Infinite-dimensional Kalman Filtering and Sensor Placement Problem

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Applied Mathematics

Waterloo, Ontario, Canada, 2016

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

One of the central problems in engineering is to estimate the state of a stochastic dynamic system from limited noisy measurements. A Kalman filter is commonly used for state estimation, which produces an unbiased optimal estimate that minimizes the variance of the estimation error. Many physical processes, such as diffusion and beam vibrations, can be described by partial differential equations. These governing equations may be reformulated mathematically as infinite-dimensional dynamic systems. In this work, the derivation of the Kalman filter for infinite-dimensional linear dynamic systems is reviewed, and the sensor placement problem for Kalman filtering is considered. The optimality criterion for sensor selection and location is to minimize the steady-state error variance, which is shown to be the nuclear norm of an operator that solves an algebraic Riccati equation.

Three partial differential equation models are examined: one-dimensional diffusion, simply supported Euler-Bernoulli beam with Kelvin-Voigt damping, and two-dimensional diffusion on an L-shaped region. Optimal sensor locations are calculated for the three models. The sensor noise effects on the state estimation are investigated with the assumption that all the selected sensors are placed optimally. Results show that using multiple low quality sensors can lead to as good an estimate as using a single high quality sensor, provided that enough sensors are used. In particular, for the one-dimensional diffusion equation, approximately proportional relations between the square root of sensor noise variance and the estimation error are observed in simulations.

Acknowledgements

First and foremost, I would like to thank my supervisor, Dr. Kirsten Morris, for her time, patience and expertise, that guide me throughout this project.

I would also like to thank my thesis defence committee member, Dr. Brian Ingalls and Dr. Marek Stastna, for their valuable comments on the thesis.

I am also deeply grateful to all the friends that I met in Waterloo, for them bringing colors to my life and sunshine to the office without a window.

Finally, I give my sincere thanks to my family, for always being there for me, giving me unconditional love and support.

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Chapter 1

Introduction

The problem of estimating the state of a stochastic dynamic system from limited noisy measurements is one of the central problems in engineering. State estimation has been considered of great value in diverse applications. For example, in monitoring a chemical process, a state estimator can be used to estimate concentrations of chemical species, which are usually challenging to measure. Estimating the dispersion of pollutants in air and ocean can be of interest for environmentalists. Another application is to estimate the position of a moving vehicle for purposes of guidance, navigation, and control. The quality of an estimate depends not only on the type of the estimator used, but also on the quality of measurements.

A Kalman filter is commonly used for state estimation, which produces an unbiased optimal estimate that minimizes the variance of the estimation error. It was first developed in [38, 39] for state estimation of finite-dimensional linear systems. Many physical processes, such as diffusion and beam vibrations, can be described by partial differential equations. These governing equations may be reformulated mathematically as differential equations on an abstract linear vector space of infinite dimensions. To estimate processes modelled by partial differential equations, a theory for infinite-dimensional Kalman filtering is necessary.

The Kalman filter has been generalized to infinite-dimensional linear systems by researchers using different approaches. For instance, in [7], the filtering problem is solved by an application of linear-quadratic (LQ) control theory and the concept of generalized random variables introduced in [32]. In [5], the infinite-dimensional Kalman filter is obtained using the theory of weak random variables and techniques of integral equations. A detailed historical development of infinte-dimensional Kalman filter can be found in [18].

In this thesis, the approach by R. F. Curtain and A. J. Pritchard in [20] will be described, which uses functional analysis methods and probability theory on Hilbert spaces.

The Kalman filter makes estimates based on measurements; different sensor placement may lead to different estimates. To obtain a desirable estimate, certain accuracy in measurements is required. However, high-quality sensors that provide accurate measurements are usually expensive and sometimes are not available. The question is, can the lack of sensor accuracy be compensated by adding number of sensors? Also, given a set of sensors, what are the best sensor locations that maximize the quality of estimates?

The sensor selection and location problem has been considered by many researchers in various contexts. For example, in [15], a general procedure for the optimal selection of sensor allocation is formulated and an example of a tubular-flow reactor system is examined, where sensors are placed either simultaneously or sequentially. In [57], a method for optimal sensor placement on a thin double-curved shell structure is presented. An optimization problem is formulated in [40] to select the optimal sensor configurations that maximize the partial observability of the dynamic system in numerical weather predictions. Optimal actuator/sensor placement problem for transport-reaction processes is considered in [2]. In [54], an optimization method is proposed, which is successfully used for determining the optimal sensor locations of a linearized continuous stirred-tank reactor model. In [11, 12, 13], the optimal filtering problem with mobile sensors is considered, and control of mobile actuator/sensor pairs is considered in [25]. Finally, in [51, 52, 53], optimizing the shape and the location of sensors with respect to observability is investigated.

In this thesis, the sensor placement problem for infinite-dimensional Kalman filtering is considered from two aspects: the optimal sensor location problem and the effect of sensor noise on the estimation. The optimality criterion for sensor selection and location is to minimize the steady-state error variance, which is shown to be the nuclear norm of an operator that solves an algebraic Riccati equation (ARE). Optimal sensor location in this context is the dual problem of the LQ optimal actuator location problem. The results on well-posedness of the optimal location problem, as well as the use of approximations in calculating the optimal locations in [48] can be applied by duality. Also, in [24], an algorithm for computation of LQ optimal actuator location problem is presented and this can be used for computation of the optimal sensor locations.

With the assumption that all the selected sensors are placed at optimal locations, the effect of the sensor noise and the number of sensors on the quality of a Kalman filter estimate is examined for three partial differential equation models: one-dimensional diffusion, simply supported Euler-Bernoulli beam with Kelvin-Voigt damping, and two-dimensional diffusion on an L-shaped region. Intuitively, using more sensors should lead to a better

estimate. The question is, whether of a larger number of inaccurate sensors, that is those with large noise variance, can provide as good an estimate as a single highly accurate sensor. Simulations with two estimators, one constructed using a single high-quality sensor and one with a number of poor quality sensors, indicates that number of sensors can compensate for high sensor noise. In particular, for the one-dimensional equation, the same approximately square root relation between the sensor noise variance and the solution to the associated ARE is observed in considering different disturbances.

The thesis is organized as follows. In Chapter 2, background materials are provided, including a brief summary of the finite-dimensional continuous-time Kalman filter, some concepts of infinite-dimensional linear systems, probability theory on Hilbert spaces, and the LQ control problem as a dual problem of Kalman filtering. Chapter 3 introduces the state estimation problem and the derivation of the Kalman filter for infinite-dimensional linear systems. The sensor placement problem is formulated in Chapter 4, with dual results from LQ actuator location problem presented and discussion of computational aspects. In Chapter 5, three partial differential equation models are examined, which are approximated by finite-dimensional systems for computational purposes. Optimal sensor locations are calculated and sensor noise effects on state estimation are investigated numerically.

Chapter 2

Background

2.1 Probability Theory

To consider systems with random disturbances, some concepts of probability theory are introduced in this section.

Suppose Ω is a nonempty set of points.

Definition 2.1.1. *[14] A* σ-algebra *of sets on* Ω *is a nonempty collection* $\mathcal{F}$ *of subsets of* Ω *that satisfies both of the following conditions:*
(1) If a sequence of subsets $\Omega_1, \Omega_2, \ldots \in \mathcal{F}$, *then the union* $\cup_{j=1}^{\infty}\Omega_j \in \mathcal{F}$;
(2) If $\Omega_0 \in \mathcal{F}$, *then the complement set* $\Omega \setminus \Omega_0 \in \mathcal{F}$.

Definition 2.1.2. *[14] Let* $\mathcal{F}$ *be a* σ*-algebra of sets on* Ω. *A* probability measure μ *on* $\mathcal{F}$ *is a real-valued set function with domain* $\mathcal{F}$, *satisfying the following conditions:*
(1) For any $\Omega_0 \in \mathcal{F}$, $\mu(\Omega_0) \geq 0$;
(2) If $\{\Omega_j\}$ *is a countable collection of pairwise disjoint sets in* $\mathcal{F}$, *then*

$$\mu(\cup_j \Omega_j) = \sum_j \mu(\Omega_j);$$

(3) $\mu(\Omega) = 1$.

The triple $(\Omega, \mathcal{F}, \mu)$ is called a probability space; Ω alone is called the sample space, and each point $\omega \in \Omega$ is called a sample point.

Example 2.1.3. *Suppose* $\Omega = (0, 1] \subseteq \mathbb{R}$, $\mathcal{F}$ *the* σ*-algebra generated by the collection of intervals*

$$\{(a, b] : 0 < a < b \leq 1\},$$

and μ_0 *is the Lebesgue measure on* $\mathcal{F}$*, then* $(\Omega, \mathcal{F}, \mu_0)$ *is a probability space.*

Definition 2.1.4. *[14] The probability space* $(\Omega, \mathcal{F}, \mu)$ *is said to be* complete *if and only if any subset of a set* $\Omega_0 \in \mathcal{F}$ *with* $\mu(\Omega_0) = 0$ *also belongs to* $\mathcal{F}$*.*

Let $\mathcal{Z}$ be a separable Hilbert space and $(\Omega, \mathcal{F}, \mu)$ be the underlying probability space, which is assumed to be complete.

Definition 2.1.5. *[20, Definition 5.1] A* $\mathcal{Z}$-valued random variable *is a measurable mapping* $\zeta : \Omega \to \mathcal{Z}$*.*

Definition 2.1.6. *[20, Definition 5.1] For a* $\mathcal{Z}$*-valued random variable* $\zeta \in L_1(\Omega; \mathcal{Z})$*, the* expectation *of* ζ *is*

$$E\zeta = \int_\Omega \zeta(\omega)\mu(d\omega).$$

A simple example of a $\mathcal{Z}$-valued random variable is as follows:

Example 2.1.7. *Suppose the Hilbert space* $\mathcal{Z} = L_2(0, 1)$*, a function* $g \in \mathcal{Z}$*, and* ξ *is a real-valued random variable. Then*

$$\zeta := \xi \cdot g$$

is a $\mathcal{Z}$*-valued random variable.*

For arbitrary $z_1, z_2 \in \mathcal{Z}$, define the operation '$\circ$' by

$$(z_1 \circ z_2)z = \langle z_2, z\rangle z_1, \ \ z \in \mathcal{Z}. \tag{2.1}$$

In particular, if $\mathcal{Z} = \mathbb{R}^n$ for some positive integer n, then $z_1 \circ z_2 = z_1 z_2^T$.

Lemma 2.1.8. *Suppose* $\zeta_1, \zeta_2 \in L_2(\Omega; \mathcal{Z})$ *are* $\mathcal{Z}$*-valued random variables. For arbitrary* $z_1, z_2 \in \mathcal{Z}$*,*

$$\langle E\{\zeta_1 \circ \zeta_2\}z_1, z_2\rangle = E\{\langle \zeta_1, z_1\rangle\langle \zeta_2, z_2\rangle\}.$$

Proof. By Definition 2.1.6 and linearity of the inner product,

$$\begin{aligned}\langle E\{\zeta_1 \circ \zeta_2\}z_1, z_2\rangle &= \Big\langle \int_\Omega \langle \zeta_2(\omega), z_1\rangle \zeta_1(\omega)\mu(d\omega), z_2\Big\rangle \\ &= \int_\Omega \langle \zeta_2(\omega), z_1\rangle\langle \zeta_1(\omega), z_2\rangle \mu(d\omega) \\ &= E\{\langle \zeta_1, z_2\rangle\langle \zeta_2, z_1\rangle\}.\end{aligned}$$

□

Definition 2.1.9. *[20, Definition 5.2] For a $\mathcal{Z}$-valued random variable $\zeta \in L_2(\Omega; \mathcal{Z})$, the* covariance operator *of ζ is*

$$Cov(\zeta) = E\{(\zeta - E\zeta) \circ (\zeta - E\zeta)\}.$$

Definition 2.1.10. *Let $\{\psi_j\}_{j=1}^\infty$ be an orthonormal basis for $\mathcal{Z}$. A self-adjoint nonnegative operator $L \in \mathcal{L}(\mathcal{Z})$ is called nuclear if*

$$\sum_{j=1}^\infty \langle L\psi_j, \psi_j\rangle < \infty.$$

The nuclear *norm of L, denoted by $||L||_1$, is*

$$||L||_1 = \sum_{j=1}^\infty \langle L\psi_j, \psi_j\rangle.$$

The value of $||L||_1$ is independent of choice of the orthonormal basis $\{\psi_j\}_{j=1}^\infty$.

Proposition 2.1.11. *[20, page 118] For a $\mathcal{Z}$-valued random variable $\zeta \in L_2(\Omega; \mathcal{Z})$, $Cov(\zeta)$ is a nuclear operator, with*

$$||Cov(\zeta)||_1 = E\{\langle \zeta - E\zeta, \zeta - E\zeta\rangle\}. \tag{2.2}$$

Proof. Let $\{\psi_j\}_{j=0}^\infty$ be an orthonormal basis for $\mathcal{Z}$. Then

$$\zeta - E\zeta = \sum_{j=0}^\infty \langle \zeta - E\zeta, \psi_j\rangle \psi_j,$$

which implies

$$\langle \zeta - E\zeta, \zeta - E\zeta \rangle = \sum_{j=0}^{\infty} \langle \zeta - E\zeta, \psi_j \rangle^2. \tag{2.3}$$

Since $\zeta \in L_2(\Omega; \mathcal{Z})$, by linearity of inner product,

$$E\{\langle \zeta - E\zeta, \zeta - E\zeta \rangle\} = E\{\langle \zeta, \zeta \rangle\} - \langle E\zeta, E\zeta \rangle < \infty. \tag{2.4}$$

Moreover, for each j, by Lemma 2.1.8,

$$\langle Cov(\zeta)\psi_j, \psi_j \rangle = E\{\langle \zeta - E\zeta, \psi_j \rangle^2\}.$$

Hence,

$$\begin{aligned} ||Cov(\zeta)||_1 &= \sum_{j=0}^{\infty} \langle Cov(\zeta)\psi_j, \psi_j \rangle \\ &= \sum_{j=0}^{\infty} E\{\langle (\zeta - E\{\zeta\}), \psi_j \rangle^2\}. \end{aligned}$$

By monotone convergence theorem,

$$\begin{aligned} ||Cov(\zeta)||_1 &= E\{\sum_{j=0}^{\infty} \langle (\zeta - E\{\zeta\}), \psi_j \rangle^2\} \\ &= E\{\langle \zeta - E\zeta, \zeta - E\zeta \rangle\} \quad \text{(by (2.3))} \\ &< \infty \quad \text{(by (2.4))}. \end{aligned}$$

□

Definition 2.1.12. *Let $\mathcal{B}(\mathbb{R})$ be the Borel σ-algebra of $\mathbb{R}$. A real-valued random variable ξ is* Gaussian*, with mean value a and variance σ^2, if for any $X \in \mathcal{B}(\mathbb{R})$,*

$$\mu(\xi \in X) = \int_X f(x)dx,$$

where

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(\frac{-(x-a)^2}{2\sigma^2})$$

is called the probability density function of ξ.

Definition 2.1.13. *[20, Definition 5.4] A $\mathcal{Z}$-valued random variable ζ is* Gaussian *if $\langle \zeta, \psi \rangle$ is a real-valued Gaussian random variable for all $\psi \in \mathcal{Z}$.*

Definition 2.1.14. *[23, Page 65] An arbitrary family $\zeta = \{\zeta(t)\}_{t \geq 0}$, of $\mathcal{Z}$-valued random variables $\zeta(t)$, $t \geq 0$, defined on Ω is called a* stochastic process. *For each $\omega \in \Omega$, $\zeta(\cdot, \omega)$ is called a* sample path *of ζ.*

Definition 2.1.15. *[23, Definition 4.2] Let Q be a nonnegative nuclear operator on $\mathcal{Z}$. A $\mathcal{Z}$-valued stochastic process $w(t)$, $t \geq 0$, is called a* Wiener process *of incremental covariance Q, if it satisfies*
(1) $w(0) = 0$;
(2) w has continuous sample paths;
(3) w has independent increments, i.e. $w(s_4) - w(s_3)$ and $w(s_2) - w(s_1)$ are independent whenever $s_4 \geq s_3 \geq s_2 \geq s_1 \geq 0$;
(4) $w(t) - w(s)$, for $t \geq s \geq 0$, is a $\mathcal{Z}$-valued Gaussian random variable with zero mean and covariance $(t - s)Q$.

Proposition 2.1.16. *[23, Proposition 4.3] Assume that $w(t)$ is a $\mathcal{Z}$-valued Wiener process of incremental covariance Q, then there exists an orthonormal basis $\{\psi_j\}_{j=0}^{\infty}$ for $\mathcal{Z}$ and a bounded sequence of nonnegative real numbers $\{\lambda_j\}_{j=0}^{\infty}$, such that*

$$Q\psi_j = \lambda_j \psi_j, \ j = 0, 1, 2, \ldots,$$

and for arbitrary $t \geq 0$, w has the expansion

$$w(t) = \sum_{j=0}^{\infty} \beta_j(t) \psi_j, \tag{2.5}$$

where $\beta_j(t)$, $j \geq 0$, are mutually independent real-valued Wiener processes of incremental covariance λ_j, and the series above is convergent in $L_2(\Omega, \mathcal{F}, P)$.

2.2 Finite-Dimensional Kalman Filter

Consider a linear system modelled by n scalar stochastic differential equations, written in the form of

$$dz(t) = Az(t)dt + Gdw(t), \ z(0) = z_0, \ t \geq 0, \tag{2.6}$$

and

$$dy(t) = Cz(t)dt + dv(t), \ y(0) = 0, \tag{2.7}$$

where $z(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times m}$ and $G \in \mathbb{R}^{n\times p}$. The function $z(t)$ is called the state of the system, and $y(t) \in \mathbb{R}^m$ is the output. The initial state $z_0 \in \mathbb{R}^n$ is known as a Gaussian random vector, with zero mean value and covariance matrix $P_0 \in \mathbb{R}^{n\times n}$. The signals $w(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^m$ are vector-valued Wiener processes, each with zero mean value and incremental covariances:

$$E\{(w(t) - w(s))(w(t) - w(s))^T\} = (t - s)\tilde{Q},$$

$$E\{(v(t) - v(s))(v(t) - v(s))^T\} = (t - s)\tilde{R},$$

for all $t \geq s \geq 0$, where $\tilde{Q} \in \mathbb{R}^{p\times p}$ is nonnegative and $\tilde{R} \in \mathbb{R}^{m\times m}$ is positive definite. Moreover, assume z_0, $w(t)$ and $v(t)$ are mutually uncorrelated.

In (2.6),

$$dw(t) = w(t + dt) - w(t),$$

and

$$dz(t) = z(t + dt) - z(t),$$

where dt is the infinitesimal time difference. Similarly, in (2.7) ,

$$dv(t) = v(t + dt) - v(t),$$

and

$$dy(t) = y(t + dt) - y(t).$$

Theorem 2.2.1. *[3, Theorem 6.2.2 & Corollary 8.2.4] The stochastic differential equation (2.6) has the unique solution*

$$z(t) = \exp(At)z_0 + \int_0^t \exp(A(t - s))Gdw(s),$$

where $\exp(At)$ *is the matrix exponential:*

$$\exp(At) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

In many cases, the internal state $z(t)$ can not be fully observed. An estimate $\tilde{z}(t)$ for the state $z(t)$ is called unbiased if

$$E\{\tilde{z}(t)\} = E\{z(t)\},\ t \geq 0.$$

Based on the output $\{y(\tau) : 0 \le \tau \le t\}$, the continuous-time Kalman filter, also known as Kalman-Bucy filter, provides system (2.6) with an optimal unbiased linear estimate $\tilde{z}(t)$ (see e.g. [10, Chapter 4]), such that the error variance

$$E\{||z(t) - \tilde{z}(t))||^2\},\ t \ge 0$$

is minimized. The Kalman-Bucy filter can be described by an n-dimensional linear dynamic system

$$d\tilde{z}(t) = A\tilde{z}(t)dt + F(t)(dy(t) - C\tilde{z}(t)dt), \quad \tilde{z}(0) = 0, \quad t \ge 0,$$

where

$$F(t) = P_e(t)C^*\tilde{R}^{-1},$$

and $P_e(t)$ is the unique solution to the differential Riccati equation

$$\begin{aligned} &\dot{P}_e(t) = AP_e + P_eA^* - P_eC^*\tilde{R}^{-1}CP_e + G\tilde{Q}G^* \\ &P_e(0) = P_0. \end{aligned} \tag{2.8}$$

Moreover, $P_e(t)$ is the estimation error covariance matrix

$$P_e(t) = E\{(z(t) - \tilde{z}(t))(z(t) - \tilde{z}(t))^T\}.$$

Under certain conditions, the matrix $P_e(t)$ reaches a steady-state matrix P_{ss} as $t \to \infty$, which solves the algebraic Riccati equation (ARE)

$$AP_{ss} + P_{ss}A^* - P_{ss}C^*\tilde{R}^{-1}CP_{ss} + G\tilde{Q}G^* = 0. \tag{2.9}$$

The Kalman filter converges to a time-invariant filter:

$$d\tilde{z}(t) = A\tilde{z}(t)dt + F_{ss}(dy(t) - C\tilde{z}(t)dt), \quad \tilde{z}(0) = 0, \quad t \ge 0,$$

where

$$F_{ss} = P_{ss}C^*\tilde{R}^{-1},$$

Definition 2.2.2. *(1) A matrix A is* Hurwitz *if all the eigenvalues of A have negative real parts.*
(2) The matrix pair (A, B) is stabilizable *if there exists K such that $A - BK$ is Hurwitz.*
(3) The matrix pair (A, C) is detectable *if there exists F such that $A - FC$ is Hurwitz.*

Theorem 2.2.3. *[43, Theorem 2.3] The ARE (2.9) has a unique positive semi-definite*

solution P_{ss} such that

$$\lim_{t\to\infty} P_e(t) = P_{ss}$$

if and only if the pair $(A, G\sqrt{\tilde{Q}})$ is stabilizable and (A, C) is detectable. Furthermore, the corresponding steady-state Kalman filter is stable. That is, the matrix $(A - F_{ss}C)$ is Hurwitz.

