Grothendieck Rings and Motivic Integration

# Grothendieck Rings and Motivic Integration

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# Abstract

This thesis consists of three parts:

In Part I we study the Burnside ring of the finite group $G$. This ring has a natural structure of a $\lambda$-ring, $\{\lambda^n\}_{n\in\mathbb{N}}$. However, a priori $\lambda^n(S)$, where $S$ is a $G$-set, can only be computed recursively, by first computing $\lambda^1(S),\dots,\lambda^{n-1}(S)$. We establish an explicit formula, expressing $\lambda^n(S)$ as a linear combination of classes of $G$-sets. This formula is derived in two ways: First we give a proof that uses the theory of representation rings in an essential way. We then give an alternative, more intrinsic, proof. This second proof is joint work with Serge Bouc.

In Part II we establish a formula for the classes of certain tori in the Grothendieck ring of varieties $\mathrm{K}_0(\mathsf{Var}_k)$. More explicitly, $\mathrm{K}_0(\mathsf{Var}_k)$ has a natural structure of a $\lambda$-ring, and we will see that if $L^*$ is the torus of invertible elements in the $n$-dimensional separable $k$-algebra $L$ then $[L^*]=\sum_{i=0}^{n}(-1)^i\lambda^i\big([\operatorname{Spec}L]\big)\mathbb{L}^{n-i}$, where $\mathbb{L}$ is the class of the affine line. This formula is suggested by the computation of the cohomology of the torus. To prove it requires some rather explicit calculations in $\mathrm{K}_0(\mathsf{Var}_k)$. To be able to make these, we introduce a homomorphism from the Burnside ring of the absolute Galois group of $k$, to $\mathrm{K}_0(\mathsf{Var}_k)$. In the process we obtain some information about the structure of the subring of $\mathrm{K}_0(\mathsf{Var}_k)$ generated by zero-dimensional varieties.

In Part III we give a version of geometric motivic integration that specializes to $p$-adic integration via point counting. This has been done before for stable sets, *cf.* [LS03]; we extend this to more general sets. The main problem in doing this is that it requires to take limits, hence the measure will have to take values in a completion of $\mathrm{K}_0(\mathsf{Var}_k)[\mathbb{L}^{-1}]$. The standard choice is to complete with respect to the dimension filtration. However, since the point counting homomorphism is not continuous with respect to this topology we have to use a stronger one. We thus begin by defining this stronger topology; we will then see that many of the standard constructions of geometric motivic integration work also in this setting. Using this theory, we are then able to give a geometric explanation of the behavior of certain $p$-adic integrals, by computing the corresponding motivic integrals.

# Contents

**Part IV: Bibliography**

## 0.1 Introduction

This thesis is concerned with two main topics. Firstly, the investigation of two Grothendieck rings: the Burnside ring, and the Grothendieck ring of varieties. Secondly, the theory of motivic integration. These topics are related by the fact that the motivic measure takes values in a certain completed localization of the Grothendieck ring of varieties. Let us first say some words about that ring:

Let $k$ be a field and let $\mathsf{Var}_k$ be the category of $k$-varieties, by which we mean the category of separated $k$-schemes of finite type. The Grothendieck ring of $k$-varieties[1], $\mathrm{K}_0(\mathsf{Var}_k)$, is by definition the free abelian group on the set of isomorphism classes of $k$-varieties $[X]$, modulo the relations $[X] = [Z] + [X \setminus Z]$ if $Z \subset X$ is closed (the scissor relations), and with a multiplication given by $[X][Y] = [X \times_k Y]$. The class of the affine line is of particular importance, and is given a special symbol: $\mathbb{L} := [\mathbb{A}^1_k] \in \mathrm{K}_0(\mathsf{Var}_k)$ (L is for Lefschetz). It follows immediately from the definition of the multiplication that $[\mathbb{A}^n_k] = \mathbb{L}^n$. For an example of how the scissor relations work, note that since we may choose a closed subscheme of $\mathbb{P}^n_k$ which is isomorphic to $\mathbb{P}^{n-1}_k$, and has complement $\mathbb{A}^n_k$, we have $[\mathbb{P}^n_k] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1 \in \mathrm{K}_0(\mathsf{Var}_k)$.

We want to use the theory of motivic integration in order to give a geometric explanation of the behavior of certain $p$-adic integrals. As an introduction to these ideas, let us illustrate how $\mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_p})$ gives a geometric way of counting solutions to polynomial equations modulo $p$. We do this with an example: For $L$ a separable $k$-algebra, define $L^*$ to be the algebraic group of invertible elements of $L$, i.e., for every $k$-algebra $R$, $L^*(R) = (L \otimes_k R)^\times$. In case $L/k = \mathbb{F}_{p^2}/\mathbb{F}_p$ we have $|L^*(\mathbb{F}_p)| = |\mathbb{F}_{p^2}^\times| = p^2 - 1$ and $|L^*(\mathbb{F}_{p^2})| = |(\mathbb{F}_{p^2}^2)^\times| = (p^2-1)^2 = (p^2)^2 - 2p^2 + 1$. On the other hand, computing in $\mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_p})$, one may show that

$$[L^*] = \mathbb{L}^2 - [\operatorname{Spec} L]\mathbb{L} + [\operatorname{Spec} L] - 1 \in \mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_p}). \tag{0.1}$$

By doing this, we have simultaneously computed $|L^*(\mathbb{F}_q)|$ for every power $q$ of $p$. Because for every such $q$ we have a point counting homomorphism $\mathrm{C}_q\colon \mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_p}) \to \mathbb{Z}$, induced by $[X] \mapsto |X(\mathbb{F}_q)|$. So by applying $\mathrm{C}_q$ to $[L^*]$ we obtain the number of $\mathbb{F}_q$-points on $L^*$. In particular, using (0.1) we again get $|L^*(\mathbb{F}_p)| = \mathrm{C}_p([L^*]) = p^2 - 1$ and $|L^*(\mathbb{F}_{p^2})| = \mathrm{C}_{p^2}([L^*]) = (p^2)^2 - 2p^2 + 1$.

Before we continue, let us mention that this example is a special case of a more general theorem. Namely, for any field $k$, $\mathrm{K}_0(\mathsf{Var}_k)$ has a natural structure of a $\lambda$-ring (a $\lambda$-ring is a ring together with a set of maps $\lambda^n$ behaving like exterior powers, see Section 0.2.1.), and using this structure

[1] First introduced by Grothendieck, *cf.* the unpublished text *Motifs*, available at *www.grothendieckcircle.org*

one may, for any separable $n$-dimensional $k$-algebra $L$, express $[L^*]$ as

$$[L^*] = \sum_{i=0}^{n} (-1)^i \lambda^i([\operatorname{Spec} L])\mathbb{L}^{n-i} \in \mathrm{K}_0(\mathsf{Var}_k). \tag{0.2}$$

The validity of this formula will be the main result of Chapter 3. In establishing it, we will be led to consider the structure of $\mathrm{K}_0(\mathsf{Var}_k)$ in more detail, and this leads us to the subject of the Burnside ring.

The Burnside ring of a profinite group $G$, denoted $\mathcal{B}(G)$, is constructed in much the same way as $\mathrm{K}_0(\mathsf{Var}_k)$, but instead of the category of varieties one uses the category of finite, continuous $G$-sets. Let $\mathcal{G}$ be the absolute Galois group of $k$. There is a natural map $\mathrm{Art}_k \colon \mathcal{B}(\mathcal{G}) \to \mathrm{K}_0(\mathsf{Var}_k)$, and since the structure of the Burnside ring is much better known than that of $\mathrm{K}_0(\mathsf{Var}_k)$, $\mathrm{Art}_k$ is useful for proving structure result about $\mathrm{K}_0(\mathsf{Var}_k)$. For an example of this, we may use $\mathrm{Art}_k$ to prove that Naumann's construction of zero divisors in $\mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_p})$ actually works in $\mathrm{K}_0(\mathsf{Var}_k)$ when $k$ is any field which is not separably closed: Let $L/k$ be a finite Galois extension of degree $n$. Then $[\operatorname{Spec} L]^2 = [\operatorname{Spec} L \otimes_k L] = [\operatorname{Spec} L^n] = [\dot\bigcup_n \operatorname{Spec} L] = n[\operatorname{Spec} L]$, hence $[\operatorname{Spec} L]\big([\operatorname{Spec} L] - n\big) = 0$. Looking at the preimage of $[\operatorname{Spec} L]$ under $\mathrm{Art}_k$, it is easy to show that $[\operatorname{Spec} L] \neq 0, n$; consequently it is a zero divisor for any $k$ (see Section 3.3).

Moreover, $\mathcal{B}(\mathcal{G})$ also has a natural structure of a $\lambda$-ring, and $\mathrm{Art}_k$ commutes with the $\lambda$-operations. This allows us to move computations of the $\lambda$-operations on $\mathrm{K}_0(\mathsf{Var}_k)$ to $\mathcal{B}(\mathcal{G})$. Actually, one of the proofs of the validity of (0.2) given in Chapter 3 relies on a theorem about Burnside rings (Theorem 1.1) which is proved in Chapter 1.

We now turn to motivic integration. Motivic integration was introduced by Kontsevich in 1995, in order to strengthen Batyrev's result that birational Calabi-Yau manifolds have the same Betti numbers, to also yield equality of Hodge numbers. It has since then ramified in many different directions; we are interested in utilizing it to give a geometric way of computing $p$-adic integrals. In fact, motivic integration is inspired by $p$-adic integration: one wants to define a measure on subsets of power series rings of the type $k[[t]]$, in much the same way as on subsets of $\mathbb{Z}_p$. However, since $k[[t]]$ is not locally compact (whenever $k$ is infinite) it is not possible to do this in the classical way. Kontsevich's method of resolving this was to let the measure take values in a certain completion of $\mathcal{M}_k := \mathrm{K}_0(\mathsf{Var}_k)[\mathbb{L}^{-1}]$. This gives the original theory of geometric motivic integration, developed in [DL99] and [DL02].

This idea may now be used also in the $p$-adic case: Let $\mathbf{W}$ be the ring scheme of Witt vectors, constructed with respect to the prime $p$. (Recall that $\mathbf{W}(\mathbb{F}_q)$, where $q = p^f$, is the integers in the unramified degree $f$ extension of $\mathbb{Q}_p$; in particular $\mathbf{W}(\mathbb{F}_p) = \mathbb{Z}_p$.) Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. $\mathbf{W}(\overline{\mathbb{F}}_p)$ then contains all the $\mathbf{W}(\mathbb{F}_q)$, $q = p^f$, as subrings.

However, similarly as $k[[t]]$, $\mathbf{W}(\overline{\mathbb{F}}_p)$ suffers from the defect of not being locally compact, hence does not come with a natural measure. This may now be resloved in the same way as for $k[[t]]$: using a similar construction as in the original theory of motivic integration, one gets a measure on certain subsets of $\mathbf{W}(\overline{\mathbb{F}}_p)$, taking values in a completion of $\mathcal{M}_{\mathbb{F}_p}$.

On certain simple subsets $A \subset \mathbf{W}(\overline{\mathbb{F}}_p)$, this now has the property that if we let the measure take values in $\mathcal{M}_{\mathbb{F}_p}$ (instead of in the completion), then, by applying the point counting homomorphism $\mathrm{C}_q \colon \mathcal{M}_{\mathbb{F}_p} \to \mathbb{Q}$ we recover the Haar measure of $A \cap \mathbf{W}(\mathbb{F}_q)$. However, a problem arises when we want to extend the property of specializing to general measurable sets: Then the integral has to take values in a completion of $\mathcal{M}_{\mathbb{F}_p}$, and we cannot use the standard one since $\mathrm{C}_q$ is not continuous with respect to this topology, hence does not extends to the completion. This problem is taken care of in Chapter 4, where we define a topology that is strong enough for the point counting homomorphism to be continuous. Let $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ be the completion with respect to this topology. In the second part of Chapter 4 we then show that many of the standard theorems of motivic integration hold also when the integral takes values in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$, and that the property of specializing to $p$-adic integration via point counting now holds for general measurable sets.

Using this modified version of geometric motivic integration we then, in Chapter 5, address the following problem: When computing $p$-adic integrals, one sometimes get a result which behaves uniformly with respect to $p$. For a simple example, we have $\int_{\mathbb{Z}_p} |X|_p d\mu_{Haar} = p/(p+1)$ for every prime $p$. More complicated examples of such integrals arise when one computes the measure of polynomials satisfying certain factorization patterns. For example, the set of all tuples $(a_1, \ldots, a_n) \in \mathbb{Z}_p^n$, with the property that $X^n + a_1 X^{n-1} + \cdots + a_n$ splits completely, has measure $I_p/n!$ where

$$I_p = \int_{\mathbb{Z}_p^n} |\prod_{1 \leq i < j \leq n} (X_i - X_j)|_p d\mu_{Haar}.$$

There is one such integral for each prime $p$, and it turns out that there exists a function $f \in \mathbb{Z}(T)$ with the property that for all primes, $I_p = f(p)$. More generally, for every $q = p^d$, let

$$I_q = \int_{\mathbf{W}(\mathbb{F}_q)^n} |\prod_{1 \leq i < j \leq n} (X_i - X_j)|_p d\mu_{Haar}$$

It is then a fact that $I_q = f(q)$ for every prime power $q$. This is certainly not true for all integrals, e.g.,

$$\int_{\mathbf{W}(\mathbb{F}_q)} |X^2+1|_p d\mu_{Haar} = \begin{cases} 1 & q = p^{2d+1} \text{ where } p \equiv 3 \pmod 4 \\ (q-1)/(q+1) & \text{otherwise.} \end{cases}$$

Here we see that the rational function occurring in the answer depends both on the prime $p$, and it also varies when we integrate over different extensions $\mathbf{W}(\mathbb{F}_q)$ for powers of a fixed prime $p$. We want to give an explanation of this phenomenon using the theory of motivic integration, and this is partly achieved in Chapter 5.

In that chapter, we use our modified theory of motivic integration to show that, for fixed $p$, the motivic integral

$$I = \int |\prod_{1 \le i < j \le n} (X_i - X_j)|_p d\mu_{\mathcal{X}} \in \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p})$$

is equal to $f(\mathbb{L})$, where $f$ is the same rational function as above. By applying the point counting homomorphisms we recover the integrals $I_q$: We have $\mathrm{C}_q\, I = I_q$ for every power $q$ of $p$, showing that $I_q = f(q)$.

Similarly, if $p \equiv 3 \pmod 4$ then

$$\int |X^2+1| d\mu_{\mathcal{X}} = 1 - [\operatorname{Spec} \mathbb{F}_{p^2}]/(\mathbb{L}+1) \in \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p}),$$

showing that

$$\int_{\mathbf{W}(\mathbb{F}_q)} |X^2+1|_p d\mu_{Haar} = \begin{cases} 1 & q = p^{2d+1} \\ 1 - 2/(q+1) & q = p^{2d} \end{cases}.$$

This is only a partial solution to the original problem, since we have to keep the prime fixed. However, for fixed prime $p$ we get the uniformity when we vary the integration set over different extensions of $\mathbb{Z}_p$. This is about how far we reached on the original problem, even though we have some ideas of how to define an integral that specializes for almost all $p$, using results of Denef and Loeser. These ideas are outlined in Chapter 5.

### 0.1.1 Overview of the thesis

The thesis consists of three main parts, Part I, II and III. We have also included two initial sections: The present introductory one, and Section 0.2 which contains definitions and basic properties of the Grothendieck rings used throughout the thesis. We refer to the introductions of the individual chapters for a more thorough description of their contents than the one given above.

The thesis is based on the paper

- Serge Bouc and Karl Rökaeus, *A note on the $\lambda$-structure on the Burnside ring*, Journal of Pure and Applied Algebra **213** (2009), 1316–1319

and on the four preprints

- Karl Rökaeus, *Computing p-adic integrals using motivic integration*, arXiv:0812.2043v1 [math.AG]
- Karl Rökaeus, *A version of geometric motivic integration that specializes to p-adic integration via point counting*, arXiv:0810.4496v1 [math.AG]
- Karl Rökaeus, *The computation of the classes of some tori in the Grothendieck ring of varieties*, arXiv:0708.4396v3 [math.AG]
- Karl Rökaeus, *A note on the $\lambda$-structure on the Burnside ring*, arXiv:0708.1470v1 [math.GR]

Part I of the thesis is based on the last of these preprints, and on the article (these share the name and main result, but use different methods in order to prove it). Part II is based on the third item in the list of preprints. Part III is based on the first two preprints.

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More specific credit is given where it is due throughout the text.

## 0.2 Background material

All the parts of this thesis are concerned with computations in different Grothendieck rings. In this section, we therefore give an introduction to these rings.

First a general remark. Many of the rings that we work with are constructed as free abelian groups on the objects of a category, subject to some relations. When defining a map from such a ring we often just give its action on an object in the category. (We then have to show that it respects the relations.) Also, when letting such a map act on the class of an object we often leave out the brackets, e.g., if $f\colon \mathrm{K}_0(\mathsf{Var}_k) \to R$ and $X$ is a scheme we write $f(X)$ for $f\big([X]\big)$.

### 0.2.1 $\lambda$-rings

For an introduction, see for example the first part of [AT69] or [Knu73]. A $\lambda$-ring is a commutative ring $R$ together with a homomorphism $\lambda_t$ from the additive group of $R$ to the multiplicative group of $R[[t]]$, taking $x \in R$ to $\sum_{n\geq 0} \lambda^n(x)t^n$, where $\lambda^0(x) = 1$ and $\lambda^1(x) = x$. A morphism of $\lambda$-rings $R \to R'$ is a ring homomorphism that commutes with the $\lambda^n$. Informally, this definition ensures that the $\lambda^n$ behave like exterior powers. The archetypal example is the representation ring of a finite group $G$. In this ring, $\lambda^n(V) = [\bigwedge^n V]$, the $n$th exterior power of the vector space $V$ with componentwise $G$-action.

Let $\sigma_t(x) = \sum_{n\geq 0} \sigma^n(x)t^n$ be a $\lambda$-structure on $R$. Then the *opposite structure of* $\sigma_t$ is the $\lambda$-structure $\lambda_t$, defined implicitly by the relation $\sigma_t(x)\lambda_{-t}(x) = 1 \in R[[t]]$. On the representation ring the natural $\lambda$-structure can be obtained as the opposite of the one coming from the symmetric powers.

A $\lambda$-ring is *special* if, in addition to the above requirements, there exist integer polynomials $P_n$ and $P_{n,m}$ with the property that for all $r, s \in R$, $\lambda^n(rs) = P_n(\lambda^1(r), \ldots, \lambda^n(r); \lambda^1(s), \ldots, \lambda^n(s))$ and $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \ldots, \lambda^{mn}(r))$. The representation ring is special, whereas we will see that the Burnside ring is not.

### 0.2.2 Representation rings

We use $\mathrm{R}_k(G)$ to denote the ring of $k$-representations of $G$, where $G$ is a profinite group. We require such a representation to be finite and continuous with respect to the profinite topology on $G$ and the discrete topology on $k$.

When $k$ has some natural topology we can use the same construction but with respect to this topology instead. We call the ring thus obtained the Grothendieck ring of $k$-representations of $G$, and denote it $\mathrm{K}_0(\mathsf{Rep}_k G)$. We have an injection $\mathrm{R}_k(G) \to \mathrm{K}_0(\mathsf{Rep}_k G)$, but this is not an isomorphism in general. For example, the cyclotomic representation is often not discrete.

As abelian groups, both these rings are free on isomorphism classes of irreducible representations. They are naturally $\lambda$-rings, the structure being given by exterior powers. A map $H \to G$ gives rise to an induction and a restriction map between the corresponding representation rings, which we denote $\operatorname{ind}_H^G$ and $\operatorname{res}_H^G$ respectively. Finally, for $g \in G$, we use $C_g$ to denote the character homomorphism from any representation ring of $G$, i.e., the map given by $V \mapsto \chi_V(g)$. Together they can be used to distinguish elements in the representation ring; $\prod_{g\in G} C_g$ is injective.

### 0.2.3 The Grothendieck ring of varieties

Let $\mathsf{Var}_k$ be the category of varieties over the field $k$. Then $\mathrm{K}_0(\mathsf{Var}_k)$ is the free abelian group generated by symbols $[X]$ for $X \in \mathsf{Var}_k$, subject to the relations that $[X] = [Y]$ if $X \simeq Y$, $[X] = [X \setminus Y] + [Y]$ if $Y$ is a closed subscheme of $X$, and with a multiplication given by $[X] \cdot [Y] := [X \times_k Y]$. The second relation is usually referred to as the *scissor relation.* [2] By the class of the $k$-scheme $X$ we mean its image $[X] \in \mathrm{K}_0(\mathsf{Var}_k)$. The class of the affine line is called the *Lefschetz class* and denoted by $\mathbb{L}$. For a quick example of how the relations work, consider the multiplicative group $\mathbb{G}_m$. It can be embedded in the affine line and its complement is then $\operatorname{Spec} k$. Hence $[\mathbb{G}_m] = \mathbb{L} - 1 \in \mathrm{K}_0(\mathsf{Var}_k)$.

The Euler characteristic gives a map to the Grothendieck ring of $\mathbb{Q}_l$-representations of the absolute Galois group of $k$,

$$\chi_c \colon \mathrm{K}_0(\mathsf{Var}_k) \to \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l} \mathcal{G}),$$

such that $\chi_c(X) = \sum(-1)^i[\mathrm{H}^i_c(X_{\overline{k}}, \mathbb{Q}_l)]$. (Here $l$ is a prime different from the characteristic of $k$.) We will use $\chi_c$ to study $\mathrm{K}_0(\mathsf{Var}_k)$, for example, its existence immediately shows that $\mathbb{Z} \subset \mathrm{K}_0(\mathsf{Var}_k)$.

Define a $\lambda$-structure on $\mathrm{K}_0(\mathsf{Var}_k)$ as the opposite structure of $\{\sigma^n\}$, where $\sigma^n(X) = [X^n/\Sigma_n]$. In [LL04] it is mentioned that this seems to be the natural $\lambda$-structure on $\mathrm{K}_0(\mathsf{Var}_k)$, since $\sigma^n$ behaves like a symmetric power map. In particular, the Euler characteristic $\chi_c$ is a $\lambda$-homomorphism.

Let $K$ be a field extension of $k$. We write $R_k^K \colon \mathrm{K}_0(\mathsf{Var}_K) \to \mathrm{K}_0(\mathsf{Var}_k)$ for the forgetful morphism, defined by the map $\mathsf{Var}_K \to \mathrm{K}_0(\mathsf{Var}_k)$ that takes the $K$-scheme $X$ to the class of $X$, viewed as a $k$-scheme via $\operatorname{Spec} K \to \operatorname{Spec} k$. $R_k^K$ is additive but not multiplicative. Also, define $E_k^K \colon \mathrm{K}_0(\mathsf{Var}_k) \to \mathrm{K}_0(\mathsf{Var}_K)$ by mapping the $k$-scheme $X$ to the class of

[2] With respect to the definition of $\mathrm{K}_0(\mathsf{Var}_k)$, the important characterization of a variety is that it is of finite type over the base field; if not we end up with the zero ring. If we also include in the definition of a variety that it be reduced, we get a canonically isomorphic ring, for every scheme $X$ has a closed subscheme $X_{red}$ that is reduced and with empty complement, hence $[X] = [X_{red}]$. In the same way one can add the conditions that a variety be separated and irreducible and still get an isomorphic ring. However, the condition that a variety be geometrically reduced probably gives a slightly smaller ring when $k$ is non-perfect.

$X_K := X \times_k \operatorname{Spec} K$, viewed as a $K$-scheme. This is a ring homomorphism.

The reason why $R_k^K$ fails to be multiplicative is that it does not preserve the multiplicative identity element, instead $R_k^K(1) = [\operatorname{Spec} K] \in \mathrm{K}_0(\mathsf{Var}_k)$. Rather than being multiplicative, $R_k^K$ has a similar property: If $X$ is $k$-scheme and $Z$ is a $K$-scheme then, from the universal property defining fibre products, $Z \times_K (\operatorname{Spec} K \times_k X) \simeq Z \times_k X$ as $k$-schemes. It follows that if we apply the restriction map to $[Z] \cdot [X_K] \in \mathrm{K}_0(\mathsf{Var}_K)$ we get $[Z] \cdot [X] \in \mathrm{K}_0(\mathsf{Var}_k)$, i.e., for $x \in \mathrm{K}_0(\mathsf{Var}_k)$ and $z \in \mathrm{K}_0(\mathsf{Var}_K)$ we have the *projection formula*

$$R_k^K\big(zE_k^K(x)\big) = R_k^K(z)x.$$

(In other words, $R_k^K$ is a morphism of $\mathrm{K}_0(\mathsf{Var}_k)$-modules, where the $\mathrm{K}_0(\mathsf{Var}_k)$-module structure on $\mathrm{K}_0(\mathsf{Var}_K)$ is given by $E_k^K$.) We will use the special case when $X = \mathbb{A}_k^n$ and $Z = \operatorname{Spec} L$, where $L$ is a finite-dimensional $K$-algebra:

$$R_k^K([\operatorname{Spec} L]\mathbb{L}^n) = [\operatorname{Spec} L]\mathbb{L}^n \in \mathrm{K}_0(\mathsf{Var}_k). \tag{0.3}$$

In particular, $R_k^K(\mathbb{L}^n) = [\operatorname{Spec} K]\mathbb{L}^n$.

### 0.2.4 Burnside rings

For an introduction to Burnside rings, as well as proofs of the statements below, see [Knu73] Chapter II, 4.

If $G$ is a profinite group, let $G-\mathsf{Sets}$ be the category with objects finite sets with continuous $G$-actions (with respect to the inverse limit topology on $G$) and morphisms $G$-equivariant maps of such sets. We will denote the set of morphisms between the $G$-sets $S$ and $T$ by $\operatorname{Hom}_G(S, T)$. The Burnside ring of $G$, $\mathcal{B}(G)$, is constructed from this category as the free abelian group generated by the symbols $[S]$, for every continuous $G$-set $S$, subject to the relations that $[S \mathbin{\dot\cup} T] = [S] + [T]$ (disjoint union), that $[S] = [T]$ if $S \simeq T$, and with a multiplication given by $[S] \cdot [T] := [S \times T]$, where $G$ acts diagonally on $S \times T$.

Since every $G$-set can be written as a disjoint union of transitive $G$-sets we see that the transitive sets generate $\mathcal{B}(G)$, and in fact it is free on the isomorphism classes of these. Moreover, every finite transitive $G$-set is isomorphic to $G/H$ where $H$ is a subgroup, and $G/H \simeq G/H'$ if and only if $H$ and $H'$ are conjugate subgroups. So every element of $\mathcal{B}(G)$ can be written uniquely as $\sum_{H \in R} a_H [G/H]$, where $R$ is a system of representatives of the set of conjugacy classes of subgroups of $G$ and where $a_H \in \mathbb{Z}$ for every $H$.

There is a natural $\lambda$-structure $\lambda_t$ on $\mathcal{B}(G)$. It is given by first defining

$$\sigma_t \colon \mathcal{B}(G) \to \mathcal{B}(G)[[t]]$$

by the map that takes the $G$-set $S$ to the power series $\sum_{n\geq 0}[S^n/\Sigma_n]t^n \in \mathcal{B}(G)[[t]]$, where $\Sigma_n$ acts on $S^n$ by permuting the entries. $\lambda_t$ is then defined as the structure opposite to $\sigma_t$. This $\lambda$-structure is non-special (see Remark 1.10). However, it should still be considered the natural $\lambda$-structure on $\mathcal{B}(G)$, a major reason for this being that there is a natural map $\mathrm{h}\colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$ which is a $\lambda$-homomorphism with respect to this structure, see Section 1.2.

Next, let $G$ and $H$ be finite groups, and let $\phi\colon H \to G$ be a group homomorphism. We associate to it two maps, restriction and induction, between the corresponding Burnside rings in the same way as for representation rings: Firstly, $\mathrm{res}^G_H\colon \mathcal{B}(G) \to \mathcal{B}(H)$ is the map induced by restricting the $G$-action on a $G$-set $S$ to a $H$-action, i.e., $S$ is considered as a $H$-set via $h{\cdot}s := \phi(h)s$ for $h \in H$ and $s \in S$. This map is well defined also when $H$ is profinite and $G$ is a finite quotient of $H$, because then the induced $H$-action on $S$ continuous. $\mathrm{res}^G_H$ is a morphism of $\lambda$-rings.

Secondly, if instead $S$ is a $H$-set then we can associate to it the $G$-set $G{\times}_H S$, i.e., the quotient of $G{\times}S$ by the equivalence relation $(g{\cdot}\phi(h), s) \sim (g, hs)$ for $(g, s) \in G{\times}S$ and $h \in H$, with a $G$-action given by $g'\cdot(g, s) := (g'g, s)$. This gives rise to the induction map $\mathrm{ind}^G_H\colon \mathcal{B}(H) \to \mathcal{B}(G)$, which is additive but not multiplicative. We will only use the induction map in the case when $H$ is a subgroup of $G$. In this case, note that if we choose a set of coset representatives of $G/H$, $R = \{g_1, \ldots, g_r\}$, then we can represent $G \times_H S$ as $R \times S$ with $G$-action given by $g \cdot (g_i, s) = (g_j, hs)$, where $gg_i = g_jh$ for $h \in H$. It follows that $\mathrm{ind}^G_H$ is defined also when $G$ is profinite and $H$ is a normal subgroup of finite index.

# Part I:
# The Burnside ring

# 1. The λ-structure on the Burnside ring

## 1.1 Introduction and statement of the main theorem

We use $\mathcal{B}(G)$ to denote the Burnside ring of the finite group $G$, see Section 0.2.4 for an introduction.

Recall that there is a λ-structure on $\mathcal{B}(G)$, $\{\lambda^n\}_{n\in\mathbb{N}}$, defined as the opposite structure of $\{\sigma^n\}_{n\in\mathbb{N}}$, where $\sigma^n(S)$ is the class of the $n$th symmetric power of $S$. It should be considered the natural λ-structure on $\mathcal{B}(G)$, one reason for this being that there is a canonical homomorphism to the ring of rational representations of $G$, $\mathrm{h}\colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$, defined by $\mathrm{h}(S) = \big[\mathbb{Q}[S]\big]$, and the given λ-structure on $\mathcal{B}(G)$ makes h into a λ-homomorphism.

The implicit nature of the definition of the λ-structure on $\mathcal{B}(G)$ makes it hard to compute with. The main result of this chapter is a closed formula for $\lambda^n(S)$, where $S$ is any $G$-set. To state it, we use the following notation: let $\mu = (\mu_1, \dots, \mu_l) \vdash n$, i.e., $\mu$ is a partition of $n$. We use $\ell(\mu) := l$ to denote the length of $\mu$, and if $\mu = (1^{\alpha_1}, 2^{\alpha_2}, \dots)$, we define the tuple $\alpha(\mu) := (\alpha_1, \dots, \alpha_{l'})$, and write $\binom{\ell(\mu)}{\alpha(\mu)}$ for $\frac{l!}{\alpha_1! \cdots \alpha_{l'}!}$. Using this notation we can express $\lambda^n(S)$, for any $G$-set $S$, as a linear combination of classes of $G$-sets:

**Theorem 1.1.** *Let $n$ be a positive integer and let $\mu = (\mu_1, \dots, \mu_l) \vdash n$. For $S$ a $G$-set, let $\mathcal{P}_\mu(S)$ be the $G$-set consisting of $\ell(\mu)$-tuples of pairwise disjoint subsets of $S$, where the first one has cardinality $\mu_1$, and so on. Then*

$$\lambda^n(S) = (-1)^n \sum_{\mu \vdash n} (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}_\mu(S)] \in \mathcal{B}(G). \tag{1.2}$$

*In particular, $\lambda^n(S) = 0$ when $n > |S|$.*

This result was first stated and proved, in a slightly different setting, in [MW97]. It was then rediscovered and proved in the preprint [Rök07b], by showing that $\lambda^n(S)$ lies in a subring of $\mathcal{B}(G)$ on which h is injective, and then that the image of (1.2) in $\mathrm{R}_{\mathbb{Q}}(G)$ is satisfied. That proof is reproduced in the present chapter.

Later, Professor Serge Bouc suggested a more intrinsic proof, using only the structure of the Burnside ring. This resulted in the joint paper [BR09]. The proof given there relies on the construction of a ring of formal power series with coefficients in Burnside rings, and an exponential map on this ring, developed in [Bou92]. Using this framework, the proof reduces to some explicit combinatorial computations. That proof is given in Chapter 2.

Theorem 1.1 originates in the paper [Rök07a] (whose content is included as Chapter 3 in this thesis), in which we compute the classes of certain tori in the Grothendieck ring of varieties, in terms of the $\lambda$-structure on that ring. This formula is suggested by the corresponding class of the cohomology of the torus, and its proof uses a map from the Burnside ring of the absolute Galois group of the base field, where formula (1.2) can be applied. Actually, it was these computations that led us to conjecture Theorem 1.1.

### Reduction to the case when $G$ is a full symmetric group

In this chapter we prove Theorem 1.1 in the special case when $G = \Sigma_i$. The general case then follows straight forward in the following way:

*Proof of Theorem 1.1.* First, recall that a group homomorphism $\phi\colon H \to G$ gives rise to a $\lambda$-homomorphism $\mathrm{res}^G_H\colon \mathcal{B}(G) \to \mathcal{B}(H)$ by restricting the action on a $G$-set $S$ to an $H$-action via $\phi$. Now let $G$ be a finite group and let $S$ be a $G$-set of cardinality $i$. By choosing an enumeration of $S$ we get a homomorphism $G \to \Sigma_i$, the symmetric group on $\{1,\dots,i\}$. Let $\mathrm{res}^{\Sigma_i}_G$ be the corresponding restriction homomorphism (which is independent of the chosen enumeration). We have that $\mathrm{res}^{\Sigma_i}_G\bigl(\{1,\dots,i\}\bigr) = [S]$, hence, since $\mathrm{res}^{\Sigma_i}_G$ is a $\lambda$-homomorphism, $\mathrm{res}^{\Sigma_i}_G\bigl(\lambda^n(\{1,\dots,i\})\bigr) = \lambda^n(S)$. Also, writing $\mathcal{P}^{(i)}_\mu$ for $\mathcal{P}_\mu(\{1,\dots,i\})$, we see that $\mathrm{res}^{\Sigma_i}_G(\mathcal{P}^{(i)}_\mu) = [\mathcal{P}_\mu(S)]$. Hence, to prove Theorem 1.1 it suffices to prove it in the special case when $G = \Sigma_i$ and $S = \{1,\dots,i\}$, i.e.,

$$\lambda^n\bigl(\{1,\dots,i\}\bigr) = (-1)^n \sum_{\mu\vdash n} (-1)^{\ell(\mu)} \tbinom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}^{(i)}_\mu] \in \mathcal{B}(\Sigma_i). \tag{1.3}$$

The validity of (1.3) will be established in Theorem 1.16. □

### Overview of the chapter

In Section 1.2 we give some basic properties of the map $\mathrm{h}_G\colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$, together with an example which will be used in Chapter 3.

The proof of (1.3) given in Theorem 1.16 uses the canonical $\lambda$-homomorphism $\mathrm{h}_{\Sigma_i}\colon \mathcal{B}(\Sigma_i) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_i)$ to move some of the computations to the rational representation ring, whose $\lambda$-structure is much easier to work with. However, there is a problem in that h is not injective for $\Sigma_i$. In Section 1.3 we will therefore introduce a subgroup $\mathit{Schur}_i \subset \mathcal{B}(\Sigma_i)$, with the property that the restriction of $\mathrm{h}_{\Sigma_i}\colon \mathcal{B}(\Sigma_i) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_i)$ to $\mathit{Schur}_i$ is injective. We then also include a brief discussion of the structure of $Schur_i$, proving that it is a ring which is not closed under the $\lambda$-operations.

In Section 1.4 we then establish (1.3). The technique of passing to the representation ring will be used at a crucial place, to prove Lemma 1.14.

## 1.2 The map from the Burnside ring to the representation ring

Let $G$ be a profinite group. There is a natural map from the Burnside ring to the rational representation ring of $G$, $\mathrm{h}_G\colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$, which is defined by associating to the $G$-set $S$ the class of the permutation representation $\mathbb{Q}[S]$. When there is no risk of confusion we write just h instead of $\mathrm{h}_G$.

The map h is a homomorphism of $\lambda$-rings, which is one of the reasons why we consider our $\lambda$-structure on $\mathcal{B}(G)$ to be the natural one. It has the property that if $S$ and $S'$ are two non-isomorphic transitive $G$-sets then

$$\mathrm{h}(S) \neq \mathrm{h}(S'). \tag{1.4}$$

However, since $\mathcal{B}(G)$ has rank equal to the number of conjugacy classes of subgroups of $G$ whereas $\mathrm{R}_{\mathbb{Q}}(G)$ has rank equal to the number of conjugacy classes of cyclic subgroups of $G$, h cannot be injective unless $G$ is cyclic. Conversely, if $G$ is procyclic (i.e., it contains a dense cyclic subgroup) then h is an isomorphism. These facts are proved for example by using the character maps $C_g$ for $g \in G$. (Usually for finite groups, but since taking the inverse limit of groups corresponds to taking the direct limit of the corresponding Burnside and representation rings, they follow immediately for any profinite group.)

The map $\mathrm{h}\colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$ commutes with the restriction maps, and also with the induction maps if $H$ is a subgroup of $G$.

We now give an example of a finite extension $k$ of $\mathbb{Q}$ with absolute Galois group $\mathcal{G}$, having the property that $\mathrm{h}_{\mathcal{G}}\colon \mathcal{B}(\mathcal{G}) \to \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ is not surjective. (This will be used in Remark 3.13, to find tori for which the main theorem of Chapter 3 does not hold.) For this we use an example of Serre of a finite group having this property, together with the following lemma:

**Lemma 1.5.** *Let $G$ be a profinite group and let $N$ be a normal subgroup of finite index. Define $H := G/N$ and let $x \in \mathrm{R}_{\mathbb{Q}}(H)$. If $\mathrm{res}_G^H x$ is contained in the image of $\mathrm{h}_G\colon \mathcal{B}(G) \to \mathrm{R}_{\mathbb{Q}}(G)$, then $x$ is contained in the image of $\mathrm{h}_H\colon \mathcal{B}(H) \to \mathrm{R}_{\mathbb{Q}}(H)$. In particular, if $\mathrm{h}_G$ is surjective, then so is $\mathrm{h}_H$.*

*Proof.* Define a map $\mathrm{R}_{\mathbb{Q}}(G) \to \mathrm{R}_{\mathbb{Q}}(H)$ by $V \mapsto [V^N]$, where $V^N$ is the elements of the $G$-representation $V$ invariant under $N$. The corresponding map $\mathcal{B}(G) \to \mathcal{B}(H)$ is defined by $S \mapsto [S/N]$, and one proves that the following diagram is commutative:

$$\begin{array}{ccccc}
\mathcal{B}(H) & \xrightarrow{\operatorname{res}^H_G} & \mathcal{B}(G) & \longrightarrow & \mathcal{B}(H) \\
\downarrow{\scriptstyle \mathrm{h}_H} & & \downarrow{\scriptstyle \mathrm{h}_G} & & \downarrow{\scriptstyle \mathrm{h}_H} \\
\mathrm{R}_{\mathbb{Q}}(H) & \xrightarrow{\operatorname{res}^H_G} & \mathrm{R}_{\mathbb{Q}}(G) & \longrightarrow & \mathrm{R}_{\mathbb{Q}}(H)
\end{array}$$

Since both the horizontal compositions equal the identity, the result follows by diagram chasing. □

**Example 1.6.** *There exists a finite extension $k$ of $\mathbb{Q}$, with absolute Galois group $\mathcal{G}$, such that $\mathrm{h}_{\mathcal{G}}\colon \mathcal{B}(\mathcal{G}) \to \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ is not surjective. For by Exercise 13.4, page 105 of [Ser77], there is a finite group $H$ such that $\mathrm{h}_H\colon \mathcal{B}(H) \to \mathrm{R}_{\mathbb{Q}}(H)$ is not surjective. (More precisely, the example in [Ser77] shows that this holds for the product of the quaternion group with the cyclic group of order 3.) Now choose a finite field extension $k$ of $\mathbb{Q}$ such that there exists a finite extension $K/k$ with the property that $\mathcal{G}\mathrm{al}(K/k) = H$. (Actually, since the group in the example of Serre is solvable we may, by a theorem of Shaferevich, choose $k$ to be any numberfield.) Since $H = \mathcal{G} \,/\, \mathcal{G}\mathrm{al}(\overline{k}/K)$, it follows from preceding lemma that $\mathrm{h}_{\mathcal{G}}$ cannot be surjective.*

## 1.3 The Schur subring of $\mathcal{B}(\Sigma_n)$

Recall that we write $\mathcal{P}_\mu^{(n)}$ for $\mathcal{P}_\mu(\{1,\ldots,n\}) \in \mathcal{B}(\Sigma_n)$.

Let $S$ be a $\Sigma_n$-set. We say that $S$ is a *Schur set* if, for every $s \in S$, the stabilizer subgroup of $s$, $(\Sigma_n)_s$, is a Schur subgroup, i.e., it stabilizes each part of some partition of $\{1,\ldots,n\}$. Equivalently, any transitive component of $S$ is isomorphic to $\mathcal{P}_\mu^{(n)}$ for some $\mu \vdash n$.

