{Com}$.

Backed by this claim, we can give a more formalized answer to our puzzle: The smile of an associative algebra is the operad Ass of associative algebras.

Let us now become a little bit more technical. Based on the claim above, let us say that a space V is an O-algebra, if we are given a morphism from the operad O to the operad $\mathsf{End}\langle V\rangle$. Thus, the claim above says that an Ass-algebra is the same thing as an associative algebra, and

a Com-algebra is the same thing as a commutative algebra. Many familiar kinds of algebras can be regarded in this way, not just associative algebras and commutative associative algebras. The most important example, apart from the two already mentioned, is probably Lie algebras. A Lie algebra is a space L equipped with an operation (usually called a "bracket") mapping a pair of elements x, y to an element $[x, y]$, and satisfying the axioms that (i) $[x, y] = -[x, y]$ and (ii)

$$[x, [y, z]] = [y, [x, z]] - [z, [x, y]]$$

for all x, y and z in L. Thus, we may recognize that Lie algebras are governed by the operad Lie, whose grin is represented by
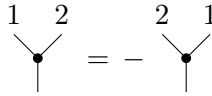
and
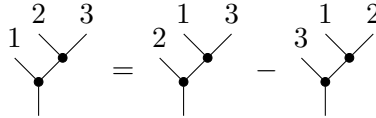
One appealing quality with operads is that if you manage to prove something about an operad O, then you automatically prove some universal statement about all O-algebras. For example, we noted in the preceeding "Claim" that there is a morphism Ass → Com corresponding to the universal fact that any commutative algebra is also an associative algebra. A slightly less trivial example is the morphism Lie → Ass which corresponds to taking an associative algebra A with product $\cdot$, and instead considering it as Lie algebra with the bracket operation given by the commutator $[a, b] = a \cdot b - b \cdot a$. This example is still rather obvious just from the ordinary perspective of algebras – operads can hardly be said to facilitate the realization that the commutator of an associative multiplication satisfies the axioms of a Lie bracket. However, most of the important results in this thesis would have been more or less impossible to guess at without adopting an operadic perspective. We shall discuss concrete examples from the thesis, but before doing so, let us digress on some further preliminary considerations.

First of all, we need to introduce what people working with operads call *colored algebras*. We shall consider a setup encompassing only two colors: *straight* and *dashed*. (The reader may rightly object that those aren't colors! We license our abuse of language by quoting Goethe, who

wrote that: "Mathematicians are like Frenchmen. They take whatever you tell them and translate it into their own language – and from then on it means something completely different.") Briefly, a colored operad is something constructed just like before, except that the inputs are now not only numbered but also colored, either *straight* or *dashed*, as is the output. Additionally, the partial compositions $\circ_i$ are only allowed to graft together things that have the same color. Let us elucidate by an example. Take a pair of spaces L, A and imagine L to be colored *straight* and A to be colored *dashed*. Given two natural numbers m and n, define $\mathsf{End}\langle L, A\rangle(m, n \mid dashed)$ to be the space of all functions

$$f : \underbrace{L \times \cdots \times L}_{m} \times \underbrace{A \times \cdots \times A}_{n} \to A,$$

and $\mathsf{End}\langle L, A\rangle(m', n' \mid straight)$ to be the space of all functions

$$g : \underbrace{L \times \cdots \times L}_{m'} \times \underbrace{A \times \cdots \times A}_{n'} \to L.$$

This should be a hopefully clear generalization of the endomorphism operad $\mathsf{End}\langle V \rangle$ discussed earlier. The only difference is that now one has two kinds of possible inputs and outputs. Clearly, the composition $f \circ_i g$, inserting the output of g into the i-th input of f, only makes sense if the color of the output matches the color of the input, i.e., it only makes sense for $i = 1, \ldots, m$ if the functions f and g are as displayed above. These colored partial composition functions $\circ_i$ give us a colored operad $\mathsf{End}\langle L, A\rangle$.

We are now ready to introduce the main object in this thesis: the colored operad NCG.[1] The letters in its name are an abbreviation for "noncommutative Gerstenhaber." Its basic operations are

The smile of relations that they satisfy is

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[1]Beware that the definition of the operad given here is not the definition we use in the thesis! The true definition of NCG differs from the definition given here by a degree-suspension on the *straight* color. To define the proper version of the operad one must, accordingly, introduce the notions of gradings and chain complexes.

and, further, also the two equations

Note how we have only grafted together the basic operations at colors that match. It is a quite complicated grin, but rather charming once acquainted. Giving an NCG-algebra

$$\mathsf{NCG} \to \mathsf{End}\langle L, A\rangle$$

is the same thing as specifying all of the following:

- An associative multiplication $\cdot$ on A.

- A Lie bracket operation $[\,,\,]$ on L.

- A function $D : L \times A \to A$ mapping a pair x, a to an element $D_x a$ in A, and satisfying the following two axioms:

  $$D_x(a \cdot b) = D_x(a) \cdot b + a \cdot D_x(b), \text{ and } D_{[x,y]}a = D_x(D_y a) - D_y(D_x a).$$

  The first axiom can be phrased succinctly by saying that, for each x in L, the function D_x from A to A is a derivation of the product. The second axiom says, in mathematically more fancy terms, that D is a *representation* of the Lie algebra L: meaning that the action $D_{[x,y]}$ of the bracketing $[x, y]$ equals the commutator bracketing $[D_x, D_y] = D_x D_y - D_y D_x$.

Let us now discuss the notion of algebras *up to homotopy*. The simplest case is given by algebras that are associative up to homotopy, so that will be our focus. Two functions $f, g : X \to Y$ between the same spaces X and Y are said to be *homotopic* if there is a function $h : [0, 1] \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$. One thinks of this as a family of functions $h_t(x) = h(t, x)$ parametrized by t, varying from the initial function $f(x) = h_0(x)$ to the function $g(x) = h_1(x)$, or, even better, one may think of it as a curve h_t from f to g inside the space of all functions. Now, let us consider a space A equipped with a binary product $a \cdot b$, but instead of assuming that the product is associative we shall assume the following. Consider the two functions

$$m_0(a, b, c) = (a \cdot b) \cdot c \text{ and } m_1(a, b, c) = a \cdot (b \cdot c).$$

To say that the product is associative is the same as saying that $m_0(a, b, c)$ equals $m_1(a, b, c)$. Instead of demanding that the two functions are equal we now demand that they are homotopic, i.e., that there is an $m_t(a, b, c)$ interpolating between the two. If we have such a thing, then we say that A is an associative algebra up to homotopy. Let us challenge our imagination and ask ourselves what the grin of such a thing is. The simplest solution is to write the smile of associativity up to homotopy in the form
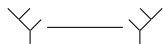
Instead of an equality we have displayed a line between the two, suggesting a homotopy $m_t(a, b, c)$. Next, consider what happens if we now want to multiply four elements. For example, look at the two expressions

$$(a \cdot (b \cdot c)) \cdot d \ \text{ and } \ a \cdot ((b \cdot c) \cdot d).$$

A moment's reflection shows that the composite $m_t(a, b \cdot c, d)$ interpolates between these two expressions. Some more serious thinking shows that there are, in total, five different ways of multiplying four elements, and that these are interpolated by five homotopies, as displayed in the following picture:
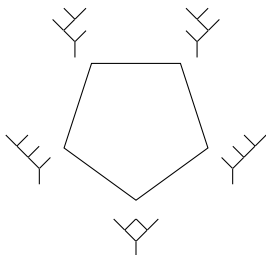
(The homotopy $m_t(a, b \cdot c, d)$ that we mentioned corresponds to the line at the top.) One then realizes, looking at this picture, that there are two ways of going from, say,

to ;
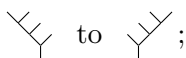
we can either follow the upper path along the pentagon or follow the lower one. There is nothing to guarantee that the two options are equal, or even related in any way. Since we have already fallen down the rabbit hole into the homotopical world, let us imagine that we have a function $m_{s,t}(a, b, c, d)$, where (s, t) is a coordinate allowed to vary inside a *solid* pentagon, such that restricting it to any boundary line of the pentagon

gives us back one of the homotopies we already have access to. In other words, $m_{s,t}$ is a two-dimensional homotopy interpolating between all our five one-dimensional homotopies.

Alas, as the reader might suspect, the fall down the rabbit hole does not stop here. There are 14 different ways to multiply together five elements, and these are interpolated by 21 one-dimensional homotopies and 9 two-dimensional homotopies. These fit together as the boundary of a three-dimensional polyhedral figure. Thus, to relate them all we should introduce a three-dimensional homotopy $m_{s,t,u}$, parametrized by that solid figure. And the story continues indefinitely.

The mathematician James Stasheff [Stasheff 1963] was the first to study these polyhedra, and the first to show how they can be constructed in an arbitrary dimension. They are nowadays called the *associahedra* and denoted K_n, where K_n is the polyhedron parametrizing all the ways to multiply together n elements. Thus, K_3 is a line and K_4 is a solid pentagon. In general, K_n is a polyhedron of dimension $n-2$. Stasheff's construction then motivated Jon Peter May to invent the general notion of an operad [May 1972].

Without going into details, the associahedra can be assembled into the components of an operad Ass_∞, with $\mathsf{Ass}_\infty(n) = K_n$. One defines a *strong homotopy associative algebra* to be an algebra

$$\mathsf{Ass}_\infty \to \mathsf{End}\langle A \rangle$$

for this operad. Thus, apart from a product, A also has a 1-dimensional homotopy relating the two ways of multiplying three elements, a 2-dimensional homotopy (corresponding to the pentagon and the ways of multiplying four elements), a 3-dimensional homotopy, etc., ad infinum in a hierarchy of homotopies that coherently relate all imaginable associativity relations.

We are now ready to state the first example of an original contribution made in this thesis.

Result. We give a geometric construction of an operad NCG_∞, bearing the same relationship to the operad NCG as the operad Ass_∞ of associahedra has to the operad Ass of ordinary associative algebras.

Note that every associative algebra can be regarded as a strong homotopy associative algebra by simply taking all homotopies to be trivial (since there is no need for them if the algebra is already associative). This means that there is a morphism

$$\mathsf{Ass}_\infty \to \mathsf{Ass}.$$

For the exact same reason, we have a morphism

$$\mathsf{NCG}_\infty \to \mathsf{NCG}.$$

Both of these morphisms are what is called "equivalences," a term which we shall not dwell on technically here. It essentially means that the introduction of strong homotopy algebras cuts you some slack, without truly altering anything. Everything fits together the same way, it just fits a little looser and more rubbery, giving you a bit more maneuverability. Almost all the results in this thesis make significant use of this extra maneuverability. To present a case in point from the thesis we need to introduce the gadgets called *polyvector fields*.

Recall that a *vector* is just another name for an element $v \in \mathbf{R}^d$, but thought of as not just a point, but, rather, as an arrow from the origin to that point. A *p-polyvector* is a sequence $v_1 v_2 \ldots v_p$ consisting of p vectors, modulu the rule that when any two neighboring vectors in the list are exhanged one picks up a minus sign:

$$v_1 v_2 \ldots v_i v_{i+1} \ldots v_p = -v_1 v_2 \ldots v_{i+1} v_i \ldots v_p.$$

This rule has a geometric origin, linked to the idea of vectors as arrows. Let us illustrate with a 2-polyvector uv.
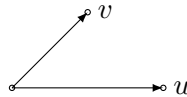
Instead of thinking of it as a pair of vectors one should think of the polyvector as the corresponding *oriented* parallelogram, with the orientation given by going from u to v:
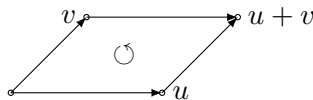
Changing the order of the two vectors flips the orientation, hence the imposed relation $uv = -vu$ just keeps track of how the parallelogram is oriented. The general rule for p-polyvectors does the exact same thing, but in a higher dimension. Define $\wedge^p(\mathbf{R}^d)$ to denote the space of all p-polyvectors. Recall that a vector field is a function $X : \mathbf{R}^d \to \mathbf{R}^d$, assigning a vector to each point. Generalizing that, a *p-polyvector field* is a function $\xi : \mathbf{R}^d \to \wedge^p(\mathbf{R}^d)$. When we speak of polyvector fields

(without any qualifying number p) it just means that we leave p unspecified. Now, note that polyvector fields can be multiplied, according to the rule

$$v_1 \ldots v_p \cdot u_1 \ldots u_q = v_1 \ldots v_p u_1 \ldots u_q.$$

(So the product of a p-polyvector field and a q-polyvector field is a $(p+q)$-polyvector field.) This product is associative. Polyvector fields also carry a natural structure of Lie algebra. The Lie bracket is traditionally called the Schouten bracket (after the Dutch mathematician Jan Arnoldus Schouten) and denoted $[\,,\,]_S$. This Lie bracket is an important object in mathematics, but somewhat technical. The grit of this discussion is that polyvector fields is a natural example of an NCG-algebra.

Define $T_{\text{poly}}(\mathbf{R}^d)$ to be the space of all polyvector fields. Then the pair

$$(L, A) = (T_{\text{poly}}(\mathbf{R}^d), T_{\text{poly}}(\mathbf{R}^d)),$$

consisting of two copies of the space of polyvector fields, is an NCG-algebra, where:

- The product $\cdot$ on $A = T_{\text{poly}}(\mathbf{R}^d)$ is the product between polyvectors that we explained above.

- The Lie bracket is the Schouten bracket $[\,,\,]_S$ on $L = T_{\text{poly}}(\mathbf{R}^d)$.

- The operation D is given by the formula $D_x a = [x, a]_S$.

Call the above the *standard* NCG-algebra structure on polyvector fields. We denote it

$$(T_{\text{poly}}(\mathbf{R}^d), T_{\text{poly}}(\mathbf{R}^d))_{\text{standard}}.$$

Result. We geometrically construct an NCG_∞-algebra structure on polyvector fields, which includes all three operations $\cdot$, $[\,,\,]_S$ and D given above, but also higher homotopies.

Call the structure promised above the *exotic* NCG_∞-algebra structure on polyvector fields.

Result. The standard structure and the exotic structure are not equivalent, meaning, intuitively, that there is no way to write down a hierarchy of coherent homotopies between the operations of the exotic one and the operations of the standard one. Moreover, up to equivalence the standard structure and exotic one are the only two possible NCG_∞-structures on polyvector fields: any NCG_∞-structure on polyvector fields must be equivalent to one of those two.

For the next result we need to introduce a little more terminology. A p-polydifferential operator is a machine C that takes p real-valued functions

$$f_1, \dots f_p : \mathbf{R}^d \to \mathbf{R}$$

and produces a new such function

$$C(f_1, \dots, f_p) : \mathbf{R}^d \to \mathbf{R},$$

in a way that satisfies some rules reminescent of the rules satisfied by derivation (analogues of the chain rule and the product rule $(fg)' = fg' + f'g$). Denote the space of all polydifferential operators by $D_{\mathrm{poly}}(\mathbf{R}^d)$. If C is a p-polydifferential operator as above and K is a q-polydifferential operator, then we can form a $(p+q)$-polydifferential operator $C \cdot K$, by letting

$$(C \cdot K)(f_1, \dots, f_{p+q}) = C(f_1, \dots, f_p) K(f_{p+1}, \dots, f_{p+q}).$$

This is an associative product on polydifferential operators. Without going into details, polydifferential operators also have a Lie bracket, called the Gerstenhaber bracket (after Murray Gerstenhaber), and a so-called differential referred to as the Hochschild differential (after Gerhard Hochschild). Just like for polyvector fields, one can recast these operations as an NCG-algebra structure on two copies of the space of polydifferential operators. Call this NCG-algebra

$$(D_{\mathrm{poly}}(\mathbf{R}^d), D_{\mathrm{poly}}(\mathbf{R}^d))_{\mathrm{standard}}.$$

Result. The two NCG$_\infty$-algebras

$$(T_{\mathrm{poly}}(\mathbf{R}^d), T_{\mathrm{poly}}(\mathbf{R}^d))_{\mathrm{exotic}} \ \text{ and } \ (D_{\mathrm{poly}}(\mathbf{R}^d), D_{\mathrm{poly}}(\mathbf{R}^d))_{\mathrm{standard}}$$

are equivalent, by an explicit geometric construction.

Remark. In particular, this result says that there is an equivalence of strong homotopy Lie algebras between the algebra of polyvector fields, with the Schouten bracket, and the algebra of polydifferential operators, with the Gerstenhaber bracket (and Hochschild differential). This result was proved by Maxim Kontsevich in 1997 (later published as [Kontsevich 2003]). Our result generalizes Kontsevich's construction by extending his strong homotopy equivalence to all the additional data that is given, such as the associative products.

One can define polyvector fields $T_{\mathrm{poly}}(M)$ and polydifferential operators $D_{\mathrm{poly}}(M)$ on any *manifold* M, not just on $\mathbf{R}^d$. (A manifold is something that can have a more intricate shape, like a sphere, or a doughnut.)

Result. In all of our constructions one may replace $\mathbf{R}^d$ by an arbitrary manifold M.

The above result is far from evident, because our construction on $\mathbf{R}^d$ relies heavily on the use of coordinates. Essentially, to obtain formulas on a manifold one needs to add further homotopies to the construction, homotopies that keep track of how the coordinates are used.

The next chapter, chapter 2, collects some preliminary theory. Chapter 3 is devoted to proving a number of novel constructions for colored operads, constructions that we use to prove our main results, but that are interesting in their own regard as well. Chapters 4, 5 and 6 spell out the details of all the results claimed in this introduction. Chapter 7 is devoted to some related questions and deepens the study of the preceding chapter. The last chapter, chapter 8, explores a relation between the main body of results and the algebraic study of multiple zeta values.

1.2 Technical introduction.

The language of operads was initially a by-product of research in stable homotopy theory. James Stasheff, building on work by John Milnor, Albrecht Dold, Richard Lashof, Masahiro Sugawara, and others, proved an elegant criteria for when a connected space has the homotopy type of a based loop space, in [Stasheff 1963]. It took almost ten more years before Jon P. May coined the term *operad* [May 1972] but, in retrospect, Stasheff's criteria can be succinctly summarized by saying that a connected space (with the homotopy type of a CW complex and with a nondegenerate base-point) has the homotopy type of a based loop space if and only if it is an algebra for the topological A_∞ operad of associahedra. J. Michael Boardman and Rainer M. Vogt, and also May, furthered Stasheff's work to analogous statements for n-fold and infinite loop spaces. The unifying theme in all these works is that of homotopy-invariant structures. Topologists had by the late 1950's and early 1960's proved many results about so-called H-spaces. (The terminology was introduced in 1951 by Jean-Pierre Serre, in honor of Heinz Hopf.) An H-space is a topological space with a continuous binary product and a two-sided unit. One class of examples is topological groups, for which the product additionally is associative and has inverses. Another class is given by loop spaces, where the product is associative only up to reparametrizing homotopies. Researchers had by the early 1960's revealed many homotopy-invariant properties about H-spaces (e.g., if a space is an H-space then one can easily deduce that its fundamental

group must be Abelian), but not all the premises for the deductions had homotopy invariant characterizations. For example, a space which is homotopy equivalent to an associative H-space need not itself be an associative H-space, yet, by definition, it must have all the homotopical properties shared by associative H-spaces. This assymmetry was highlighted by Saunders Mac Lane already in 1967 when he (according to Vogt) said that: "The disadvantage of topological groups and monoids is that they do not live in homotopy theory."[Vogt 1999] Stasheff's notion of an A_∞-space, i.e., of what we now recognize as an algebra for the topological operad of associahedra, is on the other hand a homotopy invariant notion. Thus, operads were already in their prehistory (before the general definition of an operad had been given) preeminently a means, or tool, to describe homotopy invariant algebraic structures. With the wisdom of hindsight we can recognize why operads are so suited for describing the homotopy theory of algebras. Algebra, broadly speaking, is something that Mac Lane, William Lawvere and others has taught us to phrase internal to (symmetric) monoidal categories. Homotopy theory, on the other hand, is something that Daniel Quillen and others has shown to make general sense for model categories. Homotopy theory of algebras, accordingly, naturally finds its home in so-called closed model categories. Operads sit well in such a context and, more importantly, the theory of operads distils a flavor of algebra (say, associative algebras) into a concrete object (the operad, whose algebras are, say, associative algebras). This concrete object can then be subjected to homotopical analysis, put into relation with other operads, etc. For example, the operad governing topological spaces with an associative product is not cofibrant (in the canonical model structure on topological operads), but the operad of associahedra, which is weakly equivalent to it, *is* cofibrant. This is what makes A_∞ spaces a homotopy-invariant notion. The moral for how to apply operads to do homotopy theory with algebras generalizes this example. Start with some flavour of algebra. Find the operad that governs it, and find a nice cofibrant replacement for that operad. The change in perspective that comes with distilling a flavour of algebra into a separately existing object of study is very fruitful.

Let us discuss some further problems that also motivate the study of homotopy theory for (some flavor of) algebras. The de Rham complex of a smooth manifold can be used to calculate the real cohomology of the manifold. However, the de Rham complex is more than just a complex; it is a differential graded commutative algebra, and as such it completely specifies the real homotopy type of the manifold (at least if it is simply connected). Thus, the real homotopy theory of manifolds is equivalent

to the homotopy theory of their corresponding differential graded commutative de Rham algebras. The rational homotopy theory of Quillen and Dennis Sullivan extends this to more general spaces, and rational homotopy groups, by using Sullivan's piece-wise linear polynomial differential forms. Hence the homotopy theory of differantial graded commutative algebras is, in some technical sense whose precise formulation depends on the context, equivalent to homotopy theory over a field for topological spaces.

Another and more purely algebraic motivation for studying homotopy theory of algebras is deformation-theoretic. So-called Koszul operads (in the model category of chain complexes) have canonical minimal cofibrant replacements. Such a replacement defines a functorial construction that out of an algebra for the operad produces a differential graded Lie algebra, called the deformation complex of the algebra, that governs the deformations of that algebra. This unifies several classical cohomology theories for algebras, such as Hochschild cohomology, Harrison cohomology, and Chevalley-Eilenberg cohomology. A Maurer-Cartan element in the deformation complex of an algebra is, per definition, a deformation of the algebra. If two differential graded Lie algebras are weakly equivalent, then their sets of Maurer-Cartan elements (modulu gauge equivalence) are isomorphic; implying that the deformation complex of an algebra is mainly interesting only up to weak equivalence. Thus we see homotopy theory of algebras entering in two ways: first in defining the suitable notion of deformation, and secondly in the study of the differential graded Lie algebras that govern those deformations. This brings us to the works of Maxim Kontsevich and Dimitry Tamarkin, and the field of research to which this monograph belongs.

Kontsevich conjectured in 1993 that the graded Lie algebra of polyvector fields on a manifold is weakly equivalent as a differential graded Lie algebra to the polydifferential Hochschild cochain complex of smooth functions on the manifold [Kontsevich 1993], a conjecture which he then went on to prove affirmatively in a 1997 preprint, later published as [Kontsevich 2003]. This implies that the deformation theory of smooth functions on a manifold is governed by the graded Lie algebra of polyvector fields: in particular, any Maurer-Cartan element in polyvector fields, that is, any so-called Poisson structure, defines an associative deformation. This settled the long-standing problem of deformation quantization, initiated in [Bayen et al. 1977]. Kontsevich's paper never mentions operads but, nevertheless, it is very much based on the perspective and techniques of operads. Shortly after, in 1998, Tamarkin gave a very different and explicitly operad-based proof of the same result.[Tamarkin

1998] Tamarkin's proof at one step involves choosing a Drinfel'd associator. The set (or scheme, rather) of Drinfel'd associators is a torsor for the (prounipotent) Grothendieck-Teichmüller group. This group is rather mysterious, the most important fact known concerning its structure is that it is not finitely generated (and very little is known about it apart from that), but it has far-reaching implications in several fields of mathematics. It therefore follows from Tamarkin's proof that the Grothendieck-Teichmüller group acts on the set of weak equivalences between polyvector fields and polydifferential Hochschild cochains. Exactly how this action would be visible in Kontsevich's proof was understood only very recently, when Thomas Willwacher published his preprint [Willwacher 2010], though definitive hints and partial answers were given earlier, cf. the construction by Sergei Merkulov in [Merkulov 2008]. To explain Willwacher's results, recall that a Gerstenhaber algebra is a differential graded commutative algebra with a graded Lie bracket of degree minus one, whose adjoint action is a (degree minus one) derivation of the graded commutative product. The operad of Gerstenhaber algebras is Koszul, hence has a canonical minimal reoslution. Willwacher, building on work by Tamarkin, proved that the Grothendieck-Teichmüller group is the group of connected components of the group of automorphisms of the minimal model of the Gerstenhaber operad; and hence that it acts on the set of Maurer-Cartan elements in the deformation complex of an arbitrary Gerstenhaber algebra. Polyvector fields on a manifold is naturally a Gerstenhaber algebra. However, the Gerstenhaber algebra-structure on polyvector fields is in a certain universal sense non-deformable. Using this, the action of the Grothendieck-Teichmüller group on the set of Gerstenhaber algebra structures is, in the particular case when the algebra in question is the algebra of polyvector fields, pushed to an action by weak automorphisms of polyvector fields *as a graded Lie algebra*. Moreover, these constitute the group of universal such automorphisms, if the word "universal" is taken in the same sense as in the statement that polyvector fields is universally non-deformable as a Gerstenhaber algebra. The action by the Grothendieck-Teichmüller group on weak equivalences of differential graded Lie algebras between polyvector fields and polydifferential Hochschild cochains is recovered by precomposing the weak equivalence constructed by Kontsevich with the action by weak automorphisms of polyvector fields.

The main theme in this monograph is based on a modification of the the basic set-up discussed in the preceeding paragraph. Define a two-colored noncommutative Gerstenhaber algebra (henceforth abbreviated

as an NCG-algebra) to be a pair of cochain complexes, where the first is a differential graded associative algebra, and the second is a differential graded Lie algebra with the bracket of degree minus one which is, additionally, equipped with an representation in terms of graded derivations of the product on the first cochain complex. For example, any Gerstenhaber algebra defines an NCG-algebra by taking the two cochain complexes to be copies of the complex underlying the Gerstenhaber algebra, but dividing the data of the Gerstenhaber structure into an associative product (on the first copy), a Lie bracket (on the second copy), and defining the representation to be the adjoint representation. Thus, since polyvector fields are a Gerstenhaber algebra, two copies of polyvector fields is an NCG-algebra. Another naturally occuring example of an NCG-algebra is two copies of the (polydifferential) Hochschild cochain complex, with the associative cup product on the first copy, the Gerstenhaber bracket on the second, and the action given by the so-called braces map. We prove that the operad governing NCG-algebras is Koszul, hence has a canonical minimal resolution, providing a well-behaved homotopy theory for NCG-algebras. We then prove that the two aforementioned NCG-algebras are not weakly equivalent. Our proof is based on an explicit construction, rather than a usual non-existence argument. In more detail, we extend the techniques of [Kontsevich 2003] and obtain explicit formulas for a deformation of the canonical NCG-structure on polyvector fields, which we term the *exotic NCG-structure*, together with explicit formulas for a weak equivalence from this deformation to the canonical NCG-structure on polydifferential Hochschild cochains. The deformation only involves deforming the adjoint action of the Lie bracket: the associative (commutative, in fact) product and the Lie bracket are not perturbed at all. Since only the adjoint action is deformed, the explicit construction gives both a weak equivalence of differential graded Lie algebras (which by construction coincides with Kontsevich's), and a weak equivalence of associative algebras (which is a new result). We then show that the deformation of the adjoint action is non-trivial, i.e., is not homotopic to the undeformed algebra, implying that our two canonical NCG-structures can not be weakly equivalent. Furthermore, we prove that the exotic deformation is (in a certain universal sense) the unique deformation of the canonical NCG-structure on polyvector fields. We also prove that the action of the Grothendieck-Teichmüller group, as explicated by Willwacher, induces an action in terms of gauge-equivalences between deformations of the canonical NCG-structure on polyvector fields. This action by gauge-equivalences can be regarded as a vast generalization of the Duflo auto-

25

morphisms familiar from Lie theory, to the context of Poisson complexes on manifolds. All of our explicit formulas are first constructed on Euclidean space, with reference to a fixed affine structure, but we show how to coherently modify the formulas to give them diffeomorphism-invariant sense on an arbitrary manifold. To do this we develop a very generally applicable framework, which combines elements of operad-theory with formal geometry in the sense of Isreal Gel'fand and David Kazhdan.

The last chapter gives a streamlined construction of the 1-parameter family of Drinfel'd associators that was first discovered by Willwacher and Carlo Rossi in [Rossi and Willwacher 2013]. The subject of that chapter has independent interest, but it also connects with the overall theme of the monograph via a clear (but technically not yet entirely precise) relationship between our exotic deformation and the Alekseev-Torossian Drinfel'd associator. Conjecturally, any associator should define an exotic deformation (though they should all be gauge-equivalent, via a Grothendieck-Teichmüller group action). Alternatively put, the last chapter suggest a close relationship between the coefficients of our exotic structure and the algebra of formal multiple zeta values, such that the coefficient of the lowest order term in the exotic deformation (which defines the cohomology class in the deformation complex) corresponds to the (formal) zeta-value $\zeta(2)$.

CHAPTER 2

Preliminaries

This chapter contains no new results; its purpose is only to fix notation and make our monograph (more) self-contained.

2.0.1 Finite sets.

Given a natural number $n \geq 1$, we write $[n]$ for the set $\{1, 2, \ldots, n\}$.

The cardinality of a finite set A is written $\#A$, e.g. $\#[n] = n$.

Given finite sets A, B we shall write either $A \sqcup B$ or $A + B$ for their disjoint union, and if B is a subset of A we will write either $A \setminus B$ or $A - B$ for the complement of B in A.

We say that a set is ordered if it has a total ordering; that is, if it is equipped with an antisymmetric, reflexive and total binary relation. If A is an ordered finite set, then we say that $S \subset A$ is a connected subset and write $S < A$ if $s, s'' \in S$ and $s < s' < s'' \in A$ implies that also $s' \in S$.

The group of permutations (self-bijections) of a finite set T is denoted Σ_T, except for the groups $\Sigma_{[n]}$ which are abbreviated Σ_n.

2.0.2 Differential graded vector spaces.

In this section we state our conventions regarding differential graded (hencefort abbreviated dg) vector spaces. Fix a field $\mathbf{k}$ of characteristic zero.

A **dg vector space** is defined to be synonymous with an unbounded cochain complex. A morphism of dg vector spaces $f : (V, d_V) \to (W, d_W)$

is a collection $f = \{f^p : V^p \to W^p\}_{p \in \mathbf{Z}}$ of linear maps such that $f^{p+1} \circ d_V^p = d_W^p \circ f^p$ for all p. The cohomology of a dg vector space V is the dg vector space $H(V)$ with $H(V)^p := \mathrm{Im}(d_V^{p-1})/\mathrm{Ker}(d_V^p)$, the space of degree p cocycles modulu the space of degree p coboundaries, as usual. We say a dg vector space is of finite type if it is finite-dimensional in each degree.

The space of maps from V to W is the dg vector space $\mathrm{Map}(V, W)$ with

$$\mathrm{Map}(V, W)^n := \prod_p \mathrm{Hom}_{\mathbf{k}}(V^{p-n}, W^p),$$

where $\mathrm{Hom}_{\mathbf{k}}(V^{p-n}, W^p)$ denotes the vector space of all linear maps from V^{p-n} to W^p, and differential given on $\phi \in \mathrm{Map}(V, W)^n)$ by $d_{\mathrm{Map}(V,W)}^n \phi := d_W \circ \phi - (-1)^n d_V \circ \phi$. A vector ϕ of $\mathrm{Map}(V, W)^n$ is called a map of dg vector spaces of degree n. Note that a morphism from V to W is the same thing as a cocycle of degree 0 of $\mathrm{Map}(V, W)$. We apply the Koszul sign rules to maps, which says that for homogeneous maps f, g and homogeneous vectors u, v in their respective domains, $f \otimes g$ is defined by $(f \otimes g)(u \otimes v) = (-1)^{|g||u|} f(u) \otimes g(v)$. Given dg vector spaces V and W their tensor product is the dg vector space $V \otimes W$ with $(V \otimes W)^n := \bigoplus_{p+q=n} V^p \otimes_{\mathbf{k}} W^q$ differential defined by $d_{V \otimes W} := d_V \otimes id_W + id_V \otimes d_W$ (using the Koszul sign rule for maps). The Koszul symmetry for $V \otimes W$ is the morphism

$$\sigma_{V \otimes W} : V \otimes W \to W \otimes V$$

given on vectors of homogeneous degree by $\sigma_{V \otimes W}(v \otimes w) := (-1)^{|v| \cdot |w|} w \otimes v$. The tensor product, the Koszul symmetry and the tensor unit $\mathbf{k}$ give $\mathtt{Ch}_{\mathbf{k}}$ the structure of a symmetric monoidal category. Using the space of maps and the Koszul sign rules for maps we can (and implicitly usually will do) consider $\mathtt{Ch}_{\mathbf{k}}$ as a category enriched in itself, because the space of maps and the tensor product satisfy the usual adjunction.

A graded vector space is a dg vector space (V, d_V) with $d_V = 0$. We remark that graded vector spaces form a (full) symmetric monoidal subcategory of the category of dg vector spaces.

2.0.3 Differential graded algebras.

By a (differential) graded algebra of some type we always mean an algebra in a sense internal to the category of dg vector spaces.

A **dg associative algebra** is a monoid in the category of dg vector spaces. This means that it is a dg vector space A together with a morphism $\mu : A \otimes A \to A$, called product, satisfying the associativity

constraint $(\mu \otimes id) \circ \mu = (id \otimes \mu) \circ \mu$. A unital dg associative algebra is a unital monoid. This means it is additionally equipped with a two-sided unit $1 \in A^0$ of homogeneous degree 0. A morphism of dg associative algebras is a morphism of the underlying dg vector spaces which additionally respects the products. The free dg associative algebra on a dg vector space V is $T^+(V) := \bigoplus_{n>0} V^{\otimes n}$ with the product given by concatenation of tensors. The free unital dg associative algebra on V is the full tensor space $T(V)$.

A (unital) **dg commutative algebra** is a (unital) dg associative algebra A for which the product commutes with the Koszul symmetry, i.e. satisfies $\mu \circ \sigma_{A \otimes A} = \mu$. The free dg commutative algebra on V is $S^+(V) := \bigoplus_{n>0} (V^{\otimes n})^{\Sigma_n}$, the permutation invariants taken with repsect to the action defined by the Koszul sign rules, and the free unital dg commutative algebra is the full symmetric algebra $S(V)$.

A **dg Lie algebra** is a dg vector space L together with a morphism $[,] : L \otimes L \to L$, called the bracket, which is Koszul antisymmetric $([,] \circ \sigma_{L \otimes L} = -[,])$ and satisfies the Jacobi identity

$$\sum_{\sigma \in \mathbf{Z}_3} [,] \circ (id \otimes [,]) \circ \sigma = 0.$$

(The cyclic permutations σ act according to the Koszul symmetry rule.) The free dg Lie algebra on V, $L(V)$, sits inside the free dg associative algebra $T^+(V)$ as the subspace generated by V under the bracket $[v, v'] = v \otimes v' - (-1)^{|v||v'|} v' \otimes v$. If V is a dg vector space then we define $gl(V)$ to be the dg vector space $\mathrm{Map}(V, V)$ equipped with the structure of dg Lie algebra given by the commutator of compositions of maps of dg vector spaces. A **Maurer-Cartan element** of a dg Lie algebra is an element $\pi \in V^1$ satisfying the equation $d\pi + \frac{1}{2}[\pi, \pi] = 0$. Given a Maurer-Cartan element one can define the twisted dg Lie algebra V_π, with the same underlying graded vector space and the same bracket, but with the new differential $d_\pi := d + [\pi,]$.

Morphisms of dg algebras (of any kind) are defined as morphism of dg vector spaces respecting all structure. One defines coalgebraic versions of dg associative, commutative and Lie algebras by using maps $V \to V \otimes V$ satisfying conditions dual to the respective algebra condition.

We also give the following ad hoc definitions (their conceptual motivation will be clarified in later sections):

An A_∞ **algebra** is a dg vector space A together with a nilsquare degree $+1$ coderivation

$$\nu : T^+(A[1]) \to T^+(A[1])$$

of the coalgebra $T^+(A[1])$. Since the coalgebra is cofree it is defined by its components $\nu_n : A[1]^{\otimes n} \to A[2]$. We require $\nu_1 = d_A$. That the map is a coderivation is the assumption

$$\Delta \circ \nu = (\nu \otimes id + id \otimes \nu) \circ \Delta,$$

if Δ is the coproduct. An L_∞ **algebra** is a dg vector space L together with a nilsquare degree $+1$ coderivation $\lambda = d_L + \lambda_{\geq 2}$ of the coalgebra $S^+(L[1])$.

2.0.4 Graphs.

Definition 2.0.4.1. A **graph** G is a finite set of flags F_G with an involution $\tau : F_G \to F_G$, a finite set of vertices V_G and a function $h : F_G \to V_G$. The fixed points of τ are called legs and the orbits of length two are called edges. Let E_G denote the set of edges. Let v and v' be two vertices. They are said to share an edge if there exists a flag f such that $h(f) = v$ and $h(\tau(f)) = v'$, and they are said to be connected if there exists a sequence of vertices $v = v_0, v_1, \ldots, v_k = v'$ such that v_i and v_{i+1} share an edge. A graph is called connected if any two of its vertices are connected. The valency of a vertex is the cardinality $\#h^{-1}(v)$.

A morphism of graphs $\phi : G \to G'$ is a function $\phi^* : F_{G'} \to F_G$, which is required to be bijective on legs and injective on edges, together with a function $\phi_* : V_G \to V_{G'}$, such that ϕ_* is a coequalizer of the two functions $h, h \circ \tau : F_G \setminus \phi^*(F_{G'}) \to V_G$.

A graph is called a **tree** if it is connected and $\#V_G - \#E_G = 1$. A rooted tree is a tree T together with a distinguished leg out_T, called the root. The legs not equal to the root are called the leaves of the (rooted) tree. We denote the set of leaves of a rooted tree T by In_T. Given a rooted tree T, let v' denote the unique vertex such that $h(root) = v'$. For every vertex v of T there exists a unique $v = v_0, v_1, \ldots, v_k = v'$ of miminal length k that displays v and v' as connected. Call this the distance from v to the root. Say that $f \in h^{-1}(v)$ is outgoing if the distance from $h(\tau(f))$ to the root is less than the distance from v to the root. The outgoing flag at any vertex is necessarily unique. Call a flag which is not outgoing incoming. Define In_v to be the set of incoming flags at v and out_v to be the outgoing flag at v.

Any morphism of graphs can (up to isomorphisms) be regarded as given by contracting the connected components of a subgraph into vertices. Specifically, let $\phi : G \to G'$ be a morphism of graphs which is not an isomorphism.

Lemma 2.0.4.2. Up to isomorphisms any morphism of graphs can be written as a sequence of edge contractions.

Proof. Assume $\phi : G \to G'$ is a morphism of graphs. Recall that, in particular, $\phi^* : F' \to F$ is bijective on legs and injective on edges. If it is bijective on edges then ϕ is an isomorphism. Assume it is not an isomorphism and define $F'' := F \setminus \phi^*(F')$, $\tau'' := \tau|_{F''}$, $h'' := h|_{F''}$ and $V'' := h(F'')$. This defines a new graph G'' without legs (since ϕ^* is bijective on legs), naturally pictured as a subgraph of G. Contracting each connected component of G'' into a new vertex produces a graph G/G'' equipped with a morphism $G \to G/G''$ and we can factor ϕ through $G \to G/G''$ via an isomorphism $G/G'' \cong G'$.

If ϕ^* is not an isomorphism then we can remove some edge $e = \{f, \tau(f)\}$ of G'' from F to get a new set of flags $F_{G/e} = F \setminus e$ and factor ϕ^* through $F_{G/e} \subset F$. Writing the details down one gets a factorization of ϕ through the "edge contraction" $G \to G/e$. Hence we can factor $G \to G/G''$ through $G \to G/e$. Iterating the procedure gives a factorization of ϕ as a sequence of edge-contractions and isomorphisms. $\square$

2.1   Colored operads.

Fix a symmetric monoidal category $(\mathtt{V}, \otimes, \mathbb{I})$ for the remainder of this section. We assume it to be cocomplete (by convention this includes existence of an initial object since that should be an empty colimit), to have finite limits, and that the tensor product is cocontinuous.

Definition 2.1.0.3. Fix a countable set S. An S-colored rooted tree is a rooted tree T together with a coloring, that is, together with a function $\zeta_T : F_T \to S$ such that $\zeta_T \circ \tau = \zeta_T$. We make S-colored rooted trees into a category $\mathtt{T}^S$ by declaring a morphism of S-colored rooted trees to be a morphism of the underlying graphs that maps the root leg to the root leg and commutes with colorings, $\zeta_T \circ \phi^* = \zeta_T$.

Definition 2.1.0.4. Let $S \wr \Sigma$ be the category whose objects are functions $s. : I \to S$, where I can be any finite (possibly empty) set, and whose morphisms from $s. : I \to S$ to $s' : J \to S$ are bijections $\sigma : J \to I$ such that $s. = s'_. \circ \sigma$. An S-**colored Σ-module** in $\mathtt{V}$ is a functor

$$E : S \wr \Sigma \times S \to \mathtt{V}.$$

An element $\varphi \in E(s. \mid s)$ for $s : I \to S$ is said to have **arity** $\#I$.

For an S-colored rooted tree (T, ζ_T) and vertex $v \in V_T$, define $s_{\cdot}^T :=$ $\zeta_T|_{In_T}$, $s_T := \zeta(out_T)$, $s_{\cdot}^v := \zeta_T|_{In_v}$, $s_v := \zeta(out_v)$. Note that these are objects of the form $(s. \mid s) \in S \wr \Sigma \times S$. Given an S-colored Σ-module E we then define

$$E(T) := \bigotimes_{v \in V_T} E(s_{\cdot}^v \mid s_v).$$

Together with the permutation actions on E this defines a functor

$$\mathrm{Iso}\, \mathtt{T}^S \to \mathtt{V},\ E \mapsto E(T)$$

from the category of S-colored rooted trees and isomorphisms between them. An object of $S \wr \Sigma \times S$ is equivalent to a colored rooted tree with a single vertex. Using this identification, define

$$\mathcal{F}(E)(s. \mid s) := \mathrm{colim}\big(\mathrm{Iso}(\mathtt{T}^S \downarrow (s. \mid s)) \xrightarrow{E} \mathtt{V}\big).$$

This defines an endofunctor on the category of colored Σ-modules in $\mathtt{V}$. If T is a colored rooted tree and for every vertex $u \in V_T$ we have some $T_u \to (s_{\cdot}^u \mid s_u)$, then we can build a tree T' that contains each T_u as a subtree and has the property that contracting all the T_u subtrees of T' produces the original tree T. In particular, $V_{T'} = \bigcup_{u \in V_T} V_{T_u}$, giving a canonical morphism

$$\bigotimes_{u \in V_T} \bigotimes_{v \in V_{T_u}} E(s_{\cdot}^v \mid s_v) \mapsto \bigotimes_{w \in V_{T'}} E(s_{\cdot}^w \mid s_w).$$

These maps assemble to a natural transformation $\mathcal{F} \circ \mathcal{F} \to \mathcal{F}$. The definition as a colimit gives a natural transformation $id \to \mathcal{F}$. Together these two natural transformations give $\mathcal{F}$ the structure of a monad.

Definition 2.1.0.5. An S-colored **pseudo-operad** in $\mathtt{V}$ is an algebra for this monad. Morphisms of pseudo-operads are morphisms of $\mathcal{F}$-algebras.

Remark 2.1.0.6. The above definition means that a pseudo-operad is an S-colored Σ-module $\mathtt{Q}$ together with morphisms

$$\mu_T : \mathtt{Q}(T) \to \mathtt{Q}(s_{\cdot}^T \mid s_T),$$

for every T, called compositions, satisfying certain equivariance and associativity conditions. The formula for the free pseudo-operad functor $\mathcal{F}$ can be phrased as saying that $\mathcal{F}(E)$ is the left Kan extension of $E : \mathrm{Iso}\, \mathtt{T}^S \to \mathtt{V}$ along $\mathrm{Iso}\, \mathtt{T}^S \to \mathtt{T}^S$. Thus $\mathcal{F}(E)$ is a functor $\mathtt{T}^S \to \mathtt{V}$. It follows by naturality that any pseudo-operad also defines such a functor. (But it is not true that any such functor is a pseudo-operad.) This

can be used to argue that the operations μ_T are completely determined already by those corresponding to trees with two vertices, using that any morphism of rooted trees can be written as a composition of edge-contractions, cf. 2.0.4.2.

Given $s. : [n] \to S$, $s'. : [n'] \to S$, $s, s' \in S$ and $1 \leq i \leq n$, satisfying $s_i = s'$, define

$$(s. \circ_i s'.) : [n + n' - 1] \to S$$

by

$$(s. \circ_i s'.)_k = \begin{cases} s_k & \text{if } 1 \leq k < i \\ s'_{k-i+1} & \text{if } i \leq k < i + n' \\ s_{k-n'} & \text{if } k \geq i + n'. \end{cases}$$

Define T to be the tree with two vertices v and v', set of leaves $[n+n'-1]$, v adjacent to the root, and colorings defined by $s_.^T := s. \circ_i s'.$, $s_T = s$, $In_{v'} = \{i, \dots, i + n' - 1\}$, $s_i^v = s'$ and $s_k^{v'} = s'_{k-i+1}$. If Q is an operad, then T defines a morphism

$$\circ_i := \mu_T : \mathsf{Q}(s. \mid s) \otimes \mathsf{Q}(s'. \mid s') \to \mathsf{Q}(s. \circ_i s'. \mid s).$$

These operations are called the **partial compositions**.

Definition 2.1.0.7. An S-colored **operad** in V is a pseudo-operad Q together with morphisms

$$e_{s'} : \mathbb{I} \to \mathsf{Q}(s \mid s)$$

for each $s' \in S$, called units, such that for all $(s. \mid s)$ with $s_i = s'$, the compositions

$$\mathsf{Q}(s. \mid s) \cong \mathsf{Q}(s. \mid s) \otimes \mathbb{I} \xrightarrow{id \otimes e_{s'}} \mathsf{Q}(s. \mid s) \otimes \mathsf{Q}(s' \mid s') \xrightarrow{\circ_i} \mathsf{Q}(s. \mid s)$$

and

$$\mathsf{Q}(s. \mid s) \cong \mathbb{I} \otimes \mathsf{Q}(s. \mid s) \xrightarrow{e_s \otimes id} \mathsf{Q}(s \mid s) \otimes \mathsf{Q}(s. \mid s) \xrightarrow{\circ_1} \mathsf{Q}(s. \mid s)$$

both equal the identity. Morphisms of operads are morphisms of pseudo-operads respecting the units.

Definition 2.1.0.8. Let M and N be two S-colored Σ-modules. Given $s. : I \to S$, define $M \circ N$ by

$$(M \circ N)(\tilde{s}. \mid \tilde{s}) := \bigsqcup_{p:I \to [k], s.:[k] \to S} M(s. \mid \tilde{s}) \otimes_{\Sigma_k} \mathrm{Ind}_{\Sigma_{I_1} \times \cdots \times \Sigma_{I_k}}^{\Sigma_I} \bigotimes_{i=1}^{k} N(s_.^i \mid s_i),$$

33

with $I_j := p^{-1}(j)$ and $s^j_. := \tilde{s}_.|_{I_j}$. This defines a monoidal structure $\circ$, called **plethysm**, on the category of S-colored Σ-modules in $\mathtt{V}$. The Σ-module $\mathbb{I}$ with $\mathbb{I}(s \mid s) = \mathbb{I}$ for all $s \in S$, and all other components equal to the initial object 0, is a unit for the plethysm.

Remark 2.1.0.9. An operad can be concisely defined as a (unital) monoid in the category of Σ-modules for the plethysm product. The monoid product $\gamma : \mathsf{Q} \circ \mathsf{Q} \to \mathsf{Q}$ of an operad is related to the partial compositions $\circ_i$ by $(s_. : [k] \to S, \phi \in \mathsf{Q}(s_. \mid s))$

$$\phi \circ_i \psi = \gamma(\phi; e_{s_1} \otimes \cdots \otimes e_{s_{i-1}} \otimes \psi \otimes e_{s_{i+1}} \otimes \cdots \otimes e_{s_k}).$$

The disadvantage of defining operads as monoids for the plethysm product is that it makes it difficult to describe the free operad functor in concrete terms.

Example 2.1.0.10. Assume given a set of objects $V = \{V_s\}_{s \in S}$ of $\mathtt{V}$ and assume that $\mathtt{V}$ has an internal hom-functor Map. There is then an S-colored operad $\mathsf{End}\langle V \rangle$ in $\mathtt{V}$ with

$$\mathsf{End}\langle V \rangle(s_. \mid s) := \mathrm{Map}(\bigotimes_{i \in I} V_{s_i}, V_s),$$

for $s_. : I \to S$. The maps $\circ_i$ are defined mimicking the compositions for multilinear maps, i.e.

$$\phi \circ_i \psi := \phi \circ (id_{s_1} \otimes \cdots \otimes id_{s_{i-1}} \otimes \psi \otimes id_{s_{i+1}} \otimes \cdots \otimes id_{s_n})$$

if $\phi : \bigotimes_{j=1}^n V_{s_j} \to V_s$. The units are given by $e_s = id_{V_s}$. This operad is called the **endomorphism operad** of V.

Definition 2.1.0.11. A S-colored **pseduo-cooperad** in $\mathtt{V}$ is the structure defined by reversing all arrows, i.e. it is an S-colored pseudo-operad in $\mathtt{V}^{op}$. Thus, for each tree T it has a morphism

$$\Delta_T : \mathsf{Q}(s_.^T \mid s_T) \to \mathsf{Q}(T)$$

in $\mathtt{V}$, and these satisfy certain coassociativity and equivariance conditions. Dualizing further, a pseudo-cooperad C is said to be a cooperad if it has counits $\varepsilon_s : \mathbb{I} \to \mathsf{C}(s \mid s)$ satisfying the conditions dual to those for operadic units.

2.2 Colored dg operads.

Global references for this section are the book [Loday and Vallette 2012] and the thesis [Laan 2004].

Definition 2.2.0.12. An S-colored dg Σ-module is an S-colored Σ-module in $\mathtt{Ch_k}$. For S-colored dg Σ-modules M and N we denote the set of natural transformations from M to N by $\mathrm{Hom}_\Sigma(M, N)$. The internal mapping space on dg vector spaces defines a dg vector space $\mathrm{Map}_\Sigma(M, N)$, such that $\mathrm{Hom}_\Sigma(M, N)$ is the set of degree zero cocycles in $\mathrm{Map}_\Sigma(M, N)$. In more detail,

$$\mathrm{Map}_\Sigma(M, N) = \lim(\mathrm{Map}(M, N) : (S \wr \Sigma \times S)^{op} \times (S \wr \Sigma \times S) \to \mathtt{Ch_k}).$$

A **dg operad** is a an operad in $\mathtt{Ch_k}$.

For a dg Σ-module E and an integer r, define $E\{r\}$ to be the Σ-module with

$$E\{r\}(s. \mid s) := E(s. \mid s)[r(1-n)] \otimes sgn_n^{\otimes r}.$$

for $s. : I \to S$, $n := \#I$. This is called **operadic suspension**, since if E has a dg (co)operad structure, then so will $E\{r\}$.

Remark 2.2.0.13. Operadic suspension satisfies the adjunction

$$\mathrm{Map}_\Sigma(M\{r\}, N) \cong \mathrm{Map}_\Sigma(M, N\{-r\}).$$

It also satisfies $\mathsf{End}\langle V \rangle\{r\} \cong \mathsf{End}(V[r])$ for any collection $V = \{V_s\}_{s \in S}$ of dg vector spaces, where $V[r] := \{V_s[r]\}_{s \in S}$.

Definition 2.2.0.14. The Σ-module $\mathbb{I}$ with $\mathbb{I}(s \mid s) = \mathbf{k}$ for all $s \in S$ and all other components equal to 0 has both a unique dg operad structure and a unique dg cooperad structure. A dg operad O is said to be **augmented** if it is equipped with a morphism of operads $\mathsf{Q} \to \mathbb{I}$. A dg cooperad C is said to be **coaugmented** if it is equipped with a morphism of cooperads $\mathbb{I} \to \mathsf{C}$.

A dg (pseudo-co)operad Q is said to be **reduced** if is trivial in artiy zero. In other words, if $\mathsf{Q}(\emptyset \mid s) = 0$ for all s, or, equivalently, if $\mathsf{Q}(T) = 0$ for all colored trees that have a univalent vertex.

The unit of an augmented dg operad must be a split inclusion so augmented dg operads are equivalent to dg pseduo-operads. Explicitly, the augmented dg operad Q is equivalent to the dg pseudo-operad $\mathrm{Ker}(\mathsf{Q} \to \mathbb{I})$. Similarly, coaugmented dg cooperads are equivalent to dg pseudo-cooperads.