2.3 Infinite-dimensional Linear System

Let $\mathcal{Z}$ be a separable Hilbert space. An infinite-dimensional linear system can be written similar to the state-space form (2.6), but with the state $z \in \mathcal{Z}$ instead, and where A, B, C are linear operators on Hilbert spaces. In this section, some concepts and results on infinite-dimensional linear systems are reviewed. The material here can be found in [20] and [21].

2.3.1 Semigroup Theory

Let $\mathcal{L}(\mathcal{X}, \mathcal{Z})$ denote the set of all bounded linear operators from a Hilbert space $\mathcal{X}$ to a Hilbert space $\mathcal{Z}$, and when $\mathcal{X} = \mathcal{Z}$, write $\mathcal{L}(\mathcal{Z}) = \mathcal{L}(\mathcal{Z}, \mathcal{Z})$.

Definition 2.3.1. *[21, Definition 2.1.2] A* strongly continuous semigroup *on $\mathcal{Z}$ is an operator-valued function*

$$T(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathcal{Z})$$

that satisfies the following properties:

(1) $T(t + s) = T(t)T(s)$ for $t, s \geq 0$;
(2) $T(0) = I$, where I is the identity operator on H;
(3) $\lim_{t\to 0^+} T(t)z = z$, for any $z \in \mathcal{Z}$.

Definition 2.3.2. *[21, Definition 2.1.8] The* infinitesimal generator *A of a C_0-semigroup $T(\cdot)$ on $\mathcal{Z}$ is defined by*

$$Az = \lim_{t\to 0^+} \frac{T(t)z - z}{t},$$

with the domain of A, $\mathcal{D}(A)$, being the set of elements in $\mathcal{Z}$ for which the limit exists.

For each $z_0 \in \mathcal{D}(A)$, the solution to

$$\dot{z}(t) = Az(t),\ z(0) = z_0,\ t \geq 0$$

is

$$z(t) = T(t)z_0,\ t \geq 0.$$

Theorem 2.3.3. *[21, Theorem 2.2.6] If $T(t)$ is a C_0-semigroup with infinitesimal generator A on a Hilbert space $\mathcal{Z}$, then $T^*(t)$ is a C_0-semigroup with infinitesimal generator A^* on $\mathcal{Z}$.*

Let $\mathcal{X}$, $\mathcal{Z}$ be separable Hilbert spaces and $[0, t_1]$ be a time interval. Define the set

$$\mathcal{B}_\infty([0,t_1];\mathcal{L}(\mathcal{X},\mathcal{Z})) := \left\{ D: \begin{array}{l} D(t) \in \mathcal{L}(\mathcal{X},\mathcal{Z})),\ t \in [0,t_1],\ \langle D(\cdot)z_1, z_2\rangle \text{ is measurable on} \\ [0,t_1] \text{ for all } z_1 \in \mathcal{X},\ z_2 \in \mathcal{Z} \text{ and } \operatorname{ess\,sup}_{t\in[0,t_1]} ||D(t)|| < \infty \end{array} \right\},$$

and the region

$$\triangle([0,t_1]) := \{(t,s) : t_1 \geq t \geq s \geq 0\} \subset \mathbb{R}^2.$$

Definition 2.3.4 (mild evolution operator)**.** *[21, Definition 3.2.4] The operator $S(t,s) : \triangle([0,t_1]) \to \mathcal{L}(\mathcal{Z})$ is called a mild evolution operator if it has the following properties:*
(1) $S(s,s) = I$, for $s \in [0,t_1]$;
(2) $S(t,r)S(r,s) = S(t,s),\ 0 \leq s \leq t \leq t_1$;
(3) For each fixed $s \in [0,t_1]$, $S(\cdot,s)$ is strongly continuous on $[s,\tau]$;
(4) For each fixed $t \in [0,t_1]$, $S(t,\cdot)$ is strongly continuous on $[0,t]$.

Theorem 2.3.5. *[21, Theorem 3.2.5 & Definition 3.2.6] Suppose A is the infinitesimal generator of the C_0-semigroup $T(\cdot)$. For $D(\cdot) \in \mathcal{B}_\infty([0,t_1];\mathcal{L}(\mathcal{Z}))$ and arbitrary $z \in \mathcal{D}(A)$,*

$$S(t,s)z = T(t-s)z + \int_s^t T(t-\tau)D(\tau)S(\tau,s)z d\tau,\ (s,t) \in [0,t_1]) \tag{2.10}$$

has a unique solution in the class of mild evolution operators on $\mathcal{Z}$. The solution is called the mild evolution operator generated by $A + D(\cdot)$.

For each $z_0 \in \mathcal{D}(A)$, the solution to

$$\dot{z}(t) = (A + D(t))z(t),\ z(s) = z_0,\ t \geq s \geq 0$$

is

$$z(t) = S(t,s)z_0,\ t \geq s \geq 0.$$

2.3.2 Stochastic Integral

The material in this section can be found in [20, §5.2]. Let $\mathcal{X}$, $\mathcal{Z}$ be separable Hilbert spaces and $[0, t_1]$ be a finite time interval. Define

$$\mathcal{B}_2([0,t_1];\mathcal{L}(\mathcal{X},\mathcal{Z})) = \left\{ \begin{array}{ll} \Phi: & \Phi(t) \in \mathcal{L}(\mathcal{X},\mathcal{Z}),\ t \in [0,t_1],\ \langle \Phi(\cdot)z_1, z_2\rangle \text{ is measurable on } [0,t_1] \\ & \text{for arbitrary } z_1 \in \mathcal{X},\ z_2 \in \mathcal{Z}, \text{ and } \int_0^{t_1} ||\Phi(t)||^2 dt < \infty \end{array} \right\}.$$

For an $\mathcal{X}$-valued Wiener process $w(\cdot)$ and $\Phi(\cdot) \in \mathcal{B}_2([0,t_1];\mathcal{L}(\mathcal{X},\mathcal{Z}))$, stochastic integral of the form

$$\int_0^t \Phi(s)dw(s),\ t \in [0,t_1],$$

can be defined as infinite-dimensional Itô integral (see e.g. [20, §5.2]). The construction of this integral is summarized as follows.

First, consider the integral

$$\int_0^t f(s)d\beta(s),$$

where $f(\cdot) \in L_2([0,t_1];\mathcal{Z})$ and $\beta(\cdot)$ is a real-valued Wiener process of incremental covariance λ.

Lemma 2.3.6. *[20, Lemma 5.26] Let $f(\cdot) \in L_2([0,t_1];\mathcal{Z})$ be a step function, such that*

$$f(s) = f_i \text{ on } [s_i, s_{i+1}),\ 0 = s_0 < s_1 < \ldots < s_k = t_1.$$

Define

$$\int_0^t f(s)d\beta(s) = \sum_{i=0}^{k-1} f_i(\beta(s_{i+1}) - \beta(s_i))), \tag{2.11}$$

Then the following holds:
(1)$E\{\int_0^t f(s)d\beta(s)\} = 0$;
(2)$E\{\langle \int_0^t f(s)d\beta(s), \int_0^t f(s)d\beta(s)\rangle\} = \lambda \int_0^t \langle f(s), f(s)\rangle ds$.

Since the step functions are dense in $L_2([0,t_1];\mathcal{Z})$, the integral (2.11) can be extended to arbitrary $f(\cdot) \in L_2([0,t_1];\mathcal{Z})$ by defining

$$\int_0^t f(s)d\beta(s) = \lim_{n\to\infty} \int_0^t f_n(s)d\beta(s),$$

where the limit is obtained in $L_2(\Omega, \mathcal{F}, P)$ and $\{f_n\}$ is a sequence of step functions converging to f in $L_2([0, t_1]; \mathcal{Z})$.

Using Proposition 2.1.16, the integral $\int_0^t \Phi(s)dw(s)$ can then be defined:

Definition 2.3.7. *[20, Definition 5.25] For a $\mathcal{X}$-valued Wiener process $w(\cdot)$ that has the expansion (2.5), and $\Phi(\cdot) \in \mathcal{B}_2([0, t_1]; \mathcal{L}(\mathcal{X}, \mathcal{Z}))$, define*

$$\int_0^t \Phi(s)dw(s) := \sum_{j=0}^{\infty} \int_0^t \Phi(s)\psi_j d\beta_j(s), \ t \in [0, t_1], \tag{2.12}$$

where the infinite sum converges in $L_2(\Omega, \mathcal{F}, P)$.

Theorem 2.3.8. *[20, Lemma 5.28] The indefinite integral defined by (2.12) has the following properties:*

$$E\{\int_0^t \Phi(s)dw(s)\} = 0, \ t \geq 0, \tag{2.13}$$

$$Cov\big(\int_\tau^t \Phi(s)dw(s)\big) = \int_\tau^t \Phi(s)Q\Phi^*(s)ds, \ t \geq \tau \geq 0. \tag{2.14}$$

2.3.3 Stochastic Differential Equations

The material in this section can be found in [20, §5.3]. Let $\mathcal{X}$, $\mathcal{Z}$ be separable Hilbert spaces and $[0, t_1]$ be a time interval. Consider a class of stochastic differential equations

$$dz(t) = Az(t)dt + Gdw(t), \quad z(0) = z_0, \quad t \in [0, t_1], \tag{2.15}$$

where A, with domain $\mathcal{D}(A)$, is the infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{Z}$, z_0 is an $\mathcal{Z}$-valued random variable, $G \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, and $w(t)$ represents a $\mathcal{X}$-valued Wiener process of incremental covariance Q. By definition of a Wiener process (Definition 2.1.15), Q is a nonnegative nuclear operator. Assume $\{\psi_j\}_{j\geq 1}$ is a sequence of eigenvectors of Q that forms an orthonormal basis for the space $\mathcal{X}$, and $\{\lambda_j\}_{j\geq 1}$ is the corresponding sequence of eigenvalues.

Definition 2.3.9. *[20, Definition 5.30] If $z(t) \in \mathcal{D}(A)$ with probability 1 (w.p.1), $z(t)$ satisfies the integral equation*

$$z(t) = z_0 + \int_0^t Az(s)ds + \int_0^t Gdw(s).$$

almost everywhere on $[0,t_1]\times\Omega$, *and* $z(t)$ *has continuous sample paths, then* $z(t)$ *is a strong solution of (2.15).*

Theorem 2.3.10. *[20, Theorem 5.35] If, for all* $j \geq 0$ *and* $0 \leq s \leq t \leq t_1$, $T(t-s)G\psi_j$, $T(t)z_0 \in \mathcal{D}(A)$ *w.p.1, and*

$$\sum_{j=0}^{\infty} \lambda_j \int_0^t ||AT(t-s)G\psi_j||^2 ds < \infty,$$

then

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Gdw(s) \tag{2.16}$$

is the unique strong solution of (2.15).

A more general type of differential equation, that will arise in the coming chapter, is

$$dz(t) = (A + F(t))z(t)dt + Gdw(t), \quad z(0) = z_0, \quad t \in [0,t_1], \tag{2.17}$$

where $F(t) \in \mathcal{B}_\infty([0,t_1];\mathcal{L}(\mathcal{Z}))$. By Theorem 2.3.5, $A + F(t)$ generates a mild evolution operator $S(\cdot,\cdot) : \triangle([0,t_1]) \to \mathcal{L}(\mathcal{Z})$. The solution of (2.17) can be defined analogously to Definition 2.3.9.

Definition 2.3.11. *If* $z(t) \in \mathcal{D}(A)$ *with probability 1 (w.p.1),* $z(t)$ *satisfies the integral equation*

$$z(t) = z_0 + \int_0^t (A + F(s))z(s)ds + \int_0^t Gdw(s)$$

almost everywhere on $[0,t_1]\times\Omega$, *and* $z(t)$ *has continuous sample paths, then* $z(t)$ *is a* strong solution *of (2.17).*

Theorem 2.3.12. *[20, Corollary 5.37] Under the same conditions in Theorem 2.3.10, and if additionally,* $F \in \mathcal{B}_\infty([0,t_1];\mathcal{L}(\mathcal{Z}))$ *satisfies*

$$T(t-s)F(s) : \mathcal{Z} \to \mathcal{D}(A)$$

and

$$AT(t-s)F(s) \in \mathcal{B}_2([0,t_1]\times[0,t_1];\mathcal{L}(\mathcal{Z})),$$

then

$$z(t) = S(t,0)z_0 + \int_0^t S(t,s)Gdw(s) \tag{2.18}$$

is the unique strong solution of (2.17).

2.4 White Gaussian Noise

White Gaussian noise is widely used as an idealized model for rapidly varying random fluctuations that are perceived as uncorrelated at different instants of time. A real-valued white Gaussian noise is generally understood as a stochastic process $\{\eta(t) : t \geq 0\}$, with zero mean value and covariance

$$E\{\eta(t)\eta(s)\} = Q\delta(t-s), \tag{2.19}$$

where $\delta(\cdot)$ is the Dirac delta distribution and $Q \in \mathbb{R}_+$.

However, notice that $\delta(\cdot)$ is a generalized function, $\{\eta(t) : t \geq 0\}$ cannot be treated as an ordinary stochastic process. One of the approaches to interpret white Gaussian noise is to use the concept of a generalized stochastic process, which was first introduced in [32, Chapter 3]. By considering Wiener process $w(t)$ of incremental covariance Q as a generalized stochastic process, white Gaussian noise can be interpreted as the generalized time derivative of $w(t)$ [3, §3.2]:

$$\eta(t) = \dot{w}(t). \tag{2.20}$$

To make the connection between $\eta(t)$ and $w(t)$ more explicit, for a small time difference $\triangle t > 0$, define

$$\eta_{\triangle t}(t) := \frac{w(t+\triangle t) - w(t)}{\triangle t},\ t \geq 0. \tag{2.21}$$

Since $w(t)$ has independent increments, $\eta_{\triangle t}(t)$ and $\eta_{\triangle t}(s)$ are independent whenever $|t-s| \geq \triangle t$. Moreover, $\eta_{\triangle t}$ is a Gaussian process with zero mean and covariance

$$E\{\eta_{\triangle t}(t)\eta_{\triangle t}(s)\} = Q\delta_{\triangle t}(t-s),$$

where

$$\delta_{\triangle t}(t) = \begin{cases} \frac{1}{\triangle t}(1 - \frac{|t|}{\triangle t}), & \text{if } |t| < \triangle t, \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, one may think of $\eta(t)$ as the limit of $\eta_{\triangle t}(t)$ as $\triangle t \to 0$, which is consistent with (2.20). Such a limit $\eta(t)$ only exists as a generalized stochastic process, since $w(t)$ is not differentiable under ordinary rules of calculus.

For simplicity, further discussion on generalized stochastic process is omitted. Instead,

by taking (2.20) as an intuitive notation, white Gaussian noise, $\eta(\cdot)$, can be identified as a stochastic process whose time integral

$$w(t) = \int_0^t \eta(s)ds, \ t \geq 0,$$

is a Wiener process. In stochastic differential equations, white Gaussian noise is modelled by the term $dw(t)$. For numerical simulations, the white Gaussian noise $\eta(t)$ is approximated by the Gaussian process $\eta_{\triangle t}(t)$ in (2.21):

$$\eta(t) \approx \eta_{\triangle t}(t). \tag{2.22}$$

For each $t \geq 0$, $\eta_{\triangle t}(t)$ is a real-valued Gaussian random variable, with zero mean value and variance $\frac{Q}{\triangle t}$.

2.5 Linear-Quadratic Optimal Control

Consider a linear dynamic system described by

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0, \quad t \geq 0 \tag{2.23}$$

where A, with domain $\mathcal{D}(A)$, is the infinitesimal generator of a C_0-semigroup $T(t)$ on $\mathcal{Z}$ and B is a bounded linear operator from a separable Hilbert space $\mathcal{U}$ to $\mathcal{Z}$. The function $u(\cdot)$ is the input, $z(\cdot)$ represents the state of the system, with initial state $z_0 \in \mathcal{D}(A)$.

Definition 2.5.1. *[21, Definition A.3.71] A self-adjoint operator X on the Hilbert space $\mathcal{Z}$ is* coercive *if there exists an $\epsilon > 0$ such that*

$$\langle Az, z\rangle \geq \epsilon ||z||^2$$

for all $z \in \mathcal{D}(X)$, where $\mathcal{D}(X)$ represents the domain of X.

Let $[0, t_1]$ be a finite time interval. Define a quadratic cost functional

$$J(z_0; 0, t_1, u) := \langle z(t_1), Mz(t_1)\rangle + \int_0^{t_1} \langle z(t), Qz(t)\rangle + \langle u(t), Ru(t)\rangle dt, \tag{2.24}$$

where $M, Q \in \mathcal{L}(\mathcal{Z})$ are self-adjoint and nonnegative. $R \in \mathcal{L}(\mathcal{U})$ is self-adjoint and coercive. The linear-quadratic (LQ) control problem for system (2.23) on $[0, t_1]$ is, given an initial

state z_0, to find an optimal control $u_{opt}(\cdot; z_0, 0, t_1) \in L_2([0, t_1]; \mathcal{U})$ that minimizes the cost functional $J(z_0; 0, t_1, u)$ over all state trajectories $z(\cdot)$ of (2.23). The optimal control is related to a differential Riccati equation.

Theorem 2.5.2. *[21, Theorem 6.1.13] For every self-adjoint, nonnegative operator $M \in \mathcal{L}(\mathcal{Z})$, the differential Riccati equation*

$$\begin{array}{ll} \frac{d}{dt}\langle h_1, X(t)h_2\rangle + \langle h_1, [A^*X(t) + X(t)A - X(t)BR^{-1}B^*X(t) + Q]h_2\rangle = 0, \ t \geq 0, & \\ X(0) = M, & \text{for arbitrary } h_1, h_2 \in D(A), \end{array} \tag{2.25}$$

has a unique solution $X(\cdot)$ in the class of strongly continuous, self-adjoint operators in $\mathcal{L}(\mathcal{Z})$. Furthermore, for each fixed $t_1 \geq 0$,

$$\langle z_0, X(t_1)z_0\rangle = \min_{u \in L_2([0,t_1];U)} J(z_0; 0, t_1, u).$$

On the infinite time interval $[0, \infty]$, the LQ control problem is to find an optimal control minimizing the quadratic cost functional

$$J(z_0; u) := \int_0^\infty \langle z(t), Qz(t)\rangle + \langle u(t), Ru(t)\rangle dt \tag{2.26}$$

over all state trajectories $z(\cdot)$ given by (2.23), with inputs $u \in L_2([0, +\infty); \mathcal{U})$.

Proposition 2.5.3. *[41, Theorem 9.4-2]) For any self-adjoint and nonnegative operator $Q \in \mathcal{L}(\mathcal{Z})$, there exists a unique nonnegative square root $\sqrt{Q}$, such that $\sqrt{Q} \cdot \sqrt{Q} = Q$.*

Definition 2.5.4. *[21, Definition 5.1.1 & Definition 5.2.1] (1) A C_0-semigroup on a Hilbert space $\mathcal{Z}$, say $T(t)$, is* exponentially stable *if there exist positive constants α and β such that*

$$\| T(t) \| \leq \beta e^{-\alpha t} \text{ for any } t \geq 0. \tag{2.27}$$

(2) If there exists $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ such that $A - BK$ generates an exponentially stable C_0-semigroup, then we say that the pair (A, B) is exponentially stabilizable.
(3) We say that (A, C) is exponentially detectable *if there exists $F \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that $A - FC$ generates an exponentially stable C_0-semigroup.*

Theorem 2.5.5. *[21, Theorem 6.2.4 & Theorem 6.2.7] If $\Sigma(A, B, \sqrt{Q})$ is exponentially stable and exponentially detectable, then the cost functional (2.26) has a minimum for every $z_0 \in \mathcal{Z}$. Furthermore, there exists a self-adjoint, nonnegative operator $\Pi \in \mathcal{L}(\mathcal{Z})$ such that $A - BR^{-1}B^*\Pi$ generates an exponentially stable C_0-semigroup*

$$T_\pi(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathcal{Z})$$

and

$$\langle z_0, \Pi z_0 \rangle = \min_{u \in L_2([0,+\infty);U)} J(z_0; u).$$

This operator Π *is the strong limit of* $X(t)$ *as* $t \to \infty$*, where* $X(\cdot)$ *is the unique solution of the differential Riccati equation (2.25) with* $X(0) = 0$*. Equivalently,* Π *is characterized as the unique nonnegative solution of an algebraic Riccati equation (ARE) of operators in* $\mathcal{L}(H)$*:*

$$\langle h_1, [A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q] h_2 \rangle = 0, \text{ for any } h_1, h_2 \in D(A). \tag{2.28}$$

The optimal control is a state feedback control:

$$u_{opt}(t; z_0) = -R^{-1} B^* \Pi z(t), \tag{2.29}$$

where $z(t) = T_\pi(t) z_0$ *is the corresponding state trajectory.*

Chapter 3

State Estimation

Consider an infinite-dimensional integral process

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Gdw(s) \tag{3.1}$$

and the output

$$y(t) = \int_0^t Cz(s)ds + v(t),\ t \geq 0, \tag{3.2}$$

where $T(t)$ is a C_0-semigroup, with infinitesimal generator A, on a separable Hilbert space $\mathcal{Z}$, $w(t)$ is a Wiener process of incremental covariance $\tilde{Q}$ on a separable Hilbert space $\mathcal{W}$, $v(t)$ is a Wiener process of incremental covariance $\tilde{R}$ on a separable Hilbert space $\mathcal{Y}$. The operators $G \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ and $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$. The initial state, z_0, is a $\mathcal{Z}$-valued Gaussian random variable with zero mean value and covariance P_0. Assume $\mathcal{Y}$ is finite-dimensional, $w(t)$, $v(t)$, z_0 are mutually uncorrelated, and the operator $\tilde{R} \in \mathcal{L}(\mathcal{Y})$ is coercive.