**Definition 1.7.** *$Schur_n$ is the subgroup of $\mathcal{B}(\Sigma_n)$ generated by the Schur sets.*

Equivalently, this means that $Schur_n \subset \mathcal{B}(\Sigma_n)$ is the free subgroup on $\{[\mathcal{P}_\mu]\}_{\mu\vdash n}$. The reason for us to introduce $Schur_n$ is the next theorem:

**Theorem 1.8.** *Let $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ be the canonical $\lambda$-ring homomorphism. The restriction of $\mathrm{h}$ to $Schur_n$ is injective.*

Even though this is a simple consequence of the injectivity of the character homomorphism from $\mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ to the ring of symmetric polynomials, we have chosen to give a more direct proof:

*Proof.* For every $\mu \vdash n$, let $\sigma_\mu \in \Sigma_n$ be an element in the conjugacy class determined by $\mu$ and let $C_{\sigma_\mu}\colon \mathrm{R}_{\mathbb{Q}}(\Sigma_n) \to \mathbb{Z}$ be the homomorphism defined by $V \mapsto \chi_V(\sigma_\mu)$, where $\chi_V$ is the character of $V$. This definition is independent of the choice of $\sigma_\mu$. Together the $C_{\sigma_\mu}$ give a homomorphism

$$\mathrm{R}_{\mathbb{Q}}(\Sigma_n) \to \prod_{\mu \vdash n} \mathbb{Z},$$

and it suffices to show that the composition of this with the restriction of h to $Schur_n$ is injective, i.e., that

$$\varphi\colon Schur_n \to \prod_{\mu \vdash n} \mathbb{Z}$$
$$[T] \mapsto \big(|T^{\sigma_\mu}|\big)_{\mu \vdash n}$$

is injective, where $T^{\sigma_\mu}$ is the set of points in $T$ fixed by $\sigma_\mu$. To do this, define a total ordering on the set of partitions of $n$ by $\mu > \mu'$ if $\mu_1 = \mu'_1, \ldots, \mu_{j-1} = \mu'_{j-1}$ and $\mu_j > \mu'_j$ for some $j$ (i.e., lexicographic order). We claim that $|\mathcal{P}_\mu^{\sigma_\mu}| \neq 0$, whereas $\mathcal{P}_{\mu'}^{\sigma_\mu} = \emptyset$ if $\mu > \mu'$. (Here and in the rest of this proof we write $\mathcal{P}_\mu$ for $\mathcal{P}_\mu^{(n)}$.)

For the first assertion, choose for example

$$\sigma_\mu = (1, \ldots, \mu_1)(\mu_1 + 1, \ldots, \mu_1 + \mu_2) \cdots (n - \mu_{\ell(\mu)} + 1, \ldots, n).$$

Then

$$\big(\{1, \ldots, \mu_1\}, \{\mu_1 + 1, \ldots, \mu_1 + \mu_2\}, \ldots, \{n - \mu_{\ell(\mu)} + 1, \ldots, n\}\big) \in \mathcal{P}_\mu$$

is fixed by $\sigma_\mu$.

For the second assertion, suppose $\mu' < \mu$ and $t = (T_1, \ldots, T_l) \in \mathcal{P}_{\mu'}$, where $l = \ell(\mu')$. Suppose moreover that $t$ is fixed by $\sigma_\mu$. If now $\mu_1 > \mu_2 > \cdots > \mu_{\ell(\mu)}$, then, with the same $\sigma_\mu$ as above, we must have $T_1 = \{1, \ldots, \mu_1\}, \ldots, T_l = \{n - \mu_l + 1, \ldots, n\}$. (This is because $\mu_1 \geq \mu'_j$ for every $j$ and if 1 lies in $T_j$ then so does $\sigma_\mu(1) = 2$, hence also $3, \ldots, \mu_1$. So $T_j$ has cardinality at least $\mu_1$ and the only $\mu'_j$ that can be that big is $\mu'_1$. Consequently, $j = 1$, and $\mu_1 = \mu'_1$.) But if $\mu$ and $\mu'$ differ for the first time in position $j$ it is impossible for $T_j$ to fulfill this since it has cardinality $\mu'_j < \mu_j$. In the general case, when we may have $\mu_j = \mu_{j+1}$, the above argument works the same only that we for example can have $T_1 = \{\mu_1 + 1, \ldots, \mu_1 + \mu_2\}$ and $T_2 = \{1, \ldots, \mu_1\}$ if $\mu_1 = \mu_2$.

We are now ready to prove that $\varphi$ is injective. Let $x = \sum_{\mu \vdash n} a_\mu [\mathcal{P}_\mu]$, where $a_\mu \in \mathbb{Z}$, and suppose that $x \neq 0$. Choose the maximal $\mu_0$ such that $a_{\mu_0} \neq 0$. Let $\varphi_{\mu_0}$ be the $\mu_0$th component of $\varphi$. Then

$$\varphi_{\mu_0}(x) = \sum_{\mu \vdash n} a_\mu |\mathcal{P}_\mu^{\sigma_{\mu_0}}| = a_{\mu_0} |\mathcal{P}_{\mu_0}^{\sigma_{\mu_0}}| \neq 0,$$

hence $\varphi(x) \neq 0$. □

We conclude this section by showing that $Schur_n$ is a subring of $\mathcal{B}(\Sigma_n)$.

**Proposition 1.9.** *$Schur_n$ is closed under multiplication.*

*Proof.* We want to see what happens when we multiply $[S]$ and $[T]$, where $S$ and $T$ are Schur sets. Let $s \in S$ and $t \in T$. Then the stabilizers of $s$ and $t$ equal the stabilizers of partitions of $\{1, \ldots, n\}$, which we denote $(S_1, \ldots, S_k)$ and $(T_1, \ldots, T_l)$, respectively. Then $\sigma \in \Sigma_n$ is in the stabilizer of $(s,t)$ precisely when $\sigma S_i = S_i$ and $\sigma T_j = T_j$ for each $i, j$. Equivalently, $\sigma$ must preserve $S_i \cap T_j$ for each $i, j$. Hence $(\Sigma_n)_{(s,t)}$ equals the stabilizer of the partition $\{S_i \cap T_j\}_{i,j}$. Consequently, it is a Schur subgroup, hence $S \times T$ is a Schur set. Therefore $Schur_n$ is closed under multiplication. □

**Remark 1.10.** *$Schur_n$ is in general not a $\lambda$-ring since it is not closed under the $\lambda$-operations. For example, note that when $S$ is a $\Sigma_n$-set we can represent the symmetric square of $S$ as the union of $S$ and the set of 2-subsets of $S$. Now, let $S$ be the $\Sigma_4$-set $\{1, 2, 3, 4\}$ and consider $x := \sigma^2\big(\sigma^2(S)\big) \in \mathcal{B}(\Sigma_4)$. An element of the underlying $\Sigma_4$-set is $\{\{1, 2\}, \{3, 4\}\}$ and the stabilizer $G$ of this element is generated by $\{(12), (34), (13)(24), (14)(23)\}$. The only partition that is stabilized by $G$ is the trivial one, so since $G$ does not equal $\Sigma_4$ it fails to be the stabilizer of a partition. Hence $x \notin Schur_4$. Since $\lambda^2\big(\sigma^2(S)\big) = \big(\sigma^2(S)\big)^2 - x$ it follows that this is not contained in $Schur_4$ either. But $\sigma^2(S)$ is in $Schur_4$, which is therefore not a $\lambda$-ring. This also gives an example showing that $\mathcal{B}(\Sigma_4)$ is non-special. For if it were, then $y := \lambda^2\big(\lambda^2(S)\big)$ would be a polynomial in $\lambda^i(S)$ for $i = 1, 2, 3, 4$, which all lie in $Schur_4$, so $y$ would also lie in $Schur_4$. But using the above one shows that $y \notin Schur_4$.*

## 1.4 The $\lambda$-operations on $\mathcal{B}(\Sigma_n)$

We are now ready to start the investigation of how $\lambda^i$ acts on $\{1, \ldots, n\}$, the goal being to obtain a closed formula for it. We will need some more definitions. We have only defined $\mathcal{P}_\mu^{(n)}$ when $\mu$ is a partition of $i \leq n$. More generally:

**Definition 1.11.** *If $\alpha = (i_1, \ldots, i_l)$ is any tuple of positive integers summing up to $i \leq n$ we define $\mathcal{P}_\alpha^{(n)}$ to be the $\Sigma_n$-set of $l$-tuples of disjoint subsets of $\{1, \ldots, n\}$, the first one having $i_1$ elements, and so on.*

We have $[\mathcal{P}_\alpha^{(n)}] = [\mathcal{P}_\mu^{(n)}]$, where $\mu$ is the partition of $i$ corresponding to $\alpha$. Also note that $[\mathcal{P}_\alpha^{(n)}] = [\mathcal{P}_{\alpha,n-i}^{(n)}]$, where we use $\mathcal{P}_{\alpha,n-i}^{(n)}$ to denote $\mathcal{P}_{(i_i,\ldots,i_l,n-i)}^{(n)}$. Similarly, if $\beta = (j_1, \ldots, j_k)$ we will write $\mathcal{P}_{\alpha,\beta}^{(n)}$ for $\mathcal{P}_{(i_1,\ldots,i_l,j_1,\ldots,j_k)}^{(n)}$.

Throughout this section we use the following notation:

$$\begin{aligned} s_i^{(n)} :=& \sigma^i(\{1,\dots,n\}) \\ \ell_i^{(n)} :=& \lambda^i(\{1,\dots,n\}) \in \mathcal{B}(\Sigma_n) \end{aligned}$$

We begin by giving a formula for $s_i^{(n)}$ which shows that it lies in $Schur_n$, and we then deduce from this that also $\ell_i^{(n)}$ is in $Schur_n$. Recall from the introduction that if $\mu = (\mu_1, \dots, \mu_l)$, where $\mu_1 = \cdots = \mu_{\alpha_1} > \mu_{\alpha_1+1} = \cdots = \mu_{\alpha_1+\alpha_2} > \cdots > \mu_{j-\alpha_{l'}+1} = \cdots = \mu_j$, then we define $\alpha(\mu) := (\alpha_1, \dots, \alpha_{l'})$.

**Proposition 1.12.** *We have*

$$s_i^{(n)} = \sum_{\substack{\mu \vdash i: \\ \ell(\mu) \leq n}} [\mathcal{P}_{\alpha(\mu)}^{(n)}].$$

*In particular, $s_i^{(n)}$ and $\ell_i^{(n)}$ are in $Schur_n$ for every $i$.*

*Proof.* Identify $\{1,\dots,n\}$ with $\{x_1,\dots,x_n\}$. Then the symmetric $i$th power of $\{1,\dots,n\}$, the $\Sigma_n$-set $\{1,\dots,n\}^i/\Sigma_i$, is identified with the set of monomials

$$\{x_1^{e_1}\cdots x_n^{e_n} : e_1 + \cdots + e_n = i\} = \dot{\bigcup}_{\substack{e_1+\cdots+e_n=i \\ e_1 \geq e_2 \cdots \geq e_n \geq 0}} \Sigma_n \cdot x_1^{e_1}\cdots x_n^{e_n},$$

where the index set on the disjoint union can be identified with the set of $\mu \vdash i$ such that $\ell(\mu) \leq n$. Now let $e_1 = \cdots = e_{\alpha_1} > e_{\alpha_1+1} = \cdots = e_{\alpha_1+\alpha_2} > \cdots > e_{n-\alpha_l+1} = \cdots = e_n$. Then

$$\begin{aligned} &\Sigma_n \cdot x_1^{e_1}\cdots x_n^{e_n} \\ &= \Sigma_n \cdot (x_1\cdots x_{\alpha_1})^{e_1}(x_{\alpha_1+1}\cdots x_{\alpha_1+\alpha_2})^{e_{\alpha_1+1}}\cdots(x_{n-\alpha_l+1}\cdots x_n)^{e_{n-\alpha_l+1}} \\ &\simeq \Sigma_n\big(\{x_1,\dots,x_{\alpha_1}\},\{x_{\alpha_1+1},\dots,x_{\alpha_1+\alpha_2}\},\dots,\{x_{n-\alpha_l+1},\dots,x_n\}\big) \\ &\simeq \mathcal{P}_{(\alpha_1,\dots,\alpha_l)}^{(n)} \end{aligned}$$

so the first part of the proposition follows.

To show that also $\ell_i^{(n)} \in Schur_n$ we use that, by definition,

$$-(-1)^i \ell_i^{(n)} = \sum_{j=0}^{i-1} (-1)^j \ell_j^{(n)} s_{i-j}^{(n)}.$$

Since we know that $Schur_n$ is a ring, and that all $s_j^{(n)}$ and $\ell_1^{(n)} = [\mathcal{P}_1^{(n)}]$ are in $Schur_n$, it follows by induction that $\ell_i^{(n)} \in Schur_n$. □

Recall that a group homomorphism $\phi\colon H \to G$ gives rise to an induction map $\mathrm{ind}_H^G\colon\ \mathcal{B}(H) \to \mathcal{B}(G)$, which is additive but not multiplicative (see Section 0.2.4 for its definition). Recall also that the diagram

$$\begin{array}{ccc} \mathcal{B}(G) & \xrightarrow{\mathrm{h}} & \mathrm{R}_{\mathbb{Q}}(G) \\ {\scriptstyle \mathrm{ind}_H^G}\uparrow & & \uparrow{\scriptstyle \mathrm{ind}_H^G} \\ \mathcal{B}(H) & \xrightarrow{\mathrm{h}} & \mathrm{R}_{\mathbb{Q}}(H) \end{array}$$

is commutative if $H$ is a subgroup of $G$.

In the following two lemmas we show that $\ell_i^{(n)}$ and $[\mathcal{P}_\mu^{(n)}]$, where $\mu \vdash i$, are determined by $\ell_i^{(i)}$ and $[\mathcal{P}_\mu^{(i)}]$ respectively. For this we use the map

$$\mathrm{ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{res}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_i}\colon\ \mathcal{B}(\Sigma_i) \to \mathcal{B}(\Sigma_n)$$

which is constructed in the following way: We view $\Sigma_i$ as the symmetric group on $\{1,\dots,i\}$ and embed it in $\Sigma_n$, the symmetric group on $\{1,\dots,n\}$. Moreover we view $\Sigma_{n-i}$ as the symmetric group on $\{i+1,\dots,n\}$. We then restrict from $\mathcal{B}(\Sigma_i)$ to $\mathcal{B}(\Sigma_i\times\Sigma_{n-i})$ with respect to the projection $\Sigma_i\times\Sigma_{n-i}\to\Sigma_i$ and we induce from $\mathcal{B}(\Sigma_i\times\Sigma_{n-i})$ to $\mathcal{B}(\Sigma_n)$ with respect to the inclusion $(\tau,\rho)\mapsto\tau\rho=\rho\tau\colon \Sigma_i\times\Sigma_{n-i}\to\Sigma_n$.

**Lemma 1.13.** *Let $\mu\vdash i$. For $n\ge i$, $\mathrm{ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{res}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_i}\big(\mathcal{P}_\mu^{(i)}\big) = [\mathcal{P}_\mu^{(n)}]\in\mathcal{B}(\Sigma_n)$.*

*Proof.* Let $R=\{\sigma_1,\dots,\sigma_r\}$, where $r=\binom{n}{i}$, be a system of coset representatives for $\Sigma_n/(\Sigma_i\times\Sigma_{n-i})$. We know that $\Sigma_n\times_{\Sigma_i\times\Sigma_{n-i}}\mathcal{P}_\mu^{(i)}$ can be identified with the set of pairs $(\sigma_j,t)$, where $\sigma_j\in R$ and $t=(T_1,\dots,T_l)\in\mathcal{P}_\mu^{(i)}$. From this set we define a map to $\mathcal{P}_\mu^{(n)}$ by

$$(\sigma_j,t)\mapsto\big(\sigma_jT_1,\dots,\sigma_jT_l,\sigma_j\{i+1,\dots,n\}\big).$$

This map is surjective for given $t'=(T_1',\dots,T_l',T_{l+1}')\in\mathcal{P}_\mu^{(n)}$, there is a $\sigma\in\Sigma_n$ such that $\sigma\{1,\dots,\mu_1\}=T_1',\dots,\sigma\{i-\mu_l+1,\dots,i\}=T_l'$. Let $\sigma_j\in R$ be such that $\sigma=\sigma_j\tau\rho$ where $(\tau,\rho)\in\Sigma_i\times\Sigma_{n-i}$. Then

$$\big(\sigma_j,\tau\{1,\dots,\mu_1\},\dots,\tau\{i-\mu_l+1,\dots,i\}\big)\mapsto t'.$$

Since both sets have $n!/(\mu_1!\cdots\mu_l!(n-i)!)$ elements this is a bijection. Finally, the map is $G$-equivariant, hence it is an isomorphism. □

It is the following lemma that forces us to pass to the representation ring, for we have not been able to prove it directly in the Burnside ring.

**Lemma 1.14.** *Given $i\in\mathbb{N}$. For $n\ge i$ we have*

$$\mathrm{ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n} \circ \mathrm{res}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_i}\big(\ell_i^{(i)}\big) = \ell_i^{(n)}\in\mathcal{B}(\Sigma_n).$$

*Proof.* We pass to the representation ring. Here, since h is a morphism of $\lambda$-rings that commutes with the induction and restriction maps, the image of the left hand side under h is

$$\mathrm{ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}\circ\mathrm{res}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_i}\circ\lambda^i\big(\mathbb{Q}\big[\{1,\dots,i\}\big]\big) \qquad \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n),$$

and the image of the right hand side is $\lambda^i\big(\mathbb{Q}\big[\{1,\dots,n\}\big]\big) \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$. Now, since $\ell_i^{(i)} \in Schur_i$ it follows from the preceding lemma that $\mathrm{ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}\circ\mathrm{res}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_i}\ell_i^{(i)} \in Schur_n$. Since also $\ell_i^{(n)} \in Schur_n$ and h is injective on $Schur_n$, it suffices to prove that

$$\mathrm{ind}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_n}\circ\mathrm{res}_{\Sigma_i\times\Sigma_{n-i}}^{\Sigma_i}\Big(\lambda^i\big(\mathbb{Q}\big[\{1,\dots,i\}\big]\big)\Big) \\ = \lambda^i\big(\mathbb{Q}\big[\{1,\dots,n\}\big]\big) \in \mathrm{R}_{\mathbb{Q}}(\Sigma_n),$$

i.e., we have to find a $\Sigma_n$-equivariant isomorphism of $\mathbb{Q}$-vector spaces

$$\varphi\colon \mathbb{Q}[\Sigma_n]\otimes_{\mathbb{Q}[\Sigma_i\times\Sigma_{n-i}]}\bigwedge^i\mathbb{Q}\big[\{1,\dots,i\}\big]\to\bigwedge^i\mathbb{Q}\big[\{1,\dots,n\}\big].$$

This is straightforward. (It is done explicitly in [Rök07c], Proposition 2.4.2.) □

We are now ready to prove the main theorem of this section, the formula for $\ell_i^{(n)}$. For this we first introduce a concept of degree on basis elements of $Schur_n$. Fix an $n$ and the basis $\{[\mathcal{P}_\mu^{(n)}]\}_{\mu\vdash n}$ of $Schur_n$. Let $k$ be the greatest integer such that $2k<n$. For $j=1,\dots,k$ we say that an element $[\mathcal{P}_\mu^{(n)}]$ in the basis is of degree $j$ if it is equal to $[\mathcal{P}_\nu^{(n)}]$ for some $\nu\vdash j$. Equivalently, this means that $n-j$ is an entry of $\mu$. Let the degree of $[\mathcal{P}_n^{(n)}]=1$ be zero and let the degree of the remaining elements of the basis be $k+1$. Because $n-j>k$ for $j=0,1,\dots,k$, the degree is well-defined.

**Lemma 1.15.** *Let $[\mathcal{P}_\alpha^{(n)}]$ and $[\mathcal{P}_\beta^{(n)}]$ be of degree $m$ and $m'$ respectively, where $m+m'\le n/2$. Then*

$$[\mathcal{P}_\alpha^{(n)}]\cdot[\mathcal{P}_\beta^{(n)}]=[\mathcal{P}_{\alpha,\beta}^{(n)}]+\text{ terms of degree} < m+m'.$$

*Proof.* This is a refinement of Proposition 1.9. Let $s=(S_1,\dots,S_l)\in\mathcal{P}_\alpha^{(n)}$ and $t=(T_1,\dots,T_{l'})\in\mathcal{P}_\beta^{(n)}$, where $S_l$ and $T_{l'}$ have $n-m$ and $n-m'$ elements respectively. Then the stabilizer of $(s,t)\in\mathcal{P}_\alpha^{(n)}\times\mathcal{P}_\beta^{(n)}$ equals the stabilizer of $(S_i\cap T_j)_{i,j}$. Let $m_{ij}=|S_i\cap T_j|$ and let $\gamma$ be the tuple consisting of the $m_{ij}$. Then the transitive component of $(s,t)$ is $\mathcal{P}_\gamma^{(n)}$. Since $m_{ll'}\ge n-m-m'\ge n/2$ it follows that the degree of $[\mathcal{P}_\gamma^{(n)}]$ is $n-m_{ll'}$, and this is $\le m+m'$ with equality if and only if $[\mathcal{P}_\gamma^{(n)}]=[\mathcal{P}_{\alpha,\beta}^{(n)}]$. □

**Theorem 1.16.** *Let $i$ be a positive integer. Then for any $n \geq i$,*

$$\lambda^i\big(\{1,\ldots,n\}\big) = (-1)^i \sum_{\mu\vdash i} (-1)^{\ell(\mu)} \tbinom{\ell(\mu)}{\alpha(\mu)} \big[\mathcal{P}_\mu^{(n)}\big] \in \mathcal{B}(\Sigma_n).$$

*Proof.* For $i = 1$ the formula becomes $\ell_1^{(n)} = \big[\mathcal{P}_1^{(n)}\big] = [\{1,\ldots,n\}]$, which is true for every $n$.

Given $i$, suppose the formula is true for every pair $(i', n)$ where $i' < i$ and $n$ is an arbitrary integer greater than or equal to $i'$. We want to show that it holds for $(i, n)$ where $n$ is an arbitrary integer greater than or equal to $i$.

Since $\ell_i^{(i)} \in Schur_i$, we have $\ell_i^{(i)} = \sum_{\mu\vdash i} a_\mu \big[\mathcal{P}_\mu^{(i)}\big]$, where the $a_\mu$ are uniquely determined. Since the induction and restriction maps are additive it follows from Lemma 1.13 and Lemma 1.14 that

$$\ell_i^{(n)} = \sum_{\mu\vdash i} a_\mu \big[\mathcal{P}_{\mu,n-i}^{(n)}\big] \in Schur_n \tag{1.17}$$

for every $n \geq i$. It remains to show that $a_\mu = (-1)^i(-1)^{\ell(\mu)}\binom{\ell(\mu)}{\alpha(\mu)}$ for every $\mu \vdash i$. For this, fix an $n$ greater than $2i$ and the basis $\{[\mathcal{P}_\mu]\}_{\mu\vdash n}$ of $Schur_n$. We now use the notion of degree introduced before this theorem. By (1.17), $\ell_i^{(n)}$ is a linear combination of elements of degree $i$. On the other hand, by the definition of $\ell_i^{(n)}$ we have

$$-(-1)^i \ell_i^{(n)} = \sum_{j=0}^{i-1} (-1)^j \ell_j^{(n)} s_{i-j}^{(n)}. \tag{1.18}$$

By induction and the formula for $s_j^{(n)}$ from Proposition 1.12, the right hand side of (1.18) equals

$$\sum_{\mu\vdash i} [\mathcal{P}_{\alpha(\mu)}^{(n)}] + \sum_{j=1}^{i-1} (-1)^j \left( (-1)^j \sum_{\mu\vdash j} (-1)^{\ell(\mu)} \tbinom{\ell(\mu)}{\alpha(\mu)} \big[\mathcal{P}_\mu^{(n)}\big] \right) \left( \sum_{\mu\vdash i-j} [\mathcal{P}_{\alpha(\mu)}^{(n)}] \right). \tag{1.19}$$

Since we already know that $\ell_i^{(n)}$ is zero in every degree different from $i$ it remains to compute the degree $i$ part of (1.19). In this expression, for every $j$ such that $0 < j < i$ we have a product of two sums, one consisting of elements of degree $j$ and the other one consisting of elements of degree less than or equal to $i - j$, for if $\mu \vdash i - j$ then $[\mathcal{P}_{\alpha(\mu)}]$ has degree $\leq i - j$ with equality if and only if $\mu = (1, 1, \ldots, 1)$, in which case $\alpha(\mu) = (i-j)$. If $[\mathcal{P}_\mu^{(n)}]$ has degree $j$ and $[\mathcal{P}_{\alpha(\mu')}^{(n)}]$ has degree $m \leq i - j$ then, by Lemma 1.15,

$$[\mathcal{P}_\mu^{(n)}] \cdot [\mathcal{P}_{\alpha(\mu')}^{(n)}] = [\mathcal{P}_{\mu,\alpha(\mu')}^{(n)}] + \text{ terms of degree } < j + m.$$

Hence only the degree $i-j$ part of $\sum_{\mu\vdash i-j}[\mathcal{P}^{(n)}_{\alpha(\mu)}]$, i.e., $[\mathcal{P}^{(n)}_{i-j}]$, contributes to the degree $i$ part of (1.19), which therefore equals

$$[\mathcal{P}^{(n)}_i] + \sum_{j=1}^{i-1}\sum_{\mu\vdash j}(-1)^{\ell(\mu)}\tbinom{\ell(\mu)}{\alpha(\mu)}\,[\mathcal{P}^{(n)}_{\mu,i-j}]. \tag{1.20}$$

We write this as a linear combination of elements in $\{[\mathcal{P}^{(n)}_\nu]\}_{\nu\vdash i}$. Fix $\nu\vdash i$ with $\ell:=\ell(\nu)$ and $\alpha:=\alpha(\nu)=(\alpha_1,\dots,\alpha_t)$. If $\ell=1$ then $[\mathcal{P}^{(n)}_\nu]=[\mathcal{P}^{(n)}_i]$, so $[\mathcal{P}^{(n)}_\nu]$ occurs once in (1.20). If $\ell>1$ then $[\mathcal{P}^{(n)}_\nu]$ occurs first when $i-j$ equals $\nu_1=\cdots=\nu_{\alpha_1}$; the length of $\mu$ is then $\ell-1$ and $\alpha(\mu)=(\alpha_1-1,\alpha_1,\dots,\alpha_t)$, so the coefficient in front of $[\mathcal{P}^{(n)}_{\mu,i-j}]$ is $-(-1)^\ell\alpha_1\cdot\frac{(\ell-1)!}{\alpha!}$. Also, $[\mathcal{P}^{(n)}_\nu]$ occurs in (1.20) when $i-j$ equals $\nu_{\alpha_1+1}=\cdots=\nu_{\alpha_1+\alpha_2}$, with coefficient $-(-1)^\ell\alpha_2\cdot\frac{(\ell-1)!}{\alpha!}$, and so on. Summing up, the coefficient in front of $[\mathcal{P}^{(n)}_\nu]$ is $-(-1)^\ell(\alpha_1+\cdots+\alpha_t)\frac{(\ell-1)!}{\alpha!}=-(-1)^\ell\binom{\ell}{\alpha}$; hence (1.20) equals

$$-\sum_{\nu\vdash i}(-1)^{\ell(\nu)}\tbinom{\ell(\nu)}{\alpha(\nu)}\,[\mathcal{P}^{(n)}_\nu].$$

Therefore, by (1.18), $\ell^{(n)}_i$ has the desired form and by induction we are through. □

**Remark 1.21.** *This proof starts with noting that, as a consequence of the preceding lemmas, given $i$ it suffices to compute $\ell^{(n)}_i$ for some $n$ in order to get the formula for every $n$. We then compute $\ell^{(n)}_i$ for $n$ sufficiently large. Instead we could have computed $\ell^{(i)}_i$ by using that* $\mathrm{h}(\ell^{(i)}_i)=[\mathrm{sgn}]\in\mathrm{R}_{\mathbb{Q}}(\Sigma_i)$, *where* sgn *is the signature representation. The needed expression of* [sgn] *as a linear combination of permutation representations is a classical formula in the theory of representations of $\Sigma_i$. We chose to give the above proof since it is purely combinatorial in nature.*

# 2. Intrinsic proof of Theorem 1.1

In this chapter we give a proof of Theorem 1.1 suggested by Serge Bouc and published in the joint paper [BR09]. This proof relies on the construction of a ring of formal power series with coefficients in Burnside rings, and an exponential map on this ring, developed in [Bou92]. This proof has the advantage of being intrinsic; it uses only the structure of the Burnside ring.

In Section 2.1 we give a survey of the relevant constructions and results from [Bou92]. Then in Section 2.2 we use these results to obtain a formula for $\lambda^n(S)$, which we then show to be the requested one using some combinatorial arguments.

## 2.1 Background Material

We begin by giving a quick review of some definitions and results:

### Posets

A $G$-poset $P$ is a $G$-set with a partial ordering compatible with the $G$-action, in the sense that if $s \leq t \in P$ then $gs \leq gt$ for all $g \in G$. A $G$-map of $G$-posets is a map $f\colon P \to Q$ of posets such that $gf(s) = f(gs)$ for $s \in P$ and $g \in G$. If also $f'\colon P \to Q$, then $f \leq f'$ if this holds pointwise. In connection with this, when $S$ is a $G$-set and we use it as a $G$-poset this means that we view $S$ as a $G$-poset using its discrete ordering.

Let $P$ be a $G$-poset. We recall the definition of the Lefschetz invariant of $P$: for every $i \in \mathbb{N}$, $\mathrm{Sd}_i\, P$ is the $G$-set of chains $x_0 < \cdots < x_i$ in $P$ of length $i+1$. The *Lefschetz invariant* of the $G$-poset $P$, $\Lambda_P$, is the alternating sum $\sum_{i\geq 0}(-1)^i[\mathrm{Sd}_i\, P] \in \mathcal{B}(G)$. The *reduced Lefschetz invariant* of $P$ is $\widetilde{\Lambda}_P := \Lambda_P - 1$.

We also need the notion of homotopic posets. We say that the $G$-posets $P$ and $Q$ are *simplicially homotopic*, or just *homotopic*[1], if there are $G$-maps $f\colon P \to Q$ and $g\colon Q \to P$ such that $gf \leq \mathrm{Id}_P$ or $gf \geq \mathrm{Id}_P$, and similarly for $fg$. if $P$ and $Q$ are homotopic as $G$-posets then $\widetilde{\Lambda}_P = \widetilde{\Lambda}_Q$ (see e.g. Proposition 4.2.5 in [Bou00]). In particular, if $P$ has a largest or smallest element then $\widetilde{\Lambda}_P = 0$.

[1] Note however that two non-homotopic posets may admit homotopic realizations.

### Results from [Bou92]

In this subsection we give a review of the definitions and results from [Bou92] that we use to prove Theorem 1.1. Let $G$ be a finite group. Recall that the *wreath product* of $G$ and $\Sigma_n$, denoted $G \wr \Sigma_n$, is by definition the semidirect product $G^n \rtimes \Sigma_n$, where the action of $\Sigma_n$ on $G^n$ is given by $\sigma \cdot (g_1, \dots, g_n) = (g_{\sigma^{-1}1}, \dots, g_{\sigma^{-1}n})$. We use $G_n$ to denote this wreath product, $G_n := G \wr \Sigma_n$ (by definition, $G_0 = 1$). One defines the ring $\mathbb{B}(G)$ in the folling way: as a group it is the direct product of the Burnside rings $\mathcal{B}(G \wr \Sigma_n)$, indexed over all $n \in \mathbb{N}$. We represent the elements of this group as a power series, $\sum_{i \geq 0} x_i t^i$ where $x_i \in \mathcal{B}(G \wr \Sigma_i)$. This is a ring in a natural way, see [Bou92] for the construction of the multiplication.

Let $\tilde{g} = ((g_1, \dots, g_n), \sigma)$, where $\sigma \in \Sigma_n$ and $g_i \in G$, be an element of $G_n$. When $S$ is a $G$-set we view $S^n$ as a $G_n$-set by $\tilde{g}(s_1, \dots, s_n) = (g_1 s_{\sigma^{-1}1}, \dots, g_n s_{\sigma^{-1}n})$. Moreover, let $\underline{S}$ be the poset defined by adding a smallest element 0 to $S$, and define the $G_n$-poset $S^{*n}$ as the set of maps $\{1, \dots, n\} \to \underline{S}$ which are not constant equal to zero, where the partial ordering is defined pointwise, and with the $G_n$-action defined in the same way as on $S^n$, with $G$ acting trivially on the minimal element 0.

Next one defines maps $u_i \colon \mathcal{B}(G) \to \mathcal{B}(G_i)$ by $x \mapsto \Lambda_{P^i}$, where $P$ is a $G$-poset such that $\Lambda_P = x$. Let $\mathbb{I}(G)$ be the ideal of $\mathbb{B}(G)$ consisting of those series with zero as constant coefficient. The $u_i$ are then used to define an exponential map $\mathcal{E}\mathrm{xp} \colon \mathbb{I}(G) \to \mathbb{B}(G)$ having the property that if $f, g \in \mathbb{I}(G)$ then $\mathcal{E}\mathrm{xp}(f+g) = \mathcal{E}\mathrm{xp}(f)\,\mathcal{E}\mathrm{xp}(g)$. In the case we are interested in, when $f = xt$ for $x \in \mathcal{B}(G)$, we have by definition $\mathcal{E}\mathrm{xp}(xt) = \sum_{i \geq 0} u_i(x) t^i$. (We omit the construction in the general case, see [Bou92].) In particular, when $H$ is a subgroup of $G$ we have $\mathcal{E}\mathrm{xp}([G/H]t) = \sum_{i \geq 0} [G_i/H_i] t^i$. Moreover, since, for any $G$-set $S$, $S = \Lambda_S = \widetilde{\Lambda}_{S_+}$, where $S_+ := S \,\dot{\cup}\, \{\bullet\}$, we have $\mathcal{E}\mathrm{xp}(-[S]t) = \sum_{i \geq 0} u_i(-\widetilde{\Lambda}_{S_+}) t^i$. By Lemme 4 in [Bou92] it follows that

$$\mathcal{E}\mathrm{xp}(-[S]t) = -\sum_{i \geq 0} \widetilde{\Lambda}_{(S_+)^{*i}}\, t^i. \tag{2.1}$$

For every $i \in \mathbb{N}$ we have a map $\mathrm{m}_i \colon \mathcal{B}(G_i) \to \mathcal{B}(G)$, induced by taking the $G_i$-set $S$ to the $G$-set $\Sigma_i \backslash S$. Together the $m_i$ give a homomorphism of rings $\mathrm{m} \colon \mathbb{B}(G) \to \mathcal{B}(G)[[t]]$.

## 2.2 Proof of Theorem 1.1

The property that allows us to use the above theory on our problem is the following:

**Lemma 2.2.** *For any $x \in \mathcal{B}(G)$ we have $\mathrm{m}\big(\mathcal{E}\mathrm{xp}(xt)\big) = \sigma_t(x)$ and $\mathrm{m}\big(\mathcal{E}\mathrm{xp}(-xt)\big) = \lambda_{-t}(x)$.*

*Proof.* Let $\mathrm{S}^n$ denote the $n$th symmetric power. When $x = [G/H]$, we have to show that $\Sigma_n \backslash (G_n/H_n) \simeq \mathrm{S}^n(G/H)$ as $G$-sets, for every positive integer $n$: firstly, the map

$$(g_1, \ldots, g_n, \sigma) \mapsto (\overline{g_1}, \ldots, \overline{g_n}) \colon G_n \to (G/H)^n$$

factors through $G_n/H_n$, for if $(g_1, \ldots, g_n, \sigma) \in G_n$ then, for any $(h_1, \ldots, h_n, \tau) \in H_n$, the element

$$(g_1, \ldots, g_n, \sigma)(h_1, \ldots, h_n, \tau) = (g_1 h_{\sigma 1}, \ldots, g_n h_{\sigma n}, \sigma\tau) \in G_n$$

maps to $(\overline{g_1 h_{\sigma 1}}, \ldots, \overline{g_n h_{\sigma n}}) = (\overline{g_1}, \ldots, \overline{g_n}) \in (G/H)^n$, which is also the image of $(g_1, \ldots, g_n, \sigma)$. Denote the resulting map $\phi \colon G_n/H_n \to (G/H)^n$. If we give $(G/H)^n$ the $G_n$-set structure $(g_1, \ldots, g_n, \sigma) \cdot (\overline{f_1}, \ldots, \overline{f_n}) = (\overline{g_1 f_{\sigma 1}}, \ldots, \overline{g_n f_{\sigma n}})$, then $\phi$ is $G_n$-equivariant. Moreover it is surjective. Since both $G_n/H_n$ and $(G/H)^n$ have $|G|^n/|H|^n$ elements, it follows that $\phi$ is an isomorphism of $G_n$-sets. Consequently it induces an isomorphism of $G$-sets $\Sigma_n \backslash (G_n/H_n) \to \mathrm{S}^n(G/H)$.

For arbitrary $x$ the result now follows from the properties of m and $\mathcal{E}$xp. For suppose that it holds for $x, y \in \mathcal{B}(G)$. Firstly $\mathrm{m}(\mathcal{E}\mathrm{xp}(x + y)) = \mathrm{m}(\mathcal{E}\mathrm{xp}(x))\, \mathrm{m}(\mathcal{E}\mathrm{xp}(y)) = \sigma_t(x)\sigma_t(y) = \sigma_t(x + y)$. Moreover $1 = \mathrm{m}(\mathcal{E}\mathrm{xp}(x))\, \mathrm{m}(\mathcal{E}\mathrm{xp}(-x)) = \sigma_t(x)\, \mathrm{m}(\mathcal{E}\mathrm{xp}(-x))$, it therefore follows that $\mathrm{m}(\mathcal{E}\mathrm{xp}(-x)) = \sigma_t(-x)$. Since every element of $\mathcal{B}(G)$ is a linear combination of elements $[G/H]$ we are done.

The second assertion follows immediately, since $\sigma_t(x)\lambda_{-t}(x) = 1$, so $\lambda_{-t}(x) = \sigma_t(-x)$. □

Using this lemma together with (2.1) shows that, when $S$ is a $G$-set, $\lambda_{-t}(S) = -\,\mathrm{m}\big(\sum_{n \geq 0} \widetilde{\Lambda}_{(S_+)^{*i}}\, t^n\big)$, hence that

$$\lambda^n(S) = (-1)^{n-1}\, \mathrm{m}_n\big(\widetilde{\Lambda}_{(S_+)^{*i}}\big). \tag{2.3}$$

Thus we have in some sense achieved our goal; we have expressed $\lambda^n(S)$ in a non-recursive way, without using $\lambda^i(S)$ for $i < n$. However, we want to be more concrete, and the major step is the following proposition, which allows us to express $\lambda^n(S)$ without using $\mathcal{B}(G_n)$.

**Proposition 2.4.** *For $S$ a $G$-set, let $\Omega_{\leq n}(S)$ be the $G$-poset of nonempty subsets of $S$ of cardinality $\leq n$. For any $n \in \mathbb{N}$,*

$$\mathrm{m}_n\big(\widetilde{\Lambda}_{S^{*n}}\big) = \widetilde{\Lambda}_{\Omega_{\leq n}(S)}\,.$$

*Proof.* Given the $G$-set $S$ and a positive integer $n$ we define the $G$-poset $S_n$,

$$S_n := \{\alpha \colon S \to \mathbb{N} : 1 \leq \sum_{s \in S} \alpha(s) \leq n\}$$

with the ordering given by $\alpha \leq \alpha'$ if $\alpha(s) \leq \alpha'(s)$ for every $s \in S$, and the $G$-action $(g\alpha)(s) := \alpha(g^{-1}s)$. Note that $S_n$ is $G$-homotopic to $\Omega_{\leq n}(S)$, for we have maps $\theta\colon S_n \to \Omega_{\leq n}(S)$, given by $\alpha \mapsto \alpha^{-1}(\mathbb{N} \setminus \{0\})$, and $\theta'\colon \Omega_{\leq n}(S) \to S_n$ sending $A \subseteq S$ to its characteristic function. The composition $\theta\theta'$ is the identity and $\theta'\theta \leq \mathrm{Id}_{S_n}$. Hence $\widetilde{\Lambda}_{S_n} = \widetilde{\Lambda}_{\Omega_{\leq n}(S)}$, so it suffices to show that $\mathrm{m}_n\big(\widetilde{\Lambda}_{S^{*n}}\big) = \widetilde{\Lambda}_{S_n}$. We will do this by proving that, for every $i$,

$$\Sigma_n \backslash \mathrm{Sd}_i(S^{*n}) \simeq \mathrm{Sd}_i(S_n)$$

as $G$-sets.