Convention 2.2.0.15. From now on all dg (co)operads except endomorphism operads will be assumed (co)augmented and reduced. This means we can with little risk of confusion drop the distinction between dg (co)operads and dg pseudo-(co)operads, something we will do whenever convenient.

Definition 2.2.0.16. We write dgOp_S for the category of reduced and augmented S-colored dg operads and dgCoOp_S for the category of reduced and coaugmented S-colored dg cooperads.

The category dgOp_S has a model structure induced by that on dg vector spaces. A morphism $f : \mathsf{Q} \to \mathsf{Q}'$ in dgOp_S is

- a weak equivalence, also referred to as a quasi-isomorphism, if each

$$f_{(s.\,|s)} : \mathsf{Q}(s.\mid s) \to \mathsf{Q}'(s.\mid s)$$

is a quasi-isomorphism of dg vector spaces.

- a fibration if each $f_{(s.\,|s)}$ is a fibration of dg vector spaces.

- a cofibration if it satisfies the lifting property.

For a detailed account of this model structure, see [Hinich 1997].

2.2.1   The (co)bar construction.

Definition 2.2.1.1. Let O be a dg operad. A homogeneous **derivation** of O of degree q is an endomap v of the Σ-module O, satisfying

$$v(\varphi \circ_i \varphi') = v(\varphi) \circ_i \varphi' + (-1)^{q|\varphi|}\varphi \circ_i v(\varphi'),$$

for homogeneous $\varphi \in \mathsf{O}(s.\mid s)$, $\varphi' \in \mathsf{O}(s'.\mid s')$.

Given a (coaugmented) dg cooperad C, denote by $\overline{\mathsf{C}}$ the cokernel of the coaugmentation. The free dg pseudo-operad $\mathcal{F}(\overline{\mathsf{C}}[-1])$ has a filtration $\mathcal{F}(\overline{\mathsf{C}}[-1])_{(k)}$ given by the number of vertices in a tree. The cocompositions of the cooperad structure defines

$$\delta : \overline{\mathsf{C}}[-1] \to \mathcal{F}(\overline{\mathsf{C}}[-1])_{(2)}$$

of degree 1. Extend these by the Leibniz rule with respect to the $\circ_i$-products, to a derivation

$$\delta : \mathcal{F}(\overline{\mathsf{C}}[-1])_{(k)} \to \mathcal{F}(\overline{\mathsf{C}}[-1])_{(k+1)}.$$

The coassociativity of the cocomposition of C translates to the statement that δ squares to 0.

Definition 2.2.1.2. Let C be a (coaugmented) dg cooperad. The **cobar construction** $\mathcal{C}(\mathsf{C})$ on C is the augmented dg operad defined as follows. The underlying graded operad is that corresponding to the graded pseudo-operad $\mathcal{F}(\overline{\mathsf{C}}[-1])$. The differential is that given by the differential on C, together with the differential δ defined by the cooperad structure.

Definition 2.2.1.3. Define the cofree cooperad functor $\mathcal{F}_c$ by

$$\mathcal{F}_c(E)(s. \mid s) := \lim\big(\mathrm{Iso}(\mathsf{T}^S \downarrow (s. \mid s)) \xrightarrow{E} \mathsf{V}\big).$$

The Σ-module $\mathcal{F}_c(E)$ has a canonical cooperad structure, but it is actually only cofree for a restricted class of cooperads. The cobar construction has a dual construction, given as follows. Let P be an augmented dg operad and let $\overline{\mathsf{P}}$ be the kernel of the augmentation. The composition on P defines a nilsquare coderivation ∂ on $\mathcal{F}_c(\overline{\mathsf{P}}[1])$. The **bar construction** on P is the coaugmented dg cooperad $\mathcal{B}(\mathsf{P})$ corresponding to the pseudocooperad $\mathcal{F}_c(\overline{\mathsf{P}}[1])$ and with the extra differential ∂.

Lemma 2.2.1.4. [Hinich 1997] If $\mathsf{C} \in \mathtt{dgCoOp}_S$ is such that either each component $\mathsf{C}(s. \mid s)$ is concentrated in non-negative degrees, or it has a complete filtration $F_1\mathsf{C} \subset F_2\mathsf{C} \subset \ldots$ compatible with differentials in the sense that $d(F_p) \subset F_p$ and compatible with cooperad structure in the sense that $\delta(F_p) \subset \mathcal{F}_c(F_{p-1})$, then $\mathcal{C}(\mathsf{C})$ is cofibrant in $\mathtt{dgOp}_S$.

It follows that $\mathcal{C}\mathcal{B}(\mathsf{P})$ is cofibrant for every $\mathsf{P} \in \mathtt{dgOp}_S$, because $\mathcal{B}(\mathsf{P})$ has a filtration as required, defined by the number of vertices in decorated trees.

Lemma 2.2.1.5. The bar and cobar constructions are adjoint functors.

A particular case of the bar-cobar adjunction is a canonically defined morphism $\mathcal{C}\mathcal{B}(\mathsf{P}) \to \mathsf{P}$, for every dg operad P.

Corollary 2.2.1.6. For every dg operad P, there is a canonically defined quasi-isomorphism $\mathcal{C}\mathcal{B}(\mathsf{P}) \to \mathsf{P}$.

2.2.2 Koszul duality theory.

Definition 2.2.2.1. A dg operad P is called **quadratic** if it admits a presentation as a quotient $\mathsf{P} = \mathcal{F}(E)/\mathsf{I}$, for I an operadic ideal generated by some $R \subset \mathcal{F}(E)_{(2)}$. Since R is homogeneous of degree 2 with respect to the grading by the number of vertices, P will inherit an additional grading $\mathsf{P}_{(k)} = \mathrm{Im}(\mathcal{F}(E)_{(k)} \to \mathsf{P})$.

The **Koszul dual** of a quadratic dg operad P is the cooperad

$$\mathsf{P}^{\mathsf{i}} := \mathrm{Ker}(\partial : \mathcal{F}_c(\mathsf{P}_{(1)}[1]) \to \mathcal{B}(\mathsf{P})).$$

A quadratic dg operad is called **Koszul** if the canonical morphism $\mathcal{C}(\mathsf{P}^{\mathsf{i}}) \to \mathsf{P}$ is a quasi-isomorphism.

For a Koszul dg operad P we define $\mathsf{P}_\infty := \mathcal{C}(\mathsf{P}^{\mathsf{i}})$. The operad P_∞ is always cofibrant, by the lemmata at the end of the previous subsection.

2.2.3 Deformation complexes.

Let O be a dg operad and C be a dg cooperad. Let $\mathrm{Map}_\Sigma(\mathsf{C}, \mathsf{O})$ denote the internal mapping space of Σ-modules. Take $f, g \in \mathrm{Map}_\Sigma(\mathsf{C}, \mathsf{O})$. Then define

$$(f \circ_i g)_{(s.|s)} := \sum_{s_.^1 \circ_i s_.^2 = s.} \circ_i (f_{(s_.^1|s_1)} \otimes g_{(s_.^2|s_2)}) \delta_{s_.^1 \circ_i s_.^2} : \mathsf{C}(s. \mid s) \to \mathsf{O}(s. \mid s).$$

In the above $\delta_{s_.^1 \circ_i s_.^2}$ denotes a partial cocomposition of C. These operations define a dg Lie algebra structure on $\mathrm{Map}_\Sigma(\mathsf{C}, \mathsf{O})$, by

$$[f, g] := -\sum_i f \circ_i g + (-1)^{|f||g|} \sum g \circ_j f.$$

(The sums are over all compositions which make sense.)

Definition 2.2.3.1. The space $\mathrm{Map}_\Sigma(\mathsf{C}, \mathsf{O})$, considered as a dg Lie algebra, is called the **convolution dg Lie algebra** of C and O.

Remark 2.2.3.2. A morphism of dg Σ-modules $\mathcal{C}(\mathsf{C}) \to \mathsf{O}$ is a morphism of dg operads if and only if it is a Maurer-Cartan element of the dg Lie subalgebra $\mathrm{Map}_\Sigma(\overline{\mathsf{C}}, \mathsf{O}) \subset \mathrm{Map}_\Sigma(\mathsf{C}, \mathsf{O})$.

Definition 2.2.3.3. Given a morphism $f : \mathcal{C}(\mathsf{C}) \to \mathsf{O}$ of dg operads, we define $\mathrm{Def}(f)$, or in more detailed notation,

$$\mathrm{Def}(\mathcal{C}(\mathsf{C}) \xrightarrow{f} \mathsf{O}),$$

to be the dg Lie algebra $\mathrm{Map}_\Sigma(\mathsf{C}, \mathsf{O})$, twisted by the Maurer-Cartan element f. It is called the **deformation complex** of f.

2.2.4 Algebras for dg operads.

Definition 2.2.4.1. Let $\mathsf{O} \in \mathtt{dgOp}_S$. A left O-**module** is a dg Σ-module M together with a morphism $\mathsf{O} \circ M \to M$ such that the two natural composites

$$\mathsf{O} \circ \mathsf{O} \circ M \to \mathsf{O} \circ M \to M$$

agree. A right O-module is defined as a dg Σ-module together with a morphism $M \circ \mathsf{O} \to M$, satisfying the analogous condition with the operad instead placed to the right of the plethysm.

Let $\mathsf{C} \in \mathtt{dgCoOp}_S$. A left C-comodule is a dg Σ-module M together with a morphism $\mathsf{C} \to \mathsf{C} \circ M$ such that the two natural composites

$$M \to \mathsf{C} \circ M \to \mathsf{C} \circ \mathsf{C} \circ M$$

agree. A right C-comodule is defined analogously.

Definition 2.2.4.2. Let $\mathsf{O} \in \mathtt{dgOp}_S$ be a dg operad. A **dg O-algebra** is a collection of vector spaces $V = \{V_s\}_{s \in S}$ and a morphism of dg operads

$$\mathsf{O} \to \mathsf{End}\langle V \rangle.$$

A dg O-algebra is also called a representation of O. Equivalently, we can regard V as a dg Σ-module concentrated in arity zero and an O-algebra structure on V as a module structure $\mathsf{O} \circ V \to V$.

Let $\mathsf{C} \in \mathtt{dgCoOp}_S$. A **dg C-coalgebra** is a $V = \{V_s\}_{s \in S}$ together with a left comodule structure $V \to \mathsf{C} \otimes V$.

Remark 2.2.4.3. The above definition implies that the free O-algebra on V is the dg vector space $\mathsf{O}(V) = \mathsf{O} \circ V$. More explicitly,

$$\mathsf{O}(V)_s := \bigoplus_{n \geq 1} \bigoplus_{s.:[n] \to S} \mathsf{O}(s. \mid s) \otimes_{\Sigma_n} \bigotimes_{i=1}^{n} V_{s_i}.$$

The free dg coalgebra on a dg vector space V for a dg cooperad C, denoted $\mathsf{C}(V)$, is defined by the same formula.

Definition 2.2.4.4. Let $\mathsf{O} \in \mathtt{dgOp}_S$ be a dg operad. A **morphism** of dg O-algebras $f : A \to B$ is a morphism of dg vector spaces with the property that

$$
\begin{array}{ccc}
\mathsf{O}(A) & \longrightarrow & A \\
\mathsf{O}(f) \downarrow & & \downarrow f \\
\mathsf{O}(B) & \longrightarrow & B
\end{array}
$$

commutes. Morphisms of coalgebras for a cooperad are defined analogously.

Definition 2.2.4.5. Let $\mathsf{O} \in \mathtt{dgOp}_S$ and let $\mu_A : \mathsf{O} \to \mathsf{End}\langle A \rangle$ be a O-algebra. An O-**algebra derivation** of A is a morphism $\xi : A \to A$ such that

$$\xi \circ_1 \mu_A(\phi) = \sum_{i=1}^{n} \mu_A(\phi) \circ_i \xi$$

in $\mathsf{End}\langle A \rangle$ for all n, $s. : [n] \to S$, $s \in S$ and $\phi \in \mathsf{O}(s. \mid s)$. The Lie bracket on the set of P-derivations is such that it forms a Lie subalgebra $\mathrm{Der}_{\mathsf{O}}(A)$ of $gl(A)$.

A **coderivation** of a C-coalgebra is defined dually.

Definition 2.2.4.6. Let P be a Koszul dg operad and let A be a P_∞-algebra. We define the **deformation complex of** A to be the dg Lie algebra

$$C_{\mathsf{P}}(A, A) := \mathrm{Def}(\mathsf{P}_\infty \to \mathsf{End}\langle A \rangle).$$

The cohomology of the deformation complex of A is denoted $H_{\mathsf{P}}(A, A)$. It is also called the P-cohomology of A with coefficients in A.

Remark 2.2.4.7. When A is a P-algebra concentrated in nonnegative degrees, then $H_{\mathsf{P}}(A, A)$ is a right derived functor of $A \mapsto \mathrm{Der}_{\mathsf{P}}(A)$. In more detail this means the following. With some extra care one can define the notion of a module for a P-algebra A and, for any such module M, a vector space $\mathrm{Der}_{\mathsf{P}}(A, M)$. For any cofibrant replacement $R \to A$ of the P-algebra A we have $H_{\mathsf{P}}(A, A) \cong H(\mathrm{Der}_{\mathsf{P}}(R, A))$. Hence we may think of the deformation dg Lie algebra $C_{\mathsf{P}}(A, A)$ as the dg lie algebra of homotopy derivations of the P-algebra A. In particular, for $A = \mathsf{P}(V)$ a free P-algebra concentrated in nonnegative degrees, the canonical inclusion

$$\mathrm{Der}_{\mathsf{P}}(A) \hookrightarrow \mathrm{Map}(\mathbb{I}, \mathrm{Map}(A, A)) \hookrightarrow C_{\mathsf{P}}(A, A),$$

defined by the coaugmentation $\mathbb{I} \to \mathsf{P}^{\mathsf{i}}$, is a quasi-isomorphism.

Definition 2.2.4.8. Let P be a Koszul dg operad and let A be a dg P-algebra. The morphism $\mathcal{C}(\mathsf{P}^{\mathsf{i}}) \to \mathsf{P}$ defines a degree $+1$ map

$$\mathsf{P}^{\mathsf{i}}(A) = \mathsf{P}^{\mathsf{i}} \circ A \to \mathsf{P} \circ A,$$

which by postcomposition with the algebra structure $\mathsf{P} \circ A \to A$ defines a degree $+1$ map $\mathsf{P}^{\mathsf{i}}(A) \to A$. It extends uniquely to a coderivation ∂ of degree $+1$ of the cofree P^{i}-coalgebra $\mathsf{P}^{\mathsf{i}}(A)$. Associativity of $\mathsf{P} \circ A \to A$ implies that it squares to 0. The P-**bar construction** on A is the dg P^{i}-coalgebra $\mathcal{B}_{\mathsf{P}}(A)$ equal to $\mathsf{P}^{\mathsf{i}}(A)$ but equipped with the additional differential ∂).

The P-**homology** of A is defined to be the cohomology of $\mathcal{B}_{\mathsf{P}}(A)$.

Remark 2.2.4.9. It is easy to show that $C_{\mathsf{P}}(A, A) = \mathrm{Map}(\mathcal{B}_{\mathsf{P}}(A), A)$ as a complex. By cofreeness the latter space can be identified with the space of coderivations $\mathrm{Coder}_{\mathsf{P}^{\mathrm{i}}}(\mathcal{B}_{\mathsf{P}}(A), \mathcal{B}_{\mathsf{P}}(A))$, which has a natural Lie bracket. If we transport that bracket to $\mathrm{Map}(\mathcal{B}_{\mathsf{P}}(A), A)$, then $C_{\mathsf{P}}(A, A) = \mathrm{Map}(\mathcal{B}_{\mathsf{P}}(A), A)$ is an equality of dg Lie algebras.

Lemma 2.2.4.10. Let $f : \mathcal{C}(\mathsf{C}) \to \mathsf{P}$ be a morphism of dg operads. Then for every P-algebra A we get an extra differential ∂ on the cofree coalgebra $\mathsf{C}(A)$. Let $\mathcal{B}_f(A)$ denote $\mathsf{C}(A)$ with the additional differential added. The morphism f is a quasi-isomorphism if and only if the canonical map
$$V \to \mathcal{B}_f(\mathsf{P}(V))$$
is a quasi-isomorphism for all V.

We will not give a proof of this standard result, but will indicate the argument. A morphism $f : \mathcal{C}(\mathsf{C}) \to \mathsf{P}$ defines a twisted differential d_f on the plethysm $\overline{\mathsf{C}} \circ \mathsf{P}$. The Σ-module $\overline{\mathsf{C}} \circ \mathsf{P}$ with this extra differential is called the (left) twisting composite product and is denoted $\overline{\mathsf{C}} \circ_f \mathsf{P}$. One shows that $\mathcal{B}_f(\mathsf{P}(V)) = (\mathsf{C} \circ_f \mathsf{P}) \circ V$, whence the statement of the lemma is equivalent to the statement that $\mathbb{I} \to \mathsf{C} \circ_f \mathsf{P}$ is a quasi-isomorphism. First one proves that the map $\mathbb{I} \to \mathsf{C} \circ_{id} \mathcal{C}(\mathsf{C})$ always is a quasi-isomorphism. Then it is easy to see that $id \circ f : \mathsf{C} \circ_{id} \mathcal{C}(\mathsf{C}) \to \mathsf{C} \circ_f \mathsf{P}$ is a quasi-isomorphism iff f is.

Definition 2.2.4.11. Let P be a Koszul dg operad and let A and B be two P_∞-algebras. An ∞-**morphism** (of P-algebras) $F : A \to B$ is a morphism of P^{i}-coalgebras from $\mathcal{B}_{\mathsf{P}}(A)$ to $\mathcal{B}_{\mathsf{P}}(B)$.

Part of the data of an ∞-morphism F is a morphism of dg vector spaces $F_{(0)} : A \to B$. We say F is an ∞-**(quasi-)isomorphism** if $F_{(0)}$ is a (quasi-)isomorphism of dg vector spaces.

An ∞-morphism of Lie-algebras is called an L_∞-morphism. An ∞-morphism of Ass-algebras is called an A_∞-morphism.

Remark 2.2.4.12. Note that an ∞-morphism of P-algebras is not the same thing as a morphism of P_∞-algebras.

2.2.5 Examples.

Let Lie denote the singleton-colored dg operad whose representations are dg Lie algebras. It is generated by an operation
$$\lambda \in \mathsf{Lie}(2) = sgn_2,$$

since Lie algebras are defined by a binary bracket operation. Moreover, the operad Lie is quadratic since the Jacobi identity involves (sums of) compositions of exactly two brackets. The Koszul dual cooperad is $\mathsf{Lie}^{\mathsf{i}} = \mathsf{coCom}\{-1\}$, for coCom defined as follows. For $n \geq 1$,

$$\mathsf{coCom}(n) := \mathbf{k}_n$$

is the trivial Σ_n-representation with 0 differential. The morphisms

$$\mathbf{k}_n \to \mathbf{k}_{n-n'+1} \otimes \mathbf{k}_{n'}, 1 \mapsto 1 \otimes 1$$

assemble to define a cooperad structure on coCom. The morphism $\mathcal{C}(\mathsf{coCom}\{-1\}) \to \mathsf{Lie}$ is a quasi-isomorphism, i.e. Lie is Koszul. An L_∞ algebra is thus the same thing as a representation

$$f : \mathcal{C}(\mathsf{coCom}\{-1\}) \to \mathsf{End}\langle V \rangle.$$

Such a representation is a Maurer-Cartan element in

$$\begin{aligned}
\mathrm{Map}_\Sigma(\mathsf{coCom}\{-1\}, \mathsf{End}\langle V \rangle &= \mathrm{Map}_\Sigma(\mathsf{coCom}, \mathsf{End}\langle V \rangle\{1\}) \\
&= \prod_{n \geq 1} \mathrm{Map}_\Sigma(\mathbf{k}_n, \mathrm{Map}(V[1]^{\otimes n}, V[1])) \\
&= \mathrm{Map}(S^+(V[1]), V[1])).
\end{aligned}$$

One checks the Maurer-Cartan elements are the same as nilsquare degree $+1$ coderivations of $S^+(V[1])$. Assume

$$f : \mathcal{C}(\mathsf{coCom}\{-1\}) \to \mathsf{End}\langle L \rangle$$

is an L_∞ algebra. The deformation complex $\mathrm{Def}(f) = C_{\mathsf{Lie}}(L, L)$ is, in the case that f happens to be a usual Lie algebra, easily checked to be equal to the degree-shifted and truncated Chevalley-Eilenberg cochain complex;

$$C_{\mathsf{Lie}}(L, L) = C^+_{\mathrm{CE}}(L, L)[1],$$

with its Chevalley-Eilenberg differential and bracket. Similarly, A_∞ algebras are representations of an operad $\mathsf{Ass}_\infty = \mathcal{C}(\mathsf{coAss}\{-1\})$, where $\mathsf{coAss}(n) = \mathbf{k}[\Sigma_n]$ is the Koszul dual to the Koszul dg operad controlling dg associative algebras. The operadic deformation complex of a usual associative algebra is the degree-shifted and truncated Hoschschild cochain complex

$$C_{\mathsf{Ass}}(A, A) = C^+_{\mathrm{Hoch}}(A, A)[1],$$

with its Hochschild differential and Gerstenhaber bracket.

Definition 2.2.5.1. For an A_∞ algebra A we *define* the Hochschild cochain complex by

$$C_{\mathrm{Hoch}}(A, A) := A \oplus C_{\mathsf{Ass}}(A, A)[-1] = \mathrm{Map}(T(A[1]), A).$$

The dg Lie algebra structure on the deformation complex extends to $C_{\mathrm{Hoch}}(A, A)[-1]$. Analogously, if L is an L_∞ algebra, then we *define* the Chevalley-Eilenberg cochain complex by

$$C_{\mathrm{CE}}(L, L) := L \oplus C_{\mathsf{Lie}}(L, L)[-1].$$

If A and L are algebras concentrated in degree zero, then these definitions coincide with the classical ones.

2.2.6 Deformations of algebras.

Let P be a Koszul dg operad and let

$$\mu : \mathsf{P}_\infty \to \mathsf{End}\langle A \rangle$$

be a P_∞-algebra.

Definition 2.2.6.1. A **formal deformation** of μ is a Maurer-Cartan element $\hbar\nu \in C_{\mathsf{P}}(A, A)[[\hbar]]$. Two formal deformations $\hbar\nu$ and $\hbar\nu'$ are **equivalent** if there exists an $\hbar$-linear ∞-isomorphism

$$F : (A[[\hbar]], \mu + \hbar\nu) \to (A[[\hbar]], \mu + \hbar\nu'),$$

with the property that $F_{(0)} : A[[\hbar]] \to A[[\hbar]]$ has the form $id + O(\hbar)$.

An **infinitesimal** deformation of μ is a Maurer-Cartan element of homogeneous degree one in $\hbar$ in $C_{\mathsf{P}}(A, A)[[\hbar]]/\langle\hbar^2\rangle$. Equivalence of infinitesimal deformations is defined as in the formal case, but modding out by $\hbar^2$.

Note that a formal deformation defines an infinitesimal deformation.

Proposition 2.2.6.2. The set of infinitesimal deformations of μ is in bijection with the set of degree 1 cocycles of $C_{\mathsf{P}}(A, A)$, and two infinitesimal deformations are equivalent if and only if the two corresponding cocycles are cohomologous.

Two formal deformations are equivalent if and only if their corresponding infinitesimal deformations are equivalent. In particular, if $H^1_{\mathsf{P}}(A, A) = 0$, then any formal deformation is equivalent to the trivial one.

We remark that is often possible to dispense with the parameter $\hbar$, due to nilpotency or automatic presence of a suitable filtration.

2.2.7   Homotopy transfer.

Assume W is a P_∞-algebra, for P some Koszul operad, and assume we are given a homotopy retract

$$V \underset{p}{\overset{j}{\rightleftarrows}} W \circlearrowleft \delta^{-1} \ ,$$

$$id_W - j \circ p = d_W \circ \delta^{-1} + \delta^{-1} \circ d_V.$$

Then there is an induced P_∞-algebra structure on V, such that j is a quasi-isomorphism of P_∞ algebras. The induced algebra structure is called the **homotopy transfer** to V and is defined using the following lemma.

Lemma 2.2.7.1. The homotopy retract defines a morphism of dg co-operads

$$\Psi : \mathcal{B}(\mathsf{End}\langle W \rangle) \to \mathcal{B}(\mathsf{End}\langle V \rangle).$$

Let us give the definition of Ψ. Since the bar construction is cofree, forgetting differentials, it is enough to specify Ψ as a map

$$\Psi : \mathcal{B}(\mathsf{End}\langle W \rangle) \to \mathsf{End}\langle V \rangle[1].$$

Recall

$$\mathcal{B}(\mathsf{End}\langle W \rangle)(s. \mid s) = \lim_{\mathrm{Iso}(\mathbf{T}^S \downarrow (s.\mid s))} \mathsf{End}\langle W \rangle[1](T)$$

is a limit over decorated rooted S-colored trees. The restriction of the morphism Ψ to

$$\mathsf{End}\langle W \rangle[1](T) \to \mathsf{End}\langle V \rangle[1]$$

is defined by decorating all input legs with j, all edges by δ^{-1}, the root leg with p, and keeping the vertex decorations as they are. For example, the decorated tree
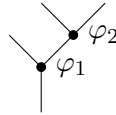
would be mapped to the operation

$$p \circ \phi_1 \circ \left(j \otimes (\delta^{-1} \circ \phi_2 \circ (j \otimes j)) \right).$$

One can verify that Ψ commutes with the bar differentials.

By the bar-cobar adjunction, the P_∞-structure

$$\mathsf{P}_\infty = \mathcal{C}(\mathsf{P}^i) \to \mathsf{End}\langle W \rangle$$

on W is equivalently given as a morphism of cooperads

$$\mathsf{P}^{\mathsf{i}} \to \mathcal{B}(\mathsf{End}\langle W \rangle).$$

Postcomposing this with Ψ defines $\mathsf{P}^{\mathsf{i}} \to \mathcal{B}(\mathsf{End}\langle V \rangle)$, which is equivalent to a morphism $\mathsf{P}_{\infty} = \mathcal{C}(\mathsf{P}^{\mathsf{i}}) \to \mathsf{End}\langle V \rangle$.

The twist construction for colored operads

Let $\mathfrak{g}$ be a dg Lie algebra. Recall that a Maurer-Cartan element is an element $\pi \in \mathfrak{g}^1$ satisfying the equation $d\pi + \frac{1}{2}[\pi, \pi] = 0$. Given a Maurer-Cartan element one can define the twisted dg Lie algebra $\mathfrak{g}_\pi$, with the same underlying graded vector space and the same bracket, but with the new differential $d_\pi := d + [\pi, \,]$. This procedure is known to generalize to L_∞ algebras, but then one has to change not just the differential: the differential changes by the same formula and the n-ary bracket λ_n is replaced by $\sum_{k \geq 0} \frac{1}{k!}\lambda_{k+n}(\pi^{\otimes k}, \,)$. Consider now the case of a (single-colored) dg operad Q equipped with a morphism of dg operads

$$\mathsf{Lie}_\infty \to \mathsf{Q}.$$

Then any Q-algebra $\mathfrak{g}$ is also an L_∞ algebra. Let $\mathfrak{g}_\pi$ be the twisted algebra corresponding to some Maurer-Cartan element. It is a new L_∞ algebra, as explained above, but it is generally not a new Q-algebra. Thomas Willwacher wrote down the definition of an operad $\mathrm{Tw}\,\mathsf{Q}$ with the property that $\mathfrak{g}_\pi$ is a $\mathrm{Tw}\,\mathsf{Q}$-algebra, in [Willwacher 2010]. The construction was later conceptually clarified in [Dolgushev and Willwacher 2012], where it was shown that it defines a comonad

$$\mathrm{Tw} : (\mathsf{Lie}_\infty \downarrow \mathtt{dgOp}) \to (\mathsf{Lie}_\infty \downarrow \mathtt{dgOp})$$

on the category of dg operads under the L_∞ operad. In this chapter we generalize the construction to colored operads. This generalization allows us to recognize the construction as a derived adjoint functor to

another simple construction, which we call the Lie module construction. The Lie-module construction can not be defined without the introduction of (more than one-)colored operads, so the adjunction is not visible when only working with single-colored operads. Thus, our work is more than just a straight-forward generalization of the single-colored twisting to the colored setting, but at the same time it clarifies the single-colored twisting.

Definition 3.0.7.2. Given a pointed set S with base point $*$, define $L_\infty\text{-mod}(S)$ to be the operad whose representations are an L_∞ algebra V_*, and a collection of L_∞ representations $V_* \to gl(V_s)$, parametrized by $s \in S \setminus \{*\}$. Note that if $S = \{*\}$ is a singleton, then $L_\infty\text{-mod}(S)$ is the operad of L_∞ algebras. Similarly, we introduce the operads $L\text{-mod}(S)$, corresponding to a strict Lie algebra and strict Lie algebra representations.

Given a operad P with set of colors X, let $L\text{-mod}(\mathsf{P})$ be the $S := X \sqcup \{*\}$-colored operad whose representations are a P-algebra $V = (V_x)_{x \in X}$, a dg Lie algebra V_*, and a dg Lie algebra representation $V_* \to \mathrm{Der}_\mathsf{P}(V)$ by P-derivations. The assignment $\mathsf{P} \mapsto L\text{-mod}(\mathsf{P})$ is a functor

$$L\text{-mod} \colon \mathsf{dgOp}_X \to (L_\infty\text{-mod}(S) \downarrow \mathsf{dgOp}_S),$$

which we term the **Lie module construction**. We define a functor

$$\mathrm{tw} : (L_\infty\text{-mod}(S) \downarrow \mathsf{dgOp}_S) \to \mathsf{dgOp}_X,$$

essentially by a Kan extension, and prove that there is an adjunction

$$\mathrm{Ho}(L\text{-mod}) \dashv \mathrm{Ho}(\mathrm{tw})$$

on homotopy categories. We call this functor the **small twist construction**. On an endomorphism operad

$$L_\infty\text{-mod}(S) \to \mathsf{End}\langle \mathfrak{g}, A \rangle$$

($\mathfrak{g}$ is in the distinguished color $*$) it has the suggestive form

$$\mathrm{tw}\,\mathsf{End}\langle \mathfrak{g}, A \rangle = C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle A \rangle),$$

using that components the of $\mathsf{End}\langle A \rangle$ have an induced $\mathfrak{g}$-module structure. Set $\tilde{S} := S \sqcup \{\tilde{*}\}$ and let $\delta : \tilde{S} \to S$ be the unique map of pointed sets which is the identity on $X \subset \tilde{S}$. We then define the **large twist construction** to be the small twist construction on $\delta^* \mathsf{Q}$:

$$\mathrm{Tw}\,\mathsf{Q} := \mathrm{tw}\,\delta^* \mathsf{Q}.$$

Our large twist construction coincides with that defined by Willwacher, when restricted to single-colored operads. On an endomorphism operad as above it equals

$$\mathrm{Tw}\,\mathsf{End}\langle \mathfrak{g}, A\rangle = C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle \mathfrak{g}, A\rangle),$$

using that $\mathfrak{g}$ always has an adjoint action on itself, as well as the action on A. Apart from the homotopy adjunction formula we also prove several properties about how these three functors behave in relation to Koszulity and deformation complexes. No part of this chapter has previously been published.

3.0.8 Model structures.

For a set X, define dgOp_X to be the category of augmented dg X-colored operads. For a pointed set S we shall call the comma category $\mathtt{Twist}_S :=$ $(L_\infty\text{-mod}(S) \downarrow \mathsf{dgOp}_S)$ the category of S-colored **operads with twist data**.

We use the standard model structure on dg vector spaces (over a field of characteristic zero), for which weak equivalences are quasi-isomorphisms and fibrations are degree-wise surjections (and cofibrations are defined by the lifting property), and the corresponding model structure on dg operads which defines quasi-isomorphisms and fibrations componentwise, cf. 2.2. Any comma category of objects over a fixed object in an ambient model category has a canonical model structure. We use this to equip the category of operads with twist data with a model structure. Explicitly, a morphism

$$f : (L_\infty\text{-mod}(S) \to \mathsf{Q}) \to (L_\infty\text{-mod}(S) \to \mathsf{Q}')$$

in $\mathtt{Twist}_S$ is defined to be a weak equivalence/fibration/cofibration if $f : \mathsf{Q} \to \mathsf{Q}'$ is a weak equivalence/fibration/cofibration in dgOp_S.

3.1 The twist construction.

3.1.1 The Lie-module construction.

Given a operad P with set of colors X, let $L\text{-mod}(\mathsf{P})$ be the $S := X \sqcup \{*\}$-colored operad whose representations are a P-algebra $V = (V_x)_{x \in X}$, a dg Lie algebra V_*, and a dg Lie algebra representation $V_* \to \mathrm{Der}_{\mathsf{P}}(V)$ by P-derivations. The assignment $\mathsf{P} \mapsto L\text{-mod}(\mathsf{P})$ is clearly functorial and

there is an obvious morphism L_∞-mod$(S) \to L$-mod(P), considering $*$ as the base-point of S, so we may regard the construction as a functor

$$L\text{-mod} : \mathtt{dgOp}_X \to \mathtt{Twist}_S.$$

Call this functor the **Lie-module construction**.

Lemma 3.1.1.1. The Lie module construction preserves weak equivalences.

Before the proof we make a preliminary remark. Let M and N be S-colored dg Σ-modules. Define the plethysm product over X, denoted $\circ_X$, by

$$(M \circ_X N)(\tilde{s}. \mid \tilde{x}) := \bigsqcup_{s.:[k]\to S, p:I\to s.^{-1}(X)} M(s. \mid \tilde{x}) \otimes_{\Sigma_k} \bigotimes_{i=1}^{k} N(s^i_. \mid s_i),$$

with $I_j := p^{-1}(j)$ and $s^j_. := \tilde{s}.|_{I_j}$. Note $s_i \in X$ in $N(s^i_. \mid s_i)$. Succinctly, the plethysm product over X allows the Σ-modules to have S-colored inputs but only uses the colors in X for composition.

Proof. Let $(f : \mathsf{P} \to \mathsf{P}') \in \mathtt{dgOp}_X$ be a weak equivalence. The Leibniz rule for derivations gives us an identification

$$L\text{-mod}(\mathsf{P}) = P \circ_X L\text{-mod}(S) + \mathsf{Lie}$$

as S-colored dg Σ-modules. It follows that we obtain

$$L\text{-mod}(f) : L\text{-mod}(\mathsf{P}) \to L\text{-mod}(\mathsf{P}')$$

by the formula $L\text{-mod}(f) = f \circ_X id_{L\text{-mod}(S)} + id_{\mathsf{Lie}}$. This is clearly a quasi-isomorphism whenever f is. $\qquad\square$

Let us record some further homotopical properties of the operads L-mod(P). We shall use the following fact when dealing with S-colored operads. Let $s.$ be a list in S that hits the base point k times. Then there is a unique list $x.$ in X, of length n if $s.$ has length $n + k$, such that $*^k \cup x.$ and $s.$ differ by a (k, n)-shuffle. Here $*^k \cup x. : [k] \sqcup I \to S$ constantly equal to the distinguished color $*$ on $[k]$, $I := s.^{-1}(X)$ and $x.$ is the restriction of $s.$ to I. This means that an S-colored operad Q can be specified by giving only its components for sequences of the form $*^k \cup x..$.

Let P be an X-colored augmented dg operad with a cobar resolution $\mathsf{P}_\infty := \mathcal{C}(\mathsf{C})$. Define L-$\mathrm{mod}(\mathsf{P})_\infty$ to be the operad whose representations are an L_∞ algebra L, a P_∞-algebra A and an L_∞ morphism

$$L \to \mathrm{Def}(\mathsf{P}_\infty \to \mathsf{End}\langle A, A \rangle).$$

It is easy to see that L-$\mathrm{mod}(\mathsf{P})_\infty = \mathcal{C}(\mathsf{LC})$ for a certain cooperad LC wich can described as follows. Its coalgebras are a cocommutative coalgebra $B[1]$, a C-coalgebra F and a B-comodule structure $\rho : F \to B \otimes F$, such that for all n, n-ary cooperations $\gamma \in \mathsf{C}$ and $1 \leq i \leq n$, $(id_B \otimes \gamma)\rho = \tau(id_F^{\otimes i-1} \otimes \rho \otimes id_F^{\otimes n-i})\gamma$ for τ the permutation isomorphism $F^{\otimes i-1} \otimes F \otimes B \otimes F^{\otimes n-i} \cong B \otimes F^{\otimes n}$. The description in terms of components is that its restriction to the color $*$ equals a copy of the cooperad Lie^i, $\mathsf{LC}(*^k \cup x. \mid x) = sgn_k[k] \otimes \mathsf{C}(x. \mid x)$ for $k \geq 0$ and $x \in X$, while all other components are 0.

Lemma 3.1.1.2. If $\mathcal{C}(\mathsf{C})$ is a resolution of P, then $\mathcal{C}(\mathsf{LC})$ is a resolution of L-$\mathrm{mod}(\mathsf{P})$.

Proof. We will verify that the LC-bar construction on a free L-$\mathrm{mod}(\mathsf{P})$-algebra is the space of generators. To this end, let $F = (U, V)$, where $U = F_*$ and $V = \{F_x\}_{x \in X}$. The free L-$\mathrm{mod}(\mathsf{P})$-algebra on F has as component in the distinguished color the free Lie algebra $L(U)$ on U, and the free P-algebra $\mathsf{P}(T(U) \otimes V)$ on the free $L(U)$-module $T(U) \otimes V$ on V in the remaining colors. The bar construction on this has component $\mathcal{B}_{\mathsf{Lie}}(L(U))$ in the distinguished color, and the Chevalley-Eilenberg homology complex $C^{CE}(L(U), \mathcal{B}_f(\mathsf{P}(T(U) \otimes V)))$ in the remaining colors, where f denotes the map $\mathsf{C} \to \mathsf{P}$.

The inclusion $U \to \mathcal{B}_{\mathsf{Lie}}(L(U))$ is a quasi-isomorphism since $L(U)$ is a free Lie algebra. The inclusion $T(U) \otimes V \to \mathcal{B}_f(\mathsf{P}(T(U) \otimes V))$ is a quasi-isomorphism since we assumed f to be a resolution of P and $\mathsf{P}(T(U) \otimes V)$ is free. But $T(U) \otimes V$ is a free $L(U)$-module, so the inclusion $V \to C^{CE}(L(U), T(U) \otimes V)$ is a quasi-isomorphism. All-in-all, this shows that the inclusion

$$(U, V) \to (\mathcal{B}_{\mathsf{Lie}}(L(U)), C^{CE}(L(U), \mathcal{B}_f(\mathsf{P}(T(U) \otimes V))))$$

is a quasi-isomorphism. □

Remark 3.1.1.3. Assume that P is quadratic. Then L-$\mathrm{mod}(\mathsf{P})$ is quadratic and the Koszul dual L-$\mathrm{mod}(\mathsf{P})^\mathsf{i}$ equals LP^i. Hence it follows by the preceeding arguments that L-$\mathrm{mod}(\mathsf{P})$ is Koszul if P is Koszul.

3.1.2 The small twist construction.

Let $p_X : S \wr \Sigma \times X \to X \wr \Sigma \times X$ be the functor that maps an object $(s. : I \to S \mid x)$ to the object $(s. : s.^{-1}(X) \to X \mid x)$ and a morphism $\sigma : (s. \mid x) \to (s'. \mid x)$ (recall this means $s'. \circ \sigma = s.$) to $\sigma|_{s'.^{-1}(I')}$. Define

$$R_X : [S \wr \Sigma \times X, \mathtt{Ch_k}] \to [X \wr \Sigma \times X, \mathtt{Ch_k}]$$

to be the right Kan extension along p_X. By first restricting output colors to X we may also consider it as defined

$$R_X : [S \wr \Sigma \times S, \mathtt{Ch_k}] \to [X \wr \Sigma \times X, \mathtt{Ch_k}].$$

We will switch freely between the two domains for R_X and rely on the context to make it clear which domain we have in mind.

Let M be an S-colored dg Σ-module. Then

$$R_X M(x. \mid x) = \lim((p_X \downarrow (x. \mid x)) \xrightarrow{M} \mathtt{Ch_k}).$$

It follows that for any object $p_X(s. \mid x) \to (x. \mid x)$ of the comma category $(p_X \downarrow (x. \mid x))$ there is a universal projection

$$R_X M(x. \mid x) \to M(s. \mid x).$$

By universality of the projections out of a limit we obtain a natural transformation

$$R_X M \circ R_X N \to R_X(M \circ_X N).$$

Any dg S-colored operad defines, by restriction of allowed outputs, an operad for the plethysm over X. The argument above shows that the functor R_X will map this to an X-colored dg operad (for the unrestricted plethysm).

Define $\mathsf{L} \in [S \wr \Sigma \times X, \mathtt{Ch_k}]$ by $\mathsf{L}(s. \mid x) := sgn_k[k]$, where $k := \#s.^{-1}(*)$, if $k \geq 1$, and $\mathsf{L}(s. \mid x) = \mathbf{k}$ if $k = 0$. It has a cooperad structure for the plethysm over X, given by the maps

$$\mathsf{L}(s. \mid x) = sgn_k[k] \to \mathsf{L}(s'. \mid x) \otimes \mathsf{L}(s''. \mid x') = sgn_{k'}[k'] \otimes sgn_{k''}[k''],$$

$$1 \mapsto 1 \otimes 1.$$

(Note $k = k' + k''$.)

Remark 3.1.2.1. Let P be a Koszul X-colored dg operad. Then according to the results of the last section, $L\text{-mod}(\mathsf{P})^{\mathsf{i}} = \mathsf{Lie}^{\mathsf{i}} + \mathsf{L} \otimes p_X^* \mathsf{P}^{\mathsf{i}}$, with the Koszul dual of the Lie operad considered as $*$-colored. Generally, $\mathcal{C}(\mathsf{LC})$, with $\mathsf{LC} := \mathsf{Lie}^{\mathsf{i}} + \mathsf{L} \otimes p_X^* \mathsf{C}$, is quasi-isomorphic to $L\text{-mod}(\mathcal{C}(\mathsf{C}))$.

Definition 3.1.2.2. Assume given an S-colored dg operad Q. Let $[\mathsf{L}, \mathsf{Q}]$ denote convolution operad. Define

$$\mathrm{tw}' \mathsf{Q} := R_X[\mathsf{L}, \mathsf{Q}].$$

It follows from our discussion that $\mathrm{tw}' \mathsf{Q}$ is a dg X-colored operad.

Remark 3.1.2.3. The category $S \wr \Sigma$ is a coproduct of categories $S^{(k)} \wr \Sigma$, where an object in $S^{(k)} \wr \Sigma$ is a sequence $s. : I \to S$ with $\#s.^{-1}(*) = k$. In particular, $S^{(0)} \wr \Sigma = X \wr \Sigma$. Accordingly, if $\mathrm{tw}' \mathsf{Q}_{(k)}$ denotes the right Kan extension of $[L, \mathsf{Q}]$ along $S^{(k)} \wr \Sigma \times X \to X \wr \Sigma \times X$, then

$$\mathrm{tw}' \mathsf{Q} = \prod_{k \geq 0} \mathrm{tw}' \mathsf{Q}_{(k)}.$$

Moreover, each object of $S^{(k)} \wr \Sigma$ over $x.$ is isomorphic to $*^k \cup x.$. Each of these objects $*^k \cup x.$ has an automorphism group over $x.$ that we can identify with Σ_k. It follows that we can model $\mathrm{tw}' \mathsf{Q}$ by

$$\mathrm{tw}' \mathsf{Q}(x. \mid x) = \prod_{k \geq 0} \mathrm{tw}' \mathsf{Q}(x. \mid x)_{(k)},$$

$$\mathrm{tw}' \mathsf{Q}(x. \mid x)_{(k)} := \mathrm{Map}_{\Sigma_k}(sgn_k[k], \mathsf{Q}(*^k \cup x. \mid x)).$$

This description is the easiest for explicit calculations, but it breaks some of the symmetry. Write 1_k for the generator of $sgn_k[k]$. The partial operadic compositions can be written as follows.

$$(\phi \circ_i \phi')(1_\ell) = \sum_{k+k'=\ell} \sum_{\sigma \in Sh(k,k')} (-1)^{|\sigma|} \big(\phi(1_k) \circ_{k+i} \phi'(1_{k'}) \big) \cdot \sigma.$$

The partial composition $\circ_{k+i}$ is a partial composition in Q.

Lemma 3.1.2.4. The convolution dg Lie algebra $\mathrm{Map}_\Sigma(\mathsf{Lie}^i, \mathsf{Q})$ (considered in the distinguished color) has a right action $\bullet$ by operadic derivations on the operad $\mathrm{tw}' \mathsf{Q}$.

Proof. The convolution operad $[\mathsf{L}, \mathsf{Q}]$ is an operadic right module for $[\mathsf{Lie}^i, \mathsf{Q}]$, using the partial compositions in the distinguished color. This transforms into an action of $\mathrm{Map}_\Sigma(\mathsf{Lie}^i, \mathsf{Q})$ by derivations of the partial $\circ_X$-compositions (that the action is by operadic derivations is equivalent to associativity of the full S-colored compositions of $[\mathsf{L}, \mathsf{Q}]$). The action is naturally induced on the right Kan extension. $\qquad\square$

Remark 3.1.2.5. The action $\bullet$ can be given in the following way. Note that $\mathrm{Map}_\Sigma(\mathsf{Lie}^{\mathsf{i}}, \mathsf{Q})$ can be decomposed as $\prod_{p \geq 1} \mathrm{Map}_\Sigma(\mathsf{Lie}^{\mathsf{i}}, \mathsf{Q})_{(p)}$, where

$$\mathrm{Map}_\Sigma(\mathsf{Lie}^{\mathsf{i}}, \mathsf{Q})_{(p)} := \mathrm{Map}_{\Sigma_p}(sgn_p[p], \mathsf{Q}|_*(p))[1].$$

Take $\gamma \in \mathrm{Map}_\Sigma(\mathsf{Lie}^{\mathsf{i}}, \mathsf{Q})$ and $\varphi \in \mathrm{tw}' \mathsf{Q}(x. \mid x)$. Mimicking the definition of compositions in $\mathrm{tw}' \mathsf{Q}$, we have

$$(\varphi \bullet \gamma)(1_\ell) = \sum_{\substack{k,k' \geq 1 \\ k+k=\ell+1}} \sum_{\sigma \in Sh(k,k')} (-1)^{|\sigma|}\big(\varphi(1_k) \circ_1 \gamma(1_{k'})\big) \cdot \sigma.$$

Note that the degree of γ is measured as a map from $sgn_p[p]$ to $\mathsf{Q}[1]$, whereas φ is a map to Q (no suspension). This is cancelled by the extra "dummy input" of φ in above formula, the input into which we insert γ.

Proposition 3.1.2.6. Let C be an augmented X-colored dg cooperad. Then, as convolution dg Lie algebras

$$\mathrm{Map}_{S \wr \Sigma \times S}(\mathsf{LC}, \mathsf{Q}) = \mathrm{Map}_{X \wr \Sigma \times X}(\overline{\mathsf{C}}, \mathrm{tw}' \mathsf{Q}) \rtimes \mathrm{Map}_\Sigma(L\text{-mod}(S)^{\mathsf{i}}, \mathsf{Q}).$$

Proof. Since we assume C to be coaugmented we can split it as $\mathbb{I} \oplus \overline{\mathsf{C}}$. This gives a direct sum decomposition $\mathsf{LC} = L\text{-mod}(S)^{\mathsf{i}} + \mathsf{L} \otimes p_X^* \overline{C}$; hence

$$\mathrm{Map}_{S \wr \Sigma \times S}(\mathsf{LC}, \mathsf{Q}) = \mathrm{Map}_{S \wr \Sigma \times X}(\mathsf{L} \otimes p_X^* \overline{\mathsf{C}}, \mathsf{Q}) \oplus \mathrm{Map}_{S \wr \Sigma \times X}(L\text{-mod}(S)^{\mathsf{i}}, \mathsf{Q}).$$

Using the adjunction for the pointwise tensor product and the adjunction defining right Kan extension:

$$\mathrm{Map}_{S \wr \Sigma \times X}(\mathsf{L} \otimes p_X^* \overline{\mathsf{C}}, \mathsf{Q}) = \mathrm{Map}_{S \wr \Sigma \times X}(p_X^* \overline{\mathsf{C}}, [\mathsf{L}, \mathsf{Q}])$$
$$= \mathrm{Map}_{X \wr \Sigma \times X}(\overline{\mathsf{C}}, \mathrm{tw}' \mathsf{Q}).$$

The only thing left to show is that the direct sum decomposition is a semidirect product of dg Lie algebras, but this is entirely obvious looking at the allowed input and output colors of respective factor and recalling the action of $\mathrm{Map}_\Sigma(\mathsf{Lie}^{\mathsf{i}}, \mathsf{Q})$ on $\mathrm{tw}' \mathsf{Q}$. $\qquad\square$

Lemma 3.1.2.7. Twist data for Q defines a Maurer-Cartan element

$$\mathcal{L} + \lambda \in \mathrm{Map}_{S \wr \Sigma \times X}(\mathsf{L} \otimes p_X^* \mathbb{I}, \mathsf{Q}) \rtimes \mathrm{Map}_\Sigma(\mathsf{Lie}^{\mathsf{i}}, \mathsf{Q})$$
$$= \mathrm{Map}_\Sigma(L\text{-mod}(S)^{\mathsf{i}}, \mathsf{Q}).$$

Proof. This is clear, since $L_\infty\text{-mod}(S) = \mathcal{C}(\mathsf{Lie}^{\mathsf{i}} \oplus p_X^* \mathbb{I})$. $\qquad\square$

Note that $\mathcal{L}$ may be considered as an operation of arity one in $\mathrm{tw}' \mathsf{Q}$, or rather, as a collection $(\mathcal{L}_x)_{x \in X}$ of operations of arity one.

Definition 3.1.2.8. Given a dg operad Q with twist data, we define the **small twist construction**, denoted $\operatorname{tw} \mathsf{Q}$, to the operad $\operatorname{tw}' \mathsf{Q}$ with the term $[\mathcal{L}, \,] + (\,) \bullet \lambda$ added to the differential.

Remark 3.1.2.9. Define $F_p \operatorname{tw} \mathsf{Q} := \prod_{k \geq p} \operatorname{tw} \mathsf{Q}_{(k)}$. This is a complete filtration

$$\mathsf{Q} = F_0 \operatorname{tw} \mathsf{Q} \supset F_1 \operatorname{tw} \mathsf{Q} \supset \ldots,$$

compatible with operadic compositions in an obvious way, and such that the differential maps $F_p \operatorname{tw} \mathsf{Q}$ into itself. In this way $\operatorname{tw} \mathsf{Q}$ can be regarded as a filtered dg operad.

Lemma 3.1.2.10. The small twist construction preserves weak equivalences.

Proof. Let $f : \mathsf{Q} \to \mathsf{Q}'$ be a quasi-isomorphism. The functor tw' preserves quasi-isomorphisms because since all dg operads are fibrant it equals a homotopy right Kan extension (there is no need for a fibrant replacement). The associated graded of the filtration $F_p \operatorname{tw} \mathsf{Q}$ is $\operatorname{tw}' \mathsf{Q}$; hence the associated graded of

$$\operatorname{tw} f : \operatorname{tw} \mathsf{Q} \to \operatorname{tw} \mathsf{Q}'$$

is a quasi-isomorphism; and hence $\operatorname{tw} f$ is a quasi-isomorphism. $\qquad \square$

Proposition 3.1.2.11. Let Q be an S-colored dg operad with twist data and let C be a coaugmented X-colored dg cooperad. There is a bijection between the set of morphisms $\mathcal{C}(\mathsf{LC}) \to \mathsf{Q}$ in the category of operads with twist data and the set of morphisms $\mathcal{C}(\mathsf{C}) \to \operatorname{tw} \mathsf{Q}$ of dg X-colored operads.

Proof. A morphism $\mathcal{C}(\mathsf{LC}) \to \mathsf{Q}$ respecting twist data is a Maurer-Cartan element

$$f + \mathcal{L} + \lambda \in \operatorname{Map}_{X \wr \Sigma \times X}(\overline{\mathsf{C}}, \operatorname{tw}' \mathsf{Q}) \rtimes \operatorname{Map}_{\Sigma}(L\text{-mod}(S)^{\mathsf{i}}, \mathsf{Q}).$$

The Maurer-Cartan equation for $f + \mathcal{L} + \lambda$ is equivalent to demanding that f satisfies the Maurer-Cartan equation in

$$\operatorname{Map}_{X \wr \Sigma \times X}(\overline{\mathsf{C}}, \operatorname{tw} \mathsf{Q}).$$

Hence f is a morphism $\mathcal{C}(\mathsf{C}) \to \mathsf{Q}$. $\qquad \square$

Corollary 3.1.2.12. The functors L-mod and tw induce an adjunction $\operatorname{Ho}(L\text{-mod}) \dashv \operatorname{Ho}(\operatorname{tw})$ between the homotopy categories.