In many practical situations, while the output is measured, the internal state, $z(t)$, cannot be fully observed, and therefore, the state estimation problem arises. An extension of the finite-dimensional Kalman filter, presented previously, is to infinite-dimensional systems. Derivation of the infinite-dimensional Kalman filter in this chapter follows the book [20, Chapter 6].

3.1 Optimal Estimation on Finite Time Horizon

Consider system (3.1)-(3.2) on a finite time interval $[0, t_1]$. The objective is to find an unbiased optimal linear estimate, $\tilde{z}_{opt}(t)$, for the state $z(t)$, based on the measured output $\{y(s) : 0 \le s \le t\}$, that minimizes the error variance,

$$J_e(t; \tilde{z}) := E\{||z(t) - \tilde{z}(t)||^2\}, \tag{3.3}$$

for each $t \in [0, t_1]$.

Define

$$\mathcal{B}_2([0,t_1]; \mathcal{L}(\mathcal{Y}, \mathcal{Z})) = \left\{ \begin{array}{ll} \Phi : & \Phi(t) \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}),\ t \in [0,t_1],\ \langle \Phi(\cdot) z_1, z_2 \rangle \text{ is measurable on } [0,t_1] \\ & \text{for arbitrary } z_1 \in \mathcal{Y},\ z_2 \in \mathcal{Z}, \text{ and } \int_0^{t_1} ||\Phi(t)||^2 dt < \infty \end{array} \right\}.$$

Consider an estimate of the form

$$\tilde{z}(t) = \int_0^t \Psi(t,s) dy(s), \tag{3.4}$$

where $\Psi(t, \cdot) \in \mathcal{B}_2([0,t]; \mathcal{L}(\mathcal{Y}, \mathcal{Z}))$ for each $t \in [0, t_1]$.

Let

$$e(t) := z(t) - \tilde{z}(t)$$

be the estimation error. By the assumption that $E\{z_0\} = 0$ and property (2.13) of stochastic integrals,

$$E\{z(t)\} = E\{T(t) z_0\} + E\{\int_0^t T(t-s) G dw(s)\} = 0,\ t \in [0.t_1].$$

Also, for the estimate $\tilde{z}(t)$ in (3.4),

$$\begin{aligned} E\{\tilde{z}(t)\} &= E\{\int_0^t \Psi(t,s) dy(s)\} \\ &= E\{\int_0^t \Psi(t,s) C z(s) ds + \int_0^t \Psi(t,s) dv(s)\} \\ &= \int_0^t \Psi(t,s) C E\{z(s)\} ds + E\{\int_0^t \Psi(t,s) dv(s)\} \\ &= 0. \end{aligned}$$

Therefore, (3.4) gives an unbiased estimate for the state $z(t)$, so that

$$E\{e(t)\} = E\{z(t)\} - E\{\tilde{z}(t)\} = 0.$$

The error covariance is

$$P_e(t) := E\{e(t) \circ e(t)\}. \tag{3.5}$$

It follows from Proposition 2.1.11 that

$$J_e(t; \tilde{z}) = ||P_e(t)||_1.$$

Define

$$J_e(t, h; \tilde{z}) := E\{\langle z(t) - \tilde{z}(t), h\rangle^2\}, \quad h \in \mathcal{Z}. \tag{3.6}$$

By Lemma 2.1.8,

$$J_e(t, h; \tilde{z}) = \langle P_e(t)h, h\rangle.$$

Suppose $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{Z}$, then

$$J_e(t; \tilde{z}) = ||P_e(t)||_1 = \sum_{j=1}^{\infty} \langle P_e(t)\psi_j, \psi_j\rangle = \sum_{j=1}^{\infty} J_e(t, \psi_j; \tilde{z}). \tag{3.7}$$

Therefore, to find an optimal estimate that minimizes the error variance $J_e(t; \tilde{z})$, it is sufficient to find an estimate that minimizes $J_e(t, h; \tilde{z})$ for each $h \in \mathcal{Z}$.

Lemma 3.1.1 (Orthogonality Projection Lemma)**.** *[20, Lemma 6.2] For each fixed $t \in [0, t_1]$, a linear estimate, $\tilde{z}(t)$, minimizes (3.6) over all estimates of the form (3.4) with $\Psi(t, \cdot) \in \mathcal{B}_2([0, t]; \mathcal{L}(\mathcal{Y}, \mathcal{Z}))$ if and only if for any σ, τ such that $0 \le \tau \le \sigma \le t_1$,*

$$E\{e(t) \circ (y(\sigma) - y(\tau))\} = 0, \tag{3.8}$$

where

$$e(t) = z(t) - \tilde{z}(t)$$

represents the estimation error, and $y(\cdot)$ represents the output process.

Proof. Let Ξ be the set of $\mathcal{Z}$-valued random variables ξ such that

$$E\{\langle \xi, \xi\rangle\} < \infty.$$

For fixed $t \in [0, t_1]$, define

$$\Xi_t := \{\eta_t \in \Xi : \eta_t = \int_0^t \Psi(t,s)dy(s), \Psi(t,\cdot) \in \mathcal{B}_2([0,t]; \mathcal{L}(\mathcal{Y}, \mathcal{Z}))\}.$$

For $h \in \mathcal{Z}$, let

$$\Xi(h) := \{\langle \xi, h\rangle : \xi \in \Xi\}$$

be a Hilbert space with inner product $\lceil \cdot, \cdot \rceil$

$$\lceil \langle \xi, h\rangle, \langle \eta, h\rangle \rceil := E\{\langle \xi, h\rangle \langle \eta, h\rangle\}.$$

Define

$$\Xi_t(h) := \{\langle \eta_t, h\rangle \in \Xi(h) : \eta_t \in \Xi_t\}$$

as a subspace of $\Xi(h)$. Then, by the well-known orthogonal projection lemma (see e.g. [50]), $E\{\langle e(t), h\rangle^2\}$ is minimized over Ξ_t by $e(t)$ if and only if $\langle e(t), h\rangle \perp \Xi_t(h)$, that is, for any $\langle \eta_t, h\rangle \in \Xi_t(h)$,

$$\lceil \langle e(t), h\rangle \langle \eta_t, h\rangle \rceil = E\{\langle e(t), h\rangle \langle \eta_t, h\rangle\} = 0.$$

Therefore, $E\{\langle e(t), h\rangle^2\}$ is minimized for all $h \in \mathcal{Z}$ if and only if for any $\eta_t \in \Xi_t$,

$$E\{e(t) \circ \eta_t\} = 0, \tag{3.9}$$

which implies (3.8) by choosing $\Psi(t,s)$ as a step function in s. On the other hand, any $\Psi(t,\cdot) \in \mathcal{B}_2([0,t]; \mathcal{L}(\mathcal{Y}, \mathcal{Z}))$ can be approximated by a sequence of step functions $\Psi_n(t,\cdot)$, $n \geq 1$ [20, page 160]. Hence (3.8) is equivalent to (3.9). □

Theorem 3.1.2 (Wiener-Hopf Equation)**.** *[17, Theorem 2.3] Let $z(\cdot)$ be the process given by (3.1). Define*

$$\Lambda(r,s) := E\{z(r) \circ z(s)\}$$

for $r, s \geq 0$. For each fixed $t \in [0, t_1]$, the estimate $\tilde{z}(t) = \int_0^t \Psi(t,s)dy(s)$ minimizes (3.6) if and only if

$$\int_0^t \Psi(t,r)C\Lambda(r,s)C^* dr + \Psi(t,s)\tilde{R} = \Lambda(t,s)C^* \tag{3.10}$$

for almost all $s \in [0,t]$.

Proof. By (3.8), for σ, τ such that $0 \le \tau \le \sigma \le t_1$,

$$\begin{aligned}
&E\{e(t) \circ (y(\sigma) - y(\tau)\} \\
&= E\{(z(t) - \textstyle\int_0^t \Psi(t,s)dy(s)) \circ (\int_\tau^\sigma Cz(s)ds + v(\sigma) - v(\tau))\} \\
&= \textstyle\int_\sigma^\tau \Lambda(t,s)C^*ds - E\{\int_0^t \Psi(t,s)Cz(s)ds \circ (\int_\tau^\sigma Cz(s)ds)\} \\
&-E\{\textstyle\int_0^t \Psi(t,s)dv(s) \circ \int_\sigma^\tau dv(s)\} \\
&\text{(by the assumed uncorrelatedness between } z(\cdot) \text{ and } v(\cdot)) \\
&= \textstyle\int_\sigma^\tau (\Lambda(t,s)C^* - \int_0^t \Psi(t,r)C\Lambda(r,s)C^*dr - \Psi(t,s)\tilde{R})ds.
\end{aligned}$$

By Lemma 3.1.1, $\tilde{z}(t) = \int_0^t \Psi(t,s)dy(s)$ minimizes (3.6) if and only if

$$E\{e(t) \circ (y(\sigma) - y(\tau)\} = 0$$

for any σ, τ such that $0 \le \tau \le \sigma \le t_1$. Equivalently, (3.10) holds for almost all $s \in [0,t]$. □

The following lemma is a special case of [17, Lemma 2.3].

Lemma 3.1.3. *The inner-product version of the infinite-dimensional differential Riccati equation*

$$\begin{array}{ll}
\frac{d}{dt}\langle P(t)h_1, h_2\rangle = \langle (AP(t) + P(t)A^* - P(t)C^*\tilde{R}^{-1}CP(t) + G\tilde{Q}G^*)h_1, h_2\rangle, \ t \ge 0, & \\
P(0) = P_0, & \textit{for arbitrary } h_1, h_2 \in \mathcal{D}(A^*),
\end{array} \tag{3.11}$$

has a unique solution $P(t) \in \mathcal{L}(\mathcal{Z})$, which is strongly continuous in t on $[0,t_1]$.

Theorem 3.1.4. *[20, Lemma 6.7] Let $P(\cdot)$ be the unique solution of (3.11), and*

$$S_p(\cdot,\cdot) : \triangle([0,t_1]) \to \mathcal{L}(\mathcal{Z})$$

indicate the mild evolution operator generated by $A - P(\cdot)C^*\tilde{R}^{-1}C$. For each fixed $t \in [0,t_1]$,

$$\Psi(t,s) = S_p(t,s)P(s)C^*\tilde{R}^{-1} \tag{3.12}$$

is the unique solution of (3.10) in the class of $\mathcal{B}_2([0,t];\mathcal{L}(\mathcal{Y},\mathcal{Z}))$.

Thus, by Theorem 3.1.2, a unique optimal estimate of the form (3.4) is obtained with $\Psi(t,s)$ given by (3.12).

Theorem 3.1.5 (Infinite-dimensional Kalman filter)**.** *[20, Theorem 6.9 & Lemma 6.12] Let $P(\cdot)$ be the unique solution of (3.11) and*

$$S_p(\cdot,\cdot) : \triangle([0,t_1]) \to \mathcal{L}(\mathcal{Z})$$

the mild evolution operator generated by $A - P(\cdot)C^*\tilde{R}^{-1}C$ *. The estimate*

$$\tilde{z}_{opt}(t) = \int_0^t S_p(t,s)P(s)C^*\tilde{R}^{-1}dy(s),\ t \in [0,t_1], \tag{3.13}$$

is the unique optimal estimate for $z(t)$ *in that for each* $h \in \mathcal{Z}$,

$$E\{\langle(z(t) - \tilde{z}_{opt}(t), h\rangle^2\} = \min_{\tilde{z}} E\{\langle(z(t) - \tilde{z}(t), h\rangle^2\} \tag{3.14}$$

where the minimum is taken over all estimates $\tilde{z}(t)$ *of the form (3.4). The error covariance*

$$P(t) = E\{(z(t) - \tilde{z}_{opt}(t)) \circ (z(t) - \tilde{z}_{opt}(t))\}. \tag{3.15}$$

Moreover,

$$\begin{aligned} ||P(t)||_1 &= E\{||z(t) - \tilde{z}_{opt}(t)||^2\} \\ &= \min_{\tilde{z}} E\{||z(t) - \tilde{z}(t)||^2\}. \end{aligned} \tag{3.16}$$

Under some conditions, (3.13) is the unique solution of a differential equation. The following theorem is a special case of [20, Theorem 6.21].

Theorem 3.1.6. *Let* $P(t)$ *be the unique solution of (3.11) and* $T(t)$ *be the* C_0*-semigroup generated by operator* A*. Define*

$$F(t) = P(t)C^*\tilde{R}^{-1}.$$

Under the additional assumptions:

$$T(t)P_0 \text{ and } T(t)G\tilde{Q}G^* : \mathcal{Z} \to D(A) \quad \text{for any } t \in [0,t_1], \tag{3.17}$$

and

$$\sum_{j=0}^{\infty} \mu_j^2 \int_0^{t_1} ||AT(t)\varphi_j||^2 dt < \infty, \tag{3.18}$$

where $\{(\mu_j, \varphi_j)\}_{j=0}^{\infty}$ *are eigenvalues and eigenvectors of the operator* $P_0 \in \mathcal{L}(\mathcal{Z})$*, the optimal estimate* $\tilde{z}_{opt}(t)$ *is the unique solution of the equation*

$$d\tilde{z}(t) = A\tilde{z}(t)dt + F(t)(dy(t) - C\tilde{z}(t)dt), \quad \tilde{z}(0) = 0, \quad t \in [0,t_1]. \tag{3.19}$$

3.2 Steady-State Kalman Filter

On a finite time interval, the Kalman filter provides system (3.1)-(3.2) with an optimal estimate. As time $t \to \infty$, in some situations, the Kalman filter converges to a time-invariant filter .

The following theorem gives conditions under which the unique solution of (3.11) converges strongly to a solution of an algebraic Riccati equation (ARE).

Theorem 3.2.1 ([20], Corollary 6.13)**.** *Let $P(\cdot)$ be the unique solution of (3.11). Suppose the pair $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable and (A, C) exponentially detectable, then the inner-product version of ARE*

$$\langle [AX + XA^* - XC^*\tilde{R}^{-1}CX + G\tilde{Q}G^*]h_1, h_2\rangle = 0, \text{ for arbitrary } h_1, h_2 \in \mathcal{D}(A^*) \quad (3.20)$$

has a unique nonnegative solution, P_{ss}, which is the strong limit of $P(t)$ as $t \to \infty$, i.e. for each $h \in \mathcal{Z}$,

$$\lim_{t\to\infty} P(t)h = P_{ss}h, \quad (3.21)$$

The theorem below follows from Theorem (2.5.5) by duality.

Theorem 3.2.2 (Steady-state Kalman filter)**.** *If $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable and (A, C) is exponentially detectable, then the ARE (3.20) has a unique nonnegative, self-adjoint solution, $P_{ss} \in \mathcal{L}(\mathcal{Z})$, such that $A - P_{ss}C^*\tilde{R}^{-1}C$ generates an exponentially stable C_0-semigroup*

$$T_p(\cdot) : \mathbb{R}_+ \to \mathcal{L}(\mathcal{Z}).$$

The estimate obtained by steady-state Kalman filter for system (3.1)-(3.2) is

$$\tilde{z}(t) = \int_0^t T_p(t-s)P_{ss}C^*\tilde{R}^{-1}dy(s). \quad (3.22)$$

Proof. By Definition 2.5.4, the exponential detectability of (A, C) and stabilizability of $(A, G\sqrt{\tilde{Q}})$ imply that $(A^*, \sqrt{\tilde{Q}}G^*)$ is exponentially detectable and (A^*, C^*) is exponentially stabilizable. By Theorem 2.3.3, A^* is the infinitesimal generator of the C_0-semigroup $T^*(\cdot)$. Then, using the duality of (2.28) and (3.20), it follows from Theorem 2.5.5 that the ARE (3.20) has a unique nonnegative solution $P_{ss} \in \mathcal{L}(\mathcal{Z})$ such that $A - P_{ss}C^*\tilde{R}^{-1}C$ generates an exponentially stable C_0-semigroup. □

According to Theorem 3.1.5, the operator $P(t)$ is the error covariance in Kalman filtering so that

$$||P(t)||_1 = \min_{\tilde{z}} E\{||z(t) - \tilde{z}(t)||^2\},$$

where the minimum is taken over all estimates $\tilde{z}(t)$ of the form (3.4). As $t \to \infty$, with the strong convergence (3.21), P_{ss} is considered as the error covariance at steady state.

A new result of convergence in nuclear norm is proved here. The following two lemmas are useful.

Lemma 3.2.3. *[16, Theorem 3.3] Suppose that A is the infinitesimal generator of a C_0-semigroup $T(t)$ on the Hilbert space $\mathcal{Z}$, $B \in \mathcal{L}(\mathbb{C}^m, \mathcal{Z})$, and $C \in \mathcal{L}(\mathcal{Z}, \mathbb{C}^p)$. If (A, B) is exponentially stabilizable, then the minimal nonnegative solution of the ARE*

$$\langle [A^*X + XA - XB\tilde{R}^{-1}B^*X + C^*C]h_1, h_2\rangle = 0, \ h_1, h_2 \in \mathcal{D}(A)$$

is nuclear.

Lemma 3.2.4. *[34, Theorem 2] If a sequence of self-adjoint operators X_n converges strongly to X, and $||X_n||_1$ converges to $||X||_1$, then*

$$\lim_{n\to\infty} ||X_n - X||_1 = 0.$$

Theorem 3.2.5. *Assume $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable and (A, C) is exponentially detectable. If both the spaces $\mathcal{W}$ and $\mathcal{Y}$ are finite-dimensional, then P_{ss} is nuclear and*

$$\lim_{t\to\infty} ||P(t) - P_{ss}||_1 = 0. \tag{3.23}$$

Proof. The fact that P_{ss} is nuclear is a straightforward consequence of Lemma 3.2 by duality and the assumption that both the input space $\mathcal{W}$ for the process noise and measurement space $\mathcal{Y}$ are finite-dimensional. Also,

$$||P(t)||_1 = E\{||z(t) - \tilde{z}_{opt}(t)||^2\} < \infty.$$

Convergence of the nuclear norm

$$\lim_{t\to\infty} ||P(t)||_1 = ||P_{ss}||_1$$

will be proved first. Then the convergence in (3.23) will follow from Lemma 3.2.4.

Results on linear-quadratic control in Section 2.5 will be applied in the proof. Let $\mathcal{D}(A^*) \subset \mathcal{Z}$ be the domain of A^*. For arbitrary $h_0 \in \mathcal{D}(A^*)$, consider the dynamic system

$$\dot{z}(t) = A^* z(t) + C^* u(t), \quad z(0) = h_0, \quad t \geq 0. \tag{3.24}$$

Define cost functionals

$$\tilde{J}(h_0; 0, t, u) := \langle z(t), P_0 z(t) \rangle + \int_0^t \langle z(s), G\tilde{Q}G^* z(s) \rangle + \langle u(s), \tilde{R}u(s) \rangle ds$$

and

$$\tilde{J}(h_0; u) := \int_0^\infty \langle z(s), G\tilde{Q}G^* z(s) \rangle + \langle u(s), \tilde{R}u(s) \rangle ds.$$

Applying the results in Theorem 2.5.2 to system (3.24), since $P(\cdot)$ is the unique solution of Riccati equation (3.11), for each fixed $t \geq 0$,

$$\langle P(t)h_0, h_0 \rangle = \min_{u \in L_2([0,t];U)} \tilde{J}(h_0; 0, t, u).$$

Also, by the dual of Theorem 2.5.5,

$$\langle P_{ss}h_0, h_0 \rangle = \min_{u \in L_2([0,+\infty);U)} \tilde{J}(h_0; u).$$

The minimum is achieved by using the state feedback control

$$u_{ss}(t) := -\tilde{R}^{-1} C P_{ss} z(t),$$

with the state trajectory $z(t) = T_p^*(t)h_0$, in which $T_p^*(\cdot)$ is the exponentially stable C_0-semigroup on H generated by $A^* - C^*\tilde{R}^{-1}CP_{ss}$. By the definition of exponentially stable C_0-semigroup (Definition 2.5.4), there exist positive constants α and β such that

$$\| T_p^*(t) \| = \| T_p(t) \| \leq \beta e^{-\alpha t} \leq \beta, \text{ for any } t \geq 0.$$

Thus,

$$\begin{aligned}\langle P(t)h_0, h_0\rangle &\leq \tilde{J}(h_0; 0, t, u_{ss}) \\ &\leq \langle T_p^*(t)h_0, P_0 T_p^*(t)h_0\rangle + \tilde{J}(h_0; u_{ss}) \\ &= ||\sqrt{P_0}T_p^*(t)h_0||^2 + \langle P_{ss}h_0, h_0\rangle \\ &\leq \langle(\beta^2 P_0 + P_{ss})h_0, h_0\rangle\end{aligned}$$

for any $h_0 \in \mathcal{D}(A^*)$. Since A^* is the infinitesimal generator of C_0-semigroup $T^*(\cdot)$, $\mathcal{D}(A^*)$ is dense in $\mathcal{Z}$. Hence, for any $h \in \mathcal{Z}$,

$$\langle P(t)h, h\rangle \leq \langle(\beta^2 P_0 + P_{ss})h, h\rangle. \tag{3.25}$$

Suppose $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{Z}$. For any $\epsilon > 0$, since both P_0 and P_{ss} are nuclear, there exists positive integer N_ϵ such that for any $N > N_\epsilon$,

$$\sum_{j=N}^{\infty}\langle\beta^2 P_0\psi_j.\psi_j\rangle < \frac{\epsilon}{2}$$

and

$$\sum_{j=N}^{\infty}\langle P_{ss}\psi_j, \psi_j\rangle < \frac{\epsilon}{2}.$$

Therefore, by (3.25)

$$\begin{aligned}||P(t)||_1 - ||P_{ss}||_1 &= \sum_{j=1}^{\infty}\langle(P(t) - P_{ss})\psi_j, \psi_j\rangle \\ &\leq \sum_{j=1}^{N_\epsilon}\langle(P(t) - P_{ss})\psi_j, \psi_j\rangle + \sum_{j=N_\epsilon+1}^{\infty}\langle\beta^2 P_0\psi_j.\psi_j\rangle \\ &< \sum_{j=1}^{N_\epsilon}\langle(P(t) - P_{ss})\psi_j, \psi_j\rangle + \frac{\epsilon}{2}.\end{aligned}$$

Since $P(t)$ converges strongly to P_{ss}, there exists $t_\epsilon > 0$ such that for any $t > t_\epsilon$,

$$\sum_{j=1}^{N_\epsilon}|\langle(P_{ss} - P(t))\psi_j, \psi_j\rangle| < \frac{\epsilon}{2}.$$

It follows that

$$||P(t)||_1 - ||P_{ss}||_1 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for any } t > t_\epsilon.$$

Also, notice that for $t > t_\epsilon$

$$\begin{aligned} ||P_{ss}||_1 &= \sum_{j=1}^{N_\epsilon} \langle (P_{ss} - P(t))\psi_j, \psi_j \rangle + \sum_{j=1}^{N_\epsilon} \langle P(t)\psi_j, \psi_j \rangle + \sum_{j=N_\epsilon+1}^{\infty} \langle P_{ss}\psi_j, \psi_j \rangle \\ &< \frac{\epsilon}{2} + ||P(t)||_1 + \frac{\epsilon}{2} = ||P(t)||_1 + \epsilon. \end{aligned}$$

Hence,

$$||P_{ss}||_1 - \epsilon < ||P(t)||_1 < ||P_{ss}||_1 + \epsilon, \text{ for any } t > t_\epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{t\to\infty} ||P(t)||_1 = ||P_{ss}||_1.$$

Moreover, since $P(t)$ converges strongly to P_{ss} and the operators are self-adjoint, Lemma 3.2.4 then implies that

$$\lim_{t\to\infty} ||P(t) - P_{ss}||_1 = 0.$$

Hence $P(t)$ converges to P_{ss} in nuclear norm. □

With the convergence of the error covariance $P(t)$ in nuclear norm, it can be furthermore shown that $||P_{ss}||_1$ represents the minimum error variance at the steady state. Therefore, the steady-state estimation error can be determined by simply solving the ARE (3.20).