We proceed to constructing this isomorphism: first, we have a map $\phi\colon S^{*n} \to S_n$, defined by $\phi(f)(s) = |f^{-1}(s)|$ for $s \in S$. One checks that this is a well-defined map of $G$-posets (where we view $S^{*n}$ as a $G$-poset via restriction). The map $\phi$ is surjective, for given $\alpha \in S_n$ one may construct an element $f$ in its preimage in the following way: for $s \in S$, choose $E_s \subseteq \{1, \dots, n\}$ such that $|E_s| = \alpha(s)$ (possibly, $E_s = \emptyset$). Since $\sum_{s \in S} \alpha(s) \leq n$ we may do this such that the $E_s$ are mutually disjoint. We now define $f \in S^{*n}$ by $f(i) = 0$ if $i \notin \cup_{s \in S} E_s$ and $f(i) = s$ if $i \in E_s$. It then follows that $\phi(f)(s) = |f^{-1}(s)| = |E_s| = \alpha(s)$ for all $s \in S$, i.e., $\phi(f) = \alpha$.

Next one shows that $\phi$ induces, for every $i$, a map of $G$-sets $\Phi\colon \mathrm{Sd}_i(S^{*n}) \to \mathrm{Sd}_i(S_n)$ defined by

$$\Phi(f_0 < \cdots < f_i) := \big(\phi(f_0) < \cdots < \phi(f_i)\big).$$

Since we already know that $\phi$ is a map of $G$-posets it suffices to show that $\Phi$ does not map chains to shorter chains, i.e., that if $f < f'$ then $\varphi(f) < \varphi(f')$. This follows since there exists an $i_0 \in \{1, \dots, n\}$ such that $f(i_0) < f'(i_0)$, i.e., $f(i_0) \notin S$ whereas $f'(i_0) = s_0 \in S$, hence $f^{-1}(s_0)$ is strictly contained in $f'^{-1}(s_0)$, i.e., $\phi(f)(s_0) < \phi(f')(s_0)$.

The map $\Phi$ is surjective, for $\phi$ is and from the construction it follows that we may choose elements in the preimages such that the chain property is not destroyed.

Finally, for $c = (f_0 < \cdots < f_i)$ and $c = (f'_0 < \cdots < f'_i)$ in $\mathrm{Sd}_i(S^{*n})$ we have $\Phi(c) = \Phi(c')$ if and only if there exists a $\sigma \in \Sigma_n$ such that $\sigma(c) = c'$. For suppose that $\Phi(c) = \Phi(c')$. Then, for every $0 \leq j \leq i$, $\phi(f_j) = \phi(f'_j)$, i.e., for every $s \in S$ we have $|f_j^{-1}(s)| = |f_j'^{-1}(s)|$. Since $f_0^{-1}(s) \subseteq \cdots \subseteq f_i^{-1}(s)$ and $f_0'^{-1}(s) \subseteq \cdots \subseteq f_i'^{-1}(s)$ this means that we may chose a bijection $\sigma_s\colon f_i^{-1}(s) \to f_i'^{-1}(s)$ such that $\sigma_s\big(f_j^{-1}(s)\big) = f_j'^{-1}(s)$ for every $0 \leq j \leq i$. Since the sets $f_j^{-1}(s)$, for $s \in S$, are mutually disjoint there exists a $\sigma \in \Sigma_n$ which, viewed as an automorphism of $\{1, \dots, n\}$, restrict to $\sigma_s$ on $f_i^{-1}(s)$ for every $s \in S$. Then, for any $1 \leq j \leq i$ and for any $m \in \{1, \dots, n\}$ and $s \in S$ we have that

$$f_j(m) = s \iff m \in f_j^{-1}(s) \iff \sigma m \in f'^{-1}(s) \iff f'_j(\sigma m) = s,$$

and also that $f_j(m) = 0 \iff f'_j(\sigma m) = 0$. Hence $\sigma f_j = f'_j$ for $0 \leq j \leq i$, i.e., $\sigma c = c'$.

It follows that $\Phi$ induces an isomorphism of $G$-sets $\Sigma_n \backslash \mathrm{Sd}_i(S^{*n}) \to \mathrm{Sd}_i(S_n)$. $\square$

Therefore, from (2.3),

$$\lambda^n(S) = (-1)^{n-1}\, \widetilde{\Lambda}_{\Omega_{\leq n}(S)}\,.$$

Theorem 1.1 therefore follows from the following computation:

**Lemma 2.5.** *Let $S$ be a $G$-set and let $S_+ := S \cup \{\bullet\}$. In $\mathcal{B}(G)$ we then have the equality*

$$\widetilde{\Lambda}_{\Omega_{\leq n}(S_+)} = -\sum_{\mu \vdash n} (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}_\mu(S)].$$

*Proof.* The inclusion $S \to S_+$ induces an inclusion

$$i\colon\ \Omega_{\leq n}(S) \to \Omega_{\leq n}(S_+).$$

By Proposition 4.2.7 of [Bou00] we have

$$\widetilde{\Lambda}_{\Omega_{\leq n}(S_+)} = \widetilde{\Lambda}_{\Omega_{\leq n}(S)} + \sum_{A \in [G \backslash \Omega_{\leq n}(S_+)]} \mathrm{ind}_{G_A}^{G} (\widetilde{\Lambda}_{i^A}\, \widetilde{\Lambda}_{]A,.[}),$$

where $i^A = \{B \in \Omega_{\leq n}(S) : B = i(B) \subseteq A\}$. However, when $A \neq \{\bullet\}$ the set $i^A$ has a largest element (namely $A \setminus \{\bullet\}$), hence $\widetilde{\Lambda}_{i^A} = 0$. Therefore the sum after the summation sign has only one non-zero element, namely the one with index $\{\bullet\}$, which equals $-\widetilde{\Lambda}_{]\bullet,.[}$ (where $]\bullet,.[$ is the set of elements of $\Omega_{\leq n}(S_+)$ containing $\bullet$). Since $]\bullet,.[$ is homotopic (more precisely isomorphic) to $\Omega_{\leq n-1}(S)$, it follows that

$$\widetilde{\Lambda}_{\Omega_{\leq n}(S_+)} = \widetilde{\Lambda}_{\Omega_{\leq n}(S)} - \widetilde{\Lambda}_{\Omega_{\leq n-1}(S)}\,.$$

It is easy to see that this last expression is the desired one: let $S$ be a $G$-set and define, for any tuple of positive integers $\alpha = (\alpha_0, \dots, \alpha_i)$, the $G$-set $\mathcal{P}_\alpha(S)$ similarly as when $\alpha$ is a partition of an integer. Then the map sending the sequence $(S_0 \subset \cdots \subset S_i) \mapsto (S_0, S_1 \setminus S_0, \dots, S_i \setminus S_{i-1})$ is an isomorphism of $G$-sets $\mathrm{Sd}_i(\Omega_{\leq n}(S)) \to \bigcup_{\substack{\alpha=(\alpha_0,\dots,\alpha_i):\\ \sum \alpha_j \leq n\\ \alpha_j > 0}} \mathcal{P}_\alpha(S)$, hence

$$[\mathrm{Sd}_i(\Omega_{\leq n}(S))] = \sum_{\substack{\alpha=(\alpha_0,\dots,\alpha_i):\\ \sum \alpha_j \leq n\\ \alpha_j > 0}} [\mathcal{P}_\alpha(S)].$$

We therefore have

$$\begin{aligned}
&[\mathrm{Sd}_i(\Omega_{\leq n}(S))] - [\mathrm{Sd}_i(\Omega_{\leq n-1}(S))] \\
&\qquad = \sum_{\substack{\alpha=(\alpha_0,\ldots,\alpha_i): \\ \sum \alpha_j = n \\ \alpha_j > 0}} [\mathcal{P}_\alpha(S)] = \sum_{\substack{\mu \vdash n \\ \ell(\mu)=i+1}} \binom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}_\mu(S)],
\end{aligned}$$

and consequently

$$\begin{aligned}
\widetilde{\Lambda}_{\Omega_{\leq n}(S)} - \widetilde{\Lambda}_{\Omega_{\leq n-1}(S)} &= \sum_{i \geq 0} (-1)^i \sum_{\substack{\mu \vdash n \\ \ell(\mu)=i+1}} \binom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}_\mu(S)] \\
&= -\sum_{\mu \vdash n} (-1)^{\ell(\mu)} \binom{\ell(\mu)}{\alpha(\mu)} [\mathcal{P}_\mu(S)]. \qquad \square
\end{aligned}$$

# Part II:
The Grothendieck ring of varieties

# 3. The computation of the classes of some tori in the Grothendieck ring of varieties

## 3.1 Introduction

The Grothendieck ring of varieties over the field $k$, $\mathrm{K}_0(\mathsf{Var}_k)$, is the free abelian group on the objects of the category of varieties, subject to so called scissor relations and with a multiplication given by the product of varieties. By construction, it is hence the universal additive and multiplicative invariant for the category of $k$-varieties. Since $\chi_c$, the Euler characteristic with compact support, (taking values for instance in a Grothendieck ring of mixed Hodge structures or Galois representations) is additive and multiplicative, it factors through $\mathrm{K}_0(\mathsf{Var}_k)$. Looking at relations among such Euler characteristics is a powerful heuristic method for guessing relations in $\mathrm{K}_0(\mathsf{Var}_k)$.

For an example of this, let $T$ be a torus defined over $k$. Then $\chi_c(T)$ can be expressed in terms of exterior powers of the first cohomology group. The latter in turn is essentially the cocharacter group of $T$. As $\mathrm{K}_0(\mathsf{Var}_k)$ is a $\lambda$-ring and as $\chi_c$ is a $\lambda$-homomorphism one can try to lift the cohomology formulas to $\mathrm{K}_0(\mathsf{Var}_k)$. There is a problem, however, in that in general one cannot find an element in $\mathrm{K}_0(\mathsf{Var}_k)$ that maps to the cocharacter representation under $\chi_c$. But when the torus is the group $L^*$ of units in a separable $k$-algebra $L$, then $[\operatorname{Spec} L]$ maps to the cocharacter representation. We are thus led to conjecture the formula

$$[L^*] = \sum_{i=0}^{n} (-1)^i \lambda^i\big([\operatorname{Spec} L]\big)\mathbb{L}^{n-i} \in \mathrm{K}_0(\mathsf{Var}_k), \tag{3.1}$$

where $n$ is the dimension of $L$. (The details of this heuristic argument are given in Section 3.4.)

The objective of this chapter is to prove (3.1), and in the process to develop some techniques for performing explicit calculations in, and prove structure results about $\mathrm{K}_0(\mathsf{Var}_k)$. These techniques are of three main types:

Firstly, one may use some homomorphism from $\mathrm{K}_0(\mathsf{Var}_k)$ to a ring with better known structure, for example the above mentioned $\chi_c$, or, in case $k = \mathbb{F}_q$ is finite, the point conting homomorphism $C_q$ induced by counting $\mathbb{F}_q$-points on the varieties. Since such homomorphisms are usually not injective, this method is mostly used in order to give heuristic suggestions of relations in $\mathrm{K}_0(\mathsf{Var}_k)$ (see Section 3.4), and also to show that elements in $\mathrm{K}_0(\mathsf{Var}_k)$ are distinct by proving that their images are. However, we will also use $C_q$ to prove the validity of relations in $\mathrm{K}_0(\mathsf{Var}_k)$, by showing that the image of the relation under $C_q$ is satisfied for sufficiently many $q$, as in the alternative proof of (3.1) given in Section 3.6.

Secondly, we construct a $\lambda$-homomorphism $\mathrm{Art}_k\colon \mathcal{B}(\mathcal{G}) \to \mathrm{K}_0(\mathsf{Var}_k)$, where $\mathcal{B}(\mathcal{G})$ is the Burnside ring of the absolute Galois group of $k$, which then may be used to prove relations in $\mathrm{K}_0(\mathsf{Var}_k)$ by proving the corresponding relations in $\mathcal{B}(\mathcal{G})$ (see Section 3.5 for examples of this). The image of $\mathrm{Art}_k$ is contained in the subring of $\mathrm{K}_0(\mathsf{Var}_k)$ generated by zero-dimensional schemes, which we call the subring of Artin classes and denote $ArtCl_k$. Using $\mathrm{Art}_k$ we may answer some questions about the structure of $ArtCl_k$. For example, when $k$ is perfect with procyclic absolute Galois group we will see that $ArtCl_k$ is free on the classes of finite separable field extensions of $k$ (Proposition 3.8). We will also give a slightly generalization of the conditions under which the zero-divisors of Naumann exists (Proposition 3.7).

Thirdly there is the technique of Galois descent. Using it we may prove the validity of a relation in $\mathrm{K}_0(\mathsf{Var}_k)$ by considering the corresponding relation in $\mathrm{K}_0(\mathsf{Var}_{\overline{k}})$, where $\overline{k}$ is a separable closure of $k$, where it may be easier to prove. This is a main tool used in the proof of (3.1) given in Section 3.5.

The structure of this chapter is as follows: In Section 3.3 we construct the $\lambda$-homomorphism $\mathrm{Art}_k\colon \mathcal{B}(\mathcal{G}) \to \mathrm{K}_0(\mathsf{Var}_k)$. We also prove the above mentioned structure results about $ArtCl_k$.

In Section 3.4 we describe the details of the heuristic arguments suggesting (3.1). We also discuss what happens when this heuristic is applied to other types of tori. Finally we give an example of a tori for which the heuristic suggests a formula that does not hold (Proposition 3.13).

In Section 3.5 we show that $[L^*] = \sum_{i=0}^{n} a_i \mathbb{L}^{n-i} \in \mathrm{K}_0(\mathsf{Var}_k)$, where $a_i \in ArtCl_k$. We also give an explicit formula for the $a_i$ in terms of elements in $\mathcal{B}(\mathcal{G})$. To derive this formula we embed $L^*$ in $\widetilde{L}$, the affine space associated to $L$, and use induction relative to the complement of $L^*$.

In Part I of the thesis we have obtained a universal formula for the $\lambda$-operations on the Burnside ring which, together with the formula obtained in Section 3.5, gives a proof of (3.1).

In Section 3.6 we give a proof of (3.1) that avoids using the universal formula for $\lambda^i$. Instead it uses one simpler result from Chapter 1, together with point counting over finite fields.

*Acknowledgment.* Thanks are due to David Bourqui for comments on the preprint [Rök07a], in particular concerning what is Proposition 3.13 in this thesis.

## 3.2 Background Material

For the necessary background on the Grothendieck ring of varieties, and on Burnside and representation rings we refer to Section 0.2. Below we give a quick review of Grothendieck's formulation of Galois theory.

### Galois theory

We use the following notation. Let $k$ be a field and let $\overline{k}$ be a separable closure of $k$. Write $\mathcal{G}$ for $\mathcal{G}\mathrm{al}(\overline{k}/k)$, the absolute Galois group of $k$. We define the *category of $\overline{k}-\mathcal{G}$-algebras* to be the category whose objects are $\overline{k}$-algebras $L$ together with $\mathcal{G}$-actions on the underlying rings such that $\overline{k} \to L$ is $\mathcal{G}$-equivariant, and whose morphisms are $\mathcal{G}$-equivariant maps of $\overline{k}$-algebras.

We then have an equivalence between the category of $k$-algebras and the category of $\overline{k} - \mathcal{G}$-algebras. This equivalence takes the $k$-algebra $L$ to $L \otimes_k \overline{k}$ with $\mathcal{G}$-action $\sigma(l \otimes \alpha) := l \otimes \sigma(\alpha)$. The map that takes the $\overline{k} - \mathcal{G}$-algebra $U$ to $U^{\mathcal{G}}$ is a pseudo-inverse.

If the $k$-algebra $L$ is finite and separable, then the corresponding $\overline{k} - \mathcal{G}$ algebra is isomorphic to $\overline{k}^S$ where $S$ is finite. The $G$-action on this must be the action on $\overline{k}$ together with a permutation of the coordinates, i.e., a $\mathcal{G}$-action on $S$. Hence as a corollary we have a contravariant equivalence between the category of finite separable $k$-algebras and the category of finite continuous $\mathcal{G}$-sets. This equivalence takes the $k$-algebra $L$ to $\mathrm{Hom}_k(L, \overline{k})$ with $\mathcal{G}$-action given by $f^\sigma(l) := \sigma \circ f(l)$. It has a pseudo-inverse that takes the $\mathcal{G}$-set $S$ to $\mathrm{Hom}_{\mathcal{G}}(S, \overline{k})$, i.e., the $\mathcal{G}$-equivariant maps of sets from $S$ to $\overline{k}$, considered as a ring by pointwise addition and multiplication and with a $k$-algebra structure given by $(\alpha \cdot f)(s) := \alpha \cdot f(s)$.

Under this correspondence, if $L$ corresponds to $S$ then the dimension of $L$ equals the number of elements in $S$. Moreover, if also $L'$ corresponds to $S'$, then $L \otimes_k L'$ corresponds to $S \times S'$ with diagonal $\mathcal{G}$-action and the algebra $L \times L'$ corresponds to $S \dot{\cup} S'$. In particular, separable field extensions of $k$ correspond to transitive $\mathcal{G}$-sets.

## 3.3 The subring of Artin classes in $\mathrm{K}_0(\mathsf{Var}_k)$

In this section we define a map from the Burnside ring of the absolute Galois group of $k$ to $\mathrm{K}_0(\mathsf{Var}_k)$. This map will be used in the computation of the class of $L^*$, but it also gives some information about the structure of $\mathrm{K}_0(\mathsf{Var}_k)$. The map will take values in the subring of $\mathrm{K}_0(\mathsf{Var}_k)$ generated by zero dimensional schemes, which we call the subring of Artin classes. We include this as a formal definition:

**Definition 3.2.** *Let $ArtCl_k$, the ring of Artin classes, be the subring of* $\mathrm{K}_0(\mathsf{Var}_k)$ *generated by zero-dimensional schemes.*

## Background

As an abelian group $ArtCl_k$ is generated by $\{[\operatorname{Spec} K]\}$, where $K$ runs over the isomorphism classes of finite field extensions of $k$. When the characteristic of $k$ is zero there is a structure theory for $\mathrm{K}_0(\mathsf{Var}_k)$ given in [LL03], which asserts that the classes of stable birational equivalence form a $\mathbb{Z}$-basis of $\mathrm{K}_0(\mathsf{Var}_k)/(\mathbb{L})$. Using this, one shows that, as an abelian group, $ArtCl_k$ is free on $\{[\operatorname{Spec} K]\}$. That $ArtCl_k$ is free on this set is also true when $k = \mathbb{F}_q$ as is shown in [Nau07], Theorem 25, using the point counting homomorphisms $C_{q^n}$, where, for any prime-power $q$, $C_q\colon \mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ is given by $X \mapsto |X(\mathbb{F}_q)|$. We will extend this result to hold for any perfect field with procyclic Galois group.

There is also a question about zero divisors. The first construction of zero divisors was given in [Poo02]; this construction works for every field of characteristic zero, also the algebraically closed ones. To show that these zero divisors really are nonzero requires the structure theory of [LL03], hence the restriction on the characteristic of $k$. Now, as was observed in [Nau07], $ArtCl_k$ often contains zero divisor: If $K$ is a finite Galois extension of degree $n$ then $[\operatorname{Spec} K]^2 = n \cdot [\operatorname{Spec} K]$ so $[\operatorname{Spec} K] \in ArtCl_k$ is a zero divisor if it is not equal to 0 or $n$. [Nau07] proves that this is the case when $k = \mathbb{F}_q$, and more generally when $k$ is finitely generated. Using the Euler characteristic $\chi_c$, we extend this result to hold for any field (Proposition 3.7), hence there are examples of zero divisors in $\mathrm{K}_0(\mathsf{Var}_k)$ whenever $k$ is not separably closed of prime characteristic.

**Remark 3.3.** $[\operatorname{Spec} K] \neq 0, n$ *also in* $\mathcal{M}_k := \mathrm{K}_0(\mathsf{Var}_k)[\mathbb{L}^{-1}]$. *This is easy to see since $\chi_c$, the invariant we use to prove that* $[\operatorname{Spec} K] \neq 0, n \in \mathrm{K}_0(\mathsf{Var}_k)$, *factors through $\mathcal{M}_k$. Recently ([Eke09], Corollary 3.5) it was shown that also the construction of Poonen yields zero divisors in $\mathcal{M}_k$, hence there are known examples of zero divisors in both* $\mathrm{K}_0(\mathsf{Var}_k)$ *and $\mathcal{M}_k$ whenever $k$ is not separably closed of prime characteristic. This question is discussed in [Nic08] Section 2.6.*

## The Artin map

Let $k$ be a field with absolute Galois group $\mathcal{G}$. In this section we use Galois theory to define a $\lambda$-homomorphism from $\mathcal{B}(\mathcal{G})$ to $\mathrm{K}_0(\mathsf{Var}_k)$, whose image is contained in $ArtCl_k$. The main purpose of doing this is that it aids the computations in $\mathrm{K}_0(\mathsf{Var}_k)$. Also it allows us to give a slightly generalization of the above mentioned results of [Nau07].

**Definition 3.4.** *Let* $\mathrm{Art}_k\colon \mathcal{B}(\mathcal{G}) \to \mathrm{K}_0(\mathsf{Var}_k)$ *be the* $\lambda$*-homomorphism defined by associating, to the* $\mathcal{G}$*-set* $S$*, the class of its image under the fully faithful, covariant Galois functor to* $\mathsf{Var}_k$*.*

Since open disjoint union is a special case of the relations in $\mathrm{K}_0(\mathsf{Var}_k)$ this is well defined. Moreover it is multiplicative since the multiplications in both rings come from the product in the respective categories. Finally, to see that it commutes with the $\lambda$-structures, note that it suffices to check this on the opposite structures, and $\sigma^n$ is defined by the same universal property in both rings (namely the $n$th symmetric power) so this is clear.

Since $\mathcal{B}(\mathcal{G})$ is free on isomorphism classes of transitive $\mathcal{G}$-sets, and $\mathrm{Art}_k$ maps these to isomorphism classes of separable field extensions of $k$, it follows that $\operatorname{Im}\mathrm{Art}_k$ is free on $\{[\operatorname{Spec} K]\}$ if and only if $\mathrm{Art}_k$ is injective. If $k$ is perfect, $\operatorname{Im}\mathrm{Art}_k = ArtCl_k$, hence the same holds for $ArtCl_k$ in this case.

Recall that we use h to denote the natural $\lambda$-homomorphism $\mathcal{B}(\mathcal{G}) \to \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$. We use $i$ for the injection $\mathrm{R}_{\mathbb{Q}}(\mathcal{G}) \to \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\mathcal{G})$. If $S$ is a $\mathcal{G}$-set and $L$ the corresponding separable $k$-algebra then the $\ell$-adic cohomology of $\operatorname{Spec} L$ is $\mathbb{Q}_l[S]$, hence we have the following commutative diagram of $\lambda$-rings:

$$\begin{array}{ccc} \mathcal{B}(\mathcal{G}) & \xrightarrow{\mathrm{Art}_k} & \mathrm{K}_0(\mathsf{Var}_k) \\ \downarrow{\scriptstyle \mathrm{h}} & & \downarrow{\scriptstyle \chi_c} \\ \mathrm{R}_{\mathbb{Q}}(\mathcal{G}) & \xrightarrow{i} & \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\mathcal{G}) \end{array} \tag{3.5}$$

Recall that for $\sigma \in \mathcal{G}$ we write $C_\sigma$ for the character homomorphism $\mathrm{R}_{\mathbb{Q}}(\mathcal{G}) \to \overline{\mathbb{Q}}$, and also that when $k = \mathbb{F}_q$ we write $C_q$ for the counting homomorphism $\mathrm{K}_0(\mathsf{Var}_k) \to \mathbb{Z}$. A special case of (3.5) is when $k = \mathbb{F}_q$ and $\mathrm{F} \in \mathcal{G}$ is the Frobenius automorphism: If $S$ is a $\mathcal{G}$-set corresponding to the variety $X$ then $C_{\mathrm{F}} \circ \chi_c(X) = |S^{\mathrm{F}}|$ and also $|X(\mathbb{F}_q)| = |\mathrm{Hom}_{\mathbb{F}_q}(\operatorname{Spec}\mathbb{F}_q, X)| = |\mathrm{Hom}_{\mathcal{G}}(\{\bullet\}, S)| = |S^{\mathrm{F}}|$, consequently

$$C_q = C_{\mathrm{F}} \circ \chi_c \qquad \text{on } \operatorname{Im}\mathrm{Art}_{\mathbb{F}_q}, \tag{3.6}$$

showing that the character maps generalize point counting, a fact that we will use in Section 3.6. (In fact, using the Lefschetz fixed point formula one can show that (3.6) holds on all of $\mathrm{K}_0(\mathsf{Var}_k)$.)

As a first application of the commutativity of (3.5) we note that if $L$ and $L'$ are two finite, non-trivial and separable field extensions of $k$, which are non-isomorphic, i.e., they correspond to two non-isomorphic transitive $\mathcal{G}$-sets $S$ and $S'$, then $[\operatorname{Spec} L] \neq [\operatorname{Spec} L']$ and $[\operatorname{Spec} L] \notin \mathbb{Z}$. For it suffices to show that this is the case for their images under $\chi_c$, i.e., that $\mathrm{h}(S) \neq \mathrm{h}(S')$ and that $\mathrm{h}(S) \notin \mathbb{Z}$, and this is a known property of h, see (1.4). In particular:

**Proposition 3.7.** *For any field $k$, the isomorphism classes of finite, non-trivial Galois extensions determine distinct zero divisors in* $\mathrm{K}_0(\mathsf{Var}_k)$.

This type of argument cannot be used to prove that $\operatorname{Im}\operatorname{Art}_k$ is free on $\{[\operatorname{Spec} K]\}$, since in general there are non-isomorphic $\mathcal{G}$-sets $S$ and $S'$ such that $\mathrm{h}(S) = \mathrm{h}(S')$. The exception is when $\mathcal{G}$ is procyclic, for then h is an isomorphism. We state this as a proposition.

**Proposition 3.8.** *If the absolute Galois group of $k$ is cyclic then* $\operatorname{Art}_k$ *is injective.*

As a corollary we get a result from [Nau07], that $ArtCl_{\mathbb{F}_q}$ is free on $\{[\operatorname{Spec} K]\}$, where $K$ runs over the finite field extensions of $\mathbb{F}_q$.

To summarize: We have a good understanding of the the additive structure of $ArtCl_k$ when $\operatorname{char} k = 0$, and also when $k$ is perfect with procyclic Galois group.

## The behavior of $\operatorname{Art}_k$ with respect to restriction of scalars

We next study how $\operatorname{Art}_k$ behaves with respect to restriction of scalars. The following proposition is due to Grothendieck but we have not been able to find a reference so we include a proof for completeness.

**Proposition 3.9.** *Fix a field $k$ together with a separable closure $\overline{k}$ and let $\mathcal{G} := \mathcal{G}\mathrm{al}(\overline{k}/k)$. Let $K$ be a finite field extension of $k$ such that $K \subset \overline{k}$. Let $L$ be a finite separable $K$-algebra and let $S$ be the corresponding $\mathcal{G}\mathrm{al}(\overline{k}/K)$-set. View $L$ as a $k$-algebra and let $S'$ be the corresponding $\mathcal{G}$-set. Then $S' \simeq \mathcal{G} \times_{\mathcal{G}\mathrm{al}(\overline{k}/K)} S$. Hence the following diagram is commutative.*

$$\begin{array}{ccc} \mathcal{B}(\mathcal{G}\mathrm{al}(\overline{k}/K)) & \xrightarrow{\operatorname{Art}_K} & \mathrm{K}_0(\mathsf{Var}_K) \\ \Big\downarrow {\scriptstyle \operatorname{ind}^{\mathcal{G}}_{\mathcal{G}\mathrm{al}(\overline{k}/K)}} & & \Big\downarrow {\scriptstyle R^K_k} \\ \mathcal{B}(\mathcal{G}) & \xrightarrow{\operatorname{Art}_k} & \mathrm{K}_0(\mathsf{Var}_k) \end{array}$$

*Proof.* Define a map $\phi\colon \mathcal{G} \times S \to S'$ by $(\sigma, f) \mapsto \sigma f$. It has the property that if $\tau \in \mathcal{G}\mathrm{al}(\overline{k}/K)$ then $\phi(\sigma\tau, f) = \sigma\tau f = \phi(\sigma, \tau f)$. Hence it gives rise to a map of $\mathcal{G}$-sets $\varphi\colon \mathcal{G} \times_{\mathcal{G}\mathrm{al}(\overline{k}/K)} S \to S'$. If $\phi(\sigma, f) = \phi(\tau, g)$ then $\tau^{-1}\sigma f = g$ so since $f$ and $g$ fixes $K$ pointwise we must have that $\tau^{-1}\sigma \in \mathcal{G}\mathrm{al}(\overline{k}/K)$. It follows that $(\tau, g) = (\tau, \tau^{-1}\sigma f) \sim (\tau\tau^{-1}\sigma, f) = (\sigma, f)$ so $\varphi$ is injective. It is also surjective, for let $d := [K : k]$ and suppose that $L$ has dimension $n$ as a $K$-algebra, i.e., $S$ has $n$ elements. Then $L$ has dimension $nd$ as a $k$-algebra so $S'$ has $nd$ elements. On the other hand, by Galois theory, $|\mathcal{G} \,/\, \mathcal{G}\mathrm{al}(\overline{k}/K)| = [K : k] = d$. Hence $\mathcal{G} \times_{\mathcal{G}\mathrm{al}(\overline{k}/K)} S$ also has $nd$ elements. Since $\varphi$ is injective it follows that it also is surjective, hence an isomorphism of $\mathcal{G}$-sets. □

## 3.4 Heuristic suggesting an expression for the class of a torus in $\mathrm{K}_0(\mathsf{Var}_k)$

Let $k$ be a field with absolute Galois group $\mathcal{G}$. As mentioned in the introduction, if $T$ is a torus of dimension $n$ and $N$ is its cocharacter group tensored with $\mathbb{Q}_l$ then

$$\chi_c(T) = \sum_{i=0}^{n} (-1)^i \lambda^i(N) \ell^{n-i} \in \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l} \mathcal{G}), \tag{3.10}$$

where $\ell := [\mathbb{Q}_l(-1)]$, the class of the dual of the cyclotomic representation. Let $x \in \mathrm{K}_0(\mathsf{Var}_k)$ be such that $\chi_c(x) = [N]$. Since $\chi_c$ is a $\lambda$-homomorphism, and since $\chi_c(\mathbb{L}) = \ell$, it follows that

$$\chi_c(T) = \chi_c\left(\sum_{i=0}^{n} (-1)^i \lambda^i(x) \mathbb{L}^{n-i}\right) \in \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l} \mathcal{G}),$$

suggesting that we should have

$$[T] = \sum_{i=0}^{n} (-1)^i \lambda^i(x) \mathbb{L}^{n-i} \in \mathrm{K}_0(\mathsf{Var}_k). \tag{3.11}$$

However, in order to make sense of this formula, we have to find an element $x$ in the preimage of $[N]$ under $\chi_c$, and it is not clear how to do that in general.

There is one case when we have an easy way of doing this, namely when $N$ is a permutation representation of $\mathcal{G}$, $N = \mathbb{Q}_l[S]$ where $S$ is a finite $\mathcal{G}$-set. In that case the element $\mathrm{Art}_k(S) \in ArtCl_k$, maps to $[N]$ under $\chi_c$. Below we examine three such cases:

- In this thesis we concentrate on the case when $T = L^*$, the torus of invertible elements in $L$. Here $L$ is a separable $k$-algebra, hence it corresponds to a $\mathcal{G}$-set $S$. Then $[N] = [\mathbb{Q}_l[S]] = \chi_c(\operatorname{Spec} L)$, hence in this case (3.11) says that (3.1) should hold. The remaining sections of this chapter (Sections 3.5-3.6) are devoted to showing that it actually does.
- The following was pointed out to me by David Bourqui: If $L$ is a separable $k$-algebra of dimension $n$, and $S$ the corresponding $\mathcal{G}$-set, then the exact sequence of $\mathcal{G}$-modules $0 \to \mathbb{Z} \to \mathbb{Z}[S] \to M \to 0$ splits over $\mathbb{Q}$, hence $[M \otimes_{\mathbb{Z}} \mathbb{Q}_l] = [\mathbb{Q}_l[S]] - 1 = \chi_c([\operatorname{Spec} L] - 1)$. So if $L^{*,1}$ is the torus with cocharacter group equal to $M$, i.e., $L^{*,1} = L^*/\mathbb{G}_m$, then the above heuristic suggests that

$$[L^{*,1}] = \sum_{i=0}^{n-1} (-1)^i \lambda^i\big([\operatorname{Spec} L] - 1\big) \mathbb{L}^{n-1-i}.$$

That this formula actually does hold can be proved by embedding $L^{*,1}$ in the projective space associated to $L$, and then proceeding in exactly same way as when proving (3.1). (It would follow directly from (3.1) if we knew that $\mathbb{L} - 1$ was not a zero-divisor, because $[L^{*,1}] \cdot (\mathbb{L} - 1) = [L^*]$.)

- We also mention one case when the above heuristic probably gives the wrong formula, namely let $T$ be the torus of elements in $L$ of norm 1. If $M$ is the cocharacter group of $T$, then the exact sequence $0 \to M \to \mathbb{Z}[S] \to \mathbb{Z} \to 0$ splits over $\mathbb{Q}$, hence by the above argument, $[T]$ would satisfy the same formula as $[L^{*,1}]$. Looking at the top dimensional term of the formula suggests that $T$ should be rational, however, by [Che54] page 322, there are field extensions $L/k$ such that $T$ is not rational. Note however that this is just a heuristic argument; it is not known whether equality of the classes of two varieties implies that they are birational.

The above heuristic suggests that $[T] \in ArtCl_k[\mathbb{L}]$ for any torus $[T]$. We conclude this section by showing, in Proposition 3.13, that this is not always the case. For this, we work in $\mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\,\mathcal{G})$. Given a field $k$ with absolute Galois group $\mathcal{G}$, let $B$ be the image of $\mathcal{B}(\mathcal{G})$ in $\mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\,\mathcal{G})$, and identify $\mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ and $\mathrm{R}_{\mathbb{Q}_l}(\mathcal{G})$ with their (injective) images in $\mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\,\mathcal{G})$. We have

$$B \subset \mathrm{R}_{\mathbb{Q}}(\mathcal{G}) \subset \mathrm{R}_{\mathbb{Q}_l}(\mathcal{G}) \subset \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\,\mathcal{G}).$$

We will use the following lemma:

**Lemma 3.12.** *Let $k$ be a field with absolute Galois group $\mathcal{G}$. Let $\ell \in \mathrm{K}_0(\mathsf{Rep}_{\mathbb{Q}_l}\,\mathcal{G})$ be the class of dual of the cyclotomic representation, $\ell = [\mathbb{Q}_l(-1)]$. Assume that $|\mathrm{Im}\,\chi_{cycl}| = \infty$. If $b_1, \dots, b_n \in \mathrm{R}_{\mathbb{Q}_l}(\mathcal{G})$ are such that $\sum_{i=1}^n b_i \ell^i = 0$, then $b_i = 0$ for every $i$.*

*Proof.* We prove this using the character maps $C_g$, $g \in \mathcal{G}$: Since the image of the cyclotomic representation is infinite we have that $\left|\{C_g(\ell)\}_{g\in\mathcal{G}}\right| = \infty$. Moreover, since every discrete representation of $\mathcal{G}$ factors through a finite quotient of $\mathcal{G}$, we have $\left|\{C_g(b_i)\}_{g\in\mathcal{G}}\right| < \infty$ for every $i$. Hence the result. □

**Proposition 3.13.** *There is a field $k$ and a $k$-torus $T$ such that $[T] \notin ArtCl_k[\mathbb{L}]$.*

*Proof.* By Example 1.6, we may choose a finite extension $k$ of $\mathbb{Q}$ such that the inclusion $B \subset \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ is proper. Moreover, since this $k$ is finitely generated, the image of the cyclotomic character is infinite. We fix this $k$. Suppose now that $[T] \in ArtCl_k[\mathbb{L}]$, where $T$ is a $k$-torus of dimension $n$. Then $[T] = \sum_{i=m}^n a_i \mathbb{L}^{n-i}$, where $a_i \in ArtCl_k$, and $m$ is a (possibly negative) integer. Then, since $\chi_c(\mathbb{L}) = \ell$, we have $\chi_c(T) = \sum_{i=m}^n b_i \ell^{n-i}$, where $b_i$ is in $B$ (by the commutativity of (3.5)).

Hence, by (3.10), $\sum_{i=0}^{n}(-1)^i\lambda^i(N)\ell^{n-i} = \sum_{i=m}^{n} b_i\ell^{n-i}$. Consequently, since $\lambda^i(N) \in \mathrm{R}_{\mathbb{Q}}(\mathcal{G}) \subset \mathrm{R}_{\mathbb{Q}_l}(\mathcal{G})$ and $b_i \in B \subset \mathrm{R}_{\mathbb{Q}_l}(\mathcal{G})$, Lemma 3.12 shows that $b_i = 0$ for $i < 0$ and that $b_i = (-1)^i\lambda^i(N)$ for every $0 \le i \le n$. In particular, $[N] = \lambda^1(N) = -b_1 \in B$.

But not all tori has this property. To see that, note that every $\mathbb{Q}$-representation of $\mathcal{G}$, $V$, comes from the cocharacter group of a torus, namely the free $\mathbb{Z}$-module $\langle ge_i\rangle_{g\in\mathcal{G}} \subset V$, where $\{e_i\}$ is a basis for $V$. (Here it is important that the representation is discrete, otherwise this group can have infinite rank.) So every $\mathbb{Q}$-representation of $\mathcal{G}$, $V$, comes from some torus $T_V$. By what was said above, if $[V]$ is not contained in $B$, then $[T_V] \notin ArtCl_k[\mathbb{L}]$. Since we have chosen $k$ such that the inclusion $B \subset \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ is proper, it follows that there are $V$ with this property. □

## 3.5 Computation of the class of $L^*$ in $\mathrm{K}_0(\mathsf{Var}_k)$

Given a field $k$ and a separable $k$-algebra $L$ of dimension $n$ we define the affine group scheme $L^*$ by letting $L^*(R) = (L \otimes_k R)^\times$ for every $k$-algebra $R$. This is a torus, because if $\overline{k}$ is a separable closure of $k$ and $R$ is a $\overline{k}$-algebra then, since $L \otimes_k \overline{k} = \overline{k}^n$, we have

$$L^*_{\overline{k}}(R) = L^*(R) = \big((L \otimes_k \overline{k}) \otimes_{\overline{k}} R\big)^\times = (R^n)^\times = \mathbb{G}_m^n(R)$$

as groups, and consequently $(L^*)_{\overline{k}} \simeq \mathbb{G}_m^n$. We call $L^*$ the torus of invertible elements in $L$. Note that if $L = k^n$ then $L^* = \mathbb{G}_m^n$, hence $[L^*] = (\mathbb{L}-1)^n \in \mathrm{K}_0(\mathsf{Var}_k)$.

The objective of this section is to compute, for an arbitrary separable $k$-algebra $L$, the class of $L^*$ in $\mathrm{K}_0(\mathsf{Var}_k)$ in terms of the Lefschetz class $\mathbb{L}$ and Artin classes. More precisely, we will show that there exist elements $a_1, \ldots, a_n \in ArtCl_k \subset \mathrm{K}_0(\mathsf{Var}_k)$ such that

$$[L^*] = \mathbb{L}^n + a_1\mathbb{L}^{n-1} + a_2\mathbb{L}^{n-2} + \cdots + a_n \in \mathrm{K}_0(\mathsf{Var}_k). \qquad (3.14)$$

In Theorem 3.32 we then derive a universal formula for the $a_i$ which, together with Theorem 1.1 proves (3.1).

### Definitions

We begin by giving a definition of $L^*$ for any free algebra of finite rank: Let $K$ be a commutative ring and let $L$ be a free $K$-algebra of rank $n$. Let $\widetilde{L/K}$, or just $\widetilde{L}$, be $L$ considered as an affine space, i.e., the set of $R$-points on $\widetilde{L}$ is $L \otimes_K R$. We have $\widetilde{L} = \operatorname{Spec} \mathrm{S}(L^\vee)$. Note in particular that $\widetilde{K}$ is the ring scheme with additive group $\mathbb{G}_a$ and multiplicative group $\mathbb{G}_m$. Also, if we choose a $K$-basis for $L$ we get an isomorphism $\mathrm{S}(L^\vee) \simeq K[X_1, \ldots, X_n]$, where $n$ is the rank of $L$. Hence $\widetilde{L} \simeq \mathbb{A}_K^n$ as schemes.

We next give the general definition of the $K$-scheme $(L/K)^*$, which we write as $L^*$ when $K$ is clear from the context. Define $L^* \subset \widetilde{L}$ as the subfunctor given by $L^*(R) = (L \otimes_K R)^\times$. To see that this is an affine group scheme, note that it is the inverse image of $K^*$ under the norm map $\widetilde{\mathrm{N}}_{L/K} \colon \widetilde{L} \to \widetilde{K}$. (Here, $\widetilde{\mathrm{N}}_{L/K}$ is defined on $R$-points as $\mathrm{N}_{L\otimes_K R/R}$.)