Proof. Any $\mathsf{P} \in \mathsf{dgOp}_X$ admits a cofibrant replacement of the form $\mathfrak{C}(\mathsf{C})$. We need only show that $\mathfrak{C}(\mathsf{LC})$ is a cofibrant replacement of $L\text{-mod}(\mathsf{P})$ in the category of operads with twist data. We know $\mathfrak{C}(\mathsf{LC}) \to L\text{-mod}(\mathsf{P})$ is a quasi-isomorphism, so the problem is to show that

$$(L_\infty\text{-mod}(S) \to \mathfrak{C}(\mathsf{LC})) \in \mathtt{Twist}_S$$

is cofibrant, i.e., that it has the lifting property with respect to any acyclic fibration

$$f : (L_\infty\text{-mod}(S) \to \mathsf{Q}) \to (L_\infty\text{-mod}(S) \to \mathsf{Q}')$$

and morphism

$$g : (L_\infty\text{-mod}(S) \to \mathfrak{C}(\mathsf{LC})) \to (L_\infty\text{-mod}(S) \to \mathsf{Q}').$$

Using the bijection from above proposition, we note that giving a lift as required is equivalent to giving a lift h fitting into the following diagram in dgOp_X:

Such a lift exists because $\mathfrak{C}(\mathsf{C})$ is cofibrant in dgOp_X, and tw f is by the preceeding lemma an acyclic fibration in dgOp_X. $\qquad\square$

Remark 3.1.2.13. Assume given twist data on $\mathsf{End}\langle \mathfrak{g} \oplus A \rangle$, $A = \{A_x\}_{x \in X}$. Thus $\mathfrak{g}$ is an L_∞ algebra and, for each x, A_x is an L_∞ $\mathfrak{g}$-module. In this case

$$\text{tw } \mathsf{End}\langle \mathfrak{g} \oplus A \rangle(x. \mid x) = C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle A \rangle(x. \mid x)),$$

where $\mathsf{End}\langle A \rangle(x. \mid x)$ has the $\mathfrak{g}$-module structure defined by that on each $A_{x'}$. More succinctly, we may write $\text{tw } \mathsf{End}\langle A \rangle = C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle A \rangle)$. The adjunction between representations $L\text{-mod}(\mathsf{P})_\infty \to \mathsf{End}\langle \mathfrak{g} \oplus A \rangle$ and representations $\mathsf{P}_\infty \to C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle A \rangle)$ is a rather obvious statment about L_∞ modules.

Moreover, assume π is a Maurer-Cartan element of $\hbar\mathfrak{g}[[\hbar]]$. The image of π under the given L_∞ action $\mathfrak{g}[[\hbar]] \to gl(A)[[\hbar]]$ defines a new differential $d_A + d_\pi$ on $A[[\hbar]]$. Denote by $A[[\hbar]]_\pi$ the algebra with this new differential. Evaluating on (products of) π defines a morphism of operads

$$C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle A \rangle) \to \mathsf{End}_\hbar\langle A[[\hbar]]_\pi \rangle.$$

In conclusion:

Given an L-mod$(\mathsf{P})_\infty$-algebra structure on $\mathsf{End}\langle\mathfrak{g}\oplus A\rangle$ and a Maurer-Cartan element of $\pi \in \hbar\mathfrak{g}[[\hbar]]$, we obtain an $\hbar$-linear P_∞-algebra structure on $A[[\hbar]]_\pi$. By functoriality, if $\mathsf{Q} \to \mathsf{End}\langle\mathfrak{g}\oplus A\rangle$ is a morphism of operads with twist data, then we obtain a representation

$$\mathrm{tw}\,\mathsf{Q} \to \mathsf{End}_\hbar\langle A[[\hbar]]_\pi\rangle.$$

Thus, the small twist construction goes some way in explaining how algebras (for some operad) behave under twisting by Maurer-Cartan elements: Q-algebras are twisted into $\mathrm{tw}\,\mathsf{Q}$-algebras.

Remark 3.1.2.14. $\mathsf{End}_\hbar\langle A[[\hbar]]\rangle = \mathsf{End}\langle A\rangle[[\hbar]]$ is filtered by

$$F_p\mathsf{End}\langle A\rangle[[\hbar]] = \hbar^p\mathsf{End}\langle A\rangle[[\hbar]].$$

Any operad $\mathrm{tw}\,\mathsf{Q}$ is likewise filtered, as we have seen, and the assumption $\pi \in \hbar\mathfrak{g}[[\hbar]]$ means that

$$\mathrm{tw}\,\mathsf{Q} \to \mathsf{End}_\hbar\langle A[[\hbar]]_\pi\rangle$$

will be a morphism of filtered operads. When $\mathfrak{g} \oplus A$ already has a complete filtration, and $\pi \in F_1\mathfrak{g}$, then we may dispense with the formal parameter $\hbar$. We may likewise dispense with the formal parameter whenever some suitable nilpotency ensures convergence.

3.1.3 The (large) twist construction.

The small twist construction does not, on the face of it, explain why a dg Lie algebra can be twisted into a new dg Lie algebra, because the small twist construction loses one color and hence does not apply to algebras in a single color. To remedy this we introduce the (large) twist construction Tw, which does not lose one color. It is defined in such a way that

$$\mathrm{Tw}\,\mathsf{End}\langle\mathfrak{g}\oplus A\rangle(s.\mid s) = C_{\mathrm{CE}}(\mathfrak{g},\mathsf{End}\langle\mathfrak{g}\oplus A\rangle(s.\mid s)),$$

where $\mathsf{End}\langle\mathfrak{g}\oplus A\rangle(s.\mid s)$ is considered as a $\mathfrak{g}$-module using the actions on the A_x, just like in the small twist construction, but also using the adjoint action of $\mathfrak{g}$ on itself. Thus, $\mathfrak{g}$ in the right hand side above comes in two incarnations: one as an L_∞ algebra, and one as a module for that L_∞ algebra. By making this "double incarnation" abstract we can formalize the (large) twist construction as a special case of the small twist construction.

Define $Y := S \cup \{\tilde{*}\}$. We consider it as a pointed set with base-point $\tilde{*}$. Let $\delta : Y \to S$ be the function that restricts to the identity on S and maps $\tilde{*}$ to $*$. It induces a functor

$$\delta^* : [S \wr \Sigma \times S, \mathtt{Ch_k}] \to [Y \wr \Sigma \times Y, \mathtt{Ch_k}].$$

We leave the following lemma without proof.

Lemma 3.1.3.1. The functor δ^* is (lax) monoidal for the plethysm, hence maps operads to operads.

There is a simple map $L_\infty\text{-mod}(Y) \to \delta^* L_\infty\text{-mod}(S)$. The restriction of $\delta^* L_\infty\text{-mod}(S)$ to the distinguished color $\tilde{*}$ equals the restriction of $L_\infty\text{-mod}(Y)$ to $*$; hence we may use the identity on these components. On components

$$L\text{-mod}(Y)(\tilde{*}^k \cup x \mid x) \to \delta^* L_\infty\text{-mod}(S)(\tilde{*}^k \cup x \mid x)$$
$$= L_\infty\text{-mod}(S)(*^k \cup x \mid x)$$

it is likewise the identity. Finally, to write down

$$L_\infty\text{-mod}(Y)(\tilde{*}^k \cup * \mid *) \to \delta^* L_\infty\text{-mod}(S)(\tilde{*}^k \cup * \mid *) = \mathsf{Lie}_\infty(k+1)$$

we use that L_∞ operations can be reinterpreted as an adjoint action:

$$L\text{-mod}(Y)^\mathrm{i}(\tilde{*}^k \cup * \mid *)[-1] = sgn_k[k-1] \to sgn_{k+1}[(k+1)-2]$$
$$= \mathsf{Lie}^\mathrm{i}(k+1)[-1].$$

Remark 3.1.3.2. The above morphism $L_\infty\text{-mod}(Y) \to \delta^* L_\infty\text{-mod}(S)$ allows us to regard δ^* as a functor $\delta^* : \mathtt{Twist}_S \to \mathtt{Twist}_Y$.

Definition 3.1.3.3. For an operad with twist data $\mathsf{Q} \in \mathtt{Twist}_S$, we define the (large) twist construction to be the small twist construction on $\delta^* \mathsf{Q}$, i.e., the dg operad

$$\mathrm{Tw}\, \mathsf{Q} := \mathrm{tw}\, \delta^* \mathsf{Q},$$

using the obvious redefinition of the small twist construction as a functor from $\mathtt{Twist}_Y$ to $\mathtt{dgOp}_S$.

There is by functoriality a morphism

$$\mathrm{tw}\, L_\infty\text{-mod}(Y) \to \mathrm{tw}\, \delta^* L_\infty\text{-mod}(S) = \mathrm{Tw}\, L_\infty\text{-mod}(S).$$

Giving a morphism $L_\infty\text{-mod}(S) \to \operatorname{tw} L_\infty\text{-mod}(Y)$ is, by our adjunction formula, in bijection with giving a morphism

$$L\text{-mod}(L_\infty\text{-mod}(S))_\infty \to L_\infty\text{-mod}(Y).$$

There is a natural such map (essentially the identity). Hence we obtain a composite

$$L_\infty\text{-mod}(S) \to \operatorname{tw} L_\infty\text{-mod}(Y) \to \operatorname{Tw} L_\infty\text{-mod}(S).$$

Using this we can regard the large twist construction as an endofunctor on $\texttt{Twist}_S$. Moreover, the definition as a (pointwise) limit gives a natural transformation

$$\eta : \operatorname{Tw} \to id_{\texttt{Twist}_S}.$$

Note that $\operatorname{Tw}\operatorname{Tw}\mathsf{Q}$ is given pointwise by

$$\operatorname{Tw}\operatorname{Tw}\mathsf{Q}(s. \mid s) = \lim_{(y.\mid s)\in(p_S\downarrow(s.\mid s))} \operatorname{Map}\Bigg(\mathsf{L}(y. \mid s),$$
$$\lim_{(y'.\mid s)\in(p_S\downarrow(\delta\circ y.\mid s))} \operatorname{Map}(\mathsf{L}(y'. \mid s), \delta^*\mathsf{Q}(y'. \mid s))\Bigg).$$

Internal homs are continuous (in the covariant argument) so the above equals the double end

$$\lim_{(y.\mid s)\in(p_S\downarrow(s.\mid s))} \lim_{(y'.\mid s)\in(p_S\downarrow(\delta\circ y.\mid s))} \operatorname{Map}(\mathsf{L}(y. \mid s), \operatorname{Map}(\mathsf{L}(y'. \mid s), \delta^*\mathsf{Q}(y'. \mid s))).$$

Finally, we can use the adjunction for the pointwise tensor product to write this as

$$\lim_{(y.\mid s)\in(p_S\downarrow(s.\mid s))} \lim_{(y'.\mid s)\in(p_S\downarrow(\delta\circ y.\mid s))} \operatorname{Map}(\mathsf{L}(y. \mid s) \otimes \mathsf{L}(y'. \mid s), \delta^*\mathsf{Q}(y'. \mid s))).$$

The partial cocompositions $\mathsf{L}(y'' \mid s) \to \mathsf{L}(y. \mid s) \otimes \mathsf{L}(y'. \mid s)$, $\delta \circ y'' = \delta \circ y'.$, define maps

$$\lim_{(y.\mid s)\in(p_S\downarrow(s.\mid s))} \lim_{(y'.\mid s)\in(p_S\downarrow(\delta\circ y.\mid s))} \operatorname{Map}(\mathsf{L}(y. \mid s) \otimes \mathsf{L}(y'. \mid s), \delta^*\mathsf{Q}(y'. \mid s)))$$
$$\to \operatorname{Map}(\mathsf{L}(y'' \mid s), \delta^*\mathsf{Q}(y'' \mid s)).$$

By universality of limits these projections must factor through a map from $\operatorname{Tw}\mathsf{Q}(s. \mid s)$ to $\operatorname{Tw}\operatorname{Tw}\mathsf{Q}(s. \mid s)$. Working out the details shows that these maps define a natural transformation

$$\mathfrak{d} : \operatorname{Tw} \to \operatorname{Tw}\operatorname{Tw}.$$

Proposition 3.1.3.4. The functor Tw is a comonad on $\mathtt{Twist}_S$, with counit η and coproduct $\mathfrak{d}$.

Remark 3.1.3.5. Thomas Willwacher introduced the functor Tw for single-colored operads in [Willwacher 2010], in a somewhat ad hoc manner. It is elementary to see that the restriction of our twist construction to the single-colored case agrees with Willwacher's original definition. In [Dolgushev and Willwacher 2012] the authors gave a thourough treatment of the single-colored twist construction and proved that it defines a comonad. Their proof can be almost literally paraphrased to give a proof of above proposition. We shall accordingly refer the reader to that paper [Dolgushev and Willwacher 2012] for a proof.

3.1.4   Comonads for the (large) twist construction.

By definition, a coalgebra for the twist comonad is a dg operad Q with twist data together with a morphism

$$(\kappa : \mathsf{Q} \to \mathrm{Tw}\,\mathsf{Q}) \in \mathtt{Twist}_S$$

such that

(i) $\kappa \circ \eta_{\mathsf{Q}} = id_{\mathsf{Q}}$, and

(ii) $\mathrm{Tw}\,\kappa \circ \kappa = \mathfrak{d}_{\mathsf{Q}} \circ \kappa$.

The meaning of these conditions is best clarified in terms of algebras, so, assume

$$\mu : \mathsf{Q} \to \mathsf{End}\langle \mathfrak{g} \oplus A \rangle \quad A = \{A_x\}_{x \in X}$$

is a Q-algebra. Then by functoriality we obtain a morphism

$$\mathrm{Tw}\,\mu : \mathrm{Tw}\,\mathsf{Q} \to \mathrm{Tw}\,\mathsf{End}\langle \mathfrak{g} \oplus A \rangle = C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle \mathfrak{g} \oplus A \rangle).$$

As before, any Maurer-Cartan element $\pi \in \hbar\mathfrak{g}[[\hbar]]$ will define a morphism

$$(\,)^{\pi} : C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle \mathfrak{g} \oplus A \rangle) \to \mathsf{End}_{\hbar}\langle \mathfrak{g}[[\hbar]]_{\pi} \oplus A[[\hbar]]_{\pi} \rangle.$$

This means that we obtain a representation

$$(\,)^{\pi} \circ \mathrm{Tw}\,\mu : \mathrm{Tw}\,\mathsf{Q} \to \mathsf{End}_{\hbar}\langle \mathfrak{g}[[\hbar]]_{\pi} \oplus A[[\hbar]]_{\pi} \rangle.$$

Now assume Q is a coalgebra for the twist comonad. Then there is a representation

$$\mu^{\pi} := (\,)^{\pi} \circ \mathrm{Tw}\,\mu \circ \kappa : \mathsf{Q} \to \mathsf{End}_{\hbar}\langle \mathfrak{g}[[\hbar]]_{\pi} \oplus A[[\hbar]]_{\pi} \rangle.$$

Thus, coalgebras for the twist construction are operads Q with the property that twisting the Q-algebras μ by Maurer-Cartan elements π will produce new Q-algebras μ^{π}. Moreover:

(a) Condition (i) says that twisting by the trivial Maurer-Cartan element has no effect: $\mu^0 = \mu$.

(b) Let $\pi' \in \hbar\mathfrak{g}[[\hbar]]_\pi$ be a second Maurer-Cartan element. Then we can iterate and twist to a representation $(\mu^\pi)^{\pi'}$ in $\mathfrak{g}[[\hbar]]_{\pi+\pi'} \oplus A[[\hbar]]_{\pi+\pi'}$ (there is no need to introduce an extra formal parameter). Condition (ii) says that twisting by Maurer-Cartan elements is linear in the sense that $(\mu^\pi)^{\pi'} = \mu^{\pi+\pi'}$.

We can collect this as an informal slogan, by saying that coalgebras for the twist comonad are exactly the operads that are well-behaved with respect to twisting by Maurer-Cartan elements.

Remark 3.1.4.1. Let P be a Koszul X-colored dg operad. Then the dg operad $L\text{-mod}(\mathsf{P})_\infty$ is a coalgebra for the twist comonad in a canonical way. To see this, note that specifying

$$\kappa : L\text{-mod}(\mathsf{P})_\infty \to \mathrm{Tw}\, L\text{-mod}(\mathsf{P})_\infty$$

is, by our adjunction equivalent to giving a $\mathsf{P}_\infty \to \mathrm{tw}\, \mathrm{Tw}\, L\text{-mod}(\mathsf{P})_\infty$. The restriction of the value of the natural transformation $\mathfrak{d}$ to the color X gives us a morphism $\mathrm{tw}\, L\text{-mod}(\mathsf{P})_\infty \to \mathrm{tw}\, \mathrm{Tw}\, L\text{-mod}(\mathsf{P})_\infty$. The identity map on $L\text{-mod}(\mathsf{P})_\infty$ corresponds under our adjunction formula to a morphism $\mathsf{P}_\infty \to \mathrm{tw}\, L\text{-mod}(\mathsf{P})_\infty$. Composing the aforementioned two morphisms gives us a morphism $\mathsf{P}_\infty \to \mathrm{tw}\, \mathrm{Tw}\, L\text{-mod}(\mathsf{P})_\infty$. It is easily checked that the κ so defined satisfies the axioms for a coalgebra.

In fact, there is no need to invoke resolutions: $L\text{-mod}(\mathsf{P})$ is a coalgebra for the twist construction for any $\mathsf{P} \in \mathtt{dgOp}_X$. But to argue this fact it is easier to look at the level of algebras than to use our adjunction. Recall that giving a $L\text{-mod}(\mathsf{P})$-algebra structure on $\mathfrak{g} \oplus A$ is equivalent to specifying a dg Lie algebra structure on $\mathfrak{g}$, a P-algebra structure on A and a morphism of dg Lie algebras $\mathfrak{g} \to \mathrm{Der}_\mathsf{P}(A)$. Assume given a Maurer-Cartan element $\pi \in \hbar\mathfrak{g}[[\hbar]]$. Its image in $\mathrm{Der}_\mathsf{P}(A)[[\hbar]]$ is a differential on $A[[\hbar]]$ that acts by P-derivations: so we obtain a representation $\mathsf{P} \to \mathsf{End}_\hbar\langle A[[\hbar]]_\pi\rangle$ by just extending the given P-algebra structure $\hbar$-linearly. Finally, the usual twisting procedure for dg Lie algebras gives a morphism $\mathfrak{g}[[\hbar]]_\pi \to \mathrm{Der}_\mathsf{P}(A)[[\hbar]]_\pi = \mathrm{Der}_{\mathsf{P},\hbar}(A[[\hbar]]_\pi)$. Taken together this specifies

$$L\text{-mod}(\mathsf{P}) \to \mathsf{End}_\hbar\langle \mathfrak{g}[[\hbar]]_\pi \oplus A[[\hbar]]_\pi\rangle.$$

It is obvious that this twisting satisfies conditions (a) and (b) above. Reinterpreting the construction on the level of operads defines a morphism $\kappa : L\text{-mod}(\mathsf{P}) \to \mathrm{Tw}\, L\text{-mod}(\mathsf{P})$ satisfying (i) and (ii).

3.1.5 Twisting and deformation complexes.

Let Q be a Koszul dg operad and assume $\kappa : \mathsf{Q}_\infty \to \mathrm{Tw}\,\mathsf{Q}_\infty$ makes Q_∞ a coalgebra for the twist comonad. Then, given a map of graded Σ-modules $\phi : \mathsf{Q}^{\mathrm{i}} \to \mathsf{P}$, we can construct a map of graded Σ-modules

$$t\phi : \mathsf{Q}^{\mathrm{i}} \xrightarrow{\kappa} \mathrm{Tw}\,\mathsf{Q}_\infty \xrightarrow{\mathrm{Tw}\,\phi} \mathrm{Tw}\,\mathsf{P}.$$

One could imagine that this defines a morphism

$$t : \mathrm{Def}(\mathsf{Q}_\infty \xrightarrow{f} \mathsf{P}) \to \mathrm{Def}(\mathsf{Q}_\infty \xrightarrow{tf} \mathrm{Tw}\,\mathsf{P})$$

of dg Lie algebras. This is not the case, however: the map does generally not respect the Lie bracket. However, we will presently show that if $\mathsf{P} = \mathsf{End}\langle V \rangle$, $V_* = \mathfrak{g}$, is an endomorphism operad and $\pi \in \hbar\mathfrak{g}[[\hbar]]$ is a Maurer-Cartan element, then there exists a subcomplex

$$C_{\mathsf{Q}}(V,V)(\pi) \subset C_{\mathsf{Q}}(V,V)$$

such that the map t postcomposed with the evaluation on π defines a morphism of dg Lie algebras

$$t^\pi : C_{\mathsf{Q}}(V,V)(\pi) \to \mathrm{Def}(\mathsf{Q}_\infty \to C_{\mathrm{CE}}(\mathfrak{g}, \mathsf{End}\langle V \rangle))$$
$$\to \mathrm{Def}(\mathsf{Q}_\infty \to \mathsf{End}_\hbar\langle V[[\hbar]]_\pi \rangle).$$

(But observe that the first map in above composite is not claimed to be a morphism of dg Lie algebras.)

Definition 3.1.5.1. In the situation above we say that $\psi \in C_{\mathsf{Q}}(V,V)$ **is Maurer-Cartan with respect to** π if $\psi(q \otimes \pi^{\otimes n} = 0$ for all $n \geq 1$, $s \in S$ and $q \in \mathsf{Q}^{\mathrm{i}}(*^n \mid s)$.

Define $C_{\mathsf{Q}}(V,V)(\pi)$ to be the subcomplex consisting of all ϕ such that both ϕ and $d\phi$ are Maurer-Cartan with respect to π.

Proposition 3.1.5.2. The morphism

$$t^\pi : C_{\mathsf{Q}}(V,V)(\pi) \to C_{\mathsf{Q}}(V[[\hbar]]_\pi, V[[\hbar]]_\pi)$$

is a morphism of dg Lie algebras.

Proof. The only thing which is not evident is that it respects the Lie brackets. Take $\phi = \phi_{(s.|s)}$ defined on a single component. First consider a sequence of the form $s. = *^k \cup x. : [k] \sqcup [n] \to S$, $n \geq 1$. Then $\mathrm{Tw}\,\phi$ is defined on the components

$$\mathrm{Tw}\,\mathsf{Q}(*^{k-p} \cup x. \mid s)_{(p)} \to \mathrm{Tw}\,\mathsf{P}(*^{k-p} \cup x. \mid s)_{(p)}, \ \ 0 \leq p \leq k.$$

If $s_. = *^\ell$, then $\operatorname{Tw}\phi$ is defined on

$$\operatorname{Tw}\mathsf{Q}(*^{\ell-r} \mid s)_{(r)} \to \operatorname{Tw}\mathsf{P}(*^{\ell-r} \mid s)_{(r)}, \ \ 0 \le r \le \ell-1.$$

Intuitively, $\operatorname{Tw}\phi$ is given by all ways of making some of the $*$-inputs "dummy" while keeping at least one input "non-dummy". In the partial compositions $t\phi \circ_i t\phi'$ (defining the Lie bracket) we must thus have at least one non-dummy input in both ϕ and ϕ'. However, in $t(\phi \circ_j \phi')$ it may happen that we make all inputs of ϕ' dummy. The discrepancy $t[\phi, \phi'] - [t\phi, t\phi']$ is given by all such expressions; all the terms where either ϕ or ϕ' has all its inputs considerd dummy. The evaluation on the Maurer-Cartan element inserts π into all dummy inputs. Hence, if both ϕ and ϕ' are in $C_{\mathsf{Q}}(V,V)(\pi)$, then the discrepancy vanishes. $\qquad\square$

The complex $C_{\mathsf{Q}}(V,V)(\pi)$ should be regarded as governing those deformations of V that induce deformations also of the twisted algebra $V[[\hbar]]_\pi$.

3.1.6 Remarks on generalizations.

Let us first mention an obvious generalization. Let $L\{r\}_\infty$-mod(S) be the operad whose representations consist of an L_∞ algebra $L[-r]$ and a collection of L_∞ morphisms $L[-r] \to gl(A_x)$, $x \in X$. It is obvious that we can repeat the small and large twist constructions for the modified version $(L\{r\}_\infty\text{-mod}(S) \downarrow \mathtt{dgOp}_S)$ of the category of operads with twist data. Define $\{r\}_*$ to be the endomorphism operad of the collection of dg vector spaces V with $V_* = \mathbf{k}[r]$ and $V_x = \mathbf{k}$ for all $x \in X$. Note $L\{r\}_\infty\text{-mod}(S) = L_\infty\text{-mod}(S) \otimes \{r\}_*$. Tensoring with $\{-r\}_*$ is accordingly a functor

$$(L\{r\}_\infty\text{-mod}(S) \downarrow \mathtt{dgOp}_S) \to \mathtt{Twist}_S.$$

Hence we can define the (large) twist construction on the category to the left by

$$\operatorname{Tw}\mathsf{Q} := (\operatorname{Tw}(\mathsf{Q} \otimes \{-r\}_*)) \otimes \{r\}_*.$$

A more interesting generalization is to extend the constructions to properads. The Lie-module construction $\mathsf{P} \mapsto L\text{-mod}(\mathsf{P})$ makes perfect sense in the more general context of a (colored) dg properad P. The theory of Koszul duality, resolutions and deformation complexes generalizes from operads to properads. Hence, if P is a Koszul properad with Koszul resolution P_∞, then we can consider the properad $L\text{-mod}(\mathsf{P})_\infty$ whose

algebras $\mathfrak{g} \oplus A$ consist of an L_∞ algebra $\mathfrak{g}$, a P_∞-algebra A, and an L_∞ morphism

$$\mathfrak{g} \to \mathrm{Def}(\mathsf{P}_\infty \to \mathsf{End}\langle A \rangle).$$

One can then look for a homotopy adjoint tw and its enlarged incarnation Tw, defined by essentially the same formulas as our operadic versions. It would appear that all our constructions can be repeated *mutatis mutandum* in this properadic setting.

Our main application of the twist construction is in formal geometry; see chapter 6. Formal geometry may be said to be the study of germs of geometric structure. To get a non-formal, global, geometric structure on a manifold one must ensure that the germs at each point vary coherently with the point. This is encoded by the action of formal diffeomorphisms on germs. Thus, in formal geometry one always has to consider the action of (the Lie algebra of) formal diffeomorphisms. In the terminology of this chapter, *all algebraic structures on germs come canonically equipped with twist data*. We will in the coming chapters only be interested in algebraic structures defined by operads, but there is a large family of germs of geometric structure, such as germs of Nijenhuis structures, that require properads for their description, hence suggesting the importance of a properadic twist construction.

CHAPTER 4

Two-colored nc Gerstenhaber (non-)formality in affine coordinates

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The lion's share of this chapter constituted sections 3-5 of [Alm 2011], but the present chapter contains several clarifying revisions of the material. Sections 4.4 and 4.9 are entirely new.

4.1   Introduction.

Let T_{poly} denote the space of polynomial polyvector fields on $\mathbf{R}^d$ and $\mathcal{A}$ denote the algebra of polynomial functions on $\mathbf{R}^d$, $d < \infty$. Maxim Kontsevich explicitly constructed an L_∞ quasi-isomorphism

$$\mathcal{U} : T_{\mathrm{poly}}[1] \to C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})[1]$$

from the space of polyvector fields equipped with the Schouten bracket to the Hochschild cochain complex equipped with Hochschild differential and Gerstenhaber bracket, extending the Hochschild-Kostant-Rosenberg quasi-isomorphism of *complexes*

$$\mathrm{HKR} : T_{\mathrm{poly}}[1] \to C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})[1].$$

For details, see [Kontsevich 2003]. Kontsevich's construction is best understood as a morphism of two-colored operads

$$\mathcal{K}(\overline{C}(\mathbf{H})) \to \mathsf{End}\langle T_{\mathrm{poly}}, \mathcal{A}\rangle,$$

65

where $\mathcal{K}(\overline{C}(\mathbf{H}))$ is the operad of fundamental chains on a cellular operad $\overline{C}(\mathbf{H})$ of compactified configuration spaces of points in the closed upper half-plane and $\mathsf{End}\langle T_{\mathrm{poly}}, \mathcal{A}\rangle$ is the standard two-colored endomorphism operad on polyvector fields and functions. The content of this map of operads is an L_∞ map from T_{poly} to the Hochschild cochain complex of $\mathcal{A}$. In this chapter we introduce a three-colored operad $\overline{CF}(\mathbf{H})$ of compactified configuration spaces of points in the closed upper half-plane equipped with a line parallel to the real axis, and, using the same techniques as Kontsevich, a representation

$$\mathcal{K}(\overline{CF}(\mathbf{H})) \to \mathsf{End}\langle T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{A}\rangle$$

of its fundamental chains. To explain the structure encoded in this representation, define the two-colored operad $\mathsf{NCG} := L\{-1\}\text{-mod}(\mathsf{Ass})$. Explicitly, an algebra for this operad is a dg Lie algebra $L[1]$, a dg associative algebra A, and a morphism of dg Lie algebras from $L[1]$ to $\mathrm{Der}_{\mathsf{Ass}}(A, A)$. We call such a structure a two-colored noncommutative Gerstenhaber algebra. Two copies of the Hochschild cochain complex carry a natural structure of algebra for this operad; namely, we can take L to be $C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})$ with Hochschild differential and Gerstenhaber bracket, A to be $C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})$ with the Hochschild differential and cup product, and the morphism to be the braces map

$$br : C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})[1] \to C_{\mathsf{Ass}}(C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}), C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})),$$

$$br(x) = (\,)\{x\}_1 + \sum_{p \geq 1} \pm x\{\dots\}_p.$$

The operad NCG is Koszul, hence has a canonical resolution NCG_∞. Our construction encodes:

- An NCG_∞-structure on $(T_{\mathrm{poly}}, T_{\mathrm{poly}})$, where the first copy of polyvector fields is a graded Lie algebra under the Schouten bracket and the second copy is an associative algebra under the usual (commutative) wedge product, while the action

$$T_{\mathrm{poly}}[1] \to C_{\mathsf{Ass}}(T_{\mathrm{poly}}, T_{\mathrm{poly}})$$

is a novel deformation of the adjoint Schouten action of T_{poly} on itself.

- Finally, the data encodes an ∞-quasi-isomorphism

$$(T_{\mathrm{poly}}, T_{\mathrm{poly}}) \to (C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}), C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}))$$

of NCG_∞-algebras, where Hochschild cochains is a two-colored non-commutative Gerstenhaber algebra in the canonical way defined by the braces map, having the property that on the Lie-part the morphism restricts to Kontsevich's formality morphism $\mathcal{U}$.

Remark 4.1.0.1. We prove in chapter 5 that the deformation of the adjoint action of the Schouten bracket is (generically) nontrivial; hence the paranthetical word "(non-)formality" in the title of this chapter, since our construction implies that the NCG-algebra $(C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}), C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}))$ is not NCG_∞-quasi-isomorphic to its cohomology (when the latter is regarded without higher homotopies).

We remark that the construction works not just for $\mathbf{R}^d$ but for an arbitrary graded vector space of finite type. This will be important in chapter 5, where we will discuss applications of our result to the Duflo isomorphism. In chapter 6, we will prove that the construction can be globalized to work on any smooth manifold.

Related results were announced in [Mochizuki 2002], which can be regarded as a precursor to our work, though that paper contains a serious error which unfortunately spoiled the main conclusion. Our work is also closely related to the more general homotopy braces formality of [Willwacher 2011]. Willwacher's results were published only shortly after our own work was announced as a preprint, but were found independently of our construction.

4.2   Configuration space models of homotopy algebras.

In this section we define four different operads in the category of cellular compact semialgebraic manifolds. The last two of the operads are our invention.

4.2.1   Semialgebraic geometry.

For a thorough treatment of the material in this subsection, see [Hardt et al. 2011].

A **semialgebraic set** (in $\mathbf{R}^n$) is a finite union of finite intersections of solution sets to polynomial equations or polynomial inequalities, for real polynomials in n variables. Semialgebraic sets are topologized as subsets. A **semialgebraic map** is a continuous map of semialgebraic sets whose graph is itself a semialgebraic set. The closure or interior of a semialgebraic set is again semialgebraic, and the inverse image of a semialgebraic map is also semialgebraic. A **semialgebraic manifold** of

dimension k is, for our purposes, a semialgebraic set in $\mathbf{R}^k$ such that each point has a semialgebraic neighbourhood semialgebraically homeomorphic to $\mathbf{R}^k$ or $\mathbf{R}_{\geq 0} \times \mathbf{R}^{k-1}$. The boundary of a semialgebraic manifold is again semialgebraic. A **smooth semialgebraic submanifold** of a semialgebraic manifold is a semialgebraic subset that is also a smooth submanifold of the ambient euclidean space.

Let $\Omega_c^p(\mathbf{R}^k)$ denote the vector space of smooth differential p-forms on $\mathbf{R}^k$ with compact support. This vector space can be topologized in a natural way, and we let $\mathcal{C}^{-p}(\mathbf{R}^k)$ be the topological dual of $\Omega_c^p(\mathbf{R}^k)$. The adjoint of the de Rham differential yields a differential graded vector space $(\mathcal{C}^{-p}(\mathbf{R}^k), \partial)$, the complex of smooth currents on $\mathbf{R}^k$.

Let X be an oriented semialgebraic manifold in $\mathbf{R}^k$ and define $\mathcal{C}(X) \subset \mathcal{C}(\mathbf{R}^k)$ to be the subspace of currents that have support contained in X. For $V_1, \ldots, V_r$ p-dimensional disjoint smooth semialgebraic submanifolds of $\mathbf{R}^k$ with each closure $\overline{V}_i$ compact and contained in X and integers $n_1, \ldots, n_r$, there is a a current $\sum_i n_i[V_i]$ in $\mathcal{C}^{-p}(X)$ (defined by integration). The **complex of semialgebraic currents** on X, denoted $\mathrm{C}_{SA}(X)$, is the subcomplex of the complex of currents spanned by all currents of that form.

The association $X \mapsto \mathrm{C}_{SA}(X)$ is a symmetric monoidal functor from semialgebraic manifolds to differential graded vector spaces; so, in particular, if X is an operad of semialgebraic manifolds, then $\mathrm{C}_{SA}(\mathsf{X})$ is a dg operad.

4.2.2 A configuration space model for Lie algebras.

For an integer $\ell \geq 2$, let $\mathrm{Conf}_\ell(\mathbf{C})$ be the manifold of all injective maps of $[\ell] := \{1, \ldots, \ell\}$ into $\mathbf{C}$. The group of translations and positive dilations of the plane, $\mathbf{C} \rtimes \mathbf{R}_{>0}$, acts on the plane and hence (by postcomposition) on $\mathrm{Conf}_\ell(\mathbf{C})$. Define $C_\ell(\mathbf{C}) := \mathrm{Conf}_\ell(\mathbf{C})/\mathbf{C} \rtimes \mathbf{R}_{>0}$. Let $\overline{\mathrm{Conf}}_\ell(\mathbf{C})$ be the real Fulton-MacPherson compactification (in the literature also called the Axelrod-Singer compactification) of $\mathrm{Conf}_\ell(\mathbf{C})$, i.e. the real oriented blow-up of $\mathbf{C}^\ell$ along all diagonals. It can be explicitly realized as the closure under the embedding

$$\mathrm{Conf}_\ell(\mathbf{C}) \to \mathbf{C}^\ell \times \prod_{i,j} S^1 \times \prod_{i,j,k} [0, \infty],$$

$$\underline{x} \mapsto \left(\underline{x}, \frac{x_j - x_i}{|x_j - x_i|}, \frac{|x_j - x_i|}{|x_i - x_k|} \right).$$

The products are, respectively, over all pairs of distinct indices $i, j \in [\ell]$ and all triples of distinct indices $i, j, k \in [\ell]$. The action by translations

and positive dilations is smooth; hence extends uniquely to a smooth action on $\overline{\mathrm{Conf}}_\ell(\mathbf{C})$. Define $\overline{C}_\ell(\mathbf{C})$ to be the quotient of $\overline{\mathrm{Conf}}_\ell(\mathbf{C})$ by this action. It is a compact semialgebraic manifold with codimension one boundary

$$\bigsqcup_S C_{\ell-|S|+1}(\mathbf{C}) \times C_S(\mathbf{C}),$$

given by products labelled by subsets $S \subset [\ell]$ (of cardinality $2 \leq |S| < \ell$). Moreover, the closure of $C_{\ell-|S|+1}(\mathbf{C}) \times C_S(\mathbf{C})$ in $\overline{C}_\ell(\mathbf{C})$ is the product $\overline{C}_{\ell-|S|+1}(\mathbf{C}) \times \overline{C}_S(\mathbf{C})$. This means that the family of spaces $\overline{C}(\mathbf{C}) = \{\overline{C}_\ell(\mathbf{C})\}$ together with the inclusions of boundary components and permutation actions by permutation of points assemble into the structure of an operad. We promote it to an operad of oriented semialgebraic manifolds as follows. Let $C_\ell^{\mathrm{std}}(\mathbf{C})$ be the submanifold of $\mathrm{Conf}_\ell(\mathbf{C})$ consisting of configurations x satisfying $\sum_{i=1}^\ell x_i = 0$ and $\sum_{i=1}^\ell |x_i|^2 = 1$. The manifolds $C_\ell(\mathbf{C})$ and $C_\ell^{\mathrm{std}}(\mathbf{C})$ are isomorphic. The manifold $\mathrm{Conf}_\ell(\mathbf{C})$ is canonically oriented; hence so is $C_\ell^{\mathrm{std}}(\mathbf{C})$. We orient $C_\ell(\mathbf{C})$ by pulling back the orientation on $C_\ell^{\mathrm{std}}(\mathbf{C})$. Requiring Stokes' formula to hold (without a sign) defines an orientation of the compactification $\overline{C}_\ell(\mathbf{C})$. It is easy to see that all permutations of $[\ell]$ preserve the orientation.

The boundary description describes a canonical stratification and the face complexes of the stratification of each component form a suboperad $\mathcal{K}(\overline{C}(\mathbf{C}))$ of the dg operad of semialgebraic chains $\mathrm{C}_{SA}(\overline{C}(\mathbf{C}))$ that is freely generated as a graded operad by the set $\{[C_\ell(\mathbf{C})] \mid \ell \geq 2\}$ of "fundamental chains". We shall regard chains in the components as semialgebraic chains, to avoid the need for simplicial subdivisions. It is well-known that representations of $\mathcal{K}(\overline{C}(\mathbf{C}))$ in a dg vector space V are in one-to-one correspondence with L_∞ structures on the suspension $V[1]$ of V; see e.g., [Getzler and Jones 1994]. In other words, $\mathcal{K}(\overline{C}(\mathbf{C}))$ is isomorphic as a dg operad to the operad $\mathsf{Lie}_\infty\{-1\}$.

Historical references for the study of configuration spaces $\mathrm{Conf}_\ell(\mathbf{C})$ and their compactifications are [Fadell and Neuwirth 1962; Arnol'd 1969], and Fred Cohen's survey of his own and other's contributions in [Cohen 1995].

4.2.3 A configuration space model for OCHAs.

Set $\mathbf{H} := \mathbf{R} \times \mathbf{R}_{\geq 0}$. For integers $m, n > 0$, with $2m + n \geq 2$, let $\mathrm{Conf}_{m,n}(\mathbf{H})$ be the manifold consisting of those injections of $[m] + [n]$ into $\mathbf{H}$ that map $[n]$ into the boundary $\mathbf{R} \times \{0\}$ of the half-plane and map $[m]$ into the interior. The group of translations along the boundary and positive dilations, $\mathbf{R} \times \mathbf{R}_{>0}$, acts (by postcomposition) on $\mathrm{Conf}_{m,n}(\mathbf{H})$,

and we let $C_{m,n}(\mathbf{H})$ be the quotient of this action. The embedding

$$\mathrm{Conf}_{m,n}(\mathbf{H}) \to \mathrm{Conf}_{2m+n}(\mathbf{C})$$

defined by sending a configuration in $[m] + [n] \hookrightarrow \mathbf{H}$ to its orbit under complex conjugation induces an embedding

$$C_{m,n}(\mathbf{H}) \to C_{2m+n}(\mathbf{C}) \subset \overline{C}_{2m+n}(\mathbf{C}).$$

The compactification $\overline{C}_{m,n}(\mathbf{H})$ of $C_{m,n}(\mathbf{H})$ was in [Kontsevich 2003] defined as the closure under this embedding. It is a semialgebraic manifold with $n!$ connected components. Let $\overline{C}_{m,n}^{+}(\mathbf{H})$ be the connected component that has the boundary points "compatibly ordered", by which we mean that if $i < j \in [n] = \{1 < \cdots < n\}$, then the point labelled by i is before the point labelled by j on the boundary for the orientation of the boundary induced by the orientation of the half-plane. This gives us a permutation-equivariant identification $\overline{C}_{m,n}(\mathbf{H}) \cong \overline{C}_{m,n}^{+}(\mathbf{H}) \times \Sigma_n$. The codimension one boundary of $\overline{C}_{m,n}^{+}(\mathbf{H})$ is

$$\bigsqcup_{I} \left(C_{m-|I|+1,n}^{+}(\mathbf{H}) \times C_I(\mathbf{C}) \right) \sqcup \bigsqcup_{S,T} \left(C_{m-|S|,n-|T|+1}^{+}(\mathbf{H}) \times C_{S,T}^{+}(\mathbf{H}) \right).$$

Here $C_{m-|I|+1,n}^{+}(\mathbf{H})$ is the interior of $\overline{C}_{m-|I|+1,n}^{+}(\mathbf{H})$, etc. The union is over all subsets $I \subset [m]$ and subsets $S \subset [m]$, $T < [n]$ such that all involved spaces are defined. This description of the boundary extends, via the identification $\overline{C}_{m,n}(\mathbf{H}) \cong \overline{C}_{m,n}^{+}(\mathbf{H}) \times \Sigma_n$, to boundary descriptions for all connected components, and defines the structure of a two-coloured operad on the collection $\overline{C}(\mathbf{H}) := \{\overline{C}_\ell(\mathbf{C}), \overline{C}_{m,n}(\mathbf{H})\}$, the points in the interior being inputs of one color and the points on the boundary being inputs of another color. The spaces $\overline{C}_{m,n}(\mathbf{H})$ are defined using embeddings into spaces of the form $\overline{C}_\ell(\mathbf{C})$, for which we have chosen orientations. We orient the spaces $\overline{C}_{m,n}(\mathbf{C})$ by the pullback orientations of these embeddings.

The dg operad of face complexes of the stratification defined by the boundary decomposition is again generated by the fundamental chains. We shall denote this operad of fundamental chains as either $\mathcal{K}(\overline{C}(\mathbf{H}))$ or as OC. A representation of it is referred to as an **open-closed homotopy algebra** [Kajiura and Stasheff 2006; Hoefel 2009], henceforth abbreviated as an OCHA. An OCHA consists of a pair of dg vector spaces V and W, an L_∞ structure on $V[1]$, an A_∞ structure on W, and an L_∞ morphism from $V[1]$ to the Hochschild cochain complex of W.

Remark 4.2.3.1. The operad OC is quasi-free but it is not formal, i.e., it is not a quasi-isomorphic resolution $\mathsf{OC} \to H(\mathsf{OC})$ of its cohomology. Thus, despite its name, OCHAs are not really to be considered as strong homotopy versions of some would-be notion of (graded) open-closed algebras, cf. [Hoefel and Livernet 2004].

We now define what we shall term "flag" versions of the operads $\overline{C}(\mathbf{C})$ and $\overline{C}(\mathbf{H})$.

4.2.4   A model for two-colored nc Gerstenhaber.

Since the affine group preserves collinearity and parallel lines it makes sense to say that some points in a configuration $x \in C_\ell(\mathbf{C})$ are collinear on a line parallel to the real axis. For integers $p \geq 0$ and $q \geq 1$ with $p + q \geq 2$, define $CF_{p,q}(\mathbf{C}) \subset C_{[p]+[q]}(\mathbf{C})$ to be the subset of configurations for which the points labelled by $[q]$ are collinear on a line parallel to the real axis. Define $\overline{CF}_{p,q}(\mathbf{C})$ to be its closure inside $\overline{C}_{p+q}(\mathbf{C})$. It has $q!$ connected components. Let $CF_{p,q}^+(\mathbf{C})$ denote the interior of the connected component that has the collinear points compatibly ordered, by which we mean that if $i < j \in [q] = \{1 < \cdots < q\}$, then the point labelled by i is before the point labelled by j on their common line for the orientation of the line induced by the orientation of the plane. Then $CF_{p,q}(\mathbf{C}) \cong CF_{p,q}^+(\mathbf{C}) \times \Sigma_q$. We deduce that the codimension one boundary of the corresponding compact connected component, $\overline{CF}_{p,q}^+(\mathbf{C})$, is

$$
\bigsqcup_I \left(CF_{p-|I|+1,q}^+(\mathbf{C}) \times C_I(\mathbf{C}) \right) \sqcup \bigsqcup_{S,T} \left(CF_{p-|S|,q-|T|+1}^+(\mathbf{C}) \times CF_{S,T}^+(\mathbf{C}) \right).
$$

The union is over all subsets $I \subset [p]$, $S \subset [p]$, $T < [q]$ for which all involved spaces are defined. One can use the inclusions of boundary components to define a two-colored operad structure on the collection

$$
\overline{CF}(\mathbf{C}) := \{ \overline{C}_\ell(\mathbf{C}), \overline{CF}_{p,q}(\mathbf{C}) \},
$$

in a way completely analogous the previously discussed operadic structure on $\overline{C}(\mathbf{H})$.

Definition 4.2.4.1. We call $\overline{CF}(\mathbf{C})$ the operad of configurations on a flag in the plane.

We orient the spaces of the form $\overline{CF}_{p,q}(\mathbf{C})$ by the pullback orientations of the defining embeddings into $\overline{C}_{p+q}(\mathbf{C})$. As before one then obtains a dg operad $\mathcal{K}(\overline{CF}(\mathbf{C}))$ of fundamental chains. It is almost identical to the operad $\mathcal{K}(\overline{C}(\mathbf{H}))$ of OCHAs: its representations also consist

of an L_∞ algebra $V[1]$, an A_∞ algebra W and an L_∞ morphism from $V[1]$ to the Hochschild cochain complex of W. The difference lies in that the latter operad contains chains $[C_{m,n}(\mathbf{H})]$ with $n = 0$ while the former operad does not contain any chain of the form $[CF_{p,q}(\mathbf{C})]$ with $q = 0$. This means that the L_∞ map of an OCHA contains components $V^{\otimes p} \to W$, so called curvature terms, whilst the L_∞ map of a $\mathcal{K}(\overline{C}(\mathbf{H}))$-representation can not, i.e. it maps into the truncated Hochschild cochain complex $C^+_{\mathrm{Hoch}}(W, W)[1] = C_{\mathsf{Ass}}(W, W)$.

Definition 4.2.4.2. We call $\mathcal{K}(\overline{CF}(\mathbf{C}))$ the operad of two-colored strong homotopy noncommutative Gerstenhaber algebras and denote it NCG_∞.

Remark 4.2.4.3. The identification NCG_∞ is canonical. Recall

$$\mathsf{NCG}_\infty = L\{-1\}\text{-mod}(\mathsf{Ass})_\infty.$$

It follows from our general results on the Lie module construction that NCG_∞ is quasi-isomorphic to its cohomology, the Koszul operad

$$\mathsf{NCG} := L\{-1\}\text{-mod}(\mathsf{Ass}).$$

We shall abbreviate "two-colored strong homotopy noncommutative Gerstenhaber algebra" as NCG_∞ algebra.

4.2.5 A model for flag OCHAs.

There is also a flag version of the operad $\overline{C}(\mathbf{H})$, defined as follows. Let $k, m, n \geq 0$ be integers with $2k + m + n \geq 1$ if $m \geq 1$ and $k + n \geq 2$ if $m = 0$. Let $CF_{k,m,n}(\mathbf{H})$ be the subspace of $C_{k+m,n}(\mathbf{H})$ consisting of all configurations wherein the points labelled by $[m]$ are collinear on a line parallel to the boundary. Denote by $\overline{CF}_{k,m,n}(\mathbf{H})$ the closure inside $\overline{C}_{k+m,n}(\mathbf{H})$. Let $CF^+_{k,m,n}(\mathbf{H})$ be the connected component of $CF_{k,m,n}(\mathbf{H})$ that has both the collinear points and the boundary points compatibly ordered, i.e. if $i < j$ in $[m]$, then $x_i < x_j$ on their common line of collinearity, and if $r < s$ in $[n]$, then $x_r < x_s$ on the boundary. The codimension one boundary of its compactification, $\overline{CF}^+_{k,m,n}(\mathbf{H})$, has the form

$$\bigsqcup_I \left(CF^+_{k-|I|+1,m,n}(\mathbf{H}) \times C_I(\mathbf{C})\right) \sqcup \bigsqcup_{P,Q} \left(CF^+_{k-|P|,m-|Q|+1,n}(\mathbf{H}) \times CF^+_{P,Q}(\mathbf{C})\right)$$

$$\sqcup \bigsqcup_{S,T,U} \left(CF^+_{k-|S|,m-|T|,n-|U|+1}(\mathbf{H}) \times CF^+_{S,T,U}(\mathbf{H})\right).$$

The union is over all subsets $I, P, S \subset [k]$, $Q, T < [m]$, $S < [n]$ for which all involved spaces are defined. These boundary factorizations define an operad structure, but now in three colors, on the collection

$$\overline{CF}(\mathbf{H}) := \{\overline{C}_\ell(\mathbf{C}), \overline{CF}_{p,q}(\mathbf{C}), \overline{CF}_{k,m,n}(\mathbf{H})\}.$$

Definition 4.2.5.1. We call $\overline{CF}(\mathbf{H})$ the operad of configurations on a flag in the half-plane.

Orient the spaces $\overline{CF}_{k,m,n}(\mathbf{H})$ by the pullback orientations of the embeddings into $\overline{C}_{k+m,n}(\mathbf{H})$. There is an associated operad $\mathcal{K}(\overline{CF}(\mathbf{H}))$ of fundamental chains.

Definition 4.2.5.2. We call $\mathcal{K}(\overline{CF}(\mathbf{H}))$ the operad of flag open-closed homotopy algebras, abbreviated as the operad of flag OCHAs, and introduce the notation $\mathsf{FOC} := \mathcal{K}(\overline{CF}(\mathbf{H}))$.

Definition 4.2.5.3. Define $\mathsf{Mor}_*(\mathsf{NCG})$ to be the three-colored operad whose representations are two NCG-algebras (L, A) and (L, A'), with the same dg Lie algebra appearing in both, and a morphism of NCG-algebras $(L, A) \to (L, A')$ that is the identity on L. In other words, the morphism is a morphism $f : A \to A'$ of associative algebras that intertwines the actions of L; $f(x \cdot a) = x \cdot f(a)$.

Lemma 4.2.5.4. The operad $\mathsf{Mor}_*(\mathsf{NCG})$ is Koszul.

Proof. We note that

$$\mathsf{Mor}_*(\mathsf{NCG}) = L\{-1\}\text{-mod}(\mathsf{Mor}(\mathsf{Ass})),$$

where $\mathsf{Mor}(\mathsf{Ass})$ is the two-colored operad governing a pair of dg associative algebras A, A' and a morphism $A \to A'$ between them. The operad $\mathsf{Mor}(\mathsf{Ass})$ is well-known to be (homotopy) Koszul, with a minimal model $\mathsf{Mor}(\mathsf{Ass})_\infty$ given by the cell-complex of the multiplihedra, see [Merkulov and Vallette 2009] and also 4.2.5.8 below. The Lie module construction preserves being Koszul. $\square$

Denote by

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty = L\{-1\}\text{-mod}(\mathsf{Mor}(\mathsf{Ass}))_\infty$$

the Koszul resolution of $\mathsf{Mor}_*(\mathsf{NCG})$.

Lemma 4.2.5.5. Let (L, B) be an OCHA. Then there is an induced structure of NCG_∞ algebra on $(L, C_{\mathrm{Hoch}}(B, B))$.

Proof. Recall that the Hochschild cochain complex $C_{\text{Hoch}}(A,A)$ of an A_∞ algebra A is, as a graded vector space, equal to $\text{Map}(T(A[1]),A)$. The brace operations on the Hochschild cochains complex are the maps

$$( )\{\dots\}_p : C_{\text{Hoch}}(A,A) \otimes \bigotimes_{i=1}^p C_{\text{Hoch}}(A,A) \to C(A,A), \ p \geq 1,$$

defined for $x \in \text{Map}(A[1]^{\otimes r}, A)$, $x_i \in \text{Map}(A[1]^{\otimes r_i}, A)$, $1 \leq i \leq p \leq r$, $n = r + r_1 + \cdots + r_p - p$, by

$$x\{x_1, \dots, x_p\}_p(a_1, \dots, a_n) = \sum_{1 \leq i_1 < \cdots < i_p < r} \pm x(a_1, \dots, a_{i_1-1}, x_1(a_{i_1}, \dots),$$

$$\dots, a_{i_p-1}, x_p(a_{i_p}, \dots), \dots, a_n).$$

Recall that $C_{\text{Ass}}(A,A)[1]$ is the subspace $\text{Map}(\bigoplus_{r \geq 1} A[1]^{\otimes r}, A)$ of the Hochschild cochain complex. Set $( )\{\dots\} := \sum_{p \geq 1} ( )\{\dots\}_p$ and define

$$br : C_{\text{Hoch}}(A,A)[1] \to C_{\text{Ass}}(C_{\text{Hoch}}(A,A), C_{\text{Hoch}}(A,A)), x \mapsto ( )\{x\}_1 + x\{\dots\}.$$

One verifies that this is a map of graded Lie algebras. Hence an L_∞ morphism $L[1] \to C_{\text{Hoch}}(B,B)[1]$ can always be postcomposed to an L_∞ action

$$L[1] \to C_{\text{Hoch}}(B,B)[1] \to C_{\text{Ass}}(C_{\text{Hoch}}(B,B), C_{\text{Hoch}}(B,B)).$$

$\square$

Proposition 4.2.5.6. A representation of the operad of flag open closed homotopy algebras, FOC, in a triple of dg vector spaces (L, A, B) is equivalent to

- an NCG_∞ algebra structure on (L, A);

- an OCHA structure on (L, B);

- and an extension of above data to a representation

$$(L, A) \to (L, C_{\text{Hoch}}(B,B))$$

of $\text{Mor}_*(\text{NCG})_\infty$, using the NCG_∞-structure on $(L, C_{\text{Hoch}}(B,B))$ induced via the braces operations, as in the preceeding lemma.