Proposition 3.2.6. *Assume $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable and (A, C) is exponentially detectable. If both the spaces $\mathcal{W}$ and $\mathcal{Y}$ are finite-dimensional, then*

$$\begin{aligned} ||P_{ss}||_1 &= \lim_{t\to\infty} E\{||z(t) - \tilde{z}_{opt}(t)||^2\} \\ &= \min_{\tilde{z}} \lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\}, \end{aligned} \tag{3.26}$$

where the minimum is taken over all estimates $\tilde{z}(t)$ of the form (3.4) such that the limit $\lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\}$ exists.

Proof. By Theorem 3.1.5,

$$\begin{aligned} ||P(t)||_1 &= E\{||z(t) - \tilde{z}_{opt}(t)||^2\} \\ &= \min_{\tilde{z}} E\{||z(t) - \tilde{z}(t)||^2\}. \end{aligned}$$

Let $t \to \infty$,

$$\begin{aligned} ||P_{ss}||_1 &= \lim_{t\to\infty} E\{||z(t) - \tilde{z}_{opt}(t)||^2\} \\ &= \lim_{t\to\infty} \min_{\tilde{z}} E\{||z(t) - \tilde{z}(t)||^2\} \end{aligned}$$

where the minimum is taken over all estimates $\tilde{z}(t)$ of the form (3.4). Hence,

$$\begin{aligned} ||P_{ss}||_1 &= \lim_{t\to\infty} E\{||z(t) - \tilde{z}_{opt}(t)||^2\} \\ &\geq \min_{\tilde{z}} \lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\}, \end{aligned}$$

where the minimum is taken over all estimates $\tilde{z}(t)$ of the form (3.4) such that the limit $\lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\}$ exists. On the other hand,

$$\begin{aligned} ||P_{ss}||_1 &= \lim_{t\to\infty} \min_{\tilde{z}} E\{||z(t) - \tilde{z}(t)||^2\} \\ &\leq \min_{\tilde{z}} \lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\}. \end{aligned}$$

Thus,

$$||P_{ss}||_1 = \min_{\tilde{z}} \lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\},$$

where the minimum is taken over all estimates $\tilde{z}(t)$ of the form (3.4) such that the limit $\lim_{t\to\infty} E\{||z(t) - \tilde{z}(t)||^2\}$ exists.

□

Chapter 4

Sensor Placement Problem

4.1 Problem Formulation

Consider an infinite-dimensional linear system

$$dz(t) = Az(t)dt + Gdw(t),\ z(0) = z_0,\ t \geq 0, \tag{4.1}$$
$$dy(t) = Cz(t)dt + dv(t),\ y(0) = 0, \tag{4.2}$$

where A, with domain $\mathcal{D}(A)$, is the infinitesimal generator of a C_0-semigroup $T(t)$ on a separable Hilbert space $\mathcal{Z}$, $w(t)$ is a Wiener process of incremental covariance $\tilde{Q}$ on a separable Hilbert space $\mathcal{W}$, $v(t)$ is a Wiener process of incremental covariance $\tilde{R}$ on a separable Hilbert space $\mathcal{Y}$. The operators $G \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ and $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$. For simplicity, assume $\mathcal{W} = \mathbb{R}$ in this chapter. This assumption is reasonable for the process noise that is random with respect to time and deterministic in spatial domain. Also, assume m sensors are used, with the measurement space $\mathcal{Y} = \mathbb{R}^m$ (for some positive integer m). The initial state, z_0, is a $\mathcal{Z}$-valued Gaussian random variable with zero mean value and covariance P_0. Assume $w(t)$, $v(t)$, and z_0 are mutually uncorrelated and the operator $\tilde{R} \in \mathcal{L}(\mathcal{Y})$ is coercive.

According to Theorem 2.3.10, if $T(t) : \mathcal{Z} \to \mathcal{D}(A)$, then (4.1) has the unique strong solution

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Gdw(s),$$

Assume that the pair $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable, and (A, C) is exponentially

detectable. Let $P_{ss} \in \mathcal{L}(\mathcal{Z})$ be the unique nonnegative solution of the inner-product version of ARE

$$\langle [AX + XA^* - XC^*\tilde{R}^{-1}CX + G\tilde{Q}G^*]h_1, h_2\rangle = 0, \text{ for all } h_1, h_2 \in \mathcal{D}(A^*).$$

By Proposition 3.2.6, $||P_{ss}||_1$ represents the limiting minimum estimation error variance:

$$||P_{ss}||_1 = \lim_{t\to\infty} E\{||z(t) - \tilde{z}_{opt}(t)||^2\}.$$

Selection and location of sensors to minimize $||P_{ss}||_1$ is therefore a reasonable design goal.

Suppose $m(\geq 1)$ sensors yield the measurement

$$y(t) = \int_0^t Cz(s)ds + v(t) \in \mathbb{R}^m, \tag{4.3}$$

where measurement noise

$$v(t) = (v_1(t), v_2(t), \ldots, v_m(t))^T \in \mathbb{R}^m$$

is characterized as an $\mathbb{R}^m$-valued Wiener process of incremental covariance $\tilde{R}$, with $v_j(t)$ represents the noise in j-th sensor, $j = 1, 2, \ldots, m$. Assume $v_1(t), v_2(t), \ldots, v_m(t)$ are mutually independent real-valued Wiener processes, with

$$E\{(v_j(t) - v_j(s))^2\} = (t - s)R_0, \ j = 1, 2, \ldots, m$$

for $t \geq s \geq 0$, where $R_0 \in \mathbb{R}_+$. If the sensors are of high quality, then $R_0 \in \mathbb{R}_+$ is small, while R_0 is larger for low-quality sensors. The overall incremental covariance of the measurement noise $v(t)$ is

$$\tilde{R} = diag(R_0, R_0 \ldots R_0) \in \mathbb{R}^{m\times m}.$$

The value of $||P_{ss}||_1$ is dependent on the measurement operator C and the matrix $\tilde{R}$.

4.2 Optimal Sensor Location

In many practical situations, the sensors' locations can be chosen within some compact set $\Omega \in \mathbb{R}^q$ ($q \leq 3$). Denoting the location of the m sensors by

$$\boldsymbol{l} := (l_1, l_2, ..., l_m) \in \Omega^m \subset \mathbb{R}^{q \times m}.$$

The output operator C is dependent on the sensor location:

$$C = C(\boldsymbol{l}).$$

Then, the solution to the ARE is also parameterized by $\boldsymbol{l}$:

$$P_{ss} = P_{ss}(\boldsymbol{l}).$$

An optimal sensor location $\hat{\boldsymbol{l}} \in \Omega^m$ is such that

$$||P_{ss}(\hat{\boldsymbol{l}}\ )||_1 = \min_{l \in \Omega^m} ||P_{ss}(\boldsymbol{l})||_1.$$

Optimal sensor location in this context is the dual problem of the LQ optimal actuator location problem, in which the nuclear norm $||\Pi||_1$ is minimized, with Π the unique nonnegative solution of the ARE

$$\langle h_1, [A^*\Pi + \Pi A - \Pi B R^{-1} B^* \Pi + Q] h_2 \rangle = 0, \text{ for any } h_1, h_2 \in D(A).$$

The existence of an optimal sensor location, $\hat{\boldsymbol{l}}$, is guaranteed by continuity of $||P_{ss}(\boldsymbol{l})||_1$ with respect to the sensor location $\boldsymbol{l}$. In [48], the existence of an optimal actuator location is proved. The theorem below follows from [48, Theorem 2.10] by duality.

Theorem 4.2.1. *Let $C(\boldsymbol{l}) \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, $\boldsymbol{l} \in \Omega^m$, be a family of output operators such that for any $\boldsymbol{l}_0 \in \Omega^m$,*

$$\lim_{\boldsymbol{l} \to \boldsymbol{l}_0} ||C(\boldsymbol{l}) - C(\boldsymbol{l}_0))|| = 0.$$

If the spaces $\mathcal{W}$ and $\mathcal{Y}$ are finite-dimensional, $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable and $(A, C(\boldsymbol{l}))$ are all exponentially detectable, then the corresponding Riccati operators $P_{ss} = P_{ss}(\boldsymbol{l})$ are continuous with respect to $\boldsymbol{l}$ in the nuclear norm:

$$\lim_{l \to l_0} ||P_{ss}(\boldsymbol{l}) - P_{ss}(\boldsymbol{l}_0)||_1 = 0,$$

and there exists an optimal sensor location $\hat{\boldsymbol{l}}$ *such that*

$$||P_{ss}(\hat{\boldsymbol{l}}\ )||_1 = \min_{\boldsymbol{l}\in\Omega^m} ||P_{ss}(\boldsymbol{l})||_1.$$

4.3 Computing Optimal Sensor Locations by Approximation

In practice, the infinite-dimensional ARE can not be solved exactly. Let $\mathcal{Z}_n \subset \mathcal{Z}$, $n = 1, 2, \ldots$, be a sequence of finite-dimensional subspaces of $\mathcal{Z}$ and Θ_n the orthogonal projection of $\mathcal{Z}$ onto $\mathcal{Z}_n$. For each $n \geq 1$, $\mathcal{Z}_n$ is an n-dimensional space equipped with the inner product inherited from $\mathcal{Z}$. Consider a sequence of operators $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$, $G_n \in \mathcal{L}(\mathcal{W}, \mathcal{Z}_n)$ and $C_n = C\mid_{\mathcal{Z}_n} \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Y})$ (the restriction of C to $\mathcal{Z}_n$), which leads to a sequence of approximations for the infinite-dimensional system (4.1):

$$\begin{aligned} dz_n(t) &= A_n z_n(t)dt + G_n dw(t),\ z_n(0) = z_{0,n} = \Theta_n z_0 \in \mathcal{Z}_n,\ t \geq 0, \\ dy_n(t) &= C_n z_n(t)dt + dv(t),\ y_n(0) = 0, \end{aligned}$$

If $(A_n, G_n\sqrt{\tilde{Q}})$ is exponentially stabilizable and (A_n, C_n) is exponentially detectable, then there exists the unique nonnegative solution $P_{ss}^{(n)} \in \mathcal{L}(\mathcal{Z}_n)$ to the finite-dimensional ARE

$$A_n P_{ss}^{(n)} + P_{ss}^{(n)} A_n^* - P_{ss}^{(n)} C_n^* \tilde{R}^{-1} C_n P_{ss}^{(n)} + G_n \tilde{Q} G_n^* = 0. \tag{4.4}$$

In [48], a standard set of assumptions for approximating controller design on the approximating systems is provided. The same assumptions should be satisfied for approximating estimator design. Let $T_n(t)$ indicate the semigroup generated by A_n. The standard assumptions are as follows:

(H1) For each $z \in \mathcal{Z}$,

- (i) $||T_n(t)P_n z - T(t)z|| \to 0$,
- (ii) $||T_n^*(t)P_n z - T^*(t)z|| \to 0$

uniformly in t on bounded intervals.

(H2) For each $z \in \mathcal{Z}$, $w \in \mathcal{W}$, $y \in \mathcal{Y}$,

- (i) $||G_n w - Gw|| \to 0$ and $||C_n \Theta_n z - Cz|| \to 0$,
- (ii) $||G_n^* \Theta_n w - G^* w|| \to 0$ and $||C_n^* z - C^* z|| \to 0$.

(H3) (i) The family of pairs $(A_n, G_n\sqrt{\tilde{Q}})$ is uniformly exponentially stabilizable, that is, there exists uniformly bounded sequence operators $K_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{W})$ such that the semigroups $T_{K_n}(t)$ generated by $A_n - G_n\sqrt{\tilde{Q}}K_n$ satisfy

$$||T_{K_n}(t)|| \leq \beta_1 e^{-\alpha_1}$$

for positive constants α_1 and $\beta_1 \geq 1$.

(ii) The family of pairs (A_n, C_n) is uniformly exponentially detectable, that is, there exists uniformly bounded sequence operators $F_n \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}_n)$ such that the semigroups $T_{F_n}(t)$ generated by $A_n - F_nC_n$ satisfy

$$||T_{F_n}(t)|| \leq \beta_2 e^{-\alpha_2}$$

for positive constants α_2 and $\beta_2 \geq 1$.

The assumptions (H1) and(H2) are required for the convergence of the systems (A_n, G_n, C_n), $n \geq 1$, as well as of the dual systems (A_n^*, G_n^*, C_n^*), $n \geq 1$, since the ajoint operators also present in the ARE; the assumption (H3) guarantees that the approximate ARE (4.4) has the unique solution $P_{ss}^{(n)}$ for each $n \geq 1$.

With assumptions (H1)-(H3), the convergence of $P_{ss}^{(n)}$ to the infinite-dimensional Riccati operator P_{ss} can be guaranteed, which follows from [48, Theorem 3.8] by duality:

Theorem 4.3.1. *Assume that $(A, G\sqrt{\tilde{Q}})$ is exponentially stabilizable, (A, C) is exponentially detectable, and the spaces $\mathcal{W}$ and $\mathcal{Y}$ are finite-dimensional. Let (A_n, G_n, C_n) be a sequence of approximations to (A, G, C) that satisfy assumptions (H1)-(H3) . Then*

$$\lim_{n\to\infty} ||P_{ss}^{(n)}\Theta_n - P_{ss}||_1 = 0, \tag{4.5}$$

where Θ_n is the orthogonal projection of $\mathcal{Z}$ onto $\mathcal{Z}_n$.

With convergence in (4.5), optimal sensor locations for approximating systems can also converge to the optimal sensor location for the infinite-dimensional system. The following result is a dual version of [48, Theorem 3.9].

Theorem 4.3.2. *Let $C(\boldsymbol{l}) \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, $\boldsymbol{l} \in \Omega^m$, be a family of output operators such that for any $\boldsymbol{l}_0 \in \Omega^m$,*

$$\lim_{\boldsymbol{l}\to\boldsymbol{l}_0} ||C(\boldsymbol{l}) - C(\boldsymbol{l}_0))|| = 0.$$

Assume that $(A, G\sqrt{\tilde{Q}})$ *is exponentially stabilizable and* $(A, C(\boldsymbol{l}))$ *are all exponentially detectable. Choose some approximation scheme such that assumptions (H1)-(H3) are satisfied for each* $\boldsymbol{l} \in \Omega^m$, $P_{ss}^{(n)}(\boldsymbol{l})$ *converges to the infinite-dimensional Riccati operator* $P_{ss}(\boldsymbol{l})$ *in nuclear norm as* $n \to \infty$. *Letting* $\hat{\boldsymbol{l}}$ *be an optimal sensor location with*

$$||P_{ss}(\hat{\boldsymbol{l}}\ )||_1 = \min_{l \in \Omega^m} ||P_{ss}(\boldsymbol{l})||_1.$$

It follows that

$$||P_{ss}(\hat{\boldsymbol{l}}\ )||_1 = \lim_{n \to \infty} ||P_{ss}^{(n)}(\hat{\boldsymbol{l}}_n)||_1,$$

and there exists a subsequence $\{\hat{\boldsymbol{l}}_{n_k}\}_{k=1}^{\infty}$ *of* $\{\hat{\boldsymbol{l}}_n\}_{n=1}^{\infty}$ *such that* $\lim_{k \to \infty} \hat{\boldsymbol{l}}_{n_k} = \hat{\boldsymbol{l}}$.

4.4 An Optimization Algorithm

In [24], an optimization algorithm is used to calculate LQ optimal actuator locations. The algorithm is presented in this section, which can be applied to search for optimal sensor locations.

Suppose an n-dimensional system is to be computed:

$$\begin{aligned} dz(t) &= Azdt + Gdw,\ z(0) = z_0,\ t \geq 0, \\ dy(t) &= Czdt + dv,\ y(0) = 0, \end{aligned}$$

where $A \in \mathbb{R}^{n \times n}$, $G \in \mathbb{R}^{n \times 1}$, and C is the output matrix that is dependent on sensor placement. The optimal sensor location problem is reformulated into a convex optimization problem as described in [33]. The reformulation relies on considering a discrete set of N possible sensor locations. Suppose that there are m sensors and N possible sensor locations. Let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_N)^T$ be a vector of N logical elements where the i-th entry $\kappa_i = 1$ if a sensor is placed at the i-th possible location and $\kappa_i = 0$ otherwise, so that

$$\sum_{i=1}^{N} \kappa_i = m.$$

Each such binary vector $\boldsymbol{\kappa}$ defines a possible set of sensor locations. Let $C_i \in \mathbb{R}^{1 \times n}$, $i = 1, 2, \ldots, N$, be the output operator with a single sensor placed at the i-th possible

location. Then the output matrix is

$$C(\boldsymbol{\kappa}) = (\kappa_1 C_1^T, \kappa_2 C_2^T, \ldots, \kappa_N C_N^T)^T$$

and the sensor noise covariance is

$$\tilde{R} = R_0 I_N,$$

where I_N is the $N \times N$ identity matrix. The operator P_{ss} is dependent on $\boldsymbol{\kappa}$:

$$P_{ss} = P_{ss}(\boldsymbol{\kappa}).$$

Define the objective function

$$\rho(\boldsymbol{\kappa}) := ||P_{ss}(\boldsymbol{\kappa})||_1.$$

Then the optimization problem is

$$\min\{\rho(\boldsymbol{\kappa}) : \boldsymbol{\kappa} \in \Upsilon\}, \tag{4.6}$$

subject to

$$\Upsilon = \{\boldsymbol{\kappa} \in \mathbb{R}^N \text{ s.t. } \kappa_i \in \{0,1\}; \sum_{i=1}^{N} \kappa_i = m\}.$$

Theorem 4.4.1. *[33, Theorem 10] Define the convex set*

$$\Upsilon_c := \{\boldsymbol{\kappa} \in \mathbb{R}^N \text{ s.t. } \kappa_i \geq 0,\ i = 1, 2, \ldots, N\}.$$

The function $\rho(\boldsymbol{\kappa}) : \Upsilon_c \to \mathbb{R}$ is a convex function. For any $\boldsymbol{\kappa} \in \Upsilon_c$, the following defines a subgradient ϱ of $\rho(\boldsymbol{\kappa})$:

$$\begin{aligned} &\varrho(\boldsymbol{\kappa}) = (\varrho_1(\boldsymbol{\kappa}), \varrho_2(\boldsymbol{\kappa}), \ldots, \varrho_n(\boldsymbol{\kappa})), \\ &\varrho_i(\boldsymbol{\kappa}) = ||L_i S(\boldsymbol{\kappa})||_1, \\ &L_i = C_i R_0^{-1} C_i^T, \quad i = 1, 2, \ldots, N, \\ &S(\boldsymbol{\kappa}) = -P_{ss}(\boldsymbol{\kappa}) \theta(\boldsymbol{\kappa}) P_{ss}(\boldsymbol{\kappa}), \end{aligned} \tag{4.7}$$

where $\theta(\boldsymbol{\kappa})$ is the solution of the Lyapunov equation

$$(A - P_{ss} C^T \tilde{R}^{-1} C)^T \theta(\boldsymbol{\kappa}) + \theta(\boldsymbol{\kappa})(A - P_{ss} C^T \tilde{R}^{-1} C) + I_n = 0.$$

Since $\rho(\boldsymbol{\kappa})$ is a convex function of $\boldsymbol{\kappa}$ on Υ_c and $\Upsilon \subset \Upsilon_c$, for arbitrary $\boldsymbol{\kappa}^0 \in \Upsilon$,

$$\rho(\boldsymbol{\kappa}) \geq \rho(\boldsymbol{\kappa}^0) + \langle \varrho(\boldsymbol{\kappa}^0), \boldsymbol{\kappa} - \boldsymbol{\kappa}^0 \rangle$$

for all $\boldsymbol{\kappa} \in \Upsilon$. Therefore, if $\hat{\boldsymbol{\kappa}} \in \Upsilon$ is such that

$$\hat{\vartheta} = \min_{\boldsymbol{\kappa} \in \Upsilon} \ \rho(\boldsymbol{\kappa}^0) + \langle \varrho(\boldsymbol{\kappa}^0), \boldsymbol{\kappa} - \boldsymbol{\kappa}^0 \rangle$$

and

$$\hat{\vartheta} = \rho(\boldsymbol{\kappa}^0) + \langle \varrho(\boldsymbol{\kappa}^0), \hat{\boldsymbol{\kappa}} - \boldsymbol{\kappa}^0 \rangle,$$

then $\hat{\boldsymbol{\kappa}}$ is the global solution to the optimization problem (4.6). The global solution $\hat{\boldsymbol{\kappa}}$ can be found by the algorithm summarized below:

Step 1: Choose an initial value $\boldsymbol{\kappa}^0$, solve the Riccati equation and calculate $\rho(\boldsymbol{\kappa}^0)$. Determine the subgragient $\varrho(\boldsymbol{\kappa}^0)$ by Theorem 4.4.1. Set $j = 0$ and choose a sufficiently small value $\epsilon > 0$.