We now turn to the problem of computing $[(L/k)^*]$ when $k$ is a field and $L$ is separable. The obvious approach would be to compute an explicit equation defining $L^*$ and then use the scissor relations. More precisely, choose a basis of $L$ over $k$. This identifies $\widetilde{L}$ with $\operatorname{Spec} k[X_1, \ldots, X_n]$ and $\widetilde{k}$ with $\operatorname{Spec} k[X]$. Then $\widetilde{\mathrm{N}}_{L/k}$ corresponds to a homomorphism of $k$-algebras $k[X] \to k[X_1, \ldots, X_n]$ sending $X$ to some polynomial, say $f(X_1, \ldots, X_n)$. Therefore,

$$\begin{aligned}\widetilde{\mathrm{N}}^{-1}_{L/K}(\mathbb{G}_m) &= \operatorname{Spec} k[X, 1/X] \otimes_{k[X]} k[X_1, \ldots, X_n] \\ &= \operatorname{Spec} k[X_1, \ldots, X_n, 1/f(X_1, \ldots, X_n)].\end{aligned}$$

When $n = 2$ this can be used to compute $[L^*]$ using the scissor relations as the following example shows.

**Example 3.15.** *Let $L$ be a separable field extension of $k$ of degree 2. We can represent $L$ as $k[T]/\bigl(f(T)\bigr)$ where $f(T) = T^2 + \alpha T + \beta$ is irreducible, in particular $\beta \neq 0$. If $\operatorname{char} k \neq 2$ we assume that $\alpha = 0$. With this notation we have $\widetilde{L}(R) = R[T]/\bigl(f(T)\bigr)$ for every $k$-algebra $R$. A basis for the $R$-algebra $\widetilde{L}(R)$ is $\{1, t\}$ where $t$ is the class of $T$ modulo $f(T)$. If $r_1, r_2 \in R$ then $\mathrm{N}_{\widetilde{L}(R)/R}(r_1 + r_2 t) = r_1^2 - r_1 r_2 \alpha + r_2^2 \beta$. So if we identify $\widetilde{L}$ with $\operatorname{Spec} k[X_1, X_2]$ then*

$$L^* = \mathrm{D}\bigl(X_1^2 - \alpha X_1 X_2 + \beta X_2^2\bigr) \subset \widetilde{L}.$$

*We now have an explicit equation describing $L^*$. To compute $[L^*]$ we first compute $[\widetilde{L} \backslash L^*]$. Now $\widetilde{L} \backslash L^* = \operatorname{Spec} k[X_1, X_2]/(X_1^2 - \alpha X_1 X_2 + \beta X_2^2) \subset \widetilde{L}$. This can be split into two parts, the closed subscheme $\operatorname{Spec} k[X_1]/(X_1^2)$, which maps to 1 in $\mathrm{K}_0(\mathsf{Var}_k)$, and its complement*

$$\operatorname{Spec} \frac{k[X_1, X_2, 1/X_2]}{\bigl(X_1^2 - \alpha X_1 X_2 + \beta X_2^2\bigr)} \simeq \operatorname{Spec} \frac{k[Y_1, Y_2, 1/Y_2]}{\bigl(Y_1^2 - \alpha Y_1 + \beta\bigr)}.$$

*Now if $\operatorname{char} k \neq 2$ then $\alpha = 0$ so $Y_1^2 - \alpha Y_1 + \beta = f(Y_1)$ and this is also true if $\operatorname{char} k = 2$ for then $-\alpha = \alpha$. Hence the above expression equals $\operatorname{Spec} k[Y_2, 1/Y_2] \times_k \operatorname{Spec} L$, and this maps to $(\mathbb{L} - 1) \cdot [\operatorname{Spec} L] \in \mathrm{K}_0(\mathsf{Var}_k)$. Putting this together gives $[L^*] = \mathbb{L}^2 - [\operatorname{Spec} L] \cdot \mathbb{L} + [\operatorname{Spec} L] - 1 \in \mathrm{K}_0(\mathsf{Var}_k)$.*

This method does not work when $L$ is a field of degree greater than 2, the cutting and pasting then becomes to complicated. The rest of this section is devoted to giving a systematic way of computing $[L^*]$.

## Reduction to case of lower dimension

We now describe a method which makes it possible to recursively compute $[L^*]$ as an element of $ArtCl_k[\mathbb{L}]$. This will be done in the following way. We first describe subschemes of $\widetilde{L}$, denoted $U_1, \ldots, U_n$, such that $[L^*] = \mathbb{L}^n - \sum_{i=1}^n [U_i]$. We are then reduced to compute $[U_i]$ for every $i$. To do that we find a subscheme $T_i$ of $U_i$ and a $T_i$-algebra $L'_i$ of rank $n-i$. More precisely, $T_i$ is the spectrum of a finite product of fields $\prod_v K_v$, and $L'_i$ equals $L_v$ on $K_v$, $L'_i = \prod_v L_v$. We then show that $U_i \simeq (L'_i/T_i)^*$ as $k$-schemes. In Lemma 3.16 we will show that $(L'_i/T_i)^* \simeq \dot\bigcup (L_v/K_v)^*$ and we are then in the situation we started with, only that the algebras have dimension less than $n$, for having computed $[(L_v/K_v)^*] \in \mathrm{K}_0(\mathsf{Var}_{K_v})$ we can find $[(L_v/K_v)^*] \in \mathrm{K}_0(\mathsf{Var}_k)$ with the help of the projection formula.

To do this we will need the following lemma.

**Lemma 3.16.** *Let $K = \prod_{v \in I} K_v$ where the $K_v$ are fields and $I$ is finite. Let $L$ be a free $K$-algebra of rank $n$, i.e., $L = \prod_{v \in I} L_v$ where, $L_v$ is a $K_v$-algebra of dimension $n$. Then $\widetilde{L/K} \simeq \dot\bigcup_v \widetilde{L_v/K_v}$ and $(L/K)^* \simeq \dot\bigcup_v (L_v/K_v)^*$ as $K$-schemes.*

*Proof.* The first part follows since $\mathrm{S}(L^\vee) \simeq \prod_{v \in I} \mathrm{S}(L_v^\vee)$ as $K$-algebras. To prove that $(L/K)^* \simeq \dot\bigcup (L_v/K_v)^*$ as $K$-schemes we prove that their functors of points are equal. Let $R$ be a $K$-algebra, i.e., $R = \prod_v R_v$ where $R_v$ is a $K_v$-algebra, possibly equal to zero. An $R$-point on $\dot\bigcup (L_v/K_v)^*$ is a morphism $f\colon \dot\bigcup_v \operatorname{Spec} R_v \to \dot\bigcup (L_v/K_v)^*$ that commutes with the structural morphisms to $\dot\bigcup_v \operatorname{Spec} K_v$. Since the image of $\operatorname{Spec} R_v$ under the structural morphism is contained in $\operatorname{Spec} K_v$ we must have $f(\operatorname{Spec} R_v) \subset L_v^*$. Therefore $f$ is determined by a set of morphisms $\{f_v\colon \operatorname{Spec} R_v \to L_v^*\}_{v \in I}$ where $f_i$ is a morphism of $K_v$-schemes. Hence we can identify $f$ with an element in $\prod L_v^*(R_v)$. The same is true for an $R$-point on $L^*$ for

$$L^*(R) = \left(\left(\prod L_v\right) \otimes_{\prod K_v} \left(\prod R_v\right)\right)^\times \simeq \prod (L_v \otimes_{K_v} R_v)^\times = \prod L_v^*(R_v).$$

So by Yoneda's lemma, $L^* \simeq \dot\bigcup L_v^*$. (This method could also have been used to prove the first part of the lemma, but there we knew the algebra representing $\widetilde{L}$ and that gave a shorter proof.) ☐

We will now give the definitions of $U_i$, $T_i$ and $L'_i$. There are two ways of doing this. The first is to construct them explicitly in much the same way as we constructed $L^*$ with the help of the norm map. The second is to just construct their images after scalar extension, as subschemes of $(\widetilde{L})_{\overline{k}}$, and then use Galois descent as described in Section 3.2. The first method requires more work but has the advantage that it works also when the base is not a field. It is carried out in [Rök07c]. In this paper we will use the second method:

Let $L$ be the separable $k$-algebra and let $S$ be the corresponding $\mathcal{G}$-set, i.e., the corresponding $\overline{k}-\mathcal{G}$-algebra is $\overline{k}^S$. Since, for every $\overline{k}$-algebra $R$,

$$\widetilde{L}_{\overline{k}}(R) = (L \otimes_k \overline{k}) \otimes_{\overline{k}} R = \overline{k}^S \otimes_{\overline{k}} R = (\overline{k} \otimes_{\overline{k}} R)^S = R^S = \mathbb{A}^S_{\overline{k}}(R),$$

it follows that the $k$-scheme $\widetilde{L}$ corresponds to the $\overline{k}-\mathcal{G}$-scheme $\mathbb{A}^S_{\overline{k}}$, i.e., $\operatorname{Spec} \overline{k}[X_s]_{s \in S}$ where $\mathcal{G}$ acts on the scalars and by permuting the indeterminates. In the same way we see that the $k$-scheme $L^*$ corresponds to the $\overline{k}-\mathcal{G}$-scheme $\mathbb{G}^S_m$. (This also gives an alternative construction of $L^*$ when the base is a field, it is the $k$-scheme corresponding to the $\overline{k}-\mathcal{G}$-scheme $\mathbb{G}^S_m \subset \mathbb{A}^S_{\overline{k}}$.)

We next define $U_i \subset \widetilde{L}$, where $i \in \{0, \dots, n\}$. For this, let $\mathcal{P}_i(S)$ be the $\mathcal{G}$-set of subsets of $S$ of cardinality $i$. We then define $U_i$ to be the $k$-scheme corresponding to the $\overline{k}-\mathcal{G}$-scheme

$$\bigcup_{T \in \mathcal{P}_i(S)} \mathbb{G}^{S \setminus T}_m \subset \mathbb{A}^S_{\overline{k}},$$

where $\mathbb{G}^{S\setminus T}_m(R)$ is the group of $n$-tuples $(r_s)_{s\in S} \in \mathbb{A}^S(R)$ such that $r_s = 0$ if $s \in T$ and $r_s \in R^\times$ if $r_s \notin T$. Since

$$\mathbb{G}^{S\setminus T}_m = \mathrm{V}\big(\{X_s\}_{s\in T}\big) \setminus \mathrm{V}\big(\{X_s\}_{s\notin T}\big) \subset \operatorname{Spec} \overline{k}[X_s]_{s\in S} = \mathbb{A}^S_{\overline{k}}$$

we see that $\bigcup_{T\in\mathcal{P}_i(S)} \mathbb{G}^{S\setminus T}_m$ is locally closed and that their union over all $i$ cover $\mathbb{A}^S_{\overline{k}}$. It hence follows that the $U_i$ are locally closed and that they cover $\widetilde{L}$. Noting that $U_0 = L^*$, we see that $[L^*] = \mathbb{L}^n - \sum_{i=1}^n [U_i] \in \mathrm{K}_0(\mathsf{Var}_k)$.

Now consider $\mathrm{V}(\{X_t - 1\}_{t\notin T}) \subset \mathbb{G}^{S\setminus T}_m$. This subscheme is isomorphic to $\operatorname{Spec} \overline{k}$. Taking the union over every $T \in \mathcal{P}_i(S)$ we get an $\overline{k}-\mathcal{G}$-subscheme

$$\operatorname{Spec} \overline{k}^{\mathcal{P}_i(S)} \subset \bigcup_{T\in\mathcal{P}_i(S)} \mathbb{G}^{S\setminus T}_m. \tag{3.17}$$

We denote the corresponding $k$-scheme with $T_i$. Moreover, the fact that $(T_i)_{\overline{k}} = \operatorname{Spec} \overline{k}^{\mathcal{P}_i(S)}$ shows (in fact is equivalent to) that $T_i$ is the spectrum of a separable algebra. Hence $T_i$ corresponds to the $\mathcal{G}$-set $\mathcal{P}_i(S)$ and it is a product of separable field extensions of $k$.

Next use the algebras $\overline{k} \to \overline{k}^{S\setminus T}$ for $T \in \mathcal{P}_i(S)$ to define a $\overline{k}^{\mathcal{P}_i(S)}$-algebra

$$\prod_{T\in\mathcal{P}_i(S)} \overline{k}^{S\setminus T}$$

of rank $|S \setminus T| = n - i$ in the category of $\overline{k} - \mathcal{G}$-algebras. Denote the corresponding $T_i$-algebra with $L_i'$, which is then also of rank $n-i$. We note that the map $T_i \to L_i'$ corresponds the projection $\{(s,T) \in S \times \mathcal{P}_i(S) : s \notin T\} \to \mathcal{P}_i(S)$ in the category of $\mathcal{G}$-sets.

Now by Proposition 3.16,

$$\left( \prod_{T \in \mathcal{P}_i(S)} \overline{k}^{S\setminus T} \Big/ \overline{k}^{\mathcal{P}_i(S)} \right)^* = \bigcup_{T \in \mathcal{P}_i(S)} (\overline{k}^{S\setminus T}/\overline{k})^*$$

and since this is isomorphic to $\bigcup_{T\in\mathcal{P}_i(S)} \mathbb{G}_m^{S\setminus T}$ it follows that the corresponding $k$-schemes are isomorphic, i.e., $(L_i'/T_i)^* \simeq U_i$.

Since $T_i$ is a finite product of separable extensions of $k$, $\prod K_v$, we must have $L_i' = \prod L_v$ where $L_v$ is a $K_v$-algebra of dimension $n-i$ (since the rank of $L_i'/T_i$ is $n-i$). Hence we see that $(L_i'/T_i)^* = \bigcup (L_v/K_v)^*$. By induction then, we may compute $[(L_v/K_v)^*] \in \mathrm{K}_0(\mathsf{Var}_{K_v})$ for each $v$ and then use the projection formula (0.3) to compute $[(L_i'/T_i)^*]$ and hence $[L^*] \in \mathrm{K}_0(\mathsf{Var}_k)$.

We summarize the results of this subsection in the following proposition:

**Proposition 3.18.** *Given a field $k$ and a separable $k$-algebra $L$ of dimension $n$. With the same notation as above we then have $[L^*] = \mathbb{L}^n - \sum_{i=1}^{n} [U_i] \in \mathrm{K}_0(\mathsf{Var}_k)$, where $[U_i] = \sum_{v\in I} [(L_v/K_v)^*]$, the index set $I$ is finite and the dimension of $L_v/K_v$ is $n-i$.*

We illustrate with an example.

**Example 3.19.** *Let $k = \mathbb{F}_q$ and $L = \mathbb{F}_{q^3}$. We then have*

$$[L^*] = \mathbb{L}^3 - [U_1] - [U_2] - 1 \in \mathrm{K}_0(\mathsf{Var}_k). \tag{3.20}$$

*Let $\mathcal{G} := \mathcal{G}\mathrm{al}(\overline{k}/k)$ and let $\mathrm{F}$ be the topological generator of $\mathcal{G}$, the Frobenius automorphism $\alpha \mapsto \alpha^q$. Then $L$ corresponds to the $\mathcal{G}$-set $S := \mathrm{Hom}_k(L, \overline{k}) = \{1, \mathrm{F}, \mathrm{F}^2\}$, where we have identified $\mathrm{F}$ with its restriction to $L$.*

*We have $\mathcal{P}_1(S) = \{\{1\}, \{\mathrm{F}\}, \{\mathrm{F}^2\}\} \simeq S$. Therefore $T_1 \simeq \mathrm{Spec}\, L$. Moreover, $L_1'$ corresponds to*

$$\{(1,\{\mathrm{F}\}), (1,\{\mathrm{F}^2\}), (\mathrm{F},\{1\}), (\mathrm{F},\{\mathrm{F}^2\}), (\mathrm{F}^2,\{1\}), (\mathrm{F}^2,\{\mathrm{F}\})\}$$

*and this is the union of two sets on which $\mathcal{G}$ acts transitive, hence it is isomorphic to $S \dot\cup S$ as a $\mathcal{G}$-set. So $L_1' \simeq L^2$. Therefore $[(L_1'/T_1)^*] = (\mathbb{L}-1)^2 \in \mathrm{K}_0(\mathsf{Var}_L)$ and hence by* (0.3)

$$[U_1] = R_k^L\big((\mathbb{L}-1)^2\big) = [\mathrm{Spec}\, L] \cdot (\mathbb{L}-1)^2 \in \mathrm{K}_0(\mathsf{Var}_k)$$

*In the same way we find that $[U_2] = [\mathrm{Spec}\, L] \cdot (\mathbb{L}-1) \in \mathrm{K}_0(\mathsf{Var}_k)$. Putting this into* (3.20) *gives that*

$$[L^*] = \mathbb{L}^3 - [\mathrm{Spec}\, L] \cdot \mathbb{L}^2 + [\mathrm{Spec}\, L] \cdot \mathbb{L} - 1 \in \mathrm{K}_0(\mathsf{Var}_k).$$

## An explicit formula

We have now showed how to compute $[L^*]$ in principle. Evolving what we already have proved will give us an explicit formula.

To get more compact formulas we use the following notation. In the last section we worked with a fixed algebra $L/k$ and defined $U_i$, $T_i$ and $L_i'$ with respect to this algebra. To translate the recursion into a closed formula we need to repeat these constructions. Hence we fix once and for all our base field, $k$. Let $K$ be a finite extension field of $k$ and let $L$ be a separable $K$-algebra. We then use $U_i(L/K)$, $T_i(L/K)$ and $L_i'(L/K)$ to denote $U_i$, $T_i$ and $L_i'$ constructed with respect to the algebra $L/K$. We have that $L_i(L/K)$ and $T_i(L/K)$ are $K$-schemes which we also can view as $k$-schemes by restriction of scalars. Similarly, $L_i'(L/K)$ is a $T_i(L/K)$-algebra which we also may view as a $k$-algebra.

We next want to generalize these definitions to the case when $K$ is a finite separable $k$-algebra. Recall that this is already done for $(L/K)^*$. However, since the definitions of $U_i$, $T_i$ and $L_i'$ use Galois descent their constructions cannot be generalized directly. We instead do the following.

**Definition 3.21.** *Let $K$ be a finite separable $k$-algebra and $L$ a finite separable $K$-algebra, so $K = \prod_v K_v$ where $K_v$ are separable extension fields of $k$ and $L = \prod_v L_v$ where $L_v$ is a separable $K_v$-algebra. Define*

$$U_i(L/K) := \dot\bigcup_v U_i(L_v/K_v).$$

*Furthermore, define*

$$T_i(L/K) := \dot\bigcup_v T_i(L_v/K_v)$$

*and define $L_i'(L/K)$ to be the $T_i(L/K)$-algebra which is $L_i'(L_v/K_v)$ on $T_i(L_v/K_v)$.*

Let $K$ be a finite separable $k$-algebra and $L$ a finite separable $K$-algebra of rank $n$, so $K = \prod_v K_v$ where $K_v$ are separable extension fields of $k$ and $L = \prod_v L_v$ where $L_v$ is a separable $K_v$-algebra of dimension $n$ as a vector space over $K_v$. Using Definition 3.21, Proposition 3.16 and the stratification of $\widetilde{L}$ over a field given in the preceding section gives the following:

**Lemma 3.22.** *We have that*

$$\widetilde{L/K} = (L/K)^* \cup \bigcup_{i=1}^{n-1} U_i(L/K) \cup \operatorname{Spec} K$$

*where the union is disjoint and open. Hence,*

$$[(L/K)^*] = [\operatorname{Spec} K] \cdot \mathbb{L}^n - \sum_{i=1}^{n-1} [U_i(L/K)] - [\operatorname{Spec} K] \in \mathrm{K}_0(\mathsf{Var}_k).$$

From the preceding section we know that $L'_i(L_v/K_v)/T_i(L_v/K_v)$ has rank $n-i$ and that $U_i(L_v/K_v) \simeq \big(L'_i(L_v/K_v)/T_i(L_v/K_v)\big)^*$. Taking the union over every $v$ and using Proposition 3.16 we get the following.

**Lemma 3.23.** *The algebra $L'_i(L/K)/T_i(L/K)$ has rank $n-i$, and*

$$U_i(L/K) \simeq \big(L'_i(L/K)/T_i(L/K)\big)^*$$

*as $k$-schemes.*

For the rest of this section, we fix a field $k$ and a separable $k$-algebra $L$ of dimension $n$. Notation: Given a sequence of positive integers $i_1, \dots, i_q$. Construct the algebra $L'_{i_1}/T_{i_1} := L'_{i_1}(L/k)/T_{i_1}(L/k)$. Define the algebra $L'_{i_2,i_1}/T_{i_2,i_1}$ as $L'_{i_2}(L'_{i_1}/T_{i_1})/T_{i_2}(L'_{i_1}/T_{i_1})$ and define inductively $L'_{i_{r+1},\dots,i_1}/T_{i_{r+1},\dots,i_1}$ as

$$L'_{i_{r+1}}(L'_{i_r,\dots,i_1}/T_{i_r,\dots,i_1})/T_{i_{r+1}}(L'_{i_r,\dots,i_1}/T_{i_r,\dots,i_1}).$$

Inductively we get that the rank of $L'_{i_r,\dots,i_1}/T_{i_r,\dots,i_1}$ is $n-(i_1+\cdots+i_r)$. Hence, as a corollary to the preceding lemmas we get the following.

**Lemma 3.24.** *Let $\alpha = (i_r, \dots, i_1)$ where $\sum_{s=1}^r i_s = i$. Then*

$$\big[(L'_\alpha/T_\alpha)^*\big] = [T_\alpha]\cdot\mathbb{L}^{n-i} - \sum_{j=1}^{n-i-1}\big[(L'_{j,\alpha}/T_{j,\alpha})^*\big] - [T_\alpha] \in \mathrm{K}_0(\mathsf{Var}_k).$$

We are now ready to prove the main theorem of this subsection.

**Theorem 3.25.** *With the same notation as above we have*

$$[L^*] = \mathbb{L}^n + a_1\mathbb{L}^{n-1} + \cdots + a_{n-1}\mathbb{L} + a_n$$

*where*

$$a_j = \sum_{r=1}^{j}(-1)^r \sum_{\substack{(i_1,\dots,i_r):\\ i_1+\cdots+i_r=j\\ i_s\geq 1}} [T_{i_r,\dots,i_1}]$$

*for $j = 1, \dots, n$.*

*Proof.* We evaluate $[L^*]$ in $n$ steps, using Lemma 3.24. In the first step we write

$$[(L/k)^*] = \mathbb{L}^n - [(L'_1/T_1)^*] - \cdots - [(L'_{n-1}/T_{n-1})^*] - 1$$

so we get the contribution $\mathbb{L}^n - 1$. We then evaluate the remaining terms, using Lemma 3.24, so in step two we get a sum consisting of two parts. First, $\big[(L'_{i_2,i_1}/T_{i_2,i_1})^*\big]$ shows up with sign $(-1)^2$, for $2 \leq i_2 + i_1 < n$ (we always have $i_s \geq 1$). These are the terms that we will take care of in step three. The second part of the sum contributes to our formula. It consists of the terms

$$(-1)^2\big(-[T_j]\cdot\mathbb{L}^{n-j}+[T_j]\big) \qquad\qquad 1\le j<n.$$

Continuing in this way we find that in step $r$ we get a sum consisting of two parts. Firstly, every term of the form $\big[(L'_{i_r,\dots,i_1}/T_{i_r,\dots,i_1})^*\big]$ with coefficient $(-1)^r$, for $\sum_{s=1}^r i_s < n$. This part is taken care of in step $r+1$. And secondly we get a contribution to our formula consisting of

$$(-1)^r\big(-[T_{i_{r-1},\dots,i_1}]\cdot\mathbb{L}^{n-j}+[T_{i_{r-1},\dots,i_1}]\big) \qquad r-1\le j<n$$

for every $r-1$-tuple $(i_{r-1},\dots,i_1)$ such that $\sum_{s=1}^{r-1} i_s = j$. This process ends in step $n$.

Collecting terms we now see that if $1\le j\le n-1$ then the coefficient in front of $\mathbb{L}^{n-j}$ becomes

$$\sum_{r=2}^{j+1}(-1)^{r+1}\sum_{\substack{(i_1,\dots,i_{r-1}):\\ i_1+\dots+i_{r-1}=j\\ i_s\ge 1}}[T_{i_{r-1},\dots,i_1}].$$

This equals

$$\sum_{r=1}^{j}(-1)^{r}\sum_{\substack{(i_1,\dots,i_{r}):\\ i_1+\dots+i_{r}=j\\ i_s\ge 1}}[T_{i_{r},\dots,i_1}]. \tag{3.26}$$

A separate computation, using that $[T_n]=1$ and that if $1\le\sum_{s=1}^{r-1} i_s = j<n$ then $T_{n-j,i_{r-1},\dots,i_1}=T_{i_{r-1},\dots,i_1}$, shows that formula (3.26) holds also for the constant coefficient, when $j=n$. □

## The formula expressed using the Burnside ring

The formula in the preceding section is not suitable for computations. In this section we make it so, by computing the object in the Burnside ring that maps to $[T_\alpha]$. We begin with some notations.

**Definition 3.27.** *Let $G$ be a profinite group. Given a $G$-set $S$ of cardinality $n$ and a positive integer $r$. Moreover, let $(i_1,\dots,i_r)$ be an $r$-tuple of positive integers such that $i_1+\dots+i_r\le n$. Then $\mathcal{P}_{i_r,\dots,i_1}(S)$ is the $G$-set of $r$-tuples $(S_r,\dots,S_1)$ where $S_j$ is a subset of $S$ of cardinality $i_j$ and the $S_j$ are pairwise disjoint. In particular $\mathcal{P}_i(S)$ has the same meaning as before (up to isomorphism).*

**Lemma 3.28.** *Let $k$ be a field and $K$ a separable $k$-algebra of dimension $t$. Let $L$ be a separable $K$-algebra of rank $n$. Let $\mathcal{G}:=\mathcal{G}\mathrm{al}(k^s/k)$*

*and let $K$ and $L$ correspond to $T$ respectively $S$ as $\mathcal{G}$-sets. Write $T = \mathrm{Hom}_k(K, k^s) = \{\tau_1, \ldots, \tau_t\}$. Let $S_j$ be the inverse image of $\tau_j$ under the map $S \to T$ corresponding to $K \to L$. Then $T_i(L/K)$ corresponds to the $\mathcal{G}$-set*

$$\bigcup_{j=1}^{t} \mathcal{P}_i(S_j)$$

*and $L_i'(L/K)$ corresponds to*

$$\left\{(f, U) \in \bigcup_{j=1}^{t} S_j \times \mathcal{P}_i(S_j) : f \notin U\right\}$$

*Proof.* Suppose first that $K$ is a field. According to (1.15), $T_i(L/K)$ corresponds to $\mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big)$ as a $\mathcal{G}\mathrm{al}(k^s/K)$-set. Hence by Proposition 3.9 it corresponds to

$$\mathcal{G} \times_{\mathcal{G}\mathrm{al}(k^s/K)} \mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big)$$

as a $\mathcal{G}$-set, with the $\mathcal{G}$-action given in that proposition. Since we assumed that $K$ is a field we may write $T$ as $\{\tau_1|_K, \ldots, \tau_t|_K\}$, where $\tau_j \in \mathcal{G}$, and this in turn can be identified with a system of coset representatives of $\mathcal{G} \,/\, \mathcal{G}\mathrm{al}(k^s/K)$. We hence want to show that we have an isomorphism of $\mathcal{G}$-sets,

$$\phi\colon T \times \mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big) \to \bigcup_{j=1}^{t} \mathcal{P}_i(S_j).$$

To construct this, define $\phi$ as $(\tau_j|_K, U) \mapsto \tau_j U$. (Note that $\tau_j$ have to be fixed for every $j$, if we replace it with $\tau_j'$ such that $\tau_j|_K = \tau_j'|_K$ we may get another $\phi$.) First $\phi$ is well defined because every element in $U$ fixes $K$, so every element of $\tau_j U$ is in $S_j$, the inverse image of $\tau_j|_K$ in $S$. Hence $\phi(\tau_j|_K, U) \in \mathcal{P}_i(S_j)$. It is also $\mathcal{G}$-equivariant, because if $\sigma \in \mathcal{G}$ is such that $\sigma\tau_j = \tau_l\tau'$, where $\tau' \in \mathcal{G}\mathrm{al}(k^s/K)$, then

$$\phi\big(\sigma(\tau_j|_K, U)\big) = \phi(\tau_l, \tau' U) = \tau_l \tau' U$$

and

$$\sigma\phi(\tau_j|_K, U) = \sigma(\tau_j U) = \sigma\tau_j U = \tau_l\tau' U.$$

Next $\phi$ is injective: If $\phi(\tau_j|_K, U) = \phi(\tau_l|_K, U')$ then they both must be in $\mathcal{P}_i(S_j)$, so $l = j$. Hence $\tau_j U = \tau_j U'$ and since $\tau_j$ is an isomorphism, $U = U'$. So $\phi$ is an injective morphism between two $\mathcal{G}$-sets of cardinality $t \cdot \binom{n}{i}$, hence an isomorphism.

For the general case when $K$ is a separable $k$-algebra of dimension $t$, note that we can identify $T$ with

$$\dot{\bigcup_v} \mathrm{Hom}_k(K_v, k^s)$$

where $K = \prod_v K_v$, by sending $f \in \mathrm{Hom}_k(K_{v_0}, k^s)$ to $(\alpha_v) \mapsto f(\alpha_{v_0}) \in T$. Denote the map $S \to T$ by $\pi$. We have that $T_i(L/K) = \dot{\bigcup}_v T_i(L_v/K_v)$. This corresponds to the $\mathcal{G}$-set

$$\dot{\bigcup_v} \bigcup_{\tau \in \mathrm{Hom}_k(K_v, k^s)} \mathcal{P}_i(\pi^{-1}\tau) = \bigcup_{\tau \in T} \mathcal{P}_i(\pi^{-1}\tau) = \dot{\bigcup_{j=1}^{t}} \mathcal{P}_i(S_j)$$

As for $L_i'(L/K)$, assume first that $K$ is a field. As a $\mathcal{G}\mathrm{al}(k^s/K)$-set, $L_i'(L/K)$ corresponds to

$$M := \{(f, U) \in \mathrm{Hom}_K(L, k^s) \times \mathcal{P}_i\big(\mathrm{Hom}_K(L, k^s)\big) : f \notin U\},$$

hence it corresponds to $T \times M$ as a $\mathcal{G}$-set. Define a map

$$T \times M \to \left\{ (f, U) \in \bigcup_{j=1}^{t} S_j \times \mathcal{P}_i(S_j) : f \notin U \right\}$$

by

$$\big(\tau_j|_K, (f, U)\big) \mapsto (\tau_j \circ f, \tau_j U).$$

As above one shows that this is an isomorphism of $\mathcal{G}$-sets. The case when $K$ is an arbitrary separable $k$-algebra is handled in the same way as $T_i$. □

**Proposition 3.29.** *Let $\alpha = (i_r, \ldots, i_1)$ be an $r$-tuple of positive integers such that $i_1 + \cdots + i_r = i$ where $1 \le i \le n$. The algebra $L'_\alpha/T_\alpha$ in the category of $k$-algebras corresponds to the $\mathcal{G}$-sets*

$$\left\{ \big(s, (S_r, \ldots, S_1)\big) \in S \times \mathcal{P}_\alpha(S) : s \notin \cup_{t=1}^{r} S_t \right\}$$

*and $\mathcal{P}_\alpha(S)$ together with the projection morphism.*

*Proof.* By construction the proposition holds for $r = 1$. Suppose the formula has been proved for $r$. We have

$$T_{i_{r+1}, i_r, \ldots, i_1} = T_{i_{r+1}}(L'_{i_r, \ldots, i_1}/T_{i_r, \ldots, i_1}).$$

By the induction hypothesis and Lemma 3.28 this corresponds to

$$\bigcup_{(S_r, \ldots, S_1) \in \mathcal{P}_{i_r, \ldots, i_1}(S)} \mathcal{P}_{i_{r+1}}\Big(\Big\{ \big(s, (S_r, \ldots, S_1)\big) : s \notin \cup_{t=1}^{r} S_t \Big\}\Big)$$

which is isomorphic to

$$\bigcup_{(S_r, \ldots, S_1) \in \mathcal{P}_{i_r, \ldots, i_1}(S)} \Big\{ \big(\{s_1, \ldots, s_{i_{r+1}}\}, S_r, \ldots, S_1\big) : s_{i_t} \notin \cup_{t=1}^{r} S_t \Big\}$$

and this in turn is equal to $\mathcal{P}_{i_{r+1},i_r,\dots,i_1}(S)$.

Moreover, $L'_{i_{r+1},i_r,\dots,i_1}$ corresponds to the set of pairs $(f,U)$ in

$$\bigcup_{(S_r,\dots,S_1)\in\mathcal{P}_{i_r,\dots,i_1}(S)} \Big\{\big(s,(S_r,\dots,S_1)\big) : s\notin \cup_{t=1}^r S_t\Big\} \times \mathcal{P}_{i_{r+1}}\Big(\Big\{\big(s,(S_r,\dots,S_1)\big) : s\notin \cup_{t=1}^r S_t\Big\}\Big)$$

such that $f\notin U$. This set is isomorphic to the union, taken over all $(S_r,\dots,S_1)\in\mathcal{P}_{i_r,\dots,i_1}(S)$, of the sets

$$\Big\{\big(s,(S_{r+1},S_r,\dots,S_1)\big)\in S\times\mathcal{P}_{i_{r+1},i_r,\dots,i_1}(S) : s\notin \cup_{t=1}^{r+1} S_t\Big\}.$$

This in turn equals

$$\Big\{\big(s,(S_{r+1},S_r,\dots,S_1)\big)\in S\times\mathcal{P}_{i_{r+1},\dots,i_1}(S) : s\notin \cup_{t=1}^{r+1} S_t\Big\}. \qquad \square$$

We are now ready to give our first closed formula for $[L^*]$. It follows from Theorem 3.25 and Proposition 3.29.

**Theorem 3.30.** *Let $L$ be a $k$-algebra of dimension $n$ and $S$ a $\mathcal{G}$-set such that* $\mathrm{Art}_k\big([S]\big)=[\mathrm{Spec}\,L]$. *Define*

$$\rho_i(S)=\sum_{t=1}^{i}\sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=i\\ i_s\geq 1}}(-1)^t[\mathcal{P}_{i_t,\dots,i_1}(S)]\in\mathcal{B}(\mathcal{G}).$$

*Then*

$$[L^*]=\mathbb{L}^n+a_1\cdot\mathbb{L}^{n-1}+\dots+a_{n-1}\cdot\mathbb{L}+a_n\in\mathrm{K}_0(\mathsf{Var}_k)$$

*where* $a_i=\mathrm{Art}_k(\rho_i(S))$.

### The universal nature of the formula

Fix a field $k$ with absolute Galois group $\mathcal{G}$. Also, fix a separable $k$-algebra $L$ of dimension $n$ corresponding to the $\mathcal{G}$-set $S$. Define a homomorphism $\phi\colon\mathcal{G}\to\Sigma_n$ as the composition of $\mathcal{G}\to\mathrm{Aut}(S)$ with an isomorphism $\mathrm{Aut}(S)\to\Sigma_n$. Let $\mathrm{res}_{\mathcal{G}}^{\Sigma_n}\colon\mathcal{B}(\Sigma_n)\to\mathcal{B}(\mathcal{G})$ be the restriction map with respect to $\phi$. Then $\mathrm{res}_{\mathcal{G}}^{\Sigma_n}$ is independent of the chosen isomorphism $\mathrm{Aut}(S)\to\Sigma_n$. We have that

$$\mathrm{res}_{\mathcal{G}}^{\Sigma_n}\big(\{1,\dots,n\}\big)=[S]\in\mathcal{B}(\mathcal{G}).$$

Using the notation that $\mathcal{P}_\alpha^{(n)} := \mathcal{P}_\alpha(\{1,\ldots,n\})$ we also have $\operatorname{res}_{\mathcal{G}}^{\Sigma_n}(\mathcal{P}_\alpha^{(n)}) = [\mathcal{P}_\alpha(S)]$. Define

$$\rho_i^{(n)} := \sum_{t=1}^{i} \sum_{\substack{(i_1,\ldots,i_t):\\ i_1+\cdots+i_t=i\\ i_s\geq 1}} (-1)^t[\mathcal{P}_{i_t,\ldots,i_1}^{(n)}] \in \mathcal{B}(\Sigma_n). \tag{3.31}$$

Then $\operatorname{res}_{\mathcal{G}}^{\Sigma_n}(\rho_i^{(n)}) = \rho_i(S)$. This discussion gives the following formulation of Theorem 3.30.

**Theorem 3.32.** *Fix a positive integer $n$. The $\rho_i^{(n)} \in \mathcal{B}(\Sigma_n)$, defined in (3.31), are universal in the sense that for every field $k$ with absolute Galois group $\mathcal{G}$ and every separable $k$-algebra of dimension $n$,*

$$[L^*] = \mathbb{L}^n + a_1\cdot\mathbb{L}^{n-1} + \cdots + a_{n-1}\cdot\mathbb{L} + a_n \in \mathrm{K}_0(\mathsf{Var}_k),$$

*where $a_i = \mathrm{Art}_k \circ \operatorname{res}_{\mathcal{G}}^{\Sigma_n}(\rho_i^{(n)})$.*

We illustrate with an example.

**Example 3.33.** *Let $L/k = \mathbb{F}_{q^4}/\mathbb{F}_q$. Since $\mathcal{G}$ is generated by the Frobenius map $\mathrm{F}$ we can identify $S$, the $\mathcal{G}$-set corresponding to $L$, with $\{1,\mathrm{F},\mathrm{F}^2,\mathrm{F}^3\}$. We have*

$$\mathcal{P}_2(S) = \big\{\{1,\mathrm{F}\},\{\mathrm{F},\mathrm{F}^2\},\{\mathrm{F}^2,\mathrm{F}^3\},\{1,\mathrm{F}^3\}\big\}\,\dot\cup\big\{\{1,\mathrm{F}^2\},\{\mathrm{F},\mathrm{F}^3\}\big\}.$$

*The first of these sets is isomorphic to $S$. The second is transitive of cardinality 2 so it corresponds to a field extension of $k$ of degree 2, i.e., $\mathbb{F}_{q^2}$. Reasoning in this way and using Theorem 3.32 we find that*

$$[L^*] = \mathbb{L}^4 - [\operatorname{Spec}\mathbb{F}_{q^4}]\cdot\mathbb{L}^3 + \big(2[\operatorname{Spec}\mathbb{F}_{q^4}] - [\operatorname{Spec}\mathbb{F}_{q^2}]\big)\cdot\mathbb{L}^2 \\ - [\operatorname{Spec}\mathbb{F}_{q^4}]\cdot\mathbb{L} + [\operatorname{Spec}\mathbb{F}_{q^2}] - 1.$$

*If instead $L/k = \mathbb{F}_{q^2}\times\mathbb{F}_{q^2}/\mathbb{F}_q$ then $S = \{e_1,\mathrm{F}\,e_1\}\,\dot\cup\{e_2,\mathrm{F}\,e_2\}$ where $e_1$ and $e_2$ are the projection maps. We then get, for example,*

$$\mathcal{P}_2(S) = \big\{\{e_1,\mathrm{F}\,e_1\}\big\}\,\dot\cup\big\{\{e_2,\mathrm{F}\,e_2\}\big\} \\ \dot\cup\big\{\{e_1,e_2\},\{\mathrm{F}\,e_1,\mathrm{F}\,e_2\}\big\}\,\dot\cup\big\{\{e_1,\mathrm{F}\,e_2\},\{\mathrm{F}\,e_1,e_2\}\big\}.$$

*This kind of computation gives that*

$$[L^*] = \mathbb{L}^4 - 2[\operatorname{Spec}\mathbb{F}_{q^2}]\cdot\mathbb{L}^3 + \big(4[\operatorname{Spec}\mathbb{F}_{q^2}] - 2\big)\cdot\mathbb{L}^2 - 2[\operatorname{Spec}\mathbb{F}_{q^2}]\cdot\mathbb{L} + 1. \tag{3.34}$$

*Note however that since $L^*(R) = (\mathbb{F}_{q^2}\otimes_k R)^\times \times (\mathbb{F}_{q^2}\otimes_k R)^\times = (\mathbb{F}_{q^2}^* \times_k \mathbb{F}_{q^2}^*)(R)$, Yoneda's lemma shows that $L^* \simeq \mathbb{F}_{q^2}^* \times_k \mathbb{F}_{q^2}^*$, so (3.34) could also have been obtained by squaring the expression for $[\mathbb{F}_{q^2}^*]$ given in Example 3.15.*

## 3.6 The class of the torus in terms of the $\lambda$-operations

In Part I of this thesis we study the $\lambda$-structure on $\mathcal{B}(\Sigma_n)$. The main result is that there is a formula for $\lambda^i$ namely that, given $n$, for $i = 1, \dots, n$ we have

$$\lambda^i\big(\{1,\dots,n\}\big) = (-1)^i \sum_{t=1}^{i} \sum_{\substack{(i_1,\dots,i_t):\\ i_1+\dots+i_t=i\\ i_s\geq 1}} (-1)^t \big[\mathcal{P}^{(n)}_{i_t,\dots,i_1}\big] \in \mathcal{B}(\Sigma_n). \quad (3.35)$$

From this and Theorem 3.30 the following theorem is immediate.