Proof. The first two items in the list are obvious. The key to the correspondence suggested in the third item is to change from the operadic perspective that the chains $[CF_{k,m,n}(\mathbf{H})]$ are represented as maps $L^{\otimes k} \otimes A^{\otimes m} \otimes B^{\otimes n} \to B$ to the perspective that they define maps

$$L^{\otimes k} \otimes A^{\otimes m} \to \mathrm{Map}(B^{\otimes n}, B).$$

This hom-adjunction argument exactly parallels the argument used for interpreting an OCHA structure $\{[C_{p,q}(\mathbf{H})] : L^{\otimes p} \otimes B^{\otimes q} \to B\}$ as an L_∞ morphism $L \to C(B,B)$, compare with [Kajiura and Stasheff 2006; Hoefel 2009]. After this reinterpretation of the chains the argument reduces to (i) recognizing the induced NCG_∞ algebra structure on $(L, C(B,B))$ and (ii) comparing the differential on the chains to the differential on $\mathsf{Mor}_*(\mathsf{NCG})_\infty$. The details are left to the reader. We work out some more explicit details in the subsequent sections. $\qquad\square$

Remark 4.2.5.7. The operad of flag open closed homotopy algebras is not formal. This is true since it contains the operad of OCHAs, which is known to not be formal.

Remark 4.2.5.8. Consider the two-colored suboperad of $\overline{CF}(\mathbf{H})$ on the components

$$\{\overline{CF}_{0,q}(\mathbf{C}), \overline{CF}_{0,m,0}(\mathbf{H}), \overline{CF}_{0,0,n}(\mathbf{H})\}.$$

It is isomorphic as an operad of compact semialgebraic manifolds to the operad of quilted holomorphic disks introduced in [Mau and Woodward 2010] as a moduli space interpretation of J. Stasheff's multiplihedra [Stasheff 1963]. Its operad of cellular chains is the operad $\mathsf{Mor}(\mathsf{Ass})_\infty$ of A_∞ morphisms of A_∞ algebras.

4.3 A method of constructing representations.

Kontsevich's proof of his formality conjecture and construction of a universal deformation quantization formula can be regarded as the construction of

- a map of cooperads $\theta : \mathsf{gra}_{\mathrm{oc}} \to \Omega(\overline{C}(\mathbf{H}))$, where $\mathsf{gra}_{\mathrm{oc}}$ is a certain cooperad of "Feynman diagrams",

- and a map of operads $D : \mathsf{Gra}_{\mathrm{oc}} \to \mathsf{End}\langle T_{\mathrm{poly}}, \mathcal{A}\rangle$ from the linearly dual operad of Feynman diagrams.

These two morphisms correspond (at least heuristically) to the Feynman rules of a perturbative path integral expansion, see [Cattaneo and Felder 2000] for a conceptual rederivation of Kontsevich's formulas in terms of physics. Dualizing the map of cooperads and composing, one gets a representation

$$D \circ \theta^* : \mathsf{OC} \to \mathsf{Gra_{OC}} \to \mathsf{End}\langle T_{\mathrm{poly}}, \mathcal{A} \rangle$$

of the fundamental chains of half-plane configurations, i.e., an OCHA structure on $(T_{\mathrm{poly}}, \mathcal{A})$. We shall show that Kontsevich's construction can be extended, essentially without any changes, to a representation

$$D \circ \theta^* : \mathsf{FOC} \to \mathsf{Gra_{FOC}} \to \mathsf{End}\langle T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{A} \rangle$$

of the operad of flag OCHAs. This is our NCG_∞ (non-)formality theorem. The new data added by extending Kontsevich's OCHA to a flag OCHA is a quasi-isomorphism $T_{\mathrm{poly}} \to C(\mathcal{A}, \mathcal{A})$ of A_∞ algebras with homotopy actions by T_{poly}.

The first construction we need for our extension of the Kontsevich representation is a suitable operad $\mathsf{Gra_{FOC}}$, extending Kontsevich's operad of Feynman diagrams.

4.4 Various graph operads.

In this section we shall introduce various operads whose operations are defined by formal sums of graphs. These operads serve as "universal," or "stable," endomorphism operads. Before giving the definitions we shall give a motivational detour to clarfiy the sense in which these graph operads are universal.

4.4.1 Stable endomorphisms of polyvectors.

Let $d \geq 1$. We define the space of polynomial polyvector fields on $\mathbf{k}^d$, to be denoted $T_{\mathrm{poly}}(\mathbf{k}^d)$, as the commutative algebra

$$S((\mathbf{k}^d)^* \oplus \mathbf{k}^d[-1])$$

and identify it with the graded commutative polynomial ring

$$\mathbf{k}[x^1, \ldots, x^d, \eta_1, \ldots, \eta_d]$$

(the generators η_μ have degree 1). The group

$$A(d) := GL((\mathbf{k}^*)^d) \ltimes ((\mathbf{k}^*)^d \oplus \mathbf{k}^d[-1])$$

acts on $T_{\mathrm{poly}}(\mathbf{k}^d)$ by algebra automorphisms, using the defining representation of GL_d on $(\mathbf{k}^*)^d$, its dual representation on $\mathbf{k}^d[-1]$ and the defining representation of $(\mathbf{k}^*)^d \oplus \mathbf{k}^d[-1]$ in terms of translations.

There are inclusions $T_{\mathrm{poly}}(\mathbf{k}^d) \hookrightarrow T_{\mathrm{poly}}(\mathbf{k}^{d+1})$ given by the canonical

$$\mathbf{k}[x^1, \ldots, x^d, \eta_1, \ldots, \eta_d] \to \mathbf{k}[x^1, \ldots, x^d, \eta_1, \ldots, \eta_d] \otimes \mathbf{k}[x^{d+1}, \eta_{d+1}]$$
$$= \mathbf{k}[x^{d+1}, \ldots, x^d, \eta_1, \ldots, \eta_{d+1}]$$

and also projections $T_{\mathrm{poly}}(\mathbf{k}^{d+1}) \to T_{\mathrm{poly}}(\mathbf{k}^d)$, given by the augmentation $\mathbf{k}[x^{d+1}, \eta_{d+1}] \to \mathbf{k}$. For every $n \geq 1$ these maps define maps

$$\mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^d)) \to \mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^{d+1})^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^{d+1}))$$

by post- and precomposition. It is clear that we get induced maps on invariants;

$$\mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^d))^{A_d} \to \mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^{d+1})^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^{d+1}))^{A_{d+1}}.$$

Define $\mathsf{G}(n) := \lim_d \mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^d))^{A_d}$ to be the limit of this diagram.

Lemma 4.4.1.1. There is an isomorphism between $\mathsf{G}(n)$ and the free graded commutative algebra $\mathbf{k}[e_{ij}]$ generated by degree -1 elements e_{ij}, $1 \leq i, j \leq n$, explicitly given by associating to e_{ij} the map

$$\frac{\partial}{\partial \eta^i_\mu} \frac{\partial}{\partial x^\mu_j} := \mu_{(n)} \circ_i \frac{\partial}{\partial \eta_\mu} \circ_j \frac{\partial}{\partial x^\mu} : T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n} \to T_{\mathrm{poly}}(\mathbf{k}^{d+1}),$$

where $\mu_{(n)} : T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n} \to T_{\mathrm{poly}}(\mathbf{k}^d)$ is the (commutative) multiplication and $\circ_i$ and $\circ_j$ denote compositions in the endomorphism operad of $T_{\mathrm{poly}}(\mathbf{k}^d)$ and a sum over $\mu = 1, \ldots, d$ is implied.

Proof. First identify

$$\mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^d))^{A_d} = \left(\mathbf{k}[x^\mu, \eta_\nu] \otimes \bigotimes_{i=1}^{n} \mathbf{k}\left[\frac{\partial}{\partial x^\mu_i}, \frac{\partial}{\partial \eta^i_\nu} \right] \right)^{A_d},$$

where, e.g., $\partial/\partial x^\mu_i := \mu_{(n)} \circ_i \partial/\partial x^\mu$, and note that this reduces to

$$\left(\bigotimes_{i=1}^{n} \mathbf{k}\left[\frac{\partial}{\partial x^\mu_i}, \frac{\partial}{\partial \eta^i_\nu} \right] \right)^{GL_d}.$$

The first fundamental theorem of invariant theory for the general linear group says that this is of the form

$$\mathbf{k}\left[\frac{\partial}{\partial \eta^i_\mu} \frac{\partial}{\partial x^\mu_j} \right] / I_d,$$

where I_d is an ideal, and $\mathrm{colim}_d I_d = 0$. $\qquad \square$

Remark 4.4.1.2. Note that we could have defined $\mathsf{G}(n)$ as a limit of $(\mathcal{S}(\mathbf{k}^d)^{\otimes n})^{GL_d}$, where

$$\mathcal{S}(\mathbf{k}^d) := \mathbf{k}\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial \eta_\nu}\right]$$

and we interpret

$$\mathcal{S}(\mathbf{k}^d)^{\otimes n} \cong \bigotimes_{i=1}^n \mathbf{k}\left[\frac{\partial}{\partial x_i^\mu}, \frac{\partial}{\partial \eta_\nu^i}\right]$$

as a subset of $\mathrm{Map}(T_{\mathrm{poly}}(\mathbf{k}^d)^{\otimes n}, T_{\mathrm{poly}}(\mathbf{k}^d))$. The spaces $(\mathcal{S}(\mathbf{k}^d)^{\otimes n})^{GL_d}$ assemble, as n varies, to operads

$$T^+(\mathcal{S}(\mathbf{k}^d))^{GL_d} \cong \mathsf{End}\langle T_{\mathrm{poly}}(\mathbf{k}^d)\rangle^{A_d} \subset \mathsf{End}\langle T_{\mathrm{poly}}(\mathbf{k}^d)\rangle.$$

The sequence $\{\mathsf{G}(n)\}_{n\geq 1}$ also defines an operad G, which we can regard as either of the two limits

$$\lim_d \mathsf{End}\langle T_{\mathrm{poly}}(\mathbf{k}^d)\rangle^{A_d} \text{ or } \lim_d T^+(\mathcal{S}(\mathbf{k}^d))^{GL_d}$$

in the category of dg operads.

Remark 4.4.1.3. Define $T_{\mathrm{poly}}(V) := S(V^* \oplus V[-1])$ and the group $A(V) := GL(V^*) \ltimes (V^* \oplus V[-1])$, for V a graded vector space of finite type. Let τ be the image of id_V under

$$V \otimes V^* \to V \otimes V^*[1] \cong (V^* \otimes V[-1])^*,$$

and then extended as a biderivation

$$T_{\mathrm{poly}}(V) \otimes T_{\mathrm{poly}}(V) \to T_{\mathrm{poly}}(V) \otimes T_{\mathrm{poly}}(V)$$

of degree -1. It is clear that there is a morphism $\mathsf{G} \to \mathsf{End}\langle T_{\mathrm{poly}}(V)\rangle^{A(V)}$, sending e_{ij} to $\mu_{(n)} \circ \tau_{ij}$, where τ_{ij} acts as τ on the ith factor tensor the jth factor and as the identity on all others.

Likewise, defining $\mathcal{S}(V) := S(V \oplus V^*[1])$ gives an operad

$$T^+(\mathcal{S}(V))^{GL(V^*)} \cong \mathsf{End}\langle T_{\mathrm{poly}}(V)\rangle^{A(V)}.$$

Definition 4.4.1.4. For any dg operad P and graded vector space of finite type V, we say that a P-algebra structure on $T_{\mathrm{poly}}(V)$ is **stable** if it factors through $\mathsf{G} \to \mathsf{End}\langle T_{\mathrm{poly}}(V)\rangle$.

4.4.2 Graphical model of the stable endomorphisms.

Elements of $\mathsf{G}(n)$ can be regarded as linear combinations of certain labeled graphs, subject to certain symmetry relations. Recall $\mathsf{G}(n)$ is a polynomial algebra $\mathbf{k}[e_{ij}]$. To every monomial $M \in \mathsf{G}(n)$ we associate a graph Γ with set of vertices $[n]$, no legs, and a directed edge connecting the vertex i and the vertex j for every $e_{ij} \in M$. Since the generators e_{ij} have degree -1 we need to order the set of edges of Γ up to an even permutation in order to be able to recover M from Γ. Moreover, the degrees imply $e_{ij} \cdot e_{ij} = 0$, meaning that Γ cannot contain a double edge. Let us make this more precise.

Definition 4.4.2.1. Define $fdgra_n^k$ to be the set of graphs Γ with set of vertices $[n]$, set of flags $[2k]$ with the involution $i \mapsto i + k \pmod{k}$. This means that the only freedom in defining Γ is the attachment map $h : [2k] \to [n]$. The set of edges is naturally identified with $[k]$ by associating to the edge $\{i, i + k\}$ the number i. We regard the edge i as directed from the vertex $h(i)$ to the vertex $h(i + k)$. Define

$$\mathsf{dGra}^{\circlearrowleft}(n) := \bigoplus_{k \geq 0} \mathbf{k}\{fgra_n^k\}[k] \otimes_{\Sigma_k} sgn_k,$$

using the action of Σ_k which permutes the edges.

Remark 4.4.2.2. There is an isomorphism of graded Σ_n-modules

$$\mathsf{dGra}^{\circlearrowleft}(n) \cong \mathsf{G}(n),$$

defined by sending a graph Γ to the monomial $\prod_{i=1}^{k} e_{h(i)\, h(i+k)}$.

Let us describe the operadic composition in terms of graphs. Take graphs $\Gamma \in fdgra_n^k$ and $\Gamma'' \in fdgra_{n''}^{k''}$. Define an embedding $\iota : \Gamma'' \hookrightarrow \Gamma$ to be a pair of injectionso $\iota_v : V'' \to V$, $\iota_f : F'' \to F$ such that $\iota_f \circ \tau'' = \tau \circ \iota_f$ and $h \circ \iota_f = \iota_v \circ h''$. Given an embedding $\iota : \Gamma'' \to \Gamma$ we define $\Gamma' \in fdgra_{n'}^{k'}$, $k' := k - k''$, $n - n'' + 1$ by letting $h' : F' \to V'$ be the composition

$$F' \cong F \setminus \iota_f(F'') \xrightarrow{h} V \to V/\iota_v(V'') \cong V'.$$

The two isomorphisms are the unique order-preserving bijections. Write $\Gamma/_i\Gamma'' = \Gamma'$ to mean an embedding $\Gamma'' \hookrightarrow \Gamma$ such that $V/\iota_v(V'') \cong V'$ maps $\iota_v(V'')$ to $i \in V'$. The operadic composition is defined by

$$\Gamma' \circ_i \Gamma'' := \sum_{\Gamma/_i\Gamma''=\Gamma'} (-1)^{|\sigma|}\Gamma,$$

for σ the unique order-preserving bijection $E' \sqcup E'' \cong E$, where $E' \sqcup E''$ has the lexicographic order defined by $E' < E''$.

4.4.3 Our graph operads.

Here we define the various operads we will use in our construction.

Definition 4.4.3.1. Let dGra be the suboperad of $\mathsf{dGra}^{\circlearrowleft}$ consisting of graphs without tadpoles (i.e., of graphs corresponding to monomials not containing any e_{ii}).

Define $\mathsf{Gra}^{\circlearrowleft} \subset \mathsf{dGra}^{\circlearrowleft}$ to be the suboperad spanned those linear combinations of graphs that are invariant under reflection of edge-directions. Thus:

$$\bigoplus_{k \geq 0} \mathbf{k}\{fgra_n^k\}[k] \otimes_{\Sigma_k \times \Sigma_2^k} sgn_k.$$

Further, we denote by $\mathsf{Gra} := \mathsf{Gra}^{\circlearrowleft} \cap \mathsf{dGra}$ the corresponding operad of graphs without tadpoles. We shall consider the vertices of graphs in Gra as colored black.

Let $p : \{\bullet, \circ\} \to \{\bullet\}$ and define the operad $\mathsf{Gra}_{\mathsf{NCG}} := p^*\mathsf{Gra}$. Hence $\mathsf{Gra}_{\mathsf{NCG}}(m, n \mid c) = \mathsf{Gra}(m + n)$, for any color $c \in \{\bullet, \circ\}$.

Let $q : \{\bullet, \circ, \square\} \to \{\bullet\}$ and let $\mathsf{Gra}_{\mathsf{FOC}}$ to be the suboperad of $q^*\mathsf{dGra}$ defined as follows. Its restriction to the first two colors is a copy of $\mathsf{Gra}_{\mathsf{NCG}}$, while $\mathsf{Gra}_{\mathsf{NCG}}(k, m, n \mid \square)$ is the subspace of $\mathsf{dGra}(k + m + n)$ spanned by only those graphs that do not have any edges directed away from a vertex labelled by $\square$. We say that the black vertices are **free interior vertices**, the white circle vertices are **collinear vertices**, and the white square vertices are **boundary vertices**.

Remark 4.4.3.2. The description of the composition of $\mathsf{dGra}^{\circlearrowleft}$ makes it clear that these operads are well-defined.

There is by definition a representation

$$D : \mathsf{Gra} \to \mathsf{End}\langle T_{\mathrm{poly}}(V)\rangle,$$

for any graded vector space V of finite type. The graph $\bullet\ \bullet$ is under this representation sent to the commutative wedge product $\wedge$, and the graph $\bullet\!\!-\!\!\bullet$ is sent to the Schouten bracket $[\,,\,]_S$. Analogously, there is a representation

$$D : \mathsf{Gra}_{\mathsf{NCG}} \to \mathsf{End}\langle T_{\mathrm{poly}}(V), T_{\mathrm{poly}}(V)\rangle.$$

Here we can mix the colors of the vertices. The triple of graphs $\bullet\!\!-\!\!\bullet$, $\circ\ \circ$ and $\genfrac{}{}{0pt}{}{\bullet}{\circ}$ define the canonical NCG-representation on polyvector fields in terms of the Schouten bracket, the wedge product, and the adjoint action of the Schouten bracket as an action by derivations of the wedge product.

Define $\mathcal{A}(V) := S(V^*)$, again for V of finite type. There is, lastly, a representation (again denoted D)

$$D : \mathsf{Gra}_{\mathsf{FOC}} \to \mathsf{End}\langle T_{\mathrm{poly}}(V), T_{\mathrm{poly}}(V), \mathcal{A}(V)\rangle,$$

defined as follows. Take a graph $\Gamma \in \mathsf{Gra}_{\mathsf{FOC}}(k, m, n \mid \square)$. A priori it by definition only defines a map

$$T_{\mathrm{poly}}(V)^{\otimes k} \otimes T_{\mathrm{poly}}(V)^{\otimes m} \otimes T_{\mathrm{poly}}(V)^{\otimes n} \to T_{\mathrm{poly}}(V).$$

Precompose this with the inclusion $\mathcal{A}(V)^{\otimes n} \subset T_{\mathrm{poly}}(V)^{\otimes n}$ on the third factor and postcompose it with the projection $T_{\mathrm{poly}}(V) \to \mathcal{A}(V)$; this gives our map

$$D_\Gamma : T_{\mathrm{poly}}(V)^{\otimes k} \otimes T_{\mathrm{poly}}(V)^{\otimes m} \otimes \mathcal{A}(V)^{\otimes n} \to \mathcal{A}(V).$$

Definition 4.4.3.3. Let $\mathsf{gra}(n)$ be the subspace of $\mathsf{Gra}(n)^*$ spanned by finite sums of (formal duals to) graphs. These assemble to a cooperad (with cocomposition dual to the operadic composition) that we denote gra. The same finiteness condition defines analogous cooperads $\mathsf{gra}_{\mathsf{NCG}}$ and $\mathsf{gra}_{\mathsf{FOC}}$.

To simplify notation we define

$$\mathsf{gra}_{\mathsf{FOC}}(k, m, n) := \mathsf{gra}_{\mathsf{FOC}}(k, m, n \mid \square), \quad \mathsf{gra}_{\mathsf{NCG}}(p, q) := \mathsf{gra}_{\mathsf{NCG}}(p, q \mid \circ).$$

This will not create any confusion since no other components will feature in our construction.

4.5 A de Rham field theory.

Given a pair of distinct indices $i, j \in [k] + [m] + [n]$ we follow Kontsevich and consider the function

$$\phi_{i,j}^h : CF_{k,m,n}(\mathbf{H}) \to S^1, \ x + \mathbf{R} \rtimes \mathbf{R}_{>0} \mapsto \mathrm{Arg}\left(\frac{x_j - x_i}{x_j - \overline{x}_i}\right).$$

Here a barred variable denotes the complex conjugate variable. The function is smooth and extends to a smooth function defined on the compactified configuration space.

Let θ be the homogeneous normalized volume form on the circle S^1. Given a graph $\Gamma \in \mathsf{dgra}(k, m, n)$ with d edges, define

$$\theta^\Gamma := \wedge_{e_{ij} \in E_\Gamma} (\phi_{i,j}^h)^* \theta.$$

The form θ^{Γ} is a smooth closed differential form of degree d on $\overline{CF}_{k,m,n}(\mathbf{H})$. We extend θ to a map of dg vector spaces

$$\theta : \mathsf{gra}_{\mathsf{FOC}}(k,m,n) \to \Omega(\overline{CF}_{k,m,n}(\mathbf{H})).$$

Define similarly, for indices $i,j \in [\ell]$, $\phi_{i,j} : C_{\ell}(\mathbf{C}) \to S^1$ by

$$\phi_{i,j} : x + \mathbf{C} \rtimes \mathbf{R}_{>0} \mapsto \mathrm{Arg}(x_j - x_i).$$

The function ϕ extends to the compactification. For a graph $\Gamma \in \mathsf{gra}$, let

$$\theta^{\Gamma} := \wedge_{e_{ij} \in E_{\Gamma}} (\phi_{i,j})^* \theta.$$

This allows us to define maps of dg vector spaces

$$\theta : \mathsf{gra}(\ell) \to \Omega(\overline{C}_{\ell}(\mathbf{C})).$$

By identifying $\overline{CF}_{p,q}(\mathbf{C})$ with a subset of $\overline{C}_{p+q}(\mathbf{C})$ and identifying the space $\mathsf{gra}_{\mathsf{NCG}}(p,q)$ with $\mathsf{gra}(p+q)$ we can use this to define maps of dg vector spaces

$$\theta : \mathsf{gra}_{\mathsf{NCG}}(p,q) \to \Omega(\overline{CF}_{p,q}(\mathbf{C}))$$

as well.

In all cases we interpret the form associated to a graph without edges as the function identically equal to 1.

Claim. The de Rham complex functor Ω is only comonoidal up to quasi-isomorphism with respect to the usual tensor product of dg vector spaces. Hence $\Omega(\overline{CF}(\mathbf{H}))$ is only a cooperad up to quasi-isomorphisms. This inconvenience can be ignored by working with a completed tensor product, regarding it, say, as a cooperad in the category of chain complexes of nuclear Fréchet spaces. Our maps $\theta : \Gamma \mapsto \theta^{\Gamma}$, for the variously colored graphs, assemble to a morphism

$$\theta : \mathsf{gra}_{\mathsf{FOC}} \to \Omega(\overline{CF}(\mathbf{H}))$$

of cooperads in this category of cooperads.

We shall not prove this statement as it is a consequence of similar statements in [Merkulov 2010] and the original arguments in [Kontsevich 2003].

4.6 Explicit (non-)formality.

Combining the previous subsections, we have a representation

$$D \circ \theta^* : \mathsf{FOC} \to \mathsf{gra}_{\mathsf{FOC}} \to \mathsf{End}\langle T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{A}\rangle.$$

Since $\mathsf{FOC} = \mathcal{K}(\overline{CF}(\mathbf{H}))$ is quasi-free the representation consists of a family of maps, one for each generator of the operad, satisfying some quadratic identities coming from the boundary differential on $\mathcal{K}(\overline{CF}(\mathbf{H}))$. We shall denote the components as follows:

- $\lambda_\ell := D \circ \theta^*([C_\ell(\mathbf{C})]) \in \mathrm{Map}^{3-2\ell}(T_{\mathrm{poly}}^{\otimes \ell}, T_{\mathrm{poly}})$, for $\ell \geq 2$.

- $\nu_p := D \circ \theta^*([CF^+_{0,q}(\mathbf{C})]) \in \mathrm{Map}^{2-q}(T_{\mathrm{poly}}^{\otimes q}, T_{\mathrm{poly}})$ for $q \geq 2$.

- $\mu_n := D \circ \theta^*([CF^+_{0,0,n}(\mathbf{H})]) \in \mathrm{Map}^{2-n}(\mathcal{A}^{\otimes n}, \mathcal{A})$ for $n \geq 2$.

- $\mathcal{V}_{p,q} := D \circ \theta^*([CF^+_{p,q}(\mathbf{C})]) \in \mathrm{Map}^{2-2p-q}(T_{\mathrm{poly}}^{\otimes p} \otimes T_{\mathrm{poly}}^{\otimes q}, T_{\mathrm{poly}})$ for $p, q \geq 1$.

- $\mathcal{U}_{k,n} := D \circ \theta^*([CF^+_{k,0,n}(\mathbf{H})]) \in \mathrm{Map}^{2-2k-n}(T_{\mathrm{poly}}^{\otimes k} \otimes \mathcal{A}^{\otimes n}, \mathcal{A})$ for $k \geq 1$, $n \geq 0$.

- Finally, there are morphisms $\mathcal{Z}_{k,m,n} := D \circ \theta^*([CF^+_{k,m,n}(\mathbf{H})])$ in $\mathrm{Map}^{1-2k-m-n}(T_{\mathrm{poly}}^{\otimes k} \otimes T_{\mathrm{poly}}^{\otimes m} \otimes \mathcal{A}^{\otimes n}, \mathcal{A})$, for $k \geq 0$, $m \geq 1$, $n \geq 0$.

Recall that the Hochschild cochain complex $C_{\mathrm{Hoch}}(A, A)$ of an A_∞ algebra A is

$$\mathrm{Map}(T(A[1]), A), \text{ where } T(A[1]) = \bigoplus_{r \geq 0} A[1]^{\otimes r},$$

while the A_∞ deformation complex is $C_{\mathsf{Ass}}(A, A) = \mathrm{Map}(T^+(A[1]), A[1])$. The brace operations on the Hochschild cochains complex are maps

$$(\ )\{\dots\}_p : C_{\mathrm{Hoch}}(A, A) \otimes \bigotimes_{i=1}^{p} C_{\mathrm{Hoch}}(A, A) \to C_{\mathrm{Hoch}}(A, A), \ p \geq 1,$$

defined for $x \in \mathrm{Map}(A[1]^{\otimes r}, A)$, $x_i \in \mathrm{Map}(A[1]^{\otimes r_i}, A)$, $1 \leq i \leq p \leq r$, $n = r + r_1 + \dots + r_p - p$, by

$$x\{x_1, \dots, x_p\}_p(a_1, \dots, a_n) = \sum_{1 \leq i_1 < \dots < i_p < r} \pm \, x(a_1, \dots, a_{i_1 - 1}, x_1(a_{i_1}, \dots),$$

$$\dots, a_{i_p - 1}, x_p(a_{i_p}, \dots), \dots, a_n).$$

The Gerstenhaber bracket on the Hochschild cochain complex is the operation
$$[x, y]_G := x\{y\}_1 \pm y\{x\}_1.$$
It is a graded Lie bracket of degree -1 in our grading on the Hochschild cochain complex, while on the deformation complex it has degree 0. Set $(){\{\dots\}} := \sum_{p\geq 1}(){\{\dots\}}_p$ and define
$$br : C_{\text{Hoch}}(A, A)[1] \to C_{\text{Ass}}(C_{\text{Hoch}}(A, A), C_{\text{Hoch}}(A, A)),$$
$$x \mapsto (){\{x\}}_1 + x\{\dots\}.$$
One verifies that this is a map of graded Lie algebras.

An A_∞ structure on A is a Maurer-Cartan element $m = d + m_2 + \dots$ in $C_{\text{Ass}}(A, A)$. The differential $[m,]_G$ makes the Hochschild cochain complex a dg Lie algebra. It is also an A_∞ algebra with A_∞ structure the Maurer-Cartan element $\cup^m := br(m)$ of $C_{\text{Ass}}(C_{\text{Hoch}}(A, A), C_{\text{Hoch}}(A, A))$. When A has a given A_∞ structure m we shall find it convenient to write $C_{\text{Hoch}}(m)$ for $C_{\text{Hoch}}(A, A)$ with differential $[m,]_G$, and, similarly, also write $C_{\text{Ass}}(m)$ for the deformation complex of (A, m).

The interpretation of the components of our representation is that

- $\lambda = \{\lambda_\ell\}$ is an L_∞ structure on $T_{\text{poly}}[1]$.

- $\nu = \{\nu_p\}$ is an A_∞ structure on T_{poly}.

- $\mu = \{\mu_n\}$ is an A_∞ structure on $\mathcal{A}$.

- $\mathcal{V} = \{\mathcal{V}_{p,q}\}$ is an L_∞ map $(T_{\text{poly}}[1], \lambda) \to (C_{\text{Ass}}(\nu), [\,,\,]_G)$.

- $\mathcal{U} = \{\mathcal{U}_{k,n}\}$ is an L_∞ map $(T_{\text{poly}}[1], \lambda) \to (C_{\text{Hoch}}(\mu)[1], [\,,\,]_G)$.

- $\mathcal{Z} = \{\mathcal{Z}_{k,m,n}\}$ is an A_∞ morphism
$$(T_{\text{poly}}, \nu, \mathcal{V}) \to (C_{\text{Hoch}}(\mu), \cup^\mu, br \circ \mathcal{U})$$
of A_∞ algebras equipped with homotopy actions by $(T_{\text{poly}}, \lambda)$.

This description is a result of the interpretation of the operad of flag open-closed homotopy algebras. All the component maps have an explicit description as sums over graphs, e.g.
$$\mathcal{V}_{p,q} = \sum_{[\Gamma]\in[\text{gra}_{\text{NCG}}(p,q)^{2p+q-2}]} \int_{\overline{CF}^+_{p,q}(\mathbf{C})} \theta^\Gamma D_\Gamma,$$
with $[\text{gra}_{\text{NCG}}(p,q)^{2p+q-2}]$ the set of equivalence classes of graphs with $2p + q - 2$ edges under the Σ_{2p+q-2}-action by permutation of edges. We shall use this description to give a more detailed description of the component maps. The main tool is "Kontsevich's vanishing lemma":

Lemma 4.6.0.4. [Kontsevich 2003] Let X be a complex algebraic variety of dimension $N \geq 1$ and $Z_1, ..., Z_{2N}$ be rational functions on X, not equal identically to 0. Let U be any Zariski open subset of X such that each function Z_α is well-defined and nowhere vanishing on U, and that U consists of smooth points. Then the integral

$$\int_{U(\mathbf{C})} \wedge_{\alpha=1}^{2N} d(\operatorname{Arg}(Z_\alpha))$$

is absolutely convergent and is equal to zero.

4.7 Descriptions of the involved structures.

4.7.1 The homotopy Lie structure.

We have

$$\lambda_\ell = \sum_{[\Gamma] \in [\mathsf{gra}(\ell)^{2\ell-3}]} \int_{\overline{C}_\ell(\mathbf{C})} \theta^\Gamma D_\Gamma.$$

For $\ell \geq 3$, $C_\ell(\mathbf{C}) \cong S^1 \times U$, with $U = (\mathbf{C} \setminus \{0,1\})^{\ell-2} \setminus diagonals$. This identification can be obtained by using the translation freedom to fix the point labelled by 1, say, at the origin of $\mathbf{C}$ and using the dilation freedom to put the point labelled by 2, say, on the unit circle S^1. Multiplying the remaining points by the inverse of the phase of the point labelled by 2 gives a point in U. Using this description we can reduce every integral

$$\int_{\overline{C}_\ell(\mathbf{C})} \theta^\Gamma$$

to an integral over a circle times an integral of the type appearing in Kontsevich's vanishing lemma. Hence all weights vanish for $\ell \geq 3$. The configuration space $\overline{C}_2(\mathbf{C})$ is a circle. The set of graphs $[\mathsf{gra}(2)^1]$ contains a single graph, namely $\bullet\!-\!\!\bullet$. It follows that λ_2 is the Schouten bracket. As all higher homotopies $\lambda_{\geq 3}$ vanish, this means λ is the usual graded Schouten Lie algebra structure on T_{poly}.

4.7.2 The first homotopy associative structure.

The A_∞ structure ν has components

$$\nu_p = \sum_{[\Gamma] \in [\mathsf{gra}_{\mathsf{NCG}}(0,p)^{p-2}]} \int_{\overline{CF}_{0,p}^+(\mathbf{C})} \theta^\Gamma D_\Gamma.$$

The angle between collinear points is constant, so the differential form associated to a graph containing an edge connecting collinear vertices will be zero; hence no such graphs can contribute. It follows that the only graph which contributes is the graph ∘ ∘ with two vertices and no edge. The associated differential form is identically equal to one and we evaluate it on the one-point space $\overline{CF}_{0,2}(\mathbf{C})$. It follows that $\nu = \nu_2 = \wedge$ is the usual (wedge) product on T_{poly}.

4.7.3 The second homotopy associative structure.

The operation μ_n is given by a sum over graphs in $\mathsf{gra}_{\mathrm{FOC}}(0,0,n)^{n-2}$. This space of graphs is empty if n is not equal to 2 since the condition that no edge begins at a boundary vertex forces a graph with only boundary vertices to have no edges; thus, the only contributing graph is ▫▫. The space $\overline{CF}_{0,0,2}^{+}(\mathbf{H})$ is a point and the differential form associated to the graph with two vertices and no edge is the function identically equal to 1. The associated operator D_Γ is the wedge product of polyvector fields, restricted to a product on functions. It follows that $\mu = \mu_2$ is the usual associative (and commutative) product on $\mathcal{A}$.

4.7.4 The homotopy action.

Since

$$\mathcal{V}_{p,q} = \sum_{[\Gamma] \in [dgra_{p,q}^{2p+q-2}]} \int_{\overline{CF}_{p,q}^{+}(\mathbf{C})} \theta^\Gamma D_\Gamma$$

and $\overline{CF}_{p,1}^{+}(\mathbf{C}) \cong \overline{C}_{p+1}(\mathbf{C})$, the argument regarding the L_∞ structure λ can be repeated to conclude that $\mathcal{V}_{p,1} = 0$ for $p \geq 2$, while

$$\mathcal{V}_{1,1} : T_{\mathrm{poly}} \otimes T_{\mathrm{poly}} \to T_{\mathrm{poly}}, X \otimes \xi \mapsto [X, \xi]_S.$$

In other words, $\mathcal{V}_{1,1}$ is the adjoint action $T_{\mathrm{poly}} \to \mathrm{Der}(T_{\mathrm{poly}})$ of T_{poly} on itself by derivations of the wedge product, corresponding to the graph ⦙.

Using the translation freedom to put the collinear point labelled by 1 at the origin and the dilation freedom to put the collinear point labelled by 2 at 1 identifies $CF_{p,2}^{+}(\mathbf{C})$ with $(\mathbf{C} \setminus \{0,1\})^p \setminus diagonals$, so that one may again use Kontsevich's vanishing lemma and conclude that $\mathcal{V}_{p,2} = 0$ for all $p \geq 1$.

Reflection of the plane in the line of collinearity induces an involution f of $\overline{CF}_{p,q}^{+}(\mathbf{C})$. (Choosing representative configurations with the collinear points on the real axis identifies f with complex conjugation.)

The map f preserves orientation if p is even and reverses it if p is odd. For $\Gamma \in \mathsf{gra}_{\mathsf{NCG}}(p,q)^{2p+q-2}$, $f^* \theta^\Gamma = (-1)^{2p+q-2} \theta^\Gamma = (-1)^q \theta^\Gamma$. Thus

$$(-1)^p \int_{\overline{CF^+_{p,q}(\mathbf{C})}} \theta^\Gamma = (-1)^q \int_{\overline{CF^+_{p,q}(\mathbf{C})}} \theta^\Gamma,$$

implying the integral is 0 whenever p and q have different parity, i.e. whenever $p + q$ is odd. This means that the first homotopy to $\mathcal{V}_{1,1}$ is given by $\mathcal{V}_{1,3}$. The angle between collinear points is constant, so the differential form associated to a graph containing an edge connecting collinear vertices will be zero. The set $[\mathsf{gra}_{\mathsf{NCG}}(1,3)^3]$ contains a unique graph without edges connecting collinear vertices, up to direction and ordering of edges, namely the graph $\mathsf{o\!\!\!\!\!\!\!\!\bullet o}$. Hence there are eight (equivalence classes of) graphs (corresponding to the 2^3 ways to direct the three edges) contributing to $\mathcal{V}_{1,3}$. Each of these eight equivalence classes has a representative with the edges ordered so that e_i connects the free vertex with the collinear vertex labelled by i, $1 \leq i \leq 3$. These representatives all have weight $1/24$. To see this one may argue as follows.

Assume given a configuration in $CF^+_{1,3}(\mathbf{C})$. Use the freedom to translate along the imaginary axis to put the line of collinearity on the real axis. Use the freedom to translate along the real axis to put the free point on the imaginary axis. We are then left with a positive dilation that can be used to put the free point either at $+i$ or at $-i$, depending on wether it lies above or below the line of collinearity, respectively. These two types of configurations are mapped to each other by the involution f in the line of collinearity, discussed above. Denote the space of configurations of the first type, i.e. the subspace of $CF^+_{1,3}(\mathbf{C})$ where the free point lies above the line of collinearity, by C. It follows from the remarks on the involution f that the weight

$$\int_{\overline{CF^+_{1,3}(\mathbf{C})}} \theta^\Gamma$$

of a graph Γ entering the operation $\mathcal{V}_{1,3}$ may be calculated as

$$\int_{\overline{CF^+_{1,3}(\mathbf{C})}} \theta^\Gamma = 2 \int_C \theta^\Gamma.$$

We can identify C with the infinite open simplex $\{-\infty < x_1 < x_2 < x_3 < \infty\}$ and, for Γ the graph $\mathsf{o\!\!\!\!\!\!\!\!\bullet o}$ with the i-th edge directed from the

free vertex to the i-th collinear vertex, we may then calculate

$$\int_C \theta^\Gamma$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty < x_1 < x_2 < x_3 < \infty} d\mathrm{Arg}(i - x_1) \wedge d\mathrm{Arg}(i - x_2) \wedge d\mathrm{Arg}(i - x_3)$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty < x_1 < x_2 < x_3 < \infty} d\arctan(x_1) \wedge d\arctan(x_2) \wedge d\arctan(x_3)$$

$$= \frac{1}{48}.$$

The total weight is $2/48 = 1/24$.

It follows that the operation has the form

$$\mathcal{V}_{1,3} = \frac{1}{24} \wedge \circ \Big(\tau_{1,4} \circ \tau_{1,3} \circ \tau_{1,2} + \tau_{1,4} \circ \tau_{1,3} \circ \tau_{2,1}$$

$$+ \tau_{1,4} \circ \tau_{3,1} \circ \tau_{1,2} + \tau_{4,1} \circ \tau_{1,3} \circ \tau_{1,2} + \tau_{4,1} \circ \tau_{3,1} \circ \tau_{1,2}$$

$$+ \tau_{4,1} \circ \tau_{1,3} \circ \tau_{2,1} + \tau_{1,4} \circ \tau_{3,1} \circ \tau_{2,1} + \tau_{4,1} \circ \tau_{3,1} \circ \tau_{2,1} \Big)$$

as a map $T_{\mathrm{poly}}^{\otimes 1+3} \to T_{\mathrm{poly}}$. (The first of the four copies of T_{poly} acts on the last three.)

4.7.5 The homotopy Lie-morphism.

The map $\mathcal{U}$ is, by construction, Kontsevich's formality map. Recall that it's first Taylor component $\mathcal{U}_1 = \sum_{n \geq 0} \mathcal{U}_{1,n}$ is the Hochschild-Kostant-Rosenberg quasi-isomorphism.

4.7.6 The (non-)formality morphism.

Since $\overline{CF}^+_{0,1,n}(\mathbf{H})$ is isomorphic to $\overline{CF}^+_{1,0,n}(\mathbf{H})$ and $\mathsf{gra}_{\mathsf{FOC}}(0,1,n)$ is isomorphic to $\mathsf{gra}_{\mathsf{FOC}}(1,0,n)$, for all n, the maps $\mathcal{Z}_{0,1,n}$ coincide with the maps $\mathcal{U}_{1,n}$. Hence the first Taylor component of $\mathcal{Z}$,

$$\sum_{n \geq 0} \mathcal{Z}_{0,1,n} : T_{\mathrm{poly}} \to C_{\mathrm{Hoch}}(\mu),$$

is the Hochschild-Kostant-Rosenberg (HKR) quasi-isomorphism. The higher components of Kontsevich's formality map $\mathcal{U}$ are homotopies measuring the failure of the HKR map to respect the Lie brackets. In the same way, the higher components of $\mathcal{Z}$ are homotopies that keep track of the failure of the HKR map to respect the associative products and the

respective actions of T_{poly} by homotopy derivations of said associative products. Since the first component is the HKR morphism, we deduce the following theorem:

Theorem 4.7.6.1. The map

$$Z = \{Z_{k,m} = \sum_{n \geq 0} \mathcal{Z}_{k,m,n}\}_{k \geq 0, m \geq 1}$$

is an explicit NCG_∞ quasi-isomorphism from $((T_{\text{poly}}, [\,,\,]_S), (T_{\text{poly}}, \wedge, \mathcal{V}))$ to $((T_{\text{poly}}, [\,,\,]_S), (C_{\text{Hoch}}(\mathcal{A}, \mathcal{A}), d_H + \cup, br \circ \mathcal{U}))$

This statement implies the following A_∞ formality theorem:

Corollary 4.7.6.2. The map $A = \{A_m := \sum_{n \geq 0} \mathcal{Z}_{0,m,n}\}_{m \geq 1}$ is an explicit A_∞ quasi-isomorphism from $(T_{\text{poly}}, \wedge)$ to $(C_{\text{Hoch}}(\mathcal{A}, \mathcal{A}), d_H + \cup)$.

This corollary has essentially already been demonstrated, but in a different way, by Boris Shoikhet; see [Shoikhet 1998].

4.8 Induced homotopy associative structure.

An NCG_∞ algebra consists of an L_∞ algebra $(L[1], \lambda)$, an A_∞ algebra (A, ν) and an L_∞ morphism $\mathcal{V} : L[1] \to C_{\text{Ass}}(\nu)$. Let $\hbar$ be a formal parameter. The map $\mathcal{V}$ induces a map on the sets of Maurer-Cartan elements,

$$\mathrm{MC}(L[1][[\hbar]]) \to \mathrm{MC}(C_{\text{Ass}}(\nu)[[\hbar]]), \pi \mapsto \sum_{p \geq 1} \frac{1}{p!} \mathcal{V}_{p,q}((\hbar\pi)^{\otimes p},).$$

This gives us, for each Maurer-Cartan element π of L, an A_∞ structure

$$\nu_q^{\mathcal{V}(\pi)} := \nu_q + \sum_{p \geq 1} \frac{1}{p!} \mathcal{V}_{p,q}((\hbar\pi)^{\otimes p},), \ q \geq 1,$$

on $A[[\hbar]]$.

If $\mathcal{Z} : (L, A, \lambda, \mathcal{V}, \nu) \to (L, B, \lambda, \mathcal{U}, \mu)$ is a morphism of NCG_∞ algebras (the same L_∞ algebra acting on both and we assume the NCG_∞ algebra morphism is the identity on the Lie-color), then, for any Maurer-Cartan element π of $L[[\hbar]]$, we get an induced map of A_∞ algebras

$$\mathcal{Z}^\pi : (A[[\hbar]], \nu^{\mathcal{V}(\pi)}) \to (B[[\hbar]], \mu^{\mathcal{U}(\pi)})$$

by $\mathcal{Z}_m^\pi := \mathcal{Z}_{0,m} + \sum_{k \geq 0} \frac{1}{k!} \mathcal{Z}_{k,m}((\hbar\pi)^{\otimes k},)$. See the preceeding chapter on the operadic twist construction for the general argument. If $\mathcal{Z}$ is a quasi-isomorphism, then $\mathcal{Z}^\pi$ is as well.

Applying this general construction to our representation $D \circ \theta^*$ produces, for any Maurer-Cartan element $\pi \in T_{\text{poly}}$ (i.e., a possibly graded Poisson structure),

- an A_∞ structure $\nu^{\mathcal{V}(\pi)}$ on $T_{\text{poly}}[[\hbar]]$ with $\nu_1^{\mathcal{V}(\pi)} + \nu_2^{\mathcal{V}(\pi)} = \hbar[\pi, \,]_S + \wedge$ as its first two Taylor components, and

$$\nu_3^{\mathcal{V}(\pi)} = \hbar \mathcal{V}_{1,3}(\pi) + O(\hbar^3);$$

- the A_∞ cup product on the Hochschild cochains of $\mathcal{A}[[\hbar]]$ that corresponds to the Kontsevich star product $\mu^{\mathcal{U}(\pi)}$ defined by π;

- and an A_∞ quasi-isomorphism

$$\mathcal{Z}^\pi : (T_{\text{poly}}[[\hbar]], \nu^{\mathcal{V}(\pi)}) \to C_{\text{Hoch}}(\mu^{\mathcal{U}(\pi)})[[\hbar]].$$

We record this fact as a corollary.

Corollary 4.8.0.3. Let $\pi \in T_{\text{poly}}$ be a Poisson structure. Then the A_∞ algebra $(T_{\text{poly}}[[\hbar]], \nu^{\mathcal{V}(\pi)})$ is quasi-isomorphic as an A_∞ algebra to the algebra of Hochschild cochains on $\mathcal{A}[[\hbar]]$ equipped with the cup product corresponding to the Kontsevich star product defined by π. The map $\mathcal{Z}^\pi$ is an explicit such quasi-isomorphism.

It has been shown that one of the integrals entering Kontsevich's L_∞ morphism $\mathcal{U}$ evaluates to a rational multiple of $\zeta(3)/\pi^2$ [Felder and Willwacher 2010], which probably is not a rational number. However, we do not know anything about the (ir)rationality of the integrals entering our exotic structure on polyvector fields.

Conjecture 4.8.0.4. Are all integrals entering the exotic NCG_∞ algebra structure on polyvector fields rational numbers?

4.9 Relationship to Tamarkin's formality.

Dimitry Tamarkin gave an alternative proof [Tamarkin 1998] of Kontsevich's result that there exists an L_∞ quasi-isomorphism from polynomial polyvector fields to the Hochschild cochain complex; a proof which is less explicit but also in a sense more general. Tamarkin's proof roughly proceeds as follows. Apart from the Schouten bracket we also have the wedge product of polyvector fields, and together they give T_{poly} the structure of a Gerstenhaber algebra. Tamarkin argues that there exists a strong homotopy Gerstenhaber structure (i.e., a Ger$_\infty$-structure) on

$C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})$ with the propoerties that (i) its homotopy trasfer to T_{poly} via the HKR map coincides with the aforementioned canonically given Gerstenhaber structure, and (ii) its L_∞-structure is the Gerstenhaber bracket. From this one deduces existence of a Ger_∞ quasi-isomorphism

$$\mathcal{T} : T_{\mathrm{poly}} \to C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}).$$

This morphism can then be restricted to an L_∞ quasi-isomorphism $T_{\mathrm{poly}}[1] \to C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})[1]$, giving an alternative proof of Kontsevich's formality. Note that neither the Ger_∞-structure on $C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})$ or the morphism $\mathcal{T}$ are (fully) explicit. Tamarkin essentially only shows existence. In particular, the induced Com_∞-structure on $C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})$ is not specified.

In a sense, Tamarkin starts with something natural on T_{poly} and lifts it to something rather mysterious on Hochschild cochains. The result we prove may be considered as doing something opposite. Recall that the braces map defines a canonical NCG-structure on the Hochschild complex of an associative algebra. The operad NCG is Koszul, hence has a canonical resolution NCG_∞. Thus, by homotopy transfer along the HKR-map there must exist an NCG_∞-structure on the pair $(T_{\mathrm{poly}}, T_{\mathrm{poly}})$, and a Gerstenhaber ∞-quasi-isomorphism

$$(T_{\mathrm{poly}}, T_{\mathrm{poly}}) \to (C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}), C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})),$$

Our construction can be read as giving an explicit construction of this NCG_∞-structure on polyvector fields, and of the morphism. Our approach is an explicit construction in the spirit of Kontsevich's construction, not actually relying on homotopy transfer, but the above discussion explains our result from the perspective of Tamarkin's work. The HKR-map has an explicit inverse, but to the author's best knowledge one can only write down a recursive definition of a contracting homotopy on the Hochschild cochain complex; hence one cannot write down the homotopy transferred NCG_∞-structure on polyvector fields in a closed form (something our explicit formula does accomplish).

We prove in the next chapter that the transferred NCG_∞-structure is not homotopic to the canonical one. In this sense we do something opposite to Tamarkin; he starts with something standard on polyvector fields and gets something exotic on Hochschild cochains, while we have something natural on Hochschild cochains and are forced to put something exotic on polyvector fields.

Homotopical properties of the exotic structure

This chapter is devoted to proving that our exotic NCG_∞-structure on polyvector fields is essentially unique. More precisely, we show the following theorem.

Theorem 5.0.0.5. $H^1(\mathrm{Def}(\mathsf{NCG}_\infty \to \mathsf{Gra}_{\mathsf{NCG}})) = \mathbf{k}\,\vcenter{\hbox{$\circ\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!$}}$.

An immediate corollary is that our exotic structure is a homotopy nontrivial deformation. Hence the following statement.

Corollary 5.0.0.6. The canonical NCG-algebra structures on polyvector fields and on Hochschild cochains can not be NCG_∞ quasi-isomorphic.

In the last section we apply the general results to give an explicit strong homotopy version of the Duflo isomorphism. This generalizes earlier work by many authors, cf. [Pevzner and Torossian 2004; Calaque and Rossi 2011; Shoikhet 1998; Kontsevich 2003] and, of course, the original work by Michel Duflo [Duflo 1969]. More specifically, we construct a universal and generically homotopy nontrivial A_∞ deformation $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))_{\mathrm{exotic}}$ of the Chevalley-Eilenberg algebra $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))$, for a graded Lie algebra $\mathbf{g}$ of finite type, and an A_∞ quasi-isomorphism $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))_{\mathrm{exotic}} \to C_{\mathrm{CE}}(\mathbf{g}, U(\mathbf{g}))$ that on the cohomology level reproduces the Duflo-Kontsevich isomorphism of Chevalley-Eilenberg cohomologies. This implies that the Duflo-Kontsevich isomorphism can *not* be lifted in a universal way to a chain-level A_∞ quasi-isomorphism $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g})) \to C_{\mathrm{CE}}(\mathbf{g}, U(\mathbf{g}))$. We thus give a negative answer to the conjectured existence of a strong homotopy Duflo quasi-isomorphism,

but our actual proof is by an explicit construction of a "best possible" substitute.

The main theorem of this chapter (above) was demonstrated by the author and Sergei Merkulov in "Grothendieck-Teichmüller group and Poisson cohomologies", which has been accepted for publication in "Journal of Noncommutative Geometry", though the application to the Duflo isomorphism was essentially contained already in [Alm 2011].

5.1   Results by Kontsevich, Tamarkin and Willwacher.

The main result of this chapter is a rather simple consequence of several remarkable theorems by Maxim Kontsevich, Dimitry Tamarkin and Thomas Willwacher. In this section we review those results.

5.1.1   A closer look at the Gerstenhaber operad.

Define a **Gerstenhaber algebra** to be a dg vector space F and two binary operations λ and μ, such that $(F[1], \lambda)$ is a dg Lie algebra, (F, μ) is a dg commutative algebra, and the adjoint action of λ is a morphism

$$F[1] \to \mathrm{Der}_{\mathsf{Com}}(F, F)$$

of dg Lie algebras; thus, for any $\xi \in F^p$ the operation $\lambda(\xi, \ )$ is a derivation of μ of degree $|\xi| - 1$. Denote the operad governing Gerstenhaber algebras by Ger.