Step 2: Solve the relaxed problem

$$\min_{\boldsymbol{\kappa} \in \Upsilon} \{\vartheta : \vartheta \geq \rho(\boldsymbol{\kappa}^i) + \langle \varrho(\boldsymbol{\kappa}^i), \boldsymbol{\kappa} - \boldsymbol{\kappa}^i \rangle, \ i = 0, 1, \ldots, j\}. \tag{4.8}$$

Let ϑ^{j+1} and $\boldsymbol{\kappa}^{j+1}$ be the optimal solution.

Step 3: Calculate $\rho^{j+1} = \rho(\boldsymbol{\kappa}^{j+1})$. If $\rho^{j+1} - \vartheta^{j+1} \leq \epsilon$, terminate. If not, calculate $\varrho^{j+1} = \varrho(\boldsymbol{\kappa}^{j+1})$ and return to Step B.

This proceduare generates a feasible sequence $\boldsymbol{\kappa}^j$, $j = 0, 1, 2, \ldots$, that will converge to the global solution $\hat{\boldsymbol{\kappa}}$.

The relaxed problem (4.8) in step 2 can be written as

$$\min_{\boldsymbol{\kappa} \in \Upsilon} \max_{0 \leq i \leq j} \ \rho(\boldsymbol{\kappa}^i) + \langle \varrho(\boldsymbol{\kappa}^i), \boldsymbol{\kappa} - \boldsymbol{\kappa}^i \rangle. \tag{4.9}$$

Since the variable $\boldsymbol{\kappa} \in \Upsilon$ is a vector of binary components, an integer optimization method, the branch and bound method, is applied to solve the problem (4.9). Initially, all the integer constraints are removed so that (4.9) is relaxed to

$$\begin{aligned} &\min_{\boldsymbol{\kappa} \in \Upsilon_0} \max_{0 \leq i \leq j} \ \rho(\boldsymbol{\kappa}^i) + \langle \varrho(\boldsymbol{\kappa}^i), \boldsymbol{\kappa} - \boldsymbol{\kappa}^i \rangle, \\ &\Upsilon_0 = \{\boldsymbol{\kappa} \in \mathbb{R}^N \text{ s.t. } 0 \leq \kappa_i \leq 1; \sum_{i=1}^{N} \kappa_i = m\}. \end{aligned} \tag{4.10}$$

The relaxed problem (4.10) is then solved. If it does not result in a binary solution, then an element of the vector is chosen to add equality binary constraints on, and the problem is solved once again. The problems with added constraints are called candidate subproblems, which form the branches of a binary tree.

Definition 4.4.2. *[29, Definition 5.3.3] A candidate subproblem will be considered that has been fathomed if one of the following two conditions takes place:*

(i) It can be ascertained that the feasible solution of the subproblem can not contain a better solution than the best solution so far;

(ii) An optimal solution of the subproblem is found.

To avoid unnecessary enumerations, fathoming tests can be performed for each candidate subproblem. Details of the branch and bound method can be found in [29, Chapter 5].

Chapter 5

Numerical Examples

In this chapter, three partial differential equation models, which depend on both time and location, are examined: two parabolic equations, one-dimensional diffusion and two-dimensional diffusion on an L-shaped region, as well as a hyperbolic equation that describes the vibration of a simply-supported Euler-Bernoulli beam. The equations are approximated by finite-dimensional systems using Galerkin methods, so that the optimal sensor locations can be calculated and the effect of sensor noise on estimation can be investigated numerically.

For simulations, the approximate algebraic Riccati equations are solved using the Matlab function *'care'*. The white Gaussian noises are approximated by Gaussian processes as in Section 2.4, with the time step size $\triangle t = 0.01$ that is used in solving the differential equations; noises are generated for each time step using the Matlab function *'randn'*, and added to the dynamic systems by linear interpolation [27, §5.3], using Matlab function *'interp1'*. Moreover, the Matlab function *'ode45'*, which is based on an explicit Runge-Kutta (4,5) formula [26], is used to solve the differential equations.

5.1 Galerkin Approximation

Consider an infinite-dimensional linear system

$$\begin{aligned} dz(t) &= Az(t)dt + Gdw(t),\ z(0) = z_0,\ t \geq 0, \\ dy(t) &= Cz(t)dt + dv(t),\ y(0) = 0, \end{aligned} \tag{5.1}$$

where A, with domain $\mathcal{D}(A)$, is the infinitesimal generator of a C_0-semigroup $T(\cdot)$ on a separable Hilbert space $\mathcal{Z}$, $w(t)$ is a Wiener process of incremental covariance $\tilde{Q}$ on a separable Hilbert space $\mathcal{W}$, $v(t)$ is a Wiener process of incremental covariance $\tilde{R}$ on a separable Hilbert space $\mathcal{Y}$. The operators $G \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ and $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$. For simplicity, assume $\mathcal{W} = \mathbb{R}$ and $\mathcal{Y} = \mathbb{R}^m$ (for some positive integer m). The initial state, z_0, is a $\mathcal{Z}$-valued Gaussian random variable with zero mean value and covariance P_0. Assume $w(t)$, $v(t)$, and z_0 are mutually uncorrelated, and the operator $\tilde{R} \in \mathcal{L}(\mathcal{Y})$ is coercive.

For computational purposes, an infinite-dimensional system is approximated by finite-dimensional systems. Suppose the approximation lies in an n-dimensional subspace $\mathcal{Z}_n$ of the state space $\mathcal{Z}$, with Θ_n the orthogonal projection of $\mathcal{Z}$ onto $\mathcal{Z}_n$. The subspace $\mathcal{Z}_n$ is equipped with the inner product inherited from $\mathcal{Z}$. A standard Galerkin approximation for (5.1) is to seek a $\mathcal{Z}_n$-valued function $z_n(t)$ that solves the weak formulation

$$\langle dz_n(t), \varphi\rangle = \langle Az_n(t), \varphi\rangle dt + \langle Gdw(t), \varphi\rangle,\ \langle z_n(0),\ \varphi\rangle = \langle z_0, \varphi\rangle,\ t \geq 0, \tag{5.2}$$

for all $\varphi \in \mathcal{Z}_n$.

Suppose $\phi_1, \phi_2 \ldots, \phi_n \in \mathcal{Z}_n$ are linearly independent so that

$$\mathcal{Z}_n = \text{span}\{\phi_1, \phi_2 \ldots, \phi_n\}.$$

The projection $\Theta_n : \mathcal{Z} \to \mathcal{Z}_n$ is orthogonal if and only if for all $h \in \mathcal{Z}$ and $j = 1, 2, \ldots, n$,

$$\langle \Theta_n h, \phi_j\rangle = \langle h, \phi_j\rangle.$$

For $h \in \mathcal{Z}$, let

$$\underline{h_n} := (h_{n,1}, h_{n,2}, \ldots, h_{n,n})^T \in \mathbb{R}^n$$

be the vector of coefficients of the projection on $\mathcal{Z}_n$. Then

$$\Theta_n h = \sum_{i=1}^{n} h_{n,i}\phi_i \in \mathcal{Z}_n,$$

and

$$\langle \Theta_n h, \phi_j\rangle = \sum_{i=1}^{n} h_{n,i}\langle \phi_i, \phi_j\rangle = \langle h, \phi_j\rangle, \quad j = 1, 2, \ldots, n. \tag{5.3}$$

Define the matrix

$$M_n := \begin{pmatrix} \langle \phi_1, \phi_1 \rangle & \dots & \langle \phi_n, \phi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_1, \phi_n \rangle & \dots & \langle \phi_n, \phi_n \rangle \end{pmatrix}.$$

By (5.3),

$$M_n \underline{h_n} = (\langle h, \phi_1 \rangle, \langle h, \phi_2 \rangle, \dots, \langle h, \phi_n \rangle)^T. \tag{5.4}$$

Since $\phi_1, \phi_2, \dots, \phi_n$ are linearly independent, the matrix M_n is nonsingular and, hence, the vector can be uniquely determined by (5.4):

$$\underline{h_n} = M_n^{-1}(\langle h, \phi_1 \rangle, \langle h, \phi_2 \rangle, \dots, \langle h, \phi_n \rangle)^T.$$

In particular, if $\{\phi_1, \phi_2, \dots, \phi_n\}$ forms an orthonormal basis for $\mathcal{Z}_n$, then $M_n \in \mathbb{R}^{n \times n}$ is the identity matrix. The orthogonal projection $\Theta_n : \mathcal{Z} \to \mathcal{Z}_n$ is well-defined:

$$\Theta_n h = (\phi_1, \phi_2, \dots, \phi_n) \cdot (M_n^{-1}(\langle h, \phi_1 \rangle, \langle h, \phi_2 \rangle, \dots, \langle h, \phi_n \rangle)^T), \tag{5.5}$$

for all $h \in \mathcal{Z}_n$. Define $G_n := \Theta_n G \in \mathcal{L}(\mathbb{R}, \mathcal{Z}_n)$ and $z_{0,n} := \Theta_n z_0 \in \mathcal{Z}_n$. By defining the approximate generator $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ such that

$$\langle A_n z_n, \varphi \rangle = \langle A z_n, \varphi \rangle$$

for arbitrary $z_n, \varphi \in \mathcal{Z}_n$, (5.2) is equivalent to

$$\langle dz_n(t), \phi_j \rangle = \langle A_n z_n(t), \phi_j \rangle dt + \langle G_n dw(t), \phi_j \rangle, \ \langle z_n(0), \ \phi_j \rangle = \langle z_{0,n}, \phi_j \rangle, \ t \geq 0, \tag{5.6}$$

for $j = 1, 2, \dots, n$.

If the operator A has eigenfunctions $\{\phi_n(x) : n \geq 1\}$ that form an orthonormal basis for the state space $\mathcal{Z}$, define

$$\mathcal{Z}_n := \text{span}\{\phi_1, \phi_2 \dots, \phi_n\}.$$

Then $\mathcal{Z}_n \subset \mathcal{D}(A)$ and the approximating generator $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ can be defined by

$$A_n \phi_j = A \phi_j$$

for $j = 1, 2, \dots, n$.

More generally, for partial differential equation models, the infinitesimal generator A can often be described in weak form as follows [47]. Let $\mathcal{V}$ be a Hilbert space that is dense

in $\mathcal{Z}$ and consider a bilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$. A linear operator A can be defined by

$$\langle -A\psi, \varphi \rangle = a(\psi, \varphi),$$

with domain $\mathcal{D}(A) = \{\psi \in \mathcal{V} : \exists h \in \mathcal{Z}, \text{ s.t. } a(\psi,\varphi) = \langle h, \varphi \rangle \text{ for all } \varphi \in \mathcal{V}\}$. On a subspace $\mathcal{Z}_n \subset \mathcal{V}$, the approximating generator $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ is defined by

$$\langle -A_n z_n, \varphi_n \rangle = a(z_n, \varphi_n)$$

for all $z_n, \varphi_n \in \mathcal{Z}_n$. This type of approximation is used in finite element methods.

Using the Galerkin approximation, a finite-dimensional system is obtained and the corresponding finite-dimensional ARE can be solved.

5.2 One-dimensional Diffusion Equation

Consider a one-dimensional diffusion equation with disturbance

$$\frac{\partial z}{\partial t} = \sigma \frac{\partial^2 z}{\partial x^2} + g(x)\eta(t),\ 0 \le x \le 1,\ t \ge 0, \tag{5.7}$$

where σ is the diffusivity, the function $g(x)$ describes the shape of the spatially distributed disturbance, and $\eta(t)$ is a time-dependent noise model. Assume the initial condition

$$z(0,x) = z_0(x), \tag{5.8}$$

where z_0 is a $L^2(0,1)$-valued Gaussian random variable with zero mean value and covariance P_0, and Neumann boundary condition

$$\frac{\partial z}{\partial x}(t,0) = 0,\ \frac{\partial z}{\partial x}(t,1) = 0. \tag{5.9}$$

The physical interpretation of z depends on the application and its value is with respect to a reference state. For example, in the case of thermal diffusion, z is temperature, and temperature in Celsius is with reference to the freezing point of water.

Suppose $\eta(t)$ is a real-valued white Gaussian noise so that the integral

$$w(t) := \int_0^t \eta(s) ds$$

is a real-valued Wiener process, with

$$E\{(w(t)-w(s))^2\} = (t-s)\tilde{Q},$$

for $t \geq s \geq 0$, where $\tilde{Q} \in \mathbb{R}_+$. Let the state space $\mathcal{Z} = L^2(0,1)$ and $A = \sigma\frac{\partial^2}{\partial x^2}$ with domain

$$\mathcal{D}(A) = \{h \in \mathcal{H}^2(0,1) : h'(0) = h'(1) = 0\} \subset \mathcal{Z}.$$

The state-space representation for the equation (5.7) is

$$dz(t) = Az(t)dt + Gdw(t),\ z(0) = z_0,\ t \geq 0, \tag{5.10}$$

where $G \in \mathcal{L}(\mathbb{R}, \mathcal{Z})$ is defined by

$$Gk = kg(x),\ \text{for } k \in \mathbb{R},$$

The operator A generates a C_0-semigroup $T(t)$, so that the solution to (5.22) can be expressed as

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Gdw(s),\ t \geq 0.$$

Each sensor measures the average state over a small interval of length $\triangle > 0$. Write

$$\boldsymbol{l} = (l_1, l_2, \ldots, l_m),$$

and define

$$c_{l_j}(x) := \begin{cases} 1/\triangle, & |x - l_j| \leq \frac{\triangle}{2} \\ 0, & \text{otherwise} \end{cases},$$

for $j = 1, 2, \ldots, m$. With m sensors centred at $x = l_1, l_2, ..., l_m$,

$$C(\boldsymbol{l})z = (\langle c_{l_1}, z\rangle, \langle c_{l_2}, z\rangle, \ldots, \langle c_{l_m}, z\rangle)^T,$$

and the measurement is

$$y(t) = \int_0^t C(\boldsymbol{l})z(s)ds + v(t),$$

where $v(t)$ is a $\mathbb{R}^m$-valued Wiener process, with

$$E\{(v(t)-v(s))(v(t)-v(s))^T\} = (t-s)\tilde{R}$$

for $t \geq s \geq 0$.

5.2.1 Finite-Dimensional Approximation

A Galerkin approximation based on the eigenfunctions of A will be used. The operator $A = \sigma \frac{\partial^2}{\partial x^2}$ has eigenfunctions

$$\phi_0(x) = 1,\ \phi_j(x) = \sqrt{2}\cos(j\pi x),\ j \geq 1,$$

with corresponding eigenvalues

$$\lambda_j := -\sigma j^2 \pi^2, j \geq 0.$$

The eigenfunctions form an orthonormal basis for the state space $\mathcal{Z} = L^2(0,1)$. Thus, for $z \in \mathcal{Z}$, there exists the expansion

$$z = \sum_{j=0}^{\infty} \langle z, \phi_j \rangle \phi_j,$$

where $\langle , \rangle$ denotes the inner product on $\mathcal{Z}$.

Let

$$\mathcal{Z}_n := \text{span}\{\phi_j(x) : 0 \leq j \leq n-1\}$$

be a subspace of the state space $\mathcal{Z}$, with the inner product inherited from $\mathcal{Z}$. Define the orthogonal projection $\Theta_n : \mathcal{Z} \to \mathcal{Z}_n$ as

$$\Theta_n(z) = \sum_{j=0}^{n-1} \langle z, \phi_j \rangle \phi_j,\ z \in \mathcal{Z}.$$

Let $A_n = \Theta_n A \mid_{\mathcal{Z}_n} \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$, $G_n = \Theta_n G \in \mathcal{L}(\mathbb{R}, \mathcal{Z}_n)$, $C_n(\boldsymbol{l}) = C(\boldsymbol{l}) \mid_{\mathcal{Z}_n} \in \mathcal{L}(\mathcal{Z}_n, \mathbb{R}^m)$, and $z_{0,n} = \Theta_n z_0 \in \mathcal{Z}_n$. A Galerkin approximation $z_n(t) \in \mathcal{Z}_n$ of the state $z(t)$ shall solve the weak formulation

$$\langle dz_n(t), \phi_j \rangle = \langle A_n z_n(t), \phi_j \rangle dt + \langle G_n dw(t), \phi_j \rangle,\ \langle z_{0,n}(0),\ \phi_j \rangle = \langle z_0, \phi_j \rangle,\ t \geq 0, \tag{5.11}$$

for $j = 0, 1, 2, \ldots, n-1$. It can be easily verified that the approximation by eigenfunctions satisfies the assumptions (H1)-(H3) listed in Section 4.3 and, therefore, the convergence can be guaranteed.

For calculations, let the vector

$$\underline{z_n}(t) := (z_{n,1}(t), z_{n,2}(t), \ldots, z_{n,n}(t))^T \in \mathbb{R}^n$$

so that

$$z_n(t) = \sum_{i=0}^{n-1} z_{n,i+1}(t)\phi_i.$$

Substituting it into (5.11) gives

$$\begin{array}{ll} \sum_{i=0}^{n-1} \langle \phi_i, \phi_j \rangle dz_{n,i+1}(t) & = \sum_{i=0}^{n-1} \langle A\phi_i, \phi_j \rangle z_{n,i+1}(t) + \langle G dw(s), \phi_j \rangle, \\ \sum_{i=0}^{n-1} \langle \phi_i, \phi_j \rangle z_{n,i+1}(0) & = \langle z_0, \phi_j \rangle, \qquad\qquad j = 0, 1, 2, \ldots, n-1. \end{array} \tag{5.12}$$

Since $\phi_0, \phi_1, \phi_2, \ldots, \phi_{n-1}$ are eigenfunctions of A that form an orthonomal basis for the subspace $\mathcal{Z}_n$,

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}, \ \langle A\phi_i, \phi_j \rangle = \lambda_i \delta_{ij},$$

for $i, j = 0, 1, 2, \ldots, n-1$, where δ_{ij} is the Kronecker delta. Define the matrix

$$\underline{A_n} := diag(\lambda_0, \lambda_1, ..., \lambda_{n-1})$$

and the vectors

$$\underline{g_n} := (\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle, \ldots, \langle g, \phi_n \rangle)^T,$$
$$\underline{z_{0,n}} := (\langle z_0, \phi_1 \rangle, \langle z_0, \phi_2 \rangle, \ldots, \langle z_0, \phi_n \rangle)^T.$$

It follows from (5.24) that

$$d\underline{z_n}(t) = \underline{A_n}\ \underline{z_n}(t)dt + \underline{g_n}dw(t), \ \underline{z_n}(0) = \underline{z_{0,n}}.$$

For each $j = 0, 1, 2, \ldots, n-1$, $C(\boldsymbol{l})\phi_j \in \mathbb{R}^m$. Define the matrix

$$\underline{C_n(\boldsymbol{l})} := (C(\boldsymbol{l})\phi_0, C(\boldsymbol{l})\phi_1, C(\boldsymbol{l})\phi_2, \ldots, C(\boldsymbol{l})\phi_{n-1}) \in \mathbb{R}^{m \times n}.$$

Then,

$$C_n(\boldsymbol{l})z_n = \underline{C_n(\boldsymbol{l})}\ \underline{z_n}.$$

Therefore, the approximate system in vector form is obtained:

$$\begin{aligned} d\underline{z_n}(t) &= \underline{A_n}\ \underline{z_n}(t)dt + \underline{g_n}dw(t),\ \underline{z_n}(0) = \underline{z_{0,n}}, \\ dy_n &= \underline{C_n(\boldsymbol{l})}\ \underline{z_n}dt + dv(t). \end{aligned}$$

5.2.2 Optimal Sensor Location

In this section, optimal sensor location that minimizes $||P_{ss}||_1$ is calculated for the case where only one sensor is used. For the simulations, the diffusivity was set to be $\sigma = 0.1$, the approximation mode number $n = 20$, the spatial domain $[0, 1]$ was equally discretized into subintervals of length $\triangle = 0.04$, and the initial condition was

$$z_0 = 2x + x^2. \tag{5.13}$$

First, consider the case where the noise on the state is evenly distributed in space:

$$g(x) = 1.$$

The value of $||P_{ss}||_1$ was computed for different sensor locations, which is constant with respect to sensor location. Hence, there does not exist a unique optimal sensor location in this case.

Next, consider a spatially localized disturbance centered at $x = 0.2$:

$$g(x) = \operatorname{sech}(100(x - 0.2)),$$

as displayed in Figure 5.1. By Figure 5.2, the value of $||P_{ss}||_1$ is minimized at around $x = 0.2$, where the process noise locates. Figure 5.3 compares two estimates with the actual state at $x = 0.5$. One estimate is obtained using a single sensor at optimal location $x = 0.2$, while the other is obtained using a sensor located at non-optimal location $x = 0.8$. The sensor placed at optimal location corresponds to a better estimate.