**Theorem 3.36.** *Let $\rho_i^{(n)}$ be the elements defined in Theorem 3.32, i.e., the elements in $\mathcal{B}(\Sigma_n)$ describing $[L^*] \in \mathrm{K}_0(\mathsf{Var}_k)$ for every separable, $n$-dimensional algebra $k \to L$. Then*

$$\rho_i^{(n)} = (-1)^i \cdot \lambda^i\big(\{1,\dots,n\}\big) \in \mathcal{B}(\Sigma_n).$$

As a corollary, (3.1) follows:

*Proof of* (3.1). Let $L$ correspond to the $\mathcal{G}$-set $S$. From Theorem 3.32 we know that the coefficient in front of $\mathbb{L}^{n-i}$ is $\mathrm{Art}_k \circ \mathrm{res}^{\Sigma_n}_{\mathcal{G}}(\rho_i^{(n)})$. By Theorem 3.36, and the fact that $\mathrm{res}^{\Sigma_n}_{\mathcal{G}}$ is a $\lambda$-homomorphism, this equals $\mathrm{Art}_k\big((-1)^i \cdot \lambda^i(S)\big)$. Since $\mathrm{Art}_k$ is a $\lambda$-morphism that maps $[S]$ to $[\operatorname{Spec} L]$ the result follows. ☐

We now give an alternative proof of Theorem 3.36. This proof is based on point counting over finite fields.

This proof does not use the universal formula (3.35) from Part I. We do however need the following results proved in Chapter 1: Write $\mathcal{P}_\mu$ for the $\Sigma_n$-set $\mathcal{P}_\mu\big(\{1,\dots,n\}\big)$. We define the Schur subring $Schur_n \subset \mathcal{B}(\Sigma_n)$ to be the subgroup generated by $\{[\mathcal{P}_\mu]\}_{\mu\vdash n}$. (Here $\mu \vdash n$ means that $\mu$ is a partition of $n$.) This is closed under multiplication, hence really a ring. It is not a $\lambda$-ring since it is not closed under the $\lambda$-operations. However, $\lambda^i(\{1,\dots,n\}) \in Schur_n$ for every $i$. Moreover the restriction of $\mathrm{h}\colon \mathcal{B}(\Sigma_n) \to \mathrm{R}_{\mathbb{Q}}(\Sigma_n)$ to $Schur_n$ is injective.

Also, we do not need to know the explicit description of the universal elements $\rho_i^{(n)}$ given in Theorem 3.32. We do however need their existence and that they lie in $Schur_n$, and once that is proved it is not such a long step to describe the elements. If we instead use Theorem 3.32 the below proof of Theorem 3.36 gives a (very *ad hoc*) proof of (3.35).

The setting for the proof is as follows. Let $L$ be a separable $\mathbb{F}_q$-algebra of dimension $n$, corresponding to the $\mathcal{G} := \mathcal{G}\mathrm{al}\big(\overline{\mathbb{F}}_q/\mathbb{F}_q\big)$-set $S$. Choose an isomorphism $\mathrm{Aut}(S) \to \Sigma_n$ and compose it with the homomorphism $\mathcal{G} \to \mathrm{Aut}(S)$ to get a homomorphism $\phi\colon \mathcal{G} \to \Sigma_n$. Let $\mathrm{F}$ be the topological generator of $\mathcal{G}$ and define $\sigma := \phi(\mathrm{F}) \in \Sigma_n$. Let $\mathrm{res}^{\Sigma_n}_{\mathcal{G}}$ denote the restriction maps with respect to $\phi$ for Burnside as well as representation rings.

Recall that we use $C_\sigma$ and $C_{\mathrm{F}}$ for the character homomorphisms with respect to $\sigma$ and F respectively. One sees that $C_\sigma = C_{\mathrm{F}} \circ \mathrm{res}_{\mathcal{G}}^{\Sigma_n}$. The corresponding map on the Grothendieck ring of varieties is the point counting homomorphism $C_q$ defined by $X \mapsto |X(\mathbb{F}_q)|$, and by (3.6) we have $C_{\mathrm{F}} \circ \mathrm{h} = C_q \circ \mathrm{Art}_{\mathbb{F}_q}$. So with this notation the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{B}(\Sigma_n) & \xrightarrow{\mathrm{res}_{\mathcal{G}}^{\Sigma_n}} & \mathcal{B}(\mathcal{G}) & & \\ \downarrow{\scriptstyle \mathrm{h}} & & \downarrow{\scriptstyle \mathrm{h}} & \searrow^{\mathrm{Art}_{\mathbb{F}_q}} & \\ \mathrm{R}_{\mathbb{Q}}(\Sigma_n) & \xrightarrow{\mathrm{res}_{\mathcal{G}}^{\Sigma_n}} & \mathrm{R}_{\mathbb{Q}}(\mathcal{G}) & & \mathrm{K}_0(\mathsf{Var}_{\mathbb{F}_q}) \\ & \searrow_{C_\sigma} & \downarrow{\scriptstyle C_{\mathrm{F}}} & \swarrow_{C_q} & \\ & & \mathbb{Z} & & \end{array} \tag{3.37}$$

Using this we are now ready to give the alternative proof.

*Proof of Theorem 3.36.* Fix a positive integer $n$. We write

$$\ell_i := \lambda^i(\{1, \ldots, n\})$$

and we want to prove that $\rho_i = (-1)^i \ell_i \in \mathcal{B}(\Sigma_n)$. Since they both lie in $Schur_n$ it suffices to show that if $R$ is a set of representatives of the conjugacy classes of $\Sigma_n$ then for every $\sigma \in R$,

$$C_\sigma\, \mathrm{h}(\rho_i) = (-1)^i C_\sigma\, \mathrm{h}(\ell_i) \in \mathbb{Z}.$$

(Since h is injective on $Schur_n$ and $\prod_{\sigma \in R} C_\sigma$ is injective.) We do this simultaneously for $i = 0, \ldots, n$ by showing that

$$\sum_{i=0}^{n} C_\sigma\, \mathrm{h}(\rho_i) X^{n-i} = \sum_{i=0}^{n} (-1)^i C_\sigma\, \mathrm{h}(\ell_i) X^{n-i} \in \mathbb{Z}[X] \tag{3.38}$$

for every $\sigma \in R$.

From now on, fix $\sigma \in R$. Let $q$ be an arbitrary prime power, let $k = \mathbb{F}_q$ and let $\mathcal{G} := \mathcal{G}\mathrm{al}(\overline{k}/k)$. Choose a $\mathcal{G}$-set $S$ and an isomorphism $\phi\colon \mathrm{Aut}(S) \to \Sigma_n$ such that $\mathrm{F} \mapsto \sigma$, where F is the topological generator of $\mathcal{G}$, the Frobenius automorphism. (Equivalently, let $S = \dot{\bigcup}_{1 \le j \le m} T_j$ such that $T_j$ is a transitive $\mathcal{G}$-set of cardinality $n_j$, where $\sigma$ has cycle-type $(n_1, \ldots, n_m)$. Such an $S$ always exists for by Galois theory it comes from $L = \prod_{j=1}^{m} K_j$ where $K_j$ is a degree $n_j$ field extension of $k$, i.e., $K_j = \mathbb{F}_{q^{n_j}}$.) Construct $\mathrm{res}_{\mathcal{G}}^{\Sigma_n}$ with respect to $\phi$. In what follows we write $\ell_i(S)$ and $\rho_i(S)$ for $\mathrm{res}_{\mathcal{G}}^{\Sigma_n}(\ell_i)$ and $\mathrm{res}_{\mathcal{G}}^{\Sigma_n}(\rho_i)$ respectively.

We begin by computing the right hand side of (3.38) in terms of $(n_1, \dots, n_m)$. Let $f$ be an endomorphism of the vector space $V$ of dimension $n$. From linear algebra we know the following expression for the characteristic polynomial of $f$:

$$\det(X \cdot E_n - f) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(\textstyle\bigwedge^i f) X^{n-i}.$$

Putting $f = \mathrm{F}$ gives

$$\det(X \cdot E_n - \mathrm{F}) = \sum_{i=0}^{n} (-1)^i \chi_{\bigwedge^i \mathbb{Q}[S]}(\mathrm{F}) X^{n-i}. \tag{3.39}$$

Since $\mathrm{h}(\ell_i(S)) = \big[\bigwedge^i \mathbb{Q}[S]\big] \in \mathrm{R}_{\mathbb{Q}}(\mathcal{G})$ it follows that

$$C_{\mathrm{F}}\, \mathrm{h}(\ell_i(S)) = \chi_{\bigwedge^i \mathbb{Q}[S]}(\mathrm{F}),$$

hence, by the commutativity of (3.37), the right hand side of (3.39) equals

$$\sum_{i=0}^{n} (-1)^i C_\sigma\, \mathrm{h}(\ell_i) X^{n-i}.$$

As for the left hand side of (3.39), since $S$ is a union of transitive $\mathcal{G}$-sets $T_j$ we have $\mathbb{Q}[S] = \oplus_{j=1}^m \mathbb{Q}[T_j]$ where $\mathbb{Q}[T_j]$ is irreducible, hence the matrix for $F$ is of the form

$$\begin{pmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_m \end{pmatrix}$$

where $M_j$ is a transitive $n_j \times n_j$ permutation matrix. Since the characteristic polynomial of such a matrix is $X^{n_j} - 1$ it follows that $\det(XE_n - F) = \prod_{j=1}^m \det(XE_{n_j} - M_j) = \prod_{j=1}^m (X^{n_j} - 1)$. From (3.39) we therefore get

$$\prod_{j=1}^{m} (X^{n_j} - 1) = \sum_{i=0}^{n} (-1)^i C_\sigma\, \mathrm{h}(\ell_i) X^{n-i}. \tag{3.40}$$

We next compute the left hand side of (3.38). By the definition of the $\rho_i$ we have

$$[L^*] = \sum_{i=0}^{n} \operatorname{Art}\big(\rho_i(S)\big) \mathbb{L}^{n-i} \in \mathrm{K}_0(\mathsf{Var}_k).$$

Applying $C_q$ to this gives

$$|L^*(k)| = \sum_{i=0}^{n} C_q \operatorname{Art}(\rho_i(S)) \cdot q^{n-i}. \tag{3.41}$$

By the commutativity of (3.37), $C_q \operatorname{Art}(\rho_i(S)) = C_\sigma \operatorname{h}(\rho_i)$, so the right hand side of (3.41) equals

$$\sum_{i=0}^{n} C_\sigma \operatorname{h}(\rho_i) q^{n-i}.$$

On the other hand, since we saw that $L = \prod_{j=1}^{m} \mathbb{F}_{q^{n_j}}$ we have $L^*(k) = L^\times = \prod_{j=1}^{m} \mathbb{F}_{q^{n_j}}^\times$ so $|L^*(k)| = \prod_{j=1}^{m}(q^{n_j} - 1)$. Hence (3.41) says that

$$\prod_{j=1}^{m}(q^{n_j} - 1) = \sum_{i=0}^{n} C_\sigma \operatorname{h}(\rho_i) q^{n-i}.$$

Since $q$ is an arbitrary prime power it follows that

$$\prod_{j=1}^{m}(X^{n_j} - 1) = \sum_{i=0}^{n} C_\sigma \operatorname{h}(\rho_i) X^{n-i}. \tag{3.42}$$

Comparing (3.40) to (3.42) now gives (3.38). □

# Part III:
# Motivic Integration

# 4. A version of motivic integration that specializes to $p$-adic integration via point counting

## 4.1 Introduction

There is a standard theory of geometric motivic integration, developed in [DL99] and [DL02], and in [Seb04] for the case of mixed characteristic. In the present chapter we give a version of that theory, with the property that it specializes to the Haar measure on $\mathbb{Z}_p^d$ in the case when we work over $\mathbb{A}^d_{\mathbb{Z}_p}$. The main difference compared to the standard theory is the value ring of the integral; normally the integral takes values in $\widehat{\mathcal{M}_k}$, i.e., $\mathcal{M}_k := \mathrm{K}_0(\mathsf{Var}_k)[\mathbb{L}^{-1}]$ completed with respect to the dimension filtration. However, for the arithmetic applications we have in mind, this does not work. The reason is that the point counting homomorphism $\mathrm{C}_q\colon \mathcal{M}_k \to \mathbb{Q}$, defined by $X \mapsto |X(\mathbb{F}_q)|$ whenever $k = \mathbb{F}_q$ is finite, is not continuous with respect to the dimension filtration, hence is not defined on $\widehat{\mathcal{M}_k}$. Since $\mathrm{K}_0(\mathsf{Var}_k)$ is the universal additive and multiplicative invariant of $\mathsf{Var}_k$, and $\mathrm{C}_q$ is one of the most fundamental examples of such invariants, it is of general interest to have it defined on the completion. In our case, we need it in order to be able to specialize the motivic integral to the corresponding $p$-adic one. So instead of using the dimension filtration, we let the measure take values in $\mathcal{M}_k$ completed with respect to a stronger topology. This topology is defined for any field $k$, and it has the property that in case $k = \mathbb{F}_q$, the point conting homomorphism is continuous.

The first part of this chapter is hence devoted to defining a topology on $\mathcal{M}_k$, where $k$ is arbitrary, in such a way that $\mathrm{C}_q$ is continuous for every prime power $q$. The construction of this topology is based on a previous construction, by Ekedahl [Eke09], of a topology on $\mathcal{M}_k$ with the property that, in case $k$ is finite, taking the trace of Frobenius on the $l$-adic cohomology is continuous. By the Lefschetz trace formula, this topology then has the property we want, that the point counting homomorphism is continuous. However, it has a drawback: the fact that the class of a variety is small does not imply that the same thing hold for its subvarieties. For example, if $X_n \subset Y_n$ are varieties such that $[Y_n]/\mathbb{L}^n \to 0$ then this does not imply that $[X_n]/\mathbb{L}^n \to 0$. For our purpose, this means that we are not able to make the definitions of geometric motivic integration work using this topology. We instead construct a weaker topology, using the topology of Ekedahl, modified using a partial ordering of $\mathcal{M}_k$ with the property that if $X \subset Y$ then $[X] \leq [Y]$. We use $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ to denote the completion of $\mathcal{M}_k$ with respect to the resulting topology. This topology is fine enough for, in case $k$ is finite, the point counting homomorphism to be continuous, and still coarse enough for the standard definitions of geometric motivic integration to work.

In the second part of the chapter we then go through the constructions of motivic integration using this stronger topology. Let us quickly outline the theory in order to point out the major differences to the classical theory: Let $\mathcal{X}$ be a variety defined over a complete discrete valuation ring, whose residue field is $k$. As in all theories of geometric motivic integration, we begin by constructing the space of arcs on $\mathcal{X}$, denoted $\mathcal{X}_\infty$. There is a Boolean algebra of stable subsets of $\mathcal{X}_\infty$. On this algebra one may define a measure, taking values in $\mathcal{M}_k$. If $\mathcal{X}$ is a smooth variety defined over $\mathbb{Z}_p$ we may specializes to the $p$-adic measure by applying the point counting homomorphism $\mathrm{C}_p\colon \mathcal{M}_{\mathbb{F}_p} \to \mathbb{Q}$, this was noted in [LS03]. Next one constructs a Boolean algebra of measurable subsets of $\mathcal{X}_\infty$, and a measure $\mu_{\mathcal{X}}$ on this algebra. The measure of a general measurable set is defined by covering it by stable sets and using a limiting process, also in the standard way. Since one needs to take limits to define the measure, it has to take values in a completion of $\mathcal{M}_k$. The standard choice is to use $\widehat{\mathcal{M}_k}$, we instead use $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. This allows us to prove that the property of specializing to the $p$-adic measure via point counting holds for general measurable sets.

Let us give some more details about the case of primary interest to us, namely when the discrete valuation ring is $\mathbb{Z}_p$, and $\mathcal{X}$ is an affine space over $\mathbb{Z}_p$: Let $\mathbf{W}(\overline{\mathbb{F}}_p)$ be the Witt vectors with coefficients in an algebraic closure of $\mathbb{F}_p$. Suppose that $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$. Then $\mathcal{X}_\infty$ can be identified with $\mathbf{W}(\overline{\mathbb{F}}_p)^d$. Moreover, for every power of $p$, $q$, we have a homomorphism $\mathrm{C}_q\colon \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p}) \to \mathbb{R}$ induced by counting $\mathbb{F}_q$-points on $\mathbb{F}_p$-varieties. The motivic measure has the property that it specialize to the (normalized) Haar measure, in the sense that for any measurable set $A \subset \mathcal{X}_\infty = \mathbf{W}(\overline{\mathbb{F}}_p)^d$ we have $\mathrm{C}_p\, \mu_{\mathcal{X}}(A) = \mu_{Haar}(A \cap \mathbb{Z}_p^d)$. More generally, $\mathrm{C}_q\, \mu_{\mathcal{X}}(A) = \mu_{Haar}(A \cap \mathbf{W}(\mathbb{F}_q)^d)$ for any power $q$ of $p$. (Recall that $\mathbb{Z}_p \subset \mathbf{W}(\mathbb{F}_{p^n}) \subset \mathbf{W}(\overline{\mathbb{F}}_p)$ is the integers in the unramified degree $n$ extension of $\mathbb{Q}_p$.)

We have a particular application in mind for this theory: Let $q$ be a power of the fixed prime $p$ and consider the set of tuples $(a_1, \ldots, a_n) \in \mathbf{W}(\mathbb{F}_q)^n$ with the property that the polynomial $a_n + a_1 X + \cdots + a_{n-1} X^{n-1} + X^n$ splits completely. The measure of of this set is equal to $I_q/n!$, where

$$I_q = \int_{\mathbf{W}(\mathbb{F}_q)^n} |\prod_{1 \leq i < j \leq n} (X_i - X_j)|_p d\mu_{Haar}.$$

Now, it turns out that there is an $f \in \mathbb{Z}(T)$ with the property that $I_q = f(q)$ for every power $q$ of the fixed prime $p$, and since this is unusual (in general the rational function should vary with $q$) we want a geometric explanation of this phenomenon. Using the theory developed in this chapter, this is achieved in Chapter 5, where we prove that the corresponding motivic integral, $\int_{\mathbf{W}(\overline{\mathbb{F}_p})^n} |\prod_{1\le i<j\le n}(X_i - X_j)| d\mu \in \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p})$ is equal to $f(\mathbb{L})$.

We now give an overview of the chapter: In Section 4.2 we define the completion of $\mathcal{M}_k$ that we work in, $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. A crucial property of this topology is that if $k = \mathbb{F}_p$ then the counting homomorphisms $\mathrm{C}_{p^n} : \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p}) \to \mathbb{R}$ are well defined and continuous.

Section 4.3 is an appendix containing results from [Eke09] which we use at a couple of places in Section 4.2. This section also serves as a motivation for the definitions given in Section 4.2.

In Section 4.4 we define the arc space and the motivic measure. The definitions given here are almost exactly the same as the ones used in the theory of geometric motivic integration, as presented for example in [Loo02] or the appendix of [DL02], and in [Seb04] for the mixed characteristic case. However, since we are using a stronger topology, the verifications that everything is well-defined are different.

It has to be remarked that this theory has some major shortcomings in its present state. Namely, when the variety has singularities, we are not able to prove that general cylinder sets are measurable (in particular, we are not able to prove that the arc space itself is measurable). That said, we are interested in utilizing the theory in the case when $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$, and in this case, and more generally for any smooth variety, everything works well.

The material in this chapter has appeared in the preprint [Rök08b].

## 4.2 The completion of $\mathcal{M}_k$

Let $k$ be a field. We use $\mathcal{M}_k$ to denote $\mathrm{K}_0(\mathsf{Var}_k)[\mathbb{L}^{-1}]$. We use the standard definition of the dimension filtration of $\mathcal{M}_k$, namely for $m \in \mathbb{Z}$ define $\mathrm{F}^m \mathcal{M}_k$ to be the subgroup generated by $[X]/\mathbb{L}^r$, where $\dim X - r \le m$. Define the *dimension* of $x \in \mathcal{M}_k$ to be the minimal $m$ (possibly equal to $-\infty$) such that $x \in \mathrm{F}^m \mathcal{M}_k$. In the theory of geometric motivic integration one completes $\mathcal{M}_k$ with respect to this filtration to obtain $\widehat{\mathcal{M}_k}$. However, we are working from an arithmetic point of view; we want to define a theory of motivic integration for which the value of a motivic integral can be specialized to the corresponding $p$-adic one. For this reason it is natural to demand, in case $k$ is finite, that the counting homomorphism is continuous. That this is not the case in the topology coming from the dimension filtration can be seen from the following simple example: Let $k = \mathbb{F}_q$ and consider the sequence $a_n = q^n/\mathbb{L}^n$. We have $\mathrm{C}_q(a_n) = 1$ for every $n$. However, $a_n \to 0$ as $n \to \infty$, and $\mathrm{C}_q(0) = 0$. Because of this we are forced to complete with respect to a stronger topology. This topology should have the property that its definition is independent of the base field, and that if the base field is finite then the counting homomorphism is continuous.

Let us outline the contents of this section: First we define a notion of weights on $\mathcal{M}_k$. This definition is from [Eke09], where the author uses it to construct a topology on $\mathcal{M}_k$ which has the property that we want, that the counting homomorphism is continuous. However, this topology is a bit too strong when we define the motivic measure, so we have to modify it slightly. For that we introduce a partial ordering on $\mathcal{M}_k$. Next we define our topology, which is stronger than the filtration topology, but weaker than the topology from [Eke09]. We conclude the section by proving some lemmas on the convergence of sums, which will be needed in later sections.

The next section, Section 4.3 is an appendix with some background material from [Eke09], which we use at some places in this section. This appendix also gives a background to some of the definitions given in the present section, which otherwise might seem somewhat unmotivated.

### Weights on $\mathcal{M}_k$

To define the topology we use a notion of weights of elements of $\mathcal{M}_k$, given in [Eke09]. We also refer to [Eke09] for the proof that the following is well-defined. For $X$ a separated $k$-scheme of finite type, we write $\mathrm{H}_c(X)$ for the $\ell$-adic cohomology with compact support, of the extension of $X$ to a separable closure of $k$.

**Definition 4.1.** *We define the notion of weights of elements in $\mathcal{M}_k$.*

- *For a scheme $X$ of finite type over the base field $k$, define for every integer $n$, $\mathrm{w}_n(X) := \sum_i \mathrm{w}_n\big(\mathrm{H}^i_c(X)\big)$, where $\mathrm{w}_n\big(\mathrm{H}^i_c(X)\big)$ is the dimension of the part of $\mathrm{H}^i_c(X)$ of cohomological weight $n$.*
- *For $x \in \mathrm{K}_0(\mathsf{Var}_k)$, let $\mathrm{w}'_n(x)$ be the minimum of $\sum_i |c_i| \mathrm{w}_n(X_i)$, where $\sum_i c_i[X_i]$ runs over all representations of $x$ as a linear combination of classes of schemes.*
- *For $x \in \mathrm{K}_0(\mathsf{Var}_k)$, define $\overline{\mathrm{w}}_n(x) := \lim_{i\to\infty} \mathrm{w}'_{n+2i}(x\mathbb{L}^i)$.*
- *Finally, extend $\overline{\mathrm{w}}_n$ to $\mathcal{M}_k$ by $\overline{\mathrm{w}}_n(x/\mathbb{L}^i) := \overline{\mathrm{w}}_{n+2i}(x)$.*

$\overline{\mathrm{w}}_n$ has the property that all $n$ and all $x \in \mathcal{M}_k$, $\overline{\mathrm{w}}_n(x) \geq \overline{\mathrm{w}}_n(\chi_c(x))$, where $\chi_c$ is the compactly supported Euler characteristic taking values in $\mathrm{K}_0(\mathsf{Coh}_k)$ and $\overline{\mathrm{w}}_n$ is the corresponding weight functions on that ring, see Section 4.3.

**Example 4.2.** *$\mathrm{H}^i_c(\mathbb{A}^1_k)$ is equal to $\mathbb{Q}_l(-1)$, the dual of the cyclotomic representation, if $i = 2$, and zero otherwise. From this one deduces, using*

$\overline{\mathrm{w}}_{2i}(\mathbb{L}^i) \geq \overline{\mathrm{w}}_{2i}(\chi_c(\mathbb{L}^i)) = \overline{\mathrm{w}}_{2i}(\mathbb{Q}_l(-i)) = 1$, *that*

$$\overline{\mathrm{w}}_n(\mathbb{L}^i) = \begin{cases} 1 & \textit{if } n = 2i \\ 0 & \textit{otherwise.} \end{cases}$$

The weight has the properties that it is subadditive, $\overline{\mathrm{w}}_n(x \pm y) \leq \overline{\mathrm{w}}_n(x) + \overline{\mathrm{w}}_n(y)$, and submultiplicative, $\overline{\mathrm{w}}_n(xy) \leq \sum_{i+j=n} \overline{\mathrm{w}}_i(x)\,\overline{\mathrm{w}}_j(y)$.

We next define the concept of uniform polynomial growth, introduced in [Eke09].

**Definition 4.3.** *We say that a sequence of elements of $\mathcal{M}_k$, $(a_i)_{i\in\mathbb{N}}$, is of* uniform polynomial growth *if there exists constants $d$, $C$ and $D$, independent of $i$ and with the property that for every integer $n$, $\overline{\mathrm{w}}_n(a_i) \leq C|n|^d + D$.*

One derives the following lemma from the fact that the the weight functions are subadditive and submultiplicative. The details are in [Eke09].

**Lemma 4.4.** *Being of uniform polynomial growth is closed under termwise addition, subtraction and multiplication: If $(a_i)$ and $(b_i)$ are of uniform polynomial growth, then so are $(a_i \pm b_i)$ and $(a_i b_i)$.*

## The partial ordering of $\mathcal{M}_k$

Before we can define our topology we also need a partial ordering on $\mathcal{M}_k$.

**Definition 4.5.** *We introduce an ordering on $\mathcal{M}_k$ in the following way: First on $\mathrm{K}_0(\mathsf{Var}_k)$ we define $x \leq y$ if there exists varieties $V_i$ such that $x + \sum[V_i] = y$. (Equivalently there exists a variety $V$ such that $x + [V] = y$.) We extend it to $\mathcal{M}_k$ by $x/\mathbb{L}^i \leq y/\mathbb{L}^j$ if there exists an $n$ such that $x\mathbb{L}^{j+n} \leq y\mathbb{L}^{i+n}$. (We see that $\mathbb{L}^{-1} > 0$.)*

In particular, if $x$ is a linear combination of non-empty varieties with positive coefficients, then $x > 0$.

**Lemma 4.6.** *The ordering given above is a partial ordering of $\mathcal{M}_k$.*

*Proof.* We first show that this is a partial ordering of $\mathrm{K}_0(\mathsf{Var}_k)$. The only nontrivial thing to prove is antisymmetry. Suppose first that $x \leq y$ and $y \leq x$, where $x, y \in \mathrm{K}_0(\mathsf{Var}_k)$. Then $x = y + \sum_i [U_i]$ and $y = x + \sum_i [V_i]$, giving together $\sum_i [V_i] + \sum_j [U_j] = 0$. Lemma 4.22 now shows that $[V_i] = [U_i] = 0$ for every $i$.

Next, the ordering is well-defined on $\mathcal{M}_k$, because if $x/\mathbb{L}^i = y/\mathbb{L}^j \in \mathcal{M}_k$, where $x, y \in \mathrm{K}_0(\mathsf{Var}_k)$, then $x\mathbb{L}^{j+n} = y\mathbb{L}^{i+n}$ for some $n$, consequently $x/\mathbb{L}^i \leq y/\mathbb{L}^j$. We finally show that we have antisymmetry also on $\mathcal{M}_k$: If $x/\mathbb{L}^i \leq y/\mathbb{L}^j$ and $x/\mathbb{L}^i \geq y/\mathbb{L}^j$ in $\mathcal{M}_k$ then for some n,

$\mathbb{L}^{j+n}x \leq \mathbb{L}^{i+n}y$ and $\mathbb{L}^{j+n}x \geq \mathbb{L}^{i+n}y$ in $\mathrm{K}_0(\mathsf{Var}_k)$. It follows from the first part that $\mathbb{L}^{j+n}x = \mathbb{L}^{i+n}y \in \mathrm{K}_0(\mathsf{Var}_k)$, hence that $x/\mathbb{L}^i = y/\mathbb{L}^j \in \mathcal{M}_k$. □

Given a variety $X$, every constructible subset of $U \subset X$ can be written as a finite disjoint union of locally closed subsets, $U = \bigcup_i U_i$. Since such a subset has a unique structure of a reduced subscheme we can take its class $[U_i] \in \mathrm{K}_0(\mathsf{Var}_k)$, hence we may define the class of $U$ as $[U] = \sum_i [U_i] \in \mathrm{K}_0(\mathsf{Var}_k)$ (this is independent of the chosen partition, hence well defined).

**Lemma 4.7.** *If $U$ and $V$ are constructible subsets of a variety, such that $V \subset U$, then $[V] \leq [U]$. Moreover, if $V \subset \bigcup_{i=1}^n U_i$ then $[V] \leq \sum_{i=1}^n [U_i]$.*

*Proof.* For the first part, if $U \subset V$ are constructible, then so is $U \setminus V$; consequently $[U] - [V] = [U \setminus V] \geq 0$.

For the second part, when $n = 2$ we have $[V] \leq [U_1 \cup U_2] = [U_1] + [U_2] - [U_1 \cap U_2] \leq [U_1] + [U_2]$. The general statement follows by induction on $n$. □

**Lemma 4.8.** *Let $x, a, b \in \mathcal{M}_k$. If $a \leq x \leq b$ then*

$$\dim x \leq \max\{\dim a, \dim b\}.$$

*Proof.* There are varieties $X$ and $Y$ such that $x = a + [X]$ and $b = x + [Y]$. This shows that $b - a = [X] + [Y]$. By Lemma 4.22, $\dim[X] \leq \dim(b - a)$. Hence $\dim x = \dim(a + [X]) \leq \max\{\dim a, \dim[X]\} \leq \max\{\dim a, \dim(b - a)\} \leq \max\{\dim a, \dim b\}$. □

## The topology on $\mathcal{M}_k$

We now define our topology on $\mathcal{M}_k$, by specifying what it means for a sequence to converge. Since we work in a group it suffices to tell what it means for a sequence to converge to zero.

**Definition 4.9.** *Let $(x_i)$ be a sequence of elements in $\mathcal{M}_k$.*

- *We say that $x_i$ is* strongly convergent to 0 *if it is of uniform polynomial growth and* $\dim x_i \to -\infty$.
- *$x_i$ converges to zero, $x_i \to 0$, if there are sequences $a_i$ and $b_i$ that converges to zero strongly, and such that $a_i \leq x_i \leq b_i$.*
- *$(x_i)$ is* Cauchy *if $x_i - x_j \to 0$ when $i, j \to \infty$.*

*We define a topology on $\mathcal{M}_k$ by stipulating that a subset is closed if it contains all its limit points with respect to this notion of convergence.*

One proves that this is a topological ring. It then follows immediately that if a sequence of elements in $\mathcal{M}_k$ is convergent then it is Cauchy. Moreover, we have the property that if $x_i \leq y_i \leq z_i$ and if $x_i$ and $z_i$

tends to zero, then $y_i$ tends to zero, a fact that we will use without further notice.

We compare this to the standard topology on $\mathcal{M}_k$:

**Lemma 4.10.** *If a sequence is convergent (respectively Cauchy), then it is convergent (respectively Cauchy) with respect to the dimension filtration. In particular, our topology is stronger than the dimension topology.*

*Proof.* Let the sequence be $x_i$ be convergent to zero. There are sequences $a_i$ and $b_i$ strongly convergent to zero and such that $a_i \leq x_i \leq b_i$. By Lemma 4.8, $\dim x_i \leq \max\{\dim a_i, \dim b_i\}$ and it follows that $\dim x_i \to -\infty$. □

In particular, if we know that a sequence $x_n$ is convergent then $x_n \to 0$ if and only if $\dim x_n \to -\infty$.

We are now ready to give the completion that let the motivic integral specialize to the corresponding $p$-adic integral. We define $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$, the completion of $\mathcal{M}_k$, in the following way: Consider the set of Cauchy sequences in $\mathcal{M}_k$. This is a ring under termwise addition and multiplication, which one proves using Lemma 4.4. Moreover it has a subset consisting of those sequences that converge to zero, and it is straight forward to prove that this subset is an ideal. We define $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ to be the quotient by this ideal. Moreover we have a completion map $\mathcal{M}_k \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ that takes $x$ to the image of the constant sequence $(x)$. We state this as a formal definition:

**Definition 4.11.** $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ *is the ring of Cauchy sequences modulo the ideal of those sequences that converges to zero.*

For any sequence $(x_i)$ of elements of $\mathcal{M}_k$ which is Cauchy with respect to dimension, $\dim x_i$ is eventually constant, or it converges to $-\infty$. Also, given $n$, $\overline{\mathrm{w}}_n(x_i)$ is eventually constant. If $x = (x_i) \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$, define $\dim x$ and $\overline{\mathrm{w}}_n(x)$ to be these constant values. These functions keep there basic properties, e.g., subadditivity for $\overline{\mathrm{w}}_n$, see [Eke09]. We may then extend the concept of being of uniform polynomial growth to $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$.

**Definition-Lemma 4.12.** *For $x, y \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ we let $x \leq y$ if for some Cauchy sequences $(x_i)$ and $(y_i)$, whose images in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ are $x$ and $y$ respectively, we have $x_i \leq y_i \in \mathcal{M}_k$ for every $i$. This is a partial ordering of $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$.*

*Proof.* It is clear that $x \leq x$. Moreover, if $x \leq y$ and $y \leq z$, let $(x_i), (y_i)$ and $(y_i'), (z_i)$ be representatives of $x, y$ and $y, z$ respectively, such that $x_i \leq y_i$ and $y_i' \leq z_i$ for all $i$. Then $x_i \leq z_i + y_i - y_i'$ for all $i$. Since $(y_i - y_i')_{i \in \mathbb{N}} = 0 \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ it follows that $x \leq z$. Finally, for antisymmetry, it suffices to show that if $x \leq 0$ and $0 \leq x$ then $x = 0$. In that case, there exists a Cauchy sequence $(x_i)$ whose image in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ is $x$, and another one, $(z_i)$ whose image in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ is zero, such that $0 \leq x_i$ and $x_i + z_i \leq 0$. It follows that $0 \leq x_i \leq -z_i$ for every $i$, consequently $x_i \to 0$, i.e., $x = 0$. □

We define the topology on $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ in the same way as on $\mathcal{M}_k$, by saying that a sequence converges to zero if and only if it is bounded from above and below by sequences strongly convergent to zero. The following result justifies the fact that we refer to $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ as the completion of $\mathcal{M}_k$.

**Lemma 4.13.** *The completion map $\mathcal{M}_k \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ is continuous and $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ is complete, in the sense that any Cauchy sequence converges. Moreover, the image of $\mathcal{M}_k$ is dense in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$.*

*Proof.* We prove that for any $x = (x_j)_{j \in \mathbb{N}} \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ we have $x_i \to x$ as $i \to \infty$. This will show that any Cauchy sequence of elements in $\mathcal{M}_k$ converges and that $\mathcal{M}_k$ is dense in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. For this we have to prove that $y_i := x_i - x = (x_i - x_j)_{j \in \mathbb{N}} \to 0$ as $i \to \infty$.

There are two sequences $a_{ij}$ and $b_{ij}$ which are strongly convergent to zero as $i, j \to \infty$, and such that $a_{ij} \leq x_i - x_j \leq b_{ij}$. Hence $(a_{ij})_{j \in \mathbb{N}} \leq y_i \leq (b_{ij})_{j \in \mathbb{N}}$. We prove that $(a_{ij})_j$ tends strongly to zero as $i \to \infty$: The dimension of $(a_{ij})_j$ is $\dim a_{i,f(i)}$, where $f(i)$ is some sufficiently large integer which we may and will assume is greater than $i$. Moreover, $\overline{\mathrm{w}}_n((a_{ij})_j) = \overline{\mathrm{w}}_n(a_{i,g(i)})$ for some $g(i)$. Since $a_{ij}$ is strongly convergent, $\dim a_{i,f(i)} \to -\infty$ as $i \to \infty$, and $\overline{\mathrm{w}}_n(a_{i,g(i)}) \leq C|n|^d + D$ for every $i$. □

Let $\widehat{\mathcal{M}_k}$ be the completion of $\mathcal{M}_k$ with respect to the dimension filtration. We have an injective, continuous homomorphism $\overline{\mathrm{K}_0}(\mathsf{Var}_k) \to \widehat{\mathcal{M}_k}$ so we may think of $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ as a subring of $\widehat{\mathcal{M}_k}$, although its topology is stronger than the subspace topology.

**Example 4.14.** *By Example 4.2 and subadditivity, $\overline{\mathrm{w}}_n(\sum_{m=i}^{j} \mathbb{L}^{-m}) \leq 1$ for every $i, j$. Furthermore $\dim \mathbb{L}^{-m} \to -\infty$ as $m \to \infty$. Together this show that the sequence $(\sum_{m=0}^{N} \mathbb{L}^{-m})_N$ is Cauchy, hence that the sum $\sum_{m \in \mathbb{N}} \mathbb{L}^{-m}$ is convergent in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. In the same way one sees that $\mathbb{L}^{-m}$ converges (strongly) to zero, hence letting $N$ tend to infinity in the equality $(1 - \mathbb{L}^{-1})(\sum_{m=0}^{N} \mathbb{L}^{-m}) = 1 - \mathbb{L}^{-(N+1)}$ shows that $\sum_{m \in \mathbb{N}} \mathbb{L}^{-m} = 1/(1 - \mathbb{L}^{-1}) \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$. In particular, since also $\mathbb{L}$ is invertible, it follows that $\mathbb{L} - 1$ is invertible.*

*Similarly one proves that if $\{e_i\}_{i \in \mathbb{N}}$ is a sequence of integers such that $e_i \to \infty$, then $\sum_{i \in \mathbb{N}} \mathbb{L}^{-e_i}$ is convergent.*

On the other hand, note that for example $1 - 3\mathbb{L}^{-1}$ is not invertible (contrary to with respect to the dimension filtration). For its formal inverse is $x = \sum_{n \geq 0} 3^n \mathbb{L}^{-n}$, and $\overline{\mathrm{w}}_{2n}(x) = 3^{-n}$, showing that $x$ is not of uniform polynomial growth. In view of Lemma 4.15, this would also follow from the fact that $\mathrm{C}_2\, x$ is not convergent (note however that $\mathrm{C}_p\, x$ is convergent for $p > 3$).

When $k$ is finite, this topology makes the point counting homomorphism continuous:

**Definition-Lemma 4.15.** *For $k = \mathbb{F}_q$, define $\mathrm{C}_q\colon \mathrm{K}_0(\mathsf{Var}_k) \to \mathbb{Z}$ by $[X] \mapsto |X(k)|$. It is a ring homomorphism and it extends to a homomorphism $\mathcal{M}_k \to \mathbb{Q}$, continuous with respect to the above constructed topology. It hence extends by continuity to a continuous homomorphism $\mathrm{C}_q\colon \overline{\mathrm{K}_0}(\mathsf{Var}_k) \to \mathbb{R}$.*

*Proof.* In [Eke09] it is proved that $\mathrm{C}_q$ is continuous with respect to the topology of $\overline{\mathrm{K}_0}^{pol}(\mathsf{Var}_k)$, i.e., if $a_n \to 0$ strongly then $\mathrm{C}_q\, a_n \to 0$. Now, if $x_n \to 0$ then there are sequences $a_n$ and $b_n$ such that $a_n \leq x_n \leq b_n$ and $a_n, b_n \to 0$ strongly. Since $\mathrm{C}_q\, a_n \leq \mathrm{C}_q\, x_n \leq \mathrm{C}_q\, b_n$ it follows that $\mathrm{C}_q\, x_n \to 0$, consequently $\mathrm{C}_q$ is continuous. □

## Various lemmas on the convergence of series

We collect here some lemmas that will be needed when working with this definition.

**Lemma 4.16.** *Suppose that $x_i \leq y_i \leq z_i$. If $\sum_{i\in\mathbb{N}} x_i$ and $\sum_{i\in\mathbb{N}} z_i$ are convergent, then so is $\sum_{i\in\mathbb{N}} y_i$. Moreover, $\sum_{i\in\mathbb{N}} x_i \leq \sum_{i\in\mathbb{N}} y_i \leq \sum_{i\in\mathbb{N}} z_i$.*

*Proof.* We have $\sum_{i=m}^n x_i \leq \sum_{i=m}^n y_i \leq \sum_{i=m}^n z_i$, and since $\sum_{i=m}^n x_i \to 0$ and $\sum_{i=m}^n z_i \to 0$ as $m, n \to \infty$, it follows that $\sum_{i=m}^n y_i$ is bounded from above and below by sequences strongly convergent to zero, hence it converges to zero. So $(\sum_{i=0}^N y_i)_N$ is Cauchy, the sum is convergent. The second assertion follows by definition, since it holds for each partial sum. □

Note that if $x_i \leq y_i \leq z_i$ it does not follow that $y_i$ is convergent when $x_i$ and $z_i$ are.