Recall the operad $\overline{C}(\mathbf{C})$ of compactified configuration spaces that we defined in 4.2.2. We noted there that its operad of cellular chains is isomorphic to the operad $\mathsf{Lie}_\infty\{-1\}$; hence the homology operad of the cellular chains, $H(\mathcal{K}(\overline{C}(\mathbf{C})))$, equals the suspended Lie operad $\mathsf{Lie}\{-1\}$. The following theorem says what the homology of *all* chains is.

Proposition 5.1.1.1. [Cohen 1995] The homology $H_\cdot(\overline{C}(\mathbf{C}))$ (with reversed grading, so that it is concentrated in negative degrees) is isomorphic to the operad of Gerstenhaber algebras, by identifying the class $[S^1] \in H_{-1}(\overline{C}_2(\mathbf{C}))$ with the Lie bracket operation and the class $[pt] \in H_0(\overline{C}_2(\mathbf{C}))$ with the commutative product operation.

Thus, the inclusion $\mathcal{K}(\overline{C}(\mathbf{C})) \to \mathrm{C}_{SA}(\overline{C}(\mathbf{C}))$ of the fundamental chains into all semialgebraic chains induces the inclusion $\mathsf{Lie}\{-1\} \to \mathsf{Ger}$ upon taking homology.

Proposition 5.1.1.2. [Arnol'd 1969] The cohomology $H^\cdot(\overline{C}_n(\mathbf{C})))$ is isomorphic as an algebra to the graded commutative algebra A_n freely

94

generated by degree 1 elements $\omega_{ij} = \omega_{ji}$ $(1 \leq i \neq j \leq n)$ modulu the relations $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ji} = 0$.

5.1.2 Kontsevich's operad of two-colored graphs.

Recall our 2-colored operad $\mathsf{Gra}_{\mathsf{NCG}}$ of graphs with black and white vertices. Kontsevich introduced the following operad (though he did so without any mention of the twist construction).

Definition 5.1.2.1. [Kontsevich 1999] Define $\mathsf{Graphs} \subset \mathrm{tw}\,\mathsf{Gra}_{\mathsf{NCG}}$ to be the dg suboperad spanned by only those graphs that have (i) all black vertices at least trivalent and (ii) have no connected component with only black vertices.

Note $\mathrm{tw}\,\mathsf{Gra}_{\mathsf{NCG}} = \mathrm{Tw}\,\mathsf{Gra}$. Polyvector fields are a Gerstenhaber algebra, with the representation factoring through

$$\mathsf{Ger} \to \mathsf{Gra}.$$

The Gerstenhaber operad is a coalgebra for the twist comonad [Dolgushev and Willwacher 2012], hence there exists a morphism

$$\mathsf{Ger} \to \mathrm{Tw}\,\mathsf{Gra},$$

and one may check that it factors through the inclusion of Graphs. Explicitly, the commutative product is represented by the graph $\circ\ \circ$ and the Schouten bracket is represented by the graph $\circ\text{-}\circ$. Kontsevich proved the following proposition.

Proposition 5.1.2.2. [Kontsevich 1999; Lambrechts and Volić 2008] The map $\mathsf{Ger} \to \mathsf{Graphs}$ is a quasi-isomorphism.

We shall not give a proof of this theorem, but will indicated the idea. Recall that we defined in 4.5 a morphism

$$\theta : \mathsf{gra} \to \Omega(\overline{C}(\mathbf{C})).$$

Define graphs the be the linearly dual cooperad to Graphs which is spanned by finite sums of (formal duals to) graphs. Take $\Gamma \in \mathsf{graphs}(n)$ with k black vertices. Considering Γ as an element of $\mathsf{gra}(k+n)$ we can consider

$$\theta^\Gamma \in \Omega(\overline{C}_{k+n}(\mathbf{C})).$$

There is a projection $\pi : \overline{C}_{k+n}(\mathbf{C})) \to \overline{C}_n(\mathbf{C}))$ that forgets the points labelled by $[k]$. Define

$$\vartheta^\Gamma := \pi^!(\theta^\Gamma) \in \Omega(\overline{C}_n(\mathbf{C}))$$

to be the fiber integration. Kontsevich proved that all the fiber integrals converge, so that these forms are well-defined, and that

$$\vartheta : \mathsf{graphs} \to \Omega(\overline{C}(\mathbf{C}))$$

is a quasi-isomorphism. In more detail, he argued that the cohomology $H(\mathsf{graphs})$ is generated multiplicatively by the graphs with a single edge connecting two vertices, say i and j, modulu exactly the Arnol'd relations 5.1.1.2, i.e., the classes of the corresponding forms $[\theta^{ij}]$ generate the cohomology algebra $H^{\cdot}(\overline{C}(\mathbf{C}))$.

5.1.3 Tamarkin and the Grothendieck-Teichmüller group.

The Lie algebra $\mathfrak{t}_n$ of infinitesimal braids on n-strands is the Lie algebra generated by $t_{ij} = t_{ji}$, $1 \leq i \neq j \leq n$ modulu the relations

$$[t_{ij}, t_{kl}] = 0 = [t_{ij} + t_{jk}, t_{ik}]$$

if $\{i, j\} \cap \{k, l\} = \emptyset$. The Arnol'd algebra A_n is Koszul dual to the Lie algebra of infinitesimal braids, in the sense that it is quasi-isomorphic as a dg commutative algebra to the Chevalley-Eilenberg cochain algebra $C_{\mathrm{CE}}(\mathfrak{t}_n)$. The infinitesimal braids are an operad in the category of Lie algebras (with direct sum as monoidal product), via

$$\circ_i : \mathfrak{t}_{n+1} \oplus \mathfrak{t}_k \to \mathfrak{t}_{n+k},$$

$$\circ_i(t_{ab}) = t_{ab}, \quad a \neq i \neq b, \ a, b \in [n+1]$$

$$= \sum_{c \in [k]} t_{ic}, \quad a = i, \ b \in [n+1]$$

$$= t_{ab}, \quad a, b \in [k].$$

This can be used as follows to define a model of the Gerstenhaber operad.

Let $\mathsf{Pa}(n)$ be the set of planar rooted trivalent trees with an isomorphism between the set of leaves and the set $[n]$. We identify it with the set of binary paranthesizations of the symbols $1, 2, \ldots, n$, e.g. $((13)2)(5(46)) \in \mathsf{Pa}(6)$. Grafting of trees (substitution of paranthesizations) defines an operad Pa with components Pa. For example,

$$((13)2) \circ_2 (1(23)) = ((15)(2(34))).$$

Let $\hat{U}(\mathfrak{t}_n)$ be the completed (with respect to word-length) universal enveloping algebra and denote by $\mathsf{PaCD}(n)$ the category with objects $\mathsf{Pa}(n)$ and morphisms between any two objects the set $\hat{U}(\mathfrak{t}_n)$. It is naturally considered as a category enriched in complete filtered cocommutative coalgebras.

Definition 5.1.3.1. [Tamarkin 2002; Bar-Natan 1998] The Grothendieck-Teichmüller group GRT_1 is the group of automorphisms of the enriched category PaCD which are the identity on objects and fix the two morphisms

$$t_{12} : (12) \rightarrow (12), \quad 1 : (12) \rightarrow (21).$$

The nerve of the enriched category $\mathsf{PaCD}(n)$ is the simplicial vector space with

$$N_r(\mathsf{PaCD}(n)) = \bigoplus_{u_0,\ldots,u_r \in Ob\,\mathsf{Pa}(n)} N_r(\hat{U}(\mathfrak{t}_n)),$$

$$N_r(\hat{U}(\mathfrak{t}_n)) = \hat{U}(\mathfrak{t}_n)^{\otimes r}.$$

The face maps d_i are for $i = 0$ and $i = r$ given by the augmentation on the first and last copy of $\hat{U}(\mathfrak{t}_n)$, respectively, while the face map d_i for $1 \leq i \leq r - 1$ is defined by the multiplication between the ith copy and the $(i+1)$th copy. The degeneracies are given by insertion of units $1 \in \hat{U}(\mathfrak{t}_n)$. Define $C(\mathsf{PaCD}(n))$ to be the associated normalized chain complex. These assemble to a dg operad $C(\mathsf{PaCD})$ and there is a sequence of quasi-isomorphisms of operads

$$\mathsf{Ger} \rightarrow \mathcal{B}(\hat{U}(\mathfrak{t})) \leftarrow C(\mathsf{PaCD}).$$

Here $\mathcal{B}(\hat{U}(\mathfrak{t}))$ is the (associative) normalized bar construction, defined as the normalized chain complex associated to $N_\bullet(\hat{U}(\mathfrak{t}))$. The only difference between that and $C(\mathsf{PaCD})$ is that all reference to objects of Pa is dropped.

The group GRT_1 is prounipotent. The action on PaCD gives a map $\mathfrak{grt}_1 \rightarrow \mathrm{Der}(\mathsf{PaCD})$ from its associated Lie algebra. Since all morphisms displayed above are quasi-isomorphisms one may, following Tamarkin, conclude that there is a (uniquely defined) morphism

$$\mathfrak{grt}_1 \rightarrow H^1(\mathrm{Def}(\mathsf{Ger}_\infty \rightarrow \mathsf{Ger})).$$

Next, Tamarkin proved the following results.

Proposition 5.1.3.2. [Tamarkin 2002] The dg vector spaces $C^{\mathrm{CE}}(\mathfrak{t}_n)$ form an operad, quasi-isomorphic to the Gerstenhaber operad, and the first cohomology group of $\mathrm{Def}(\mathsf{Com}_\infty \rightarrow C^{\mathrm{CE}}(\mathfrak{t}))$ equals the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$. Moreover, the composite

$$\mathfrak{grt}_1 \rightarrow H^1(\mathrm{Def}(\mathsf{Ger}_\infty \rightarrow \mathsf{Ger})) \rightarrow H^1(\mathrm{Def}(\mathsf{Com}_\infty \rightarrow C^{\mathrm{CE}}(\mathfrak{t}_n))) = \mathfrak{grt}_1$$

is the identity.

Corollary 5.1.3.3. The map $\mathfrak{grt}_1 \to H^1(\mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Ger}))$ is injective.

We shall not prove this proposition but only make the following remark. The complex

$$\mathrm{Def}(\mathsf{Com}_\infty \to C^{\mathrm{CE}}(\mathfrak{t})) = \prod_{n \geq 1} \mathrm{Map}_{\Sigma_n}(\mathsf{Com}^{\mathsf{i}}(n), C^{\mathrm{CE}}(\mathfrak{t}_n))$$

has a subcomplex

$$C' := \prod_{n \geq 1} \mathrm{Map}_{\Sigma_n}(\mathsf{Com}^{\mathsf{i}}(n), \mathfrak{t}_n[1]) \cong \prod_{n \geq 1} \mathsf{Lie}(n) \otimes_{\Sigma_n} \left(sgn_n \otimes \mathfrak{t}_n[2-n]\right).$$

An elment of degree 1 in C' is a $\psi \in \mathfrak{t}_3$ satisfying the following two symmetry conditions:

$$\big((213) - (231) - (123)\big) \cdot \psi = 0,$$
$$\big((132) - (312) - (123)\big) \cdot \psi = 0.$$

It follows from the defining relations of the Lie algebra of infinitesimal braids that $\mathfrak{t}_3 = \mathbf{k}z \rtimes \mathfrak{lie}(x, y)$ can be decomposed as a semidirect product of an Abelian Lie algebra on a central element z and a free Lie algebra on two generators. Using this, above symmetry conditions mean that we can write $\psi = \psi(x, y) \in \mathfrak{lie}(x, y)$, satisfying the two symmetry relations

$$\psi(x, y) + \psi(x, -x - y) + \psi(-x - y, x) = 0,$$
$$\psi(x, y) + \psi(y, x) = 0.$$

Demanding that $\psi \in C'$ is a cocycle adds the so-called pentagon condition

$$0 = \psi(t_{12}, t_{23}) - \psi(t_{13} + t_{23}, t_{34}) + \psi(t_{12} + t_{13}, t_{24} + t_{34})$$
$$- \psi(t_{12}, t_{23} + t_{24}) + \psi(t_{23}, t_{34}) \in \mathfrak{t}_4.$$

There are no exact degree 1 elements in C'. The two symmetry relations and the pentagon condition are the defining equations of the Grothendieck-Teichmüller Lie algebra; thus:

Remark 5.1.3.4. $H^1(C') = \mathfrak{grt}_1$. (The reader may take $H^1(C') = \mathfrak{grt}_1$, i.e. the relations given above, as the definition of the Grothendieck-Teichmüller Lie algebra.)

The first part of Tamarkin's result, accordingly, amounts to the statement that the degree 1 cohomology of the deformation complex

$$\mathrm{Def}(\mathsf{Com}_\infty \to C^{\mathrm{CE}}(\mathfrak{t}))$$

is concentrated in the subcomplex C'.

5.1.4 Willwacher's theorems.

It follows from Tamarkin's results that

$$H^1(\text{Def}(\mathsf{Com}_\infty \to \mathsf{Graphs})) \cong \mathfrak{grt}_1,$$

since Kontsevich showed that Graphs is quasi-isomorphic to the Gerstenhaber operad, which, in turn, is quasi-isomorphic to $C^{\text{CE}}(\mathfrak{t})$. It also follows that there is an injection

$$\mathfrak{grt}_1 \to H^1(\text{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs})).$$

Definition 5.1.4.1. Define the full Kontsevich's graph complex to be the deformation complex

$$\text{fGC} := \text{Def}(\mathsf{Lie}\{-1\}_\infty \to \mathsf{Gra})$$
$$= \prod_{n \geq 1} \mathsf{Gra}(n)_{\Sigma_n}[2 - 2n].$$

Its elements are linear combinations of graphs with symmetrized vertex-labels, and we define **Kontsevich's graph complex** to be the subcomplex $\text{GC} \subset \text{fGC}$ that is spanned by connected graphs with all vertices of valency 3 or higher.

The differential on Kontsevich's graph complex has the pictorial form

$$\delta\gamma = [\,\substack{\bullet\\\bullet}\,, \gamma].$$

General properties of twisting imply a morphism

$$\text{Def}(\mathsf{Lie}\{-1\}_\infty \to \mathsf{G})[-1] \to \text{Def}(\mathsf{Ger}_\infty \to \text{Tw}\,\mathsf{G}).$$

The operad G was defined a kind of universal endomorphism operad of polynomial polyvector fields:

$$\mathsf{G} := \lim_d \mathsf{End}\langle T_{\text{poly}}(\mathbf{k}^d)\rangle^{A_d}.$$

The morphism above is in this sense a universal incarnation of the morphism

$$C_{\mathsf{Lie}}(T_{\text{poly}}(\mathbf{k}^d)[1], T_{\text{poly}}(\mathbf{k}^d)[1])[-1]$$
$$\to C_{\text{CE}}(T_{\text{poly}}(\mathbf{k}^d)[1], C_{\mathsf{Ger}}(T_{\text{poly}}(\mathbf{k}^d), T_{\text{poly}}(\mathbf{k}^d)))$$

that sends a cochain $\gamma : T_{\text{poly}}(\mathbf{k}^d)[1])^{\otimes n} \to T_{\text{poly}}(\mathbf{k}^d)[1])$ to the cochain that maps n polyvector fields $\xi_1, \ldots, \xi_n$ to the cochain

$$\gamma(\xi_1, \ldots, \xi_n) \wedge (\,) + [\gamma(\xi_1, \ldots, \xi_n), \,]_S.$$

The map on deformation complexes induces a morphism

$$\mathrm{GC} \to \mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs}) \simeq \mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Ger}),$$

that plays a very essential rôle in the work of Willwacher. Let us spend some time on the conceptual properties of this morphism, since it will be important also for us.

Let $W := \mathbf{k}^d \oplus (\mathbf{k}^d[-1])^*$, so $T_{\mathrm{poly}}(\mathbf{k}^d)$ can be identified with the algebra $\mathcal{O}_W := S(W^*)$ of polynomial functions on W. The space W can be regarded as an odd symplectic manifold and the corresponding Poisson Lie bracket on $\mathcal{O}_W[1]$ is the Schouten bracket on $T_{\mathrm{poly}}(\mathbf{k}^d)$. Finally, define $\mathcal{X}_W := \mathrm{Der}_{\mathsf{Com}}(\mathcal{O}_W, \mathcal{O}_W)$.

Remark 5.1.4.2. The inclusion $S^+_{\mathcal{O}_W}(\mathcal{X}_W[-2])[2] \to C_{\mathsf{Ger}}(\mathcal{O}_W, \mathcal{O}_W)$ is a quasi-isomorphism, if the complex to the left is equipped with the Poisson-Lichnerowitz differential defined by the odd symplectic structure.

Proof. We can put a filtration on

$$C_{\mathsf{Ger}}(\mathcal{O}_W, \mathcal{O}_W) = \mathrm{Map}(S^+(L(\mathcal{O}_W)[1])[1])[-2], \mathcal{O}_W)$$

that at the first step only sees the differential increasing the length of Lie words, i.e., only sees the differential on $L(\mathcal{O}_W[1]) = \mathcal{B}_{\mathsf{Com}}(\mathcal{O}_W)[1]$. Since $\mathcal{O}_W$ is free commutative the inclusion $W^*[1] \subset \mathcal{B}_{\mathsf{Com}}(\mathcal{O}_W)[1]$ is a quasi-isomorphism. Whenever a spectral sequence at the first step reduces to a subcomplex, the inclusion of that subcomplex must be a quasi-isomorphism. Hence

$$\mathrm{Map}(S^+(W^*[1][1])[-2], \mathcal{O}_W) \hookrightarrow C_{\mathsf{Ger}}(\mathcal{O}_W, \mathcal{O}_W)$$

is a quasi-isomorphism. □

The association

$$X : \mathcal{O}_W[1] \to \mathcal{X}_W, \ f \mapsto X_f := \{f, \}$$

of a Hamiltonian vector field to every function defines a morphism of cochain complexes

$$X : (\mathcal{O}_W[1], d = 0) \to S^+_{\mathcal{O}_W}(\mathcal{X}_W[-2])[2],$$

where the complex on the right is equipped with the Poisson-Lichnerowicz differential. The isomorphism between 1-forms and vector fields given by the symplectic structure, defines an isomorphism of complexes

$$\omega^\sharp : \Omega^+_W[2] = S^+_{\mathcal{O}_W}(\Omega^1_W[-1])[2] \to S^+_{\mathcal{O}_W}(\mathcal{X}_W[-2])[2].$$

The map X may be written as the composite $\omega^\sharp \circ d_{dR}$ between this map and the de Rham differential $d_{dR} : \mathcal{O}_W[1] \to \Omega_W^+[2]$. The truncated de Rham complex Ω_W^+ and the Poisson complex $S_{\mathcal{O}_W}^+(\mathcal{X}_W[-2])[2]$ are both modules for the Lie algebra $\mathcal{O}_W$, with the action of $f \in \mathcal{O}_W$ given in both cases the Lie derivative L_{X_f}. In particular, both d_{dR} and $\omega^\sharp$ are morphisms of Lie modules.

Define $\mathcal{O}_W^+ := S^+(W^*)$. The de Rham differential restricts to a quasi-isomorphism from $\mathcal{O}_W^+$ to Ω_W^+. From this and the preceeding lemma we can conclude the following corollary.

Corollary 5.1.4.3. The inclusion $X : \mathcal{O}_W^+ \to C_{\mathsf{Ger}}(\mathcal{O}_W, \mathcal{O}_W)$ is a quasi-isomorphism.

It follows form the preceeding discussion that

$$\mathrm{Def}(\mathsf{Lie}\{-1\}_\infty \to \mathsf{G})[-1] \to \mathrm{Def}(\mathsf{Ger}_\infty \to \mathrm{Tw}\,\mathsf{G})$$

corresponds to the evident map

$$C_{\mathsf{Lie}}(\mathcal{O}_W[1], \mathcal{O}_W[1])[-1] \to C_{\mathrm{CE}}(\mathsf{O}_W, \mathcal{O}_W^+).$$

There are problems with this correspondence because the group A_d is not reductive, so our cohomology computations can not simply be commuted with taking invariants. The space $\mathcal{O}_W^+$ is not even a representation of A_d. However, we can reduce to

$$X : C_{\mathsf{Lie}}(\mathcal{O}_W[1], \mathcal{O}_W[1])[-1] \to C_{\mathrm{CE}}(\mathsf{O}_W, S_{\mathcal{O}_U}^+(\mathcal{X}_U[-2])[2])$$

without any problems. A universal version of that step is a reduction to the subcomplex

$$fC \subset \mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs})$$

spanned by graphs with only univalent white vertices.

Theorem 5.1.4.4. [Willwacher 2010] The morphism

$$\mathrm{GC}[-1] \to \mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs})$$

induces an isomorphism

$$H^0(\mathrm{GC}) \oplus \mathbf{k} \cong H^1(\mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs})).$$

The summand $\mathbf{k}$ is essentially a copy of $H(\mathrm{Def}(\mathsf{Ger} \to \mathsf{G}))$. A representative class is the graph with two white vertices and an edge.

Again, the map from the graph complex can be restricted to lie in the subcomplex fC of graphs with univalent white vertices. The morphism then has the form

$$X : \mathrm{GC}[-1] \to fC, \ \gamma \mapsto X_\gamma := \overset{\bullet}{\underset{\circ}{}} \bullet \gamma.$$

Recall $H^1(\mathrm{Def}(\mathsf{Com}_\infty \to \mathsf{Graphs})) \cong \mathfrak{grt}_1$. If we replace Graphs by the full operad $\mathrm{Tw}\,\mathsf{G}$, then on the level of algebras we consider

$$\mathrm{Def}(\mathsf{Com}_\infty \to \mathrm{Tw}\,\mathsf{End}\langle \mathcal{O}_W, \mathcal{O}_W \rangle) = C_{\mathrm{CE}}(\mathcal{O}_W[1], C_{\mathsf{Com}}(\mathcal{O}_W, \mathcal{O}_W)).$$

Since $\mathcal{O}_W$ is free as graded commutative algebra, we have a quasi-isomorphism

$$C_{\mathrm{CE}}(\mathcal{O}_W[1], \mathcal{X}_W) \to C_{\mathrm{CE}}(\mathcal{O}_W[1], C_{\mathsf{Com}}(\mathcal{O}_W, \mathcal{O}_W)).$$

Graphically, this implies that we can replace $\mathrm{Def}(\mathsf{Com}_\infty \to \mathsf{Graphs})$ by the subcomplex C spanned by graphs with a univalent single white vertex. Note that $X : \mathrm{GC}[-1] \to fC$ actually factors as map into C.

Lemma 5.1.4.5. [Willwacher 2010] The morphism $X : \mathrm{GC}[-1] \to C$ is injective on cohomology.

We now have almost all ingredients necessary to deduce $H^0(\mathrm{GC}) \cong \mathfrak{grt}_1$. It first follows from Tamarkin's result and the theorem above by Willwacher that there is an inclusion

$$\mathfrak{grt}_1 \to H^1(\mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs})) \cong H^0(\mathrm{GC}) \oplus \mathbf{k},$$

and that the composite

$$\mathfrak{grt}_1 \to H^1(\mathrm{Def}(\mathsf{Ger}_\infty \to \mathsf{Graphs}))$$
$$\cong H^0(\mathrm{GC}) \oplus \mathbf{k} \to H^1(\mathrm{Def}(\mathsf{Com}_\infty \to \mathsf{Graphs})) = \mathfrak{grt}_1$$

is the identity. However, the injective map $X : H^0(GC) \to H^1(C) = \mathfrak{grt}_1$ allows one to deduce that there is a composite of injections

$$\mathfrak{grt}_1 \to H^0(\mathrm{GC}) \to H^1(C) = \mathfrak{grt}_1$$

which equals the identity; hence $\mathfrak{grt}_1 \cong H^0(\mathrm{GC})$ as a vector space. (But note that Willwacher proves the isomorphism is compatible with the Lie brackets.)

5.1.5 The Furusho-Willwacher theorem.

We noted in 5.1.3.4 that $H^1(\mathrm{Def}(\mathsf{Com}_\infty \to C^{\mathrm{CE}}(\mathfrak{t})))$ is spanned by Lie series $\psi \in \mathfrak{lie}(x, y)$ satisfying the two symmetry equations

$$\psi(x, y) + \psi(x, -x - y) + \psi(-x - y, x) = 0$$
$$\psi(x, y) + \psi(y, x) = 0.$$

and the pentagon equation

$$0 = \psi(t_{12}, t_{23}) - \psi(t_{13} + t_{23}, t_{34}) + \psi(t_{12} + t_{13}, t_{24} + t_{34})$$
$$- \psi(t_{12}, t_{23} + t_{24}) + \psi(t_{23}, t_{34}) \in \mathfrak{t}_4.$$

Hidekazu Furusho proved the following very remarkable result.

Proposition 5.1.5.1. [Furusho 2010] If $\psi \in \mathfrak{lie}(x, y)$ satisfies the pentagon equation and, additionally, the coefficient of $[x, y]$ in ψ is zero, then ψ also satisfies the two symmetry equations.

Changing from the operad Com_∞ to the operad Ass_∞ exactly has the effect that the symmetry conditions are dropped. Hence the following corollary.

Corollary 5.1.5.2. $H^1(\mathrm{Def}(\mathsf{Ass}_\infty \to C^{\mathrm{CE}}(\mathfrak{t}))) = \mathfrak{grt}_1 \oplus \mathbf{k}[t_{12}, t_{23}]$.

We not that Willwacher has given an independ direct proof of this fact, in the following form.

Proposition 5.1.5.3. [Willwacher 2010]

$$H^1(\mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})) \cong H^1(\mathrm{Def}(\mathsf{Com}_\infty \to \mathsf{Graphs})) \oplus \mathbf{k} \, \text{⊶}.$$

5.2 Homotopical uniqueness.

Before we prove our main theorem of this chapter we recall yet another result by Willwacher. Write $\mathrm{fGC}_c \subset \mathrm{fGC}$ for the subcomplex spanned by connected graphs. Recall that Kontsevich's graph complex GC is defined by additionally requiring all vertices to be at least trivalent.

Theorem 5.2.0.4. [Willwacher 2010] The cohomology connected graph complex satisfies

$$H(\mathrm{fGC}_c) = H(\mathrm{GC}) \oplus \bigoplus_{j \geq 0} \mathbf{k}[-3 - 4j].$$

The class spanning $\mathbf{k}[-3 - 4j]$ is represented by a wheel with $5 + 4j$ edges and only bivalent vertices.

The negative cohomology of Kontsevich's graph complex vanishes, $H^{<0}(\mathrm{GC}) = 0$.

To simplify notation we define the complexes

$$f\mathscr{C} := \mathrm{Def}(\mathsf{NCG}_\infty \to \mathsf{Gra}_{\mathsf{NCG}});$$
$$f\mathscr{D} := \mathrm{Def}(\mathsf{Ass}_\infty \to \mathrm{Tw}\,\mathsf{Gra}).$$

It follows from our result 3.1.2.6 on the twist and Lie module constructions that

$$f\mathscr{C} = \mathrm{Cone}(X : \mathrm{fGC}[-1] \to f\mathscr{D}) = \mathrm{fGC} \oplus f\mathscr{D}.$$

For completeness, let us record the pictorial forms of the differentials. The differential on $f\mathscr{D}$ is of the form $\partial + d_H$, where ∂ is the internal differential on $\mathrm{Tw}\,\mathsf{Gra}$ and d_H is a Hochschild-type differential.

$$\partial\Gamma = [\,\substack{\bullet\\\circ}\,, \Gamma] + \Gamma \bullet \substack{\bullet\\\circ}, \quad d_H\Gamma = [\circ\,\circ, \Gamma].$$

Denote the full differential on $\mathscr{C}$ by d, viz.

$$d(\Gamma, \gamma) = \big(\partial\Gamma + d_H\Gamma - X_\gamma, \delta\gamma\big),$$

for $\delta\gamma = [\,\substack{\bullet\\\bullet}\,, \gamma]$ the differential on Kontsevich's graph complex. Recall also that the morphism X has the form

$$\gamma \mapsto X_\gamma := \substack{\bullet\\\circ} \bullet \gamma.$$

Lemma 5.2.0.5. *The morphism $X : \mathrm{fGC}[-1] \to f\mathscr{D}$ is injective on cohomology in degree 1 and 2.*

Proof. Since any graph can be written as a disjoint union of connected components, we may write $\mathrm{fGC} = \hat{S}^+(\mathrm{fGC}_c[-2])[2]$, where $\mathrm{fGC}_c \subset \mathrm{fGC}$ the subcomplex of connected graphs. Analogously if we write $\mathsf{fGraphs}_c \subset \mathsf{Graphs}$ for the dg suboperad spanned by graphs without a connected component with only black vertices, then

$$\mathrm{Tw}\,\mathsf{Gra} = \hat{S}(\mathrm{fGC}_c[-2]) \otimes \mathsf{fGraphs}_c.$$

With this decomposition in place, X is for each $p \geq 1$ a map

$$S^p(\mathrm{fGC}_c[-2])[1] \to S^{p-1}(\mathrm{fGC}_c[-2]) \otimes \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{fGraphs}_c).$$

In particular, X is completely determined by the Leibniz rule and its restriction

$$X : \mathrm{fGC}_c \to \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{fGraphs}_c).$$

The proposed injectivity now follows from Willwacher's result 5.1.4.5 since, (i) $H^i(\mathrm{fGC}_c) = H^i(\mathrm{GC})$ in degrees $i = 0, 1$ by the theorem cited above, and (ii) the inclusion $\mathsf{Graphs} \subset \mathsf{fGraphs}_c$ is a quasi-isomorphism [Willwacher 2010]. $\square$

Proposition 5.2.0.6. $H^1(f\mathscr{C}) = \mathbf{k}\,\vcenter{\hbox{graph}}$.

Proof. Consider the long exact sequence defined by the mapping cone:

$$\cdots \to H^i(\text{fGC}) \to H^{i+1}(f\mathscr{D}) \to H^{i+1}(f\mathscr{C}) \to \cdots .$$

Applying the preceeding lemma to this sequence, we extract a short exact sequence

$$0 \to H^0(\text{fGC}) \to H^1(f\mathscr{D}) \to H^1(f\mathscr{C}) \to 0.$$

We now note that

$$H^1(f\mathscr{D}) = H^1\left((\text{fGC}[-2] \oplus \mathbf{k}) \otimes \text{Def}(\mathsf{Ass}_\infty \to \mathsf{fGraphs}_c) \right).$$

The quasi-isomorphism $\mathsf{Graphs} \subset \mathsf{fGraphs}_c$ implies

$$H^i(\text{Def}(\mathsf{Ass}_\infty \to \mathsf{fGraphs}_c)) \cong H^i(\text{Def}(\mathsf{Ass}_\infty \to C^{\text{CE}}(\mathfrak{t}))).$$

Since the results of Tamarkin and Willwacher say that the complex on the right has cohomology concentrated in the subcomplex

$$\prod_{n\geq 1} \mathfrak{t}_n[2-n],$$

which has no cohomology in strictly negative degrees, it follows that

$$H^{<0}(\text{Def}(\mathsf{Ass}_\infty \to \mathsf{fGraphs}_c)) = 0.$$

Applying the Künneth formula to the above tensor decomposition of $H^1(f\mathscr{D})$, and using 5.2.0.4, we conclude

$$H^1(f\mathscr{C}) \cong H^1(\text{Def}(\mathsf{Ass}_\infty \to \mathsf{fGraphs}_c))/H^0(\text{fGC}_c)$$
$$\cong H^1(\text{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs}))/H^0(\text{GC}).$$

The combination of Willwacher's theorem (that $H^0(\text{GC})$ equals $\mathfrak{grt}_1$) and the Furusho-Willwacher theorem now implies that $H^1(f\mathscr{C})$ is isomorphic to the quotient $(\mathfrak{grt}_1 \oplus \mathbf{k}\,\vcenter{\hbox{graph}})/\mathfrak{grt}_1 = \mathbf{k}\,\vcenter{\hbox{graph}}$. $\qquad\square$

Lemma 5.2.0.7. The graph $\vcenter{\hbox{graph}}$ also represents the exotic cohomology class.

Proof. We first note that

$$d_H\left(\ \raisebox{-0.5em}{[graph]}\ +\ \raisebox{-0.5em}{[graph]}\ \right) = -2\ \raisebox{-0.5em}{[graph]}\ .$$

Here we have written out the order on edges so that the signs are unambiguous, and we regard the white vertices to be numbered from left to right. Then we observe that

$$\partial\left(\ \raisebox{-0.5em}{[graph]}\ +\ \raisebox{-0.5em}{[graph]}\ \right) = 2\ \raisebox{-0.5em}{[graph]}\ .$$

These two equations imply that the two cochains

 and
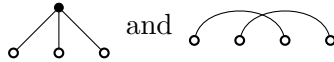
in $\mathsf{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})$ are cohomologous. $\qquad\square$

Remark 5.2.0.8. The image of $\raisebox{-0.2em}{[graph]}$ in the Hochschild cochain complex of polyvector fields is the map

$$X \otimes Y \otimes Z \otimes W \mapsto [X, Z]_S \wedge [Y, W]_S.$$

5.3 The Duflo isomorphism.

Kontsevich's paper [Kontsevich 2003] contained a proof that the tangential morphism of his formality morphism, applied to a finite dimensional Lie algebra, defined an isomorphism $H(\mathbf{g}, S(\mathbf{g})) \to H(\mathbf{g}, U(\mathbf{g}))$ of Chevalley-Eilenberg cohomology algebras. This result was later given a detailed proof and generalized to an arbitrary graded Lie algebra of finite type, see [Pevzner and Torossian 2004; Calaque and Rossi 2011]. In this section we discuss a homotopy generalization of this theorem.

Let $\mathbf{g}$ be a graded real vector space which is concentrated in non-negative degrees and finite dimensional in each degree. Let T_{poly} be the polyvector fields on $\mathbf{g}[1]$, so $T_{\mathrm{poly}} = S(\mathbf{g}^*[-1]) \otimes S(\mathbf{g})$. Identify T_{poly} with $\mathrm{Map}(S(\mathbf{g}[1]), S(\mathbf{g}))$. The wedge product on polyvector fields is under this identification equal to the convolution product on the space of maps $\mathrm{Map}(S(\mathbf{g}[1]), S(\mathbf{g}))$ from a coalgebra into an algebra. The graded Lie algebra

$$C_{\mathsf{Lie}}(\mathbf{g}, \mathbf{g}) = \mathrm{Map}(S^+(\mathbf{g}[1])), \mathbf{g}[1])$$

embeds into $T_{\mathrm{poly}}[1]$ as a Lie subalgebra. Denote by $\mathcal{A} := S(\mathbf{g}^*[-1])$ the algebra of functions on $\mathbf{g}[1]$.

The following result is a straight-forward corollary to the main theorem of the preceeding chapter.

Lemma 5.3.0.9. The representation

$$D \circ \theta^* : \mathsf{FOC} \to \mathsf{End}\langle T_{\mathrm{poly}}, T_{\mathrm{poly}}, \mathcal{A}\rangle$$

restricts to a representation $\mathsf{FOC} \to \mathsf{End}\langle C_{\mathsf{Lie}}(\mathbf{g}[1], \mathbf{g}[1])[-1], T_{\mathrm{poly}}, \mathcal{A}\rangle$.

Note that the Hoschschild cochain complex of $\mathcal{A}$ is

$$C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A}) = \mathrm{Map}(\mathcal{B}(S(\mathbf{g}^*[-1])), S(\mathbf{g}^*[-1]))$$
$$\cong \mathrm{Map}(S(\mathbf{g}[1]), \mathcal{C}(S(\mathbf{g}[1]))).$$

Here $\mathcal{B}$ denotes the classical (coassociative) bar construction: $\mathcal{B}(A)$ is the coalgebra $T(A[1])$ with the product on A turned into a differential; and $\mathcal{C}$ denotes the classical (associative) cobar construction, dually defined. In the isomorphism we use that $\mathbf{g}$ concentrated in degrees ≥ 0 and finite-dimensional in each degree. These two assumptions ensure that $S(\mathbf{g}^*[-1])^* \cong S(\mathbf{g}[1])$. Note that both mapping spaces above carry convolution products, since both are mapping spaces from a coalgebra into an algebra, and the isomorphisms respect these products. Combining this with the above lemma defines an explicit quasi-isomorphism

$$\big(C_{\mathsf{Lie}}(\mathbf{g}, \mathbf{g})[-1], \mathrm{Map}(S(\mathbf{g}[1]), S(\mathbf{g}))\big)$$
$$\to \big(C_{\mathsf{Lie}}(\mathbf{g}[1], \mathbf{g}[1])[-1], \mathrm{Map}(S(\mathbf{g}[1]), \mathcal{C}(S(\mathbf{g}[1])))\big)$$

of NCG_∞-algebras.

We can twist the construction by Maurer-Cartan elements. A Maurer-Cartan element Q of $C_{\mathsf{Lie}}(\mathbf{g}, \mathbf{g})$ is precisely an L_∞ structure on $\mathbf{g}$. Assume given a Q without differential part (so $Q = Q_2 + Q_3 + \ldots$) and interpret it as a coderivation of $S(\mathbf{g}[1])$. Denote the dg coalgebra $(S(\mathbf{g}[1]), Q)$ by $C^{\mathrm{CE}}(\mathbf{g})$, as usual. The cobar construction

$$\mathcal{C}(C^{\mathrm{CE}}(\mathbf{g})) =: U_\infty(\mathbf{g})$$

is the derived universal enveloping algebra of the L_∞ algebra $(\mathbf{g}, Q)$ introduced in [Baranovsky 2008]. (That it is quasi-isomorphic to the usual universal enveloping algebra in the special case that $\mathbf{g}$ is a Lie algebra is a classical case of Koszul duality.) Kontsevich's formality map $\mathcal{U}$ quantizes Q to a differential on $\mathcal{A}$ and we may identify $\mathcal{A}$ equipped with this differential with the Chevalley-Eilenberg cochain complex $C_{\mathrm{CE}}(\mathbf{g})$

with trivial coefficients. If follows that after twisting by Q the A_∞ structure on the Hochschild cochain complex of $\mathcal{A}$ will be the algebra

$$C_{\mathrm{Hoch}}(C_{\mathrm{CE}}(\mathbf{g}), C_{\mathrm{CE}}(\mathbf{g})) \cong \mathrm{Map}(C^{\mathrm{CE}}(\mathbf{g}), U_\infty(\mathbf{g}))$$
$$= C_{\mathrm{CE}}(\mathbf{g}, U_\infty(\mathbf{g})).$$

However, the induced A_∞ structure on polyvector fields after twisting is not simply

$$C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g})) = \mathrm{Map}(C^{\mathrm{CE}}(\mathbf{g}), S(\mathbf{g})).$$

Instead, we obtain an A_∞ algebra $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))_{\mathrm{exotic}}$, which is a deformation of the usual algebra $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))$. The first term deforming the usual product is defined by the graph ⚬⚫⚬. Nevertheless, we have an explicit A_∞ quasi-isomorphism

$$\mathcal{Z}^Q : C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))_{\mathrm{exotic}} \to C_{\mathrm{CE}}(\mathbf{g}, U_\infty(\mathbf{g})).$$

Proposition 5.3.0.10. (i) The cohomologies of $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))_{\mathrm{exotic}}$ and $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))$ are isomorphic as associative algebras and the map on cohomology induced by $\mathcal{Z}^Q$ coincides with the Duflo-Kontsevich isomorphism.

(ii) There does not, generically, exist a quasi-isomorphism

$$C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g})) \to C_{\mathrm{CE}}(\mathbf{g}, U_\infty(\mathbf{g}))$$

of A_∞ algebras. In other words, it is impossible to find a universal A_∞ lift of the Duflo-Kontsevich isomorphism on Chevalley-Eilenberg cohomologies to the Chevalley-Eilenberg cochain algebras.

Proof. Item (i) is true by construction. For the second point we invoke our theorem 5.0.0.5 and make the following remark. Define

$$\mathsf{Graphs}' := \mathsf{Graphs}/\langle \, \substack{\bullet \\ \bullet} \, \rangle$$

to be the quotient by the ideal operadically generated by the displayed graph. Decorating all black vertices by Q defines a representation

$$D^Q : \mathsf{Graphs} \to \mathsf{End}\langle (T_{\mathrm{poly}}, d_\pi) \rangle,$$

and it naturally factors through the quotient projection onto Graphs'. (The quotient can be regarded as a universal way of imposing the Maurer-Cartan equation $[Q, Q] = 0$.) There is an induced map

$$\mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs}) \to \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs}')$$

of dg Lie algebras. The cohomology class defined by [graph] maps to a nonzero cohomology class, because it is cohomologous to [graph], which, since it has no black vertices, is exact in $\mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs}')$ if and only if it is exact to begin with. Thus the exotic deformation is homotopy trivial only when this class is mapped to a coboundary under

$$D^Q : \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs}') \to C_{\mathsf{Ass}}((T_{\mathrm{poly}}, d_\pi), (T_{\mathrm{poly}}, d_\pi)).$$

We now finish the proof by cheating and define the mathematical meaning of the word "generically" that appears in the proposition to mean "whenever the exotic cohomology class is not mapped to zero under the map displayed above," making the statement true by default. □

Remark 5.3.0.11. Note, however, that the above definition of "generically" is not completely stupid. There are essentially only two reasons that can make the class vanish. The first is dimensional, but

$$\mathsf{Graphs} \subset \mathrm{Tw}\,\mathsf{Gra} \subset \lim_d C_{\mathrm{CE}}(T_{\mathrm{poly}}(\mathbf{k}^d), \mathsf{End}\langle T_{\mathrm{poly}}(\mathbf{k}^d)\rangle)^{A_d}$$

implies that if the dimension is high enough, then for most Q the exotic structure is nontrivial. The other reason that can make the class vanish is that Q is Abelian. This can be argued in at least two ways. The first one is direct and graphical. If $Q = 0$, then [graph] is represented by a map which is identically equal to zero, hence it is a trivial cohomology class. (The other representative, [graph], is mapped to a nonzero cochain, but since the graph is d_H-exact the corresponding cochain will be exact when $Q = 0$.) The second way to argue that the cohomology class is trivial when the Lie algebra is Abelian is to note that $U(\mathbf{g})$ is isomorphic to $S(\mathbf{g})$ as an algebra when the the Lie algebra happens to be Abelian, implying $C_{\mathrm{CE}}(\mathbf{g}, U(\mathbf{g}))$ and $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))$ are isomorphic.

Remark 5.3.0.12. There is a canonical isomorphism between T_{poly} on $\mathbf{g}[1]$ and T_{poly} on $\mathbf{g}^*$. Above we used the first graded vector space, for which $\mathcal{A} = S(\mathbf{g}^*[-1])$. Application of Kontsevich's formality to the second case, for which $\mathcal{A} = S(\mathbf{g})$, quantizes an L_∞ structure $Q \in T_{\mathrm{poly}}$ to a (flat) A_∞ structure $\star$ on $S(\mathbf{g})[[\hbar]]$. In [Calaque et al. 2011] the authors constructed a nontrivial but explicit A_∞ $(S(\mathbf{g})[[\hbar]], \star) - C_{\mathrm{CE}}(\mathbf{g})[[\hbar]]$-bimodule structure $K_\hbar$ on $\mathbf{R}[[\hbar]]$ and they proved that the derived left action

$$L : (S(\mathbf{g})[[\hbar]], \star) \to \mathrm{Map}_\hbar(K_\hbar[1] \otimes \mathcal{B}(C_{\mathrm{CE}}(\mathbf{g}))[[\hbar]], K_\hbar[1])$$

is a quasi-isomorphism of A_∞ algebras. Here $\mathrm{Map}_\hbar$ denotes the mapping space of maps which are linear in $\hbar$. In the present case one may formally

set $\hbar = 1$ in this quasi-isomorphism, and then identify the term on the right (above) with the cobar construction $\mathcal{C}(C^{CE}(\mathbf{g}))$. Thus the result of [Calaque et al. 2011] implies that the quantization of the symmetric algebra on the L_∞ algebra $\mathbf{g}$, i.e. $(S(\mathbf{g}), \star)$, is quasi-isomorphic as an A_∞ algebra to Baranovsky's derived universal enveloping algebra of $\mathbf{g}$. Together with our result this quasi-isomorphism implies that the A_∞ algebras $C_{\mathrm{CE}}(\mathbf{g}, S(\mathbf{g}))_{\mathrm{exotic}}$ and $C_{\mathrm{CE}}(\mathbf{g}, (S(\mathbf{g}), \star))$ are quasi-isomorphic, though the quasi-isomorphism is presently not explicit.

The material in this chapter has not been previously published.

6.1 Algebraic structures and formal geometry.

Let us start with some motivational discussion. The infinitesimal neighborhood of a point x in a smooth manifold M of dimension d looks (noncanonically) like the infinitesimal neighborhood of $0 \in \mathbf{R}^d$. The germs at x of sections of some "naturally/universally" defined sheaf FM will look like the germs $V := J_0^\infty F\mathbf{R}^d$ at 0 of the corresponding sheaf on $\mathbf{R}^d$. Consider the following question: what algebraic structures on V are sufficiently natural that they are induced on the global sections $\Gamma(M, FM)$ of F, for any manifold M? For example, if $F = T$ is the tangent space functor, then V has the structure of a Lie algebra and this Lie algebra structure is well-known to be universal in the sense that it can be regarded as inducing a Lie algebra structure on the global vector fields on any manifold. A sufficient condition is of course that the formulae defining the algebraic operations are equivariant with respect to all coordinate changes on $\mathbf{R}^d$. (This is the case for the Lie bracket on germs of vector fields.) It is not a neccessary condition however, at least not if we are suitably homotopical in what we mean when we require the structure on V to globalize.

In this section we make both this question and its answer more precise.

6.1.1   Geometric background.

We begin by briefly recalling the notion of profinite smooth manifolds. Consider the directed system of topological spaces

$$\cdots \to \mathbf{R}^{k+1} \to \mathbf{R}^k \to \cdots \to \mathbf{R} \to \{*\},$$

where $\mathbf{R}^{k+1} \to \mathbf{R}^k$ sends $(x_1, \ldots, x_k, x_{k+1})$ to $(x_1, \ldots, x_k)$. Let $\mathbf{R}^\infty$ be the limit of this diagram. Denote by $\pi_k : \mathbf{R}^\infty \to \mathbf{R}^k$ the projection. If U and V are open subsets of $\mathbf{R}^\infty$ then we say that a function $f : U \to V$ is smooth if for every $x \in U$ there exists an open neighborhood $W \subset U$ of x, a natural number $k \geq 0$ and, for every $\ell \geq 0$, a smooth function $f_{k,\ell} : \pi_k(W) \to \pi_\ell(V)$ with the property that $\pi_\ell \circ f|_W$ can be represented as $f_{k,\ell} \circ \pi_k$. One defines a profinite manifold just like one defines ordinary manifolds, except that the local charts and transition functions are open subsets of $\mathbf{R}^\infty$ and smooth maps between them. A profinite Lie group is the evident generalization of Lie groups to the context of profinite smooth manifolds. From now on "manifold" will refer to a possibly profinite smooth manifold, and "Lie group," "fiber bundle," etc. will refer to these objects in the category of (possibly) profinite smooth manifolds.

We shall similarly allow Lie algebras, vector spaces, etc., to be profinite (limits of directed systems of finite-dimensional counterparts). In the case of profinite linear objects we shall tacitly assume that all linear maps between them are continuous with respect to the projective topologies.

Let $d < \infty$.

Definition 6.1.1.1. [Kolar, Michor, and Slovak 1993] A **natural bundle functor** on d-dimensional manifolds is a functorial assignment of a graded vector bundle $FM \to M$ to every d-dimensional manifold M and of a vector bundle morphism $Ff : FM \to F(N)$ to every local diffeomorphism $f : M \to N$, such that (i) Ff covers f and is a fibrewise isomorphism of graded vector spaces, and (ii) if $\gamma : \mathbf{R} \times M \to N$ is a smooth 1-parameter family of local diffeomorphisms, then the map $\mathbf{R} \times FM \to F(N)$ defined by $(t,p) \mapsto F\gamma_t(p)$ is smooth.

6.1.2   Harish-Chandra torsors.

A **Harish-Chandra pair** is the datum of a Lie group G, a Lie algebra $\mathfrak{h}$, a linear action of G on $\mathfrak{h}$ and a G-equivariant embedding of the Lie algebra $\mathfrak{g}$ of G into $\mathfrak{h}$ which is compatible with the G-action in the sense that the differential of the G-action coincides with the adjoint action of $\mathfrak{g}$ on $\mathfrak{h}$. We denote a Harish-Chandra pair as $(G, \mathfrak{h})$.

112

Let $(G, \mathfrak{h})$ be a Harish-Chandra pair. A $(G, \mathfrak{h})$-**module** is a dg vector space V that carries a representation $\mathfrak{h} \to gl(V)$ of $\mathfrak{h}$ with the property that the induced action of $\mathfrak{g}$ integrates to an action of G. This terminology was introduced in [Beilinson and Bernstein 1993].

Definition 6.1.2.1. [Bezrukavnikov and Kaledin 2003] Let M be a smooth finite-dimensional manifold. A $(G, \mathfrak{h})$-**torsor** on M is a principal G-bundle $P \to M$ together with a flat $\mathfrak{h}$-valued G-equivariant connection $\vartheta : T(P) \to P \times \mathfrak{h}$ with the property that the composite map

$$P \times \mathfrak{g} \to T(P) \xrightarrow{\vartheta} P \times \mathfrak{h},$$

of bundles on P, coincides with the map $P \times \mathfrak{g} \to P \times \mathfrak{h}$ defined by the given $\mathfrak{g} \to \mathfrak{h}$.

If $(P \to M, \vartheta)$ is a $(G, \mathfrak{h})$-torsor and V is a $(G, \mathfrak{h})$-module, then one can use the $\mathfrak{h}$-action on V to turn ϑ into a flat connection on the associated bundle $P \times_G V \to M$. In this way the torsor defines an exact symmetric monoidal functor from the category of $(G, \mathfrak{h})$-modules to the category of bundles on M with flat connections. The idea of formal geometry is to use this to reduce general constructions on manifolds to constructions with Harish-Chandra modules. The most important Harish-Chandra torsor is the torsor of formal coordinate systems, which we will now define.

6.1.3 Formal geometry.

Given finite-dimensional manifolds M and N and points $x \in M$, $y \in N$, denote by $J_x^\infty(M, N)_y$ the space of infinite jets of based smooth maps $(M, x) \to (N, y)$. We shall write inv $J_x^\infty(M, N)_y$ for the subspace of jets with invertible differential, i.e. the jets corresponding to maps which are local diffeomorphisms near x. We introduce the Lie group G_d of formal diffeomorphisms of $\mathbf{R}^d$ that preserve the origin,

$$G_d := \text{inv } J_0^\infty(\mathbf{R}^d, \mathbf{R}^d)_0.$$

and denote its Lie algebra by g_d. It sits inside the Lie algebra $W_d := J_0^\infty(T(\mathbf{R}^d))$ of formal vector fields as the subalgebra of vector fields vanishing at the origin.

For a manifold M, of finite dimension d, we shall denote the bundle of formal coordinate systems on M by $M^{\text{coor}} \to M$. It is given as the space of infinite jets at $0 \in \mathbf{R}^d$ of smooth maps $\mathbf{R}^d \to M$ with invertible differential at 0, i.e., the fiber above $x \in M$ is

$$M_x^{\text{coor}} := \text{inv } J_0^\infty(\mathbf{R}^d, M)_x.$$

It is clearly a principal (right) G_d-bundle. Each tangent space of M^{coor} is canonically isomorphic to the Lie algebra W_d, and these isomorphisms define a flat connection form $\vartheta \in \Omega(M^{\mathrm{coor}}, W_d)$. In fact, $(M^{\mathrm{coor}} \to M, \vartheta)$ is a (G_d, W_d)-torsor. This torsor has the following useful property.

Remark 6.1.3.1. Let F be a natural bundle functor and denote the projection from M^{coor} to M by π. The pullback by π of the bundle of infinite jets of sections of FM is canonically trivial:

$$\pi^* J^\infty FM \cong M^{\mathrm{coor}} \times J_0^\infty F\mathbf{R}^d.$$

Equivalently, one can realize $J^\infty FM$ as

$$M^{\mathrm{coor}} \times_{G_d} J_0^\infty F\mathbf{R}^d \cong J^\infty FM,$$

and the isomorphism is one of bundles with flat connections. The map of taking jets,

$$\Gamma(M, FM) \to \Omega(M, J^\infty FM)_\theta,$$

where the cochain complex on the right is equipped with the differential defined by the canonical flat connection on jets, is a quasi-isomorphism.

Define $M^{\mathrm{aff}} := M^{\mathrm{coor}}/GL_d$, the bundle of formal affine coordinate systems on M, to be the quotient of M^{coor} by all linear changes of coordinates. The projection $M^{\mathrm{coor}} \to M^{\mathrm{aff}}$ is a (GL_d, W_d)-torsor.

6.2 Globalization of formal algebraic structures.

6.2.1 Descent to associated bundles.

Pick a Harish-Chandra pair $(G, \mathfrak{h})$.