For a mixed disturbance function that combines a localized disturbance with a evenly distributed disturbance

$$g(x) = 0.5 \operatorname{sech}(100(x - 0.2)) + 0.5,$$

as shown in Figure 5.4, the unique optimal sensor location is where the process noise peaks (Figure 5.5). Figure 5.6 compares two estimates with the actual state at $x = 0.5$. One estimate is obtained using a single sensor at optimal location $x = 0.2$, while the other is

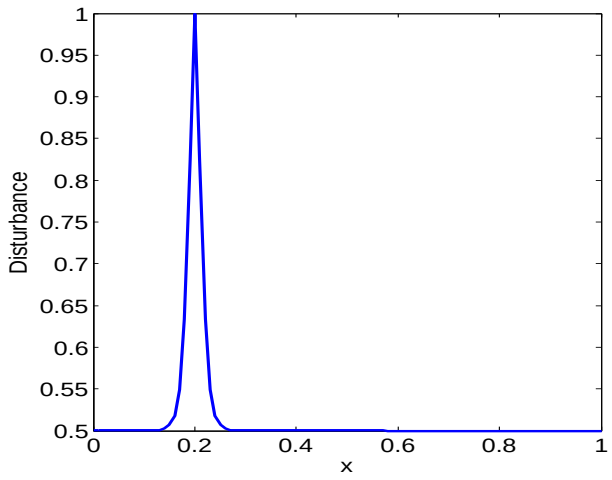

Figure 5.1: Disturbance spatially localized around $x = 0.2$.

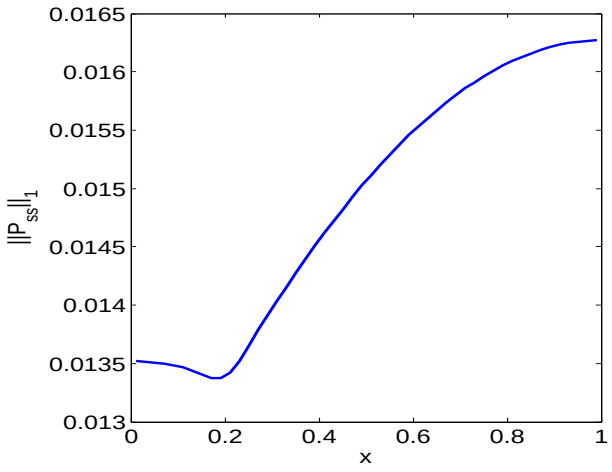

Figure 5.2: Spatially localized disturbance; $||P_{ss}||_1$ reaches minimum value for sensor location at around $x = 0.2$.

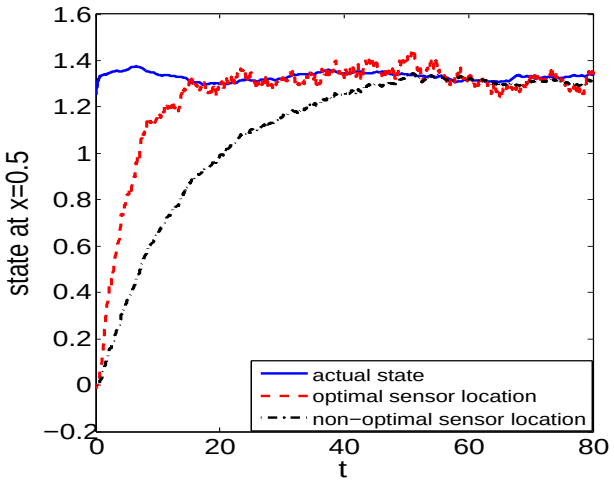

Figure 5.3: Spatially localized disturbance. Comparison of the actual and estimated state at $x = 0.5$. One estimate is obtained using a single sensor at optimal location $x = 0.2$, while the other is obtained using a sensor located at non-optimal location $x = 0.8$.

obtained using a sensor located at non-optimal location $x = 0.8$. Again, the sensor placed at optimal location corresponds to a better estimate.

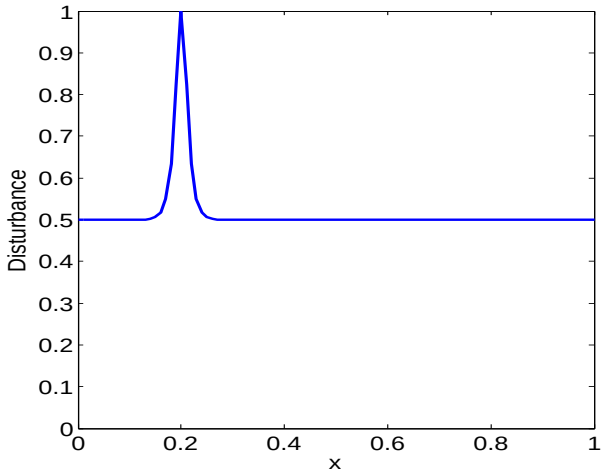

Figure 5.4: Mixed disturbance $g(x) = 0.5\,\mathrm{sech}(100(x - 0.2)) + 0.5$.

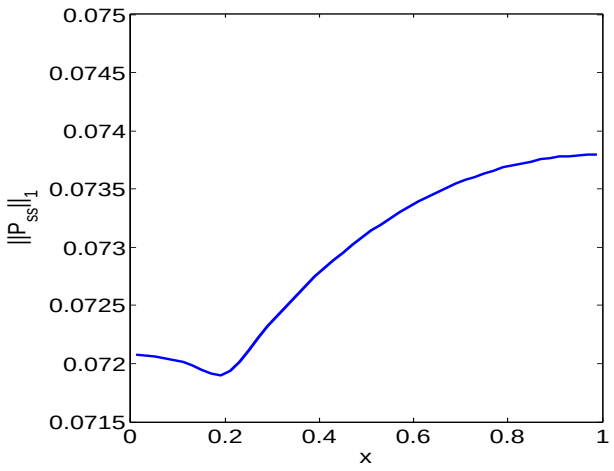

Figure 5.5: Mixed disturbance; $||P_{ss}||_1$ reaches minimum value for sensor location at around $x = 0.2$.

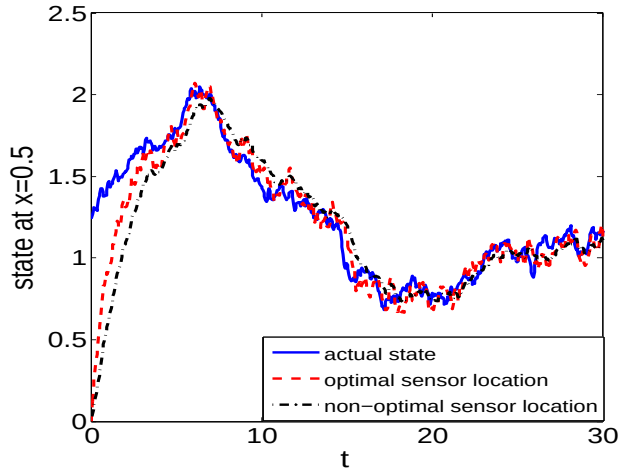

Figure 5.6: Mixed disturbance. Comparison of the actual and estimated state at $x = 0.5$. One estimate is obtained using a single sensor at optimal location $x = 0.2$, while the other is obtained using a sensor located at non-optimal location $x = 0.8$.

5.2.3 Effect of Sensor Noise on Estimation

In this section, the effect of sensor noise on estimation is investigated by considering the dependence of $||P_{ss}||_1$ on the sensor noise variance R_0 and using different number of sensors.

As mentioned previously, if there are $m(\geq 1)$ sensors, the measurement noise is

$$v(t) = (v_1(t), v_2(t), \ldots, v_m(t))^T \in \mathbb{R}^m$$

where $v_j(t)$ $(j = 1, 2, \ldots, m)$ represents the noise in j-th sensor, and $v_1(t), v_2(t), \ldots, v_m(t)$ are mutually independent real-valued Wiener processes, with

$$E\{(v_j(t) - v_j(s))^2\} = (t - s)R_0, \ j = 1, 2, \ldots, m$$

for $t \geq s \geq 0$, where $R_0 \in \mathbb{R}_+$. If the sensors are of high quality, then $R_0 \in \mathbb{R}_+$ is small, while R_0 is larger for low-quality sensors. Both the vaule of R_0 and m can be varied.

For the simulations, the diffusivity was set to $\sigma = 0.1$, the approximation mode number $n = 20$ and the coefficient of process noise covariance was $\tilde{Q} = 10$, the spatial domain $[0, 1]$ was equally discretized into subintervals of length $\triangle = 0.04$, and the initial condition was

$$z_0 = a(2x + x^2). \tag{5.14}$$

where a has normal distribution $\mathcal{N}(0, 10)$ and was chosen using the Matlab function 'randn'.

Three different disturbances were considered.

Spatially distributed disturbance

First, consider the case where the noise on the state is evenly distributed in space:

$$g(x) = 1.$$

The initial condition (5.14) was scaled by $a = 0.2722$.

When using a single sensor, the sensor noise covariance $\tilde{R} = R_0$. The value of $||P_{ss}||_1$ was computed for R_0 ranging from 0.1 to 2. As shown in Figure 5.7(a), $||P_{ss}||_1$ is an increasing function of R_0. Motivated by the presence of a quadratic term $XC^*\tilde{R}^{-1}CX$ in ARE (3.20), a square root function was used to match the value of $||P_{ss}||_1$:

$$f_1(R_0) := C_1\sqrt{R_0},$$

in which the constant $C_1 \approx 1.00$.

Now suppose a number of poor-quality sensors (say $R_0 = 2$) are used. Figure 5.7(b) shows that $||P_{ss}||_1$ decreases as the sensor number m increases. Let

$$g_1(m) := C_2/\sqrt{m} \quad (C_2 \approx 1.41)$$

be the function defined continuously on the interval $[0, 20]$. As shown in Figure 5.7(b), the curve of function g_1 fits with the value of $||P_{ss}||_1$ for $m = 1, 2, ..., 20$. It appears that $||P_{ss}||_1$ is approximately proportional to the value of $\sqrt{R_0/m}$:

$$||P_{ss}||_1 \approx C\sqrt{R_0/m} \tag{5.15}$$

with $C = C_1 \approx C_2/\sqrt{2}$. Therefore, a smaller value of R_0/m implies a better estimate.

The two estimates obtained by using a single sensor of noise scaled by $R_0 = 0.2$ and by using 15 sensors of noise scaled by $R_0 = 2$, are compared in Figure 5.8, which shows a comparison of the two estimates with the actual system state $z(t)$ at the middle point $x = 0.5$ for $t \in [0, 20]$. Because of the process and sensor noise, no estimate will be perfect; however, using multiple relatively poor-quality sensors led to a more accurate estimate.

Spatially localized disturbance

Now, consider a localized disturbance as illustrated in Figure 5.1:

$$g(x) = \text{sech}(100(x - 0.2)).$$

The initial condition (5.14) was scaled by $a = 2.1064$.

By computing the value of $||P_{ss}||_1$ for the case that using a single sensor ($m = 1$) with varied $R_0 \in [0.1, 2]$, an ascending curve is derived, as shown in Figure 5.9(a). It again coincides well with a square root function:

$$f_2(R_0) := C_3\sqrt{R_0} \quad (C_3 \approx 0.09).$$

For the multi-sensor case (with $R_0 = 2$), the value of $||P_{ss}||_1$ fits well with the curve of a decreasing function

$$g_2(m) := C_4/\sqrt{m} \quad (C_4 \approx 0.14),$$

as shown in Figure 5.9(b). Hence, the same proportional relation (5.15), but with constant $C = C_3 \approx C_4/\sqrt{2}$ appears to hold.

Comparisons of the actual system state with estimates made by a single sensor of noise scaled by $R_0 = 0.2$ and by 15 sensors of noise scaled by $R_0 = 2$ are displayed in Figure 5.10. The errors are similar for estimation obtained using either type of measurement.

Spatially mixed disturbance

A third disturbance, as shown in Figure 5.4, that includes both the evenly distributed disturbance and the spatially localized disturbance:

$$g(x) = 0.5\,\mathrm{sech}(100(x - 0.2)) + 0.5,$$

was considered. The initial condition (5.14) was scaled by $a = -0.3404$.

Like the previous two disturbances, the same computations were carried out. The value of $||P_{ss}||_1$ as the sensor noise covariance is changed are shown in Figure 5.11. Two curves

$$f_3(R_0) := C_5\sqrt{R_0} \quad (C_5 \approx 1.57),$$

$$g_3(m) := C_6/\sqrt{m} \quad (C_6 \approx 2.25).$$

were fit to the plots in Figure 5.11(a) and 5.11(b) respectively. The same square root relation as for other disturbances seems to hold with constant $C = C_5 \approx C_6/\sqrt{2}$.

The estimate obtained with one sensor of noise scaled by $R_0 = 0.2$ is compared with that obtained using 15 sensors of noise scaled by $R_0 = 2$ in Figure 5.12. The accuracy of the estimate obtained with many inaccurate sensors is better than that obtained with one accurate sensor.

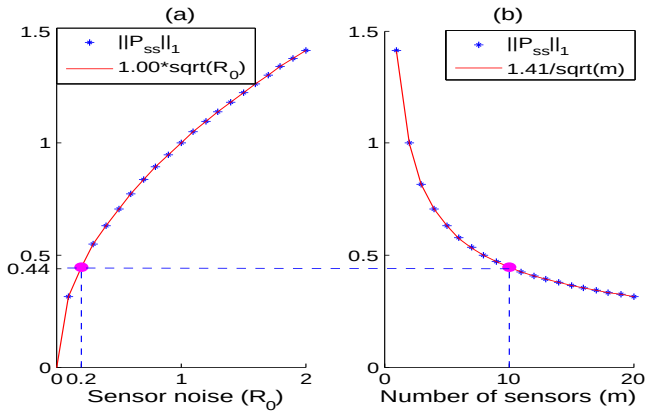

Figure 5.7: Uniform disturbance $g(x) = 1$. (a)$||P_{ss}||_1$ is an increasing function of $\tilde{R} = R_0$, coincides well with the curve $f_1(R_0) = C_1\sqrt{R_0}$, $C_1 \approx 1.00$; (b)$||P_{ss}||_1$ is an decreasing function of the number of sensors (m) with $R_0 = 2$. The curve is matched by $g_1(m) = C_2/\sqrt{m}$, $C_2 \approx 1.41$

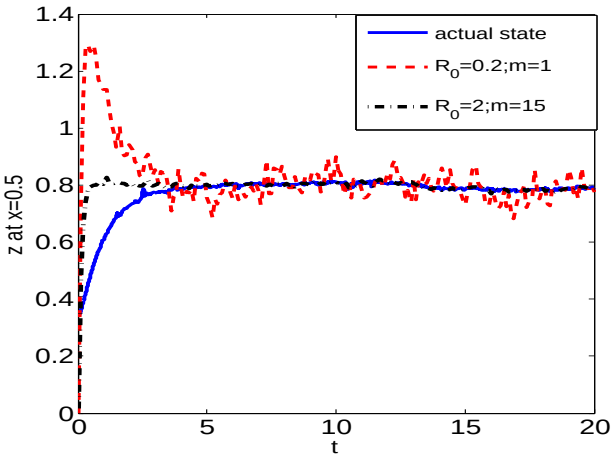

Figure 5.8: Comparison of the actual state at $x = 0.5$, $t \in [0, 20]$ for $g(x) = 1$ with estimates using a single sensor with noise variance $R_0 = 0.2$, and using 15 sensors each with $R_0 = 2$.

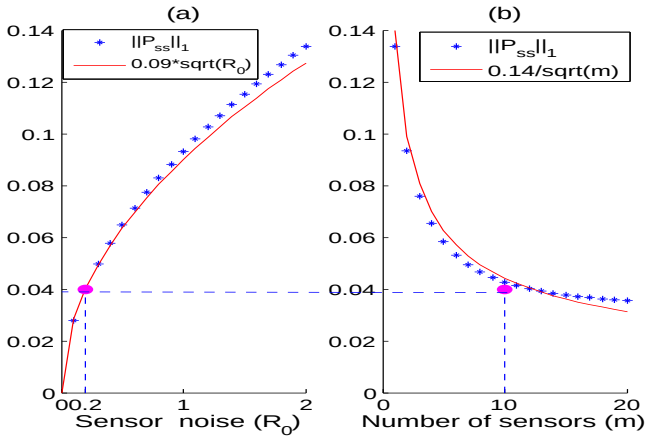

Figure 5.9: Spatially localized disturbance. (a)$||P_{ss}||_1$ is an increasing function of $\tilde{R} = R_0$. It coincides $f_2(R_0) = C_3\sqrt{R_0}$, $C_3 \approx 0.09$; (b)$||P_{ss}||_1$ is an decreasing function of the number of sensors (m) with fixed $R_0 = 2$. It is matched by the curve $g_2(m) = C_4/\sqrt{m}$, $C_4 \approx 0.14$.

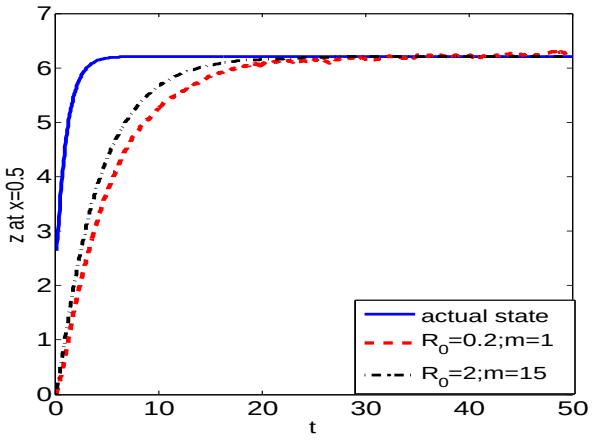

Figure 5.10: Comparison of the actual and estimated state at $x = 0.5$ with $g(x)$ a spatially localized disturbance. One estimate is obtained using a single sensor with noise variance $R_0 = 0.2$, while the other uses 15 sensors each with $R_0 = 2$.

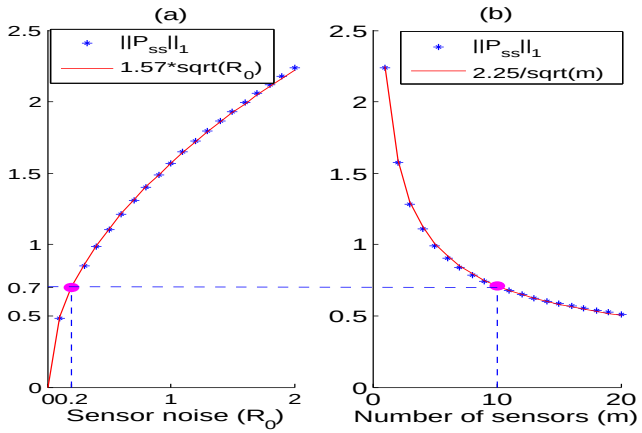

Figure 5.11: Mixed disturbance. (a)$||P_{ss}||_1$ is an increasing function of $\tilde{R} = R_0$ and the behaviour matches the curve $f_3(R_0) = C_5\sqrt{R_0}$, $C_5 \approx 1.57$; (b)$||P_{ss}||_1$ is an decreasing function of the number of sensors (m) and matches the curve $g_3(m) = C_6/\sqrt{m}$, $C_6 \approx 2.25$

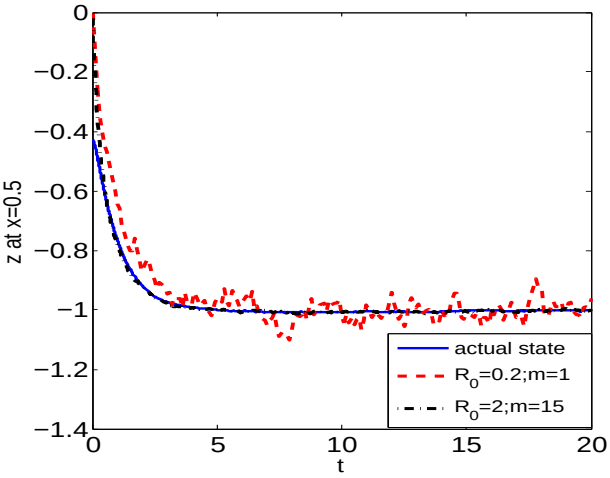

Figure 5.12: Mixed disturbance. Comparison of the actual and estimated state at $x = 0.5$. The two estimates are obtained from a single sensor with noise variance $R_0 = 0.2$ and 15 sensors each with $R_0 = 2$.

5.3 Simply Supported Euler-Bernoulli Beam

Consider a simply supported Euler-Bernoulli beam of length 1, with Kelvin-Voigt damping. Let $f(t,x)$ denote the deflection of the beam at time t and position x. The beam deflection is described by the partial differential equation

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^4 f}{\partial x^4} + c_d \frac{\partial^5 f}{\partial x^4 \partial t} = g(x)\eta(t), \quad t \geq 0,\ 0 < x < 1, \tag{5.16}$$

where c_d is the damping parameter, $g(x)$ models the shape of the spatially distributed disturbance, and $\eta(t)$ is a time-dependent noise model. Assume initial condition

$$f(0,x) = f_0(x),\ \frac{\partial}{\partial t} f(0,x) = f_1(x),$$

and boundary conditions

$$f(t,0) = f(t,1) = 0,$$
$$\frac{\partial^2}{\partial x^2} f(t,0) = \frac{\partial^2}{\partial x^2} f(t,1) = 0.$$

Suppose $\eta(t)$ is a real-valued white Gaussian noise so that the integral

$$w(t) := \int_0^t \eta(s) ds$$

is a real-valued Wiener process, with

$$E\{(w(t) - w(s))^2\} = (t-s)\tilde{Q},$$

for $t \geq s \geq 0$, where $\tilde{Q} \in \mathbb{R}_+$. Let

$$\mathcal{H}_s(0,1) = \{f \in \mathcal{H}^2(0,1) : f(0) = f(1) = 0\}$$

and the state space $\mathcal{Z} = \mathcal{H}_s(0,1) \times L^2(0,1)$, with state $\mathbf{z} := (f, f_t)^T \in \mathcal{Z}$. Equation (5.16) can be written in the state-space form

$$d\mathbf{z}(t) = A\mathbf{z}(t)dt + G dw(t),\ \mathbf{z}(0) = \mathbf{z}_0,\ t \geq 0, \tag{5.17}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} & -c_d \frac{\partial^4}{\partial x^4} \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}, \quad \mathbf{z}_0 = \begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix},$$

with domain

$$\mathcal{D}(A) = \{\mathbf{z} = (z_1, z_2) \in \mathcal{Z} : z_1'' \in H_s(0,1), z_2'' \in H_s(0,1)\}.$$

The solution to (5.22) can be expressed as

$$\mathbf{z}(t) = T(t)\mathbf{z}_0 + \int_0^t T(t-s)Gdw(s),\ t \geq 0,$$

where $T(t)$ is the C_0-semigroup generated by A.