Let $a_i \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$. In general it is not true that $\sum a_i$ is convergent if and only if $a_i \to 0$, a property that holds for $\widehat{\mathcal{M}_k}$. However, some of the consequences of this are true in a special case:

**Lemma 4.17.** *If $a_i \geq 0$ for every $i$ and if $\sum_i a_i$ is convergent then every rearrangement of the sum is convergent, and to the same limit.*

*Proof.* If $(b_i)_{i\in\mathbb{N}}$ is a rearrangement of $(a_i)_{i\in\mathbb{N}}$ then, for some $N_n$, the elements $a_0, \ldots, a_n$ are among $b_0, \ldots, b_{N_n}$. Therefore $\sum_{i\leq N_n}(b_i - a_i)$ is an alternating sum of $a_j$, with $j > n$. Since every $a_i \geq 0$, this sum is between $-\sum_{j=n+1}^M a_i$ and $\sum_{j=n+1}^M a_i$ for some $M$. Since $\sum_{i\in\mathbb{N}} a_i$ is Cauchy, both these sums tend to 0 as $n$ tends to $\infty$. Hence $\sum_{i\in\mathbb{N}}(b_i - a_i) = 0$. □

For this reason, we will write $\sum_{i,j\in\mathbb{N}} a_{ij}$ to mean the sum over some unspecified enumeration of $\mathbb{N}^2$.

**Lemma 4.18.** *Assume that all $a_{ij} \geq 0$. If the sum $\sum_{i,j\in\mathbb{N}} a_{ij}$ is convergent then the same holds for $\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{ij}$, and the two sums are equal.*

*Proof.* For every $i \in \mathbb{N}$, it follows from Lemma 4.16, and the convergence of $\sum_{i,j\in\mathbb{N}} a_{ij}$, that $\sum_{j\in\mathbb{N}} a_{ij}$ is convergent. We then have

$$\begin{aligned}\sum_{i\leq n}\sum_{j\in\mathbb{N}} a_{ij} - \sum_{i,j\leq n} a_{ij} &= \sum_{i\leq n}\sum_{j>n} a_{ij}\\ &= \sum_{j>n}\sum_{i\leq n} a_{ij},\end{aligned}$$

Now $\sum_{j\in\mathbb{N}}(\sum_{i=0}^{j} a_{ij})$ is convergent, because of Lemma 4.16 and since every rearrangement of $\sum_{i,j\in\mathbb{N}} a_{ij}$ is. Hence $\lim_{n\to\infty}\sum_{j>n}(\sum_{i\leq n} a_{ij}) = 0$ (because if $\sum_{i\in\mathbb{N}} b_i = b$ then $\sum_{i>n} b_i = b - \sum_{i\leq n} b_i \to 0$). The result follows. □

**Lemma 4.19.** *Suppose that $a_i \geq 0$, $b_i \geq 0$. The sum $\sum_{n\in\mathbb{N}}\sum_{i+j=n} a_i b_j$ is convergent if and only if $\sum_{i\in\mathbb{N}} a_i \sum_{i\in\mathbb{N}} b_i$ is. In this case they are equal.*

*Proof.*

$$\left(\sum_{0\leq i\leq N} a_i\right)\left(\sum_{0\leq i\leq N} b_i\right) - \sum_{0\leq n<N}\sum_{i+j=n} a_i b_j = \sum_{I(N)} a_i b_j,$$

where $I(N)$ is a set of indices $(i,j)$, all of which fulfill $i+j \geq N$. Now, since every $a_i$ and $b_i$ is $\geq 0$, it follows that $0 \leq \sum_{I(N)} a_i b_j \leq \sum_{n\geq N}\sum_{i+j=n} a_i b_j$ for every $N$. Since $\sum_{n\in\mathbb{N}}(\sum_{i+j=n} a_i b_j)$ is Cauchy it follows that $\sum_{n\geq N}\sum_{i+j=n} a_i b_j \to 0$ as $N\to\infty$, hence $\sum_{I(N)} a_i b_j \to 0$. □

**Lemma 4.20.** *If $a_i \geq 0$, $b_i \geq 0$, and if $\sum_{i\in\mathbb{N}} a_i$ and $\sum_{i\in\mathbb{N}} b_i$ are convergent, then $\sum_{i\in\mathbb{N}} a_i b_i$ is convergent and $\leq \sum_{i\in\mathbb{N}} a_i \sum_{i\in\mathbb{N}} b_i$.*

*Proof.* Lemma 4.19 shows that $\sum_{n\in\mathbb{N}}(\sum_{i+j=n} a_i b_j)$ is convergent. Since $a_n b_n \leq \sum_{i+j=2n} a_i b_j$, and $0 \leq \sum_{i+j=2n+1} a_i b_j$ it hence follows from Lemma 4.16 that $\sum_{i\in\mathbb{N}} a_i b_i$ is convergent, and less than or equal to $\sum_{n\in\mathbb{N}}(\sum_{i+j=n} a_i b_j)$. □

## 4.3 Appendix to Section 4.2: Results from [Eke09]

In this section we give a quick review of the construction of the topology from [Eke09]. First, recall that we have an Euler characteristic map from $\mathcal{M}_k$, taking values in $\mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_l)$. The first step is to modify this map, by letting the recipient ring be $\mathrm{K}_0(\mathsf{Coh}_k)$, which is defined as follows: Firstly, for $k$ finitely generated, we use $\mathsf{Coh}_k$ to denote the category of mixed Galois $\mathbb{Q}_l$-representations. This is the full subcategory of $\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_l$ consisting of those representations that have a weight filtration. We then have an injection $\mathrm{K}_0(\mathsf{Coh}_k) \hookrightarrow \mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_l)$, the image being generated by modules of pure cohomological weight. (When $k = \mathbb{F}_q$ a representation is of pure weight $n$ if all the archimedean absolute values of all eigenvalues of the geometric Frobenius are in $\overline{\mathbb{Q}}$ and have absolute value $q^{n/2}$.) The image of $\chi_c$ is contained in this subring, i.e., we have $\chi_c\colon \mathcal{M}_k \to \mathrm{K}_0(\mathsf{Coh}_k)$. We refer to [Eke09] for the definition of this ring in case $k$ is not finitely generated, but let us note that if $\{k_\alpha\}$ is the collection of finitely generated subfields of $k$ then we have an isomorphism $\lim_{\to\alpha} \mathrm{K}_0(\mathsf{Coh}_{k_\alpha}) \to \mathrm{K}_0(\mathsf{Coh}_k)$, and that if $X$ is a $k$-scheme of finite type then it is defined over some finitely generated subfield $k_{\alpha_0}$, hence $\chi_c(X)$ is defined in $\mathrm{K}_0(\mathsf{Coh}_{k_{\alpha_0}})$ which maps to $\mathrm{K}_0(\mathsf{Coh}_k)$.

The reason for us to work in $\mathrm{K}_0(\mathsf{Coh}_k)$ rather than in $\mathrm{K}_0(\mathsf{Rep}_{\mathcal{G}}\mathbb{Q}_l)$ is that it is graded by weight: If $V$ is any mixed representation then there is a Jordan-Hölder sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$. Here $V_{i+1}/V_i$ is simple, hence of pure weight. Since $[V] = \sum_i [V_{i+1}/V_i]$, and since this sum is independent of the chosen sequence, it follows that any $v \in \mathrm{K}_0(\mathsf{Coh}_k)$ may be written uniquely as

$$v = \sum_n v_n, \text{ where } v_n = \sum_i c_{ni}[V_{ni}] \tag{4.21}$$

and the $V_{ni}$ are irreducible, pairwise non-isomorphic representations of pure weight $n$. Since this also respects the multiplication, it follows that $\mathrm{K}_0(\mathsf{Coh}_k)$ is graded by weight. When working with this grading, the following result of Deligne is useful: For any separated scheme of finite type, $\mathrm{H}^i_c(X)$ is of mixed weight $\leq i$, and if $i > 2\dim X$ then $\mathrm{H}^i_c(X) = 0$. Furthermore, if $\dim X = n$ then $\mathrm{H}^{2n}_c(X)$ is of pure weight $2n$, and its dimension equals the number of geometric components of $X$ of dimension $n$. Using it, one sees that the Euler characteristic $\chi_c\colon \mathcal{M}_k \to \mathrm{K}_0(\mathsf{Coh}_k)$ is continuous with respect to the dimension filtration on $\mathcal{M}_k$ and the weight grading on $\mathrm{K}_0(\mathsf{Coh}_k)$ (where a sequence tends to zero if the degree of its highest nonzero part tends to $-\infty$). For suppose that $\dim x_i = d$, where $x_i \in \mathcal{M}_k$. Then $\chi_c(x_i)$ is zero in degree $> 2d$. Hence, if $x_i \to 0$, then $\chi_c(x_i) \to 0$. Therefore $\chi_c$ extends by continuity to a continuous homomorphism $\chi_c\colon \widehat{\mathcal{M}_k} \to \mathrm{K}_0\widehat{(\mathsf{Coh}_k)}$. Since the topology on $\mathrm{K}_0(\mathsf{Coh}_k)$ comes from a grading, its completion is very nice; elements can be represented uniquely as sums, infinite in the negative direction:

$$\mathrm{K}_0\widehat{(\mathsf{Coh}_k)} = \Big\{ \sum_{n \leq N} v_n : \text{infinite sums} \Big\}.$$

We use this grading to prove the following structure result about $\mathcal{M}_k$, which we need in the proof of Lemma 4.6.

**Lemma 4.22.** *Let $n_i$ be non-negative integers and let $X_i$ be $k$-varieties. Let $x = \sum_i n_i[X_i] \in \mathcal{M}_k$. If $x = 0$ then $n_i = 0$ for every $i$. And if $x \in \mathrm{F}^m \mathcal{M}_k$, then $[X_i] \in \mathrm{F}^m \mathcal{M}_k$ for every $i$.*

*Proof.* The first part follows from [Nic08], Corollary 2.11. The second part can be proved similarly, alternatively, we give a proof here. We use the above mentioned results of Deligne. Let $m'$ be the maximal dimension of the $X_i$ and suppose that $m' > m$. Since the dimension of $x$ is $m$, the weight $2m'$ part of $\chi_c(x)$ is 0. On the other hand

$$\chi_c(x) = \sum_{\dim X_i = m'} n_i[\mathrm{H}_c^{2m'}(X_i)] + \big(\text{terms of weight} < 2m'\big),$$

and the weight $2m'$-part of this is non-zero, a contradiction. □

This now makes it transparent why the point counting homomorphism is not continuous. For, as was noted in Section 3.3, when $k = \mathbb{F}_q$ we have, by the Lefschetz trace formula, a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_k & \xrightarrow{\chi_c} & \mathrm{K}_0(\mathsf{Coh}_k) \\ & \searrow^{\mathrm{C}_q} & \downarrow \mathrm{C}_\mathrm{F} \\ & & \overline{\mathbb{Q}} \end{array} \tag{4.23}$$

where $\mathrm{C}_\mathrm{F}$ is defined by taking the trace of Frobenius, i.e., by mapping the representation $V$ to the character of $V$ evaluated at the geometric Frobenius automorphism F. We then first need to complete $\mathrm{K}_0(\mathsf{Coh}_k)$ in such a way that it is possible to extend $\mathrm{C}_\mathrm{F}$ to the completion in case $k$ is finite, and for this we need a stronger topology on $\mathrm{K}_0(\mathsf{Coh}_k)$ than the above mentioned. For define, for $v = \sum_n v_n$ as in (4.21),

$$\overline{\mathrm{w}}_n(v) := \sum_i |c_{ni}| \dim V_{ni}.$$

Then, if $k = \mathbb{F}_q$, the only possible extension of $\mathrm{C}_\mathrm{F}$ to $\mathrm{K}_0(\mathsf{Coh}_k)$ would be by continuity, i.e., to define $\mathrm{C}_\mathrm{F}\, v = \sum_n \mathrm{C}_\mathrm{F}\, v_n$. However, $\sum_n |\mathrm{C}_\mathrm{F}\, v_n| \approx \sum_n q^{n/2}\, \overline{\mathrm{w}}_n(v)$. So in case $\overline{\mathrm{w}}_n$ grows too fast, $\mathrm{C}_\mathrm{F}\, v$ is not convergent. (For an example, let $v = \sum_{0 \le n} q^n[\mathbb{Q}_l(n)]$. This is convergent with respect to the grading, but $\mathrm{C}_\mathrm{F}\, q^n[\mathbb{Q}_l(n)] = 1$ for all $n$.) We therefore need a stronger topology on $\mathrm{K}_0(\mathsf{Coh}_k)$:

Let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathrm{K}_0(\mathsf{Coh}_k)$. We say that the sequence is of *uniform polynomial growth* is there exist constants $D$ and $d$ such that for every $i$ and $n$ we have

$$\overline{\mathrm{w}}_n(v_i) \le |n|^d + D. \tag{4.24}$$

We now define the sequence to be convergent if it is of uniform polynomial growth, and convergent with respect to the weight filtration. This topology does not come from a filtration anymore, neither from a metric. However, we may define the completion as the ring of all Cauchy sequences (meaning that $v_i - v_j \to 0$ as $i, j \to \infty$) modulo those converging to zero. It has the properties of a completion: $\mathrm{K}_0(\mathsf{Coh}_k)$ is dense in it, and every Cauchy sequence is convergent. Denote this completion by $\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k)$. It has a very simple description as

$$\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k) = \left\{ v = \sum_{n \leq N} v_n \,:\, v \text{ of uniform polynomial growth} \right\}.$$

If $k = \mathbb{F}_q$ we may now define $\mathrm{C}_\mathrm{F}$, the trace of Frobenius, on $\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k)$. For if $v = \sum_{n \leq N} v_n \in \overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k)$ then

$$|\mathrm{C}_\mathrm{F}\, v| \leq \sum_{n \leq N} |\mathrm{C}_\mathrm{F}\, v_n| \leq \sum_{n \leq N} q^{n/2}\, \overline{\mathrm{w}}_n(v) \leq \sum_{n \leq N} q^{n/2}(|n|^d + D)$$

which is convergent. Similarly one proves that $\mathrm{C}_\mathrm{F}$ is continuous.

If we want to use (4.23) to make $\mathrm{C}_q$ continuous, we need to introduce a stronger topology on $\mathcal{M}_k$ as well, for $\chi_c \colon \mathcal{M}_k \to \overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k)$ is not continuous with respect to the dimension filtration. In [Eke09], this is resolved by introducing weight functions $\overline{\mathrm{w}}_n$ also on $\mathcal{M}_k$, see Section 4.2. These are compatible with the weight functions on $\mathrm{K}_0(\mathsf{Rep}_\mathcal{G}\mathbb{Q}_l)$ in that $\overline{\mathrm{w}}_n(x) \geq \overline{\mathrm{w}}_n(\chi_c(x))$. One then defines a sequence in $\mathcal{M}_k$ to converge to zero if it is of uniform polynomial growth with respect to this notion of weight, and converges to zero with respect to the dimension filtration. We write $\overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Var}_k)$ to denote $\mathcal{M}_k$ completed with respect to this topology. Now, for $k = \mathbb{F}_q$, we get a commutative diagram of continuous maps, where the extension of $\mathrm{C}_q$ is defined by continuity, or equivalently as just the composition $\mathrm{C}_\mathrm{F} \circ \chi_c$:

$$\begin{array}{ccc} \overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Var}_k) & \xrightarrow{\chi_c} & \overline{\mathrm{K}_0}^{\mathrm{pol}}(\mathsf{Coh}_k) \\ & \searrow^{\mathrm{C}_q} & \downarrow{\scriptstyle \mathrm{C}_\mathrm{F}} \\ & & \mathbb{C} \end{array}$$

However, this topology is too strong for our purpose of motivic integration; in Section 4.2 we therefore define a slightly coarser topology which fits out purpose, but which has the disadvantage that we cannot tell whether the Euler characteristic is still continuous.

## 4.4 Definition of the motivic measure

Before we start, let us remark that all the definitions in this section are the standard ones, used in the appendix of [DL02], in [Loo02] and in [Seb04]. (There are some minor differences between these expositions; in these cases we have followed the path of [Loo02]. In particular, our arc spaces will be sets, rather than schemes.) However, since we use a stronger topology, the proofs that everything works is slightly different, even though they are based on the corresponding existing proofs. In the end of the section we also prove that we now have the property that we want, that we may specialize to $p$-adic integration via point counting.

Given a complete discrete valuation ring $\mathcal{O}$, which we assume to be absolutely unramified in the mixed characteristic case, with perfect residue field $k$. Let $\mathcal{X}$ be a scheme over $\mathcal{O}$, of finite type and of pure relative dimension $d$, and define $\mathcal{X}_n$ to be the its $n$th Greenberg scheme over $k$. In the mixed characteristic case, $\mathcal{X}_n$ is characterized by the property that if $R$ is a $k$-algebra then $\mathcal{X}_n(R) = \mathcal{X}\big(\mathbf{W}_n(R)\big)$, where $\mathbf{W}$ is the Witt vectors (see Section 5.2 for the an introduction). We then have projection maps, $\pi_n^{n+1}\colon \mathcal{X}_{n+1} \to \mathcal{X}_n$, defined on $R$-points by the projection $\mathbf{W}_{n+1}(R) \to \mathbf{W}_n(R)$. Next, in the equal characteristic case, $\mathcal{X}_n$ is characterized by the property that if $R$ is a $k$-algebra then $\mathcal{X}_n(R) = \mathcal{X}\big(R[T]/(T^n)\big)$. The projections $\pi_n^{n+1}$ are now defined by $R[T]/(T^{n+1}) \to R[T]/(T^n)$. Moreover, in both the equal and mixed characteristic case, we see that $\mathcal{X}_1$ is the base extension of $\mathcal{X}$ to $k$, $\mathcal{X}_1 = \mathcal{X} \times_{\mathcal{O}} \operatorname{Spec} k$.

Fix an algebraic closure $\overline{k}$ of $k$ and define $\mathcal{X}_\infty$ to be the projective limit of the sets $\mathcal{X}_n(\overline{k})$. Let $\pi_n\colon \mathcal{X}_\infty \to \mathcal{X}_n(\overline{k})$ be the projection maps. We are going to define the measure of certain subsets of $\mathcal{X}_\infty$.

We say that a subset $S \subset \mathcal{X}_n(\overline{k})$ is *constructible* if there are a finite number of locally closed subschemes $V_i \subset \mathcal{X}_n$ such that $S = \bigcup_i V_i(\overline{k})$.

If $A \subset \mathcal{X}_\infty$, then it is a *cylinder* if the following holds: For some $n$, $\pi_n(A)$ is constructible in $\mathcal{X}_n(\overline{k})$, and defined over $k$, and $A = \pi_n^{-1}\pi_n(A)$. In this case there are mutually disjoint subschemes $V_i \subset \mathcal{X}_n$ such that $\pi_n(A) = \bigcup_i V_i(\overline{k})$; we define $[\pi_n(A)] := \sum_i [V_i] \in \mathrm{K}_0(\mathsf{Var}_k)$. It follows that $[\pi_n(A)] \geq 0$.

If, in addition, for all $m \geq n$, the projection $\pi_{m+1}(A) \to \pi_m(A)$ is a piecewise trivial fibration, with fiber an affine space of dimension $d$, then we say that the cylinder $A$ is *stable of level* $n$. A cylinder is *stable* if it is stable of some level $n$. It is a fact that every cylinder is stable in case $\mathcal{X}$ is smooth.

If $A$ is stable of level $n$ we see that $\dim \pi_m(A) - md$ is independent of the choice of $m \geq n$, define the *dimension* of $A$ to be this number.

We also see that $\tilde{\mu}_{\mathcal{X}}(A) := [\pi_m(A)]\mathbb{L}^{-md} \in \mathcal{M}_k$ is independent of the choice of $m \geq n$. This is an additive measure on the set of stable subsets. It is immediate that $\tilde{\mu}_{\mathcal{X}}(A) \geq 0$.

We want to define a measure on a bigger collection of subsets of $\mathcal{X}_\infty$. We will use the standard construction, except that we let the measure take values in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ instead of $\widehat{\mathcal{M}_k}$. So let $\mu_{\mathcal{X}}(A)$, the measure of $A$, be the image of $\tilde{\mu}_{\mathcal{X}}(A)$ in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. Note that $\dim \mu_{\mathcal{X}}(A) = \dim A$.

**Example 4.25.** *The case that we are particularly interested in is when $\mathcal{O} = \mathbb{Z}_p$ (so $k = \mathbb{F}_p$) and $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$. Then*

$$\mathcal{X}_\infty = \varprojlim \mathbb{A}^d_{\mathbb{Z}_p}\big(\mathbf{W}_n(\overline{k})\big) = \mathbf{W}(\overline{k})^d.$$

*By the standard construction of the (normalized) Haar measure on $\mathbb{Z}_p^d$ we see that if $A \subset \mathcal{X}_\infty$ is stable then $\mu_{Haar}(A \cap \mathbb{Z}_p^d) = \mathrm{C}_p(\mu_{\mathcal{X}}(A))$ (*cf. *[LS03], Lemma 4.6.2). Below we define a concept of measurability of subsets of $\mathcal{X}_\infty$, for general $\mathcal{X}$, in such a way that in the special case when $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$, if $A \subset \mathcal{X}_\infty$ is measurable then $\mu_{Haar}(A \cap \mathbb{Z}_p^d) = \mathrm{C}_p(\mu_{\mathcal{X}}(A))$.*

We next extend the motivic measure to a bigger collection of subsets: We want to do this in a way similar to that of the ordinary Haar measure. The problem is that the standard construction involves taking sup or inf, which we cannot do in $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$. We therefore use the same method as is used in [Loo02], except that we use $\overline{\mathrm{K}_0}(\mathsf{Var}_k)$ instead of $\widehat{\mathcal{M}_k}$:

**Definition 4.26.** *The subset $A \subset \mathcal{X}_\infty$ is* measurable *if the following holds:*

- *For every positive integer $m$, there exists a stable subset $A_m \subset \mathcal{X}_\infty$ and a sequence of stable subsets $(C_i^m \subset \mathcal{X}_\infty)_{i=1}^\infty$, with $\sum_i \mu_{\mathcal{X}}(C_i^m)$ convergent, such that $A \,\Delta\, A_m \subset \bigcup_i C_i^m$.*
- $\lim_{m\to\infty} \sum_i \mu_{\mathcal{X}}(C_i^m) = 0$.

*The measure of $A$ is then defined to be $\mu_{\mathcal{X}}(A) := \lim_{m\to\infty} \mu_{\mathcal{X}}(A_m)$.*

Below we will prove that this is well-defined. But let us first note that by definition $\mu_{\mathcal{X}}(A) \geq 0$ for every measurable $A$, because this is true for stable sets.

**Remark 4.27.** *This definition works well when $\mathcal{X}$ is smooth, since in that case all cylinders are stable. In the general case, it is probably better to first define the measure of an arbitrary cylinder $A$ as $\lim_{e\to\infty} \mu_{\mathcal{X}}(\mathcal{X}_\infty^{(e)} \cap A) \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$, where $\mathcal{X}_\infty^{(e)} = \mathcal{X}_\infty \setminus \pi_e^{-1}\big((\mathcal{X}_{sing})_e\big)$, and then replace "stable sets" with "cylinders" in the definition of a measurable set. However, since we are not able to prove that this limit exists in general, we use the present definition, which in any case works for our purposes.*

To prove that this is well-defined we will use the following well known lemma (*cf.* Lemma 3.5.1 in [LS03]).

**Lemma 4.28.** *Let $A \subset \mathcal{X}_\infty$ be stable. Then every countable covering of $A$ with stable subsets $C_i$ has a finite subcovering.*

**Proposition 4.29.** *The measure $\mu_{\mathcal{X}}$ is well-defined. The measurable subsets form a Boolean ring on which $\mu_{\mathcal{X}}$ is additive.*

*Proof.* We first prove that the limit exists. For this we have to prove that the sequence $(\mu_{\mathcal{X}}(A_m))_{m=1}^{\infty}$ is Cauchy, i.e., that $\mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(A'_m) \to 0$ as $m, m' \to 0$. Let $m$ and $m'$ be any positive integers. Since $A_m$ and $A_{m'}$ are stable,

$$\mu_{\mathcal{X}}(A_m \,\Delta\, A_{m'}) = \mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(A_{m'}) + 2\mu_{\mathcal{X}}(A_{m'} \setminus A_m).$$

Moreover, $\mu_{\mathcal{X}}(A_{m'} \setminus A_m) \geq 0$ so it follows that $\mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(A_{m'}) \leq \mu_{\mathcal{X}}(A_m \,\Delta\, A_{m'})$. Next, $A_m \,\Delta\, A_{m'} \subset \bigcup_i C_i^m \cup C_i^{m'}$. By Lemma 4.28 a finite number of the $C_i^m \cup C_i^{m'}$ suffices, hence the right hand side is stable so $\mu_{\mathcal{X}}(A_m \,\Delta\, A_{m'}) \leq \mu_{\mathcal{X}}(\bigcup_{i=0}^N C_i^m \cup C_i^{m'})$. From Lemma 4.7 we then see that $\mu_{\mathcal{X}}(A_m \,\Delta\, A_{m'}) \leq \sum_{i=0}^N \big(\mu_{\mathcal{X}}(C_i^m) + \mu_{\mathcal{X}}(C_i^{m'})\big)$. Hence

$$\mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(A_{m'}) \leq \sum_{i=0}^{N} \big(\mu_{\mathcal{X}}(C_i^m) + \mu_{\mathcal{X}}(C_i^{m'})\big).$$

Similarly, $-\sum_{i=0}^N \big(\mu_{\mathcal{X}}(C_i^m) + \mu_{\mathcal{X}}(C_i^{m'})\big) \leq \mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(A_{m'})$. Since, by assumption, $\sum_{i=0}^N \mu_{\mathcal{X}}(C_i^m)$ converges to zero as $m$ tends to infinity, it follows that $\mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(A_{m'})$ is bounded from above and below by sequences strongly convergent to zero.

Next, suppose that $A \,\Delta\, B_m \subset \bigcup_i D_i^m$ is another sequence that defines $\mu_{\mathcal{X}}(A)$. In the same way as above we see that for some $N$,

$$-\sum_{i=0}^{N} \big(\mu_{\mathcal{X}}(C_i^m) + \mu_{\mathcal{X}}(D_i^m)\big) \leq \mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(B_m) \leq \sum_{i=0}^{N} \big(\mu_{\mathcal{X}}(C_i^m) + \mu_{\mathcal{X}}(D_i^m)\big)$$

hence $\mu_{\mathcal{X}}(A_m) - \mu_{\mathcal{X}}(B_m) \to 0$ as $m \to \infty$. □

The following proposition is sometimes useful when proving that a set is measurable.

**Proposition 4.30.** *Let $A \subset \mathcal{X}_\infty$ be a union of stable sets $A_i$, $A = \bigcup_{i \in \mathbb{N}} A_i$, such that the sum $\sum_{i \in \mathbb{N}} \mu_{\mathcal{X}}(A_i)$ is convergent. Then $A$ is measurable and $\mu_{\mathcal{X}}(A) = \lim_{n \to \infty} \mu_{\mathcal{X}}(\bigcup_{i \leq n} A_i)$. If furthermore the $A_i$ are pairwise disjoint, then $\mu_{\mathcal{X}}(A) = \sum_{i \in \mathbb{N}} \mu_{\mathcal{X}}(A_i)$.*

*Proof.* We have $A \,\Delta \bigcup_{i \le n} A_i = \bigcup_{i > n} A_i$. Here $\sum_{i>n} \mu_{\mathcal{X}}(A_i)$ is convergent since $\sum_{i \in \mathbb{N}} \mu_{\mathcal{X}}(A_i)$ is, and it then follows that

$$\lim_{n\to\infty} \sum_{i>n} \mu_{\mathcal{X}}(A_i) = \lim_{n\to\infty} \Big(\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i) - \sum_{i\le n} \mu_{\mathcal{X}}(A_i)\Big)$$
$$= \sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i) - \lim_{n\to\infty} \sum_{i\le n} \mu_{\mathcal{X}}(A_i) = 0.$$

Therefore, by definition, $A$ is measurable and

$$\mu_{\mathcal{X}}(A) = \lim_{n\to\infty} \mu_{\mathcal{X}}(\bigcup_{i\le n} A_i).$$

In the case when the $A_i$ are disjoint we have $\mu_{\mathcal{X}}(\bigcup_{i\le n} A_i) = \sum_{i\le n} \mu_{\mathcal{X}}(A_i)$, hence $\mu_{\mathcal{X}}(A) = \sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i)$. □

We will use the following lemma to apply the proposition.

**Lemma 4.31.** *Let $(A_i)$ and $(B_i)$ be two sequences of stable subsets of $\mathcal{X}_\infty$, such that for any $i$, $A_i \subset B_i$. If $\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(B_i)$ is convergent, then so is $\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i)$.*

*Proof.* For some $n$, both $A_i$ and $B_i$ are stable of level $n$. We have $\pi_n(A_i) \subset \pi_n(B_i)$, hence by Lemma 4.7, $[\pi_n(A_i)] \le [\pi_n(B_i)]$. Therefore $0 \le \mu_{\mathcal{X}}(A_i) \le \mu_{\mathcal{X}}(B_i)$, it follows from Lemma 4.16 that the sum is convergent. □

The following proposition shows that this definition generalizes the $p$-adic measure.

**Proposition 4.32.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$. If $A \subset \mathcal{X}_\infty$ is measurable then $\mu_{Haar}(A \cap \mathbb{Z}_p^d) = \mathrm{C}_p(\mu_{\mathcal{X}}(A))$.*

*Proof.* By definition, we have that $\mathrm{C}_p\, \mu_{\mathcal{X}}(A) = \mathrm{C}_p \lim_{m\to\infty} \mu_{\mathcal{X}}(A_m)$. Since $\mathrm{C}_p$ is continuous, this equals

$$\lim_{m\to\infty} \mathrm{C}_p\, \mu_{\mathcal{X}}(A_m) = \lim_{m\to\infty} \mu_{Haar}(A_m \cap \mathbb{Z}_p^d).$$

If we now can show that $\mu_{Haar}\big((A \,\Delta\, A_m) \cap \mathbb{Z}_p^d\big) \to 0$ when $m \to \infty$, then by standard measure theory it follows that $\lim_{m\to\infty} \mu_{Haar}(A_m \cap \mathbb{Z}_p^d) = \mu_{Haar}(A \cap \mathbb{Z}_p^d)$, and we are done.

Since, by definition, there are stable sets $C_i^m$ such that

$$(A \,\Delta\, A_m) \cap \mathbb{Z}_p^d \subset (\bigcup_i C_i^m) \cap \mathbb{Z}_p^d,$$

it suffices to show that $\sum_i \mu_{Haar}(C_i^m \cap \mathbb{Z}_p^d) \to 0$ as $m \to \infty$. Now this equals $\lim_{m\to\infty} \sum_i \mathrm{C}_p\, \mu_{\mathcal{X}}(C_i^m)$. Using two times the fact that $\mathrm{C}_p$ is continuous, together with the assumption on the $C_i$ gives that this equals $\mathrm{C}_p \lim_{m\to\infty} \sum_i \mu_{\mathcal{X}}(C_i^m) = \mathrm{C}_p\, 0 = 0$ □

We next define the integrals of functions $\mathcal{X}_\infty \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$. For this we note that every $x \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ may be written as $x = a - b$, where $a, b \geq 0$. (This follows straight forward from that the fact that the same thing is true for elements of $\mathrm{K}_0(\mathsf{Var}_k)$.)

**Definition-Lemma 4.33.** *We say that* $f\colon \mathcal{X}_\infty \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ *is* positively integrable *if its image is contained in* $\{a \geq 0\} \subset \mathrm{K}_0(\mathsf{Var}_k)$*, it has measurable fibers, and the property that the sum* $\sum_{a\in\overline{\mathrm{K}_0}(\mathsf{Var}_k)} \mu_{\mathcal{X}}(f^{-1}(a))a$ *is convergent, (in particular it has only countably many nonzero terms). We then define* $\int f d\mu_{\mathcal{X}}$ *to be this limit.*

*We say that* $f$ *is* integrable *if it has measurable fibres, and we can write every* $a \in \mathrm{K}_0(\mathsf{Var}_k)$ *as* $a = a' - a''$*, where* $a', a'' \geq 0$*, in such a way that the sums* $S_1 = \sum_{a\in\overline{\mathrm{K}_0}(\mathsf{Var}_k)} \mu_{\mathcal{X}}(f^{-1}(a))a'$ *and* $S_2 = \sum_{a\in\overline{\mathrm{K}_0}(\mathsf{Var}_k)} \mu_{\mathcal{X}}(f^{-1}(a))a''$ *are convergent. In that case we define* $\int f d\mu_{\mathcal{X}} = S_1 - S_2$*.*

*Proof.* Since $\mu_{\mathcal{X}}(f^{-1}(a)) \geq 0$, the notion of positive integrability is well-defined by Lemma 4.17.

That a function is integrable is well-defined, because if there exist such convergent sums $S_1$ and $S_2$, then (by definition of addition in a ring of Cauchy sequences) their difference equals $\sum_{a\in\overline{\mathrm{K}_0}(\mathsf{Var}_k)} \mu_{\mathcal{X}}(f^{-1}(a))a$, which is then independent of the choice of $a'$ and $a''$. □

If $A \subset \mathcal{X}_\infty$, let $\chi_A$ be the characteristic function of $A$ and define $\int_A f d\mu_{\mathcal{X}} := \int f \cdot \chi_A d\mu_{\mathcal{X}}$. If $\int_A f d\mu_{\mathcal{X}}$ and $\int_B f d\mu_{\mathcal{X}}$ exists and $A$ and $B$ are disjoint then $\int_{A\cup B} f d\mu_{\mathcal{X}}$ exists and is equal to $\int_A f d\mu_{\mathcal{X}} + \int_B f d\mu_{\mathcal{X}}$.

**Proposition 4.34.** *Let* $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$*. If* $A \subset \mathcal{X}_\infty$ *is measurable and if* $f\colon \mathcal{X}_\infty \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ *is integrable, then*

$$\mathrm{C}_p \int_A f d\mu_{\mathcal{X}} = \int_{A\cap\mathbb{Z}_p^d} \mathrm{C}_p \circ f d\mu_{Haar}.$$

*Proof.* As we have set things up, this is straight forward:

$$\begin{aligned}
\mathrm{C}_p \int_A f d\mu_{\mathcal{X}} &= \mathrm{C}_p \sum_{a\in\overline{\mathrm{K}_0}(\mathsf{Var}_k)} \mu_{\mathcal{X}}(f^{-1}a \cap A)a \\
&= \sum_{a\in\overline{\mathrm{K}_0}(\mathsf{Var}_k)} \mu_{Haar}(f^{-1}a \cap A \cap \mathbb{Z}_p^d)\,\mathrm{C}_p\, a \\
&= \sum_{r\in\mathbb{R}} \mu_{Haar}\big((\mathrm{C}_p \circ f)^{-1} r \cap A \cap \mathbb{Z}_p^d\big) r \\
&= \int_{A\cap\mathbb{Z}_p^d} \mathrm{C}_p \circ f d\mu_{Haar}.
\end{aligned}$$ □

For a first illustration of how this work, see Example 5.11 where we compute explicit some simple integrals.

# 5. Computing motivic integrals

## 5.1 Introduction

Fix a prime $p$. In [Sko09] the author computes the integral

$$I_p = \int_{\mathbb{Z}_p^n} |\prod_{1\le i<j\le n} (X_i - X_j)|_p d\mu_{Haar}.$$

The method is recursive and finds a rational function $f$ such that $I_p = f(p)$ (with respect to the normalized absolute value). Looking at these computations it is immediate that instead of integrating over $\mathbb{Z}_p^n$ we may as well integrate over $\mathbf{W}(\mathbb{F}_q)^n$, where $\mathbf{W}$ is the Witt vectors and $q$ is any power of $p$, to obtain the value of

$$I_q = \int_{\mathbf{W}(\mathbb{F}_q)^n} |\prod_{1\le i<j\le n} (X_i - X_j)|_p d\mu_{Haar}.$$

It turns out that for every $q$, $I_q = f(q)$. Here $f$ is the same rational function as above, it is hence independent of $q$.

This kind of behavior is unusual, one would expect that $f$ should vary with $q$ (*cf.* Example 5.11). When it occurs one could suspect that there is some geometric explanation; the aim of this chapter is to give such an explanation.

For this we use the version of geometric motivic integration developed in Chapter 4 to prove that if $\mathcal{X} = \mathbb{A}^n_{\mathbb{Z}_p}$ then

$$I = \int_{\mathcal{X}_\infty} |\prod_{1\le i<j\le n} (X_i - X_j)|_p d\mu_{\mathcal{X}} \in \overline{\mathrm{K}_0}(\mathsf{Var}_{\mathbb{F}_p})$$

is a rational function in $\mathbb{L}$ with integer coefficients, more precisely $I = f(\mathbb{L})$ where $f$ is the same as above (this follows from Theorem 5.27). By applying the point counting homomorphism we recover the original integrals:

$$I_q = \mathrm{C}_q\, I = \mathrm{C}_q\, f(\mathbb{L}) = f(q)$$

for every power $q$ of $p$.

Here is an overview of the chapter: Since we will do a lot of computations in the Witt vectors, we begin with some necessary background about them, in Section 5.2.

When performing the computations of this chapter, we cannot work in the full generality for which the integration theory was developed in Chapter 4, we need to impose some extra condition on the variety $\mathcal{X}$ with respect to which the integrals are constructed (most importantly that it is an affine space, but we also need some additional assumptions on the discrete valuation ring). In Section 5.3 we describe these conditions. We then prove some general result about our motivic integral in this case, in particular that all the integrals that we are interested in exist.

In Section 5.4 we give three fundamental results, about change of variables and separation of variables. We need these in the last section, to perform the recursions which are the ultimate goal of the chapter.

In Section 5.5 we do the work that allows us to compute the integral $I$ defined above. In fact, we compute more general integrals; we show that for $p$ sufficiently large, the motivic integral of the absolute value of any product of linear forms is a rational function in $\mathbb{L}$, with coefficients in $\mathbb{Z}$. Then when $p$ is arbitrary we compute the motivic integral of the absolute value of a product of more special linear forms, in particular the integral $I$. These computations also make it possible to give an explicit formula for the integral. (If one is only interested in the $p$-adic integrals, it is possible to translate these computations to that setting, with no reference to motivic integration.) We will also see that all these computations work also in the equal characteristic case.

Finally, it turns out that the rational function $f$, discussed above, is independent also of the prime $p$: $I_p = f(p)$ for every prime $p$. In Section 5.7 we discuss how one could give a motivic explanation also of this fact.

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## 5.2 Background material about the Witt vectors

In this section we give the basic definitions in connection with the Witt vectors, $\mathbf{W}$. This material is essentially in [Ser79] pp. 40-44 and in [Dem72].

### Definitions

Fix a prime $p$. Define, for every $n \in \mathbb{N}$, the polynomial

$$W_n = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X].$$

The ring of Witt vectors with coefficients in the commutative ring $A$, $\mathbf{W}(A)$, is by definition $A^{\mathbb{N}}$ with the ring operations defined by requiring that the map

$$\begin{aligned} W_*(A) : \mathbf{W}(A) &\to A^{\mathbb{N}} \\ \mathbf{a} &\mapsto \big(W_0(\mathbf{a}), \dots, W_n(\mathbf{a}), \dots\big) \end{aligned}$$

should be a homomorphism. $\mathbf{W}(A)$ is a commutative ring with identity element $(1, 0, 0, \dots)$. The ring scheme of Witt vectors is the functor $\mathbf{W}\colon \mathsf{Rings} \to \mathsf{Rings}$ that takes the commutative ring $A$ to $\mathbf{W}(A)$. $W_*\colon \mathbf{W} \to \mathbb{A}^{\mathbb{N}}_{\mathbb{Z}}$ is then a morphism of ring schemes. If $p$ is invertible in $A$, $W_*(A)$ is an isomorphism, hence $\mathbf{W}_{\mathbb{Z}[1/p]} \simeq \mathbb{A}^{\mathbb{N}}_{\mathbb{Z}[1/p]}$ as ring schemes.

One defines the Witt vectors of length $n$, $\mathbf{W}_n$, to be the functor that takes the ring $A$ to the projection of $\mathbf{W}(A)$ onto its $n$ first coordinates. This scheme is of finite type over $\operatorname{Spec} \mathbb{Z}$ . One has that $\mathbf{W}_1$ is the identity functor, that is $\mathbf{W}_1(A) = A$. We also have that the ring $\mathbf{W}(A)$ is the inverse limit of the rings $\mathbf{W}_n(A)$ as $n \to \infty$. We define the projection map $\pi_n\colon \mathbf{W} \to \mathbf{W}_n$ by

$$(a_0, a_1, \dots) \mapsto (a_0, \dots, a_{n-1})\colon \mathbf{W}(A) \to \mathbf{W}_n(A)$$

for every ring $A$.

If $A$ is a perfect ring of characteristic $p$ (meaning that $x \mapsto x^p$ is surjective) then $p$ is not a zero-divisor in $\mathbf{W}(A)$, which is Hausdorff and complete with respect to the filtration $\{p^n\mathbf{W}(A)\}_{n\in\mathbb{N}}$. Moreover, the residue ring of $\mathbf{W}(A)$ is $A$. In particular, $\mathbf{W}(\mathbb{F}_p) = \mathbb{Z}_p$ and if $q = p^n$ then $\mathbf{W}(\mathbb{F}_q)$ is the integral closure of $\mathbb{Z}_p$ in the unique unramified degree $n$ extension of $\mathbb{Q}_p$ (in a fixed algebraic closure of $\mathbb{Q}_p$).