Let X be a set of colors and let $*$ be a color not in X. Let O be an $S := \{*\} \sqcup X$-colored dg operad with twist data and let $V = \{V_s\}_{s \in S}$ be an S-colored collection of dg vector spaces with $\mathfrak{h} \subset V_*$.

Definition 6.2.1.1. Consider the situation of a representation

$$\mathsf{O} \to \mathsf{End}\langle V \rangle,$$

with the properties that (i) $\mathfrak{h} \subset V_*$ is a dg Lie subalgebra and (ii) each V_x, $x \in X$, is a $(G, \mathfrak{h})$-module. When these conditions are satisfied we say that $\mathsf{O} \to \mathsf{End}\langle V \rangle$ is a **Harish-Chandra** representation.

Remark 6.2.1.2. Observe that we do *not* require the L_∞ actions $\mathfrak{h} \to gl(V_x)$ defined by the twist data of O to equal the Harish-Chandra structures.

Let $(P \to M, \vartheta)$ be some given $(G, \mathfrak{h})$-torsor and assume that O is a coalgebra for the twist comonad. By the assumption that O is a Tw-coalgebra, we obtain an induced representation

$$\mathsf{O} \to \mathsf{Tw}\,\mathsf{O} \to \mathsf{End}\langle \Omega(P, V)_\vartheta \rangle,$$

where $\Omega(P, V)_\vartheta$ denotes $\{\Omega(P, V_s)\}_{s \in S}$ with differentials given by the flat connection ϑ. We are interested in conditions that allow us to descend this to a representation

$$\mathsf{O} \to \mathsf{End}\langle \Omega(M, P \times_G V)_\vartheta \rangle$$

on the associated bundle(s) on M.

Let $\Omega_{\mathrm{hor}}(P, V)$ denote the subspace of $\Omega(P, V)$ consisting of horizontal forms with respect to the projection $P \to M$. In other words, it is the intersection of the kernels of all the contractions i_{ζ_X}, $X \in \mathfrak{g}$, for ζ_X the vector field on P associated to X. Define the subspace of basic forms by

$$\Omega_{\mathrm{basic}}(P, V) := \Omega_{\mathrm{hor}}(P, V)^G.$$

By G-invariance of ϑ this is a subcomplex with respect to d_ϑ. The following lemma is well-known.

Lemma 6.2.1.3. Pullback $\Omega(M, P \times_G V)_\vartheta \to \Omega(P, V)_\vartheta$ along the projection $P \to M$ factors through an isomorphism onto the the subcomplex $\Omega_{\mathrm{basic}}(P, V)_\vartheta$.

Proposition 6.2.1.4. The representation of $\mathsf{Tw}\,\mathsf{O}$ descends to a representation in

$$\Omega(M, P \times_G V)_\vartheta$$

if and only if the following two conditions are satisfied for all $n \geq 1$, $k \geq 0$, $s. : [n] \to S$, $s \in S$ and $\varphi \in \mathsf{Tw}\,\mathsf{O}(s. \mid s)_{(k)}$:

(1) $\varphi : \mathfrak{h}^{\otimes k} \otimes \bigotimes_{i=n}^{n} V_{s_i} \to V_s$ is G-equivariant.

(2) If $k \geq 1$, then $\varphi(X, \dots) : \mathfrak{h}^{\otimes k-1} \otimes \bigotimes_{i=n}^{n} V_{s_i} \to V_s$ is 0 for all insertions of an $X \in \mathfrak{g}$.

Proof. Take $\varphi \in \mathsf{Tw}\,\mathsf{O}(s. \mid s)_{(k)}$ and forms $\alpha_i \in \Omega_{\mathrm{basic}}^{q_i}(P, V_{s_i})$, $1 \leq i \leq n$. Write $q := q_1 + \cdots + q_n$ and $\alpha := \alpha_1 \wedge \cdots \wedge \alpha_n$. That the representation of $\mathsf{Tw}\,\mathsf{O}$ is given by operations that preserve G-invariance, i.e. that

$g_*\varphi(\vartheta^{\wedge k},\alpha_1,\dots,\alpha_n)=g^{-1}\varphi(\vartheta^{\wedge k},\alpha_1,\dots,\alpha_n)$ for $g\in G$, is equivalent to that the outer square below commutes.

$$
\begin{array}{ccccc}
\wedge^{k+q}T_p(P) & \xrightarrow{\;\vartheta^{\wedge k}\,\wedge\,\alpha\;} & \mathfrak{h}^{\otimes k}\otimes\bigotimes_{i=n}^{n}V_{s_i} & \xrightarrow{\;\varphi\;} & V_{s_0}\\[2pt]
{\scriptstyle g_*}\downarrow & & {\scriptstyle (g^{-1})^{\otimes k+n}}\downarrow & & \downarrow{\scriptstyle g^{-1}}\\[2pt]
\wedge^{k+q}T_p(P) & \xrightarrow[\;\vartheta^{\wedge k}\,\wedge\,\alpha\;]{} & \mathfrak{h}^{\otimes k}\otimes\bigotimes_{i=n}^{n}V_{s_i} & \xrightarrow[\;\varphi\;]{} & V_{s_0}.
\end{array}
$$

The left square commutes by the G-invariance of $\vartheta,\alpha_1,\dots,\alpha_n$. Hence commutativity of the outer square is equivalent to commutativity of the right square, which is equivalent to G-equivariance of φ.

Next we consider

$$
i_{\zeta_X}\varphi(\vartheta^{\wedge k},\alpha_1,\dots,\alpha_n)=\varphi(i_{\zeta_X}(\vartheta^{\wedge k}),\alpha_1,\dots,\alpha_n).
$$

Since $i_{\zeta_X}\alpha_i=0$ for $i=1,\dots,n$ by assumption, the above is automatically 0 if $k=0$. So assume $k\geq 0$. We have to show that $\varphi(i_{\zeta_X}\vartheta,\dots)=0$. Since $i_{\zeta_X}\vartheta=X$ this finishes the proof. $\qquad\square$

6.2.2 Globalization on smooth manifolds.

This subsection contains few new technical result, its purpose is mainly to explain the typical application of the technical result of the previous section. To this end, we start by assuming we have a family of natural bundle functors $F=\{F_s\}_{s\in S}$, such that F_* contains the tangent bundle as a graded subbundle. Define $\mathcal{F}^d:=J_0^\infty F\mathbf{R}^d$. Thus the Lie algebra W_d of formal vector fields is a subspace of $\mathcal{F}_*^d$, as is the Lie algebra $gl_d\subset W_d$ of linear vector fields. Denote by $\mathsf{End}_{\mathrm{basic}}\langle\mathcal{F}^d\rangle$ the dg suboperad of multilinear maps that (i) are GL_d-equivariant and (ii) vanish whenever we insert an $X\in gl_d$ as an input in the distinguished color. Assume further that the family of natural bundles F exists for manifolds of an arbitrary finite dimension d and that we are given an operad G_F, which for every d has a morphism

$$
\mathsf{G}_F\to\mathsf{End}_{\mathrm{basic}}\langle\mathcal{F}^d\rangle.
$$

In typical situations the operads $\mathsf{End}_{\mathrm{basic}}\langle\mathcal{F}^d\rangle$ assemble in a diagram

$$
\cdots\to\mathsf{End}_{\mathrm{basic}}\langle\mathcal{F}^d\rangle\to\mathsf{End}_{\mathrm{basic}}\langle\mathcal{F}^{d+1}\rangle\to\cdots
$$

and we take G_F to be (some suboperad of) the limit $\lim_d\mathsf{End}_{\mathrm{basic}}\langle\mathcal{F}^d\rangle$. Moreover, the operad G_F typically has a combinatorial description in terms of graphs.

The assumption that F is a family of natural bundle functors automatically ensures that each $\mathcal{F}_s^d$ ($s \in S$) is a (G_d, W_d)-module. Assume given twist data for G_F. Take M to be a manifold of dimension d. By 6.2.1.4 there is an induced representation

$$\mathrm{Tw}\, \mathsf{G}_F \to \mathsf{End}\langle \Omega(M^{\mathrm{aff}}, M^{\mathrm{coor}} \times_{GL_d} \mathcal{F}^d)_\vartheta \rangle.$$

The bundle $M^{\mathrm{aff}} \to M$ always admits a section because the fibers are contractible. Pick such a section φ. Then we can pull back above morphism to a representation

$$\mathrm{Tw}\, \mathsf{G}_F \to \mathsf{End}\langle \Omega(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} \mathcal{F}^d))_{\varphi^* \vartheta} \rangle.$$

There is a canonical isomorphism

$$\Omega(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} \mathcal{F}^d))_{\varphi^* \vartheta} \cong \Omega(M, J^\infty FM)_\theta,$$

where θ denotes the jet bundle connection. The map of taking jets

$$j : \Gamma(M, FM) \to \Omega(M, J^\infty FM)_\theta$$

is a quasi-isomorphism. Thus, we have a representation of $\mathrm{Tw}\, \mathsf{G}_F$ in $\Gamma(M, FM)$, up to homotopy.

This is rather uninteresting in itself since G_F does not code any kind of particular algebraic structure: it is just a dimension-independent incarnation of $\mathsf{End}_{\mathrm{basic}}\langle \mathcal{F}^d \rangle$. Therefore, let us assume that $\mathsf{Q}_\infty = \mathcal{C}(\mathsf{C})$ is a coalgebra for the twist comonad and that we have a representation $\mathsf{Q}_\infty \to \mathsf{G}_F$. Since Q_∞ is a comonad for the twist construction, we then obtain

$$\mathsf{Q}_\infty \to \mathrm{Tw}\, \mathsf{G}_F \to \mathsf{End}\langle \Omega(M, J^\infty FM)_\theta \rangle.$$

A choice of retract data

$$\Gamma(M, FM) \mathrel{\substack{\longleftarrow \\ \longrightarrow}} \Omega(M, J^\infty FM)_\theta \circlearrowleft$$

will then, via homotopy transfer formulae, define an explicit representation

$$\mathsf{Q}_\infty \to \mathsf{End}\langle \Gamma(M, FM) \rangle.$$

We remark that the only choices we made to get this representation where the choice of a section φ of the bundle M^{aff} and the choice of an extension of the map of taking jets, $j : \Gamma(M, FM) \to \Omega(M, J^\infty FM)$, to a contraction and homotopy-diagram.

Proposition 6.2.2.1. The representation $\mathsf{Q}_\infty \to \mathsf{End}\langle \Gamma(M, FM)\rangle$ is up to homotopy independent of the choices made.

Let us only sketch a proof since we do not want to get into detail about the precise meaning of homotopy between representations, and then also only of the independence of the choice of section φ.

Proof. (Sketch.) Let φ_0 and φ_1 be two given sections of the bundle M^{aff}. One may find a path φ_t in the space of sections $\Gamma(M, M^{\mathrm{aff}})$ from φ_0 to φ_1 depending polynomially on t. For example, if φ_i is defined by the jets of the exponential map of an affine torsion-free connection ∇_i, $i = 0, 1$, then we can take φ_t to be defined by the spray of the connection $(1 - t)\nabla_0 + t\nabla_1$; a choice which is manifestly polynomial. From this we obtain a family of representations

$$\rho_t : \mathsf{Q}_\infty \to \mathsf{End}\langle \Omega(M, J^\infty FM)_\theta\rangle$$

depending polynomially on t, i.e., a polynomial path ρ_t in the space of Maurer-Cartan elements of

$$\mathrm{Map}(Q^{\mathsf{i}}, \mathsf{End}\langle \Omega(M, J^\infty FM)_\theta\rangle),$$

connecting the representation defined by the choice φ_0 and the representation defined by the choice φ_1. $\qquad\square$

6.2.3   Universal deformation complexes.

Let the notation be as in the last subsection. There is a morphism of dg Lie algebras

$$\mathrm{Def}(\mathsf{Q}_\infty \to \mathrm{Tw}\,\mathsf{G}_F) \to C_\mathsf{Q}(\Omega(M, J^\infty FM)_\theta, \Omega(M, J^\infty FM)_\theta)).$$

Yet, we claim, it is incorrect to regard $\mathrm{Def}(\mathsf{Q}_\infty \to \mathrm{Tw}\,\mathsf{G}_F)$ as a universal deformation complex for the Q_∞-stucture on $\Gamma(M, FM)$. Recall that G_F is supposed to be a universal version of $\mathsf{End}_{\mathrm{basic}}\langle \mathcal{F}^d\rangle$. Thus $\mathrm{Tw}\,\mathsf{G}_F$ relates to

$$\mathrm{Tw}\,\mathsf{End}\langle \mathcal{F}^d\rangle = C_{CE}(\mathcal{F}^d_*, \mathsf{End}\langle \mathcal{F}^d\rangle),$$

so that, by our adjunction formula for the twist and Lie module constructions, $\mathrm{Def}(\mathsf{Q}_\infty \to \mathrm{Tw}\,\mathsf{G}_F)$ relates to

$$\mathrm{Def}(L\text{-mod}(\mathsf{Q})_\infty \to \delta^* \mathsf{End}\langle \mathcal{F}^d\rangle),$$

where $\delta : Y = S \sqcup \{\tilde{*}\} \to S$ sends $\tilde{*}$ to $*$ and is the identity on S. This means that the Q_∞-deformations of $\Omega(M, J^\infty FM)_\theta$ coming from

$\mathrm{Def}(\mathsf{Q}_\infty \to \mathrm{Tw}\,\mathsf{G}_F)$ typically involve deforming the effect of twisting by the Maurer-Cartan element, by way of deforming the $L_\infty\text{-mod}(S)$-structure. In the applications this is something we want to avoid.

Definition 6.2.3.1. Say that $\gamma \in \mathrm{Def}(\mathsf{Q}_\infty \to \mathsf{G}_F)$ is Maurer-Cartan with respect to vector fields if, for every d, the image of ψ under $p_d :$ $\mathsf{G}_F \to \mathsf{End}\langle \mathcal{F}^d \rangle$ has the property that $p_d(\psi)(q)(X_1,\ldots,X_n) = 0$ for all $n \geq 1$, $q \in \mathsf{Q}^{\mathrm{i}}(*^n \mid s)$, $s \in S$ and vector fields $x_1,\ldots,X_n \in W_d$.

Define $\mathrm{Def}(\mathsf{Q}_\infty \to \mathrm{Tw}\,\mathsf{G}_F)(W_d)$ to be the subcomplex of the deformation complex spanned by all ϕ such that both ϕ and $d\phi$ are Maurer-Cartan with respect to vector fields.

Remark 6.2.3.2. The above definition mimics our related definition in 3.1.5.1.

The given morphism

$$\mathrm{Def}(\mathsf{Q}_\infty \to \mathsf{G}_F) \to C_{\mathsf{Q}}(\Omega(M,J^\infty FM),\Omega(M,J^\infty FM))$$

restricts to a morphism

$$\mathrm{Def}(\mathsf{Q}_\infty \to \mathsf{G}_F)(W_d) \to C_{\mathsf{Q}}(\Omega(M,J^\infty FM),\Omega(M,J^\infty FM))(\theta),$$

since θ corresponds to a vector-valued differential form. Recall that there is a morphism

$$C_{\mathsf{Q}}(\Omega(M,J^\infty FM),\Omega(M,J^\infty FM))(\theta)$$
$$\to C_{\mathsf{Q}}(\Omega(M,J^\infty FM)_\theta,\Omega(M,J^\infty FM)_\theta),$$

from which we get

$$\mathrm{Def}(\mathsf{Q}_\infty \to \mathsf{G}_F)(W_d) \to C_{\mathsf{Q}}(\Omega(M,J^\infty FM)_\theta,\Omega(M,J^\infty FM)_\theta).$$

The complex $\mathrm{Def}(\mathsf{Q}_\infty \to \mathsf{G}_F)(W_d)$ is more appropriately considered as a universal deformation complex for the Q_∞-stucture on $\Gamma(M,FM)$. The deformations it classifies do not inadvertently use the Maurer-Cartan element. It classifies, simply, those deformations of $\mathcal{F}^d$ as a Q_∞-algebra that survive the globalization to become deformations of $\Gamma(M,FM)$.

6.3   Global two-colored (non-)formality.

Fix a d-dimensional smooth manifold M. We define the space of polyvector fields on M to be the space of global sections of the exterior powers of the tangent bundle; that is, we set $T_{\mathrm{poly}}(M) := \Gamma(M,S(TM[-1]))$.

Note that this clashes with our notation $T_{\mathrm{poly}}(\mathbf{R}^d)$ for the polynomial polyvector fields. To remove the ambiguity we adopt the convention that $T_{\mathrm{poly}}(\mathbf{R}^d)$ always refers to the polynomial vector fields; only for manifolds $M \neq \mathbf{R}^d$ do we adopt the other definition. The polydifferential Hochschild cochain complex on M is defined as the subcomplex $D_{\mathrm{poly}}(M)$ of the Hochschild cochain complex of $C^\infty(M)$ that is spanned by those maps $C^\infty(M)^{\otimes n} \to C^\infty(M)$ which are polydifferential. In the Chapter 4 we constructed a representation

$$\mathsf{FOC} \to \mathsf{End}\langle T_{\mathrm{poly}}(\mathbf{R}^d), T_{\mathrm{poly}}(\mathbf{R}^d), \mathcal{A}(\mathbf{R}^d)\rangle.$$

Recall that this representation defines a representation

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty \to \mathsf{End}\langle T_{\mathrm{poly}}(\mathbf{R}^d), T_{\mathrm{poly}}(\mathbf{R}^d), C_{\mathrm{Hoch}}(\mathcal{A}(\mathbf{R}^d), \mathcal{A}(\mathbf{R}^d))\rangle.$$

We will show that this can be globalized to a representation

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty \to \mathsf{End}\langle T_{\mathrm{poly}}(M), T_{\mathrm{poly}}(M), D_{\mathrm{poly}}(M)\rangle.$$

The reason that we globalize the representation of $\mathsf{Mor}_*(\mathsf{NCG})_\infty$ instead of the representation of FOC is that the latter operad does not have twist data, nor is it formal, while

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty = L\{-1\}\text{-mod}(\mathsf{Mor}(\mathsf{Ass}))_\infty$$

implies that the former is both formal and a coalgebra for the twist construction.

Definition 6.3.0.3. Recall that in coordinates y^i on $\mathbf{R}^d$,

$$\mathcal{A}(\mathbf{R}^d) = \mathbf{R}[y^1, \dots y^d],$$

$$T_{\mathrm{poly}}(\mathbf{R}^d) = \mathbf{R}[y^1, \dots, y^d, \partial/\partial y^1, \dots, \partial/\partial y^d], \quad |\partial/\partial y^i| = 1.$$

Define the completions

$$\hat{\mathcal{A}}(\mathbf{R}^d) := \mathbf{R}^d[[y^1, \dots, y^d]], \quad \hat{T}_{\mathrm{poly}}(\mathbf{R}^d) := \hat{\mathcal{A}}(\mathbf{R}^d)[\partial/\partial y^1, \dots, \partial/\partial y^d].$$

Introduce also the formal polydifferential Hochschild cochain complex $\hat{\mathcal{D}}(\mathbf{R}^d) \subset C_{\mathrm{Hoch}}(\hat{\mathcal{A}}, \hat{\mathcal{A}})$ as the complex of polydifferential cochains, i.e., of cochains of the form

$$\sum D^{I_1, \dots, I_k} \frac{\partial}{\partial y^{I_1}} \cdots \frac{\partial}{\partial y^{I_n}}, \ D^{I_1, \dots, I_k} \in \hat{\mathcal{A}}(\mathbf{R}^d),$$

with multiindices I_j. We may identify $\hat{\mathcal{D}}(\mathbf{R}^d) = C_{\mathrm{Hoch}}(\mathcal{A}, \hat{\mathcal{A}})$.

Note that $\hat{T}_{\mathrm{poly}}(\mathbf{R}^d) \cong J_0^\infty S(T\mathbf{R}^d[-1])$ and similarly for the other completions.

Remark 6.3.0.4. There is an induced representation

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty \to \mathsf{End}\langle \hat{T}_{\mathrm{poly}}(\mathbf{R}^d), \hat{T}_{\mathrm{poly}}(\mathbf{R}^d), \hat{\mathcal{D}}(\mathbf{R}^d)\rangle.$$

The general philosophy explained in the previous section tells us that we should look for a universal version G_F of an operad of the form $\mathsf{End}_{\mathrm{basic}}\langle \mathcal{F}\rangle$, where subscript "basic" refers to GL-equivariance vanishing whenever a linear vector field is inserted in the distinguished color. Our operad G_F is, in the present case, the operad $\mathsf{Gra}_{\mathsf{FOC}}$. The operations defined by graphs in this operad are GL-equivariant, but the operations do *not* vanish when a linear vector field is inserted at a black input. To get around this we need to following series of lemmata, for which we have borrowed the proofs either in part or in full from [Kontsevich 2003].

Lemma 6.3.0.5. Let Γ be a graph in either $\mathsf{gra}_{\mathsf{NCG}}(p,q)$ or $\mathsf{gra}_{\mathsf{FOC}}(k,m,n)$, respectively, with at least three vertices in total. If Γ has a univalent black vertex, then the integral of θ^Γ over $\overline{CF}_{p,q}(\mathbf{C})$ or $\overline{CF}_{k,m,n}(\mathbf{H})$, respectively, vanishes.

Proof. Let v be the univalent black vertex. The short proof is that v has a two-dimensional freedom while θ^Γ has only a 1-form dependence on v, since only one edge connects to v. In more detail, we argue as follows. If $\Gamma \in \mathsf{gra}_{\mathsf{NCG}}(p,q)$ then, since we cannot have edges between collinear vertices, but have at least three vertices in total, we must have at least one other black vertex for the form to be top-dimensional. Let u be the (single) vertex adjacent to v and let e be the edge connecting them. We have $2\pi\theta^\Gamma = \theta^{\Gamma\setminus e} \wedge d\arg(x_v - x_u)$. Use the gauge-freedom to put this other black vertex at i and the collinear points on the real axis. Then it is clear that v encodes two degrees of freedom. However, the level sets of $\arg(x_v - x_u)$ foliate the domain of integration and θ^Γ restricts to zero on any one of them.

The argument if $\Gamma \in \mathsf{graphs}_{\mathsf{FOC}}(k,m,n)$ is a repetition of essentially the same argument. $\square$

Lemma 6.3.0.6. Let Γ be a graph in $\mathsf{gra}_{\mathsf{NCG}}(p,q)$. If Γ has a bivalent black vertex, then the integral of θ^Γ over $\overline{CF}_{p,q}(\mathbf{C})$ vanishes.

Proof. Let v be the bivalent black vertex and let it be connected to vertices u and w. Since we cannot have double edges $u \neq w$. Consider the

function f defined on $CF_{p,q}(\mathbf{C})$ which reflects the point x_v (corresponding to the bivalent vertex) in the barycenter $(x_u + x_w)/2$, and leaves all other points unchanged. This is not a map

$$f : CF_{p,q}(\mathbf{C}) \to CF_{p,q}(\mathbf{C}),$$

because $f(x_v) = x_u + x_w - x_u$ might equal one of the other points x. Nevertheless, f can be extended to the compactification by replacing the image in that special case with the boundary point given by letting x and $\epsilon(x_u + x_w - x_u)$ collapse as ϵ tends to 0. We note that f preserves orientation. Let e_1 be the edge connecting v and u and let e_2 be the edge connecting v and w. Then $f^*\theta^{e_1} = \theta^{e_2}$, $f^*\theta^{e_2} = \theta^{e_1}$, while $f^*\theta^e = \theta^e$ for any other edge e. Hence $f^*\theta^\Gamma = -\theta^\Gamma$. Thus the integral is zero. $\qquad\square$

Lemma 6.3.0.7. Let Γ be a graph in $\mathsf{gra}_{\mathsf{FOC}}(k,m,n)$. If Γ has a bivalent vertex with one outgoing edge and one incoming edge, then the integral of θ^Γ over $\overline{CF}_{k,m,n}(\mathbf{H})$ vanishes.

Proof. Let $\pi : \overline{CF}_{k,m,n}(\mathbf{H}) \to \overline{CF}_{k-1,m,n}(\mathbf{H})$ be the projection which forgets the point x_v that corresponds to the bivalent vertex v. We can apply Fubini in the form

$$\int_{\overline{CF}_{k,m,n}(\mathbf{H})} \theta^\Gamma = \int_{\overline{CF}_{k-1,m,n}(\mathbf{H})} \pi^!(\theta^\Gamma).$$

Let $e_1 = (u,v)$ be the incoming edge at v and let $e_2 = (v,w)$ be the outgoing edge. Then

$$\pi^!(\theta^\Gamma) = \pm\theta^{\Gamma\setminus\{e_1,e_2\}} \wedge \pi^!(\theta^{e_1} \wedge \theta^{e_2}).$$

The sign depends on which order the edges are in. We shall show that the function $h := \pi^!(\theta^{e_1} \wedge \theta^{e_2})$ is constantly equal to zero. First we argue that it is constant. By fibrewise Stokes':

$$dh = \pi^!(d(\theta^{e_1} \wedge \theta^{e_2})) \pm \pi^!_\partial(\theta^{e_1} \wedge \theta^{e_2}) = \pm\pi^!_\partial(\theta^{e_1} \wedge \theta^{e_2}),$$

if $\pi^!_\partial$ denotes integration along the fibrewise boundary. The fibrewise boundary has three kinds of strata. First of all there are strata where the point x_u collapses to a point on the boundary. This point can be a new boundary point or an existing one. In either case the resulting differential form to be integrated over the strata is zero, because since v has an outgoing edge e_2 the form will contain a factor corresponding to an edge emanating from a boundary point. The second kind of strata are given by x_u tending to infinity, or, equivalently, by all the other

points collapsing together to a single boundary point. In this case the form is also zero, because in this limit configuration the edge e_1 becomes an edge emanating from a boundary point. Finally, the third kind of strata is when x_u collapses to one the points not on the boundary. In all such cases the strata is a circle $S^1 = \overline{C}_2(\mathbf{C})$. We then distinguish two possibilities: either x_u collapses to one of the points x_v or x_w, or it collapses to some other point x. In the latter case, when x_u collapses to x, the form is zero. If x_u collapses to x_v or x_w the form is the normalized volume form on the circle, so the integral does not vanish; however, those two contributions cancel each other. This proves that h is constant.

Finally, we argue that h is zero. Here we have to distinguish two cases. In the first case neither v nor w is a boundary vertex. In that case, assume $x_v = i$ and $x_w = 2i$. For this particular case the involution given by reflection in the imaginary axis reverses orientation of the fiber but preserves the integrand. Thus $h(i, 2i) = 0$, which since we know it is a constant function implies it vanishes everywhere. In the second case w is a boundary point. In this case we look at $x_w = 0$ and $x_u = i$, apply the same involution, and make the same conclusion. □

Corollary 6.3.0.8. Let $\pi \in \hbar T_{\mathrm{poly}}[[\hbar]]$ be a Maurer-Cartan element. All the operations in the π-twisted representation

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty \to \mathsf{End}\langle T_{\mathrm{poly}}[[\hbar]]_\pi, T_{\mathrm{poly}}[[\hbar]]_\pi, C_{\mathrm{Hoch}}(\mathcal{A}, \mathcal{A})[[\hbar]]_\pi \rangle$$

are given by sums over graphs $\Gamma \in \mathsf{gra}_{\mathsf{NCG}}$, with π decorating at least trivalent black vertices, and graphs $\Gamma \in \mathsf{gra}_{\mathsf{FOC}}$, that have π decorating black vertices that might be bivalent, but can not have just one incoming and one outgoing edge.

Proposition 6.3.0.9. The θ-twisted representation

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty$$
$$\to \mathsf{End}\langle \Omega(M^{\mathrm{coor}}, \hat{T}_{\mathrm{poly}}(\mathbf{R}^d))_\vartheta, \Omega(M^{\mathrm{coor}}, \hat{T}_{\mathrm{poly}}(\mathbf{R}^d))_\vartheta, \Omega(M^{\mathrm{coor}}\hat{\mathcal{D}}(\mathbf{R}^d))_\vartheta \rangle$$

descends to a representation in the associated bundles on M^{aff}.

Proof. We need to check the conditions of 6.2.1.4. The GL_d equivariance is clear since all $\mathsf{Mor}_*(\mathsf{NCG})_\infty$-operations are represented as linear combinations of operations of the form $D_\Gamma(\vartheta^{\wedge k}, \dots)$, and all operations defined by graphs are GL_d-equivariant. The necessary vanishing on gl_d follows from the preceeding corollary. The form ϑ takes values in formal vector fields, and a formal vector field can handle at most one outgoing

edge. Hence, if a vertex of a graph in $\mathsf{gra}_{\mathsf{NCG}}$ that decorated by ϑ is trivalent, then two of the three edges must be bivalent. This means that the operation vanishes if the vector field is linear, since a second derivative of something linear vanishes. If the graph is in $\mathsf{gra}_{\mathsf{FOC}}$, then any vertex decorated by ϑ must have exactly one outgoing edge, since ϑ is vector field-valued. Since it cannot have only one outgoing and one incoming edge, it must then have at least two incoming edges, and we will again be taking the second derivative of something linear. □

Choose a section φ of $M^{\mathrm{aff}} \to M$. Define

$$\mathfrak{T}_{\mathrm{poly}}(M) := \Omega(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{T}_{\mathrm{poly}}(\mathbf{R}^d))),$$

$$\mathfrak{D}_{\mathrm{poly}}(M) := \Omega(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{\mathcal{D}}(\mathbf{R}^d))),$$

and $B := \varphi^*\vartheta \in \Omega^1(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} W_d))$.

Lemma 6.3.0.10. In the B-twisted representation

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty \to \mathsf{End}\langle \mathfrak{T}_{\mathrm{poly}}(M)_B, \mathfrak{T}_{\mathrm{poly}}(M)_B, \mathfrak{D}_{\mathrm{poly}}(M)_B \rangle$$

the L_∞-structure on $\mathfrak{T}_{\mathrm{poly}}(M)[1]$ is the one with differential $d+[B,\ ]_S$ and fibrewise Schouten bracket, the A_∞-structure on $\mathfrak{T}_{\mathrm{poly}}(M)$ is nonstandard but has differential $d+[B,\ ]_S$, and the A_∞-structure on $\mathfrak{D}_{\mathrm{poly}}(M)$ is the one with differential $d+[B,\ ]_G$ and fibrewise standard cup product.

Proof. The statement about the Lie algebra $\mathfrak{T}_{\mathrm{poly}}(M)$ follows from noting that it, before twisting, is a dg Lie algebra (so there can be no higher operations after twisting either).

The statement about the differential on the A_∞ algebra $\mathfrak{T}_{\mathrm{poly}}(M)$ follows from noting that the differential is given by

$$[B,\ ]_S + \sum_{p \geq 2} \frac{1}{p!} \mathcal{V}_{p,1}(B^{\wedge p},\ ),$$

but by the trivalency condition on free interior vertices the second term vanishes.

The statement about $\mathfrak{D}_{\mathrm{poly}}(M)$ is argued as follows. It has the A_∞-structure corresponding to the fibrewise star product $\mathcal{U}(B)$. But $\mathcal{U}(B) = B$ because the graphs used in all other potential contributions to $\mathcal{U}(B)$ are at least trivalent, hence vanish when we decorate all vertices with B. □

124

Lemma 6.3.0.11. The following identifications are canonical:

$$\varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{T}_{\mathrm{poly}}(\mathbf{R}^d)) \cong \hat{S}(T^*M) \otimes S(TM[-1]);$$

$$\varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{\mathcal{D}}(\mathbf{R}^d)) \cong \hat{S}(T^*M) \otimes \hat{T}(S(TM)[-1]);$$

$$\varphi^*(M^{\mathrm{coor}} \times_{GL_d} W_d) \cong \hat{S}(T^*M) \otimes TM.$$

Proof. Let p denote the projection $p : M^{\mathrm{aff}} \to M$ and let $J^\infty(M)$ denote the bundle of infinite jets of functions on M. The identifications are more or less formal consequences of the canonical identification $p^*J^\infty(M) \cong p^*\hat{S}(T^*M)$, so we settle for proving that. Let π be the projection $M^{\mathrm{coor}} \to M$ and take a ϕ with $\pi(\phi) = x$. The formal coordinate system ϕ comes from some local diffeomorphism $\tilde{\phi} : \mathbf{R}^d \to M$. That local diffeomorphism defines an isomorphism

$$\pi^*J^\infty(M)_\phi = J_x^\infty(M) \cong J_0^\infty(\mathbf{R}^d) = \mathbf{R}[[y^1, \ldots, y^d]].$$

It similarly defines an isomorphism

$$\pi^*\hat{S}(T^*M)_\phi \cong \hat{S}(T_0^*\mathbf{R}^d) = \mathbf{R}^d[[dy^1, \ldots, dy^d]], \ |dy^i| = 0.$$

It follows that

$$p^*J^\infty(M) \cong M^{\mathrm{coor}} \times_{GL_d} \mathbf{R}[[y^1, \ldots, y^d]],$$

$$p^*\hat{S}(T^*M) \cong M^{\mathrm{coor}} \times_{GL_d} \mathbf{R}[[dy^1, \ldots, dy^d]].$$

The coordinates y^i and the covectors dy^i transform the same under linear coordinate changes, hence the two bundles are isomorphic. $\quad\square$

Remark 6.3.0.12. The preceeding lemma lets us write everything locally in terms of coordinates x^i on M and the corresponding 1-forms dx^i, a frame y^i of T^*M and the corresponding dual frame $\partial/\partial y^i$ of TM. For example, a bivector field in $\mathfrak{T}_{\mathrm{poly}}(M)$ will have a local expression

$$\xi = \sum \xi_I^{ij}(x, dx) y^I \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j},$$

with I a multi-index, and each coefficient $\xi_I^{ij}(x, dx)$ a differential form on M.

There universal property of M^{coor} with respect to natural bundle functors gives an isomorphism

$$\varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{T}_{\mathrm{poly}}(\mathbf{R}^d)) \cong J^\infty S(TM[-1]).$$

This isomorphism, together with the map of taking jets, defines a quasi-isomorphism of dg vector spaces

$$j : T_{\mathrm{poly}}(M) \to \mathfrak{T}_{\mathrm{poly}}(M)_B = (\mathfrak{T}_{\mathrm{poly}}(M), d + [B,]_S).$$

Similarly, we have a quasi-isomorphism $j : D_{\mathrm{poly}}(M) \to \mathfrak{D}_{\mathrm{poly}}(M)_B$. Define inverse projections

$$p : \mathfrak{T}_{\mathrm{poly}}(M)_B, \mathfrak{D}_{\mathrm{poly}}(M)_B \to T_{\mathrm{poly}}(M), D_{\mathrm{poly}}(M)$$

by $p(x^i) = x^i$, $p(\partial/\partial y^i) = \partial/\partial x^i$, and $p(dx^i) = 0 = p(y^i)$. We complete the data with a homotopy δ^{-1} defined by

$$\delta^{-1} f(x, dx, y) := y^k \iota_{\partial/\partial x^k} \int_0^1 f(x, t\, dx, ty) \frac{dt}{t}.$$

We now have a homotopy retract diagram

$$T_{\mathrm{poly}}(M) \rightleftarrows \mathfrak{T}_{\mathrm{poly}}(M)_B \circlearrowleft$$

and a similar one for polydifferential Hochschild cochains.

Lemma 6.3.0.13. The A_∞-structure on $T_{\mathrm{poly}}(M)$ obtained by homotopy transfer from $\mathfrak{T}_{\mathrm{poly}}(M)_B$ is the usual structure of graded associative algebra given by the wedge product.

Proof. Call the components of the transferred structure μ_n and the components of the structure we transfer ν_q^B. Recall

$$\nu_2^B = \wedge + \sum_{p \geq 1} \frac{1}{p!} \mathcal{V}_{p,2}(B^{\wedge p},), \quad \nu_q^B = \sum_{p \geq 1} \frac{1}{p!} \mathcal{V}_{p,q}(B^{\wedge p},), \ q \geq 3.$$

The transferred structure is given by the formula $\mu_n = \sum_{T \in PT_n} \nu_T^B$, the sum being over all planar rooted trees with at least bivalent vertices, and with ν_T^B denoting the operation obtained by decorating the vertices of T with the matching operations ν_q^B, all edges with δ^{-1}, the root leg with p and all input legs with j, and interpreting the resulting decorated tree as a composition diagram.

The projection p kills everything of form-degree ≥ 1. This means that the only terms that can contribute to μ_n have $\mathcal{V}_{0,2} = \wedge$ decorating the vertex closest to the root.

Assume, then, that T has a bivalent vertex u decorated by $\wedge$ closest to the root, and at least one other vertex v connected by an edge to u. The edge between u and v is decorated with δ^{-1}. Anything not of

form-degree ≥ 1 is killed by δ^{-1}, so at the vertex v we must place a decoration containing at least one B. But on applying the homotopy we then obtain an input for the wedge product at u which is of degree ≥ 1 in y. Since $\wedge$ can not reduce the degree in y we then end up feeding something of degree ≥ 1 in y to the projection p at the root; getting something that vanishes.

We conclude that the only nonzero μ_n is $\mu_2(X, Y) = p(jX \wedge jY) = X \wedge Y$, the first wedge denoting the fibrewise wedge-product and the second one denoting the usual wedge-product on polyvector fields. $\qquad \square$

Theorem 6.3.0.14. There is an explicit homotopy transferred structure

$$\mathsf{Mor}_*(\mathsf{NCG})_\infty \to \mathsf{End}\langle T_{\mathrm{poly}}(M), T_{\mathrm{poly}}(M), D_{\mathrm{poly}}(M)\rangle$$

with

(i) the Schouten graded Lie algebra structure on $T_{\mathrm{poly}}(M)[1]$.

(ii) the associative wedge product on $T_{\mathrm{poly}}(M)$.

(iii) the associative cup product on $D_{\mathrm{poly}}(M)$.

(iv) the action of polyvector fields on the polydifferential Hochschild cochain complex being of the form $br \circ \overline{\mathcal{U}}$, where

$$\overline{\mathcal{U}} : T_{\mathrm{poly}}(M) \to D_{\mathrm{poly}}(M)$$

is a quasi-isomorphism of L_∞ algebras. (The globalized Kontsevich formality.)

(v) the morphism of NCG_∞-algebras being a quasi-isomorphism.

Proof. The map of taking jets $j : T_{\mathrm{poly}}(M) \to \mathfrak{T}_{\mathrm{poly}}(M)_B$ is a quasi-isomorphism of dg Lie algebras, if both are equipped with the Schouten bracket. Similarly, the map of taking jets of polydifferential operators is also a quasi-isomorphism of dg associative algebras. This proves (i) and (iii). Point (ii) is the statement of the previous lemma. Point (iv) is argued as follows. The action before homotopy transfer is given by the fibrewise B-twisted $br \circ \mathcal{U}$. Since br is a strict morphism of dg Lie algebras, twisting and homotopy transfer has to produce a morphism of the form $br \circ \overline{\mathcal{U}}$. It has to be a quasi-isomorphism since homotopy transfer preserves quasi-isomorphisms. Point (v) again follows from the fact that the homotopy transfer of a quasi-isomorphism is a quasi-isomorphism. $\qquad \square$

6.3.1 The global exotic action.

Part of the globalized data constructed in 6.3.0.14 is an L_∞ morphism

$$\overline{\mathcal{V}} : T_{\text{poly}}(M)[1] \to C_{\mathsf{Ass}}(T_{\text{poly}}(M), T_{\text{poly}}(M)).$$

Remark 6.3.1.1. The morphism $\overline{\mathcal{V}}$ maps into a subcomplex of polydifferential Hochschild cochains.

Our task in this section is to give explicit formulas for the simplest terms of $\overline{\mathcal{V}}$.

Proposition 6.3.1.2. The first two terms of $\overline{\mathcal{V}}$ are $\overline{\mathcal{V}}_{1,1} = [\,,\,]_S$ and

$$\overline{\mathcal{V}}_{1,3} = p \circ \left(\mathcal{V}_{1,3} + [\,,\,]_S \circ \big(id \otimes (\delta^{-1} \circ \mathcal{V}_{1,3}(B))\big) \right) \circ (j \otimes j^{\otimes 3}).$$

Proof. The morphism $\overline{\mathcal{V}}$ is defined by transfer of NCG_∞-structure, from $\mathfrak{T}_{\text{poly}}(M)_B$ to $T_{\text{poly}}(M)$. Denote by

$$\mathcal{V}^B : \mathfrak{T}_{\text{poly}}(M)_B[1] \to C_{\mathsf{Ass}}(\mathfrak{T}_{\text{poly}}(M)_B, \mathfrak{T}_{\text{poly}}(M)_B)$$

the B-twisted fibrewise structure. Recall that

$$\mathcal{V}^B_{p,q} = \sum_{k \geq 0} \frac{1}{k!} \mathcal{V}_{k+p,q}(B^{\wedge k}, \dots).$$

The transferred structure is given by a formula

$$\overline{\mathcal{V}}_{r,s} = \sum_T \mathcal{V}^B_T,$$

where we sum over rooted trees T with at least bivalent vertices and with two kinds of legs/edges, corresponding to the two operadic colors. The legs in the Lie color are non-planar while the legs in the associative color are planar. Vertices are decorated by matching components $\mathcal{V}^B_{p,q}$, λ^B_k or ν^B_n, of the B-twisted fibrewise structures, edges are decorated by the homotopy δ^{-1}, the root by the projection p, and all inputs by j.

Let us start with $\overline{\mathcal{V}}_{1,1}$. Simply by constraint on the number of inputs,

$$\overline{\mathcal{V}}_{1,1} = p \circ \mathcal{V}^B_{1,1} \circ j \otimes j.$$

Since p is 0 on anything of form-degree ≥ 1 we cannot have a B as input at the bottom vertex. Thus

$$\overline{\mathcal{V}}_{1,1} = p \circ \mathcal{V}_{1,1} \circ j \otimes j.$$

Since $\mathcal{V}_{1,1} = [\,,\,]_S$ is the fibrewise Shouten-bracket we deduce $\overline{\mathcal{V}}_{1,1} = [\,,\,]_S$ is the (global) Schouten bracket, considered as an adjoint action.

Similarly, we cannot have a B as input at the bottom vertex in any tree contributing to $\overline{\mathcal{V}}_{1,3}$. This means that a tree contributing to this operation must either have $\overline{V}_{1,3}$ or $\overline{V}_{1,1}$ at the bottom vertex. If we put $\overline{V}_{1,3}$ at the bottom vertex then we cannot have any further vertices, since that would have to increase the number of inputs beyond $(1,3)$. Hence we have a contribution

$$p \circ \mathcal{V}_{1,3} \circ (j \otimes j^{\otimes 3})$$

to $\overline{\mathcal{V}}_{1,3}$. Assume we put $\overline{V}_{1,1}$ at the bottom vertex. Then we must have at least one other vertex v. The homotopy δ^{-1} kills everything of form-degree zero. Since $\mathcal{V}_{1,2} = 0$, this (together with the constraint on the number of inputs) then requires that we put ν_3^B on v. The homotopy reduces form-degree by 1 so there can be at most one B at the inputs of the operation decorating v, and that singles out the term $\mathcal{V}_{1,3}(B)$ of ν_3^B. This gives the only other contribution to $\overline{\mathcal{V}}_{1,3}$. $\qquad\square$

Up to taking jets and using the contracting homotopy, the only operations used to compute $\overline{\mathcal{V}}_{1,3}$ are the fibrewise Schouten bracket and the fibrewise $\mathcal{V}_{1,3}$. Recall that up to orientations of edges there is a single graph contributing to $\mathcal{V}_{1,3}$; the graph $\circ\!\!\!\raisebox{1pt}{$\bullet$}\!\!\!\circ$, with three collinear vertices and a single trivalent free interior vertex. Still, the general formula for $\overline{\mathcal{V}}_{1,3}$ is very complicated. We shall write it out explicitly only for $\overline{\mathcal{V}}_{1,3}(\pi)(f, Y, Z)$, where π is a bivector, f is a function, and Y and Z are vector fields.

Over a parallelizable open subset $U \subset M$ we can always find a lift of φ to a section ϕ of M^{coor}. Recall that $\phi(x)$ is the jet of a local diffeomorphism $\tilde{\phi} : \mathbf{R}^d \to M$ taking 0 to x. If we give $\mathbf{R}^d$ coordinates y^i and take the subset U to have coordinates x^i, then ϕ is represented as the Taylor expansion of $\tilde{\phi}$:

$$\phi^i(x, y) = x^i + \phi^i_j y^j + \frac{1}{2}\phi^i_{jk} y^j y^k + \cdots + \frac{1}{r!}\phi^i_{j_1 \ldots j_r} y^{j_1} \ldots y^{j_r} + \ldots.$$
$$= \sum \frac{1}{r!} \frac{\partial^r \tilde{\phi}}{\partial y^{j_1} \ldots \partial y^{j_r}} y^{j_1} \ldots y^{j_r}.$$

We shall find it convenient to not distinguish between $\tilde{\phi}$ and ϕ. The condition that ϕ is a local diffeomorphism at $0 \in \mathbf{R}^d$ amounts to demanding that ϕ^i_j is an invertible matrix. Moreover, since $M^{\mathrm{aff}} = M^{\mathrm{coor}}/GL_d$ we may assume the representative ϕ of φ to start with the unit matrix: $\phi^i_j = \delta^i_j$. Finally, let us drop also the distinction between φ and ϕ in all local calculations.

6.3.2 The Maurer-Cartan element.

Let $X \in T_\phi M^{\mathrm{coor}}$ be a tangent vector and let $\phi_t : I \to M^{\mathrm{coor}}$ be a path of formal coordinate systems, with $\phi_0 = \phi$ and tangent vector X at ϕ. The connection form ϑ on M^{coor} is defined locally by

$$\vartheta(X) = -\sum \frac{1}{r!} \frac{\partial^r}{\partial y^{j_1} \dots \partial y^{j_r}} \left((d\phi(y))^{-1} \frac{d}{dt} \phi_t(y)|_{t=0} \right) \in W_d.$$

It follows that $\varphi^* \vartheta \in \Omega^1(M, W_d)$ has the local expression

$$\varphi^* \vartheta = B^i_j(x,y) dx^j \frac{\partial}{\partial y^i},$$

where

$$B^i_j(x,y) := -\left(\left(\frac{\partial \varphi}{\partial y} \right)^{-1} \right)^i_k \frac{\partial \varphi^k}{\partial x^j}.$$

Here $(\partial \varphi / \partial y)^{-1}$ denotes the matrix inverse of the matrix of formal power series $\partial \varphi^i / \partial y^j$. Let us caculate B up to second order. Define

$$\Psi := \left(\frac{\partial \varphi}{\partial y} \right)^{-1}, \quad \Psi^i_j = \sum \frac{1}{r!} \psi^i_{j a_1 \dots a_r} y^{a_1} \dots y^{a_r}.$$

The equation $\Psi^i_k (\partial \varphi^k / \partial y^j) = \delta^i_j$ can be solved iteratively, using

$$\psi^i_j = \delta^i_j,$$

$$\frac{1}{r!} \psi^i_{j a_1 \dots a_r} y^{a_1} \dots y^{a_r} = -\sum_{p=1}^r \frac{1}{p!(r-p)!} \varphi^i_{k(a_1 \dots a_p|} \psi^k_{j|a_{p+1} \dots a_r)} y^{a_1} \dots y^{a_r}.$$

Note that the indices a_s to the right are symmetrized since the corresponding variables y^{a_s} commute. Our convention for symmetrization is to normalize, so that $A_{(ij)} = (1/2)(A_{ij} + A_{ji})$ and, e.g., $\varphi^i_{(ab)} = \varphi^i_{ab}$. It follows that

$$-B^i_j = \left(\delta^i_k - \varphi^i_{ka} y^a + \left(\varphi^i_{k(a|} \varphi^k_{j|b)} - \frac{1}{2} \varphi^i_{kab} \right) y^a y^b + \dots \right)$$

$$\cdot \left(\delta^k_j + \frac{1}{2} \frac{\partial}{\partial x^j} \varphi^k_{rs} y^r y^s + \dots \right)$$

$$= \delta^i_j - \varphi^i_{ja} y^a + \left(\frac{1}{2} \partial_j \varphi^i_{ab} + \varphi^i_{k(a|} \varphi^k_{j|b)} - \frac{1}{2} \varphi^i_{jab} \right) y^a y^b + O(y^3).$$

Let us now specialize to the case when φ is defined as the jet of the exponential map of an affine torsion-free connection on M. In this case

we get the local expression of φ as follows. Let $\Phi^i(x, y, t)$ be smooth in x and $t \in [0, 1]$ and formal in y. The formal geodesic equation

$$\ddot{\Phi}^i + \Gamma^i_{jk}\dot{\Phi}^j\dot{\Phi}^k = 0, \quad \Phi(x, y, 0) = x, \quad \dot{\Phi}(x, y, 0) = y,$$

where Γ^i_{jk} are the Christoffel symbols of the connection, can be solved and $\varphi(x, y) = \Phi(x, y, 1)$. The first orders of the solution are

$$\varphi^i = x^i + y^i - \frac{1}{2}\Gamma^i_{ab}y^a y^b + \frac{1}{3!}\left(2\Gamma^i_{k(a}\Gamma^k_{bc)} - \partial_{(a}\Gamma^i_{bc)}\right)y^a y^b y^c + O(y^4).$$

Hence $\varphi^i_{ab} = -\Gamma^i_{ab}$ and $\varphi^i_{abc} = 2\Gamma^i_{k(a}\Gamma^k_{bc)} - \partial_{(a}\Gamma^i_{bc)}$. Inserting this into our expression for the quadratic coefficient B^i_{jab} of B^i_j gives[1]

$$
\begin{aligned}
-B^i_{jab} &= \frac{1}{2}\left(\frac{1}{3}\left(\partial_{[a}\Gamma^i_{j]b} + \partial_{[b}\Gamma^i_{j]a}\right) + \frac{1}{3}\left(\Gamma^i_{ka}\Gamma^k_{jb} + \Gamma^i_{kb}\Gamma^k_{ja} - 2\Gamma^i_{kj}\Gamma^k_{ab}\right)\right) \\
&= \frac{1}{6}\left(R^i_{baj} + R^i_{abj}\right) \\
&= \frac{1}{3}R^i_{(ab)j}.
\end{aligned}
$$

The components of the Riemann curvature tensor are here defined according to the standard convention $R(u, v)w = R^i_{jkl}u^k v^l w^j$. Bracketed indices denote normalized antisymmetrization. It follows that

$$B^i_j = -\delta^i_j - \Gamma^i_{aj}y^a - \frac{1}{3}R^i_{(ab)j}y^a y^b + O(y^3).$$

6.3.3   Explicit computation of a term.

We can by assumption invert $\varphi(x) : \mathbf{R}^d \to M$ in a neighborhood of x. Taking a polyvector field ξ on M we may then take the push-forward $(\varphi(x)^{-1})_*\xi$ and get a polyvector field on some small neighborhood of $0 \in \mathbf{R}^d$. Taking the Taylor expansion of that then gives us the formula for the jet $j\xi$. For a bivector field π this gives

$$
\begin{aligned}
j\pi^{ij} &= \pi^{kl}(\varphi(x, y))\Psi^i_k\Psi^j_l \\
&= \pi^{ij} + \nabla_a\pi^{ij}y^a + \left(\nabla_{(a}\nabla_{b)}\pi^{ij} + \frac{2}{3}R^{[i}_{(ab)k}\pi^{j]k}\right)y^a y^b + O(y^3),
\end{aligned}
$$

after a simplification analogous to the one that gave us the quadratic part of the Maurer-Cartan element. For the vector fields and functions we only need up to linear in y: $jX^i = X^i + \nabla_a X^i y^a + O(y^2)$, analogously for jY, and $jf = f + \partial_a f y^a + O(y^2)$.

[1]I am grateful to Malin Göteman for helping me with this calculation.