Each sensor measures average deflection over a small interval of length $\triangle > 0$. Write

$$\boldsymbol{l} = (l_1, l_2, \ldots, l_m)$$

and define

$$c_{l_j}(x) := \begin{cases} 1/\triangle, & |x - l_j| \leq \frac{\triangle}{2} \\ 0, & \text{otherwise} \end{cases},$$

for $j = 1, 2, \ldots, m$. With m sensors centred at $x = l_1, l_2, ..., l_m$,

$$C(\boldsymbol{l})\boldsymbol{z}(t) = (\int_0^1 c_{l_1}(x)f(t,x)dx, \int_0^1 c_{l_2}(x)f(t,x)dx, \ldots, \int_0^1 c_{l_n}(x)f(t,x)dx)^T,$$

and the measurement is

$$y(t) = \int_0^t C(\boldsymbol{l})\boldsymbol{z}(s)ds + v(t),$$

where $v(t)$ is a $\mathbb{R}^m$-valued Wiener process, with

$$E\{(v(t) - v(s))(v(t) - v(s))^T\} = (t-s)\tilde{R}$$

for $t \geq s \geq 0$.

5.3.1 Finite-dimensional Approximation

Define

$$A_0 := \frac{\partial^4}{\partial x^4}$$

with domain

$$\mathcal{D}(A_0) = \{z \in \mathcal{H}_s(0,1) : z'' \in \mathcal{H}_s(0,1)\}.$$

The operator A_0 has eigenfunctions

$$\phi_j(x) = \sqrt{2}\sin(j\pi x),\ j \geq 1$$

that form an orthonormal basis for $L^2(0,1)$, with corresponding eigenvalues $\{\lambda_j = j^4\pi^4 : j \geq 1\}$. Let

$$\mathcal{Z}_n = \text{span}\{\phi_1, \phi_2 \ldots, \phi_n\}.$$

The approximating operator $A_{0,n} \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ is defined by

$$A_{0,n}\phi_j = \lambda_j \phi_j$$

for $j = 1, 2, \ldots, n$.

The infinitesimal generator is

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -c_d A_0 \end{pmatrix},$$

where I is the identity operator. Then (5.17) can be written as

$$\begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -A_0 z_1 - c_d A_0 z_2 \end{pmatrix} dt + \begin{pmatrix} 0 \\ g \end{pmatrix} dw(t),$$

where $z_1 = f$ and $z_2 = f_t$. Let

$$A_n := \begin{pmatrix} 0 & I_n \\ -A_{0,n} & -c_d A_{0,n} \end{pmatrix},$$

where I_n is the identity operator on $\mathcal{Z}_n$. A Galerkin approximation $\mathbf{z}_n := (z_{1,n}, z_{2,n}) \in \mathcal{Z}_n \times \mathcal{Z}_n$ solves the weak formulation

$$\begin{pmatrix} \langle dz_{1,n}, \phi_j \rangle \\ \langle dz_{2,n}, \phi_j \rangle \end{pmatrix} = \begin{pmatrix} \langle z_{2,n}, \phi_j \rangle \\ \langle -A_{0,n} z_{1,n} - c_d A_{0,n} z_{2,n}, \phi_j \rangle \end{pmatrix} dt + \begin{pmatrix} 0 \\ \langle g, \phi_j \rangle \end{pmatrix} dw(t) \tag{5.18}$$

for $j = 1, 2, \ldots, n$. The convergence of this approximation scheme is shown in [48], where the LQ optimal actuator location problem is considered. The convergence of optimal sensor location then follows by duality.

For calculations, let the vector

$$\underline{\boldsymbol{z}_n}(t) := (z_{n,1}(t), z_{n,2}(t), \ldots, z_{n,2n}(t))^T \in \mathbb{R}^{2n},$$

so that

$$z_{n,1}(t) = \sum_{i=1}^{n} z_{n,i}(t)\phi_i, \;\; z_{n,2}(t) = \sum_{i=n+1}^{2n} z_{n,i}(t)\phi_i.$$

Write the matrix

$$\underline{A_{0,n}} := \mathrm{diag}(\pi^4, 16\pi^4, \ldots, n^4\pi^4)$$

and the vectors

$$\underline{g_n} := (\langle g, \phi_1\rangle, \langle g, \phi_2\rangle, \ldots, \langle g, \phi_n\rangle)^T,$$

$$\underline{\boldsymbol{z}_{0,n}} := (\langle f_0, \phi_1\rangle, \langle f_0, \phi_2\rangle, \ldots, \langle f_0, \phi_n\rangle, \langle f_1, \phi_1\rangle, \langle f_1, \phi_2\rangle, \ldots, \langle f_1, \phi_n\rangle)^T.$$

It follows from (5.18)

$$d\underline{\boldsymbol{z}_n}(t) = \underline{A_n}\;\underline{\boldsymbol{z}_n}(t)dt + \underline{G_n}dw(t), \;\; \underline{\boldsymbol{z}_n}(0) = \underline{\boldsymbol{z}_{0,n}}, \;\; t \geq 0,$$

where

$$\underline{A_n} = \begin{pmatrix} 0 & \underline{I_n} \\ -\underline{A_{0,n}} & -c_d\underline{A_{0,n}} \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

and

$$\underline{G_n} = \begin{pmatrix} 0 \\ \underline{g_n} \end{pmatrix} \in \mathbb{R}^{2n},$$

The measurement is approximated by

$$y_n(t) = \underline{C_n}(\boldsymbol{l})\underline{\boldsymbol{z}_n}(t) + v(t),$$

where the sensor location $\boldsymbol{l} = (l_1, l_2, \ldots, l_m) \in \mathbb{R}^m$, the matrix

$$\underline{C_n}(\boldsymbol{l}) = \left(\begin{array}{ccc|ccc} \int_0^1 c_{l_1}\phi_1 dx & \ldots & \int_0^1 c_{l_1}\phi_n dx & 0 & \ldots & 0 \\ \int_0^1 c_{l_2}\phi_1 dx & \ldots & \int_0^1 c_{l_2}\phi_n dx & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_0^1 c_{l_m}\phi_1 dx & \ldots & \int_0^1 c_{l_m}\phi_n dx & 0 & \ldots & 0 \end{array}\right) \in \mathbb{R}^{m\times 2n}.$$

5.3.2 Optimal Sensor Location

In this section, optimal sensor location that minimizes $||P_{ss}||_1$ is calculated for the case where only one sensor is used. For simulations, the parameters used are: $c_d = 0.0001$,

$\triangle = 0.02$, the approximation mode number $N = 15$, and the initial condition

$$f_0(x) = 0.25 - (x - 0.5)^2, \ f_1(x) = 0, \ 0 < x < 1.$$

Consider the evenly distributed disturbance: $g(x) = 1$. The value of $||P_{ss}||_1$ was computed for different sensor locations with the varied process noise variance. Figure 5.13 shows that the optimal sensor location is dependent on the variance of the process noise: when $\tilde{Q}$ is small, the optimal sensor location is at the center of the beam; as $\tilde{Q}$ gets larger, the optimal sensor location slips to the side.

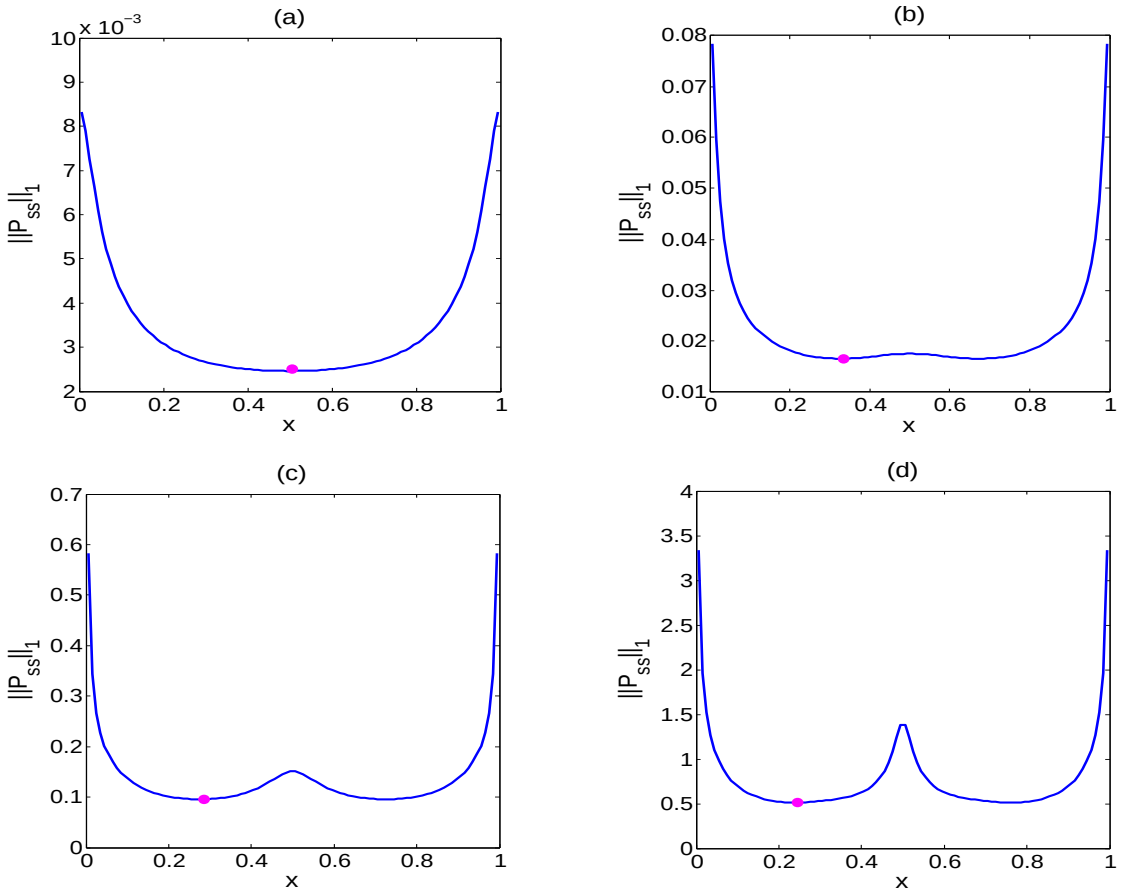

Figure 5.13: The value of $||P_{ss}||_1$ for different sensor location, with noise parameters $R_0 = 0.0001$, and (a) $\tilde{Q} = 0.1$, (b) $\tilde{Q} = 1$, (c) $\tilde{Q} = 10$, (d) $\tilde{Q} = 100$; the magenta dots indicate the optimal sensor locations.

5.3.3 Effect of Sensor Noise on Estimation

Here, estimates obtained using a single high-quality sensor are compared with estimates obtained using multiple relatively low-quality sensors. Again, the three different disturbances were considered.

First, consider the case where the noise on the state is evenly distributed in space:

$$g(x) = 1.$$

The estimates obtained by using a single sensor with $R_0 = 0.02$ and by using 15 sensors each with $R_0 = 0.2$ are compared in Figure 5.14 with the actual system state $z(t)$ at the middle point $x = 0.5$ for $t \in [0, 10]$.

For a localized disturbance

$$g(x) = \operatorname{sech}(100(x - 0.2)),$$

comparisons of the actual system state with estimates made by a single sensor with $R_0 = 0.02$ and by 15 sensors with $R_0 = 0.2$ are displayed in Figure 5.15. The estimation errors are similar.

Then, consider a mixed disturbance that includes both the evenly distributed disturbance and the spatially localized disturbance:

$$g(x) = 0.5 \operatorname{sech}(100(x - 0.2)) + 0.5,$$

The estimate obtained using one sensor with $R_0 = 0.02$ is compared with that obtained using 15 sensors each with $R_0 = 0.2$ in Figure 5.16. The accuracy of the estimate obtained with many poor sensors is similar to that obtained with one accurate sensor.

All the three figures suggest that increasing the number of sensors can compensate for high sensor noise.

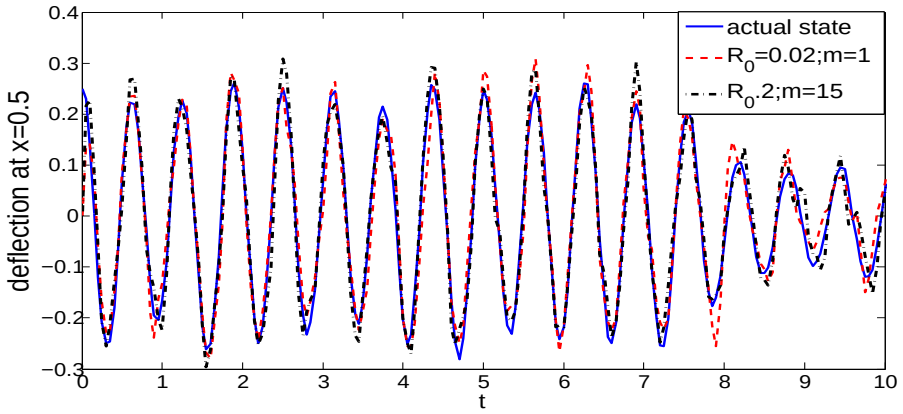

Figure 5.14: Uniform disturbance $g(x) = 1$. Comparison of the actual state at $x = 0.5$, $t \in [0, 10]$ with estimates using a single sensor with noise variance $R_0 = 0.02$ and 15 sensors each with $R_0 = 0.2$.

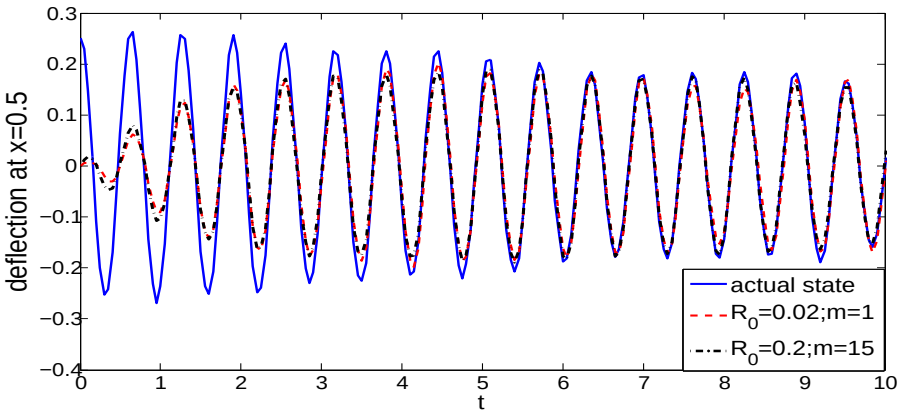

Figure 5.15: Spatially localized disturbance. Comparison of the actual state at $x = 0.5$, $t \in [0, 10]$ with estimates using a single sensor with noise variance $R_0 = 0.02$ and 15 sensors each with $R_0 = 0.2$.

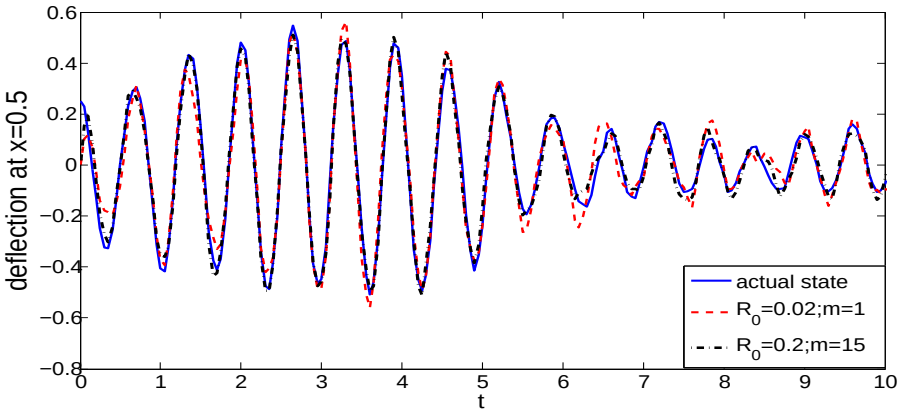

Figure 5.16: Mixed disturbance. Comparison of the actual state at $x = 0.5$, $t \in [0, 10]$ with estimates using a single sensor with noise variance $R_0 = 0.02$ and 15 sensors each with $R_0 = 0.2$.

5.4 Two-Dimensional Diffusion Equation

Let Ω be an L-shaped region in $\mathbb{R}^2$, as shown in Figure 5.17. Consider a two-dimensional diffusion equation

$$z_t = \sigma\left(\frac{\partial^2 z}{\partial x_1^2} + \frac{\partial^2 z}{\partial x_2^2}\right) + g(x_1, x_2)\eta(t), \ (x_1, x_2) \in \Omega, \tag{5.19}$$

where σ represents the diffusivity, the function $g(x_1, x_2)$ describes the shape of the spatially distributed disturbance, and $\eta(t)$ is a time-dependent noise model. Assume the initial condition

$$z(0, x_1, x_2) = z_0(x_1, x_2), \tag{5.20}$$

where z_0 is a $L^2(\Omega)$-valued Gaussian random variable with zero mean value and covariance P_0, and Dirichlet boundary condition

$$z(t, x_1, x_2) \mid_{\partial\Omega} = 0. \tag{5.21}$$

In applications, the case $z = 0$ represents a reference state, such as the freezing point of water when considering temperature in Celsius in thermal diffusion, and the value of $z(t, x_1, x_2)$ represents the departure from the reference state at time t and location (x_1, x_2).

Suppose $\eta(t)$ is a real-valued white noise so that the integral

$$w(t) := \int_0^t \eta(s)ds$$

is a real-valued Wiener process, with

$$E\{(w(t) - w(s))^2\} = (t - s)\tilde{Q},$$

for $t \geq s \geq 0$, where $\tilde{Q} \in \mathbb{R}_+$. Let the state space $\mathcal{Z} = L^2(\Omega)$ and $A = \sigma\nabla^2$ with domain

$$\mathcal{D}(A) = \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega) \subset \mathcal{Z}.$$

The state-space representation for the equation (5.7) is

$$dz(t) = Az(t)dt + Gdw(t), \ z(0) = z_0, \ t \geq 0, \tag{5.22}$$

where $G \in \mathcal{L}(\mathbb{R}, \mathcal{Z})$ is such that for $k \in \mathbb{R}$,

$$Gk = kg(x).$$

The solution to (5.22) can be expressed as

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Gdw(s),\ t \geq 0,$$

where $T(t)$ is the C_0-semigroup generated by A.

Each sensor measures the average state $z(t)$ over a small square of side length $\triangle > 0$. Write

$$\boldsymbol{l} = (l_1, l_2, \ldots, l_m)$$

and define

$$c_{l_j}(x_1, x_2) := \begin{cases} 1/\triangle^2, & \text{if } |x_1 - l_j(1)| \leq \frac{\triangle}{2} \text{ and } |x_2 - l_j(2)| \leq \frac{\triangle}{2} \\ 0, & \text{otherwise} \end{cases},$$

for $j = 1, 2, \ldots, m$. With m sensors centred at $x = l_1, l_2, ..., l_m$,

$$C(\boldsymbol{l})z = (\langle c_{l_1}, z\rangle, \langle c_{l_2}, z\rangle, \ldots, \langle c_{l_m}, z\rangle)^T,$$

and the measurement is

$$y(t) = \int_0^t C(\boldsymbol{l})z(s)ds + v(t),$$

where $v(t)$ is a $\mathbb{R}^m$-valued Wiener process, with

$$E\{(v(t) - v(s))(v(t) - v(s))^T\} = (t-s)\tilde{R}$$

for $t \geq s \geq 0$.

5.4.1 Finite-Dimensional Approximation

Here, the equation is approximated by a standard finite element method with linear basis functions [4]. Higher order polynomial basis can also be used to improve the accuracy.