### Operations on $\mathbf{W}$

Define $\mathrm{V}\colon \mathbf{W} \to \mathbf{W}$ by $\mathrm{V}\,\mathbf{a} = (0, a_0, \dots, a_{n-1}, \dots)$. V is short for "Verschiebung". It is not a morphism of ring schemes but it is additive. Note that $\mathbf{W}(A)/\,\mathrm{V}^n\,\mathbf{W}(A) \simeq \mathbf{W}_n(A)$ for every ring $A$.

Next we define the map $\mathrm{r}\colon \mathbf{W}_1 \to \mathbf{W}$ by $a \mapsto (a, 0, \dots, 0, \dots)$. The map r is multiplicative. Moreover, for any $\mathbf{a} = (a_0, a_1, ,\dots,) \in \mathbf{W}(A)$ we have

$$\mathbf{a} = \sum_{i=0}^{\infty} \mathrm{V}^i\, \mathrm{r}(a_i). \tag{5.1}$$

When $A$ is perfect of characteristic $p$, $\mathrm{r}(A)$ is the unique system of multiplicative representatives of $A$ in $\mathbf{W}(A)$. In particular, $\mathrm{r}(\mathbb{F}_p)$ is the subset of $\mathbb{Z}_p$ consisting of 0 and the $(p-1)$st roots of unity.

Finally, over $\mathbb{F}_p$ (where $p$ is the prime that was fixed in the beginning of this section) we define the Frobenius morphism $\mathrm{F}\colon \mathbf{W}_{\mathbb{F}_p} \to \mathbf{W}_{\mathbb{F}_p}$ by $\mathrm{F}\,\mathbf{a} = (a_0^p, \dots, a_n^p, \dots)$. It is a morphism of ring schemes.

**Proposition 5.2.** *If $A$ is an $\mathbb{F}_p$-algebra and $\mathbf{a}, \mathbf{b} \in \mathbf{W}(A)$ the following formulas hold:*

$$\begin{aligned}\mathrm{V}\,\mathrm{F}\,\mathbf{a} &= \mathrm{F}\,\mathrm{V}\,\mathbf{a} = p\mathbf{a}\\ \mathbf{a}\cdot\mathrm{V}\,\mathbf{b} &= \mathrm{V}(\mathrm{F}\,\mathbf{a}\cdot\mathbf{b}).\end{aligned}$$

*Proof.* For the first formula see [Ser79]. For the second formula it suffices to prove this when $A$ is perfect so we may assume that $\mathbf{b} = \mathrm{F}\,\mathbf{c}$. The first formula, the distributive law and the fact that F is a ring homomorphism then give

$$\mathrm{V}(\mathrm{F}\,\mathbf{a}\cdot\mathbf{b}) = \mathrm{V}(\mathrm{F}\,\mathbf{a}\cdot\mathrm{F}\,\mathbf{c}) = \mathrm{V}\,\mathrm{F}(\mathbf{a}\cdot\mathbf{c}) = p(\mathbf{a}\cdot\mathbf{c}) = \mathbf{a}\cdot(p\mathbf{c}) = \mathbf{a}\cdot\mathrm{V}\,\mathrm{F}\,\mathbf{c} = \mathbf{a}\cdot\mathrm{V}\,\mathbf{b}.$$

□

It follows that if $A$ is an $\mathbb{F}_p$-algebra, $\mathbf{a}, \mathbf{b} \in \mathbf{W}(A)$ and $i, j \in \mathbb{N}$ then

$$\mathrm{V}^i\,\mathbf{a}\cdot\mathrm{V}^j\,\mathbf{b} = \mathrm{V}^{i+j}\big(\mathrm{F}^j\,\mathbf{a}\cdot\mathrm{F}^i\,\mathbf{b}\big). \tag{5.3}$$

We need the following consequence of the the proposition:

**Corollary 5.4.** *Let $k$ be a perfect $\mathbb{F}_p$-algebra, let $A$ be a $k$-algebra and let $\Delta \in \mathbf{W}(k)[X_1, \dots, X_n]$ be a form of degree $d$. If $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbf{W}(A)$ then*

$$\Delta(\mathrm{V}\,\mathbf{a}_1, \dots, \mathrm{V}\,\mathbf{a}_n) = \mathrm{F}^{d-1}\,\mathrm{V}^d(\mathrm{F}\,\Delta)(\mathbf{a}_1, \dots, \mathbf{a}_n).$$

*In particular, if $\Delta \in \mathbb{Z}_p[X_1, \dots, X_n]$ then*

$$\Delta(\mathrm{V}\,\mathbf{a}_1, \dots, \mathrm{V}\,\mathbf{a}_n) = \mathrm{F}^{d-1}\,\mathrm{V}^d\,\Delta(\mathbf{a}_1, \dots, \mathbf{a}_n).$$

*Proof.* Let $\Delta = X_1^d$. The formula is true for $d = 1$. Suppose that it is true for $d - 1$. Then with the help of Corollary 5.3,

$$\begin{aligned}\Delta(\mathrm{V}\,\mathbf{a}) &= (\mathrm{V}\,\mathbf{a})(\mathrm{V}\,\mathbf{a})^{d-1}\\ &= (\mathrm{V}\,\mathbf{a})(\mathrm{F}^{d-2}\,\mathrm{V}^{d-1}\,\mathbf{a}^{d-1})\\ &= \mathrm{V}^d(\mathrm{F}^{d-1}\,\mathbf{a}\cdot\mathrm{F}^{d-1}\,\mathbf{a}^{d-1})\\ &= \mathrm{F}^{d-1}\,\mathrm{V}^d\,\Delta(\mathbf{a}).\end{aligned}$$

Next, let $d$ and $n$ be arbitrary and suppose the formula is proved for every $X_1^{d_1}\cdots X_{n-1}^{d_{n-1}}$ with $d_1 + \cdots + d_{n-1} \le d$. Let $\Delta = X_1^{d_1}\cdots X_n^{d_n}$ with $d_1 + \cdots + d_n = d$. Then

$$\begin{aligned}\Delta(\mathrm{V}\,\mathbf{a}_1, \dots, \mathrm{V}\,\mathbf{a}_n) &= (\mathrm{V}\,\mathbf{a}_1)^{d_1}\prod_{i=2}^{n}(\mathrm{V}\,\mathbf{a}_i)^{d_i}\\ &= \mathrm{F}^{d_1-1}\,\mathrm{V}^{d_1}\,\mathbf{a}_1^{d_1}\cdot\mathrm{F}^{d-d_1-1}\,\mathrm{V}^{d-d_1}\prod_{i=2}^{n}\mathbf{a}_i^{d_i}.\end{aligned}$$

Since F and V commute we can use Corollary 5.3 on this expression to get

$$\mathrm{V}^d\left(\mathrm{F}^{d-1}\,\mathbf{a}_1^{d_1}\cdot\mathrm{F}^{d-1}\prod_{i=2}^{n}\mathbf{a}_i^{d_i}\right)$$

and because F is a homomorphism this equals $\mathrm{F}^{d-1}\,\mathrm{V}^d\,\Delta(\mathbf{a}_1,\ldots,\mathbf{a}_n)$.

Finally, let $\Delta\in\mathbf{W}(k)[X_1,\ldots,X_n]$ be an arbitrary form of degree $d$: $\Delta=\sum_{\alpha\in I}c_\alpha\Delta_\alpha$, where the $\Delta_\alpha$ are monomials of degree $d$ and $c_\alpha\in\mathbf{W}(k)$. Since $k$ is perfect we may, for every $\alpha\in I$, chose $c'_\alpha$ such that $\mathrm{F}^{d-1}\,c'_\alpha=c_\alpha$. We then have

$$\begin{aligned}\Delta(\mathrm{V}\,\mathbf{a}_1,\ldots,\mathrm{V}\,\mathbf{a}_n)&=\sum_{\alpha\in I}\mathrm{F}^{d-1}\,c'_\alpha\,\mathrm{F}^{d-1}\,V^d\Delta_\alpha(\mathbf{a}_1,\ldots,\mathbf{a}_n)\\&=\mathrm{F}^{d-1}\sum_{\alpha\in I}\mathrm{V}^d(\mathrm{F}^d\,c'_\alpha\cdot\Delta_\alpha)\\&=\mathrm{F}^{d-1}\,\mathrm{V}^d\sum_{\alpha\in I}\mathrm{F}\,c_\alpha\cdot\Delta_\alpha\\&=\mathrm{F}^{d-1}\,\mathrm{V}^d(\mathrm{F}\,\Delta)(\mathbf{a}_1,\ldots,\mathbf{a}_n).\end{aligned}$$

In particular, if $c_\alpha\in\mathbb{Z}_p=\mathbf{W}(\mathbb{F}_p)$ then $\mathrm{F}\,c_\alpha=c_\alpha$, hence $\mathrm{F}\,\Delta=\Delta$. □

## 5.3 Assumptions and general results

In Chapter 4 we developed a version of motivic integration, valid for any variety $\mathcal{X}$ defined over a complete discrete valuation ring, $\mathcal{O}$. In this chapter we will utilize this theory in the case when $\mathcal{X}$ is an affine space, $\mathcal{X}=\mathbb{A}^d_{\mathcal{O}}$. Actually, what interests us the most is when $\mathcal{O}=\mathbb{Z}_p$, for then the computations of certain $p$-adic integrals can be done in the motivic setting, and then be obtained by applying $\mathrm{C}_p$ to the result.

In the present section first we go through the assumptions on $\mathcal{X}$ and $\mathcal{O}$ used throughout this chapter; we then prove that all the integrals we are interested in actually exist, and prove some basic facts about them.

The computations of measures and integrals given in this section are all standard. We include them anyway, firstly to establish notation and give examples, and also to check that everything works also with respect to our stronger topology

### The setup

Throughout this chapter, we assume that $\mathcal{X}=\mathbb{A}^d_{\mathcal{O}}$ where $\mathcal{O}$ is a complete discrete valuation ring with residue field $k$, of one of the following types:

- $k$ is a perfect field, of prime characteristic $p$, and $\mathcal{O}=\mathbf{W}(k)$, where $\mathbf{W}$ is the ring scheme of Witt vectors constructed with respect to the prime $p$.

- $k$ is a field, and $\mathcal{O} = k[[t]]$.

(We are mainly interested in the cases when $\mathcal{O}$ is either $\mathbb{Z}_p$ or $\mathbb{Q}[[t]]$ so that $k = \mathbb{F}_p$ and $\mathbb{Q}$ respectively.) Then the space of arcs $\mathcal{X}_\infty$ of $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$ is the set

$$\mathcal{X}_\infty = \begin{cases} \mathbf{W}(\overline{k})^d, & \mathcal{O} = \mathbf{W}(k) \\ \overline{k}[[t]]^d, & \mathcal{O} = k[[t]] \end{cases},$$

where $\overline{k}$ is an algebraic closure of $k$. We use the standard terminology in connection with the Witt vectors, see Section 5.2. Moreover, in order to get a uniform notation, we will write $(x_i)_{i\geq 0}$ for $\sum_{i\geq 0} x_i t^i \in \overline{k}[[t]]$.

For every $n$, if $\mathcal{O} = \mathbf{W}(k)$ then $\mathcal{X}_n$ is the $k$-scheme whose $R$-points is $\mathbf{W}^d_n(R)$ for every $k$-algebra $R$ ($\mathbf{W}_n$ is the Witt vectors of length $n$). If instead $\mathcal{O} = k[[t]]$ then $\mathcal{X}_n(R) = (R[t]/(t^n))^d$ for every $k$-algebra $R$. We use $\pi_n$ to denote the projection $\mathcal{X}_\infty \to \mathcal{X}_n(\overline{k})$.

## Convergence of integrals

We continue to use $\mathcal{O}$ to denote a complete discrete valuation ring with residue field $k$, of the type defined above.

To show that the integrals we are interested in exist we need the following simple approximation:

**Lemma 5.5.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$ and let $A$ be a measurable subset of $\mathcal{X}_\infty$. Then $0 \leq \mu_{\mathcal{X}}(A) \leq 1$ and $\dim \mu_{\mathcal{X}}(A) \leq 0$.*

*Proof.* Suppose first that $A$ is stable of level $n$. We have $\pi_n(A) \subset \mathcal{X}_n = \mathbb{A}^{dn}_k$, giving immediately that $\dim A = \dim \pi_n(A) - nd \leq nd - nd = 0$, i.e., $\dim \mu_{\mathcal{X}}(A) \leq 0$. Also $0 \leq [\pi_n(A)] \leq \mathbb{L}^{dn}$, hence $0 \leq \mu_{\mathcal{X}}(A) \leq 1$.

In the general case there are, by definition, stable subsets $A_i$ such that $\mu_{\mathcal{X}}(A) = \lim_{i\to\infty} \mu_{\mathcal{X}}(A_i)$, i.e., $\mu_{\mathcal{X}}(A) = (\mu_{\mathcal{X}}(A_i))_{i\in\mathbb{N}}$. Since the equalities holds for stable sets it follows by definition that $0 = (0)_{i\in\mathbb{N}} \leq \mu_{\mathcal{X}}(A) \leq (1)_{i\in\mathbb{N}} = 1$. The same reasoning goes for the statement about dimension. □

**Lemma 5.6.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$ and let $A_i$ be measurable subsets of $\mathcal{X}_\infty$. If $e_i \to \infty$ as $i \to \infty$, then the sum $\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i)\mathbb{L}^{-e_i}$ is convergent.*

*Proof.* By Lemma 5.5, $0 \leq \mu_{\mathcal{X}}(A_i)\mathbb{L}^{-e_i} \leq \mathbb{L}^{-e_i}$. By Example 4.14 the sum $\sum_{i\in\mathbb{N}} \mathbb{L}^{-e_i}$ is convergent if and only if $\dim \mathbb{L}^{-e_i} \to -\infty$, i.e., $e_i \to \infty$. In this case it follows from Lemma 4.16 that $\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i)\mathbb{L}^{-e_i}$ is convergent. □

## The motivic measure of $\{\operatorname{ord} f \geq n\}$

Let $\mathcal{X}^1 = \mathbb{A}^1_{\mathcal{O}}$ and let $\mathcal{X}^1_\infty$ be its arc space. We have a function $\operatorname{ord}\colon \mathcal{X}^1_\infty \to \mathbb{N} \cup \{\infty\}$, mapping $x$ to the biggest power of the uniformizer dividing

$x$, and with the properties that $\operatorname{ord} ab = \operatorname{ord} a + \operatorname{ord} b$ and $\operatorname{ord}(a+b) \geq \min\{\operatorname{ord} a, \operatorname{ord} b\}$. We continue to write $\mathcal{X}$ for $\mathbb{A}^d_{\mathcal{O}}$. If $f \in \mathcal{O}[X_1, \ldots, X_d]$ it defines a function $\mathcal{X}_\infty \to \mathcal{X}^1_\infty$. So for $n \in \mathbb{N}$ we may consider the subset

$$\{\operatorname{ord} f \geq n\} := \{(a_1, \ldots, a_d) \in \mathcal{X}_\infty : \operatorname{ord} f(a_1, \ldots, a_d) \geq n\}.$$

When $\mathcal{O} = \mathbb{Z}_p$ this set has the property that $\{\operatorname{ord} f \geq n\} \cap \mathbb{Z}_p^d = \{(a_1, \ldots, a_d) \in \mathbb{Z}_p^d : \operatorname{ord}_p f(a_1, \ldots, a_d) \geq n\}$.

**Lemma 5.7.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$. The subset $\{\operatorname{ord} f \geq n\} \subset \mathcal{X}_\infty$ is stable of level $n$ and $\mu_{\mathcal{X}}(\{\operatorname{ord} f \geq n\}) = [\pi_n(\{\operatorname{ord} f \geq n\})]\mathbb{L}^{-dn}$.*

*Proof.* To simplify the notation, we prove this for the special case when $\mathcal{O} = \mathbf{W}(k)$; the case when $\mathcal{O} = k[[t]]$ is similar. When we view $\mathcal{X}_\infty$ as the $\overline{k}$-points on the ring scheme $\mathbf{W}^d$, we see that $\{\operatorname{ord} f \geq n\}$ is actually the $\overline{k}$-points on a closed subscheme. For let $k[X_{i0}, X_{i1}, \ldots, X_{iN}, \ldots]_{i=1}^d$ represent $\mathbf{W}^d$. Let $f_0, f_1, \ldots$ be the universal polynomials defining $f$ in the Witt vectors, i.e., $f_n \in k[X_{i0}, \ldots, X_{in}]_{i=1}^d$ and if $R$ is any $k$-algebra and $r_i = (r_{i0}, r_{i1}, \ldots) \in \mathbf{W}(R)$ for $i = 1, \ldots, d$, then $f(r_1, \ldots, r_d) = (f_0(r_1, \ldots, r_d), f_1(r_1, \ldots, r_d), \ldots) \in \mathbf{W}(R)$. (We write $f_n(r_1, \ldots, r_d)$ for $f_n(r_{10}, \ldots, r_{d0}; \ldots; r_{1n}, \ldots, r_{dn}) \in R$.) We then see that $\{\operatorname{ord} f \geq n\}$ is identified with

$$\{(x_1, \ldots, x_d) \in \mathbf{W}(\overline{k})^d : f(x_1, \ldots, x_d) \equiv 0 \pmod{\mathrm{V}^n)}\} \subset \mathcal{X}_\infty,$$

i.e.,

$$\{(x_1, \ldots, x_d) \in \mathbf{W}(\overline{k})^d : f_i(x_1, \ldots, x_d) = 0 \text{ for } i = 0, \ldots, n-1\} \subset \mathcal{X}_\infty,$$

This in turn is the $\overline{k}$-points on the close subscheme

$$\operatorname{Spec} \frac{k[X_{i0}, X_{i1}, \ldots, X_{iN}, \ldots]_{i=1}^d}{(f_0, \ldots, f_{n-1})} \subset \mathbf{W}^d.$$

Now $\pi_m(\mathcal{X}_\infty) = \mathbf{W}^d_m(\overline{k})$. Hence, for $m \geq n$, we see that $\pi_m(\{\operatorname{ord} f \geq n\})$ is the $\overline{k}$-points on the $k$-scheme

$$\operatorname{Spec} \frac{k[X_{i0}, \ldots, X_{i,m-1}]_{i=1}^d}{(f_0, \ldots, f_{n-1})}.$$

In what follows we identify $\pi_m(\{\operatorname{ord} f \geq n\})$ with its underlying scheme. We then see that $\pi_m(\{\operatorname{ord} f \geq n\}) = \pi_n(\{\operatorname{ord} f \geq n\}) \times_k \mathbb{A}_k^{d(m-n)}$. The result follows. $\square$

When $\{f = 0\} \subset \mathbb{A}^d_{\mathcal{O}}$ is smooth we have the following refinement of the preceding lemma:

**Lemma 5.8** (Motivic Newton's Lemma). *Let $f \in \mathcal{O}[X_1, \ldots, X_d]$ be non-constant. Assume that $\{f = 0\} \subset \mathbb{A}^d_{\mathcal{O}}$ is smooth. Consider the subset $\{\operatorname{ord} f \geq n\} \subset \mathcal{X}_\infty$, where $n \geq 1$. Then $[\pi_n(\{\operatorname{ord} f \geq n\})] = \mathbb{L}^{(d-1)(n-1)}[\pi_1(\{\operatorname{ord} f \geq n\})] \in \mathrm{K}_0(\mathsf{Var}_k)$. In particular,*

$$\mu_{\mathcal{X}}(\{\operatorname{ord} f \geq n\}) = \left[\operatorname{Spec} \tfrac{k[X_{10}, \ldots, X_{d0}]}{(f_0)}\right] \mathbb{L}^{-n-d+1}.$$

*Proof.* Let $\mathcal{Z} := \{f = 0\} \subset \mathcal{X}$. Note that, for every $n \geq 1$,

$$\pi_n\{\operatorname{ord} f \geq n\} = \{x \in \mathbf{W}_n(\overline{k})^d : \ f_0(x) = \cdots = f_{n-1}(x) = 0\} = \mathcal{Z}_n(\overline{k}).$$

Since $\mathcal{Z}$ is smooth, $\mathcal{Z}_\infty$ is stable of level 1, hence $[\mathcal{Z}_n] = \mathbb{L}^{(d-1)(n-1)}[\mathcal{Z}_1]$. The result follows. □

## Motivic integrals of absolute values of polynomials

Next consider the function $\underline{a} \mapsto \mathbb{L}^{-\operatorname{ord} f(\underline{a})} \colon \mathcal{X}_\infty \to \overline{\mathrm{K}_0}(\mathsf{Var}_k)$. For $a \in \mathbf{W}(\overline{k})$ we write $|a| := \mathbb{L}^{-\operatorname{ord} a}$ and we want to compute the integral $\int_{\mathcal{X}_\infty} |f| d\mu_{\mathcal{X}}$. The following proposition shows that the integral exists.

**Proposition 5.9.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$. Let $A$ be a measurable subset of $\mathcal{X}_\infty$, and $f \in \mathcal{O}[X_1, \ldots, X_d]$. The integral $\int_A |f| d\mu_{\mathcal{X}} = \int_A \mathbb{L}^{-\operatorname{ord} f} d\mu_{\mathcal{X}}$ exists. Moreover, when $\mathcal{X} = \mathbb{A}^d_{\mathbb{Z}_p}$ we have, for $q$ any power of $p$, $\mathrm{C}_q \int_A |f| d\mu_{\mathcal{X}} = \int_{A \cap \mathbf{W}(\mathbb{F}_q)^d} |f|_p d\mu_{Haar}$.*

*Proof.* By definition the integral equals $\mu_{\mathcal{X}}(f = 0) \cdot 0 + \sum_{i \in \mathbb{N}} \mu_{\mathcal{X}}(A \cap \{\operatorname{ord} f = i\})\mathbb{L}^{-i}$. By Lemma 5.7, $\{\operatorname{ord} f = i\}$ is stable, hence $A \cap \{\operatorname{ord} f = i\}$ is measurable. The integral therefore exists by Lemma 5.6. Next, by Proposition 4.34, $\mathrm{C}_q \int_A \mathbb{L}^{-\operatorname{ord} f} d\mu_{\mathcal{X}} = \int_{A \cap \mathbf{W}(\mathbb{F}_q)^d} |f|_p d\mu_{Haar}$. □

When $\{f = 0\} \subset \mathbb{A}^d_{\mathcal{O}}$ is smooth we may compute the integral more explicitly:

**Proposition 5.10.** *Let $f \in \mathcal{O}[X_1, \ldots, X_d]$ and assume that $\{f = 0\} \subset \mathbb{A}^d_{\mathcal{O}}$ is smooth. Then*

$$\int |f| d\mu_{\mathcal{X}} = 1 - [\operatorname{Spec} k[X_{10}, \ldots, X_{d0}]/(f_0)] \frac{\mathbb{L}^{1-d}}{\mathbb{L} + 1} \in \overline{\mathrm{K}_0}(\mathsf{Var}_k).$$

*Proof.* By definition we have

$$\int_{\mathcal{X}_\infty} |f| d\mu_{\mathcal{X}} = \sum_{m \geq 0} \mathbb{L}^{-m} \mu_{\mathcal{X}}\{\operatorname{ord} f = m\}.$$

Since $\{\operatorname{ord} f = m\} = \{\operatorname{ord} f \geq m\} \setminus \{\operatorname{ord} f \geq m + 1\}$ we have

$$\mu_{\mathcal{X}}\{\operatorname{ord} f = m\} = \left[\operatorname{Spec} \tfrac{k[X_{10}, \ldots, X_{d0}]}{(f_0)}\right] \cdot \mathbb{L}^{1-d}(\mathbb{L}^{-m} - \mathbb{L}^{-(m+1)})$$

for $m \geq 1$. For $m = 0$ we have $\mu_{\mathcal{X}}(\operatorname{ord} f = 0) = \mu_{\mathcal{X}}(\mathcal{X}_\infty \setminus \{\operatorname{ord} f \geq 1\}) = 1 - \left[\operatorname{Spec} \frac{k[X_{10},\ldots,X_{d0}]}{(f_0)}\right]\mathbb{L}^{1-d}\mathbb{L}^{-1}$. Therefore, using Example 4.14,

$$\begin{aligned}
&\int_{\mathcal{X}_\infty} |f| d\mu_{\mathcal{X}} \\
&\quad = 1 + \left[\operatorname{Spec} \frac{k[X_{10},\ldots,X_{d0}]}{(f_0)}\right] \cdot \mathbb{L}^{1-d}\Big(-\mathbb{L}^{-1} + \sum_{m\geq 1} \mathbb{L}^{-m}\big(\mathbb{L}^{-m} - \mathbb{L}^{-(m+1)}\big)\Big) \\
&\quad = 1 - \left[\operatorname{Spec} \frac{k[X_{10},\ldots,X_{d0}]}{(f_0)}\right] \cdot \frac{\mathbb{L}^{1-d}}{\mathbb{L}+1}. \qquad \square
\end{aligned}$$

**Example 5.11.** *We look at the case when $\mathcal{O} = \mathbb{Z}_p$: If $f = aX + b$, where $a \in \mathbb{Z}_p^\times$, then $\frac{k[X_0]}{(f_0)} = k$. Since $[\operatorname{Spec} k] = 1$ we have $\int_{\mathcal{X}_\infty} |aX + b| d\mu_{\mathcal{X}} = \frac{\mathbb{L}}{\mathbb{L}+1}$, showing in particular that if $q$ is a power of $p$ then $\int_{\mathbf{W}(\mathbb{F}_q)} |aX + b| dX = \frac{q}{q+1}$*

*More generally, assume that $f$ is such that $f_0$ is irreducible of degree $d$. Then $\operatorname{Spec} k[X_0]/(f_0) \simeq \mathbb{F}_{p^d}$, hence $\int_{\mathcal{X}_\infty} |f| d\mu_{\mathcal{X}} = 1 - \frac{[\operatorname{Spec} \mathbb{F}_{p^d}]}{\mathbb{L}+1}$. Applying $\mathrm{C}_q$ for different powers of $p$ shows that*

$$\int_{\mathbf{W}(\mathbb{F}_q)} |f|_p d\mu_{Haar} = \begin{cases} 1 - d/(q+1) & q = p^i \text{ where } d \mid i \\ 1 & q = p^i \text{ where } d \nmid i \end{cases}.$$

## Partitions of integrals

The primary purpose of this chapter is to show that the integral of the absolute value of a certain polynomial in many variables is a rational function in $\mathbb{L}$, with coefficients in $\mathbb{Z}$. For this we begin with some lemmas about general integrals of this kind of functions.

**Lemma 5.12.** *Let $A = \bigcup_{i\in\mathbb{N}} A_i$ be a disjoint union of stable subsets and suppose that $\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(A_i)$ is convergent (so that $A$ is measurable). Then for any $f \in \mathcal{O}[X_1, \ldots, X_d]$, we have $\int_A |f| d\mu_{\mathcal{X}} = \sum_{i\in\mathbb{N}} \int_{A_i} |f| d\mu_{\mathcal{X}}$.*

*Proof.* By Proposition 5.9 the integral exists. Since $A_i \cap \{\operatorname{ord} f = m\}$ is stable, and $A_i \cap \{\operatorname{ord} f = m\} \subset A_i$, it follows from Lemma 4.31 that the sum $\sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}\big(A_i \cap \{\operatorname{ord} f = m\}\big)$ is convergent. Hence, since the union $A \cap \{\operatorname{ord} f = m\} = \bigcup_{i\in\mathbb{N}} A_i \cap \{\operatorname{ord} f = m\}$ is disjoint it follows from Proposition 4.30 that $\mu_{\mathcal{X}}\big(A \cap \{\operatorname{ord} f = m\}\big) = \sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}\big(A_i \cap \{\operatorname{ord} f = m\}\big)$. We may therefore write

$$\begin{aligned}
\int_A |f| d\mu_{\mathcal{X}} &= \sum_{m\in\mathbb{N}} \mu_{\mathcal{X}}\big(A \cap \{\operatorname{ord} f = m\}\big)\mathbb{L}^{-m} \\
&= \sum_{m\in\mathbb{N}} \sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}\big(A_i \cap \{\operatorname{ord} f = m\}\big)\mathbb{L}^{-m}.
\end{aligned}$$

Because of Lemma 5.6, if we do the above summation over an enumeration of $\mathbb{N}^2$, it is convergent. Hence by Lemma 4.18 it equals

$$\sum_{i\in\mathbb{N}}\sum_{m\in\mathbb{N}} \mu_{\mathcal{X}}\big(A_i \cap \{\operatorname{ord} f = m\}\big)\mathbb{L}^{-m} = \sum_{i\in\mathbb{N}} \int_{A_i} |f| d\mu_{\mathcal{X}}. \qquad \square$$

Let $f_1, \ldots, f_r \in \mathcal{O}[X_1, \ldots, X_d]$. For $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ we write $\{\operatorname{ord} f_i = \alpha_i\}_{i=1}^r$ for the subset $\{a \in \mathcal{X}_\infty : \operatorname{ord} f_i(a) = \alpha_i\}_{i=1}^r \subset \mathcal{X}_\infty$.

**Lemma 5.13.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$. For $I \subset (\mathbb{N} \cup \{\infty\})^d$ (finite or infinite), let $U_I := \bigcup_{\alpha\in I} \{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d$. Let $f \in \mathcal{O}[X_1, \ldots, X_d]$. Then $\int_{U_I} |f| d\mu_{\mathcal{X}} = \sum_{\alpha\in I} \int_{\{\operatorname{ord} X_i = \alpha_i\}} |f| d\mu_{\mathcal{X}}$. (In particular the integral exists.)*

*Proof.* We show that $U_I$ is measurable, the result then follows from Lemma 5.12. Since the union $U_I = \bigcup_{\alpha\in I}\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d$ is disjoint it suffices, by Proposition 4.30, to prove convergence of the sum

$$\sum_{\alpha\in I} \mu_{\mathcal{X}}(\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d).$$

Let $N$ be a large integer. We have $\pi_N(\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d) \subset \pi_N(\{\operatorname{ord} X_i \geq \alpha_i\}_{i=1}^d)$. The underlying scheme of this latter set is

$$\operatorname{Spec} \frac{k[X_{i0}, \ldots, X_{iN}]_{i=1}^d}{(X_{i0}, \ldots, X_{i,\alpha_i - 1})_{i=1}^d} = \operatorname{Spec} k[X_{i\alpha_i}, \ldots, X_{iN}]_{i=1}^d.$$

Hence

$$[\pi_N(\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d)] \leq \mathbb{L}^{Nd - \sum_{i=1}^d \alpha_i}$$

and consequently $\mu_{\mathcal{X}}(\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d) \leq \mathbb{L}^{-\sum_{i=1}^d \alpha_i}$. Now, when $I$ is infinite we see that as $\alpha$ varies over $I$, $\max\{\alpha_i\}_{i=1}^d \to \infty$, hence that $-\sum_{i=1}^d \alpha_i \to -\infty$. So by Example 4.14, $\sum_{\alpha\in I} \mathbb{L}^{-\sum_{i=1}^d \alpha_i}$ is convergent, hence by Lemma 4.16, $\sum_{\alpha\in I} \mu_{\mathcal{X}}(\{\operatorname{ord} X_i = \alpha_i\}_{i=1}^d)$ is convergent. $\square$

## 5.4 Change of variables

We prove three theorems about manipulation of these kind of integrals. Recall that we use $\mathcal{O}$ to denote a complete discrete valuation ring with perfect residue field $k$.

### Linear change of variables

A linear change of variables is easy to do also in the motivic case:

**Proposition 5.14.** *Let $\mathcal{X} = \mathbb{A}^d_{\mathcal{O}}$ and let $a_{ij} \in \mathcal{O}$ be such that the determinant of $M = (a_{ij})$ is in $\mathcal{O}^\times$. Given $f \in \mathcal{O}[X_1, \ldots, X_d]$, define $g(X_1, \ldots, X_d) := f\big((X_1, \ldots, X_d)M\big)$. Then $\int_{\mathcal{X}_\infty} |f| d\mu_{\mathcal{X}} = \int_{\mathcal{X}_\infty} |g| d\mu_{\mathcal{X}}$.*

*Proof.* We first prove that $\mu_{\mathcal{X}}\{\operatorname{ord} f \geq n\} = \mu_{\mathcal{X}}\{\operatorname{ord} g \geq n\}$ for every $n$. We have a map

$$\{\operatorname{ord} f \geq n\} \to \{\operatorname{ord} g \geq n\},$$

given by $(x_1, \ldots, x_d) \mapsto (x_1, \ldots, x_d)M^{-1}$. This is a bijection, for it is well defined since $g(\underline{x}M^{-1}) = f(\underline{x}M^{-1}M) = f(\underline{x}) \equiv 0 \pmod{\mathrm{V}^n}$, and it has a well defined inverse $\underline{x} \mapsto \underline{x}M$. Therefore $\pi_n\{\operatorname{ord} f \geq n\}$ and $\pi_n\{\operatorname{ord} g \geq n\}$ are isomorphic (viewed as subschemes of $\mathcal{X}_n$), consequently $[\pi_n\{\operatorname{ord} g \geq n\}] = [\pi_n\{\operatorname{ord} f \geq n\}]$.

It follows that $\mu_{\mathcal{X}}\{\operatorname{ord} f = n\} = \mu_{\mathcal{X}}\{\operatorname{ord} g = n\}$ for every $n$, hence that the integrals are equal. □

(The proposition holds more generally when $\mathcal{X}$ is any smooth $\mathcal{O}$-scheme and $M\colon \mathcal{X} \to \mathcal{Y}$ is an isomorphism of $\mathcal{O}$-schemes.)

## Separation of variables

Throughout this subsection, let $\mathcal{X} := \mathbb{A}^d_{\mathcal{O}}$ and $\mathcal{Y} := \mathbb{A}^e_{\mathcal{O}}$. Moreover, let $\mathcal{Z} := \mathbb{A}^{d+e}_{\mathcal{O}} = \mathcal{X} \times_{\mathcal{O}} \mathcal{Y}$. We may then identify $(\mathcal{Z})_\infty$ with $\mathcal{X}_\infty \times \mathcal{Y}_\infty$. Our aim is to show the separation of variables result, Theorem 5.17. We do this using two partial results, which we state as the following two lemmas:

**Lemma 5.15.** *If $A \subset \mathcal{X}_\infty$ and $B \subset \mathcal{Y}_\infty$ are stable, then $A \times B \subset \mathcal{Z}_\infty$ is stable, and $\mu_{\mathcal{Z}}(A \times B) = \mu_{\mathcal{X}}(A)\mu_{\mathcal{Y}}(B)$.*

*Proof.* Since $A$ and $B$ are stable there is an integer $n$ with the property that there are a finite number of $k$-varieties $V_i$ such that $\pi_n(A) = \bigcup_i V_i(\overline{k})$, and a finite number of $k$-varieties $U_i$ such that $\pi_n(A) = \bigcup_i U_i(\overline{k})$. We have

$$\begin{aligned}\pi_n(A \times B) =& \pi_n(A) \times \pi_n(B) \\ =& \bigcup_i V_i(\overline{k}) \times \bigcup_j U_j(\overline{k}) \\ =& \bigcup_{i,j} V_i(\overline{k}) \times U_j(\overline{k}) \\ =& \bigcup_{i,j} (V_i \times_k U_j)(\overline{k}),\end{aligned}$$

hence $[\pi_n(A \times B)] = \sum_{i,j}[V_i \times_k U_j] = \sum_{i,j}[V_i][U_j] = (\sum_i [V_i])(\sum_j [U_j]) = [\pi_n(A)][\pi_n(B)]$. Therefore $\mu_{\mathcal{X}\times\mathcal{Y}}(A \times B) = [\pi_n(A \times B)]\mathbb{L}^{-n(d+e)} = ([\pi_n(A)]\mathbb{L}^{-nd})([\pi_n(B)]\mathbb{L}^{-ne}) = \mu_{\mathcal{X}}(A)\mu_{\mathcal{Y}}(B)$. □

**Lemma 5.16.** *If $A \subset \mathcal{X}_\infty$ and $B \subset \mathcal{Y}_\infty$ are measurable, then $A \times B \subset \mathcal{Z}_\infty$ is measurable, and $\mu_{\mathcal{Z}}(A \times B) = \mu_{\mathcal{X}}(A)\mu_{\mathcal{Y}}(B)$.*

*Proof.* Let $A_m$ and $C_i^m$ be stable subsets of $\mathcal{X}_\infty$ such that $A \,\Delta\, A_m \subset \bigcup_{i\in\mathbb{N}} C_i^m$. Let $u_m := \sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(C_i^m)$ be convergent and $\lim_{m\to\infty} u_m = 0$. Let $B_m$ and $D_i^m$ be stable subsets of $\mathcal{Y}_\infty$ such that $B \,\Delta\, B_m \subset \bigcup_{i\in\mathbb{N}} D_i^m$, where $v_m := \sum_{i\in\mathbb{N}} \mu_{\mathcal{X}}(D_i^m)$ is convergent and $\lim_{m\to\infty} v_m = 0$. Then $(A \times B) \,\Delta (A_m \times B_m) = (A \,\Delta\, A_m) \times (B \,\Delta\, B_m) \subset \bigcup_{i\in\mathbb{N}} C_i^m \times D_i^m$. By Lemma 5.15, $\mu_{\mathcal{Z}}(C_i^m \times D_i^m) = \mu_{\mathcal{X}}(C_i^m)\mu_{\mathcal{Y}}(D_i^m)$, hence by Lemma 4.20, the sum $s_m := \sum_{i\in\mathbb{N}} \mu_{\mathcal{Z}}(C_i^m \times D_i^m)$ is convergent, and $s_m \le u_m v_m$. Since $u_m$ and $v_m$ tends to zero the same holds for $u_m v_m$ and consequently also for $s_m$. Hence, since $A_m$ and $B_m$ are stable,

$$\begin{aligned}\mu_{\mathcal{Z}}(A \times B) &= \lim_{m\to\infty} \mu_{\mathcal{Z}}(A_m \times B_m) \\ &= \lim_{m\to\infty} \mu_{\mathcal{X}}(A_m)\mu_{\mathcal{Y}}(B_m) = \mu_{\mathcal{X}}(A)\mu_{\mathcal{Y}}(B). \quad \square\end{aligned}$$

**Theorem 5.17** (Separation of variables)**.** *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be defined as above. If $A \subset \mathcal{X}_\infty$ and $B \subset \mathcal{Y}_\infty$ are measurable, and $f \in \mathcal{O}[X_1, \ldots, X_d]$, $g \in \mathcal{O}[Y_1, \ldots, Y_e]$, then $\int_{A\times B} |fg| d\mu_{\mathcal{Z}} = \int_A |f| d\mu_{\mathcal{X}} \cdot \int_B |g| d\mu_{\mathcal{Y}}$.*

*Proof.* By Proposition 5.9 the integral is convergent:

$$\int_{A\times B} |fg| d\mu_{\mathcal{Z}} = \sum_{\xi\in\mathbb{N}} \mu_{\mathcal{Z}}((A \times B) \cap \{\operatorname{ord} fg = \xi\})\mathbb{L}^{-\xi}$$

Since $\{\operatorname{ord} fg = \xi\} = \bigcup_{\mu+\nu=\xi} \{\operatorname{ord} f = \mu\} \times \{\operatorname{ord} g = \nu\}$ we have

$$(A \times B) \cap \{\operatorname{ord} fg = \xi\} = \bigcup_{\mu+\nu=\xi} \big(A \cap \{\operatorname{ord} f = \mu\}\big) \times \big(B \cap \{\operatorname{ord} g = \nu\}\big)$$

and since this is a disjoint union of measurable sets it follows from the previous lemma that

$$\begin{aligned}&\int_{A\times B} |fg| d\mu_{\mathcal{Z}} \\ &\quad = \sum_{\xi\in\mathbb{N}} \Bigg( \sum_{\mu+\nu=\xi} \mu_{\mathcal{X}}\big(A \cap \{\operatorname{ord} f = \mu\}\big) \cdot \mu_{\mathcal{Y}}\big(B \cap \{\operatorname{ord} g = \nu\}\big) \Bigg) \mathbb{L}^{-\xi}.\end{aligned}$$

Since this sum is convergent, Lemma 4.19 says that we may rearrange it to obtain

$$\begin{aligned}&\left( \sum_{\mu\in\mathbb{N}} \mu_{\mathcal{X}}\big(A \cap \{\operatorname{ord} f = \mu\}\big)\mathbb{L}^{-\mu} \right) \left( \sum_{\nu\in\mathbb{N}} \mu_{\mathcal{Y}}\big(B \cap \{\operatorname{ord} f = \nu\}\big)\mathbb{L}^{-\nu} \right) \\ &\qquad = \int_A |f| d\mu_{\mathcal{X}} \cdot \int_B |g| d\mu_{\mathcal{Y}} \quad \square\end{aligned}$$

## Multiplication by the uniformizer

In this subsection we prove a motivic version of the change of variables induced by multiplication by the uniformizer of the discrete valuation ring.