Lemma 6.3.3.1. In local coordinates as above,

$$24p(\mathcal{V}_{1,3}(j\pi)(jX, jY, jf))$$

$$= 4\left(\nabla_{(a}\nabla_{b)}\pi^{ci} + \frac{2}{3}R^{[c}_{(ab)k}\pi^{i]k}\right)X^aY^b\partial_c f\frac{\partial}{\partial x^i}$$

$$+ 2\nabla_a\pi^{bc}\nabla_b Y^i X^a\partial_c f\frac{\partial}{\partial x^i} + 2\nabla_a\pi^{bc}Y^a\nabla_b X^i\partial_c f\frac{\partial}{\partial x^i}.$$

Proof. Recall that $\mathcal{V}_{1,3}$ is given by the 8 differential operators corresponding to the 2^3 ways of orienting the edges of the graph ⊶, and that each of these graphs have weight $1/24$. Since π is a bivector and f is a function we can have at most two outgoing edges from the free vertex, and must have an incoming edge at the vertex decorated with f. This reduces the calculation from 8 terms to 3. That p sends all y^i's to zero gives us constraints on the order of the jets (Taylor expansions). The rest is a straight-forward computation (no terms cancel each other, no Bianchi identities are used, or etc.) $\qquad\square$

Lemma 6.3.3.2. In local coordinates as above,

$$24p\big([j\pi, \delta^{-1}\mathcal{V}_{1,3}(B)(jX, jY, jf)]_S\big) = -\frac{4}{3}\pi^{ki}R^c_{(ab)k}X^aY^b\partial_c f\frac{\partial}{\partial x^i}.$$

Proof. Since the fibrewise Schouten bracket $[j\pi,]_S$ reduces the degree in y by one, δ^{-1} increases the degree in y by one, and p vanishes on y, we are looking for the term of $\mathcal{V}_{1,3}(B)(jX, jY, jf)$ of y-degree zero. Since B is a vector field and jf must have an incoming edge, there is a single directed graph contributing to $\mathcal{V}_{1,3}(B)(jX, jY, jf)$; that with the edges directed from jX and jY to B and from B to f. The degree condition means that the only term of B that contributes is the quadratic term

$$-\frac{1}{3}dx^j R^i_{(ab)j}y^ay^b\frac{\partial}{\partial y^i}.$$

A simple calculation gives

$$24\mathcal{V}_{1,3}(B)(jX, jY, jf) = -\frac{2}{3}dx^j R^c_{(ab)j}y^ay^b X^aY^b\partial_c f\frac{\partial}{\partial y^i} + O(y).$$

Applying δ^{-1} replaces dx^j by y^j. Finally applying $p \circ [j\pi,]_S$ contracts indices and multiplies by 2 for the polyvector-degree of π. $\qquad\square$

Proposition 6.3.3.3. In local coordinates,

$$24\overline{\mathcal{V}}_{1,3}(\pi)(X, Y, f)$$

$$= -\frac{4}{3}R^i_{(ab)c}X^aY^b\pi^{ck}\partial_k f\frac{\partial}{\partial x^i} + 4X^aY^b\nabla_{(a}\nabla_{b)}\pi^{ki}\partial_k f\frac{\partial}{\partial x^i}$$

$$- 2X^a\nabla_a\pi^{kb}\partial_k f\nabla_b Y^i\frac{\partial}{\partial x^i} - 2Y^a\nabla_a\pi^{kb}\partial_k f\nabla_b X^i\frac{\partial}{\partial x^i}.$$

Proof. Combine the two preceeding lemmata and note that two terms cancel each other. $\qquad\square$

Remark 6.3.3.4. If π is a Poisson bivector, then the global $\overline{\mathcal{V}}$ defines an A_∞ structure $\overline{\nu}^\pi$ on $(T_{\text{poly}}(M)$ with differential the Poisson complex differential $d_\pi = [\pi,]_S$. The term $\overline{\mathcal{V}}_{1,3}(\pi)(X, Y, f)$ is the lowest order term in π of $\overline{\nu}_3^\pi(X, Y, f)$.

Remark 6.3.3.5. Assume π is a symplectic Poisson structure and take ∇ to be a symplectic connection. Then all covariant derivaties of π vanish and, moreover, $\pi^{ck}\partial_k f$ are the components of the Hamiltonian vector field H corresponding to f. Thus, in this special case the above formula reduces to

$$\overline{\mathcal{V}}_{1,3}(\pi)(X, Y, f) = -\frac{1}{36}\big(R(Y, H)X + R(X, H)Y\big).$$

This showcases an interesting (but not entirely surprising) feature of the globalization. Namely, applying the non-global construction in formal Darboux coordinates (so that the symplectic Poisson structure is constant) gives operations that vanish. Still, the actual global formula involves correction terms whose prescription is constrained by the global topology of the manifold.

Duflo automorphisms of Poisson cohomology

7.1 Summary of the results.

Let M be a smooth finite-dimensional manifold. It is known that the Grothendieck-Teichmüller group GRT_1 acts by L_∞ automorphisms of the Lie algebra of polyvector fields with Schouten bracket. (The action is strictly speaking defined only up to homotopy, as one needs to pick representative cocycles in the graph complex GC of elements in $\mathfrak{grt}_1$.) Recall the canonical (strict) NCG-structure on polyvector fields, given by the Schouten bracket, the wedge product, and the adjoint action of the Schouten bracket by derivations of the wedge product. The main result of this chapter is the following:

Theorem 7.1.0.6. Let $g \in GRT_1$ be an element of the Grothendieck-Teichmüller group. Then:

(i) There is an NCG_∞ deformation

$$\big((T_{\mathrm{poly}}(M), [\,,\,]_S), (T_{\mathrm{poly}}(M), \mu^g), \mathcal{V}^g\big)$$

of the standard structure on polyvector fields, where $\mu^g = \wedge + \mu^g_{\geq 5}$ and $\mathcal{V}^g = ad_S + \mathcal{V}_{\geq 3, \geq 5}$ have higher homotopies only for more than five inputs in the associative color.

(ii) There is an NCG_∞-isomorphism

$$\big((T_{\mathrm{poly}}(M), [\,,\,]_S), (T_{\mathrm{poly}}(M), \wedge), ad_S\big)$$
$$\to \big((T_{\mathrm{poly}}(M), [\,,\,]_S), (T_{\mathrm{poly}}(M), \mu^g), \mathcal{V}^g\big),$$

extending the L_∞ automorphism defined by g.

Let $\hbar\pi \in T_{\text{poly}}(M)[[\hbar]]$ be a Maurer-Cartan element. The image of $\hbar\pi$ under the L_∞ automorphism defined by g is another Maurer-Cartan element $\hbar g(\pi)$. By twisting we obtain from the preceeding theorem an A_∞ isomorphism

$$(T_{\text{poly}}(M)[[\hbar]], d_{\hbar\pi}, \wedge) \to (T_{\text{poly}}(M)[[\hbar]], d_{\hbar g(\pi)}, \nu^{g,\hbar g(\pi)}).$$

The following is an immediate corollary of above theorem since $\nu^{g,\hbar g(\pi)} = \wedge + \nu^{g,\hbar g(\pi)}_{\geq 5}$.

Corollary 7.1.0.7. There is an algebra isomorphism

$$(H(T_{\text{poly}}(M)[[\hbar]], d_{\hbar\pi}), \wedge) \to (H(T_{\text{poly}}(M)[[\hbar]], d_{\hbar g(\pi)}), \wedge)$$

on cohomology.

As a special case, consider the case when $M = \mathbf{g}[1]$ is the suspension of a graded Lie algebra $\mathbf{g}$ of finite type and π is defined as the tensor giving the Lie bracket on $\mathbf{g}$. In this case we may set the formal parameter $\hbar = 1$ and identify $(T_{\text{poly}}(M), d_\pi)$ as a complex with the Chevalley-Eilenberg cochain complex $C_{\text{CE}}(\mathbf{g}, S(\mathbf{g}))$. Moreover, it is known that $g(\pi) = \pi$ for a Maurer-Cartan element of this form. (The reason is the elementary observation that in this situation π is a vectorfield; so vertices decorated by π can have at most one outgoing edge, but $g(\pi)$ is defined by decorating all vertices of some sum of graphs $\gamma \in GC^0$ with at least trivalent vertices. Such a graph can not be directed without some vertex getting more than one outgoing edge.) Hence:

Corollary 7.1.0.8. Every $g \in GRT_1$ defines an algebra automorphism of $H_{\text{CE}}(\mathbf{g}, S(\mathbf{g}))$.

This last corollary is a slight generalization of Kontsevich's remark on the Duflo automorphism, cf. [Kontsevich 1999].

The above results are contained in "Grothendieck-Teichmüller group and Poisson cohomologies", by the author and S. Merkulov. However, that paper does not give a fully detailed discussion of the globalized statements, unlike the present chapter, which explains everything in detail.

7.2 Proof of the main theorem.

7.2.1 An equality in graph complexes.

Recall the operads Gra and Gra$_{\mathsf{NCG}}$ and that the deformation complex Def(NCG$_\infty \to$ Gra$_{\mathsf{NCG}}$) may be regarded either as the semidirect product (explains the Lie bracket)

$$\mathrm{Def}(\mathsf{Ass}_\infty \to \mathrm{tw}\,\mathsf{Gra}_{\mathsf{NCG}}) \rtimes \mathrm{Def}(\mathsf{Lie}_\infty\{-1\} \to \mathsf{Gra}).$$

of the right action $(\Gamma, \gamma) \mapsto \Gamma \bullet \gamma$, or as the mapping cone (makes the differential transparent) of the morphism

$$X : \mathrm{Def}(\mathsf{Lie}_\infty\{-1\} \to \mathsf{Gra})[-1] \to \mathrm{Def}(\mathsf{Ass}_\infty \to \mathrm{tw}\,\mathsf{Gra}_{\mathsf{NCG}}),$$

$$\gamma \mapsto X_\gamma := \overset{\bullet}{\underset{\circ}{}} \bullet\, \gamma.$$

We draw vertices of graphs in Def(Lie$_\infty\{-1\} \to$ Gra) and the dummy vertices of graphs in Def(Ass$_\infty \to$ tw Gra$_{\mathsf{NCG}}$) as black, and the others white. Since we shall be interested in globalizations, let us restrict attention right away to graphs that have at least trivalent black vertices and look at the subcomplex

$$\mathscr{C} := \mathrm{Cone}\big(X : \mathrm{GC}[-1] \to \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})\big),$$

discussed at some length in chaper 5, e.g. 5.2.0.6. Denote the differential on GC by δ and the differential on the complex to the right by $\partial + d_H$. In pictures,

$$\delta\gamma = [\overset{\bullet}{\underset{\bullet}{}}, \gamma], \quad \partial\Gamma = [\overset{\bullet}{\underset{\circ}{}}, \Gamma] + \Gamma \bullet \overset{\bullet}{\underset{\bullet}{}}, \quad d_H\Gamma = [\circ\,\circ, \Gamma].$$

Denote the full differential on $\mathscr{C}$ by d, viz.

$$d(\Gamma, \gamma) = \big(\partial\Gamma + d_H\Gamma - X_\gamma, \delta\gamma\big).$$

Define a map

$$\mathrm{GC} \to \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs}), \gamma \mapsto \gamma_{(1)}$$

by coloring the vertex labeled by, say, 1 white. (Remember that vertex labels of graphs in GC are symmetrized, so it does not matter which label we recolor.)

Lemma 7.2.1.1. For all graphs $\gamma \in \mathrm{GC}$, $X_\gamma = \partial(\gamma_{(1)}) - (\delta\gamma)_{(1)}$.

Proof. The statement is easiest to verify on the level of representations, recalling that the graphical operads can be regarded as certain universal endomorphism operads. Let V be a finite-dimensional vector space, set $\mathcal{O}_W = S^*(V^* \oplus V[-1])$ and consider it as the polynomial functions on the odd symplectic space $V \oplus V^*[1]$. The shifted algebra $\mathcal{O}_W[1]$ is a Lie algebra under the Poisson bracket $\{\,,\,\}$. In this setting our morphisms correspond to the map

$$X : C_{\mathsf{Lie}}(\mathcal{O}_W[1], \mathcal{O}_W[1])[-1] \to C_{\mathrm{CE}}(\mathcal{O}_W[1], C_{\mathsf{Ass}}(\mathcal{O}_W, \mathcal{O}_W)),$$

that sends a cochain $\gamma : S^n(\mathcal{O}_W[2])[-1] \to \mathcal{O}_W$ to the map that takes n functions $f_1, \ldots, f_n$ to the Hamiltonian vector field $X_{\gamma(f_1,\ldots,f_n)} = \{\gamma(f_1, \ldots, f_n), \}$ associated to the function $\gamma(f_1, \ldots, f_n)$, and the map

$$C_{\mathsf{Lie}}(\mathcal{O}_W[1], \mathcal{O}_W[1]) \to C_{\mathrm{CE}}(\mathcal{O}_W[1], C_{\mathsf{Ass}}(\mathcal{O}_W, \mathcal{O}_W))$$

which sends a cochain γ as above to the map which sends $n-1$ functions $f_1, \ldots, f_{n-1}$ to the map sending a function g to $\gamma_{(1)}(f_1, \ldots, f_{n-1})(g) = \gamma(f_1, \ldots, f_{n-1}, g)$. The differentials δ and ∂ are the respective Chevalley-Eilenberg differentials while d_H is the differential on $C_{\mathsf{Ass}}(\mathcal{O}_W, \mathcal{O}_W)$. We leave the actual verification of the formula to the reader. $\qquad\square$

The lemma has the following consequence: if γ is a cocycle in GC, then $X_\gamma = \partial\gamma_{(1)}$ is a cocycle in $\mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})$, because, clearly, $d_H X_\gamma = 0$. Recall the result 5.1.2.2 by Kontsevich that the morphism $\mathsf{Ger} \to \mathsf{Graphs}$ is a quasi-isomorphism. A consequence of that result is that all cocycles in

$$\mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})$$

can be represented as graphs without dummy vertices. Using this one can prove the following statement.

Lemma 7.2.1.2. [Lambrechts and Volić 2008] Any ∂-cocycle Γ in

$$\mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})$$

with at least one dummy vertex is ∂-exact.

Corollary 7.2.1.3. Let $\gamma \in \mathrm{GC}$ be a cocycle with each term having k black vertices. Then there are graphs $\gamma_{(\ell)}$ with ℓ white vertices and $k-\ell$ white vertices for $2 \leq \ell \leq k$, such that

$$(\partial + d_H)(\gamma_{(1)} + \cdots + \gamma_{(k)}) = \partial\gamma_{(1)} + d_H\gamma_{(k)}.$$

Proof. If $k = 1$, then we can take the $\gamma_{(1)}$ already defined. If $k \geq 2$ we note that, since $\partial d_H \gamma_{(1)} = 0$, there must exist a graph $\gamma_{(2)}$ with two white vertices and the property that $d_H \gamma_{(1)} = -\partial \gamma_{(2)}$. If $k \geq 3$ we can apply the same argument again, obtaining at the final step a graph $\gamma_{(k)}$ without dummy vertices. $\qquad\square$

Definition 7.2.1.4. In the situation as above, define $\gamma_{\max} := d_H \gamma_{(k)}$. It is a graph with $k + 1$ white vertices and no dummy vertices.

Corollary 7.2.1.5. Let $\gamma \in \mathrm{GC}$ be a cocycle. Then

$$d(\gamma_{(1)} + \cdots + \gamma_{(k)}, \gamma) = (\gamma_{\max}, 0)$$

in the complex $\mathscr{C}$.

7.2.2   The main theorem on affine space.

Let us apply the corollary at the end of the last subsection to a degree 0 cocycle γ and interpret the result in terms of deformations. The degree 1 element $(\gamma_{\max}, 0)$ represents a universal infinitesimal deformation of the standard NCG_∞ structure on polyvector fields. By its very form, involving no black vertices, we deduce that it involves deforming only the asociative wedge-product, which is given by the graph $\circ\,\circ$, to a product given infinitesimally by the sum $\circ\,\circ + \gamma_{\max}$. Since degree 0 cocycles in Kontsevich's graph complex have ≥ 4 vertices, $\gamma_{\max}$ has ≥ 5 white vertices.

The degree 0 element $(\gamma_{(1)} + \cdots + \gamma_{(k)}, \gamma)$ defines a universal infinitesimal NCG_∞-isomorphism from the standard structure to this deformation.

Proposition 7.2.2.1. The element $(\gamma_{(1)} + \cdots + \gamma_{(k)}, \gamma)$ can be exponentiated to define an NCG_∞-isomorphism H^γ from the standard NCG_∞-structure on polynomial polyvector fields to one which deforms the standard one by adding homotopies involving only ≥ 5 inputs in the associative color. In short, the main theorem is true on affine space.

Proof. We must only prove that all operations defined by exponentiation converge, without the need to introduce a formal parameter. This is clear because there are always a finite number of terms having a fixed number of black and a fixed number of white inputs. In more detail, the gauge equation

$$(\Gamma, \text{\textbullet}) = \exp\big[-ad_{\mathscr{C}}(\gamma_{(1)} + \cdots + \gamma_{(k)}, \gamma)\big] \cdot \big(\circ\,\circ + \text{\textbullet}, \text{\textbullet}\big)$$

has a solution $\Gamma \in \mathrm{Def}(\mathsf{Ass}_\infty \to \mathsf{Graphs})$ of the form

$$\Gamma = \circ\,\circ + \gamma_{\max} + \overset{\bullet}{\underset{\circ}{\vert}} + \Gamma',$$

where terms in Γ' have ≥ 5 white vertices and a total number of vertices ≥ 8, since its first contributions are $[\gamma_{(1)} + \cdots + \gamma_{(k)}, \gamma_{\max}]$. Restricted to the Lie color the gauge equation says that the cocycle γ exponentiates to an L_∞ automorphism e^γ of the Schouten bracket. $\qquad\square$

7.2.3 The main theorem on a smooth manifold.

Fix a smooth manifold M of finite dimension d for the remainder of this section. As in the Chapter 6, choose a section φ of $M^{\mathrm{aff}} \to M$ and define

$$\mathfrak{T}_{\mathrm{poly}}(M) := \Omega(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{T}_{\mathrm{poly}}(\mathbf{R}^d))),$$

$$B := \varphi^*\vartheta \in \Omega^1(M, \varphi^*(M^{\mathrm{coor}} \times_{GL_d} W_d)).$$

As in 6.3.0.11, we identify

$$\varphi^*(M^{\mathrm{coor}} \times_{GL_d} \hat{T}_{\mathrm{poly}}(\mathbf{R}^d)) \cong \hat{S}(T^*M) \otimes S(TM[-1]),$$

and, accordingly, $\mathfrak{T}_{\mathrm{poly}}(M) \cong \Omega(M, \hat{S}(T^*M) \otimes S(TM[-1]))$.

Lemma 7.2.3.1. The canonical morphism

$$D : \mathrm{Def}(\mathsf{NCG}_\infty \to \mathsf{Gra}_{\mathsf{NCG}}) \to C_{\mathsf{NCG}}(\mathfrak{T}_{\mathrm{poly}}(M), \mathfrak{T}_{\mathrm{poly}}(M))$$

maps the complex $\mathscr{C}$ (from the preceeding section) into the subcomplex $C_{\mathsf{NCG}}(\mathfrak{T}_{\mathrm{poly}}(M), \mathfrak{T}_{\mathrm{poly}}(M))(B)$ (defined in 3.1.5.1).

Proof. The Maurer-Cartan element is a fibrewise vector-valued. Hence the statement follows from the fact that all black vertices of graphs are at least trivalent; such a graph with only black vertices can not be directed in such a way that all vertices have at most one outgoing edge. $\qquad\square$

The above lemma says that the following is well-defined.

Definition 7.2.3.2. Define Φ to be the composition

$$\mathscr{C} \xrightarrow{D} C_{\mathsf{NCG}}(\mathfrak{T}_{\mathrm{poly}}(M), \mathfrak{T}_{\mathrm{poly}}(M))(B) \xrightarrow{t^B} C_{\mathsf{NCG}}(\mathfrak{T}_{\mathrm{poly}}(M)_B, \mathfrak{T}_{\mathrm{poly}}(M)_B).$$

Proposition 7.2.3.3. The main theorem is true.

Proof. The morphism Φ allows us to transfer the gauge equation

$$(\circ\,\circ + \gamma_{\max} + \text{⧖} + \Gamma', \text{⦚}) = \exp\!\big[-ad_{\mathscr{C}}(\gamma_{(1)} + \cdots + \gamma_{(k)}, \gamma)\big] \cdot \big(\circ\,\circ + \text{⧖}, \text{⦚}\big)$$

in $\mathscr{C}$, which gives the affine version of the construction promised in the main theorem, to an analogous equation in the deformation complex

$$C_{\mathsf{NCG}}(\mathfrak{T}_{\mathrm{poly}}(M)_B, \mathfrak{T}_{\mathrm{poly}}(M)_B).$$

The morphism Φ informally acts by decorating an arbitrary number of black vertices by B; in particular, Φ does not alter the fact that the higher homotopies in the NCG_∞ structure represented by the Maurer-Cartan element to the left have at least 5 inputs in the associative color. This proves that the main theorem holds in the form of an NCG_∞ isomorphism

$$H^{\gamma,B} : \big((\mathfrak{T}_{\mathrm{poly}}(M)_B, [\,,\,]_S), (\mathfrak{T}_{\mathrm{poly}}(M)_B, \wedge), ad_S\big)$$
$$\to \big((\mathfrak{T}_{\mathrm{poly}}(M)_B, [\,,\,]_S), (\mathfrak{T}_{\mathrm{poly}}(M)_B, \mu^\gamma), \mathcal{V}^\gamma\big).$$

Homotopy transfer creates new higher homotopies by summing over compositions of the operations in the structure that is transfered, hence one deduces that the new higher homotopies also have ≥ 5 inputs in the associative color. Thus, the main theorem holds on M. $\qquad\square$

7.2.4 Remarks on the global morphism.

Define $G^\gamma := e^\gamma$ to be the fibrewise L_∞ automorphism of the Schouten bracket and $F^\gamma = \{F^\gamma_{n,m}\}$ the remaining terms, such that $H^\gamma = (G^\gamma, F^\gamma)$ is the fibrewise NCG_∞ morphism on $\mathfrak{T}_{\mathrm{poly}}(M)$ (note no twisting by B!). We shall find it convenient to drop the superscript γ whenever no confusion arises, and, e.g., write just F instead of F^γ.

Define

$$G^i_k := \frac{1}{i!}G_{i+k}(B^{\wedge i}, \dots), \quad F^i_{n,m} := \frac{1}{i!}F_{i+k}(B^{\wedge i}, \dots).$$

Note that the B-twisted fibrewise L_∞ morphism G^B accordingly has components $G^B_k = \sum_{i\geq 0} G^i_k$ and similarly for the components of $H^B = (G^B, F^B)$. Define $\overline{H} = (\overline{G}, \overline{F})$ to be the homotopy transfer of H^B.

Lemma 7.2.4.1. The following formula holds:

$$\overline{F}_{n,1} = pF^0_{n,1}(j^{\otimes n} \otimes j) + \sum p[G^0_{n_r}, \delta^{-1}[\delta^{-1}G^{j_{r-1}}_{n_{r-1}},$$
$$\dots, \delta^{-1}[\delta^{-1}G^{j_2}_{n_2}, \delta^{-1}F^{j_1}_{n_1,1}]_S \dots]_S(j^{\otimes n} \otimes j),$$

summing over all $r \geq 1$, $j_1, \dots, j_{r-1} \geq 1$, and $n_1, \dots, n_r \geq 1$ such that $n_1 + \dots n_r = n$.

Proof. Clearly, all such expressions may contribute. The question is why no other compositions may contribute. The argument is similar to that given in 6.3.1.2. Recall that the homotopy δ^{-1} reduces differential form-degree by 1 and that p vanishes on everything of form-degree ≥ 1. This means there is some redundancy in our formula above since most choices of indices j_i give a vanishing result.

A priori the homotopy transfer also includes expressions such as that given, but with some Lie word in the Schouten bracket inserted into a Lie-input, suitably inserspersed with copies of δ^{-1}. These can not contribute because they do not include any B and δ^{-1} vanishes on something of form-degree 0. A priori we may also have terms such as that given, but precomposed with a composition of homotopies and terms $\mathcal{V}_{p,1}$ of the NCG_∞-action. However, the only part of the action with a single input in the associative color is the adjoint action ad_S, but then there is no room to place the necessary copy of B. $\qquad\square$

Note that the fibrewise $\sum_{n\geq 1} F_{n,1}$ is given by

$$e^{-(\gamma_{(1)},\gamma)} = \circ - \gamma_{(1)} + \frac{1}{2}\big([\gamma_{(1)},\gamma_{(1)}] + \gamma_{(1)} \bullet \gamma\big)$$

$$+ \frac{1}{6}\big(-[\gamma_{(1)},[\gamma_{(1)},\gamma_{(1)}]] - [\gamma_{(1)},\gamma_{(1)} \bullet \gamma] - \gamma_{(1)} \bullet \gamma \bullet \gamma\big) + \dots$$

It follows from this that $\overline{F}_{0,1} = id$ and the lowest order term beyond that is

$$\overline{F}_{k-1,1} = -p \circ D_{\gamma_{(1)}} \circ j^{\otimes k-1} \otimes j,$$

if γ has k vertices.

Remark 7.2.4.2. The formula $\overline{F}_1 := \sum_{n\geq 0} \frac{\hbar^n}{n!} \overline{F}_{n,1}(\pi^{\otimes n},)$, with π a Poisson bivector, defines an isomorphism of complexes

$$\overline{F}_1 : (T_{\mathrm{poly}}(M)[[\hbar]], d_{\hbar\pi}) \to (T_{\mathrm{poly}}(M)[[\hbar]], d_{\hbar\gamma(\pi)}),$$

where $\hbar\gamma(\pi) := \overline{G}(\hbar\pi)$, that on the level of cohomology is an isomorphism of *algebras*.

Example 7.2.4.3. The simplest nontrivial cocycle in Kontsevich's graph complex is the tetrahedron graph, given schematically as:

For this choice of γ, and f a function, it is relatively easy to verify that

$$\overline{F}_1(f) = f + \hbar^3 C \nabla_i \pi^{jk} \nabla_k \pi^{lm} \nabla_m \pi^{ni} \nabla_j \nabla_l \partial_n f + O(\hbar^6),$$

where $C \in \mathbf{Q}$ is some combinatorial prefactor.

Remark 7.2.4.4. If $\pi = 0$ or if π is symplectic, then $\overline{F}_1 = id$. This follows from the observation that 7.2.4.1 implies that all terms of $\overline{F}_1$ except the identity have to involve a factor corresponding to a graph that has a vertex decorated by π. Since black vertices are at least trivalent that factor must involve a (covariant) derivative of π. However, the higher homotopies extending $\overline{F}_1$ to an A_∞ morphism are not all zero, even for a symplectic Poisson structure. This is true because in the higher homotopies there terms given by graphs with only white vertices, and graphs that have a single black vertex (e.g., the graph $\gamma_{(k-1)}$) which may be decorated by B.

CHAPTER 8

Graph-free Rossi-Willwacher Drinfel'd associators and multiple zeta values

A Drinfel'd associator is usually defined as a formal series $\Phi(x, y)$ in two noncommutative variables satisfying certain equations. Alternatively, one may also see it defined as a series $\Phi(t_{12}, t_{23})$ in the completion of the universal enveloping algebra of the Lie algebra of infinitesimal braids on three strands. For a long time only two examples of associators where known, the Knizhnik-Zamolodchikov associator Φ_{KZ} and its sibling the anti-Knizhnik-Zamolodchikov associator $\Phi_{\overline{KZ}}$, both introduced by Vladimir Drinfel'd [Drinfel'd 1991]. The set of Drinfel'd associators is a torsor for the Grothendieck-Teichmüller group and appear naturally in such diverse fields as knot theory and multiple zeta values. In 2009, Anton Alekseev and Charles Torossian [Alekseev and Torossian 2010] gave a construction which they conjectured gave a third example of an associator. Their conjecture was affirmatively answered shortly after in [Ševera and Willwacher 2011]. This third associator is since called the Alekseev-Torossian associator and denoted Φ_{AT}. Let us briefly remark on how it is constructed.

The components of Kontsevish's operad Graphs have subspaces, denoted $\mathsf{TCG}(n)$, that are spanned by graphs satisfying certain degree-restrictions, connectedness assumptions and cocycle conditions. The spaces $\mathsf{TCG}(n)$ are L_∞ algebras, equipped with quasi-isomorphic projections $\mathsf{TCG}(n) \to \mathfrak{t}_n$ onto the Lie algebra of the corresponding number

of infinitesimal braids. The formula

$$\sum_{\Gamma \in \mathsf{TCG}(3)} \vartheta^\Gamma \otimes \Gamma \in \Omega(\overline{C}_3(\mathbf{C}), \mathsf{TCG})$$

defines a flat L_∞ connection. Here ϑ^Γ is the form associated to Γ in Kontsevich's proof of the formality of the little disks operad [Kontsevich 1999]; it is defined as the fiber integral $\pi^! \theta^\Gamma$ of the form

$$\theta^\Gamma := \wedge_{(i,j) \in E_\Gamma} \frac{d \arg(x_j - x_i)}{2\pi}$$

along the projection $\pi : \overline{C}_{\#V_\Gamma}(\mathbf{C}) \to \overline{C}_3(\mathbf{C})$ that forgets all points except the three distinguished ones. Using the projection $\mathsf{TCG}(3) \to \mathfrak{t}_3$ gives a flat $\mathfrak{t}_3$-connection. Taking the holonomy along the path from 0 to 1 along the real axis defines a series $\Phi_{AT} \in \hat{U}(\mathfrak{t}_3)$. A geometric argument then proves it satisfies all the equations required from a Drinfel'd associator.

Degree zero cocycles in Kontsevich's graph complex should act on associators, since $H^0(\mathrm{GC}) \cong \mathfrak{grt}_1$ and associators are a torsor for the group GRT_1. The construction of the associator Φ_{AT} makes this action transparent, using the action of GC on Kontsevich's operad Graphs. The construction of Φ_{AT} also makes it transparent how associators fit into Kontsevich's graphically constructed formality morphisms, cf. [Ševera and Willwacher 2011]. But, the construction makes it difficult to extract the coefficients of Φ_{AT} as a series in two-noncommutative variables. The difficulty arises from fact that the projection $\mathsf{TCG}(3) \to \mathfrak{t}_3$ has a rather extensive kernel. Carlo Rossi and Thomas Willwacher recently made a remarkable preprint available [Rossi and Willwacher 2013], where they, among other things, construct a whole family of associators Φ_t. At $t = 0$ the associator equals the KZ associator (up to normalizing prefactors); at $t = 1/2$ it equals the AT associator; and at $t = 1$ it equals the $\overline{KZ}$ associator (up to normalizing prefactors). Their associators are constructed using the same graphical technique as that used to define Φ_{AT} and, hence, all associators in their family share the problem that it is somewhat difficult to extract the coefficients. This chapter explains how their construction can be modified, so as to side-step the use of a projection from graphs to Lie words and, as a result, obtain simpler explicit integral formulas for the coefficients of all the associators. All the steps we take are more or less implicit in the existing literature but we believe it is useful to cohesively put everything together. Let us summarize our central construction in the form of a theorem.

Let $\mathscr{L}$ denote the set of Lyndon words on the two letters $0, 1$ and for a Lyndon word λ, let x_λ denote the corresponding basis element of

the free Lie algebra $\mathfrak{lie}(x_0, x_1)$; thus, the bracket on this free Lie algebra is defined by structure constants $c_{\mu\nu}^\lambda$, for $\lambda, \mu, \nu \in \mathscr{L}$.

Theorem 8.0.4.5. Define differential forms

$$\vartheta_t^0 := \frac{1-t}{2\pi i} d\log(w) + \frac{t}{2\pi i} d\log(\bar{w});$$

$$\vartheta_t^1 := \frac{1-t}{2\pi i} d\log(1-w) + \frac{t}{2\pi i} d\log(1-\bar{w}),$$

and for a Lyndon word λ of length greater than one, we recursively define

$$\vartheta_t^\lambda = \frac{1}{2\pi i} \sum_{\mu < \nu} c_{\mu\nu}^\lambda \pi_w^! \left(\left((1-t)\frac{dw}{w-z} - t\frac{d\bar{w}}{\bar{w}-\bar{z}} \right) \pi_z^* \vartheta_t^\mu \pi_z^* \vartheta_t^\nu \right).$$

Here $\pi_x : \operatorname{Conf}_2(\mathbf{C} \setminus \{0, 1\}) \to \mathbf{C} \setminus \{0, 1\}$ is the projection which forgets the point not equal to x and a superscript ! means integration along the fiber.

Then $\Theta_t := \sum_\lambda \vartheta_t^\lambda x_\lambda$ is a flat connection on $\mathbf{C} \setminus \{0, 1\}$ and its holonomy along the path from 0 to 1 along the real axis is Rossi's and Willwacher's Drinfel'd associator Φ_t.

There is some regularization procedure needed to define the holonomy of the connection but this regularization problem is solved by Rossi and Willwacher. It essentially repeats *mutatis mutandum* the regularization of the Knizhnik-Zamolodchikov connection [Knizhnik and Zamolodchikov 1982].

The main application of knowing explicit formulas for the coefficients in the family of associators is that it allows us to write down an equally explicit associated family of evaluations ($\mathbf{Q}$-algebra morphisms) $\zeta_t : FZ \to \mathbf{C}$ on the algebra of formal multiple zeta values, by using the fundamental result of [Furusho 2011]. Papers about multiple zeta values tend to include a lot of combinatorics with Lyndon words, iterative constructions, etc., and our graph-free formulation of the Rossi-Willwacher associators is specifically tailored to fit in such a context.

Proposition 8.0.4.6. Define $\zeta_t(n_1, \dots, n_d)$ to be the value on the formal multiple zeta value $\mathbf{z}(n_1, \dots, n_d)$ under the evaluation defined by Φ_t. Then $\zeta_t(n_1, \dots, n_d) = (-1)^{n_1 + \cdots + n_d} \zeta_{1-t}(n_1, \dots, n_d)$; so in particular, $\zeta_{1/2}$ is zero on all multiple zeta values of odd weight.

The Alekseev-Torossian associator was known to be even before it was known to actually be a Drinfel'd associator [Alekseev and Torossian 2010]. The main result of [Furusho 2011] then immediately implies that

$\zeta_{1/2}$ vanishes on multiple zeta values of odd weight, so the second part of the proposition above is not really new.

Another application is that we can give a slightly more explicit formula for the conjectured generators τ_{2j+1} of $\mathfrak{grt}_1$ defined by Rossi and Willwacher. Lastly, our formulas show that the differential forms entering the integrals giving the coefficients (both of the associators and Grothendieck-Teichmüller elements) satisfy certain recurrence relations which are obscured in the original graphical construction. These new recurrence relations could, maybe, simplify the calculations of the integrals used to compute the coefficients, help in recognizing which coefficients are rational, or etc.

The material in this chapter has not been previously published.

8.1   Graphs and Lie words.

8.1.1   Internally connected graphs and Lie graphs.

This section contains some overlap with earlier chapters but we spell out the details, for completeness. Let $fgraphs_{n,k}^d$ be the set of graphs Γ with d (undirected) edges, no tadpoles (edges connected at both ends to the same vertex), no legs, $n + k$ vertices and the following extra data:

- The edges are ordered by a specified isomorphism $E(\Gamma) \cong [d]$.

- n of the vertices are labelled "white" and k of the vertices are labelled "black".

Define $graphs_{n,k}^d$ to be the subset consisting only of those graphs that have no tadpoles (edges with both ends connecting to the same vertex), have no connected component with only black vertices, and have all black vertices $\geq$ 3-valent. We define

$$\mathsf{graphs}(n) := \bigoplus_{d \geq 0, k \geq 1} \mathbf{Q}(graphs_{n,k}^d)_{\Sigma_k}[d - 2k] \otimes_{\Sigma_d} sgn_d.$$

We remark that $\mathsf{graphs}(n) \subset \mathsf{Graphs}(n)$ sits inside Kontsevich's operad of graphs with black and white vertices as the subspace spanned by finite sums. These subspaces form a dg suboperad; in particular, $\mathsf{graphs}(n)$ is for each $n \geq 1$ a complex under the edge-insertion differential

$$\partial \Gamma = [\begin{smallmatrix}\bullet\\\circ\end{smallmatrix}, \Gamma] + \Gamma \bullet \begin{smallmatrix}\bullet\\\bullet\end{smallmatrix}.$$

For a graph $\Gamma \in graphs_{n,k}^d$ we define the corresponding internal graph, to be denoted Γ^{int}, to be the graph with legs obtained by deleting the

white vertices and turning edges previously connected to white vertices into legs.

Definition 8.1.1.1. A graph Γ is **internally connected** if either (i) it consists of a single edge connecting two white vertices or (ii) it has no edge connecting two white vertices and the associated internal graph Γ^{int} is connected.

Define $\mathsf{ICG}(n)[1] \subset \mathsf{graphs}(n)$ to be the subspace spanned by all graphs that are internally connected. Clearly,

$$\mathsf{graphs}(n) = S(\mathsf{ICG}(n)[1]),$$

since any graph can be regarded as a superposition of internally connected graphs. The differential ∂ is a coderivation of the coalgebra $S(\ )$, hence, by definition, $\mathsf{ICG}(n)$ is an L_∞ algebra.

Definition 8.1.1.2. We define the truncation $\mathsf{TCG}(n) \subset \mathsf{ICG}(n)$ as follows. $\mathsf{TCG}^{\leq -1} := \mathsf{ICG}^{\leq -1}$, $\mathsf{TCG}^{\geq 1} := 0$ and TCG^0 is the degree 0 cocycles of ICG.

It is rather immediate that $\mathsf{TCG}(n)$ is an L_∞ subalgebra.

Definition 8.1.1.3. Define $\mathsf{L}_n \subset \mathsf{TCG}(n+1)$ to be the subspace spanned by the graphs that have the white vertex labelled by $n + 1$ of valency one or higher. For a graph $\Gamma \in \mathsf{L}_n$, we call the vertex $n + 1$ the root vertex.

Again, it is obvious that $\mathsf{L_n}$ is an L_∞ subalgebra. Let $\mathfrak{lie}_n$ denote the free Lie algebra over $\mathbf{Q}$ generated by variables $y_1, \ldots, y_n$.

Definition 8.1.1.4. Define $(\ ) : \mathsf{L}_n \to \mathfrak{lie}_n$ by the following recipe. Take a graph $\Gamma \in \mathsf{L}_n$. If the root vertex has valency exactly one and Γ^{int} is a tree with all vertices exactly trivalent, then Γ^{int} is canonically a *rooted* trivalent tree with the input legs (legs not equal to the root) labelled by $[n]$. Regard the rooted tree as an operadic composition diagram for a Lie bracket and insert y_i into every input leg that is labelled by i. This produces a Lie word $(\Gamma) \in \mathfrak{lie}_n$. If the root vertex of Γ has valency ≥ 2 or if Γ^{int} is not a trivalent tree, then we define $(\Gamma) := 0$.

Proposition 8.1.1.5. The map $(\ ) : \mathsf{L}_n \to \mathfrak{lie}_n$ is a quasi-isomorphism.

The proof of this statement is sketched at the end of [Ševera and Willwacher 2011].

Proof. The only graphs in L_n that do not have a black vertex are the n single-edge graphs on edges $\{n+1, i\}$, $1 \leq i \leq n$. these are mapped to the corresponding generators y_i. Hence we may redefine L_n to be spanned by graphs with ≥ 1 black vertex, and prove that this is quasi-isomorphic to the space $\mathfrak{lie}_n^{\geq 2}$ spanned by Lie words containing ≥ 1 bracket.

Introduce the following two-term filtration on L_n. Define $\mathcal{F}^1\mathsf{L}_n$ to be spanned by all graphs that have a univalent root vertex and define $\mathcal{F}^2\mathsf{L}_n$ to be spanne by all graphs that have a root vertex of valency ≥ 2. Consider the corresponding spectral sequence, which has as only nonzero terms

$$E_0^{-1,q} = \mathcal{F}^1\mathsf{L}_n^{q-1}, \quad E_0^{-2,q} = \mathcal{F}^2\mathsf{L}_n^{q-2}.$$

The differential, d_0, is zero on $E_0^{-1,q}$, while on $E_0^{-2,q}$ is has the schematic form
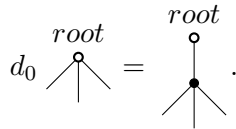
It follows that E_1 can be identified with the subspace of $\mathcal{F}^1\mathsf{L}_n$ that is spanned those graphs Γ with the property that Γ^{int} becomes disconnected when one removes the root and the unique edge connected to the root. Alternatively put, $\Gamma \in E_1$ if contracting the root edge gives a

$$\Gamma' = \Gamma_1 \wedge \cdots \wedge \Gamma_r \in S(\mathsf{L}_n[1]).$$

Using this, we can identify $E_1 \cong S^{\geq 2}(\mathsf{L}_n[1])$, with differential induced by that on L_n. By induction on the number of vertices we may conclude $E_2 = S^{\geq 2}(\mathfrak{lie}_n[1])$, equipped with the Chevalley-Eilenberg differential. The cohomology of the full Chevalley-Eilenberg complex $S(\mathfrak{lie}_n[1])$ is $\mathbf{Q}^n[1]$ since the algebra is free. Hence

$$E_3 = S^2(\mathfrak{lie}_n[1])/d_3(S^2(\mathfrak{lie}_n[1])).$$

We can identify this as formal antisymmetric bracketings $[a, b]$ of Lie words $a, b \in \mathfrak{lie}_n$ modulu relations $[[x, y], z] - [[x, z], y] + [[y, z], x]$, where the outer brackets are formal bracketings and the inner ones are actual Lie brackets. The relations say exactly that the formal bracketings satisfy the Jacobi identity, i.e., can be identified with actual Lie brackets. Hence $E_3 \cong \mathfrak{lie}_n^{\geq 2}$. $\square$

The morphism $\Gamma \mapsto (\Gamma)$ is actually a quasi-isomorphism of L_∞ algebras. To prove that we first introduce a binary operation

$$\curlywedge : graphs_{n,k}^d \times graphs_{n,k'}^{d'} \to graphs_{n,k+k'+1}^{d+d'+1}.$$

Take $\Gamma \in graphs_k^d$ and $\Gamma' \in graphs_{n,k'}^{d'}$. First form the disjoint union $\Gamma \sqcup \Gamma'$. Then form a graph $\Gamma \wedge \Gamma'$ by superimposing the respective white vertices (the two vertices labelled $i \in [n+1]$ are identified into a single new white vertex labelled i) and ordering the edge-set lexicographically by $E(\Gamma) < E(\Gamma')$. Finally, recolor the vertex labelled by $n+1$ black, and connect that black vertex to a new white vertex labelled $n+1$ via a new edge e, which we order to be the first edge. The resulting graph is $\Gamma \curlywedge \Gamma'$. Below is a schematic picture.

$$\Gamma \curlywedge \Gamma' = \quad \overset{\displaystyle root}{\underset{\Gamma \qquad \Gamma'}{\wedge}} \quad .$$
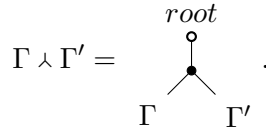
Remark 8.1.1.6. The operation $\curlywedge$ induces a bilinear operation (of degree 0) on L_n.

Proposition 8.1.1.7. The map $(\ ) : \mathsf{L}_n \to \mathfrak{lie}_n$ is a quasi-isomorphism of L_∞ algebras.

Proof. Recall the edge-splitting differential ∂ on $\mathsf{graphs}(n)$. The ℓ-ary bracket $\lambda_\ell(\Gamma_1, \ldots, \Gamma_\ell)$ on L_n is given by projecting $\partial(\Gamma_1 \wedge \cdots \wedge \Gamma_\ell)$ onto L_n. This means that

$$(\lambda_2(\Gamma_1, \Gamma_2)) = (\Gamma_1 \curlywedge \Gamma_2),$$

since all other edge-splittings of $\Gamma_1 \wedge \Gamma_2$ have a root vertex of valency ≥ 2. Moreover, the black vertex attached to the root vertex in $\Gamma \curlywedge \Gamma'$ is trivalent if and only if Γ_1 and Γ_2 both have a univalent root vertex. Since $(\Gamma) = 0$ unless Γ^{int} is a trivalent tree, this allows us to conclude inductively that $(\Gamma_1 \curlywedge \Gamma_2) = 0$ unless Γ_1 and Γ_2 are generated by the single-edged graphs $\{i, n+1\}$ ($i \in [n]$) under $\curlywedge$. These single-edge graphs are mapped to the generators y_i. Thus

$$(\lambda_2(\Gamma_1, \Gamma_2)) = [(\Gamma_1), (\Gamma_2)].$$

Note $(\lambda_\ell(\Gamma_1, \ldots, \Gamma_\ell)) = 0$ if $\ell \geq 3$ because all terms in $\partial(\Gamma_1 \wedge \cdots \wedge \Gamma_\ell)$ must contain a vertex which is more than trivalent. $\qquad \square$

8.1.2 The Lyndon basis.

In this subsection we recollect some known results about Lyndon words and free Lie algebras.

Define $\mathscr{W}$ to be the set of (arbitrarily long) words in the letters 0 and 1. We order it using the lexicographical ordering defined by $0 < 1$ and denote the length of a word w by $|w|$. Recall that a **Lyndon word** is a nonempty word λ that either has length one or has the property that if $\lambda = \mu\nu$ with $\nu \neq \emptyset$, then $\lambda < \nu$. Define $\mathscr{L} \subset \mathscr{W}$ to be the set of Lyndon words. The set of nontrivial factorizations $\lambda = \mu'\nu'$ $(\mu', \nu' \neq \emptyset)$ of a Lyndon word contains an element $\mu\nu$ with ν as small as possible, i.e. $\nu < \nu'$ for all other factorizations $\mu'\nu'$. This factorization $\lambda = \mu\nu$ is called the **standard factorization**. A fundamental theorem about Lyndon words is that if $\lambda = \mu\nu$ is the standard factorization of a Lyndon word λ, then both μ and ν are themselves Lyndon words.

Define $\mathfrak{l}$ to be the free Lie algebra over $\mathbf{Q}$ on generators x_0 and x_1. Define a function $\mathscr{L} \to \mathfrak{l}, \lambda \mapsto x_\lambda$ inductively by $x_0 := x_0$, $x_1 := x_1$ and for a Lyndon word λ with standard factorization $\mu\nu$, $x_\lambda := [x_\mu, x_\nu]$. For example,

$$x_{001} = [x_0, x_{01}] = [x_0, [x_0, x_1]].$$

Note that $|\lambda| - 1$ equals the number of brackets in the Lie word x_λ. The set of Lie words $\{x_\lambda \mid \lambda \in \mathscr{L}\}$ is a vector space basis for $\mathfrak{l}$.

The Lyndon basis and the Poincaré-Birkhoff-Witt (PBW) isomorphism combine to give a basis of the universal enveloping algebra $U(\mathfrak{l})$ of $\mathfrak{l}$. The definition of the universal enveloping algebra leads to the identification, as algebras, of $U(\mathfrak{l})$ with the algebra of noncommutative polynomials in two variables, $\mathbf{Q}\langle x_0, x_1 \rangle$. The PBW isomorphism gives an identification as vector spaces between $U(\mathfrak{l})$ and the commutative polynomial ring $\mathbf{Q}[x_\lambda]$ generated by Lyndon words. Since Lyndon words are ordered lexicographically we can fix the convention that monomials in $\mathbf{Q}[x_\lambda]$ are always written with the smallest element to the right:

$$x_{\lambda_1} \dots x_{\lambda_p}, \quad \lambda_1 \geq \dots \geq \lambda_p.$$

There is a simple algorithm for rewriting any noncommutative monomial in $\mathbf{Q}\langle x_0, x_1 \rangle$ in terms of expressions like the one above, cf. [Melanchon and Reutenauer 1989].

Define $U(\mathfrak{l})' = \mathbf{Q}\langle x^0, x^1 \rangle$ to be the finite dual of $U(\mathfrak{l})$. It is equipped with a product dual to the coproduct on $U(\mathfrak{l})$. This product is known as the shuffle product and traditionally denoted ш. The algebra $U(\mathfrak{l})'$ is freely generated by Lyndon dual basis elements $x^\lambda \in \mathfrak{l}^*$ under the ш-product.

Given a word $w = a_1 \dots a_r$ with letters $a_j \in \{0, 1\}$, denote $\underline{x}^w := x^{a_1} \dots x^{a_r} \in U(\mathfrak{l})'$ and $\underline{x}_w := x_{a_1} \dots x_{a_r} \in U(\mathfrak{l})$. Note that $x_\lambda \neq \underline{x}_\lambda$ for a Lyndon word λ, e.g. $x_{01} = [x_0, x_1] = x_0 x_1 - x_1 x_0 \in U(\mathfrak{l})$ but $\underline{x}_{01} = x_0 x_1$.

The paper by Melanchon and Reutenauer [Melanchon and Reutenauer 1989] proves the identity

$$\sum_w \underline{x}^w \otimes \underline{x}_w = \prod_\lambda \exp(x^\lambda \otimes x_\lambda) \in \hat{U}(\mathfrak{l})' \otimes \hat{U}(\mathfrak{l}),$$

where the product over Lyndon words is taken in decreasing order and $\hat{U}(\mathfrak{l})'$ and $\hat{U}(\mathfrak{l})$ are the completed algebras of noncommutative formal series. Spelled out in more detail the exponential above is the formal sum

$$\sum_{\lambda_1 > \cdots > \lambda_k} \sum_{i_1,\ldots,i_k \geq 1} \frac{1}{i_1!\ldots i_k!} (x^{\lambda_1})^{\text{⧢} i_1} \text{⧢} \ldots \text{⧢} (x^{\lambda_k})^{\text{⧢} i_k} \otimes x_{\lambda_1}^{i_1} \ldots x_{\lambda_k}^{i_k}.$$

Remark 8.1.2.1. The monomials $\underline{x}^\lambda$ also constitute a multiplicative basis of $U(\mathfrak{l})'$, called the Radford basis [Radford 1979]). The Radford basis is better suited for computations relating to multiple zeta values, since it makes regularization easier to describe. We will not develop regularization, and will, accordingly, not use the Radford basis.

8.1.3 Lyndon words and graphs.

Based on the Lyndon basis it is possible to write the Lie bracket on $\mathfrak{l}$ in the form

$$[x_\mu, x_\nu] = \sum_{\lambda \in \mathscr{L}} c_{\mu\nu}^\lambda x_\lambda.$$

For reasons that will be clear later, consider the white vertices of L_2 as labelled by 0, 1 and w, with w labelling the root vertex. Using the structure constants $c_{\mu\nu}^\lambda$, we define a function $\mathscr{L} \to \mathsf{L}_2, \lambda \mapsto \Gamma^\lambda$ inductively by letting Γ^0 be the graph with only the edge $\{0, w\}$, Γ^1 be the graph with only the edge $\{1, w\}$ and, for a λ of length ≥ 2,

$$\Gamma^\lambda := \sum_{\mu < \nu \in \mathscr{L}} c_{\mu\nu}^\lambda \Gamma^\mu \curlywedge \Gamma^\nu.$$

This function is clearly injective. We can regard the function as a map $\mathfrak{l}^* \to \mathsf{L}_2^*$, since Lyndon words can be regarded as a basis $\{x^\lambda\}$ of the dual space $\mathfrak{l}^*$

Remark 8.1.3.1. The map $\lambda \mapsto \Gamma^\lambda$ is the map on dual spaces induced by $(\ ) : \mathsf{L}_2 \to \mathfrak{lie}_2$, up to the obvious identification $\mathfrak{l} \cong \mathfrak{lie}(y_1, y_2)$.

153

Below are some examples.

$$\Gamma^{001} = \quad , \quad \Gamma^{011} = \quad , \quad \Gamma^{0011} = \quad + \quad .$$

The fact that Γ^{0011} is a sum of two terms is a consequence of the equality $[x_0, x_{011}] = x_{0011} = [x_{001}, x_1]$ in the free Lie algebra.

8.1.4 Special derivations and infinitesimal braids.

The splitting $\mathsf{TCG}(n+1) = \mathsf{TCG}(n) \oplus \mathsf{L}_n$ is a semi-direct product

$$\mathsf{TCG}(n+1) = \mathsf{TCG}(n) \ltimes \mathsf{L}_n$$

of L_∞ algebras. It follows from the quasi-isomorphism $\mathsf{L}_n \simeq \mathfrak{lie}_n$ that

$$H(\mathsf{TCG}(n+1)) = H(\mathsf{TCG}(n)) \ltimes \mathfrak{lie}_n.$$

Inductively, this proves that $H(\mathsf{TCG}(n)) \cong \mathfrak{t}_n$, cf. [Ševera and Willwacher 2011]. Here $\mathfrak{t}_n$ is the so called Lie algebra of infinitesimal braids on n strands. It has generators $t_{ij} = t_{ji}$, $1 \le i \ne j \le n$ and relations

$$[t_{ij}, t_{kl}] = 0 = [t_{ij} + t_{jk}, t_{ik}]$$

if $\{i, j\} \cap \{k, l\} = \emptyset$. The relations imply

$$\mathfrak{t}_{n+1} = \mathfrak{t}_n \rtimes \mathfrak{lie}(t_{1,n+1}, \ldots, t_{n,n+1}),$$

and hence we can recursively use $\mathsf{TCG}(n+1) = \mathsf{TCG}(n) \ltimes \mathsf{L}_n$ to extend our L_∞ quasi-isomorphism ( ) to an L_∞ quasi-isomorphism

$$(\ ) : \mathsf{TCG}(n) \to \mathfrak{t}_n.$$

Definition 8.1.4.1. A **tangential derivation** of $\mathfrak{lie}_n$ is a derivation u of $\mathfrak{lie}_n$ which acts on each y_i by an inner derivation $y_i \mapsto [y_i, u_i]$, $u_i \in \mathfrak{lie}_n$. Thus, a tangential derivation is uniquely represented by an n-tuple $(u_1, \ldots, u_n)$, where for each $i = 1, \ldots, n$ the term of order 1 with respect to y_i in $u_i \in \mathfrak{lie}_n$ is zero. The space of tangential derivations of $\mathfrak{lie}_n$ is denoted $\mathfrak{tder}_n$. It is a Lie algebra under the bracket on derivations.

A tangential derivation u is called **special** if $u(\sum_{i=1}^n y_i) = 0$. The special derivations form a Lie subalgebra $\mathfrak{sder}_n$ of the tangential derivations.