Let $\mathcal{V} = \mathcal{H}_0^1(\Omega) \subset \mathcal{Z}$. Define a bilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ such that for $\psi, \varphi \in \mathcal{V}$,

$$a(\psi,\varphi) := \sigma(\langle \frac{\partial \psi}{\partial x_1}, \frac{\partial \varphi}{\partial x_1} \rangle + \langle \frac{\partial \psi}{\partial x_2}, \frac{\partial \varphi}{\partial x_2} \rangle),$$

where $\langle , \rangle$ denotes the inner product on $\mathcal{Z} = L^2(\Omega)$. For $\varphi \in \mathcal{V}$, write

$$\mathbf{F} := (\frac{\partial z}{\partial x_1}\varphi, \frac{\partial z}{\partial x_2}\varphi)^T.$$

By the divergence theorem,

$$\int_\Omega div\ \mathbf{F} = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n}\ ds = 0,$$

where $\mathbf{n}$ represents the outward pointing unit normal vector of $\partial\Omega$. Hence, with the homogeneous Dirichlet boundary condition,

$$\langle \nabla^2 z, \varphi \rangle = -\big(\langle \frac{\partial z}{\partial x_1}, \frac{\partial \varphi}{\partial x_1} \rangle + \langle \frac{\partial z}{\partial x_2}, \frac{\partial \varphi}{\partial x_2} \rangle\big).$$

It follows that

$$\langle -Az, \varphi \rangle = a(z,\varphi).$$

A triangular mesh of the L-shaped region Ω can be created and refined using Matlab functions *'initmesh'* and *'refinemesh'*. The mesh is shown in Figure 5.17. Suppose there are n interior nodes $\{\nu_j : j = 1, 2, \ldots, n\}$ in the mesh. Basis functions $\{\phi_j : j = 1, 2, \ldots, n\} \subset \mathcal{V}$ are chosen to be linear spline functions such that

$$\phi_j(\nu_k) = \delta_{jk},\ \ j,k = 1, 2, \ldots, n,$$

where δ_{jk} is the Kronecker delta. Let

$$\mathcal{Z}_n = \text{span}\{\phi_1, \phi_2 \ldots, \phi_n\} \subset \mathcal{V}.$$

The approximating generator A_n is defined by

$$\langle -A_n z_n, \varphi_n \rangle = a(z_n, \varphi_n)$$

for $z_n, \psi_n \in \mathcal{Z}_n$. A Galerkin approximation $z_n(t) \in \mathcal{Z}_n$ of the state $z(t)$ shall solve the

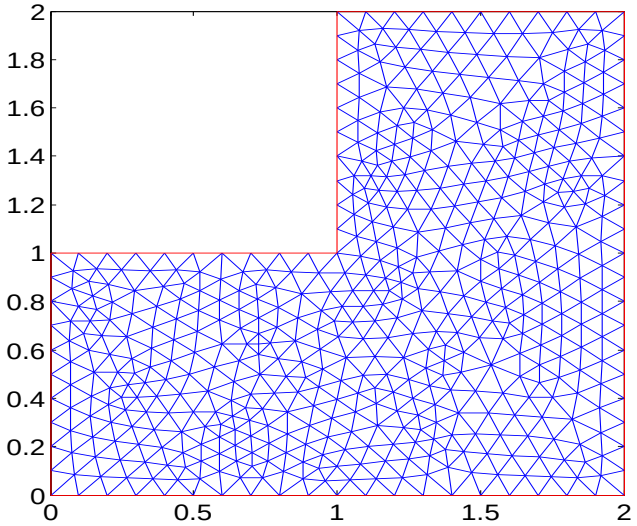

Figure 5.17: Triangular mesh of Ω

weak formulation

$$\langle dz_n(t), \phi_j \rangle = \langle A_n z_n(t), \phi_j \rangle dt + \langle G_n dw(t), \phi_j \rangle, \ \langle z(0), \ \phi_j \rangle = \langle z_0, \phi_j \rangle, \ t \geq 0, \tag{5.23}$$

for $j = 1, 2, \ldots, n$. The convergence of the finite element approximation is shown in [6, Chapter 12].

For calculations, let the vector

$$\underline{z_n}(t) := (z_{n,1}(t), z_{n,2}(t), \ldots, z_{n,n}(t))^T \in \mathbb{R}^n,$$

so that

$$z_n(t) = \sum_{i=1}^{n} z_{n,i}(t)\phi_i.$$

Substituting it into (5.23),

$$\begin{aligned} \sum_{i=1}^{n} \langle \phi_i, \phi_j \rangle dz_{n,i}(t) = & \sum_{i=1}^{n} \langle A_n \phi_i, \phi_j \rangle z_{n,i}(t) + \langle G_n dw(s), \phi_j \rangle, \\ \sum_{i=1}^{n} \langle \phi_i, \phi_j \rangle z_{n,i}(0) = & \langle z_0, \phi_j \rangle, \qquad\qquad j = 1, 2, \ldots, n. \end{aligned} \tag{5.24}$$

Define the mass matrix

$$M_n := \begin{pmatrix} \langle \phi_1, \phi_1 \rangle & \dots & \langle \phi_n, \phi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle \phi_1, \phi_n \rangle & \dots & \langle \phi_n, \phi_n \rangle \end{pmatrix},$$

and the stiffness matrix

$$K_n := \begin{pmatrix} \langle A_n \phi_1, \phi_1 \rangle & \dots & \langle A_n \phi_n, \phi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle A_n \phi_1, \phi_n \rangle & \dots & \langle A_n \phi_n, \phi_n \rangle \end{pmatrix}.$$

Also, since $G \in \mathcal{L}(\mathbb{R}, \mathcal{Z})$, there exists $g \in \mathcal{Z}$, such that for any $k \in \mathbb{R}$,

$$Gk = kg.$$

Hence, for $j = 1, 2, \dots, n$,

$$\langle G dw(t), \phi_j \rangle = \langle g, \phi_j \rangle dw(t).$$

Define the vector

$$\underline{g_n} := (\langle g, \phi_1 \rangle, \langle g, \phi_2 \rangle, \dots, \langle g, \phi_n \rangle)^T.$$

It follows from (5.24) that the vector $\underline{z_n}(t)$ is determined by

$$\begin{aligned} M_n d\underline{z_n}(t) &= K_n \underline{z_n}(t) dt + \underline{g_n} dw(t), \\ M_n \underline{z_n}(0) &= (\langle z_0, \phi_1 \rangle, \langle z_0, \phi_2 \rangle, \dots, \langle z_0, \phi_n \rangle)^T. \end{aligned}$$

The measurement $y(t)$ is approximated by

$$y_n(t) = \underline{C_n}(\boldsymbol{l}) \underline{\boldsymbol{z}_n}(t) + v(t),$$

where the matrix

$$\underline{C_n}(\boldsymbol{l}) := (C(\boldsymbol{l})\phi_1, C(\boldsymbol{l})\phi_1, C(\boldsymbol{l})\phi_2, \dots, C(\boldsymbol{l})\phi_n) \in \mathbb{R}^{m \times n}.$$

5.4.2 Optimal Sensor Location

For simulations, the triangular mesh shown in Figure 5.17 was used, which contains $n = 526$ nodes. The mesh size is 0.1, that is, the maximal length of any triangle side. The

parameters are $\sigma = 0.1$, $\tilde{Q} = 0.01$, $\triangle = 0.1$, and the initial condition

$$z_0(x_1, x_2) = \text{sech}((10(x_1 - 1.5)^2 + 50(x_2 - 0.3)^2)),$$

as shown in Figure 5.18.

Consider a spatially localized disturbance centered at $x = (1.5, 1.5)$ (see Figure 5.19):

$$g(x) = \text{sech}(100((x_1 - 1.5)^2 + (x_2 - 1.5)^2)).$$

When using only one sensor, as marked in Figure 5.20, the optimal sensor location is around where the local disturbance is. Figure 5.22 shows the actual state and the estimate obtained by the Kalman filter at $t = 0.5$, $t = 1.0$, $t = 1.5$, with the sensor placed at the optimal location. Two estimates, one obtained when the sensor is optimally placed and the other obtained when the sensor is placed at the non-optimal location $x = (1.35, 1.45)$ are compared in Figure 5.21, with the actual state at two points $x = (1.74, 1.36)$ and $x = (1.51, 1.02)$. Putting sensor at optimal location leads to a more accurate estimate for both of the sample points.

5.4.3 Effect of Sensor Noise on Estimation

When using 25 sensors, the optimal sensor locations are still concentrated at around the location of the disturbance (see Figure 5.23). The estimates obtained by using a single sensor with $R_0 = 0.00002$, and by using 25 sensors each with $R_0 = 0.0002$, are compared in Figure 5.24. Again, the two estimates are similar, which indicates that using more sensors can compensate for larger noise in sensors.

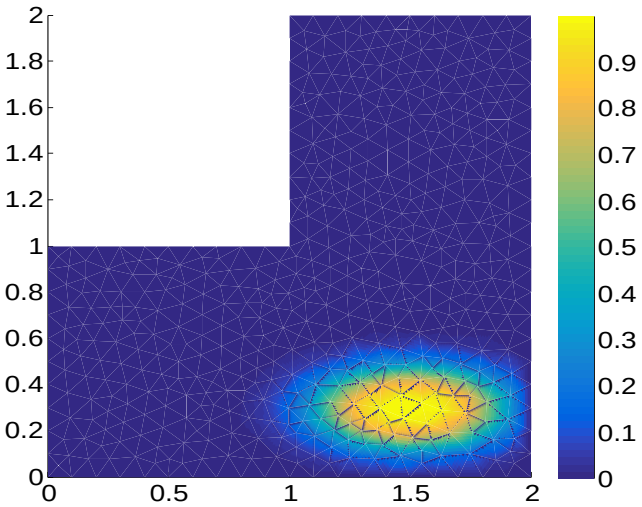

Figure 5.18: Initial condition

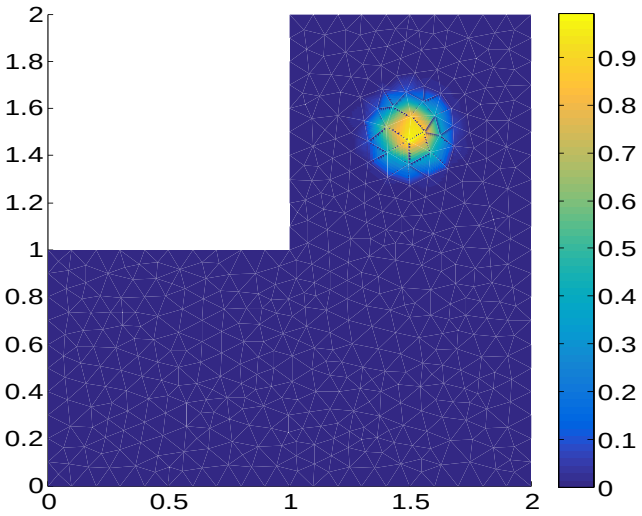

Figure 5.19: Disturbance spatially localized around $x = (1.5, 1.5)$

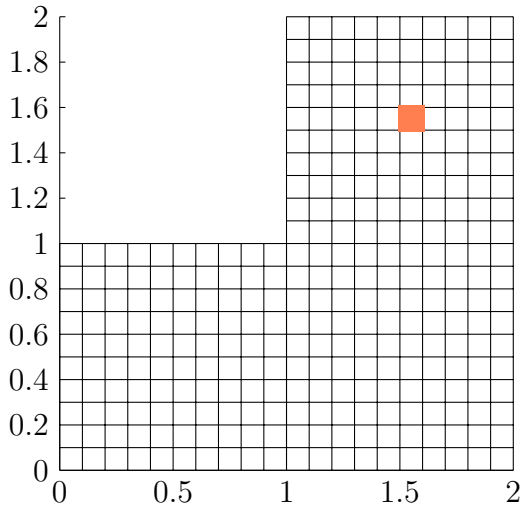

Figure 5.20: Optimal location for a single sensor

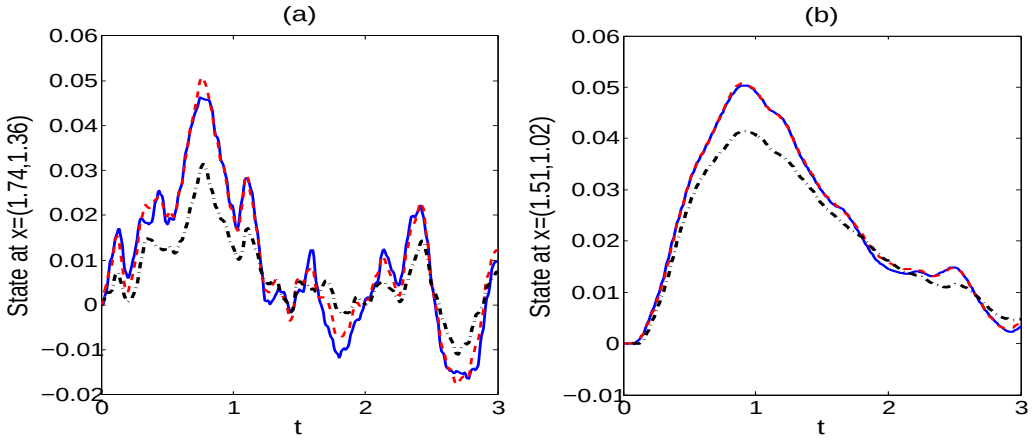

Figure 5.21: Comparison of two estimates: one obtained when the sensor is optimally placed (red dashed line) and the other obtained when the sensor is placed at the non-optimal location $x = (1.35, 1.45)$ (black dotted line), with the actual state (blue solid line) at points (a)$x = (1.74, 1.36)$, (b)$x = (1.51, 1.02)$

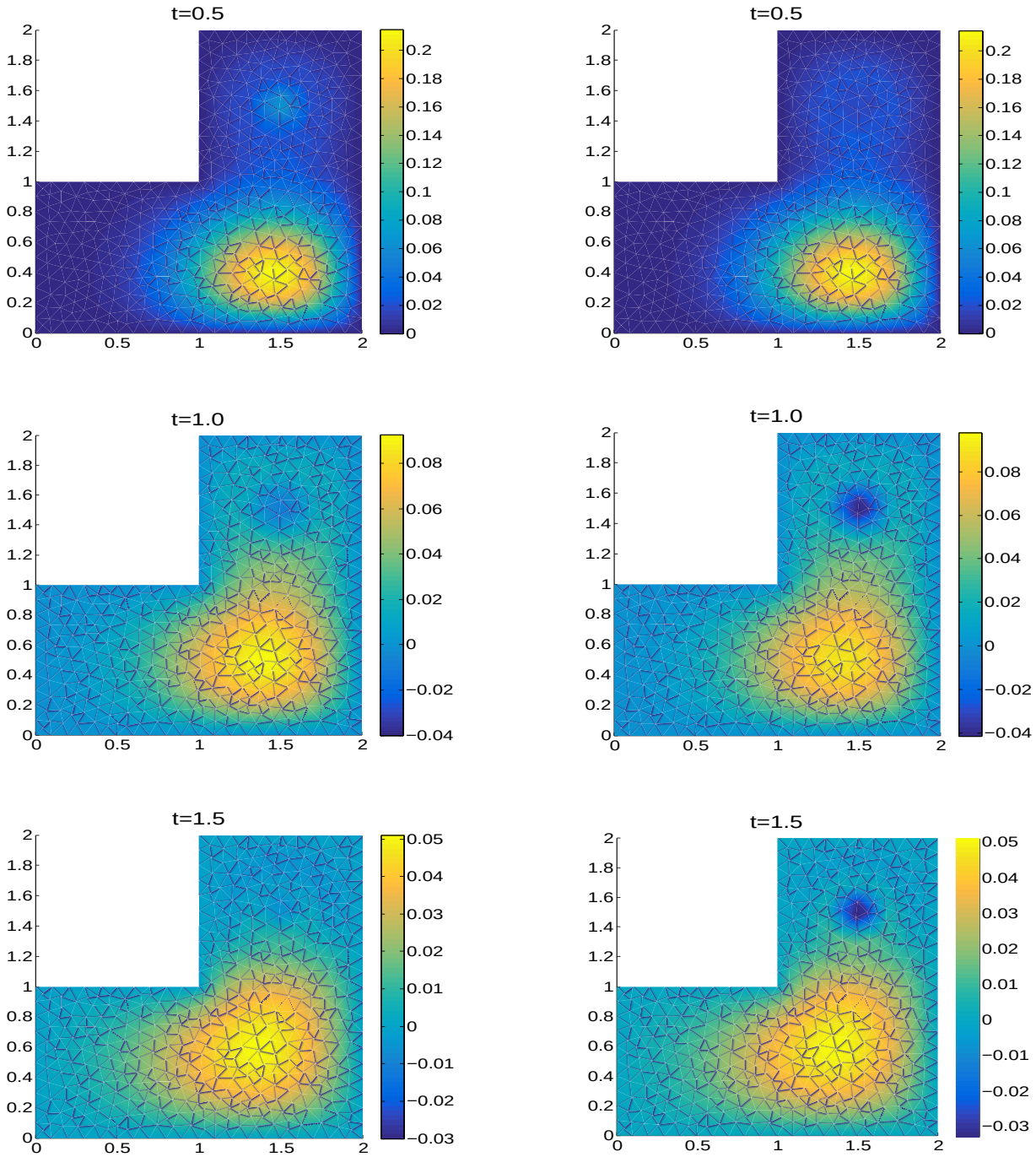

Figure 5.22: Comparison of the actual state (left) with the estimate (right) at t=0.5, 1.0, 1.5

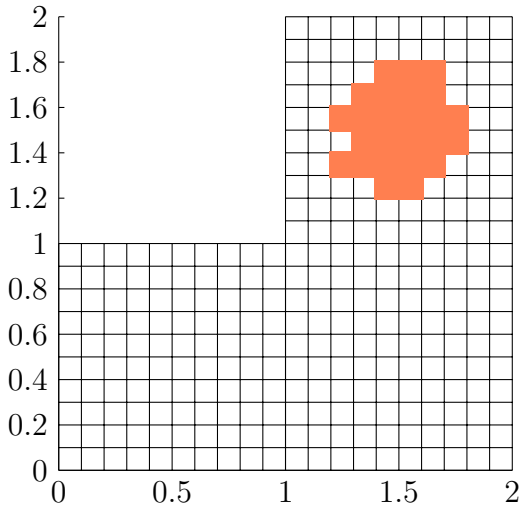

Figure 5.23: Optimal location for 25 sensors

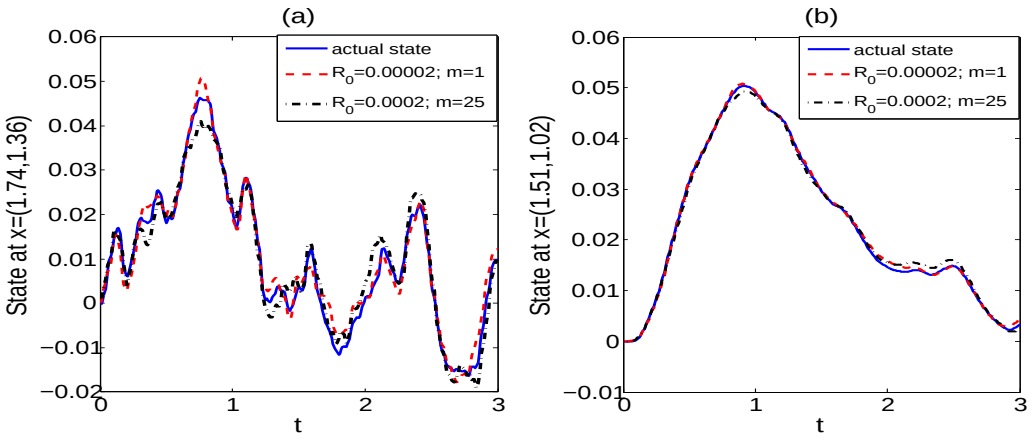

Figure 5.24: Comparison of two estimates: one obtained when using a single optimally placed sensor with $R_0 = 0.00002$ (red dashed line) and the other obtained when using 25 optimally placed sensors with $R_0 = 0.0002$ (black dotted line), with the actual state (blue solid line) at points (a)$x = (1.74, 1.36)$, (b)$x = (1.51, 1.02)$.

Chapter 6

Conclusion and Future Work

In this work, the theory of Kalman filtering for infinite-dimensional linear systems is reviewed. It is known that a Kalman filter produces an optimal linear estimate that minimizes the variance of the estimation error and that the error covariance, $P(t)$, is a solution to a differential Riccati equation. As $t \to \infty$, under certain conditions, the Kalman filter converges to a time-invariant filter. The convergence of error covariance, $P(t)$, in the nuclear norm is proved, with the limit, P_{ss}, the unique nonnegative solution to an algebraic Riccati equation. Furthermore, the nuclear norm, $\|P_{ss}\|_1$, represents the minimum steady-state error variance obtained by a Kalman filter. The value of $\|P_{ss}\|_1$ is dependent on measurements, and a smaller $\|P_{ss}\|_1$ indicates a better estimate at the steady state.

Using $\|P_{ss}\|_1$ as the optimality criterion, the sensor placement problem for Kalman filtering is considered. Optimal sensor location in this context is the dual problem of the LQ optimal actuator location problem. The results on well-posedness of the optimal location problem, as well as the use of approximations in calculating the optimal locations in [48] are applied by duality. The optimization algorithm used to calculate LQ optimal actuator locations in [24] is introduced for computation of the optimal sensor locations.

Three partial differential equation models are examined: one-dimensional diffusion, simply supported Euler-Bernoulli beam with Kelvin-Voigt damping, and two-dimensional diffusion on an L-shaped region. For computational purposes, the equations are approximated by finite-dimensional systems using Galerkin methods, and optimal sensor locations that minimizes the value of $\|P_{ss}\|_1$ were calculated. With a spatially evenly-distributed disturbance, for the one-dimensional diffusion equation, $\|P_{ss}\|_1$ is constant at different sensor locations, and hence no preference for sensor location was observed. While for the simply supported Euler-Bernoulli beam equation, the optimal sensor location is dependent on the

variance of the process noise: when the variance is small, the optimal sensor location is at the center of the beam; as the variance gets larger, the optimal sensor location slips to the side. By considering spatially localized disturbances, the optimal sensor locations for the diffusion equations are around the location of disturbance.

Also, sensor noise effects on state estimation were investigated by simulations. Both the quality and the quantity of sensors were varied. For the one-dimensional diffusion equation, three different disturbances were considered: spatially evenly-distributed disturbance, spatially localized disturbance and mixed disturbance that combines the evenly distributed disturbance with a spatially localized disturbance. Assuming that all the selected sensors are optimally placed, accuracy of the estimation depends on sensor accuracy which is expected. However, using a number of poor quality sensors, that is those with large noise variance, leads to an estimator with accuracy comparable to that with a single good quality sensor. Similar results are also observed for the simply supported Euler-Bernoulli beam and the two-dimensional diffusion equation. In particular, for the one-dimensional diffusion equation, the same approximately square root relation between $\|P_{ss}\|_1$ and the noise variance, as well as between $\|P_{ss}\|_1$ and the number of sensors, holds for all the calculations.

Other partial differential equation models can be examined in the future, for example, a convection-diffusion equation. Estimation problem for nonlinear partial differential equations can also be considered, using techniques such as the extended Kalman filter [9] and the unscented Kalman filter [37]. Another interesting future project could be investigating sensor noise effects on state estimation from a theoretical perspective.

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