To simplify notation, we write $k[X_{\bullet 0}, \dots, X_{\bullet N}]$ for the polynomial ring $k[X_{i0}, \dots, X_{iN}]_{i=1}^{n}$. As usual, we use $Q_N$ to denote the universal polynomials defining $Q \in \mathcal{O}[X_1, \dots, X_n]$, so that $Q_N \in k[X_{\bullet 0}, \dots, X_{\bullet N}]$. (See the discussion in the proof of Lemma 5.7.)

**Lemma 5.18.** *Let $\mathcal{X} = \mathbb{A}^n_{\mathcal{O}}$. Let $Q \in \mathcal{O}[X_1, \dots, X_n]$ be a form of degree $s$. If $\operatorname{ord} x_i > 0$ for $i = 1, \dots, n$, then $\operatorname{ord} Q(x_1, \dots, x_n) \geq s$. Moreover for every $\xi \in \mathbb{N}$*

$$\mu_{\mathcal{X}}(\{\operatorname{ord} Q > \xi + s, \operatorname{ord} x_i > 0\}_{i=1}^{n}) = \mathbb{L}^{-n} \mu_{\mathcal{X}}(\{\operatorname{ord} Q > \xi\}).$$

*Proof.* For $N$ sufficiently large, $\pi_{N+1}(\operatorname{ord} Q > \xi + s, \operatorname{ord} x_i > 0)$ is the spectrum of the algebra

$$\frac{k[X_{\bullet 0}, \dots, X_{\bullet N}]}{(Q_0, \dots, Q_{\xi+s}, X_{\bullet 0})}.$$

The class of this in $\mathrm{K}_0(\mathsf{Var}_k)$ equals the class of the spectrum of

$$\frac{k[X_{\bullet 1}, \dots, X_{\bullet N}]}{(Q_0(X_{\bullet 1}), \dots, Q_{\xi}(X_{\bullet 1}, \dots, X_{\bullet \xi+1}))}.$$

In the mixed characteristic case, this is proved using Corollary 5.4 (from the section about the Witt vectors), together with the fact that $[X] = [X_{red}]$ for any scheme $X$. In the equal characteristic case it is straight forward to prove.

Now, using the change of variables $X_{\bullet i} \mapsto X_{\bullet i-1}$, the spectrum of this algebra is $\pi_N(\operatorname{ord} Q > \xi)$. The result follows. □

**Theorem 5.19.** *Let $\mathcal{X} = \mathbb{A}^n_{\mathcal{O}}$, and let $Q \in \mathcal{O}[X_1, \dots, X_n]$ be a form of degree $s$. Define $A = \{\operatorname{ord} x_i > 0\}_{i=1}^{n} \subset \mathcal{X}_\infty$. Then $\int_A |Q| d\mu_{\mathcal{X}} = \mathbb{L}^{-s-n} \int_{\mathcal{X}_\infty} |Q| d\mu_{\mathcal{X}}$.*

*Proof.* By the first part of Lemma 5.18, $\{\operatorname{ord} Q = \xi, \operatorname{ord} x_i > 0\}_{i=1}^{n} = \emptyset$ for $\xi < s$, hence

$$\int_A |Q| d\mu_{\mathcal{X}} = \sum_{\xi \geq 0} \mu_{\mathcal{X}}\{\operatorname{ord} Q = \xi + s, \operatorname{ord} x_i > 0\} \mathbb{L}^{-(\xi+s)}.$$

Using the second part of the lemma it follows that this equals

$$\sum_{\xi \geq 0} \mathbb{L}^{-n} \mu_{\mathcal{X}}(\operatorname{ord} Q = \xi) \mathbb{L}^{-(\xi+s)} = \mathbb{L}^{-s-n} \int_{\mathcal{X}_\infty} |Q| d\mu_{\mathcal{X}}. \qquad \square$$

## 5.5 The integral of the absolute value of a product of special linear forms

As before, we let $\mathcal{O}$ be a complete discrete valuation ring with residue field $k$, of one of the types described in Section 5.3. We define, for any $n \in \mathbb{N}$, $\mathcal{X}^n = \mathbb{A}^n_{\mathcal{O}}$, and we let $\mathcal{X}^n_\infty$ be its space of arcs.

The main result of this section is Theorem 5.27. In Theorem 5.27 we give a recursive method to compute $I = \int_{\mathcal{X}^n_\infty} |\prod \ell_i| d\mu_{\mathcal{X}^n}$, where $\ell_i \in \mathbb{Z}[X_1, \ldots, X_n]$ are linear forms. When $\mathcal{O} = k[[t]]$, these forms are arbitrary; when $\mathcal{O} = \mathbf{W}(k)$ we need the forms to be of a rather special type. The restriction in the second case is taken care of in Section 5.6, in Theorem 5.32, when we give a recursion that works for general forms in case $\mathcal{O} = \mathbf{W}(k)$, provided that the characteristic of $k$ is sufficiently large.

When applicable, these theorems also give a function $f \in \mathbb{Z}(T)$ such that $I = f(\mathbb{L})$. If we let $\mathcal{O} = \mathbb{Z}_p$ this also gives a motivic explanation to the phenomenon discussed in the introduction. For by applying $\mathrm{C}_q$ to $I$, for different powers $q$ of $p$, we get $\int_{\mathbf{W}(\mathbb{F}_q)} |\prod \ell_i| d\mu_{Haar} = f(q)$.

**Remark 5.20.** *As mentioned in the introduction, it is not true in general that the motivic integral of the absolute value of a polynomial is equal to $f(\mathbb{L})$, with $f \in \mathbb{Z}(T)$. This can be seen from the integral $\int_{\mathcal{X}^1_\infty} |x^2+1| d\mu_{\mathcal{X}^1}$: When $\mathcal{O} = \mathbb{Z}_p$ with $p \equiv 3 \mod 4$, then by Example 5.11 this integral is equal to $1 - [\operatorname{Spec} \mathbb{F}_{p^2}]/(\mathbb{L}+1)$, and by applying the point counting homomorphism for different powers of p we see that this cannot be equal to $f(\mathbb{L})$ for $f \in \mathbb{Z}(T)$.*

### The crucial step in the recursion

The main result of this subsection is a change of variables result, Theorem 5.26, which is the crucial step in performing the recursion of Theorem 5.27. However, since the motivic recursion is inspired by a $p$-adic computation, but requires a different, more complicated, method, we begin by briefly discuss the $p$-adic method, and why it does not translate to the motivic case. This is done in the following example:

**Example 5.21.** *Suppose we want to compute $I = \int_A |\alpha_1 - \alpha_2| d\underline{\alpha}$, where we integrate over the set $A \subset \mathbb{Z}_p^3$ of tuples such that $\alpha_1 \equiv \alpha_2 \mod p$ and $\alpha_1 \not\equiv \alpha_3 \mod p$. One way to do this is to express the integral in terms of an integral of the same function, but now integrated over all of $\mathbb{Z}_p^3$. This is what is done for general integrals, and in the motivic setting in Theorem 5.26. We illustrate this method on the integral $I$: Choosing a set of representatives of the cosets of $p\mathbb{Z}_p$, $x \mapsto \tilde{x} \colon \mathbb{F}_p \to \mathbb{Z}_p$, we may write each such $\alpha_i$ uniquely as $\alpha_i = \tilde{x}_i + p\beta_i$. Moreover, $\alpha_i \equiv \alpha_j \mod p$*

*if and only if $x_i = x_j$ in $\mathbb{F}_p$. Hence*

$$I = \sum_{\substack{\underline{x}\in\mathbb{F}_p^3:\\ x_1=x_2\neq x_3}} \int_{\alpha_i=\tilde{x}_i+p\beta_i} |\alpha_i-\alpha_j|d\underline{\alpha}.$$

*On each of these integrals we perform the change of variables $\alpha_i = \tilde{x}_i+p\beta_i$ (the $x_i$ are fixed). The Jacobian of this has absolute value $p^{-3}$, hence each of the integrals in the sum equal $p^{-3}\int_{\mathbb{Z}_p^3}|p\beta_1 - p\beta_2|d\underline{\beta}$. (This step do not generalize to arbitrary linear forms, since then the $\tilde{x}_i$ will not cancel.) Since there are $p(p-1)$ terms in the sum, we find that $I = p(p-1)p^{-3-1}\int_{\mathbb{Z}_p^3}|\beta_1-\beta_2|d\underline{\beta}$.*

*The main problem in translating this to the motivic setting is that it is not possible to partition the integration set in this manner. However, this partition is not visibly in the result of the computation, so the result might still be possibly to translate into the motivic setting. This is done in Theorem 5.26. Also, note that in that theorem we work in the most general case possibly, contrary to in this example.*

To prove Theorem 5.26 we first needs two lemmas about the Witt vectors. See Section 5.2 for notation used in connection with the Witt vectors.

**Lemma 5.22.** *Let $k$ be a perfect field of characteristic $p$, where $p$ is a prime different from 2, and let $\ell = aX + bY \in \mathbf{W}(k)[X,Y]$ be a linear form in two variables whose coefficients are multiplicative representatives. For $x = (x_0, x_1, \dots) \in \mathbf{W}(A)$, where $A$ is a $k$-algebra, define $\tilde{x} = (x_1, x_2, \dots) \in \mathbf{W}(A)$. Let $y \in \mathbf{W}(A)$ and define $\tilde{y}$ similarly. Suppose that $\ell(x,y) \equiv 0 \mod \mathrm{V}$. Then $\ell(x,y) = \mathrm{V}\,\ell(\tilde{x},\tilde{y})$. If $p = 2$ the result holds provided $\ell = X - Y$.*

*Proof.* When $p \neq 2$ we have for every ring $A$ that $-1 = -\mathrm{r}(1) = \mathrm{r}(-1) \in \mathbf{W}(A)$. This follows in the standard way by first proving it when $p$ is invertible in $A$. In this case $W^*$ is an isomorphism, and since $W^*\mathrm{r}(x) = (x, x^p, x^{p^2}, \dots)$ we have $W^*\mathrm{r}(1) + W^*\mathrm{r}(-1) = 0$ so the result follows. But if it this result is true for a ring $A$, it holds for every sub- and quotient ring, hence for every ring.

Let $a = \mathrm{r}\,a_0$ and $b = \mathrm{r}\,b_0$. The condition $\ell(x,y) \equiv 0 \mod \mathrm{V}$ then means that $a_0x_0 + b_0y_0 = 0$. From what was said above it follows that $a\,\mathrm{r}\,x_0 + b\,\mathrm{r}\,y_0 = \mathrm{r}(a_0x_0) + \mathrm{r}(b_0y_0) = \mathrm{r}(a_0x_0) + \mathrm{r}(-a_0x_0) = 0$. We then have

$$\begin{aligned}\ell(x,y) =& \ell(\mathrm{r}\,x_0 + \mathrm{V}\,\tilde{x}, \mathrm{r}\,y_0 + \mathrm{V}\,\tilde{y})\\ =&(a\,\mathrm{r}\,x_0 + b\,\mathrm{r}\,y_0) + (a\,\mathrm{V}\,\tilde{x} + b\,\mathrm{V}\,\tilde{y})\\ =&\mathrm{V}(\mathrm{F}\,a\cdot\tilde{x}) + \mathrm{V}(\mathrm{F}\,b\cdot\tilde{y})\\ =&\mathrm{V}\,\ell(\tilde{x},\tilde{y}).\end{aligned}$$

When $p = 2$ we may still prove the result when $\ell = X - Y$, for if $\ell(x,y) \equiv 0 \mod \mathrm{V}$ then $x_0 = y_0$, hence $\mathrm{r}(x_0) = \mathrm{r}(y_0)$. □

Recall the notation used in Section 5.4: We write $k[X_{\bullet 0}, \dots, X_{\bullet N}]$ for the polynomial ring $k[X_{i0}, \dots, X_{iN}]_{i=1}^{n}$, and use $Q_N$ to denote the universal polynomials defining $Q \in \mathcal{O}[X_1, \dots, X_n]$.

**Lemma 5.23.** *Fix a perfect field $k$ of characteristic $p$. Let $P = \prod_{i=1}^{d} \ell_i \in \mathbf{W}(k)[X_1, \dots, X_n]$, where the $\ell_i$ are linear forms, all of whose coefficients are multiplicative representatives, at most two of which are non-zero. Moreover, if $p = 2$, assume that all the forms are of the type $X_i - X_j$. Let $x_1, \dots, x_n \in \mathbf{W}(A)$, where $A$ is a $k$-algebra, be such that $\ell_i(x_1, \dots, x_n) \equiv 0 \mod \mathrm{V}$ for every $i$, and define $\tilde{x_i}$ as in Lemma 5.22. Then*

$$P(x_1, \dots, x_n) = \mathrm{F}^{d-1}\,\mathrm{V}^{d}\,P(\tilde{x_1}, \dots, \tilde{x_n}).$$

*In particular, working in $\mathbf{W}(k[X_{\bullet n}]_{n \in \mathbb{N}})$, we have $P_\xi = 0$ for $\xi < d$, and $P_{\xi+d} = P_\xi(X_{\bullet 1}, \dots, X_{\bullet \xi+1})^{p^{d-1}}$ for $\xi \in \mathbb{N}$.*

*Proof.* Using the preceding lemma and Corollary 5.3 we get

$$\begin{aligned} P(x_1, \dots, x_n) &= \prod_{i=1}^{d} \ell_i(x_1, \dots, x_n) \\ &= \prod_{i=1}^{d} \mathrm{V}\,\ell_i(\tilde{x_1}, \dots, \tilde{x_n}) \\ &= \mathrm{F}^{d-1}\,\mathrm{V}^{d} \prod_{i=1}^{d} \ell_i(\tilde{x_1}, \dots, \tilde{x_n}) \\ &= \mathrm{F}^{d-1}\,\mathrm{V}^{d}\,P(\tilde{x_1}, \dots, \tilde{x_n}) \end{aligned}$$

□

In the case of equal characteristic, this lemma also holds, but for any set of linear forms. Its proof is straight forward, we just state the result:

**Lemma 5.24.** *Let $k$ be a field, and let $\mathcal{O} = k[[t]]$. Let $P = \prod_{i=1}^{d} \ell_i \in \mathcal{O}[X_1, \dots, X_n]$, where the $\ell_i$ are linear forms. Let $x_1, \dots, x_n \in A[[t]]$, where $A$ is a $k$-algebra, be such that $\ell_i(x_1, \dots, x_n) \equiv 0 \mod t$ for every $i$, and define $\tilde{x_i}$ as in Lemma 5.22 (recall that we write elements of power series rings as tuples). Then*

$$P(x_1, \dots, x_n) = t^d P(\tilde{x_1}, \dots, \tilde{x_n}).$$

*In particular, working in $k[X_{\bullet n}]_{n \in \mathbb{N}}[[t]]$, we have $P_\xi = 0$ for $\xi < d$, and $P_{\xi+d} = P_\xi(X_{\bullet 1}, \dots, X_{\bullet \xi+1})$ for $\xi \in \mathbb{N}$.*

**Lemma 5.25.** *Let $S$ be a finite set. Let $Q = \prod_{i \in S} \ell_i$, where the $\ell_i \in \mathcal{O}[X_1, \dots, X_n]$ are linear forms satisfying the following conditions:*

- *If $\mathcal{O} = k[[t]]$ then the $\ell_i$ are arbitrary*
- *If $\mathcal{O} = \mathbf{W}(k)$, where $k$ is a field of prime characteristic different from 2, then each $\ell_i$ is linear form in at most two variables, and its coefficients are multiplicative representatives.*
- *If $\mathcal{O} = \mathbf{W}(k)$, where $k$ is a field of characteristic 2, assume that all the forms are of the type $X_i - X_j$*

*Let $T$ be a subset of $S$. If $\operatorname{ord}\ell_i > 0$ for $i \in T$, then $\operatorname{ord} Q \geq |T|$. Moreover, for every integer $\xi \geq -1$ we have*

$$\mu_{\mathcal{X}^n}(\operatorname{ord} Q > \xi + |T|, \operatorname{ord}\ell_i > 0, i \in T, \operatorname{ord}\ell_i = 0, i \in S \setminus T) \\ = \tfrac{[H_T]}{\mathbb{L}^n}\mu_{\mathcal{X}^n}(\operatorname{ord} Q > \xi),$$

*where $H_T$ is the subvariety of $\mathcal{X}_1^n = \mathbb{A}_k^n$ given by $(\ell_i)_0 = 0$ for $i \in T$ and $(\ell_i)_0 \neq 0$ for $i \in S \setminus T$.*

*Proof.* For $N$ sufficiently large, $\pi_{N+1}(\operatorname{ord} Q > \xi + |T|, \operatorname{ord}\ell_i > 0, i \in T, \operatorname{ord}\ell_i = 0, i \notin T)$ is the spectrum of the algebra

$$\frac{k[X_{\bullet 0}, \ldots, X_{\bullet N}][(\ell_i)_0^{-1}]_{i \notin T}}{\left(Q_0, \ldots, Q_{\xi+|T|}, (\ell_i)_0\right)_{i \in T}}.$$

By Lemma 5.23 or 5.24 the class of this equals the class the spectrum of

$$\frac{k[X_{\bullet 0}, \ldots, X_{\bullet N}][(\ell_i)_0^{-1}]_{i \notin T}}{\left(Q_0(X_{\bullet 1}), \ldots, Q_\xi(X_{\bullet 1}, \ldots, X_{\bullet \xi+1}), (\ell_i)_0\right)_{i \in T}}.$$

Now, since the $(\ell_i)_0$ only involves the variables $X_{\bullet 0}$, we may write this as

$$\frac{k[X_{\bullet 0}][(\ell_i)_0^{-1}]_{i \notin T}}{((\ell_i)_0)_{i \in T}} \otimes_k \frac{k[X_{\bullet 1}, \ldots, X_{\bullet N}]}{(Q_0(X_{\bullet 1}), \ldots, Q_\xi(X_{\bullet 1}, \ldots, X_{\bullet \xi+1}))}$$

The spectrum of the first factor is $H_T$ whereas, using the change of variables $X_{\bullet i} \mapsto X_{\bullet i-1}$, the spectrum of the second factor is $\pi_N(\operatorname{ord} Q > \xi)$. The result follows. □

**Theorem 5.26.** *Let $S$ be a finite set. For $i \in S$, let $\ell_i \in \mathcal{O}[X_1, \ldots, X_n]$ be a linear forms, satisfying the conditions of Lemma 5.25. Let $Q = \prod_{i \in S} \ell_i$. For $T \subset S$, define $Q_T = \prod_{i \in T} \ell_i$. Let $H_T$ be the subvariety of $\mathcal{X}_1^n = \mathbb{A}_k^n$ given by $(\ell_i)_0 = 0$ for $i \in T$ and $(\ell_i)_0 \neq 0$ for $i \in S \setminus T$. Then*

$$\int_{\substack{\operatorname{ord}\ell_i \neq 0, i \in T \\ \operatorname{ord}\ell_i = 0, i \notin T}} |Q| d\mu_{\mathcal{X}^n} = [H_T]\mathbb{L}^{-|T|-n} \int_{\mathcal{X}_\infty^n} |Q_T| d\mu_{\mathcal{X}^n}.$$

*Proof.* By the first part of the lemma,

$$\{\operatorname{ord} Q = \xi, \operatorname{ord}\ell_i > 0, i \in T, \operatorname{ord}\ell_i = 0, i \notin T\} = \emptyset$$

for $\xi < |T|$, hence

$$\int_{\substack{\operatorname{ord}\ell_i \neq 0, i\in T\\ \operatorname{ord}\ell_i = 0, i\notin T}} |Q| d\mu_{\mathcal{X}^n} = \int_{\substack{\operatorname{ord}\ell_i \neq 0, i\in T\\ \operatorname{ord}\ell_i = 0, i\notin T}} |Q_T| d\mu_{\mathcal{X}^n}$$

equals

$$\sum_{\xi \geq 0} \mu_{\mathcal{X}^n}(\operatorname{ord} Q_T = \xi + |T|, \operatorname{ord}\ell_i > 0, i \in T, \operatorname{ord}\ell_i = 0, i \notin T)\mathbb{L}^{-(\xi+|T|)}.$$

Using the second part of the lemma it follows that this equals

$$\sum_{\xi \geq 0} [H_T]\mathbb{L}^{-n}\mu_{\mathcal{X}^n}(\operatorname{ord} Q_T = \xi)\mathbb{L}^{-(\xi+|T|)} = [H_T]\mathbb{L}^{-|T|-n} \int_{\mathcal{X}^n_\infty} |Q_T| d\mu_{\mathcal{X}^n}.$$

□

## The recursion

We are finally ready to give the motivic version of the recursion. We will then see in Example 5.30 how to use this in practice.

**Theorem 5.27.** *Let $\mathcal{O}$ be a complete discrete valuation ring. Let $Q = \prod_{i\in S} \ell_i$, where $\ell_i \in \mathcal{O}[X_1, \dots, X_n]$ are linear forms, satisfying the conditions of Lemma 5.25. Then $\int_{\mathcal{X}^n_\infty} |Q| d\mu_{\mathcal{X}^n}$ is a rational function in $\mathbb{L}$, i.e., there is an $f \in \mathbb{Z}(T)$ such that the integral equals $f(\mathbb{L})$. Moreover, $f$ may be computed explicitly by recursion.*

*Proof.* This recursion is immediate, using Theorem 5.26: Write, for $T \subset S$, $Q_T := \prod_{i\in T} \ell_i$. We have

$$\int_{\mathcal{X}^n_\infty} |Q| d\mu_{\mathcal{X}} = \int_{\bar{\ell}_i = 0,\, i\in S} |Q| d\mu_{\mathcal{X}} + \sum_{T \subsetneq S} \int_{\substack{\bar{\ell}_i = 0,\, i\in T\\ \bar{\ell}_i \neq 0,\, i\in S\setminus T}} |Q_T| d\mu_{\mathcal{X}}.$$

Theorem 5.26 now shows that

$$(1 - [H_S]\mathbb{L}^{-|S|-n}) \int_{\mathcal{X}^n_\infty} |Q| d\mu_{\mathcal{X}} = \sum_{T \subsetneq S} [H_T]\mathbb{L}^{-|T|-n} \int_{\mathcal{X}^n_\infty} |Q_T| d\mu_{\mathcal{X}}. \quad (5.28)$$

The right hand side is known inductively. Moreover, since $\dim H_S \leq n$ it follows that $[H_S]\mathbb{L}^{-|S|-n} \in \mathrm{F}^{\leq -|S|}\,\mathcal{M}_k$, hence it is not equal to 1. Also, by the first part of the proof of Theorem 5.29, $H_S$ is an affine space, hence equal to $\mathbb{L}^m$ for some $m$. Therefore, by Example 4.14, $1 - [H_S]\mathbb{L}^{-|S|-n}$ is invertible. Hence $\int_{\mathcal{X}^n_\infty} |Q| d\mu_{\mathcal{X}}$ is as a rational function in $\mathbb{L}$ and the classes of various hyper plane arrangements. Because of the next theorem, Theorem 5.29, it follows that the integral is a rational function in $\mathbb{L}$. □

The following theorem is of course already well known. We provide a proof for completeness.

**Theorem 5.29.** *Let $V$ be a finite dimensional $k$-space, $I$ a finite set and for every $i \in I$, let $\ell_i \colon V \to k$ be a linear function. For $S \subset I$, let $H_S = \{\ell_i = 0, i \in S, \ell_i \neq 0, i \notin S\} \subset \mathbb{A}_V$, i.e, $H_S(R) = \{x \in R \otimes_k V : \ell_i(x) = 0, i \in S, \ell_i(x) \in R^\times, i \notin S\}$ for every $k$-algebra $R$. Then there is a polynomial $p \in \mathbb{Z}[X]$ such that $[H_S] = p(\mathbb{L}) \in \mathrm{K}_0(\mathsf{Var}_k)$.*

*Proof.* First, let $U = \cap_{i \in S} \ker \ell_i$. Then $H_S = \{\ell_i|_U \neq 0, i \notin S\} \subset \mathbb{A}_U$. We may therefore assume that $S = \emptyset$. We now prove the claim by induction on the number of hyperplanes: First, let $\ell \colon V \to k$ be non-zero. We may then choose a basis of $V$, $\{e_1, \ldots, e_d\}$ such that $\ell(e_1) = 1$ and $\ell(e_i) = 0$ for $i > 1$. Hence $\{\ell \neq 0\} = \operatorname{Spec} k[X_1, \ldots, X_d][X_1^{-1}]$, and consequently $[\{\ell \neq 0\}] = \mathbb{L}^{d-1}(\mathbb{L} - 1) \in \mathrm{K}_0(\mathsf{Var}_k)$.

Assume now that the claim holds for any $k$-space $V$ and for any collection of less than $n$ hyperplanes. Let $H = \{\ell_i \neq 0\}_{i=1}^n \subset \mathbb{A}_V$. Define $U = \ker \ell_n$. Then $\{\ell_i \neq 0\}_{i=1}^{n-1} \subset \mathbb{A}_V$ is the disjoint union of $H$ and $\{\ell_i|_U \neq 0\}_{i=1}^{n-1} \subset \mathbb{A}_U \subset \mathbb{A}_V$, hence $[H] = [\{\ell_i \neq 0\}_{i=1}^{n-1}] - [\{\ell_i|_U \neq 0\}_{i=1}^{n-1}] \in \mathrm{K}_0(\mathsf{Var}_k)$, and we are done by induction. □

We illustrate this with an example.

**Example 5.30.** *We apply this to the example that motivated this computations, the integral $V^n := \int_{\mathcal{X}_\infty^n} \left|\prod_{1 \le i < j \le n}(X_i - X_j)\right| d\mu_{\mathcal{X}^n}$. By* (5.28) *we have*

$$\left(1 - [\{X_1 = X_2\}]\mathbb{L}^{-2-1}\right)V^2 = [\{X_1 \neq X_2\}]\mathbb{L}^{-2}.$$

*Since $[\{X_1 = X_2\}] = \mathbb{L}$ and $[\{X_1 \neq X_2\}] = \mathbb{L}^2 - \mathbb{L}$ it follows that $V^2 = \mathbb{L}/(\mathbb{L}+1)$.*

*Note that $V^2$ may also be computed using change- and separation of variables, together with the result of Example 5.11: $V^2 = \int_{\mathcal{X}_\infty^2} |X_1 - X_2| d\mu_{\mathcal{X}^2} = \int_{\mathcal{X}_\infty^2} |X_1| d\mu_{\mathcal{X}^2} = \int_{\mathcal{X}_\infty^1} |X_1| d\mu_{\mathcal{X}^1} \cdot \int_{\mathcal{X}_\infty^1} d\mu_{\mathcal{X}^1} = \mathbb{L}/(\mathbb{L}+1)$.*

*Next we compute $V^3$. Here we really need our recursion, there is no straight forward way to compute $V^3$, as was the case with $V^2$. So we use* (5.28) *to obtain*

$$\begin{aligned}(1 - [\{X_1 = X_2 = X_3\}]\mathbb{L}^{-3-3})V^3 & \\ &= 3[\{X_1 = X_2 \neq X_3\}]\mathbb{L}^{-1-3}\int_{\mathcal{X}^3} |X_1 - X_2| d\mu_{\mathcal{X}} \\ &\quad + [\{X_1 \neq X_2, X_2 \neq X_3, X_1 \neq X_3\}]\mathbb{L}^{-3}.\end{aligned}$$

*Here the classes of the two first hyperplane arrangements are straight forward to compute, $[\{X_1 = X_2 = X_3\}] = \mathbb{L}$, and $[\{X_1 = X_2 \neq X_3\}] = \mathbb{L}(\mathbb{L}-1)$. For the third one we use the method of Theorem 5.29 to obtain (the expected result) $[H_\emptyset] = \mathbb{L}(\mathbb{L}-1)(\mathbb{L}-2)$. Finally, by separation of variables, the integral in the right hand side equals $V^2$. Putting things together we obtain $V^3 = \frac{(1-\mathbb{L}^{-1})(1-\mathbb{L}^{-1}+\mathbb{L}^{-2})}{(1+\mathbb{L}^{-1})(1-\mathbb{L}^{-5})}$.*

**Remark 5.31.** *The integrals $V^n$ comes up in connection with the problem of computing the density of the monic nth degree polynomials whose Galois group is the full symmetric group and whose nontrivial inertia groups are generated by a transposition. It is possible to do the same thing for general Weyl groups, and the integrals appearing in these computations are also possible to handle using the recursion.*

## 5.6 The integral of the absolute value of a product of arbitrary linear forms

In this section, we give a way to compute the integrals for arbitrary forms that works also in the mixed characteristic case. However, we have to assume that the characteristic of the residue field is sufficiently large, and that the forms have coefficients in $\mathbb{Z}$. Also, this method is rather complicated to use in practice. (This method works also in the case if equal characteristic, but since we already have Theorem 5.27 in that case, we state the following theorem in the case of mixed characteristic only.)

**Theorem 5.32.** *Let $\mathcal{X} = \mathbb{A}^n_{\mathcal{O}}$, where $\mathcal{O} = \mathbf{W}(k)$ and $k$ is a perfect field of characteristic $p$. Let $Q = \prod_{i \in S} \ell_i$, where $\ell_i \in \mathbb{Z}[X_1, \ldots, X_n]$ are linear forms. Then, if $p$ is sufficiently large, $\int_{\mathcal{X}_\infty} |Q| d\mu_{\mathcal{X}^n} \in \overline{\mathrm{K}_0}(\mathsf{Var}_k)$ is a rational function in $\mathbb{L}$, i.e., there is an $f \in \mathbb{Z}(T)$ such that the integral equals $f(\mathbb{L})$. Moreover, there is an algorithm for computing $f$.*

*Proof.* We do the recursion over the number of forms, the cardinality of $S$. Also, for the recursion to work, we compute $\int_{\overline{m}_j=0,\, j\in M} |Q| d\mu$ for any finite set of linear functions $m_j$, $j \in M$ (in fact, it would suffice to compute it for the two cases when $M = \emptyset$, and when all the $\ell_j$ are contained among the $m_j$):

So assume that all such integrals are known when $Q$ is any product of less that $|S|$ linear forms. We then want to compute $\int_{\overline{m}_j=0,\, j\in M} |Q| d\mu$, where $Q = \prod_{i \in S} \ell_i$

As a first reduction, note that we may assume that

$$K := \bigcap_{i \in S} \ker \ell_i \cap \bigcap_{j \in M} \ker m_j$$

is equal to 0. For let $K'$ be a linear complement of $K$. Let $\mathcal{Y} = \mathbb{A}_K$ and $\mathcal{Z} = \mathbb{A}_{K'}$. Then, by separation of variables, Theorem 5.17,

$$\int_{\overline{m}_j=0,\, j\in M} |Q| d\mu_{\mathcal{X}} = \int_{\mathcal{Y}_\infty} d\mu_{\mathcal{Y}} \int_{\overline{m}_j=0,\, j\in M} |Q| d\mu_{\mathcal{Z}} = \int_{\overline{m}_j=0,\, j\in M} |Q| d\mu_{\mathcal{Z}}.$$

Hence, after a linear change of variables, Proposition 5.14, we may assume that all the elements of the dual basis is contained among the $\ell_i$

and $m_j$, i.e., $\ell_i = x_i$ for $i = 1, \dots, n'$ and $m_j = x_{n'+j}$ for $j = 1, \dots, n-n'$. For this to work, we need the Jacobian to be invertible in $\mathbb{Z}_p$, which it is for $p$ sufficiently large.

For $T \subset S$, write $Q_T := \prod_{i \in T} \ell_i$. Note that, for $T \subsetneq S$, the integral

$$I_T := \int_{\substack{\overline{\ell}_i = 0,\, i \in T \\ \overline{\ell}_i \neq 0,\, i \in S \setminus T}} |Q_T| d\mu$$

is known by induction. For we may eliminate the $\overline{\ell}_i \neq 0$ conditions in the following way: Choose $t \in S \setminus T$. Then

$$I_T + \int_{\substack{\overline{\ell}_i = 0,\, i \in T \\ \overline{\ell}_i \neq 0,\, i \in S \setminus (T \setminus \{t\}) \\ \overline{\ell}_t = 0}} |Q_T| d\mu = \int_{\substack{\overline{\ell}_i = 0,\, i \in T \\ \overline{\ell}_i \neq 0,\, i \in S \setminus (T \setminus \{t\})}} |Q_T| d\mu,$$

so inductively, $I_T$ may be expressed as an alternating sum of

$$\int_{\substack{\overline{\ell}_i = 0,\, i \in T \\ \overline{\ell}_i = 0,\, i \in T'}} |Q_T| d\mu,$$

for different $T' \subset S \setminus T$.

We now compute $\int_{\mathcal{X}_\infty^n} |Q| d\mu$ in terms of things that are already known by induction:

$$\int_{\mathcal{X}_\infty^n} |Q| d\mu = \int_{\overline{\ell}_i = 0,\, i \in S} |Q| d\mu + \sum_{T \subsetneq S} \int_{\substack{\overline{\ell}_i = 0,\, i \in T \\ \overline{\ell}_i \neq 0,\, i \in S \setminus T}} |Q_T| d\mu.$$

Since we may assume that $\ell_i = x_i$ for $i = 1, \dots, n$, we have $\{\overline{\ell}_i = 0\}_{i \in S} = \{\overline{x}_i = 0\}_{i=1}^n$. Hence first term of the sum is, by Theorem 5.19, equal to $\mathbb{L}^{-n-s} \int_{\mathcal{X}_\infty^n} |Q| d\mu$. The rest of the terms (after the summation sign), is already known by induction. Denote this second sum with $\Sigma$. It then follows that $\int_{\mathcal{X}_\infty} |Q| d\mu = (1 - \mathbb{L}^{-s-n})^{-1} \Sigma$. (That $1 - \mathbb{L}^{-s-n}$ is invertible follows from Example 4.14.)

Consider now an arbitrary integral:

$$\int_{\overline{m}_j = 0,\, j \in M} |Q| d\mu = \int_{\substack{\overline{\ell}_i = 0,\, i \in S \\ \overline{m}_j = 0,\, j \in M}} |Q| d\mu + \sum_{T \subsetneq S} \int_{\substack{\overline{\ell}_i = 0,\, i \in T \\ \overline{\ell}_i \neq 0,\, i \in S \setminus T \\ \overline{m}_j = 0,\, j \in M}} |Q_T| d\mu$$

The terms after the summation sign is again taken care of by the induction assumption. The first term is, because of Theorem 5.19, equal to $\mathbb{L}^{-n-s} \int_{\mathcal{X}_\infty^n} |Q| d\mu$. By the first part of the induction step, this is already known. (It would suffice to compute this integral in the case when all $\ell_i$ are contained among the $m_i$, and in this case we get $\int_{\overline{m}_j = 0,\, j \in M} |Q| d\mu = \mathbb{L}^{-n-s} \int_{\mathcal{X}_\infty^n} |Q| d\mu$.) $\square$

## 5.7 Varying the prime

Let us mention one remaining question about these integrals: From the computations performed in Theorem 5.27 and 5.32 it is clear that $f \in \mathbb{Z}(T)$, the rational function with the property that $\int_{\mathcal{X}^n_\infty} |\prod \ell_i| d\mu_{\mathcal{X}^n} = f(\mathbb{L})$, is independent of $\mathcal{O}$, provided that we choose $\mathcal{O}$ among the rings

$$\{\mathbf{W}(k) : p \text{ sufficiently large}\} \cup \{k[[t]]\}.$$

In particular, we have $\int_{\mathbb{Z}_p^n} |\prod \ell_i| d\mu_{Haar} = f(p)$ for $p$ big enough. It would be desirable to have a motivic explanation also for this fact. This can probably be achieved using the theory of motivic integration developed in [CL08]. Alternatively, we indicate in the following paragraph how the problem could be handled using geometric motivic integration:

### Motivic computation of the rational function

By Theorem 6.1 of [DL01], if $\mathcal{O} = \mathbb{Q}[[t]]$, if $\mathcal{Y} = \{P = 0\} \subset \mathcal{X}^n$ where $P$ is a polynomial, and if $J(T) = \sum_{i \geq 0} [\mathcal{Y}_{i+1}] T^i \in \mathcal{M}_{\mathbb{Q}}[[T]]$, then the following holds: Firstly, $J(T)$ is rational, with denominator consisting of factors of the form $1 - \mathbb{L}^a T^b$. Moreover, if we choose representatives for the coefficients of $J(T)$, defined over $\mathbb{Z}$, and then count $\mathbb{F}_p$-points on them, then for $p$ sufficiently large we get the power series $J_p(T) = \sum_{i \geq 0} |\{\underline{x} \in (\mathbb{Z}/(p^{i+1}))^n : P(\underline{x}) = 0\}| T^i$.

The process of choosing representatives for elements of $\mathcal{M}_{\mathbb{Q}}$, and then counting $\mathbb{F}_p$-points on them for all $p$, defines a homomorphism $\mathrm{C}\colon \mathcal{M}_{\mathbb{Q}} \to \prod_p \mathbb{Q}/{\sim}$, where $(a_p)$ and $(b_p)$ are equivalent if $a_p = b_p$ for almost all $p$. (The filter product with respect to the Fréchet filter.) Since $\int_{\mathbb{Z}_p^n} |P| d\mu_{Haar} = 1 + p^{-n-1}(1-p) J_p(p^{-1-n})$, one could define the motivic integral of $P$ by first computing $J$ as a rational function, and then define the integral to be

$$I = 1 + \mathbb{L}^{-n-1}(1 - \mathbb{L}) J(\mathbb{L}^{-1-n}) \in \mathcal{M}_{\mathbb{Q}}[(1 - \mathbb{L}^i)^{-1}]_{i \geq 1}.$$

This integral then has the property that $\mathrm{C} I = \left(\int_{\mathbb{Z}_p^n} |P| d\mu_{Haar}\right)_p \in \prod_p \mathbb{Q}/{\sim}$. For example, let $P = X^2 + 1$, and let $m = [\operatorname{Spec} \mathbb{Q}[X]/(P(X))]$. Using Theorem 5.8 one sees that $J(T) = m/(1-T)$, so the integral of $P$ is $1 - m/(\mathbb{L}+1)$. Hence, for $p$ sufficiently large the value of $\int_{\mathbb{Z}_p} |P| d\mu_{Haar}$ is 1 if $p \equiv 1 \mod 4$ and $(p-1)/(p+1)$ if $p \equiv 3 \mod 4$ (a result that of course is true for all $p$).

Probably the method used to prove Theorem 5.27 can be used also to compute $J(T)$ when $P$ is a product of linear forms, showing that the integral equals $f(\mathbb{L}) \in \mathcal{M}_{\mathbb{Q}}[(1 - \mathbb{L}^i)^{-1}]_{i \geq 1}$, hence that $\int_{\mathbb{Z}_p^n} |\prod \ell_i| d\mu_{Haar} = f(p)$ for $p$ big enough.

### Small $p$

We now give an example showing that $f$ is not independent of $p$ for all $p$, only for $p$ sufficiently large. For this, fix a prime $l$. Let $p$ be any prime different from $l$, and let $\mathcal{X}^2 = \mathbb{A}^2_{\mathbb{Z}_p}$. Then, using Proposition 5.14, Proposition 5.17, and Proposition 5.10, we see that

$$\int_{\mathcal{X}^2_\infty} |(x_1 + x_2)(x_1 - (l-1)x_2)| d\mu_{\mathcal{X}^2}$$
$$= \int_{\mathcal{X}^2_\infty} |y_1 y_2| d\mu_{\mathcal{X}^2} = \left( \int_{\mathcal{X}^1_\infty} |y| d\mu_{\mathcal{X}^1} \right)^2 = \tfrac{\mathbb{L}^2}{(\mathbb{L}+1)^2}.$$

If this formula were true for $p = l$, then it would follow that $\int_{\mathbb{Z}_p^2} |(x_1 + x_2)(x_1 - (p-1)x_2)| dx_1 dx_2 = \frac{p^2}{(p+1)^2}$, contradicting the following example.

**Example 5.33.** *Consider the linear mapping*

$$(x_1, x_2) \mapsto (x_1 + x_2, x_1 - (p-1)x_2) \colon \mathbb{Z}_p^2 \to \mathbb{Z}_p^2.$$

*It is easy to check that it is injective, that its image is* $\bigcup_{a=0}^{p-1} (a + p\mathbb{Z}_p)^2$, *and that its Jacobian is constant of absolute value* $1/p$. *Hence*

$$\int_{\mathbb{Z}_p^2} |(x_1 + x_2)(x_1 - (p-1)x_2)| dx_1 dx_2 = p \sum_{a=0}^{p-1} \int_{(a+p\mathbb{Z}_p)^2} |y_1 y_2| dy_1 dy_2.$$

*Now if* $a \neq 0$, $\int_{(a+p\mathbb{Z}_p)^2} |y_1 y_2| dy_1 dy_2 = (\int_{a+p\mathbb{Z}_p} dy)^2 = 1/p^2$, *whereas* $\int_{(p\mathbb{Z}_p)^2} |y_1 y_2| dy_1 dy_2 = (\int_{p\mathbb{Z}_p} |y| dy)^2 = (\int_{\mathbb{Z}_p} |y| dy - \int_{\mathbb{Z}_p^\times} dy)^2 = (p/(p+1) - (p-1)/p)^2 = 1/(p(p+1))^2$, *hence*

$$\int_{\mathbb{Z}_p^2} |(x_1 + x_2)(x_1 - (p-1)x_2)| dx_1 dx_2 = \tfrac{p^2+p-1}{(p+1)^2}.$$

# Part IV:
# Bibliography

# 6. References


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