154

Remark 8.1.4.2. One may identify the Lie algebra of infinitesimal braids with the subalgebra of $\mathfrak{sder}_n$ generated by

$$t_{ij} = (0, \ldots, 0, y_j, 0, \ldots, 0, y_i, 0, \ldots, 0) \in \mathfrak{sder}_n,$$

where y_j is in the ith position and y_i is in the jth position, cf. [Alekseev and Torossian 2012].

The inclusion $\mathfrak{t}_n \hookrightarrow \mathfrak{sder}_n$ has a graphical counterpart wich is important in the construction by Rossi and Willwacher. Define $\mathsf{TCG}(n)_{\mathrm{tree}} := \mathsf{TCG}(n)/\mathsf{TCG}(n)_{\mathrm{loop}}$ to be the quotient by modding out all graphs that contain a cycle, i.e., the quotient may be considered as spanned by TCG-graphs that are internal (but not necessarily trivalent) trees. It inherits an L_∞ algebra structure from $\mathsf{TCG}(n)$. Its degree 0 cohomology can be identified with $\mathfrak{sder}_n$, via the following combinatorial procedure due to [Alekseev and Torossian 2010].

Take a degree zero Γ in $\mathsf{TCG}(n)_{\mathrm{tree}}$. For degree reasons Γ^{int} must be a trivalent tree. Consider its legs as labelled by remembering which white vertex they where attached to. Assume that the white vertex i of Γ has valency d_i and label the edges attached to i by the set $[d_i]$. For each i and each $1 \leq r \leq d_i$ we obtain a way of regarding Γ^{int} as a rooted tree, by declaring the leg corresponding to r to be the root. The remaining legs now become input legs. The resulting rooted labeled tree defines a word $u_{i,r}$ in $\mathfrak{lie}_n$, just like we how we define the word Γ corresponding to a $\Gamma \in \mathsf{L}_n$. Define the tangential derivation u by

$$u := (u_1, \ldots, u_n), \quad u_i := \sum_{r=1}^{d_i} u_{i,r}.$$

The statement of Alekseev and Torossian [Alekseev and Torossian 2010] is that u is a special derivation. If $\Gamma \in \mathsf{TCG}(n)_{\mathrm{tree}}$ does not have degree 0, then the associated special derivation is defined to be zero.

Lemma 8.1.4.3. The following diagram commutes.

$$
\begin{array}{ccccc}
\mathsf{L}_{n-1} & \longrightarrow & \mathsf{TCG}(n) & \longrightarrow & \mathsf{TCG}(n)_{\mathrm{tree}} \\
{\scriptstyle ()} \downarrow & & \downarrow & & \downarrow {\scriptstyle u} \\
\mathfrak{lie}_{n-1} & \longrightarrow & \mathfrak{t}_n & \longrightarrow & \mathfrak{sder}_n.
\end{array}
$$

Note that $\mathsf{L}_{n-1} \to \mathsf{TCG}(n) \to \mathsf{TCG}(n)_{\mathrm{tree}}$ is injective on degree zero.

Proof. (Sketch.) The left square commutes since $\mathsf{TCG}(n) \to \mathfrak{t}_n$ was defined recursively using the semidirect product decomposition. By an analogous recursive argument we conclude that it is enough to check that the outer square commutes. First note that the outer square commutes on the single-edged graphs $e_{i,n} := \{i, n\}$. There are two possible orientations of these, corresponding to

$$(0, \ldots, y_n, \ldots, 0) + (0, \ldots, y_i) = t_{in} \in \mathfrak{sder}_n.$$

We noted in 8.1.1.7, above, that $(\Gamma) = 0$ unless $\Gamma \in \mathsf{L}_{n-1}$ can be generated from the single-edged graphs $e_{i,n}$ using the $\curlywedge$-operation. As $(\Gamma_1 \curlywedge \Gamma_2) = [(\Gamma_1), (\Gamma_2)]$ it is by induction enough to show that $e_{i,n} \curlywedge e_{j,n}$ is mapped to $[t_{in}, t_{jn}] \in \mathfrak{sder}_n$. There are three ways of orienting

corresponding to the sum

$$(0, \ldots, [y_n, y_j], \ldots, 0) + (0, \ldots, [y_i, y_n], \ldots, 0) + (0, \ldots, [y_i, y_j])$$

of tangential derivations. (Here $[y_n, y_j]$ is in the ith position and $[y_i, y_n]$ is in the jth.) One verifies that this sum equals the bracket $[t_{in}, t_{jn}]$ in $\mathfrak{sder}_n$. $\qquad\square$

8.1.5 Lyndon words and special derivations.

We here define two different maps

$$\gamma_0, \gamma_1 : \mathscr{L}_{\geq 3} \to \mathsf{TCG}(2)^*_{\text{tree}}$$

from Lyndon words of length ≥ 3. We define them as follows. Let $\tilde{\gamma}_0^\lambda$ be the (sum of) graph(s) obtained by identifying identifying the vertices 0 and w in Γ^λ into a single vertex 0. This sum of graphs possibly contains graphs with a double edge. Define γ_0^λ to be $\tilde{\gamma}_0^\lambda$ minus all terms in $\tilde{\gamma}_0^\lambda$ containing a double edge; i.e., the part of $\tilde{\gamma}_0^\lambda$ not containing double edges. Below are two examples.

We define γ_1^λ just like we defined γ_0^λ, except we now collapse w and 1 (into a new 1). For example:

$$\gamma_1^{0011} = \quad , \quad \gamma_1^{001} = \quad .$$
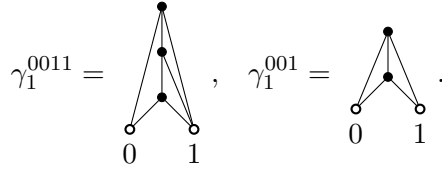
It is clear from these examples that the graphs we construct possess certain symmetries, relating graphs associated to different Lyndon words to each other. To phrase the most crucial symmetry we recall a fact about the involution of the free Lie algebra on two variables that exchanges the generators. Let $\rho : \{0,1\} \to \{0,1\}$ be the self-bijection mapping 0 to 1 and 1 to 0. It induces a self-bijection ρ of the set of words $\mathscr{W}$. Given a word $w = a_1 \dots a_p \in \mathscr{W}$, let $\overleftarrow{w}$ be the reverese word $a_p \dots a_1$ and put $i(w) := \rho(\overleftarrow{w})$. The operation $w \mapsto i(w)$ restricts to an involution on the set of Lyndon words. (For example, $i(00101) = 01011$.) This involution has the following interpretation in terms of the free Lie algebra: Let $\varrho : \mathfrak{l} \to \mathfrak{l}$ be the map that sends a Lie word $a(x_0, x_1)$ to the word $(\varrho a)(x_0, x_1) := a(x_1, x_1)$. Then, for all Lyndon words λ,

$$\varrho(x_\lambda) = (-1)^{|\lambda|-1} x_{i(\lambda)}.$$

Denote by ϱ^* the map on linear combinations of graphs that interchanges the vertices 0 and 1 and multiplies a graph with k internal vertices by $(-1)^k$. Then, clearly

Lemma 8.1.5.1. For all Lyndon words λ of length ≥ 3 the equation $\varrho^*(\gamma_1^\lambda) = (-1)^{|\lambda|-1}\gamma_0^{i(\lambda)}$ holds in $\mathsf{TCG}(2)_{\text{tree}}^*$.

8.1.6   Drinfel'd associators.

Let $\mathbf{k} \supset \mathbf{Q}$ be a field containing the rationals and let $\kappa \in \mathbf{k}$.

Definition 8.1.6.1. A κ-**Drinfel'd associator** over $\mathbf{k}$ is an element $\Phi \in \mathbf{k}\langle\langle x_0, x_1 \rangle\rangle$ satisfying the following axioms.

  (i) It is group-like, $\Delta\Phi = \Phi \hat{\otimes} \Phi$.

  (ii) $\Phi(x_1, x_0) = \Phi(x_0, x_1)^{-1}$.

  (iii) $e^{\kappa a/2}\Phi(c, a)e^{\kappa c/2}\Phi(b, c)e^{\kappa b/2}\Phi(a, b) = 1$ whenever $a + b + c = 1$.

(iv) It satisfies the pentagon equation

$$\Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23})$$
$$= \Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13}, +t_{23}, t_{34})$$

in the completed universal enveloping algebra of $\mathfrak{t}_4$.

The set of κ-Drinfel'd associators over $\mathbf{k}$ is denoted $Assoc_\kappa(\mathbf{k})$.

The Grothendieck-Teichmüller group $GRT_1(\mathbf{k})$ can be defined as $Assoc_0(\mathbf{k})$. The formula

$$(\Phi \cdot \Psi)(x_0, x_1) := \Phi(\Psi(x_0, x_1)x_0\Psi(x_1, x_0), x_1)\Psi(x_0, x_1),$$

defines both a group multiplication on GRT_1 and a right action of it on the set of associators.

8.2 Configuration spaces and differential forms.

For $n \geq 2$ and a manifold M we denote by $\mathrm{Conf}_n(M)$ the space of all injective maps $x : [n] \to M$, considered as a submanifold of M^n. Define $\mathfrak{M}_{0,n+1} := \mathrm{Conf}_{n-2}(\mathbf{C}\setminus\{0,1\})$. The space $\mathrm{Conf}_n(\mathbf{C})$ has a (real) Fulton-MacPherson-Axelrod-Singer compactification $\overline{\mathrm{Conf}}_n(\mathbf{C})$. Let $\overline{\mathfrak{M}}_{0,n+1}$ be the closure of the embedding $\mathfrak{M}_{0,n+1} \to \overline{\mathrm{Conf}}_n(\mathbf{C})$ sending $(w_1, \ldots, w_{n-2})$ ($w_i \neq 0, 1, w_j$) to $(x_1, \ldots, x_n) = (0, w_1, \ldots, w_{n-2}, 1)$. (Note that this is not the Deligne-Mumford compactification.)

Given $i, j \in [n]$ and a variable t, let

$$\theta_t^{ij} := \frac{1-t}{2\pi i}d\log(x_j - x_i) - \frac{t}{2\pi i}d\log(\bar{x}_j - \bar{x}_i) \in \mathscr{E}^1(\mathrm{Conf}_n(\mathbf{C})).$$

Here we use $\mathscr{E}$ to denote the analytical de Rham complex. Pull it back to a form on $\mathfrak{M}_{0,n+1}$. It can alternatively be written

$$\frac{1}{2\pi}d\arg(x_j - x_i) + \frac{1-2t}{2\pi i}d\log|x_j - x_i|.$$

Given a graph Γ with 3 white vertices, k black vertices and d edges, we define a differential form θ_t^Γ on $\mathfrak{M}_{4+k}$ as follows. Label the white vertices according to the order-preserving bijection $\{1 < 2 < 3\} \cong \{0 < w < 1\}$ and label the black vertex i ($1 \leq i \leq k$) by z_i. Thus, the set of vertices is labelled by the list

$$(0, w, z_1, \ldots, z_k, 1).$$

Regard this formally as a point $(x_1, \ldots, x_{k+3}) \in \text{Conf}_{k+3}(\mathbf{C})$. If under this identification the edge $e \in E(\Gamma)$ is $\{i, j\}$, then we put $\theta_t^e := \theta_t^{ij}$, and

$$\theta_t^\Gamma := \wedge_{e \in E(\Gamma)} \theta_t^e \in \mathscr{E}^d(\mathfrak{M}_{4+k}).$$

The point-forgetting projection $\pi : \mathfrak{M}_{0,4+k} \to \mathfrak{M}_{0,4}$ that only remembers $(0, w, 1)$ induces a projection

$$\pi : \overline{\mathfrak{M}}_{0,4+k} \to \overline{\mathfrak{M}}_{0,4}.$$

Define $\vartheta_t^\Gamma := \pi^!(\theta_t^\Gamma)$ to be the fiber integration of the form previously defined. It is a nontrivial analytical fact, proved in [Rossi and Willwacher 2013], that the fiber integration converges for all graphs $\Gamma \in \mathsf{TCG}(2)^*_{\text{tree}}$ of degree zero. Given a Lyndon word λ we define define an associated differential form by

$$\vartheta_t^\lambda := \vartheta_t^{\Gamma^\lambda} \in \mathscr{E}^1(\mathfrak{M}_{0,4}).$$

This defines a map of graded algebras

$$\vartheta_t : C_{\text{CE}}(\mathfrak{l}) \to \mathscr{E}(\mathfrak{M}_{0,4}), x^{\lambda_1} \wedge \cdots \wedge x^{\lambda_p} \mapsto \vartheta_t^{\lambda_1} \wedge \ldots \vartheta_t^{\lambda_p}.$$

Proposition 8.2.0.2. The map $\vartheta_t : C_{\text{CE}}(\mathfrak{l}) \to \mathscr{E}(\mathfrak{M}_{0,4})$ is a morphism of dg algebras.

Proof. Take a Lyndon word λ of length ≥ 2. Recall that

$$\Gamma^\lambda = \sum_{\mu < \nu} c_{\mu\nu}^\lambda \Gamma^\mu \curlywedge \Gamma^\nu.$$

If λ has length two or more then the vertex w is connected to a single edge $e_1 = \{w, z_1\}$ The form $d\vartheta_t^\lambda$ has the following appearance (see [Rossi and Willwacher 2013])

$$d\vartheta_t^\lambda = \sum_{e \in E(\Gamma^\lambda)} \pm \pi^!(\theta_t^{\Gamma^\lambda/e}),$$

where Γ^λ/e is a the graph formed by contracting the edge e to a single vertex. Contracting the edge e_1 of Γ^λ gives exactly

$$\sum_{\mu < \nu} c_{\mu\nu}^\lambda \Gamma^\mu \wedge \Gamma^\nu,$$

by the definition of $\curlywedge$. It follows that what we need to show is that the differential form associated to

$$d\Gamma^\lambda - \Gamma/e_1 = - \sum_{e \in E(\Gamma^\lambda) \setminus \{e_1\}} (-1)^{|e|} \Gamma/e$$

vanishes. This follows from degree reasons. The form associated to $d\Gamma^\lambda - \Gamma/e_1$ should be a 2-form. However, the vertex w has valency 1 in each of the graphs Γ/e ($e \neq e_1$). This means that the none of the indecomposable tensors in the form corresponding to Γ/e can depend on both dw and $d\bar{w}$, so the form is zero.

If λ has length one, then it is either the word 0 or the word 1. In either case the associated differential form is closed, as are the corresponding words in $C_{\mathrm{CE}}(\mathfrak{l})$. $\qquad\qquad\square$

Remark 8.2.0.3. As is implicit in above proof, note that the obvious map $C_{\mathrm{CE}}(\mathfrak{l}) \to C_{\mathrm{CE}}(\mathsf{TCG}(3))$ does *not* respect differentials.

8.3   A family of associators.

Since Maurer-Cartan elements of a dg Lie algebra $\mathfrak{g}$, with coefficients in an algebra A, are in bijective correspondence with morphisms of dg algebras $C_{\mathrm{CE}}(\mathfrak{g}) \to A$ we deduce the following corollary.

Corollary 8.3.0.4. The element $\Theta_t := \sum_\lambda \vartheta_t^\lambda x_\lambda \in \mathscr{E}^1(\mathfrak{M}_{0,4}, \mathfrak{l})$ is a flat connection.

Chen's theory identifies homotopy invariant iterated integrals on a manifold M with the degree 0 cohomology of the (reduced) bar construction on the de Rham complex of M. By functoriality we have an induced morphism

$$\vartheta_t : \overline{\mathcal{B}}(C_{\mathrm{CE}}(\mathfrak{l})) \to \overline{\mathcal{B}}(\mathscr{E}(\mathfrak{M}_{0,4}))$$

which allows us to view closed degree 0 elements of $\overline{\mathcal{B}}(C(\mathfrak{l}))$ as homotopy invariant iterated integrals.

Note that $\overline{\mathcal{B}}(C_{\mathrm{CE}}(\mathfrak{l}))$ is concentrated in degrees ≥ 0 so $H^0(\overline{\mathcal{B}}(C(\mathfrak{l})))$ is the same as the space of degree 0 cocycles. General Koszul duality theory provides an isomorphism $I : U(\mathfrak{l})' \to H^0(\overline{\mathcal{B}}(C_{\mathrm{CE}}(\mathfrak{l})))$. The product on $U(\mathfrak{l})' = \mathbf{Q}\langle x^0, x^1 \rangle$ is the so-called shuffle product ш. It is commutative and $U(\mathfrak{l})'$ is a free commutative algebra generated by $\mathfrak{l}^*$. The algebra $H^0(\overline{\mathcal{B}}(C_{\mathrm{CE}}(\mathfrak{l})))$ is also equipped with a product called the shuffle product, which we we also denote ш. The map I is an isomorphism of algebras. Define $\mathscr{B} := H^0(\overline{\mathcal{B}}(\mathscr{E}(\mathfrak{M}_{0,4})))$. Put $I_t := \vartheta_t \circ I$ and define $\Pi_t \in \mathscr{B} \otimes \hat{U}(\mathfrak{l})$ to be the image under I_t of $id \in \hat{U}(\mathfrak{l})' \otimes \hat{U}(\mathfrak{l})$. We interpret Π_t as the holonomy operator of the flat connection Θ_t. It follows that

$$\Pi_t = \prod_\lambda \exp(I_t(x^\lambda) \otimes x_\lambda),$$

with the product over Lyndon words taken in decreasing order. This expression is an algebraic counterpart of expressing the holonomy as a path ordered exponential. Expanding this expression leads to

$$\Pi_t = \sum_{\lambda_1 > \cdots > \lambda_k} \sum_{i_1, \ldots, i_k \geq 1} \frac{1}{i_1! \ldots i_k!} I_t(x^{\lambda_1})^{\text{Ш} i_1} \text{Ш} \ldots \text{Ш} I_t(x^{\lambda_k})^{\text{Ш} i_k} \otimes x_{\lambda_1}^{i_1} \ldots x_{\lambda_k}^{i_k}.$$

In this formula $x_{\lambda_1}^{i_1} \ldots x_{\lambda_k}^{i_k}$ is considered as an element of $\hat{U}(\mathfrak{l})$ by the Poincaré-Birkhoff-Witt theorem. Note that Π_t takes values in the set of group-like elements of the universal enveloping algebra.

Remark 8.3.0.5. The parallel translation $\Pi_{1/2}$ is convergent for paths with endpoints in the compactification $\overline{\mathfrak{M}}_{0,4}$. This follows from the fact that the forms

$$\theta_{1/2}^{ij} = \frac{1}{2\pi} d \arg(x_j - x_i)$$

are minimal algebraic forms and their push-forwards ϑ_t are, hence, semi-algebraic PA-forms on $\overline{\mathrm{Conf}}_n(\mathbf{C})$.[Hardt et al. 2011] The associated holonomy

$$\Pi_{1/2}(\overline{01})$$

along the interval $[0, 1]$ is the Alekseev-Torossian Drinfel'd associator.

Let us explain in more detail why our formula is equivalent to that given in [Rossi and Willwacher 2013]. They define Φ_{AT} as follows. First they introduce the $\mathfrak{sder}_3$-valued flat connection

$$\tilde{\Theta}_{1/2} := \sum_{\Gamma} \vartheta_{1/2}^{\Gamma} u_{\Gamma}.$$

Here the sum is over all degree zero graphs $\Gamma \in \mathsf{TCG}(3)^*_{\text{tree}}$ and u_{Γ} is the corresponding special derivation defined by the graph. Then, they know a priori from the article [Ševera and Willwacher 2011] that it in fact must be a connection with values in the subalgebra $\mathfrak{t}_3 \subset \mathfrak{sder}_3$, because the connection is a restriction/projection of an L_∞ connection

$$\sum_{\Gamma \in \mathsf{TCG}(3)^*} \vartheta_{1/2}^{\Gamma} \Gamma$$

with values in the L_∞ algebra $\mathsf{TCG}(3)$, which comes with a quasi-isomorphic projection $\mathsf{TCG}(3) \to \mathfrak{t}_3$. Finally, the resulting associator can by a standard symmetry argument, using $\mathfrak{t}_3 = \mathbf{k} t_{13} \oplus \mathfrak{lie}(t_{12}, t_{23})$, be taken have values only in $\mathfrak{lie}(t_{12}, t_{23})$. By the commutative diagram

of 8.1.4.3 we may, instead, right from the beginning work with the L_∞ connection

$$\sum_{\Gamma \in \mathsf{L}_2^*} \vartheta_{1/2}^\Gamma \Gamma$$

and project that to a connection with values in $\mathfrak{l} \cong \mathfrak{lie}(t_{12}, t_{23})$. But note,

$$\sum_{\Gamma \in \mathsf{L}_2^*} \vartheta_{1/2}^\Gamma(\Gamma) = \sum_{\Gamma \in \mathsf{L}_2^*} \vartheta_{1/2}^\Gamma \sum_\lambda \langle x^\lambda, (\Gamma) \rangle x_\lambda$$

$$= \sum_\lambda \sum_{\Gamma \in \mathsf{L}_2^*} \langle \Gamma^\lambda, \Gamma \rangle \vartheta_{1/2}^\Gamma x_\lambda$$

$$= \sum_\lambda \vartheta_{1/2}^\lambda x_\lambda,$$

because $\lambda \mapsto \Gamma^\lambda$ is dual to $\mathsf{L}_2 \to \mathfrak{l}$.

Define p_ϵ to be the path following the real axis from ϵ to $1 - \epsilon$, with $\epsilon > 0$. Rossi and Willwacher work with $\mathfrak{sder}_3$-valued flat connections

$$\tilde{\Theta}_t := \sum_{\Gamma \in \mathsf{TCG}(3)_{\text{tree}}^*} \vartheta_t^\Gamma u_\Gamma,$$

and their associated parallel transport operators $\tilde{\Pi}_t$. They show that $\tilde{\Pi}_t(p_\epsilon)$ has an asymptotic expansion with respect to $\log \epsilon$. Formally setting $\log \epsilon = 0$ gives a regularized value

$$\tilde{\Phi}_t := \lim_{\epsilon \to 0} (\Pi_t(p_\epsilon)|_{\log \epsilon = 0}).$$

They then prove:

Theorem 8.3.0.6. [Rossi and Willwacher 2013] The $\tilde{\Phi}_t$'s are Drinfel'd associators for all t.

The argument showing equality between our $\Phi_{1/2}$ and their $\tilde{\Phi}_{1/2}$ implies we have an analogous asymptotic expansion of $\Pi_t(p_\epsilon)$ and that the regularized limits Φ_t equal their Drinfel'd associators. From now on we discard the tilde-symbol on top of their associators.

8.3.1 Associated Grothendieck-Teichmüller elements.

First, recall the following constructions from [Rossi and Willwacher 2013].

Define functions h^{ij} on $\mathrm{Conf}_n(\mathbf{C})$ by

$$h^{ij} := \frac{i}{\pi} \log |x_j - x_i|.$$

If e is an edge of a graph that corresponds to $\{x_i, x_j\}$, then we put $h^e := h^{ij}$. For a degree zero graph $\Gamma \in \mathsf{TCG}(3)^*_{\mathrm{tree}}$, define

$$\phi^\Gamma_t := - \sum_{e \in E(\Gamma)} (-1)^{|e|} h^e \theta^{\Gamma^\lambda \backslash e}_t \in \mathscr{E}^{d-1}(\mathfrak{M}_{4+k}).$$

Here $\theta^{\Gamma\backslash e}_t = \wedge_{e' \neq e} \theta^{e'}_t$. Then, similarly to how ϑ^λ_t was defined, define

$$\varphi^\Gamma_t := \pi^!(\phi^\Gamma_t) \in \mathscr{E}^0(\mathfrak{M}_{0,4}).$$

That these functions exist (that the fiber integrals converge) is proved in [Rossi and Willwacher 2013]. Rossi and Willwacher also define (but in different notation)

$$\tilde{f}_t := \sum_{\Gamma \in \mathsf{sg}^*_3} \varphi^\Gamma_t u_\Gamma \in \mathscr{E}^0(\mathfrak{M}_{0,4}, \mathfrak{sder}_3),$$

and prove that

$$\partial_t \tilde{\Theta}_t = d\tilde{f}_t + [\tilde{\Theta}_t, \tilde{f}_t] \in \mathscr{E}^1(\mathfrak{M}_{0,4}, \mathfrak{sder}_3).$$

The gauge-transformations $\tilde{f}_t$ relate the parallel transport operators $\tilde{\Pi}_t$ associated to the $\tilde{\Theta}_t$. To make this precise we introduce some notation. For a point $w \in \mathfrak{M}_{0,4}$, and $t < t'$ define the path-ordered exponential

$$E^{t'}_t[\tilde{f}_s(w)] := \int_t^{t'} \left(1 + \tilde{f}_s(w)ds + [\tilde{f}_s(w)ds | \tilde{f}_s(w)ds] \right.$$

$$\left. + [\tilde{f}_s(w)ds | \tilde{f}_s(w)ds | \tilde{f}_s(w)ds] + \dots \right) \in \exp(\mathfrak{sder}_3).$$

Then, for any path $p : [0,1] \to \mathfrak{M}_{0,4}$ from $x = p(0)$ to $y = p(1)$, we have

$$\tilde{\Pi}_{t'}(p) = E^{t'}_t[\tilde{f}_s(y)]\tilde{\Pi}_t(p)E^{t'}_t[-\tilde{f}_s(x)] \in \exp(\mathfrak{sder}_3).$$

They argue that after regularized evaluation on the path $\overline{01}$ this equation translates to an equality of associators

$$\Phi_{t'} = E^{t'}_t[\psi_s] \cdot \Phi_t, \quad E^{t'}_t[\psi_s] \in GRT_1,$$

where

$$\psi_s := \lim_{\delta \nearrow 1}\left(\tilde{f}_s(\delta) - \frac{i}{\pi} \log(1-\delta)t_{23}\right) - \lim_{\epsilon \searrow 0}\left(\tilde{f}_s(\epsilon) - \frac{i}{\pi} \log \epsilon t_{12}\right)$$

is an element that, by a result of Alekseev and Torossian [Alekseev and Torossian 2012] lies in $\mathfrak{grt}_1 \subset \mathfrak{lie}(t_{12}, t_{23}) \subset \mathfrak{sder}_3$.

Remark 8.3.1.1. Note that neither of the two limits in ψ_s lie in $\mathfrak{lie}(t_{12}, t_{23})$; only when they are added together does one get an element in the free Lie algebra.

Our present goal is to show how using Lyndon words and Lyndon graphs simplifies the formula for ψ_s. The simplification is, we believe, significant.

Define $f_s := \sum_\lambda \varphi_s^\lambda x_\lambda$, where $\varphi_s^\lambda := \varphi_s^{\Gamma^\lambda}$. We must have

$$\psi_s = \lim_{\delta \nearrow 1} \big(f_s(\delta) - \frac{i}{\pi} \log(1-\delta)t_{23}\big) - \lim_{\epsilon \searrow 0} \big(f_s(\epsilon) - \frac{i}{\pi} \log \epsilon t_{12}\big),$$

because no terms in $\tilde{f}_s - f_s$ can contribute since $\psi_s \in \mathfrak{lie}(t_{12}, t_{23})$. Moreover, both limits in above expression must both exist separately. Since $-i \log(1-\delta)/\pi \cdot t_{23} = \phi_s^1(\delta)x_1$ and $-i \log(\epsilon)/\pi \cdot t_{12} = \phi_s^0(\epsilon)x_0$ under the identfication $x_0 = t_{12}$, $x_1 = t_{23}$, it follows that

$$\psi_s = \lim_{\delta \nearrow 1} \sum_{|\lambda| \geq 2} \varphi_s^\lambda(\delta)x_\lambda - \lim_{\epsilon \searrow 0} \sum_{|\lambda| \geq 2} \varphi_s^\lambda(\epsilon)x_\lambda.$$

Recall the graphs γ_0^λ and γ_1^λ. Using them we define, for every Lyndon word λ of length ≥ 2,

$$\alpha_s^\lambda := - \sum_{e \in E(\gamma_0^\lambda)} (-1)^{|e|} h^e \wedge_{e \neq e' \in E(\gamma_0^\lambda)} \theta_t^{e'} \in \mathscr{E}(\mathfrak{M}_{0,3+k}),$$

$$\beta_s^\lambda := - \sum_{e \in E(\gamma_1^\lambda)} (-1)^{|e|} h^e \wedge_{e \neq e' \in E(\gamma_1^\lambda)} \theta_t^{e'} \in \mathscr{E}(\mathfrak{M}_{0,3+k}),$$

where $k = |\lambda| - 1$, and the corresponding integrals

$$a_s^\lambda := \int_{\mathfrak{M}_{0,3+k}} \alpha_s^\lambda, \quad b_s^\lambda := \int_{\mathfrak{M}_{0,3+k}} \beta_s^\lambda.$$

Lemma 8.3.1.2. The following limits hold: $a_s^\lambda = \lim_{\epsilon \searrow 0} \varphi_s^\lambda(\epsilon)$, and $b_s^\lambda = \lim_{\delta \nearrow 1} \varphi_s^\lambda(\delta)$.

Proof. The limits can be obtained by formally setting w to 0 or 1 in the function $\varphi_s^\lambda(w)$. This is the same as formally setting $w = 0, 1$ in the form ϕ_s^λ, giving us the forms α_s^λ or β_s^λ, respectively, and then applying integration. $\qquad\square$

Lemma 8.3.1.3. For all λ, $b_s^{i(\lambda)} = (-1)^{|\lambda|-1} a_s^\lambda$.

Proof. Let ϱ be the coordinate change $z \mapsto 1 - z$ of $\mathbf{C}$ and extend it factor-wise to a diffeomorphism of $\mathfrak{M}_{0,3+k}$, with $k = |\lambda| - 1$. Note that it preserves the orientation of $\mathfrak{M}_{0,3+k}$. The combinatorial equality $\varrho^*(\gamma_1^\lambda) = (-1)^{|\lambda|-1}\gamma_0^{i(\lambda)}$ translates to the equality $\varrho^*(\beta_s^\lambda) = (-1)^{|\lambda|-1}\alpha_s^\lambda$ for differential forms. Hence the claim. $\qquad\square$

Corollary 8.3.1.4. The expressions $\psi_s^\lambda := b_s^\lambda - a_s^\lambda$ are the explicit coefficients of the family of $\mathfrak{grt}_1 \subset \mathfrak{l}$-elements found in [Rossi and Willwacher 2013].

The following lemma is proved by Rossi and Willwacher.

Lemma 8.3.1.5. [Rossi and Willwacher 2013] For all s,

$$\psi_s = \sum_{|\lambda|\geq 3} 2^{2|\lambda|-2}(s(1-s))^{|\lambda|-1}\psi_{1/2}^\lambda x_\lambda.$$

The proof is based on noting that θ_s^{ij} is a sum of a holomorphic part scaled by $1 - s$ and an antiholomorphic part scaled by s. The terms of the differential form ϕ_s^λ contributing to the integral φ_s^λ must contain an equal number of holomorphic and anti-holomorphic parts, and that number is the complex dimension $|\lambda| - 1$ of the space over which we integrate.

Lemma 8.3.1.6. The coefficients ψ_s^λ are 0 whenever $|\lambda|$ is even.

Rossi and Willwacher do not prove this claim in detail, so let us do it. It follows from the lemma expressing the s-dependence of ψ_s and the following:

Lemma 8.3.1.7. $\varphi_s^\lambda(w) = (-1)^{|\lambda|-1}\varphi_{1-s}^\lambda(w)$ for real $0 < w < 1$.

Proof. Let $f : \text{Conf}_{|\lambda|-1}(\mathbf{C} \setminus \{0, w, 1\}) \to \text{Conf}_{|\lambda|-1}(\mathbf{C} \setminus \{0, w, 1\})$ be the map which acts as complex conjugation on all points, where w is real. It acts on orientation of each fiber above $\mathbf{C} \setminus \{0, w, 1\}$ by $-1)^{|\lambda|-1}$. On the other hand $f^*\phi_s^\lambda = \phi_{1-s}^\lambda$. Applying integration then proves the claim. $\qquad\square$

Rossi's and Willwacher's formula for the s-dependence implies $\psi_s^\lambda = \psi_{1-s}^\lambda$, while our lemma above implies $\psi_s^\lambda = (-1)^{|\lambda|-1}\psi_s^\lambda$. The two are compatible only if $\psi_s^\lambda = 0$ if λ has even length. Let us summarize the discussion so far in the form of a theorem.

Theorem 8.3.1.8. Let $j \geq 1$. The Lie-series

$$\tau_{2j+1} := \sum_{|\lambda|=2j+1} 2^{4j}\big(b^{\lambda}_{1/2} + (-1)^{|\lambda|}b^{i(\lambda)}_{1/2}\big)x_\lambda$$

are elements of the Grothendieck-Teichmüller Lie algebra $\mathfrak{grt}_1$. The family of elements ψ_s has the form

$$\psi_s = \sum_{j\geq 1}(s(1-s))^{2j}\tau_{2j+1}.$$

Moreover, each Drinfel'd associator Φ_t is obtained as $E^t_{1/2}[\psi_s]\cdot\Phi_{AT}$ where $\Phi_{AT} = \Phi_{1/2}$.

We take no credit for the actual mathematical content of above theorem, our contribution is only a formulaic simplification of its constituents.

Remark 8.3.1.9. The Deligne-Drinfel'd-Ihara conjecture may be phrased as saying that the morphism

$$\mathfrak{lie}(\sigma_3, \sigma_5, \ldots, \sigma_{2j+1}, \ldots) \to \mathfrak{grt}_1$$

from the degree-completed free Lie algebra on generators σ_{2j+1} to $\mathfrak{grt}_1$ defined by sending $\sigma_{2j+1} \mapsto \tau_{2j+1}$ is an isomorphism of Lie algebras. A recent theorem by Francis Brown [Brown 2012] implies that this morphism is injective.

8.4   Recurrence relations.

A big part of the difficulty in the calculations of the integrals defining the coefficients of the associators Φ_t or the Grothendieck-Teichmüller elements ψ_s is the calculation of the differential forms entering the integrals. We show here that these differential forms satisfy interesting recurrences, coming from the recursive definition the Lie word associated to a Lyndon word.

Denote by $\pi_x : \mathrm{Conf}_k(\mathbf{C} \setminus \{0,1\}) \to \mathrm{Conf}_1(\mathbf{C} \setminus \{0,1\})$ the projection which forgets all points except the point x.

Proposition 8.4.0.10. The differential form ϑ^{λ}_t satisfies

$$\vartheta^{\lambda}_t = \frac{1}{2\pi i}\sum_{\mu<\nu}c^{\lambda}_{\mu\nu}\pi^!_w\left(\left((1-t)\frac{dw}{w-z} - t\frac{d\bar{w}}{\bar{w}-\bar{z}}\right)\pi^*_z\vartheta^{\mu}_t\pi^*_z\vartheta^{\nu}_t\right).$$

Remark 8.4.0.11. In other words, the differential forms defining the Drinfel'd associator can be defined by above recurrence formula. The formula $\vartheta_t^\lambda = \pi^!(\theta_t^{\Gamma^\lambda})$ is a solution to the recurrence, but maybe it is possible to write the solution in another way.

Proof. The equality $\Gamma^\lambda = \sum_{\mu < \nu} c_{\mu\nu}^\lambda \Gamma^\mu \wedge \Gamma^\nu$ implies that

$$\theta_t^\lambda = \sum_{\mu < \nu} c_{\mu\nu}^\lambda \theta_t(w, z_1) \theta_t^\mu(z_1, z_2, \ldots, z_r) \theta_t^\nu(z_1, z_{r+1}, \ldots, z_k).$$

We then integrate out $z_1, \ldots, z_k$ to obtain ϑ_t^λ. Repeated fiber integration is associative, so we may first integrate out $z_2, \ldots, z_k$, after which we are left with (renaming $z = z_1$)

$$\sum_{\mu < \nu} c_{\mu\nu}^\lambda \theta_t(w, z) \theta_t^\mu(z) \theta_t^\nu(z).$$

The end result, i.e. ϑ_t^λ, must be a form in dw and $d\bar{w}$. Recalling that

$$\theta_t(w, z) = \frac{1 - t}{2\pi i} d \log(z - w) - \frac{t}{2\pi i} d \log(\bar{z} - \bar{w})$$

then gives the proposed formula. $\qquad \square$

Let us introduce the convention that a lack of subscript s means $s = 1/2$. The following two results are proved in a way completely analogous to the preceeding proposition.

Proposition 8.4.0.12. The function φ^λ satisfies

$$\varphi^\lambda = -\sum_{\mu < \nu} c_{\mu\nu}^\lambda \pi_w^! \left(\frac{i}{\pi} \log |z - w| \pi_z^* \vartheta^\mu \pi_z^* \vartheta^\nu \right.$$
$$\left. + \frac{1}{2\pi} d \arg(w - z) \left(\pi_z^* \varphi^\mu \pi_z^* \vartheta^\nu - \pi_z^* \vartheta^\mu \pi_z^* \varphi^\nu \right) \right),$$

and the constant ψ^λ satisfies

$$\psi^\lambda = -\sum_{\mu < \nu} c_{\mu\nu}^\lambda \int_{\mathfrak{M}_{0,4}} \left(\frac{i}{\pi} \log \frac{|1 - w|}{|w|} \vartheta^\mu \vartheta^\nu \right.$$
$$\left. + \frac{1}{2\pi} d \arg \frac{1 - w}{w} \left(\varphi^\mu \vartheta^\nu - \vartheta^\mu \varphi^\nu \right) \right).$$

Note that the dependence on s of φ_s and ψ_s is known so one may reduce to $\varphi = \varphi_{1/2}$ and $\psi = \psi_{1/2}$.

8.5   Multiple zeta values.

A theorem due to Hidekazu Furusho states that every Drinfel'd associator with coefficients in a field $\mathbf{k} \supset \mathbf{Q}$ defines a morphism of $\mathbf{Q}$-algebras $FZ \to \mathbf{k}$ from the algebra of formal multiple zeta values [Furusho 2011]. The example which served as the starting point for this result was the original associator due to Drinfel'd, the Knizhnik-Zamolodchikov associator. Its associated algebra morphism is the evaluation of a formal multiple zeta value as an actual multiple zeta value. Since we have achieved a thoroughly explicit formula for the Rossi-Willwacher associators Φ_t, we can write down the associated morphisms $FZ \to \mathbf{C}$ in equally explicit form, and prove some properties of them.

8.5.1   The double shuffle group.

Denote by $\mathbf{Q}\langle\langle Y \rangle\rangle$ the completed tensor algebra on the countably many symbols $y_0 = 1$, y_i, $i \geq 1$, and define $P_Y : \mathbf{Q}\langle\langle x_0, x_1 \rangle\rangle \to \mathbf{Q}\langle\langle Y \rangle\rangle$ to be the linear map that sends a word ending in x_0 to 0 and $x_0^{n_1-1}x_1 \ldots x_0^{n_r-1}x_1$ to $(-1)^r y_{s_1} \ldots y_{s_r}$. We consider both spaces as coalgebras, by equipping $\mathbf{Q}\langle\langle Y \rangle\rangle$ with the stuffle coproduct

$$\Delta_* y_n := \sum_{k+l} y_k \otimes y_l,$$

(note $y_0 = 1$) and $\mathbf{Q}\langle\langle x_0, x_1 \rangle\rangle$ with the usual coproduct $\Delta_{\mathrm{III}} x_i = 1 \otimes x_i + 1 \otimes x_i$, $i = 0, 1$. The following definition is due to Georges Racinet, though we use slightly different sign conventions.

Definition 8.5.1.1. [Racinet 2000] Define DMR to be the set of $\Phi \in \mathbf{Q}\langle\langle x_0, x_1 \rangle\rangle$ such that

- the coefficient of 1 is 1, i.e. $(\Phi|1) = 1$,

- $(\Phi|x_0) = 0 = (\Phi|x_1)$,

- $\Delta_{\mathrm{III}}\Phi = \Phi \hat{\otimes} \Phi$,

- and $\Delta_* \Phi_* = \Phi_* \hat{\otimes} \Phi_*$,

where

$$\Phi_* := \exp\left(\sum_{n \geq 1} \frac{(-1)^n}{n}\left(\Phi|x_0^{n-1}x_1\right)y_1^n\right) P_Y(\Phi).$$

The set DMR is an affine variety defined over the rationals. The algebra of formal multiple zeta values FZ is defined to be the ring of functions on DMR.

168

Define DMR_0 to be the the subvariety defined by the additional equation $(\Phi|x_0 x_1) = 0$. More generally, for $\kappa \in \mathbf{k} \supset \mathbf{Q}$, define $DMR_\kappa(\mathbf{k})$ to be the set of $\mathbf{k}$-valued points Φ with $(\Phi|x_0 x_1) = -\kappa$.

The following theorem is due to Racinet.

Theorem 8.5.1.2. [Racinet 2000] DMR_0 is a group under the multiplication

$$(\Phi \cdot \Psi)(x_0, x_1) := \Phi(\Psi(x_0, x_1) x_0 \Psi^{-1}(x_0, x_1), x_1) \Psi(x_0, x_1),$$

for Ψ^{-1} defined as the inverse of Ψ in the group of grouplike elements of $\mathbf{Q}\langle\langle x_0, x_1 \rangle\rangle$. Moreover, DMR is a torsor for DMR_0 under the right action given by the same formula as the group multiplication.

The following theorem is due to Hidekazu Furusho.

Theorem 8.5.1.3. [Furusho 2011] The trivial map $\Phi \mapsto \Phi$ is an injective morphism of groups from GRT_1 to DMR_0, and it extends to an inclusion of the set of associators $Assoc(\mathbf{k})$ over $\mathbf{k} \supset \mathbf{Q}$ into the set of $\mathbf{k}$-valued points of DMR.

Given a sequence of integers $(n_1, \ldots, n_r)$, with $n_1 \geq 2$, define

$$\mathbf{z}(n_1, \ldots, n_r) \in FZ$$

to be the function given by evaluating $(-1)^r (\Phi | x_0^{n_1-1} x_1 \ldots x_0^{n_r-1} x_1)$. These elements generate FZ (but with a lot of redundancy, of course). It follows that we can identify $DMR_\kappa(\mathbf{k})$ with the set of $\mathbf{Q}$-algebra morphisms $FZ \to \mathbf{k}$ sending $\mathbf{z}(2)$ to κ. Generally, Furusho's theorem may be phrased as saying that if Φ is a Drinfel'd associator over $\mathbf{k}$, then the map $FZ \to \mathbf{k}$ that sends $\mathbf{z}(n_1, \ldots, n_r)$ to the coefficient $(-1)^r (\Phi | x_0^{n_1-1} x_1 \ldots x_0^{n_r-1} x_1)$ is a morphism of $\mathbf{Q}$-algebras.

Remark 8.5.1.4. The algebra FZ is defined by imposing only the so-called regularized double shuffle relations. They are conjectured to form a complete list of relations for the classical multiple zeta values. There is also a relation called the (Hoffman) duality relation, and as far as the present author knows it is still unknown wether the regularized shuffle relations imply it or not; however, the duality relation is satisfied by any evaluation defined by a Drinfel'd associator, see [Soudères 2013]. In the present case the duality relations can be verified also more directly, using the representation as iterated integrals.

Remark 8.5.1.5. The t-dependence of the forms ϑ_t^λ is polynomial. Thus, the family of evaluations should maybe more properly be considered as a map of $\mathbf{Q}$-algebras $FZ \to \mathbf{C}[t]$.

8.5.2 Properties of the family of evaluations.

Define $\zeta_t : FZ \to \mathbf{C}$ to be the family of algebra morphisms corresponding to the family of Drinfel's associators Φ_t and set $\zeta_t(n_1, \ldots, n_r) := \zeta_t(\mathbf{z}(n_1, \ldots, n_r))$.

Remark 8.5.2.1. The associator Φ_0 is, up to normalizing prefactors, equal to the Knizhnik-Zamolodchikov associator. Hence, and in more detail,

$$\zeta_0(n_1, \ldots, n_r) = \frac{1}{(2\pi i)^n} \zeta(n_1, \ldots, n_r), \quad n := n_1 + \ldots n_r,$$

where the ζ on the right refers to the classical multiple zeta value. For example,

$$\zeta_0(2) = -\frac{1}{4\pi^2} \frac{\pi^2}{6} = -\frac{1}{24}.$$

Proposition 8.5.2.2. For all t, $\zeta_t(2) = -1/24$.

The result is equivalent to saying that all associators Φ_t are $\kappa = 1$-associators. It is known that the GRT_1-action on associators does not change κ; hence $\zeta_0(2) = -1/24$ already implies the claim. Let us give a direct proof as well.

Proof. The formal zeta $\mathbf{z}(2)$ is given by evaluating the coefficient of $-x_0 x_1$. We first rewrite this in the PBW Lyndon basis, as $x_0 x_1 = x_{01} + x_1 x_0$. The dual is $x^{01} + x^1 \text{\san} x^0$, which gives us the iterated integral

$$I_t(x^{01}) + I_t(x^1) \text{\san} I_t(x^1) = [\vartheta_t^{01}] + [\vartheta_t^1 | \vartheta_t^0] + [\vartheta_t^1] \text{\san} [\vartheta_t^0]$$
$$= [\vartheta_t^{01}] + 2[\vartheta_t^1 | \vartheta_t^0] + [\vartheta_t^0 | \vartheta_t^1].$$

The iterated integral $2[\vartheta_t^1 | \vartheta_t^0]$ diverges on $\overline{01}$ but is regularized away in the associator. (The regularization discards all iterated integrals that start with ϑ_t^1 and/or end with ϑ_t^0.) Hence, the proposition is that evaluation of the iterated integral $[\vartheta_t^{01}] + [\vartheta_t^0 | \vartheta_t^1]$ on the path $\overline{01}$ equals $1/24$ for all t. The differential form ϑ_t^{01} was calculated in [Rossi and Willwacher 2013] and equals

$$\frac{t(1-t)^2}{2\pi^2} \left(\frac{\log|1-w|}{w} + \frac{\log|w|}{1-w} \right) dw + \frac{t^2(1-t)}{2\pi^2} \left(\frac{\log|1-w|}{\bar{w}} + \frac{\log|w|}{1-\bar{w}} \right) d\bar{w}.$$

Restricted to real $0 < w < 1$ it simplifies to

$$\frac{t(1-t)}{2\pi^2} \left(\frac{\log(1-w)}{w} + \frac{\log(w)}{1-w} \right) dw.$$

Note

$$\int_0^1 \log(1-w)\frac{dw}{w} = \int_0^1 \int_0^{w_1} d\log(1-w_2)d\log(w_1) = \zeta(2) = \frac{\pi^2}{6}.$$

The other term gives another $\zeta(2)$, and hence

$$\int_{\overline{01}} [\vartheta_t^{01}] = \frac{t(1-t)}{6}.$$

Since the iterated integral $[\vartheta_t^0|\vartheta_t^1]$ on $\overline{01}$ does not depend on the angular part of the differentials one deduces

$$\int_{\overline{01}} [\vartheta_t^0|\vartheta_t^1] = \frac{(1-2t)^2}{2\pi^2}\zeta(2) = \frac{(1-2t)^2}{24}.$$

Collecting terms,

$$\frac{t(1-t)}{6} + \frac{(1-2t)^2}{24} = \frac{1}{24}.$$

$\square$

Remark 8.5.2.3. As a consequence, the formal version

$$\mathbf{z}(2n) = (-1)^{n+1}\frac{B_{2n}}{2}\frac{(24)^n}{(2n)!}\mathbf{z}(2)^n$$

of Euler's formula implies that

$$\zeta_t(2n) = -\frac{B_{2n}}{2(2n)!} \in \mathbf{Q}.$$

Proposition 8.5.2.4. For all Lyndon words, $\vartheta_t^\lambda = (-1)^{|\lambda|}\vartheta_{1-t}^\lambda$ when restricted to $\overline{01}$.

Proof. This is a copy of the analogous relation that we proved in 8.3.1.7. The only difference is that $f^*\theta_t^\lambda = -\theta_{1-t}^\lambda$. $\square$

Corollary 8.5.2.5. The evaluations satisfy the the following time-reflection symmetry

$$\zeta_t(n_1,\ldots,n_d) = (-1)^{n_1+\cdots+n_d}\zeta_{1-t}(n_1,\ldots,n_d).$$

We remark that the number $n_1 + \cdots + n_d$ is called the weight of $\mathbf{z}(n_1,\ldots,n_d)$.

171

Proof. If $x_{\lambda_1} \ldots x_{\lambda_k}$ appears as a summand when writing the monomial $x_0^{n_1-1} x_1 \ldots x_0^{n_d-1} x_1$ in the PBW Lyndon basis, then $|\lambda_1| + \cdots + |\lambda_k| = n_1 + \cdots + n_d$. Hence, the corresponding contributing iterated integrals satisfy

$$\int_{\overline{01}} [\vartheta_t^{\lambda_1} | \ldots | \vartheta_t^{\lambda_k}] = (-1)^{n_1 + \cdots + n_d} \int_{\overline{01}} [\vartheta_{1-t}^{\lambda_1} | \ldots | \vartheta_{1-t}^{\lambda_k}].$$

$\square$

Corollary 8.5.2.6. For $t = 1/2$ all multiple zetavalues of odd weight are zero.

For example, the odd zetavalues $\zeta_{1/2}(2n + 1) = 0$ and, of course, many more. Let us give a direct verification of this latter fact, since the argument has some elucidating features.

Proof. The odd zeta $\zeta_t(2n+1)$ equals $-(\Phi_t | x_0^{2n} x_1)$. In the PBW Lyndon basis,

$$x_0^{2n} x_1 = x_{0^{2n}1} + c_1 x_{0^{2n-1}1} x_0 + \cdots + c_{2n-1} x_{01} x_0^{2n-1} + x_1 x_0^{2n},$$

for some integer coefficients $c_1, \ldots, c_{2n-1}$. Note that

$$\theta_{1/2}^{ij} = \frac{1}{2\pi} d \arg(x_j - x_i)$$

is zero when the points are constrained to be collinear; it follows immediately that all iterated integrals involving $\vartheta_{1/2}^0$ or $\vartheta_{1/2}^1$ along $\overline{01}$ are zero, and hence

$$\zeta_{1/2}(2n + 1) = -\int_{\overline{01}} \vartheta_{1/2}^{0^{2n}1}.$$

Consider $CF_{2n,3} := \{(z_1, \ldots, z_{2n}, w) \in \mathfrak{M}_{0,2n+1} \mid 0 \leq w \leq 1\}$. We may identify

$$\int_{\overline{01}} \vartheta_{1/2}^{0^{2n}1} = \int_{CF_{2n,3}} \theta_{1/2}^{0^{2n}1}.$$

Let f be the diffeomorphism of $CF_{2n,3}$ induced by complex conjugation. It acts on orientation by $(-1)^{2n}$, i.e., it preserves orientation. On the other hand $f^* \theta_{1/2}^{0^{2n}1} = (-1)^{2(2n+1)-1} \theta_{1/2}^{0^{2n}1} = -\theta_{1/2}^{0^{2n}1}$, so the integral vanishes. $\square$

8.5.3 Questions.

It is a famous conjecture that the standard evaluation $FZ \to \mathbf{R}$ is injective (possibly adding to FZ also the duality relations). Since the standard evaluation corresponds to the Knizhnik-Zamolodchikov associator, the conjecture is equivalent to conjecturing that ζ_0 is injective up to the "gauge-fix" that $\zeta_0(2) = -1/24$. The evaluation for $t = 1$ is given by $\zeta_1(n_1, \ldots, n_d) = (-1)^{n_1 + \cdots + n_d}\zeta_0(n_1, \ldots, n_d)$, so it equals ζ_0 on multiple zetas of even weight and is minus ζ_0 on multizetas of odd weight. Hence the conjecture also implies that ζ_1 is essentially injective. On the other hand, $\zeta_{1/2}$ has a very large kernel because, as we have seen, it vanishes on all multizetas of odd weight. It is tempting to conjecture that $\zeta_{1/2}$ is rational. Since the various evaluations ζ_t are all given by acting on $\zeta_{1/2}$ with a Grothendieck-Teichmüller element

$$G_t = E_{1/2}^t \left[\sum_{j \geq 1} (s(1-s))^{2j} \tau_{2j+1} \right]$$

that conjecture would (somewhat informally) be equivalent to saying that all the transcendent numbers come from the coefficients of the τ_{2j+1}'s. The conjecture is probably too optimistic. Since there are no relations in FZ between multiple zetas of different weight one could, to the other extreme, conjecture that $\zeta_{1/2}$ is injective mod $\mathbf{z}(2)$ on the part of even weight.

The graph Γ^{01} is the graph defining the operation $\mathcal{V}_{1,3}$ in our exotic NCG_∞-structure on polyvector fields. The weight of the graph is $1/24$ in that construction as well. Hence, it is natural to conjecture that there is a family of exotic NCG_∞-structures on polyvector fields, corresponding to the family Φ_t, such that our exotic structure from chapter 4 corresponds to the Alekseev-Torossian associator, and such that all the NCG_∞-structures in the family have a first exotic correction term with weight $1/24$. Ideally such a construction should be possible by simply redefining the differential forms associated to graphs by decorating edges with θ_t; however, there are convergence issues that require regularization and it is not clear how to regularize and still respect all operadic structure.

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